On graphs with smallest eigenvalue at least -3

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Outline

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Defintion

Graph: $\Gamma = (V, E)$ where V vertex set, $E \subseteq \binom{V}{2}$ edge set.

- All graphs in this talk are simple.
- $x \sim y$ if $xy \in E$.
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- The adjacency matrix A of a graph Γ is the matrix whose rows and columns are indexed by its vertices such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.
- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.
- In this talk I will be mainly interested in the smallest eigenvalue of Γ, denoted by $\lambda_{\min}$.

A structure theory for graphs with fixed smallest eigenvalue?

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- I will give some ideas for this theory in this talk.

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We say a connected graph with smallest eigenvalue at least -2 and adjacency matrix A is a generalised line graph if there exists an integral matrix N such that $A + 2I = NN^T$.

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Note that if I can take N a matrix with only $0'$s and $1'$s then the graph is a line graph. So a generalized line graph is a generalization of a line graph.

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We give now a sketch of proof for this result.

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- Then Λ is an even lattice, generated by norm square root of two vectors, so it is a root lattice and it is irreducible as Γ is connected.

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- The irreducible root lattices were classified by Witt, and are of type A_n, D_n or E_6, E_7, E_8.
- The first two cases give us generalised line graphs, and for the last three lattices one can show that the number of vertices is at most 36.

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- We do not have a similar classification for lattices generated by square root 3 vectors.
- Note that if Γ has $\lambda_{\min} \geq -\lambda$ for λ a positive integer, then Γ can not contain an induced $(\lambda^2 + 1)$-claw.
- Let $\tilde{K}_{2t}$ be a K_{2t} with one extra vertex adjacent to half of the vertices of the K_{2t}.
- Then it is easy to see that $\lim_{t\to\infty} \lambda_{\min}(\tilde{K}_{2t}) = -\infty$. (Use the equitable partition with quotient matrix
$$Q = \begin{bmatrix} t-1 & t & 0 \\ t & t-1 & 1 \\ 0 & t & 0 \end{bmatrix})$$

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Theorem

Let Γ be a graph with smallest eigenvalue $\lambda_{\min}$. Then the following hold.

1. For a real number $\lambda \geq 1$ there exists a positive integer $t = t(\lambda)$ such that Γ contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph if the minimal eigenvalue of Γ satisfies $\lambda_{\min}(\Gamma) \geq -\lambda$.
2. For a positive integer t there exists a positive real number $\lambda = \lambda(t)$ such that if Γ contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph, then $\lambda_{\min}(\Gamma) \geq -\lambda$.

The main idea is that in order to bound the smallest eigenvalue, you need to obtain some structure in the graph. This structure is of independent interest. But first I will discuss another result of Hoffman which proof used the structure as described above.

Smallest eigenvalue $-1-\sqrt{2}$

Hoffman (1977) also showed the following result:

Theorem

Let $2 < \lambda < 1+\sqrt{2}$. Then there is constant $K = K(\lambda)$ such that if Γ is a connected graph with minimal valency at least K and smallest eigenvalue $\lambda_{\text{min}} \geq -\lambda$, then Γ is a generalised line graph. In particular $\lambda_{\text{min}} \geq -2$.

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- Woo and Neumaier (1995) generalised this result by Hoffman by going slightly below $-1-\sqrt{2}$.
- K., Yang and Yang obtained a result for graphs with smallest eigenvalue at least -3. We will see this below.

Outline

Hoffman Graphs 1

Hoffman graphs were introduced by Woo and Neumaier (1995) formalising the concepts Hoffman used for his 1977-result.

Hoffman Graph

- A **Hoffman Graph** $\mathfrak{G} = (G = (V, E), \ell : V \to \{f, s\})$, such that any two vertices with label f are non-adjacent. In other words, it is a graph with a distinguished independent set $F = \{v \in V \mid \ell(v) = f\}$ of vertices.

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- The vertices in the independent set F, we will call **fat** and the rest of the vertices we will call **slim**.
- A Hoffman graph $\mathfrak{H}$ is called **fat** if every slim vertex has at least one fat neighbour.
- The subgraph induced on $S := \{v \in V \mid \ell(v) = s\}$ is called the slim subgraph of $\mathfrak{H}$.

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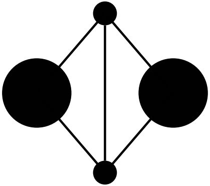
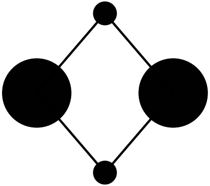
- Hoffman graphs and especially fat Hoffman graphs give a good way to construct graphs with unbounded number of vertices such that the smallest eigenvalue is at least a fixed number.
- We will later construct fat Hoffman graphs from graphs by representing some dense subgraphs by fat vertices.

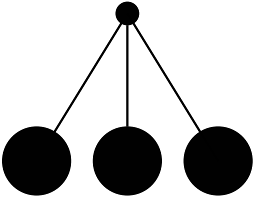
Examples



$\mathfrak{H}_3, \lambda_{\min} = -3$



$\mathfrak{H}_6, \lambda_{\min} = -4$



$\mathfrak{H}_4, \lambda_{\min} = -3$

Eigenvalues

Eigenvalues of Hoffman graphs

- Let $\mathfrak{H}$ be a Hoffman graph with fat vertex set F and slim vertex set S.
- The adjacency matrix A of $\mathfrak{H}$ can be written in the following form:

$$A := \left(\begin{array}{c|c} B & C \\ \hline C^T & 0 \end{array} \right),$$

where the block B corresponds to the adjacency matrix on the set S, and so on.

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where the block B corresponds to the adjacency matrix on the set S, and so on.
- The eigenvalues of $\mathfrak{H}$ are the eigenvalues of the special matrix $Sp := B - CC^T$.
- As CC^T is a positive semidefinite matrix $\lambda_{\min}(B) \geq \lambda_{\min}(\mathfrak{H})$.

Replacing fat vertices by cliques

One reason for the definition of the smallest eigenvalue of a Hoffman graph is the following theorem of Hoffman and Ostrowski (1960's):

Theorem

Let $\mathfrak{H}$ be a Hoffman graph with at least one fat vertex. Define the graph G_n as follows: Replace the fat vertices with complete graphs $C_f (f \in F)$ with n vertices and each vertex of C_f has the same neighbours in S as f. Then $\lim_{n\to\infty} \lambda_{\min}(G_n) = \lambda_{\min}(\mathfrak{H})$.

Outline

Direct sums

In order to state our second result we need to introduce direct sums.

Direct sum

Let $\mathfrak{H}$ have special matrix

$$Sp = \left(\begin{array}{c|c} Sp_1 & 0 \\ \hline 0 & Sp_2 \end{array} \right).$$

Let $\mathfrak{H}_i$ be the induced Hoffman subgraph of $\mathfrak{H}$ with special matrix Sp_i for $i = 1, 2$. We say that $\mathfrak{H}$ is the direct sum of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and write $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$.

A more combinatorial (but equivalent) definition is as follows:

Direct sums

Let $\mathfrak{H}' = (F' \cup S', E')$ and $\mathfrak{H}'' = (F'' \cup S'', E'')$ be two Hoffman graphs, such that

- $S' \cap S'' = \emptyset$;
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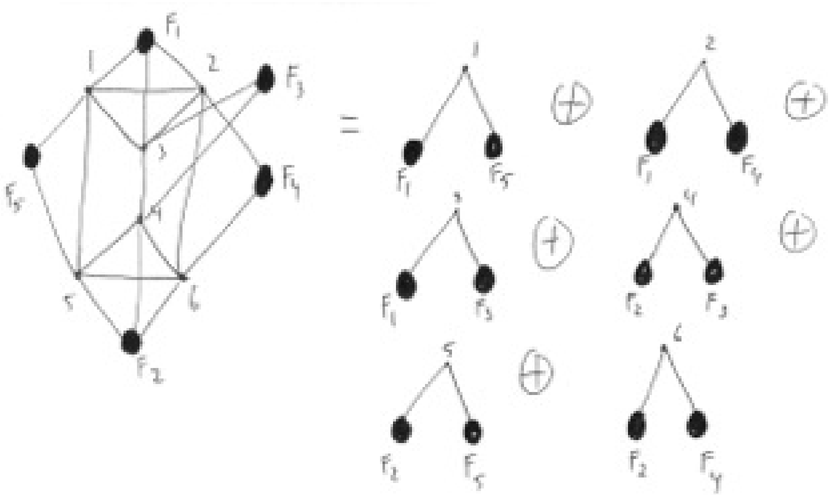
- $s' \in S'$ and $s'' \in S''$ have at most one common fat neighbour in $F' \cap F''$.
- The Hoffman graph $\mathfrak{H}' \oplus \mathfrak{H}''$ has as vertex set $S \cup F$ where $S = S' \cup S''$ and $F = F' \cup F''$.
- The induced subgraphs on $S' \cup F'$ resp. $S'' \cup F''$ are $\mathcal{H}'$ resp. $\mathcal{H}''$.
- $s' \in S'$ and $s'' \in S''$ are adjacent if and only if they have exactly one common fat neighbour.

Example

Decomposing a line graph.



Theorem (Woo & Neumaier)

- Let $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ where $\mathfrak{H}'$ and $\mathfrak{H}''$ are Hoffman graphs.
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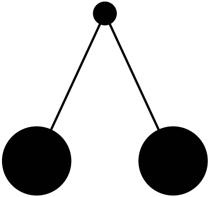
This means that I can construct large graphs with smallest eigenvalue at least a fixed number using the direct sum construction.

$\mathcal{F}$-line graph

Let $\mathcal{F}$ be a family of Hoffman graphs. A graph is called **$\mathcal{F}$-line graph** if it is an induced subgraph of the slim subgraph of $\bigoplus_{i=1}^{t} \mathfrak{F}_i$ where $\mathfrak{F}_i \in \mathcal{F}$.

Line and generalised line graphs

- A $\{\mathfrak{H}_1\}$-line graph is exactly the same as a line graph.
- A $\{\mathfrak{H}_1, \mathfrak{H}_2\}$-line graph is exactly the same as a generalised line graph. (You can also take this as the definition of a generalised line graph)



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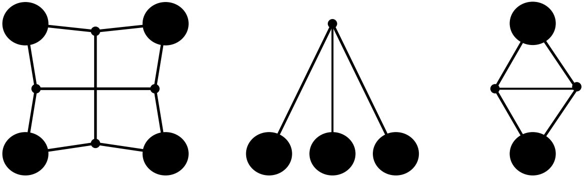
$\mathfrak{H}_2, \lambda_{\min} = -2$

We need the following fat Hoffman graphs for the next result:



$\mathfrak{H}_5$ $\mathfrak{H}_4$ $\mathfrak{H}_3$

ℓ-plex

A ℓ-plex is a graph whose complement has maximal valency at most ℓ. They are studied in network theory to understand these networks better.

Theorem

- Let G be a connected graph with smallest eigenvalue at least -3.
- There exist positive integers ℓ and C such that if
  - the valency k_x of any vertex x is at least ℓ;
  - and the order of any 10-plex containing a vertex x is at most $k_x - C$,

then G is a $\{\mathfrak{H}_3, \mathfrak{H}_4, \mathfrak{H}_5\}$-line graph.

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We can generalise this result to $-4, -5, \ldots$

A similar result as above.

Theorem

- Let G be a connected graph with smallest eigenvalue at least -3.
- There exist positive integers ℓ and C such that if
  - the valency k_x of any vertex x is at least ℓ;
  - and the average valency of the local graph in vertex x is at most $k_x - C$,

  then G is a $\{\mathfrak{H}_3, \mathfrak{H}_4, \mathfrak{H}_5\}$-line graph.

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- One reason is that there are infinitely many -3-irreducible fat Hoffman graphs with smallest eigenvalue -3.
- In Jang, K., Munemasa and Taniguchi (2014) we did some work towards the classification of these fat Hoffman graphs.
- I am working with Yan Ran Li to complete the work of Jang et al.

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- If you know your graph is regular (you can see this from the spectrum) and the second largest eigenvalue is not too large then by a similar argument as for the Hoffman coclique bound, it is sometimes possible to obtain a good upper bound for the number of vertices of a *t*-plex. I will give an example below.

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- If you know your graph is regular (you can see this from the spectrum) and the second largest eigenvalue is not too large then by a similar argument as for the Hoffman coclique bound, it is sometimes possible to obtain a good upper bound for the number of vertices of a t-plex. I will give an example below.
- If you graph is regular and has at most 4 distinct eigenvalues, then it is walk-regular. This means that the number of triangles through a vertex x does not depend on the vertex x. We will see examples below.

The Hamming graph $H(3, q)$

- The Hamming graph $H(D, q)$ has as vertex set Q^D where Q is a set with cardinality q.
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- Applying our theorem gives that G is locally $3 \times K_{q-1}$ if q is very large.
- Bang et al. (2008) showed earlier that this is the case for $q \geq 36$, and that they are determined by their spectrum if $q \geq 36$.

The Johnson graph $J(n, 3)$

- The Johnson graph $J(n, t)$ has as vertex set $\binom{N}{t}$ where N is a set with cardinality n.
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- $J(n,3)$ has spectrum $[3(n-3)]^1, [2(n-4)-1]^{n-1}, [n-7]^{n(n-1)/2}, [-3]^{n(n-1)(n-5)/6}$.

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- The Johnson graph $J(n,t)$ has as vertex set $\binom{N}{t}$ where N is a set with cardinality n.
- Two t-sets A and B are adjacent if $\#A \cap B = t-1$.
- $J(n,3)$ has spectrum $[3(n-3)]^1, [2(n-4)-1]^{n-1}, [n-7]^{n(n-1)/2}, [-3]^{n(n-1)(n-5)/6}$.
- Hence any graph G cospectral with $J(n,3)$ is walk-regular and the local graph has average valency $n-2$.
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- Using our result shows that $J(n, 3)$ is the point graph of a partial linear space with three lines through any point, if n is very large.
- Van Dam et al. (2006) gave two constructions to obtain graphs cospectral with $J(n, 3)$, one used Godsil-McKay switching, the other construction used partial linear spaces.

2-clique extension of a grid graph

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- Then we obtain that H is a $\{\mathfrak{H}_3, \mathfrak{H}_4, \mathfrak{H}_5\}$-line graph.
- Using this fact, Aida Abiad, QianQian Yang and myself showed that the 2-clique extension of the $t \times t$-grid is determined by its spectrum if t large enough.

Outline

Grassmann graphs

- The Grassmann graph $J_q(n, D)$ is the graph with vertex set the set of the D-dimensional subspaces of an n-dimensional vector space over the finite field with q elements, where q is a prime power and $n \geq 2D$ are positive integers.

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- Metsch showed that the Grassmann graph $J_q(n, D)$ is characterised as a distance-regular graph if $n \geq 2D + 2$, unless $q \leq 3$.
- Van Dam and K. constructed the twisted Grassmann graphs in 2005, which have the same intersection numbers as $J_q(2D + 1, D)$. So the Grassmann graph $J_q(2D + 1, D)$ is not characterised by its intersection numbers.

Grassmann graphs, 2

- What do we know for $J_q(2D, D)$?
- With Gavrilyuk (201?) we showed that the local subgraph (that is, the graph induced on the neighbours of a fixed vertex) of a distance-regular graph with the same intersection numbers as $J_q(2D, D)$, has the same spectrum as the q-clique extension of a certain square grid.

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- If we know that q-clique extension of a square $(t \times t)$-grid is characterised by its spectrum we can show that the corresponding Grassman graph is determined by its intersection numbers.
- For t small compared to q, these q-clique extensions are NOT characterised by their spectrum, but I suspect they are if t is large compared to q.

2-clique extension of a square grid

- We have seen: The 2-clique extension of the $(t \times t)$-grid is characterized by its spectrum if $t >> 0$.

2-clique extension of a square grid

- We have seen: The 2-clique extension of the $(t \times t)$-grid is characterized by its spectrum if $t >> 0$.
- This implies that $J_2(2D, D)$ is determined by its intersection numbers if D is large enough.

Thank you for your attention.