Space-efficient Decision Procedures for Three Interpolable Propositional Intermediate Logics

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Abstract

In this paper we present duplication-free tableau calculi for three propositional intermediate interpolable logics, namely the logic characterized by rooted Kripke models with depth two at most, the logic characterized by rooted Kripke models with two final elements at most and depth two at most and the logic characterized by rooted Kripke models with a final element at most (also known as Jankov Logic). Using such calculi we define a \(O(n)\)-SPACE decision procedure for the second logic and a \(O(n \log n)\)-SPACE decision procedure for each of the other two logics.

Keywords: Tableau systems, intermediate logics, decision procedures.

1 Introduction

The development of efficient proof strategies is a fundamental topic for people working on theorem proving and automated deduction. For this reason there is a lot of interest in Tableau Systems, a goal-oriented method which seems to be useful to automate deductions.

Recently many papers [12, 16, 17, 24, 1, 13] have improved the proof-search of well-known sequent and tableau calculi for Intuitionistic Propositional Logic and intermediate propositional logics, and attention has been given to computational complexity questions. In particular, in [12, 16, 17] loop-free sequent calculi for Intuitionistic Propositional Logic are given, where a sequent calculus is loop-free when defined as a well founded complexity measure on sequents, in every rule of the calculus the complexity of the sequent in the conclusion is greater than the complexity of every sequent in the premiss. Calcuclus with this property guarantee that in deduction every sequent is used at most once. In [17] this property, together with the fact that the complexity function is linearly bounded by the length of the sequent to be proved, give rise to a \(O(n \log n)\)-SPACE decision procedure for Propositional Intuitionistic Logic. Among the intermediate logics, Gödel-Dummett Logic [11] (also known as LC because it is characterized by Kripke models whose elements form linear chains) has been extensively studied both in the propositional and in the predicate case [9, 4, 10, 1, 3, 6, 13]; in particular, in [13] a loop-free sequent calculus is provided that allows one to avoid back-tracking among the applications of the rules.

On the tableau side re-using formulas in deductions is known as duplication. In [22, 23, 24] calculi for Intuitionistic Logic are presented in which duplications are progressively reduced and a new technique, which is an improvement of the one presented in [14] where the duplication problem is not taken into account, has been developed for the construction of the counter model. In [1] Interpolable Propositional Intermediate Logics are considered and
duplication-free tableau calculi and loop-free sequent calculi are given for them.

Intermediate logics are used in computer science in the frameworks of concurrency, program synthesis, abstract data type specification and formal verification [4, 21, 2, 20]. Among the propositional intermediate logics the interpolable ones were the first examples of logics lying between propositional Intuitionistic and Classical logics; they are well known and investigated for their simple and natural semantics and for their relationships with nonmonotonic reasoning, many valued and fuzzy logics [26, 5, 25, 15, 8]. In this paper we consider three Interpolable Propositional Intermediate Logics (following [1] we call them $\text{BD}_2$, $\text{GS}_e$ and $\text{J}_n$) and we give duplication-free tableau calculi in which (as for the calculus for the Intuitionistic Propositional Logic given in [17]) proofs have depth which is linearly bounded by the length of the formula to be proved, every rule has a constant number of conclusions and the number of formulas in every conclusion is linearly bounded by the length of the formula to be proved; these properties allow space-efficient decision procedures. Moreover we give attention to the degree of nondeterminism of the related decision procedures. Giving attention both to lower the computational space and to shrink the proof-search space, our aim is to improve the time efficiency of the decision procedures of the logics in hand, although these procedures are not time efficient in the sense of computational complexity. These logics are studied in [1] without taking into account the computational space complexity problem. In this sense this work follows the lines of [17] and [19], where space-efficient procedures are given, and [27] where Intuitionistic Propositional Logic is shown to be PSPACE-complete.

For every calculus we give a semantical proof of Soundness and Completeness theorems. Our semantical methods are in line with those used in [1] and are an adaptation of the techniques used in [14]. In our decision procedures the permutation among noninvertible rules is restricted to some of them; we show that our strategy is complete by a careful construction in the proof of the Completeness Theorem.

2 Basic definitions

In this section we give notions and notation we will use in the paper. A comprehensive presentation of all notions regarding intermediate logics and Kripke models can be found in [7] and [14].

Given an enumerable set of propositional variables and the connectives $\neg, \land, \lor, \to$, a well formed formula (wff for short) is defined as usual. Given a wff $A$, we say that $A$ is an atom if it coincides with a propositional variable and we say that $\neg A$ is a negated wff.

In the following, we use $\text{Int}$ to denote both an arbitrary Hilbert-style calculus for Intuitionistic Propositional Logic and the set of intuitionistically valid wffs.

If $\mathcal{A}$ is a set of axiom schemata, $\text{Int} + \mathcal{A}$ will denote both the Hilbert-style calculus obtained by adding to $\text{Int}$ the axiom schemata of $\mathcal{A}$, and the set of theorems of such a deductive system. We use the notation $\text{Int} + (A)$ when the system is characterized by the single axiom schemata ($A$). An intermediate propositional logic is any consistent set of wffs $\text{Int} + \mathcal{A}$.

A Kripke model is a triple $\mathfrak{K} = \langle P, \leq, \models \rangle$ where:

(i) $P$ is a nonempty set which elements are called worlds;
(ii) $\leq$ is a reflexive, antisymmetric and transitive binary relation on $P$;
(iii) $\models$ is a binary relation between $P$ and the propositional variables, called forcing, with the following property: let $p$ be an atom and $\Gamma \in P$, if $\Gamma \models p$ then, for every $\Theta \in P$ such that $\Gamma \leq \Theta$, $\Theta \models p$. The relation $\models$ is extended to the wffs as follows:
(a) $\Gamma \vdash A \land B$ iff $\Gamma \vdash A$ and $\Gamma \vdash B$;
(b) $\Gamma \vdash A \lor B$ iff $\Gamma \vdash A$ or $\Gamma \vdash B$;
(c) $\Gamma \vdash A \rightarrow B$ iff, for every $\Delta \in P$ such that $\Gamma \leq \Delta$, $\Delta \not\models A$ or $\Delta \models B$;
(d) $\Gamma \vdash \neg A$ iff for every $\Delta \in P$ such that $\Gamma \leq \Delta$, $\Delta \not\models A$.

From the above definition it is easy to prove that if $\Gamma \models A$ and $\Gamma \leq \Delta$ then $\Delta \models A$.

Let $\Gamma$ be a world of $P$ and let $A$ be a wff, if $\Gamma \models A$ we say that $A$ is forced in $\Gamma$ (or in a world of $K$).

To characterize the class of models we will consider in this paper, we will need the following notions.

A wff $A$ is valid in a Kripke model $K$ if it is forced in every world of $K$. If $\mathcal{K}$ is a set of Kripke models, with $\mathcal{L}(\mathcal{K})$ we denote the set of wffs that are valid in every Kripke model $K$ in the class $\mathcal{K}$.

For any logic $\mathcal{L}$ we consider in this paper, we know at least a class of Kripke models $\mathcal{K}_\mathcal{L}$ such that $\mathcal{L}(\mathcal{K}_\mathcal{L}) = \mathcal{L}$. We call $\mathcal{L}$-model any Kripke model $K$ such that $K \in \mathcal{K}_\mathcal{L}$.

Given a Kripke model $K = \langle P, \leq, \models \rangle$ and $\Gamma, \Delta \in P$, $\Gamma \leq \Delta$ means $\Gamma \leq \Delta$ and $\Gamma \not\leq \Delta$ and $\Delta \geq \Gamma$ means $\Gamma \leq \Delta$; moreover, $\Delta$ is an immediate successor of $\Gamma$ iff $\Gamma < \Delta$ and for all $\Theta \in P$ such that $\Gamma \leq \Theta \leq \Delta$, $\Gamma = \Theta$ or $\Delta = \Theta$. We call root of $K = \langle P, \leq, \models \rangle$ a world $\Phi \in P$ (if it exists) such that, for every $\Gamma \in P$, $\Phi \leq \Gamma$ and we call final (in $K$) a world $\Delta \in P$ such that for any $\Theta \in P$, $\Delta \leq \Theta$ iff $\Theta = \Delta$.

Given $K = \langle P, \leq, \models \rangle$, $\Gamma \in P$ has depth at least $h$ in $K$ if there exists a sequence $\Delta_1, \ldots, \Delta_h$ of worlds of $P$ such that $\Delta_h = \Gamma$ and $\Delta_{i+1}$ is immediate successor of $\Delta_i$, $i = 1, \ldots, h-1$; $\Gamma$ has depth at most $h$ in $K$ if it has not depth at least $h + 1$ and $\Gamma$ has depth $h$ if it has depth at least $h$ and depth at most $h$.

Every calculus $\mathcal{L}$-T given in the next sections, where $\mathcal{L} \in \{ \text{Bd}_2, \text{GS}_e, \text{J}_n \}$, has its signed language $\mathcal{L}_\mathcal{L}$ which is built on a set of signs $S_\mathcal{L} = \{ S_1, \ldots, S_n \}$ and the set of wffs. Every member of $\mathcal{L}_\mathcal{L}$ is a signed wff (swff for short) whose syntax is $SA$, with $S \in S_\mathcal{L}$ and $A$ wff.

The length of a wff $A$ (respectively swff $SA$), denoted by $\|A\|$ (respectively $\|SA\|$), is the number of symbols in $A$ (respectively the number of symbols in $A$ plus one). The length of a set $S$ of wffs or swffs is the sum of the lengths of its elements. Given two wffs or two swffs $A, B$ with $A \equiv B$ we mean that $A$ and $B$ are syntactically identical. Finally, given a set $S$ of wffs or swffs, by $\text{Pr}(S)$ we mean the set of the atoms appearing in the members of $S$.

3 The logic $\text{Bd}_2$

In this section we will study the logic

$$\text{Bd}_2 = \text{Int} + (p \lor (p \rightarrow q \lor \neg q))$$

semantically characterized by the class of rooted Kripke models which depth is at most two.

The signed language of the calculus $\text{Bd}_2$-T (as in [1]) is built on the signs $T, F, \neg, \rightarrow$, whose meaning is explained in terms of $\text{Bd}_2$-realizability: given a $\text{Bd}_2$-model $K = \langle P, \leq, \models \rangle$ and a swff $H$, we say that $\Gamma \in P \text{Bd}_2$-realizes $H$, and we write $\Gamma \triangleright H$, if the following conditions hold:

- if $H \equiv TA$ then $\Gamma \models A$;
- if $H \equiv FA$ then $\Gamma \not\models A$;
such that the notation kind does not reappear in the conclusion of the rule. The notation in [1] lacks this property. The rule \( \text{T} \rightarrow \text{Atom} \), given in [28, 12, 16], is invertible whereas the related rule \( \text{T} \rightarrow \text{AN} \) in [1] does not enjoy this property.

For every rule given in the following tables, \( S \) is a set of swffs, \( H_1, \ldots, H_t \) are swffs and the notation \( S, H_1, \ldots, H_t \) means \( S \cup \{ H_1, \ldots, H_t \} \). In every rule we distinguish two parts: the premiss, the set of swffs above the line, and the conclusion, the sequence of sets of swffs below the line. We call main set of swffs of a rule the set of swffs that are in evidence in the premiss, e.g. the main set of swffs of the rule \( \text{T} \rightarrow \text{Atom} \) is \( \{ \text{T} \cdot \text{A} | \text{A} \rightarrow \text{B} \} \). All the rules we give hereafter will be applied in a duplication-free style: a rule \( \mathcal{R} \) with main set of swffs \( \{ H_1, \ldots, H_n \} \) applies to a set \( U \) of swffs if it is possible to choose \( S \) and \( H_1, \ldots, H_n \) in such a way that \( U = S \cup \{ H_1, \ldots, H_n \} \), with \( S \neq U \setminus \{ H_1, \ldots, H_n \} \). This implies that the main set of swffs does not reappear in the conclusion of the rule.

**Table 1. The \( \text{Bd}_2 \cdot \text{T} \) calculus**

\[
\begin{align*}
S, \text{F} | \text{T} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \\
S, \text{F} | \text{T} \cdot (A \rightarrow B) & \quad \text{F} \rightarrow \quad S, \text{F} | \text{T} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \\
S, \text{F} | \text{T} \cdot (A \lor B) & \quad \text{F} \rightarrow \quad S, \text{F} | \text{T} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land
\end{align*}
\]

see Table 2

\[
\begin{align*}
S, \text{F} | \text{T} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \\
S, \text{F} | \text{T} \cdot (A \rightarrow B) & \quad \text{F} \rightarrow \quad S, \text{F} | \text{T} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \quad S, \text{F} | \text{F} \cdot (A \lor B) & \quad \text{T} \lor \\
S, \text{F} | \text{T} \cdot (A \lor B) & \quad \text{F} \rightarrow \quad S, \text{F} | \text{T} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land \quad S, \text{F} | \text{F} \cdot (A \land B) & \quad \text{T} \land
\end{align*}
\]

The following terminology will be used throughout the paper.

**Definition 3.1**

Let \( \mathcal{L} \) be one of the logics treated in this paper and let \( \mathcal{L} \cdot \text{T} \) be the related tableau calculus.

1. A world \( \Gamma \) of a \( \mathcal{L} \)-model \( K \) \( \mathcal{L} \)-realizes a set \( S \) of swffs of \( \mathcal{L} \) (and we write \( \Gamma \models S \)) iff \( \Gamma \) \( \mathcal{L} \)-realizes every swff in \( S \), where this notion depends on the logic in hand. A set \( S \) of swffs of \( \mathcal{L} \) is \( \mathcal{L} \)-realizable iff there is a world \( \Gamma \) of a \( \mathcal{L} \)-model \( K \) such that \( \Gamma \models S \).

2. A configuration is any finite sequence \( S_1 | \ldots | S_j | \ldots | S_n \) (with \( n \geq 1 \)), where every \( S_j \) is a set of swffs; a configuration is \( \mathcal{L} \)-realizable iff at least a \( S_j \) is \( \mathcal{L} \)-realizable; we refer to \( S_j \) as an element of \( S_1 | \ldots | S_n \).
Table 2. The rules $T \rightarrow$

\[
\begin{array}{l}
S, T, A, T(A \rightarrow B), T \rightarrow \text{Atom with } A \text{ atom} \quad S, T, ((A \land B) \rightarrow C), T \rightarrow \land \\
S, T, A, T(B \rightarrow C), T \rightarrow \lor \text{ with } p \text{ new variable} \\
S, T, (A \rightarrow B) \rightarrow C, T \rightarrow \rightarrow \\
S, F, (\neg A \rightarrow B), T \rightarrow \neg \\
\end{array}
\]

3. A $L$-proof table is a finite sequence of configurations $C_1, \ldots, C_i, C_{i+1}, \ldots, C_n$, where the configuration $C_{i+1}$ is obtained from $C_i = S_1 \ldots S_k$ by applying to each non-$L$-contradictory $S_j$ of $C_i$ a rule of the calculus $L\cdot T$ and copying to $C_{i+1}$ all $S_l \neq S_j$ of $C_i$ which are $L$-contradictory or contain signed atoms only. Moreover, an $L$-proof table is closed iff all the sets $S_j$ in its final configuration are $L$-contradictory, where the notion of $L$-contradictory set depends on the logic at hand. Finally, the depth of an $L$-proof table is the number of its configurations.

4. A proof of a wff $B$ in $L\cdot T$ is a closed $L$-proof table starting from the configuration $\{FB\}$.

5. A finite set of swffs $S$ of $L$ is $L$-consistent iff no $L$-proof table starting from $S$ is closed.

6. Let $U$ be a main set of swffs in the object language of $L\cdot T$. We call $L$-extension(s) of $U$ the set(s) of swffs $R^1_U, \ldots, R^n_U$ ($n \geq 1$) such that $R^1_U \ldots R^n_U$ is the configuration obtained by applying the rule related to $U$ in $L\cdot T$ to the set of swffs $U$. If $U$ is a main set of swffs of the kind $\{H\}$ with $H$ swff, then we call $L$-extension(s) of $H$ the $L$-extension(s) of $\{H\}$ and we denote them $R^1_H, \ldots, R^n_H$ ($n \geq 1$).

A set $S$ of swffs is $BD_{2}\text{-contradictory}$ if one of the following conditions hold:

- $TA \in S$ and $FA \in S$;
- $TA \in S$ and $F_eA \in S$;
- $T_{cl}A \in S$ and $F_eA \in S$.

It is easy to prove the following fact.

**Proposition 3.2**

If a set of swffs is $BD_{2}\text{-contradictory}$, then it is not $BD_{2}\text{-realizable}$

For our purposes the initial configuration of every $L$-proof table has one element. In an obvious way it is possible to build a correspondence between an $L$-proof table and a tree, we call it an $L$-proof tree, whose nodes are the elements in the configurations of the $L$-proof table.

Some rules of the calculus $BD_{2}\cdot T$, namely $F \rightarrow, F_{\neg}, T_{cl} \lor, T_{cl} \rightarrow, F_{e} \land$ and $T \rightarrow \rightarrow$, narrow the set $S$ to $S_{c}$ (the certain part of $S$) in at least one of the sets in the final configuration of the rule. Following [1], we call these rules $BD_{2}\cdot c\text{-rules}$ and $BD_{2}\cdot c\text{-swff}$ every swff $H$
such that \( \{ H \} \) is a main set of swffs of some \( \text{Bd}_2 \)-c-rule. We call \( \text{Bd}_2 \)-regular rule any rule of the \( \text{Bd}_2 \)-T calculus which is not a \( \text{Bd}_2 \)-c-rule and \( \text{Bd}_2 \)-regular swff any swff to which a \( \text{Bd}_2 \)-regular rule can be applied. Finally, a \( \text{Bd}_2 \)-basic rule is a \( \text{Bd}_2 \)-regular rule different from \( T \rightarrow \text{Atom} \) and \( T \rightarrow \neg \), and a \( \text{Bd}_2 \)-basic swff is a swff to which a \( \text{Bd}_2 \)-basic rule is applicable.

In the following definition we introduce the function \( \text{deg}_{\text{nat}_2} \), the complexity measure on wffs and swffs that will be used to study the properties of the \( \text{Bd}_2 \)-T calculus.

**Definition 3.3**

1. The degree of a set \( A \), denoted by \( \text{deg}_{\text{nat}_2}(A) \), where \( A \) is a wff, is defined as follows: if \( A \equiv p \), where \( p \) is an atom, then \( \text{deg}_{\text{nat}_2}(p) = 0 \); if \( A \equiv \alpha \rightarrow \beta \), then \( \text{deg}_{\text{nat}_2}(\alpha \rightarrow \beta) = \text{deg}_{\text{nat}_2}(\alpha) + \text{deg}_{\text{nat}_2}(\beta) + 1 \); if \( A \equiv \alpha \land \beta \), then \( \text{deg}_{\text{nat}_2}(\alpha \land \beta) = \text{deg}_{\text{nat}_2}(\alpha) + \text{deg}_{\text{nat}_2}(\beta) \); if \( A \equiv \alpha \lor \beta \), then \( \text{deg}_{\text{nat}_2}(\alpha \lor \beta) = \text{deg}_{\text{nat}_2}(\alpha) + \text{deg}_{\text{nat}_2}(\beta) + 3 \); if \( A \equiv \neg \alpha \), then \( \text{deg}_{\text{nat}_2}(\neg \alpha) = \text{deg}_{\text{nat}_2}(\alpha) + 1 \).
2. The degree of a set \( S \) (where \( S \in \{ T, F, F_c, T_c \} \) ), denoted by \( \text{deg}_{\text{nat}_2}(S) \) coincides with the degree of \( A \).
3. The degree of a set \( S \), denoted by \( \text{deg}_{\text{nat}_2}(S) \), whose elements are wffs or swffs coincides with the sum of the degrees of its elements.

**Proposition 3.4**

For every rule of the calculus \( \text{Bd}_2 \)-T with main set of swffs \( U \), for every \( \text{Bd}_2 \)-extension \( R^1_U \) of \( U \) \((1 \leq i \leq n, n \geq 1)\), \( \text{deg}_{\text{nat}_2}(R^1_U) < \text{deg}_{\text{nat}_2}(U) \).

**Proof.** We consider three rules.

- **Rule** \( T \rightarrow \forall \): the main set of swffs of the rule \( T \rightarrow \forall \) is \( U = \{ T(A \lor B) \rightarrow C \} \) and the \( \text{Bd}_2 \)-extension of \( U \) is \( R^1_U = \{ T(A \rightarrow p), T(B \rightarrow p), T(p \rightarrow C) \} \). Now, \( \text{deg}_{\text{nat}_2}(R^1_U) = \text{deg}_{\text{nat}_2}(T(A \rightarrow p)) + \text{deg}_{\text{nat}_2}(T(B \rightarrow p)) + \text{deg}_{\text{nat}_2}(T(p \rightarrow C)) = \text{deg}_{\text{nat}_2}(A) + \text{deg}_{\text{nat}_2}(B) + \text{deg}_{\text{nat}_2}(C) + 3 < \text{deg}_{\text{nat}_2}(U) = \text{deg}_{\text{nat}_2}(A) + \text{deg}_{\text{nat}_2}(B) + \text{deg}_{\text{nat}_2}(C) + 4 \).

- **Rule** \( T \rightarrow \rightarrow \): the main set of swffs of the rule \( T \rightarrow \rightarrow \) is \( U = \{ T((A \rightarrow B) \rightarrow C) \} \), the \( \text{Bd}_2 \)-extensions of \( U \) are \( R^1_U = \{ T_cA, F_cB \} \) and \( R^2_U = \{ T, T(B \rightarrow C) \} \). Now, \( \text{deg}_{\text{nat}_2}(R^1_U) = \text{deg}_{\text{nat}_2}(\{ T_cA, F_cB \}) < \text{deg}_{\text{nat}_2}(T((A \rightarrow B) \rightarrow C)) \) and \( \text{deg}_{\text{nat}_2}(\{ T, T(B \rightarrow C) \}) < \text{deg}_{\text{nat}_2}(T((A \rightarrow B) \rightarrow C)) \) hold.

- **Rule** \( T \rightarrow \text{Atom} \): the main set of swffs of the rule \( T \rightarrow \text{Atom} \) is \( U = \{ T, A, (T \rightarrow B) \} \), the \( \text{Bd}_2 \)-extension of \( U \) is \( R^1_U = \{ T, A, T(B) \} \), and the inequality \( \text{deg}_{\text{nat}_2}(\{ T, A, T(B) \}) < \text{deg}_{\text{nat}_2}(\{ T, A, T(A \rightarrow B) \}) \) holds.

The proposition above and the fact \( \text{deg}_{\text{nat}_2}(S_c) \leq \text{deg}_{\text{nat}_2}(S) \) can be taken as the formal counterpart of the notion of ‘duplication free’ (see [1]) and guarantee that, starting from a configuration \( H \), the number of the resulting configurations is finite.

Moreover, by the proposition below, we get a linear bound, with respect to the length of the wff to be proved, on the depth of the deductions in \( \text{Bd}_2 \)-T.

**Proposition 3.5**

Let \( S \) be a set of swffs, \( \text{deg}_{\text{nat}_2}(S) \leq 3||S|| \).

**Proof.** We prove that the complexity of every wff \( A \) is bounded by the length of \( A \). The proof is by induction on the number of connectives in \( A \).
**Basis:** if the number of connectives is zero then \( A \equiv p \), with \( p \) an atom, and we have 
\[ \deg_{\text{na}_2}(p) = 0 < 3||p|| = 3; \]

**Step:** suppose the proposition hold for the swffs \( A' \) with \( n - 1 \) connectives at most. Let us consider the case where \( A \equiv B \lor C \), then 
\[ \deg_{\text{na}_2}(B \lor C) = \deg_{\text{na}_2}(B) + \deg_{\text{na}_2}(C) + 3. \]

By induction hypothesis \( \deg_{\text{na}_2}(B) \leq 3||B|| \) and \( \deg_{\text{na}_2}(C) \leq 3||C|| \), thus 
\[ \deg_{\text{na}_2}(B \lor C) \leq 3||B|| + 3||C|| + 3 = 3(||B|| + ||C|| + 1) = 3||B \lor C||; \] the other cases are similar.

Now, using the last two propositions we immediately get:

**Theorem 3.6**

Given a wff \( A \), every \( \text{Bd}_2 \)-proof table starting from the configuration \( \{ FA \} \) has \( 3||A|| \) configurations at most.

The calculus \( \text{Bd}_2 \)-\( T \) given in [1] does not enjoy the property stated in the previous theorem since in \( T \to \lor \) and \( T \to \rightarrow \) a subformula of the main set of swffs appears twice in an element of the conclusion. This implies that some deductions may have a depth which is quadratic in the length of the wff to be proved. The rule \( T \to \lor \) is rewritten using a new atom, a standard technique for this case. The new atom is repeated twice in the conclusion of the rule. The advantage is that the length of the atom does not depend on the length of the premiss and this guarantees that, by choosing the constant values in a suitable way, the function \( \deg_{\text{na}_2} \) enjoys the property given in Proposition 3.5. Our rule \( T \to \rightarrow \) has a conclusion with three elements and, differently from [1], the subformula \( B \) of the main set of swffs of \( T \to \rightarrow \) is not repeated in any element of the conclusion.

The following proposition is the main step towards the soundness theorem.

**Proposition 3.7**

The rules of the calculus \( \text{Bd}_2 \cdot T \) preserve \( \text{Bd}_2 \)-realizability

**Proof.** By definition of \( \text{Bd}_2 \)-realizability we have to prove that, if a configuration is \( \text{Bd}_2 \)-realized in a world of a rooted Kripke model \( \mathcal{K} = \langle P, \leq, \models \rangle \) whose depth is at most two, then the configuration obtained by applying to the former configuration one of the rules of \( \text{Bd}_2 \cdot T \) is \( \text{Bd}_2 \)-realized in a possibly different world of \( \mathcal{K} \). The proof requires one to analyse the rules of \( \text{Bd}_2 \cdot T \). We will consider only some of the rules.

- Rule \( T \to \rightarrow \): let \( S \) be a set of swffs containing the \( T((A \to B) \to C) \), let \( \mathcal{K} = \langle P, \leq, \models \rangle \) be a \( \text{Bd}_2 \)-model and let \( \Gamma \in P \) be a world of \( \mathcal{K} \) such that \( \Gamma \models S \). By definition of \( \text{Bd}_2 \)-realizability \( \Gamma \models (A \to B) \to C \); we have three cases:
  1. if \( \Gamma \models C \), then \( \Gamma \models TC; \)
  2. if \( \Gamma \models A \) and \( \Gamma \not\models B \), then since \( \Gamma \not\models A \to B \) and \( \Gamma \models (A \to B) \to C \), we get 
    \( \Gamma \models B \to C; \)
  3. if there is a world \( \Delta \in P \) such that \( \Delta \triangleright \Gamma \), \( \Delta \models A \) and \( \Delta \not\models B \), then \( \Delta \models \neg A \) and \( \Delta \models \neg B \) because \( \Delta \) is a final world; this implies \( \Delta \models T_{\text{cl}}A \) and \( \Delta \models F_{\text{cl}}B \). Moreover, by the persistence of the forcing relation, if \( \Gamma \models H \), then \( \Delta \models H \), with \( H \) wff, and this implies \( \Delta \triangleright S \).

- Rule \( T \to \lor \): let \( S \) be a set of swffs containing the \( T((A \lor B) \to C) \), let \( \mathcal{K} = \langle P, \leq, \models \rangle \) be a \( \text{Bd}_2 \)-model and let \( \Gamma \in P \) be a world of \( \mathcal{K} \) such that \( \Gamma \models S \). By definition of \( \text{Bd}_2 \)-realizability \( \Gamma \models (A \lor B) \to C \). Now we can define a new Kripke model \( \mathcal{K}^* = \langle P, \leq, \models \rangle \) such that the forcing relation between worlds of \( P \) and wffs is the following:
  1. for every atom \( q \in P \lor(S) \) and for every world \( \Delta \in P \) we define \( \models \) as follows: \( \Delta \models q \) iff \( \Delta \models q \);
2. let $p$ be an atom such that $p \notin P_\nu(S)$, for every $\Lambda \in P$ we define $\models^p$ as follows: $\Lambda \models^p p$ if $\Lambda \models A$ or $\Lambda \models B$;
3. for every other atom $q'$, such that $q' \neq p$ and $q' \notin P_\nu(S)$, and for every world $\Lambda \in P$ we define $\models^{q'}$ as follows: $\Lambda \models^{q'} q'$.

Let $\models^r$ be the $\text{Bd}_2$-realizability relation with respect to $\models^\nu$; it is easy to check that for every wff $C$ such that $P_\nu(C) \subseteq P_\nu(S)$ and, for every $\Gamma \in P$, $\Gamma \models C$ iff $\Gamma \models^\nu C$.

Moreover:
1. $\Gamma \models^r T(A \rightarrow p)$, because if $\Delta \in P$ is such that $\Delta \supseteq \Gamma$ and $\Delta \models^\nu A$ then $\Delta \models A$ and by definition of $\models^r$, $\Delta \models^\nu p$;
2. $\Gamma \models^r T(B \rightarrow p)$, see the above point;
3. $\Gamma \models^r T(p \rightarrow C)$, because if $\Delta \in P$ is such that $\Delta \supseteq \Gamma$ and $\Delta \models^{q'} p$ then, by definition of $\models^{q'}$, $\Delta \models^{q'} A$ or $\Delta \models^{q'} B$; in every case, since $\Delta \models^{q'} (A \lor B) \rightarrow C$, it is guaranteed that $\Delta \models^{q'} C$.

Some $\text{Bd}_2$-extensions of the main sets of swffs of $\text{Bd}_2$-c-rules are $\text{Bd}_2$-realized in a world $\Delta$ above $\Gamma$. This explains why in some elements $S_\nu$ appears, in fact, if $FH \in S$ and $\Gamma \models S$ we have $\Gamma \models FH$ and $\Gamma \models H$; now we cannot conclude $\Delta \models^r H$, but, by the persistence of the forcing relation, we can only conclude that the wffs forced in $\Gamma$ are forced in $\Delta$ too.

Now we can give the soundness theorem for the calculus $\text{Bd}_2\cdot T$.

**THEOREM 3.8 (Soundness of $\text{Bd}_2\cdot T$)**

If a $\text{Bd}_2$-proof table starting from the configuration $\{FA\}$ is closed, then $A$ is valid in every $\text{Bd}_2$-model.

**PROOF.** Assume that the theorem is not true, then there exist a $\text{Bd}_2$-model $K = \langle P, \leq, \models \rangle$ and a world $\Delta \in P$ such that $\Delta \models \neg A$. Then $\Gamma \models FA$. By assumption there is a closed $\text{Bd}_2$-proof table starting from the configuration $\{FA\}$; hence, by Proposition 3.7, its final configuration is $\text{Bd}_2$-realizable. But this means that a $\text{Bd}_2$-contradictory set is $\text{Bd}_2$-realizable, contradicting Proposition 3.2.

Now we start to study the completeness of $\text{Bd}_2\cdot T$. We have to prove that every wff in the logic $\text{Bd}_2$ has a closed $\text{Bd}_2$-proof table. The statement of the Completeness Theorem is: if a wff $A$ is valid in every $\text{Bd}_2$-model, then there is a closed $\text{Bd}_2$-proof table starting from the configuration $\{FA\}$. What is usually proved is the contrapositive: if there does not exist closed $\text{Bd}_2$-proof tables starting from the configuration $\{FA\}$ then there exists a $\text{Bd}_2$-model $K$ that $\text{Bd}_2$-realizes $FA$ (hence a world of $K$ does not force $A$). The proof of the Completeness Theorem consists of showing that there exists a procedure allowing one to build a $\text{Bd}_2$-model $K$ that $\text{Bd}_2$-realizes $FA$ for every swff $FA$ such that all the $\text{Bd}_2$-proof tables starting from the configuration $\{FA\}$ are not closed. Thus the proof emphasizes that the set of all nonclosed $\text{Bd}_2$-proof tables starting from $\{FA\}$ has enough information to build a $\text{Bd}_2$-model that realizes $FA$.

Given a $\text{Bd}_2$-consistent set $S$ of swffs, the construction of the model $\widehat{K}_\text{Bd}_2(S)$ follows the technique used in [1], which is an adaptation of Fitting’s technique [14]. In the first stage we construct the set of swffs $S^*$, called the $\text{Bd}_2$-saturated set of $S$, and the set of swffs $\widehat{S}$, called the $\text{Bd}_2$-node set of $S$. The reader who knows Fitting’s technique may consider $S^*$ a kind of Hintikka set (whose definition is adapted to the object language at hand). The set $\widehat{S}$ contains the swffs of $S^*$ that are not considered in the saturation process described below; moreover, $\widehat{S}$ will be the root of the model $\widehat{K}_\text{Bd}_2(S)$, and the signed atoms in $\widehat{S}$ will determine the forcing
relation in $\overline{S}$. In the second stage we construct the $\text{Bd}_2$-successor sets of $\overline{S}$. Again, the reader accustomed to Fitting’s technique may consider this step analogous to the one that builds the associated sets in [14] (p. 34, lines 10–13). The model $K_{\text{Bd}_2}(S)$ will be constructed by applying the first step to the $\text{Bd}_2$-successor sets of $\overline{S}$.

Let $A_1, \ldots, A_n$ be any listing of the swffs of $S$ (without repetitions of swffs). Starting from this enumeration we construct the following sequence $\{S_i\}_{i \in \omega}$ of sets of swffs.

- $S_0 = S$;
- let $S_i = \{H_1, \ldots, H_n\}$; then
  \[ S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i), \]

where, setting $S'_j = \mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup H_j, \ldots, H_n$ we have

1. if $H_j$ is a $\text{Bd}_2$-basic swff, then $\mathcal{U}(H_j, i)$ is any $\text{Bd}_2$-extension $\mathcal{R}_{H_j}$ of $H_j$ such that $(S'_j \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $\text{Bd}_2$-consistent;
2. if $H_j \equiv T(A \to B)$, with $A$ an atom, and $TA \in S'_j$ then $\mathcal{U}(H_j, i) = \{TB\}$;
3. if $H_j \equiv T(\neg A \to B)$ and $(S'_j \setminus \{H_j\}) \cup \{TB\}$ is $\text{Bd}_2$-consistent, then $\mathcal{U}(H_j, i) = \{TB\}$, otherwise $\mathcal{U}(H_j, i) = H_j$;
4. if the rule $T \to \to$ is applicable to $H_j$ and $(S'_j \setminus \{H_j\}) \cup \{TC\}$ is $\text{Bd}_2$-consistent then $\mathcal{U}(H_j, i) = \{TC\}$, otherwise if $(S'_j \setminus \{H_j\}) \cup \{TA, T(B \to C)\}$ is $\text{Bd}_2$-consistent then $\mathcal{U}(H_j, i) = \{TA, T(B \to C)\}$. If $(S'_j \setminus \{H_j\}) \cup \{TC\}$ is not $\text{Bd}_2$-consistent and $(S'_j \setminus \{H_j\}) \cup \{TA, T(B \to C)\}$ is not $\text{Bd}_2$-consistent then $\mathcal{U}(H_j, i) = \{H_j\}$;
5. if the rule $T_{\text{et}} \to$ is applicable to $H_j$ and $(S'_j \setminus \{H_j\}) \cup \{F_{\text{e}A}\}$ is $\text{Bd}_2$-consistent then $\mathcal{U}(H_j, i) = \{F_{\text{e}A}\}$ otherwise $\mathcal{U}(H_j, i) = \{H_j\}$;
6. if the rule $F \to$ is applicable to $H_j$ and $(S'_j \setminus \{H_j\}) \cup \{TA, FB\}$ is $\text{Bd}_2$-consistent then $\mathcal{U}(H_j, i) = \{TA, FB\}$ otherwise $\mathcal{U}(H_j, i) = \{H_j\}$;
7. if one among the rules $F \neg, T_{\text{et}} \lor, F_{\text{e}A}$ is applicable to $H_j$ then $\mathcal{U}(H_j, i) = \{H_j\}$.

Now, by induction on $i \geq 0$, it is easy to prove that if $S$ is $\text{Bd}_2$-consistent, then any $S_i$ is $\text{Bd}_2$-consistent. This implies that also the set

\[ S^* = \bigcup_{i \geq 0} S_i \]

is $\text{Bd}_2$-consistent. Following [1] we call $S^*$ a $\text{Bd}_2$-saturated set of $S$.

A swff $H \in S$ is $\text{Bd}_2$-final in $S$ if one of the following conditions holds:

- $H$ is a signed atom;
- $H$ is one among the swffs of the kind $F(\neg A)$, $T_{\text{et}}(A \lor B)$ or $F_{\text{e}}(A \land B)$;
- $H$ is of the kind $T(A \to B)$, with $A$ an atom or negated, and $TB \notin S$;
- $H$ is of the kind $T((A \to B) \to C)$ and $TC \notin S$ and $\{TA, T(B \to C)\} \notin S$;
- $H$ is of the kind $T_{\text{et}}(A \to B)$ and $F_{\text{e}}A \notin S$;
- $H$ is of the kind $F(A \to B)$ and $\{TA, FB\} \notin S$. 

Let $\overline{S}$, the $\mathsf{Bd}_2$-node set of $S$, be the set of swffs that are $\mathsf{Bd}_2$-final in $S^*$. The set $\overline{S}$ is a subset of $S^*$, hence $\overline{S}$ is $\mathsf{Bd}_2$-consistent. To build the $\mathsf{Bd}_2$-successor set $U$ of $\overline{S}$ we take into account two main cases:

(S1) if $\overline{S}$ contains a swff of the kind $\mathbf{F}(\neg A)$, $\mathbf{T}(\neg A \rightarrow B)$, $\mathbf{T}((A \rightarrow B) \rightarrow C)$ or $\mathbf{F}(A \rightarrow B)$, then for every such a swff $H \in \overline{S}$, we define the $\mathsf{Bd}_2$-successor sets of $\overline{S}$ as follows:
(a) if $H \in \overline{S}$ is of the kind $\mathbf{F}(\neg A)$ then $U = (\overline{S} \setminus \{H\}) \cup \{\mathbf{T}_d A\}$ is a $\mathsf{Bd}_2$-successor set of $\overline{S}$;
(b) if $H \in \overline{S}$ is of the kind $\mathbf{T}(\neg A \rightarrow B)$ then $U = (\overline{S} \setminus \{H\}) \cup \{\mathbf{T}_d A\}$ is a $\mathsf{Bd}_2$-successor set of $\overline{S}$;
(c) if $H \in \overline{S}$ is of the kind $\mathbf{T}((A \rightarrow B) \rightarrow C)$ then $U = (\overline{S} \setminus \{H\}) \cup \{\mathbf{T}_d A, \mathbf{F}_e B\}$ is a $\mathsf{Bd}_2$-successor set of $\overline{S}$;
(d) if $H \in \overline{S}$ is of the kind $\mathbf{F}(A \rightarrow B)$ then $U = (\overline{S} \setminus \{H\}) \cup \{\mathbf{T}_d A, \mathbf{F}_e B\}$ is a $\mathsf{Bd}_2$-successor set of $\overline{S}$.

(S2) Otherwise, if $H \in \overline{S}$ is a swff either of the kind $\mathbf{T}_d A(A \lor B)$ or $\mathbf{F}_e (A \land B)$ or $\mathbf{T}_d A(A \rightarrow B)$, then we define the $\mathsf{Bd}_2$-successor set $U$ of $\overline{S}$ as follows:
(a) if $H \in \overline{S}$ is a swff either of the kind $\mathbf{T}_d A(A \lor B)$ or $\mathbf{F}_e (A \land B)$ and $R_{H_j}$ is an extension of $H$ such that $(\overline{S} \setminus \{H\}) \cup R_{H_j}$ is $\mathsf{Bd}_2$-consistent, then $U = (\overline{S} \setminus \{H\}) \cup R_{H_j}$ is the $\mathsf{Bd}_2$-successor set of $\overline{S}$;
(b) if $H \in \overline{S}$ is of the kind $\mathbf{T}_d A(A \rightarrow B)$ then $U = (\overline{S} \setminus \{H\}) \cup \{\mathbf{T}_d B\}$ is the $\mathsf{Bd}_2$-successor set of $\overline{S}$.

Given $U$, a $\mathsf{Bd}_2$-successor of $\overline{S}$, we build the following sequence:

- $U_0 = U$;
- let $U_i = H_1, \ldots, H_n$; we define
  \[ U_{i+1} = \bigcup_{H_j \in U_i} \mathcal{U}(H_j, i), \]

where, setting $U'_j = \mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup H_j, \ldots, H_n, \mathcal{U}(H_j, i)$ is one of the extensions $R_{H_j}$ of $H_j$ such that $(U'_j \setminus \{H_j\}) \cup R_{H_j}$ is consistent.

By induction on $i \geq 0$, it is easy to prove that if $U$ is $\mathsf{Bd}_2$-consistent, then any $U_i$ is $\mathsf{Bd}_2$-consistent. It also follows that the set
\[ U^* = \bigcup_{i \geq 0} U_i \]
is $\mathsf{Bd}_2$-consistent. We define $\mathcal{U}$, the $\mathsf{Bd}_2$-node set of $U$, as the following set of swffs:
\[ \mathcal{U} = \{ H \in S^* | H \text{ is a signed atom } \}. \]

Starting from a $\mathsf{Bd}_2$-consistent set $S$ of swffs, $K_{\mathsf{Bd}_2}(S) = \langle P, \leq, \ll \rangle$ is defined as follows.

1. The root of $K_{\mathsf{Bd}_2}(S)$ is a node set $\overline{S}$ of $S$.
2. For every $\mathsf{Bd}_2$-successor set $U$ of $\overline{S}$ let $\overline{U}$ be a $\mathsf{Bd}_2$-node set of $U$. Then $\overline{U}$ is member of $P$ and $\overline{U}$ is an immediate successor of $\overline{S}$ in $K_{\mathsf{Bd}_2}(S)$.
3. $\leq$ is the transitive and reflexive closure of the relation immediate successor.
4. For every atom \( p \), \( S \models p \) iff \( T_p \in S \) and, for every \( \Gamma \in P \), \( \Gamma \not\models S \), \( \Gamma \models p \) iff \( T_{\text{cl}p} \in \Gamma \).

By the construction of \( K_{\text{bd}_2}(S) \) it follows that (i) for every \( Tp \in S \), \( T_{\text{cl}p} \) is in every \( \text{bd}_2\)-successor set of \( S \), thus \( K_{\text{bd}_2}(S) \) is a Kripke model, (ii) \( K_{\text{bd}_2}(S) \) has depth two at most; hence by (i) and (ii) \( K_{\text{bd}_2}(S) \) is a \( \text{bd}_2\)-model. We highlight that in every \( \text{bd}_2\)-successor set \( U \) of \( S \), there are \( \text{wffs} \) signed only with \( T_{\text{cl}} \) and \( F_c \); the rules of \( \text{bd}_2 \cdot T \) change \( T_{\text{cl}} \) and \( F_c \) \( \text{wffs} \) into \( T_{\text{cl}} \) and \( F_c \) \( \text{wffs} \), thus, in the construction of \( K_{\text{bd}_2}(S) \), the \( \text{wffs} \) signed with \( T \) and \( F \) may appear only in \( S^* \), the saturated related to \( S \).

**Lemma 3.9**

Let \( S \) be a \( \text{bd}_2\)-consistent set of \( \text{wffs} \) and let \( K_{\text{bd}_2}(S) = \langle P, \leq, \models \rangle \) be defined as above. For every \( \Gamma \in P \) and for every \( H \in \Gamma^*, \Gamma \Vdash H \) in \( K_{\text{bd}_2}(S) \).

**Proof.** The proof is by induction on the complexity of \( H \) measured with respect to the function \( \text{deg}_{\text{na}_2} \).

**Basis:** If \( \text{deg}_{\text{na}_2}(H) = 0 \), then \( H \equiv S p \), with \( p \) atom, and \( S p \in \Gamma \). If \( S \equiv T \) then \( \Gamma = S \) and \( S \Vdash Tp \) by definition of forcing; if \( S \equiv F \) then \( \Gamma = S \) and by \( \text{bd}_2\)-consistency of \( \Gamma \), \( Tp \not\in \Gamma \), hence, by definition of forcing, \( \Gamma \models F_p \); if \( S \equiv T_c \) then for every \( \Delta \supseteq \Gamma \), \( F_c \in \Delta \), and by \( \text{bd}_2\)-consistency of \( \Delta \), \( Tp \not\in \Delta \) and \( T_{\text{cl}p} \not\in \Delta \) hence \( \Delta \not\models p \) and \( \Gamma \Vdash F_c p \); if \( S \equiv T_{\text{cl}} \) then, by construction, in every final world \( \Delta \supseteq \Gamma \), \( T_{\text{cl}p} \in \Delta \). By definition of forcing in the final worlds, \( \Delta \models p \), thus \( \Delta \Vdash \neg p \) and hence \( \Gamma \not\models T_{\text{cl}p} \).

**Step:** suppose the lemma holds for every \( H' \) such that \( \text{deg}_{\text{na}_2}(H') < \text{deg}_{\text{na}_2}(H) \). The proof proceeds by structural induction on \( H \). We give some illustrative examples.

- **Case** \( H \equiv T(\neg A \rightarrow B) \); if \( H \in \Gamma^* \) then by construction of \( K_{\text{bd}_2}(S) \), \( \Gamma \) is the root of the \( \text{bd}_2\)-model. We have two cases:
  1. \( H \) is \( \text{bd}_2\)-final in \( \Gamma^* \), thus \( H \in \Gamma \). By construction of \( K_{\text{bd}_2}(S) \) there exists a \( \text{bd}_2\)-successor set \( \Delta \) of \( \Gamma \) such that \( T_{\text{cl}A} \in \Delta \). By induction hypothesis \( \Delta \supseteq T_{\text{cl}A} \), hence \( \neg A \in \Delta \). In every \( \text{bd}_2\)-successor set \( \Lambda \not\supseteq \Delta \), \( T_{\text{cl}B}(\neg A \rightarrow B) \in \Lambda \), therefore, by construction of \( K_{\text{bd}_2}(S) \), \( F_c \in \Lambda \) or \( T_{\text{cl}B} \in \Lambda \) and, by induction hypothesis, \( \Lambda \not\models \neg A \) or \( \Lambda \Vdash B \). Thus, for every \( \Theta \in P \) with \( \Theta \supseteq \Gamma, \Theta \not\models \neg A \) or \( \Theta \Vdash B \);
  2. \( H \) is not \( \text{bd}_2\)-final in \( \Gamma^* \), thus \( TB \in \Gamma^* \) and by induction hypothesis \( \Gamma \Vdash TB \); hence \( \Gamma \not\Vdash T(\neg A \rightarrow B) \).

- **Case** \( H \equiv T(A \rightarrow B) \), with \( A \) an atom; if \( H \in \Gamma^* \) then, by construction of \( K_{\text{bd}_2}(S) \), \( \Gamma \) is the root of the \( \text{bd}_2\)-model. We have two cases:
  1. \( H \) is \( \text{bd}_2\)-final in \( \Gamma^* \), thus \( H \in \Gamma \). This implies \( TA \not\in \Gamma^* \), hence \( TA \not\in \Gamma \) and, by definition of forcing, \( \Gamma \not\models A \). Moreover, by construction of \( K_{\text{bd}_2}(S) \), \( T_{\text{cl}A} \not\in \Gamma \) and \( \text{bd}_2\)-successor set \( \Delta \) of \( \Gamma \) in every \( \text{bd}_2\)-successor set \( \Lambda \not\supseteq \Delta \). Thus \( F_c \in \Lambda \) or \( T_{\text{cl}B} \in \Delta \) and, by induction hypothesis, the final world \( \Delta \) realizes one of them. Therefore for every \( \Lambda \in P \), with \( \Lambda \not\supseteq \Gamma, \Lambda \not\models A \) or \( \Lambda \Vdash B \) and we get \( \Gamma \not\Vdash T(A \rightarrow B) \);
  2. \( H \) is not \( \text{bd}_2\)-final in \( \Gamma^* \), thus \( TB \in \Gamma^* \). By induction hypothesis \( \Gamma \Vdash TB \), hence \( \Gamma \not\Vdash T(A \rightarrow B) \).

- **Case** \( H \equiv T((A \rightarrow B) \rightarrow C) \); if \( H \in \Gamma^* \) then by construction of \( K_{\text{bd}_2}(S) \), \( \Gamma \) is the root of the \( \text{bd}_2\)-model. By construction we have three cases:
  1. \( TC \in \Gamma^* \), then by induction hypothesis \( \Gamma \not\Vdash C \), hence, for every world \( \Delta \supseteq \Gamma, \Delta \not\Vdash C \); therefore, \( \Gamma \not\Vdash (A \rightarrow B) \rightarrow C \);
2. \{TA, T(B \rightarrow C)\} \subseteq \Gamma^* then, by induction hypothesis, \(\Gamma \triangleright TA\) and \(\Gamma \triangleright T(B \rightarrow C)\).

Hence, by definition of \(\text{Bd}_2\)-realizability, \(\Gamma \models A\) and \(\Gamma \models B \rightarrow C\). If for a world \(\Delta \in P\) such that \(\Delta \models \Gamma\), \(\Delta \models B\) holds, then \(\Delta \models B \rightarrow C\) implies \(\Delta \models C\).

3. \(H\) is \(\text{Bd}_2\)-final in \(\Gamma^*\), then by construction there exists a \(\text{Bd}_2\)-successor set \(\Delta\) of \(\Gamma\) containing \(\text{T}_{\text{cl}}A\) and \(\text{F}_{\text{cl}}B\), while the other \(\text{Bd}_2\)-successor sets \(\Lambda\) of \(\Gamma\) contain \(\text{T}_{\text{cl}}((A \rightarrow B) \rightarrow C)\). Then, by induction hypothesis, the swffs \(\text{T}_{\text{cl}}A\) and \(\text{F}_{\text{cl}}B\) are \(\text{Bd}_2\)-realized in \(\Delta\). Moreover, \(\Delta\) is a final world, hence \(\Delta \models A\) and \(\Delta \not\models A \rightarrow B\).

\(\text{T}_{\text{cl}}((A \rightarrow B) \rightarrow C)\) belongs to the saturated set \(\Lambda^*\) related to the \(\text{Bd}_2\)-successor sets \(\Lambda\), then, by construction, either \(\text{T}_{\text{cl}}C\) or \(\text{F}_{\text{cl}}((A \rightarrow B)\rightarrow C)\) is a member of \(\Lambda^*\). By induction hypothesis either \(\Delta \models \neg C\) or \(\Delta \models (A \rightarrow B)\), hence either \(\Delta \models C\) or \(\Delta \models (A \rightarrow B)\), therefore \(\Delta \models A\) or \(\Delta \models B\) implies \(\Delta \models C\).

- Case \(H \equiv \text{F}(-A)\): if \(H \in \Gamma^*\) then \(\Gamma\) is the root of \(\text{K}_{\text{Bd}_2}(S)\) and \(H\) is \(\text{Bd}_2\)-final in \(\Gamma^*\).

  By construction there exists a \(\text{Bd}_2\)-successor set \(\Delta\) containing \(\text{T}_{\text{cl}}A\). Since \(\Delta \subseteq \Delta^*\), we get, by induction hypothesis, \(\Delta \triangleright \text{T}_{\text{cl}}A\), thus \(\Delta \not\models -A\) and then \(\Gamma \triangleright \text{F}(-A)\).

- Case \(H \equiv \text{T}_{\text{cl}}(A \land B)\): if \(H \in \Gamma^*\) then we have two cases:

  1. \(H \in \Gamma\), thus \(H\) is \(\text{Bd}_2\)-final in \(\Gamma^*\) and, by construction of \(\text{K}_{\text{Bd}_2}(S)\), \(\Gamma\) is the root of the \(\text{Bd}_2\)-model. For every \(\text{Bd}_2\)-successor set \(\Delta\) of \(\Gamma\), \(\text{T}_{\text{cl}}A \in \Delta\) or \(\text{T}_{\text{cl}}B \in \Delta\); since \(\Delta \subseteq \Delta^*\) and by induction hypothesis we get \(\Delta \models A\) or \(\Delta \models B\). Thus \(\Gamma \models A \land B\) and this implies \(\Gamma \triangleright H\).

  2. \(H \not\in \Gamma\), then \(\Gamma\) is a final world of \(\text{K}_{\text{Bd}_2}(S)\). By construction of \(\text{K}_{\text{Bd}_2}(S)\), \(\text{T}_{\text{cl}}A \in \Gamma^*\) or \(\text{T}_{\text{cl}}B \in \Gamma^*\), thus we get \(\Gamma \triangleright H\).

- \(H \equiv \text{T}_{\text{cl}}(A \rightarrow B)\): if \(H \in \Gamma^*\) then we have two cases:

  1. \(H \in \Gamma\), thus \(H\) is \(\text{Bd}_2\)-final in \(\Gamma^*\) and, by construction of \(\text{K}_{\text{Bd}_2}(S)\), \(\Gamma\) is the root of the model. For every \(\text{Bd}_2\)-successor set \(\Delta\) of \(\Gamma\), \(\text{F}_{\text{cl}}A \in \Delta^*\) or \(\text{T}_{\text{cl}}B \in \Delta^*\) and, by induction hypothesis, \(\Delta \not\models A\) or \(\Delta \not\models B\). Thus \(\Delta \models \neg A\) and \(\Delta \models \neg B\) and this implies \(\Gamma \triangleright H\).

  2. \(H \not\in \Gamma\), then if \(\Gamma = \text{S}\) by construction of \(\text{K}_{\text{Bd}_2}(S)\) we have \(\text{F}_{\text{cl}}A \in \Gamma^*\) and, by induction hypothesis, \(\Gamma \models \neg A\); this implies that in every final world \(\Delta \supseteq \Gamma\), \(\Delta \models A \rightarrow B\), thus \(\Gamma \triangleright H\) holds. If \(\Gamma \neq \text{S}\), then \(\Gamma\) is a final world of \(\text{K}_{\text{Bd}_2}(S)\) and by construction \(\text{F}_{\text{cl}}A \in \Gamma^*\) or \(\text{T}_{\text{cl}}B \in \Gamma^*\), and we get \(\Gamma \triangleright H\).

\section*{Theorem 3.10 (Completeness of \(\text{Bd}_2\)-T)}

If \(A\) is valid in every \(\text{Bd}_2\)-model, then there exists a closed \(\text{Bd}_2\)-proof table starting from the configuration \(\{\text{F}A\}\).

\section*{Proof}

Suppose the theorem is not true, then \(\{\text{F}A\}\) is a \(\text{Bd}_2\)-consistent set of swffs. By the Lemma 3.9 this implies that \(\text{F}A\) is \(\text{Bd}_2\)-realizable and we get a contradiction.

\section*{Remark 3.11}

Our construction of the \(\text{Bd}_2\)-successors sets is different from the one given in [1]. Indeed, our construction does not use the rules \(\text{T}_{\text{cl}}A \lor \text{F}_{\text{cl}}A\), and \(\text{T}_{\text{cl}}A \rightarrow \text{F}_{\text{cl}}A\) if at least one of the rules \(\text{F} \rightarrow \text{F} \lor \text{F} \rightarrow \neg \text{F} \lor \text{F} \rightarrow \neg \text{F}\) is applicable. If none of them is applicable, we choose a swff of the kind \(\text{T}_{\text{cl}}(A \land B)\), \(\text{F}_{\text{cl}}(A \lor B)\), and \(\text{T}_{\text{cl}}(A \rightarrow B)\) (if any) and we apply the related rule. This strategy is complete and it has the advantage that the number of noninvertible rules we apply is less than the number of noninvertible rules applied in [1], thus we have a lower degree of nondeterminism in the application of the rules. We emphasize that if \(S = \text{S}\) the rules \(\text{T}_{\text{cl}}A \lor \text{F}_{\text{cl}}A\), and \(\text{T}_{\text{cl}}A \rightarrow \text{F}_{\text{cl}}A\) are invertible, thus if they are applied to a set of swff \(S\) such that \(S = \text{S}\), we can avoid the backtracking mechanism related to the order in which the swffs
of the kind \( \text{T}_{cl}(A \lor B) \), \( \text{F}_c(A \land B) \), and \( \text{T}_{cl}(A \rightarrow B) \) are treated. The construction related to the Completeness Theorem suggests a strategy to search for a closed \( \text{Bd}_2 \)-proof table for a set of swffs \( \Gamma \):

(a) The procedure picks a swff \( H \in \Gamma \) which is \( \text{Bd}_2 \)-basic or of the kind \( \text{T}(A \rightarrow B) \) with \( A \) an atom, if any, and apply to \( \Gamma \) the rule related to \( H \) (to apply the rule \( \text{T} \rightarrow \text{Atom} \) the procedure checks that \( \text{T}A \in \Gamma \)). The \( \text{Bd}_2 \)-basic rules are invertible, thus the procedure does not need a backtracking mechanism. Hence, if it is not possible to find a closed \( \text{Bd}_2 \)-proof table starting from \( \Gamma \) by applying the \( \text{Bd}_2 \)-basic rule related to \( H \), then a closed \( \text{Bd}_2 \)-proof table for \( \Gamma \) does not exist.

(b) If step (a) cannot be performed then, for every swff \( H \in \Gamma \) of the kind \( \text{F}(A \rightarrow B) \), \( \text{F}(\neg A) \), \( \text{T}(\neg A \rightarrow B) \) and \( \text{T}((A \rightarrow B) \rightarrow B) \), the procedure applies to \( \Gamma \) the rule related to \( H \). These rules are not invertible, thus the procedure needs a backtracking mechanism. Hence, for each \( H \in \Gamma \) of the kind \( \text{F}(A \rightarrow B) \), \( \text{F}(\neg A) \), \( \text{T}(\neg A \rightarrow B) \) and \( \text{T}((A \rightarrow B) \rightarrow B) \) the procedure searches a closed \( \text{Bd}_2 \)-proof table starting from \( \Gamma \), where \( H \) is the first treated swff.

(c) If steps (a) and (b) cannot be performed, then the procedure picks a swff \( H \in \Gamma \) of the kind \( \text{T}_{cl}(A \lor B) \), \( \text{F}_c(A \land B) \), and \( \text{T}_{cl}(A \rightarrow B) \) and applies to \( \Gamma \) the rule related to \( H \). These rules are invertible if \( \mathcal{S} = \mathcal{S}_c \); in this case the procedure does not need a backtracking mechanism. If \( \mathcal{S} \neq \mathcal{S}_c \) and the steps (a) and (b) cannot be performed, then Point (S3) in the construction of \( \overline{K}_\text{int} \mathcal{S} \) ensures that the procedure does not need a backtracking mechanism. Indeed in Point (S3) a swff \( H \in \mathcal{S} \) of the kind \( \text{T}_{cl}(A \lor B) \), \( \text{F}_c(A \land B) \), or \( \text{T}_{cl}(A \rightarrow B) \) is taken and the rule related to \( H \) is applied to \( \mathcal{S} \).

4 The logic \( \text{GS}_c \)

In this section we will treat the intermediate propositional logic

\[
\text{GS}_c = \text{Int} + \{ (p \lor (p \lor q \lor \neg q)), ((p \rightarrow q) \lor (q \rightarrow p)) \lor ((p \rightarrow \neg q) \land (\neg q \rightarrow p)) \},
\]

which is semantically characterized by the class of rooted Kripke models with depth at most two and with at most two final worlds.

Here we improve the calculus given [1] from the point of view of the depth of the deductions. We do not use the rules \( \text{T} \rightarrow \lor \) and \( \text{T} \rightarrow \rightarrow \) of [1] which do not allow one to linearly bound the depth of the deductions, but we treat the swff of the kind \( \text{T}(A \rightarrow B) \) with a single rule. This is possible because we enlarge the signed language of [1] by introducing the new signs \( \text{T}^{1,2}_{cl} \) and \( \text{F}_c^{1,2} \). Thus the signed language is built on the signs \( \text{T}, \text{F}, \text{T}^{1,2}_{cl}, \text{F}_c^{1,2}, \text{T}_{cl}, \text{F}_c, \text{T}^{1,2}, \text{F}_c^{1,2} \) and the propositional language is built on the connectives \( \lor, \land, \rightarrow \) and the propositional constant \( \bot \). The meaning of the signs is explained in terms of \( \text{GS}_c \)-realizability as follows: given a \( \text{GS}_c \)-model \( K = (\mathcal{P}_c, \leq, \models) \) and a swff \( H \), we say that \( \Gamma \models P \text{ GS}_c \)-realizes \( H \), and we write \( \Gamma \vdash H \), if one of the following conditions holds:

1. \( H \equiv \text{T}A \) and \( \Gamma \models A \);
2. \( H \equiv \text{F}A \) and \( \Gamma \not\models A \);
3. \( H \equiv \text{T}^{2,1}_{cl}A \) and there exists a final world \( \phi_j, \Gamma \leq \phi_j \), such that \( \phi_j \models A, j \in \{1, 2\} \);
4. \( H \equiv \text{F}^{1,2}_{cl}A \) and there exists a final world \( \phi_j, \Gamma \leq \phi_j \), such that \( \phi_j \models \neg A, j \in \{1, 2\} \);
5. \( H \equiv \text{T}^{1,2}_{cl}A \) and \( \Gamma \) is the root of \( K \) and \( \phi_1 \not\models A \) and \( \phi_2 \not\models A \), \( \phi_1 \) and \( \phi_2 \) being distinct final worlds of \( K \).
6. \( H \equiv F_{cl}^{1,2}A \) and \( \Gamma \) is the root of \( \mathcal{K} \) and \( \phi_1 \models \neg A \) and \( \phi_2 \models \neg A \), \( \phi_1 \) and \( \phi_2 \) being distinct final worlds of \( \mathcal{K} \).

We remark that in Points 3 and 4 of the above definition the two final worlds \( \phi_1 \) and \( \phi_2 \) may coincide.

A set of swffs \( S \) is \( \mathbf{GS}_c \)-\textit{contradictory} if one of the following conditions holds:

- \( TA \in S \) and \( FA \in S \);
- \( T \perp \in S \);
- \( T^{j}_A \in S \) and \( F^{j}_A \in S \) \((j \in \{1, 2\})\);
- \( TA \in S \) and \( F^{j}_A \in S \) \((j \in \{1, 2\})\);
- \( T^{1,2}_A \in S \) and \( F^{j}_A \in S \) \((j \in \{1, 2\})\);
- \( F^{1,2}_A \in S \) and \( T^{j}_A \in S \) \((j \in \{1, 2\})\);
- \( T^{1,2}_A \in S \) and \( F^{1,2}_A \in S \).

It is easy to prove the following proposition.

**Proposition 4.1**

If a set of swffs \( S \) is \( \mathbf{GS}_c \)-\textit{contradictory}, then \( S \) is not \( \mathbf{GS}_c \)-\textit{realizable}.

The tableau calculus \( \mathbf{GS}_c \cdot T \) is given in the Table 3. The complexity measure on wffs and swffs related to the calculus \( \mathbf{GS}_c \cdot T \) is the function \( de g_{gne} \).

**Definition 4.2**

1. The degree of \( A \), denoted by \( de g_{gne} (A) \), where \( A \) is a wff, is defined as follows: if \( A \equiv p \), where \( p \) is an atom, then \( de g_{gne} (p) = 0 \); if \( A \equiv \alpha \rightarrow \beta \), or \( A \equiv \alpha \land \beta \), or \( A \equiv \alpha \lor \beta \), then \( de g_{gne} (\alpha \rightarrow \beta) = de g_{gne} (\alpha \land \beta) = de g_{gne} (\alpha \lor \beta) = de g_{gne} (\alpha) + de g_{gne} (\beta) + 1 \);  
2. The degree of \( SA \) (where \( S \in \{ T, F, T^{1,2}_A, T_A, F_A, F^{1,2}_A, T^{1,2}_A \} \)), denoted by \( de g_{gne} (SA) \), coincides with the degree of \( A \);  
3. The degree \( de g_{gne} (S) \) of a set \( S \) which elements are wffs or swffs, coincides with the sum of the degrees of its elements.

The following proposition is the analogous of the Proposition 3.4.

**Proposition 4.3**

For every rule of the calculus \( \mathbf{GS}_c \cdot T \) with main set of swffs \( U \), for every \( \mathbf{GS}_c \)-extension \( R^i_U \) of \( U \) \((1 \leq i \leq n, n \geq 1)\), \( de g_{gne} (R^i_U) < de g_{gne} (U) \).

**Proof.** We prove the proposition for the rule \( T \rightarrow \), the other cases are similar. The main set of swffs of \( T \rightarrow \) is \( \{ T(A \rightarrow B) \} \) and the \( \mathbf{GS}_c \)-extensions of \( U \) are \( R^i_U = \{ F^{1,2}_A, T^{1,2}_A \}, R^2_U = \{ T^{1,2}_A, F^{1,2}_A \}, R^3_U = \{ T^{1,2}_A, F^{1,2}_A \}, R^4_U = \{ TB \} \) and \( R^5_U = \{ F^{1,2}_A \} \). Now, \( de g_{gne} (R^1_U) = de g_{gne} (R^2_U) = de g_{gne} (R^3_U) = de g_{gne} (R^4_U) = de g_{gne} (R^5_U) = de g_{gne} (A) + de g_{gne} (B) < de g_{gne} (U) = de g_{gne} (A) + de g_{gne} (B) + 1 \).

The following proposition is the analogous of the Proposition 3.5.

**Proposition 4.4**

Let \( S \) be a set of swffs and \( || S || \) the sum of the symbols of the swffs in \( S \); then \( de g_{gne} (S) \leq || S || \).
Theorem 4.5

The following proposition will be used to prove the Soundness Theorem.

Proposition 4.6

The rules of the calculus $G_{S_c}$ preserve $G_{S_c}$-realizability.

Proof. By definition of $Bd_2$-realizability we have to prove that, if a configuration is $G_{S_c}$-realized in a world of a rooted Kripke model $K = \langle P, \leq, \cdot \rangle$ with depth at most two and at most two final worlds, then the configuration obtained by applying to the former configuration one of the rules of $G_{S_c}$ preserves $G_{S_c}$-realizability in a (possibly different) world of $K$. We will consider only some significant rules.
Theorem 4.7 (Soundness of $\mathsf{规则 Rule Rule Rule Rule Rule Rule Rule Rule Rule Rule}$)

If a $\mathsf{规则 Rule Rule Rule Rule Rule Rule Rule Rule Rule Rule}$

Let $\phi_1$ and $\phi_2$ be final worlds of $K$, such that $\phi_1 \models A \lor B$ and $\phi_2 \models A \lor B$, hence either $\phi_1 \not\models A$ or $\phi_2 \not\models A$. We have the following cases:

1. $\phi_1 \not\models A$ and $\phi_2 \models B$, hence $\Gamma \models T_{cl}A$;
2. $\phi_1 \not\models A$ and $\phi_2 \models B$, hence $\Gamma \models T_{cl}B$;
3. $\phi_1 \models A$ and $\phi_2 \not\models A$, hence $\Gamma \models T_{cl}A$;
4. $\phi_1 \models A$ and $\phi_2 \not\models B$, hence $\Gamma \models T_{cl}B$.

Now we can give the Soundness Theorem for the calculus $\mathsf{GS_c-T}$ with respect to the Kripke semantics for the logic $\mathsf{GS_c}$.

Theorem 4.7 (Soundness of $\mathsf{GS_c-T}$)

If a $\mathsf{GS_c-proof table}$ starting from the configuration $\{F_A\}$ is closed, then $A$ is valid in every $\mathsf{Bdl_c-model}$.

Proof. Assume that the theorem is not true, then there exist a $\mathsf{GS_c-model K_c = \langle P, \subseteq, \models \rangle}$ and a world $\Gamma \in P$ such that $\Gamma \not\models A$. Then $\Gamma \models F_A$ and there is a closed $\mathsf{GS_c-proof table}$ starting from the configuration $\{F_A\}$; hence, by the previous proposition, its final configuration is $\mathsf{GS_c-realizable}$. But this means that a $\mathsf{GS_c-contradictory set}$ is $\mathsf{GS_c-realizable}$, contradicting Proposition 4.1.

To show the completeness of the calculus $\mathsf{GS_c-T}$, given a $\mathsf{GS_c-consistent}$ set $S$ of swffs, we build up a $\mathsf{GS_c-model K_{GS_c}(S)}$ which root $\mathsf{GS_c-realizes}$ the swffs in $S$.

Let $A_1, \ldots, A_n$ be any listing of swffs of $S$ (without repetitions of swffs). Starting from this enumeration we construct the following sequence $\{S_i\}_{i \in \omega}$ of sets of swffs:

$S_0 = S$;

- Rule $T \rightarrow$: let $S$ be a set of swffs containing $TA \rightarrow B$ and $\mathsf{GS_c-realized}$ in a world $\Gamma$ of a $\mathsf{GS_c-model K_c = \langle P, \subseteq, \models \rangle}$. By definition of $\mathsf{GS_c-realizability}$ $\Gamma \models TA \rightarrow B$, then for every world $\Delta \in P$ such that $\Delta \supseteq \Gamma$, either $\Delta \not\models A$ or $\Delta \models B$. We have the following cases:

  1. $\Gamma \models B$, thus $\Gamma \models TB$;
  2. $\Gamma \models \neg A$, thus $\Gamma \models F_{cl}^{1,2}A$;
  3. $\Gamma \not\models A$, $\phi_1 \models B$ and $\phi_2 \models B$, $\phi_1$ and $\phi_2$ being the final worlds of $K$, thus $\Gamma \models FA$ and $\Gamma \models T_{cl}B$;
  4. $\Gamma \not\models A$, either $\phi_1 \not\models A$ and $\phi_2 \models B$, or $\phi_1 \not\models B$ and $\phi_2 \not\models A$, $\phi_1$ and $\phi_2$ being the final worlds of $K$, thus either $\Gamma \models \{F_{cl}A, T_{cl}B\}$ or $\Gamma \models \{T_{cl}A, T_{cl}B\}$.

- Rule $T_{cl}^{1,2} \rightarrow$: let $S$ be a set of swffs containing $T_{cl}^{1,2}A \rightarrow B$ and $\mathsf{GS_c-realized}$ in a world $\Gamma$ of a $\mathsf{GS_c-model K_c = \langle P, \subseteq, \models \rangle}$. By definition of $\mathsf{GS_c-realizability}$ there means that there exist two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models A \lor B$ and $\phi_2 \models A \lor B$, thus either $\phi_1 \not\models A$ or $\phi_1 \not\models B$, and either $\phi_2 \not\models A$ or $\phi_2 \not\models B$. We have the following cases:

  1. $\phi_1 \not\models A$ and $\phi_2 \models B$, hence $\Gamma \models T_{cl}A$;
  2. $\phi_1 \not\models A$ and $\phi_2 \not\models B$, hence $\Gamma \models T_{cl}B$;
  3. $\phi_1 \models A$ and $\phi_2 \not\models A$, hence $\Gamma \models T_{cl}A$;
  4. $\phi_1 \models A$ and $\phi_2 \not\models B$, hence $\Gamma \models T_{cl}B$.

- Rule $T_{cl}^{1,2}$ \rightarrow: let $S$ be a set of swffs containing $T_{cl}^{1,2}A \rightarrow B$ and $\mathsf{GS_c-realized}$ in a world $\Gamma$ of a $\mathsf{GS_c-model K_c = \langle P, \subseteq, \models \rangle}$. By definition of $\mathsf{GS_c-realizability}$ there means that there exist two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models A \rightarrow B$ and $\phi_2 \models A \rightarrow B$, hence either $\phi_1 \not\models A$ or $\phi_1 \not\models B$, and either $\phi_2 \not\models A$ or $\phi_2 \not\models B$. We have the following cases:

  1. $\phi_1 \not\models A$ and $\phi_2 \models B$, hence $\Gamma \models T_{cl}A$;
  2. $\phi_1 \not\models A$ and $\phi_2 \not\models B$, hence $\Gamma \models F_{cl}A$ and $\Gamma \models T_{cl}B$;
  3. $\phi_1 \models A$ and $\phi_2 \not\models A$, hence $\Gamma \models T_{cl}A$;
  4. $\phi_1 \not\models A$ and $\phi_2 \not\models B$, hence $\Gamma \models T_{cl}B$.
let \( S_i = \{ H_1, \ldots, H_n \} \), then

\[
S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i),
\]

where \( \mathcal{U}(H_j, i) \) is a GS\(_c\)-extension \( R_{H_j} \) of \( H_j \) such that \( \mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup R_{H_j} \cup \{ H_j, \ldots, H_n \} \) is GS\(_c\)-consistent.

It is easy to prove, by induction on \( i \), that any \( S_i \) is GS\(_c\)-consistent; it follows that the set

\[
S^* = \bigcup_{i \geq 0} S_i
\]

is GS\(_c\)-consistent. Following [1] we call \( S^* \) a GS\(_c\)-saturated set of \( S \).

Given a set of swffs \( S \) and a swff \( H \), we say that \( H \) is GS\(_c\)-final in \( S \) if \( H \in S \) and \( H \) is a signed atom. We define:

\[
\begin{align*}
\mathcal{S} & = \{ H \mid H \text{ GS\(_c\)-final in } S^* \}; \\
S_0 & = \{ H \in \mathcal{S} \mid H \equiv T \lambda \}; \\
S_1 & = \{ H \in \mathcal{S} \mid H \equiv T \lambda A \text{ or } H \equiv T^{1,2}_d A \text{ or } H \equiv T^{1,2}_d A \}; \\
S_2 & = \{ H \in \mathcal{S} \mid H \equiv T \lambda A \text{ or } H \equiv T^{1,2}_d A \text{ or } H \equiv T^{1,2}_d A \}.
\end{align*}
\]

Given a GS\(_c\)-consistent set of swffs \( S \), we define a structure \( K_{\text{GS\(_c\)}}(S) = \langle P, \leq, \models \rangle \) as follows:

1. \( P = \{ S_0, S_1, S_2 \} \);
2. \( S_0 \leq S_1, S_0 \leq S_2 \) and \( S_i \leq S_i \), for \( i \in \{ 0, 1, 2 \} \);
3. for every atom \( p \), \( S_0 \models p \) iff \( Tp \in S_0 \). For \( j \in \{ 1, 2 \} \), and for every atom \( p \), \( S_j \models p \) iff \( Tp \in S_j \) or \( T^{1,2}_d p \in S_j \) or \( T^{1,2}_d p \in S_j \).

By 3 above it follows that if an atom is forced in \( S_0 \), then it is forced in \( S_1 \) and \( S_2 \), this means that the forcing is preserved; moreover the structure \( K_{\text{GS\(_c\)}}(S) \) defined above has depth two and two final worlds, namely \( S_1 \) and \( S_2 \), hence \( K_{\text{GS\(_c\)}}(S) \) is a GS\(_c\)-model. In the following we show that a GS\(_c\)-consistent set \( S \) of swffs is GS\(_c\)-realized by the root of \( K_{\text{GS\(_c\)}}(S) \). The proof proceeds by induction on the degree of the swffs measured with respect to the function \( deg_{\text{GS\(_c\)}} \).

**Lemma 4.8.**

Let \( S \) be a GS\(_c\)-consistent set of swffs and let \( K_{\text{GS\(_c\)}}(S) = \langle P, \leq, \models \rangle \) be defined as above. For every \( H \in S^* \), \( S_0 \models H \) in \( K_{\text{GS\(_c\)}}(S) \).

**Proof.** The proof is by induction on the complexity of swffs, measured with respect to the function \( deg_{\text{GS\(_c\)}} \).

**Basis:** If \( deg_{\text{GS\(_c\)}}(H) = 0 \), then \( H \equiv S_p \), with \( p \) an atom. If \( S \equiv T \) then \( Tp \in S_0 \) and \( S_0 \models Tp \) by definition of forcing; if \( S \equiv F \) then, by GS\(_c\)-consistency of \( S^* \), \( Tp \notin S_0 \), hence, \( S_0 \models Fp \); if \( S \equiv T^{1,2}_d \), with \( j \in \{ 1, 2 \} \), then, by construction, \( T^{1,2}_d p \in S_j \) and by definition of forcing \( S_j \models p \), thus \( S_0 \models T^{1,2}_d p \); if \( S \equiv F^{1,2}_d \), with \( j \in \{ 1, 2 \} \), then, by GS\(_c\)-consistency of \( S^* \), \( Tp \notin S^* \), \( T^{1,2}_d p \notin S^* \) and \( T^{1,2}_d p \notin S^* \), thus \( S_j \models p \) and \( S_0 \models F^{1,2}_d p \); if \( S \equiv T^{1,2}_d \), then, by construction \( T^{1,2}_d p \in S_1 \) and \( T^{1,2}_d p \in S_2 \) and by definition of forcing, \( S_1 \models p \) and \( S_2 \models p \).
thus $S_0 \vdash T_{cl}^{1,2} p$; if $S \equiv F_{cl}^{1,2}$ then, by $\text{GS}_c$-consistency of $S^*$, $T_p \not\in S^*$, $T_d^{j,p} \not\in S^*$, with $j \in \{1,2\}$, and $T_{cl}^{1,2} p \not\in S^*$, thus $S_1 \not\subseteq p$, $S_2 \not\subseteq p$ and $S_0 \vdash F_{cl}^{1,2} p$.

Step: suppose the lemma holds for every $H'$ such that $\deg_{\text{nes}}(H') < \deg_{\text{nes}}(H)$. We prove it holds for $H$. We give only a few cases:

- Case $H \equiv T(A \to B)$: if $H \in S^*$ then by construction of the saturated $S^*$ we have the following cases:

1. $TB \in S^*$, then, by induction hypothesis, $S_0 \vdash TB$, thus $S_0 \models B$ and $S_0 \models T(A \to B)$;
2. $F_{cl}^{1,2} A \in S^*$, then, by induction hypothesis, $S_0 \vdash F_{cl}^{1,2} A$, thus $S_0 \models \neg A$, and $S_0 \models T(A \to B)$;
3. $\{F_{cl}^{1,2} A, T_{cl}^{2} B\} \subseteq S^*$, then, by induction hypothesis, $S_0 \vdash F_{cl}^{1,2} A$ and $S_0 \vdash T_{cl}^{2} B$. Thus, there exist two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models \neg A$ and $\phi_2 \models B$. Moreover, since $\phi_1 \models \neg A$ we deduce that $S_0 \not\models A$, and hence $S_0 \models T(A \to B)$;
4. $\{T_{cl}^{1} B, F_{cl}^{2} A\} \subseteq S^*$, similar to the previous point;
5. $\{T_{cl}^{1,2} B, FA\} \subseteq S^*$, then by induction hypothesis $S_0 \vdash T_{cl}^{1,2} B$, and $S_0 \vdash FA$. Thus, $S_0 \not\models A$ and there exist two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models B$ and $\phi_2 \models B$. By definition of forcing of implication we get $S_0 \models T(A \to B)$.

- Case $H \equiv T_{cl}^{1,2} (A \to B)$: by construction of $S^*$, there are four cases:

1. $F_{cl}^{1,2} A \in S^*$, then, by induction hypothesis $S_0 \vdash F_{cl}^{1,2} A$, hence there are two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models \neg A$ and $\phi_2 \models \neg A$. Thus $\phi_1 \models A \to B$ and $\phi_2 \models A \to B$ and hence $S_0 \models T_{cl}^{1,2} (A \to B)$;
2. $T_{cl}^{1,2} B \in S^*$, then by induction hypothesis $S_0 \vdash T_{cl}^{1,2} B$, hence there are two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models B$ and $\phi_2 \models B$, therefore $S_0 \models T_{cl}^{1,2} (A \to B)$;
3. $\{F_{cl}^{1,2} A, T_{cl}^{2} B\} \subseteq S^*$, then by induction hypothesis $S_0 \vdash F_{cl}^{1,2} A$ and $S_0 \vdash T_{cl}^{2} B$. Thus there are two final worlds $\phi_1$ and $\phi_2$ such that $\phi_1 \models \neg A$ and $\phi_2 \models B$, hence $\phi_1 \models A \to B$ and $\phi_2 \models A \to B$;
4. $\{T_{cl}^{1} B, F_{cl}^{2} A\} \subseteq S^*$, similar to the previous point.

The proof of the Completeness Theorem for the calculus $\text{GS}_c-T$ uses the previous lemma and is analogous to Theorem 3.10.

**Theorem 4.9 (Completeness of $\text{GS}_c-T$)**

If $A$ is valid in every $\text{GS}_c$-model then there exists a closed $\text{GS}_c$-proof table starting from the configuration $\{FA\}$.

**Remark 4.10**

We emphasize that the rules of Table 3 are invertible. This means that searching for a closed $\text{GS}_c$-proof table it is possible to avoid the backtracking related to the order in which the wffs are treated. Therefore, since that order is not relevant, to decide if a wff $A$ is in $\text{GS}_c$ it is sufficient to build a single $\text{GS}_c$-proof table starting from the configuration $\{FA\}$ and ending in a configuration where every non-$\text{GS}_c$-contradictory set of wffs contains only signed atoms. We will call the $\text{GS}_c$-proof tables of this kind normalized. A normalized and nonclosed $\text{GS}_c$-proof table has enough information to build a $\text{GS}_c$-model whose root realizes $FA$.

**Remark 4.11**

We point out that by the invertible rules, it follows that if a world of a $\text{GS}_c$-model $\text{GS}_c$-realizes the conclusion of the rule then the same world $\text{GS}_c$-realizes the premiss. We could show the completeness of our calculus by the following theorem.
**Theorem 4.12**
A wff $A$ is $\text{GS}_e$-valid iff there exists a closed and normalized $\text{GS}_e$-proof table starting from the configuration $\{F,A\}$.

**Proof.** Let us suppose that $A$ is $\text{GS}_e$-valid and the statement does not hold, then every normalized $\text{GS}_e$-proof table we consider is not closed. Then in the last configuration there is a non-$\text{GS}_e$-contradictory set $S_j$ containing only signed atoms. $S_j$ is $\text{GS}_e$-realizable by the $\text{GS}_e$-model $\mathcal{K}_{\text{GS}_e}(S_j)$. By the fact that the rules are invertible we get that the first configuration $\{F,A\}$ is $\text{GS}_e$-realizable, which is absurd. The other part of the claim is Theorem 4.7. $lacksquare$

5 Jankov logic

In this section we consider Jankov Logic, the propositional intermediate logic characterized by the axiom schema of the weak law of the excluded middle [18], that is

$$J_n = \text{Int} + (\neg p \lor \neg \neg p).$$

This logic is semantically characterized by the class of rooted Kripke models with a single final world.

We consider the signed language with the connectives $\land, \lor, \rightarrow, \neg$ and the signs $T, F, \text{F}_e, T_d$. The meaning of these signs is explained in terms of $J_n$-realizability: given a $J_n$-model $K = (P, \preceq, \models)$ and a wff $H$, we say that $\Gamma \in P$ $J_n$-realizes $H$, and we write $\Gamma \vdash H$, if one of the following conditions holds:

1. $H \equiv T_e A$ and $\Gamma \models A$;
2. $H \equiv F_e A$ and $\Gamma \not\models A$;
3. $H \equiv F_e A$ and $\Gamma \models \neg A$.
4. $H \equiv T_e A$ and $\Gamma \not\models \neg \neg A$.

The rules of the calculus $J_n$-T are given in Tables 4–6. In order to get a calculus whose deductions have depth linearly bounded by the length of the wff to be proved, we have modified the rules $T \rightarrow \lor$ and $T \rightarrow \rightarrow$ of [1]. In both cases we have introduced a new atom in those elements of the conclusion in which a subformula of the main set of wffs appear twice.

A set $S$ of wffs is $J_n$-contradictory if one of the following conditions holds:

- $T_e A \in S$ and $F_e A \in S$;
- $T_e A \in S$ and $F_e A \in S$;
- $T_e A \in S$ and $F_e A \in S$.

It is easy to prove the following fact.

**Proposition 5.1**

If a set of wffs is $J_n$-contradictory then it is not $J_n$-realizable.

The function $\text{deg}_{J_n}$ we are going to define is a complexity measure on wffs and sets of wffs.

**Definition 5.2**

1. The degree of $A$, denoted by $\text{deg}_{J_n}(A)$, where $A$ is a wff, is defined as follows: if $A \equiv p$, where $p$ is an atom, then $\text{deg}_{J_n}(p) = 0$; if $A \equiv \alpha \rightarrow \beta$, then $\text{deg}_{J_n}(\alpha \rightarrow \beta) = \text{deg}_{J_n}(\alpha) + \text{deg}_{J_n}(\beta) + 1$; if $A \equiv \alpha \land \beta$, then $\text{deg}_{J_n}(\alpha \land \beta) = \text{deg}_{J_n}(\alpha) + \text{deg}_{J_n}(\beta) + 2$; if $A \equiv \alpha \lor \beta$, then $\text{deg}_{J_n}(\alpha \lor \beta) = \text{deg}_{J_n}(\alpha) + \text{deg}_{J_n}(\beta) + 3$; if $A \equiv \neg \alpha$, then $\text{deg}_{J_n}(\neg \alpha) = \text{deg}_{J_n}(\alpha) + 1$. 

Space-efficient Decision Procedures 973
2. The degree of \( SA \) (where \( S \in \{ T, F, F_e, T_{cl} \} \)), denoted by \( deg_{J_n}(SA) \) coincides with the degree of \( A \).

3. The degree \( deg_{J_n}(S) \) of a set \( S \) of swffs coincides with the sum of the degrees of its elements.

The following proposition is analogous to Proposition 3.5.

**Proposition 5.3**

Let \( H \) be a swff, then \( deg_{J_n}(H) \leq 3||H|| \).

In the following discussion we do not consider the rules in Table 6. Differently from the previous calculi, from Proposition 5.3 we cannot immediately get the linear bound on the depth of the deductions with \( J_n \). T. Indeed, it easy to check that the degree of one of the \( J_n \)-extensions of \( T((A \rightarrow B) \rightarrow C) \) is not lowered. Therefore, by applying the rule \( T \rightarrow \) to a set \( S \) of swffs, the degree of one of the elements in the obtained configuration is equal to the degree of \( S \). The following proposition shows that this may happen when \( deg_{J_n}((A \rightarrow B) \rightarrow C) \)
times, at most. This allows us to show that in every branch of every $J_n$-proof table, the
class number of times that the degree of a set of swffs is not lowered is linearly bounded by the
degree of the first configuration which is, by Proposition 5.3, linearly bounded by the length
of the first configuration.

**Proposition 5.4**
Let $S$ be a set of swffs; in every branch of every $J_n$-proof table starting from the configuration
$S$ (and in which the rules in Table 6 are not applied) the rule $T \rightarrow \rightarrow$ is applicable $deg_{J_n}(S)$
times, at most.
PROOF. The proof is by induction on \( \text{deg}_{s_m}(S) \).

**Basis:** \( \text{deg}_{s_m}(S) = 0 \), hence \( S = \{ SA \} \), with \( SA \) signed atom, and the proposition holds.

**Step:** suppose the proposition holds for every set of swffs \( U \) such that \( \text{deg}_{s_m}(U) < n \). We prove the proposition holds for \( \text{deg}_{s_m}(S) = n \). The only nontrivial case is when the rule \( T \rightarrow \) is applied to \( S = S' \cup \{ T(A \rightarrow B) \rightarrow C \} \). In this case we get a configuration with two sets of swffs, namely \( S' \cup \{ TA, \neg p, T(B \rightarrow p) \} \) and \( S' \cup \{ TC \} \). We study the former set, which is the only difficult case. Now, \( \text{deg}_{s_m}(S) = \text{deg}_{s_m}(S') + \text{deg}_{s_m}(A) + \text{deg}_{s_m}(B) + \text{deg}_{s_m}(C) + 2 = n \) and the number of times that the rule \( T \rightarrow \) is applied to \( S' \cup \{ TA, \neg p, T(B \rightarrow p) \} \) coincides with the number of times that it is applied to the set of swffs \( S' \cup \{ TA, \neg p, T(B \rightarrow p), p \rightarrow C \} \) and by induction hypothesis this number is bounded by \( \text{deg}_{s_m}(S') + \text{deg}_{s_m}(A) \), thus the complexity of a set of swffs, measured with respect to \( \text{deg}_{s_m} \), may be doubled at most.

Now, using the last two propositions we get that the depth of the deductions obtained by using the rules in Tables 4 and 5 is linearly bounded by the length of the wff to be proved.

A special-swff is a swff \( H \) such that \( \{ H \} \) is the main set of swffs of some special-rule (one of the rules in Table 6). A regular-rule is a rule which is not a special-rule. A regular-swff is a swff which is not a special-swff.

The special-rules given in Table 6 give rise to duplication, but by the Completeness Theorem it will be clear that the special-rules need to be applied only once in each branch of a \( J_n \)-proof table, thus the complexity of a set of swffs, measured with respect to \( \text{deg}_{s_m} \), may be doubled at most.

Let \( A \) be a swff, by the above discussion we get that every \( J_n \)-proof table starting from the configuration \( \{ FA \} \) has at most a linear number of configurations.

Now, to get the soundness of the calculus \( J_n \cdot T \), we show that \( J_n \cdot T \) preserve the \( J_n \)-realizability.

**Proposition 5.5**

The rules of the calculus \( J_n \cdot T \) preserve \( J_n \)-realizability.

**Proof.** By definition of \( J_n \)-realizability we have to prove that, if a configuration is \( J_n \)-realized in a world \( \Gamma \) of a Kripke model \( K = \{ P, \leq, \models \} \) built on a rooted set \( \{ P, \leq \} \) with at most a final element, then the configuration obtained by applying to the former configuration one of the rules of \( J_n \cdot T \) is \( J_n \)-realized in a possibly different world of \( K \). The proof requires one to analyse the rules of \( J_n \cdot T \). We consider some rules.

- **Rule** \( T \rightarrow \) : let \( S \) be a set of swffs containing the swff \( T(A \rightarrow B) \rightarrow C \) and let \( \Gamma \in P \) be a world of \( K \) such that \( \Gamma \models S \); by definition of \( \models \), \( \Gamma \models (A \rightarrow B) \rightarrow C \). It is possible to define a new \( J_n \)-model \( K' = \{ P, \leq, \models' \} \) such that the forcing relation between worlds of \( P \) and wffs is the following:
  1. for every atom \( q \in \text{Pr} \) and for every world \( \Lambda \in P \) we define \( \models' \) as follows: \( \Lambda \models' q \) iff \( \Lambda \models q \)
  2. if \( p \) is an atom such that \( p \notin \text{Pr} \), then, for every \( \Lambda \in P \) we define \( \models' \) as follows: \( \Lambda \models' p \) iff \( \Lambda \models B \)
  3. for every other atom \( q' \), such that \( q' \neq p \) and \( q' \notin \text{Pr} \), and for every world \( \Lambda \in P \) we define \( \models' \) as follows: \( \Lambda \models' q' \).
Now it is possible to define a new $\mathbf{J}_n$-realizability relation $\triangleright'$ with respect to $\models'$ as we did for $\triangleright$ with respect to $\models$.

By definition of $\models'$ it is easy to show that if $H$ is a wff such that $Prv(H) \subseteq Prv(S)$ and, if $\Gamma \not\models' H$ iff $\Gamma \not\models C$ or $\Gamma \not\models A \rightarrow B$. If $\Gamma \not\models' A \rightarrow B$ then there exists $\Delta \in P$, with $\Delta \supseteq \Gamma$, such that:

1. $\Delta \not\models' A, \Delta \not\models' B$ and, by definition of $\models'$, $\not\models' p$;
2. $\Delta \models' p \rightarrow C$, because if $\Lambda \in P$ is such that $\Lambda \supseteq \Delta$ and $\Lambda \not\models' p$ then, by definition of $\models'$, $\Lambda \models' B$ and by the fact $\Lambda \supseteq \Delta$ we get $\Lambda \models' A$ and $\Lambda \models' (A \rightarrow B) \rightarrow C$; therefore $\Delta \models' C$;
3. $\Delta \models' B \rightarrow p$, by definition of $\models'$.

* Rule $\mathbf{T} \rightarrow \triangleright$: let $T$ be a set of wffs containing the wff $T(\neg A \rightarrow B)$, let $\Gamma \in P$ be a world of $K$ such that $\Gamma \triangleright S$, then $\Gamma \models \neg A \rightarrow B$; this means $\Gamma \not\models \neg A$ or $\Gamma \models B$. If $\Gamma \models B$, then we get $\Gamma \triangleright T B$. If $\Gamma \not\models \neg A$ then there exists a world $\Delta \in P$, such that $\Delta \supseteq \Gamma$ and $\Delta \models A$. Let $\Phi \in P$ be the final world of $K$, then $\Phi \models A$. Hence $\Gamma \models \neg \neg A$ and we get $\Gamma \triangleright T \triangleright A$.

* Rule special $\rightarrow \rightarrow$: let $S$ be a set of wffs containing the wff $T((A \rightarrow B) \rightarrow C)$ and let $\Gamma \in P$ be a world of $K$ such that $\Gamma \triangleright S$, thus $\Gamma \models (A \rightarrow B) \rightarrow C$. This means that either $\Gamma \not\models A \rightarrow B$ and $\Gamma \models B \rightarrow C$ or $\Gamma \models C$:

1. if $\Gamma \models C$, then $\Gamma \models \neg \neg C$, moreover:
   (a) if $\Gamma \models A \rightarrow B$, then $\Gamma \models \neg \neg (A \rightarrow B)$;
   (b) if $\Gamma \not\models A \rightarrow B$, then there exists a world $\Delta \in P$, $\Delta \supseteq \Gamma$ such that $\Delta \models A$, this implies $\Gamma \models \neg \neg A$ and $\Delta \not\models B$, thus there are two possibilities:
      i. $\Gamma \models \neg \neg B$;
      ii. $\Gamma \models \neg B$;
2. if $\Gamma \not\models A \rightarrow B$ and $\Gamma \models B \rightarrow C$ then there exists a world $\Delta \in P$, $\Delta \supseteq \Gamma$ such that $\Delta \models A$, this implies $\Gamma \models \neg \neg A$, and $\Delta \not\models B$; thus there are two possibilities:
   (a) $\Gamma \models \neg \neg B$, this implies $\Gamma \models \neg \neg C$;
   (b) $\Gamma \models \neg B$, thus there are two possibilities:
      i. $\Gamma \models \neg C$;
      ii. $\Gamma \models \neg \neg C$.

By 1(a) we get $\Gamma \mathbf{J}_n$-realizes one among the sets labelled 1, 2, 3 in the conclusion of the rule; by 1(b)i we get $\Gamma \mathbf{J}_n$-realizes the set labelled 3; by 1(b)ii $\Gamma \mathbf{J}_n$-realizes the set labelled 4 in the conclusion of the rule; if 2(a) is the case then $\Gamma \mathbf{J}_n$-realizes the set labelled 3 in the conclusion of special $\rightarrow \rightarrow$; finally, if 2(b) holds then $\Gamma \mathbf{J}_n$-realizes one among the sets labelled 4, 5 in the conclusion of the rule.

**Theorem 5.6 (Soundness of $\mathbf{J}_n$-)**

If a $\mathbf{J}_n$-proof table starting from the configuration $\{FA\}$ is closed, then $A$ is valid in every $\mathbf{J}_n$-model.

Now we discuss the procedure that, given a $\mathbf{J}_n$-consistent set of wff $S$, allows one to build a $\mathbf{J}_n$-model $K_{\mathbf{J}_n}(S)$ whose root $\mathbf{J}_n$-realizes $S$.

Let $A_1, \ldots, A_n$ be any listing of wffs of $S$ (without repetitions of wffs). Starting from this enumeration we construct the following sequence $\{S_i\}_{i \in \omega}$ of sets of wffs:

- $S_0 = S$;
let $S_i = \{H_1, \ldots, H_n\}$, then

$$S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i),$$

where setting $S_j' = \mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_j, \ldots, H_n\}$. We have:

1. if $H_j$ is a regular-swoff, then $\mathcal{U}(H_j, i)$ is one of the $J_n$-extensions $\mathcal{R}_{H_j}$ of $H_j$ such that $(S_j' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $J_n$-consistent;

2. if $H_j \equiv T(A \rightarrow B)$ with $A$ an atom and $TA \in S_j'$ and there exists an atom $p \in PV(H_j)$ such that $p \in PV(S)$ and $p$ does not occur in a $F_c$ or $T_{cl}$ swff of $S_j'$, then $\mathcal{U}(H_j, i)$ is one of the $J_n$-extensions $\mathcal{R}_{H_j}$ of $H_j$ with respect to the rule \special-rule, such that $(S_j' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $J_n$-consistent; otherwise, if such $p \in PV(H_j)$ does not exist, $\mathcal{U}(H_j, i) = \{H_j\}$;

3. if $H_j \equiv T((A \rightarrow B) \rightarrow C)$ then, if $(S_j' \setminus \{H_j\}) \cup \{TC\}$ is $J_n$-consistent we get $\mathcal{U}(H_j, i) = \{TC\}$. Otherwise if there is an atom $p \in PV(H_j)$ such that $p \in PV(S)$ and $p$ does not occur in a $F_c$ or $T_{cl}$ swff of $S_j'$, then $\mathcal{U}(H_j, i)$ is one of the $J_n$-extensions $\mathcal{R}_{H_j}$ of $H_j$ with respect to the rule \special-rule, such that $(S_j' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $J_n$-consistent. If such a $p \in PV(H_j)$ does not exist, $\mathcal{U}(H_j, i) = \{H_j\}$;

4. if $H_j \equiv T(A \rightarrow B)$ and there is an atom $p \in PV(H_j)$ such that $p \in PV(S)$ and $p$ does not occur in a $F_c$ or $T_{cl}$ swff of $S_j'$, then $\mathcal{U}(H_j, i)$ is one of the $J_n$-extensions $\mathcal{R}_{H_j}$ of $H_j$ with respect to the \special-rule related to $H$ such that $(S_j' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $J_n$-consistent, otherwise, if such an atom $p \in PV(H_j)$ does not exist, $\mathcal{U}(H_j, i) = \{H_j\}$. It is easy to prove, by induction on $i$, that every $S_i$ is $J_n$-consistent, and this implies that

$$S^* = \bigcup_{i \geq 0} S_i$$

is $J_n$-consistent too. Now, a swff $H$ is $J_n$-final in a set $S$ of swffs if one of the following conditions holds:

- $H$ is a signed atom;
- $H \equiv T(A \rightarrow B)$ with $A$ an atom and $TB \not\in S$ and $F_cA \not\in S$;
- $H \equiv F(A \rightarrow B)$ and $\{T_{cl}A, F_cB\} \not\in S$;
- $H \equiv T((A \rightarrow B) \rightarrow C)$ and $TC \not\in S$ and $\{T_{cl}A, F_cB\} \not\in S$.

We call $J_n$-node set of $S$ related to $S^*$ the set $\overline{S}$ of swffs that are $J_n$-final $S^*$. Given a $J_n$-node set $\overline{S}$, we define the $J_n$-successor sets of $\overline{S}$ as follows:

- if $H \equiv F(A \rightarrow B) \in \overline{S}$ then $U = (\overline{S} \setminus \{H\}) \cup \{TA, FB\}$ is a $J_n$-successor set of $\overline{S}$;
- if $H \equiv T((A \rightarrow B) \rightarrow C) \in \overline{S}$ then $U = (\overline{S} \setminus \{H\}) \cup \{TA, Fp, TB \rightarrow p\}, T(p \rightarrow C)$ is a $J_n$-successor set of $\overline{S}$.

Given a finite $J_n$-consistent set of swffs $S$, the finite structure $K_{J_n}(S) = \langle P, \leq, \parallel \rangle$ is defined as follows:

- $\overline{S} \in P$, where $\overline{S}$ is any $J_n$-node set of $S$. 

• for every \( \Gamma \in P \) and for every \( J_n \)-successor set \( U \) of \( \Gamma \), \( U \in P \) and it is an immediate successor of \( \Gamma \) in \( K_{J_n}(S) \);

• the set \( F = \{ T_{cl}p | T_{cl}p \in S \} \cup \{ Tp | Tp \in S \} \cup \{ Tp | p \notin P \}, \) and there exists \( U \in P \) such that \( U \) has no immediate successors and \( Tp \in \overline{U} \cup \{ T_{cl}p | p \notin P \}, \) and there exists \( U \in P \) such that \( U \) has no immediate successors and \( T_{cl}p \notin \overline{U} \) is a world of \( P \);

• for every \( \Gamma \in P \) such that \( \Gamma \) has no immediate successors, \( F \) is an immediate successor of \( \Gamma \);

• \( \leq \) is the reflexive and transitive closure of the relation immediate successor;

• for every atom \( p \) and for every \( \Gamma \in P \), \( \Gamma \models p \iff \Gamma \neq \overline{F} \) and \( Tp \in \overline{F} \), or \( \Gamma \equiv F \) and \( T_{cl}p \in \overline{F} \) or \( Tp \in \overline{F} \).

By the construction of \( K_{J_n}(S) \) it follows that: (i) \( K_{J_n}(S) \) has a single final world, namely \( F \), (ii) the forcing relation is preserved, indeed if \( S \Delta A \in S \), with \( A \) an atom and \( S \in \{ T, T_{cl}, F_c \} \), then \( \Delta A \) is in every \( J_n \)-successor set of \( \Delta \). Moreover, by the special-rules, if \( T \Delta A \notin S \), with \( A \in P \), then either \( T_{cl}A \in S \) or \( F_c \notin S \), thus either \( T_{cl}A \in F \) or \( F_c \) is in \( ( \text{in particular, if } T \Delta A \notin S \) and \( T \Delta A \in S \) then \( T_{cl}A \in S \)). By (i) and (ii) we get \( K_{J_n}(S) \) is a \( J_n \)-model.

The definition of \( F \) takes into account that atoms may appear in the construction of \( K_{J_n}(S) \) that are not in \( S^* \). The \( J_n \)-consistency of \( F \) follows from the fact that \( S \) is \( J_n \)-consistent and the new signed atoms are all different among them. We emphasize that in the construction of a \( J_n \)-node set \( \Delta \) different from \( S \) the special-rules are not used and every new atom \( p \) is introduced in a swff of the kind \( Fp \) or \( T(A \rightarrow p) \) or \( T(p \rightarrow A) \); by these facts and by the structure of the conclusions of the rules in Table 5 it follows that subsequently to the construction of the root \( S \) of \( K_{J_n}(S) \), every new atom \( p \) cannot be signed with \( T_{cl} \) or \( F_c \).

**Lemma 5.7**

Let \( S \) be a \( J_n \)-consistent set of swffs and let \( K_{J_n}(S) = \langle P, \leq, \models \rangle \) be defined as above. For every \( \Gamma \in P \) and every \( H \in \Gamma \), \( \Gamma \models H \) in \( K_{J_n}(S) \).

**Proof.** The proof is by induction on the complexity of the swffs, measured with respect to the function \( \text{deg}_{J_n} \):

**Basis:** If \( \text{deg}_{J_n}(H) = 0 \) then \( H \equiv S \), with \( p \) an atom. If \( H \in \Gamma \) then \( H \) is \( J_n \)-final in \( \Gamma \), hence \( H \in \Gamma \). If \( H \equiv \overline{F} \) by definition of forcing \( \Gamma \models \overline{F} \); if \( H \equiv \overline{F}p \) then by \( J_n \)-consistency of \( \overline{F} \), \( Fp \notin \overline{F} \), hence, by definition of forcing, \( \Gamma \models Fp \); if \( H \equiv \overline{F}_{cl}p \) then, by construction, for every world \( \Delta \neq F \) such that \( \overline{\Delta} \leq \Gamma \), \( T_{cl}p \in \overline{\Delta} \) and (i) if \( p \in P \), then \( F_{cl}p \in \overline{\Delta} \) and \( F_{cl}p \in \overline{\Delta} \), hence, by \( J_n \)-consistency, \( Tp \notin \Delta \), \( Tp \notin \overline{F} \) and \( T_{cl}p \notin \overline{F} \), therefore \( F \) is not \( p \), thus \( \Gamma \models T_{cl}p \); (ii) if \( p \notin P \), then, as remarked above, a new atom appears signed with \( F_c \), only if it is introduced in the construction of the root \( S \) of \( K_{J_n}(S) \). Thus \( F_{cl}p \notin \overline{\Delta} \) and we can apply the argument used in the previous case. If \( H \equiv \overline{F}_{cl}p \) then, by construction, \( T_{cl}p \in \overline{\Delta} \) and \( T_{cl}p \in \overline{\Delta} \), thus we get \( \Gamma \models T_{cl}p \); (iii) if \( p \notin P \), then, for every \( \overline{\Delta} \leq \Gamma \), \( T_{cl}p \in \overline{\Delta} \) and thus \( \Gamma \models T_{cl}p \).

**Step:** Suppose the lemma holds for every \( H' \) such that \( \text{deg}_{J_n}(H') < \text{deg}_{J_n}(H) \). We prove it holds for \( H \). We give only a few cases.

1. If \( H \equiv \overline{F}(A \rightarrow B) \in \Gamma \) then we have two cases:
   (a) \( H \) is not \( J_n \)-final in \( \Gamma \); this implies that the swffs \( T_{cl}A \) and \( F_cB \) are in \( \Gamma \); by induction hypothesis they are \( J_n \)-realized. Thus \( F \vDash A \) and \( F \vDash B \), hence \( F \vDash A \rightarrow B \);
(b) $H$ is \( J_n \)-final in \( \Gamma^* \); by construction $H \in \Gamma$ and there exists a \( J_n \)-successor set $\Delta$ related to $H$ with the swffs $TA$ and $FB$ that are also in \( \Delta^* \); by induction hypothesis they are \( J_n \)-realized, hence $\Delta \vDash A$ and $\Delta \not\vDash B$; this implies $\Gamma \vDash H$.

2. If $H \equiv \mathbf{T}(A \rightarrow B) \in \Gamma^*$, with $A$ an atom, we show that for every world $\Sigma \in P$ such that $\Sigma \supseteq \Gamma$, either $\Sigma \vDash B$ or $\Sigma \not\vDash A$. We have two cases:
   (a) $H \not\vDash \Delta$, this means that $H$ is \( J_n \)-final in \( \Delta^* \), thus $\mathbf{T}B \not\vDash \Delta^*$ and, by construction of $K_n(S)$, $TA \not\vDash \Delta^*$; this implies $\Sigma \not\vDash A$;
   (b) $H \not\vDash \Delta$, this implies that there exists $\Theta \in P$ such that $\Gamma \subseteq \Theta \subseteq \Delta$, $H \in \Theta^*$ but $H \not\vDash \Theta$. Thus $\mathbf{T}B \in \Theta^*$ and, by induction hypothesis, $\Theta \vDash B$. Since $\Theta \subseteq \Delta$, it follows $\Sigma \vDash B$.

3. If $H \equiv \mathbf{T}(\langle A \rightarrow B \rangle) \rightarrow C) \in \Gamma^*$, we have to prove that for every world $\Sigma \in P$ such that $\Sigma \supseteq \Gamma$, $\Sigma \not\vDash A \rightarrow B$ or $\Sigma \vDash C$. We have two cases:
   (a) $H \in \Sigma$, then by construction there exists a \( J_n \)-successor set $\Theta$ of $\Sigma$ that contains the swffs $TA$, $\mathbf{F}p$, $\mathbf{T}(B \rightarrow p)$ and $\mathbf{T}(p \rightarrow C)$. By induction hypothesis they are \( J_n \)-realized, thus $\Theta \vDash A$, $\Theta \not\vDash p$ and $\Theta \vDash B \rightarrow p$, hence $\Sigma \not\vDash A \rightarrow B$;
   (b) if $H \not\vDash \Sigma$, there exist two worlds $\Theta_1$, $\Theta_2 \in P$ such that $\Gamma \subseteq \Theta_1 \subseteq \Theta_2 \subseteq \Delta$, $H \in \Theta_1$ and $H \not\vDash \Theta_2$, with $\Theta_2$, immediate successor of $\Theta_1$. We have two subcases:
      i. $\Theta_2$ is a \( J_n \)-successor set of $\Theta_1$ related to $H$, then $\{TA, \mathbf{F}p, \mathbf{T}(B \rightarrow p), \mathbf{T}(p \rightarrow C)\} \subseteq \Theta_2$; hence, by induction hypothesis $\Theta_2 \vDash A$, $\Theta_2 \vDash B \rightarrow p$ and $\Theta_2 \not\vDash p \rightarrow C$; this guarantees $\Sigma \not\vDash A \rightarrow B$ or $\Sigma \vDash C$;
      ii. $\mathbf{T}C \in \Theta_2$ or $\{\mathbf{T}cA, \mathbf{F}cB\} \subseteq \Theta_2$ and by induction hypothesis we get $\Sigma \vDash C$ or $\Sigma \not\vDash A \rightarrow B$.

By the previous lemma we get:

\textbf{Theorem 5.8 (Completeness of $J_n \cdot T$)}

If $A$ is valid in every $J_n$-model then there exists a closed $J_n$-proof table starting from the configuration $\{FA\}$.

We want to emphasize the role of the \textit{special-rules}. The \textit{special-rules} are related to the \( J_n \)-final swffs of the kind $\mathbf{F}(A \rightarrow B)$, $\mathbf{T}(\langle A \rightarrow B \rangle \rightarrow C)$ and $\mathbf{T}(p \rightarrow B)$, with $p$ an atom. These rules are used in the construction of $K_{\Gamma_n}(S)$ to guarantee that if a $J_n$-final swff $H$ belongs to $S$, then, for every atom $p \in P_{\text{v}}(H)$ occurring in $S$, $\mathbf{F}p \in S$ or $\mathbf{F}c p \in S$ hold. We are interested in the atoms of the $J_n$-final swffs because that atoms may not be considered in the construction of the $J_n$-saturated $S^*$, and they may become forced in a subsequent world of $S$. The $J_n$-models have a single final world, hence when we build the saturated set $\Delta^*$ related to the world $\Delta$ of $K_{\Gamma_n}(S)$, we must take into count which atoms are forced in every world of $K_{\Gamma_n}(S)$; indeed, given a $J_n$-model $K_{\Gamma_n} = \langle P, \Delta^*, \not\vDash \rangle$, if an atom $p$ is forced in $P \vDash p$, then $\neg p$ is forced in every world of $K_{\Gamma_n}$; thus, to build a well-defined $J_n$-model $K_{\Gamma_n}(S)$, we need to know if an atom that is not forced in the root $S$ becomes forced in a subsequent world. The atoms that may become forced in a subsequent world of the root are those occurring in the $J_n$-final swffs. Therefore, the purpose of the \textit{special-rules} is to fix, in the construction of $S^*$, which of the atoms occurring in $S$ are forced in the final world of $K_{\Gamma_n}(S)$; the information about the forcing of these atoms is taken to the $J_n$-successor sets of $S$ and it is essential to the construction of $K_{\Gamma_n}(S)$ to prove Lemma 5.7. Moreover, the construction guarantees that the sum of the degrees of the swffs to which the \textit{special-rules} are applied does not exceed the degree of $S$; this means the construction takes into account only the $J_n$-proof tables whose depth is linearly bounded by the degree of the first configuration.
In this section we will show that, starting from the calculus $BZ$, we can define a $O(n)$-SPACE decision procedure for $GS_c$. On the other hand, starting from the calculi $Bd_2$ and $J_n$, we can define $O(n \log n)$-SPACE decision procedures for the related logics.

The following are the main properties of our calculi that allow one to obtain procedures with such a complexity:

(i) deductions in the calculi $Bd_2$-$T$, $GS_c$-$T$ and $J_n$-$T$ have depth which is linearly bounded by the length of the wff to be proved;

(ii) for every rule of the above calculi, the number of elements in the conclusion does not depend on the length of the premiss of the rule;

(iii) for every rule of $Bd_2$-$T$, $GS_c$-$T$ and $J_n$-$T$, each element in the conclusion has a number of symbols which is bounded by the number of symbols in the premiss plus a constant.

Property (i) is discussed in the previous sections, while Properties (ii) and (iii) can be easily checked by inspecting the rules of the calculi. By Properties (i) and (ii) it follows that, given any set $S$ in any configuration of any $Bd_2$ or $GS_c$ or $J_n$-proof table, the number of wffs of $S$ is linearly bounded by the length of the wff to be proved.

Now we describe $GS_c$-$P$, the decision procedure for the logic $GS_c$. $GS_c$-$P$ is a depth-first left-to-right procedure that visits a single $GS_c$-proof table $T$ (or the corresponding $GS_c$-proof tree that we denote with $T$ too), the one obtained by the following strategy. Let $S$ be...
Table 7. The procedure $\text{GS}_e$-$P$

\begin{verbatim}
$\text{GS}_e$-$Test()$
1  READ(A)
2  $S \leftarrow \{F,A\}$
3  $\text{GS}_e$-$branch\ list \leftarrow \emptyset$
4  $\text{START} \leftarrow S$
5  $v \leftarrow \text{GS}_e$-$P()$

$\text{GS}_e$-$P()$
1  if $S$ is $\text{GS}_e$-contradictory
2    then return closed
3  if $S$ contains only signed atoms
4    then return open
5  $H \leftarrow$ the first swff of $S$ different from a signed atom
6  $j \leftarrow$ number of extensions $R_H$ of $H$
7  for $i \leftarrow 1$ to $j$
8     do
9       $\text{GS}_e$-$branch\ list \leftarrow \text{GS}_e$-$branch\ list \odot i$
10      $S \leftarrow (S \setminus \{H\}) \cup R_H^i$
11      $v \leftarrow \text{GS}_e$-$P()$
12  if $v =$ open
13    then
14       $\text{GS}_e$-$branch\ list \leftarrow \text{GS}_e$-$\text{REMOVE}\ LAST(\text{GS}_e$-$branch\ list)$
15       return open
16  $H \leftarrow \text{GS}_e$-$\text{REBUILD\ NODE}()$
17  $i \leftarrow$ the first swff of $S$ different from a signed atom
18  $j \leftarrow \text{GS}_e$-$\text{COPY}\ LAST(\text{GS}_e$-$branch\ list)$
19  $\text{GS}_e$-$branch\ list \leftarrow \text{GS}_e$-$\text{REMOVE}\ LAST(\text{GS}_e$-$branch\ list)$
21  return closed
\end{verbatim}

a noncontradictory set of a configuration in $\mathcal{T}$ containing a swff $H$ different from a signed atom, then $\text{GS}_e$-$P$ applies to $S$ the rule of $\text{GS}_e$-$T$ related to $H$; $\text{GS}_e$-$P$ proceeds recursively in this way for every set of swffs of the obtained configuration. As we discussed in Section 4 this procedure is complete because the rules are invertible. $\text{GS}_e$-$P$ implements this strategy, always choosing in the set $S$ the first swff different from a signed atom. To visit $\mathcal{T}$, $\text{GS}_e$-$P$ needs to know the last visited successor of each node of the branch it is visiting. This information can be stored in a list we call the $\text{GS}_e$-$branch\ list$. Such a list contains as many elements as the longest branch in $\mathcal{T}$, which is linearly bounded by the wff to be proved. Moreover, by property (ii), each element of $\text{GS}_e$-$branch\ list$ is an integer whose length does not depend on the length of the wff to be proved (indeed three bits are enough).
Table 8. The procedure $\text{Bd}_2\cdot P$

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>READ($A$)</td>
</tr>
<tr>
<td>2</td>
<td>$S \leftarrow {F,A}$</td>
</tr>
<tr>
<td>3</td>
<td>$\text{Bd}_2\cdot \text{branch list} \leftarrow \emptyset$</td>
</tr>
<tr>
<td>4</td>
<td>$\text{Bd}_2\cdot \text{generation list} \leftarrow \emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>$\text{START} \leftarrow S$</td>
</tr>
<tr>
<td>6</td>
<td>$v \leftarrow \text{Bd}_2\cdot P()$</td>
</tr>
</tbody>
</table>

$\text{Bd}_2\cdot P()$

1. if $S$ is $\text{Bd}_2\cdot \text{contradictory}$
   2. then return $\text{closed}$
3. if $S$ contains only signed atoms
   4. then return $\text{open}$
5. $\text{Bd}_2\cdot I()$
6. $\text{Bd}_2\cdot II()$
7. $\text{Bd}_2\cdot III()$
8. $\text{Bd}_2\cdot IV()$

implement the backtracking mechanism it is sufficient to know the first configuration and the path to reach a node $S$ from the root; hence $\text{GS}_c\cdot P$ does not need to keep the swffs of every node between the root and $S$ because this information may be rebuilt by inspecting the $\text{GS}_c\cdot \text{branch list}$. Finally, we point out that every symbol in $S$ may be coded by a constant number of bits with respect to the length of the wff to be proved, hence the work space needed to store the $\text{GS}_c\cdot \text{branch list}$ and the set of swffs representing the last visited node, is linearly bounded by the length of the wff to be proved. As a consequence of this discussion:

**Theorem 6.1**

The logic $\text{GS}_c$ is decidable in $O(n)\cdot \text{SPACE}$.

Now we describe $\text{Bd}_2\cdot P$, the decision procedure for the logic $\text{Bd}_2$. Since the rules $\text{Bd}_2\cdot c\text{-rules}$ are noninvertible, the order in which this rules are applied is relevant and this gives rise to a search space containing many $\text{Bd}_2\cdot \text{proof tables}$. Thus a backtracking mechanism is necessary to explore the search space of $\text{Bd}_2\cdot \text{proof tables}$ (or, in other words, a mechanism is necessary to generate all possible $\text{Bd}_2\cdot \text{proof tables}$ starting from $\{F,A\}$, with $A$ wff to be proved). Moreover, as in the case of $\text{GS}_c\cdot P$, a second backtracking mechanism is necessary to explore the branches of a given $\text{Bd}_2\cdot \text{proof table}$. $\text{Bd}_2\cdot P$ is a depth-first left-to-right procedure that visits a sequence of $\text{Bd}_2\cdot \text{proof tables}$ using the idea sketched in Remark 3.11. Every $\text{Bd}_2\cdot \text{proof table}$ (or $\text{Bd}_2\cdot \text{proof tree}$) $T$ is obtained by the following strategy. Let $S$ be a set of swffs. If $S$ is $\text{Bd}_2\cdot \text{contradictory}$ then $S$ is not $\text{Bd}_2\cdot \text{consistent}$ and $\text{Bd}_2\cdot P$ returns this information; if all swffs of $S$ are signed atoms, then $S$ is $\text{Bd}_2\cdot \text{consistent}$ and $\text{Bd}_2\cdot P$ returns this information:

(I) if $S$ contains a $\text{Bd}_2\cdot \text{basic}$ swff $H$, then $\text{Bd}_2\cdot T$ applies to $S$ the $\text{Bd}_2\cdot \text{basic}$ rule related to
TABLE 9. The subroutines $\text{Bd}_2$-I and $\text{Bd}_2$-II

\begin{verbatim}
$\text{Bd}_2$-I()
if S contains a $\text{Bd}_2$-basic swff
then
H ← the first $\text{Bd}_2$-basic swff in S
j ← position of H in S
$\text{Bd}_2$-generation list ← $\text{Bd}_2$-generation list ◦ j
j ← number of extensions $R_H$ of H
for i ← 1 to j
  do
    $\text{Bd}_2$-branch list ← $\text{Bd}_2$-branch list ◦ i
S ← (S \ {H}) \cup $R_H^i$
v ← $\text{Bd}_2$.P()
if v = open
  then
    $\text{Bd}_2$-generation list ←
    $\text{Bd}_2$-REMOVE_LAST($\text{Bd}_2$-generation list)
    $\text{Bd}_2$-branch list ←
    $\text{Bd}_2$-REMOVE_LAST($\text{Bd}_2$-branch list)
return open

S ← $\text{Bd}_2$.REBUILD_NODE()
H ← the first $\text{Bd}_2$-basic swff in S
i ← $\text{Bd}_2$.COPY_LAST($\text{Bd}_2$-branch list)
j ← number of extensions $R_H$ of H
$\text{Bd}_2$-branch list ← $\text{Bd}_2$.REMOVE_LAST($\text{Bd}_2$-branch list)
$\text{Bd}_2$-generation list ← $\text{Bd}_2$.REMOVE_LAST($\text{Bd}_2$-generation list)
return closed

$\text{Bd}_2$-II()
if S contains a couple of swffs of the kind T(A \rightarrow B) and T.A, with A an atom
then
H ← the first swff of the kind T.A \rightarrow B in S which related T.A is in S
j ← position of H in S
$\text{Bd}_2$-generation list ← $\text{Bd}_2$-generation list ◦ j
$\text{Bd}_2$-branch list ← $\text{Bd}_2$-branch list ◦ 1
S ← (S \ {H}) \cup \{TB\}
v ← $\text{Bd}_2$.P()
$\text{Bd}_2$-generation list ← $\text{Bd}_2$.REMOVE_LAST($\text{Bd}_2$-generation list)
$\text{Bd}_2$-branch list ← $\text{Bd}_2$.REMOVE_LAST($\text{Bd}_2$-branch list)
return v
\end{verbatim}
TABLE 10. The subroutine \textbf{Bd}_2-III

\begin{verbatim}
\textbf{Bd}_2-III():
1  if \( S \) contains a wff of the kind \( \text{F}(A \rightarrow B) \) or \( \text{F}(\neg A) \) or \( \text{T}(\neg A \rightarrow B) \) or \( \text{T}((A \rightarrow B) \rightarrow C) \) 
2     then 
3         for \( k \leftarrow 1 \) to \( \text{number of wffs in } S \) 
4             do 
5                 \( H \leftarrow \text{the } k \text{-th wff in } S \) 
6                 if \( H \) of the kind \( \text{F}A \rightarrow B \) or \( \text{F}\neg A \) or \( \text{T}\neg A \rightarrow B \) or \( \text{T}(A \rightarrow B) \rightarrow C \) 
7                     then 
8                         \( j \leftarrow \text{number of extensions } \mathcal{R}_H \text{ of } H \) 
9                             for \( i \leftarrow 1 \) to \( j \) 
10                               do 
11                                   \( \text{Bd}_2\)-generation list \( \leftarrow \text{Bd}_2\)-generation list \( \circ k \) 
12                                   \( \text{Bd}_2\)-branch list \( \leftarrow \text{Bd}_2\)-branch list \( \circ i \) 
13                                   \( S \leftarrow (S \setminus \{H\}) \cup \mathcal{R}_H \) 
14                                   \( v \leftarrow \text{Bd}_2\cdot P() \) 
15                                   if \( v = \text{open} \) 
16                                       then 
17                                           \( k \leftarrow \text{Bd}_2\cdot\text{COPY}\_\text{LAST}(\text{Bd}_2\cdot\text{generation list}) \) 
18                                           \( \text{Bd}_2\)-generation list \( \leftarrow \) 
19                                           \( \text{Bd}_2\)-branch list \( \leftarrow \) 
20                                           \( \text{Bd}_2\cdot\text{REBUILD\_NODE}() \) 
21                                           \( S \leftarrow \text{Bd}_2\)-REBUILD\_NODE() \) 
22                                           \( k \leftarrow \text{Bd}_2\cdot\text{COPY}\_\text{LAST}(\text{Bd}_2\cdot\text{generation list}) \) 
23                                           \( i \leftarrow \text{Bd}_2\cdot\text{COPY}\_\text{LAST}(\text{Bd}_2\cdot\text{branch list}) \) 
24                                           \( H \leftarrow \text{the } k \text{-th wff in } S \) 
25                                           \( j \leftarrow \text{number of extensions } \mathcal{R}_H \text{ of } H \) 
26                                           \( \text{Bd}_2\)-branch list \( \leftarrow \text{Bd}_2\)-\text{REMOVE}\_\text{LAST}(\text{Bd}_2\)-branch list) 
27                                           \( \text{Bd}_2\)-generation list\( \leftarrow \) 
28                                           \( \text{Bd}_2\)-\text{REMOVE}\_\text{LAST}(\text{Bd}_2\)-generation list) 
29                               return \textit{densed} 
30                           return \textit{open} 
\end{verbatim}
Table 11. The subroutine $\text{Bd}_2$-IV

<table>
<thead>
<tr>
<th>$\text{Bd}_2$-IV()</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 if $S$ contains a wff of the kind $T_{e}(A \lor B), F_{e}(A \land B), T_{e}(A \rightarrow B)$</td>
</tr>
<tr>
<td>2 then</td>
</tr>
<tr>
<td>3 $j \leftarrow$ position in $S$ of the first swff of the kind $T_{e}(A \lor B), F_{e}(A \land B), T_{e}(A \rightarrow B)$</td>
</tr>
<tr>
<td>4 $H \leftarrow$ the $j$-th swff in $S$</td>
</tr>
<tr>
<td>5 $\text{Bd}_2$-generation list $\leftarrow \text{Bd}_2$-generation list $\circ j$</td>
</tr>
<tr>
<td>6 $j \leftarrow$ number of extensions $\mathcal{R}_{H}$ of $H$</td>
</tr>
<tr>
<td>7 for $i \leftarrow 1$ to $j$</td>
</tr>
<tr>
<td>8 do</td>
</tr>
<tr>
<td>9 $\text{Bd}_2$-branch list $\leftarrow \text{Bd}_2$-branch list $\circ i$</td>
</tr>
<tr>
<td>10 $S \leftarrow (S \setminus {H}) \cup \mathcal{R}_{H}$</td>
</tr>
<tr>
<td>11 $v \leftarrow \text{Bd}_2$-P()</td>
</tr>
<tr>
<td>12 if $v = \text{open}$</td>
</tr>
<tr>
<td>13 then</td>
</tr>
<tr>
<td>14 $\text{Bd}_2$-branch list $\leftarrow \text{Bd}_2$-REMOVALAST($\text{Bd}_2$-branch list)</td>
</tr>
<tr>
<td>15 $\text{Bd}_2$-generation list $\leftarrow \text{Bd}_2$-REMOVALAST($\text{Bd}_2$-generation list)</td>
</tr>
<tr>
<td>16 return $\text{open}$</td>
</tr>
<tr>
<td>17 $S \leftarrow \text{Bd}_2$-REBUILDNODE()</td>
</tr>
<tr>
<td>18 $i \leftarrow \text{Bd}_2$-COPYLAST($\text{Bd}_2$-branch list)</td>
</tr>
<tr>
<td>19 $k \leftarrow \text{Bd}_2$-COPYLAST($\text{Bd}_2$-generation list)</td>
</tr>
<tr>
<td>20 $H \leftarrow$ the $k$-th swff in $S$</td>
</tr>
<tr>
<td>21 $j \leftarrow$ number of extensions $\mathcal{R}_{H}$ of $H$</td>
</tr>
<tr>
<td>22 $\text{Bd}_2$-branch list $\leftarrow \text{Bd}_2$-REMOVALAST($\text{Bd}_2$-branch list)</td>
</tr>
<tr>
<td>23 $\text{Bd}_2$-generation list $\leftarrow \text{Bd}_2$-REMOVALAST($\text{Bd}_2$-generation list)</td>
</tr>
<tr>
<td>24 return $\text{closed}$</td>
</tr>
</tbody>
</table>

$H$. $\text{Bd}_2$-P recursively analyse every set of swffs of the obtained configuration; $S$ is not $\text{Bd}_2$-consistent if every element of the obtained configuration is not $\text{Bd}_2$-consistent.

(II) if Step (I) cannot be applied to $S$, then if $S$ contains two swffs of the kind $T(A \rightarrow B)$ and $T_{A}$, with $A$ an atom, then $\text{Bd}_2$-P applies the rule $T \rightarrow \text{Atom}$ to $S$ and proceed recursively on the obtained set of swffs. $S$ is not $\text{Bd}_2$-consistent if the set of the obtained configuration is not $\text{Bd}_2$-consistent.

(III) if Steps (I) and (II) cannot be applied to $S$, then for every swff $H \in S$ of the kind $F(A \rightarrow B), F(\neg A), T(\neg A \rightarrow B)$ and $T((A \rightarrow B) \rightarrow C)$, $\text{Bd}_2$-P applies to $S$ the rule related to $H$. $\text{Bd}_2$-P proceeds recursively for every set of swffs of the obtained configuration; $S$ is not $\text{Bd}_2$-consistent if for some $H$ every set of the obtained configuration is not $\text{Bd}_2$-consistent.
(IV) if Steps (I) − (III) cannot be applied to $S$, then $\text{Bd}_2$-$P$ take out a swff $H \in S$ of the kind $T_{\alpha}(A \lor B)$, $F_{c}(A \land B)$, and $T_{c_1}(A \rightarrow B)$ and applies to $S$ the rule related to $H$. $\text{Bd}_2$-$P$ recursively analyses every set of swffs of the obtained configuration; $S$ is not $\text{Bd}_2$-consistent if every set of the obtained configuration is not $\text{Bd}_2$-consistent.

The following two lists of integers are used by $\text{Bd}_2$-$P$ to implement the two backtracking mechanisms: the first list, $\text{Bd}_2$-$branch list$, keeps the branching points in the branch of the $\text{Bd}_2$-proof tree that $\text{Bd}_2$-$P$ is visiting, the second list, $\text{Bd}_2$-$generation list$, takes into account the visited $\text{Bd}_2$-proof trees. This second list is related to Point (III) of the above strategy, in which $S$ is $\text{Bd}_2$-consistent if for all swffs of the prescribed kind it is not possible to get a closed $\text{Bd}_2$-proof table for $S$. Each choice of $H$ give rise to a different $\text{Bd}_2$-proof table for $S$ and the $\text{Bd}_2$-generation list takes into account which $\text{Bd}_2$-proof tables have already been visited by $\text{Bd}_2$-$P$. The two lists have an element for every node in the visited branch, thus they have a number of elements linearly bounded by the wff to be proved. Each element in the $\text{Bd}_2$-branch list needs a constant number of bits (two are enough). Each element in the $\text{Bd}_2$-generation list is a pointer to the wff treated last among those of the kind considered in (III); by property (iii) above, each element of such a list needs a number of bits which is logarithmic in the length of the wff to be proved. The two lists allow $\text{Bd}_2$-$P$ to rebuild the swffs in every node $S$. Hence to implement the backtracking mechanism, $\text{Bd}_2$-$P$ does not need to keep the swffs of every node between the root and $S$. In the set $S$ new atoms may appear but their number is linearly bounded by the length of the wff to be proved; thus every variable may be coded using a logarithmic number of bits. The number of bits to codify the other symbols of $S$ does not depend on the length of the wff to be proved. Summarizing,
TABLE 13. The subroutine $J_n$-III

\begin{verbatim}
J_n-III() 
1  if $S$ contains a special-swff with an atom that 
does not occur in some $T_{ct}$ or $F_c$ swff in $S$
2      then
3          $H \leftarrow$ the first special-swff with an atom that does
4              not occur in some $T_{ct}$ or $F_c$ swff in $S$
5          $j \leftarrow$ position of $H$ in $S$
6          $J_n$-generation list $\leftarrow J_n$-generation list $\circ j$
7          $j \leftarrow$ number of extensions $\mathcal{R}_H$ of $H$
8          for $i \leftarrow 1$ to $j$
9          do
10             $J_n$-branch list $\leftarrow J_n$-branch list $\circ i$
11             $S \leftarrow (S \setminus \{H\}) \cup \mathcal{R}^i_H$, with $\mathcal{R}^i_H$ taken with respect
12                 the special-rule related to $H$
13             $v \leftarrow J_n$-P() 
14             if $v = \text{open}$
15             then
16                $J_n$-generation list $\leftarrow$
17                $J_n$-REMOVE-LAST($J_n$-generation list)
18                $J_n$-branch list $\leftarrow J_n$-REMOVE-LAST($J_n$-branch list)
19                return open
20             $S \leftarrow J_n$-REBUILD-NODE()
21             $H \leftarrow$ the first $J_n$-basic swff in $S$
22             $i \leftarrow J_n$-COPY-LAST($J_n$-branch list)
23             $j \leftarrow$ number of extensions $\mathcal{R}_H$ of $H$
24             $J_n$-branch list $\leftarrow J_n$-REMOVE-LAST($J_n$-branch list)
25             $J_n$-generation list $\leftarrow J_n$-REMOVE-LAST($J_n$-generation list)
26             return closed
\end{verbatim}

the work space needed for the $Bd_2$-branch list, the $Bd_2$-generation list and the set of swffs
representing the visited node is bounded by $O(n \log n)$, with $n$ length of the wff to be proved.

The recursive procedure $GS_c$-P described above is sketched in Table 7. The variables $S$, $GS_c$-branch list, $START$, $i$, $j$ and $v$ are global. Using the list $GS_c$-branch list and the variable $START$, the subroutine $GS_c$-REBUILD-NODE can rebuild every node in the branch. The subroutine $GS_c$-REMOVE-LAST erase the last element of a list and the subroutine $GS_c$-COPY-LAST return the last element of a list. By $GS_c$-branch list $\leftarrow GS_c$-branch list $\circ i$ we mean that the integer $i$ is added to the end of the list. Since the depth of the $GS_c$-proof tables
starting from the configuration $\{FA\}$ is linearly bounded by the length of $A$, the recursive
depth of $GS_c$-P is linear, thus the work space for the stack of the recursive calls is linearly
bounded too. However, we emphasize that a recursion-free procedure equivalent to $GS_c$-P
\textbf{Table 14. The subroutine $J_n$-IV}

```
$J_n$-IV()
1 if $S$ contains a special-swff different from $T(A \rightarrow B)$ with $A$ an atom
2 then
3 for $k \leftarrow 1$ to number of swffs in $S$
4 do
5 $H \leftarrow$ the $k$-th swff in $S$
6 if $H$ is of the kind $F(A \rightarrow B)$ or $T((A \rightarrow B) \rightarrow C)$
7 then
8 $j \leftarrow$ number of extensions $\mathcal{R}_H$ of $H$ with respect the
9 related non special-rule
10 for $i \leftarrow 1$ to $j$
11 do
12 $J_n$-generation list $\leftarrow J_n$-generation list $\circ k$
13 $J_n$-branch list $\leftarrow J_n$-branch list $\circ i$
14 $S \leftarrow (S \setminus \{H\}) \cup \mathcal{R}_H$, with $\mathcal{R}_H$ taken with respect the
15 non special-rulerealated to $H$
16 $v \leftarrow J_n$-$P()$
17 if $v = \text{open}$
18 then
19 $k \leftarrow J_n$-$COPY$ \_\_\_LAST($J_n$-generation list)
20 $J_n$-generation list$\leftarrow$
21 $\leftarrow J_n$-REMOVE \_\_\_LAST($J_n$-generation list)
22 $J_n$-branch list$\leftarrow$
23 $\leftarrow J_n$-REMOVE \_\_\_LAST($J_n$-branch list)
24 GOTO 3
25 $S \leftarrow J_n$-REBUILD \_\_\_NODE()
26 $k \leftarrow J_n$-$COPY$ \_\_\_LAST($J_n$-generation list)
27 $i \leftarrow J_n$-$COPY$ \_\_\_LAST($J_n$-branch list)
28 $H \leftarrow$ the $k$-th swff in $S$
29 $j \leftarrow$ number of extensions $\mathcal{R}_H$ of $H$
30 $J_n$-branch list$\leftarrow$
31 $\leftarrow J_n$-REMOVE \_\_\_LAST($J_n$-branch list)
32 $J_n$-generation list$\leftarrow$
33 $\leftarrow J_n$-REMOVE \_\_\_LAST($J_n$-generation list)
34 return \_\_\_closed
35 return \_\_\_open
```
may be given using the GS_c-branch list to simulate the stack.

In Tables 8–11 the procedure Bd_d-P is sketched. All its variables are global and an equiva-

lent recursion-free procedure may be given using the two lists Bd_d-generation list and Bd_d-

branch list to simulate the stack.

The decision procedure J_n-P for J_n is analogous to the one given for Bd_d: since some rules of J_n-T are not invertible, J_n-P needs two backtracking mechanisms and its strategy is

similar to the one given for Bd_d. Indeed with such a procedure we try to lower the nonde-

terminism related to the application of noninvertible rules by applying the noninvertible rules

only when the invertible rules are not applicable.

Thus, J_n-P is a depth-first left-to-right procedure visiting a sequence of J_n-proof tables.

Every J_n-proof table (or J_n-proof tree) T is obtained by the following strategy. Let S be

a set of swffs; if S is J_n-contradictory then S is not J_n-consistent and J_n-P returns this

information; if all swffs of S are signed atoms, then S is J_n-consistent and Bd_d-P returns

this information:

(I) if S contains a J_n-regular swff H, then J_n-P applies to S the J_n-regular rule related to

H. J_n-P proceeds recursively for every set of swffs of the current configuration; S is not

J_n-consistent if every set of the current configuration is not J_n-consistent;

(II) if Step (I) cannot be applied to S, then if S contains two swffs of the kind T(A → B)

and T.A, with A an atom, then J_n-P applies the rule T → Atom to S and recursively

proceeds on the set of swffs obtained; S is not J_n-consistent if the set in the obtained

configuration is not J_n-consistent;

(III) if Steps (I)–(II) cannot be applied to S then if S contains a special-swff H such that some

atoms in it do not occur in some T_c or F_c swff H' ∈ S, then J_n-T applies to S the

special-rule related to H. J_n-P recursively applies to every set of swffs of the obtained

configuration; S is not J_n-consistent if every set of the obtained configuration is not

J_n-consistent;

(IV) if Steps (I)–(III) cannot be applied to S, then, for every special-swff H ∈ S different

from T(A → B) with A an atom, J_n-P applies to S the nonspecial-rule related to H. 

J_n-P recursively applies to every set of swffs of the obtained configuration. S is not

J_n-consistent if for some H every set in the obtained configuration is not J_n-consistent.

The computational-space analysis for J_n-P is analogous to the one given for Bd_d-P. J_n-P

is sketched in Tables 12–14. The subroutine J_n-II is similar to Bd_d-II and is not given here,

the other subroutines are analogous to those given for Bd_d-P: the data structures are used in

the same manner and the final swffs are treated following the same ideas.

Theorem 6.2

The logics Bd_d and J_n are decidable in O(n log n) space.

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References


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