

A Prologue to the Theory of Deduction

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Abstract

This is an introduction to general proof theory in terms of some philosophical considerations. The central problem of this theory is the problem of identity criteria for deductions. When it is treated in categorial proof theory, this problem is closely tied to categorial coherence results, which may be understood as completeness results for systems of equations between deductions with respect to simple model categories. With the help of the categorial notion of isomorphism, the categorial perspective enables us to formulate in precise mathematical terms criteria for propositional identity.

Keywords: general proof theory, proof, deduction, consequence relation, implication, identity of deductions, propositional identity, isomorphism of propositions

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1 Introduction

The central question of general proof theory as defined by Prawitz in [23] may be understood as the question “What is a deduction?”, and the central problem of this theory is the problem of identity criteria for deductions. This problem is treated either in the typed lambda calculus or in terms of category theory. These two approaches are up to a point convergent, the latter one providing sometimes a deeper perspective. The categorial approach to proof theory—categorial proof theory—is also closely tied to what in category theory is called coherence results, which may be understood as completeness results for systems of equations between deductions with respect to manageable models, often of a graphical kind. With the help of the categorial notion of isomorphism, the categorial perspective enables us to formulate in precise mathematical terms criteria for propositional identity, i.e. identity of meaning for propositions. This paper consists in some introductory philosophical considerations concerning these matters of general proof theory.

2 Proofs and deductions

The word “proof” may refer to a proof with no hypothesis (or assumption). This is what this word does in its marked usage. In its unmarked usage, it may refer also to a hypothetical proof, i.e. a proof from hypotheses. (In the terminology of the Prague Linguistic Circle, “fox” in its *marked* usage refers to a male animal, the male partner of a vixen, while in its *unmarked* usage it refers to a male or female animal.)

General proof theory ought to be concerned with proofs in the unmarked usage. In order to make this clear, it is wiser to opt for the synonymous term “deduction”, which cannot be mistaken for “proof” in the marked usage. So general proof theory ought to be about deductions.

That this precaution is not exaggerated is corroborated by the fact that proof-theoretical investigations of intuitionistic logic are inclined to understand “proof” in the marked usage. This is shown by the Curry-Howard correspondence.

There we find typed lambda terms t as codes of natural deduction derivations. If t codes the derivation that ends with the formula B as the last conclusion, then this may be written $t : B$, and we say that t is of *type* B .

(Formulae are of course of the grammatical category of propositions.) Our derivation may have uncanceled hypotheses. That will be seen by t 's having possibly a free variable x , which codes an occurrence of a formula A as hypothesis; i.e. we have $x : A$, an x of type A .

All this makes conclusions prominent, while hypotheses are veiled. Conclusions are clearly there to be seen as types of terms, while hypotheses are hidden as types of free variables, which are cumbersome to write always explicitly when the variables occur as proper subterms of terms. The desirable terms are closed terms, which code derivations where all the hypotheses have been cancelled. These closed terms are supposed to play a key role in understanding intuitionistic logic.

Another asymmetry is brought by the usual format of natural deduction, where there can be more than one premise, but there is always a single conclusion. This format favours intuitionistic logic, and in this logic the coding with typed lambda terms works really well with implication and conjunction, while with disjunction there are problems.

An alternative to this coding would be a coding of derivations that would allow hypotheses to be as visible as conclusions, and to be treated on an equal footing also with respect to multiplicity. With such a coding we could hope to deal too with classical logic, with all its Boolean symmetries, and with disjunction as well as with conjunction.

Such an alternative coding exists in categorial proof theory. There one writes $f : A \vdash B$ as a code for a derivation that starts with premise A and ends with conclusion B . The arrow term f is an arrow term of a category—a *cartesian closed category* if we want to cover the conjunction-implication fragment of intuitionistic logic. The *type* of f is now not a single formula, but an ordered pair of formulae (A, B) . The notation $A \vdash B$ serves just to have something more suggestive than (A, B) . (In categories one usually writes $f : A \rightarrow B$ instead of $f : A \vdash B$, but \rightarrow is sometimes used in logic for implication, and we should not be led to confuse deduction with implication just because of notation; cf. Section 4 below.)

If B happens to be derived with all hypotheses cancelled, then we will have $f : \top \vdash B$, with the constant formula \top standing for the empty collection of hypotheses. If we happen to have more than one hypothesis, but as usual a finite number of them, then we will assume that with the help of conjunction all these hypotheses have been joined into a single hypothesis. So the categorial notation $f : A \vdash B$ with a single premise does not introduce

a cramping limitation; at least not for the things intended to be said here.

The typed lambda coding of the Curry-Howard correspondence, involving finite product types and functional types, and the categorial coding in cartesian closed categories are equivalent in a very precise sense. This has been first shown by Lambek (see [20], [21], [7] and [9]). The import of the two formalisms is however not exactly the same. The typed lambda calculus may suggest something different about the subject matter than category theory. It makes prominent the *proofs* $t: B$ —and we think immediately of the marked ones, without hypotheses—while category theory is about the *deductions* $f: A \vdash B$.

Logic is concerned not with any deductions, but first of all with *formal* deductions. Perhaps it is concerned only with such deductions. These are deductions that hold in virtue of the *form* exhibited by keeping constant some particular expressions—the logical constants (the standard list is made of the connectives and quantifiers of first-order logic plus the binary predicate of equality)—and replacing everything else by variables, of appropriate grammatical categories. The expressions of the grammatical category of propositions that exhibit this form are called *formulae*.

If general proof theory is described as the theory of formal deduction, it might seem it would cover the whole field of logic, as traditionally conceived. Formal deductions are however hardly the subject matter of model theory, computability theory and set theory. (In [17] it is found that model theory could fall under the heading of *theory of definition*, which should be a traditional concern of logicians.) The theory of formal deduction would not cover the whole field of logic as it is conceived since the last century. Once it is properly developed, it could perhaps pretend to occupy a place more central than today.

3 Deductions in general

Following Prawitz's suggestion in [23], the primary concern of general proof theory should then be the question "What is a deduction?". Let us consider this question in all its generality. So we are not concerned only with formal deductions, where premises and conclusions are formulae, but with any deductions, where premises and conclusions are propositions. What is then a deduction in general, in this broad sense?

There is a strong tendency to answer this question by relying on the notion of proposition as more fundamental. It is as if Frege's recommendation from the *Grundlagen der Arithmetik* to look after meaning in the context of a proposition was understood to apply not only to bits of language narrower than propositions, which should be placed in the broader propositional context, but also to something broader than propositions, as deductions, in which propositions partake, which should be explained in terms of the narrower notion of proposition.

A deduction is usually taken as something involving propositions. Restricting ourselves to deductions with single premises, as we agreed to do above, for ease of exposition, we may venture to say that a deduction consists in *passing* from a proposition called premise to a proposition called conclusion.

What could "passing" mean here? Another teaching of Frege from *Grundlagen der Arithmetik* would not let us understand this passing as something happening in our head. Such an understanding would expose us to being accused of psychologism. No, this passing should be something objective, something done or happening independently of any particular thinking subject, something sanctioned by language and the meaning it has.

The temptation of psychologism is particularly strong here, but as a proposition is not something mental that comes into being when one asserts a sentence, so a deduction should not be taken as a mental activity of passing from sentences to sentences or from propositions to propositions. Such a mental activity exists, as well as the accompanying verbal and graphical activities, but the deduction we are interested in is none of these activities. It is rather something in virtue of which these activities are judged to be correctly performed or not. It is something tied to rules governing the use of language, something based on these rules, which are derived from the meaning of language, or which confer meaning to it.

A deduction could be compared to a move in a game like chess. We envisage here possible moves in a particular situation on the board in a particular game. Such moves are not moves in the physical world made with the help of a brain and fingers. Moves in the physical world may also be incorrectly performed. We are however concerned with the correct abstract moves, sanctioned by the rules of the game, whose existence is objective, and which do not owe anything to particular chess players. There are such moves that no chess player has ever envisaged.

This chess terminology suggests saying, as a Wittgenstein would perhaps do, that deduction is a *move* in a language game. However, “move” does not fare much better than “passing”. One smacks of psychologism, but the other is very metaphorical, and neither is very explanatory.

4 Deductions and consequence relations

Can deductions be reduced to consequence relations? So that having a deduction from A to B means just that B is a consequence of A . This would square well with the objective character of deductions we have just talked about, because B 's being a consequence of A is something objective. “Consequence” should presumably be understood here in a syntactical manner, but because of the completeness of classical first-order logic it could even mean semantical consequence. If our view of logic is dictated by semantics, as it is understood in model-theoretical terms, then the objectivity of consequence has semantical grounds.

Since B is a consequence of A whenever the implication $A \rightarrow B$ is true or correct, there would be no essential difference between the theory of deduction and the theory of implication. A deduction is often written vertically, with the premise above the conclusion,

$$\frac{A}{B}$$

and an implication is written horizontally $A \rightarrow B$, but besides that, and purely grammatical matters, there would not be much difference.

This reduction of deduction to implication accords well with the point of view, which we mentioned above, where propositions are taken as a more fundamental notion. And this is indeed the point of view of practically all of the philosophy of logic and language in the twentieth century. Propositions do not only play the leading role in language, but everything is reduced to them.

This applies not only to classically minded theories where the essential, and desirable, quality of propositions is taken to be truth, but also to other theories, like constructivism in mathematics, or verificationism in science, where this quality is something different. It may be deemed strange that even in constructivism, where the quality is often described as *provability*,

deductions are not more prominent. Rather than speak about deductions, constructivists, such as intuitionists, tend to speak about something more general covered by the magical word “construction”. (Constructions produce mathematical objects as well as proofs of mathematical propositions, which are about these objects.) Where above we spoke about *passing* and *moves*, a constructivist would presumably speak about *constructions*.

By reducing deductions from A to B to ordered pairs (A, B) in a consequence relation we would lose the need for the categorial point of view. The f in $f: A \vdash B$ would become superfluous. There would be at most one arrow with a given source and target, which means that our categories would be preordering relations (i.e. reflexive and transitive relations). These pre-orderings are consequence relations.

With that we would achieve something akin to what has been achieved for the notion of function. This notion has been *extensionalized*. It has been reduced to a set of ordered pairs. If before one imagined functions as something like a *passing* from an argument to the value, or a *move* from an argument to the value, now a function is just a set of ordered pairs made of arguments and values. Analogously, deductions would be the ordered pairs made of premises and conclusions.

This extensionalizing might represent a progress, but it has left in its vicinity, in the theory of computing, an unresolved matter concerning the notion of algorithm. While it might be desirable to extensionalize the notion of function, is it desirable in the same manner to extensionalize the notion of algorithm? Are all algorithms with the same input and output the same? Common sense says no, but no developed or widely accepted theory exists concerning this matter (see [22] and [2]).

Gentzen’s results in general proof theory concerning the normal forms of natural deduction and sequent derivations demonstrate the technical advantage of taking deductions seriously. Not so much as something irreducible to a consequence relation, but as something vertical, certainly irreducible to the horizontal implication.

5 Identity of deductions

The extensionalizing of the notion of deduction which consists in its reduction to the notion of consequence relation can be called into question if we are

able to produce examples of two different deductions with the same premise and the same conclusion. Here are first two such examples of formal, logical, deductions, which involve conjunction, the simplest and most basic of all logical connectives.

From $p \wedge p$ to p there are two deductions, one obtained by applying the first rule of conjunction elimination, the first projection rule,

$$\frac{A \wedge B}{A}$$

and the other obtained by applying the second projection rule

$$\frac{A \wedge B}{B}$$

The other example is given by the two deductions from $p \wedge p$ to $p \wedge p$, one being the identity deduction, an instance of

$$\frac{A \wedge B}{A \wedge B}$$

and the other an instance of permutation

$$\frac{A \wedge B}{B \wedge A}$$

These and other such examples from logic redeem the categorial point of view (see [13] and [14]). In the first example we have $\pi^1: p \wedge p \vdash p$ and $\pi^2: p \wedge p \vdash p$ with $\pi^1 \neq \pi^2$, and in the second $\mathbf{1}: p \wedge p \vdash p \wedge p$ and $c: p \wedge p \vdash p \wedge p$ with $\mathbf{1} \neq c$.

A category where these arrows are exemplified is \mathcal{C} , which is the category with binary product freely generated by a set of objects. The category \mathcal{C} models deductions involving only conjunction. It does so for both classical and intuitionistic conjunction, because the deductions involving this connective do not differ in the two alternative logics. This is a common ground of these two logics.

The notion of binary product codified in \mathcal{C} is one of the biggest successes of category theory. The explanation of the extremely important notion of ordered pair in terms of this notion is the most convincing corroboration of the point of view that mathematical objects should be characterized only up

to isomorphism. It is remarkable that the same matter should appear at the very beginning of what category theory has to say about deductions in logic, in connection with the connective of conjunction.

For the category \mathcal{C} there exists a kind of completeness theorem, which categorists call a *coherence* result. There is namely a faithful functor from \mathcal{C} to the model category \mathcal{M} , which is the opposite of the category of finite ordinals with functions as arrows. With this functor, π^1 and π^2 above correspond respectively to

$$\pi^1 \quad \begin{array}{c} p \wedge p \\ \searrow \\ p \end{array} \qquad \pi^2 \quad \begin{array}{c} p \wedge p \\ \nearrow \\ p \end{array}$$

while $\mathbf{1}$ and c correspond respectively to

$$\mathbf{1} \quad \begin{array}{c} p \wedge p \\ | \qquad | \\ p \wedge p \end{array} \qquad c \quad \begin{array}{c} p \wedge p \\ \times \\ p \wedge p \end{array}$$

Another example of two different formal deductions with the same premise and the same conclusion, which involves graphs of a slightly more complicated kind, is given by

$$\begin{array}{c} p \wedge (p \rightarrow p) \\ \searrow \\ p \end{array} \qquad \begin{array}{c} p \wedge (p \rightarrow p) \\ \frown \\ p \end{array}$$

The first deduction is made by conjunction elimination, while the second by modus ponens.

Finally we give an example where the deductions are not formal in the logical sense (at least they are not such at first glance, before further analyzing). Whoever has heard of Achilles, and knows what “faster” means, knows that (1) Achilles is faster than anybody else, and whoever has heard of the Tortoise, and knows what “slower” means, knows that (2) the Tortoise is slower than anybody else. Then from the premise that Achilles and the Tortoise have started their famous race one may deduce (*pace* Zeno) that

Achilles will overtake the Tortoise. Two different deductions however lead from our premise to this conclusion: one relies on (1), and the other on (2).

Coherence is one of the main inspirations of categorial proof theory (see [13]). The other, older, inspiration, which works for deductions in intuitionistic logic, comes ultimately from the notion of adjunction (see [19], [8], [9] and [11]).

In model categories such as we find in coherence results we have models of equational theories axiomatizing identity of deductions. These are not models of the theorems of logic. This is often unclear, partly because of uncertainties related to matters considered in Section 2. Moreover, some authors seem to think that models are better if they are complicated. (They may indeed be more likely to impress the public, and hence be better for marketing purposes.) The primary purpose of these models is to serve to obtain simple decision procedures for identity of deductions. So they should be manageable and as simple as possible, as the model category \mathcal{M} above is.

The arrows of the model categories are hardly what deductions really are. It is not at all clear that these categories provide a real semantics of deductions (cf. [11]).

Invoking now another principle of Frege's *Grundlagen der Arithmetik*, we might look for an answer to the question "What is a deduction?" by looking for a criterion of identity between deductions. We would strive to define a significant and plausible equivalence relation on derivations as coded by arrow terms of our syntactical categories, and equivalence classes of derivations, or equivalence classes of arrow terms, which are the arrows of our syntactical categories, would stand for deductions.

A deduction $f: A \vdash B$ would be something sui generis, that does not reduce to its type, the ordered pair (A, B) . It would be represented by an arrow in a category, to which is tied a criterion of identity given by the system of equations that hold in the category. The category should not be a preorder.

In the arrow term f of $f: A \vdash B$, the rules of inference, the rules of deduction, involved in building a deduction are made manifest as operations for building this term. The theory of deduction is as a matter of fact the theory of such operations (usually partial). It is an algebraic theory codifying with equations the properties of these operations.

It is rather to be expected that the theory of deduction should be the theory of the rules of deduction, as arithmetic, the theory of natural num-

bers, is the theory of arithmetic operations (addition, multiplication etc.). Extensionalizing the notion of deduction by reducing it to consequence (as in the preceding section) makes us forget the rules of deduction, which are prominent in categorial proof theory.

This would be the way to approach the question “What is a deduction?”, and it is a mathematical way. Let us consider some philosophical aspects of this mathematical approach.

From a classical point of view, the desirable quality of propositions, their correctness, is their truth. If the notion of deduction is something *sui generis*, not reducible to the notion of proposition, the desirable quality of deductions, their correctness, need not be reducible to the desirable quality of propositions, which from a classical point of view is their truth, as we have just said. A correct deduction would not be just one that preserves truth—a correct consequence relation is that. A deduction $f: A \vdash B$ is not just $A \vdash B$; we also have the f . As a matter of fact, the deduction is f . It is a necessary condition for a correct $f: A \vdash B$ that B be a consequence of A , but this is not sufficient for the correctness of f . This is not what the correctness of f consists in. The correctness of a deduction would be as the notion of deduction itself something *sui generis*, not to be explained by the desirable quality of propositions.

We might perhaps even try to turn over the positions, and consider that the desirable quality of propositions should be explained in terms of the correctness of deductions. This is presumably congenial to a point of view like that found in intuitionism, where the desirable quality of propositions is taken to be provability, i.e. deducibility from an empty collection of premises. We could however take an analogous position with a classical point of view, where the *truth* of analytic propositions would be guaranteed by the correctness of some deductions (cf. [16], p. 26). The correctness of the deductions underlies the truth, and not the other way round.

If deductions are objective, as we have recommended in Section 3, do incorrect deductions exist? The subjective activity of deducing may be performed incorrectly, but how can the objective paradigm be incorrect? This question seems to be analogous to the question whether propositions that are not true exist objectively. Does the proposition $2 + 2 = 5$ exist objectively? These questions have a typically philosophical, suspect, ring; with them we might be stepping into an ontological quagmire. Without trying to survey this whole, dangerous, ground, let us try to see what difference there may be

here between deductions and propositions.

A correct deduction is like a well-formed syntactical object—for example, a formula—and an incorrect one is like a badly formed syntactical object. (In logic, derivations, like other syntactical objects, are defined inductively, and deductions are equivalence classes of derivations.) An incorrect deduction does not exist as a badly formed syntactical object does not belong to the language. On the other hand, $2 + 2 = 5$ is not badly formed syntactically; it is a well-formed formula. Such a formula may be considered incorrect for syntactical reasons, because it is not a theorem, i.e. because it is underivable in a formal system. From a classical point of view, it is however usually branded as incorrect, as being untrue, for semantical reasons. For semantical reasons, other formulae would be considered correct, i.e. true.

It seems that a deduction $f: A \vdash B$ too may be considered incorrect for semantical reasons. This may be done simply because B is not a consequence of A . It is not clear whether this may be done also because f does not satisfy something required by a putative semantics of deduction, something formulated perhaps in terms of models of deduction, like our category \mathcal{M} above.

Can f be branded as correct for semantical reasons? We said already that this may not be done simply because B is a consequence of A . Are there then other semantical reasons that might induce us to do that? This question is unclear. Many things are unclear about the semantics of deduction (see [11]). Many things are however unclear about semantics in general.

6 Isomorphism of propositions

We will now consider another philosophical aspect of our mathematical approach to the question “What is a deduction?”. This is something that may be interesting for semantics. It has to do with the meaning of propositions.

Isomorphism between formulae should be an equivalence relation stronger than mutual implication. This is presumably the relation underlying the relation that holds between propositions that have the same meaning just because of their logical form. Any propositions that are instances, with the same substitution, of isomorphic formulae would have the same meaning, which presumably need not be the case for formulae that are just equivalent, i.e. which just imply each other.

One may try to characterize isomorphic formulae by looking only into the inner structure of formulae. This is the way envisaged by Carnap and Church (see [4], Sections 14-15, where related work by Quine and C.I. Lewis is mentioned, [5], [1], Section 2, and references therein).

Another way is to try to characterize isomorphism between formulae by looking also at the outer structure in which formulae are to be found. This outer structure may be a deductive structure, characterized in terms of categories in categorial proof theory. These are the syntactical categories envisaged in the preceding sections of this paper: their objects are formulae and their arrows are deductions.

Isomorphism between formulae may then be understood exactly as isomorphism between objects is understood in category theory. The formulae A and B are isomorphic when there is a deduction $f: A \vdash B$, and another deduction $g: B \vdash A$, such that f composed with g , i.e. $g \circ f: A \vdash A$ is equal to the identity deduction from A to A , namely $\mathbf{1}_A: A \vdash A$, while $f \circ g: B \vdash B$ is equal to $\mathbf{1}_B: B \vdash B$. This analysis of isomorphism presupposes a criterion of identity for deductions, which is formalized by equality between arrows in our syntactical categories.

That A and B are isomorphic means here intuitively that they function in the same manner in deductions. In a deduction one can replace one by the other, either as premise or as conclusion, so that nothing is lost, nor gained. The replacements, which are made by composing our deduction with the deductions f and g , are such that they enable us to return to our original deduction by further composing with g and f , since $g \circ f$ and $f \circ g$ are identity deductions, and hence may be cancelled. (For this view concerning isomorphic formulae and its relationship with propositional identity, see [6], Section 9, [11], Section 5, and [15].)

The study of isomorphic formulae first started in intuitionistic logic, for which it is widely believed that we have solid nontrivial criteria of identity of deductions. These criteria are provided either in terms of equality in the typed lambda calculus, via the Curry-Howard correspondence, or in terms of equality between arrows in categories based on cartesian closed categories (as we mentioned in Section 2).

A result exists in this area for the conjunction-implication fragment of intuitionistic logic (see [24]), but the problem of characterizing formulae isomorphic in the whole of intuitionistic propositional logic (which is related to Tarski's high-school algebra problem; see [3]) seems to be still open. There

is a further result characterizing isomorphic formulae in the analogous multiplicative fragment of linear logic, which corresponds to symmetric monoidal closed categories, and is common to classical and intuitionistic linear logic (see [12]).

Results characterizing isomorphic formulae in classical propositional logic may be found in [15]. These results cover also a fragment of classical linear propositional logic. For these results to be significant, we need for the logics we want to cover a plausible and nontrivial notion of equality of arrows in categories formalizing identity of deductions in these logics. A consensus for classical linear propositional logic may be found around the multiplicative fragment of that logic caught by proof nets, which leads to notions of category closely related to star-autonomous category (see [14] and references therein).

For classical propositional logic, it is on the contrary widely believed that no nontrivial notion of category would do the job. It is believed that no nontrivial notion of Boolean category may be found. This is indeed the case if one wants these Boolean categories to be cartesian closed (see [10], Section 5, [13], Section 14.3, and references therein). But, whereas on the level of theorems classical logic is an extension of intuitionistic logic, it is not clear that this should be so at the level of deductions and of criteria concerning their identity.

If one does not require that Boolean categories be cartesian closed categories, and bases identity of deductions in Boolean categories on coherence results analogous to those available for classical linear propositional logic, a nontrivial notion may arise. The coherence results in question are categorial results analogous to the classical coherence result of Kelly and Mac Lane for symmetric monoidal closed categories (see [18]). They reduce equality of arrows in the syntactical category to equality of arrows in a graphical model category.

Such a nontrivial notion of Boolean category may be found in [13] (Chapter 14), and [15] deals with isomorphism of formulae engendered by that particular notion, and by another notion motivated more consistently by generality of deductions (see [10]). These results give a complete characterization of isomorphic formulae in classical and classical linear propositional logic. These characterizations are such that they easily lead to decision procedures for the isomorphisms in question.

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