Graphs for Small Multiprocessor Interconnection Networks*

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Let D be the diameter of a graph G and let λ_1 be the largest eigenvalue of its (0,1)-adjacency matrix. We give a proof of the fact that there are exactly 69 non-trivial connected graphs with $(D+1)\lambda_1 \leqslant 9$. These 69 graphs all have up to 10 vertices and were recently found to be suitable models for small multiprocessor interconnection networks. We also examine the suitability of integral graphs to model multiprocessor interconnection networks, especially with respect to the load balancing problem. In addition, we classify integral graphs with small values of $(D+1)\lambda_1$ in connection with the load balancing problem for multiprocessor systems.

Keywords: graph theory, graph spectra, integral graphs, diameter, multiprocessor interconnection networks

AMS Classification: 05C50, 68M07, 68M10, 68M14.

1. Introduction

Let G be a simple graph on n vertices with (0,1)-adjacency matrix A. The eigenvalues of A (i.e. the zeros of $\det(xI-A)$) and the spectrum of A (which consists of n eigenvalues) are also called the *eigenvalues* of G and the *spectrum* of G, respectively. The eigenvalues of G are real because A is symmetric (thus can be ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$). The largest eigenvalue $\lambda_1 = \lambda_1(G)$ is called the *index* of G; $m = m(G)$ denotes the number of distinct eigenvalues of G.

Let $D = \operatorname{diam}(G)$ be the diameter of a (connected) graph G. Graphs with small product $(D+1)\lambda_1$ (tightness) appear to be suitable for designing multiprocessor interconnection networks [7,9]. It was proved that there are exactly 69 non-trivial connected graphs with $(D+1)\lambda_1 \leqslant 9$. The aim of this paper is to extend mathematical arguments used in [9], and to offer some related results.

The rest of this paper is organized as follows: in Section 2 we give some preparatory results; in Section 3 we prove theorems related to the 69 graphs identified in [9]; in

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Section 4 we consider integral graphs in connection to the load balancing problem for multiprocessor systems; in Section 5 we study integral graphs with small value of tightness.

2. Preliminaries

As usual, K_n, C_n, S_n and P_n denote the *complete graph*, *cycle*, *star* and *path* on n vertices, respectively. $K_{m,n}$ denotes the *complete bipartite graph* on $m+n$ vertices (in particular, $K_{1,n-1} = S_n$).

Maximum (minimum) vertex degree of a graph G is denoted by $\Delta = \Delta(G)$ (resp. $\delta = \delta(G)$); $\bar{d} = \bar{d}(G)$ denotes the average vertex degree of G.

It is well-known (see, for example, [11], p. 85) that

$$\delta \leqslant \bar{d} \leqslant \lambda_1 \leqslant \Delta. \tag{1}$$

In addition we have (cf. [11], p. 112 and p. 85):

$$\sqrt{\Delta} \leqslant \lambda_1 \leqslant \Delta. \tag{2}$$

We also have (see, for example, [11] Theorem 3.13):

$$D \leqslant m - 1. \tag{3}$$

The following inequality is well-known (for any connected graph on n vertices)

$$n \leqslant 1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots + \Delta(\Delta-1)^{D-1}. \tag{4}$$

This upper bound for the number of vertices is obtained by adding maximal numbers of neighbours of a particular vertex at distances $1, 2, \ldots, D$.

Definition 2.1. Let G be a graph of diameter D, maximum (vertex) degree Δ, index λ_1 and with m distinct eigenvalues. Then we have the following types of tightness for G

(i) $t_1(G) = m\Delta$ is the *first type mixed tightness*;

(ii) $stt(G) = (D+1)\Delta$ is the *structural tightness*;

(iii) $spt(G) = m\lambda_1$ is the *spectral tightness*;

(iv) $t_2(G) = (D+1)\lambda_1$ is the *second type mixed tightness*.

If the type of tightness is not relevant in some discussion, then any of them is addressed, for short, by *tightness*; if the graph in question is understood from context we suppress its name from our notation (so write only t_1, t_2, stt and spt). Definition 2.1 stems from [7]. It was proved there that the number of graphs with a bounded tightness of any type is finite.

Several arguments were given in [7,9] supporting the claim that graphs with small tightness t_2 are well suited for multiprocessor interconnection networks.

Applying (1) and (2) we get:

$$t_1 \geqslant stt, \quad t_1 \geqslant spt \quad \text{and} \quad t_2 \leqslant stt, \quad t_2 \leqslant spt.$$

Let $\mathcal{G}_c$ be the set of connected graphs with at least two vertices. Let us introduce the following notation:

$$T_1^a = \{G : G \in \mathcal{G}_c,\ t_1(G) \leqslant a\}, \quad T_{stt}^a = \{G : G \in \mathcal{G}_c,\ stt(G) \leqslant a\},$$

$$T_{spt}^a = \{G : G \in \mathcal{G}_c,\ spt(G) \leqslant a\} \ , \qquad T_2^a = \{G : G \in \mathcal{G}_c,\ t_2(G) \leqslant a\}.$$

It is obvious that $T_1^a \subseteq T_{stt}^a \subseteq T_2^a$ and $T_1^a \subseteq T_{spt}^a \subseteq T_2^a$.

A graph is called *integral* if its spectrum consists entirely of integers (see [2], for a general survey on integral graphs). As noted in [6], the important feature of integral graphs is that each eigenspace has a basis consisting of integral eigenvectors. This fact could be relevant in managing load balancing in multiprocessor systems. It is noteworthy that integral graphs have already several applications: in quantum computing (perfect state transfer [22], in quantum spin networks with periodic dynamics [4]) and in theoretical chemistry (hyperenergetic and equienergetic graphs [21]). We expect that some further applications can be found in load balancing (see Section 4).

Graphs with small index are relevant in studying graphs with small tightness. Connected graphs with $\lambda_1 = 2$ (known as *Smith graphs*), can play here an important role (as is generally the case in the whole spectral graph theory). All of them are given on Fig. 1. There are 6 types of Smith graphs. Four of them are concrete graphs, while the remaining types are cycles C_n (on $n \geqslant 3$ vertices) and *double-head snakes* W_n (on $n \geqslant 6$ vertices; note $W_5 = S_5$).

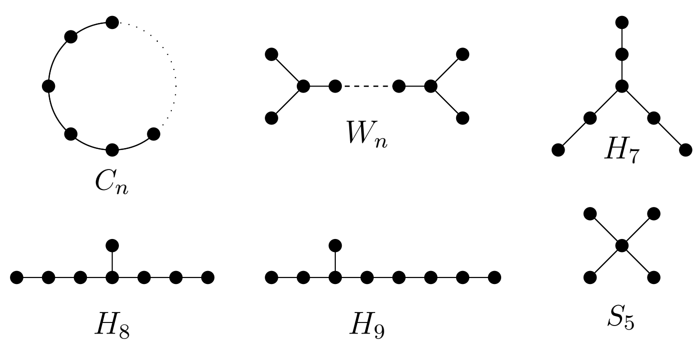

Figure 1. Smith graphs

Connected graphs with $\lambda_1 < 2$ are also relevant to us. They are (connected) subgraphs of Smith graphs. By removing vertices out of Smith graphs, we obtain paths P_n ($n \geqslant 2$) and *single-head snakes* Z_n ($n \geqslant 4$; note $Z_3 = P_3$) (see Fig. 2); other 3 graphs (given in the second row of Fig. 2) are denoted by E_6, E_7 and E_8. It is enough to consider only one vertex removal; removing further vertices leads to the graph already obtained in another way.

Eigenvalues of Smith graphs and of their connected subgraphs are explicitly calculated in [13].

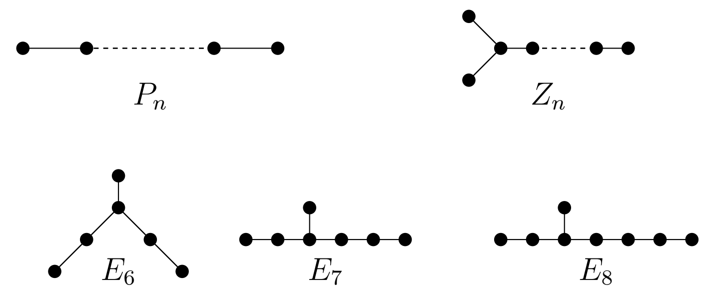

Figure 2. Connected subgraphs of Smith graphs

3. New proofs of some theorems related to graph tightness

We are interested in the 69 graphs given in Figs. 3-8 under names $\Omega_{n,k}$, where n, $2 \leqslant n \leqslant 10$ (denotes the number of vertices) and $k \geqslant 1$ (being a counter).

In Appendix, we give in Table 1 some data on these 69 graphs.

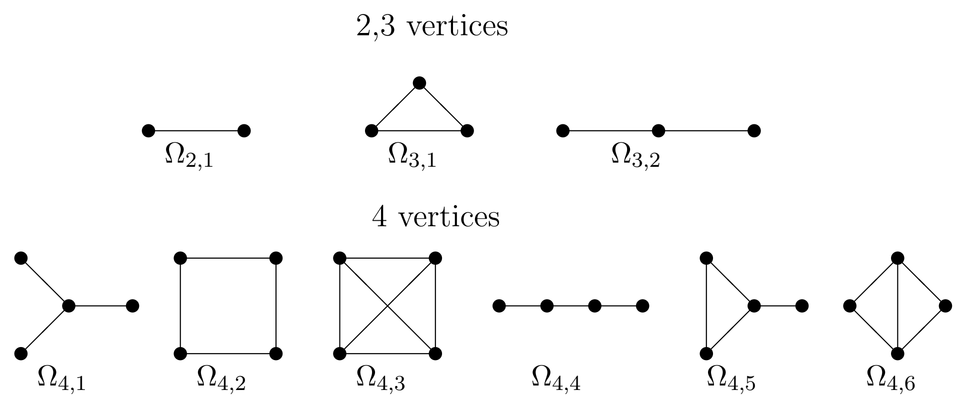

Figure 3. Graphs up to 4 vertices with small tightness

The main result of [9] is the next theorem. However, in [9] only a sketch of a proof is given. Here we provide a complete proof.

Theorem 3.1. *The only non-trivial connected graphs G such that $t_2(G) \leqslant 9$ are the 69 graphs $\Omega_{n,k}$, depicted on Figs. 3-8.*

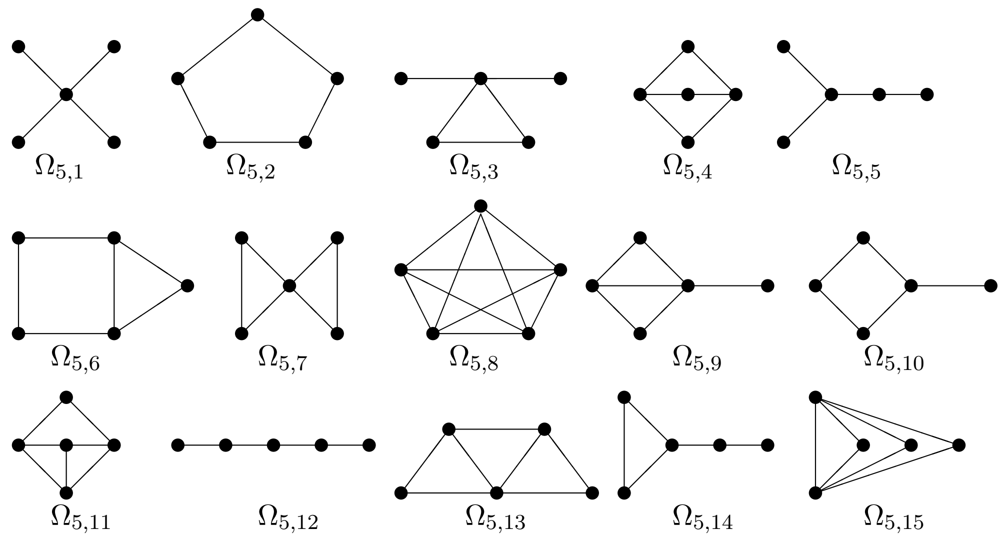

Figure 4. Graphs on 5 vertices with small tightness

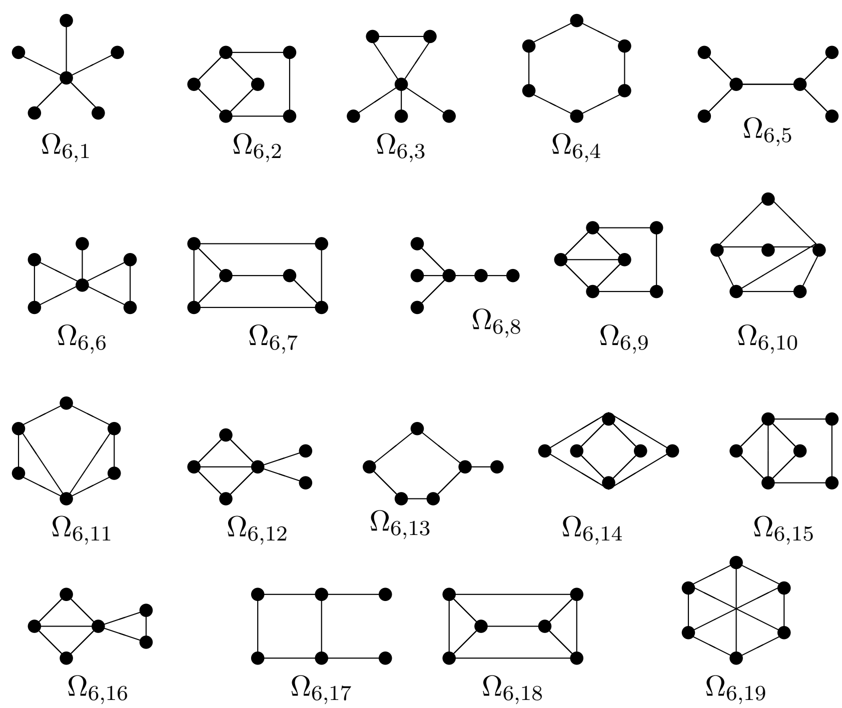

Figure 5. Graphs on 6 vertices with small tightness

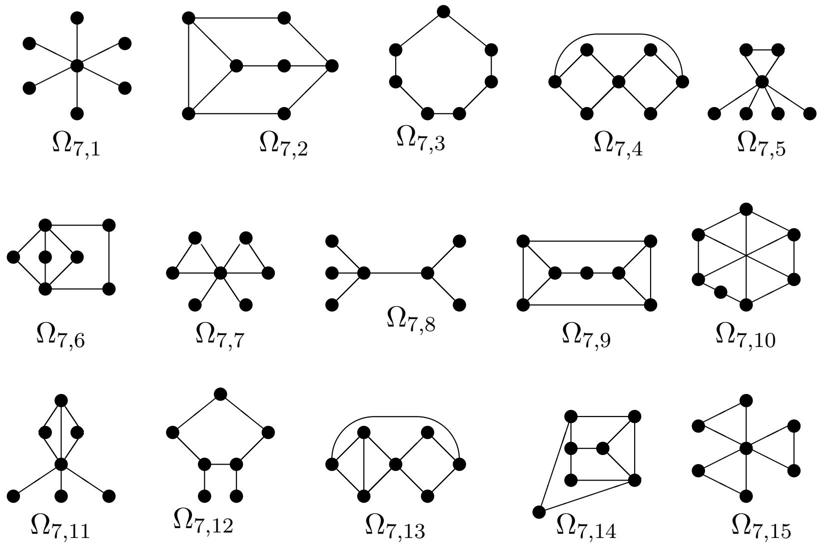

Figure 6. Graphs on 7 vertices with small tightness

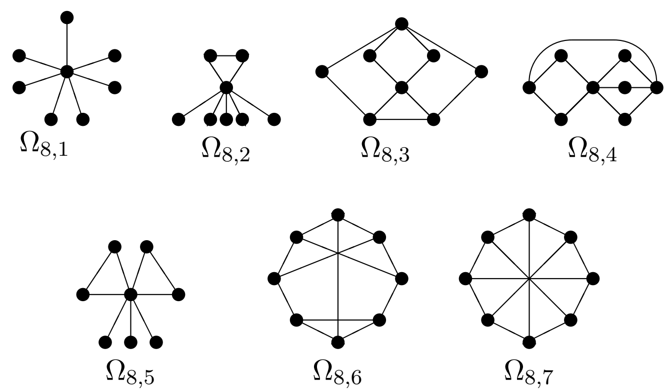

Figure 7. Graphs on 8 vertices with small tightness

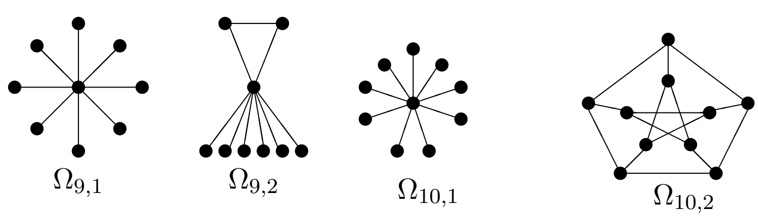

Figure 8. Graphs on 9 and 10 vertices with small tightness

Proof of Theorem 3.1. We have the following cases:

a° $D = 1$, $\lambda_1 \leqslant 4.5$. We have complete graphs $\Omega_{2,1}$, $\Omega_{3,1}$, $\Omega_{4,3}$, $\Omega_{5,8}$.

b° $D = 2$, $\lambda_1 \leqslant 3$. Denote the set of graphs satisfying these conditions by $\mathcal{A}_1$. According to (2) we have $\Delta \leqslant \lambda_1^2 \leqslant 9$ and by formula (4) we get $n \leqslant 1 + 9 + 9 \cdot 8 = 82$. For example, the star $\Omega_{10,1} \in \mathcal{A}_1$. The set $\mathcal{A}_1$ is completely determined in Lemma 3.2.

c° $D = 3$, $\lambda_1 \leqslant 2.25$. Denote the set of graphs satisfying these conditions by $\mathcal{A}_2$. Now, $\Delta \leqslant 5$ since $\lambda_1^2 < 6$, and we have $n \leqslant 1 + 5 + 5 \cdot 4 + 5 \cdot 4^2 = 106$. Graphs belonging to the set $\mathcal{A}_2$ are listed in Lemma 3.3.

d° $D = 4$, $\lambda_1 \leqslant 1.8$. It is easy to see that the only graph in this case is the path $\Omega_{5,12}$.

e° $D \geqslant 5$, $\lambda_1 \leqslant 1.5$. There are no graphs satisfying these conditions.

To treat the cases b° and c° in Lemmas 3.2 and 3.3 we need an auxiliary result.
Let R be the set of graphs satisfying the conditions $D = 2$, $\Delta = 3$.

Lemma 3.1. *The set R consists of the following 17 graphs:* $\Omega_{4,1}$, $\Omega_{4,5}$, $\Omega_{4,6}$, $\Omega_{5,4}$, $\Omega_{5,6}$, $\Omega_{5,11}$, $\Omega_{6,2}$, $\Omega_{6,7}$, $\Omega_{6,9}$, $\Omega_{6,18}$, $\Omega_{6,19}$, $\Omega_{7,2}$, $\Omega_{7,9}$, $\Omega_{7,10}$, $\Omega_{8,6}$, $\Omega_{8,7}$ *and* $\Omega_{10,2}$.

Sketch of the proof. (The complete proof is given in [9].) By formula (4) graphs from R have at most 10 vertices. Consider a graph $G \in R$. It has a vertex v of degree 3. Let f be the number of edges in the subgraph of G induced by the three neighbours of v. We have the following possibilities:

If $f = 3$, we have $G = \Omega_{4,3}$ which is excluded since $D = 1$.

Consider $f = 2$. Now we start from vertex v and its neighbours and add new vertices and edges in such a way that conditions $D = 2$, $\Delta = 3$ are not violated. We readily get $G = \Omega_{4,6}$, or $G = \Omega_{5,11}$ given on Fig. 4, or G is isomorphic to $\Omega_{6,9}$ from Fig. 5.

In the case $f = 1$ the obtained graphs up to 7 vertices are presented on Fig. 9. Finally, we get the graph $\Omega_{8,6}$ from Fig. 7 on $n = 8$ vertices.

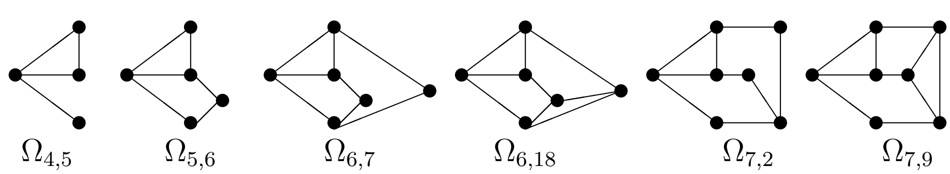

Figure 9. Some graphs from the set R

If $f = 0$, we first have complete bipartite graphs $\Omega_{4,1}$, $\Omega_{5,4}$, and $\Omega_{6,19}$, and $\Omega_{6,2}$. For $n = 7$ we again come across graph $\Omega_{7,2}$, and the graph $\Omega_{7,10}$. For $n = 8$ the graphs $\Omega_{8,6}$, $\Omega_{8,7}$ from Fig. 7 appear. The Petersen graph $\Omega_{10,2}$ on 10 vertices belongs here. There are no graphs on 9 vertices. □

Lemma 3.2. *The set $\mathcal{A}_1$ consists of 52 graphs given below.*

$n = 3:$ $\Omega_{3,2}$;

$n = 4:$ $\Omega_{4,1},\ \Omega_{4,2},\ \Omega_{4,6},\ \Omega_{4,5}$;

$n = 5:$ $\Omega_{5,1},\ \Omega_{5,2},\ \Omega_{5,3},\ \Omega_{5,4},\ \Omega_{5,6},\ \Omega_{5,7},\ \Omega_{5,9},\ \Omega_{5,11},\ \Omega_{5,13},\ \Omega_{5,15}$;

$n = 6:$ $\Omega_{6,1},\ \Omega_{6,2},\ \Omega_{6,3},\ \Omega_{6,6},\ \Omega_{6,7},\ \Omega_{6,9},\ \Omega_{6,10},\ \Omega_{6,11},\ \Omega_{6,12},\ \Omega_{6,14},\ \Omega_{6,15},$
$\Omega_{6,16},\ \Omega_{6,18},\ \Omega_{6,19}$;

$n = 7:$ $\Omega_{7,1},\ \Omega_{7,2},\ \Omega_{7,4},\ \Omega_{7,5},\ \Omega_{7,6},\ \Omega_{7,7},\ \Omega_{7,9},\ \Omega_{7,10},\ \Omega_{7,11},\ \Omega_{7,13},\ \Omega_{7,14},\ \Omega_{7,15}$;

$n = 8:$ $\Omega_{8,1},\ \Omega_{8,2},\ \Omega_{8,3},\ \Omega_{8,4},\ \Omega_{8,5},\ \Omega_{8,6},\ \Omega_{8,7}$;

$n = 9:$ $\Omega_{9,1},\ \Omega_{9,2}$;

$n = 10:$ $\Omega_{10,1},\ \Omega_{10,2}$ (the Petersen graph).

Proof. (The paper [9] contains only a sketch of a proof of the Lemma.) We shall first prove that there are no graphs on $n > 10$ vertices with diameter 2 and index less than or equal to 3.

Assume to the contrary that G is a graph on $n > 10$ vertices such that $\mathrm{diam}(G) = 2$ and $\lambda_1(G) \leqslant 3$.

We first claim that $\Delta(G) \leqslant 9$, for otherwise $\lambda_1(G) \geqslant \lambda_1(S_{\Delta+1}) = \sqrt{\Delta} > 3$, a contradiction. If $\delta(G) = 1$, let v be a pendant vertex G, and w its neighbour. Since the eccentricity of v is at most 2, w must be adjacent to all vertices of G, but then $n \leqslant 10$, a contradiction.

Therefore, we can assume further on that $\delta(G) > 1$. Let e be the number of edges of G. Then, $\lambda_1(G) \geqslant \frac{2e}{n}$ $(= \bar{d}$, see (1)), with equality if and only if G is regular. By (4) G can have at most 10 $(= 1+3+3\cdot 2)$ vertices, a contradiction. So the average (vertex) degree of G is less than 3, and since none of them is of degree 1, nor all are of degree 3, there exists at least one vertex in G, say u, of degree 2. Denote with v and w its neighbours. Let the remaining vertices ($n-3$ in total) be partitioned as follows: A contains the vertices that are adjacent only to v; B contains the vertices that are adjacent only to w; C contains the vertices that are adjacent to both, v and w. If so

$$|A| + |C| \leqslant 7 \qquad \text{and} \qquad |B| + |C| \leqslant 7.$$

Since $|A| + |B| + |C| = n - 3$ and $n > 10$, we have $|A| > 0$ and $|B| > 0$.

Let all edges incident to v or w be coloured in blue, while the other edges, non-incident to v or w (but incident only to vertices from $A \cup B \cup C$) be coloured in red. Let e_b and e_r be the number of blue and red edges (in G), respectively. Clearly, $e_b \geqslant n - 1 + |C|$.

We now claim that $e_r \geqslant |A| + |B| - 1$. To see this, assume first that $H = \langle A \cup B \cup C \rangle$ (the subgraph induced by the vertex set $A \cup B \cup C$) is connected. But then $e_r \geqslant |A| + |B| + |C| - 1$ $(\geqslant |A| + |B| - 1)$ and we are done. Let x and y be the vertices belonging to different components of H. Since G is of diameter 2, there is a vertex z (in G) adjacent to both, x and y. Clearly, $z \neq u$ (otherwise, $\mathrm{diam}(G) > 2$). If $z \in A \cup B \cup C$, then x and y are not in different components of H. So $z = v$ or w. If $z = v$ then $x, y \in A \cup C$; otherwise, if $z = w$ then $x, y \in B \cup C$. Assume first, that $x \in A$ and $y \in B$. But then we have (in G) that x and y are at distance greater than 2, a contradiction. Otherwise, we have that all vertices from A and B are in the same component of H, and therefore $e_r \geqslant |A| + |B| - 1$, as required.

Consequently, we have

$$\frac{3n}{2} = e = e_b + e_r \geqslant (n - 1 + |C|) + (|A| + |B| - 1) = 2n - 5.$$

But this is equivalent to $n \leqslant 10$, a contradiction.

Hence, there are no graphs on $n > 10$ vertices with diameter 2 and index less than or equal to 3.

By an exhaustive search of connected graphs up to ten vertices one can verify that only the 52 graphs, quoted in the statement of the lemma fulfill the requirements. □

Remark 3.1. (i) The exhaustive search in [9] was performed by the program **nauty**.
(ii) Another possibility to find the 52 graphs from Lemma 3.2 is to use computer assisted reasoning.

Graphs up to 7 vertices can be found using existing graph tables [14,18] (up to 6 vertices), [12] (7 vertices).

Using an interactive graph package we follow the effect of adding vertices and edges to the largest eigenvalue λ_1. (We have used the package newGRAPH available at the address `http://www.mi.sanu.ac.rs/newgraph/`.)

If $\Delta = k$, then there exists a subgraph in the form of the star S_{k+1}.

If $\Delta = 9$, the only solution is $\Omega_{10,1} = S_{10}$, in all other cases $\lambda_1 > 3$.

If $\Delta = 8$, only one edge can be added and we get $\Omega_{9,1} = S_9$ and $\Omega_{9,2}$. Adding a vertex yields $\lambda_1 > 3$.

If $\Delta = 7$, at most two edges can be added and we get $\Omega_{8,1} = S_8$, $\Omega_{8,2}$ and $\Omega_{8,5}$.

If $\Delta = 6$, addition of at most three edges is possible and we get $\Omega_{7,1} = S_7$, $\Omega_{7,5}$, $\Omega_{7,7}$, $\Omega_{7,11}$, $\Omega_{7,15}$.

If $\Delta = 5$, again by adding at most three edges we get $\Omega_{6,1} = S_6$, $\Omega_{6,3}$, $\Omega_{6,6}$, $\Omega_{6,12}$, $\Omega_{6,15}$. Now adding vertices in a specific way is possible and we get $\Omega_{8,4}$.

If $\Delta = 4$, we get $\Omega_{8,3}$ and graphs with less than 8 vertices can be found by graph tables.

The case $\Delta = 3$ is covered by Lemma 3.1, while the cases $\Delta < 3$ are trivial. ◊

Lemma 3.3. *The set $\mathcal{A}_2$ consists of 12 graphs listed below.*

$n = 4: \quad \Omega_{4,4}$;

$n = 5: \quad \Omega_{5,5},\ \Omega_{5,10},\ \Omega_{5,14}$;

$n = 6: \quad \Omega_{6,4},\ \Omega_{6,5},\ \Omega_{6,8},\ \Omega_{6,13},\ \Omega_{6,17}$;

$n = 7: \quad \Omega_{7,3},\ \Omega_{7,8},\ \Omega_{7,12}$.

Proof. (Reproduced from [9].) By Table 1 given in Appendix the above 12 graphs clearly belong to the the set $\mathcal{A}_2$. We shall show that no other graphs H belong to $\mathcal{A}_2$.

Maximal degree of H cannot be 5 since in this case H would contain S_6 with an additional vertex (since $D = 3$). Such a subgraph would have $\lambda_1 > 2.25$ which is forbidden.

If $\Delta = 4$, H contains a subgraph isomorphic to S_5. We can not add an edge to S_5, since then we obtain $\Omega_{5,3}$ with $\lambda_1 > 2.25$ (see Table 1). However, S_5 can be extended with new vertices to graphs $\Omega_{6,8}$ and $\Omega_{7,8}$. No other extensions of vertices and edges are feasible.

Next we have to consider the case $\Delta \leqslant 3$. Now formula (4) gives that H can have at most 10 vertices which completes the proof using Lemma 3.1. □

This completes the proof of Theorem 3.1. ■

Recall from Section 2 that $T_1^9 \subseteq T_{stt}^9 \subseteq T_2^9$ and $T_1^9 \subseteq T_{spt}^9 \subseteq T_2^9$. Using Table 1 from Appendix we can immediately verify the following corollaries of Theorem 3.1.

Corollary 3.1. *The only non-trivial connected graphs G such that $t_1(G) \leqslant 9$ are 14 graphs $\Omega_{i,j}$, where (i,j) is:*

$$(2,1),\ (3,1),\ (3,2),\ (4,j)\ (j \in \{1,\dots,4\}),$$
$$(5,j)\ (j \in \{2,4,8\}),\ (6,4),\ (6,19),\ (7,3),\ (10,2).$$

Corollary 3.2. *The only non-trivial connected graphs G such that $stt(G) \leqslant 9$ are 27 graphs $\Omega_{i,j}$, where (i,j) is:*

$$(2,1),(3,1),(3,2),\ (4,j)\ (j \in \{1,\dots,6\}),\ (5,j)\ (j \in \{2,4,6,8,11\},$$
$$(6,j)\ (j \in \{2,4,7,9,18,19\}),\ (7,j)\ (j \in \{2,3,9,10\}),\ (8,6),(8,7),(10,2).$$

Corollary 3.3. *The only non-trivial connected graphs G such that $spt(G) \leqslant 9$ are 21 graphs $\Omega_{i,j}$, where (i,j) is:*

$$(2,1),\ (3,1),\ (3,2),\ (4,j)\ (j \in \{1,\dots,5\}),\ (5,j)\ (j \in \{1,2,4,8,\}),$$
$$(6,j)\ (j \in \{1,4,14,19\}),\ (7,1),\ (7,3),\ (8,1),\ (10,2).$$

Corollaries 3.1–3.3. have been proved in [9] in another way.

Remark 3.2. In fact in [9] we have proved that $T_2^9 = Q \cup R' \cup S' \cup V'$ where $T_1^9 = Q$, $T_{stt}^9 = Q \cup R'$, $T_{spt}^9 = Q \cup S'$ and $|T_2^9| = 69$.

Here we have

$Q = \{K_2, K_3, K_4, K_5, P_3, P_4, C_4, C_5, C_6, C_7, K_{1,3}, K_{2,3}, K_{3,3}, PG\}$,
$S' = \{P_5, K_{1,4}, K_{1,5}, K_{1,6}, K_{1,7}, K_{1,8}, K_{1,9}\}$,
$R' = \{\Omega_{4,5}, \Omega_{4,6}, \Omega_{5,6}, \Omega_{5,11}, \Omega_{6,2}, \Omega_{6,7}, \Omega_{6,9}, \Omega_{6,18}, \Omega_{7,2}, \Omega_{7,9}, \Omega_{7,10}, \Omega_{8,6}, \Omega_{8,7}\}$ and V' consists of the remaining 35 graphs. Here, PG denotes the Petersen graph. We see that the sets Q and S' (related to tightness t_1 and spt) contain only the standard graphs. When considering stt and t_2, the graphs with non-standard names occur. ◊

4. Load balancing problem and integral graphs

Among graphs with small tightness integral graphs deserve special attention because of their suitability for solving the load balancing problems in multiprocessors. The purpose of this section is to present some arguments in this direction.

Note first that each integral eigenvalue admits a basis consisting of integral eigenvectors (see [6]); so for integral graphs, there exists an integral eigenbasis for the whole space. Therefore, in integral graphs load balancing algorithms for multiprocessors (based on spectral techniques) can be executed in integer arithmetic. This possibility was mentioned in [9], and here we elaborate it more.

We shall first describe the load balancing problem in some details. Let us assume that a multiprocessor system is modelled by an integral graph G. The vertices of G correspond to processors while edges represent links between processors.

The job which has to be executed by a multiprocessor system is divided into parts assigned to particular processors to handle them. In other words, the whole job consists of a number of elementary jobs (modules, items) so that each processor gets a number of such items to execute. Mathematically, the item distribution among processors can be viewed as a vector $\mathbf{x}$ whose coordinates are non-negative integers. If the coordinates are associated to vertices of G (as their weights), then they indicate how many items are assigned to the corresponding processors.

Vector $\mathbf{x}$ is usually changed during the work of the system because some items are executed, while new items are arriving for execution. Of course, it would be optimal that the number of items assigned to any processor is the same, i.e. that the vector $\mathbf{x}$ is an integer multiple of all-1 vector $\mathbf{j}$. Since this is not always possible, it is reasonable that processors with more items assigned (or more load) send some of them to adjacent processors to make the item distribution more uniform. This gives raise to the so called (dynamic) *load balancing problem*, which turns to be very important in managing multiprocessor systems. The load balancing problem requires creation of algorithms for moving items among processors in order to achieve the uniform distribution (see, for example, [19,20] for further information on the load balancing problem).

We shall present an *ad hoc* algorithm for the load balancing problem which is based on eigenvalues and eigenvectors of the adjacency matrix of a regular integral graph. At this stage we do not claim that our algorithm is any better than the existing ones. We just want to suggest that calculations in integer arithmetic could offer some advantages.

One idea for generating *integral eigenbases* (i.e. those consisting of integral eigenvectors) could be taken from the theory of star partitions of a graph (see [17], Chapter 7).

Definition 4.1. *Let G be a graph on n vertices with m distinct eigenvalues $\mu_1, \ldots, \mu_m$. A partition $X_1 \cup \cdots \cup X_m$ of the vertex set $V(G)$ of G is a* star partition *of G if for each $i \in \{1, \ldots, m\}$, μ_i is not an eigenvalue of $G - X_i$.*

It is known (see [15]) that for any graph at least one star partition exists (a polynomial time algorithm for constructing star partitions is described in [16]). Therefore, the following algorithm for generating integral eigenbases can be proposed:

Algorithm 4.1. For any vertex $v \in X_i$ the subgraph of G induced by the vertex set $(V(G) \setminus X_i) \cup \{v\}$ (call it $H_i + v$) has μ_i as a simple eigenvalue. If G is an integral graph, then clearly μ_i is an integer, and the corresponding eigenvector of $H_i + v$ can be chosen to be integral. Extending it with zeros for coordinates corresponding to vertices from $X_i \setminus \{v\}$, we obtain an n-dimensional integral vector which is an eigenvector of G for μ_i. In this way n independent integral eigenvectors of G can be found, as required. ◯

Remark 4.1. The constructed eigenvectors are not necessarily mutually orthogonal within the fixed eigenspace. However, the eigenvectors for the eigenvalue μ_i found by algorithm have at least $k_i - 1$ coordinates equal to 0, where k_i ($= |X_i|$) is the multiplicity of μ_i. Needless to say, integral coordinates, including a lot of zeros in many cases, represent an advantage of this approach. The constructed basis can be transformed into an orthogonal integral one, but then the total number of non-zero coordinates in the resulting eigenbasis vectors could increase. $\diamondsuit$

Before describing the load balancing algorithm, we illustrate it by an example.

Example 4.1. All star partitions of the Petersen graph are given in [17], pp. 180–181. Using the first of them we have constructed 10 independent integral eigenvectors of the Petersen graph. They are displayed in Fig. 10.

Recall that $3, 1^5, (-2)^4$ is the spectrum of the Petersen graph. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_9$ denote an integral eigenbasis for the Petersen graph, where $\mathbf{v}_0$ is the all-1 vector belonging to eigenvalue 3, while $\mathbf{v}_1, \ldots, \mathbf{v}_5$ and $\mathbf{v}_6, \ldots, \mathbf{v}_9$ are eigenvectors belonging to eigenvalues 1 and -2, respectively. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_9$ are given on first 9 copies of the Petersen graph on Fig. 10. The sum of coordinates of each of these vectors is equal to 0, since they are orthogonal to all-1 vector. Only non-zero entries of the vectors are indicated as vertex labels. The labelling of vertices by numbers $1, 2, \ldots, 10$ can be arbitrary but it is understood that the labelling of vertices is the same in all copies of the Petersen graph.

Assume that *a balancing flow* shows how to move the commodity from the sources (vertices with positive weights) to the sinks (vertices with negative weights) in order to transform the corresponding eigenvector to 0-vector.

For each eigenvector $\mathbf{v}_1, \ldots, \mathbf{v}_9$ one of the simplest balancing flow is indicated by directed thick edges connecting non-zero entries of the corresponding eigenvector (see the first 9 copies of the Petersen graph on Fig. 10). In our example, the value of the flow through each thick edge is equal to 1 and the direction of the flow is indicated by an arrow. There is no flow through other edges.

A balancing flow can be viewed as a vector whose dimension is equal to the number of edges in G ($= 15$ in our case), provided the edges are numbered and each edge is directed in an arbitrary (but fixed) way. Let $\mathbf{b}_1, \ldots, \mathbf{b}_9$ be the balancing flow vectors that correspond to eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_9$.

The tenth copy of the Petersen graph on Fig. 10 contains an initial load distribution among processors given by weights on vertices. The difference between the corresponding load vector $\mathbf{x}$ and the vector of uniform load distribution $30\mathbf{v}_0$ can be represented as linear combination

$$\mathbf{x} - 30\mathbf{v}_0 = 11\mathbf{v}_1 + 4\mathbf{v}_2 + 5\mathbf{v}_3 - 12\mathbf{v}_5 - 5\mathbf{v}_8 - 3\mathbf{v}_9.$$

The resulting balancing flow vector $\mathbf{b}$ is given on the same copy of the Petersen graph by edge weights. It is obtained by the above linear combination of balancing flows vectors $\mathbf{b}_1, \ldots, \mathbf{b}_9$, i.e.

$$\mathbf{b} = 11\mathbf{b}_1 + 4\mathbf{b}_2 + 5\mathbf{b}_3 - 12\mathbf{b}_5 - 5\mathbf{b}_8 - 3\mathbf{b}_9.$$

When the flow is realized each vertex has a load equal to 30.

In this way we have defined a load flow on the edge set of the Petersen graph. Particular amounts of load flow should be considered algebraically, i.e. having in mind their sign. The flow through an edge ij at the end has a nonnegative value which is sent either from i to j or vice versa. $\triangledown$

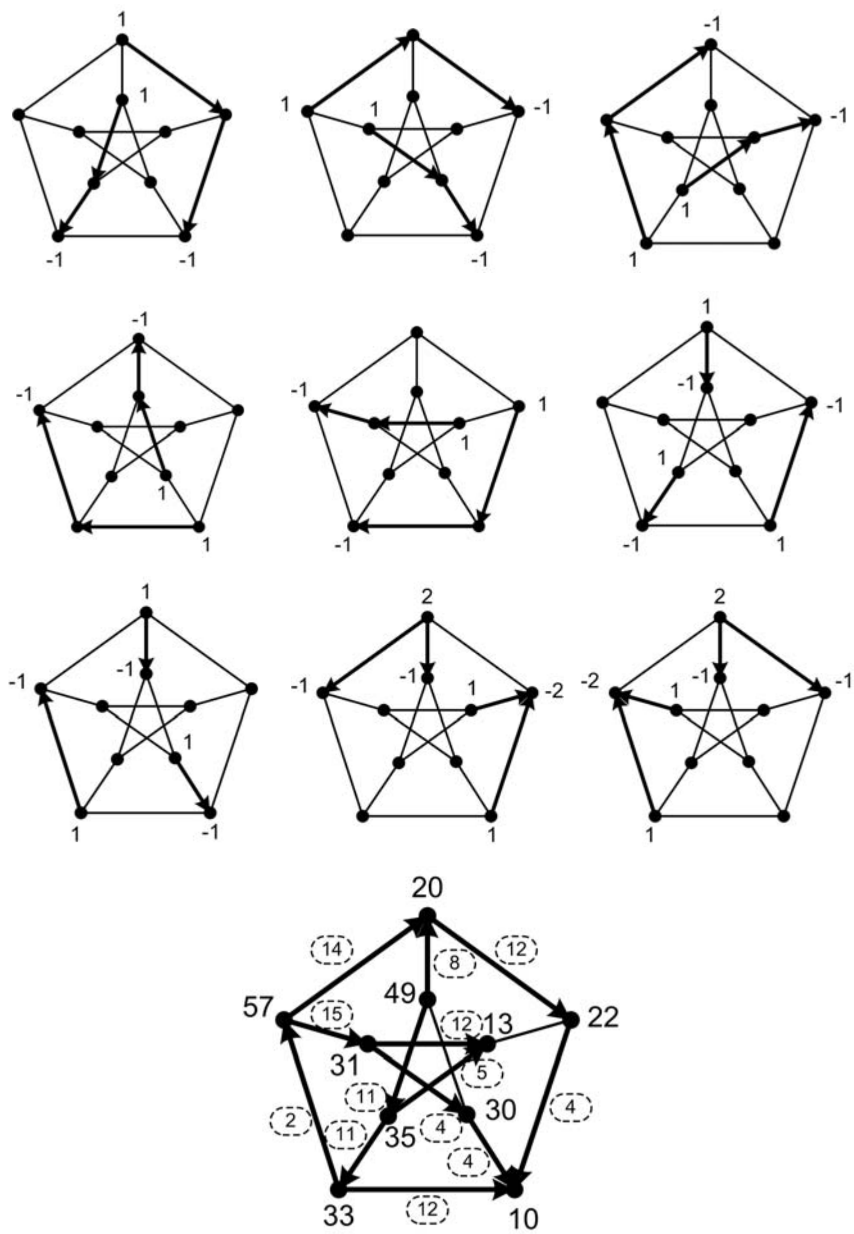

Figure 10. Integral eigenvectors and the load balancing of the Petersen graph

Example 4.1. can be generalized to all regular integral graphs, since all–1 vector is an eigenvector for the largest eigenvalue; thus the sum of coordinates of eigenvectors for other eigenvalues is equal to 0 (by orthogonality to all–1 vector).

Algorithm 4.2. Given an integral regular graph G on n vertices and e edges, find by Algorithm 4.1. an integral eigenbasis $B=\{\mathbf{v}_0,\dots,\mathbf{v}_{n-1}\}$ of G. For each eigenvector $\mathbf{v}_i \in B$ other than $\mathbf{v}_0 = \mathbf{j}$ (all–1 vector), define *ad hoc* an e-dimensional balancing flow vector $\mathbf{b}_i$. Represent the difference between the load vector and the vector of uniform load distribution as a linear combination of the vectors from B. Form the same linear combination of balancing flow vectors $\mathbf{b}_i$ to obtain the resulting load balancing flow $\mathbf{b}$.◯

Remark 4.2. When choosing the balancing flow vectors $\mathbf{b}_i$ in Example 4.1. we were concerned to minimize the number of edges with non-zero flow, instead of ℓ_2-optimality (cf. [19]). However, one can use various integral bases of eigenspaces and for each eigenvector the balancing flow could be defined in several ways. These facts indicate that one should further study the load balancing in integral graphs and find, if possible, flow plans which would be optimal according to some criteria. ◊

The suggested approach via eigenvectors can be extended to interconnection networks constructed by some operations on simpler networks. For example, in [24] the Cartesian product (known also as the sum) of two Petersen graphs, and similar complex networks, have been studied. In the sum of graphs the eigenvectors are the Kronecker products of the eigenvectors of the starting graphs (see, for example, [11], pp. 70–71) and the balancing flow can be easily constructed.

For non-regular graphs, we should use the Laplacian instead of the adjacency matrix. Recall that, given a graph, the matrix $L = D - A$ is called the *Laplacian*, where A is the adjacency matrix and D is the diagonal matrix of vertex degrees. Now all-1 vector is an eigenvector for the smallest Laplacian eigenvalue (in any graph).

A graph is called *Laplacian integral* if all its Laplacian eigenvalues are integral. In Laplacian integral graphs, we again have an eigenbasis consisting of integral vectors. Laplacian integral graphs are generally more frequent than integral graphs.

The further study of integral graphs in connection to multiprocessor topologies seems to be a promising subject for future research.

5. Integral graphs with small tightness

In this section we present a classification of integral graphs with small tightness with emphasis on regular graphs, indicating also the open enumeration problems.

We shall need the fact that $t_2(K_n) = 2n - 2$.

Proposition 5.1. *We have:*
(i) *if $p > 2$ is a prime, there are no graphs G such that $t_2(G) = p$;*
(ii) *integral Smith graphs are the following graphs: K_3, C_4, $K_{1,4}$, C_6, W_6, H_7;*
(iii) *the only graph G with $t_2(G) = 2k$, k being a prime greater than 5, is the graph K_{k+1}.*

Proof. (i) As known, a rational eigenvalue of a graph is an integer. Therefore, if $t_2(G) = p$, then either $\lambda_1 = p$ and $D = 0$ or $\lambda_1 = 1$ and $D = p - 1$. In both cases G does not exist.

(ii) By inspecting spectra of Smith graphs from [13] we easily get the graphs quoted.

(iii) If we put $D + 1 = 2$ and $\lambda_1 = k$, then clearly $G = K_{k+1}$. In the case $D + 1 = k$ and $\lambda_1 = 2$ we have to find integral Smith graphs with $D \geqslant 6$. By (ii) such graphs do not exist. ■

By Proposition 5.1(i), there are no integral graphs G with $t_2(G) = 3, 5, 7, 11, 13, 17, 19$.

Based on Proposition 5.1, we shall give a survey of integral graphs with $t_2 \leqslant 20$.

$t_2(G) = 2$. We have $G = K_2$.

$t_2(G) = 4$. We have $G = K_3$.

$t_2(G) = 6$. We have $G = K_4, C_4, K_{1,4}$ (by Proposition 5.1(ii)).

$t_2(G) = 8$. We have $G = K_5, C_6, W_6$ (by Proposition 5.1(ii)).

$t_2(G) = 9$. All connected graphs G with $t_2(G) \leqslant 9$ have been determined in Theorem 3.1. There are 69 such graphs and among them exactly 14 are integral (see Table 1 in Appendix). Those with $t_2(G) = 9$ are the following graphs: regular graphs $\Omega_{6,18}$, $\Omega_{6,19} = K_{3,3}$ and the Petersen graph, and non-regular graphs $\Omega_{10,1} = K_{1,9}$, $\Omega_{5,15}$ and $\Omega_{7,15}$.

$t_2(G) = 10$. We have $G = K_6, H_7$ (by Proposition 5.1(ii-iii)).

$t_2(G) = 12$. Here we have K_7 for $\lambda_1 = 6$ and by Proposition 5.1(ii), there are no integral graphs for $\lambda_1 = 2$. The following two cases remain.

$\lambda_1 = 3$ and $D = 3$. All integral cubic graphs are well known [5]. Those with $D = 3$ are graphs denoted in [5] by G_4, G_7, G_8, G_{11} and they have 8, 10, 12, 10 vertices respectively. G_4 is the cube graph. Non-regular case is not yet completely explored; only non-regular connected integral graphs with $\lambda_1 = 3$ and $\Delta = 4$ are found.

$\lambda_1 = 4$ and $D = 2$. Integral regular graphs of degree 4 up to 24 vertices are listed in [23]. Those which fulfill the requirements are $B_1 = K_{4,4}$, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_{11} (graph names as in [23]). There are only sporadic data in the literature for non-regular case.

$t_2(G) = 14$. We have only $G = K_8$ (by Proposition 5.1(iii)).

$t_2(G) = 15$. $\lambda_1 = 3$ and $D = 4$. Out of 13 cubic integral graphs [5] there are two which fulfill these requirements G_7 (Tutte's 8-cage on 30 vertices with girth 8) and G_{12} (6-sided prism). Non-regular case is not explored yet.

$\lambda_1 = 5$ and $D = 2$. We have no data on such graphs.

$t_2(G) = 16$. Here we have K_9 for $\lambda_1 = 8$ and by Proposition 5.1(ii), there are no integral graphs for $\lambda_1 = 2$. The following case remains.

$\lambda_1 = 4$ and $D = 3$. Among integral regular graphs of degree 4 up to 24 vertices, listed in [23], there are 16 graphs which fulfill the requirements (B_2, B_3, B_4, B_5, B_{11}, B_{12}, B_{13}, B_{20}, B_{22}, B_{25}, B_{26}, D_9, D_{10}, D_{12}, D_{13}, D_{14}). We have no data for non-regular case.

$t_2(G) = 18$. Here we have K_{10} for $\lambda_1 = 9$ and by Proposition 5.1(ii), there are no integral graphs for $\lambda_1 = 2$. The following two cases remain.

$\lambda_1 = 3$ and $D = 5$. Out of 13 cubic integral graphs [5] there are two which fulfill these requirements (G_9 and G_{10}, i.e. the Desargues graph and a graph cospectral to it). As noted above, the non-regular case is still unexplored.

$\lambda_1 = 6$ and $D = 2$. An example is the complement of the Petersen graph.

$t_2(G) = 20$. Here we have K_{11} for $\lambda_1 = 10$ and by Proposition 5.1(ii), there are no integral graphs for $\lambda_1 = 2$. The following two cases remain.

$\lambda_1 = 4$ and $D = 4$. Among integral regular graphs of degree 4 up to 24 vertices (see [23]) there are 34 graphs which fulfill the requirements. We have no data for non-regular case.

$\lambda_1 = 5$ and $D = 3$. We have no data on such graphs.

Basic data on connected integral graphs up to 10 vertices can be found in [1]. According to [3] there are several integral graphs on 13 vertices which fall into the above classification.

We conclude our considerations by some remarks and data on Laplacian integral graphs with small tightness. Note that they are more frequent than integral graphs with adjacency spectrum, but much less studied in the literature. Among the 69 graphs that we consider in this paper the following ones are Laplacian integral (37 graphs):

$$\Omega_{2,1},\ \Omega_{3,1},\ \Omega_{3,2},\ \Omega_{4,j}\ (j \in \{1,\dots,6\}),\ \Omega_{5,j}\ (j \in \{1,3,4,7,8,9,11,15\}),$$
$$\Omega_{6,j}\ (j \in \{1,3,4,6,12,14,16,18,19\}),\ \ \Omega_{7,j}\ (j \in \{1,5,7,11,15\}),$$
$$\Omega_{8,j}\ (j \in \{1,2,5\}),\ \Omega_{9,1},\ \Omega_{9,2},\ \Omega_{10,1},\ \ \Omega_{10,2}.$$

Laplacian spectra of connected graphs up to 5 vertices can be found in [18], Table A1. For other graphs we used system newGRAPH.

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Appendix

The Table 1 given below contains some relevant data about 69 graphs with second type mixed tightness not exceeding 9.

Graphs are ordered first by n (the number of vertices), and within the groups with fixed n, by t_2. Columns of the table provide graph name, the number of vertices n, the number of edges e, the name(s) under which the graph appeared in [9], diameter D, maximum vertex degree Δ, the number of distinct eigenvalues m, the spectrum starting with the largest eigenvalue λ_1. Last four columns contain the values of the four types of tightness t_1, stt, spt, t_2.

As "the old names" we used different notation. First we distinguish the well known graphs such as complete graphs, circuits, stars, complete bipartite graphs, and so on. For graphs up to 5 vertices we used the notation from [11], while graphs on $n = 6$ vertices are marked primarily as in [14]. $N(n, j)$ denotes the j-th graph on n vertices generated by program **nauty**. PG denotes the well known Petersen graph.

Table 1: Graphs on up to 10 vertices with small tightness

graph	n	e	old name(s)	D	Δ	m	$\lambda_1, \lambda_2, \ldots, \lambda_n$	t_1	stt	spt	t_2
$\Omega_{2,1}$	2	1	$G_1 = K_2 = P_2$	1	1	2	**1**, -1	2	2	2	2
$\Omega_{3,1}$	3	3	$G_2 = K_3 = C_3$	1	2	2	**2**, -1, -1	4	4	4	4
$\Omega_{3,2}$		2	$G_3 = P_3 = S_3 = K_{1,2}$	2	2	3	**1.41**, 0, -1.41	6	6	4.23	4.23
$\Omega_{4,1}$	4	3	$G_8 = S_4 = K_{1,3}$	2	3	3	**1.73**, 0, 0, -1.73	9	9	5.19	5.19
$\Omega_{4,2}$		4	$G_7 = C_4 = K_{2,2}$	2	2	3	**2**, 0, 0, -2	6	6	6	6
$\Omega_{4,3}$		6	$G_4 = K_4$	1	3	2	**3**, -1, -1, -1	6	6	6	6
$\Omega_{4,4}$		3	$G_9 = P_4$	3	2	4	**1.62**, 1.62, 0.62, -0.62, -1.62	8	8	6.47	6.47
$\Omega_{4,5}$		4	G_6	2	3	4	**2.17**, 0.31, -1, -1.48	12	9	8.68	6.51
$\Omega_{4,6}$		5	G_5	2	3	4	**2.56**, 0, -1, -1.56	12	9	10.24	7.68
$\Omega_{5,1}$	5	5	$G_{28} = S_5 = K_{1,4}$	2	4	3	**2**, 0, 0, 0, -2	12	12	6	6
$\Omega_{5,2}$		5	$G_{27} = C_5$	2	2	3	**2**, 0.62, 0.62, -1.62, -1.62	6	6	6	6
$\Omega_{5,3}$		5	G_{23}	2	4	5	**2.34**, 0.47, 0, -1, -1.81	20	12	11.71	7.03
$\Omega_{5,4}$		6	$G_{22} = K_{2,3}$	2	3	3	**2.45**, 0, 0, 0, -2.45	9	9	7.35	7.35
$\Omega_{5,5}$		4	G_{29}	3	3	5	**1.85**, 0.77, 0, -0.77, -1.85	15	12	9.24	7.39
$\Omega_{5,6}$		6	G_{21}	2	3	5	**2.48**, 0.69, 0, -1.17, -2	15	9	12.41	7.44
$\Omega_{5,7}$		6	G_{20}	2	4	4	**2.56**, 1, -1, -1, -1.46	16	12	10.25	7.68
$\Omega_{5,8}$		10	$G_{10} = K_5$	1	4	2	**4**, -1, -1, -1, -1	8	8	8	8
$\Omega_{5,9}$		6	G_{18}	2	4	5	**2.69**, 0.33, 0, -1.27, -1.75	20	12	13.43	8.06
$\Omega_{5,10}$		5	G_{26}	3	3	5	**2.14**, 0.66, 0, -0.66, -2.14	15	12	10.68	8.54
$\Omega_{5,11}$		7	G_{17}	2	3	5	**2.86**, 0.32, 0, -1, -2.18	15	9	14.28	8.57
$\Omega_{5,12}$		4	$G_{30} = P_5$	4	2	5	**1.73**, 1, 0, -1, -1.73	10	10	8.66	8.66
$\Omega_{5,13}$		7	G_{16}	2	4	5	**2.94**, 0.62, -0.46, -1.47, -1.62	20	12	14.68	8.81
$\Omega_{5,14}$		5	G_{25}	3	3	5	**2.21**, 1, -0.54, -1, -1.67	15	12	11.07	8.86
$\Omega_{5,15}$		7	G_{15}	2	4	4	**3**, 0, 0, -1, -2	16	12	12	9

Table 1: Graphs on up to 10 vertices with small tightness (cont.)

graph	n	e	old name(s)	D	Δ	m	$\lambda_1, \lambda_2, \ldots, \lambda_n$	t_1	stt	spt	t_2
$\Omega_{6,1}$	6	5	$S_6 = K_{1,5} = CP(107) = N(6,1)$	2	5	3	**2.24**, 0, 0, 0, 0, -2.24	15	15	6.71	6.71
$\Omega_{6,2}$		7	$CP(93) = N(6,35)$	2	3	6	**2.39**, 0.77, 0.62, 0, -1.62, -2.16	18	9	14.35	7.17
$\Omega_{6,3}$		6	$CP(94) = N(6,3)$	2	5	5	**2.51**, 0.57, 0, 0, -1, -2.09	25	15	12.57	7.54
$\Omega_{6,4}$		6	$C_6 = CP(106) = N(6,49)$	3	2	4	**2**, 1, 1, -1, -1, -2	8	8	8	8
$\Omega_{6,5}$		5	$CP(109) = N(6,5)$	3	3	5	**2**, 1, 0, 0, -1, -2	15	12	10	8
$\Omega_{6,6}$		7	$CP(79) = N(6,17)$	2	5	5	**2.71**, 1, 0.19, -1, -1, -1.90	25	15	13.55	8.13
$\Omega_{6,7}$		8	$CP(72) = N(6,89)$	2	3	6	**2.74**, 0.71, 0.62, -0.23, -1.62, -2.22	18	9	16.45	8.22
$\Omega_{6,8}$		5	$CP(108) = N(6,2)$	3	4	5	**2.07** , 0.84, 0, 0, -0.84, -2.07	20	16	10.37	8.30
$\Omega_{6,9}$		8	$CP(69) = N(6,90)$	2	3	6	**2.79**, 1, 0.62, -1, -1.62, -1.79	18	9	16.75	8.37
$\Omega_{6,10}$		8	$CP(71) = N(6,36)$	2	4	5	**2.80**, 0.85, 0, 0, -1.20, -2.45	20	12	13.98	8.39
$\Omega_{6,11}$		8	$CP(68) = N(6,57)$	2	4	6	**2.81**, 1, 0.53, -1, -1.34, -2	24	12	16.88	8.44
$\Omega_{6,12}$		7	$CP(75) = N(6,8)$	2	5	5	**2.81**, 0.53, 0, 0, -1.34, -2	25	15	14.07	8.44
$\Omega_{6,13}$		6	$CP(105) = N(6,19)$	3	3	6	**2.11**, 1, 0.62, -0.25, -1.62, -1.86	18	12	12.69	8.46
$\Omega_{6,14}$		8	$K_{2,4} = CP(73) = N(6,13)$	2	4	3	**2.83**, 0, 0, 0, 0, -2.83	12	12	8.49	8.49
$\Omega_{6,15}$		8	$CP(66) = N(6,39)$	2	4	5	**2.90**, 0.81, 0, 0, -1.71, -2	20	12	14.52	8.71
$\Omega_{6,16}$		8	$CP(61) = N(6,32)$	2	5	6	**2.95**, 1.16, 0, -1, -1.29, -1.82	30	15	17.68	8.84
$\Omega_{6,17}$		6	$CP(102) = N(6,18)$	3	3	6	**2.25**, 0.80, 0.55, -0.55, -0.80, -2.25	18	12	13.48	8.99
$\Omega_{6,18}$		9	$CP(51) = N(6,93)$	2	3	4	**3**, 1, 0, 0, -2, -2	12	9	12	9
$\Omega_{6,19}$		9	$K_{3,3} = CP(52) = N(6,71)$	2	3	3	**3**, 0, 0, 0, 0, -3	9	9	9	9
$\Omega_{7,1}$	7	6	$S_7 = K_{1,6} = N(7,1)$	2	6	3	**2.45**, 0, 0, 0, 0, 0, -2.45	18	18	7.35	7.35
$\Omega_{7,2}$		9	$N(7,337)$	2	3	5	**2.66**, 1.21, 0.62, 0.62, -1.62, -1.62, -1.87	15	9	13.28	7.97
$\Omega_{7,3}$		7	$C_7 = N(7,292)$	3	2	4	**2**, 1.255, 1.25, -0.45, -0.45, -1.80, -1.80	8	8	8	8
$\Omega_{7,4}$		9	$N(7,156)$	2	4	6	**2.68**, 1, 0.64, 0, 0, -2, -2.32	24	12	16.09	8.04
$\Omega_{7,5}$		7	$N(7,3)$	2	6	5	**2.68**, 0.64, 0, 0, 0 ,-1, -2.32	30	18	13.41	8.04

Table 1: Graphs on up to 10 vertices with small tightness (cont.)

graph	n	e	old name(s)	D	Δ	m	$\lambda_1, \lambda_2, \ldots, \lambda_n$	t_1	stt	spt	t_2
$\Omega_{7,6}$	7	9	$N(7,75)$	2	4	6	**2.75**, 0.84, 0.62, 0, 0, -1.62, -2.59	24	12	16.51	8.25
$\Omega_{7,7}$		8	$N(7,23)$	2	6	6	**2.86**, 1, 0.32, 0, -1, -1, -2.18	36	18	17.13	8.57
$\Omega_{7,8}$		6	$N(7,5)$	3	4	5	**2.18**, 1.13, 0, 0, 0, -1.13, -2.18	20	16	10.88	8.70
$\Omega_{7,9}$		10	$N(7,624)$	2	3	7	**2.90**, 1.41, 0.81, 0, -1.41, -1.71, -2	21	9	20.32	8.71
$\Omega_{7,10}$		10	$N(7,514)$	2	3	6	**2.90**, 0.81, 0.73, 0, 0, -1.71, -2.73	18	9	17.42	8.71
$\Omega_{7,11}$		8	$N(7,8)$	2	6	5	**2.94**, 0.66, 0, 0, 0, -1.37, -2.24	30	18	14.72	8.83
$\Omega_{7,12}$		7	$N(7,92)$	3	3	6	**2.21**, 1, 1, 0, -0.54, -1.68, -2	18	12	13.29	8.86
$\Omega_{7,13}$		10	$N(7,448)$	2	4	7	**2.98**, 1.33, 0.65, 0, -1, -1.77, -2.19	28	12	20.86	8.94
$\Omega_{7,14}$		9	$N(7,324)$	2	4	7	**2.97**, 0.80, 0.70, 0.45, -0.55, -2.12, -2.25	28	12	20.77	8.90
$\Omega_{7,15}$		9	$N(7,219)$	2	6	4	**3**, 1, 1, -1, -1, -1, -2	24	18	12	9
$\Omega_{8,1}$	8	7	$S_8 = K_{1,7} = N(8,1)$	2	7	3	**2.65**, 0, 0, 0, 0, 0, 0, -2.65	21	21	7.94	7.94
$\Omega_{8,2}$		8	$N(8,3)$	2	7	5	**2.54** , 0.69, 0, 0, 0, 0, -1, -2.84	35	21	14.22	8.53
$\Omega_{8,3}$		11	$N(8,1039)$	2	4	8	**2.90**, 1.30, 0.81, 0.62, 0, -1.62, -1.71, -2.30	32	12	23.23	8.71
$\Omega_{8,4}$		11	$N(8,342)$	2	5	6	**2.98**, 1.13, 0.65, 0, 0, 0, -2.07, -2.68	30	15	17.86	8.93
$\Omega_{8,5}$		9	$N(8,30)$	2	7	6	**3**, 1, 0.41, 0, 0, -1, -1, -2.41	42	21	18	9
$\Omega_{8,6}$		12	$N(8,8469)$	2	3	6	**3**, 1.56, 0.62, 0.62, 0, -1.62, -1.62, -2.56	18	9	18	9
$\Omega_{8,7}$		12	$N(8,6660)$	2	3	5	**3**, 1, 1, 0.41, 0.41, -1, -2.41,-2.41	15	9	15	9
$\Omega_{9,1}$	9	8	$S_9 = K_{1,8} = N(9,1)$	2	8	3	**2.83**, 0, 0, 0, 0, 0, 0, 0, -2.83	24	24	8.49	8.49
$\Omega_{9,2}$		9	$N(9,3)$	2	8	5	**3**, 0.73, 0, 0, 0, 0, 0, -1, -2.73	40	24	15	9
$\Omega_{10,1}$	10	9	$S_{10} = K_{1,9} = N(10,1)$	2	9	3	**3**, 0, 0, 0, 0, 0, 0, 0, 0, -3	27	27	9	9
$\Omega_{10,2}$	10	15	$PG = N(10,8027956)$	2	3	3	**3**, 1, 1, 1, 1, 1, -2, -2, -2, -2	9	9	9	9