Mixed Integer Quadratically Constrained Programming (MIQCP) model for VSP-HV

In order to present mathematical formulation of the problem, the following notation is introduced:

J : The set of locations; I : The set of vehicles; R : The set of tours;
n : The total number of locations; m : The total number of vehicles;
r_{max} : The maximum number of tours a vehicle can make during working day;
q_j : The quantity of goods collected at location $j \in J$;
d_j : The distance between the factory and location $j \in J$;
c_i : Capacity of vehicle $i \in I$; D : Daily factory needs;
v : The average speed of a vehicle;
W_i : The time needed for loading a vehicle $i \in I$;
U_i : The time that a vehicle $i \in I$ spends in factory area between two tours;
$p_{i,j}$: The time needed for serving location $j \in J$ by a vehicle i, calculated as the sum of driving time in both directions and loading and unloading time, i.e., $p_{i,j} = 2\frac{d_j}{v} + U_i + W_i$;
h_j : The number of days that the goods are kept in the open at location $j \in J$;
h_0 : The maximum number of days that the goods can stay in the open without losing quality;
H_j**:** The binary value indicating if location $j \in J$ is urgent, defined as: $H_j = 1$ if $h_j > h_0$, and $H_j = 0$ otherwise;
p : The maximal number of vehicles that can be unloaded in the same time at the factory area;
t_{start} : Starting time; t_{end} : The end of working day;
ε : Small positive number; M : Large positive number.

Although the maximal number of tours of a vehicle during the working day is set to r_{max}, it may happen that some vehicles make less than r_{max} tours, having in mind that the daily transport finishes when the required amount of goods is transported to the factory and all urgent location are emptied. In order to simplify mathematical formulation, we equalize the number of tours for all vehicles to r_{max}, by allowing a vehicle to make one or more *virtual tours*. It is assumed that during a virtual tour a vehicle stays in the factory area, meaning that the duration of a virtual tour is equal to zero. Therefore, virtual tours do not affect the objective function value and the problem constraints, as they are simply discarded from a solution during the objective function calculation.

The following decision variables are used in mathematical formulations:

- Binary variables $x^j_{i,r}$ take the value of 1 if a vehicle $i \in I$ visits location $j \in J$ in the tour $r \in R$, and 0 otherwise. If the tour $r \in R$ of a vehicle $i \in I$ is virtual, $\sum\limits_{j \in J} x^j_{i,r} = 0$ holds;

- Real variables $t_{i,r}$ indicate the departure time of a vehicle $i \in I$ from the factory in the tour $r \in R$;
- Binary variables y_j indicate whether a location $j \in J$ is emptied or not. If all goods collected at $j \in J$ are transported to the factory, then $y_j = 0$, otherwise $y_j = 1$ holds. By using variables y_j, we are able to keep the track on the total amount of goods that arrive to the factory from location $j \in J$. Namely, if $y_j = 0$, location j is emptied, and the amount of goods transported from the location j to the factory is equal to q_j. Otherwise, this amount is obtained by summing the capacities c_i of all vehicles $i \in I$ that have visited the location j during the working day;
- Real variable π represents the objective function value, the very last moment of time when all vehicles finish their last tours.
- Binary variables $b_{i,r}^{l,s}$, $i, l \in I$, $r, s \in R$ are used in the set of constraints ensuring that more than one vehicle cannot be loaded at each location in the same time. Namely, $b_{i,r}^{l,s} = 1$ if and only if $t_{i,r} - t_{l,s} < 0$, while $b_{i,r}^{l,s} = 0$ if and only if $t_{i,r} - t_{l,s} \geq 0$, where $t_{i,r}$ and $t_{l,s}$ represent the departure times of vehicle $i \in I$ in tour r and vehicle l in tour s, respectively, to the same location. The role of variables $b_{i,r}^{l,s}$ is to avoid the expressions containing absolute values;
- Several sets of variables are needed in order to formulate the condition that only p vehicles can be unloaded in the same time at the factory area. Binary variables $M_{i,r}^{l,s}$ take the value of 1 if vehicle $i \in I$ in tour $r \in R$ arrives earlier than vehicle $l \in I$ in tour $s \in R$ in the factory area, and 0 otherwise. More precisely, $M_{i,r}^{l,s} = 1$ if $\sum_{j \in J} p_{i,j} x_{i,r}^j - U_i \leq \sum_{j \in J} p_{i,j} x_{l,s}^j - U_l$, and 0 otherwise. Real nonnegative variables $a_{i,r}^{l,s}$, $i, l \in I$, $r, s \in R$ are defined as absolute values of difference in arrival times to the factory area of vehicles $i, l \in I$ in tours $r, s \in R$, respectively, i.e., $a_{i,r}^{l,s} = |\sum_{j \in J} p_{i,j} x_{i,r}^j - U_i - \sum_{j \in J} p_{i,j} x_{l,s}^j + U_l|$. Binary variables $z_{i,r}^{l,s}$ take the value 1 if $a_{i,r}^{l,s} < M_{i,r}^{l,s} U_i + (1 - M_{i,r}^{l,s}) U_l$ holds, meaning that the difference of the arrival times of vehicles $i, l \in I$ to the factory in tours $r, s \in R$, respectively, is less than the duration of unloading of the first vehicle that arrived to the factory. Otherwise, the value of $z_{i,r}^{l,s}$ is set to 0. Binary variables $g_{i,r}^{l,s}$, $i, l \in I$, $r, s \in R$ are introduced to formulate this condition by avoiding absolute values.
- Binary variables $\rho_{i,r,l,s}^{j,h}$ are used to substitute the product of binary variables $x_{i,r}^j$ and $x_{l,s}^h$, for each $j, h \in J$, $i, l \in I$, $r, s \in R$.

By using the above notation and decision variables, the considered problem is formulated as a Mixed Integer Quadratically Constraint Programming (MIQCP) model:

$$\min \quad \pi \tag{1}$$

$$\sum_{j \in J} x_{i,r}^j \leq 1 \qquad \forall i \in I, \quad \forall r \in R, \tag{2}$$

$$\sum_{(i,r) \in I \times R} c_i \rho_{i,r,l,s}^{j,j} - c_l x_{l,s}^j + \varepsilon \leq q_j \qquad \forall j \in J, \quad \forall l \in I, \quad \forall s \in R, \tag{3}$$

$$H_j + y_j \leq 1 \qquad \forall j \in J, \tag{4}$$

$$\sum_{j\in J}(1-y_j)q_j + \sum_{(i,j,r)\in I\times J\times R} c_i y_j x^j_{i,r} \geq D, \tag{5}$$

$$\sum_{(i,r)\in I\times R} c_i x^j_{i,r} \geq q_j(1-y_j) \qquad \forall j \in J, \tag{6}$$

$$\sum_{(i,r)\in I\times R} c_i y_j x^j_{i,r} + \varepsilon \leq q_j \qquad \forall j \in J, \tag{7}$$

$$t_{i,r} + \sum_{j\in J} p_{i,j} x^j_{i,r} \leq t_{i,r+1} \qquad \forall i \in I, \ \ \forall r \in R \setminus \{r_{max}\}, \tag{8}$$

$$t_{i,r_{max}} + \sum_{j\in J} p_{i,j} x^j_{i,r_{max}} \leq \pi \qquad \forall i \in I, \tag{9}$$

$$t_{i,r} - t_{l,s} + M b^{l,s}_{i,r} \geq W_l \rho^{j,j}_{i,r,l,s} \qquad \forall j \in J, \ \ \forall i \neq l \in I, \ \ \forall r,s \in R, \tag{10}$$

$$-t_{i,r} + t_{l,s} + M(1-b^{l,s}_{i,r}) \geq W_i \rho^{j,j}_{i,r,l,s} \qquad \forall j \in J, \ \ \forall i \neq l \in I, \ \ \forall r,s \in R, \tag{11}$$

$$\rho^{j,h}_{i,r,l,s} - \frac{1}{2}(x^j_{i,r} + x^h_{l,s}) \leq 0 \qquad \forall i,l \in I, \ \ \forall j,h \in J, \ \ \forall r,s \in R, \tag{12}$$

$$\rho^{j,h}_{i,r,l,s} - x^j_{i,r} - x^h_{l,s} + 1 \geq 0 \qquad \forall i,l \in I, \ \ \forall j,h \in J, \ \ \forall r,s \in R, \tag{13}$$

$$z^{i,s}_{i,r} = 0 \qquad \forall i \in I, \ \ \forall r,s \in R, \tag{14}$$

$$z^{l,s}_{i,r} \leq \sum_{j\in J}\sum_{h\in J} \rho^{j,h}_{i,r,l,s} \qquad \forall i,l \in I, \ \ \forall r,s \in R, \tag{15}$$

$$\begin{aligned} &2z^{l,s}_{i,r} M^{l,s}_{i,r}(U_i - U_l) + 2z^{l,s}_{i,r}(U_l - \varepsilon) + a^{l,s}_{i,r} - 2z^{l,s}_{i,r} a^{l,s}_{i,r} + \varepsilon \geq \\ &\sum_{j\in J}\sum_{h\in J} \rho^{j,h}_{i,r,l,s} M^{l,s}_{i,r}(U_i - U_l) + \sum_{j\in J}\sum_{h\in J} \rho^{j,h}_{i,r,l,s} U_l \ \ \forall i \neq l \in I, \ \ \forall r,s \in R, \end{aligned} \tag{16}$$

$$\begin{aligned} &- t_{i,r} + t_{l,s} - \sum_{j\in J} p_{i,j} x^j_{i,r} + \sum_{j\in J} p_{l,j} x^j_{l,s} + U_i - U_l <= M^{l,s}_{i,r}(-t_{i,r} + t_{l,s} - \\ &\sum_{j\in J} p_{i,j} x^j_{i,r} + \sum_{j\in J} p_{l,j} x^j_{l,s} + U_i - U_l) \qquad \forall i \neq l \in I, \ \ \forall r,s \in R, \end{aligned} \tag{17}$$

$$\begin{aligned} &t_{i,r} - t_{l,s} + \sum_{j\in J} p_{i,j} x^j_{i,r} - \sum_{j\in J} p_{l,j} x^j_{l,s} - U_i + U_l <= (1 - M^{l,s}_{i,r})(t_{i,r} - t_{l,s} + \\ &\sum_{j\in J} p_{i,j} x^j_{i,r} - \sum_{j\in J} p_{l,j} x^j_{l,s} - U_i + U_l) \qquad \forall i \neq l \in I, \ \ \forall r,s \in R, \end{aligned} \tag{18}$$

$$t_{i,r} - t_{l,s} + \sum_{j\in J} p_{i,j} x^j_{i,r} - \sum_{j\in J} p_{l,j} x^j_{l,s} - U_i + U_l \leq a^{l,s}_{i,r} \qquad \forall i,l \in I, \ \ \forall r,s \in R, \tag{19}$$

$$-t_{i,r} + t_{l,s} - \sum_{j\in J} p_{i,j} x^j_{i,r} + \sum_{j\in J} p_{l,j} x^j_{l,s} + U_i - U_l \leq a^{l,s}_{i,r} \qquad \forall i,l \in I, \ \ \forall r,s \in R, \tag{20}$$

$$\begin{aligned} &a^{l,s}_{i,r} \leq t_{i,r} - t_{l,s} + \sum_{j\in J} p_{i,j} x^j_{i,r} - \sum_{j\in J} p_{l,j} x^j_{l,s} - U_i + U_l + 2(t_{end} - t_{start})(1 - g^{l,s}_{i,r}) \\ &\qquad \forall i,l \in I, \ \ \forall r,s \in R, \end{aligned} \tag{21}$$

$$\begin{aligned} &a^{l,s}_{i,r} \leq -t_{i,r} + t_{l,s} - \sum_{j\in J} p_{i,j} x^j_{i,r} + \sum_{j\in J} p_{l,j} x^j_{l,s} + U_i - U_l + 2(t_{end} - t_{start}) g^{l,s}_{i,r} \\ &\qquad \forall i,l \in I, \ \ \forall r,s \in R, \end{aligned} \tag{22}$$

$$\sum_{(l,s)\in I\times K} z_{i,r}^{l,s} \le p-1, \qquad \forall i\in I \ \ \forall r\in R, \tag{23}$$

$$a_{i,r}^{l,s} \ge 0 \qquad \forall i,l\in I, \ \ \forall r,s\in R, \tag{24}$$

$$g_{i,r}^{l,s} \in \{0,1\} \qquad \forall i,l\in I, \ \ \forall r,s\in R, \tag{25}$$

$$x_{i,r}^{j} \in \{0,1\} \quad \forall i\in I, \ \ \forall j\in J, \ \ \forall r\in R, \tag{26}$$

$$y_j \in \{0,1\} \qquad \forall j\in J, \tag{27}$$

$$t_{i,r} \in [t_{start}, t_{end}] \qquad \forall i\in I, \ \ \forall r\in R, \tag{28}$$

$$z_{i,r}^{l,s} \in \{0,1\} \qquad \forall i,l\in I, \ \ \forall r,s\in R, \tag{29}$$

$$b_{i,r}^{l,s} \in \{0,1\} \qquad \forall i,l\in I, \ \ \forall r,s\in R, \tag{30}$$

$$\rho_{i,r,l,s}^{j,h} \in \{0,1\} \quad \forall i,l\in I, \ \ \forall r,s\in R, \ \ \forall j,h\in J, \tag{31}$$

$$M_{i,r}^{l,s} \in \{0,1\} \quad \forall i,l\in I, \ \ \forall r,s\in R. \tag{32}$$

The objective function (1) together with constraints (9) minimize the very last moment of time when all vehicles finish their last tours. In each tour, a vehicle either serves only one location or a tour is virtual one, which is provided by constraints (2). Constraints (3) ensure that the transported amount of sugar beet from a location $j\in J$ to the factory cannot exceed q_j. The subtrahend $c_l x_{l,s}^j$ on the left hand side of constraints (3) is needed to cover the case when the last vehicle that visits location j is not full with goods. The role of small positive number ε on the left hand side of (3) is to provide the strict inequality between the left and right hand side, thus preventing a potential unnecessary tour to location j.

The constraints (4) provide that all goods collected at an urgent location $j\in J$ must be transported during the working day. The daily factory needs must be satisfied, which is ensured by constraints (5). The first sum on the left hand side of (5) represents the amount of sugar beet from emptied locations, while the second sum stands for the quantities delivered to the factory from the unemptied ones.

The values of variables y_j are defined by constraints (6) and (7). If a location $j\in J$ is emptied, y_j is equal to 0 and the constraints (6) ensure that the amount of goods transported to the factory from location j is at least q_j, while constraints (7) will be satisfied, having in mind that ε is a small positive number. If location j is not emptied, y_j is set to 1, and constraints (6) obviously hold. The constraints(7) provide that the amount of transported goods from each unemptied location $j\in J$ is strictly less than q_j times.

Each vehicle $i\in I$ can start the next tour only if the previous one is finished, which is provided by constraints (8). The starting time of each tour must not be smaller than the starting time of the previous one increased by the time needed to serve the location visited in the previous tour. In the case of virtual tour $r\in R\setminus\{r_{max}\}$ of vehicle $i\in I$, the sum on the left hand side of constraints (8)is equal to zero.

The constraints (10) and (11) provide that two different vehicles cannot be loaded at the same location in the same time. If two different vehicles visit the same location, the absolute value of difference between their departure times must be at least equal to the duration of loading for a vehicle that arrives first to the considered location. Note that the absolute value of the difference between the arrival times at the location is calculated as the absolute value of the difference between the departure times from the factory, as it is assumed that the average velocity of vehicles is the same. The purpose of variables $\rho_{i,r,l,s}^{j,h}$ on the right hand side of hand side (10) and (11) is to substitute the product of binary variables $x_{i,r}^{j}$ and $x_{l,s}^{h}$. These variables must satisfy constraints (12) and (13) in order to be correctly defined.

The purpose of constraints (14)-(23) provide that at most p vehicles can be unloaded at the factory area in the same time. The case when we deal with two tours of the same vehicle is ignored by constraints (14), while constraints (15) are imposed to ignore the case when at least one of the two observed vehicles has a virtual tour. The constraints (16) provide that $z_{i,r}^{l,s} = 1$ if the absolute difference between the arrival times at the factory area of vehicles i and l in tours r and s, respectively, is smaller then the duration of unloading of the vehicle that arrived first to the factory, otherwise $z_{i,r}^{l,s} = 0$. The purpose of two sums on the right hand side of constraints (16) is to ignore the case when at least one of the two observed vehicles has a virtual tour.

Constraints (17) and (18) provide the value of variable $M_{i,r}^{l,s}$ (equal to 1 if vehicle i in its tour r arrives to the factory before vehicle l in its tour s, and 0 otherwise), while constraints (19)-(22) are used to set the value of variable $a_{i,r}^{l,s}$ to the absolute value of the difference between the arrival times of vehicles i and l in tours r and s to the factory, respectively. At most $p-1$ vehicles can be unloaded in the same time as the observed vehicle $i \in I$ in the tour $r \in R$ in the factory area, which is ensured by constraints (23). Finally, the type of decision variables is defined by constraints (24)-(32).