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Monoids, Segal's condition and bisimplicial spaces

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Abstract

A characterization of simplicial objects in categories with finite products obtained by the reduced bar construction is given. The condition that characterizes such simplicial objects is a strictification of Segal's condition guaranteeing that the loop space of the geometric realization of a simplicial space X and the space X_1 are of the same homotopy type. A generalization of Segal's result appropriate for bisimplicial spaces is given. This generalization gives conditions guaranteing that the double loop space of the geometric realization of a bisimplicial space X and the space X_{11} are of the same homotopy type.

Mathematics Subject Classification (*2010*): 18G30, 57T30, 55P35

Keywords: reduced bar construction, simplicial space, loop space

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1 Introduction

This paper is based on the author talks delivered in 2014 at the Fourth Mathematical Conference of the Republic of Srpska and at the CGTA Colloquium of the Faculty of Mathematics in Belgrade. Its first part gives a condition which is necessary and sufficient for a simplicial object to be obtained by the reduced bar construction. It turns out that this condition is a strictification of Segal's condition guaranteeing that the loop space of the geometric realization of a simplicial space X and the space X_1 are of the same homotopy type.

The second part of this paper is devoted to a generalization of Segal's result. This generalization gives conditions guaranteing that the double loop space of the geometric realization of a bisimplicial space X and the space X_{11} are of the same homotopy type. We refer to [8] for a complete generalization of Segal's result.

2 Monoids and the reduced bar construction

A *strict monoidal* category $(\mathcal{M}, \otimes, \mathcal{I})$ is a category $\mathcal{M}$ with an associative bifunctor $\otimes \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$,

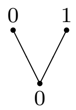
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad \text{and} \quad (f \otimes g) \otimes h = f \otimes (g \otimes h),$$

and an object I, which is a left and right unit for $\otimes$,

$$A \otimes I = A = I \otimes A \quad \text{and} \quad f \otimes \mathbf{1}_I = f = \mathbf{1}_I \otimes f.$$

A *strict monoidal functor* between strict monoidal categories is a functor that preserves strict monoidal structure "on the nose", i.e., $F(A \otimes B) = F(A) \otimes F(B)$, $F(I) = I$, etc.

Algebraist's *simplicial category* Δ is an example of strict monoidal category. The objects of Δ are all finite ordinals $0 = \emptyset$, $1 = \{0\}$, $\dots, n = \{0, \dots, n-1\}$, etc. The arrows of Δ from n to m are all order preserving functions from the set n to the set m, i.e., $f : n \to m$ satisfying: if $i < j$ and $i, j \in n$, then $f(i) \leq f(j)$. We use the standard graphical presentation for arrows of Δ. For example, the unique arrows from 2 to 1 and from 0 to 1 are graphically presented as:

$2 \to 1$  $0 \to 1$ 

A bifunctor $\otimes : \Delta \times \Delta \to \Delta$ is defined on objects as addition and on arrows as placing "side by side", i.e., for $f : n \to m$ and $f' : n' \to m'$

$$(f \otimes f')(i) = \begin{cases} f(i), & \text{when} \quad 0 \leq i \leq n-1, \\ m + f'(i-n), & \text{when} \quad n \leq i \leq n + n' - 1, \end{cases}$$

and 0 serves as the unit I.

A *monoid* in a strict monoidal category $(\mathcal{M}, \otimes, I)$ is a triple $(M, \mu : M \otimes M \to M, \eta : I \to M)$ such that

$$\mu \circ (\mu \otimes \mathbf{1}_M) = \mu \circ (\mathbf{1}_M \otimes \mu) \quad \text{and} \quad \mu \circ (\mathbf{1}_M \otimes \eta) = \mathbf{1}_M = \mu \circ (\eta \otimes \mathbf{1}_M).$$

For example, $(1, \bigvee, \bullet)$ is a monoid in Δ, where $\bigvee$ and $\bullet$ are the above graphical presentations of the arrows of Δ from 2 to 1, and from 0 to 1. The following result, taken over from [3, VII.5, Proposition 1], shows the "universal" property of this monoid.

Proposition 2.1. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$, there is a unique strict monoidal functor $F : \Delta \to \mathcal{M}$ such that $F(1) = M$, $F(\bigvee) = \mu$ and $F(\bullet) = \eta$.*

Let Δ_{par} be the category with the same objects as Δ, whose arrows are order preserving partial functions. Then $(1, \bigvee, \bullet)$ is a monoid in the strict monoidal category Δ_{par} with the same tensor and unit as Δ. The empty partial function from 1 to 0 is graphically presented as $^{\bullet}$. By [7, Proposition 6.2] we have the following universal property of this monoid.

Proposition 2.2. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, there is a unique strict monoidal functor $F : \Delta_{par} \to \mathcal{M}$ such that $F(1) = M$, $F(\bigvee) = \mu$, $F(\,_{\bullet}\,) = \eta$ and $F(\,^{\bullet}\,)$ is the unique arrow from M to the unit (a terminal object of $\mathcal{M}$).*

Topologist's simplicial category is the full subcategory of Δ on nonempty ordinals as objects. We identify this category with the subcategory of *Top*. The object $n+1$ is identified with the standard ordered simplex

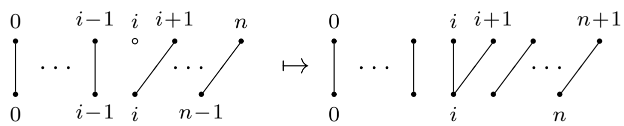
$$\Delta^n = \{(t_0, \dots, t_n) \mid t_0, \dots, t_n \geq 0, \sum_i t_i = 1\},$$

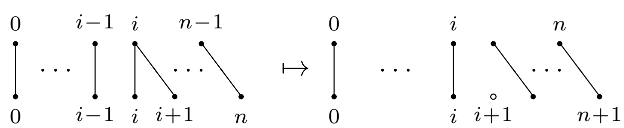
and an arrow $f\colon n+1 \to m+1$ is identified with the affine map defined by

$$f(t_0, \dots, t_n) = (s_0, \dots, s_m), \text{ where } s_j = \sum_{f(i)=j} t_i.$$

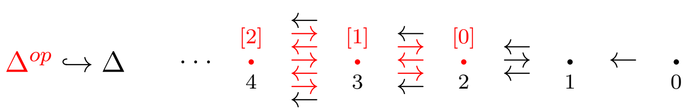
We denote by Δ^{op} the opposite of topologist's simplicial category and rename its objects so that the ordinal $n+1$ is denoted by $[n]$, i.e., $[n] = \{0, \dots, n\}$. Let Δ_{Int} be the subcategory of Δ whose objects are finite ordinals greater or equal to 2 and whose arrows are interval maps, i.e., order-preserving functions, which preserve, moreover, the first and the last element.

The categories Δ^{op} and Δ_{Int} are isomorphic by the functor $\mathcal{J}$ (see [7, Section 6]). This functor maps the object $[n]$ to $n+2$ and it maps the generating arrows of Δ^{op} in the following way.





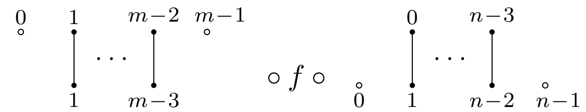
The functor $\mathcal{J}$ may be visualized as the following embedding of Δ^{op} into Δ. (I am grateful to Matija Bašić for this remark.)



(1)

Throughout this paper, we represent the arrows of Δ^{op} by the graphical presentations of their $\mathcal{J}$ images in Δ_{Int}.

We have a functor $\mathcal{H}\colon \Delta_{Int} \to \Delta_{par}$ defined on objects as $\mathcal{H}(n) = n-2$, and on arrows, for $f\colon n \to m$, as

$\mathcal{H}(f) =$ 

(Intuitively, $\mathcal{H}(f)$ is obtained by omitting the vertices 0, $n-1$ from the source, and 0, $m-1$ from the target in the graphical presentation of f together with all the edges incident to these vertices.) It is not difficult to see that $\mathcal{H}(\mathbf{1}_n) = \mathbf{1}_{n-2}$, and that for a pair of arrows $f\colon n \to m$ and $g\colon m \to k$ of Δ_{Int} we have

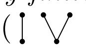
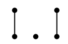
$$\mathcal{H}(g)\circ\mathcal{H}(f)(i) = \begin{cases} g(f(i+1))-1, & f(i+1) \notin \{0, m-1\} \wedge g(f(i+1)) \notin \{0, k-1\} \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and

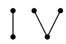
$$\mathcal{H}(g\circ f)(i) = \begin{cases} g(f(i+1))-1, & g(f(i+1)) \notin \{0, k-1\} \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Since $g(f(i+1)) \notin \{0, k-1\}$ implies $f(i+1) \notin \{0, m-1\}$, we have that $\mathcal{H}(g) \circ \mathcal{H}(f)(i) = \mathcal{H}(g \circ f)(i)$, and $\mathcal{H}$ so defined is indeed a functor.

A *simplicial object* X in a category $\mathcal{M}$ is a functor $X : \Delta^{op} \to \mathcal{M}$. The following proposition is a corollary of Proposition 2.2.

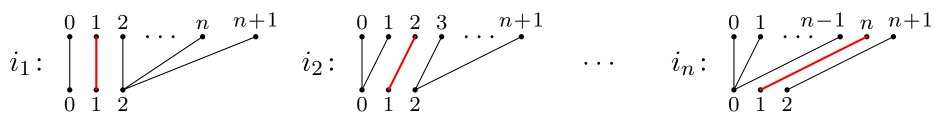
Proposition 2.3. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, there is a simplicial object X in $\mathcal{M}$ such that $X([n]) = M^n$, $X($*  *$) = \mu$, $X($*  *$) = \eta$.*

Proof. Take X to be the composition $F \circ \mathcal{H} \circ \mathcal{J}$, for F as in Proposition 2.2. ⊣

Note that both  and  are mapped by X to the unique arrow from M to the unit M^0 (a terminal object of $\mathcal{M}$). We say that a simplicial object in $\mathcal{M}$ obtained as the composition $F \circ \mathcal{H} \circ \mathcal{J}$, for F as in Proposition 2.2, is the *reduced bar construction based* on M (see [10] and [7]).

For X a simplicial object, we abbreviate $X([n])$ by X_n. Also, for f an arrow of Δ^{op}, we abbreviate $X(f)$ by f whenever the simplicial object X is determined by the context.

For $n \geq 2$, consider the arrows $i_1, \ldots, i_n : [n] \to [1]$ of Δ^{op} graphically presented as follows.



(It would be more appropriate to denote these arrows by $i_1^n, \ldots, i_n^n$, but we omit the upper indices taking them for granted.)

For arrows $f\colon C \to A$ and $g\colon C \to B$ of a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, we denote by $\langle f, g\rangle : C \to A \times B$ the arrow obtained by the universal property of product in $\mathcal{M}$. For X a simplicial object in $\mathcal{M}$, we denote by p_0 the unique arrow from X_0 to the unit, i.e., a terminal object $(X_1)^0$ of $\mathcal{M}$, and we denote by p_1 the identity arrow from X_1 to X_1. For $n \geq 2$ and the above mentioned arrows $i_1, \ldots, i_n : [n] \to [1]$ of Δ^{op}, we denote by p_n the arrow

$$\langle i_1, \ldots, i_n\rangle\colon X_n \to (X_1)^n,$$

where by our convention, i_j abbreviates $X(i_j)$.

Let X be the reduced bar construction based on a monoid M. Since X_0 is the unit M^0 and for $n \geq 2$, the arrow $i_j : M^n \to M$ is the jth projection, we have that for every $n \geq 0$, the arrow p_n is the identity. We show that this property characterizes the reduced bar construction based on a monoid in $\mathcal{M}$.

Proposition 2.4. *Let $\mathcal{M}$ be a strict monoidal category whose monoidal structure is given by finite products. A simplicial object X in $\mathcal{M}$ is the reduced bar construction based on a monoid in $\mathcal{M}$ if and only if for every $n \geq 0$, the arrow $p_n\colon X_n \to (X_1)^n$ is the identity.*

Proof. The "only if" part of the proof is given in the paragraph preceding this proposition. For the "if" part of the proof, let us denote X_1 by M. By our convention, the X images of arrows of Δ^{op} are denoted just by their names or graphical presentations. We show that

$$(M, \quad , \quad)$$

is a monoid in $\mathcal{M}$. Let $k^1_{M^2,M} : M^2 \times M \to M^2$ and $k^2_{M^2,M} : M^2 \times M \to M$ be the first and the second projection respectively. Since $p_3 = \langle i_1, i_2, i_3\rangle\colon M^3 \to M^3$ is the identity, we have that $k^1_{M^2,M} = \langle i_1, i_2\rangle$ and $k^2_{M^2,M} = i_3 =$.

For arrows $f : C \to A$, $g : C \to B$, $h: D \to C$, $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ and projections $k^1_{A_1,A_2}\colon A_1 \times A_2 \to A_1$ and $k^2_{A_1,A_2}\colon A_1 \times A_2 \to A_2$, the following equations hold in $\mathcal{M}$

$$\langle f \circ h, g \circ h\rangle = \langle f, g\rangle \circ h, \qquad f_1 \times f_2 = \langle f_1 \circ k^1_{A_1,A_2}, f_2 \circ k^2_{A_1,A_2}\rangle.$$

We have

$$k^1_{M^2,M} = \langle i_1, i_2\rangle = \langle \quad , \quad \rangle = \langle \quad , \quad \rangle = p_2 \circ \quad .$$

Hence, $k^1_{M^2,M} =$. Analogously, we prove that $k^2_{M,M^2} =$. Also,

$$\mu \times \mathbf{1} = \langle \mu \circ k^1_{M^2,M}, k^2_{M^2,M}\rangle = \langle \quad , \quad \rangle = \langle \quad , \quad \rangle = p_2 \circ \quad .$$

Hence, $\mu \times \mathbf{1} =$. Analogously, we prove that $\mathbf{1} \times \mu =$. Now, $\mu \circ (\mu \times \mathbf{1}) = \mu \circ (\mathbf{1} \times \mu)$, since

$$=$$

That $k^1_{M,M^0} = \mathbf{1} =$, and $k^2_{M,M^0} =$ follows from the fact that M^0 is the strict unit and a terminal object of $\mathcal{M}$. Hence,

$$\mathbf{1}\times\eta = \langle k^1_{M,M^0}, \eta\circ k^2_{M,M^0}\rangle = \langle \ , \ \rangle = \langle \ , \ \rangle = p_2 \circ \ = \ .$$

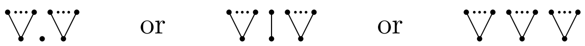
Now, $\mu \circ (\mathbf{1} \times \eta) = \mathbf{1}$, since

$$= \ = \mathbf{1}.$$

Analogously, we prove that $\mu \circ (\eta \times \mathbf{1}) = \mathbf{1}$, and conclude that M is a monoid in $\mathcal{M}$.

Let Y be the reduced bar construction based on M. We show that $X = Y$. It is clear that the object parts of the functors X and Y coincide. We prove that for every arrow $f\colon [m] \to [n]$ of Δ^{op}, the arrows $X(f)$ and $Y(f)$ are equal in $\mathcal{M}$.

If $n = 0$, then this is trivial since X_0, which is equal to M^0, is a terminal object of $\mathcal{M}$. If $n = 1$, then f has one of the following forms



In the first case, $f =$ and the X and Y images of the upper part are equal as in the case $n = 0$, while $X(\ ) = Y(\ )$ by the definition of Y.

In the second case, f is either identity and $X(f) = Y(f)$ holds, or f is i_j for some $1 \leq j \leq m$. From $\langle X(i_1), \ldots, X(i_m)\rangle = \mathbf{1}$, we conclude that $X(i_j)$ is the jth projection from M^m to M. On the other hand, by the definition of Y, we have that $Y(i_j)$ is the jth projection from M^m to M. Hence $X(f) = Y(f)$.

In the third case, when f is $\overbrace{\ \cdots\ }^{l}$, we proceed by induction on $l \geq 2$. In the proof we use the fact that two arrows $g, h\colon C \to M^2$ are equal in $\mathcal{M}$ iff $k^1_{M,M} \circ g = k^1_{M,M} \circ h$ and $k^2_{M,M} \circ g = k^2_{M,M} \circ h$, where $k^1_{M,M}$ and $k^2_{M,M}$ are the first and the second projection from M^2 to M. Also, we know from above that

$$k^1_{M,M} = X(\ ) = Y(\ ), \quad k^2_{M,M} = X(\ ) = Y(\ ).$$

If $l = 2$, then f is equal to . Since $X(\ ) = Y(\ )$, in order to prove that $X(f) = Y(f)$, it suffices to prove that $g = X(\ )$ is equal to $h = Y(\ )$. By relying on the second case for †, we have that

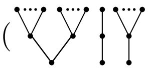
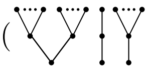
$$k^1_{M,M} \circ g = X(\ ) \stackrel{\dagger}{=} Y(\ ) = k^1_{M,M} \circ h.$$

Analogously, we prove that $k^2_{M,M} \circ g = k^2_{M,M} \circ h$. Hence, $g = h$.

If $l > 2$, then f is equal to , and it suffices to prove that $g = X(\ )$ is equal to $h = Y(\ )$. By relying on the induction hypothesis for †, we have that

$$k^1_{M,M} \circ g = X(\ ) \stackrel{\dagger}{=} Y(\ ) = k^1_{M,M} \circ h.$$

By relying on the second case for †, we have that

$k^2_{M,M} \circ g = X($  $) \overset{\dagger}{=} Y($  $) = k^2_{M,M} \circ h.$

Hence, $g = h$. This concludes the case when f maps $[m]$ to $[1]$.

Suppose now that $f\colon [m] \to [n]$ is an arrow of Δ^{op} and $n \geq 2$. As in the case when $n = 1$, we conclude that for every $1 \leq j \leq n$,

$$X(i_j \circ f) = Y(i_j \circ f).$$

Since,

$$\langle X(i_1), \ldots, X(i_n)\rangle = \langle Y(i_1), \ldots, Y(i_n)\rangle = \mathbf{1}_{M^n},$$

we have that

$$\begin{aligned} X(f) &= \langle X(i_1), \ldots, X(i_n)\rangle \circ X(f) = \langle X(i_1) \circ X(f), \ldots, X(i_n) \circ X(f)\rangle \\ &= \langle Y(i_1) \circ Y(f), \ldots, Y(i_n) \circ Y(f)\rangle = \langle Y(i_1), \ldots, Y(i_n)\rangle \circ Y(f) \\ &= Y(f). \end{aligned}$$

⊣

3 Segal's simplicial spaces

Let *Top* be the category of compactly generated Hausdorff spaces. For a simplicial object in *Top*, i.e., a *simplicial space* X, a relaxed form of the condition

$$\text{for every } n,\ p_n\colon X_n \to (X_1)^n \text{ is the identity},$$

reads

$$\text{for every } n,\ p_n\colon X_n \to (X_1)^n \text{ is a homotopy equivalence.}$$

Segal, [9], used simplicial spaces satisfying this relaxed condition for his delooping constructions and we call them *Segal's simplicial spaces*. (Note that, for the sake of simplicity, this notion is weaker than the one defined in [8].) Essentially as in the proof of Proposition 2.4, one can show the following.

Proposition 3.1. *If $X : \Delta^{op} \to Top$ is Segal's simplicial space, then X_1 is a homotopy associative* H*-space whose multiplication is given by the composition*

$$(X_1)^2 \xrightarrow{p_2^{-1}} X_2 \xrightarrow{d_1^2} X_1,$$

where p_2^{-1} is an arbitrary homotopy inverse to p_2, and whose unit is $s_0^1(x_0)$, for an arbitrary $x_0 \in X_0$.

(A complete proof of this proposition is given in [8, Appendix, Proof of Lemma 3.1].)

The *realization* of a simplicial space X, is the quotient space

$$|X| = \left(\coprod_n X_n \times \Delta^n\right) \Big/ \sim,$$

where $\sim$ is the smallest equivalence relation on $\coprod_n X_n \times \Delta^n$ such that for every $f\colon [n] \to [m]$ of Δ, $x \in X_m$ and $t \in \Delta^n$

$$(f^{op}(x), t) \sim (x, f(t)).$$

A *simplicial map* is a natural transformation between simplicial spaces. Note that the realization is *functorial*, i.e., it is defined also for simplicial maps. For simplicial spaces X and Y the *product* $X \times Y$ is defined so that $(X \times Y)_n = X_n \times Y_n$ and $(X \times Y)(f) = X(f) \times Y(f)$. The nth component of the first projection $k^1\colon X \times Y \to X$ is the first projection $k^1_n : X_n \times Y_n \to X_n$ and analogously for the second projection. The realization functor preserves products of simplicial spaces (see [4, Theorem 14.3], [2, III.3, Theorem] and [5, Corollary 11.6]) in the sense that

$$\langle |k^1|, |k^2| \rangle\colon |X \times Y| \to |X| \times |Y|$$

is a homeomorphism.

The following two propositions stem from [9, Proposition 1.5 (b)] and from [6, Appendix, Theorem A4 (ii)] (see also [8, Lemma 2.11]).

Proposition 3.2. *Let $X : \Delta^{op} \to Top$ be Segal's simplicial space such that for every m, X_m is a CW-complex. If X_1 with respect to the* H*-space structure is grouplike, then $X_1 \simeq \Omega|X|$.*

Proposition 3.3. *Let $f : X \to Y$ be a simplicial map of simplicial spaces such that for every m, X_m and Y_m are CW-complexes. If each $f_m : X_m \to Y_m$ is a homotopy equivalence, then $|f|\colon |X| \to |Y|$ is a homotopy equivalence.*

4 Segal's bisimplicial spaces

A *bisimplicial space* is a functor $X\colon \Delta^{op} \times \Delta^{op} \to Top$ and it may be visualized as the following graph (see the red subgraph of (1)).

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots \\
\cdots & X_{22} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{12} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{02} \\
 & \downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow \\
\cdots & X_{21} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{11} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{01} \\
 & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
\cdots & X_{20} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{10} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{00}
\end{array}$$

Let Y_n, for $n \geq 0$, be the realization of the nth column, i.e., $Y_n = |X_{n\_}|$. Since the realization is functorial, we obtain the simplicial space Y.

$$\begin{array}{cccccc}
\cdots & Y_2 & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & Y_1 & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & Y_0
\end{array}$$

The *realization* $|X|$ of the bisimplicial space X is the realization $|Y|$ of the simplicial space Y.

If the simplicial space $X_{1\_}$ is Segal's, then, by Proposition 3.1, X_{11} is a homotopy associative H-space and this is the H-space structure we refer to in the following proposition.

Proposition 4.1. *If $X\colon \Delta^{op} \times \Delta^{op} \to Top$ is a bisimplicial space such that $X_{1\_}$ is Segal's, X_{11} with respect to the* H*-space structure is grouplike, for every $m \geq 0$, $X_{\_m}$ is Segal's, and for every $n, m \geq 0$, X_{nm} and Y_n are CW-complexes, then $X_{11} \simeq \Omega^2|X|$.*

Proof. Since $X_{1\_}$ is Segal's simplicial space such that for every m, X_{1m} is a CW-complex and X_{11} with respect to the H-space structure is grouplike, by Proposition 3.2 we have that $X_{11} \simeq \Omega|X_{1\_}| = \Omega Y_1$.

For every m, $X_{\_m}$ is Segal's. Hence, for every n, the map $p_{nm} : X_{nm} \to (X_{1m})^n$, is a homotopy equivalence. The map p_{0m} is the unique map from X_{0m} to $(X_{1m})^0$, the map p_{1m} is the identity on X_{1m}, and for $n \geq 2$, the map p_{nm} is

$$\langle (i_1, m), \ldots, (i_n, m)\rangle \colon X_{nm} \to (X_{1m})^n.$$

Also, for every $f\colon [m] \to [m']$ of Δ^{op} and every n the following diagram commutes:

$$\begin{tikzcd}
X_{nm} \arrow[r, "p_{nm}"] \arrow[d, "{(n,f)}"'] & (X_{1m})^n \arrow[d, "{(1,f)^n}"] \\
X_{nm'} \arrow[r, "p_{nm'}"] & (X_{1m'})^n
\end{tikzcd}$$

Hence, for every n, $p_{n\_}$ is a simplicial map.

$$\begin{array}{ccc}
\vdots & & \vdots \\
X_{n2} & \xrightarrow{\quad p_{n2} \quad} & (X_{12})^n \\
\downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow \\
X_{n1} & \xrightarrow{\quad p_{n1} \quad} & (X_{11})^n \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
X_{n0} & \xrightarrow{\quad p_{n0} \quad} & (X_{10})^n
\end{array}$$

Every $(X_{1m})^n$ is a CW-complex since the product of CW-complexes in *Top* is a CW-complex. By Proposition 3.3, for every n, $|p_{n\_}| : Y_n \to |(X_{1\_})^n|$ is a homotopy equivalence. Since $|(X_{1\_})^0|$ is a singleton it is homeomorphic to $(Y_1)^0$ and we have that $p_0 : Y_0 \to (Y_1)^0$, as a composition of a homeomorphism with $|p_{0\_}|$, is a homotopy equivalence. The map $p_1 \colon Y_1 \to Y_1$ is the identity. For $n \geq 2$, $\langle |k^1|, \ldots, |k^n|\rangle : |(X_{1\_})^n| \to |X_{1\_}|^n$ is a homeomorphism and for $1 \leq j \leq n$, $|(i_j, \_)| = |X(i_j, \_)| = Y(i_j)$. Hence, the map

$$p_n = \langle Y(i_1), \ldots, Y(i_n)\rangle = \langle |k^1|, \ldots, |k^n|\rangle \circ |\langle (i_1, \_), \ldots, (i_n, \_)\rangle|,$$

as a composition of a homeomorphism with $|p_{n\_}|$, is a homotopy equivalence between Y_n and $(Y_1)^n$. We conclude that Y is Segal's, and by Proposition 3.1, Y_1 is a homotopy associative H-space.

If a simplicial space is Segal's, then its realization is path-connected. This is because its value at [0] is contractible and therefore path-connected (see [5, Lemma 11.11]). Since $X_{1\_}$ is Segal's, we conclude that Y_1 is path-connected. Moreover, it is grouplike since every path-connected homotopy associative H-space, which is a CW-complex, is grouplike (see [1, Proposition 8.4.4]).

Applying Proposition 3.2 to Y, we obtain that $Y_1 \simeq \Omega|Y|$. Hence,

$$X_{11} \simeq \Omega Y_1 \simeq \Omega(\Omega|Y|) = \Omega^2|X|. \qquad \dashv$$

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