SOME REMARKS ON SPECTRAL RECOGNITION OF MUSIC MELODIES

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This is a joint work with Professor Dragoš Cvetković

Spectral graph theory in computer sciences

Cvetković D., Simić S.K., *Graph spectra in computer science*, Linear Algebra Appl., 434(2011), 1545-1562.

Arsić B., Cvetković D., Simić S.K., Škarić M., *Graph spectral techniques in computer sciences*, Applicable Analysis and Discrete Mathematics, 6(2012), No. 1, 1-30.

First paper:

1. Expanders and combinatorial optimization,
2. Complex networks and the Internet topology,
3. Data mining,
4. Computer vision and pattern recognition,
5. Internet search,
6. Load balancing and multiprocessor interconnection networks,
7. Anti-virus protection versus spread of knowledge,
8. Statistical databases and social networks,
9. Quantum computing,
10. Bio-informatics,
11. Coding theory,
12. Control theory.

The second paper contains, among others, the following sections and subsections:
3. Significant eigenvalues, 3.1. Largest eigenvalue, 3.2. Algebraic connectivity, 3.3. The second largest eigenvalue, 3.4. The least eigenvalue, 3.5. Main eigenvalues,
4. Eigenvector techniques, 4.1. Principal eigenvector, 4.2. The Fiedler eigenvector, 4.3. Other eigenvectors,
5. Spectral recognition problems, 6. Spectra of random graphs, 7. Miscellaneous topics, 7.1. The Hoffman polynomial, 7.2. Integral graphs, 7.3. Graph divisors.

Spectral recognition problems

The whole spectral graph theory is related in some sense to the recognition of graphs since spectral graph parameters contain a lot of information on the graph structure (both global and local). However, we shall treat here the problems of recognizing entire graphs, or some parts of them, both in an exact manner and in an approximative way.

In particular, we shall consider

- characterizations of graphs with a given spectrum
- exact or approximate constructions of graphs with a given spectrum,
- similarity of graphs,
- perturbations of graphs.

Queries for databases and the subgraph isomorphism problem

In several databases the data are often represented as graphs. Very frequently graphs are indexed by their spectra.

In

Pinto A., Leuken R.H. van, Demirci M.F., Wiering F., Veltkamp R.C., Indexing music collections through graph spectra, *Proc. 8th Internat. Conf. Music Information Retrivial, ISMIR 2007, Vienna, September 23 - 27, 2007, 153-156.*

a spectral graph theory approach is presented for representing melodies as graphs, based on intervals between the notes they are composed of. These graphs are then indexed using their Laplacian spectrum. This makes it possible to find melodies similar to a given melody.

The query for such a database is given by a graph. To find similar data in the database it is necessary to compare subgraphs of the query graph with subgraphs of the graphs stored in the database. One should efficiently select a small set of database graphs, which share a subgraph with the query.

Instead of comparing subgraphs one can compare their spectra. While the subgraph isomorphism problem is NP-complete, comparing spectra can be done in polynomial time.
The so called Interlacing Theorem plays an important role in problems of spectral graph recognition and in spectral graph theory and its applications in general.
Recall that the matrix A with complex entries a_{ij} is called *Hermitian* if $A^T = \overline{A}$, i.e. $a_{ji} = \overline{a}_{ij}$ for all i, j.

Theorem. Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let B be one of its principal submatrices. If the eigenvalues of B are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ then $\lambda_{n-m+1} \leq \mu_i \leq \lambda_i$ $(i = 1, \ldots, m)$

We associate the graph G_M to the melody M as follows:
The pitch classes are denoted by the natural numbers and these numbers are the multi-digraph vertices. For example, the set $V = \{1, \dots, 12\}$ can be the set of graph vertices for the standard octave with 12 pitch classes.
A *melody* M is a finite sequence of pitches (or corresponding notes) $p_1, p_2, \dots, p_m$. A melody M considered as a sequence of vertices determines a closed walk consisting of arcs $(p_1, p_2), (p_2, p_3), \dots, (p_m, p_1)$.
The weight of the arc (p_i, p_{i+1}) would denote the duration of the pitch p_i.
We represent the melody as the closed walk of these vertices. Some vertices can be repeated and the last edge does not represent an interval in melody.
The arcs occuring in the path are directed edges of the multi-digraph. For some vertices v_1 and v_2 multiple directed edges from v_1 to v_2 are possible.

It is possible to select different indices for G_M. Examples are adjacency spectra, Laplacian spectra and signless Laplacian spectra.
The adjacency matrix A of the multi-digraph G is a square matrix of order equal to the number of vertices and where in i-th row and j-th column we have the number of edges from i-th vertex to the j-th vertex.
The degree matrix D of G is diagonal matrix with the number of edges terminating in i-th vertex as the i-th element of the diagonal.
The spectrum of the G is the spectrum of A, the Laplacian spectrum of G is the spectrum of $D - A$ and the signless Laplacian spectrum of G is the spectrum of $D + A$.

In general these spectra are complex because these matrices are not symmetric for digraphs. It makes some difficulties. Therefore we suggested in

Cvetković D., Manojlović V., *Spectral recognition of music melodies*, SYM-OP-IS 2013, 269-271.

to consider another spectrum of a digraph.

Consider the matrices $A^T A$ and AA^T where A is the adjacency matrix of a digraph. They are real and symmetric. Therefore their spectra are real and non-negative. This spectrum is called the non-negative spectrum of the digraph.

Square roots of these eigenvalues are called singular values of A.

The non-negative spectrum of a digraph has been studied in

1. Jovanović I., *Spectral recognition of graphs and networks*, (Serbian), PhD Thesis, School of Mathematics, University of Belgrade.
2. Jovanović I., *Non-negative spectrum of a digraph*, Ars Mathematica Contemporanea, to appear.

Melodies we call similar if these non-negative spectra, considered as real vectors are close in the Euclidean metric.

Some examples:

Ia. A. Vivaldi: "Io son quel gelsomino", aria from the opera Arsilda, regina di Ponto https://youtu.be/gXoh8Q38xBM
Ib. A. Vivaldi: "Si fulgida per te", aria di Abra from oratorio Juditha triumphans https://youtu.be/RRaTIF-dQ8g

IIa.P. I. Tchaikovsky: Piano Concerto No. 1 B-flat minor part I https://youtu.be/kzoPBj5NKRg
IIb. P.I.Tchaikovsky: "Denj li tsarit" https://youtu.be/W3XcmBcryiU

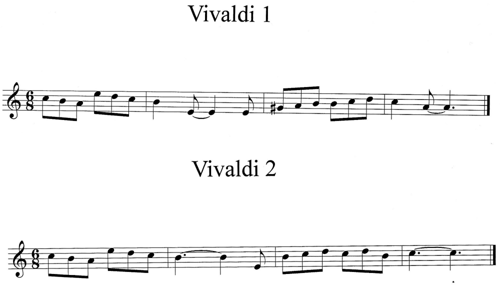

Figure: Vivaldi

Thank You for Your attention!