The natural deduction normal form and coherence

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This talk is about coherence, a notion originated in category theory, and its proof theoretical counterpart. Everything will be explained through an example recently obtained in a joint work with Kosta Došen.

Turning disjunction into conjunction

$$\vee$$

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$$\vee$$

The same holds for derivations

$$\frac{\Phi}{\Phi \vee \Theta}$$

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The goal

$$
\cfrac{\cfrac{\cfrac{\cfrac{p_1 \wedge p_2}{p_1} \qquad \cfrac{p_1 \wedge p_2}{p_2}}{p_1 \wedge p_2}}{p_1} \qquad \cfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}
$$

is faithfully represented by (where $\Pi = p \vee p \vee p$)

$$
\cfrac{{\color{red}\Pi} \vee {\color{blue}\Pi} \qquad \cfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \cfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \cfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \cfrac{\color{blue}p}{\Pi \vee {\color{blue}p}} \quad \cfrac{\color{blue}p}{\Pi \vee {\color{blue}p}} \quad \cfrac{\color{blue}p}{\Pi \vee {\color{blue}p}}}{p \vee p \vee p \vee p} \; \text{5 times } \vee \text{ elim.}
$$

In the language of category theory

A skeleton of the category with finite coproducts freely generated by a single object has a subcategory isomorphic to a skeleton of the category with finite products freely generated by a countable set of objects.

The conjunctive system

Consider conjunction separated from other connectives.

alphabet: $p_1, p_2, \ldots, \wedge$

rules of inference: $$\frac{A \quad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

reductions:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} & \mathcal{E}\\ A & B\end{matrix}}{A \wedge B}}{A} \xrightarrow{\beta} \begin{matrix}\mathcal{D}\\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix} \xrightarrow{\eta} \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{A} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{B}}{A \wedge B}$$

Equality of derivations

Single premise and single conclusion derivations. The reductions are turned into equalities.

The following derivations from $p_1 \wedge p_2$ to $p_1 \wedge p_1$ are equal.

$$
\dfrac{\dfrac{\dfrac{\dfrac{{\color{red}p_1 \wedge p_2}}{{\color{red}p_1}} \quad \dfrac{p_1 \wedge p_2}{p_2}}{p_1 \wedge p_2}}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}
=
\dfrac{\dfrac{{\color{red}p_1 \wedge p_2}}{{\color{red}p_1}} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}
$$

The disjunctive system

Consider disjunction separated from other connectives.

alphabet: $p, \vee$

rules of inference: as usual

reductions: as usual

The formulae (up to associativity) may be identified with finite ordinals.

The representation of formulae

Let F be a mapping from conjunctive formulae to disjunctive formulae:

$$p_i \mapsto \underbrace{p \vee \ldots \vee p}_{\mathbf{p}_i},$$

where $\mathbf{p}_i$ is the i-th prime number, and if A and B are mapped respectively to

$$\underbrace{p \vee \ldots \vee p}_{m} \text{ and } \underbrace{p \vee \ldots \vee p}_{n},$$

then $A \wedge B$ is mapped to

$$\underbrace{p \vee \ldots \vee p}_{m \cdot n}.$$

Examples

$$p_1 \wedge p_2 \mapsto \underbrace{p \vee p \vee p \vee p \vee p \vee p}_{2 \cdot 3},$$

$$p_1 \wedge p_1 \mapsto \underbrace{p \vee p \vee p \vee p}_{2 \cdot 2}.$$

Derivability

For $m, n \geq 1$ it is always the case that

$$\underbrace{p \vee \ldots \vee p}_{m} \vdash \underbrace{p \vee \ldots \vee p}_{n}.$$

If we are interested just in derivability, then our mapping F is not conclusive since it is not true that

$$A \vdash B \Leftrightarrow FA \vdash FB.$$

For example, let A be p_1 and let B be p_2—there is a derivation from $p \vee p$ to $p \vee p \vee p$, but there is no derivation from p_1 to p_2. Hence, when one starts representing the derivations, it will not be the case that every disjunctive derivation represents a conjunctive derivation.

Derivability

The following definition of $\vdash$ at the right-hand side of

$$A \vdash B \Leftrightarrow FA \vdash FB.$$

makes this equivalence true. For $m, n \geq 1$,

$$\underbrace{p \vee \ldots \vee p}_{m} \vdash \underbrace{p \vee \ldots \vee p}_{n},$$

when every prime that divides n divides m, too. This gives an arithmetical characterization of derivability in the conjunctive system.

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{\color{red}{p_1}} \qquad \dfrac{p_1 \wedge p_2}{\color{blue}{p_1}}}{p_1 \wedge p_1}$$

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{\color{red}{p_1}} \qquad \dfrac{p_1 \wedge p_2}{\color{blue}{p_1}}}{\color{red}{p_1} \wedge \color{blue}{p_1}}$$

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Coherence

$$\dfrac{\dfrac{\color{red}{p_1} \wedge p_2}{\color{red}{p_1}} \qquad \dfrac{\color{blue}{p_1} \wedge p_2}{\color{blue}{p_1}}}{\color{red}{p_1} \wedge \color{blue}{p_1}}$$

For every conjunctive derivation in normal form there is a function (from the letter occurrences in the conclusion to the letter occurrences in the premise). Two different normal forms correspond to different functions. Hence, our derivation is completely determined by the following picture.

$$\color{purple}{p_1} \wedge p_2 \vdash \color{purple}{p_1} \wedge \color{purple}{p_1}.$$

Coherence

Dually, every disjunctive derivation from

$$\underbrace{p \vee p \vee p \vee p \vee p \vee p}_{m} \quad \text{to} \quad \underbrace{p \vee p \vee p \vee p \vee p \vee p}_{n}$$

is identified with a function from the ordinal m to the ordinal n.

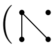
For every function $f : m \to n$ there is a derivation identified with f.

Extending F to derivations

We have to find a function from $2\cdot 3$ to $2\cdot 2$ that faithfully represents our derivation

$$p_1 \wedge p_2 \vdash p_1 \wedge p_1,$$

which can be determined also by the following triple

( $, p_1 \wedge p_2, p_1 \wedge p_1).$

Faithfulness means that two different derivations should be mapped to different functions.

Brauerian representation

Brauerian representation of

$$(\;, p_1 \wedge p_2, p_1 \wedge p_1)$$

is a function from $2 \cdot 3$ to $2 \cdot 2$ defined as follows.

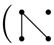
Identify the elements of the ordinal 6 with the elements of cartesian product 2×3 lexicographically ordered. Do the same with 4.

$$00 \;\; 01 \;\; 02 \;\; 10 \;\; 11 \;\; 12$$

$$00 \;\; 01 \;\; 10 \;\; 11$$

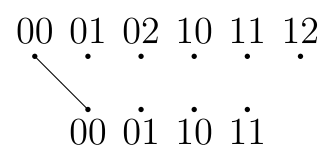
Brauerian representation

Brauerian representation of

$$\Big($$  $$, p_1 \wedge p_2, p_1 \wedge p_1\Big)$$

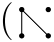
is a function from the ordinal $2 \cdot 3$ to the ordinal $2 \cdot 2$ defined as follows.

Identify the elements of the ordinal 6 with the elements of cartesian product 2×3 lexicographically ordered. Do the same with 4.



Brauerian representation

Brauerian representation of

$$\Big($$  $$, p_1 \wedge p_2, p_1 \wedge p_1\Big)$$

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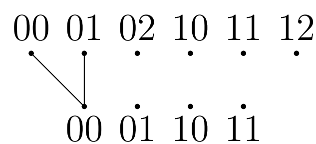

Brauerian representation

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$$(\ , p_1 \wedge p_2, p_1 \wedge p_1)$$

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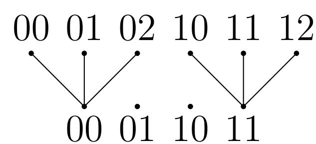


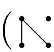
Different triples are represented by different functions.

Representing conjunctive by disjunctive derivations

Let $\mathcal{D}$ be a conjunctive derivation. Use the following steps in order to represent it by a disjunctive derivation.

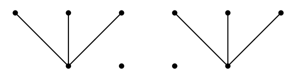
(1) Normalize $\mathcal{D}$.

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

(2) Find its triple. $($$, p_1 \wedge p_2, p_1 \wedge p_1)$

Representing conjunctive by disjunctive derivations

(3) Transform it into a function using brauerian representation.



(4) Find a disjunctive derivation identified with that function.

$$\dfrac{{\color{red}\Pi} \vee {\color{blue}\Pi} \qquad \dfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \dfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \dfrac{\color{red}p}{{\color{red}p} \vee \Pi} \quad \dfrac{\color{blue}p}{\Pi \vee {\color{blue}p}} \quad \dfrac{\color{blue}p}{\Pi \vee {\color{blue}p}} \quad \dfrac{\color{blue}p}{\Pi \vee {\color{blue}p}}}{p \vee p \vee p \vee p} \; \text{5 times} \vee \text{elim.}$$

Representing conjunctive by disjunctive derivations

It is not the case that the conjunctive inference rules are derivable from the disjunctive inference rules.

“Composition”, which corresponds to the cut rule in sequent systems is preserved by this representation.

Composition

Take the derivations

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1} \qquad \text{and} \qquad \dfrac{p_1 \wedge p_1}{p_1}$$

and paste them together

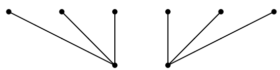
$$\dfrac{\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}}{p_1}$$

Composition

The normal form of the result is $\dfrac{p_1 \wedge p_2}{p_1}$.

The corresponding triple is ( $, p_1 \wedge p_2, p_1)$.

Its brauerian representation is given by:



A disjunctive derivation identified with this function is

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p}\ \text{5 times} \vee \text{elim.}$$

Composition

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p}$$

is equal to the derivation obtained by pasting together

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p}}{p \vee p \vee p \vee p} \text{ 5 times } \vee \text{ elim.}$$

and

$$\dfrac{p \vee p \vee p \vee p \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p} \text{ 3 times } \vee \text{ elim.}$$

Substitution

How can we treat $F(p_1) = p \vee p$ and $F(p_2) = p \vee p \vee p$ as variables in the formula $p \vee p \vee p \vee p \vee p \vee p$?

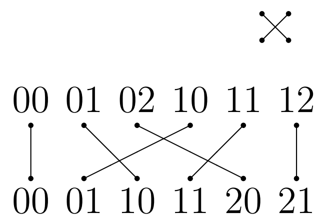
How to substitute $F(A)$ for $F(p_1)$ and $F(B)$ for $F(p_2)$ in the representation of our derivation

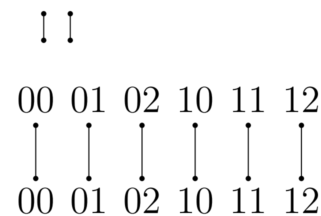
$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Substitution

The image of our conjunctive system in the disjunctive system has the universal property with respect to $\{F(p_1), F(p_2), \ldots\}$ in the sense that every mapping of that set to the set of disjunctive formulae extends in a unique way to a function that maps all the disjunctive formulae to the disjunctive formulae and all the derivations in the image of our representation to the disjunctive derivations. This function imitates substitution. However, it is not the operation of replacing words by words.

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_2} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_2 \wedge p_1} \qquad\qquad \dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_2}}{p_1 \wedge p_2}$$





The talk was based on: K. Došen and Z. Petrić, *Representing conjunctive deductions by disjunctive deductions*, (available at: arXiv)