KOSTA DOŠEN AND ZORAN PETRIĆ

MODAL FUNCTIONAL COMPLETENESS

INTRODUCTION

This paper is a companion to [3], where it was shown how a modal version of the deduction theorem induces modal extensions of substructural logics. Here we shall prove some of the results announced in the concluding section of [3]. Namely, we shall prove modal functional completeness for categories corresponding to the main modal substructural propositional logics. This modal functional completeness is related to the modal version of the deduction theorem of [3] as functional completeness for bicartesian closed categories is related to the deduction theorem for intuitionistic propositional logic. Roughly speaking, the deduction theorem says that the system is strong enough to express its own deductive metatheory. Functional completeness says that the deductive metatheory can be embedded in the system.

Functional completeness for categories is a property of the same kind as combinatorial completeness for systems of combinators. Combinatorial completeness permits us to define functional abstraction and demonstrate the equivalence with systems of lambda terms. Functional completeness permits us similarly to find systems of typed lambda terms as internal languages of closed categories. A well-known use of typed lambda terms in proof theory, which goes by the name of the Curry–Howard correspondence, is to serve as codes of natural-deduction proofs. If categories are conceived as logical systems with an equivalence relation among proofs, which is induced by normalization, this coding and extracting the internal language boil down to the same thing.

We learned about these matters from [6] and refer to that book for a demonstration of the importance of functional completeness. We shall try to follow the style of [6], so that our results may be compared with the standard functional completeness results proved there for cartesian, cartesian closed and bicartesian closed categories. An acquaintance with [6] may also help to grasp the proof-theoretical import of what we will present here—a matter about which we don't have space to say much. Further logical motivation may be found in the aforementioned paper [3] and in [4]. We refer to these works only for motivation. Otherwise, our paper will be rather self-contained.

For our functional completeness results we concentrate on categories corresponding to what we take to be the minimal substructural logic: namely, Lambek's nonassociative calculus. The modal postulates assumed for these categories are those of S4 plus modalized versions of the missing structural rules—the same kind of modal postulates one finds in usual presentations of linear logic. The modal functional completeness theorem proved for these categories can easily be extended to categories corresponding to better-known substructural logics, like linear and relevant logic, as we indicate in Section 6. Working with the minimal substructural logic has technical advantage and should help dispelling the impression that this sort of theorem, as well as some others, is exclusively tied to a particular substructural logic. Results about linear logic are often presented in a fashion that fosters such misleading impressions. For example, modal translation results (which are not foreign to the matter we are treating) are presented as if they had to do with linear logic specifically, though analogous results may be obtained as easily for other substructural logics.

We shall devote a section to a transformation of our proof of modal functional completeness into a proof of ordinary, nonmodal, functional completeness for cartesian, cartesian closed and bicartesian closed categories axiomatized in a nonstandard manner. Namely, instead of having as primitives projection arrows and a pairing operation on arrows, we have arrows corresponding to structural rules, and we show how with arrows corresponding to the structural rules of thinning and contraction, Mac Lane's axiomatization of symmetric monoidal categories, which among other things has arrows corresponding to association and permutation, can be extended to an axiomatization of cartesian categories. This means putting cartesian categories in a substructural perspective, where we envisage rejecting structural rules.

We produce such nonmodal substructural categories when in a separate section, at the end, we consider restricted nonmodal functional completeness for them. The restrictions in question are obtained by having special notions of what in the terminology of [6] is called polynomials in polynomial categories. This is related to having restricted classes of typed lambda terms as codes of proofs in substructural logics. We leave for the future a more thorough investigation of this and some other matters mentioned in the concluding section.

We also devote a section to justifying the assumptions made for our categories, and, in particular, assumptions related to structural rules. We shall show for a number of them, and in particular the more abstruse, that they are not only sufficient, but also necessary for functional completeness. Most of these assumptions are quite well known and can be found in Mac Lane's book [7]. There they are motivated differently: they appear in connection with coherence problems (cf. the very end of our paper). We motivate the same assumptions by functional completeness. Though coherence may also be related to logic, we believe the sort of motivation we provide does more justice to the logical character of our categories. Anyway, it should be better than just taking over these assumptions from category theory, as it is sometimes done when categories are presented as 'models' for logics.

However, we didn't find in the literature all of these assumptions: some of those tied to the structural rule of contraction may be new. Among them we single out something we shall call *octagonal equation*, a principle we shall use quite often, which is comparable to Mac Lane's pentagonal and hexagonal diagrams of natural associativity and commutativity.

To end this introduction, let us mention a notational and terminological matter. Though we shall in general imitate the style of [6], we shall diverge somewhat in notation. In particular, we write arrows in categories with a turnstile, as $f : A \vdash B$, instead of $f : A \rightarrow B$. This we do because we reserve $\rightarrow$ for implication, an operation on objects, for which category theory uses an exponential notation or square brackets. We prefer to stay close to logical notation because our motivation is in logic. We want to suggest that objects in categories correspond to formulae, operations on objects to connectives, arrows to sequents, special arrows to axioms, operations on arrows to rules of inference, and equations between arrows to conversions of proofs, like those we make in normalization. However, we don't want to diverge from category theory also in terminology. So we call $f : A \vdash B$ an *arrow*, rather than a *sequent*. This hybrid between logical notation and categorial terminology may be slightly awkward, both from the point of view of logic and of category theory, but it just reflects the nature of our work. We use categories to say something about logical systems.

Because of its connection with the deduction theorem and proof theory, functional completeness could perhaps be called *deductive completeness*. But, to follow the policy exposed in the previous paragraph, and to prevent misunderstanding, we prefer to stick to the established terminology.

1 NL CATEGORIES

A *graph* consists of a class of *arrows* and a class of *objects*, together with the functions that to every arrow assign the objects that are its *source* and *target*. We use $f, g, h, \ldots$, possibly with indices, for arrows, and $A, B, C, \ldots$, possibly with indices, for objects. We write $f : A \vdash B$ to say that A is the source of f and B the target of f. For such an f we say that it is of type $A \vdash B$.

A *deductive system* is a graph in which we have a special arrow for every object

A:

$$1_A : A \vdash A$$

and the binary (partial) operation of composition of arrows:

$$\frac{f : A \vdash B \quad g : B \vdash C}{gf : A \vdash C}$$

A *category* is a deductive system in which the following equations between arrows are satisfied:

(cat 1) For $f : A \vdash B$, $\quad 1_B f = f, \quad f 1_A = f.$

(cat 2) For $f : A \vdash B, g : B \vdash C$ and $h : C \vdash D$, $\quad h(gf) = (hg)f.$

An NL deductive system ('NL' stands for 'nonassociative Lambek') has the following in addition to what every deductive system must have:

binary operations on objects: $\bullet, \rightarrow, \leftarrow, \wedge, \vee$

special objects: $\mathrm{I}, \top, \bot$

special arrows for every object A and B:

$$\sigma_A : \mathrm{I} \bullet A \vdash A \qquad \delta_A : A \bullet \mathrm{I} \vdash A$$

$$\sigma^i_A : A \vdash \mathrm{I} \bullet A \qquad \delta^i_A : A \vdash A \bullet \mathrm{I}$$

$$\varepsilon^{\rightarrow}_{A,B} : A \bullet (A \rightarrow B) \vdash B \qquad \varepsilon^{\leftarrow}_{A,B} : (B \leftarrow A) \bullet A \vdash B$$

$$\pi_{A,B} : A \wedge B \vdash A \qquad \pi'_{a,B} : A \wedge B \vdash B$$

$$\tau_A : A \vdash \top$$

$$\kappa_{A,B} : A \vdash A \vee B \qquad \kappa'_{A,B} : B \vdash A \vee B$$

$$\iota_A : \bot \vdash A$$

('σ' is to be associated with 'sinister', 'δ' with 'dexter', 'τ' with 'terminal', 'ι' with 'initial', and the superscript 'i' stands for 'inverse' or, maybe, 'introduction'; the remaining notation for arrows and operations on them is modelled after [6]) *operations on arrows:*

$$\frac{f : A \vdash B \quad g : C \vdash D}{f \bullet g : A \bullet C \vdash B \bullet D}$$

$$\frac{f : A \bullet C \vdash B}{{}^{*}f : C \vdash A \rightarrow B} \qquad \frac{f : C \bullet A \vdash B}{f^{*} : C \vdash B \leftarrow A}$$

$$\frac{f : C \vdash A \quad g : C \vdash B}{\langle f, g \rangle : C \vdash A \wedge B}$$

$$\frac{f : A \vdash C \quad g : B \vdash C}{[f, g] : A \vee B \vdash C}$$

The minimal NL deductive system is an extension of Lambek's nonassociative calculus with the propositional constant I, the lattice connectives $\wedge, \vee, \top$ and $\bot$ (lattice connectives are called 'additive' in the terminology of linear logic), and the associated arrows and operations on arrows. Note that the extension with I and the σ, σ^i, δ and δ^i arrows amounts to adding some structural rules; namely, rules for dealing with the empty collection of premises. Hence, it shouldn't be surprising that this extension is not conservative (it enables us to prove, for example, $(A \rightarrow A) \rightarrow B \vdash B$). We shall call the σ, σ^i, δ and δ^i arrows *$\sigma\delta$ arrows*.

An NL category is an NL deductive system that is a category in which the following equations between arrows are satisfied:

- equations

  ($\bullet$) For $f_1 : A_1 \vdash B_1, g_1 : B_1 \vdash C_1, f_2 : A_2 \vdash B_2$ and $g_2 : B_2 \vdash C_2$, $\quad (g_1 \bullet g_2)(f_1 \bullet f_2) = (g_1 f_1) \bullet (g_2 f_2)$.

  ($\bullet 1$) $1_A \bullet 1_B = 1_{A \bullet B}$

  $\sigma\delta$ equations

  (σ) For $f : A \vdash B$, $\quad f\sigma_A = \sigma_B(1_{\mathrm{I}} \bullet f)$.

  (δ) For $f : A \vdash B$, $\quad f\delta_A = \delta_B(f \bullet 1_{\mathrm{I}})$.

  ($\sigma\sigma^i$) $\sigma_A \sigma^i_A = 1_A, \qquad \sigma^i_A \sigma_A = 1_{\mathrm{I} \bullet A}$

  ($\delta\delta^i$) $\delta_A \delta^i_A = 1_A, \qquad \delta^i_A \delta_A = 1_{A \bullet \mathrm{I}}$

  ($\sigma\delta$) $\sigma_{\mathrm{I}} = \delta_{\mathrm{I}}$

  closure equations

$(\rightarrow \beta)$ For $f : A \bullet C \vdash B$, $\varepsilon^{\rightarrow}_{A,B}(1_A \bullet {}^*f) = f$.

$(\rightarrow \eta)$ For $g : C \vdash A \rightarrow B$, ${}^*(\varepsilon^{\rightarrow}_{A,B}(1_A \bullet g)) = g$.

$(\leftarrow \beta)$ For $f : C \bullet A \vdash B$, $\varepsilon^{\leftarrow}_{A,B}(f^* \bullet 1_A) = f$.

$(\leftarrow \eta)$ For $g : C \vdash B \leftarrow A$, $(\varepsilon^{\leftarrow}_{A,B}(g \bullet 1_A))^* = g$.

bicartesian equations

$(\wedge\beta)$ For $f : C \vdash A$ and $g : C \vdash B, \pi_{A,B}\langle f, g\rangle = f$, $\pi'_{A,B}\langle f, g\rangle = g$.

$(\wedge\eta)$ For $h : C \vdash A \wedge B$, $\langle \pi_{A,B}h, \pi'_{A,B}h\rangle = h$.

$(\top)$ For $f : A \vdash \top$, $\tau_A = f$.

$(\vee\beta)$ For $f : A \vdash C$ and $g : B \vdash C, [f, g]\kappa_{A,B} = f$, $[f, g]\kappa'_{A,B} = g$.

$(\vee\eta)$ For $h : A \vee B \vdash C$, $[h\kappa_{A,B}, h\kappa'_{A,B}] = h$.

$(\bot)$ For $f : \bot \vdash A$, $\iota_A = f$.

With the help of either $(\rightarrow \beta)$ or $(\leftarrow \beta)$ we can derive $(\bullet 1)$: we have

$$\begin{aligned} 1_A \bullet 1_B &= \varepsilon^{\rightarrow}_{A,A\bullet B}(1_A \bullet {}^*1_{A\bullet B})(1_A \bullet 1_B), \text{ with (cat 1) and } (\rightarrow \beta) \\ &= 1_{A\bullet B}, \text{ with } (\bullet), \text{ (cat 1) and } (\rightarrow \beta) \end{aligned}$$

and can proceed analogously with $(\leftarrow \beta)$. However, we have preferred to include $(\bullet 1)$ among the $\bullet$ equations so that we may have it even in the absence of $\rightarrow$ and $\leftarrow$ (see Section 4).

With the definitions

$$D \bullet f =_{\text{df}} 1_D \bullet f$$
$$f \bullet D =_{\text{df}} f \bullet 1_D$$

we can derive the following equations by using (cat 1) and $(\bullet)$:

$(\bullet 2)$ $D \bullet (gf) = (D \bullet g)(D \bullet f)$, $(gf) \bullet D = (g \bullet D)(f \bullet D)$

$(\bullet\text{bifunctor})$ $(f_1 \bullet B_2)(A_1 \bullet f_2) = (B_1 \bullet f_2)(f_1 \bullet A_2)$

Conversely, if the unary operations on arrows $D \bullet \_$ and $\_ \bullet D$ are primitive instead of the binary operation on arrows $\_ \bullet \_$, we can define the latter by either of the following two definitions

$$f_1 \bullet f_2 =_{\text{df}} (f_1 \bullet B_2)(A_1 \bullet f_2)$$
$$f_1 \bullet f_2 =_{\text{df}} (B_1 \bullet f_2)(f_1 \bullet A_2)$$

and derive ($\bullet$) by using ($\bullet$2) and ($\bullet$bifunctor). With these unary operations primitive, ($\bullet$1)is replaced by

$$D \bullet 1_B = 1_{D \bullet B}, \qquad 1_A \bullet D = 1_{A \bullet D}.$$

When it is more convenient to work with $D \bullet \_$ and $\_ \bullet D$ instead of $\_ \bullet \_$, we may freely avail ourselves of this opportunity.

The first six $\sigma\delta$ equations assert that σ and δ are natural isomorphisms. They can be replaced by the four equations

$$\begin{array}{lll} (\mathrm{I}\beta) & \sigma_B(1_\mathrm{I} \bullet f)\sigma^i_A = f, & \delta_B(f \bullet 1_\mathrm{I})\delta^i_A = f \\ (\mathrm{I}\eta) & \sigma^i_B f \sigma_A = 1_\mathrm{I} \bullet f, & \delta^i_B f \delta_A = f \bullet 1_\mathrm{I} \end{array}$$

which are more parallel with the closure and bicartesian equations. The letters 'β' and 'η' in the names of all these equations should point towards the analogy with β and η conversion in the lambda calculus, which are themselves analogous to the conversions of two types of detours in natural deduction: introduction followed by elimination and elimination followed by introduction. However, the $\sigma\delta$ equations as we have given them are more transparent. These equations are assumed by Mac Lane [7, VII.1] for monoidal categories. The difference is that for $\bullet$ in NL categories we need not have a natural associativity isomorphism.

For an NL category $\mathcal{C}$, the operations $\_ \bullet \_$ on objects and arrows determine a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, i.e. a bifunctor, whereas $D \bullet \_$ and $\_ \bullet D$ are functors from $\mathcal{C}$ to $\mathcal{C}$. With the definitions

$$\begin{array}{l} D \rightarrow f =_{\mathrm{df}} {}^*(f\varepsilon_{\overrightarrow{D},A}) \\ f \leftarrow D =_{\mathrm{df}} (f\varepsilon_{\overleftarrow{D},A})^* \end{array}$$

we obtain the functors $D \rightarrow \_$ and $\_ \leftarrow D$ from $\mathcal{C}$ to $\mathcal{C}$. We also have the definitions

$$\begin{array}{ll} \varepsilon_{\overrightarrow{D}}(A) =_{\mathrm{df}} \varepsilon_{\overrightarrow{D},A} & \varepsilon_{\overleftarrow{D}}(A) =_{\mathrm{df}} \varepsilon_{\overleftarrow{D},A} \\ \eta_{\overrightarrow{D}}(A) =_{\mathrm{df}} {}^*1_{D \bullet A} & \eta_{\overleftarrow{D}}(A) =_{\mathrm{df}} 1_{A \bullet D}{}^* \end{array}$$

(conversely, we can define ${}^*f$ as $(D \rightarrow f)\eta_{\overrightarrow{D}}(A)$, and f^* as $(f \leftarrow D)\eta_{\overleftarrow{D}}(A)$).

The $\bullet$ and closure equations amount to asserting that $\_ \bullet \_$ is a bifunctor, that for every object D the functors $D \bullet \_$ and $D \rightarrow \_$ are adjoined, the first being left adjoined and the second right adjoined, with the natural transformations $\varepsilon_{\overrightarrow{D}}$ and $\eta_{\overrightarrow{D}}$ being the counit and unit of the adjunction, and analogously with the functors $\_ \bullet D$

and $\_ \leftarrow D$ and the natural transformations $\varepsilon^{\leftarrow}_D$ and $\eta^{\leftarrow}_D$. All this may be expressed by saying that NL categories are biclosed. The bicartesian equations amount to asserting that NL categories are bicartesian with respect to the operations $\wedge$ and $\vee$, their terminal object being $\top$ and their initial object $\bot$.

The equations $(\rightarrow \eta)$ and $(\leftarrow \eta)$ can be replaced by the equations

$$
\begin{array}{lll}
(^*) & \text{For } f : A \bullet D \vdash B \text{ and } g : C \vdash D, & {}^*(f(1_A \bullet g)) = ({}^*f)g. \\
 & \text{For } f : D \bullet A \vdash B \text{ and } g : C \vdash D, & (f(g \bullet 1_A))^* = (f^*)g. \\
 & & \\
(^*\varepsilon) & {}^*\varepsilon^{\rightarrow}_{A,B} = 1_{A\rightarrow B}, & {\varepsilon^{\leftarrow}_{A,B}}^* = 1_{B\leftarrow A}.
\end{array}
$$

The style of these equations is comparable to the style of the equation (σ) and the second $(\sigma\sigma^i)$ equation (or (δ) and the second $(\delta\delta^i)$ equation): the equations $(^*)$ say how the operations * permute with composition, whereas the equations $(*\varepsilon)$ exhibit the result of eliminating and then reintroducing an implication. Similarly, the equations $(\wedge\eta)$ and $(\vee\eta)$ can be replaced by the equations:

$$
\begin{array}{ll}
\langle fh, gh\rangle = \langle f, g\rangle h, & \langle \pi_{A,B}, \pi'_{A,B}\rangle = 1_{A\wedge B} \\
[hf, hg] = h[f, g], & [\kappa_{A,B}, \kappa'_{A,B}] = 1_{A\vee B}.
\end{array}
$$

The equations $(\top)$ and $(\bot)$ can be replaced by the equations:

$$
\begin{array}{lll}
\text{For } f : A \vdash B, & \tau_A = \tau_B f, & \tau_\top = 1_\top. \\
\text{For } f : B \vdash A, & \iota_A = f\iota_B, & \iota_\bot = 1_\bot.
\end{array}
$$

2 NL□ CATEGORIES

To define the modal NL□ deductive systems and NL□ categories we need the following piece of terminology. First we define inductively the *factors* of an object in a deductive system that has the binary operation $\bullet$ on objects: A is a factor of A; if $B \bullet C$ is a factor of A, then B and C are factors of A. An *atomic factor* of A is a factor of A that has no factors save itself. Let us consider deductive systems that have the binary operation $\bullet$ and a unary operation $\Box$ on objects, and also the special object I. We shall say that an object $\Box A$ in such a deductive system is *boxed.* An object is *modalized* if and only if each of its atomic factors is either boxed or else it is I.

An NL□ deductive system is an NL deductive system that, moreover, has the following:

unary operation on objects: $\Box$

special arrows:

$\mathbf{r}_A : \Box A \vdash A,$	for every object A,
$\mathbf{b}^{\rightarrow}_{A,B,C} : A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C,$	provided A or B or C is modalized,
$\mathbf{b}^{\leftarrow}_{A,B,C} : (A \bullet B) \bullet C \vdash A \bullet (B \bullet C),$	provided A or B or C is modalized,
$\mathbf{c}_{A,B} : A \bullet B \vdash B \bullet A,$	provided A or B is modalized.
$\mathbf{k}_A : A \vdash \mathrm{I},$	provided A is modalized,
$\mathbf{w}_A : A \vdash A \bullet A,$	provided A is modalized,

operation on arrows:

$$\frac{f : B \vdash A}{f^{\Box} : B \vdash \Box A} \qquad \text{provided } B \text{ is modalized.}$$

Note immediately that for modalized objects A we have the arrows $1^{\Box}_A$: $A \vdash \Box A$. These and the $\mathbf{r}$ arrows would enable us to formulate the provisos for the $\mathbf{b}^{\rightarrow}, \mathbf{b}^{\leftarrow}, \mathbf{c}, \mathbf{k}$ and $\mathbf{w}$ arrows by restricting ourselves to boxed objects A, B or C, rather than any modalized objects. However, for technical reasons, it is more convenient to have the provisos in the equivalent form above.

The $\mathbf{b}^{\rightarrow}, \mathbf{b}^{\leftarrow}, \mathbf{c}, \mathbf{k}$ and $\mathbf{w}$ arrows are related to the combinators usually named with the corresponding capital letters. A $\mathbf{b}^{\rightarrow}$ arrow is used to obtain the arrow

$$^{**}(\varepsilon^{\rightarrow}_{A,B}(\varepsilon^{\rightarrow}_{C,A} \bullet 1_{A\rightarrow B})\mathbf{b}^{\rightarrow}_{C,C\rightarrow A,A\rightarrow B}) : A \rightarrow B \vdash (C \rightarrow A) \rightarrow (C \rightarrow B)$$

which corresponds to the functional type of the combinator **B** taking arguments on the left-hand side, whereas $\mathbf{b}^{\leftarrow}$ is used for

$$(\varepsilon^{\leftarrow}_{A,B}(1_{B\leftarrow A} \bullet \varepsilon^{\leftarrow}_{C,A})\mathbf{b}^{\leftarrow}_{B\leftarrow A,A\leftarrow C,C})^{**} : B \leftarrow A \vdash (B \leftarrow C) \leftarrow (A \leftarrow C)$$

which corresponds to the functional type of **B** taking arguments on the right-hand side. This explains the upper indices of $\mathbf{b}^{\rightarrow}$ and $\mathbf{b}^{\leftarrow}$. The $\mathbf{b}^{\rightarrow}$ and $\mathbf{b}^{\leftarrow}$ arrows will be called **b** *arrows*.

The **b**, **c**, **k** and **w** arrows are related to structural rules, too: **b** arrows to association, **c** arrows to permutation, **k** arrows to thinning and **w** arrows to contraction (to give the full force of thinning, **k** arrows have to cooperate with $\sigma\delta$ arrows). So we shall call these arrows *structural arrows*. We have said in the previous section that the $\sigma\delta$ arrows may be taken as structural. Such are also the arrows 1_A, which correspond to the combinator **I**. However, when we say here *structural arrows*, we mean the **b**, **c**, **k** and **w** arrows.

An NL$\Box$ deductive system has modalized forms of the structural rules missing from NL deductive systems; moreover, it has S4 modal postulates. We can replace the operation on arrows $^{\Box}$ by special arrows of types

$$\Box(A \rightarrow B) \vdash \Box A \rightarrow \Box B$$
$$\Box A \vdash \Box\Box A$$

and the restricted form of $^{\Box}$ where B is I; this form of $^{\Box}$ corresponds to the modal rule of necessitation. An alternative is to replace $^{\Box}$ by special arrows of types

$$\Box A \bullet \Box B \vdash \Box(A \bullet B)$$
$$\mathrm{I} \vdash \Box\mathrm{I}$$

and the restricted form of $^{\Box}$ where B is boxed (cf. the derivation of $^{\Box}$ in Section 5). Still another alternative is to replace $^{\Box}$ by special arrows of the last two types plus $\Box A \vdash \Box\Box A$ and the operation on arrows

$$\frac{f : A \vdash B}{\Box f : \Box A \vdash \Box B}$$

An NL$\Box$ category is an NL$\Box$ deductive system that is an NL category in which the following equations between arrows are satisfied:

$\Box$ *equations*:

$(\Box\beta)$ For $f : B \vdash A$, $\quad \mathbf{r}_A f^{\Box} = f$.
$(\Box\eta)$ For $f : B \vdash \Box A$, $\quad (\mathbf{r}_A f)^{\Box} = f$.

b *equations*:

(b) For $f : A \vdash D, g : B \vdash E$ and $h : C \vdash F$ $\quad ((f \bullet g) \bullet h)\mathbf{b}^{\rightarrow}_{A,B,C} = \mathbf{b}^{\rightarrow}_{D,E,F}(f \bullet (g \bullet h))$.

(bb) $\mathbf{b}^{\rightarrow}_{A,B,C}\mathbf{b}^{\leftarrow}_{A,B,C} = 1_{(A \bullet B)\bullet C}$, $\quad \mathbf{b}^{\leftarrow}_{A,B,C}\mathbf{b}^{\rightarrow}_{A,B,C} = 1_{A\bullet(B \bullet C)}$

$(\sigma\delta\mathbf{b})$ $(\delta_A \bullet 1_B)\mathbf{b}^{\rightarrow}_{A,\mathrm{I},B} = 1_A \bullet \sigma_B$

(b5) $\mathbf{b}^{\rightarrow}_{A\bullet B,C,D}\mathbf{b}^{\rightarrow}_{A,B,C\bullet D} = (\mathbf{b}^{\rightarrow}_{A,B,C} \bullet \mathbf{1}_D)\mathbf{b}^{\rightarrow}_{A,B\bullet C,D}(\mathbf{1}_A \bullet \mathbf{b}^{\rightarrow}_{B,C,D})$

c *equations:*

(c) For $f : A \vdash C$ and $g : B \vdash D$, $\quad (g \bullet f)\mathbf{c}_{A,B} = \mathbf{c}_{C,D}(f \bullet g)$.

(cc) $\mathbf{c}_{B,A}\mathbf{c}_{A,B} = \mathbf{1}_{A\bullet B}$

($\sigma\delta$c) $\sigma_A\mathbf{c}_{A,\mathrm{I}} = \delta_A$

(bc6) $\mathbf{b}^{\rightarrow}_{C,A,B}\mathbf{c}_{A\bullet B,C}\mathbf{b}^{\rightarrow}_{A,B,C} = (\mathbf{c}_{A,C} \bullet \mathbf{1}_B)\mathbf{b}^{\rightarrow}_{A,C,B}(\mathbf{1}_A \bullet \mathbf{c}_{B,C})$

k *equations*:

(k) For $f : A \vdash B$, $\quad \mathbf{k}_A = \mathbf{k}_B f$.

(1k) $\mathbf{k}_\mathrm{I} = \mathbf{1}_\mathrm{I}$

w *equations*:

(w) For $f : A \vdash B$, $\quad (f \bullet f)\mathbf{w}_A = \mathbf{w}_B f$.

($\sigma\delta$w) $\sigma_\mathrm{I}\mathbf{w}_\mathrm{I} = \mathbf{1}_\mathrm{I}$

(bw) $\mathbf{b}^{\rightarrow}_{A,A,A}(\mathbf{1}_A \bullet \mathbf{w}_A)\mathbf{w}_A = (\mathbf{w}_A \bullet \mathbf{1}_A)\mathbf{w}_A$

(cw) $\mathbf{c}_{A,A}\mathbf{w}_A = \mathbf{w}_A$

(bcw8) If $\mathbf{c}^m_{A,B,C,D} =_{\mathrm{df}} \mathbf{b}^{\rightarrow}_{A,C,B\bullet D}(\mathbf{1}_A \bullet (\mathbf{b}^{\leftarrow}_{C,B,D}(\mathbf{c}_{B,C} \bullet \mathbf{1}_D)\mathbf{b}^{\rightarrow}_{B,C,D}))\mathbf{b}^{\leftarrow}_{A,B,C\bullet D}$,
$\mathbf{c}^m_{A,B,A,B}\mathbf{w}_{A\bullet B} = \mathbf{w}_A \bullet \mathbf{w}_B$.

(σkw) $\sigma_A(\mathbf{k}_A \bullet \mathbf{1}_A)\mathbf{w}_A = \mathbf{1}_A$ (δkw) $\delta_A(\mathbf{1}_A \bullet \mathbf{k}_A)\mathbf{w}_A = \mathbf{1}_A$

Of course, the arrows f in ($\Box\beta$) and ($\Box\eta$) must have B modalized, and, likewise, the other equations involve arrows with provisos for modalized objects. The equation ($\Box\eta$) can be replaced by the two equations

($^\Box$) $(fg)^\Box = f^\Box g$
($^\Box$r) $\mathbf{r}^\Box_A = \mathbf{1}_{\Box A}$

which is quite parallel to replacing $(\rightarrow\eta)$ and $(\leftarrow\eta)$ by $(^*)$ and $(^*\varepsilon)$; the **r** arrows are analogous to the ε arrows, and the operation on arrows $^\Box$ is analogous to *. For the sake of example, let us derive $(^\Box)$:

$$
\begin{aligned}
& fg = fg \\
& \mathbf{r}_A(fg)^\Box = \mathbf{r}_A f^\Box g, \text{ with } (\Box\beta) \\
& (\mathbf{r}_A(fg)^\Box)^\Box = (\mathbf{r}_A f^\Box g)^\Box \\
& (fg)^\Box = f^\Box g, \text{ with } (\Box\eta).
\end{aligned}
$$

It follows immediately from $(\Box\beta)$, $(^\Box)$ and $(^\Box\mathbf{r})$ that for modalized A the arrow $1_A^\Box : A \vdash \Box A$ is an isomorphism, its inverse being $\mathbf{r}_A$.

If for $f : A \vdash B$ we define $\Box f$ as $(f\mathbf{r}_A)^\Box$ and $\mathbf{t}_A$ as $1_{\Box A}^\Box$, then the functor $\Box\_$, the **r** arrows and the **t** arrows make a comonad (or cotriple). It is easy to check that the **r** and **t** arrows are natural transformations and that we have equations corresponding to the three commutative diagrams of [7, VI.1, p. 135] (**r** corresponds to Mac Lane's ε, and **t** to Mac Lane's δ). However, the **r** and **t** arrows and the operation on arrows $\Box\_$ don't suffice to define our operation on arrows $\_^\Box$. As we have already noted, we need moreover arrows of types $\Box A \bullet \Box B \vdash \Box(A \bullet B)$ and $\mathrm{I} \vdash \Box\mathrm{I}$ (which in the axiomatization of [8, 9.7, pp. 87–90] must be recuperated in a roundabout way, via isomorphisms between $\Box A \bullet \Box B$ and $\Box(A \wedge B)$, and between I and $\Box\top$; in the absence of **k** and **w** arrows, which are recuperated similarly, such an axiomatization becomes impracticable).

The equations **(b)** and **(bb)** can be replaced by the two equations

$$
\begin{aligned}
& \mathbf{b}^{\rightarrow}_{D,E,F}(f \bullet (g \bullet h))\mathbf{b}^{\leftarrow}_{A,B,C} = (f \bullet g) \bullet h \\
& \mathbf{b}^{\leftarrow}_{D,E,F}((f \bullet g) \bullet h)\mathbf{b}^{\rightarrow}_{A,B,C} = f \bullet (g \bullet h)
\end{aligned}
$$

(analogous to $(\mathrm{I}\beta)$ and$(\mathrm{I}\eta)$ from Section 1), but **(b)** and **(bb)** make it more transparent we are dealing with a natural isomorphism. Similarly, **(c)** and **(cc)** can be replaced by

$$\mathbf{c}_{D,C}(g \bullet f)\mathbf{c}_{A,B} = f \bullet g$$

but, again, **(c)** and **(cc)** make it more transparent we are dealing with a natural isomorphism. Of course, from **(b)** and **(bb)** we obtain immediately

$$(f \bullet (g \bullet h))\mathbf{b}^{\leftarrow}_{A,B,C} = \mathbf{b}^{\leftarrow}_{D,E,F}((f \bullet g) \bullet h)$$

and this equation could replace **(b)**.

The equation (**b**) can be replaced by the three equations

$$
\begin{array}{ll}
(\mathbf{b}_1) & ((f \bullet B) \bullet C)\mathbf{b}^{\rightarrow}_{A,B,C} = \mathbf{b}^{\rightarrow}_{D,B,C}(f \bullet (B \bullet C)) \\
(\mathbf{b}_2) & ((A \bullet g) \bullet C)\mathbf{b}^{\rightarrow}_{A,B,C} = \mathbf{b}^{\rightarrow}_{A,E,C}(A \bullet (g \bullet C)) \\
(\mathbf{b}_3) & ((A \bullet B) \bullet h)\mathbf{b}^{\rightarrow}_{A,B,C} = \mathbf{b}^{\rightarrow}_{A,B,F}(A \bullet (B \bullet h))
\end{array}
$$

where $D \bullet f$ is $1_D \bullet f$ and $f \bullet D$ is $f \bullet 1_D$, as we have defined them in Section 1. Similarly, (**c**) can be replaced by either of the two equations

$$
\begin{array}{ll}
(\mathbf{c}_1) & (B \bullet f)\mathbf{c}_{A,B} = \mathbf{c}_{C,B}(f \bullet B) \\
(\mathbf{c}_2) & (g \bullet A)\mathbf{c}_{A,B} = \mathbf{c}_{A,D}(A \bullet g).
\end{array}
$$

It is natural to assume these substitute equations if the unary operations on arrows $D \bullet \_$ and $\_ \bullet D$ are primitive instead of the binary operation on arrows $\_ \bullet \_$, though (**b**) and (**c**) would do as well. The equation (**w**) then reads

$$(f \bullet B)(A \bullet f)\mathbf{w}_A = \mathbf{w}_B f.$$

From $(\sigma\delta\mathbf{b})$, which amounts to

$$\mathbf{b}^{\rightarrow}_{A,\mathrm{I},B} = \delta^i_A \bullet \sigma_B$$

we can derive (without using **c** arrows) the following two analogous equations:

$$
\begin{array}{ll}
(\sigma\mathbf{b}) & (\sigma_A \bullet 1_B)\mathbf{b}^{\rightarrow}_{\mathrm{I},A,B} = \sigma_{A \bullet B} \\
(\delta\mathbf{b}) & \delta_{A \bullet B}\mathbf{b}^{\rightarrow}_{A,B,\mathrm{I}} = 1_A \bullet \delta_B
\end{array}
$$

[7, VII.1, p. 161, Exercise 1]. We call these three equations, and those derived from them with $(\sigma\sigma^i)$, $(\delta\delta^i)$ and (**bb**), *triangular equations*, because Mac Lane assumes $(\sigma\delta\mathbf{b})$ for monoidal categories as a triangular commutative diagram [7, VII.1, p. 159].

The equation (**b5**) is Mac Lane's pentagonal diagram for monoidal categories [7, VII.1, p. 158]. We call this equation, and equations derived from it with (**bb**), *pentagonal equations*.

The equation $(\sigma\delta\mathbf{c})$, assumed by Mac Lane for symmetric monoidal categories [7, VII.7, p. 180], enables us to define the δ and δ^i arrows in terms of the σ and σ^i arrows, or the other way round. With this equation, it is superfluous to assume in Section 1 the equations (δ) and $(\delta\delta^i)$, or (σ) and $(\sigma\sigma^i)$. From $(\sigma\delta\mathbf{c})$ and $(\sigma\sigma^i)$ it follows that $\mathbf{c}_{A,\mathrm{I}} = \sigma^{i}_{A}\delta_{A}$, which with $(\sigma\delta)$ and $(\sigma\sigma^{i})$ yields $\mathbf{c}_{\mathrm{I},\mathrm{I}} = \mathbf{1}_{\mathrm{I}\bullet\mathrm{I}}$. This last equation with $(\sigma\delta\mathbf{c})$ immediately yields $(\sigma\delta)$.

The equation (**bc6**) is Mac Lane's hexagonal diagram for symmetric monoidal categories [7, VII.7, p. 180]. It says, intuitively, that permutation of products like $A \bullet B$ in $\mathbf{c}_{A\bullet B,C}$ can be replaced by permutation of the factors A and B in $\mathbf{c}_{A,C}$ and $\mathbf{c}_{B,C}$. We call (**bc6**), and equations derived from it with (**bb**) and (**cc**), *hexagonal equations*.

If we forget about the provisos for modalized objects, the **b** equations together with the $\bullet$ and $\sigma\delta$ equations of Section 1 axiomatize monoidal categories. If to that we add the **c** equations, we obtain symmetric monoidal categories. The closure equations of Section 1 transform the former into monoidal biclosed categories and the latter into symmetric monoidal closed categories (cf. Section 6).

The **k** equations can be replaced by the single equation:

$$\text{For } f : A \vdash \mathrm{I}, \quad \mathbf{k}_A = f.$$

At the end of Section 1 we have noted something quite analogous concerning the equation $(\top)$. The **k** equations say that I is a terminal object when we restrict ourselves to arrows from modalized objects. Formulating these equations as we did makes clearer the parallelism with the **b**, **c** and **w** equations. The equation (**k**) corresponds to (**b**), (**c**) and (**w**): when written $\mathbf{1}_{\mathrm{I}}\mathbf{k}_A = \mathbf{k}_B f$, it says that **k** is a natural transformation from the identity functor to the constant functor that maps objects into I and arrows into $\mathbf{1}_{\mathrm{I}}$. Of course, **k** need not be an isomorphism as **b** and **c** are. The equation (1**k**) should be compared with

$\mathbf{b}^{\rightarrow}_{A,\mathrm{I},B} = \delta^{i}_{A} \bullet \sigma_B$ and $\mathbf{b}^{\rightarrow}_{\mathrm{I},\mathrm{I},\mathrm{I}} = \sigma^{i}_{\mathrm{I}} \bullet \sigma_{\mathrm{I}}$, which are related to $(\sigma\delta\mathbf{b})$ and $(\sigma\delta)$,

$\mathbf{c}_{A,\mathrm{I}} = \sigma^{i}_{A}\delta_{A}$ and $\mathbf{c}_{\mathrm{I},\mathrm{I}} = \mathbf{1}_{\mathrm{I}\bullet\mathrm{I}}$, which are related to $(\sigma\delta\mathbf{c})$ and $(\sigma\delta)$,

$\mathbf{w}_{\mathrm{I}} = \sigma^{i}_{\mathrm{I}}$ and $\mathbf{w}_{\mathrm{I}} = \delta^{i}_{\mathrm{I}}$, which are related to $(\sigma\delta\mathbf{w})$ and $(\sigma\delta)$,

but it may also be compared with (**bb**) and (**cc**).

The equation (**w**) says that **w** is a natural transformation from the identity functor to the square functor that maps objects A into $A \bullet A$ and arrows f into $f \bullet f$. However, **w** need not be an isomorphism.

The equation $(\sigma\delta\mathbf{w})$ has 'δ' in its name because through $(\sigma\delta)$ it involves δ_{I}. The equations $(\sigma\delta\mathbf{w})$, $(\mathbf{bw})$, $(\mathbf{cw})$, $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$ say, intuitively, how σ, δ, **b**, **c** and **k** arrows behave if they are composed with **w** arrows. Though $\mathbf{c}_{\mathrm{I},\mathrm{I}}$ equals $\mathbf{1}_{\mathrm{I}\bullet\mathrm{I}}$, the arrow $\mathbf{c}_{A,A}$ need not be equal to $\mathbf{1}_{A\bullet A}$; but it behaves like $\mathbf{1}_{A\bullet A}$ if composed with $\mathbf{w}_A$. (We can derive $\mathbf{c}_{\mathrm{I},\mathrm{I}} = \mathbf{1}_{\mathrm{I}\bullet\mathrm{I}}$ from **(cw)**, $(\sigma\delta\mathbf{w})$ and $(\sigma\sigma^i)$, which is different from the derivation we gave above.) Similarly, $\mathbf{b}^{\rightarrow}_{A,A,A}$ need not be equal to $\mathbf{c}_{A,A\bullet A}$, but from **(bw)**, **(cw)** and **(c)** it follows that

$$\mathbf{b}^{\rightarrow}_{A,A,A}(\mathbf{1}_A \bullet \mathbf{w}_A)\mathbf{w}_A = \mathbf{c}_{A,A\bullet A}(\mathbf{1}_A \bullet \mathbf{w}_A)\mathbf{w}_A.$$

From that with $(\sigma\delta\mathbf{w})$ and $(\sigma\sigma^i)$ we obtain, however, $\mathbf{b}^{\rightarrow}_{\mathrm{I},\mathrm{I},\mathrm{I}} = \mathbf{c}_{\mathrm{I},\mathrm{I}\bullet\mathrm{I}}$ (we have said already that $\mathbf{b}^{\rightarrow}_{\mathrm{I},\mathrm{I},\mathrm{I}}$ is equal to $\sigma^i_{\mathrm{I}} \bullet \sigma_{\mathrm{I}}$). The equation **(bw)** is obtained by reversing the arrows of the first diagram for monoids in [7, VII.3, p. 166, diagram (1)].

The arrow

$$\mathbf{c}^m_{A,B,C,D} : (A \bullet B) \bullet (C \bullet D) \vdash (A \bullet C) \bullet (B \bullet D)$$

for which, with **(•)**, **(b)** and **(c)**, we can prove

$$((f \bullet h) \bullet (g \bullet j))\mathbf{c}^m_{A,B,C,D} = \mathbf{c}^m_{F,G,H,J}((f \bullet g) \bullet (h \bullet j))$$

is, like $\mathbf{c}_{A,B}$, a natural isomorphism (the upper index 'm' stands for 'middle'). In the equation **(bcw8)** it enables us, intuitively, to replace contraction of products like $A \bullet B$ in $\mathbf{w}_{A\bullet B}$ by contraction of the factors A and B in $\mathbf{w}_A$ and $\mathbf{w}_B$. In this sense, this equation is analogous to Mac Lane's hexagonal equations. We shall call **(bcw8)**, and equations derived from it with **(bb)** and **(cc)**, *octagonal equations*, because the commutative diagram corresponding to

$$\mathbf{b}^{\rightarrow}_{A,A,B\bullet B}(\mathbf{1}_A \bullet \mathbf{b}^{\leftarrow}_{A,B,B})(\mathbf{1}_A \bullet (\mathbf{c}_{B,A} \bullet \mathbf{1}_B))(\mathbf{1}_A \bullet \mathbf{b}^{\rightarrow}_{B,A,B})\mathbf{b}^{\leftarrow}_{A,B,A\bullet B}\mathbf{w}_{A\bullet B} = (\mathbf{w}_A \bullet \mathbf{1}_{B\bullet B})(\mathbf{1}_A \bullet \mathbf{w}_B)$$

has eight sides. There are two main octagonal equations: **(bcw8)** and

$$\mathbf{c}^m_{A,A,B,B}(\mathbf{w}_A \bullet \mathbf{w}_B) = \mathbf{w}_{A\bullet B}.$$

As **(bw)** is obtained by reversing the arrows of a diagram for monoids, so the equations $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$ are obtained from the remaining diagrams for monoids in [7, VII.3, p. 166, diagram (2)] by the same procedure (which involves replacing σ by σ^i and δ by δ^i). These equations can be replaced by the equations:

$$\text{For} f : A \vdash B, \quad \sigma_B(\mathbf{k}_A \bullet f)\mathbf{w}_A = f, \quad \delta_B(f \bullet \mathbf{k}_A)\mathbf{w}_A = f.$$

If $\mathbf{p}_{A,B} : A \bullet B \vdash A$ and $\mathbf{p}'_{A,B} : A \bullet B \vdash A$ are defined by

$$\mathbf{p}_{A,B} =_{\mathrm{df}} \delta_A(\mathbf{1}_A \bullet \mathbf{k}_B), \quad \mathbf{p}'_{A,B} =_{\mathrm{df}} \sigma_B(\mathbf{k}_A \bullet \mathbf{1}_B)$$

then $(\sigma\mathbf{kw})$ becomes $\mathbf{p}'_{A,A}\mathbf{w}_A = \mathbf{1}_A$ and $(\delta\mathbf{kw})$ becomes $\mathbf{p}_{A,A}\mathbf{w}_A = \mathbf{1}_A$. It follows from $(\mathbf{1k})$ that $\sigma_A = \mathbf{p}'_{\mathrm{I},A}$ and $\delta_A = \mathbf{p}_{A,\mathrm{I}}$ (cf. Section 4). In the presence of **c** arrows, $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$ are not independent: one can be derived from the other.

The equation $(\sigma\delta)$, i.e. $\sigma_\mathrm{I} = \delta_\mathrm{I}$, becomes superfluous in the presence of (**k**) and (1**k**). Likewise, $(\sigma\delta\mathbf{w})$ follows from (1**k**) and $(\sigma\mathbf{kw})$. However, these superfluous equations may be needed even if we dont have **k** arrows (see Section 6), and because of that we keep them as primitive. In the same spirit, we have assumptions for both σ and δ arrows, though these assumptions are not mutually independent in the presence of the **c** arrows, as we have noted above. From the beginning, we also have the superfluous equation $(\bullet 1)$.

3 MODAL FUNCTIONAL COMPLETENESS FOR NL□ CATEGORIES

Given an NL□ category $\mathcal{C}$ and an object A of $\mathcal{C}$, we build the *polynomial* NL□ category $\mathcal{C}[x]$ with an 'indeterminate' arrow $x : \mathrm{I} \vdash A$ by a procedure described in [6, I.5, p. 57]. Namely, we add a new arrow $x : \mathrm{I} \vdash A$ to the underlying graph of $\mathcal{C}$ and then build the NL□ category freely generated by the new graph. That this will succeed is guaranteed by the fact that NL□ categories are equationally presented, i.e. that all our assumptions about arrows are equations. We shall not rehearse here this procedure, which is explained in sufficient detail in [6]. It is also covered by theorems in universal algebra concerning equationally presented algebras with partial operations [5, Section 5, p. 124, corollary to Lemma 3], and [1, Section 7, p. 129, Corollary 1 to Proposition 18]. The name 'polynomial category' is explained by thinking about the arrows of $\mathcal{C}[x]$ as polynomials in x.

We can now state the central result of this paper:

MODAL FUNCTIONAL COMPLETENESS THEOREM.
For every arrow $\varphi : B \vdash C$ of the polynomial NL□ category $\mathcal{C}[x]$ built over the NL□ category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, there is a unique arrow $f : \Box A \bullet B \vdash C$ of $\mathcal{C}$ such that $f(x^\Box \bullet \mathbf{1}_B)\sigma^i_B = \varphi$ holds in $\mathcal{C}[x]$.

We also have the following:

COROLLARY
For every arrow $\varphi : \mathrm{I} \vdash C$ of the polynomial NL□ category $\mathcal{C}[x]$ built over the NL□ category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, there is a unique arrow $g : \Box A \vdash C$ of $\mathcal{C}$ such that $gx^\Box = \varphi$ holds in $\mathcal{C}[x]$.

If B is I in the Modal Functional Completeness Theorem, then in $\mathcal{C}[x]$ we have

$$f(x^{\Box} \bullet 1_{\mathrm{I}})\sigma^{i}_{\mathrm{I}} = \varphi$$
$$f\delta^{i}_{\Box A} x^{\Box} = \varphi, \text{ with } (\sigma\delta), (\delta) \text{ and } (\delta\delta^{i}).$$

So, in the Corollary, we shall take g to be $f\delta^{i}_{\Box A}$. Actually, as with cartesian closed categories in [6, I.6, p. 61], the Corollary entails the Theorem, since the arrows $\varphi : B \vdash C$ of $\mathcal{C}[x]$ are in one-one correspondence with the arrows ${}^{*}(\varphi\delta_B) : \mathrm{I} \vdash B \rightarrow C$ (or $(\varphi\sigma_B)^{*} : \mathrm{I} \vdash C \leftarrow B$). This one–one correspondence also entails that our restriction of x to arrows of type $\mathrm{I} \vdash A$ is surmountable (cf. [6, I.2, p. 52, Exercise 1]). However, the Theorem has a better form than the Corollary for the proof we are going to present. The remainder of this section will be devoted to this proof of the Modal Functional Completeness Theorem.

If φ ranges over arrows of the polynomial NL$\Box$ category $\mathcal{C}[x]$ built over the NL$\Box$ category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, whereas f ranges over the arrows of type $\Box A \bullet B \vdash C$ of $\mathcal{C}$, for some objects B and C, and

$$f\text{`}x =_{\mathrm{df}} f(x^{\Box} \bullet 1_B)\sigma^{i}_{B}$$

then the Modal Functional Completeness Theorem asserts that ‘x is an *onto* and *one-one* function from the f arrows to the φ arrows (from the definition of ‘x it is clear that it is a function). Our proof of the Modal Functional Completeness Theorem (inspired by the proof of functional completeness for cartesian closed categories in [6, I.6]) will proceed by defining a function μ_x from the φ arrows to the f arrows and showing that μ_x is the inverse of ‘x. Applying μ_x to φ is related to applying the functional abstraction operator λx. Syntactically, μ_x binds the variables x that may occur in the polynomial φ. Semantically, it extracts from φ a function that may be applied to x in the sense of ‘: if μ_x is like functional abstraction, ‘x is like application to x. The analogue of μ_x in [6, I.2, I.6] is $\kappa_{x \in A}$, whereas the analogue of ‘x is $\langle x \bigcirc_B, 1_B \rangle$.

We define the function μ_x by the following inductive clauses, which cover all possible forms an arrow of $\mathcal{C}[x]$ may have:

(μ0.1) $\mu_x x = \mathbf{r}_A \delta_{\Box A}$

(μ0.2) For $h : D \vdash E$ an arrow of $\mathcal{C}$,

$$\begin{aligned} \mu_x h &= h\sigma_D(\mathbf{k}_{\Box A} \bullet 1_D) \\ &= h\mathbf{p}'_{\Box A, D}. \end{aligned}$$

(μ1) For $\psi : D \vdash E$ and $\xi : E \vdash F$,

$$\mu_x(\xi\psi) = \mu_x\xi(\mathbf{1}_{\Box A} \bullet \mu_x\psi)\mathbf{b}^{\leftarrow}_{\Box A,\Box A,D}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D).$$

(μ2) For $\psi : D \vdash E$ and $\xi : F \vdash G$,

$$\mu_x(\psi \bullet \xi) = (\mu_x\psi \bullet \mu_x\xi)\mathbf{c}^{m}_{\Box A,\Box A,D,F}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_{D\bullet F}).$$

(μ3.1) For $\psi : E \bullet D \vdash F$,

$$\mu_x({}^*\psi) = {}^*(\mu_x\psi\mathbf{b}^{\leftarrow}_{\Box A,E,D}(\mathbf{c}_{E,\Box A} \bullet \mathbf{1}_D)\mathbf{b}^{\rightarrow}_{E,\Box A,D}).$$

(μ3.2) For $\psi : D \bullet E \vdash F$,

$$\mu_x(\psi^*) = (\mu_x\psi\mathbf{b}^{\leftarrow}_{\Box A,D,D})^*.$$

(μ4) For $\psi : D \vdash E$ and $\xi : D \vdash F$,

$$\mu_x\langle\psi,\xi\rangle = \langle\mu_x\psi,\mu_x\xi\rangle.$$

(μ5) For $\psi : E \vdash D, \xi : F \vdash D$ and $\mathbf{d}_{G,E,F} : G \bullet (E \vee F) \vdash (G \bullet E) \vee (G \bullet F)$ defined by $\mathbf{d}_{G,E,F} =_{\text{df}} \varepsilon^{\rightarrow}_{G,(G\bullet E)\vee(G\bullet F)}(\mathbf{1}_G \bullet [{}^*\kappa_{G\bullet E,G\bullet F}, {}^*\kappa'_{G\bullet E,G\bullet F}])$,

$$\mu_x[\psi,\xi] = [\mu_x\psi,\mu_x\xi]\mathbf{d}_{\Box A,E,F}.$$

(μ6) For $\psi : D \vdash E$ with D modalized,

$$\mu_x(\psi^{\Box}) = (\mu_x\psi)^{\Box}.$$

Let us first deduce that the following equations hold in $\mathcal{C}$:

(μ1.1) For $\psi : D \vdash E$ and $h : E \vdash F$ an arrow of $\mathcal{C}$,

$$\mu_x(h\psi) = h\mu_x\psi.$$

(μ1.2) For $h : D \vdash E$ an arrow of $\mathcal{C}$ and $\xi : E \vdash F$,

$$\mu_x(\xi h) = \mu_x\xi(\mathbf{1}_{\Box A} \bullet h).$$

We could as well have taken ($\mu 1.1$) and ($\mu 1.2$) as clauses in the definition of μ_x, but then we would have to show that they are compatible with ($\mu 1$) and ($\mu 0.2$), and this compatibility is demonstrated by deducing them from ($\mu 1$) and ($\mu 0.2$). For ($\mu 1.1$) we have

$$\begin{aligned}
\mu_x(h\psi) &= h\sigma_E(\mathbf{k}_{\Box A} \bullet \mathbf{1}_E)(\mathbf{1}_{\Box A} \bullet \mu_x\psi)\mathbf{b}^{\leftarrow}_{\Box A,\Box A,D}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D), && \text{with } (\mu 1) \text{ and } (\mu 0.2)\\
&= h\sigma_E(\mathbf{1}_{\mathrm{I}} \bullet \mu_x\psi)(\mathbf{k}_{\Box A} \bullet \mathbf{1}_{\Box A \bullet D})\mathbf{b}^{\leftarrow}_{\Box A,\Box A,D}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D), && \text{with } (\bullet)\\
&= h\mu_x\psi\sigma_{\Box A \bullet D}\mathbf{b}^{\leftarrow}_{\mathrm{I},\Box A,D}((\mathbf{k}_{\Box A} \bullet \mathbf{1}_{\Box A}) \bullet \mathbf{1}_D)(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D), && \text{with } (\sigma) \text{ and } (\mathbf{b})\\
&= h\mu_x\psi((\sigma_{\Box A}(\mathbf{k}_{\Box A} \bullet \mathbf{1}_{\Box A})\mathbf{w}_{\Box A}) \bullet \mathbf{1}_D), && \text{with } (\sigma\mathbf{b}), (\mathbf{bb}) \text{ and } (\bullet).
\end{aligned}$$

Then we apply ($\sigma\mathbf{kw}$) and ($\bullet 1$). For ($\mu 1.2$) we have

$$\begin{aligned}
\mu_x(\xi h) &= \mu_x\xi(\mathbf{1}_{\Box A} \bullet (h\sigma_D(\mathbf{k}_{\Box A} \bullet \mathbf{1}_D)))\mathbf{b}^{\leftarrow}_{\Box A,\Box A,D}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D), && \text{with } (\mu 1) \text{ and } (\mu 0.2)\\
&= \mu_x\xi(\mathbf{1}_{\Box A} \bullet h)(\mathbf{1}_{\Box A} \bullet \sigma_D)\mathbf{b}^{\leftarrow}_{\Box A,\mathrm{I},D}((\mathbf{1}_{\Box A} \bullet \mathbf{k}_{\Box A}) \bullet \mathbf{1}_D)(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D), && \text{with } (\bullet) \text{ and } (\mathbf{b})\\
&= \mu_x\xi(\mathbf{1}_{\Box A} \bullet h)((\delta_{\Box A}(\mathbf{1}_{\Box A} \bullet \mathbf{k}_{\Box A})\mathbf{w}_{\Box A}) \bullet \mathbf{1}_D), && \text{with } (\sigma\delta\mathbf{b}), (\mathbf{bb}) \text{ and } (\bullet).
\end{aligned}$$

It remains to apply ($\delta\mathbf{kw}$) and ($\bullet 1$).

In a rather similar manner, with the help of ($\mu 0.2$), ($\sigma\mathbf{kw}$) and ($\delta\mathbf{kw}$), we derive from ($\mu 2$):

($\mu 2.1$) For $\psi : D \vdash F$,

$$\mu_x(E \bullet \psi) = (E \bullet \mu_x\psi)\mathbf{b}^{\leftarrow}_{E,\Box A,D}(\mathbf{c}_{\Box A,E} \bullet D)\mathbf{b}^{\rightarrow}_{\Box A,E,D}.$$

($\mu 2.2$) For $\psi : D \vdash F$,

$$\mu_x(\psi \bullet E) = (\mu_x\psi \bullet E)\mathbf{b}^{\rightarrow}_{\Box A,D,E}.$$

In these clauses we write $D \bullet f$ for $\mathbf{1}_D \bullet f$ and $f \bullet D$ for $f \bullet \mathbf{1}_D$. If the unary operations on arrows $D \bullet \_$ and $\_ \bullet D$ are primitive instead of the binary operation on arrows $\_ \bullet \_$, then we can derive ($\mu 2$) from ($\mu 2.1$) and ($\mu 2.2$). These substitute clauses, which are often simpler to work with than ($\mu 2$), will also serve for the results of Sections 5 and 6.

Then we have to check that $\mu_x h$ is well-defined for arrows h in $\mathcal{C}$. For example, we must check for $h = ts$, $s : D \vdash E$ and $t : E \vdash F$, that an equation corresponding to clause ($\mu 1$), namely,

$$h\mathbf{p}'_{\Box A,D} = t\mathbf{p}'_{\Box A,E}(\mathbf{1}_{\Box A} \bullet (s\mathbf{p}'_{\Box A,D}))\mathbf{b}^{\leftarrow}_{\Box A,\Box A,D}(\mathbf{w}_{\Box A} \bullet \mathbf{1}_D)$$

holds in $\mathcal{C}$. This amounts to the deduction of $(\mu 1.1)$. We have to check similar equations corresponding to clauses $(\mu 2)$–$(\mu 6)$. We shall not go into the details of this lengthy, but rather straightforward, exercise. However, let us note as a hint that it may be easier to check such equations with $D \bullet \_$ and $\_ \bullet D$, rather than with $\_ \bullet \_$. In that case, instead of $(\mu 2)$ we use $(\mu 2.1)$ and $(\mu 2.2)$. Let us note as another hint that when we check the equation corresponding to $(\mu 5)$, we use the fact that the distribution arrow $\mathbf{d}_{G,E,F}$ is an isomorphism (actually, a natural isomorphism), its inverse being $[1_G \bullet \kappa_{E,F}, 1_G \bullet \kappa'_{E,F}]$.

Next we check that μ_x is indeed a function:

LEMMA 1. *If $\varphi = \psi$ holds in $\mathcal{C}[x]$, then $\mu_x \varphi = \mu_x \psi$ holds in $\mathcal{C}$.*

Proof. From the inductive definition of μ_x it follows immediately that if $\mu_x \varphi = \mu_x \psi$ holds in $\mathcal{C}$, then $\mu_x(\xi\varphi) = \mu_x(\xi\psi)$ and $\mu_x(\varphi\xi) = \mu_x(\psi\xi)$ hold in $\mathcal{C}$. We have analogous implications for the other operations on arrows of NL□ categories.

If φ and ψ are arrows of $\mathcal{C}$ and $\varphi = \psi$ holds in $\mathcal{C}[x]$, then $\varphi = \psi$ holds in $\mathcal{C}$. So, we have in $\mathcal{C}$

$$\varphi \mu_x 1_B = \psi \mu_x 1_B$$
$$\mu_x \varphi = \mu_x \psi, \text{ by } (\mu 1.1) \text{ and (cat 1)}$$

(we could as well have used $(\mu 1.2)$).

It remains to check that for all the equations $\varphi = \psi$ we have assumed for NL□ categories, in which arrows of $\mathcal{C}[x]$ not in $\mathcal{C}$ may occur, $\mu_x \varphi = \mu_x \psi$ holds in $\mathcal{C}$.

The equations of (cat 1) are covered by $(\mu 1.1)$ and $(\mu 1.2)$. For (cat 2) we have

$$\begin{aligned}
&\mu_x(\xi(\psi\varphi)) \\
&= \mu_x\xi(1_{\Box A} \bullet (\mu_x\psi(1_{\Box A} \bullet \mu_x\varphi)\overleftarrow{\mathbf{b}}_{\Box A,\Box A,B}(\mathbf{w}_{\Box A} \bullet 1_B)))\overleftarrow{\mathbf{b}}_{\Box A,\Box A,B}(\mathbf{w}_{\Box A} \bullet 1_B) \\
&= \mu_x\xi(1_{\Box A} \bullet \mu_x\psi)(1_{\Box A} \bullet (1_{\Box A} \bullet \mu_x\varphi))(1_{\Box A} \bullet \overleftarrow{\mathbf{b}}_{\Box A,\Box A,B})\overleftarrow{\mathbf{b}}_{\Box A,\Box A \bullet \Box A,B} \\
&\qquad ((1_{\Box A} \bullet \mathbf{w}_{\Box A}) \bullet 1_B)(\mathbf{w}_{\Box A} \bullet 1_B), \text{ with } (\bullet) \text{ and } (\mathbf{b});
\end{aligned}$$

then we have the pentagonal equation

$$(1_{\Box A} \bullet \overleftarrow{\mathbf{b}}_{\Box A,\Box A,B})\overleftarrow{\mathbf{b}}_{\Box A,\Box A \bullet \Box A,B} = \overleftarrow{\mathbf{b}}_{\Box A,\Box A,\Box A \bullet B}\overleftarrow{\mathbf{b}}_{\Box A \bullet \Box A,\Box A,B}(\overrightarrow{\mathbf{b}}_{\Box A,\Box A,\Box A} \bullet 1_B)$$

which with the help of (**bw**) and (**b**) yields

$$\begin{aligned}
\mu_x(\xi(\psi\varphi)) &= \mu_x\xi(1_{\Box A} \bullet \mu_x\psi)\overleftarrow{\mathbf{b}}_{\Box A,\Box A,C}(1_{\Box A \bullet \Box A} \bullet \mu_x\varphi)(\mathbf{w}_{\Box A} \bullet 1_{\Box A \bullet B}) \\
&\qquad \overleftarrow{\mathbf{b}}_{\Box A,\Box A,B}(\mathbf{w}_{\Box A} \bullet 1_B) \\
&= \mu_x((\xi\psi)\varphi), \text{ with } (\bullet).
\end{aligned}$$

For ($\bullet$) it is again easier (and more instructive for what we do in Sections 5 and 6) to work with $D \bullet \_$ and $\_ \bullet D$ instead of $\_ \bullet \_$, i.e. with (μ2.1) and (μ2.2) instead of (μ2), and check ($\bullet$2) and ($\bullet$ bifunctor). For the first equation of ($\bullet$2) we need pentagonal and hexagonal equations, whereas for the second we need only a pentagonal equation. For ($\bullet$ bifunctor) we need (**cw**) besides pentagonal and hexagonal equations in order to check that

$$\mathbf{c}^m_{\Box A,A_1,\Box A,A_2}\mathbf{b}^{\leftarrow}_{\Box A\bullet A_1,\Box A,A_2}(\mathbf{c}_{\Box A,\Box A\bullet A_1}\bullet A_2)\mathbf{b}^{\rightarrow}_{\Box A,\Box A\bullet A_1 A_2}(\Box A\bullet \mathbf{b}^{\rightarrow}_{\Box A,A_1,A_2})$$
$$\mathbf{b}^{\leftarrow}_{\Box A,\Box A,A_1\bullet A_2}(\mathbf{w}_{\Box A}\bullet(A_1\bullet A_2)) = \mathbf{w}_{\Box A}\bullet(A_1\bullet A_2).$$

For (σ) and (δ) we apply (μ1.1) and (μ1.2). Among the remaining equations for NL categories that need checking, namely, the closure and bicartesian equations, let us consider ($\bot$).

For $\varphi : \bot \vdash B$, we have to show $\mu_x \iota_B = \mu_x \varphi$. With ($\bot$) we have

$$\begin{aligned}
{}^*(\mu_x \iota_B) &= \iota_{\Box A\rightarrow B}\\
{}^*(\mu_x \iota_B) &= {}^*(\mu_x \varphi)\\
\varepsilon^{\rightarrow}_{\Box A,\Box A\rightarrow B}(1_{\Box A}\bullet {}^*(\mu_x \iota_B)) &= \varepsilon^{\rightarrow}_{\Box A,\Box A\rightarrow B}(1_{\Box A}\bullet {}^*(\mu_x \varphi))\\
\mu_x \iota_B &= \mu_x \varphi, \text{ with } (\rightarrow \beta).
\end{aligned}$$

A similar trick involving * is applied when we show that the distribution arrow **d** is an isomorphism, and this we need when we check the lemma for ($\vee\eta$).

It remains to check ($\Box\beta$), ($\Box\eta$), (**b**), (**c**), (**k**) and (**w**). The cases with ($\Box\beta$) and ($\Box\eta$) follow readily by applying (μ1.1) and (μ6). For the case with (**b**) it is easier and more instructive to break the checking into checking ($\mathbf{b}_1$), ($\mathbf{b}_2$) and ($\mathbf{b}_3$) with the clauses (μ2.1) and (μ2.2). Then for ($\mathbf{b}_1$) we need the pentagonal equation (**b5**) (see the justification of pentagonal equations in Section 5). For ($\mathbf{b}_2$) we use two pentagonal equations, and for ($\mathbf{b}_3$) one hexagonal equation besides three pentagonal equations. To check ($\mathbf{c}_1$) or ($\mathbf{c}_2$) we need a hexagonal equation (see the justification of hexagonal equations in Section 5). For (**k**) we just use the fact that the object I is terminal for arrows from modalized objects. Finally, for (**w**) we have

$$\begin{aligned}
\mu_x((\varphi\bullet\varphi)\mathbf{w}_B &= (\mu_x\varphi\bullet\mu_x\varphi)\mathbf{c}^m_{\Box A\Box A,B,B}(\mathbf{w}_{\Box A}\bullet\mathbf{w}_B),\\
&\qquad \text{with } (\mu 2) \text{ and } (\mu 1.2)\\
&= (\mu_x\varphi\bullet\mu_x\varphi)\mathbf{w}_{\Box A\bullet B}, \text{ with an octagonal equation}\\
&= \mu_x(\mathbf{w}_C\varphi), \text{ with } (\mathbf{w}) \text{ and } (\mu 1.1).
\end{aligned}$$

■

The following two lemmata assert that μ_x is the inverse of ${}^{\lq}x$. As μ_x corresponds to functional abstraction and ${}^{\lq}x$ to application, so Lemma 2 corresponds to

β-conversion and Lemma 3 to η-conversion.

LEMMA 2. *For* $\varphi : B \vdash C$ *an arrow of* $\mathcal{C}[x]$, *the equation* $(\mu_x\varphi)`x = \varphi$ *holds in* $\mathcal{C}[x]$.

Proof. We proceed by induction on the complexity of φ. For the basis we have:

(0.1) If φ is x, then

$$\begin{aligned}(\mu_x x)`x &= \mathbf{r}_A\delta_{\Box A}(x^{\Box} \bullet 1_{\mathrm{I}})\sigma^i_{\mathrm{I}}, \text{ by definition}\\ &= \mathbf{r}_A x^{\Box}\delta_{\mathrm{I}}\sigma^i_{\mathrm{I}}, \text{ with } (\delta)\\ &= x, \text{ with } (\Box\beta), (\sigma\delta) \text{ and } (\sigma\sigma^i)\end{aligned}$$

(0.2) If φ is an arrow h of $\mathcal{C}$, then, by definition,

$$(\mu_x h)`x = h\sigma_B(\mathbf{k}_{\Box A} \bullet 1_B)(x^{\Box} \bullet 1_B)\sigma^i_B$$

and we use $(\bullet)$, $\mathbf{k}_{\Box A}x^{\Box} = 1_{\mathrm{I}}$ and $(\sigma\sigma^i)$ to show that the right-hand side is equal to h.

In the induction step we shall only consider the following case as an example:

(1) If φ is $\xi\psi$, with $\psi : B \vdash E$ and $\xi : E \vdash C$, then by the induction hypothesis we have $(\mu_x\psi)`x = \psi$ and $(\mu_x\xi)`x = \xi$. We need to prove $(\mu_x(\xi\psi))`x = \xi\psi$, which amounts to $\mu_x\xi(1_{\Box A}\bullet\mu_x\psi)\overleftarrow{\mathbf{b}}_{\Box A,\Box A,B}(\mathbf{w}_{\Box A}\bullet 1_B)(x^{\Box}\bullet 1_B)\sigma^i_B = \mu_x\xi(x^{\Box} \bullet 1_E)\sigma^i_E\mu_x\psi(x^{\Box} \bullet 1_B)\sigma^i_B$. For the left-hand side (lhs) of this equation we have

$$\begin{aligned}\text{lhs} &= \mu_x\xi(1_{\Box A} \bullet \mu_x\psi)\overleftarrow{\mathbf{b}}_{\Box A,\Box A,B}((x^{\Box} \bullet x^{\Box}) \bullet 1_B)(\mathbf{w}_{\mathrm{I}} \bullet 1_B)\sigma^i_B,\\ &\qquad\qquad \text{with } (\bullet) \text{ and } (\mathbf{w})\\ &= \mu_x\xi(1_{\Box A} \bullet \mu_x\psi)(x^{\Box} \bullet (x^{\Box} \bullet 1_B))\overleftarrow{\mathbf{b}}_{\mathrm{I,I},B}(\mathbf{w}_{\mathrm{I}} \bullet 1_B)\sigma^i_B, \text{ with } (\mathbf{b})\\ &= \mu_x\xi(x^{\Box} \bullet 1_E)(1_{\mathrm{I}} \bullet (\mu_x\psi(x^{\Box} \bullet 1_B)))\overleftarrow{\mathbf{b}}_{\mathrm{I,I},B}(\sigma^i_{\mathrm{I}} \bullet 1_B)\sigma^i_B,\\ &\qquad\qquad \text{with } (\bullet), (\sigma\delta\mathbf{w}) \text{ and } (\sigma\sigma^i)\end{aligned}$$

which, with (σ) and a triangular equation derived from $(\sigma\mathbf{b})$ with $(\sigma\sigma^i)$ and $(\mathbf{bb})$, is equal to the right- hand side.

As a lengthy exercise, it remains to check cases corresponding to clauses $(\mu2)$–$(\mu6)$ (for ease, it may be preferable to work with $(\mu2.1)$ and $(\mu2.2)$ instead of $(\mu2)$; see the justification of $(\sigma\mathbf{b})$ and $(\sigma\delta\mathbf{b})$ in Section 5). ∎

LEMMA 3. *For* $f : \Box A \bullet B \vdash C$ *an arrow of* $\mathcal{C}$, *the equation* $\mu_x(f`x) = f$ *holds in* $\mathcal{C}$.

Proof. By using (μ1.1), (μ1.2), (μ2.2), (μ6) and (μ0.1) we obtain

$$\begin{aligned}\mu_x(f`x) &= f((\mathbf{r}_A\delta_{\Box A})^{\Box} \bullet 1_B)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I},B}(1_{\Box A} \bullet \sigma^i_B)\\ &= f(\delta_{\Box A} \bullet 1_B)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I},B}(1_{\Box A} \bullet \sigma^i_B), \text{ with } (\Box\eta);\end{aligned}$$

with the triangular equation ($\sigma\delta$b), ($\bullet$) and ($\sigma\sigma^i$), the right-hand side is equal to f. ■

As a corollary of Lemma 2 we can easily obtain a more general statement; namely, for every arrow $a : \mathrm{I} \vdash A$ of $\mathcal{C}[x]$ the equation $(\mu_x\varphi)`a = \varphi^x_a$ holds in $\mathcal{C}[x]$. In this equation, which corresponds exactly to $\beta$-conversion, $f`a$ is defined analogously to $f`x$, and φ^x_a is obtained from φ by uniformly substituting a for x. We shall not define this substitution with more precision since we don't need the corollary here. Lemma 2 suffices; i.e., Lemmata 1–3 immediately give the Modal Functional Completeness Theorem.

4 FUNCTIONAL COMPLETENESS FOR CARTESIAN, CARTESIAN CLOSED AND BICARTESIAN CLOSED CATEGORIES

An NL□ category in which every object A is isomorphic with $\Box A$ is a cartesian category with respect to $\bullet$ and I, a cartesian closed category with respect to $\bullet$, $\rightarrow$ and I, and a bicartesian closed category with respect to $\bullet$, $\rightarrow$, I, $\vee$ and $\perp$. From our axiomatization of NL□ categories in Sections 1 and 2 one can easily extract non-standard axiomatizations of these sorts of category by selecting the assumptions tied with the mentioned operations and objects (of course, we always assume that we have the 1_A arrows, composition, (cat 1) and (cat 2); i.e. that we are in a category).

As a matter of fact, to axiomatize bicartesian closed categories in such a non-standard way, all we have to do is forget about □ in Section 2, without making any selection among our assumptions. Namely, let us assume whatever is assumed in Section 1 for NL categories, and let us, moreover, assume the structural arrows **b**, **c**, **k** and **w** *without* provisos concerning modalized objects. We forget about the **r** arrows and the operation on arrows $^{\Box}$: the arrow $\mathbf{r}_A$ may be identified with 1_A and $f^{\Box}$ is just f. For the structural arrows without provisos we assume the equations of Section 2.

Remember that we have defined the **p** and **p′** arrows at the end of Section 2 by

$$\mathbf{p}_{A,B} =_{\mathrm{df}} \delta_A(1_A \bullet \mathbf{k}_B), \qquad \mathbf{p}'_{A,B} =_{\mathrm{df}} \sigma_B(\mathbf{k}_A \bullet 1_B).$$

In the same spirit, we define the binary operation on arrows

$$\frac{f : C \vdash A \quad g : C \vdash B}{\{f, g\} : C \vdash A \bullet B}$$

by

$$\{f, g\} =_{\mathrm{df}} (f \bullet g)\mathbf{w}_C.$$

Then we can show first that $\langle \mathbf{p}_{A,B}, \mathbf{p}'_{a,B}\rangle$ is a natural isomorphism from $A \bullet B$ to $A \wedge B$, its inverse being $\{\pi_{A,B}, \pi'_{A,B}\}$. The essential step in checking

$$\{\pi_{A,B}, \pi'_{A,B}\}\langle \mathbf{p}_{A,B}, \mathbf{p}'_{A,B}\rangle = \mathbf{1}_{A\bullet B}$$

is to show $\{\mathbf{p}_{A,B}, \mathbf{p}'_{A,B}\} = \mathbf{1}_{A\bullet B}$, i.e.

$$(\sigma\delta\mathbf{kw}) \quad ((\delta_A(\mathbf{1}_A \bullet \mathbf{k}_B)) \bullet (\sigma_B(\mathbf{k}_A \bullet \mathbf{1}_B)))\mathbf{w}_{A\bullet B} = \mathbf{1}_{A\bullet B}.$$

In this equation is hidden the octagonal principle: with an octagonal equation, the left-hand side is equal to

$$(\delta_A \bullet \sigma_B)\mathbf{c}^m_{A,\mathrm{I},\mathrm{I},B}((\mathbf{1}_A \bullet \mathbf{k}_A) \bullet (\mathbf{k}_B \bullet \mathbf{1}_B))(\mathbf{w}_A \bullet \mathbf{w}_B)$$

which with $\mathbf{c}^m_{A,\mathrm{I},\mathrm{I},B} = \mathbf{1}_{(A\bullet \mathrm{I})\bullet(\mathrm{I}\bullet B)}$ (an equation related to $\mathbf{c}_{\mathrm{I},\mathrm{I}} = \mathbf{1}_{\mathrm{I}\bullet\mathrm{I}}$, mentioned in Section 2), $(\delta\mathbf{kw})$, $(\sigma\mathbf{kw})$, $(\bullet)$ and $(\bullet 1)$ is equal to $\mathbf{1}_{A\bullet B}$.

We can also prove equations that correspond exactly to $(\wedge\beta)$ and $(\wedge\eta)$:

$$\begin{array}{lll} (\bullet\beta) & \mathbf{p}_{A,B}\{f, g\} = f, & \mathbf{p}'_{A,B}\{f, g\} = g \\ (\bullet\eta) & \text{For } h : C \vdash A \bullet B, & \{\mathbf{p}_{A,B}h, \mathbf{p}'_{A,B}h\} = h. \end{array}$$

For $(\bullet\eta)$ we use $(\sigma\delta\mathbf{kw})$. Note that, with $(\bullet)$, the equation $(\sigma\delta\mathbf{kw})$ yields also

$$\{f\mathbf{p}_{A,B}, g\mathbf{p}'_{A,B}\} = f \bullet g.$$

To show all that we don't need $\rightarrow, \leftarrow, \top, \vee, \bot$ and the assumptions tied with them. The equation last displayed permits us to give $(\leftarrow \beta)$ and $(\leftarrow \eta)$ the form they have in [6, I.3, p. 53, E4a, E4b]. Of course, we can analogously rewrite $(\rightarrow\ \beta)$ and $(\rightarrow\ \eta)$.

Next we show that τ_{I} is a natural isomorphism from I to $\top$, its inverse being $\mathbf{k}_{\top}$. For that we need only $(\top)$ and the $\mathbf{k}$ equations.

Finally, we show that we can keep $\rightarrow$ and reject $\leftarrow$; actually, it does not matter which one of the two implications we keep while rejecting the other. This follows from the fact that $(\varepsilon^{\rightarrow}_{a,B}\mathbf{c}_{A\rightarrow B,A})^{*}$ is a natural isomorphism from $A \rightarrow B$ to $B \leftarrow A$, its inverse being ${}^{*}(\varepsilon^{\leftarrow}_{A,B}\mathbf{c}_{A,B\leftarrow A})$. To demonstrate that, we need only $(\bullet)$, closure equations, **(c)** and **(cc)**.

So we have indeed an axiomatization of cartesian closed categories, which, given that we also have the assumptions for $\vee$ and $\perp$, amounts to an axiomatization of bicartesian closed categories. We can forget in this nonstandard axiomatization about $\leftarrow$, $\wedge$, $\top$ and the assumptions tied with them.

Our proof of the Modal Functional Completeness Theorem then yields an alternative proof of ordinary, nonmodal, functional completeness for bicartesian closed categories. We only have to forget about $\Box$. That means that in the statement of the Functional Completeness Theorem we shall have $f : A \bullet B \vdash C$ and $f(x \bullet 1_B)\sigma^i_B = \varphi$, i.e. $f`x$ is $f(x \bullet 1_B)\sigma^i_B$ (cf. the Substructural Functional Completeness Theorem in Section 6). In the definition of μ_x, in clause $(\mu 0.1)$ we shall have

$$\begin{aligned} \mu_x x &= 1_A \delta_A \\ &= \delta_A \end{aligned}$$

whereas in the other clauses we just delete $\Box$ wherever it occurs (clause $(\mu 6)$ is omitted). Then it may be checked that our new μ_x may be identified with $\kappa_{x \in A}$ of [6, I.2, p. 51]. For example, since

$$\kappa_{x\in A}(\xi\psi) = \kappa_{x\in A}\xi\langle \pi_{A,D}, \kappa_{x\in A}\psi\rangle$$

the connection with clause $(\mu 1)$ in the new definition of μ_x is achieved by verifying

$$(1_A \bullet \mu_x\psi)\mathbf{b}^{\leftarrow}_{A,A,D}(\mathbf{w}_A \bullet 1_D) = \{\mathbf{p}_{A,D}, \mu_x\psi\}$$

which is done with the help of an octagonal equation.

It should be clear that we also have an alternative proof of ordinary functional completeness for cartesian and cartesian closed categories. As we have already remarked, from our nonstandard axiomatization of bicartesian closed categories we obtain a nonstandard axiomatization of cartesian closed categories by rejecting the assumptions tied with the bicartesian equations. For cartesian categories we reject moreover the assumptions tied with the closure equations. In our nonstandard axiomatization of bicartesian closed categories, the essentially nonstandard part is the cartesian part, which is given by the operation $\bullet$ on objects, the object I, the arrows 1_A, the $\sigma\delta$ arrows, the structural arrows **b**, **c**, **k** and **w** without provisos for modalized objects, the operations composition and $\bullet$ on arrows, and the equations between arrows: (cat 1), (cat 2), the $\bullet$ equations, the $\sigma\delta$ equations and the **b**, **c**, **k** and **w** equations.

In the same style, we can make as nonstandard the axiomatization of the part involving the coproduct $\vee$ and the initial object $\bot$ in bicartesian categories. Namely, we would have special arrows analogous to the $\sigma\delta$, **b** and **c** arrows in which $\bullet$ is replaced everywhere by $\vee$ and I by $\bot$. Instead of **k** and **w** arrows we would have $\iota_A : \bot \vdash A$ and $\mathbf{w}^{\vee}_A : A \vee A \vdash A$. We would assume for these arrows equations analogous to the $\bullet$ equations, the $\sigma\delta$ equations and the **b**, **c**, **k** and **w** equations. In the equations involving $\mathbf{w}^{\vee}$ and ι the order would have to be reversed and the analogues of σ and δ replaced by the analogues of σ^i and δ^i (the equations corresponding to the **k** equations would state that $\bot$ is an initial object; we have written down these equations at the very end of Section 1).

In our nonstandard axiomatization of cartesian categories there are redundancies among the special arrows (there are also redundancies among the equations, as we have noted at the end of Section 2; however, ($\bullet$1) is not redundant now: we use it to derive ($\sigma\delta\mathbf{kw}$)). Actually, since the definition of **p** is given in terms of δ and **k**, the definition of $\mathbf{p}'$ in terms of σ and **k**, and the definition of the curly brackets in terms of **w**, we can keep only the special arrows $1_A, \sigma, \delta, \mathbf{k}$ and **w**, and define all the others with $\mathbf{p}, \mathbf{p}'$ and the curly brackets, as this is done with the standard axiomatization of cartesian categories. Namely, we would have

$$\sigma^i_A =_{\text{df}} \{\mathbf{k}_A, 1_A\}, \quad \delta^i_A =_{\text{df}} \{1_A, \mathbf{k}_A\}$$
$$\mathbf{b}^{\rightarrow}_{A,B,C} =_{\text{df}} \{\{\mathbf{p}_{A,B\bullet C}, \mathbf{p}_{B,C}\mathbf{p}'_{A,B\bullet C}\}, \mathbf{p}'_{B,C}\mathbf{p}'_{A,B\bullet C}\}$$
$$\mathbf{b}^{\leftarrow}_{A,B,C} =_{\text{df}} \{\mathbf{p}_{A,B}\mathbf{p}_{A\bullet B,C}, \{\mathbf{p}'_{A,B}\mathbf{p}_{A\bullet B,C}, \mathbf{p}'_{A\bullet B,C}\}\}$$
$$\mathbf{c}_{A,B} =_{\text{df}} \{\mathbf{p}'_{A,B}, \mathbf{p}_{A,B}\}.$$

The equations behind these definitions all hold in our nonstandard axiomatization of cartesian categories (for the first two we use ($\sigma\mathbf{kw}$) and ($\delta\mathbf{kw}$), whereas for the last three we use octagonal equations). Note that the last definition says that in Gentzen's sequent systems we can derive permutation from thinning and contraction, provided we are allowed to contract sequences of formulae, rather than single formulae only. Here is a permutation obtained by two thinnings followed by a contraction:

$$\frac{\frac{\frac{B, A \vdash C}{A, B, A \vdash C}}{A, B, A, B \vdash C}}{A, B \vdash C}$$

To obtain cartesian categories with this reduced stock of special arrows we have to assume (cat 1), (cat 2), $(\bullet)$, (σ), (δ), $(\mathbf{k})$, $(1\mathbf{k})$, $(\mathbf{w})$, $(\sigma\mathbf{kw})$, $(\delta\mathbf{kw})$ and $(\sigma\delta\mathbf{kw})$. The octagonal equations enter into this axiomatization via $(\sigma\delta\mathbf{kw})$. By the way, the $\mathbf{c}^m$ arrows have the following symmetrical definition in terms of $\mathbf{p}, \mathbf{p}'$ and the curly brackets:

$$\mathbf{c}^m_{A,B,C,D} =_{\mathrm{df}}$$
$$\{\{\mathbf{p}_{A,B}\mathbf{p}_{A\bullet B,C\bullet D}, \mathbf{p}_{C,D}\mathbf{p}'_{A\bullet B,C\bullet D}\}, \{\mathbf{p}'_{A,B}\mathbf{p}_{A\bullet B,C\bullet D}, \mathbf{p}'_{C,D}\mathbf{p}'_{A\bullet B,C\bullet D}\}\}.$$

We cannot economize similarly on $\sigma^i, \delta^i, \mathbf{b}$ and $\mathbf{c}$ arrows with the restricted, modalized, versions of the arrows $\mathbf{k}$ and $\mathbf{w}$ in Section 2. With the definitions of $\sigma^i, \delta^i, \mathbf{b}$ and $\mathbf{c}$ given above, σ^i_A and δ^i_A would be lacking if A is not modalized, whereas $\mathbf{b}^{\rightarrow}_{A,B,C}$, $\mathbf{b}^{\leftarrow}_{A,B,C}$ and $\mathbf{c}_{A,B}$ would be lacking if A, B and C are not *all* modalized. For example, $\mathbf{w}_{A\bullet B}$ hidden in the curly brackets of $\{\mathbf{p}'_{A,B}, \mathbf{p}_{A,B}\}$ is not available if A and B are not *both* modalized, whereas in Sections 2 and 3 we need $\mathbf{c}_{A,B}$ even in cases where only A or only B is modalized, and similarly with $\mathbf{b}$ arrows (the arrows σ^i_B with B not necessarily modalized are involved in the formulation of the Modal Functional Completeness Theorem).

Another axiomatization of cartesian categories may be obtained by taking the special arrows $\mathbf{p}$ or $\mathbf{p}'$ as primitive instead of $\mathbf{k}$. With $\mathbf{p}$ primitive, we define $\mathbf{k}_A$ as $\mathbf{p}_{\mathrm{I},A}\sigma^i_A$ and $\mathbf{p}'_{A,B}$ as $\mathbf{p}_{B,A}\mathbf{c}_{A,B}$. A further possibility is to define δ_A as $\mathbf{p}_{A,\mathrm{I}}$ and σ_A as $\mathbf{p}_{A,\mathrm{I}}\mathbf{c}_{\mathrm{I},A}$. We shall not investigate here what reshuffling of our equations this change of primitives would require. Let us only note that in Section 2 we may have taken $\mathbf{p}_{A,B}$ as primitive instead of $\mathbf{k}_A$ *provided* B is modalized. Still another possibility is to take the curly brackets operation on arrows as primitive instead of the $\mathbf{w}$ arrows and define $\mathbf{w}_A$ as $\{1_A, 1_A\}$. In Section 2, we would have to require that with $f : C \vdash A$ and $g : C \vdash B$ we have $\{f, g\}$ only if C is modalized. However, as we have explained in the previous paragraph, having $\mathbf{p}$ and these restricted curly brackets primitive would not permit us to economize on $\mathbf{b}$ and $\mathbf{c}$ arrows in Section 2.

5 NECESSITY OF ASSUMPTIONS FOR NL$\Box$ CATEGORIES

We have seen that what we have assumed for NL$\Box$ categories is sufficient to demonstrate modal functional completeness. We want now to address the question whether these assumptions are also necessary. This is not a question we can answer in an absolute sense, but only relatively to some presuppositions. These presuppositions are contained in the particular notion of category without functional completeness that we extend to obtain a notion of category with functional completeness, as our notion of NL category was extended to the notion of NL$\Box$ category. But, foremost, the exact form of the functional completeness theorem carries presuppositions about what are polynomial arrows φ and about the type of the arrow f (the arrow f is of type $\Box A \bullet B \vdash C$ rather than $B \bullet \Box A \vdash C$, or some other type). Moreover, we require that a particular function from the f arrows to the φ arrows be *onto* and *one–one*. We shall see in the next section that we may understand functional completeness in different ways by restricting the notion of polynomial (still another way to restrict functional completeness is briefly mentioned in the concluding section).

For the time being we assume we have NL categories, axiomatized as in Section 1, and we shall try to see to what extent the additional assumptions for NL$\Box$ categories, in Section 2, are necessary for proving modal functional completeness as this is done in Section 3. After that we shall try to see whether at least some assumptions about NL categories in Section 1 are necessary in the same sense.

That in NL$\Box$ categories we must have the special arrows **r**, the modalized structural arrows **b**, **c**, **k** and **w**, and the operation on arrows $^{\Box}$, follows, as explained in [3], from the *Modal Deduction Theorem* and its converse, which are both consequences of the Modal Functional Completeness Theorem. The Modal Deduction Theorem says:

($\Box\downarrow$) For every arrow $\varphi : B \vdash C$ of the polynomial NL$\Box$ category $\mathcal{C}[x]$ built over the NL$\Box$ category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, there is an arrow $f : \Box A \bullet B \vdash C$ of $\mathcal{C}$.

It is clear that to prove ($\Box\downarrow$) it is enough to take for f the arrow $\mu_x \varphi$. The converse of the Modal Deduction Theorem says:

($\Box\uparrow$) For every arrow $f : \Box A \bullet B \vdash C$ of the NL$\Box$ category $\mathcal{C}$, there is an arrow $\varphi : B \vdash C$ of the polynomial NL$\Box$ category $\mathcal{C}[x]$ built over $\mathcal{C}$ with $x : \mathrm{I} \vdash A$.

It is clear that to prove ($\Box\uparrow$) it is enough to take for φ the arrow f‘x. It is also clear that though we may infer $(\Box\downarrow)$ and $(\Box\uparrow)$ from the Modal Functional Completeness Theorem, this theorem does not follow from $(\Box\downarrow)$ and $(\Box\uparrow)$ alone. In $(\Box\downarrow)$ and $(\Box\uparrow)$ nothing is said about μ_x and ${}^{\text{`}}x$, and their functional character.

Although this is something proven in [2, 3], we shall show again in more detail, and in a way adapted to the present context, how $(\Box\downarrow)$ and $(\Box\uparrow)$ deliver the special modal arrows and the operation on arrows ${}^{\Box}$ of NL$\Box$ categories. For that it will be useful to have the abbreviations:

$$\text{For } g : C \vdash A \rightarrow B, \quad {}^{\circ}g =_{\text{df}} \varepsilon^{\rightarrow}_{A,B}(1_A \bullet g).$$
$$\text{For } g : C \vdash A \leftarrow B, \quad g^{\circ} =_{\text{df}} \varepsilon^{\leftarrow}_{A,B}(1_A \bullet g).$$

r) Since in $\mathcal{C}[x]$ we have $x : \mathrm{I} \vdash A$, by $(\Box\downarrow)$ we must have $f : \Box A \bullet \mathrm{I} \vdash A$ in $\mathcal{C}$, and we take $f\delta^{i}_{\Box A}$ to be $\mathbf{r}_A$.

b) Since in $\mathcal{C}$ we have

$$1_{(\Box A \bullet B)\bullet C}{}^{*} : \Box A \bullet B \vdash ((\Box A \bullet B) \bullet C) \leftarrow C$$

by $(\Box\uparrow)$ we must have in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash A$,

$$\varphi : B \vdash ((\Box A \bullet B) \bullet C) \leftarrow C$$

and hence also $\varphi^{\circ} : B \bullet C \vdash (\Box A \bullet B) \bullet C$. Then, by $(\Box\downarrow)$, we must have in $\mathcal{C}$

$$f : \Box A \bullet (B \bullet C) \vdash (\Box A \bullet B) \bullet C$$

and we take this f to be $\mathbf{b}^{\rightarrow}_{\Box A,B,C}$.

Since in $\mathcal{C}$ we have

$${}^{*}(1_{(A\bullet\Box B)\bullet C}{}^{*})\delta_{\Box B} : \Box B \bullet \mathrm{I} \vdash A \rightarrow (((A \bullet \Box B) \bullet C) \leftarrow C)$$

by $(\Box\uparrow)$ we must have in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash B$,

$$\varphi : \mathrm{I} \vdash A \rightarrow (((A \bullet \Box B) \bullet C) \leftarrow C)$$

and hence also ${}^{*}(({}^{\circ}\varphi\delta^{i}_{A})^{\circ}) : C \vdash A \rightarrow ((A \bullet \Box B) \bullet C)$. Then, by $(\Box\downarrow)$, we must have in $\mathcal{C}$

$$f : \Box B \bullet C \vdash A \rightarrow ((A \bullet \Box B) \bullet C)$$

and we take ${}^{\circ}f$ to be $\mathbf{b}^{\rightarrow}_{A,\Box B,C}$.

Since in $\mathcal{C}$ we have

$$^{*}1_{(A\bullet B)\bullet\Box C}\delta_{\Box C} : \Box C \bullet \mathrm{I} \vdash (A \bullet B) \rightarrow ((A \bullet B) \bullet \Box C)$$

by ($\Box\uparrow$) we must have in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash C$,

$$\varphi : \mathrm{I} \vdash (A \bullet B) \rightarrow ((A \bullet B) \bullet \Box C)$$

and hence also $^{*}(^{*}(^{\circ}\varphi\delta^{i}_{A\bullet B})\delta_{B}) : \mathrm{I} \vdash B \rightarrow (A \rightarrow ((A \bullet B) \bullet \Box C))$. Then, by ($\Box\downarrow$), we must have in $\mathcal{C}$

$$f : \Box C \bullet \mathrm{I} \vdash B \rightarrow (A \rightarrow ((A \bullet B) \bullet \Box C))$$

and we take ${}^{\circ\circ}(f\delta^{i}_{\Box C})$ to be $\mathbf{b}^{\rightarrow}_{A,B,\Box C}$. We proceed analogously for the $\mathbf{b}^{\leftarrow}$ arrows.

c) Since in $\mathcal{C}$ we have

$$^{*}1_{B\bullet\Box A}\delta_{\Box A} : \Box A \bullet \mathrm{I} \vdash B \rightarrow (B \bullet \Box A)$$

by ($\Box\uparrow$) we must have in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash A$,

$$\varphi : \mathrm{I} \vdash B \rightarrow (B \bullet \Box A)$$

and hence also $(^{\circ}\varphi\delta^{i}_{B}\sigma_{B})^{*} : \mathrm{I} \vdash (B \bullet \Box A) \leftarrow B$. Then, by ($\Box\downarrow$), we must have in $\mathcal{C}$

$$f : \Box A \bullet \mathrm{I} \vdash (B \bullet \Box A) \leftarrow B$$

and we take $(f\delta^{i}_{\Box A})^{\circ}$ to be $\mathbf{c}_{A,\Box B}$. We proceed analogously for $\mathbf{c}_{A,\Box B}$.

k) Since in $\mathcal{C}[x]$ we have 1_{I}, by ($\Box\downarrow$) in $\mathcal{C}$ we must have $f : \Box A \bullet \mathrm{I} \vdash \mathrm{I}$. We take $f\delta^{i}_{\Box A}$ to be $\mathbf{k}_{\Box A}$.

w) Since in $\mathcal{C}$ we have $\delta_{\Box A}$, by ($\Box\uparrow$) we must have $\varphi : \mathrm{I} \vdash \Box A$ in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash A$. Then in $\mathcal{C}[x]$ we have $(\varphi \bullet \varphi)\sigma^{i}_{\mathrm{I}} : \mathrm{I} \vdash (\Box A \bullet \Box A)$, and by ($\Box\downarrow$), in $\mathcal{C}$ we must have $f : \Box A \bullet \mathrm{I} \vdash (\Box A \bullet \Box A)$. We take $f\delta^{i}_{\Box A}$ to be $\mathbf{w}_{\Box A}$.

$^{\Box}$) Since in $\mathcal{C}$ we have $\delta_{\Box A}$, by ($\Box\uparrow$) we must have $\varphi : \mathrm{I} \vdash \Box A$ in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash A$. We take this φ to be $x^{\Box}$. So we have $^{\Box}$ applying to arrows of type $\mathrm{I} \vdash A$.

We want to show that we have also $^{\Box}$ applying to arrows of type $B \vdash A$ where B is any modalized object.

Since in the polynomial NL$\Box$ category $\mathcal{C}[x][y]$ built with $y : \mathrm{I} \vdash B$ over the polynomial NL$\Box$ category $\mathcal{C}[x]$, which was itself built over the NL$\Box$ category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, we have

$$((x \bullet y)\sigma^{i}_{\mathrm{I}})^{\Box} : \mathrm{I} \vdash \Box(A \bullet B)$$

by $(\Box\downarrow)$ we must have $g : \Box B \bullet \mathrm{I} \vdash \Box(A \bullet B)$ in $\mathcal{C}[x]$, and hence also

$$g\delta^{i}_{\Box B} : \Box B \vdash \Box(A \bullet B).$$

Then, again by $(\Box\downarrow)$, we must have in $\mathcal{C}$

$$f : \Box A \bullet \Box B \vdash \Box(A \bullet B).$$

We also have $1^{\Box}_{\mathrm{I}}$ in $\mathcal{C}$.

Suppose that we have in $\mathcal{C}$ an arrow $h : \Box C \vdash A$, and hence also $h\delta_{\Box C}$. Then by $(\Box\uparrow)$ we must have $\varphi : \mathrm{I} \vdash A$ in $\mathcal{C}[x]$, where $x : \mathrm{I} \vdash C$. Hence we have $\varphi^{\Box}$ in $\mathcal{C}[x]$, and by $(\Box\downarrow)$ we must have $f : \Box C \bullet \mathrm{I} \vdash \Box A$ and $f\delta^{i}_{\Box C} : \Box C \vdash \Box A$ in $\mathcal{C}$. Taking $f\delta^{i}_{\Box C}$ to be $h^{\Box}$, we have $^{\Box}$ applying to arrows of type $\Box C \vdash A$. To have full $^{\Box}$, applying to arrows of type $B \vdash A$ where B is any modalized object, it remains to show that for modalized B we have in $\mathcal{C}$ an arrow of type $B \vdash \Box B$, and this we do by induction on the complexity of B, using the arrows and operations on arrows we have already secured.

The fact that for modalized A we have an arrow of type $A \vdash \Box A$, as well as $\mathbf{r}_A$, gives us the structural arrows **b**, **c**, **k** and **w** with the provisos involving any modalized objects, as we have assumed them in Section 2, and not only boxed objects, as we have inferred them above.

Now we shall justify to a certain extent the equations of Section 2. In this justification we presuppose the equations (**b**), (**bb**), (**c**), (**cc**), (**k**) and (**w**); i.e., we presuppose that the **b** and **c** arrows are natural isomorphisms and that the **k** and **w** arrows are natural transformations. We also presuppose the equation $(^{\Box})$, which is of the same sort as (**b**), (**c**), (**k**) and (**w**). Proof- theoretically, it amounts to permuting the rule $^{\Box}$ with cut. We presuppose that modalized objects A are isomorphic with $\Box A$; i.e., for modalized A we presuppose the equations

$$\mathbf{r}_A 1^{\Box}_A = 1_A, \qquad 1^{\Box}_A \mathbf{r}_A = 1_{\Box A}.$$

These equations are of the same sort as **(bb)** and **(cc)**. They amount to the conversion of some detours in proofs. Furthermore, we presuppose that the definitions of ${}^{\text{‘}}x$ and μ_x are given and that μ_x satisfies also clauses $(\mu 1.1), (\mu 1.2)$, $(\mu 2.1)$ and $(\mu 2.2)$. Finally, we presuppose Lemmata 1–3, i.e. that μ_x is a function and that it is the inverse of ${}^{\text{‘}}x$. Our derivation of the remaining equations of Section 2 from modal functional completeness will depend on all these presuppositions.

In some cases our derivation will fall short of obtaining the required equation in full generality in which it holds. This will happen with the pentagonal equations, the hexagonal equations, the triangular equation $(\sigma\delta\mathbf{b})$ and the equation $(\sigma\delta\mathbf{c})$. For them we shall push our derivation only up to instances of these equations where particular indices are modalized. We shall indicate at what places other instances are required, if such is the case.

First we derive $(\Box\beta)$. As a consequence of Lemma 2 (see case (0.1) in the proof) we have

$$(\Box\beta x) \quad \mathbf{r}_A x^{\Box} = x.$$

Then for $f : B \vdash A$ with B modalized and $y : \mathrm{I} \vdash B$ we have

$$\begin{aligned}
&\mathbf{r}_A f^{\Box} y = \mathbf{r}_A (fy)^{\Box}, \text{ with } ({}^{\Box})\\
&\mathbf{f}_A f^{\Box} y = fy, \text{ with } (\Box\beta x)\\
&\mu_y(\mathbf{r}_A f^{\Box} y) = \mu_y(fy), \text{ by Lemma 1}\\
&\mathbf{r}_A f^{\Box} \mathbf{r}_B \delta_{\Box B} = f\mathbf{r}_b \delta_{\Box B}, \text{ with } (\mu 1.1) \text{ and } (\mu 0.1)
\end{aligned}$$

which with $(\delta\delta^i)$ and $\mathbf{r}_B 1_B^{\Box} = 1_B$ yields $(\Box\beta)$. The equation $(\Box\eta)$ amounts to $({}^{\Box})$ and $1_A^{\Box}\mathbf{r}_A = 1_{\Box A}$, which we have presupposed.

Next we justify the pentagonal equations. From Lemma 1 and from the equation **(b)**, more precisely **(b$_1$)**, it follows that we must have

$$\mu_x(((\psi \bullet 1_C) \bullet 1_D)\mathbf{b}^{\rightarrow}_{B,C,D}) = \mu_x(\mathbf{b}^{\rightarrow}_{E,C,D}(\psi \bullet (1_C \bullet 1_D)))$$

which by using $(\mu 1.1), (\mu 1.2), (\mu 2.2)$ and **(b)** gives

$$\begin{aligned}
&((\mu_x\psi \bullet 1_C) \bullet 1_D)(\mathbf{b}^{\rightarrow}_{\Box A,B,C} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box A,B\bullet C,D}(1_{\Box A} \bullet \mathbf{b}^{\rightarrow}_{B,C,D}) =\\
&\quad ((\mu_x\psi \bullet 1_C) \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box A\bullet B,C,D}\mathbf{b}^{\rightarrow}_{\Box A,B,C\bullet D}.
\end{aligned}$$

Then we substitute $1_{\Box A\bullet B}{}^{\text{‘}}x$ for ψ and, by Lemma 3, obtain the pentagonal equation **(b5)** with A boxed. Since a modalized object A is isomorphic with $\Box A$, we have the pentagonal equation (**b**5) for modalized A, too. This is not the full pentagonal equation (**b**5), because, in the full one, A need not be modalized provided two objects among B, C and D are modalized. Other forms of pentagonal equations, not covered by (**b**5) with A modalized, are involved in the proof of Lemma 1 when we derive the equations obtained by prefixing μ_x to the two sides of ($\mathbf{b}_2$) and ($\mathbf{b}_3$), instead of ($\mathbf{b}_1$) as above.

At this place we can make the following remark. By going carefully over our proof of the Modal Functional Completeness Theorem, one finds that we never need the arrows $\mathbf{b}^{\rightarrow}_{A,B,C}$ and $\mathbf{b}^{\leftarrow}_{A,B,C}$ where C is modalized, but only where A or B is modalized, except when in the proof of Lemma 1 we derive the equation obtained by prefixing μ_x to the two sides of ($\mathbf{b}_3$). This has to do with the fact that $\mu_x\varphi$ is taken to be of type $\Box A \bullet B \vdash C$, rather than $B \bullet \Box A \vdash C$. We might as well have defined an analogous $\mu_x\varphi$ of this other type. Then for the first index of the **b** arrows and the equation ($\mathbf{b}_1$) we would have the same thing that we have now for the third index and ($\mathbf{b}_3$). However, with our definition of μ_x, we could not exclude the **b** arrows in which only the object in the third index is modalized because these are definable as follows in terms of **c** arrows and the remaining **b** arrows:

$$\mathbf{b}^{\rightarrow}_{A,B,C} =_{\mathrm{df}} \mathbf{c}_{C,A\bullet B}\mathbf{b}^{\leftarrow}_{C,A,B}(\mathbf{c}_{A,C} \bullet \mathbf{1}_B)\mathbf{b}^{\rightarrow}_{A,C,B}(\mathbf{1}_A \bullet \mathbf{c}_{B,C})$$

where only C is modalized. The equation installed by this definition is a hexagonal equation.

Next we justify the hexagonal equations. From Lemma 1 and the equation (**c**), more precisely ($\mathbf{c}_1$), it follows that we must have

$$\mu_x((\psi \bullet \mathbf{1}_C)\mathbf{c}_{C,B}) = \mu_x(\mathbf{c}_{C,D}(\mathbf{1}_C \bullet \psi))$$

which by using (μ1.1), (μ1.2), (μ2.1), (μ2.2) and (**c**) gives

$$(\mu_x\psi \bullet \mathbf{1}_C)\mathbf{b}^{\rightarrow}_{\Box A,B,C}(\mathbf{1}_{\Box A} \bullet \mathbf{c}_{C,B}) = (\mu_x\psi \bullet \mathbf{1}_C)\mathbf{c}_{C,\Box A\bullet B}\mathbf{b}^{\leftarrow}_{C,\Box A,B}(\mathbf{c}_{\Box A,C} \bullet \mathbf{1}_B)\mathbf{b}^{\rightarrow}_{\Box A,C,B}.$$

Then we substitute $\mathbf{1}_{\Box A\bullet B}{}^{\backprime}x$ for ψ and, by Lemma 3, obtain a hexagonal equation. The same hexagonal equation is induced by prefixing μ_x to the two sides of ($\mathbf{c}_2$). As before, $\Box A$ can be replaced by modalized A. This doesn't yet amount to the full hexagonal equation (**bc**6), because, there, A need not be modalized if C is (it is not enough that only B be modalized). A hexagonal equation where only C is modalized is involved in the proof of Lemma 1 when we derive the equation obtained by prefixing μ_x to the two sides of ($\mathbf{b}_3$). Such an equation is also installed by the definition of $\mathbf{b}^{\rightarrow}_{A,B,C}$ where only C is modalized, which we gave in the preceding paragraph.

To derive the octagonal equations it is enough to consider the case with (**w**) in the proof of Lemma 1 and proceed as for the pentagonal and hexagonal equations. The octagonal equations are completely justified by that (unlike the pentagonal and hexagonal equations, whose justification we have pushed only up to a point).

Let us now justify the triangular equations and ($\sigma\delta$**c**). As a consequence of Lemma 2 we have that

$$\mu_x(\psi \bullet 1_E)`x = (\mu_x\psi`x) \bullet 1_E$$

which, by substituting $1_{\Box A \bullet D}`x$ for ψ, and by using (μ2.2) and Lemma 3, reduces to

$$\mathbf{b}^{\rightarrow}_{\Box A,D,E}(x^{\Box} \bullet 1_{D \bullet E})\sigma^{i}_{D \bullet E} = ((x^{\Box} \bullet 1_D)\sigma^{i}_{D}) \bullet 1_E$$
$$((x^{\Box} \bullet 1_D) \bullet 1_E)\mathbf{b}^{\rightarrow}_{\mathrm{I},D,E}\sigma^{i}_{D \bullet E} = ((x^{\Box} \bullet 1_D) \bullet 1_E)(\sigma^{i}_{D} \bullet 1_E), \text{ with } (\mathbf{b}) \text{ and } (\bullet).$$

By prefixing μ_x to the two sides of the last equation, as Lemma 1 allows, with (μ1.2), (μ2.2), (μ6), (μ0.1) and ($\Box\eta$) we obtain

$$(((\delta_{\Box A} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I},D}) \bullet 1_E)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I}\bullet D,E}(1_{\Box A} \bullet (\mathbf{b}^{\rightarrow}_{\mathrm{I},D,E}\sigma^{i}_{D \bullet E})) =$$
$$(((\delta_{\Box A} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I},D}) \bullet 1_E)\mathbf{b}^{\rightarrow}_{\Box A,\mathrm{I}\bullet D,E}(1_{\Box A} \bullet (\sigma^{i}_{D} \bullet 1_E)).$$

With ($\delta\delta^i$), (**bb**) and ($\bullet$) this reduces to

$$1_{\Box A} \bullet (\mathbf{b}^{\rightarrow}_{\mathrm{I},D,E}\sigma^{i}_{D \bullet E}) = 1_{\Box A} \bullet (\sigma^{i}_{D} \bullet 1_E)$$

from which, by taking A to be I and by using ($\bullet$) and $\mathbf{r}_{\mathrm{I}}1_{\Box\mathrm{I}}1^{\Box}_{\mathrm{I}} = 1_{\mathrm{I}}$, we obtain

$$1_{\mathrm{I}} \bullet (\mathbf{b}^{\rightarrow}_{\mathrm{I},D,E}\sigma^{i}_{D \bullet E}) = 1_{\mathrm{I}} \bullet (\sigma^{i}_{D} \bullet 1_E).$$

By prefixing $\sigma_{(\mathrm{I}\bullet D)\bullet E}$ to both sides, with (σ) and ($\sigma\sigma^i$) we derive the triangular equation

$$(\sigma^i\mathbf{b}) \qquad \mathbf{b}^{\rightarrow}_{\mathrm{I},D,E}\sigma^{i}_{D \bullet E} = \sigma^{i}_{D} \bullet 1_E$$

which, with ($\sigma\sigma^i$) and ($\bullet$), amounts to (σ**b**). (The triangular equation (δ**b**), in which, however, A is modalized, is derivable by prefixing μ_x to the two sides of (δ).)

As a consequence of Lemma 2 we also have

$$\mu_x(1_E \bullet \psi)`x = 1_E \bullet (\mu_x \psi`x)$$

which, by substituting $1_{\Box \mathrm{I} \bullet D}`x$ for ψ, and by using (μ2.1) and Lemma 3, reduces to

$$\begin{aligned}
&\mathbf{b}^{\leftarrow}_{E,\Box\mathrm{I},D}(\mathbf{c}_{\Box\mathrm{I},E} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box\mathrm{I},E,D}(x^{\Box} \bullet 1_{E\bullet D})\sigma^i_{E\bullet D} = 1_E \bullet ((x^{\Box} \bullet 1_D)\sigma^i_D)\\
&(1_E \bullet (x^{\Box} \bullet 1_D))\mathbf{b}^{\leftarrow}_{E,\mathrm{I},D}(\mathbf{c}_{\mathrm{I},E} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\mathrm{I},E,D}\sigma^i_{E\bullet D} = (1_E \bullet (x^{\Box} \bullet 1_D))(1_E \bullet \sigma^i_D),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } (\mathbf{b}), (\mathbf{c}), \text{ and } (\bullet).
\end{aligned}$$

By prefixing μ_x to both sides and proceeding as in the last paragraph, with (σ^i**b**) and ($\bullet$) we obtain

$$\mathbf{b}^{\leftarrow}_{E,\mathrm{I},D}((\mathbf{c}_{\mathrm{I},E}\sigma^i_E) \bullet 1_D) = 1_E \bullet \sigma^i_D$$

which, with ($\sigma\sigma^i$), (**bb**), (**cc**) and ($\bullet$), amounts to

$$(\sigma\mathbf{c}\sigma\mathbf{b}) \qquad ((\sigma_E\mathbf{c}_{E,\mathrm{I}}) \bullet 1_D)\mathbf{b}^{\rightarrow}_{E,\mathrm{I},D} = 1_E \bullet \sigma_D.$$

This would be the triangular equation ($\sigma\delta$**b**) if we had $\sigma_E\mathbf{c}_{E,\mathrm{I}} = \delta_E$, i.e. ($\sigma\delta$**c**).

Now, let us see what we can do for ($\sigma\delta$**c**). From Lemma 3 (see the proof of the lemma) we can derive

$$\begin{aligned}
&(\delta_{\Box E} \bullet 1_D)\mathbf{b}^{\rightarrow}_{\Box E,\mathrm{I},D} = 1_{\Box E} \bullet \sigma_D\\
&\delta_{\Box E} \bullet 1_D = (\sigma_{\Box E}\mathbf{c}_{\Box E,\mathrm{I}}) \bullet 1_D, \text{ with } (\sigma\mathbf{c}\sigma\mathbf{b}) \text{ and } (\mathbf{bb}).
\end{aligned}$$

By taking D to be I and prefixing $\delta_{\Box E}$ to both sides, with (δ) and ($\delta\delta^i$) we obtain

$$\delta_{\Box E} = \sigma_{\Box E}\mathbf{c}_{\Box E,\mathrm{I}}.$$

So, we have ($\sigma\delta$**c**) for modalized A.

In our proof of modal functional completeness, we actually dont need ($\sigma\delta$**b**) and ($\sigma\delta$**c**) except when A is modalized, provided we have assumed other triangular and pseudotriangular equations, like (σ**b**) and (σ**c**σ**b**). However, such a reduced stock of triangular and pseudotriangular equations works only because we have taken $\mu_x\varphi$ to be of type $\Box A \bullet B \vdash C$, rather than $B \bullet \Box A \vdash C$. With this other type we would need other equations. (With the first type, σ is preponderant over δ; with the second, it is the other way round.) Our tidier, more symmetric, axiomatization of NL$\Box$ categories, in which $(\sigma\delta\mathbf{b})$ and $(\sigma\delta\mathbf{c})$ hold for nonmodalized A too, is insensitive to this change of type for $\mu_x\varphi$. It works equally well with either type. This slight generalization has repercussions on the underlying nonmodal part of NL$\Box$ categories, i.e. the NL part, because σ and δ, which were not linked in it, are now linked through $\mathbf{c}$. Formerly independent assumptions for σ and δ can now be derived from each other, as we noted in Section 2.

To derive $(1\mathbf{k})$, we have as a consequence of Lemma 2 (see case (0.2) in the proof)

$$1_{\mathrm{I}}\sigma_{\mathrm{I}}(\mathbf{k}_{\Box A}\bullet 1_{\mathrm{I}})(x^{\Box}\bullet 1_{\mathrm{I}})\sigma^i_{\mathrm{I}} = 1_{\mathrm{I}}$$

and then we use $(\sigma\delta)$, (δ), $(\sigma\sigma^i)$ and $(\mathbf{k})$.

It remains to derive $(\sigma\delta\mathbf{w})$, $(\mathbf{bw})$, $(\mathbf{cw})$, $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$. We derive $(\sigma\delta\mathbf{w})$ from the case we have considered in the induction step of the proof of Lemma 2 (case (1)), together with equations we already have, in particular $(\sigma\mathbf{b})$, which we have derived in full generality. We derive $(\mathbf{bw})$ by prefixing μ_x to the two sides of (cat 2) and using equations we already have, in particular a pentagonal equation with A modalized (see the proof of Lemma 1). Similarly, we derive $(\mathbf{cw})$ by prefixing μ_x to the two sides of ($\bullet$bifunctor) and using equations we already have, in particular pentagonal and hexagonal equations with A modalized. Finally, to derive $(\sigma\mathbf{kw})$, take ψ to be $1_{\Box A\bullet \mathrm{I}}`x$ and h to be $1_{\Box A\bullet\mathrm{I}}$ in the last equation displayed in our derivation of $(\mu 1.1)$ in Section 3, so as to obtain

$$(\sigma_{\Box A}(\mathbf{k}_{\Box A}\bullet 1_{\Box A})\mathbf{w}_{\Box A})\bullet 1_{\mathrm{I}} = 1_{\Box A}\bullet 1_{\mathrm{I}}.$$

Then we prefix $\delta_{\Box A}$ to the two sides of this equation; with (δ) and $(\delta\delta^i)$ this gives $(\sigma\mathbf{kw})$. (We could as well have taken ψ to be $\delta_{\Box A}`x$). For $(\delta\mathbf{kw})$ we proceed analogously using the last equation displayed in the derivation of $(\mu 1.2)$. Note that we already have all the equations used in the derivations of $(\mu 1.1)$ and $(\mu 1.2)$ up to the last displayed equations; in particular, we have $(\sigma\mathbf{b})$ and $(\sigma\delta\mathbf{b})$ with A modalized. (We use $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$ also in the derivation of $(\mu 2.1)$ and $(\mu 2.2)$ from $(\mu 2)$.) With these derivations, $(1\mathbf{k})$, $(\sigma\delta\mathbf{w})$, $(\mathbf{bw})$, $(\mathbf{cw})$, $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$ are completely justified, and we have accomplished our partial justification of the assumptions made for NL$\Box$ categories in Section 2.

Can we justify in a similar manner the assumptions of Section 1? The existence of an operation $\bullet$ on objects and arrows follows from the formulation of the Modal Functional Completeness Theorem. That there is a functor behind these operations is something we have to presuppose, as we presupposed about the structural arrows that they are natural isomorphisms or natural transformations (proof-theoretically, the equation ($\bullet$) amounts to permuting the rule $\bullet$ with cut). The assumptions concerning $\bullet$ can be understood as structural assumptions, too. Such are also the assumptions about the arrows 1_A and composition. (The equations (cat 1) imply that the arrows 1_A are natural isomorphisms from the identity functor to the identity functor; proof-theoretically, these equations amount to eliminating some cuts, whereas (cat 2) amounts to permuting cut with itself.)

Since they involve the modalized object I, and are related to structural rules, the $\sigma\delta$ arrows might be understood as modal structural arrows, on a par with **b**, **c**, **k** and **w** arrows. So, perhaps the assumptions concerning $\sigma\delta$ arrows could be shifted to Section 2, and would need to be justified as much as the modal assumptions of that section. (We put, however, $\sigma\delta$ arrows in Section 1 because they don't involve $\Box$, and because I makes sense in the absence of $\Box$, too.) The arrows σ^i are needed for the definition of the function 'x (they are in the formulation of the Modal Functional Completeness Theorem), whereas σ and δ arrows are needed for the definition of μ_x: the first for (μ0.2) and the second for (μ0.1). The δ^i arrows are not absolutely needed (as we have remarked above, σ is preponderant over δ), but this is only because we take $\mu_x\varphi$ to be of type $\Box A \bullet B \vdash C$, rather than $B \bullet \Box A \vdash C$. The equations ($\sigma$) and ($\delta$) are comparable to (**b**), (**c**), (**k**) and (**w**), whereas ($\sigma\sigma^i$) and ($\delta\delta^i$) are comparable to (**bb**) and (**cc**). They have the same proof-theoretical import. We presuppose these $\sigma\delta$ equations as we presupposed the corresponding **b**, **c**, **k** and **w** equations earlier in this section.

We can, however, derive ($\sigma\delta$) from modal functional completeness. As a consequence of Lemma 2 (see case (0.1) in the proof) we have

$$\begin{array}{l}
\mathbf{r}_{\mathrm{I}}\delta_{\Box\mathrm{I}}(x^{\Box} \bullet 1_{\mathrm{I}})\sigma^i_{\mathrm{I}} = x \\
\mathbf{r}_{\mathrm{I}}x^{\Box}\delta_{\mathrm{I}}\sigma^i_{\mathrm{I}} = x, \text{ with } (\delta) \\
\mu_x(\mathbf{r}_{\mathrm{I}}x^{\Box}\delta_{\mathrm{I}}\sigma^i_{\mathrm{I}}) = \mu_x x, \text{ by Lemma 1} \\
\mathbf{r}_{\mathrm{I}}(\mathbf{r}_{\mathrm{I}}\delta_{\Box\mathrm{I}})^{\Box}(1_{\Box\mathrm{I}} \bullet (\delta_{\mathrm{I}}\sigma^i_{\mathrm{I}})) = \mathbf{r}_{\mathrm{I}}\delta_{\Box\mathrm{I}}, \text{ with } (\mu 1.1), (\mu 1.2), (\mu 6) \text{ and } (\mu 0.1) \\
\delta_{\Box\mathrm{I}}(1_{\Box\mathrm{I}} \bullet (\delta_{\mathrm{I}}\sigma^i_{\mathrm{I}})) = \delta_{\Box\mathrm{I}}, \text{ with } 1^{\Box}_{\mathrm{I}}\mathbf{r}_{\mathrm{I}} = 1_{\Box\mathrm{I}} \text{ and } (^{\Box})
\end{array}$$

which, with ($\delta\delta^i$), ($\bullet$), $\mathbf{r}_{\mathrm{I}}1_{\Box\mathrm{I}}1^{\Box}_{\mathrm{I}} = 1_{\mathrm{I}}$, ($\sigma$) and ($\sigma\sigma^i$) yields ($\sigma\delta$). (Note that we have used ($\sigma\delta$) in the justification of ($\Box\beta$); here we dont use ($\Box\beta$), but only the isomorphism between I and $\Box$I.)

The assumptions concerning $\rightarrow, \leftarrow, \wedge, \top, \vee$ and $\bot$ are obviously independent from modal functional completeness. Note, however, that we wouldn't have functional completeness with $\vee$ if we didnt have $\rightarrow$, or without a primitive distribution arrow **d** (with $\mu_x\varphi$ of type $B \bullet \Box A \vdash C$ instead of $\Box A \bullet B \vdash C$, we would need $\leftarrow$, or a distribution arrow of type $(E \vee F) \bullet G \vdash (E \bullet G) \vee (F \bullet G)$). A related fact (about which we heard from Djordje Čubrić) is that a bicartesian category is functionally complete in the ordinary sense if and only if it is distributive; in that case $\wedge$, usually written $\times$, plays the role of $\bullet$ for formulating functional completeness.

To conclude this section, let us note that for a weakened form of modal functional completeness where we would be happy with asserting that 'x is *onto*, without necessarily being *one–one*, we would need far less assumptions for our categories. In the justification above we could appeal only to Lemma 2, and for proving this lemma we dont need $1^{\Box}_{A}\mathbf{r}_A = 1_{\Box A}$, the pentagonal, hexagonal and octagonal equations, (**bw**), (**cw**), $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$.

6 SUBSTRUCTURAL FUNCTIONAL COMPLETENESS

Let us now take NL deductive systems with the unary operations on arrows $D\bullet\_$ and $\_\bullet D$ primitive instead of the binary operation on arrows $\_\bullet\_$, and let us consider the following hierarchy of nonmodal deductive systems, obtained by assuming, in addition to what we have for NL deductive systems, the structural arrows mentioned in parentheses *without* provisos concerning modalized objects:

AL *deductive systems* (**b** arrows)
M *deductive systems* (**b** and **c** arrows)
BCK *deductive systems* (**b**, **c** and **k** arrows)
R *deductive systems* (**b**, **c** and **w** arrows)
H *deductive systems* (**b**, **c**, **k** and **w** arrows).

For easier comparison, we use the labels introduced in [3]. The label 'AL' stands for 'associative Lambek', 'M' for 'multiset' (in the light of latter-day developments, it would be more intelligible if we called these systems *linear*—more precisely, *intuitionistic linear*), 'R' for 'relevant' (intuitionistic and without distribution of $\wedge$ over $\vee$) and 'H' for 'Heyting'. The label 'BCK' is pretty standard (the linear logic trade has recently produced some ersatz names for BCK systems; the BCK systems we consider here are of an intuitionistic sort).

Let us write S for NL, AL, M, BCK, R or H (the variable 'S' stands for 'substructural') and let S *categories* be S deductive systems that are NL categories and satisfy moreover the equations between arrows of Section 2, which, of course, apply only if the arrows in question are present in the deductive system. So, AL categories must satisfy **b** equations, M categories **b** and **c** equations, BCK categories **b**, **c** and **k** equations, R categories **b**, **c** and **w** equations except $(\sigma\mathbf{kw})$ and $(\delta\mathbf{kw})$, and H categories all the **b**, **c**, **k** and **w** equations.

For all S categories except NL and AL categories, we have that $A \rightarrow B$ is naturally isomorphic to $B \leftarrow A$, as in Section 4. However, for none except H categories, which are the bicartesian closed categories of Section 4, we need have that $A \bullet B$ is isomorphic to $A \wedge B$. Only for BCK and H categories I must be naturally isomorphic to $\top$. The AL categories are monoidal biclosed with respect to $\bullet$, I, $\rightarrow$ and $\leftarrow$, and M categories are symmetric monoidal closed with respect to $\bullet$, I and $\rightarrow$. However, they are all also bicartesian with respect to $\wedge$ and $\vee$, their terminal object being $\top$ and their initial object $\bot$.

We can extend the axiomatization of S categories to the axiomatization of the corresponding S$\Box$ categories as we extended the axiomatization of NL categories to the axiomatization of NL$\Box$ categories: we just add the missing modal assumptions from Section 2. For example, the axiomatization of M categories is extended to the axiomatization of M$\Box$ categories by adding the operation on objects $\Box$, the **r** arrows, the operation on arrows $^{\Box}$ and the modal structural arrows $\mathbf{k}_A$ and $\mathbf{w}_A$, *with* the proviso for modalized A, together with the $\Box$, **k** and **w** equations (the **b** and **c** equations are already assumed for the unrestricted structural arrows of M categories). The M$\Box$ categories correspond to intuitionistic modal linear propositional logic. For them we can prove modal functional completeness as we did for NL$\Box$ categories, and similarly with other S$\Box$ categories. For H$\Box$ categories we have to add just the **r** arrows, the operation on arrows $^{\Box}$ and the $\Box$ equations, which gives categories corresponding to intuitionistic S4 propositional logic. For these categories we can prove modal functional completeness, but ordinary, non-modal, functional completeness fails, though we have it for H categories, as shown in Section 4 (otherwise we would always have arrows of type $A \vdash \Box A$ in H$\Box$ categories). We can also infer the necessity of assumptions for our modal categories as we did in Section 5.

We want now to state a general functional completeness theorem, which as a special case covers ordinary functional completeness for H categories, i.e. bicartesian closed categories, and in other cases yields restricted functional completeness for other S categories. To state this theorem we need special notions of polynomials in polynomial categories, which we proceed to define.

Given an S category $\mathcal{C}$, we build the polynomial S category $\mathcal{C}[x]$ with $x : \mathrm{I} \vdash A$ as before. This we can always do because, for every S, the S categories are equationally presented. If all arrows of $\mathcal{C}[x]$ are called *polynomials*, then our notion of polynomial satisfies the following clauses (parallel with the inductive clauses for μ_x in Section 3):

(P0.1) The arrow x is a polynomial.

(P0.2) Every arrow of $\mathcal{C}$ is a polynomial.

(P1) If $\psi : D \vdash E$ and $\xi : E \vdash F$ are polynomials, then $\xi\psi$ is a polynomial.

(P1.1) If $\psi : D \vdash E$ is a polynomial and $h : E \vdash F$ is an arrow of $\mathcal{C}$, then $h\psi$ is a polynomial.

(P1.2) If $h : D \vdash E$ is an arrow of $\mathcal{C}$ and $\xi : E \vdash F$ is a polynomial, then ξh is a polynomial.

(P2.1) If ψ is a polynomial, then $E \bullet \psi$ is a polynomial.

(P2.2) If ψ is a polynomial, then $\psi \bullet E$ is a polynomial.

(P3.1) If $\psi : E \bullet D \vdash F$ is a polynomial, then ${}^*\psi$ is a polynomial.

(P3.2) If $\psi : D \bullet E \vdash F$ is a polynomial, then ψ^* is a polynomial.

(P4) If $\psi : D \vdash E$ and $\xi : D \vdash F$ are polynomials, then $\langle \psi, \xi \rangle$ is a polynomial.

(P5) If $\psi : E \vdash D$ and $\xi : F \vdash D$ are polynomials, then $[\psi, \xi]$ is a polynomial.

Clauses (P1.1) and (P1.2) are redundant in the presence of (P0.2) and (P1), but we have listed them nevertheless because we need them for the inductive definitions of restricted notions of polynomials. These notion are obtained by assuming the clauses mentioned in parentheses:

M *polynomial* (all clauses save (P0.2) and (P1))
BCK *polynomial* (all clauses save (P1))
R *polynomial* (all clauses save (P0.2))
H *polynomial* (all clauses)

(NL and AL polynomials will be considered below). Of course, H polynomials are not restricted: they coincide with all the arrows of $\mathcal{C}[x]$. (As we have just noted above, we may omit clauses (P1.1) and (P1.2) from their definition.) However, the other notions of polynomial reject some arrows of $\mathcal{C}[x]$.

Before looking into that, let us make a point concerning the nature of the arrow x in $\mathcal{C}[x]$. This arrow must be *new* to $\mathcal{C}$ (otherwise, with NL□ categories we would need the equation $x^{\Box}\mathbf{k}_{\Box A} = 1_{\Box A}$ to make (μ0.1) and (μ0.2) match; with H categories we would need $x\mathbf{k}_A = 1_A$, which doesn't hold necessarily in cartesian categories). However, nothing prevents us from introducing a new arrow x of type $\mathrm{I} \vdash \mathrm{I}$, which in the course of constructing $\mathcal{C}[x]$ will be identified with 1_I for some categories (in NL□ categories we have $1_\mathrm{I}^{\Box}\mathbf{k}_{\Box\mathrm{I}} = 1_{\Box\mathrm{I}}$ because $\mathbf{k}_{\Box\mathrm{I}} = \mathbf{r}_\mathrm{I}$; in BCK and H categories, and NL□ categories as well, we have $\mathbf{k}_\mathrm{I} = 1_\mathrm{I}$). So, in some S categories, an arrow of $\mathcal{C}[x]$ may qualify as a polynomial on more than one ground:

in BCK, and hence also H, categories $x : \mathrm{I} \vdash \mathrm{I}$ will be a polynomial both by (P0.1) and (P0.2).

It can easily be checked that if we exclude the operations on arrows mentioned in clauses (P4) and (P5), an M polynomial is an arrow of $\mathcal{C}[x]$ in whose construction x occurs *exactly* once, a BCK polynomial an arrow of $\mathcal{C}[x]$ in whose construction x occurs *at most* once, and an R polynomial an arrow of $\mathcal{C}[x]$ in whose construction x occurs *at least* once.

Note that for R and H polynomials we obtain as a derived clause:

(P2) If ψ and ξ are polynomials, then $\psi \bullet \xi$ is a polynomial.

This is because $\psi \bullet \xi$ is equal by definition to $(\psi \bullet E)(F \bullet \xi)$ in all S categories (we have taken the unary operations on arrows $D \bullet \_$ and $\_ \bullet D$ as primitive), and the arrow $(\psi \bullet E)(F \bullet \xi)$ is an R polynomial by (P2.1), (P2.2) and (P1). Since we lack (P1) for M and BCK polynomials, we shall also lack (P2). This is parallel with the fact that if we replace clause (μ2) by (μ2.1) and (μ2.2), the **w** arrows enter into the definition of μ_x only via clause (μ1). The **k** arrows enter into this definition only via clause (μ0.2).

Let now the function 'x be defined as it was defined in Section 4:

$$\text{For } x : \mathrm{I} \vdash A \text{ and } f : A \bullet B \vdash C, \quad f\text{'}x =_{\mathrm{df}} f(x \bullet 1_B)\sigma^i_B.$$

For μ_x as in Section 4, we replace clause (μ0.1) by

$$\mu_x x = \delta_A$$

and assume (μ1.1), (μ1.2), (μ2.1), (μ2.2), (μ3.1), (μ3.2), (μ4) and (μ5) with $\Box$ deleted everywhere. Clause (μ0.2) with $\Box$ deleted will be assumed only in the presence of **k** arrows, which in the present context means only for S being BCK or H. Similarly, clause (μ1) with $\Box$ deleted will be assumed only in the presence of **b** and **w** arrows, which in the present context means only for S being R or H. When we refer to the μ clauses from now on, we assume these are the newly introduced, nonmodal, clauses. If S is R, clause (μ1) is independent from (μ1.1) and (μ1.2), because arrows of $\mathcal{C}$ dont qualify as polynomials. If S is H, we have to show this clause is compatible with (μ1.1), (μ1.2) and (μ0.2); this we do by deriving (μ1.1) and (μ1.2) from (μ1) and (μ0.2), quite analogously to what we did in Section 3. Since $D \bullet \_$ and $\_ \bullet D$ are primitive, instead of $\_ \bullet \_$, we shall have clause (μ2) with $\Box$ deleted only as a derived clause when **b**, **c** and **w** arrows are present, and, again, this will happen only for S being R or H. That (μ2) can actually replace (μ2.1) and (μ2.2) will be the case only for H (we use (μ0.2), (σ**kw**) and (δ**kw**) to derive (μ2.1) and (μ2.2) from (μ2)).

We can now state our general functional completeness theorem:

Substructural Functional Completeness Theorem If S is M, BCK, R or H, then for every S polynomial $\varphi : B \vdash C$ of the polynomial S category $\mathcal{C}[x]$ built over the S category $\mathcal{C}$ with $x : \mathrm{I} \vdash A$, there is a unique arrow $f : A \bullet B \vdash C$ of $\mathcal{C}$ such that $f(x \bullet 1_B)\sigma^i_B = \varphi$ holds in $\mathcal{C}[x]$.

In other words, the function 'x is an *onto* and *one–one* function from the arrows $f : A \bullet B \vdash C$ of $\mathcal{C}$ to the S polynomials of $\mathcal{C}[x]$.

We can prove this theorem by a straightforward adaptation of the argument in Section 3. For example, for M categories, we have to check that for an M polynomial φ the **k** and **w** arrows are not involved in $\mu_x\varphi$, and that Lemmata 1–3 can be demonstrated for M polynomials φ and ψ, and M categories $\mathcal{C}$. For other S categories covered by the theorem, we similarly have to check that for an S polynomial φ the structural arrows rejected in S categories are not involved in $\mu_x\varphi$, and that the proofs of Lemmata 1–3 work.

In analogy with the demonstration of necessity of Section 5, we can show that structural arrows and equations we have assumed for them are necessary. This is now simpler than in Section 5, since the complications involving modalized objects are eschewed. In particular, we can completely justify pentagonal, hexagonal and triangular equations.

In [3], our restrictions concerning polynomials are matched by restrictions concerning structural rules in the deductive metalogic, and it is demonstrated that if the metalogic is appropriately restricted, BCK logic is minimal in the presence of $\wedge$ for proving the deduction theorem and its converse. The discrepancy between that and the Substructural Functional Completeness Theorem, which covers also M and R, is explained by the fact that [3] always allows the polynomials $\langle \psi, h \rangle$ and $\langle h, \psi \rangle$ where ψ is a polynomial of $\mathcal{C}[x]$ and h an arrow of $\mathcal{C}$. Here, we allow that for BCK and H polynomials, but not for M and R polynomials.

The Substructural Functional Completeness Theorem does not cover NL and AL categories. The corresponding notions of polynomial should presumably be

NL *polynomial*	(all clauses save (P0.2), (P1), (P2.1), (P2.2), (P3.1) and (P3.2))
AL *polynomial*	(all clauses save (P0.2), (P1), (P2.1) and (P3.1))

because, without ($\mu 2$), the **b** arrows enter into the definition of μ_x via clauses ($\mu 1$), ($\mu 2.1$), ($\mu 2.2$), ($\mu 3.1$) and ($\mu 3.2$), and the **c** arrows via clauses ($\mu 2.1$) and ($\mu 3.1$). If for $\varphi : B \vdash C$, the arrow $\mu_x \varphi$ were redefined so as to be of type $B \bullet A \vdash C$ instead of $A \bullet B \vdash C$, then for AL polynomials we would reject (P2.2) and (P3.2) instead of (P2.1) and (P3.1). With NL and AL polynomials we shall run into trouble in the proof of Lemma 1, because, for example, the NL and AL polynomial $x\sigma_{\mathrm{I}}$ is equal to $\sigma_A(\mathrm{I} \bullet x)$, by ($\sigma$), and $\mu_x(\sigma_A(\mathrm{I} \bullet x))$ is, by ($\mu 1.1$), ($\mu 2.1$) and ($\mu 0.1$), equal to

$$\sigma_A(\mathrm{I} \bullet \delta_A)\mathbf{b}^{\leftarrow}_{\mathrm{I},A,\mathrm{I}}(\mathbf{c}_{A,\mathrm{I}} \bullet \mathrm{I})\mathbf{b}^{\rightarrow}_{A,\mathrm{I},\mathrm{I}}$$

which involves both **b** and **c** arrows. The problem is that for S being NL or AL, the class of S polynomials considered as arrows in an S deductive system is not closed under equality of arrows in S categories (since we lack (P2.1), the arrow $\sigma_A(\mathrm{I} \bullet x)$ is neither an NL nor an AL polynomial), whereas for S being M, BCK, R or H, this is the case. The proof of Lemma 3, too, does not work for NL categories, because of an essential use of **b** arrows: f‘x is not an NL polynomial. However, the proof of Lemma 2 works, and we can demonstrate a weakened version of the Substructural Functional Completeness Theorem with NL or AL substituted for S if we dont require that f be unique; i.e. ‘x is *onto*, but not necessarily *one–one*.

7 CONCLUSION

We conclude this paper by brief indications about matters related to our results that we intend to treat in the future.

There is another way of restricting functional completeness, different from the way of the Substructural Functional Completeness Theorem. We may redefine μ_x so that for an arbitrary polynomial $\varphi : B \vdash C$ (i.e. an H polynomial), the arrow $\mu_x\varphi$ is not necessarily of type $A \bullet B \vdash C$ (nor $B \bullet A \vdash C$), but of some type $B[A] \vdash C$, where $B[A]$ is obtained from B by replacing factors D of B by $A \bullet D$ or $D \bullet A$, there are as many A's in $B[A]$ as there are x's in φ, and these A's are distributed in $B[A]$ in a way matching the distribution of x's in φ. This distribution requirement becomes unnecessary in the presence of **b** and **c** arrows, whereas there can be more A's in $B[A]$ than x's in φ in the presence of **k** arrows, and less in the presence of **w** arrows. With such a μ_x we may also be able to prove restricted functional completeness for various S categories.

Next, as functional completeness for cartesian closed categories enables us to extract systems of typed lambda terms as the internal languages of these categories, so modal functional completeness should lead to a kind of system of lambda terms with modalized types, such a system being the internal language of an S□ category. On the other hand, the restricted functional completeness of Section 6 leads to systems of typed lambda terms with restricted functional abstraction (for example, we may bind with a lambda operator exactly one variable, or not more than one, or at least one).

At the end of [3] it is supposed that freely extending a nonmodal S category to an S□ category might result in the former being a full subcategory of the latter. We suppose this could be demonstrated by a normalization technique, perhaps inspired by Gentzen's methods, or lambda conversion.

Another matter we will try to consider is the relationship between functional completeness and coherence. It is remarkable that assumptions about categories that Mac Lane needed to prove coherence for monoidal and symmetric monoidal categories should reappear as necessary for proving functional completeness. We conjecture that some sort of equivalence between coherence and functional completeness could be established. The reason for this equivalence should be that in functional completeness we transform a polynomial $\varphi : B \vdash C$ into $\mu_x \varphi : A \bullet B \vdash C$ irrespectively of where x occurs in φ. This requires that certain diagrams whose nodes are obtained from B by replacing factors D of B by $A \bullet D$ of $D \bullet A$ should commute. And the commuting of these diagrams should be sufficient for functional completeness.

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NOTE ADDED IN PRINT

The proof of Lemma 1 is incomplete as it stands, because it is not necessarily the case, as claimed in the second paragraph of that proof, that if φ and ψ are arrows of $\mathcal{C}$ and $\varphi = \psi$ holds in $\mathcal{C}\{x\}$, then $\varphi = \psi$ holds in $\mathcal{C}$. When this is not the case, the proof should be phrased as the corresponding part of the proof of Proposition 6.1 in [6, chapter I.6]. The calculations of the proof of Lemma 1 are sufficient for this rephrasing.

In the comments about the bijection 'x one should bear in mind that if $\Box A \bullet \_$ is not one-one on objects, the bijection is only local; i.e., it exists only between the arrows $f : \Box A \bullet B \vdash C$ and the polynomials $\psi : B \vdash C$ for B given in advance (and the same with $A \bullet \_$ instead of $\Box A \bullet \_$).

Kosta Došen
Université Paul Sabatier, France, and
Mathematical Institute of Belgrade, Yugoslavia

Zoran Petrić
University of Belgrade, Yugoslavia

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