

# Inferential Semantics

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## Abstract

Prawitz's views concerning inferences and their validity are examined in the light of opinions about general proof theory and proof-theoretic semantics inspired by categorial proof theory. The frame for these opinions, and for the ensuing examination of those of Prawitz, is provided by what has been qualified as a dogmatic position that proof-theoretic semantics shares with model-theoretic semantics.

*Keywords:* inference, deduction, proposition, name, consequence, truth, provability, validity, ground, proof-theoretic semantics, general proof theory, categorial proof theory

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## 1 Introduction

The expression that makes the title of this paper, “inferential semantics”, may be understood either as a synonym of “proof-theoretic semantics”, the

explanation of meaning through inference (see Section 1 below for references), or as the semantics of inferences, i.e. the explanation of the meaning of inferences, which for Prawitz consists primarily in defining the notion of validity of an inference. Both of these readings cover the subject-matter of the paper.

The goal is to examine Prawitz's views concerning inferences and their validity, views that are based on the notions of ground and operation. I will concentrate on a section of a rather recent paper of Prawitz [28], which I chose because it is self-contained and, I believe, fairly representative of his current views on proof-theoretic semantics and the semantics of inference.

My examination will be made in the light of some opinions I have concerning general proof theory and proof-theoretic semantics, which are inspired partly by categorical proof theory. I will spend more time in summarizing these opinions than in exposing those of Prawitz (much better known than mine). His opinions will not be examined before the last section. I have exposed the philosophical side of my opinions in a paper I have published rather recently [8], and I will repeat much of what is in that short paper. This is done in order not to oblige the reader to keep returning to that previous paper. Concerning more technical matters I cannot do better than give a number of references.

The exposition of these opinions I give here differs however from the preceding one in being organized with respect to what Schroeder-Heister in [32] and [30] qualified as a dogmatic position that proof-theoretic semantics shares with model-theoretic semantics (see Section 3 below). My opinions are shown to go against these dogmas. I did not know of this idea of Schroeder-Heister when I wrote [8], and I was happy to find how well what I said may be framed within the context provided by this idea. Besides two dogmas that I derive from [32] and [30], I add a third one, of the same kind, and with that I obtain a position from which, armed with a poignant terminology, I can examine Prawitz's opinions. Though he propounds a new proof-theoretic semantics, and though some of his ideas may point towards something else, his opinions are found to be in accordance with the dogmas.

## 2 Propositions, names and inferences

As a legacy of the last century, and to a certain extent of the nineteenth century too, the primary notion in the philosophy of language is nowadays

the notion of proposition. With propositions we perform what is considered, as part of the same legacy, the main act of language, the main act of speech: namely, we assert something. Concurrently, the key to a theory of meaning would be the quality propositions have when it is legitimate to assert them; namely, their correctness, which may be either their truth, as in classical semantics, or something of the same kind like their provability, as in the constructivist understanding of mathematics. From a classical semantical perspective, inspired by logical model theory, the central notion is the notion of truth. The motto that may be put over this legacy is Frege's principle from the introduction of the *Grundlagen der Arithmetik* "never to ask for the meaning of a word in isolation, but only in the context of a proposition" [16]. The notion of proposition is primary in what Dummett calls the order of explanation of the function of language (see [13], Chapter 1).

In the previous tradition, that should be traced back to Aristotle, the main act of speech and the key to a theory of meaning was in the act of naming. The primary notion in the philosophy of language was the notion of name. This old tradition is the one so forcibly called into question at the beginning of Wittgenstein's second book (see [35], Section 1). At the same place however Wittgenstein calls into question the primacy of asserting over other speech acts. It is usually taken that in his first book [34] Wittgenstein took asserting as the main speech act, but the *Tractatus* allows many, if not contradictory, readings, and it has been found in [23] that the young Wittgenstein may even be understood as a precursor of proof-theoretic semantics. The views of the later Wittgenstein are however closer to the ideas of this semantics, and it seems right to say that the young Wittgenstein is on the trace of these ideas to the extent that in his first book he anticipates the second.

Proof-theoretic semantics (the term is due to Schroeder-Heister; see [18]), which is inspired by Gentzen's work on natural deduction (and to a lesser degree sequent systems; see [17], in particular, Section II.5.13), covers ideas about meaning like those propounded by Prawitz and Dummett since the 1970s (see [26] and [13], Appendix to Chapter 12). This is primarily an approach to logical semantics that looks for the meaning of logical constants in the role that they play in inference. (Although I prefer the term *deduction*, following Prawitz's usage I will speak of *inference* in this text; the two terms should mean the same.) This logical semantics may lead to analogous ideas about meaning outside logic. The proof-theoretic semantic conceptions of

Prawitz and Dummett are constructivist; they stem from natural deduction for intuitionistic logic.

With proof-theoretic semantics, one could perhaps assume that the notion of inference should replace the notion of proposition as the primary notion of the philosophy of language. In the order of explanation of language, in the sense of Dummett, the notion of inference should now presumably precede that of proposition. The main act of language should not be any more the act of asserting, but the act of inferring (or deducing). This should perhaps be so, but it is not exactly so in the texts of Prawitz and Dummett.

### 3 Two dogmas of semantics

Categorical notions are those that are not hypothetical. (This use of *categorical* should not be confused with *categorial*, which, according to the *Oxford English Dictionary* [24], means “relating to, or involving, categories”; unfortunately, in mathematical category theory *categorical* dominates in the sense of *categorial*.) The best example we may give for the distinction, an example taken from our subject-matter, is with categorical and hypothetical proofs. The latter is a proof under hypotheses, while the former depends on no hypothesis.

Schroeder-Heister (together with P. Contu in [32], Section 4, in [30], Section 3, and in [31]) found that constructivist proof-theoretic semantics shares with classical semantics based on model theory two *dogmas*, which one may formulate succinctly as follows:

- (1) Categorical notions have primacy over hypothetical notions.
- (2) The correctness of the hypothetical notions reduces to the preservation of the correctness of the categorical ones.

The formulations of these dogmas in English in [30] and [31] is not the same as ours: Schroeder-Heister speaks of *consequence* in both dogmas, and in the second dogma he has *transmission* of correctness instead of *preservation* of correctness (which is not an essential difference); he does not speak about the correctness of consequence, but just about consequence. Our formulations are however close enough.

The second dogma may be understood as a corollary of the first one, and Schroeder-Heister spoke first of a single dogma. In [31] he speaks however

of two dogmas, and adds a third one, which we will mention in Section 6 below (where another third dogma is discussed). It is preferable (as Quine suggests) to have two dogmas, which makes referring to them easier.

The first dogma (1) is accepted if we take the notion of proposition, a categorical notion, to have primacy over the notion of inference, a hypothetical notion. Let us also illustrate the second, preservation, dogma (2). In the classical, model-theoretical, case the correctness, i.e. validity, of a consequence relation, which is something hypothetical, is defined in terms of the preservation of the correctness, i.e. truth, of propositions, which are categorical. In the constructivist case, a correct hypothetical proof should preserve categorical provability when one passes from the hypotheses to the conclusion.

## 4 Concerning the first dogma

In [8] I have argued myself against the first dogma (1) in general proof theory, and I will repeat here what I said on that matter. The first dogma is present in general proof theory in the Curry-Howard correspondence, where we find typed lambda terms  $t$  as codes of natural deduction derivations. If  $t$  codes the derivation that ends with the formula  $B$  as the last conclusion, then this may be written  $t: B$ , and we say that  $t$  is of *type*  $B$ . (Formulae are of course of the grammatical category of propositions.) Our derivation may have uncanceled hypotheses. That will be seen by  $t$ 's having possibly a free variable  $x$ , which codes an occurrence of a formula  $A$  as hypothesis; i.e. we have  $x: A$ , an  $x$  of type  $A$ .

All this makes conclusions prominent, while hypotheses are veiled. Conclusions are clearly there to be seen as types of terms, while hypotheses are hidden as types of free variables, which are cumbersome to write always explicitly when the variables occur as proper subterms of terms. The desirable terms are closed terms, which code derivations where all the hypotheses have been cancelled. These closed terms are supposed to play a key role in understanding intuitionistic logic. The categorical has precedence over the hypothetical.

An alternative to this coding would be a coding of derivations that would allow hypotheses to be as visible as conclusions, and such an alternative coding exists in categorial proof theory. There one writes  $f: A \vdash B$  as a code

for a derivation that starts with premise  $A$  and ends with conclusion  $B$ . The arrow term  $f$  is an arrow term of a category (a *cartesian closed category* if we want to cover the conjunction-implication fragment of intuitionistic logic; see [21], Part I). The *type* of  $f$  is now not a single formula, but an ordered pair of formulae  $(A, B)$ . The notation  $A \vdash B$  serves just to have something more suggestive than  $(A, B)$ . (In categories one usually writes  $f: A \rightarrow B$  instead of  $f: A \vdash B$ , but  $\rightarrow$  is sometimes used in logic for implication, and we should not be led to confuse inference with implication just because of notation.)

If  $B$  happens to be derived with all hypotheses cancelled, then we will have  $f: \top \vdash B$ , with the constant formula  $\top$  standing for the empty collection of hypotheses. If we happen to have more than one hypothesis, but as usual a finite number of them, then we will assume that with the help of conjunction all these hypotheses have been joined into a single hypothesis. So the categorial notation  $f: A \vdash B$  with a single premise does not introduce a cramping limitation; at least not for the things intended to be said here.

The typed lambda coding of the Curry-Howard correspondence, involving finite product types and functional types, and the categorial coding in cartesian closed categories are equivalent in a very precise sense. This has been first shown by Lambek (see [20], [21], [2] and [5]). The import of the two formalisms is however not exactly the same. The typed lambda calculus suggests something different about the subject matter than category theory. It makes prominent the *proofs*  $t: B$ —and we think immediately of the categorial ones, without hypotheses—while category theory is about the *inferences*  $f: A \vdash B$ .

Another asymmetry is brought by the usual format of natural deduction, where there can be more than one premise, but there is always a single conclusion. This format favours intuitionistic logic, and in this logic the coding with typed lambda terms works really well with implication and conjunction, while with disjunction there are problems.

The categorial coding of derivations allows hypotheses to be treated on an equal footing with conclusions also with respect to multiplicity. With such a coding we could hope to deal too with classical logic, with all its Boolean symmetries, and with disjunction as well as with conjunction.

As an aside, let us note that the asymmetry of natural deduction with respect to premises and conclusions is most unfortunate when one has to formulate precisely what Prawitz calls the Inversion Principle (see [25], Chapter II), which, following Gentzen's suggestion of [17](Section II.5.13), relates the

introduction and elimination rules for the logical constants. Dummett encounters analogous problems with his principle of harmony (see [15]).

With Gentzen's plural (or multiple-conclusion) sequents we overcome this asymmetry, and we may formulate rules for the logical constants as *double-line rules*, i.e. invertible rules, going both ways, from the premises to the conclusion and back. The inversion of the Inversion Principle is now really inversion. I believe Gentzen was aware of double-line rules, because they are implicit in [19], a doctoral thesis he supervised. (Older references concerning these rules are listed in [1], Section 25, and [3], end of Section 7; see also [29].)

What one will not realize by passing simply to sequents, without introducing the categorial coding mentioned above, is that with double-line rules we have just a superficial aspect of adjoint situations. This matter, which explains how the Inversion Principle is tied to a most important mathematical phenomenon, would however lead us too far afield, and I will here just list some references: [22], [4], [5] (see in particular Section 9), [7] (Section 4), [11] (Section 7) and [12] (Section 1.4).

Let us return now to the dogmas of semantics. The first dogma (1) is manifested also in the tendency to answer the question what is an inference by relying on the notion of proposition as more fundamental. It is as if Frege's recommendation from the *Grundlagen der Arithmetik* to look after meaning in the context of a proposition (see Section 2) was understood to apply not only to bits of language narrower than propositions, which should be placed in the broader propositional context, but also to something broader than propositions, as inferences, in which propositions partake, which should be explained in terms of the narrower notion of proposition.

An inference is usually taken as something involving propositions. Restricting ourselves to inferences with single premises, as we agreed to do above, for ease of exposition, we may venture to say that an inference consists in *passing* from a proposition called premise to a proposition called conclusion.

What could "passing" mean here? Another principle of Frege from the introduction of the *Grundlagen der Arithmetik* would not let us understand this passing as something happening in our head. Such an understanding would expose us to being accused of psychologism. No, this passing should be something objective, something done or happening independently of any particular thinking subject, something sanctioned by language and the meaning it has.

The temptation of psychologism is particularly strong here, but as a proposition is not something mental that comes into being when one asserts a sentence, so an inference should not be taken as a mental activity of passing from sentences to sentences or from propositions to propositions. Such a mental activity exists, as well as the accompanying verbal and graphical activities, but the inference we are interested in is none of these activities. It is rather something in virtue of which these activities are judged to be correctly performed or not. It is something tied to rules governing the use of language, something based on these rules, which are derived from the meaning of language, or which confer meaning to it.

When inspired by categorial proof theory we reject the first dogma (1) we do not take that categorial and hypothetical notions are on an equal footing, but we give priority to the hypothetical notions. This is related to the priority that category theory gives to arrows over objects. When in the category with sets as objects and functions as arrows, which is the paradigmatic example of a category, one has to explain what is an ordered pair, i.e. one has to characterize the operation of cartesian product on sets, one does not look *inside* the cartesian product of two sets, but one characterizes cartesian product from the *outside*. This may be achieved in terms of some special functions—namely, the projection functions—next, in terms of an operation on functions—namely, the operation of pairing that applied to the functions  $f: C \rightarrow A$  and  $g: C \rightarrow B$  gives the function  $\langle f, g \rangle: C \rightarrow A \times B$ —and, finally, in terms of equations between functions concerning these special functions and operation on functions (see [21], Section I.3, and [9], Section 9.1).

In categorial proof theory inferences are taken as arrows and propositions as objects, and inferences have priority over propositions. When one has to characterize the connective of conjunction, one may do it in terms of the inferences of conjunction elimination, which correspond to projection functions, in terms of the rule of conjunction introduction, which corresponds to the pairing operation on functions, and in terms of equations between inferences concerning implication elimination and implication introduction. These equations, the same as the equations of cartesian product mentioned above, correspond to  $\beta$  and  $\eta$  reduction rules, which one encounters in reductions to normal form in natural deduction and sequent systems.

## 5 Concerning the second dogma

We shall now consider matters, taken again from [8], that lead to dissent with the second dogma of semantics (2).

Can inferences be reduced to consequence relations? So that having an inference from  $A$  to  $B$  means just that  $B$  is a consequence of  $A$ . This would square well with the objective character of inferences we have just talked about, because  $B$ 's being a consequence of  $A$  is something objective. "Consequence" here can be understood as semantical consequence, and the objectivity of consequence would have semantical grounds.

Since  $B$  is a consequence of  $A$  whenever the implication  $A \rightarrow B$  is true or correct, there would be no essential difference between the theory of inference and the theory of implication. An inference is often written vertically, with the premise above the conclusion,

$$\frac{A}{B}$$

and an implication is written horizontally  $A \rightarrow B$ , but besides that, and purely grammatical matters, there would not be much difference.

This reduction of inference to implication, which squares well with the second dogma of semantics, is indeed the point of view of practically all of the philosophy of logic and language in the twentieth century. This applies not only to classically minded theories where the essential, and desirable, quality of propositions, their correctness, is taken to be truth, but also to other theories, like constructivism in mathematics, or verificationism in science, where this quality is something different. It may be deemed strange that even in constructivism, where the quality is often described as *provability*, inferences are not more prominent. Rather than speak about inferences, constructivists, such as intuitionists, tend to speak about something more general covered by the portmanteau word "construction". (Constructions produce mathematical objects as well as proofs of mathematical propositions, which are about these objects.) Where above we spoke about *passing*, a constructivist would presumably speak about *constructions*.

By reducing inferences from  $A$  to  $B$  to ordered pairs  $(A, B)$  in a consequence relation we would lose the need for the categorial point of view. The  $f$  in  $f: A \vdash B$  would become superfluous. There would be at most one arrow with a given source and target, which means that our categories would be

preordering relations (i.e. reflexive and transitive relations). These preorderings are consequence relations.

With that we would achieve something akin to what has been achieved for the notion of function. This notion has been *extensionalized*. It has been reduced to a set of ordered pairs. If before one imagined functions as something like a *passing* from an argument to the value, now a function is just a set of ordered pairs made of arguments and values. Analogously, inferences would be the ordered pairs made of premises and conclusions.

The extensionalizing of the notion of inference which consists in its reduction to the notion of consequence relation can be called into question if we are able to produce examples of two different inferences with the same premise and the same conclusion. Here is such an example of formal, logical, inferences, which involve conjunction, the simplest and most basic of all logical connectives.

From  $p \wedge p$  to  $p$  there are two inferences, one obtained by applying the first rule of conjunction elimination, the first projection rule,

$$\frac{A \wedge B}{A}$$

and the other obtained by applying the second projection rule

$$\frac{A \wedge B}{B}$$

This and other such examples from logic redeem the categorial point of view (see [9] and [10]). In this example we have  $\pi^1: p \wedge p \vdash p$  and  $\pi^2: p \wedge p \vdash p$  with  $\pi^1 \neq \pi^2$ .

A category where these arrows are exemplified is  $\mathcal{C}$ , which is the category with binary product freely generated by a set of objects. The category  $\mathcal{C}$  models inferences involving only conjunction. It does so for both classical and intuitionistic conjunction, because the inferences involving this connective do not differ in the two alternative logics. This is a common ground of these two logics.

The notion of binary product codified in  $\mathcal{C}$  is one of the biggest successes of category theory. The explanation of the extremely important notion of ordered pair in terms of this notion is the most convincing corroboration of the point of view that mathematical objects should be characterized only up to isomorphism. It is remarkable that the same matter should appear at

the very beginning of what category theory has to say about inferences in logic, in connection with the connective of conjunction (see the end of the preceding section).

For the category  $\mathcal{C}$  there exists a kind of completeness theorem, which categorists call a *coherence* result. There is namely a faithful functor from  $\mathcal{C}$  to the model category that is the opposite of the category of finite ordinals with functions as arrows. With this functor,  $\pi^1$  and  $\pi^2$  above correspond respectively to

$$\pi^1 \quad \begin{array}{c} p \wedge p \\ \searrow \\ p \end{array} \qquad \pi^2 \quad \begin{array}{c} p \wedge p \\ \nearrow \\ p \end{array}$$

Another example of two different formal inferences with the same premise and the same conclusion, which involves graphs of a slightly more complicated kind, is given by

$$\begin{array}{c} p \wedge (p \rightarrow p) \\ \searrow \\ p \end{array} \qquad \begin{array}{c} p \wedge (p \rightarrow p) \\ \frown \\ p \end{array}$$

The first inference is made by conjunction elimination, while the second by modus ponens.

Coherence is one of the main inspirations of categorial proof theory (see [9]). The other, older, inspiration, which works for inferences in intuitionistic logic, comes ultimately from the notion of adjunction (for which we gave references in Section 4).

In model categories such as we find in coherence results we have models of equational theories axiomatizing identity of inferences. These are not models of the theorems of logic. The arrows of the model categories are however hardly what inferences really are. It is not at all clear that these categories provide a real semantics of inferences (cf. [7]).

It is not clear what, from the point of view of proof-theoretical semantics, should be the semantics of inferences, i.e. the explanation of meaning of inferences. If inferences provide meaning, how can their meaning be reduced to something more primitive, in the style of the dogmas?

Invoking now another principle of Frege’s *Grundlagen der Arithmetik*, we might look for an answer to the question “What is an inference?” by looking for a criterion of identity of inferences. Prawitz introduced in [26] the field of general proof theory with that question. (It is remarkable that the same question was raised by Hilbert in his discarded 24th problem; see [33].)

We would strive to define a significant and plausible equivalence relation on derivations as coded by arrow terms of our syntactical categories, and equivalence classes of derivations, or equivalence classes of arrow terms, which are the arrows of our syntactical categories, would stand for inferences.

An inference  $f: A \vdash B$  would be something *sui generis*, that does not reduce to its type, the ordered pair  $(A, B)$ . It would be represented by an arrow in a category, to which is tied a criterion of identity given by the system of equations that hold in the category. The category should not be a preorder.

In the arrow term  $f$  of  $f: A \vdash B$ , the inference rules involved in building an inference are made manifest as operations for building this term. The theory of inference is as a matter of fact the theory of such operations (usually partial). It is an algebraic theory codifying with equations the properties of these operations.

It is rather to be expected that the theory of inference should be the theory of inference rules, as arithmetic, the theory of natural numbers, is the theory of arithmetic operations (addition, multiplication etc.). Extensionalizing the notion of inference by reducing it to consequence (as in the preceding section) makes us forget the inference rules, which are prominent in categorial proof theory.

From a classical point of view, the desirable quality of propositions, their correctness, is their truth. If the notion of inference is something *sui generis*, not reducible to the notion of proposition, why should the desirable quality of inferences, their correctness, be reducible to the desirable quality of propositions?

If we abandon the second dogma of semantics, a correct inference would not be just one that preserves truth—a correct consequence relation could be that. An inference  $f: A \vdash B$  is not just  $A \vdash B$ ; we also have the  $f$ . As a matter of fact, the inference is  $f$ . It may be a necessary condition for a correct  $f: A \vdash B$  that  $B$  be a consequence of  $A$ , but this is not sufficient for the correctness of  $f$ . This is not what the correctness of  $f$  consists in. The correctness of an inference would be, as the notion of inference itself,

something *sui generis*, not to be explained in terms of the correctness of propositions.

We might perhaps even try to turn over the positions, and consider that the correctness of propositions should be explained in terms of the correctness of inferences. This is presumably congenial to a point of view like that found in intuitionism, where the correctness of propositions is taken to be provability, i.e. deducibility from an empty collection of premises. We could however take an analogous position with a classical point of view, where the *truth* of analytic propositions would be guaranteed by the correctness of some inferences (cf. [14], p. 26). The correctness of the inferences underlies the truth, and not the other way round.

## 6 A third dogma of semantics

In [31] Schroeder-Heister considers a third dogma of semantics, which he formulates as follows:

Consequence means the same as correctness of inference.

To understand properly this and the alternative view of proof-theoretic semantics Schroeder-Heister proposes would require entering into the technical notions of [31], where the matter is presented with respect to an understanding of consequence within the framework of a specific theory of definition. (Among these notions are definitional clause, definitional closure, definitional reflection, and others.) We will not do that here, and will not try to determine how much Schroeder-Heister's views and terminology accord with ours. Our purpose is not that, but to examine Prawitz's views in the light of the two dogmas previously formulated, and a third one, which, though maybe related, is not Schroeder-Heister's. Before I try to formulate this third dogma, which I think constructivist proof-theoretic semantics shares with classical model-theoretic semantics, some explanations are needed.

In classical semantics, it is not the categorical notion of proposition that is really central. Its place is taken by the notion of correctness of a proposition, namely *truth*. Similarly, the hypothetical notion of consequence is really less central than the notion of correctness of a consequence, i.e. its *validity*, in terms of which consequence is defined. The notion of consequence reduces to its validity.

In proof-theoretic semantics the situation is not much different. The notion of proposition seems to be less central than its correctness, namely *provability*, and for consequences, or inferences (if these two notions are distinguished), one concentrates again on their validity.

I will formulate the third dogma of semantics as follows:

- (3) The notions of correctness of the notions mentioned in (2) are more important than these notions themselves.

The third dogma stems probably from the old, venerable and general, denotational perspective on semantics, which Wittgenstein has criticized at the beginning of the *Philosophical Investigations* (mentioned in Section 2 above). As names stand for objects, as they refer to them, so propositions stand for truth-values, and the correct ones stand for truth, i.e. the value *true*. They need not exactly behave as names (as the later Frege thought), and they do not exactly refer to the truth-values, but still they somehow stand for them, in a manner not quite foreign to the manner in which a name stands for the object it denotes. The insistence on validity when speaking of consequence should have the same denotational roots.

I think that the influence of the third dogma on proof-theoretic semantics is more pernicious when we talk of inferences than when we talk of propositions. The notion of inference should be here more important than the notion of proposition, and the notion of inference itself should be more important than the notion of correctness of an inference, which Prawitz calls validity, and on whose definition he has worked for a long time.

It is questionable whether in proof-theoretic semantics we need at all the notion of validity of an inference. What I have just said may be surprising. How come—I will be told—that valid inferences are not our subject matter? I would reply that invalid inferences do not exist in a certain sense.

Some people may pass from  $A$  to  $B$  when this is not sanctioned by an inference  $f: A \vdash B$ , and we might say that we have here an invalid inference, but this is only a manner of speaking, and not a good one, because it leads us astray. Invalid inferences may exist as interesting psychological entities, but in logic they cannot have this status. Instead of saying that we have an invalid inference from  $A$  to  $B$ , we should say that for the type  $A \vdash B$  we have no inference. It is as if all inferences are valid, and since they are all valid, their validity need not be mentioned expressly.

To make an analogy, illegal chess moves may exist in the physical world, but they do not exist in the world of chess. To characterize all the legal chess moves is to characterize all the possible moves. Impossible moves are excluded.

The situation is similar with the notion of formula. There used to be a time when logicians spoke much of *well-formed* formulae, as if there were formulae that are not well formed. This bad usage has, fortunately, died out, and nowadays the attribute “well-formed” is much less often applied to formulae. A formula that is not well formed is better characterized as a word in our alphabet that is not a formula. All formulae are well formed. There are no other formulae in the perspective of logic. There might be a badly formed word an individual would take wrongly for a formula, and call so. This is something that might perhaps be interesting for psychology, but need hardly concern us as logicians. The notion of *well-formedness* of a formula is hardly something separate from a formula, which could serve to make an important semantical point. A semantical theory based on the abstract notion of well-formedness seems as suspect as a physiological theory concerning opiates based on the notion of *virtus dormitiva*.

I surmise that the notion of validity of an inference is very much like the notion of well-formedness of a formula. A valid inference is like a well-formed syntactical object. In logic, derivations, like other syntactical objects, are defined inductively, and inferences are equivalence classes of derivations.

A term  $f$ , of type  $A \vdash B$ , is defined inductively as well, and usually it stands only for valid inferences, which yields that the implication  $A \rightarrow B$  is correct. We may envisage, for technical reasons, into which we cannot go here (see [6], Section 7, [9], Section 1.6, and Chapters 12 and 13), arrow terms  $f: A \vdash B$ , where  $A \rightarrow B$  is not correct, but this is not what is usually done.

Even if we assume that the notion of validity of an inference is legitimate and, by following the second dogma, assume that this validity is to be defined in terms of the correctness of propositions, which amounts to the correctness of implications, why would this oblige us to follow the third dogma too, and take that the notion of validity of an inference should be more central in our proof-theoretic semantical theory than the notion of inference itself? It seems that the second dogma need not entail the third dogma, but since they probably have similar roots, those following one would be inclined to follow the other.

## 7 Inferences as operations on grounds

The title of the present section is taken from the title of Section 7 of Prawitz’s paper [28]. Our goal is to examine Prawitz’s views expressed at that particular place in the light of our critique of the three dogmas of semantics. As I said in the introduction, I chose that particular paper, and that particular section, because it is a fairly recent and self-contained piece, which is I believe representative of his current views (he may have written similar things elsewhere).

Prawitz’s goal in that section is to reconsider the concept of inference “to get a fresh approach to the concept of valid inference”. In the second paragraph Prawitz mentions the “intuitive understanding of an inference as consisting of something more than just a conclusion and some premises”. This is something that accords well with the critique from Section 5 above of the extensionalizing of inference by its reduction to consequence.

Prawitz says that his “main idea is thus to take an inference as given not only by its premises and conclusion (...), but also by an operation defined for grounds for the premises”. Prawitz does not define the term “operation” more precisely, and it is not clear at the beginning that he has something very mathematical in mind here, but it is natural to suppose that these operations correspond to our arrows  $f$  from Section 4. Prawitz’s first example of such an operation is mathematical induction. The other examples, which we will mention below, involve rules for conjunction.

The term *ground for a sentence* is more difficult to understand (*sentence* here should mean what we meant by *proposition* throughout this text). It is something very wide (approaching in the width of its scope the term *construction* of the intuitionists). Prawitz says it is “what a person needs to be in possession of in order to be justified in holding the sentence true”. Although at the beginning of the section he suggests that grounds are premises—hence sentences, i.e. propositions as we would say—he concludes that “the premises from which a conclusion is inferred do not constitute grounds for the conclusion—rather the premises have their grounds, and it is by operating on them that we get a ground for the conclusion”. Grounds are perhaps not linguistic at all. Could they be, for example, sense perceptions? But how does one operate with inferences on sense perceptions?

Prawitz does not mention the Curry-Howard correspondence in the present context, but his exposition accords rather well with its assumptions, includ-

ing the first dogma of semantics (1). The categorical is primary, and a ground is like a typed lambda term coding a natural deduction derivation. Even if the term is not closed, it is of a different kind from the operation on typed lambda terms that stands for an inference. Examples of such operations are surjective pairing and projections, which Prawitz mentions on pp. 21-22, without calling them so. (In that context he gives two equations for inferences—he says grounds—which may be understood as the equations of the lambda calculus with product types derived from  $\beta$ -reduction; there are corresponding equations of categories with products.)

The alternative point of view of categorial proof theory would take grounds to be of the same kind as the operations that correspond to inferences (intuitionists would use the word “construction” for both). Instead of Prawitz’s ground  $t: A$ , i.e. ground  $t$  for  $A$ , to which we apply the inference, i.e. operation,  $o: A \vdash B$  in order to get the ground  $o(t): B$ , we would have the inference  $f: \top \vdash A$ , which we compose with the inference  $o: A \vdash B$  in order to get the inference  $o \circ f: \top \vdash B$ . We would understand grounds as something inferential too. They may be distinguished from the hypothetical inferences by their type—they have a  $\top$  on the left—but otherwise they are of the same nature. (One may treat axioms of logical systems just as rules with no premises.) Operations on grounds correspond to inferences, but we have also to deal with rules of inference that correspond to operations on inferences (see the ends of Sections 4 and 5 above).

Speaking about *individual inferences* Prawitz expresses some quite psychologistic views, but he soon moves to *inference forms*, in which psychologistic ingredients disappear. His *inference figures*, or *schemata*, or *schemes*, which “abstract from the operation left in an inference form”, are something like consequence; we would say they are the *type*  $A \vdash B$  of an inference obtained from  $o: A \vdash B$  by forgetting the  $o$ .

Then Prawitz gives (on p. 19) a characterization of the validity of an inference in which the basic notion is validity of an individual inference:

“An individual inference is valid if and only if the given grounds for the premises are grounds for them and the result of applying the given operation to these grounds is in fact a ground for the conclusion.”

The notion of individual inference is psychologistic, but the definition of its validity does not seem to depend very much on that, and the characteriza-

tion of the validity of an inference form is essentially the same, except that instances are mentioned. An inference figure  $A \vdash B$  is taken to be valid when there is an  $o$  such that  $o: A \vdash B$  is a valid inference form.

Prawitz's discussion is intermingled with epistemological considerations, which are very important for his concerns. (A paper contemporaneous with [28] where this is even more clear is [27].) I think this has to do with the psychologistic ingredients in his position.

Prawitz's opinions accord well with the second dogma of semantics (2). His characterization of validity of an inference, sketched above, as well as his older views upon that matter, clearly give precedence to something tied to propositions, here called grounds. The validity of an inference consists in the preservation of *groundedness*.

I think that Prawitz's opinions accord pretty well with the third dogma of semantics (3), too. From a semantical perspective, the notion of validity of an inference seems to be for him more important than the notion of inference. The main task of semantics seems to be for him the definition of this validity.

He makes what may be interpreted as a move away from the dogmas by taking that an inference is an operation, understood non-extensionally. Take a typical operation like addition of natural numbers. One may speak about the correctness of an individual application of addition, an individual performance of it by a human being (or perhaps a machine), but one does not usually speak about the correctness of the abstract notion of addition, independently of its applications. How would correct addition differ from addition tout court? One does not speak either about the validity of addition. What would that be? Were it not for his psychologistic inclinations, we could then surmise that the understanding of inferences as operations might have led Prawitz to question the third, and the other dogmas, too.

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