Cospectral digraphs from locally line digraphs

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Outlook

1. Introduction

- Aim:
  Construct digraphs with the same spectrum (or the same algebraic properties).
- How?
  By local modifications of digraphs with some properties (as Godsil-McKay switching leads to cospectral graphs).
- Which properties?
  Being a locally line digraph.

1. Introduction: Locally line digraphs

- In the line digraph $L\Gamma$ of a digraph Γ, each vertex represents an arc of Γ, $V(L\Gamma) = \{uv : (u,v) \in E(G)\}$, and a vertex uv is adjacent to a vertex wz when the arc (u,v) is adjacent to the arc (w,z): $u \to v(=w) \to z$.
- Heuchenne's condition (1964): A digraph Γ is a line digraph if and only if, for every pair of vertices u, v, either

$$\Gamma^+(u) = \Gamma^+(v) \quad \text{or} \quad \Gamma^+(u) \cap \Gamma^+(v) = \emptyset.$$

- As $L\overline{\Gamma} = \overline{L\Gamma}$, Heuchenne's condition can be restated in terms of the in-neighborhoods $\Gamma^-(u)$ and $\Gamma^-(v)$.
- A digraph is a (U-)locally line digraph if there is a vertex subset U, with $|U| > 1$, such that

$$\Gamma^-(u) = \Gamma^-(v) \quad \text{for every} \quad u, v \in U.$$

1. Introduction: Some locally line digraphs

With a few exceptions, all known ‘dense’ digraphs are locally line digraphs. These include:

- De Bruijn digraphs (1946)
- Kautz digraphs (1968)
- Imase-Ito digraphs (1981, 1983)
- Alegre digraph (Fiol, Alegre, Yebra, 1984)
- Partial line digraphs (Fiol, Lladó, 1992)
- Faber-Moore-Chen digraphs (1993)
- All almost Moore digraphs of diameter two (Gimbert, 2001)
- Multipartite Moore digraphs (Fiol, Yebra, 1990; Fiol, Gimbert, Gómez, Wu, 2003; Fiol, Gimbert, Miller, 2006)
- Cyclic Kautz digraphs (Böhmová, Dalfó, Huemer, 2014)
- ...

2. Main result

Theorem. Let $\Gamma = (V, E)$ be a locally line digraph with diameter $D \geq 2$. Let $X = \{x_1, \ldots, x_r\} \subset V$ such that $Y = \Gamma^-(x_i)$ for some $Y \subset V$, and $i = 1, \ldots, r$. Let $Z = \Gamma^+(X)$. Let Γ' be the modified digraph obtained from Γ by changing $e(X, Z)$ to $e'(X, Z)$:

(i) Loops in $e(Y, X)$ (and in $e(X, Z)$) remain unchanged.

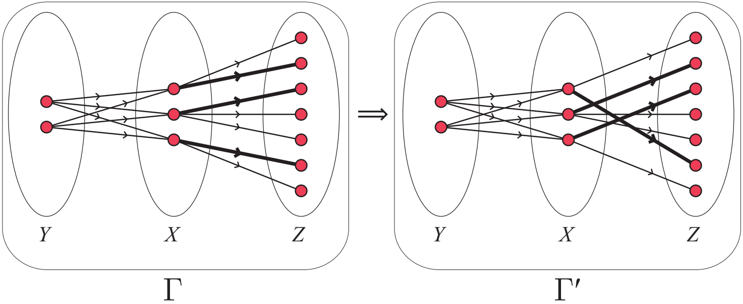
(ii) For the other arcs, every vertex of X has some out-going arc to a vertex of Z, and every vertex of Z gets some in-going arc from a vertex of X.

Assume that there is a walk of length $\ell \geq 2$ from u to v $(u, v \in V)$ in Γ. Then,

(a) If $u \notin X$, then there is also a walk of length ℓ from u to v in Γ'.

(b) If $u \in X$, then there is a walk of length at most $\ell + 1$ from u to v in Γ'.

2. Main result: Scheme of the Theorem



The arcs that change from Γ to Γ' are represented with a thick line.

2. Main result: Considering shortest walks. . .

- **Corollary.** If Γ is a digraph with diameter D, the modified digraph Γ' (as in the Theorem) has diameter D' satisfying

$$D - 1 \leq D' \leq D + 1.$$

- The case $D' = D - 1$ could happen when, in Γ, all vertices not in X have eccentricity $D - 1$ and in Γ' all vertices in X result with the same eccentricity $D - 1$.

- Examples of the case when the diameter remains unchanged, $D' = D$, are provided by the modified De Bruijn digraphs (discussed later).

3. Cospectral digraphs

- **Proposition.** Assume that in the modified digraph Γ' from Γ, every vertex of Z gets the same in-going arcs as in Γ, $|\Gamma'^{-}(v)| = |\Gamma^{-}(v)|$ for every $v \in Z$. Let $\boldsymbol{A} = (a_{uv})$ and $\boldsymbol{A}' = (a'_{uv})$ be the adjacency matrices of Γ and Γ', respectively. Then, for any polynomial $p \in \mathbb{R}[x]$ without constant term, say, $p(x) = xq(x)$, with $\deg q = \deg p - 1$, we have

$$p(\boldsymbol{A}') = \boldsymbol{A}' q(\boldsymbol{A}).$$

- **Corollary.** The digraphs Γ and Γ' are cospectral.
  Proof. Γ and Γ' have the same characteristic polynomial.

- ... but not necessarily with the same Jordan normal form (see an example later).

3. Cospectral digraphs

Proof of the Proposition. We only need to prove that $\boldsymbol{A'A} = \boldsymbol{A'A'}$.
Since the only modified arcs are those adjacent from the vertices of X, we have

$$\begin{aligned}
(\boldsymbol{A'A})_{uv} &= \sum_{x\in X} a'_{ux}a_{xv} + \sum_{x\notin X} a'_{ux}a_{xv} = |X\cap\Gamma^-(v)| + \sum_{x\notin X} a'_{ux}a_{xv} \\
&= |X\cap\Gamma'^-(v)| + \sum_{x\notin X} a'_{ux}a_{xv} = \sum_{x\in X} a'_{ux}a'_{xv} + \sum_{x\notin X} a'_{ux}a'_{xv} \\
&= (\boldsymbol{A'A'})_{uv},
\end{aligned}$$

where we used that every vertex of Z in Γ' gets the same in-going arcs as in Γ.

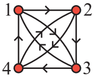
4. Examples: Equi-reachable and UPP digraphs

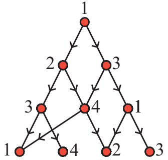
- ℓ-reachable or equi-reachable digraph: A digraph $\Gamma = (V, E)$ with diameter D is *ℓ-reachable* if, for every pair of vertices $u, v \in V$, there is a walk of length $\ell (\leq D)$ from u to v (ℓ is the smallest integer).

- UPP digraph (Mendelsohn, 1970): A digraph $\Gamma = (V, E)$ with diameter D is UPP (Unique Path Property) if it is ℓ-reachable and it has d^ℓ vertices.

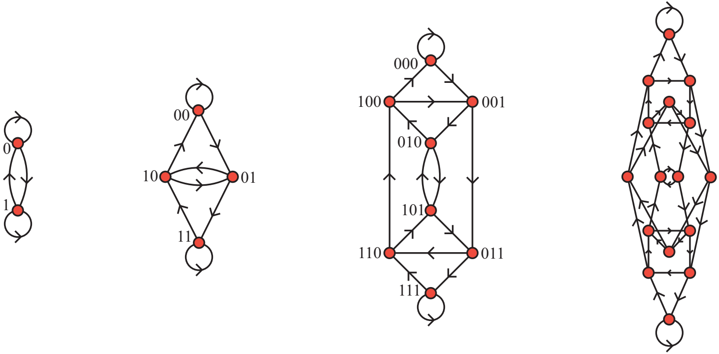
- If Γ is ℓ-reachable and has maximum out-degree d, then its order is at most $N = d^\ell$. Then, $\boldsymbol{A}^\ell = \boldsymbol{J}$, and, therefore, Γ is d-regular (Hoffman and McAndrew, 1965).

- **Example.** A 3-reachable but not UPP digraph ($n = 4 \neq 2^3$):





4. Examples: UPP digraphs: De Bruijn digraphs



De Bruijn digraphs $B(2,1)$, $B(2,2)$, $B(2,3)$, and $B(2,4)$.

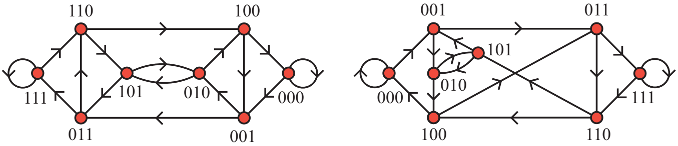
4. Examples: Modified De Bruijn digraphs

Proposition. Let $\Gamma = B(d,\ell)$. For some fixed values $x_i \in \mathbb{Z}_d$, $i = 1, 2, \ldots, \ell-1$, not all of them being equal (to avoid loops), consider the vertex set $X = \{x_1 x_2 \ldots x_{\ell-1} k : k \in \mathbb{Z}_d\}$. Let α_j, $j \in \mathbb{Z}_d$, be d permutations of $0, 1, \ldots, d-1$. Let $\Gamma' = B'(d,\ell)$ the modified digraph obtained by changing the out-going arcs of X is such a way that every vertex $x_1 x_2 \ldots x_{\ell-1} k \in X$ is adjacent to the d vertices

$$x_2 x_3 \ldots x_{\ell-1} \alpha_j(k) j, \qquad k = 0, 1, \ldots, d-1.$$

Then, Γ' is a d-regular digraph with diameter $D' = \ell$, and it is ℓ-reachable.

4. Examples: A modified De Bruijn digraph



De Bruijn digraph $B(2,3)$ and the modified De Bruijn digraph $B'(2,3)$.

- $B'(2,3)$: Fiol, Alegre, Yebra, and Fàbrega (1985).
- $B'(2,3) \not\cong B(2,3)$.
- $\operatorname{sp} B'(2,3) = \operatorname{sp} B(2,3) = \{0^7, 2^1\}$.
- A computer exploration shows that the only nonisomorphic 3-reachable 2-regular digraphs are

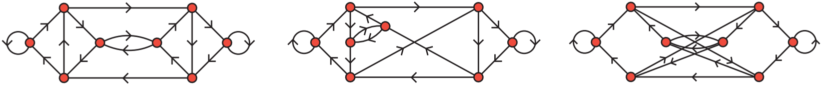
$$B(2,3), \quad B'(2,3), \quad \text{and} \quad \overline{B'(2,3)}.$$

4. Examples: From $B(2,3)$ to $B'(2,3)$

$$
\boldsymbol{A} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & {\color{orange}1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & {\color{orange}1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad
\boldsymbol{A}' = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & {\color{orange}1} & 0 & 0 & 0 & 0 \\
0 & {\color{orange}1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

$$
J(\boldsymbol{A}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
J(\boldsymbol{A}') = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

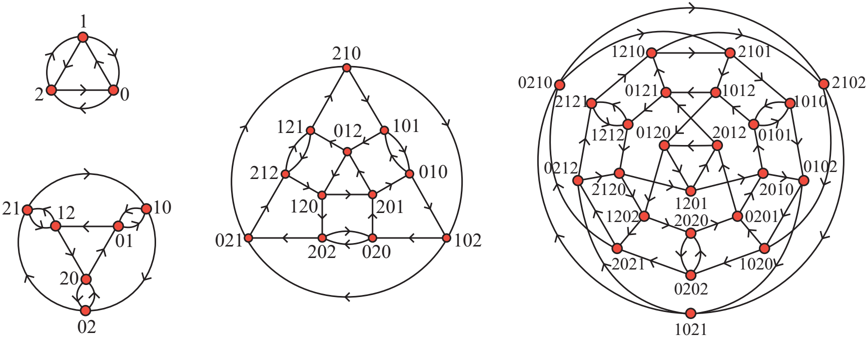
4. Examples: A double modified De Bruijn digraph



De Bruijn digraph $B(2,3)$, the modified De Bruijn digraph $B'(2,3)$, and the double modified De Bruijn digraph $B''(2,3)$.

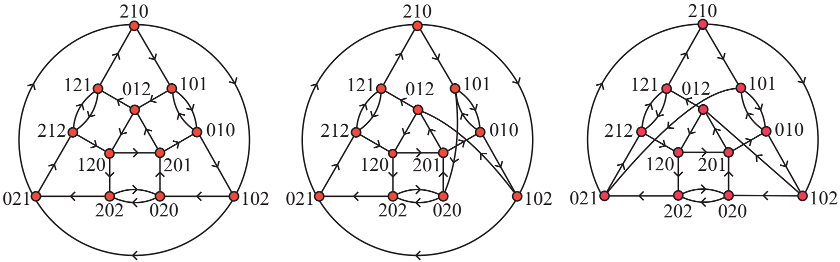
- $\operatorname{sp} B''(2,3) = \operatorname{sp} B'(2,3) = \operatorname{sp} B(2,3) = \{0^7, 2^1\}$.
- $B''(2,3)$ is not a UPP digraph, in contrast with $B(2,3)$ and $B'(2,3)$.

4. Examples: Kautz digraphs



The Kautz digraphs $K(2,1)$, $K(2,2)$, $K(2,3)$, and $K(2,4)$.

4. Examples: Modified Kautz digraphs

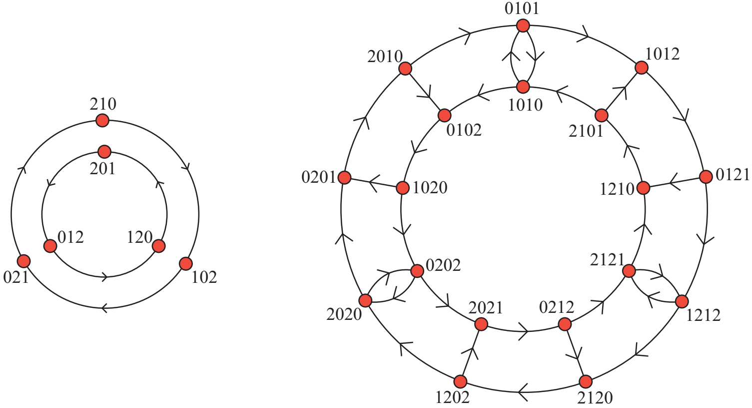


Kautz digraphs $K(2,3)$, and the modified Kautz digraphs $K'(2,3)$ and $K''(2,3)$.

- $\operatorname{sp} K(2,3) = \operatorname{sp} K'(2,3) = \operatorname{sp} K''(2,3) = \{-1^2, 0^9, 2^1\}$.
- In $B(2,3)$: $D' = D (= \ell)$.
- In $K(2,3)$: Computer exploration seems to show that all the modified Kautz digraphs have diameter $D' = D + 1 (= \ell + 1)$.

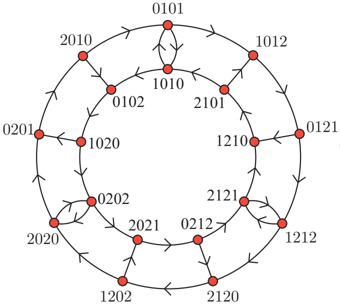
4. Examples: Cyclic Kautz digraphs

(Böhmová, Dalfó, Huemer, 2015)



Cyclic Kautz digraphs $CK(2,3)$ and $CK(2,4)$.

4. Examples: Modified cyclic Kautz digraph $CK'(2,4)$



Cyclic Kautz digraphs $CK(2,4)$ and the modified cyclic Kautz digraph $CK'(2,4)$.

- $\operatorname{sp} CK(2,4) = \operatorname{sp} CK'(2,4)$.
- In $B(2,3)$: $D' = D (= \ell)$.
- In $K(2,3)$: Computer explorations seem to show that all the modified Kautz digraphs have diameter $D' = D + 1 (= \ell + 1)$.
- In $CK(2,4)$: Computer explorations seem to show that all the modified cyclic Kautz digraphs have diameter $D' = D + 1 (= 2\ell)$.

Open problems

- Can all UPP digraphs be obtained as modified De Bruijn digraphs?
- Does Heuchenne's condition show up in almost all dense digraphs?
- Is the diameter of all the modified Kautz digraphs $D' = D + 1$?
- Is there any condition that decide whether the modified digraphs are not isomorphic with the original ones?

References

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Hvala na pažnji

Thank you for your attention

Gràcies per la vostra atenció