Zoran Petrić

270 Minutes on Categorial Proof Theory

Abstract. The aim of these notes is to provide an introduction of basic categorial notions to a reader interested in logic and proof theory. The first part is devoted to justification of these notions through a cut elimination procedure. In the second part a classification of formulae up to isomorphism, and an example of coherence are given.

Mathematics Subject Classification (2010): Primary: 18-01, 03F07, 18A15; Secondary 03F05, 18D10.

Keywords: sequent system, cut elimination, functor, natural transformation, adjunction, isomorphism, coherence.

Contents

The cut-elimination procedure preserves the equality of derivations.

$$\dfrac{\dfrac{C \wedge A \vdash B}{A \vdash C \to B} \qquad \dfrac{D \vdash C \quad B \vdash E}{D \wedge (C \to B) \vdash E}}{D \wedge A \vdash E} \rightsquigarrow \dfrac{D \vdash C \qquad \dfrac{C \wedge A \vdash B \quad B \vdash E}{C \wedge A \vdash E}}{D \wedge A \vdash E}$$

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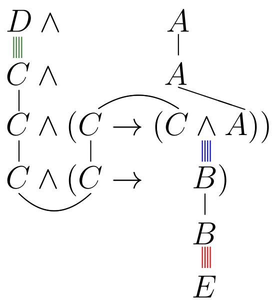

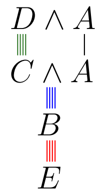

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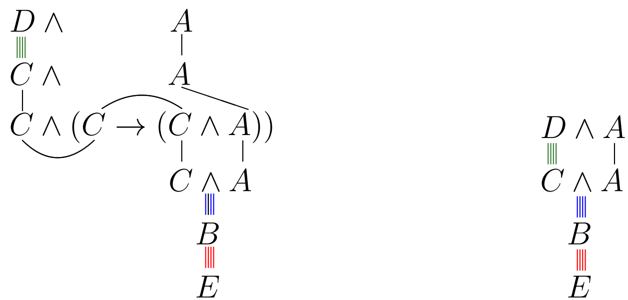

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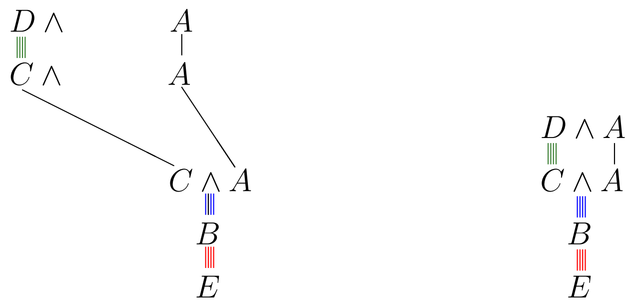

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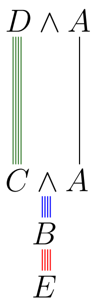

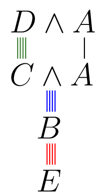

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Boolean algebra Set

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If $P = \{p, q\}$, $B = \mathcal{P}(\{1, 2, 3, 4, 5\})$ and f is the function from the set P to the set $G(B)$ defined by

$$f(p) = \{1, 3, 5\}, \qquad f(q) = \{2, 3, 4\},$$

then the corresponding homomorphism g from the Boolean algebra $F(P)$ to the Boolean algebra B is such that

$$g(p \wedge q) = \{3\}, \quad g(p \vee q) = \{1, 2, 3, 4, 5\}, \quad g(p^c) = \{2, 4\}, \text{ etc.}$$

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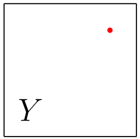
Space



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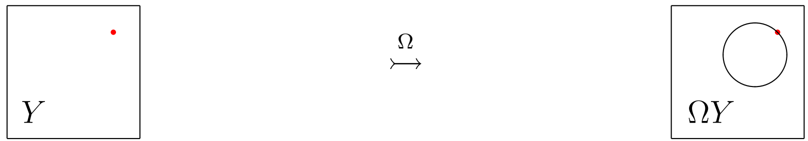

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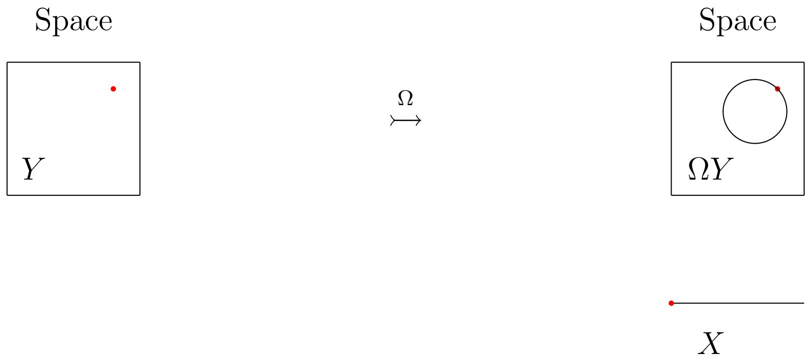

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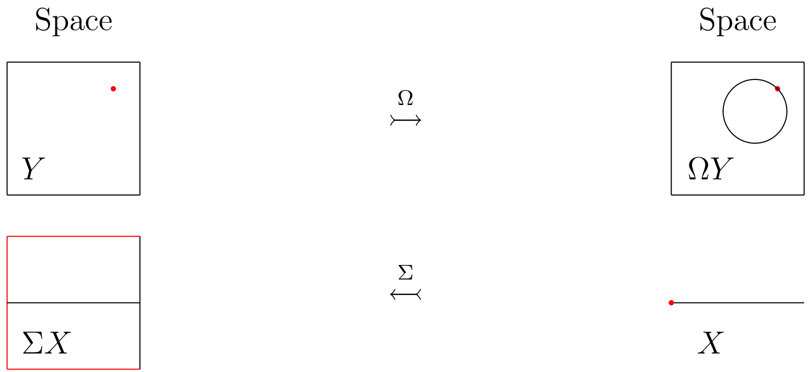

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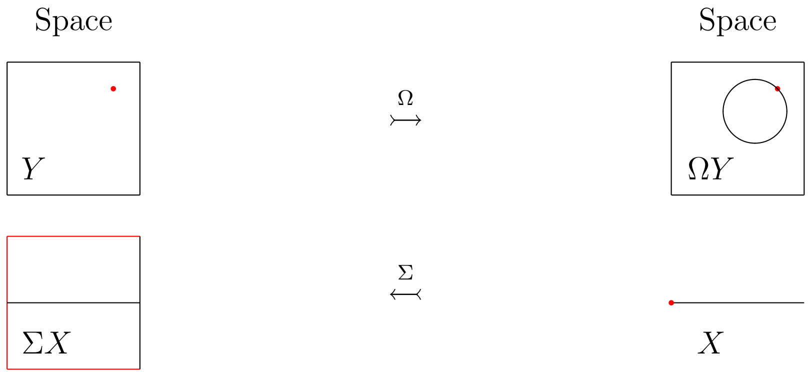

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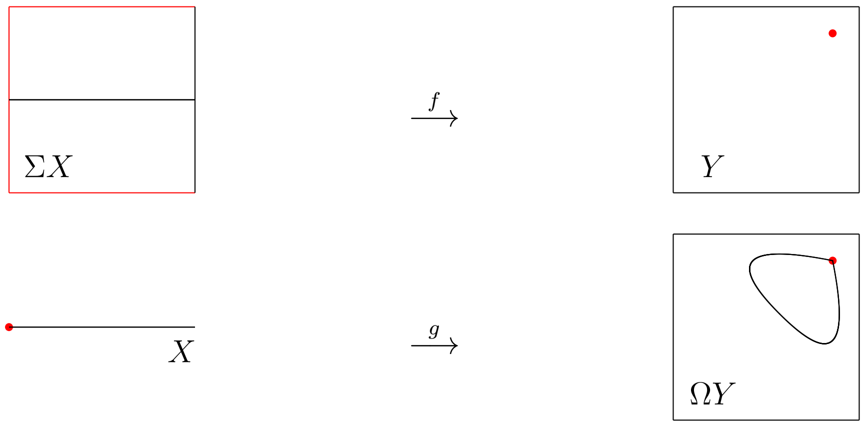

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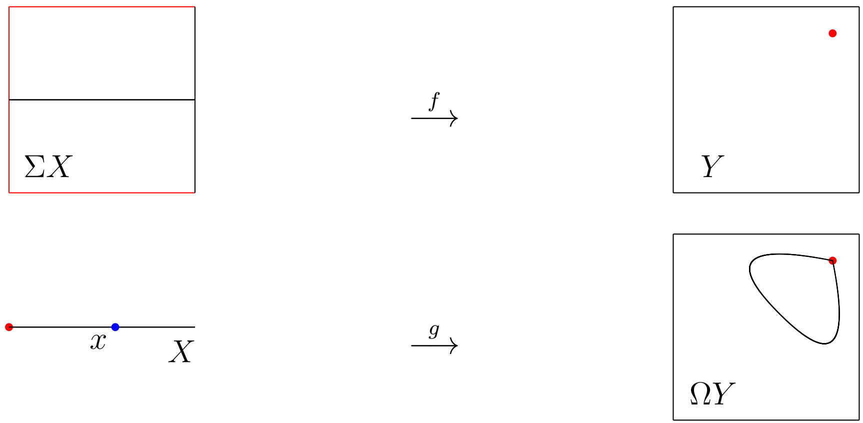

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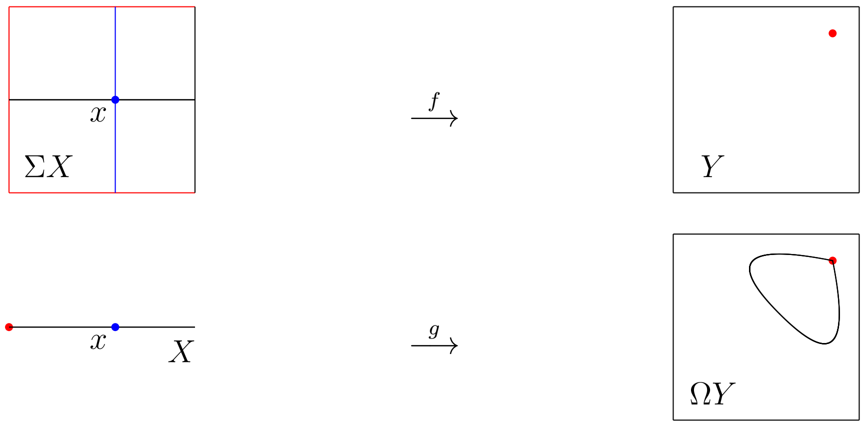

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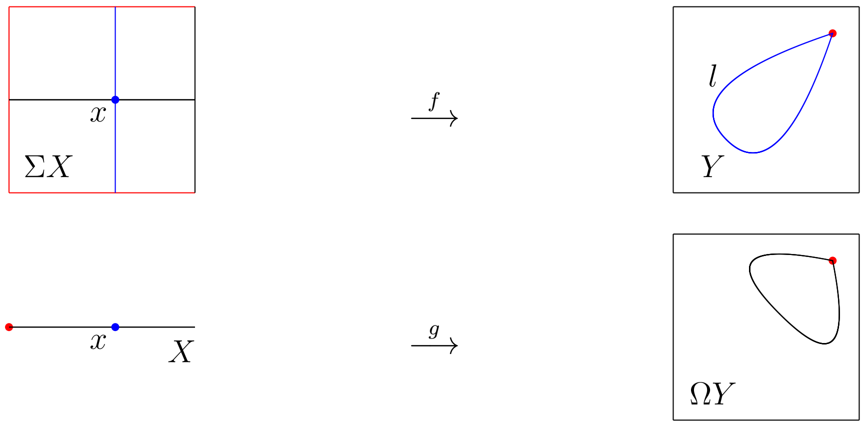

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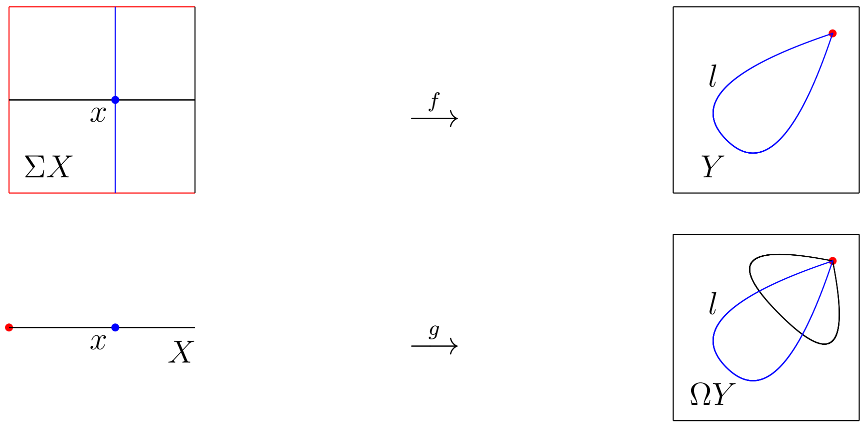

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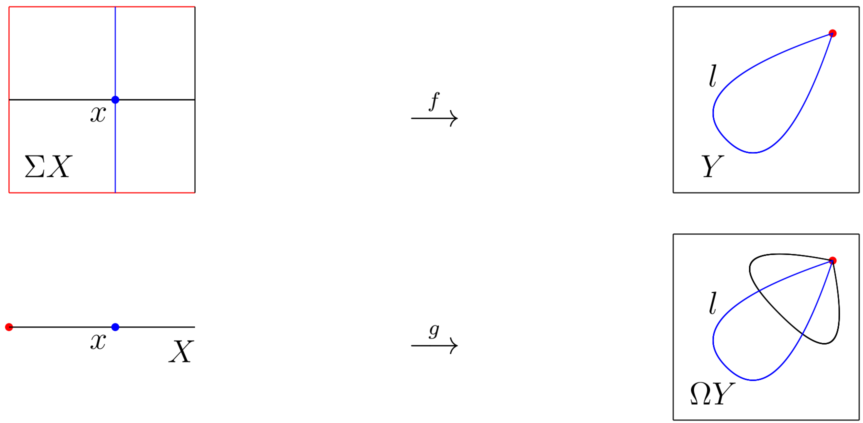

$$g(x) = l$$

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Formula

B

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$$
\begin{tikzcd}
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B \arrow[rr, tail, "C\to"] & & C \to B
\end{tikzcd}
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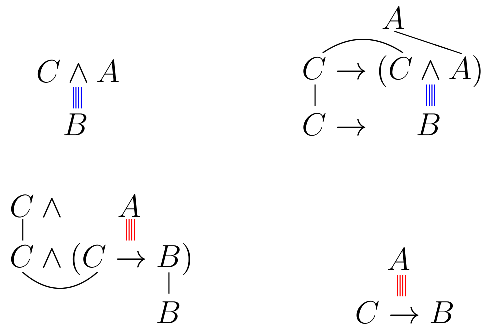

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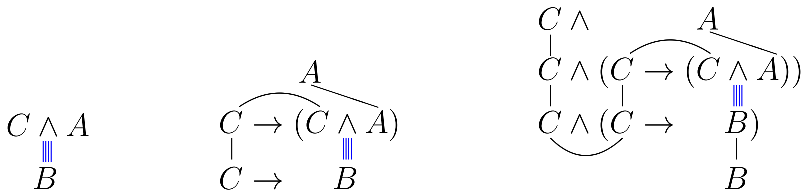

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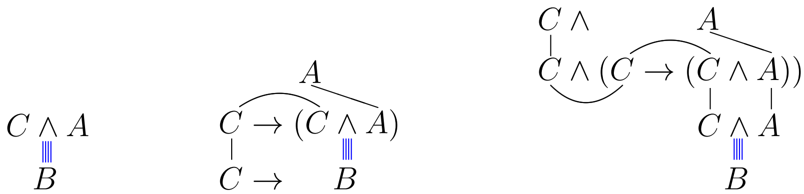

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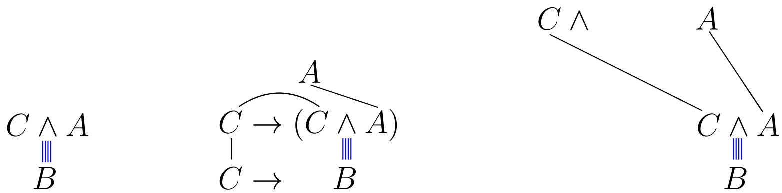

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