On the Normalized Laplacian Energy(Randić Energy)

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Outline

Some Fundamental Definitions

In this talk, we report some bounds for the normalized Laplacian energy and Randić energy of a connected (bipartite) graph. Firstly, we give some fundamental definitions which are used in our results.

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Let G be undirected and simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. Furthermore, for $i = 1, 2, \cdots, n$, the *degree* of a vertex v_i in $V(G)$ will be denoted by d_i.

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In this talk, we report some bounds for the normalized Laplacian energy and Randić energy of a connected (bipartite) graph. Firstly, we give some fundamental definitions which are used in our results.

Let G be undirected and simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. Furthermore, for $i = 1, 2, \cdots, n$, the *degree* of a vertex v_i in $V(G)$ will be denoted by d_i.

If any vertices v_i and v_j are adjacent, then we use the notation $v_i{\sim}v_j$.

Some Fundamental Definitions

It is known that we also have the Laplacian matrix related to the adjacency and diagonal matrices. In fact, for a diagonal matrix $D(G)$ whose (i,i)-entry is d_i, the Laplacian matrix $L(G)$ of G is defined as $L(G) = D(G) - A(G)$. Since $A(G)$ and $L(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So we assume that

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G)$$

$$(\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G))$$

are the adjaceny (Laplacian) eigenvalues of G.

Some Fundamental Definitions

Because the graph G is assumed to be connected, it has no isolated vertices and therefore the matrix $D(G)^{-1/2}$ is well defined. Then

$$L^* = L^*(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}$$

is called the normalized Laplacian matrix of the graph G. Its eigenvalues are

$$\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) \geq \rho_n(G).$$

It is convenient to write the normalized Laplacian matrix as $I_n - R$, where R is the so-called Randić matrix , whose (i,j)-entry is

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & , \textit{ if } v_i \sim v_j \\ 0 & , \textit{otherwise} \end{cases}$$

[Maden et al.-2010]

The Randić eigenvalues $q_1(G)$, $q_2(G)$,..., $q_n(G)$ of the graph G are the eigenvalues of its Randić matrix. Since R is real symmetric matrix, its eigenvalues are real number. So we can order them so that

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G).$$

Some Fundamental Definitions

M-energy of G is

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i(M) - \frac{tr(M)}{n} \right|,$$

where $tr(M)$ is the trace of M. The energy of a graph was introduced by Gutman in 1978 as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Recently, the adjacency enery, Laplacian energy, Randić energy and normalized Laplacian energy of a graph has received much interest. Along the some lines, the energy of more general matrices and sequences has been studied.

Some Fundamental Definitions

Using the above equality with M taken to be L^*, the normalized Laplacian energy and Randić energy of a graph G is

$$E_L * (G) = \sum_{i=1}^{n} |\rho_i - 1| and E_R(G) = \sum_{i=1}^{n} |q_i|,$$

respectively. Since $L^* = I_n - R$, it easy to see that this is equivalent to

$$E_L * (G) = \sum_{i=1}^{n} |q_i| = E_R(G).$$

In the literature, some basic properties of $E_L * (G)$ may be found.

Some Fundamental Definitions

Now, recall that the *Randić index* of a graph G is defined as

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$$R_\alpha = R_\alpha(G) = \sum_{v_i \sim v_j} (d_i d_j)^\alpha,$$

where the summation is over all edges $v_i v_j$ in G, and $\alpha \neq 0$ is a fixed real number.

The general Randić index when $\alpha = -1$ is

$$R_{-1} = R_{-1}(G) = \sum_{v_i \sim v_j} \frac{1}{d_i d_j},$$

Preliminary Results

Now, we recall some results from spectral graph theory and state a few analytical spectral inequalities for our work.

Lemma (2.2)

[F. Chung -1997] Let the normalized Laplacian eigenvalues of G be given as

$$\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n = 0.$$

Then

$$0 \leq \rho_i \leq 2.$$

Morover $\rho_1 = 2$ if and only if G has a connected bipartite nontrivial component.

Preliminary Results

Lemma (2.3)

[P. Zumstein -2005] Let G be a graph with n vertices and normalized Laplacian matrix L^* without isolated vertices. Then

$$\sum_{i=1}^{n} \rho_i = n$$

and

$$\sum_{i=1}^{n} {\rho_i}^2 = n + 2R_{-1}.$$

Preliminary Results

Lemma (2.4)

[L. Shi -2009] Let G be a graph of order n with no isolated vertices. Suppose that G has minimum verwerte degree equal to d_{min} and maximum vertex degree equal to d_{max}. Then

$$\frac{n}{2d_{max}} \leq R_{-1} \leq \frac{n}{2d_{min}}$$

Equality occurs in both bounds if and only if G is a regular graph.

Main Results

After all above materials, we are ready to present our main results. The following results are also valid for Randić energy.

Main Results

Theorem (3.1)

Let G *be undirected , simple and connected graph with* n,$n \geq 3$ *vertices . Then*

$$1+\sqrt{2R_{-1}+(n-1)(n-2)\Delta^{\frac{2}{n-1}}} \leq E_{L^*}(G) = E_R(G)$$

$$\leq 1+\sqrt{(n-2)(2R_{-1}-1)+(n-1)\Delta^{\frac{2}{n-1}}} \qquad (1)$$

where $\Delta = det(I_n - L^*)$.

Main Results

Remark

In [Hakimi-Nezhaad et al.-2014], Hakimi-Nezhaad et al. obtained the following lower bound for the normalized Laplacian energy :

$$E_{L^*}(G) \geq 1 + \sqrt{\frac{n}{d_{max}} - 1 + 2\left(\frac{n-1}{2}\right)\Delta^{\frac{2}{n-1}}}. \quad (2)$$

From Lemma (2.4), the lower bound (1) is better than (2).

Main Results

Considering Lemma (2.4) and the inequality (1), we arrive at the following result.

Corollary

Let G be a graph of order n with no isolated vertices. Suppose that G has minimum vertex degree equal to d_{min} and maximum vertex degree equal to d_{max}. Then

$$1+\sqrt{\frac{n}{d_{max}}-1+(n-1)(n-2)\Delta^{\frac{2}{n-1}}} \leq E_{L^*}(G)=E_R(G)$$

$$\leq 1+\sqrt{(n-2)\left(\frac{n}{d_{min}}-1\right)+(n-1)\Delta^{\frac{2}{n-1}}} \qquad (3)$$

where $\Delta = det(I_n - L^*)$.

Main Results

Remark

It can be easily to see that the bound (1) *is better than all results which was obtained for* $E_{L^*}(G)$ *in [Gutman et al. -2015] and [Cavers et al.-2010] on many examples.*

We consider the graph $G = (V, E)$ *with vertex set* $V = \{v_1, v_2, v_3, v_4\}$ *and the edge set* $E = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4\}$*. For this graph,* $E_{L^*}(G) = 2.4574$*. While the bound* (1) *gives* $E_{L^*}(G) \geq 2.406$*, the lower bounds in [Gutman et al. -2015, (3.8)] and [Cavers et al.-2010, Theorem 16] give* $E_{L^*}(G) \geq 1$ *and* $E_{L^*}(G) \geq 2.3016$*, respectively. Similarly, while the upper bound* (1) *gives* $E_{L^*}(G) \leq 2.59$*, the upper bound in [Cavers et al.-2010, Lemma 1] gives* $E_{L^*}(G) \leq 2.708$*.*

Main Results

If G has k connected components, in particular, $G_1, G_2, ..., G_k$, then

$$E_{L^*}(G) = \sum_{i=1}^{k} E_{L^*}(G_i)$$

Now, we present a bound on the normalized Laplacian energy of a graph with k connected components.

Main Results

Theorem (3.2)

Let G be a graph of order n with k connected components and no isolated vertices. Then

$$k+\sqrt{2R_{-1}-k+(n-k-1)(n-k)\Delta^{\frac{2}{n-k}}} \leq E_{L^*}(G)=E_R(G)$$

$$\leq k+\sqrt{(n-k-1)(2R_{-1}-k)+(n-k)\Delta^{\frac{2}{n-k}}}$$

where $\Delta = det(I_n - L^*)$.

Main Results

Taking $k = 2$ in Theorem (3.2), we obtain the following result for the normalized Laplacian energy (Randić energy) of connected bipartite graphs.

Main Results

Corollary

Let G be a connected bipartite graph with $n \geq 3$ vertices. Then

$$2+\sqrt{2R_{-1}-2+(n-3)(n-2)\Delta^{\frac{2}{n-2}}} \leq E_{L^*}(G) = E_R(G)$$

$$\leq 2+\sqrt{(n-3)(2R_{-1}-2)+(n-2)\Delta^{\frac{2}{n-2}}}$$

where $\Delta = det(I_n - L^*)$.

Main Results

Recently, the concept of Randić energy was studied intensively in the literature. One can easily see that the bound (1) is better than the some previous results.

For example, the lower bound which was obained for Randić energy in [Das et al. -2014] is same with the bound (3). But as we mentioned in the begining of this work, the lower bound (1) is better than (3). Again, in [Bozkurt et al. -2013] and [Li et al. -2015], it was presented the following upper bound for Randić energy

$$E_R(G) \leq 1 + \sqrt{(n-1)(2R_{-1}-1)}. \tag{4}$$

Main Results

Using the arithmetic-geometric mean inequality, it follows that the upper bound (1) is better than the upper bound (4).
Also, for the other results which was obtained over Randić energy previously, it can be seen that the bound (1) is better on many examples.

THANK YOU FOR YOUR ATTENTION...

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