Final Coalgebras are Ideal Completions of Initial Algebras

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Abstract

For $\omega$-continuous endofunctors of Set an ordering of a final coalgebra $T$ is exhibited which makes $T$ a CPO. Moreover, an initial algebra, considered as a canonical subobject of $T$, has $T$ as its ideal completion. In more generality, for $\omega$-continuous endofunctors of locally finitely presentable categories the analogous result holds: here the ordering is considered on the hom-sets $\text{horn}(B, T)$ for all finitely presentable objects $B$.

Keywords: Final coalgebra, initial algebra, complete partial order, ideal completion, algebraically complete, locally presentable category.

1 Introduction

1.1 Functors over Set

Initial algebras $I$ and final coalgebras $T$ of functors $F : \mathcal{K} \to \mathcal{K}$ represent an important tool for specification of data types as objects of the base category $\mathcal{K}$. Much depends on the structure of the category $\mathcal{K}$; e.g. if $\mathcal{K}$ is enriched over CPO (the category of all complete partial orders, i.e. posets with $\bot$ and joins of directed subsets, and all continuous maps) then all enriched endofunctors fulfil

$$ I \cong T. $$

That is, given an initial algebra $FI \xrightarrow{\gamma} I$, then the inverse is a final coalgebra $I \xleftarrow{\nu} FI$. In the present paper we show that CPOs play an important role in the theory of final coalgebras also for categories lacking any structure, e.g. for the category Set of sets. We prove that every $\omega$-continuous (i.e. $\omega^{op}$-limits preserving) endofunctor $F : \text{Set} \to \text{Set}$ with $F\emptyset \neq \emptyset$ carries a ‘natural’ ordering on its final coalgebra $T$ such that

(a) $T$ is a CPO,
and
(b) $T$ is an ideal completion of the initial algebra $I$ is considered as a canonical subobject of $T$.

The ordering of $T$ depends on a choice of an element $p$ of $F\emptyset$. In the important example of a polynomial functor, e.g.

$$ FX = A + B \times X \times X \quad \text{(thus, } p \in A) $$

the ordering can be explained simply: it is well known that $T$ is the coalgebra of all properly labelled trees (in our concrete case these are all binary trees with leaves labelled in $A$ and
inner nodes labelled in $B$), and $I$ is the algebra of all finite properly labelled trees. Given
trees $x, y \in T$, put

$$x \leq y \text{ iff } x \text{ is obtained from } y \text{ by cutting }$$
$$y \text{ at depth } n \text{ and relabelling nodes }$$
$$\text{of depth } n \text{ by } p \text{ (for some } n < \omega).$$

It is easy to see that this is a CPO. The reason why $T$ is an ideal completion of $I$ is that every
infinite tree $y$ is an $\omega$-join of its cuttings $y_{(n)} \ (n < \omega)$. Example:

Our result is reminiscent of the fact established by M. Barr [7] that $\omega$-continuous and
finitary set functors $F$ with $F^0 \neq \emptyset$ have a complete metric on $T$ such that $T$ is a Cauchy
completion of $I$. (Finitarity is actually not needed, see [3].) In the present example of trees,
the distance $d(x, y)$ is $2^{-k}$ for the largest $k$ such that cutting $x$ and $y$ at level $k$ results in the
same tree.

For general $\omega$-continuous set functors, the ordering of $T$ uses the well-known fact that $T$
is a limit of the sequence

$$1 \xrightarrow{t} F1 \xrightarrow{F t} F^2 1 \xrightarrow{F^2 t} \cdots$$

(where $t$ is the unique morphism into the chosen terminal object 1). The limit projections
$t_n : T \to F^n 1$ have a splitting $e_n : F^n 1 \to T$ (with $t_n e_n = \text{id}$) based on the given choice of
an element of $F^0$, and the ordering of $T$ is defined by

$$x \leq y \text{ iff } x = y \text{ or } x = e_n t_n y \text{ for some } n.$$ 

We observe that, then, not only is $T$ a CPO, but also $t_n$ and $e_n$ form an embedding-projection
pair. Also the sequence above is formed by embedding-projection pairs if each $F^n 1$ is ordered
analogously to $T$.

### 1.2 Functors over $\text{Pos}$ and beyond

In a recent paper by D. Pavlović and V. Pratt [9] real numbers have been described as a final
coaLGebra of a simple endofunctor $F_2$ of $\text{Pos}$. This gives rise to interest in data types defined
in $\text{Pos}$ — or, more generally, in any locally finitely presentable (LFP) category of Gabriel
and Ulmer, see [8] or [6]. Recall that a cocomplete category $\mathcal{K}$ is called LFP provided that it
has a set $\mathcal{B}$ of objects $B$ that are finitely presentable (i.e. such that $\text{hom}(B, -) : \mathcal{K} \to \text{Set}$
preserves filtered colimits) such that every object is a filtered colimit of objects in $\mathcal{B}$. In such
a category every object $K$ is fully described by the hom-sets $\text{hom}(B, K)$ for $B \in \mathcal{B}$. For
example, in the category $\text{Pos}$, every object $K$ is fully described by (i) the set of all elements
of $K$, which is $\text{hom}(1, K)$ for the one-element poset 1, and (ii) the set of all ordered pairs in
$K$, which is $\text{hom}(2, K)$ for the two-element chain 2. The idea of ‘mimicking’ the situation with ordering of $T$ for set functors in our category $\mathcal{K}$ is simple: we order the sets

$$\text{hom}(B, T) \quad \text{for all finitely presentable object } B.$$

Assuming that $F : \mathcal{K} \to \mathcal{K}$

(a) is $\omega$-continuous
(b) preserves monomorphisms,

and

(c) has an element in $F^0$

(i.e. a morphism $p : 1 \to F^0$ can be chosen), we conclude that $\text{hom}(B, T)$, ordered quite analogously to the above case $\mathcal{K} = \text{Set}$, is a CPO. Furthermore, we prove that $F$ has an initial algebra which is a canonical subobject

$$u : I \to T$$

of the final coalgebra. This defines an embedding

$$\text{hom}(B, I) \hookrightarrow \text{hom}(B, T) \text{ by } x \mapsto u.x$$

and we prove, again, that $\text{hom}(B, T)$, as a subposet of $\text{hom}(B, T)$, has the ideal completion $\text{hom}(B, T)$.

Returning to $\mathcal{K} = \text{Pos}$, we conclude that a final coalgebra carries two different orderings, one as an object of $\text{Pos}$, and one given by $\text{hom}(1, T)$ ordered by our general rule. These two orderings are different, in general (e.g. in the Pavlović-Pratt example of $F_2 : \text{Pos} \to \text{Pos}$ our ordering is not even linear!). And the latter determines the former via ideal completions as follows: we also define an ordering on $\text{hom}(1, I)$ and $\text{hom}(2, I)$ by our general rule, with $\text{hom}(B, T)$ being an ideal completion of $\text{hom}(B, I)$ for $B = 1, 2$. The initial algebra $I$ in the above example $F_2 : \text{Pos} \to \text{Pos}$ is the poset of all dyadic rational numbers. This has been observed in [3] where the above result of M. Barr on Cauchy completion has been generalized to LFP categories similarly to our treatment of ideal completions in the present paper.

In [4] we have considered a different CPO structure related to final coalgebras of set functors: there a final coalgebra consisted of ideal points, only, and carried the trivial (discrete) ordering.

2 Grounded endofunctors

2.1 Standing assumptions

Throughout the paper, we assume that a category $\mathcal{K}$ is given which has

(1) an initial object 0 and a terminal object 1

and

(2) colimits of $\omega$-chains and limits of $\omega^{op}$-chains.
2.2 Terminology

An endofunctor $F : \mathcal{K} \to \mathcal{K}$ is called
(a) $\omega$-continuous if it preserves limits of $\omega^\text{op}$-chains,
(b) $\omega$-cocontinuous if it preserves colimits of $\omega$-chains,
and
(c) bicontinuous if it is both $\omega$-continuous and $\omega$-cocontinuous.

An element of $F^0$, i.e. a morphism $p : 1 \to F^0$, is called a ground $F$. By a grounded endofunctor we mean an endofunctor with a chosen ground.

**Notation 2.1**

(a) We denote by $u : 0 \to 1$ the unique morphism.

(b) Given an endofunctor $F$, we also have unique morphisms $i : 0 \to F^0$ and $t : F^1 \to 1$.

These can be iterated to yield an $\omega$-sequence

$$0 \xrightarrow{i} F^0 \xrightarrow{Fi} F^20 \xrightarrow{F^2i} \ldots$$

whose colimit is denoted by

$$0 \xrightarrow{i} F^0 \xrightarrow{Fi} F^20 \xrightarrow{F^2i} \ldots$$

(2.1)

and, dually, an $\omega^\text{op}$-sequence

$$1 \xleftarrow{t} F^1 \xleftarrow{F^1t} F^21 \xleftarrow{F^21} \ldots$$

(2.2)

whose limit is denoted by

$$1 \xleftarrow{t} F^1 \xleftarrow{F^1t} F^21 \xleftarrow{F^21} \ldots$$

(2.3)

It is well known that for $\omega$-continuous functors we have

$$T = F^\omega 1.$$  

(2.5)

More precisely, $F^\omega 1$ is a terminal coalgebra whose coalgebra morphism

$$\psi : T \to FT$$

(2.6)
is the canonical isomorphism between $T = \lim_{n < \omega} F^n 1$ and
\[
F_T \cong \lim_{n \leq m \leq \omega} F^{m+1} 1 = T.
\]
Dually, for $\omega$-cocontinuous functors we have
\[
I = F^{\omega} 0.
\]
(c) For grounded endofunctors we also define $e : 1 \to F 1$ by the following commutative triangle
\[
\begin{array}{ccc}
1 & \xrightarrow{c} & F 1 \\
\downarrow{p} & & \downarrow{F \mu} \\
F 0 & \xrightarrow{\mu} & F 1
\end{array}
\]  
(2.8)
Observe that
\[
te = \text{id}
\]
(2.9)
since $1$ is terminal. Thus, the following $\omega$-chain
\[
1 \overset{c}{\to} F 1 \xrightarrow{F \mu} F^2 1 \xrightarrow{F \mu^2} \cdots
\]
(2.10)
is formed by splittings of the morphisms of the $\omega^{op}$-chain (2.3).

**PROPOSITION 2.2**
For every grounded $\omega$-continuous functor an initial algebra is a colimit of the $\omega$-chain (2.10).

**PROOF.** We are going to prove that the following cocone
\[
i_{n+1}(F^n p) : F^n 1 \to I
\]
(2.11)
is a colimit of (2.10). In fact, it is easy to verify that since
\[
pu = i : 0 \to F 0,
\]
(2.12)
the morphisms $F^n u : F^n 0 \to F^n 1$ form a natural transformation from (2.1) to (2.10), and the morphisms $F^n p : F^n 1 \to F^{n+1} 0$ form a natural transformation from (2.10) to the $\omega$-chain obtained from (2.1) by forgetting the first arrow. Since these two natural transformations compose as follows:
\[
(F^n p)(F^n u) = F^n i \quad (n < \omega)
\]
due to (2.12), and
\[
(F^{n+1} u)(F^n p) = F^n e \quad (n < \omega),
\]
this shows that colim $F^n u$ is inverse to colim $F^n p$; more precisely, that (2.11) is indeed a colimit of (2.10).
Example 2.3

Polynomial endofunctors of $\textbf{Set}$. These are the functors obtained from $\text{Id}$ via products and coproducts. This includes

\[ C_1 \quad (\text{constant functors with value } 1) \]

as the empty product (i.e. terminal object of the category of all endofunctors of $\textbf{Set}$) and

\[ C_A \quad (\text{constant functors with value } A) \]

as a coproduct of $A$ copies of $C_1$. Thus, a typical polynomial functor has the form

\[ FX = A_0 + (A_1 \times X) + (A_2 \times X^2) + \cdots + A_k \times X^k + \cdots \]

where $k$ is either restricted to natural numbers (those are the finitary polynomial functors), or can in general be an infinite cardinal. More precisely,

\[ F \cong C_{A_0} + (C_{A_1} \times \text{Id}) + (C_{A_2} \times \text{Id} \times \text{Id}) + \cdots + (C_{A_k} \times \text{Id}^k) + \cdots \]

A ground of $F$ is a choice of $p$ in $A_0$. As a typical representative of finitary polynomial functors, consider

\[ FX = A + (B \times X^2). \]

The sequence (2.1) has the following form:

\[
\begin{align*}
0 &= \emptyset, \\
F0 &= A, \\
F^20 &= A + (B \times A \times A) \\
&\vdots \\
\end{align*}
\]

and we denote the elements of $F^{n+1}0 = A + (B \times F^n0 \times F^n0)$ by trees

\[
\begin{align*}
\text{for } a \in A & \quad \text{and} \\
\text{for } b \in B & \\
\text{for } x, y \in F^n0
\end{align*}
\]

Then $F^n0$ is the set of all binary trees of depth $< n$ whose leaves are labelled in $A$ and inner nodes are labelled in $B$. And the connecting maps $F^n_i : F^n0 \to F^{n+1}0$ are the inclusions. We conclude that $I = \text{colim}F^n0$ is the set of all finite binary trees with the above labelling.

The sequence (2.2) has the following form:

\[
\begin{align*}
1 &= \{*, \} \quad (\text{where } \ast \notin A) \\
F1 &= A + (B \times 1 \times 1) \\
&\vdots 
\end{align*}
\]
and we again denote the elements of $F^{n+1}1 = A + (B \times F^n1 \times F^n1)$ by binary trees as above. Then $F^n1$ is the set of all binary trees of depth $\leq n$ whose leaves of depth $n$ are labelled by $\ast$ and other leaves are labelled in $A$, whereas inner nodes are labelled in $B$. Example: $F^1$ is the set of all trees

![Diagram](image)

and $F^21$ is the set of all trees

![Diagram](image)

The connecting map $F^n1 : F^{n+1}1 \to F^n1$ cuts a tree $x$ at level $n$ and relabels nodes of depth $n$ by $\ast$. Thus $T = \lim F^n1$ is the set of all finite and infinite binary trees with leaves labelled in $A$ (no label $\ast$) and inner nodes labelled in $B$.

Finally,

$$e : 1 \to F^1 = A + (B \times 1 \times 1)$$

maps $\ast$ to the chosen element $p$ of $A = F0$. Thus,

$$F^n1 \to F^{n+1}1$$

takes a tree $x$ and relabels all nodes of depth $n$ from $\ast$ to $p$. A colimit of this sequence is indeed $I$ again.

**Lemma 2.4**

For every endofunctor $F$ then exists a unique morphism $\tilde{u} : F^\omega0 \to F^\omega1$ for which the following diagrams

![Diagram](image)

commute.

**Proof.** Denote by

$$t_{k,n} = (F^k t)(F^{k-1} t) \cdots (F^1 t) : F^k1 \to F^n1 \quad (k \geq n) \quad (2.13)$$

the connecting maps of (2.3). For each number $k$ the morphisms

$$t_{n,k}(F^n u) : F^n0 \to F^k1 \quad (n \geq k)$$

form a cocone of the $\omega$-chain (2.1) with the first $k$ members deleted. In fact, observe first that $u = t(Fu) t$, since $0$ is initial; therefore

$$F^n u = (F^n t)(F^{n+1} u)(F^n t).$$
Since $t_{n+1,k} = t_{n,k}t_{n+1,n} = t_{n,k}(F^nt)$, we conclude
\[
\begin{align*}
t_{n,k}(F^n u) &= t_{n,k}(F^n t)(F^{n+1} u)(F^{n+1} i) \\
&= t_{n+1,k}(F^{n+1} u)(F^{n+1} i)
\end{align*}
\]
as requested. Thus, we have a unique morphism
\[
f_k : F^\omega 0 \rightarrow F^k 1
\]
with
\[
f_k i_n = t_{n,k}(F^n u) \quad \text{for all } n \geq k.
\]
These morphisms form a cone of (2.1): to prove $f_k = (F^{k+1} t) f_{k+1}$ observe that for all $n \geq k + 1$ we have
\[
[(F^{k+1} t)f_{k+1}] i_n = (F^{k+1} t)t_{n,k+1}(F^n u) = t_{n,k}(F^n u) = f_k i_n.
\]
Denote by $\tilde{u} : F^\omega 0 \rightarrow T = \text{colim}_{n<\omega} F^n 1$ the unique morphism with
\[
f_k = t_k \tilde{u} \quad (k < \omega).
\]
Then the statement above holds:
\[
t_k \tilde{u}_i_k = f_k i_k = t_{k,k}(F^k u) = F^k u \quad (k < \omega).
\]
Conversely, the above implies $f_k = t_k \tilde{u}$ for all $k < \omega$ because we have, for all $n \geq k$,
\[
(t_k \tilde{u}) i_n = t_{n,k} t_{n} \tilde{u}_i_n = t_{n,k}(F^n u) = f_k i_n.
\]

**Example 2.5**

For the functor $FX = A + (B \times X \times X)$ above, $\tilde{u} : I \rightarrow T$ is the inclusion map.

More generally, whenever $F$ is bicontinuous, then
\[
\tilde{u} : I \rightarrow T
\]
is the usual canonical morphism: recall that the coalgebra morphism $\psi : T \rightarrow FT$ is an isomorphism, so that $T$ is also an algebra, $\psi^{-1} : FT \rightarrow T$. And $\tilde{u}$ is the unique homomorphism from the initial algebra $I$ to the last algebra $T$.

**Remark 2.6**

Suppose that
(i) $u : 0 \rightarrow 1$ is a monomorphism,
(ii) $F$ preserves monomorphisms
and
(iii) for every cocone of an $\omega$-sequence in $\mathcal{X}$ formed by monomorphisms the factoring map from the colimit is also a monomorphism.

Then $\tilde{u} : F^\omega 0 \rightarrow F^\omega 1$ is a monomorphism. This follows from the fact that each $F^n u = t_n \tilde{u} i_n$ is a monomorphism, thus, each $\tilde{u} i_n$ is a monomorphism.
3 Ordering of final coalgebras

Throughout this section we work with a grounded \( \omega \)-continuous functor \( F : \text{Set} \to \text{Set} \). Without loss of generality we can assume that

\[ F \text{ preserves monomorphisms} \]

In fact, as shown in [4, Remark 3], for every grounded set functor \( F \) by changing the value of \( F \) at the empty set (and empty maps) we can obtain a set functor \( F' \) which (a) has the same initial algebra and the same final coalgebra as \( F \) and (b) preserves monomorphisms. Thus, all our results are valid for \( \omega \)-continuous grounded functors, but in the proofs we sometime use the preservation of monomorphisms. Later we generalize the results of the present section to other base categories — there the assumption that monomorphisms be preserved has to be made explicitly.

For each grounded \( \omega \)-continuous endofunctor \( F \) denote by \( e_n : F^{n+1}1 \to T \) the morphisms defined by the following commutative diagram (see Lemma 2.4):

\[
\begin{array}{c}
F^{n+1}1 \\
\downarrow e_n \\
T
\end{array} \quad \begin{array}{c}
\xrightarrow{F^n p} \\
\downarrow \alpha \\
\xrightarrow{(n<\omega)} F^{\omega}0
\end{array}
\]

We define the ordering of \( T \) by the following rule: given elements \( x, y \) of \( T \), put

\[ x \leq y \text{ iff } x = y \text{ or } x = e_n t_n y \text{ for some } n < \omega. \]

**Remark 3.1**

1. Observe that

\[ t_n e_n = \text{id}_T \]

for all \( n < \omega \). In fact, from

\[ t(Fu)p = \text{id} : 1 \to 1 \]

we get

\[
\begin{align*}
t_ne_n &= t_n \bar{u}e_{n+1}F^n p \\
&= (F^{n+1} t_{n+1} \bar{u}) e_{n+1}F^n p \quad \text{by compatibility} \\
&= (F^{n+1})(F^{n+1}u)F^n p \quad \text{by Lemma 2.4} \\
&= \text{id}.
\end{align*}
\]

2. The above ordering of \( T \) obviously yields

\[ e_n t_n \leq \text{id}_T. \]

We will see later that \( e_n \) and \( t_n \) form an embedding -projection pair w.r.t. an ordering on \( F^{n+1}1 \) analogous to that of \( T \).

**Example 3.2**

Consider

\[ FX = A + (B \times X \times X) \]
of Proposition 2.2. The morphism \( e_n : F^n 1 \to T \) takes a tree \( x \) in \( F^n 1 \) and relabels all nodes of depth \( n \) from \( * \) to \( p \). Thus
\[
e_n t_n : T \to F^n 1
\]
cuts a tree at level \( n \) and relabels nodes of depth \( n \) by \( p \). Therefore
\[
x \leq y
\]
iff \( x = y \) or \( x \) is obtained from \( y \) by cutting \( y \) at some level \( n \) relabeling the new leaves by \( p \).

**Remark.** This is reminiscent of the construction of free continuous algebras (i.e. algebras defined on CPOs such that all operations are continuous), see [5]. A free continuous algebra can be described as the algebra of all finite and infinite trees (properly labelled), where \( \psi \) iff \( \phi \) is obtained from \( \phi \) by cutting away some branches and relabelling the new leaves by \( \bot \), the smallest element. However, as we shall next see, our poset \( T \) is much more ‘primitive’ than the ordering of free continuous algebras, being algebraically \( \omega \)-complete. This means that every order-preserving function \( f : T \to T \) has the least fixed point \( \bigvee_{n < \omega} f^n(\bot) \). It has been shown in [2] that a poset is algebraically \( \omega \)-complete iff

(a) it has a least element

and

(b) every strictly increasing \( \omega \)-sequence has a unique upper bound.

**Theorem 3.3**

For every grounded \( \omega \)-continuous set functor a final coalgebra with the ordering (3.2) is an algebraically \( \omega \)-complete CPO.

**(Proof)** (a) We first establish the equality
\[
(e_m t_m)(e_n t_n) = e_j t_j \quad \text{where} \quad j = \min(m, n) \tag{3.4}
\]
for all natural numbers \( n \) and \( m \). Let us abbreviate
\[
r_n = e_n t_n : T \longrightarrow T. \tag{3.5}
\]

For \( m \leq n \) the statement (3.4) is derived as follows: observe that \( t_n r_n = t_n \) (due to (3.3)), thus
\[
r_m r_n = \tilde{u} i_{m+1} (F^m p) t_{m+1} r_n = \tilde{u} i_{m+1} (F^m p) t_{m+1} t_n = \tilde{u} i_{m+1} (F^m p) t_m t_n \quad \text{by compatibility}
\]
\[
= \tilde{u} i_{m+1} (F^m p) t_m = t_n r_n = t_n \quad \text{by compatibility}
\]
\[
= r_m.
\]

For \( m > n \) we only have to establish the case of \( m = n + 1 \):
\[
r_{n+1} r_n = \tilde{u} i_{n+2} (F^{n+1} p) t_{n+1} i_{n+1} (F^n p) t_n = \tilde{u} i_{n+2} (F^{n+1} p) t_{n+1} (F^n p) t_n = \tilde{u} i_{n+2} (F^{n+1} p) t_n = \tilde{u} i_{n+1} (F^n p) t_n = r_n.
\]
(b) For each pair \( x, y \) of elements of \( T \) we have
\[ x = r_n y \Rightarrow x = r_n x. \]
This follows from \( r_n r_n = r_n \), established in (a).

(c) As a final preparation remark we establish that
\[ e_n \]
is a monomorphism
for each \( n < \omega \). Recall our tacit assumption that \( F \) preserve monomorphisms. Since \( i : 0 \to F 0 \) is a monomorphism, we conclude that (2.1) is a sequence of monomorphisms, thus, a colimit cocone is also formed by monomorphisms \( i_n : F^n 0 \to F^{n+1} 0 \). Also, \( \bar{u} \) is a monomorphism, see Remark 2.6. And \( p : 1 \to F 0 \) is a monomorphism, thus, so is each \( F^n p \). Consequently, \( e_n \) is a composite of three monomorphisms, see (3.1).

(d) Let us verify that the relation (3.2) is a partial order. Reflexivity is clear; for antisymmetry, consider
\[ x \leq y \quad \text{and} \quad y \leq x \]
in \( T \). If one of those relations is equality, we have \( x = y \), else there exist \( n, m \) with \( x = r_n y \) and \( y = r_m x \) and if for example \( n \geq m \), we conclude from (a) that
\[ x = r_n y = r_n r_m x = r_m x = y. \]
For transitivity, consider
\[ x \leq y \leq z \]
and assume again that there exist \( n, m \) with
\[ x = r_n y \quad \text{and} \quad y = r_m z. \]
Then
\[ x = r_n r_m z = r_j z \quad \text{for} \ j = \min(n, m), \]
which shows \( x \leq z \).

Observe that the element
\[ e_0 : 1 \longrightarrow T \]
is the least one in \( T \). In fact, for every element \( x : 1 \to T \) we have
\[ e_0 = e_0 \circ_0 x \]
(since \( t_0 x = \text{id}_1 \)) and this yields
\[ e_0 \leq x \]
by the formula (3.2).

(e) Every strictly increasing sequence
\[ x_0 < x_1 < x_2 < \cdots \]
in \( T \) has an upper bound (proved unique below). In fact, since for each \( n < \omega \) we have \( x_n < x_{n+1} \), we are given natural numbers \( k(n) \) with
\[ x_n = r_{k(n)} x_{n+1} \quad (n < \omega). \]
We first observe that the sequence \( k(n) \) is unbounded. In fact, assuming the contrary we could (by thinning) suppose that \( k(n) \) is constant, \( k(n) \equiv k \), but then \( x_n = r_k x_n \), see (b), which proves \( x_{n-1} = r_n x_n = x_n \), a contradiction. Again, by thinning, we thus can suppose that 
\[
k_0 < k_1 < k_2 < \cdots
\]

Then the elements 
\[
y_n = t_{k(n)} x_n
\]
of \( F^{k(n)}1 \) are compatible with the chain (2.3). In fact, for each \( n \) we have the connecting map 
\[
t_{k(n+1), k(n)} y_{n+1} = t_{k(n+1), k(n)} t_{k(n+1)} x_{n+1} = t_{k(n)} x_{n+1}
\]
which, multiplied by \( e_{k(n)} \), yields 
\[
r_{k(n)} x = r_{k(n)} x_n.
\]
Now (3.6) implies \( x_n = r_{k(n)} x_n \), see (b), which proves 
\[
x_n = r_{k(n)} x
\]
and thus 
\[
x_n \leq x
\]
for all \( n < \omega \). We have found an upper bound, \( x \).

(f) Uniqueness of upper bounds. Suppose \( x' \) is another upper bound of the sequence in (e), and let \( k'(n) \) be numbers with 
\[
x_n = r_{k'(n)} x'
\]
Arguing as in (e) we see that sequence \( k'(n) \) is unbounded. Thus, so is the sequence \( j(n) = \min(k(n), k'(n)) \). From (a) we conclude 
\[
r_{j(n)} x = r_{j(n)} x'
\]
for all \( n < \omega \). Now the sequence \( j(n) \) is unbounded, thus, \( t_m x = t_m x' \) for all \( m < \omega \), and this proves \( x = x' \).
REMARK 3.4
We have claimed in Remark 3.1 that

\[ F^k \xrightarrow{t_k} T \]

are embedding-projection pairs. We are now going to show this; in fact, also the pairs

\[ F^k \xrightarrow{F^k t} F^{k+1} \]

are embedding-projection pairs provided that each \( F^k \) is ordered analogously to \( T \) above.

We can introduce morphisms

\[ e_{k,n} : F^n \rightarrow F^k \]

for all \( n < k \) by the following commutative diagram analogous to (3.1):

\[
\begin{array}{c}
F^{n+1} \xrightarrow{t_{n+1}} F^{n+2} \\
F^n \xrightarrow{e_{n,n}} F^k
\end{array}
\]

Here we use the notation for the connecting maps of the \( \omega \)-chain (2.1):

\[ \iota_{n,k} = (F^{k-1} \iota)(F^{k-2} \iota) \cdots (F^n \iota) : F^n \rightarrow F^k \quad (n < k) \]

(3.8)

analogous to the connecting maps of the \( \omega^p \)-chain (2.2) in (2.13). Put also \( \iota_{n,n} = \text{id} \), and \( t_{nn} = \text{id} \). The above notation (3.8) is consistent with the present one:

LEMMA 3.5
For all \( n < k \) we have

\[ e_{k,n} = t_k e_n = (F^{k-1} e)(F^{k-2} e) \cdots (F^n e) : F^n \rightarrow F^k \]

(the connecting maps of the \( \omega \)-chain (2.10)), and

\[ t_{k,n} = t_n e_k : F^k \rightarrow F^n \]

PROOF. Use Lemma 2.4 and (2.8) to conclude that the following diagram

\[
\begin{array}{c}
F^{n+1} \xrightarrow{t_{n+1}} F^{n+2} \\
F^n \xrightarrow{e_{n,n}} F^k
\end{array}
\]

commutes, establishing the first equation.

Analogously, use Lemma 2.4 and \( t(Fu)p = \text{id} : 1 \rightarrow 1 \) to establish the latter equation:
Remark 3.6
Ordering of $F^k1$ is defined as follows (compare with (3.2)):

$$x \leq y \text{ iff } x = y \text{ or } x = e_{k,n}t_{k,n}y \text{ for some } n < k. \quad (3.10)$$

We are going to prove that the formula

$$T = \lim_{k < \omega} F^k1$$

holds not only in $\text{Set}$, but also in $\text{Pos}$ (the category of posets and order-preserving maps) and in $\text{CPO}$ (the category of CPOs and continuous maps). In fact, it also holds in $\text{CPO}^*$

the category of CPOs and embedding-projection pairs. We establish the last one since the previous ones then follow: $\text{CPO}^*$ is closed under limits of $\omega^\rho\pi$-chain in $\text{CPO}$ and this, in turn, is closed under these limits in $\text{Pos}$.

Theorem 3.7
Let $F$ be a grounded $\omega$-continuous set functor. Then each $F^k1$ is a CPO, and each pair

$$F^k1 \xrightarrow{F^k\epsilon} F^{k+1} \quad (k < \omega) \quad (3.11)$$

is an embedding-projection pair. In $\text{CPO}^*$ a limit of the $\omega$-chain (3.11) is $T$ with the limit cone formed by the following embedding-projection pairs

$$F^k1 \xrightarrow{t_k} T \quad (k < \omega). \quad (3.12)$$

Proof. (a) $F^k1$ is partially ordered by (3.10). The proof is completely analogous to that of Theorem 3.3.

(b) $F^k1$ is a CPO by default: it has no strictly increasing $\omega$-sequences (thus, every directed subset contains a largest element). In fact, given a strictly increasing sequence $x_0 < x_1 < x_2 \ldots$ in $F^k1$, there exist numbers $q(n) < k$ with $x_n = e_{k(q(n))}t_{k(q(n))}x_{n+1}$, and by thinning we can assume that $q(n) = q$ is independent of $n$:

$$x_n = e_{kq}t_{kq}x_{n+1}.$$
which proves \( x_{n-1} = e_k t_k x_n = x_n \) by the same argument as in Theorem 3.3 (part (e) of the proof).

(c) Each \( F^{k-1}: F^k \to F^{k-1} \) is order-preserving (thus, continuous). In fact, given \( x < y \) in \( F^k \), we have \( n < k \) with

\[
x = e_{k,n}t_{k,n}y.
\]  

(3.13)

In the case \( n < k - 1 \) we have

\[
(F^{k-1}t)x = (F^{k-1})t \epsilon_n t_{k,n} y = t_{k-1} \epsilon_n t_{k,n} y \quad \text{by Lemma 3.5}
\]

\[
= e_{k-1,n}t_{k,n}y \quad \text{by compatibility}
\]

\[
= e_{k-1,n}t_{k-1,n}(F^{k-1}t)y \quad \text{by compatibility}
\]

which proves

\[
(F^{k-1}t)x \leq (F^{k-1}t)y.
\]

In the case \( x = e_{k,n}t_{k,n}y = (F^{k-1}t)(F^{k-1}t)y \) we have, by (2.9)

\[
(F^{k-1}t)x = F^{k-1}(t \epsilon t)y = (F^{k-1}t)y.
\]

(d) Each \( t_k : T \to F^{k-1} \) is order-preserving (thus, continuous, since on each increasing \( \omega \)-chain \( t_k \) is eventually constant). In fact, given \( x < y \) in \( T \), we have \( n < \omega \) with

\[
x = e_{n}t_{n}y.
\]

If \( n < k \), then

\[
t_{k}x = t_{k}e_{n}t_{n}y = e_{k,n}t_{n}y \quad \text{by Lemma 3.5}
\]

\[
= e_{k,n}t_{k,n}y \quad \text{by compatibility},
\]

thus \( t_{k}x \leq t_{k}y \). If \( n \geq k \), then

\[
t_{k}x = t_{n,k}t_{n}e_{n}t_{n}y \quad \text{by compatibility}
\]

\[
= t_{n,k}t_{n}y \quad \text{by (3.3)}
\]

\[
= t_{k}y \quad \text{by compatibility}.
\]

(e) Each \( e_k : F^k \to T \) is order-preserving (thus continuous). We first observe that

\[
e_{k}e_{k,n} = e_{n} \quad \text{for all } n < k
\]  

(3.14)

which follows from (2.12):

Thus, given \( x < y \) in \( F^k \), we have

\[
x = e_{k,n}t_{k,n}y \quad \text{for some } n < k.
\]
and we conclude

\[ e_k x = e_n t_{k,n} y \quad \text{by (3.14)} \]
\[ = e_n t_n e_k y \quad \text{by Lemma 3.5} \]

which proves \( e_k x \leq e_k y \) in \( T \).

(f) Each \( F^k e : F^k 1 \to F^{k+1} 1 \) is order-preserving (thus, continuous).

In fact, given \( x < y \) in \( F^k 1 \) we have \( n < k \) with

\[ x = e_{k,n} t_{k,n} y \]

which implies (since \( t_{k,n} = t_{k+1,n} F^k e \), due to (2.9)) that

\[ (F^k e)x = (F^k e)(F^k u) t_{k,n+1} (F^n p) t_{k,n} y \quad \text{by (3.8)} \]
\[ = (F^{k+1} u) t_{k+1,n+1} (F^n p) t_{k,n} y \quad e u = (F u) t \]
\[ = e_{k+1,n} t_{k,n} y \quad \text{by (3.8)} \]
\[ = e_{k+1,n} t_{k+1,n} (F^k e)y \quad t_{k,n} = t_{k+1,n} F^k e \]

and thus \( (F^k e)x \leq (F^k e)y \).

(g) The pair (3.11) is an embedding-projection pair: we have established continuity of \( F^k e \) and \( F^k t \) above. We have

\[ (F^k t)(F^k e) = \text{id} \]

by (2.9), and

\[ (F^k e)(F^k t) \leq \text{id} \]

by the definition of ordering (3.10) since

\[ F^k e = e_{k+1,k} \quad \text{and} \quad F^k t = t_{k+1,k}. \]

Also the pair (3.12) is an embedding-projection, see Remark 3.1.

(h) To prove that \( T \) is a limit of \( F^k 1 \) in the category CPO*, it is sufficient to prove that (i) the pairs (3.12) form a cone of the \( \omega \)-sequence (3.11) — this follows from (3.14) and (2.4) — and (ii) given elements \( x, y \) of \( T \) then

\[ t_k x \leq t_k y \quad \text{for all } k < \omega \quad \Rightarrow \quad x \leq y. \]

This is clear if \( x = y \), so assume that \( x \neq y \), i.e.

\[ t_{k_0} x \neq t_{k_0} y \]

for some \( k_0 < \omega \). For each \( k \geq k_0 \) there exists \( n < k \) with

\[ t_k x = e_{k,n} t_{k,n} (t_k y) = t_k e_n t_n y \quad \text{(by Lemma 3.5)}. \]

This implies, by (3.3),

\[ t_n x = t_{k,n} t_k x \]
\[ = t_{k,n} t_k e_n t_n y \]
\[ = t_n e_n t_n y \]
\[ = t_n y. \]
Consequently, \( n < k_0 \). Thus, we see that for each \( k \geq k_0 \) the chosen \( n \) is among \( 0, \ldots, k_0 - 1 \), and there exists \( n \) such that the above equation

\[
t_k x = t_k e_n t_n y
\]

holds for arbitrary large \( k \)'s. From the fact that \((t_k)_{k<\omega}\) is a limit cone of an \( \omega^{op}\)-chain in \( \textbf{Set} \) we now conclude

\[
x = e_n t_n y,
\]

thus, \( x \leq y \). □

4 Ordering of initial algebras — bicontinuous case

Suppose that \( F : \textbf{Set} \to \textbf{Set} \) is a grounded bicontinuous functor. The morphism

\[
\bar{u} : I \to T
\]

of Lemma 2.4 is a monomorphism (see Remark 2.6) and thus \( I \) can be considered as a subposet of \( T \):

\[
x \leq y \text{ iff } x = y \text{ or } \bar{u} x = e_n t_n (\bar{u} y) \text{ for some } n < \omega.
\]

Now \( e_n = \bar{u} i_{n+1} F^n p \) and \( \bar{u} \) is a monomorphism, thus, the last equality is equivalent to \( x = i_{n+1} (F^n p) t_n \bar{u} y \). Each element \( y \) of \( I = \text{colim} F^k 0 \) has the form \( y = i_k y' \) for some element \( y' \) of \( F^k 0 \), where \( k \) can always be chosen to be larger than a given \( n \). Thus, the last equality takes the following form, where \( k > n \) and \( y' \) is an element of \( F^k 0 \):

\[
x = i_{n+1} (F^n p) t_n \bar{u} i_k y' \\
= i_{n+1} (F^n p) t_{k,n} i_k \bar{u} i_k y' \text{ by compatibility} \\
= i_{n+1} (F^n p) t_{k,n} (F^k u) y' \text{ by Lemma 2.4}.
\]

We are led to introduce notation for the following morphisms \( q_{k,n} : F^k 0 \to I \) for each \( n < k < \omega \):

\[
\begin{align*}
\begin{array}{c}
F^k 0 \\
\downarrow \phi_{k,n} \\
F^k 1 \\
\downarrow i_{k,n} \\
F^n 1
\end{array}
\end{align*}
\]

Then we have the following ordering of \( I \): given elements \( x, y \) of \( I \) put

\[
x \leq y \text{ iff } x = y \text{ or } x = q_{k,n} y' \text{ for some numbers } n < k < \omega \text{ and some element } y' \text{ of } F^k 0 \text{ with } y = i_k y'. \tag{4.2}
\]

The above derivation shows that \( I \) is a subposet of \( T \), i.e. for elements \( x, y \) of \( I \) we have

\[
x \leq y \text{ in } I \text{ iff } \bar{u} x \leq \bar{u} y \text{ in } T.
\]

Observe that (4.2) does not use the structure of a final coalgebra at all. In fact, \( T \) can be derived from \( I \) by the following
PROPOSITION 4.1
For every grounded bicontinuous set functor a final coalgebra is an ideal completion of an
initial algebra ordered by (4.2).

PROOF. For every element \( x \) of \( T \) we denote

\[ x_{(n)} = e_n t_n x \]  \( (n < \omega) \).

This is an increasing \( \omega \)-sequence:

\[ x(0) \leq x(1) \leq x(2) \leq \ldots \]

because by (3.4) we have

\[ x_{(n)} = (e_n t_n)(e_{n+1} t_{n+1}) x = e_n t_n x_{n+1} \]

thus, \( x_{(n)} \leq x_{(n+1)} \). Since \( e_n \) factors through \( a \), see (3.1), we have found an increasing
\( \omega \)-sequence in \( I \). And since \( x \) is obviously an upper bound of that sequence, we conclude that

\[ x = \bigvee _{n < \omega} x_{(n)} . \]

In fact, either the sequence \( x_{(n)} \) has a strictly increasing subsequence, then \( x \) is the unique upper bound of that subsequence (recall that \( T \) is algebraically \( \omega \)-complete). Or the sequence is eventually constant and then \( x = x_{(n)} \) for some \( n \), establishing the above formula, too.

Thus, \( I \) is \( \omega \)-dense in \( T \). Moreover, there are no ideals (i.e. downward closed upwards directed subset) of \( I \) other than those of the form \( \{ x_{(n)} \}_{n < \omega} \) for some \( x \in T \). In fact, given an ideal \( D \subseteq I \), let \( x = \bigvee D \) in \( T \), then every element of \( D \) is smaller or equal to \( x \), thus, assuming \( x \notin D \), every element of \( D \) has the form \( x_{(n)} \) for some \( n \) — which proves \( D = \{ x_{(n)} \}_{n < \omega} \). Thus, \( T \) is an ideal completion of \( I \).

EXAMPLE 4.2
For the functor \( FX = A + (B \times X^2) \) the morphism \( q_{k,n} \) takes a tree \( x \) in \( F^k X \) (i.e. of depth
\( < k \)), cuts it at level \( n \), and relabels the leaves of depth \( n \) by \( p \). Thus, the ordering (4.2) is indeed a suposet of \( T \), i.e. \( x \leq y \) iff \( x \) and \( y \) are finite trees (the constant \( k \) in (4.2) being
an upper bound of their depths) such that \( x \) is obtained from \( y \) by cutting \( y \) at depth \( n \) and relabelling the leaves of depth \( n \) by \( p \).

It is true that \( T \) is an ideal completion of \( I \): infinite trees are pairwise incompatible, and
every tree is an \( \omega \)-join of its finite cuttings. Example:

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Now observe that, whereas in the category \textbf{Set} the \(\omega\)-sequences \(F^k i : F^k 0 \to F^{k+1} 0\) and \(F^k e : F^k 1 \to F^{k+1} 1\) have the same limit \(I\) (see Proposition 2.2), their colimits in \textbf{Pos} and in CPO* respectively are fundamentally distinct: by Theorem 3.7 the latter sequence has colimit \(T\) in CPO*. In contrast:

**Proposition 4.4**

For every grounded bicontinuous set functor an initial algebra is a colimit, in the category \textbf{Pos}, of each of the \(\omega\)-sequences (2.1) and (2.10).

**Proof.** I. The sequence (2.1). The following diagram

\[
\begin{array}{c}
F^k 0 \\
F^k u \\
F^k 1 \\
\end{array}
\begin{array}{c}
\downarrow i_k \\
\downarrow F^k i \\
\downarrow F^k u \\
\end{array}
\begin{array}{c}
I \\
F^{k+1} 0 \\
T \\
\end{array}
\end{array}
\]

commutes, see (2.12) and (3.1). Thus, the outward square commutes, which proves that, since \(e_k\) is order-preserving by Theorem 3.7, the map \(i_k\) is order-preserving.

Since \(I = \text{colim} F^k 0\) in \textbf{Set}, it remains to show that given \(x < y\) in \(I\) there exist \(\bar{x} \leq \bar{y}\) in \(F^k 0\) for some \(k < \omega\) with \(x = i_k \bar{x}\) and \(y = i_k \bar{y}\). By (4.2), \(x = q_k n y'\) and \(y = i_k y'\) for some \(n < k < \omega\) and some \(y'\) in \(F^k 0\). Put

\[\bar{x} = i_{n+1, k} (F^n p) t_{k,n} (F^k u) \bar{y}\]

and

\[\bar{y} = y'.\]

Then \(x = i_k \bar{x}\), see (4.1), and \(y = i_k y'\). To show that \(\bar{x} \leq \bar{y}\), i.e. that \((F^k u) \bar{x} \leq (F^k u) \bar{y}\) according to (3.10), we observe that the following diagram

\[
\begin{array}{cccccccc}
F^n 1 & \rightarrow & F^n 0 \\
\downarrow F^n e & & \downarrow F^n i & & \downarrow F^n u \\
F^{n+1} 1 & \rightarrow & F^{n+1} 0 & \rightarrow & F^{n+2} 0 & \rightarrow & \cdots & \rightarrow & F^{k-1} 1 \\
\downarrow F^{n+1} e & & \downarrow F^{n+1} i & & \downarrow F^{n+2} i & & \cdots & & \downarrow F^{k-1} i & & \downarrow F^k i \\
\end{array}
\]

commutes, due to (2.8) and (2.12), and it establishes

\[(F^k u) i_{n+1, k} (F^n p) = e_{k,n}.
\]

Thus,

\[(F^k u) \bar{x} = (F^k u) i_{n+1, k} (F^n p) t_{k,n} (F^k u) \bar{y} = e_{k,n} t_{k,n} (F^k u) \bar{y},\]

which proves \((F^k u) \bar{x} \leq (F^k u) \bar{y}\).

II. The sequence (2.10). We know from 3.7 that

\[F^k e = (F^{k+1} u)(F^k p) : F^k 1 \rightarrow F^{k+1} 1\]
is order-preserving, thus, also
\[ F^k p : F^k 1 \to F^{k+1} 0 \]
is order-preserving (since the ordering of \( F^{k+1} 0 \) is defined via \( F^{k+1} 1 \)). Consequently, each
\[ i_{k+1}(F^k p) : F^k 1 \to I \]
is order-preserving. Since this is a colimit of the \( \omega \)-sequence (2.10) in Set, see the proof of 2.2, it remains to show that given \( x < y \) in \( I \) there are \( \tilde{x} \leq \tilde{y} \) in \( F^k 1 \) with \( x = i_{k+1}(F^k p)\tilde{x} \) and \( y = i_{k+1}(F^k p)\tilde{y} \). This is analogous to the previous part.

**Remark 4.5**

For functors which are \( \omega \)-cocontinuous but not \( \omega \)-continuous we can consider the above ordering of \( I = \text{colim} F^\omega 0 \), too. However, an ideal completion of \( I \) is no longer a final algebra.

In fact, the \( \omega \)-continuity of \( F \) has not been used in section 3 at all except to conclude \( T = F^\omega 1 = (\text{colim}_k \leq 1 F^\omega 1) \). In other words, for any grounded set functor we can define an ordering on \( F^\omega 1 \) by (3.2) and prove that \( F^\omega 0 \), considered as a subposet via \( \tilde{a} : F^\omega 0 \to F^\omega 1 \), has the ideal completion \( F^\omega 1 \). Unless \( F \) is \( \omega \)-continuous, our expectation is that \( T \neq F^\omega 1 \) -- thus, this ideal completion of \( I \) is different from \( T \).

**Example:** Consider the functor
\[ P_{\text{fin}} : \text{Set} \to \text{Set} \]
assigning to every set the set of all finite subsets (as a subfunctor of the usual power-set functor). This functor is finitary, thus, \( I = P_{\text{fin}}^\omega 0 \), but not \( \omega \)-continuous. As illustrated by J. Worrell [10], \( P_{\text{fin}}^\omega 1 \) is a semifinal coalgebra of all ‘strongly extensional’ trees (an ideal completion of the subset \( P_{\text{fin}}^\omega 0 \) of all finite such trees ordered by (4.2), i.e. by cutting a tree at level \( k \) and inserting the single element of \( P_{\text{fin}}^\omega 0 = \{\emptyset\} \) as leaf labels). In contrast, a final coalgebra, \( T \), is the subcoalgebra of \( P_{\text{fin}}^\omega 1 \) consisting of all finitely branching strongly extensional trees.

### 5 Ordering of initial algebras — continuous case

Also for grounded \( \omega \)-continuous functors \( F : \text{Set} \to \text{Set} \) we are going to show that a final coalgebra is an ideal completion of an initial algebra. However, we must be more careful here: we first have to establish the existence of initial algebra \( I \) and of the canonical monomorphism \( \tilde{a} : I \to T \) making it a subobject of \( T \). Then we show that the ordering of \( I \) (as a subposet of \( T \)) can be defined by essentially the same formula as (4.2) above, and yields \( T \) via ideal completion.

#### 5.1 Existence of initial algebra

Following [1], we are going to work with an infinitary generalization of the chain (2.1): we define a transfinite chain with objects \( F^n 0 \) where \( n \) is an ordinal number, and with connecting morphisms \( i_{n,m} : F^n 0 \to F^m 0 \) \( (n \leq m) \) generalizing the case \( i = i_{0,1} : 0 \to F^0, F i = i_{1,2} : F^0 \to F^2 0 \), etc. of (2.1). It has been proved in [1] that whenever the transfinite chain below converges in \( n \) steps, i.e. whenever \( i_{n,n+1} \) is an isomorphism, then
\[ I = F^n 0 \]
is an initial algebra (whose algebra morphism $FI \to I$ is the inverse of $i_{n,n+1}$).

This transfinite chain is defined by the following transfinite induction:

**Initial step:**

\[
F^0 0 = 0, \\
F^1 0 = F0, \\
i_{01} : 0 \to 1.
\]

**Isolated step:**

\[
F^{m+1} 0 = F(F^m 0) \\
i_{n+1,m+1} = F(i_{n,m}) : F(F^m 0) \to F(F^n 0).
\]

**Limit step:**

\[
F^m 0 = \text{colim}_{n < m} F^n 0 \text{ for every limit ordinal } m, \text{ with} \quad a \text{ colimit cocone } i_{n,m} : F^n 0 \to F^m 0 \text{ (for all } n < m).\]

All this holds for every endofunctor of Set. Now if $F$ is $\omega$-continuous, we have a final coalgebra $T = \text{colim}_{n < \omega} F^n 1$ (with $\psi : T \to FT$ as described in (2.6)). We will show that the above chain $F^n 0 (n \in \text{Ord})$ is a transfinite chain of subobjects of $T$ — and then we use the fact that Set is wellpowered to conclude that the chain above indeed converges.

Thus, we define monomorphisms

\[
u_n : F^n 0 \to T \quad (n \in \text{Ord})
\]

forming a cocone of the chain $(F^n 0)$. We use transfinite induction again: the initial step

\[
u_0 : 0 \to T
\]

is the unique morphism, the isolated step is defined by the commutativity of the following triangle

\[
\begin{array}{ccc}
F(F^n 0) & \rightarrow & T \\
\downarrow F\nu_n & & \downarrow \psi^{-1} \\
FT & \rightarrow & T
\end{array}
\]

and the limit step is uniquely determined: $u_m : F^n 0 \to T$ is the unique morphism with $u_n = u_m i_{nm}$ for all $n < m$.

**Lemma 5.1**

The above morphisms $u_n : F^n 0 \to T (n \in \text{Ord})$ are monomorphisms forming a cocone of the transfinite chain $F^n 0 (n \in \text{Ord})$.

**Proof.** I. We first prove the compatibility, i.e.

\[
u_n = u_m j_{nm} \text{ for all } n < m
\]

by transfinite induction on $n$. The initial step is trivial, and the limit step follows from the definition of $u_m$ above. For the isolated step, it is sufficient to prove

\[
u_n = u_{n+1} i_{n,n+1}
\]

for all ordinals $n$. We do this by transfinite induction. The initial step is trivial. In the isolated step we observe that from $u_{n-1} = u_n i_{n-1,n}$ it follows that $F\nu_{n-1} = (F\nu_n)(F i_{n-1,n})$ which is precisely the equality we are proving. Let $n$ be a limit ordinal. Then for each $k < n$ we have

\[
i_{n+1,k+1,n} i_{k+1,n+1} = F i_{k,n}
\]
and thus $u_n = u_{n+1}^i n_{n+1}$ follows from the equations

$$
\begin{align*}
u_{n+1} i_{n+1} &= u_{n+1} \\
&= \psi^{-1} Fu_k \\
&= \psi^{-1} F(u_n i_{k,n}) \\
&= u_{n+1} i_{k,n} \\
&= (u_{n+1} i_{n,n+1}) i_{n+1,n}.
\end{align*}
$$

The proof that each $u_i$ is a monomorphism now follows by easy transfinite induction: $u_0 : 0 \to T$ is a monomorphism; if $u_i$ is a monomorphism then so is $u_{n+1} = \psi^{-1} Fu_n$ since $F$ preserves monomorphisms (see beginning of section 3); and the limit step is clear: a cocone of monomorphisms is always factored by a monomorphism.

**Corollary 5.2**

Every $\omega$-continuous set functor $F$ has an initial algebra

$$
I = F^r 0 \quad \text{for some ordinal } r
$$

which is a ‘canonical’ subobject of a final coalgebra: we denote by

$$
\hat{u} : I \to T
$$

the monomorphism $u_r$ above.

**Remark 5.3**

For bicontinuous functors we have $r = \omega, I = F^\omega 0$ and $\hat{u} = \bar{u}$ (see Lemma 2.4). For general functors, the ordinal $r$ above can be arbitrarily large: consider, for example, the polynomial functor

$$
FX = 1 + X^C
$$

where $C$ is an infinite set (of cardinality $n$). Then $I \neq F^k 0$ for any $k \leq n$.

**Remark 5.4**

Ordering of $I$ is now defined analogously to the bicontinuous case: let

$$
I = F^r 0 \quad \text{for some ordinal } r \leq \omega.
$$

Recall that $\hat{u} : I \to T$ is $u_r$ and $\bar{u} : F^\omega 0 \to T$ is $u_\omega$, thus, $\hat{u} i_{r,\omega} = \bar{u}$. Analogously to (4.1) introduce morphisms

$$
\begin{align*}
\begin{array}{c}
I \\
\downarrow \hat{u} \\
T \\
\downarrow i_n \\
F^n 1
\end{array}
\quad \begin{array}{c}
\downarrow q_n \\
F^{n+1} 0 \\
\downarrow i_{n+1,r} \\
F^{n+1} 1
\end{array}
\end{align*}
$$

(5.1)

Then we define, for elements $x, y$ of $I$:

$$
x \leq y \text{ iff } x = y \text{ or } x = q_n y \text{ for some } n < \omega.
$$

(5.2)
This is a subposet of \( T \). In fact, if \( x \leq y \) in \( I \) then \( \hat{u}x \leq \hat{u}y \) in \( T \) because from \( x = q_n y \) we derive
\[
\hat{u}x = \hat{u}q_n y = \hat{u}i_{r,\omega} t_n \hat{u}y = i_{r,\omega}^* t_n^* i_{r,\omega} \hat{u}y = i_{r,\omega}^* t_n^* i_{r,\omega} \hat{u}y = i_{r,n} \hat{u}y
\]
by compatibility with \( i_{r,\omega} \) implies
\[
\hat{u}x = \hat{u}i_{r,\omega} t_{n+1,\omega} (F^n p) t_n \hat{u}y = \hat{u} i_{r,n+1} (F^n p) t_n \hat{u}y
\]
and since \( \hat{u} \) is a monomorphism, this means
\[
x = i_{r,n+1} (F^n p) t_n \hat{u} = q_n y.
\]

**Proposition 5.5**
For every grounded \( \omega \)-continuous set functor a final coalgebra is an ideal completion of an initial algebra ordered by (5.2).

The proof is precisely the same as in Proposition 4.1: since \( I = F^\omega 0 \) contains \( F^\omega 0 \), for each \( x \in T \) the sequence \( x_{(n)} = e_n t_n x \) lies in \( I \). And every ideal \( D \subseteq I \) with \( x = \bigvee D \) in \( T \), where \( x \not\in D \), has all elements of the form \( q_n x \) (in \( I \)) or equivalently \( \hat{u} q_n x \) (in \( T \)). Thus, it remains to show that these are precisely the elements \( x_{(n)} \), i.e. to establish the equality
\[
\hat{u} q_n = e_n t_n \hat{u}\quad \text{for all } n < \omega.
\]
In fact,
\[
\begin{align*}
\hat{u} q_n &= \hat{u} i_{n+1, \omega} (F^n p) t_n \hat{u} \\
&= i_{n+1} (F^n p) t_n \hat{u} \quad \text{by compatibility}\ \\
&= i_{n+1, \omega} (F^n p) t_n \hat{u} \quad \text{by compatibility}\ \\
&= e_n t_n \hat{u} \quad \text{by (3.1)}.
\end{align*}
\]

**Example 5.6**
Consider the infinitary polynomial functor
\[
FX = A + B \times X^N
\]
Its final co-algebra \( T \) is the set of all trees in which every inner node is labelled in \( B \) and has countably many successors, and each leaf is labelled in \( A \). An initial algebra is the subset \( I \) of all trees with finite branches. And \( F^\omega 0 \) is the subset of \( I \) of all trees of finite depth. We see that \( T \) (ordered by \( x \leq y \) if \( x = y \) or \( x \) is obtained from \( y \) by cutting at depth \( n \) and labelling the leaves of depth \( n \) by \( p \)) is an ideal completion of \( F^\omega 0 \), and consequently of \( I \).

### 6 Endofunctors of general categories

As mentioned in the Introduction, all the results of the previous sections can be generalized to all LFP categories \( \mathcal{K} \) (e.g. to \( \text{Pos} \), to the category of groups — or, indeed to any variety of finitary algebras, to the category of graphs, etc.) We only request the following, rather harmless, side condition:

an initial object \( 0 \) of \( \mathcal{K} \) be simple.
i.e. have no proper quotient (in other words, every epimorphism with domain 0 be invertible). Although this is true in the majority of important LFP categories, it does not hold generally: consider the category \textbf{Rng} of rings with unit. Here 0 is the usual ring of integers, which is not simple. Since morphisms in LFP categories factor as epimorphisms followed by monomorphisms, the simplicity of 0 makes the unique morphism \( 0 \to X \) a monomorphism for every object \( X \).

**Remark 6.1**
(1) In LFP categories every object \( K \) is determined by the hom-sets

\[
\text{hom}(B, K) \quad \text{for } B \text{ finite presentable.}
\]

In fact, \( K \) is a canonical colimit of the diagram whose objects are indexed by morphisms \( f : B \to K \) with \( B \) finitely presentable and whose morphisms are indexed by the commutative triangles

\[
\begin{array}{ccc}
B & \xrightarrow{b} & B' \\
\downarrow{f} & & \downarrow{f'} \\
K & & K
\end{array}
\]

\((B, B' \in \mathcal{B})\)

(2) In every LFP category \( \mathcal{K} \), colimits of \( \omega \)-chains of monomorphisms are formed by monomorphisms, and for any cocone of monomorphisms the factorizing morphism is also a monomorphism, see [6, 1.62]. Thus, if \( F : \mathcal{K} \to \mathcal{K} \) preserves monomorphisms, then from Remark 2.6 we derive that

\[
\tilde{u} : F^\omega 0 \to F^\omega 1
\]

is a monomorphism - except that this requests \( u : 0 \to 1 \) to be a monomorphism. To guarantee the last, we assume that 0 is simple.

**Remark 6.2 (Orderings for final coalgebras)**
Assume that \( \mathcal{K} \) is an LFP category with a single initial object 0. Let \( F \) be a grounded, \( \omega \)-continuous endofunctor preserving monomorphisms. Then \( F \) has a final coalgebra \( T = \lim F^\eta 1 \) with limit cone, as above, \( t_n : T \to F^\eta 1 \); we also use the notation \( e_n : F^\eta 1 \to T \) of (3.1).

Each hom-set \( \text{hom}(B, T) \), where \( B \) is an arbitrary object, is ordered by the above rule (3.2); i.e. given morphisms \( x, y : B \to T \) then

\[
x \leq y \iff x = y \text{ or } x = e_n t_n y \text{ for some } n < \omega.
\]

This makes \( \text{hom}(B, T) \) an algebraically \( \omega \)-complete CPO.

The proof is identical to that of Theorem 3.3 (in fact, the notation of that proof has been chosen so that no changes are needed — just the domain of \( x \) and \( y \) is now \( B \), whereas there we worked with \( x, y : 1 \to T \)). Note that since 0 is assumed to be simple, not only \( u : 0 \to 1 \) but also \( i : 0 \to F0 \) is a monomorphism; and \( p : 1 \to F0 \) is a monomorphism in any category.

**Remark 6.3 (Orderings for initial algebras)**
First, add to the assumption of Remark 6.2 the bicontinuity of \( F \). By Remark 2.6, we have a monomorphism in \( \mathcal{K} \)

\[
\tilde{u} : I \to T
\]
which, for every object $B$, leads to an embedding
\[
\text{hom}(B, I) \hookrightarrow \text{hom}(B, T), \quad x \mapsto \bar{a}x
\]
and thus makes $\text{hom}(B, I)$ a subposet of the CPO $\text{hom}(B, T)$.

Analogously to the beginning of section 4, each
\[
y : B \to I = \text{colim} \, F^k 0
\]
with $B$ finitely presentable factors through some $i_k$ (because $\text{hom}(B, I) \cong \text{colim} \, \text{hom}(B, F^k 0)$), i.e. $y$ has the form
\[
y = i_k y' \quad \text{for some } y' : B \to F^k 0.
\]
Thus, if we denote by $q_{h,n} : F^k 0 \to I$ the morphisms defined by (4.1), we have the definition of ordering of
\[
\text{hom}(B, I) \quad \text{for any finitely presentable object } B
\]
precisely as in (4.2):
\[
x \leq y \text{ iff } x = y \text{ or } x = q_{h,n} y' \text{ for some numbers } n < k < \omega
\]
\[
\text{and some } y' : B \to F^k 0 \text{ with } y = i_k y'.
\]
The above ordering does not use the structure of $T$.

Now for $\omega$-continuous functors in general, we first have to establish that an initial algebra exists and is a canonical subobject $\hat{u} : I \hookrightarrow T$ — this is done precisely as in section 5 above. Denote by $q_n : I \to I$ the morphisms defined by (5.1) (which now of course do depend on the structure of $T$). Then (5.2) defines an ordering on each hom-set $\text{hom}(B, I)$. And the following result is proved precisely as Propositions 4.1 and 5.4.

**Theorem 6.4**

Let $\mathcal{K}$ be an LFP category with a simple initial object. Given a grounded, $\omega$-continuous end-ofunctor preserving monomorphisms, then an initial algebra $I$ determines a final coalgebra via ideal completion: each $\text{hom}(B, I)$, where $B$ is finitely presentable, is a poset with ideal completion $\text{hom}(B, T)$.

**Example 6.5**

Real numbers. Recall the two functors $F_1, F_2 : \mathbf{Pos} \to \mathbf{Pos}$ presented in [9] such that a final coalgebra for both $F_1$ and $F_2$ is formed by the poset of real numbers (more precisely, by the real interval $[0, 1]$). The functor $F_1$ is defined by
\[
F_1 X = X \cdot \omega
\]
where $X \cdot Y$ denotes the ordinal product, i.e. the Cartesian product of posets with order augmented by $(x, y) < (x', y')$ whenever $y < y'$ in $Y$. Analogously, $X; Y$ denotes the ordinal sum, i.e. the disjoint union of posets with order augmented by $x \leq y$ for all $x \in X$ and $y \in Y$. The functor $F_2$ is defined by
\[
F_2 X = 1; (X^{\text{op}} \cdot \omega)
\]
where $X \mapsto X^{\text{op}}$ is the functor reverting the ordering and leaving morphisms unchanged.
Final Coalgebras are Ideal Completions of Initial Algebras

It is easy to see that both $F_1$ and $F_2$ are bicontinuous and preserve monomorphisms. Since $F_0 = 0$, our theory applies to $F_2$ only. A final coalgebra for $F_2$ can be described as the set of all finite and infinite sequences of positive natural numbers ordered lexicographically so that the odd members form the usual ordering of natural numbers and the even ones the opposite ordering. Example

$$ ( ) < (1) < (2) ; (2, 1) < (2). $$

The isomorphism between this poset and $[0, 1)$ is explained in [9, see 3.2: alternating dyadics].

The order of (3.2) above is easy to identify for $F_2$: given sequences $x$ and $y$ of natural numbers, $x \leq y$ iff $x = y$ or $x$ is a prefix of $y$. Example:

$$ ( ) < (2) < (2, 1). $$

An initial algebra for $F_2$ the subposet of $T$ formed by all finite sequences; these correspond precisely to the dyadic rational numbers $\frac{m}{2^n}$ of $[0, 1)$. Thus $I$ is dense in $T$ both under the ordering of $T$ as an object of $\mathsf{Pos}$, and under the ordering of (3.2); in each case for completely different reasons.

Acknowledgements

Supported by the Grant Agency of Czech Republic, Grant No. 201/99/0310.

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Received September 2000