Mathematical modeling in music theory - example of counterpoint

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Abstract

We present the overview of some areas of interconnectedness between mathematics and music. Specifically, we focus on the problem of construction of certain type of counterpoint.

1 Introduction

The theme of interaction between mathematics and music is ancient and well established topic, begining with Pythagoras[1], the first philosopher, great mathematician and music theorist. Pythagoras was primarily concerned with tuning theory, specifically just inotation and what we now call Pythagorean tuning. However, Pythagoras connection between mathematics and music was mostly philosophical and mystical and it will remain as such up until the Age of Enlightenment when many great mathematicians, having put aside philosophical and mystical bases for the connection of mathematics and music, were trying to apply mathematical methods, by then much more advanced then that of the ancient Greeks, to problems in music theory, in particular, its foundation. Among those mathematicians were Marin Mersenne[2], sometimes referred to as "father of acoustics" due to his book *LHarmonie Universelle* (1637) in which he described the frequency of oscillation of a stretched string. It is also important to note that he was *de facto* the center of the world of mathematics and science due to his extensive correspondence with many mathematicians and other scientists. d'Alembert[3] was fascinated with Rameau's[4] *magnum opus Démonstration du principe de l'harmonie* (1750), and tried to emphasizing Rameau's claim that music was a mathematical science which has a certain set of principals (axioms) from which all elements of music can be deduced; he explained his theory in *Eléments de musique théorique et pratique suivant les principes de M. Rameau* (1752). However, the most important mathematician that made contributions to music theory in this period was Euler[5], one of the most important and prolific mathematician of all time. In 1739 he wrote *Tentamen novae theoriae musicae* hoping to establish musical theory as a mathematical discipline. Unfortunately, his work did not receive wide attention it deserved; it was considered too mathematical for musicians and too musical for mathematicians of the time. However, his work is fundamental for among other things it introduced Tonnetz[6], a way of representing the tonal space. More than a century after Euler, Hugo Riemann[7] would reinvent the Tonnetz and make it a foundation of his music theory. His work was subsequently improved and current musical theory that derived from it is now called neo-Riemannian, one of whose modern propo-

[1] Pythagoras of Samos (570 B.C. - 495 B.C.) - Ancient Greek philosopher, mathematician and founder of religious/philosophical movement that bears his name.

[2] Marin Mersenne (1588 - 1648) - French theologian, philosopher, mathematician and music theorist. Mersenne primes are named after him.

[3] Jean le Rond d'Alembert (1717 - 1783) - French mathematician, physicist, philosopher, and music theorist. Know for finding the general solution to the one-dimensional wave equation.

[4] Jean-Philippe Rameau (1683 - 1764) - French composers and music theorists of the Baroque era.

[5] Leonhard Euler (1707 - 1783) - Swiss mathematician and physicist.

[6] German for "tone-network".

[7] Karl Wilhelm Julius Hugo Riemann (1849 - 1919) - German music theorist and composer.

nents and developers was David Lewin[8]. The advancement in computer technology also stimulated interest in mathematical modeling of music. Currently, the problems connecting mathematics and music range form the traditional questions of tuning theory, group theory, algorithmic composition etc.
In this article we will illustrate how mathematical models can be used to solve one particular problem, creating a counterpoint for a given cantus firmus.

2 Counterpoint

Counterpoint is a type of compositional technique best defined as the art of combining two or more independent melodic lines (voices) in a harmonic (i.e. musically satisfying) way, most actively developed and used during the Renaissance and Baroque. It achieved its highest point in the works of J. S. Bach[9] such as The Art of The Fugue[10] and The Well-Tempered Clavier[11]. In this article we consider a specific form of counterpoint called species counterpoint. It was defined by Johann Fux[12] in his famous treatise on counterpoint, *Gradus ad Parnassum*[13], that became the single most influential book on counterpoint, specifically Palestrina[14] style of Renaissance polyphony. Many great composers such as J. S. Bach, Mozart, Beethoven, Haydn and many other, studied from it and held it in high regard.

[8]David Lewin (1933 - 2003) - American music theorist, music critic and composer.

[9]Johann Sebastian Bach (13 March 1685 - 28 July 1750) - German composer, organist, harpsichordist, violist, and violinist. Wildly considered the greatest composer of all time.

[10]Die Kunst der Fuge, BWV 1080

[11]Das Wohltemperierte Klavier, BWV 846-893

[12]Johann Joseph Fux (1660 - 13 February 1741) - Austrian composer, music theorist and pedagogue.

[13]Lat. Steps to Parnassus.

[14]Giovanni Pierluigi da Palestrina (2 February 1526 - 2 February 1594) - Italian Renaissance composer of sacred music. His work is wildly regarded as high point of Renaissance polyphony.

3 Preliminary definitions

Fux defined several types of species counterpoint. In this article we consider the fist one. Assuming equal-temperment, we identify the set of notes with the set $\mathbb{Z}$ of integers by assigning to each note the number of semitones it is distant from chosen fixed note, say C_4. The pitch space is a space obtained by identifying all pitches that are separated by a whole number of octaves. Since the octave has 12 semitones the pitch space is isomorphic to $\mathbb{Z}_{12}$. By distance between the two notes we shall consider their distance in $\mathbb{Z}$ and by interval between them, their distance in $\mathbb{Z}_1 2$. For example, the distance between the notes C_4 and F_5 is 17 (they are 17 semitones apart) while the interval between them is 5 (or perfect fourth). The intervals: unison, 3rd, 5th, 6th and octave are called consonances. The other intervals are called dissonances. Since in first species counterpoint all of the notes are of the same length, we can consider canus firmus and contrapunct as two arrays of equal lenght. Let us call $(cp(i), cf(i)), (cp(i+1), cf(i+1))$ a basic segment. To each such basic segment we can assign the 4-tuple $(cp(i) - cf(i), cp(i+1) - cp(i), cf(i+1) - cf(i), cp(i+1) - cf(i+1))$ that we shall call d-bloc. It is obvious that both cf and cp are uniquely determined by any one of their notes and the series of corresponding d-blocks. Contrapuntal motions are types of progression that can occur within a segment of counterpoint. They can be defined in terms of d-blocks (l, u, d, r) of a basic segment as follows:

Type of motion	Condition on d-block
similar	$ud > 0$
paralel	$u = d$
obleque	$ud = 0 \wedge u^2 + d^2 \neq 0$
contrary	$ud < 0$

Since Contrapuntal composition is uniquely determined by any one of its notes and a sequence of its d-blocks, the problem of constructing the contrapuntal composition is equivalent to construction of the sequence of d-block that satisfy given set of rules.

4 Construction of counterpoint graph

We have divided the rules Fux gave in three categories, R, E and S. The R-rules will be used to determine the middle part of counterpoint, i.e. the part of counterpoint that respect all the rules except possibly the rules concerning the beginning and ending segment; we shall also refer to those as open counterpoints. E-rules will be used for determining the beginning and ending segment. Finlay, S-rules will be used for the construction of the function of aesthetic measure, i.e. they will be used to order the set of all possible solutions thus helping us find the most desirable (most musically pleasing) ones. We shall call d-blocks that satisfy the R-rules, admissible.
The following is the algorithm that decides if a given d-block (l, u, d, r) is admissible if *cons* is set of consonances and *inc* the set of admissible melodic motions:
(1) Check if either u or d is a step, i.e. weather $u \vee d \in \{1, 2\}$.
(2) Check if l and r belong to a given set of consonances and if u and d belong to a given set of admissible melodic movements.
(3) If l is a prefect consonance check if motion if not parallel.
(4) if r is a prefect consonance check if motion is contrary.
(5) d-block if admissible iff all (1)-(4) are valuated as true.
From the set of admissible d-block we can construct a directed graph Γ in the following way. The vertices of Γ are pairs (x, y) such that $i(x, y)$ is a consonance. For each two vertices (x_1, y_1) and (x_2, y_2), if $(x_1 - y_1, x_2 - x_1, y_2 - y_1, x_2 - y_1)$ is admissible d-block we add a directed edge from (x_1, y_1) to (x_2, y_2) and label it $(x_2 - x_1, y_2 - y_1)$. By construction, it is obvious that the following holds.

Theorem 4.1. *If* $b_0, b_1, ..., b_n$, $b_i = (l_i, u_i, d_i, r_i)$, $r_i = l + i + 1$ *are d-block of some counterpoint then* $l_1(u_1, d_1)r_1(u_1, d_1)$ *is a path in* Γ.

Now for a given cantus firmus we construct a suitable counterpoint recursively in the following way:
(1) Construct the last segment using the E-rules, if possible.
(2) Recursively do the following:
If possible select the interval between cf and cp such that there is a directed edge in Γ from it to the one selected in the previous step. If that is not possible, go one step back. Repeat until reaching the first segment or exhausting all possibilities.
(3) If the first segment is reached construct the first segment using E-rules.
Since Γ is finite the process find all admissible solutions in a finite time. One all the solutions have been found, sort them using the function of aesthetic measure constructed form S-rules.

5 Example

We conclude by giving an one of Fux's examples of first species counterpoint and some of our alternative solutions constructed using the method previously described. Fux's example[15]:

Some our alternative solutions with the maximal score in respect to our function of aesthetic measure:

[15] All of the notes in the first species counterpoint are semibreves in the original. Here we use crotchets instead.

References

[1] Johann Joseph Fux, *Gradus ad Parnassum*, 1741

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[3] Nigel Gwee, Complexity and heuristics in rule-based algorithmic music composition, 2002

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