On the maximum and minimum Zagreb indices of some classes of trees

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- P_n *the path*, $K_{1,n-1}$ *the star*

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On the maximum and minimum Zagreb indices of trees with a given number of maximum degree vertices

Theorem (Gutman, Das)

Among n-vertex trees, the star $K_{1,n-1}$ has the maximum, and the path P_n has the minimum value of the first (second) Zagreb index. If T_n is an n-vertex tree, different from the star or the path, then

$$M_1(P_n) < M_1(T_n) < M_1(K_{1,n-1})$$

and

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Trees with a given number of maximum degree vertices whose M_1 is minimum

Theorem (B. Borovićanin, T.A. Lampert)

Let $T \in \mathcal{T}_{n,k}$, where $1 \leq k \leq \frac{n}{2} - 1$. Then

$$M_1(T) \geqslant 2k + 4n - 6 \tag{1}$$

with equality if and only if T has the degree sequence

$$(\underbrace{3, \ldots, 3}_{k}, \underbrace{2, \ldots, 2}_{n-2k-2}, \underbrace{1, \ldots, 1}_{k+2}) \tag{2}$$

Trees with a given number of maximum degree vertices whose M_1 is maximum

Theorem (B. Borovićanin, T.A. Lampert)

Let $T \in \mathcal{T}_{n,k}$, where $1 \leq k \leq \frac{n}{2} - 1$. Then

$$M_1(T) \leqslant k\Delta^2 + p(\Delta-1)^2 + \mu^2 + n - k - p - 1, \tag{3}$$

with equality if and only if T has the degree sequence

$$(\underbrace{\Delta,\ldots,\Delta}_{k},\underbrace{\Delta-1,\ldots,\Delta-1}_{p},\mu,\underbrace{1,\ldots,1}_{n-k-p-1}), \tag{4}$$

where $\Delta = \lfloor \frac{n-2}{k} \rfloor + 1$, $p = \lfloor \frac{n-2-k(\Delta-1)}{\Delta-2} \rfloor$ and
$\mu = n - 1 - k(\Delta-1) - p(\Delta-2)$.

Theorem (B. Borovićanin, T.A. Lampert)

Let $T \in \mathcal{T}_{n,k}$, where $1 \leq k \leq \frac{n}{2} - 1$. Then

$$M_2(T) \geqslant \begin{cases} 3k + 4n - 10, & \text{if } n \geq 3k+1 \\ 6k + 3n - 9, & \text{if } n < 3k+1. \end{cases} \tag{5}$$

The equality holds if and only if the following three conditions are satisfied.

(i) The tree T has the vertex degree sequence $(\underbrace{3,\ldots,3}_{k},\underbrace{2,\ldots,2}_{n-2k-2},\underbrace{1,\ldots,1}_{k+2})$.

(ii) Between any two vertices of degree 3 in T there should be at least one vertex of degree 2, if possible.

(iii) The remaining vertices of degree 2 (if they exist) in T are placed between two vertices of degree 2, or between a vertex of degree 2 and a vertex of degree 3.

Theorem (B. Borovićanin, T.A. Lampert)

Let $T \in \mathcal{T}_{n,k}$, where $1 \le k \le \frac{n}{2} - 1$. Then

$$M_2(T) \le (k-1)\Delta^2 + 2p(\Delta-1)^2 + \mu(\Delta+\mu-1) + \Delta(n-k-(\Delta-1)p-\mu), \tag{6}$$

where $\Delta = \lfloor \frac{n-2}{k} \rfloor + 1$, $p = \lfloor \frac{n-2-k(\Delta-1)}{\Delta-2} \rfloor$ and $\mu = n-1-k(\Delta-1)-p(\Delta-2)$. The equality holds if and only if the following conditions are satisfied.

(i) The tree T has the vertex degree sequence $(\underbrace{\Delta, \ldots, \Delta}_{k}, \underbrace{\Delta-1, \ldots, \Delta-1}_{p}, \mu, \underbrace{1, \ldots, 1}_{n-k-p-1})$.

(ii) Every vertex of degree $\Delta - 1$ is adjacent to a vertex of degree Δ and to $\Delta - 2$ pendent vertices.

(iii) The vertex of degree μ (when $\mu > 1$) is adjacent to a vertex of the degree Δ and to $\mu - 1$ pendent vertices.

(iv) The remaining pendent vertices are attached to the vertices of degree Δ.

- A subset D of $V(G)$ such that each vertex of G that is not contained in D is adjacent to at least one vertex of D is called *dominating set* of G

On the extremal trees with a given domination number

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- B. Borovićanin, B. Furtula, On extremal Zagreb indices of trees with given domination number, *Appl. Math. Comput.* **279** (2016), 208218.

On the maximum Zagreb indices of trees with a given domination number

Theorem (B. Borovićanin, B. Furtula)

Let T be a tree with domination number γ. Then

$$M_1(T) \leqslant (n-\gamma)(n-\gamma+1) + 4(\gamma-1) \tag{7}$$

and

$$M_2(T) \leqslant 2(n-\gamma+1)(\gamma-1) + (n-\gamma)(n-2\gamma+1). \tag{8}$$

Equality in both cases hold if and only if $T \cong S_{n,n-\gamma}$, where $S_{n,n-\gamma}$ is a spur obtained from the star $K_{1,n-\gamma}$ by attaching a pendent edge to its $\gamma-1$ pendent vertices.

Trees with a given domination number whose M_1 is minimum

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- $\mathcal{D}(n, \gamma)$ is a set of n-vertex trees T with domination number γ such that T consists of the stars of orders $\lfloor \frac{n-\gamma}{\gamma} \rfloor$ and $\lceil \frac{n-\gamma}{\gamma} \rceil$ with exactly $\gamma - 1$ pairs of adjacent leaves in neighboring stars

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Theorem (B. Borovićanin, B. Furtula)

Let T be a tree on n vertices with domination number γ, where $1 \leqslant \gamma \leqslant \frac{n}{3}$. Then,

$$M_1(T) \geqslant -\gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor^2 + (2n-\gamma) \left\lfloor \frac{n-1}{\gamma} \right\rfloor + 6(\gamma - 1)\,. \tag{9}$$

The equality holds if and only if $T \in \mathcal{D}(n,\gamma)$.

Trees with a given domination number whose M_1 is minimum

- $\frac{n}{3} \leqslant \gamma \leqslant \frac{n}{2}$
- $\mathcal{G}(n, \gamma)$ is a set of trees T on n vertices with domination number γ, such that every vertex from T has at most one pendent neighbor and
  (i) there exists a minimum dominating set D of T containing $3\gamma - n - 2$ vertices of degree 3 and $2n - 4\gamma$ vertices of degree 2, while the set $\overline{D}$ contains $n - 2\gamma + 2$ vertices of degree 2 and $3\gamma - n$ pendent vertices, or
  (ii) there exists a minimum dominating set D of T containing $n - 2\gamma$ vertices of degree 2 and $3\gamma - n$ pendent vertices, while the set $\overline{D}$ contains $2n - 4\gamma + 2$ vertices of degree 2, $3\gamma - n - 2$ vertices of degree 3 and every vertex from $\overline{D}$ has exactly one neighbor in D.

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Trees with a given domination number whose M_1 is minimum

Theorem (B. Borovićanin, B. Furtula)

Let T be a tree on n vertices with domination number γ, where $\frac{n}{3} \leqslant \gamma \leqslant \frac{n}{2}$. Then,

$$M_1(T) \geqslant \begin{cases} 4n-6 & \text{if } \gamma = \left\lceil \frac{n}{3} \right\rceil \\ 2n+6\gamma = 10 & \text{if } \frac{n+3}{3} \leqslant \gamma \leqslant \frac{n}{2} \end{cases} \tag{10}$$

with equality if and only if $T \cong P_n$, for $\gamma = \lceil \frac{n}{3} \rceil$, or $T \in \mathcal{G}(n, \gamma)$, otherwise.

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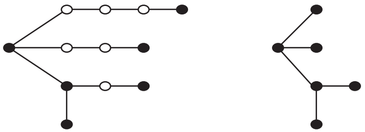

T $\qquad\qquad$ $S(T)$

Let $S(T)$ be a tree obtained from T by replacing each segment of T by an edge, it is called *the squeeze* of T

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- $\mathcal{ST}_{n,k}$ for $3 \leq k \leq n-2$
- A tree is said to be a *starlike of degree k* if it contains exactly one vertex of degree greater than two (the central vertex), and the central vertex has degree k ($k \geq 3$)

The upper bound for M_2 of trees with a given number of sements

Theorem (B. Borovićanin)

Let $T \in \mathcal{ST}_{n,k}$, where $3 \leq k \leq n-2$, then

$$M_2(T) \leqslant \begin{cases} 2k^2 - 6k + 4n - 4\,, & n \geq 2k+1 \\ k(n-3) + 2n - 2\,, & n < 2k+1 \end{cases} \tag{11}$$

The upper bound is attained if and only if T is an n-vertex starlike tree of degree k, such that an arbitrary pendent vertex is adjacent to a vertex of degree 2, for $2k+1 \leq n$, or the central vertex of degree k has exactly $2k+1-n$ pendent neighbors, for $n < 2k+1$.

The lower bound for M_2 of trees with a given number of segments

Denote by $\mathbb{ST}_O(n,k)$, for odd k, the set of all n-vertex trees with the degree sequence

$$(\underbrace{3,\ldots,3}_{\frac{k-1}{2}},\underbrace{2,\ldots,2}_{n-k-1},\underbrace{1,\ldots,1}_{\frac{k+3}{2}}) \tag{12}$$

such that there is at least one vertex of degree 2 between any two vertices of degree 3, and the remaining vertices of degree 2 (if exist) can be placed arbitrarily either between two vertices of degree 2 or between a vertex of degree 2 and a vertex of degree 3.

The lower bound for M_2 of trees with a given number of segments

Denote by $\mathbb{ST}_E(n,k)$, for even k, the set of all n-vertex trees with the degree sequence

$$(4,\underbrace{3,\ldots,3}_{\frac{k-4}{2}},\underbrace{2,\ldots,2}_{n-k-1},\underbrace{1,\ldots,1}_{\frac{k+4}{2}}) \tag{13}$$

such that the unique vertex of degree 4 has three pendent neighbors and a neighbor of degree 2. The vertices of degree 2 are placed between the vertices of degree 3 (at least one vertex between any two vertices of degree 3, if possible) and the remaining vertices of degree 2 are placed either between two vertices of degree 2 or between the vertex of degree 4 and a vertex of degree 2, or between a vertex of degree 3 and a vertex of degree 2.

The lower bound for M_2 of trees with a given number of segments

Theorem (B. Borovićanin)

Let $T \in \mathcal{ST}_{n,k}$, where $3 \le k \le n-2$, then

$$M_2(T) \geqslant \begin{cases} \dfrac{8n+3k-23}{2}\,, & n \ge \frac{3k-1}{2} \text{ and k odd} \\ 3n+3k-12\,, & n < \frac{3k-1}{2} \text{ and k odd} \\ \dfrac{8n+3k-18}{2}\,, & n \ge \frac{3k-2}{2} \text{ and k even} \\ 3n+3k-10\,, & n < \frac{3k-2}{2} \text{ and k even} \end{cases} \tag{14}$$

The equality holds if and only if $T \in \mathbb{ST}_O(n,k)$, for odd k, or $T \in \mathbb{ST}_E(n,k)$, for even k.

THANK YOU FOR YOUR ATTENTION!