LOGICAL CONSTANTS, An essay in proof theory

Kosta Došen

ABSTRACT

     The main goal of this work is to give philosophically significant
analyses of those expressions which, in most accounts of logic, are
the only expressions called "logical constants". These analyses will
be given in a language in which we speak about what we shall call
"the deductive meta-language of the object language to which logical
constants belong", and they will have a particular form. The achievement
of this goal presupposes an understanding of the context in which our
analyses will be made and also a certain account of philosophical
analyses and their purpose.

     A secondary goal is to consider the following two theses, which
provide a motivation for seeking our analyses,

     Thesis [I] A constant is logical iff it can be ultimately
     analyzed in purely structural terms.

     Thesis [II] Two logical systems are alternative iff they
     differ only in their assumptions on structural deductions.

We shall try to give a partial reconstruction of the import of these
theses by making more precise the terms in which they are phrased.
(The term "structural" in them is used in the sense in which some
rules in sequent systems are usually called "structural".) We shall
also try to give some grounds for their justification. If Thesis [I]
could be justified, it would provide us with means to answer the
question "Where are the limits of logic?". If Thesis [II] could be
justified, it would help us in understanding the problems generated
by the existence of alternative logics.

     The general plan of this work is the following.

     Chapter 1 consists of some preliminary considerations on languages
and systems of provable sentences.

     In Chapter 2 we present the deductive meta-language which
provides a context for our analyses. Next we consider structural
systems in this language. Essentially, these are a type of hier-
archical sequent systems in which sequents of various levels are
distinguished. In particular, we show that in these structural
systems a generalized form of the Cut rule is not eliminable.

     In Chapter 3 we define the notion of structural analysis.

The main ingredient of this notion is the notion of <u>analytic rules</u>.
An analytic rule is of the form

$$\frac{B_1 \ \ldots \ B_k}{\underline{A}}$$

which is short for

$$\frac{B_1 \ \ldots \ B_k}{A} \ , \quad \frac{A}{B_1} \ , \quad \ldots \ , \quad \frac{A}{B_k} \ ,$$

where only one constant of the object language occurs in A, and no
such constant occurs in B_i.

In Chapter 4 we consider analytic rules for the <u>propositional
constants</u> of <u>classical logic</u>. We show that these analytic rules are
necessary and sufficient in a certain sense, and we also consider
whether they can be replaced by some other analytic rules or postulates
in the object language. We proceed analogously in Chapters 5-8.

In Chapter 5 we use the same analytic rules to analyze the
<u>propositional constants</u> of <u>intuitionistic logic</u> and of a relevant
logic which we shall call "<u>intuitionistic relevant logic</u>". Systems
with analytic rules used for these analyses differ only in their
structural part with respect to a generalized form of the Thinning
rule. We also consider a logic dual to intuitionistic logic which
we propose to interpret as a <u>logic of refutation</u>.

In Chapter 6 we give analytic rules for <u>modal propositional
constants</u> with which we shall analyze the constants of S5 and S4.
Again, the analytic rules for both S5 and S4 will be the same, and
the respective systems will differ only in their structural part
with respect to a generalized form of the Thinning rule. These
analytic rules will be based on sequents of higher levels, and pro-
vide the main application of our hierarchy of sequents in this work.

In Chapter 7 we consider analytic rules for <u>first-order quantifiers</u>.
In this context we make some comments on an alternative approach
which would apparently make quantification and modality incompatible.

In Chapter 8 we consider analytic rules for <u>identity</u>. First
we give such a rule in a second-order context, and then in a first-
-order context. We also consider some single axiom-schemata for
identity which are connected with the second analytic rule above.

In Chapter 9 we consider the notion of <u>uniqueness</u> of expressions,
which amounts to synonymity with everything with identical syntactical
characterization. Uniqueness is another ingredient of the notion of
structural analysis. We show that, under certain natural assumptions, our
characterizations with analytic rules guarantee uniqueness, whereas
postulates in the object language for implication, and either postulates
in the object language or some usual sequent rules for the necessity
operator, do not guarantee uniqueness.

In Chapter 10 we summarize the results of our analyses and then

consider the relation of these analyses to Thesis [I]. We shall
also try to provide some grounds for this thesis and to consider
the general notion of analysis of which structural analysis is an
instance. Next, we introduce the notion of structurally alternative
systems, and consider the relation of this notion, together with
those of our results which can be expressed with the help of this
notion, to Thesis [II]. We shall also try to provide some grounds
for this thesis. Finally, we shall briefly compare our enquiry to
some other enquiries which have a similar goal or are in general
similar in spirit.

LOGICAL   CONSTANTS

An essay in proof theory

Kosta Došen

La science (comme toute chose humaine)
est sur plusieurs plans verticaux.
L'algèbre met tout sur le même plan.

<div align="right">Émile Novis</div>

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Chapter 1

INTRODUCTION

The main goal of this work is to give philosophically
significant analyses of the expressions which, in most accounts of
logic, are the only expressions called "logical constants".   These
analyses will be given in a language in which we speak about what we
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The achievement of this goal presupposes an understanding of the
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> ultimately analyzed in purely structural terms.

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If Thesis [I] could be justified, it would provide us with means to
answer the question, "Where are the limits of logic?". If Thesis [II]
could be justified, it would help us in understanding the problems
generated by the existence of alternative logics.

The general plan of this work is the following.   After some
preliminaries on languages and systems /in the remainder of this chapter],

we consider the context in which our analyses will take place
[Chapter 2] and the form of these analyses [Chapter 3]. Next we
consider the analyses of propositional constants [Chapters 4,5],
propositional modal constants [Chapter 6], first-order quantifiers
[Chapter 7] and identity [Chapter 8]. After considering whether
our analyses give unique characterizations of the constants analyzed,
in a sense to be made precise [v.§22; Chapter 9], we try to give an
account of the general notion of analysis with which we have worked,
to show its connection with Theses [I] and [II], and to provide some
grounds for these theses [Chapter 10].

Logical constants are expressions of a language. Therefore we
shall try to set forth some of the things we are assuming about
languages for our enquiry.

We take the term "language" in a sense which is in accordance
with common usage and which precludes constructions of uninterpreted
formal objects being called "languages". That is, "language" will
be synonymous with "interpreted language". Semantical questions about
the nature of this interpretation, or about meaning, will not be
considered here. Our discussion will be on the syntactical level, but
we shall be concerned with the syntax of languages in the above sense.

A language is made up of expressions . We shall say that a language
has a certain expression, or that a certain expression belongs to a
language, or something synonymous with that. Expressions can be
divided exhaustively into elementary and complex expressions. Complex
expressions are constructed out of at least two other elementary or complex
expressions; they are ultimately arrangements of elementary expressions.
The expressions from which a complex expression is constructed occur
in this complex expression. The relation "occurs in" is reflexive

and transitive.

A language L_1 is <u>included</u> in a language L_2 iff every expression of L_1 is an expression of L_2. L_1 <u>coincides</u> with L_2 iff L_1 and L_2 are included in each other. L_1 is <u>properly included</u> in L_2 (or is a <u>fragment</u> of L_2) iff L_1 is included in L_2 but does not coincide with L_2. The <u>union</u> of L_1, ..., L_k, $k \geqslant 2$, is the language which includes L_i, for every i, $1 \leqslant i \leqslant k$, and has no expression which does not belong to L_j, for some j, $1 \leqslant j \leqslant k$.

3   Expressions can also be divided exhaustively according to the <u>grammatical categories</u>, or <u>categories</u>, <u>tout court</u>, to which they belong. There are two basic categories: the category of <u>sentences</u>, named "δ", and the category of <u>singular terms</u>, named "t". Besides expressions of these two basic categories we can have in a language expressions which serve to construct complex expressions of a certain category from simple or complex expressions of some categories. These expressions are <u>functors</u> from expressions of this language to expressions of this language.

We shall use the fractional notation of categorial grammar to name the categories of these functorial expressions. To the category $\frac{a}{b}$ will belong an expression which is a functor having as an argument an expression of the category b and as a value an expression of the category a, where a and b are not necessarily basic categories. A functor having as arguments expressions of the categories b_1, ..., b_k, $k \geqslant 1$, and as a value an expression of the category a, will belong to the category

$$\cfrac{\cfrac{\cfrac{a}{b_1}}{\vdots}}{b_k} \quad ;$$

as an abbreviation for the name of this category we shall use

"$\dfrac{a}{b_1 \ldots b_k}$" . (This notation will also occasionally be used with

$k \geqslant 0$; if $k = 0$, then "$\dfrac{a}{b_1 \ldots b_k}$" is "$a$".)   An expression of the

category $\dfrac{a}{b_1 \ldots b_i \ldots b_k}$ can have as an argument not only an expression

of the category b_i, in which case its value is an expression of the

category $\dfrac{a}{b_1 \ldots b_{i-1} b_{i+1} \ldots b_k}$, but also an expression of the category

$\dfrac{b_i}{c_1 \ldots c_\ell}$, $\ell \geqslant 1$, in which case its value is of the category

$\dfrac{a}{b_1 \ldots b_{i-1} c_1 \ldots c_\ell b_{i+1} \ldots b_k}$.

     For example, an expression of the category $\dfrac{s}{ss}$ can have arguments

of the categories $\dfrac{s}{t}$ and $\dfrac{s}{tt}$, in which case its value is of the

category $\dfrac{s}{ttt}$; or an expression of the category $\dfrac{s}{\frac{s}{t}}$ can have an argument

of the category $\dfrac{s}{tt}$, in which case its value is of the category $\dfrac{s}{t}$.

     These remarks should not be taken as giving a comprehensive

treatment of categorial grammar, or even a comprehensive treatment

of every possible case which might arise in our work. We only assume

that a treatment of the kind offered here should enable us to treat

the expressions we shall be concerned with in this work as being of

the categories specified once for all when these expressions were intro-

duced. For example,"and", which is of the category $\dfrac{s}{ss}$, will be of

this category also in the expression "__ is even and __ is divisible by __",

which is of the category $\dfrac{s}{ttt}$. It is also possible to consider that

"__ is even and __ is divisible by __" was not constructed by joining

"__ is even" and "__ is divisible by __" with "and", but by filling the

empty places with expressions of the category t, joining with "and",

and then taking out the expressions of the category t. Our treatment

tries to follow in a certain sense such "step-by-step constructions" in other

cases too /cf. §61/.

Expressions of both basic categories and functors can be

either elementary or complex.

§4   Expressions can also be divided into <u>schemata</u> and <u>constants</u>.

An expression of a language L is elementary or complex, or is of a

certain grammatical category, only relative to L. But an expression

of L can be a schema relative to a language L_1 which can either

coincide or not with L.

Schemata of a language L relative to a language L_1 are given

in the following way. We first give constructively some expressions of

L, not necessarily elementary, of a category a, which will be

called "<u>basic schemata</u> for expressions of L_1 satisfying the proviso

P". The exact nature of the proviso P, which expressions of L_1

can satisfy, will depend on the particular form of L_1; however, P

must specify that the expressions of L_1 in question are of the

category a. In the simplest case P will specify only that; basic

schemata are usually in that case elementary expressions of L and are

called "<u>schematic letters</u>". Next we can give other such expressions

changing the category a and the proviso P. Expressions of L of

a category b in which occur: (1) at least some basic schemata for

expressions of L_1, (2) possibly some expressions of L_1, provided

these are also expressions of L, and (3) no other expressions, will

be called "<u>schemata</u> for expressions of L_1 satisfying the proviso Q".

Q must specify that the expressions of L_1 in question are of the

category b. A schema can be either an elementary or a complex

expression of L.

We assume that a basic schema for expressions of L_1 is of

the category c in L iff it is a basic schema for expressions of L_1 of

the category c /cf. however §68/, and that an expression of L_1 which

is also an expression of L is of the category c in L_1 iff it is of the

category c in L. Whenever L_1 is included in L, we can assume that

this second requirement is met automatically. In this work we shall

consider only cases where L_1 is included in L.

Next, we must assume rules which will permit us to substitute

uniformly for every occurrence of a basic schema for expressions of

L_1 satisfying the proviso P, which occurs in an expression of L,

either a schema for expressions of L_1 satisfying the proviso P, or an

expression of L_1 satisfying the proviso P, salva a certain property.

We shall call rules of this form, where the property in question is

specified, "rules of substitution". Expressions obtained from an

expression e by applying a rule of substitution will be called

"instances of e" (relative to that rule). Applications of a rule

of substitution are empty iff in e no basic schema of the required

kind occurs. (N.B. Applications in which we substitute for a basic

schema this same basic schema are not empty in this sense .) However, if

the relation "is an instance of" is tied to the notion of a schema,

so that only an expression in which a schema occurs can have instances

relative to some particular basic schema which occurs in it, empty

applications of a rule of substitution will not be permitted

(otherwise every expression would be a kind of schema and not only

the proper schemata we have considered). Whether or not to allow

empty applications of a rule of substitution depends on the specific

purposes we have in mind /̄cf. Chs.7,8; sp. §73_7. In general, in

this work a rule of substitution can be applied only to an expression

of L in which the basic schema involved in the rule occurs at least

once.

Some rule of substitution must be assumed, at least implicitly,

for every expression we want to use as a schema. In some cases we

shall have to specify the property involved and to phrase rules of

substitution more precisely than suggested above. However, in many

other cases where we use schemata, we shall presuppose an apprehension

of such a rule in which all the details are implicitly supplied

/̄indeed we have already used some schemata in §2, §3 and in this section,

e.g. "L", with subscripts, for names of languages_7.

An expression of L which is a schema for expressions of L_1

will be called "a schema relative to L_1". An expression of L which

is a schema relative to some language included in L will be called

"a schema of L", tout court.

Definition of constants of a language

An expression of a language L is a constant of L iff it is not

a schema of L.

As an immediate consequence of this definition we have

that an expression of L is a constant iff for any language included

in L it is not a schema relative to that language.   We also have

that constants can be either elementary or complex.   In principle,

a constant of L can be a schema relative to a language L_1 which is

not included in L; but since we shall only consider cases where L_1

is included in L, this will be of no concern here. From the

assumption that L_1 is included in L it also follows that every

expression of a language L which is a schema relative to some

language L_1 will be a schema of L in cases we shall consider.

If some expressions of L_1 are called "ℓ's", a schema of L

for these expressions will be called "an ℓ-schema of L".

The expressions of a language can be divided exhaustively into

schemata and constants.

§5   Next, we assume that there are certain deducibility relations

between collections of sentences of a language. A collection of

sentences can be either a set of sentences, or a sequence of sentences,

or possibly something else [v. §16]. However, in practically the

whole of this work we shall be concerned only with finite (possibly empty)

sets of sentences. The deducibility relations we have in mind are

syntactical, not semantical.If a collection of sentences is deducible

from another, this will mean that there is a deduction, i.e. a

syntactical construction, in which the members of the second collection

are premises, and the members of the first conclusions. Since we

shall allow the first collection to have more than one member, these

deductions are not always the familiar constructions exemplified by

proofs, which have a single conclusion; they will sometimes represent

a more general notion of multiple-conclusion deductions, in which

the conclusions are taken alternatively [v.§17].

The syntactical notion of deduction can correspond to a semantical

notion of consequence, in such a way that whenever two collections

are in a deducibility relation, they are also in a general

consequential relation (and possibly also conversely). However, this

semantical notion will not be treated in this work.

A language from which we take sentences which are members of

collections in deducibility relations will be called "an object

language". For the name of an object language we shall use the

schematic letter "O" (possibly with subscripts).

The language in which we have sentences expressing that two

collections of sentences of O are in a deducibility relation will be

called "the deductive meta-language of O of level 1". The language O

itself will be called "the deductive meta-language of O of level 0".

The deductive meta-language of level 1 of O_n, where O_n is the deductive

meta-language of O of level n, or possibly the union of the deductive

meta-languages of O of all levels $\leqslant n$, will be called "the deductive

meta-language of O of level $n + 1$". The language which is the union of

the deductive meta-languages of O of all levels $\geqslant 0$ will be called "the

deductive meta-language of O", tout court.   For the name of the deductive

meta-language of a language O we shall use the schematic letter "D"

(possibly with indices), where we assume that it is understood from

the context what is the language O with respect to which D is the

deductive meta-language. We shall say that the language D of O is built

on O. A sentence of the deductive meta-language of O of level n will

be called "a sentence of D of level n ".

We must also mention the language used for communication in this

work, which in particular serves to communicate investigations

concerning O and D. This language will be called "U". It is not

enough to say that U is English, for it is a specific fragment of

English, having some features additional to the linguistic apparatus

which is supposed to be generally understood.   One of these features
is the presence in U of schemata for expressions of O and D, and
other schemata.   In general, these features are the presence in U
of technical terms and symbols.   These features of U will not be
determined once and for all.   As our work progresses we shall
introduce additional technical apparatus and symbolism, and change U
accordingly.   U is not absolutely precise, but with care any
desired limited degree of precision can be attained with it. (The
form of rules of substitution mentioned above was mainly intended
for the use of schemata of U, and this justifies the imprecise
form in which we have left it.)   When we are reasoning in U we rely
on principles of classical logic, but often our reasoning could easily
be shown compatible with principles of more stringent logics.
Some of this reasoning belongs to what would be the deductive meta-language
of D, if D were taken to be an object language and the appropriate
fragment of U were characterized explicitly.   Whenever we take the
deductive metalanguage of O of some finite level $n \geqslant 0$, the deductive
meta-language of level 1 of this language (i.e. the deductive meta-
-language of O of level $n+1$) will match a fragment of U, and a logic
systematized in it will be the same as the one used in the corresponding
fragment of U if it is, in principle, classical.

Quotation marks of U serve various purposes.   No attempt is
made here to distinguish them, and in general we rely on the context
of U to convey what exactly is intended.   For example, when a schema
occurs in an expression which is quoted, this does not necessarily
mean that it ceases to be a schema.   In some cases, where the use of
quotation marks would have been too cumbersome, and no confusion is
likely, we shall try to avoid using them.

The constants we want to analyze will be treated as expressions

of a language L included in O (i.e. O itself or a language

properly included in O), so we assume that they belong to O. We

also assume that every expression of O belongs to one, and only one,

category and that it can effectively be decided for any given

arrangement of elementary expressions of O whether it is unambiguously

an expression of O, whether it is elementary or complex (if the

latter, out of what expressions it is consructed), to what category

it belongs, and whether it is a schema or a constant. In this sense

this language O will be _formal_. Next, we must assume that in O

we have sentences, i.e. expressions of the category δ. For a

great part of our work [Chapters 24-6] we practically need not assume

anything else about O. But some additional assumptions need to be

made concerning O for the analyses of some constants [v. Chapters 7,8].

In general O can be any language which satisfies these conditions: e.g. the

language of formal arithmetic, or the language of the propositional calculus

(using the propositional variables $"p"$, $"q"$, $"\hbar"$), or a fragment of

ordinary English (sometimes even a degenerate language with only one

sentence [Chapter 2]; D built on an O with no sentences can be

investigated out of curiosity, but this will not be attempted here).

O will be properly included in D. In D we must have in

addition expressions of the categories $\dfrac{t}{\underbrace{\delta\ldots\delta}_{k}}$, $k \geqslant 0$, for

constructing singular terms which will stand for collections of

sentences, and expressions of the category $\dfrac{\delta}{tt}$ for expressing

deducibility relations between collections of sentences. If D is

built on a formal language O, D will also be a formal language.

D will be properly included in U. As we have remarked above, we shall have in U schemata relative to D. The question whether U can also be formal depends on an attempted reconstruction of U.

A more precise characterization of a kind of language D built on a given language O, and of the relevant features of U, will be given in the following chapter. As a heuristic example, take O to be the language of formal arithmetic. Then

$$2 + 3 = 5$$

will be a sentence of O,

$$\{2 + 3 = 5, 2 + 3 = 6\} \vdash \{2 + 3 = 6\}$$

will be a sentence of D of level 1, and

$$\{A, B\} \vdash \{B\}$$

will be a sentence of U , where "A" and "B" are schematic letters of U for sentences of O, and the other symbols are interpreted in the familiar way.

Since D and O are included in U, they can be looked upon as specific fragments of English, separated from the main body of the language, in certain respects precisely characterized, and able to serve some precisely defined purposes.

§6   One of the purposes for which D, or O, is introduced is to construct systems of provable sentences of D, or of O. Instead of "systems of provable sentences of L" we shall also say "systems in L", tout court.   In general, the systems we shall consider are given by specifying

(1)   a formal language L to which the provable sentences belong, and

(2)   a non-empty set of expressions of U of the form $"\frac{\Pi}{C}"$ called "directives", where Π is a <u>finite (possibly empty) set of occurrences</u> of sentences of L and C is a sentence of L.

Once we have (1) and (2), the set of provable sentences of a system is always specified in the same manner. Using the schematic letter $"S"$ (possibly with indices) for names of systems, and taking for granted certain simple notions concerning trees, we give the following

Definition of proofs

A finite tree of occurrences of sentences of a language L is a proof in a system S in L iff for each occurrence of sentence C in the tree there is a directive $\frac{\Pi}{C}$ of S, where Π is the set of all the occurrences of sentences immediately above C in the tree.

N.B. A sentence can occur more than once in Π. If, for example, in a proof we have a configuration of the form

$$\frac{A \qquad B \qquad A}{C}$$
,
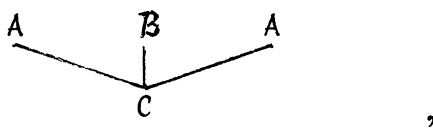
where $"A"$, $"B"$ and $"C"$ are sentences of L, Π is $\{A,A,B\}$ and not $\{A,B\}$. A stricter notion of directives is obtained by requiring that Π be a <u>sequence of occurrences</u> of sentences.

A proof is a proof <u>of</u> a sentence C iff C is the sentence occurring at the bottom of this proof.

Definition of provable sentences

A sentence of L is provable in a system S in L iff there is a proof of this sentence in S.

The length of proofs is measured in steps. A sentence is proved in k steps iff the number of occurrences of sentences in the proof of this sentence is k. The minimal length is obtained when $k = 1$. We assume that for any system in L and any expression $\frac{\Pi}{C}$, where Π is a finite (possibly empty) set of occurrences of sentences of L and where C is a sentence of L, it can be decided effectively whether $\frac{\Pi}{C}$ is a directive of this system. Hence, it can be decided effectively whether a tree of occurrences of sentences of L is a proof in this system.

A sentence C is called "an axiom of the system S" iff there is a directive $\frac{\emptyset}{C}$ of S. Every axiom is derivable in one step.

A directive $\frac{\Pi}{C}$ of a system S is called "a primitive transition of S" iff $\Pi \neq \emptyset$.

Every directive corresponds exactly to one axiom or one primitive transition. So we can give the directives by giving the axioms and the primitive transitions, the latter as usual in the form

$$\frac{P_1 \ \cdots \ P_k}{C} \quad ,$$

where $\Pi = \{P_1, \ldots, P_k\}$. More specifically, clause (2) for a system S will be given in the standard way by specifying

(2.1)   a finite non-empty set of postulates, where a postulate
         is an axiom, or an axiom-schema, or a primitive transition,
         or a primitive transition-schema.

We shall be concerned only with systems given in the standard way. The implicit rule of substitution assumed in connection with axiom-schemata and transition-schemata of U is that uniform substitution for basic schemata will preserve the property of being an axiom-schema or axiom, or the property of being a primitive transition-schema or primitive transition. Likewise, if we have a schema of U for sentences provable in a system S, substitution will preserve the property of being a provable sentence-schema or provable sentence.

The basic schemata occurring in axiom-schemata and transition--schemata need not be elementary expressions of U.   These basic schemata will not necessarily be used only for expressions of L of a certain category, but also for expressions of L of a certain category, which in addition satisfy a certain proviso. Since the property of being a directive must be effectively decidable, we assume that given an axiom-schema, or a transition-schema, it can be decided effectively for any expression of L, or any expression of U of the form $\frac{\Pi}{C}$ (specified as above), whether or not it is an instance of this schema, i.e. whether it can be obtained from this schema by uniform substitution for basic schemata. For this to be possible, the proviso of the basic schema must refer only to effectively decidable properties of expressions of L. L is a formal language, and we shall assume that besides the property of being of a certain category, the properties of being elementary or complex, being constructed in a specified way from some expressions, being a schema or a constant and related syntactical properties, are also effectively decidable.

An axiom or an axiom-schema will be called "an axiom-postulate"; and a primitive transition or a primitive transition-schema will be

called "a <u>primitive rule</u>".   In general we have

Definition of transitions

An expression of U of the form $"\frac{\Pi}{C}"$, where Π is a finite non-
-empty set of occurrences of sentences of L, and C is a sentence of L,
is called "a transition in L".

Definition of rules

A transition in L or a transition in L-schema is called "a rule
in L".

As usual, in a rule $\frac{\Pi}{C}$ the members of Π are called "the premises",
and C "the conclusion".   We shall use "R" (possibly with subscripts)
as a schema for names of rules.

Definition of formulae

A sentence of L or a sentence of L-schema is called "a formula
of L".

Definition of extensions of a system

A system S_2 in L_2 is an extension of a system S_1 in L_1 iff
every directive of S_1 is a directive of S_2 and L_1 is included in L_2.

An extension S_2 of S_1 is <u>proper</u> iff there is a sentence provable
in S_2 which is not provable in S_1.   Adding directives to S_1 need not
give rise to a proper extension.

Definition of subsystems of a system

A system S_1 in L_1 is a subsystem of a system S_2 in L_2 iff S_2
is an extension of S_1.

A subsystem S_1 of S_2 is <u>proper</u> iff S_2 is a proper extension of S_1.

When an extension S_2 of S_1 is obtained by giving a postulate P in addition to the postulates of S_1, we shall say that S_2 is an extension of S_1 with P (sometimes we shall also say simply that S_2 is S_1 with P). When a subsystem S_1 of S_2 is obtained by rejecting a postulate P from the postulates of S_2, we shall say that S_1 is a subsystem of S_2 without P (sometimes we shall also say simply that S_1 is S_2 without P).

Definition of systems contained in other systems

A system S_1 in L_1 is contained in a system S_2 in L_2 iff every sentence provable in S_1 is provable in S_2 and L_1 is included in L_2.

S_1 is properly contained in S_2 iff S_1 is contained in S_2 and there is a sentence provable in S_2 which is not provable in S_1.

The clause "L_1 is included in L_2" in the definitions above may be redundant under some assumptions.

Every system S_1 is contained in an extension S_2 of S_1, but if S_1 is contained in S_2, S_2 is not necessarily an extension of S_1. This shows that a system should not be identified with the set of its provable sentences. If it were identified with it, it would be separated from the main body of U; but our account shows that a system depends on U.

Additional technical apparatus pertaining to systems will be introduced in the course of our work.

§7   [§3]   A more comprehensive discussion of categorial grammar needed for logic, but not necessarily in complete accordance with our remarks, can be found in Curry 1963 and Belnap 1975. The fractional

notation of categorial grammar was introduced by Ajdukiewicz
1935.   Our remarks on the value of e.g. $\frac{a}{b}$ when its argument is $\frac{b}{c}$
are influenced by Geach 1970.   The notion of a "step-by-step
construction" mentioned in §3 is treated in Dummett 1973 (Chapter 2).

[§5]   Our understanding of O,D, and U and their interrelations
is connected with Curry's conceptions concerning formal languages
and systems; an account of U (and the name "U") can be found in
Curry 1950 (pp.11ff) and Curry 1963.   This does not mean that we
fully endorse Curry's elaborate theory concerning this topic, which
in many respects we think is not strictly relevant to our goals,
and which is not always easy to understand (v.Curry 1950a, 1951,
1952,1958, 1959, 1963; and also Curry & Feys 1958; Curry, Hindley
& Seldin 1972).

[§6]   The unifying notion of directives of a system can be
derived from Carnap 1934 (§§31,47).

Chapter 2

STRUCTURAL SYSTEMS

§8   In this chapter we shall specify more precisely the kind of

language D built on some language O which will be of central concern

to us throughout this work. Next, we shall consider a type of systems

in this language D called "structural systems". We shall investigate

in this context the properties of "admissibility", "deducibility" and

"derivability" of sentences of D or of rules in D, and their inter-

connections. In particular we shall show that a rule which is an

extended form of Cut of sequent calculi is not eliminable from our

systems, whereas another rule is. We shall comment on the importance

of eliminating this other rule, where it can be eliminated. After

some considerations connected with the question of the soundness and

completeness of our structural systems, we shall consider some

alternative conceptions of the language D and the understanding of

sentences of our language D.

§9   Suppose a language O is given which has at least one sentence.

The language D built on O we shall consider will be called "D1" .

D1 is constructed as follows

Elementary expressions of D1

(1)   all the elementary expressions of O;

(2)   an infinite number of expressions

$$\{ - \}; \{-,-\}; \{-,-,-\}; \quad \dots$$

with k argument places represented by dashes, and $k-1$ empty

spaces between argument places represented by commas, where

k is finite and $\geqslant 1$; these expressions are of the categories

$$\frac{t}{\underbrace{s...s}_{k}} \;,$$

and are called "curly brackets";

(3)   the expression

$$\emptyset$$

of the category t; called autonymously "$\emptyset$";

(4)   an infinite number of expressions

$$\_ \overset{1}{\vdash} \_ \; ; \_ \overset{2}{\vdash} \_ \; ; \_ \overset{3}{\vdash} \_ \; ; \; \cdots$$

where the superscript can be any $n \geqslant 1$, and the dashes stand for

two argument places; these expressions are of the category $\frac{s}{tt}$,

and are called "turnstiles".

Complex expressions of D1

(1)   All the complex expressions of O are complex expressions of

D1. The sentences of O are sentences of D1 of level O.

(2)   The elementary expression $\emptyset$ is a singular term of D1 of any

level n, $n \geqslant 1$. Let

$$A_1^{m_1} \; ; \; \cdots \; ; \; A_k^{m_k}$$

where $k \geqslant 1$, be sentences of levels $m_1, \ldots, m_k$; then the

expression of the following form

$$\{ \; A_1^{m_1}, \; \cdots \; , \; A_k^{m_k} \; \}$$

is a complex singular term of D1 of level $n + 1$, provided

(2.1) for every i, $1 \leqslant i \leqslant k$, $m_i = n$, where n is a given number $\geqslant 0$,

(2.2)   no sentence occurs more than once among

$$A_1^{m_1} \ ; \ \ldots \ ; \ A_k^{m_k} \qquad , \qquad \text{and}$$

(2.3)   if

$$B_1^{\ell_1} \ ; \ \ldots \ ; \ B_k^{\ell_k}$$

are the sentences

$$A_1^{m_1} \ ; \ \ldots \ ; \ A_k^{m_k}$$

taken in any order, the expression of the form

$$\{ B_1^{\ell_1} \ , \ \ldots, \ B_k^{\ell_k} \}$$

is the same singular term as the expression of the form

$$\{ A_1^{m_1}, \ \ldots, \ A_k^{m_k} \} \qquad .$$

(3)   Let

$$\Gamma \ ; \ \Delta$$

be singular terms of level n, where n is a given number $\geqslant 1$;

then the expression of the form

$$\Gamma \vdash^n \Delta$$

is a sentence of D1 of level n .

(4)   Nothing is a complex expression of D1 save if it can be obtained

by (1) - (3).

It should be clear from this specification that a singular

term of D1 of level n, $n \geqslant 1$, is in a one-one correspondence with

a finite (possibly empty) set of sentences of D1 of level $n - 1$, $\emptyset$

corresponding to the empty set.

We shall use the following basic schemata of U for sentences
of an unspecified level n , $n \geqslant 0$,

$$A^n, \ B^n, \ C^n, \ \ldots, \ G^n; \ A_1^n, \ B_1^n; \ \ldots$$

If for n we substitute: 0, 1, 2, ... we get basic schemata for
sentences of levels: 0, 1, 2, ..., but we shall in general omit
the superscript "0". For example, the following expression of U

$$\{A^1, B^1\}$$

will be a schema for a singular term of level 2. Substitution
for these basic schemata is subject to the proviso that the resulting
expression is an expression of D1 or a schema for an expression of
D1. This means that e.g. in

$$\{A^1, B^1\} \vdash^2 \emptyset$$

we can substitute only <u>different</u> expressions for "A^1" and "B^1".
But in

$$\{A^1\} \vdash^2 \{B^1\}$$

we can substitute any expressions of the required category and level
for "A^1" and "B^1". The level of a sentence-schema or singular
term-schema is the same as the level of those of its instances which
are expressions of D1, provided these are all of the same level.

The following basic schemata of U will be used for singular
terms of any level $\geqslant 1$

$$\Gamma, \Delta, \Theta, \Xi, \Pi, \Sigma, \Phi, \Psi, \Gamma_1, \Delta_1, \ \ldots$$

Substitution for these schemata is subject to the same proviso   as
above. The allowable substitutions for a schema like "Γ" can be
inferred from the sentence-schema in which it occurs, i.e. they will be

restricted to some levels in sentence-schemata. Though in principle we could also note that level by a superscript, no ambiguities can arise in a sentence-schema without that.

Since a singular term "Γ", or "$\{A^1, B^1\}$", corresponds exactly to a set, we shall say that it <u>stands</u> for a set, and that Γ, or $\{A^1, B^1\}$, are sets. We shall use the set-theoretical **sym-bol** "$\cup$" to construct the following complex basic schemata for singular terms of any level $\geqslant 1$

$$\Gamma \cup \Delta; \; \Gamma \cup \Delta \cup \Theta ; \; \dots$$

A basic schema of the form

$$\Gamma_1 \cup \dots \cup \Gamma_k, \; k \geqslant 2$$

is used for a singular term which stands for the union of the sets $\Gamma_1, \dots, \Gamma_k$. $\Gamma_1, \dots, \Gamma_k$ need not be disjoint, but we must bear in mind that for the expression for which the basic schema is used (2.1), (2.2) and (2.3) are satisfied.

We shall also use complex basic schemata of the following form for singular terms of an unspecified level $n, n \geqslant 0$,

$$\Gamma_1 \cup \dots \cup \Gamma_k \cup \{A^n_1, \dots, A^n_\ell\}$$

where $k \geqslant 1$, $\ell \geqslant 1$. If for n we substitute: $0, 1, 2, \dots$, we get basic schemata for singular terms of levels: $0, 1, 2, \dots$ The following convention will be essential for this work. A basic schema of the form above is used for a singular term which stands for the union of the sets $\Gamma_1, \dots, \Gamma_k, \{A^n_1, \dots, A^n_\ell\}$ <u>provided</u> that for every i, $1 \leqslant i \leqslant k$, Γ_i and $\{A^n_1, \dots, A^n_\ell\}$ <u>are disjoint.</u>

The symbol "$\cup$" will not be used in a singular term-schema otherwise
than as specified above. So, for example, the following

$$\Gamma \cup \{A^n\} \cup \Delta; \{A^1\} \cup \{B^1, C^1\}; \quad \emptyset \cup \Gamma \cup \{A^n\}$$

are not singular term-schemata.

We cannot substitute freely for the letters which occur in a
basic schema but only for the whole basic schema, and in doing that
we must keep in mind the provisions we have stated, and the particular
requirements pertaining to the occurrence of other basic schemata
and superscripts in this basic schema. In principle it is possible
to give a complex basic schema as a schematic letter with a proviso
which dictates what requirement must be satisfied by the expressions
which can be substituted for this schematic letter.

When we speak about all the singular terms
"$\{A^n_1, \ldots, A^n_k\}$", $n \geqslant 0$, $k \geqslant 0$, if $k = 0$, the singular term in
question is $\emptyset$.

The following

$$\vdash^n; \quad \vdash^{n+1}; \quad \ldots$$

will be basic schemata for turnstiles. [Examples for sentence-
-schemata constructed with basic schemata, and curly brackets and
$\emptyset$ as the only expressions of D1, can be found in the primitive
rules of §10. Note that the curly brackets in $\underline{A}$ and $\underline{D}$ are expressions
of D1, whereas those in $\underline{C}$ are not.]

In a less formal context of $\cup$, as distinguished from the context
of D1 and its schemata, we shall use some other familiar set-
-theoretical notation, like

$$\Gamma \subseteq \Delta; \ \Gamma = \Delta; \ \Gamma \neq \Delta; \ A^n \in \Gamma; \ A^n \notin \Gamma.$$

We shall also use in this context

$$A^n = B^n$$

to mean that the sentences involved are the same sentence.   All

this is, of course, translatable into plain language which refers

only to the syntactical features of Dl.

We shall use schemata like

$$A^n; \ A^1; \Gamma \vdash^n \Delta; \ \Gamma \cup \{A^n\} \vdash^{n+1} \Delta \ ; \ \text{etc.}$$

to speak about formulae of D in general, i.e. both sentences

and sentence-schemata, conveying what is meant by the context.

We shall also omit quotation marks in many cases, writing e.g.

$$A^n \ \text{ is a sentence.}$$

Definition of sequents

A formula A^n is a sequent iff $n \geqslant 1$.

In the sequent $\Gamma \vdash^{n+1} \Delta, n \geqslant 0$, Γ is called "the left set" and Δ "the right

set". Since $\Gamma \vdash^{n+1} \Delta$ can be a schema, the left or right set

can be the set of sentence-schemata which is in one-one

correspondence with the singular term-schema "Γ" or "Δ".

Definition of levels of rules

A rule of Dl is of level n iff the highest level of formulae

occurring in it is n.

Definition of level-preserving rules

The rule $\dfrac{A_1^{m_1} \cdots A_k^{m_k}}{B^{m_0}}$ is level-preserving iff for every

i, $0 \leqslant i \leqslant k$, $m_i = n$, for some $n \geqslant 0$.

Definition of horizontalizations of rules

The sequent $\{A_1^n, \ldots, A_k^n\} \vdash^{n+1} \{B^n\}$, $k \geqslant 1$, is a horizontali-

zation of the level-preserving rule $\dfrac{\Pi}{B^n}$ iff, for every i, $1 \leqslant i \leqslant k$,

an occurrence of the formula A_i^n belongs to Π and if an occurrence of

a formula C^n belongs to Π, either $C^n = A_j^n$, for some j, $1 \leqslant j \leqslant k$, o:

A_j^n, for some j, $1 \leqslant j \leqslant k$, is an instance of C^n.

N.B. A formula can occur only once in $\{A_1^n, \ldots, A_k^n\}$ but more

than once in Π. So a certain contraction can be involved in

producing a horizontalization. Accordingly, the correspondence

between transitions and their horizontalizations is many-one.

A transition-schema can have more than one horizontalization:

e.g. to

$$\frac{A \quad B}{A}$$

correspond both $\{A,B\} \vdash^1 \{A\}$ and $\{A\} \vdash^1 \{A\}$ if B can be either

different from or equal to A [cf. §§16,28]. Every level-preserving

rule has at least one and at most a finite number of horizontalizations.

§10   Definition of expressions essential for a system

An expression of a language included in L is essential for a system S in L iff it occurs at least once in a postulate of S or it occurs, or is referred to, at least once in a proviso of a postulate of S.

Definition of structural systems

A system in a language D is structural iff no constant of O is essential for it.

A postulate will be called "structural" iff if can be given for a structural system.

For the structural systems we shall consider we shall give at least some of the following primitive rules. They are all transition-schemata.

Primitive rules

Ascending $(\underline{A})$: $\underline{A}^{n+1} \dfrac{A^n}{\emptyset \vdash^{n+1}\{A^n\}}, n \geqslant 0;$

Descending $(\underline{D})$: $\underline{D}^{n+1} \dfrac{\emptyset \vdash^{n+1}\{A^n\}}{A^n} ,\quad n \geqslant 0 ;$

Iteration $(\underline{I})$: $\underline{I}^n \dfrac{A^n}{A^n} ,\ n \geqslant 0;$

Cut $(\underline{C})$: $\underline{C}^{n+1} \dfrac{\Gamma \vdash^{n+1} \Delta \cup \{A^n\} \qquad \Theta \cup \{A^n\} \vdash^{n+1} \Xi}{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi} ,\ n \geqslant 0 ,$

provided either $\Gamma \neq \Theta \cup \{A^n\}$ or $\Delta \cup \{A^n\} \neq \Xi$;

<u>Thinning</u> (<u>T</u>): $\underline{T}^{n+1}$ $\dfrac{\Gamma \vdash^{n+1} \Delta}{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi}$, $n \geqslant 0$,

provided either $\Gamma \cup \Theta \neq \Gamma$ or $\Delta \cup \Xi \neq \Delta$.

N.B. According to our convention, with $\underline{C}$ we have also the proviso: $A^n \notin \Delta$ and $A^n \notin \Theta$.

On the left side of the horizontal line of an instance of a rule we write the name of the rule and the level of the instance superscribed on this name, as we have done schematically above. The clause "$n \geqslant 0$" on the right is a proviso.

We shall also consider these rules restricted in some ways, and in particular the following:

(1) These rules can be given only for some levels. In that context, e.g. "$\underline{A}^{\leqslant n+1}$" will be the name of the rule $\underline{A}$ given only for <u>all</u> levels k, $1 \leqslant k \leqslant n+1$.

(2) Adding to $\underline{T}$ the proviso on the right (in addition to the proviso it already has) gives

$\underline{T}_{er}$: Ξ must be empty;

$\underline{T}_{el}$: Θ must be empty;

$\underline{T}_{sr}$: $\begin{cases} \text{if } \Delta = \emptyset, \ \Xi \text{ must be a singleton or empty,} \\ \text{if } \Delta \neq \emptyset, \ \Xi \text{ must be empty;} \end{cases}$

$\underline{T}_{sl}$: $\begin{cases} \text{if } \Gamma = \emptyset, \ \Theta \text{ must be a singleton or empty,} \\ \text{if } \Gamma \neq \emptyset, \ \Theta \text{ must be empty} \end{cases}$

(the subscript "e" stands for "empty", "s" for "single", "r" for "right", and "l" for "left").

If O has basic schemata, some rules of substitution should
be given for structural systems in D1.   We shall consider such rules
later when we explicitly assume that O is such a language
[v.Chapters 6 and 7].

The provisos for $\underline{C}$ and $\underline{T}$ after "$n \geqslant 0$" are given to make the
rules strictly independent from each other and to forestall some
trivial considerations.

Axiom-postulates for the structural systems and other systems
in D1 we shall consider will be systematically generated from the
rules according to the following principle.

Horizontalizing of primitive rules ($\underline{h}$)

All the horizontalizations of the primitive level-preserving
rule R are axiom-postulates.

Not every primitive level-preserving rule need be a rule R
mentioned by $\underline{h}$, i.e. in the scope of h.

Using $\underline{h}$ is mainly a matter of economy.   In principle, all the
needed axiom — postulates obtained by applying $\underline{h}$ could be listed,
their number being finite for systems given in the standard way.   But $\underline{h}$
also helps to make the articulation of our systems transparent.

The application of $\underline{h}$ can be restricted to some levels of the
rules R.   In that context, "$\underline{h}^{\leqslant n}$" will be the name of $\underline{h}$ assumed only
for all levels k, $0 \leqslant k \leqslant n$.

For some systems we shall consider, axiom-postulates will not
necessarily be obtained by $\underline{h}$.

Canonically we shall name a system S by listing the names of all of its postulates. When we assume $\underline{h}$ for some primitive rules, we shall not list the names of the horizontalizations, but we shall write "$\underline{h}$" in the name of the system with the understanding that every level-preserving rule to the right of "$\underline{h}$" is in the scope of $\underline{h}$. Since $\underline{A}$ and $\underline{D}$ are not level-preserving, it makes no difference whether we write them to left or right of "$\underline{h}$". Thus, the first structural system in D1 we shall consider will be named by

<div align="center">

$\underline{h}$ A D I C T   .

</div>

To designate the horizontalizations of R we shall use "$\underline{h}(R)$". Sometimes we shall superscribe to "$\underline{h}$" and "R" the level of R or the instance of R in question. In a proof the designation of a horizontalization will be written on its right.

§11   We have already defined the provable sentences $[v.\S6]$. A sentence-schema will be called "provable" iff all its instances which are sentences are provable. We now give some related definitions.

Definition of admissible sequents

The sequent $\Gamma \vdash^{n+1} \Delta$ is admissible in S iff [if all the formulae in Γ are provable in S, a formula in Δ is provable in S].

Definition of deducible sequents

A sequent $\Gamma \vdash^{n+1} \Delta$ is deducible in S iff $\Gamma \vdash^{n+1} \Delta$ is admissible in the extension of S with all the members of Γ (i.e. a member of Δ is provable in this extension).

Definition of horizontalizable rules

A rule in D1 is horizontalizable in the system S in D1 iff all its horizontalizations are provable in S.

Definition of admissible rules

A rule R in L is admissible in the system S in L iff [if all the premises of R are provable in S, the conclusion of R is provable in S].

Definition of deducible rules

A rule R in L is deducible in the system S in L iff R is admissible in the extension of S with the premises of R (i.e. the conclusion of R is provable in this extension).

Definition of derivable transitions

A transition T in L is derivable in the system S in L iff a proof of the conclusion of T can be exhibited in the extension of S with the premises of T in which all the premises of T are among the axioms, with repetitions among these premises either omitted or not.

Definition of derivable rules

A rule in L is derivable in the system S in L iff it is either a transition derivable in S or a transition-schema whose every instance which is a transition is derivable in S.

As an immediate consequence of definitions, we have that a level--preserving rule is admissible (deducible) for a system S in D1 iff all

its horizontalizations are admissible (deducible). We can also

easily show the following

Lemma 1   1.1 If a rule is horizontalizable in an extension S

of h A D C, then it is derivable in S.

1.2   If a rule is derivable in a system, it is deducible

in that system.

1.3   If a rule is deducible in a system, it is admissible in

that system.

Demonstration: 1.1 Suppose $\Gamma \vdash^{n+1} \{A^n\}$ is provable in S . Let

Γ be $\{B_1^n, \ldots, B_k^n\}$, $k \geqslant 1$.   We have in the extension of S with the

members of Γ

$$\underline{A}^{n+1} \frac{B_1^n}{\emptyset \vdash^{n+1} \{B_1{}^n\}} \quad \cdots \quad \underline{A}^{n+1} \frac{B_k^n}{\emptyset \vdash^{n+1} \{B_k^n\}} \quad \Gamma \vdash^{n+1} \{A^n\}$$

$$\overline{\qquad\qquad k \text{ applications of } \underline{C}^{n+1} \qquad\qquad}$$

$$\underline{D}^{n+1} \frac{\emptyset \vdash^{n+1} \{A^n\}}{A^n} .$$

The demonstration of 1.2 is obvious.

1.3 Suppose a rule $\frac{\Pi}{C}$ is deducible in S_1; then C is prov-

able in the extension of S_1 with the members of Π.   But this

extension is not proper if all the members of Π are provable in

S_1. Hence C is provable in S_1.

Q.E.D.

The converse of 1.1, or 1.2, or 1.3, is not necessarily true.
[cf.§22].

We can easily show that if a rule is derivable in a system, it is
derivable in any extension of that system. The same holds when we
replace "derivable" by "deducible". For suppose $\frac{\Pi}{C}$ is deducible
in a system S; then C is provable in S with Π.   A fortiori, it
will be provable in any extension with Π of an extension of S .
Hence, $\frac{\Pi}{C}$ is deducible in any extension of S. On the other hand,
if a rule is admissible in a system, it is not necessarily admissible
in an extension of that system [v.§14].

In this chapter we shall investigate some of the interconnections
between the notions defined above in the context of structural systems.

§12   Definition of eliminable postulates

A postulate is eliminable from a system S iff S is
contained in the subsystem of S without this postulate.

We can show

Lemma 2   The rule R in L is admissible in a system S in L iff
R is eliminable from the extension of S with R .

Demonstration: Take an application of R in a proof in the extension
S_1 of S with R such that there are no applications of R above it.
Then the premises are provable in S, and by the left-hand side of the Lemma,
the conclusion is too. Then replace the proof of this conclusion in
S_1 by its proof in S . The Lemma from left to right follows by an

induction on the number of applications of R in a proof in S_1.

For the other direction suppose that the premises of R are provable in S; then they are provable in S_1, and hence the conclusion is provable in S_1. Then we use the right-hand side of the Lemma.

<div align="right">Q.E.D.</div>

Lemma 3   If the rule R in D1 is admissible in a system S in D1, then a sequent is admissible in S iff it is admissible in the extension of S with R.

Demonstration: Assume the left-hand side. Then suppose that $\Gamma \vdash^{n+1} \Delta$ is admissible in S. If all the formulae in Γ are provable in the extension S_1 of S with R, then by Lemma 2 they are provable in S, and so a formula in Δ is provable in S. This formula is a fortiori provable in S_1. Next, suppose that $\Gamma \vdash^{n+1} \Delta$ is admissible in S_1. If all the formulae in Γ are provable in S, then a fortiori they are provable in S_1, and so a formula in Δ is provable in S_1. This formula is by Lemma 2 provable in S.

<div align="right">Q.E.D.</div>

Lemma 3 shows that an eliminable primitive rule does not increase the stock of admissible sequents.

It can easily be shown that A is not eliminable from hADICT, and that the rule I (not h(I)) is eliminable from any system. The rule T will be eliminable from hADICT (where we retain h(T) as a postulate), for we have

$$\underline{A}^{n+2}\ \dfrac{\Gamma \vdash^{n+1} \Delta}{\underline{C}^{n+2}\ \emptyset \vdash^{n+2}\{\Gamma \vdash^{n+1}\Delta\}}\quad \dfrac{\{\Gamma \vdash^{n+1}\Delta\}\vdash^{n+2}\{\Gamma \cup \Theta \vdash^{n+1}\Delta \cup \Xi\}\ \underline{h}^{n+1}(\underline{T}^{n+1})}{\underline{D}^{n+2}\ \dfrac{\emptyset \vdash^{n+2}\{\Gamma \cup \Theta \vdash^{n+1}\Delta \cup \Xi\}}{\Gamma \cup \Theta \vdash^{n+1}\Delta \cup \Xi.}}$$

$\underline{C}$ of any particular level $n + 1$, $n \geqslant 0$, could be shown to be
eliminable in the same way, using $\underline{h}^{n+1}$ $(\underline{C}^{n+1})$ and $\underline{C}$ of level $n + 2$.
But in general, with $\underline{C}$ of all levels, we have

Lemma 4. $\underline{C}$ is not eliminable from $\underline{hADICT}$.

Demonstration: We have a proof of the following form in $\underline{hADICT}$

$$\underline{A}^{n+2}\ \dfrac{\{A^n\}\vdash^{n+1}\{A^n\}\ \underline{h}^n(\underline{I}^n)}{\underline{A}^{n+3}\ \dfrac{\emptyset \vdash^{n+2}\{\{A^n\}\vdash^{n+1}\{A^n\}\}}{\underline{C}^{n+3}\ \emptyset \vdash^{n+3}\{\emptyset \vdash^{n+2}\{\{A^n\}\vdash^{n+1}\{A^n\}\}\}}}\qquad \underline{h}^{n+2}(\underline{C}^{n+2})$$
$$\dfrac{\{\emptyset \vdash^{n+2}\{\{A^n\}\vdash^{n+1}\{A^n\}\},\{\{A^n\}\vdash^{n+1}\{A^n\}\vdash^{n+2}\emptyset\}\vdash^{n+3}\{\emptyset \vdash^{n+2}\emptyset\}}{\{\{\{A^n\}\vdash^{n+1}\{A^n\}\}\vdash^{n+2}\emptyset\}\vdash^{n+3}\{\emptyset \vdash^{n+2}\emptyset\}.}$$

But we can show that a sentence proved in such a way is not provable
in $\underline{hADIT}$ with $\underline{h}(\underline{C})$.

First we show that $\underline{D}$ is eliminable from this system. For
suppose $\emptyset \vdash^{n+1}\{B^n\}$ is provable in $\underline{hAIT}$ with $\underline{h}(\underline{C})$.   It cannot be
an axiom; $\underline{I}$ is eliminable; and if it is got by $\underline{T}$, $\emptyset \vdash^{n+1}\emptyset$ would be
provable in this system, which by inspection of the postulates can be
seen to be impossible. Hence, it can only be got by $\underline{A}$. So $\underline{D}$ is
admissible and we use Lemma 2.

That $\{\{\{A^n\} \vdash^{n+1} \{A^n\}\} \vdash^{n+2} \emptyset\} \vdash^{n+3} \{\emptyset \vdash^{n+2} \emptyset\}$ is not provable

in hAIT with h(C) is shown as follows: it cannot be an axiom;

it cannot be got by A; I is eliminable; and it cannot be got by T:

otherwise either $\emptyset \vdash^{n+3} \{\emptyset \vdash^{n+2} \emptyset\}$, and hence $\emptyset \vdash^{n+2} \emptyset$, would be provable,

or $\{\{\{A^n\} \vdash^{n+1} \{A^n\}\} \vdash^{n+2} \emptyset\} \vdash^{n+3} \emptyset$ would be provable, which again by

inspection of the postulates can be shown to be impossible.

Q.E.D.

More precisely, this demonstration shows that it is impossible

to eliminate C of all levels in any system $\underline{h^{\leqslant n} A^{\leqslant n+1} D^{\leqslant n+1} I^{\leqslant n} C^{\leqslant n+1} T^{\leqslant n+1}}$,

where $n \geqslant 2$, i.e. any system which has provable sentences of at

least level 3. But we can show also that in an extension of

$\underline{h^{\leqslant 1} A^{\leqslant 2} D^{\leqslant 2} I^{\leqslant 1} C^{\leqslant 2} T^{\leqslant 2}}$ with a sentence of level 0, $\underline{C}^2$ is not

eliminable: we can simply start the proof of the demonstration above

with this sentence instead of $\{A^n\} \vdash^{n+1} \{A^n\}$. That is, in any

extension with sentences of the object language which has provable

sentences of at least level 2, C of all levels is not eliminable.

In an extension whose provable sentences are at most of level 2, C

of level 1 can be eliminated, but C of level 2 cannot.

We have also that $\underline{C}^2$ is not eliminable from $\underline{h^{\leqslant 1} A^{\leqslant 2} D^{\leqslant 2} I^{\leqslant 1} C^{\leqslant 2} T^{\leqslant 2}}$

if there are at least two (different) sentences of level 0. In

this system the following is provable only with $\underline{C}^2$ and not without

$$\{\emptyset \vdash^1 \{A\}, \{A\} \vdash^1 \{B\}, \{B\} \vdash^1 \emptyset\} \vdash^2 \{\emptyset \vdash^1 \emptyset\}.$$

It can be shown that when 0 has only one sentence, $\underline{C}^{\leqslant 2}$ is eliminable

from this system (there are only four sentences of level 1 in this

case, and a finite number of sentences at each level). For the

other facts about the eliminability of $\underline{C}$ or its level relativized

forms we assumed only that O has at least one sentence.

"Cut-elimination" usually consists in showing that a rule

corresponding to $\underline{C}^1$ is eliminable from a system corresponding to

$\underline{h^0A^1D^1I^0C^1T^1}$ or an extension of it [v.also §13].

$\underline{C}$ of an infinite number of levels must be given in extending

$\underline{hADIT}$ plus $\underline{h(C)}$, for otherwise there would be a highest level, and

the system above would be incomplete.

That a rule is eliminable does not mean that its horizontalizations

are eliminable. For example, $\underline{h(I)}$, $\underline{h(T)}$ and $\underline{h(C)}$ are, of course,

not eliminable from $\underline{hADICT}$. If we take the system $\underline{hADIT}$, without

$\underline{h(C)}$, $\underline{C}$ can be shown admissible, but $\underline{ChADIT}$ would be incomplete

in a certain sense.

§13   Next we shall show that $\underline{D}$ is eliminable from $\underline{hADICT}$ as a

corollary of a more general result which is important for the rest

of our work. From now on we assume the eliminability of $\underline{I}$ to

simplify demonstrations.

<u>Lemma 5</u>   If $\Gamma \vdash^{n+1} \Delta$ is provable in $\underline{hAICT}$, $\Gamma \vdash^{n+1} \Delta$ is

admissible in every extension of $\underline{hAICT}$.

<u>Demonstration</u>: We shall make an induction on the length of the

proof of $\Gamma \vdash^{n+1} \Delta$.

If $\Gamma \vdash^{n+1} \Delta$ is an axiom-postulate the demonstration is trivial.

For the induction step suppose that if $\Gamma \vdash^{n+1} \Delta$ is proved in

$\leqslant k$ steps, the Lemma holds. Let it be proved in $k+1$ steps. The last

step can be:

(1) $\underline{A}^{n+1} \dfrac{A^n}{\emptyset \vdash^{n+1}\{A^n\}}$, $\Gamma = \emptyset$, $\Delta = \{A^n\}$.

Then A^n is provable in every extension of <u>hAICT</u>, and hence $\Gamma \vdash^{n+1} \Delta$ is admissible in every such extension;

(2) $\underline{C}^{n+1} \dfrac{\Gamma_1 \vdash^{n+1} \Delta_1 \cup \{A^n\} \qquad \Gamma_2 \cup \{A^n\} \vdash^{n+1} \Delta_2}{\Gamma \vdash^{n+1} \Delta}$,

$\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Delta_1 \cup \Delta_2 = \Delta$. Then if all members of Γ are provable in an extension S of <u>hAICT</u>, all of Γ_1 are provable in S, and so a member of $\Delta_1 \cup \{A^n\}$ is, by the induction hypothesis. If this is a member of Δ_1, a member of Δ is provable in S. On the other hand, suppose A^n is provable in S. Then if all of Γ are provable in S, all of Γ_2 are; and since by the induction hypothesis we have that if all of $\Gamma_2 \cup \{A^n\}$ are provable in S, a member of Δ_2 is, we get that a member of Δ is provable in S;

(3) $\underline{T}^{n+1} \dfrac{\Gamma_1 \vdash^{n+1} \Delta_1}{\Gamma \vdash^{n+1} \Delta}$, $\Gamma_1 \subseteq \Gamma, \Delta_1 \subseteq \Delta$.

Then if all members of Γ are provable in S, all of Γ_1 are, and so a member of Δ_1 is, by the induction hypothesis; hence, a member of Δ is provable in S.

Q.E.D.

Then we have

Lemma 6 $\underline{D}$ is eliminable from $\underline{hADICT}$.

Demonstration: Suppose $\emptyset \vdash^{n+1} \{A^n\}$ is provable in $\underline{hAICT}$.   It follows

by Lemma 5 that A^n is provable.   Hence, $\underline{D}$ is admissible in this

system, and we use Lemma 2.

<div align="right">Q.E.D.</div>

However, $\underline{D}$ of all levels will not be eliminable from any

extension of $\underline{hADICT}$ [we shall consider this question in §21]. It

is easily shown that $\underline{D}$ is not a deducible rule of $\underline{hAICT}$. Later

[§14] we shall make some comments on the importance of eliminating

$\underline{D}$. In that context we shall consider mainly the system $\underline{hAICT}$. We

note first something concerning eliminability for this system.

By adapting the demonstration of Lemma 4 we could show the following

> for every $n \geqslant 3$, $\underline{C}$ of level n is not eliminable from
> $\underline{hAICT}$.

This is because without $\underline{D}$ we cannot use $\underline{C}$ of some level $k + 1$ to get

the effect of $\underline{C}$ of level k. Whereas in $\underline{hADICT}$, $\underline{C}$ of any particular

level was eliminable, provided we had $\underline{C}$ of a higher level (i.e.

$\underline{C}^{\leqslant 1}$, $\underline{C}^{\leqslant 2}$, $\underline{C}^{\leqslant 3}$, ... were all eliminable if we kept in every case the

rest of $\underline{C}$), here $\underline{C}$ of any particular level $\geqslant 3$ is not eliminable. It

remains possible only that $\underline{C}^{\leqslant 2}$ is eliminable. We could also show that

> for every $n \geqslant 2$, and every extension S of $\underline{hAICT}$
> with sentences of level 0, $\underline{C}$ of level n is not eliminable
> from S; for every $n \geqslant 2$, $\underline{C}$ of level n is not eliminable
> from $\underline{hAICT}$, provided there are at least two sentences of
> level 0;

> $\underline{C}^1$ is eliminable from the S above and from $\underline{hAICT}$;

$C^{\leqslant 2}$ is eliminable from hAICT if there is only one

sentence of level 0.

Also, T ceases to be eliminable in the absence of D.

Note that our considerations on eliminability did not depend

essentially on having T and h(T) unrestricted, nor on having them

at all. In particular we could show that

D is eliminable from hADICT$_{\delta\imath}$, hADICT$_{e\imath}$ and hADIC.

Also, all the considerations on the non-eliminability of C would hold

true with T$_{\delta\imath}$, T$_{e\imath}$ or no T at all.

§14   Now we shall give some results on the interconnections between

provable, deducible and admissible sequents, and also on provable

sequents and derivable rules, which are relevant to the soundness

and completeness of structural systems.

Lemma 7 $\Gamma \vdash^{n+1} \Delta$ is provable in hAICT iff $\Gamma \vdash^{n+1} \Delta$ is deducible

in hAICT.

Demonstration: From left to right we have by Lemma 5 that $\Gamma \vdash^{n+1} \Delta$

is admissible in the extension S of hAICT with the members of Γ.

For the other direction we shall make an induction on the length

of the proof of any A^n to show that if A^n is provable in S, $\Gamma \vdash^{n+1} \{A^n\}$

is provable in hAICT.

For the basis we have that if A^n is an axiom-postulate of

hAICT:

$$\frac{A^{n+1}}{T^{n+1}} \quad \frac{A^n}{\frac{\emptyset \vdash^{n+1} \{A^n\}}{\Gamma \vdash^{n+1}\{A^n\}}} \ .$$

If A^n is a proper axiom-postulate of S, then it is identical with a member of Γ, and $\Gamma \vdash^{n+1}\{A^n\}$ is provable in <u>hIT</u>.

Suppose for the induction step that if A^n is proved in S in $\leq k$ steps, $\Gamma \vdash^{n+1}\{A^n\}$ is provable in <u>hAICT</u>. Let it be proved in $k + 1$ steps.

The last step can be:

(1) $\dfrac{A^n \qquad B^{n-1}}{\emptyset \vdash^n \{B^{n-1}\}}$, $\quad A^n = \emptyset \vdash^n \{B^{n-1}\}$. Then no proper axiom of S could have been used in the proof, since they are all of level n , and A^n is provable in <u>hAICT</u>. Then as in the basis we get that $\Gamma \vdash^{n+1}\{A^n\}$ is too;

(2) $\dfrac{C^n \quad B^n \qquad C^n}{A^n}$. Then by the induction hypothesis $\Gamma \vdash^{n+1}\{B^n\}$ and $\Gamma \vdash^{n+1}\{C^n\}$ are provable in <u>hAICT</u>, and we have

$$\frac{C^{n+1} \quad \dfrac{\Gamma \vdash^{n+1}\{B^n\} \qquad \{B^n,C^n\} \vdash^{n+1}\{A^n\}}{C^{n+1} \quad \dfrac{\Gamma \vdash^{n+1}\{C^n\} \qquad \Gamma_1 \cup \{C^n\} \vdash^{n+1}\{A^n\}}{\Gamma \vdash^{n+1}\{A^n\}}}}{} \quad \underline{h}^n(\underline{C}^n)$$

in this proof $\Gamma_1 = \Gamma$ if $C^n \notin \Gamma$, and $\Gamma_1 \cup \{C^n\} = \Gamma$ if $C^n \in \Gamma$;

(3) $\dfrac{T^n \quad B^n}{A^n}$. Then by the induction hypothesis $\Gamma \vdash^{n+1}\{B^n\}$ is provable in <u>hAICT</u>, and we have

$$\underline{C}^{n+1} \quad \frac{\Gamma \vdash^{n+1}\{B^n\} \qquad \{B^n\}\vdash^{n+1}\{A^n\} \quad \underline{h}^n(\underline{T}^n)}{\Gamma \vdash^{n+1}\{A^n\} \ .}$$

This concludes the induction. Then let $A^n \in \Delta$.   We have

$$\underline{T}^{n+1} \quad \frac{\Gamma \vdash^{n+1}\{A^n\}}{\Gamma \vdash^{n+1}\Delta \ .}$$

<div align="right">Q.E.D.</div>

As a corollary, we have that $\emptyset \vdash^{n+1} \Delta$ is provable in <u>hAICT</u> iff a member of Δ is provable in <u>hAICT</u>.

In Lemma 7 is given from left to right a "verticalizing principle" (corresponding loosely to <u>modus ponens</u>), and from right to left a "horizontalizing principle" (corresponding loosely to the Deduction Theorem). If we take Γ to be non-empty and Δ a singleton, Lemma 7 says that <u>hAICT</u> is sound and complete with respect to the deducible level-preserving rules of <u>hAICT</u>. Verticalizing corresponds to soundness and horizontalizing to completeness. <u>D</u> corresponds to verticalizing and <u>A</u> to horizontalizing. The horizontalizing of primitive level--preserving rules (i.e. <u>h</u>) is just a particular case of this general horizontalizing principle. To show the eliminability of <u>D</u> means to show the eliminability of a verticalizing element of <u>hADICT</u>, and corresponds somehow to the elimination of a certain form of Cut. (It has also a similar effect: it shows that there need not be detours in proofs which consist in ascending to a higher level and then descending to a lower one, and it makes practicable the inductions on the length of proofs.)

Lemma 5 shows that every provable sequent is admissible in <u>hAICT</u>, and by Lemma 3 the same holds for <u>hADICT</u>. The converse is

not the case: there are sequents admissible in hAICT which are not

admissible in an extension of hAICT, and hence by Lemma 5 are not

provable in hAICT. (Also a sequent not admissible in hAICT can

become admissible in an extension of hAICT.) For example, every

sequent $\Gamma \vdash^1 \Delta$, where $\Gamma \neq \emptyset$, is admissible in hAICT; this is

because the clause "all members of Γ are provable" is always

false, no sentence of level 0 being provable. On the other hand,

not every $\Gamma \vdash^1 \Delta$, where $\Gamma \neq \emptyset$, is provable in hAICT. For $\Gamma \vdash^1 \Delta$

is not necessarily admissible in an extension in which some, but

not all, sentences of level 0 are provable. Another example, which

is not somehow "vacuous" like the preceding, is

$$\left\{\left\{\left\{A^n\right\} \vdash^{n+1} \left\{B^n\right\}\right\} \vdash^{n+2} \left\{\left\{C^n\right\} \vdash^{n+1} \left\{D^n\right\}\right\}\right\} \vdash^{n+3} \left\{\left\{\left\{A^n, E^n\right\} \vdash^{n+1} \left\{B^n\right\}\right\} \vdash^{n+2} \left\{\left\{C^n, E^n\right\} \vdash^{n+1} \left\{D^n\right\}\right\}\right\}\right\} ,$$

which is admissible in hAICT but not provable [for a demonstration

v. Lemma 50].

But we can show that hAICT is sound and complete with respect

to the admissible sequents mentioned in Lemma 5:

Lemma 8 $\Gamma \vdash^{n+1} \Delta$ is provable in hAICT iff $\Gamma \vdash^{n+1} \Delta$ is admissible

in every extension of hAICT.

Demonstration: From left to right we have Lemma 5.

For the other direction suppose $\Gamma \vdash^{n+1} \Delta$ is not provable in

hAICT. We shall show that in that case there is an extension of

hAICT in which $\Gamma \vdash^{n+1} \Delta$ is not admissible.

If $\Gamma \vdash^{n+1} \Delta$ is not admissible in hAICT, the Lemma is shown.

If, on the other hand, it is admissible in hAICT, then either not

all members of Γ are provable, or a member of Δ is provable. But

no member of Δ is provable, for otherwise $\Gamma \vdash^{n+1} \Delta$ would be provable

with $\underline{A}$ and $\underline{T}$. So not all members of Γ are provable. Let Γ_1 be

the set of these members and let S be the extension of hAICT with

members of Γ_1. Then all members of Γ are provable in S, but no

member of Δ is. For suppose A^n, where $A^n \in \Delta$, is provable in S.

Then by Lemma 7, $\Gamma_1 \vdash^{n+1} \Delta$ is provable in hAICT, and so is

$\Gamma \vdash^{n+1} \Delta$, contrary to our supposition. Hence, $\Gamma \vdash^{n+1} \Delta$ is not

admissible in S.   Q.E.D.

If $\Delta \neq \emptyset$, it is possible to show that $\Gamma \vdash^{n+1} \Delta$ is provable in

hAICT iff it is admissible in every extension of hAICT consistent at

all levels, i.e. such that for every level $n \geqslant 0$, there is a sentence

A^n which is not provable in this extension (which amounts to the

improvability of a sentence of level 0 and $\emptyset \vdash^{n+1} \emptyset$, for every $n \geqslant 0$).

By analogous arguments it is possible to show that Lemmata

7 and 8 hold when for "hAICT" we substitute "hAICT$_{\delta\hbar}$" or "hAICT$_{e\hbar}$",

and require that Δ be at most a singleton. If $\Gamma \vdash^{n+1} \Delta$ is provable

in these latter systems, Δ must be a singleton. But $\underline{T}$ on the left is

needed for the results of these lemmata for we have that e.g.

$\{B^1\} \vdash^2 \{\{A\} \vdash^1 \{A\}\}$ is admissible in the extension of hAIC with B^1, or

in every extension of hAIC , but is not provable in hAIC. What

we can show for hAIC is the following

Lemma 9   If $\Gamma \neq \emptyset, \Gamma \vdash^{n+1} \Delta$ is provable in hAIC iff it is a

horizontalization of a derivable rule of hAIC; and if $\Gamma = \emptyset$, it is

provable in hAIC iff the only member of Δ is provable in hAIC.

Demonstration: It can easily be shown that $\Gamma \vdash^{n+1} \Delta$ is provable

in hAIC only if $\Delta = \{A^n\}$, for some A^n. Suppose that $\Gamma \vdash^{n+1} \{A^n\}$ is provable

in hAIC. If it is an axiom, the Lemma from left to right is shown.

Suppose for the induction step that if it is proved in $\leqslant k$ steps, the

Lemma in this direction is shown. The last step can be:

1) $\underline{A}^{n+1} \dfrac{A^n}{\emptyset \vdash^{n+1} \{A^n\}}$;   or

2) $\underline{C}^{n+1} \dfrac{\Gamma_1 \vdash \{B^n\} \quad \Gamma_2 \cup \{B^n\} \vdash^{n+1} \{A^n\}}{\Gamma \vdash^{n+1} \{A^n\}}$, $\Gamma_1 \cup \Gamma_2 = \Gamma$;

and in both cases the result easily follows. To show the other

direction suppose $\Gamma \neq \emptyset$. There is no application of $\underline{A}$ in the proof

of A^n from the axioms among which are all of Γ. So every step is an

application of $\underline{C}^n$ and can be horizontalized. Then we use $\underline{C}^{n+1}$.

If $\Gamma = \emptyset$, we use $\underline{A}$.

                                                                    Q.E.D.

Lemma 9 cannot hold for "hAICT", or "hAICT$_{sr}$", or "hAICT$_{er}$",

substituted for "hAIC" , for then we have e.g. $\{B^1\} \vdash^2 \{\{A\} \vdash^1 \{A\}\}$.

$\underline{D}$ is not necessarily admissible in an extension of hAICT, so

that we must be careful in connecting the above results with hADICT,

or its extensions. The same holds for the other systems with $\underline{T}$
restricted.

§15   In this work we shall often use a kind of translation of D1
into O.   We shall here present one such translation, without,
however, assuming that O is necessarily the language on which D1 is
built.

Let O be the language of the propositional calculus with the
propositional variables $p, q, \hbar, \dots$, and the constants $\rightarrow, \&, \vee, \perp$ and T.
Then the o-translation of D1 into O is obtained as follows

$o(A_{i}^{0}) = A_{i}$, where A_{i}^{0} is the i-th sentence in an
enumeration of sentences of level 0 and A_{i} is the i-th variable
in an enumeration of propositional variables;

$$\hat{o}(\Gamma) = \begin{cases} o(A_1^n) \& \dots \& o(A_k^n), \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, \ k \geqslant 1, \\ \qquad T \qquad , \text{if } \Gamma = \emptyset \ ; \end{cases}$$

$$\check{o}(\Gamma) = \begin{cases} o(A_1^n) \vee \dots \vee o(A_k^n), \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, \ k \geqslant 1, \\ \qquad \perp \qquad , \text{if } \Gamma = \emptyset; \end{cases}$$

$$o(\Gamma \vdash^{n+1} \Delta) = \hat{o}(\Gamma) \rightarrow \check{o}(\Delta).$$

We use "$A_1 \& \dots \& A_n$" as an abbreviation for
"$(\dots (A_1 \& A_2) \& \dots \& A_{n-1}) \& A_n$" and homologously with $\vee$.

Next we can define a distinctive levelled wff of the language
of the propositional calculus as follows. The formulae

$$\left(A_1 \& \dots \& A_n \right) \rightarrow \left(B_1 \vee \dots \vee B_m \right), n \geqslant 1 \ , \ m \geqslant 1;$$
$$\left(A_1 \& \dots \& A_n \right) \rightarrow \ \perp \qquad\qquad , n \geqslant 1 \ ;$$
$$\qquad T \qquad \rightarrow \left(B_1 \vee \dots \vee B_m \right), m \geqslant 1 \ ;$$
$$\qquad T \rightarrow \perp \quad ,$$

are:

(1)   distinctive levelled wffs of level 1, if $A_1, \ldots, A_n$ is a
        sequence of propositional variables without repetition, and
        the same for $B_1, \ldots, B_m$;

(2)   distinctive levelled wffs of level $k + 1$, if $A_1, \ldots, A_n$
        is a sequence of distinctive levelled wffs of level k without
        repetition, and the same for $B_1, \ldots, B_m$;

(3)   nothing else is a distinctive levelled wff.

It can easily be shown that $o(A^n)$ is always a distinctive levelled
wff, and also that if A^n is provable in hADICT, $o(A^n)$ is a distinctive
levelled tautology [cf. Lemma 19]. But not conversely: there are
distinctive levelled tautologies $o(A^n)$ such that A^n is not
provable in hADICT, e.g.

$$((p \to q) \to (\hbar \to \delta)) \to (((p \& t) \to q) \to ((\hbar \& t) \to \delta))$$

[v. §14 and Lemma 50].

        This translation can incidentally also serve to prove the
consistency of hADICT at all levels.

§16   We have seen [v. Lemma 7] that every level-preserving rule
deducible in hAICT is horizontalizable in hAICT.   This relation
between deducible rules and horizontalizable rules will not be found
in every extension of hADICT.   For example, we shall consider
extensions in which the following level-preserving rules are
derivable, and hence deducible [the notation is familiar, but it will
also be explained later]

$$\frac{A}{\Box A} \qquad ; \qquad \frac{A(x)}{\forall x A(x)} \qquad ,$$

whereas $\{A\}\vdash^1\{\Box A\}$ and $\{A(x)\}\vdash^1\{\forall x A(x)\}$ are not provable. This is

not necessarily a weakness of D1 and systems in it. This shows only

that sequents cannot stand for any kind of deduction exemplified

in proofs, nor even for any kind of level-preserving deduction.

Later we shall discuss for what kind of deduction sequents may be

taken to stand *[v.§17]*. In this section we shall indicate briefly

how the "expressive power" of D1 could be increased.

In the specification of complex expressions of D1 any of the

clauses (2.1), (2.2) and (2.3) for singular terms could be changed.

First, we shall consider only replacing in (2.1) "$m_i = n$" by

"$m_i \leqslant n$". In such a language D - let us call it "DC" ("C" stands

for "cumulative") - we could envisage horizontalizing deducible rules

which are not level-preserving. Incidentally, this could also

take care of some previously non-horizontalizable level-preserving

deducible rules. For example, we shall see that the following rule

will be horizontalizable even with our language D1

$$\frac{\emptyset\vdash^1\{A\}}{\emptyset\vdash^1\{\Box A\}} \quad ;$$

i.e. $\{\emptyset\vdash^1\{A\}\}\vdash^2\{\emptyset\vdash^1\{\Box A\}\}$ should be provable in some system. In

the language DC we could also have the following horizontalizations

of instances of Ascending and Descending

$$\{A\}\vdash^2\{\emptyset\vdash^1\{A\}\}; \ \{\emptyset\vdash^1\{\Box A\}\}\vdash^2\{\Box A\} \quad ,$$

so that with Cut of level 2 we could presumably get

$$\{A\} \vdash^2 \{\Box A\} \ .$$

But this sequent should be distinguished from $\{A\} \vdash^1 \{\Box A\}$, which

must remain unprovable as we shall see. It is however questionable

whether even in this context we should have $\{A(\dot{x})\} \vdash^2 \{\forall x \ A(x)\}$

[v.§§60,65].

We shall not consider in detail how this more comprehensive

language DC and systems in it could be formulated. First, concerning

sequents of level 1 this language would not make any difference, and

sequents of level 1 are in a certain sense the most important in a

great part of this work. Second, with our more uniform treatment

of sequents we are able to achieve all the goals we have set to

ourselves, so that it is not clear what immediate gain we could

obtain with this more comprehensive language D. If this language

is needed for some other purposes, as it may be, we could always

try to formulate it and then consider our language D1 a fragment

of this language. The results we shall present in this work should

in principle hold in this wider context too.

Next, we can consider rejecting only (2.2) or (2.3) in order

to get singular terms which can be taken to stand for collections

of sentences which are not finite (possibly empty) sets of sentences.

One notion of singular terms which we could thus obtain would be based

on finite (possibly empty) sequences of occurrences of sentences.

That these sequences are made of occurrences of sentences, and not

sentences, means that a sentence can occur more than once in a sequence.

This would involve rejecting both (2.2) and (2.3). For the rest, this new language - let us call it "D2" - would be constructed exactly as D1.

Let us use the same basic sentence-schemata as for D1, and expressions of the form "$\{L_1, \ldots, L_{k_1}, A_1^n, \ldots, A_\ell^n, L_{k_1+1}, \ldots, L_{k_1+k_2}\}$" $k_1 \geqslant 0$, $k_2 \geqslant 0$, $\ell \geqslant 0$, as basic schemata for singular terms, where each L stands for a sequence of occurrences of sentences of D2 of the appropriate level. Then a system amounting in a certain sense to hADICT will be obtained with the following primitive rules, together with h applicable to all of those rules which are level-preserving,

Ascending $\quad \dfrac{A^n}{\emptyset \vdash^{n+1} \{A^n\}}$; $\qquad$ Descending $\quad \dfrac{\emptyset \vdash^{n+1} \{A^n\}}{A^n}$;

Iteration $\quad \dfrac{A^n}{A^n}$; $\qquad$ Cut $\quad \dfrac{\{L_1\} \vdash^{n+1} \{L_2, A^n\} \quad \{L_3, A^n\} \vdash^{n+1} \{L_4\}}{\{L_1, L_3\} \vdash^{n+1} \{L_2, L_4\}}$;

Thinning $\quad \dfrac{\{L_1\} \vdash^{n+1} \{L_2\}}{\{L_1, L_3\} \vdash^{n+1} \{L_2, L_4\}}$;

Permutation $\quad \dfrac{\{L_1, A^n, B^n, L_2\} \vdash^{n+1} \{L_3\}}{\{L_1, B^n, A^n, L_2\} \vdash^{n+1} \{L_3\}}$; $\qquad \dfrac{\{L_1\} \vdash^{n+1} \{L_2, A^n, B^n, L_3\}}{\{L_1\} \vdash^{n+1} \{L_2, B^n, A^n, L_3\}}$;

Contraction $\quad \dfrac{\{L_1, A^n, A^n, L_2\} \vdash^{n+1} \{L_3\}}{\{L_1, A^n, L_2\} \vdash^{n+1} \{L_3\}}$; $\qquad \dfrac{\{L_1\} \vdash^{n+1} \{L_2, A^n, A^n, L_3\}}{\{L_1\} \vdash^{n+1} \{L_2, A^n, L_3\}}$;

Repetition $\quad \dfrac{\{L_1, A^n, L_2\} \vdash^{n+1} \{L_3\}}{\{L_1, A^n, A^n, L_2\} \vdash^{n+1} \{L_3\}}$; $\qquad \dfrac{\{L_1\} \vdash^{n+1} \{L_2, A^n, L_3\}}{\{L_1\} \vdash^{n+1} \{L_2, A^n, A^n, L_3\}}$.

Everywhere we add $n \geqslant 0$, and possibly a proviso for Thinning to
obtain independence.

Repetition is superfluous in the presence of Thinning, but not
in its absence [v.§18, cf.Ch.5]. Note that with D1 a certain
function of Contraction is taken up by C.

It is also possible to envisage a notion of singular terms
based on finite (possibly empty) sets of occurrences of sentences,
which would involve rejecting (2.2) and keeping (2.3). Let us
call the corresponding language "D3". For systems in D3 we could
then dispense with giving Permutation, but we should explicitly
give or reject Contraction and Repetition.

Another notion of singular terms would be based on
finite (possibly empty) sequences of sentences; this would involve
rejecting (2.3) and keeping (2.2). Let us call the corresponding
language "D4". For systems in D4 we could then dispense with giving
Contraction and Repetition, and Cut would be modified, but we should
explicitly give or reject Permutation.

Finally, we could envisage singular terms based on infinite collections
of sentences, which would involve some obvious changes in the syntax
of D.

We can also have either of the modifications involving (2.2) and
(2.3) together with a modification of (2.1).

Working with D1 has some disadvantages. First, we have seen
that substitution for basic schemata is given with a proviso which
does not allow for complete freedom. Second, we shall see that the
analysis of some constants of O cannot be made in the context of D1.

In a certain sense D1 pictures incompletely the deductive
framework of U. This is immediately clear from our specification
of rules in which the premises are sets of occurrences of formulae.
With an appropriate notion of horizontizations, in a language like
D3 we could establish a one-one correspondence between level-
-preserving transitions and their horizontalizations, in contrast
to the many-one correspondence of D1. Indeed we could find in D3
a one-one correspondence between level-preserving rules and the
appropriate form of their horizontalizations. With an appropriate
notion of horizontalizations, we could have a one-one correspondence
between rules and their horizontalizations in D2, provided we
worked with another notion of rules based on sequences of occurrences
of premises. Our notion of rules was chosen because it seems to be
the best suited with classical logic holding in U.

On the other hand, working with D1 has some advantages. Though
substitution for schemata is somewhat more complicated, demonstrations
by induction on the length of proofs are simpler if we don't have
to consider Permutation, Contraction and Repetition (these
considerations would often be somewhat trivial). And though D1
"pictures" incompletely the deductive framework of U, it pictures a
definite and in a certain sense minimal aspect of it. To show that
some customary constants can be analyzed in this minimal context
seems to be of some philosophical interest. Also, in principle,
many of the results we have established and which we shall establish
are translatable in the context of a different language D.

§17   In this section we shall try to consider what kind of

deducibility relations can be expressed by sequents provable in

a system in Dl.

In general there is a many-one correspondence between <u>deductions</u>

and sequents, such that to the left set of the sequent belong

all the formulae which occur as <u>premises in a deduction</u> and **to** the

right set all the formulae which occur as <u>conclusions in a deduction</u>.

The appropriate notion of deduction hence must allow for more than

one conclusion (where the conclusions are taken <u>alternatively</u>), for

an empty set of premises, and for an empty set of conclusions. To proofs

of a system can correspond a kind of deduction for which only the second

of these things is allowed. We shall not attempt here to give an

account of deductions involving their graph-theoretical or other

similar properties. Assuming that such an account can be given, we

shall treat only of those features of deductions which can be represented

by sequents of Dl. This means: that only the premises and conclusions

of a deduction will be noted, all the formulae in between being

omitted; that a formula which eventually occurs more than once as

a premise (or a conclusion) will be noted only once; and that the

order of premises and conclusions is not taken into account.

The general notion of deduction involving an empty set of

premises or conclusions, and in particular a set of conclusions with

more than one member, i.e. <u>multiple conclusions</u>, is to a large extent

a construction of logicians and need not be particularly intuitive.

However, a coherent account of these deductions can be given.

A deduction represented by $\Gamma \vdash^n \Delta$ will be <u>valid</u> relative to

a certain notion of <u>correctness</u> of sentences of Dl iff [if all the

sentences in Γ are correct, some sentence in Δ is correct].
We shall not treat of this notion of correctness and the
corresponding notion of validity, which belong to semantics. But
we note that in particular cases this notion of correctness can be
interpreted syntactically. We have shown e.g. [v. §14] that the
structural system hADICT can be shown sound and complete with respect
to all the sequents which correspond to valid deductions with premises
and conclusions of the same level, where the notion of correctness
involved is identified with provability in an extension of hAICT.
These notions of correctness and validity will not be appropriate
for extensions of hADICT we shall consider.

Multiple-conclusion deductions are not exemplified in that
fragment of U which corresponds to the deductive meta-theory of a system
of provable sentences of D1. It is this, basically, which prevents
the interpretation of any sequent of D1 in this meta-theory. On the
other hand, sequents with singleton or empty right sets are in
principle interpretable in this meta-theory. Sequents of hADICT
could be interpreted within this meta-theory because they, so to say,
only set the limits within which we can look for a single-conclusion
deduction.

A deduction with an empty set of premises and a single conclusion
corresponds to a proof, provided all the steps in the deduction are
made according to directives. In such a deduction the "premises"
of directives which correspond to axioms are counted as premises of the
deduction.

A deduction with an empty set of conclusions can correspond to
a refutation of one of the premises. A refutation need not be made

of steps with single conclusions. In Chapter 5 we shall consider

systems in which provable sequents with empty right sets could

perhaps be interpreted as corresponding to refutations in which we

want to refute only one premise, and in which the deduction tree

branches downward towards the conclusions, the set of which is empty.

These systems will be based on the use of $\underline{T}_{e\ell}$ or $\underline{T}_{\delta\ell}$ [v.§§40-44].

In general, we shall see that provable sequents of extensions

of structural systems which have, ceteris paribus ,   (1) $\underline{T}$, (2) $\underline{T}_{\delta\hbar}$

or $\underline{T}_{e\hbar}$, (3) $\underline{T}_{\delta\ell}$ or $\underline{T}_{e\ell}$, or (4) no $\underline{T}$ at all, could be considered

as corresponding to

(1)   deductions of classical logic,

(2)   deductions of intuitionistic logic,

(3)   decuctions of a logic of refutation, or

(4)   deductions of an intuitionistic relevant logic.

However, not all deductions of these logics will be represented

by sequents of D1, but only those in which either the premises and

the conclusions are of the same level, or the set of premises or

the set of conclusions is empty. Also, only those deductions which

can be represented by sequents of a level immediately above the level

of the premises or conclusions will be representable in D1. When

considering sequents of level 1 these deductions can be considered

to be deductions in a single-conclusion or multiple-conclusion

natural deduction calculus to which implication-introduction may be

applied to discharge the premises. This is why we shall have provable

sequents like

$$\{A\&B\} \vdash^{1} \{A \vee C\}   ,$$

but no provable sequents like

$$\{A\} \vdash^1 \{\Box A\} \quad \text{or} \quad \{A(x)\} \vdash^1 \{\forall \hat{x} A(x)\} .$$

In general our systems will be interpretable as "being about" natural deduction calculi.

That deductions in classical logic can have multiple conclusions is far from being generally acknowledged, though sequent systems of level 1 appropriate for classical logic, which have sequents with arbitrary finite right sets, are acknowledged as a legitimate representation of classical logic. We suggest the following as one possible explanation of why multiple conclusions in classical logic are not recognized. Although classical deductions have multiple conclusions, we usually consider them only in an enthymematic form , i.e., in a form in which are omitted all alternative conclusions save one, as well as those parts of the deduction which "stem" from the omitted conclusions. For example, a deduction of classical logic like

$$\frac{\neg \neg A}{A}$$

would be the enthymematic form of

$$\frac{\text{no premises}}{(\text{or } \neg \neg A)}$$

$$\frac{\neg \neg A \qquad\qquad \neg A \qquad\qquad A}{\text{no conclusions}}$$
$$(\text{or } A) .$$

Usually it is taken that the form of a deduction is enthymematic if some correct premises are omitted; but we could as well say that in the

enthymematic form of a deduction are omitted some incorrect
(alternative) conclusions.

§18   [§§9-10] The notion of sequents we have presented is an extension
and modification of Gentzen's notion of a sequent (Sequenz)
introduced in Gentzen 1934 (Gentzen used "→" as the turnstile).
Hertz's work in the 1920's on expressions of the form $A_1, \ldots, A_k \vdash B$
apparently influenced Gentzen (v. Gentzen 1932). Gentzen's sequents
would correspond to our sequents of level 1. In Gentzen's sequents
on the left-hand side (Antezedens) and right-hand side (Sukzedens)
there are finite (possibly empty) sequences of occurrences of sentences.
Sequents for which finite (possibly empty) sets of sentences or
occurrences of sentences are used, can be found in Lyndon 1966 (p.64);
Smullyan 1968; Scott 1971, 1974, 1974a; and Shoesmith & Smiley 1978.
The rules I, C and T are derived from the structural part of Gentzen's
calculus LK. The structural part of this calculus would correspond
to the system $\underline{h^0 I^0 C^1 T^1}$.

      [§11] The notion of admissible (zulässig) rules (or also
sometimes "permissible rules") can be derived from Lorenzen 1955
(pp.19-20). The distinction between what we have called "deducible"
and "derivable" rules does not seem to be always pointed out. In
particular, it does not seem to be generally realized that the notion
of a derivable rule (or also sometimes "derived rule"), which can
well seem common, is a notion of relevant logic, and that deducible
rules are natural in the context of classical or intuitionistic logic.
We have thought it useful to collect here various references to the
notion of admissible rules and its connection with other notions

concerning rules: v. Kreisel & Putnam 1957; Hiż 1958, 1959;

Anderson & Belnap 1959, 1975 (pp.54, 299); Schütte 1960 (pp.40-41),

1977(pp.15-17); Porte 1960; Leblanc 1961, 1962; Lorenzen 1962;

Belnap, Leblanc & Thomason 1963; Curry 1963 (p.97); Harrop 1965;

Wang 1965; Pogorzelski 1968, 1971, 1975; Mints 1972; Wasserman 1974.

The distinction between what we have called "horizontalizable"

rules and deducible or **de**rivable rules which are not horizontalizable,

is hinted at in some of the works mentioned above. Some remarks on

this topic can also be found in Dummett 1973 (pp.435-436), 1977

(pp.168-170); Scott 1974; and Anderson & Belnap 1975 (pp.235-236).

Concerning admissible rules, a further distinction can be

drawn between those which are <u>constructively admissible</u> (i.e. such

that given a proof of the premises, there is an effective procedure

to find a proof of the conclusion), and those which are not; v.

Anderson & Belnap 1975 (pp.298-299). In a number of our

considerations concerning admissibility, "admissible" could be

replaced by "constructively admissible".

[§12]   Our results on the non-eliminability of Cut could

perhaps be compared to some remarks of Scott 1971 (pp.793-794),

and   1974a(p.414), which sound similar.

[§16]   The notion of sequents appropriate to the language

D2, and the corresponding structural rules are derived from Gentzen

1934. The rule of Repetition is not mentioned by Gentzen; it is

superfluous in the presence of Thinning, but not in its absence.

In relevant logic calculi corresponding to our systems of level 1

have been studied which don't have Thinning but have Repetition.

These are essentially the _mingle_ systems; v. Anderson & Belnap

1975 and Smirnov 1979. Also, systems which involve rejecting

or restricting in various ways either Thinning, or Permutation,

or Contraction, or Repetition,have been studied in relevant logic.

[§17]   As Shoesmith & Smiley 1978 remark, the interpretation

of sequents <<has been a matter not so much of dispute as tacit

disagreement.>> (p.33) What Shoesmith & Smiley call "the material

interpretation" is exemplified in Gentzen 1934, where there are no

allusions to a distinction between the object language and the

deductive meta-language, but only an object language with sequents

is considered.   Sequents in this language are to be taken as

abbreviations for other more conventional expressions, like those

of the o-translation of §15 (cf. however Gentzen 1936, II.5.1,

and 1938, 1.2). Church 1956 (§29), for example, takes this inter-

pretation for granted (cf. also Popper 1948, p.181, fn.7, and

Kleene 1949 for a misunderstanding along these lines). The "meta-

-linguistic" interpretation,which we have taken for granted,is

exemplified   in Curry 1963 (pp.184ff) and Prawitz 1965 (pp.90-91).

The multiple-conclusion aspects of the general notion of deduction

to which we have appealed are treated extensively in Shoesmith &

Smiley 1978, where there are also references to their origins and

some philosophical discussion of them.

Our idea concerning enthymemes can be derived from Meyer 1973,

which presupposes Anderson & Belnap 1961.

Chapter 3

STRUCTURAL ANALYSIS

§19   Our purpose in this chapter is to define the notion of

structural analysis with which we shall work from now on.

In order to give this definition we shall first characterize

some notions presupposed by it: in particular the notions of

analytic rules and uniqueness. We shall also consider some

related matters.

Our principal definitions will mention a language D in general,

and not any particular language D like D1. But our discussion will

be precise only if we substitute "D1" for "D" everywhere. With other

languages D, like DC, D2, etc., some notions should be redefined

implicitly, by analogy with what we have in D1, so that the notions

defined here can be applicable in contexts with them [e.g. the notion

of "horizontalizations" in § 24].

§20   Let $B_1^{m_1}$, ..., $B_k^{m_k}$, A^n, $k \geqslant 1$, $m_i \geqslant 0$, $1 \leqslant i \leqslant k$, $n \geqslant 0$,

be formulae of a language D; then all the rules

$$\frac{B_1^{m_1} \ldots B_k^{m_k}}{A^n} \quad ; \quad \frac{A^n}{B_1^{m_1}} \quad ; \ldots ; \quad \frac{A^n}{B_k^{m_k}}$$

will be given by the expression of U

$$\frac{B_1^{m_1} \ldots B_k^{m_k}}{A^n}$$

called "a double-line rule".

If "R" is the name of this double-line rule, "$R\!\downarrow$" will be the
name of the rule

$$\frac{B_1^{m_1} \quad \cdots \quad B_k^{m_k}}{A^n}$$

and "$R\!\uparrow$" will be a designation for any of the rules

$$\frac{A^n}{B_1^{m_1}} \quad ; \quad \cdots \quad ; \quad \frac{A^n}{B_k^{m_k}} \; .$$

Double-line rules are only an abbreviatory device of U, and are in
principle dispensable. For example, the rules $\underline{A}$ and $\underline{D}$ could have
been given by the double-line rule R

$$\frac{A^n}{\varnothing \vdash^{n+1} \{A^n\}} \quad ,$$

where $R\!\downarrow$ is $\underline{A}$ and $R\!\uparrow$ is $\underline{D}$.   In fact, a system with exactly the same
provable sentences as $\underline{hADICT}$ could have been given only with double-
-line rules, viz. the following rules and the horizontalizations of
the primitive level-preserving rules given by them

$$\frac{A^n}{\varnothing \vdash^{n+1}\{A^n\}} \quad ; \quad \frac{A^n}{A^n} \quad ; \quad \frac{\Gamma \vdash^{n+1} \Delta \cup \{A^n\} \qquad \Gamma \cup \{A^n\} \vdash^{n+1} \Delta}{\Gamma \vdash^{n+1} \Delta} \; .$$

The last of these **double-line** rules is in a certain sense a conflation of Cut
and Thinning.

Definition of analytic rules

The double-line rule of the form

$$\frac{B_1^n \quad \ldots \quad B_k^n}{A^n} \quad ,$$

where a $k \geqslant 1$, and an $n \geqslant 0$ are given, is analytic iff

(1) only one constant of 0 occurs at only one place in the
formula A^n, **and**

(2) no constant of 0 occurs in the formulae B_1^n, ..., B_k^n.

Every rule given by an analytic rule is level-preserving.

For names of expressions of 0 we shall use the schemata

$$\alpha, \beta, \gamma, \alpha_1, \beta_1, \ldots \quad ,$$

occasionally also with some special indices. The constant α which
is the only constant of 0 to occur in an analytic rule R will be
called "the constant analyzed by R". The standard name of the analytic
rule R will be an expression of the form

$$(\alpha); \quad \text{or} \quad (\alpha)_k \quad ,$$

where k is a subscript $\geqslant 0$, to distinguish analytic rules in which
the constant analyzed is identical.

The level of an analytic rule will be the same as the level
of the rules given by it.

Extensions of a structural system S with analytic rules and
eventually the horizontalizations of some or all of the rules given
by these analytic rules will be called "analytic extensions of S".

§21

The _level of an analytic extension_ will be the same as the highest

level of its analytic rules. An analytic rule is in the

scope of $\underline{h}$ iff all the rules given by it are in the scope of $\underline{h}$.

The horizontalizations of the rules given by an analytic rule (α)

will be called "the horizontalizations of (α)". We shall say that

(α) is horizontalizable iff all the rules given by (α) are horizontalizable.

§21   We shall now consider the eliminability of $\underline{D}$, of at least some

levels, from analytic extensions of structural systems.

Lemma 10   Let S be $\underline{hADICT}$, $\underline{hADICT}_{\delta\hbar}$, $\underline{hADICT}_{e\hbar}$, $\underline{hADICT}_{\delta\ell}$, $\underline{hADICT}_{e\ell}$,

or $\underline{hADIC}$, and let S_1 be an analytic extension of S of level k, for

some $k \geq 0$. Then $\underline{D}$ of all levels $\geq k + 1$ is eliminable from S_1.

Demonstration: Let S_2 be the subsystem of S_1 without $\underline{D}$ of all

levels $\geq k + 1$; hence, S_2 has $\underline{D}^{\leq k}$. We shall show that $\underline{D}$ is admissible

for S_2. For all levels $\leq k$ the demonstration is trivial. Suppose

now that $\Gamma \vdash^n \Delta$ is provable in S_2 , where $n \geq k + 1$. We can show that

$\Gamma \vdash^n \Delta$ is admissible in S_2 by an induction of the length of the proof

of $\Gamma \vdash^n \Delta$. The only addition to be made in an induction adapted from

the one in the demonstration of Lemma 5 is in the basis where $\Gamma \vdash^n \Delta$

is a horizontalization of a primitive rule.

Then if $\emptyset \vdash^n \{A^{n-1}\}$ is provable in S_2, it is admissible in S_2 ,

and hence A^{n-1} is provable in S_2. Then we use Lemma 2.

                                        Q.E.D.

We note that if S_1 is an analytic extension of level 0, $\underline{D}$ is completely eliminable from S_1. $\underline{D}$ will also be completely eliminable from an extension of S with sentences of level 0 and rules of level 0, eventually in the scope of $\underline{h}$.

§22   In this section we shall consider the notions of synonymity and uniqueness of expressions.

Let ξ be a schematic letter of U for an arbitrary expression of O of some category and let α and β be arbitrary expressions of O of the same category as ξ. ("ξ" is the name in U of the schematic letter ξ of U, whereas "α" is the name in U of the expression α of O.) We shall use basic schemata of the form

$$A^n(\xi)$$

for sentence-schemata of D in which ξ occurs at least once, and basic schemata of the form

$$S_\alpha^\xi \, A^n(\xi)|$$

for formulae of D obtained from $A^n(\xi)$ by substituting α at the place of every occurrence of ξ in $A^n(\xi)$. If ξ was the only expression in $A^n(\xi)$ which is not an expression of O,

$$S_\alpha^\xi A^n(\xi)|$$

will be a basic schema for sentences of D. We are here interested in ξ only as a "place-marker", i.e. an auxilliary of the notation for substitution instances.

Definition of inter-admissible (inter-deducible, inter-derivable)

formulae

The formulae A^n and B^n of D are inter-admissible (inter-
-deducible, inter-derivable) iff the rules given by

$$\frac{A^n}{B^n}$$

are admissible (deducible, derivable).

Definition of admissibly (deducibly, derivably) synonymous expressions

The expressions α and β of O are adm.-synonymous (ded.-synonymous,

der.-synonymous) in a system S in D iff, for every $A^n(\xi)$ such that

$S_\alpha^\xi A^n(\xi)|$ and $S_\beta^\xi A^n(\xi)|$ are formulae of D , $S_\alpha^\xi A^n(\xi)|$ and $S_\beta^\xi A^n(\xi)|$ are

inter-admissible (inter-deducible, inter-derivable) in S.

N.B.   All of these definitions of synonymity permit both

uniform and non-uniform replacements of α by β and vice versa, salva

provability.

To show the difference between the three notions of

synonymity take a language O which has only two functors: α and

β , of the category $\frac{\Lambda}{\Lambda\Lambda}$, and then take in this language the

system which has the axiom-schemata

$$A\,\alpha\,A \quad ; \quad A\,\beta\,A \quad ,$$

provided neither α nor β occurs in A, and the primitive rules

$$\frac{B}{B\,\alpha\,C} \quad ; \quad \frac{C}{B\,\alpha\,C} \quad ; \quad \frac{B}{B\,\beta\,C} \quad ; \quad \frac{C}{B\,\beta\,C} \quad .$$

In this system α and β are adm.-synonymous but not ded.-
-synonymous; and $A\,\alpha\,A$ and $A\,\beta\,A$, for any A such that neither

α nor β occurs in A, are ded.-synonymous but not der.-

-synonymous.

If α and β are <u>der.</u>-synonymous in a system, they are <u>ded.</u>-
-synonomyous in this system; and if they are <u>ded.</u>-synonymous they
are <u>adm.</u>-synonymous [v. Lemma 1].

Which of these notions of synonymity we shall choose will
depend on the particular purposes we have in mind. However, we
shall see that for the particular systems with which we shall be
concerned, α and β will be synonymous in one of these senses iff
they are synonymous in one of the other senses [v.§§77,78].

From now on, by "synonymous", <u>tout court</u>, we shall understand
"<u>adm.</u>-synonymous". This notion embodies the intuitive notion of
interreplaceability <u>salva</u> provability.

Let S be a system in D and let $S^{\#}$ be the system we obtain
when we replace <u>everywhere</u> in the postulates of S the expression α of
O, which occurs at least once in these postulates, by the expression $\alpha^{\#}$
of O of the same category as α and different from α. $SS^{\#}$ will be
the system obtained by giving both the postulates of S and $S^{\#}$, in the language with
both α and $\alpha^{\#}$. N.B. $SS^{\#}$ is the result of introducting <u>only one</u> expression $\alpha^{\#}$: no
simultaneous introduction of more than one $\alpha^{\#}$ is envisaged.

We have an example of a system $SS^{\#}$ if in the system above we
rewrite "β" as "$\alpha^{\#}$".

<u>Definition of uniqueness of expressions</u>

The expression α of O is unique in a system S in D iff α and $\alpha^{\#}$
are synonymous in $SS^{\#}$.

Later [v. Ch.9] we shall consider in more detail questions concerning uniqueness in a particular context.

Homologous definitions of synonymity and uniqueness are obtained when for "D" we subsitute "O", above.

§23   We shall say that an expression α of O is explicitly definable in terms of the expressions $\beta_1,\ldots,\beta_k$, $k \geqslant 1$, of O iff for every sentence A^n of D in which α occurs at least once, a sentence B^n of D can effectively be found in which there occur only: (1) all the expressions $\beta_1,\ldots,\beta_k$ and (2) all the expressions occurring in A^n which remain after α has everywhere been deleted, and such that A^n and B^n are synonymous in every system S in D.

We allow the case that $k = 1$ and α is identical with β_1: i.e. every expression of O is explicitly definable in terms of itself.

An expression of U which describes a procedure which enables us to find the sentence B^n whenever a sentence A^n is given is called "an explicit definition of α". An explicit definition is proper iff α is different from β_i, for every i, $1 \leqslant i \leqslant k$.

If α is explicitly definable in terms of $\beta_1,\ldots,\beta_k$, $k \geqslant 1$, and β_i, $1 \leqslant i \leqslant k$, is explicitly definable in terms of $\gamma_1, \ldots, \gamma_\ell$, $\ell \geqslant 1$, then α is explicitly definable in terms of $\beta_1,\ldots,\beta_{i-1},\gamma_1,\ldots,\gamma_\ell,\beta_{i+1},\ldots,\beta_k$.

Our notion of explicit definition satisfies

Pascal's Condition

A definition must enable us to replace in every context a sentence in which occurs a defined expression by a sentence provided

by the definition, _salva_ provability.

§24   Definition of structurally analyzed constants

The expression α is structurally analyzed by the double-line

rule (α) iff

(0) for any language L in which α is a constant there is a

language O in which L is included and which has no constant which is not a

constant of L,

(1) (α) is an analytic rule in which α is the constant analyzed,

(2) there is a system S in D built on O for which (α) and eventually

the horizontalizations of (α) are given, for which without these postulates α

is not essential, and in which are provable all those, and only those,

sentences of O which are correct (provided these correct sentences

coincide with the provable sentences of a system), and

(3) α is unique in S (provided O is deductively monotonic

with respect to $SS^{\#}$).

The notion of deductive monotonicity, used do state the

proviso of clause (3), will be defined later _[§77]_. The treatment

of this notion is postponed for heuristic reasons. The import of

the proviso will be that if $\{A\} \vdash^1 \{B\}$ and $\{B\} \vdash^1 \{A\}$ are provable in $SS^{\#}$,

and C_2 is obtained from C_1 by replacing A by B at some or all places

where A occurs in C_1, then $\{C_1\} \vdash^1 \{C_2\}$ and $\{C_2\} \vdash^1 \{C_1\}$ are provable

in $SS^{\#}$ _[v. also §87]_.

A structural analysis of α consists in specifying the languages

O and D, and the analytic rule (α) with which α can be structurally analyzed.

In the definition of structurally analyzed constants (1) states
the _form of a structural analysis_,   (2) requires that a structural
analysis characterizes _soundly_ and _completely_ the constant analyzed,
and (3) requires that this characterization be _unique_. The point
of clause (0) is that for L we need to assume only that it has the
analyzed constant α.   For 0 we have to make additional assumptions:
e.g. the assumption that it has sentences, but we shall also make
more substantial assumptions later _[v. Chapters 7,8]_. If L
satisfies these assumptions, it can be taken to coincide with 0; if
not, then 0 is obtained from L by adding schemata for the expressions
required by the assumptions, but no new constant _[v.§87, cf.§37]_.

The _correct_ sentences of 0 can be specified independently of S,
as those which are provable in some other system, or are correct in
a semantical framework, or are possibly given in some other way.
But these correct sentences can eventually also be defined as those
which are provable in S: in this latter case S is trivially sound
and complete with respect to these sentences _[cf. §92]_.

In principle S should also be at least sound, but not necessarily
complete, with respect to the correct sentences of levels higher than
0, and not only with respect to those of level 0. But deciding which of
these sentences of higher levels are correct will depend on the
exact interpretation of sequents. A system S will be trivially sound
and complete if S is to codify that part of the deductive meta-
-language which is, so to speak, horizontalizable in it. But we have
seen _[v. Lemma 1]_ that in some particular cases S can be shown at
least sound in an independent manner, and it would also be possible
to show that S is sound with respect to a semantical notion of

consequence. In principle, the soundness and completeness of S

should be shown with respect to a natural deduction framework, and

where in S we have provable sequents with arbitrary finite collections

on the right, this natural deduction framework will involve multiple-

-conclusion deductions.

The proviso in (2) that correct sentences of O coincide with the

provable sentences of a system seems to imply that the definition of

structurally analyzed constants will be inapplicable where O does

not satisfy this proviso. Such a language O will be e.g.

a   language which includes the language of formal arithmetic. However,

in this case we can say the following. Let us distinguish two

kinds of correct sentences of formal arithmetic: those that are correct

in the standard interpretation and those that are provable in a given

system S . Then we can show that some constants of formal arithmetic

are structurally analyzed if "correct" is taken in the second sense.

That these constants are the same as those which occur in correct

sentences where "correct" is taken in the first sense, would be

shown by semantical considerations.

Definition of structural constants

A constant of O is structural iff it is explicitly definable

in terms of some structurally analyzed constants of O.

We do not require the explicit definition of α to be proper,

i.e. α can be explicitly defined in terms of itself. Hence, a

structurally analyzed constant is structural. If a constant is

established as structural by means of a structural analysis, we

shall call it "a primary structural constant", and if it is established

as structural by a proper explicit definition, we shall call it "a
secondary structural constant".   A structural constant can be
both primary and secondary.

    Our main purpose in the following chapters is to show that the
customary constants of classical first-order logic with identity, and
some non-classical logics, and the constants of the modal logics
S5 and S4,are structural, where the context of analysis is provided
by D1. We shall be concerned first with clauses (0), (1) and (2) of
the definition of structurally analyzed constants *[Chapters 4-8]*
and then with clause (3) *[Chapter 9]*.

§25   [§20] The general notion of analytic rules is in a certain
sense implicit in Gentzen 1934. This notion was more explicitly
suggested by Ketonen 1944 (cf. also Bernays 1945 and Curry 1963,
pp.199-203). It might also be a part of what Popper wanted to
express in his work in the 1940's *[v. §99]*. It is exemplified explicitly
in Kneale 1956 and Kneale & Kneale 1962 (p.561); Scott 1971, 1973,
1974a; and Smirnov 1972, 1979.

    [§22] Historical remarks on uniqueness will be given in §85.
Here we note only that the system used for distinguishing various
**notions of synonymity can be compared to a system of Schütte
1977 (p.15) which serves a somewhat similar purpose.**

    [§23] One of the laws to be obeyed in demonstrations given
by Pascal 1658   is : <<Substituer   toujours mentalement les
definitions à la place des definis, pour ne pas se tromper par

l'equivoque des termes que les definitions ont restreints.>> (p.280;
cf. also pp.244, 279-282). Pascal's Condition is derived from
this law (v. also Beth 1959, pp.504-505; cf. Aristotle Topica,
Z.4, 142b).

Chapter 4

CLASSICAL PROPOSITIONAL CONSTANTS

§26   First we introduce the propositional constants which we shall

consider in this and the following chapter. Next we give the

analytic rules of level 1 and explicit definitions for these

constants, which will constitute the canonical set of analytic

rules and explicit definitions for these constants in both classical

and non-classical logics.

We next consider the analytic rules which could be used to replace

the analytic rules of this set, in the context of classical logic.

Then we introduce an axiomatization of classical propositional

logic (and also of Heyting and Kolmogorov-Johansson propositional

logic), and we show that the analytic extensions of hADICT with the

analytic rules for propositional constants can serve to demonstrate

the soundness and completeness of our analyses of the classical

propositional constants. Finally, we consider whether this set of

analytic rules is replaceable by postulates of this axiomatization

in O. This topic is treated because it shows to what extent an

axiomatization in O can provide a certain sort of analysis of logical

constants when sentences of levels higher than O are also considered.

In general, results on replaceability are not our main concern in

this work, but they shed some light on a number of questions connected

with our analyses.

§27   <u>Definition of propositional constants</u>

A constant of O is propositional iff it is of the category

$$\frac{\delta}{\underbrace{\delta...\delta}_{k}} \quad , \text{ for some } k \geqslant 0.$$

We assume that O has the following propositional constants

implication: →,
conjunction: &,
disjunction: v,
co-implication: ↔,

which are of the category $\frac{\delta}{\delta\delta}$,

the <u>fal</u>sum: ⊥ ,
the <u>verum</u>:   T ,

which are of the category δ, and

negation: ¬ ,

which is of the category $\frac{\delta}{\delta}$.   These are not necessarily the
only constants of O, but those we assume must be in it. Occasionally
we can assume that we have in O some other propositional constants
which are not in this list.

If O has the constants above, O will have an infinite
number of sentences.

A constant of O will be named autonymously, in a standard way;
for example, we shall say

→ is a constant.

Anticipating our conclusions [v.§§94-95], we shall use the
same expression of O (that is, the same symbol) for different constants
when these constants are to be shown equivalent in a sense to be made
precise later. So we shall use all of the above symbols, and also
some symbols we shall introduce later, for constants of both classical
and non-classical logics.

Strictly speaking we need not assume that we have parentheses in O: they are only an auxiliary grammatical notation of U for showing the argument-places of functors. We shall use them in a standard way.

§28   We give the following analytic rules in D1 for the constants above

$$(\rightarrow) \quad \frac{\Gamma \cup \{A\} \vdash^A \Delta \cup \{B\}}{\Gamma \vdash^A \Delta \cup \{A \rightarrow B\}} \quad ;$$

$$(\&) \quad \frac{\Gamma \vdash^A \Delta \cup \{A\} \qquad \Gamma \vdash^A \Delta \cup \{B\}}{\Gamma \vdash^A \Delta \cup \{A \& B\}} \quad ;$$

$$(\vee) \quad \frac{\Gamma \cup \{A\} \vdash^A \Delta \qquad \Gamma \cup \{B\} \vdash^A \Delta}{\Gamma \cup \{A \vee B\} \vdash^A \Delta} \quad ;$$

$$(\leftrightarrow) \quad \frac{\Gamma \cup \{A\} \vdash^A \Delta \cup \{B\} \qquad \Gamma \cup \{B\} \vdash^A \Delta \cup \{A\}}{\Gamma \vdash^A \Delta \cup \{A \leftrightarrow B\}} \quad ;$$

$$(\perp) \quad \frac{\Gamma \vdash^A \varnothing}{\Gamma \vdash^A \{\perp\}} \quad ;$$

$$(\top) \quad \frac{\varnothing \vdash^A \Delta}{\{\top\} \vdash^A \Delta} \quad .$$

All these analytic rules are of level 1.

We shall not consider the analytic rule ($\leftrightarrow$), since, in the presence of $\rightarrow$ and &, we have the following explicit definition ("$=_{df}$" will indicate that the expression on the left is explicitly definable by the expression on the right)

$$A \leftrightarrow B =_{df} \quad (A \to B) \& (B \to A).$$

For $\neg$ we give the following explicit definition

$$\neg A =_{df} \quad A \to \perp.$$

All these analytic rules and explicit definitions will be used without any change for both classical and non-classical constants. We shall see later that it is possible to give analytic rules for negation, but not an analytic rule which will serve for the structural analysis of negation in every context we shall consider.

We note that substitution for "A" and "B" in the analytic rules and explicit definitions above is free, so that A and B can be the same formula. Hence, there will be two horizontalizations of (&)↓ and (v)↓ , viz. $\{\Gamma \vdash^1 \Delta \cup \{A\}, \Gamma \vdash^1 \Delta \cup \{B\}\} \vdash^2 \{\Gamma \vdash^1 \Delta \cup \{A\&B\}\}$ and $\{\Gamma \vdash^1 \Delta \cup \{A\}\} \vdash^2 \{\Gamma \vdash^1 \Delta \cup \{A\&A\}\}$; and homologously with (v)↓. We note also that according to our conventions, in (→): $A \notin \Gamma$, $B \notin \Delta$, $A \to B \notin \Delta$; in (&): $A \notin \Delta$, $B \notin \Delta$, $A\&B \notin \Delta$; and in (v): $A \notin \Gamma$, $B \notin \Gamma$, $AvB \notin \Gamma$. But we can show that the following rules

$$\frac{\Gamma \cup \{A\} \vdash \Delta \cup \{A \to B, B\}}{\Gamma \vdash \Delta \cup \{A \to B\}} \quad ; \quad \frac{\Gamma \cup \{A\} \vdash \Delta \cup \{B, A \to B\}}{\Gamma \cup \{A\} \vdash \Delta \cup \{B\}} \quad ;$$

$$\frac{\Gamma \vdash \Delta \cup \{A\&B, A\} \quad \Gamma \vdash \Delta \cup \{A\&B, B\}}{\Gamma \vdash \Delta \cup \{A\&B\}} \quad ; \quad \frac{\Gamma \vdash \Delta \cup \{A, A\&B\}}{\Gamma \vdash \Delta \cup \{A\}} \quad ;$$

$$\frac{\Gamma \vdash \Delta \cup \{B, A\&B\}}{\Gamma \vdash \Delta \cup \{B\}} \quad ; \quad \frac{\Gamma \cup \{AvB, A\} \vdash \Delta \quad \Gamma \cup \{AvB, B\} \vdash \Delta}{\Gamma \cup \{AvB\} \vdash \Delta} \quad ;$$

$$\frac{\Gamma \cup \{A, AvB\} \vdash \Delta}{\Gamma \cup \{A\} \vdash \Delta} \quad ; \quad \frac{\Gamma \cup \{B, AvB\} \vdash \Delta}{\Gamma \cup \{B\} \vdash \Delta}$$

are horizontalizable in $\underline{hAIC}(\rightarrow)(\&)(\vee)$. As an example we give

the following proof

$$(\rightarrow)\!\uparrow\frac{\{A\rightarrow B\}\vdash\{A\rightarrow B\}\ \underline{h}^{\circ}(\underline{I}^{\cdot})}{\{A\rightarrow B,A\}\vdash\{B\}}$$

$$\underline{A}^{2}\ \frac{}{\underbrace{\emptyset\vdash^{2}\{\{A\rightarrow B,A\}\vdash\{B\}\}}_{①}}\qquad\underbrace{\{\Gamma\cup\{A\}\vdash\Delta\cup\{B\}\}\vdash^{2}\{\Gamma\vdash\Delta\cup\{A\rightarrow B\}\}}_{②}\ \underline{h}^{1}((\rightarrow)\!\downarrow)$$

$$\underbrace{\{\{\Gamma\cup\{A\}\vdash\Delta\cup\{A\rightarrow B,B\},\{A\rightarrow B,A\}\vdash\{B\}\}\vdash^{2}\{\Gamma\cup\{A\}\vdash\Delta\cup\{B\}\}}_{③}\ \underline{h}^{\circ}(\underline{C}^{\cdot})$$

$$\underline{C}^{2}\ \underline{\underline{C}}^{2}\ \frac{\overset{①}{}\qquad\qquad\qquad\overset{③}{}}{\{\Gamma\cup\{A\}\vdash\Delta\cup\{A\rightarrow B,B\}\}\vdash^{2}\{\Gamma\cup\{A\}\vdash\Delta\cup\{B\}\}}\overset{②}{}$$

$$\frac{}{\{\Gamma\cup\{A\}\vdash\Delta\cup\{A\rightarrow B,B\}\}\vdash^{2}\{\Gamma\vdash\Delta\cup\{A\rightarrow B\}\}}\ ,$$

which shows that the first of the rules above is horizontalizable.

We note that analogous remarks could be made for the analytic

rules we shall consider later, and we shall not comment on this topic

any more.

§29   We shall now consider the analytic extensions of $\underline{hADICT}$ with

$(\rightarrow)$, $(\&)$, $(\vee)$, $(\perp)$ and (T) . In these extensions the analytic rules

will be in the scope of $\underline{h}$. $\underline{D}$ of all levels $\geqslant 2$ is eliminable from

these extensions according to Lemma 10.

First we present the following analytic rules.

$$(\rightarrow)_1 \ \frac{\Gamma \vdash^A \Delta \cup \{A\} \quad \Gamma \cup \{B\} \vdash \Delta}{\Gamma \cup \{A \rightarrow B\} \vdash \Delta} \ ; \quad (\&)_0 \ \frac{A \qquad B}{A \& B} \ ;$$

$$(\&)_{1.1} \ \frac{\Gamma \cup \{A, B\} \vdash \Delta}{\Gamma \cup \{A \& B\} \vdash \Delta} \ ; \quad (\&)_{1.2} \ \frac{\Gamma \cup \{A\} \vdash \Delta}{\Gamma \cup \{A \& A\} \vdash \Delta} \ ;$$

$$(\vee)_{1.1} \ \frac{\Gamma \vdash^A \Delta \cup \{A, B\}}{\Gamma \vdash \Delta \cup \{A \vee B\}} \ ; \quad (\vee)_{1.2} \ \frac{\Gamma \vdash^A \Delta \cup \{A\}}{\Gamma \vdash \Delta \cup \{A \vee A\}} \ ;$$

$$(\perp)_1 \ \frac{\Gamma \vdash^A \Delta}{\Gamma \vdash^A \Delta \cup \{\perp\}} \ ; \quad (T)_1 \ \frac{\Gamma \vdash^A \Delta}{\Gamma \cup \{T\} \vdash \Delta} \ .$$

Definition of replaceability

A set of postulates Π_1 of a system S is replaceable by
a set of rules or formulae Π_2 iff the subsystem of S without Π_1
extended with Π_2 gives a system which has the same provable sentences
as S _and_ the same deducible rules as S.

If in the definition above we delete the clause beginning
with "_and_", we get the definition of _weak replaceability_. If two
systems have the same provable sentences, they have the same admissible
rules, but not so for deducible (or derivable) rules.

When we say that an analytic rule (α) in the scope of _h_ is
replaceable by a set of postulates Π, this will mean that all the
postulates given by (α), and (α) and _h_, are replaceable by Π. If a
level-preserving rule is in Π, we shall presuppose that it is in the scope
of _h_ if not stated otherwise.

Also,when we speak of an extension of a system with $\underline{h}$ with some postulates Π, we shall presuppose that any level-preserving rule in Π is in the scope of $\underline{h}$ if not stated otherwise.

N.B.   These presuppositions are essential for the correct understanding of what follows.

Now we shall give a number of lemmata omitting their demonstrations, which are a matter of routine. We shall only give a demonstration of one part of the first of these lemmata as an example.

<u>Lemma 11</u>   11.1 $(\rightarrow)_1$ is horizontalizable in $\underline{hADICT}(\rightarrow)$.

11.2 $(\rightarrow)$ is horizontalizable in $\underline{hADICT}(\rightarrow)_1$.

<u>Demonstration of 11.1:</u>

$$(\rightarrow)\uparrow \quad \frac{\{A\rightarrow B\} \vdash \{A\rightarrow B\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\rightarrow B, A\} \vdash \{B\}}$$

$$\frac{A^2}{C^2} \quad \frac{\varnothing \vdash^2 \{\{A\rightarrow B, A\} \vdash \{B\}\}}{}$$

$$\frac{\{\Gamma \vdash \Delta \cup \{A\}, \{A\rightarrow B, A\} \vdash \{B\}\} \vdash^2 \{\Gamma \cup \{A\rightarrow B\} \vdash \Delta \cup \{B\}\} \quad \underline{h}^{\wedge}(\underline{C}^{\wedge})}{\underbrace{\{\Gamma \vdash \Delta \cup \{A\}\} \vdash^2 \{\Gamma \cup \{A\rightarrow B\} \vdash \Delta \cup \{B\}\}}_{①}}$$

$$\frac{C^2 \; ① \quad \{\Gamma \cup \{A\rightarrow B\} \vdash \Delta \cup \{B\}, \Gamma \cup \{B\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{A\rightarrow B\} \vdash \Delta\} \quad \underline{h}^{\wedge}(\underline{C}^{\wedge})}{\underline{\{\Gamma \vdash \Delta \cup \{A\}, \Gamma \cup \{B\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{A\rightarrow B\} \vdash \Delta\}}} \; ;$$

$$T^{\wedge} \; \frac{\{A\} \vdash \{A\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\} \vdash \{A, B\}} \quad (\rightarrow)\downarrow \; \frac{\{A\} \vdash \{A\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\varnothing \vdash \{A\rightarrow A\}} \quad T^{\wedge} \; \frac{\{B\} \vdash \{B\} \quad \underline{I}^{\circ}(\underline{I}^{\circ})}{\{A, B\} \vdash \{B\}} \quad (\rightarrow)\downarrow \; \frac{\{B\} \vdash \{B\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\varnothing \vdash \{B\rightarrow B\}}$$

$$(\rightarrow)\downarrow \; \frac{}{\underbrace{\varnothing \vdash \{A, A\rightarrow B\}}_{②}} \quad T^{\wedge} \; \frac{}{\underbrace{\varnothing \vdash \{A, A\rightarrow A\}}_{②.1}} \quad (\rightarrow)\downarrow \; \frac{}{\underbrace{\{B\} \vdash \{A\rightarrow B\}}_{③}} \quad T^{\wedge} \; \frac{}{\underbrace{\{B\} \vdash \{B\rightarrow B\}}_{③.1}}$$

$$\frac{A^2}{C^2} \; \underline{\textcircled{2} \; \text{or} \; \textcircled{2.1}}$$

$$\emptyset \vdash^2 \{ \emptyset \vdash \{A, A \to B\}\}$$

$$\underline{\{\emptyset \vdash \{A, A \to B\}, \Gamma \cup \{A \to B\} \vdash \Delta\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A\}\}} \quad \underline{h}^1(\underline{C}^1)$$

$$\{\Gamma \cup \{A \to B\} \vdash \Delta\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A\}\} \; ;$$

$$\frac{A^2}{C^2} \; \underline{\textcircled{3} \; \text{or} \; \textcircled{3.1}}$$

$$\emptyset \vdash^2 \{ \{B\} \vdash \{A \to B\}\}$$

$$\underline{\{\{B\} \vdash \{A \to B\}, \Gamma \cup \{A \to B\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{B\} \vdash \Delta\}} \quad \underline{h}^1(\underline{C}^1)$$

$$\{\Gamma \cup \{A \to B\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{B\} \vdash \Delta\} \; .$$

$$\text{Q.E.D.}$$

It follows that $(\to)$ is replaceable by $(\to)_1$ in the context of
hADICT.   It can also easily be shown that in this context $(\to)$ is
replaceable by the formulae

$$\emptyset \vdash^1 \{A \to B, A\} \; ; \; \{B\} \vdash^1 \{A \to B\} \; ; \; \{A \to B, A\} \vdash^1 \{B\} \quad ,$$

where substitution for "A" and "B" is free.

Lemma 12.   12.01 $(\&)_0$ is horizontalizable in hADICT$(\&)$.

   12.02 $(\&)$ is horizontalizable in hADICT$(\&)_0$.

   12.1 $(\&)_{1.1}$ and $(\&)_{1.2}$ are horizontalizable in hADICT$(\&)$.

   12.2 $(\&)$ is horizontalizable in hADICT$(\&)_{1.1}(\&)_{1.2}$.

So, (&) is replaceable by $(\&)_0$, or by $(\&)_{1.1}$ and $(\&)_{1.2}$, in the context of hADICT. From 12.01 and 12.02 it follows that in this context (&) is replaceable by the formulae

$\{A,B\} \vdash^1 \{A\&B\}$; $\{A\} \vdash^1 \{A\&A\}$; $\{A\&B\} \vdash^1 \{A\}$; $\{A\&B\} \vdash^1 \{B\}$,

where substitution for "A" and "B" is free, except in the first formula [cf. Lemma 20.11, 20.12].

In a language like D3 an analytic rule corresponding to $(\&)_{1.1}$ would be sufficient to replace (&), $(\&)_{1.2}$ being superfluous ($\{A\} \vdash^1 \{A\&A\}$ would also be superfluous).

Lemma 13   13.1 $(v)_{1.1}$ and $(v)_{1.2}$ are horizontalizable in hADICT(v).

13.2 (v) is horizontalizable in $\underline{hADICT(v)}_{1.1}(v)_{1.2}$.

So, (v) is replaceable by $(v)_{1.1}$ and $(v)_{1.2}$ in the context of hADICT. It can also easily be shown that in this context (v) is replaceable by the formulae

$\{A\} \vdash^1 \{AvB\}$; $\{B\} \vdash^1 \{AvB\}$; $\{AvB\} \vdash^1 \{A,B\}$; $\{AvA\} \vdash^1 \{A\}$,

where substitution for "A" and "B" is free, except in the third formula.

In a language like D3 an analytic rule corresponding to $(v)_{1.1}$ would be sufficient to replace (v), $(v)_{1.2}$ being superfluous ($\{AvA\} \vdash^1 \{A\}$ would also be superfluous).

Lemma 14   14.1 $(\perp)_1$ is horizontalizable in $\underline{hADICT}(\perp)$.

   14.2 $(\perp)$ is horizontalizable in $\underline{hADICT}(\perp)_1$.

So, $(\perp)$ is replaceable by $(\perp)_1$ in the context of $\underline{hADICT}$. It can also easily be shown that in this context $(\perp)$ is replaceable by the formula

$$\{\perp\} \vdash^1 \emptyset \ .$$

Lemma 15   15.1 $(T)_1$ is horizontalizable in $\underline{hADICT}(T)$.

   15.2 (T) is horizontalizable in $\underline{hADICT}(T)_1$.

So, (T) is replaceable by $(T)_1$ in the context of $\underline{hADICT}$. It can also easily be shown that in this context (T) is replaceable by the formula

$$\emptyset \vdash^1 \{T\} \qquad \text{(or the formula T)} \ .$$

Although $(\rightarrow)$, $(\&)$, (v), $(\perp)$ and (T) are replaceable in the context of $\underline{hADICT}$ by the above analytic rules or formulae, this will not be the case for all these analytic rules in all the analytic extensions of structural systems we shall consider *[v. Chapter 5]*.

§30   Consider the following analytic rules

$$(\neg)_1 \ \frac{\Gamma \cup \{A\} \vdash^1 \Delta}{\Gamma \vdash^1 \Delta \cup \{\neg A\}} \quad ; \quad (\neg)_2 \ \frac{\Gamma \vdash^1 \Delta \cup \{A\}}{\Gamma \cup \{\neg A\} \vdash^1 \Delta} \ .$$

We state the following, omitting the demonstrations,

Lemma 16   If $A =_{df} A \rightarrow \bot$, $(\neg)_1$ and $(\neg)_2$ are horizontalizable in hADICT($\rightarrow$)($\bot$).

Lemma 17   17.1 $(\neg)_2$ is horizontalizable in hADICT($\neg$)$_1$.

17.2 $(\neg)_1$ is horizontalizable in hADICT($\neg$)$_2$.

So, $(\neg)_1$ and $(\neg)_2$ are mutually replaceable in the context of hADICT. It can also easily be shown that in this context $(\neg)_1$ or $(\neg)_2$ are replaceable by the formulae

$$\emptyset \vdash^1 \{A, \neg A\} \ ; \quad \{A, \neg A\} \vdash^1 \emptyset.$$

We shall also consider analytic extensions with $(\neg)_1$. [For other remarks on analytic rules for classical $\neg$ v. §37].

§31   Consider the following rule

$$(\underline{mp}) \quad \frac{A \quad A \rightarrow B}{B} \ ;$$

and the following formulae

$\underline{c_1}$ $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;

$\underline{c_2}$ $A \rightarrow (B \rightarrow A)$;

$\underline{c_3}$ $((A \rightarrow B) \rightarrow A) \rightarrow A$;

$\underline{k_1}$ $(A\&B) \rightarrow A$;

$\underline{k_2}$ $(A\&B) \rightarrow B$;

$\underline{k_3}$ $(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A\&B)))$;

$\underline{a_1}$ $A \rightarrow (A \lor B)$;

$\underline{a_2}$ $B \rightarrow (A \lor B)$;

$\underline{a_3}$ $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \lor B) \rightarrow C))$;

$\underline{\emptyset}$ $\bot \rightarrow A$;

$\underline{v}$ T ;

*

$\underline{n_1}$ $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$;

$\underline{n_2}$ $A \rightarrow \neg\neg A$;

$\underline{n_3}$ $\neg\neg A \rightarrow (\neg A \rightarrow A)$

(substitution is everywhere free).

As an abbreviation of e.g. "$\underline{c_1 c_2 c_3}$" we shall use "$\underline{c_{1-3}}$".

The system

$$(\text{mp}) \quad \underline{c_{1-3} k_{1-3} a_{1-3} \delta v}$$

will be called "CP/0" ("C" stands for "classical", "P" for

"propositional", and "0" for "level 0").

The system

$$(\text{mp}) \quad \underline{c_{1-2} \; k_{1-3} \; a_{1-3} \; \delta v}$$

will be called "HP/0" ("H" stands for "Heyting").

The system

$$(\text{mp}) \quad \underline{c_{1-2} \; k_{1-3} \; a_{1-3} \; v}$$

will be called "KP/0" ("K" stands for "Kolmogorov-Johansson").

In CP/0, HP/0 and KP/0, $\underline{v}$ is replaceable by $A \rightarrow T$.

Systems in 0, like CP/0, HP/0 and KP/0, will be called

"$\underline{\text{axiomatizations}}$ in 0" or "axiomatizations", $\underline{\text{tout court}}$.

If the language 0 is the language of the propositional

calculus, CP/0 will be an axiomatization of the classical propositional

calculus. Since, however, we have left O undetermined, CP/O

for us should be a classical propositional system and not the

classical propositional calculus. CP/O, HP/O and KP/O are

separative. That these axiomatizations are separative means that

any provable formula in which occur certain constants is provable with

the use of only the postulates for $\rightarrow$ and these constants. The

extensions of CP/O and HP/O with n_{1-3} and the extension of KP/O with

n_{1-2} preserve separativeness. When $\neg$ is explicitly defined, n_{1-3}

are provable in CP/O and HP/O, and n_{1-2} in KP/O. The subsystems

of these extensions without $\oint$ have the same provable formulae in

which $\perp$ occurs only in $\neg$, as CP/O, HP/O and KP/O *[v.§34 for*

references on these axiomatizations and extensions].

A system in O is, of course, also a system in D built on O.

§32   Definition of n-equivalence

The systems S_1 and S_2 in D are n-equivalent, for a given

$n \geqslant 0$, iff for every sentence A^n of level n, A^n is provable in

S_1 iff A^n is provable in S_2.

Our main purpose in this section is to show the O-equivalence

of CP/O and the system

hADICT ($\rightarrow$) (&) ($\vee$) ($\perp$) (T)

called "CP/D". This will be done in stages in order to show the

separativeness of CP/D, and its extension with $(\neg)_1$.

<u>Lemma 18</u>   18.1 (<u>mp</u>) is horizontalizable

and $\underline{c}_{1-3}$ are provable in <u>hADICT</u>($\rightarrow$).

$\quad$ 18.2 $\underline{k}_{1-3}$ are provable in <u>hADICT</u>($\rightarrow$)(&) .

$\quad$ 18.3 $\underline{a}_{1-3}$ are provable in <u>hADICT</u>($\rightarrow$)(v) .

$\quad$ 18.4 $\underline{f}$ $\quad$ is provable in <u>hADICT</u>($\rightarrow$)($\perp$) .

$\quad$ 18.5 $\underline{v}$ $\quad$ is provable in <u>hADICT</u>(T) .

$\quad$ 18.6 $\underline{n}_{1-3}$ are provable in <u>hADICT</u>($\rightarrow$)($\neg$)$_1$.

<u>Demonstration</u>:

18.1 $\quad (\rightarrow)\uparrow \dfrac{\{A \rightarrow B\} \vdash^\wedge \{A \rightarrow B\} \quad \underline{h}^0(\underline{I}^\circ)}{\{A \rightarrow B, A\} \vdash^\wedge \{B\}}$;

$(\rightarrow)\uparrow \underline{c}^\wedge \dfrac{\dfrac{\{A \rightarrow (B \rightarrow C)\} \vdash^\wedge \{A \rightarrow (B \rightarrow C)\} \quad \underline{h}^0(\underline{I}^\circ)}{\{A \rightarrow (B \rightarrow C), A\} \vdash^\wedge \{B \rightarrow C\}} \quad (\rightarrow)\uparrow \dfrac{\{B \rightarrow C\} \vdash^\wedge \{B \rightarrow C\} \quad \underline{h}^0(\underline{I}^\circ)}{\{B \rightarrow C, B\} \vdash^\wedge \{C\}}}{\underbrace{\{A \rightarrow (B \rightarrow C), A, B\} \vdash^\wedge \{C\}}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ①$

if $A = B$, instead of ① we get ①.① : $\{A \rightarrow (B \rightarrow C), B\} \vdash^\wedge \{C\}$

$(\rightarrow)\uparrow \underline{c}^\wedge \dfrac{\dfrac{\{A \rightarrow B\} \vdash^\wedge \{A \rightarrow B\} \quad \underline{h}^0(\underline{I}^\circ)}{\{A \rightarrow B, A\} \vdash^\wedge \{B\}} \qquad ① \text{ or } ①.①}{\dfrac{\{A \rightarrow (B \rightarrow C), A \rightarrow B, A\} \vdash^\wedge \{C\}}{3 \text{ applications of } (\rightarrow)\downarrow, \text{ and } \underline{D}^\wedge}}$

$\qquad\qquad\qquad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;

$\underline{T}^\wedge \dfrac{\{A\} \vdash^\wedge \{A\} \quad \underline{h}^0(\underline{I}^\circ)}{\{A, B\} \vdash^\wedge \{A\}}$ $\qquad\qquad$ $(\rightarrow)\downarrow \dfrac{\{A\} \vdash^\wedge \{A\} \quad \underline{h}^0(\underline{I}^\circ)}{\varnothing \vdash^\wedge \{A \rightarrow A\}}$

$\qquad$ 2 applications of $(\rightarrow)\downarrow$, and $\underline{D}^\wedge$ $\qquad\qquad$ $\underline{T}^\wedge \dfrac{}{\{A\} \vdash^\wedge \{A \rightarrow A\}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\rightarrow)\downarrow$ and $\underline{D}^\wedge$

$\qquad A \rightarrow (B \rightarrow A)$ $\qquad$ or $\qquad\qquad$ $A \rightarrow (A \rightarrow A)$;

$$T' \frac{\{A\}\vdash\{A\} \quad \underline{h}°(\underline{I}°)}{(→)↓\frac{\{A\}\vdash\{A,B\}}{\varnothing\vdash\{A,A→B\}}}$$

$$\underbrace{\qquad}_{②}$$

$$(→)↓\frac{\{A\}\vdash\{A\} \quad \underline{h}°(\underline{I}°)}{\varnothing\vdash\{A→A\}}$$
$$T' \frac{}{\varnothing\vdash\{A,A→A\}}$$

$$\underbrace{\qquad}_{②.1}$$

$$\underline{C}' \frac{② \text{ or } ②.1}{} \qquad (→)↑\frac{\{(A→B)→A\}\vdash\{(A→B)→A\} \quad \underline{h}°(\underline{I}°)}{\{(A→B)→A, A→B\}\vdash\{A\}}$$
$$\frac{\{(A→B)→A\}\vdash\{A\}}{(→)↓ \text{ and } \underline{D}'}$$
$$((A→B)→A)→A .$$

$$18.2 \quad (\&)↑\frac{\{A\&B\}\vdash\{A\&B\} \quad \underline{h}°(\underline{I}°)}{\{A\&B\}\vdash\{A\}}$$
$$\frac{(→)↓ \text{ and } \underline{D}'}{(A\&B)→A} \qquad ; \text{ and homologously for } \underline{k_2};$$

$$(→)↑\frac{\{C→A\}\vdash\{C→A\} \quad \underline{h}°(\underline{I}°)}{\{C→A,C\}\vdash\{A\}} \qquad (→)↑\frac{\{C→B\}\vdash\{C→B\} \quad \underline{h}°(\underline{I}°)}{\{C→B,C\}\vdash\{B\}}$$

$$\underbrace{\qquad}_{①} \qquad\qquad\qquad \underbrace{\qquad}_{②}$$

if $A ≠ B$:

$$T' \frac{①}{(\&)↓\; \{C→A,C→B,C\}\vdash\{A\}} \qquad T' \frac{②}{\{C→A,C→B,C\}\vdash\{B\}}$$
$$\frac{\{C→A,C→B,C\}\vdash\{A\&B\}}{3 \text{ applications of } (→)↓, \text{ and } \underline{D}'}$$
$$(C→A)→((C→B)→(C→(A\&B))) ,$$

if $A = B$:

$$(\&)\downarrow \;\frac{\textcircled{1}\qquad\qquad\textcircled{2}}{\{C\to B, C\}\vdash\{A\&B\}}$$

$$\frac{}{2\text{ applications of }(\to)\downarrow}$$

$$T^{1}\;\frac{\varnothing\vdash^{1}\{(C\to B)\to(C\to(A\&B))\}}{\{C\to A\}\vdash^{1}\{(C\to B)\to(C\to(A\&B))\}}$$

$$\frac{}{(\to)\downarrow\;\text{ and }\;\underline{D}^{1}}$$

$$(C\to A)\to((C\to B)\to(C\to(A\&B))).$$

$$18.3\quad (\vee)\uparrow\;\frac{\{A\vee B\}\vdash^{1}\{A\vee B\}\;\underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\}\vdash^{1}\{A\vee B\}}$$

$$\frac{}{(\to)\downarrow\;\text{ and }\;\underline{D}^{1}}$$

$$A\to(A\vee B)\text{; and homologously for }\underline{\alpha}_{2};$$

$$(\to)\uparrow\;\frac{\{A\to C\}\vdash^{1}\{A\to C\}\;\underline{h}^{\circ}(\underline{I}^{\circ})}{\underbrace{\{A\to C, A\}\vdash^{1}\{C\}}_{\textcircled{1}}}\qquad (\to)\uparrow\;\frac{\{B\to C\}\vdash^{1}\{B\to C\}\;\underline{h}^{\circ}(\underline{I}^{\circ})}{\underbrace{\{B\to C, B\}\vdash^{1}\{C\}}_{\textcircled{2}}}$$

if $A\neq B$:

$$\begin{array}{c}T^{1}\\(\vee)\downarrow\end{array}\;\frac{\textcircled{1}}{\{A\to C, B\to C, A\}\vdash^{1}\{C\}}\qquad T^{1}\;\frac{\textcircled{2}}{\{A\to C, B\to C, B\}\vdash^{1}\{C\}}$$

$$\frac{}{\{A\to C, B\to C, A\vee B\}\vdash^{1}\{C\}}$$

$$\frac{}{3\text{ applications of }(\to)\downarrow,\text{ and }\underline{D}^{1}}$$

$$(A\to C)\to((B\to C)\to((A\vee B)\to C)),$$

if $A=B$:

$$(\vee)\downarrow\;\frac{\textcircled{1}\qquad\qquad\textcircled{2}}{\{B\to C, A\vee B\}\vdash^{1}\{C\}}$$

$$\frac{}{2\text{ applications of }(\to)\downarrow}$$

$$T^{1}\;\frac{\varnothing\vdash^{1}\{(B\to C)\to((A\vee B)\to C)\}}{\{A\to C\}\vdash^{1}\{(B\to C)\to((A\vee B)\to C)\}}$$

$$\frac{}{(\to)\downarrow\;\text{ and }\;\underline{D}^{1}}$$

$$(A\to C)\to((B\to C)\to((A\vee B)\to C)).$$

$$18.4 \quad (\bot)\!\uparrow \frac{\{\bot\} \vdash \{\bot\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underline{T}\uparrow \dfrac{\{\bot\} \vdash \varnothing}{\dfrac{\{\bot\} \vdash \{A\}}{\dfrac{(\to)\downarrow \text{ and } \underline{D}\uparrow}{\bot \to A \, .}}}}$$

$$18.5 \quad (T)\!\uparrow \frac{\{T\} \vdash \{T\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underline{D}\uparrow \dfrac{\varnothing \vdash \{T\}}{T \, .}}$$

$$18.6 \quad (\to) \frac{\{A \to B\} \vdash \{A \to B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underbrace{\{A \to B, A\} \vdash \{B\}}_{①}}$$

$$(\neg)\!\uparrow \frac{\{\neg B\} \vdash \{\neg B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underbrace{\{B, \neg B\} \vdash \varnothing}_{②}}$$

if $A \neq \neg B$:

$$\underline{C}\uparrow \frac{① \qquad\qquad ②}{(\neg)\!\downarrow \dfrac{\{A \to B, A, \neg B\} \vdash \varnothing}{\dfrac{\{A \to B, \neg B\} \vdash \{\neg A\}}{\dfrac{2 \text{ applications of } (\to)\downarrow, \text{ and } \underline{D}\uparrow}{(A \to B) \to (\neg B \to \neg A),}}}}$$

if $A = \neg B$:

$$\underline{C}\uparrow \frac{① \qquad\qquad (\neg)_{\uparrow}\!\downarrow \dfrac{②}{\{B\} \vdash \{\neg A\}}}{\dfrac{\{A \to B, \neg B\} \vdash \{\neg A\}}{\dfrac{2 \text{ applications of } (\to)\downarrow, \text{ and } \underline{D}\uparrow}{(A \to B) \to (\neg B \to \neg A);}}}$$

$$(\neg)_{\uparrow}\!\uparrow \frac{\{\neg A\} \vdash \{\neg A\} \quad \underline{h}^\circ(\underline{I}^\circ)}{(\neg)_{\uparrow}\!\downarrow \dfrac{\{\neg A, A\} \vdash \varnothing}{\dfrac{\{A\} \vdash \{\neg\neg A\}}{\dfrac{(\to)\downarrow \text{ and } \underline{D}\uparrow}{A \to \neg\neg A \, ;}}}}$$

$$(\neg)_{\uparrow}\!\uparrow \frac{\{\neg\neg A\} \vdash \{\neg\neg A\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underline{T}\uparrow \dfrac{\{\neg\neg A, \neg A\} \vdash \varnothing}{\dfrac{\{\neg\neg A, \neg A\} \vdash \{A\}}{\dfrac{2 \text{ applications of } (\to)\downarrow, \text{ and } \underline{D}\uparrow}{\neg\neg A \to (\neg A \to A) \, .}}}}$$

Q.E.D.

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Lemma 19   If a sentence of level 0 is provable in CP/D, it is provable in CP/O.

Demonstration :   Consider the following o_1-translation:

$$o_1(A^O) = A^O;$$

$\widehat{o}_1(\Gamma)$, $\widecheck{o}_1(\Gamma)$ and $o_1(\Gamma \vdash^{n+1} \Delta)$ are obtained by

substituting everywhere "o_1" for "o" in the clauses for $\widehat{o}(\Gamma)$, $\widecheck{o}(\Gamma)$ and $o(\Gamma \vdash^{n+1} \Delta)$ [v.§15].

Suppose now that a sentence $A^n, n \geqslant 0$, is provable in CP/D. We can show by an induction on the length of the proof of A^n that $o_1(A^n)$ is provable in CP/O.

For the basis we have:

1) $\underline{h}^{n-1}(\underline{I}^{n-1})$; then we use A → A;

2) $\underline{h}^{n-1}(\underline{C}^{n-1})$; then we use

((A → (B∨E))&((C&E) → D)) → ((A&C) → (B∨D)),

((A → (B∨E))&(E → D)) → ((A → (B∨D)),

((A → E)&((C&E) → D)) → ((A&C) → D),

((A → E)&(E → D)) → (A → D),

if A is T, (A&C) ↔ C, and if D is ⊥, (B∨D) ↔ B;

3) $\underline{h}^{n-1}(\underline{T}^{n-1})$; then we use

(A → B) → ((A&C) → (B∨D)),

(A → B) → ((A&C) → B),

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$(A \rightarrow B) \rightarrow (A \rightarrow (B \vee D))$,

if A is T, $(A \& C) \leftrightarrow C$, and if B is $\bot$, $(B \vee D) \leftrightarrow D$;

4) $\underline{h}^1((\rightarrow)\downarrow)$ and $\underline{h}^1((\rightarrow)\uparrow)$; then we use

   $((C \& A) \rightarrow (B \vee D)) \leftrightarrow (C \rightarrow (D \vee (A \rightarrow B)))$,

   $((C \& A) \rightarrow B) \leftrightarrow (C \rightarrow (A \rightarrow B))$,

   $(A \rightarrow (B \vee D)) \leftrightarrow (T \rightarrow (D \vee (A \rightarrow B)))$,

   $(A \rightarrow B) \leftrightarrow (T \rightarrow (A \rightarrow B))$;

5) $\underline{h}^1((\&)\downarrow)$ and $\underline{h}^1((\&)\uparrow)$; then we use

   $((C \rightarrow (D \vee A)) \& (C \rightarrow (D \vee B))) \leftrightarrow (C \rightarrow (D \vee (A \& B)))$,

   $((C \rightarrow A) \& (C \rightarrow B)) \leftrightarrow (C \rightarrow (A \& B))$;

6) $\underline{h}^1((\vee)\downarrow)$ and $\underline{h}^1((\vee)\uparrow)$; then we use

   $(((C \& A) \rightarrow D) \& ((C \& B) \rightarrow D)) \leftrightarrow ((C \& (A \vee B)) \rightarrow D)$,

   $((A \rightarrow D) \& (B \rightarrow D)) \leftrightarrow ((A \vee B) \rightarrow D)$;

7) $\underline{h}^1((\bot)\downarrow)$ and $\underline{h}^1((\bot)\uparrow)$; then we use

   $A \rightarrow A$;

8) $\underline{h}^1((T)\downarrow)$ and $\underline{h}^1((T)\uparrow)$; then we use

   $A \rightarrow A$.

For the induction step suppose that if A^n is proved in $\leqslant k$ steps, $o_1(A^n)$ is provable in CP/0. Let it be proved in $k + 1$ steps. The last step can be:

   1) $\underline{A}^n$; then we use $A \rightarrow (T \rightarrow A)$ and (m̲p̲);
   2) $\underline{D}^{n+1}$; then we use $(T \rightarrow A) \rightarrow A$ and (m̲p̲);

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for the other cases we use the formulae of the basis, k_{1-3} and (mp). This concludes the induction.

If $n = 0$, $\mathcal{O}_1(A^n) = A^n$, and the Lemma follows.

Q.E.D.

From Lemmata 18 and 19 we get immediately

Theorem 1   CP/D and CP/O are O-equivalent.

§33   We shall now consider whether the analytic rules of CP/D are replaceable by postulates of CP/O. We shall consider in principle the possibility of replacing all these analytic rules by postulates in O, and not these analytic rules taken one by one. We can show

Lemma 20   20.11 $(\&)\downarrow$ is horizontalizable in $hAC(mp)c_{1-2}k_3$.

20.12 $(\&)\uparrow$ are horizontalizable in $hAC(mp)k_{1-2}$.

20.21 $(T)\downarrow$ is horizontalizable in $hT_{e h}$.

20.22 $(T)\uparrow$ is horizontalizable in $hAC\nu$.

20.3 $(\rightarrow)\uparrow$ is horizontalizable in $hAC(mp)$.

20.4 $(v)\uparrow$ are horizontalizable in $hAC(mp)a_{1-2}$.

20.5 $(\bot)\downarrow$ is horizontalizable in hT_{el} or hT_{sh} .

Demonstration:

20.11 $\underset{C^1}{\overset{A^1}{\rule{0pt}{0pt}}}$ $\cfrac{A \rightarrow (B \rightarrow (A\&B))\ \text{provable in } (mp)c_{1-2}k_3 \qquad h^0((mp))}{\cfrac{\varnothing \vdash \{A\rightarrow(B\rightarrow(A\&B))\}\quad \{A\rightarrow(B\rightarrow(A\&B)),A\}\vdash\{B\rightarrow(A\&B)\}}{\cfrac{\{A\}\vdash\{B\rightarrow(A\&B)\}\quad \{B\rightarrow(A\&B),B\}\vdash\{A\&B\}\ h^0((mp))}{\underbrace{\{A,B\}\vdash\{A\&B\}}_{\textcircled{1}}}}}$

C^1

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if $A=B$, instead of ①, we get ⓵⒈⓵ : $\{A\} \vdash \{A \& A\}$

$$\frac{A^2 \quad ①}{C^2 \quad \emptyset \vdash^2 \{①\}} \qquad \{\Gamma \vdash \Delta \cup \{A\}, ①\} \vdash^2 \{\Gamma \cup \{B\} \vdash \Delta \cup \{A \& B\}\} \quad \underline{h}^1(\underline{C}^1)$$

$$\frac{}{C^2 \quad \{\Gamma \vdash \Delta \cup \{A\}\} \vdash^2 \{\Gamma \cup \{B\} \vdash \Delta \cup \{A \& B\}\}} \qquad \underline{h}^1(\underline{C}^1)$$

$$\frac{\{\Gamma \vdash \Delta \cup \{B\}, \Gamma \cup \{B\} \vdash \Delta \cup \{A \& B\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A \& B\}\}}{\{\Gamma \vdash \Delta \cup \{A\}, \Gamma \vdash \Delta \cup \{B\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A \& B\}\}}, \text{ or}$$

$$\frac{A^2 \quad ⓵⒈⓵}{C^2 \quad \emptyset \vdash^2 \{⓵⒈⓵\}} \qquad \{\Gamma \vdash \Delta \cup \{A\}, ⓵⒈⓵\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A \& A\}\} \quad \underline{h}^1(\underline{C}^1)$$

$$\frac{}{\{\Gamma \vdash \Delta \cup \{A\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A \& A\}\}}.$$

20.12 $\frac{A \quad (A \& B) \to A \quad \underline{k}_1}{C^1 \quad \emptyset \vdash \{(A \& B) \to A\}} \qquad \{(A \& B) \to A, A \& B\} \vdash \{A\} \quad \underline{h}^0(\!(\text{mp})\!)$

$$\frac{A^2 \quad \{A \& B\} \vdash \{A\}}{C^2 \quad \emptyset \vdash^2 \{\{A \& B\} \vdash \{A\}\}}$$

$$\frac{\{\Gamma \vdash \Delta \cup \{A \& B\}, \{A \& B\} \vdash \{A\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A\}\} \quad \underline{h}^1(\underline{C}^1)}{\{\Gamma \vdash \Delta \cup \{A \& B\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{A\}\}} ;$$

and homologously with $\underline{k}_2$.

20.21 $\{\emptyset \vdash \Delta\} \vdash^2 \{\{\top\} \vdash \Delta\} \quad \underline{h}^1(\top_{er}^1)$.

20.22 $\dfrac{\top}{A^1} \text{ and } \dfrac{v}{A^2}$

$$\frac{C^2 \quad \emptyset \vdash^2 \{\emptyset \vdash \{\top\}\} \qquad \{\emptyset \vdash \{\top\}, \{\top\} \vdash \Delta\} \vdash^2 \{\emptyset \vdash \Delta\} \quad \underline{h}^1(\underline{C}^1)}{\{\{\top\} \vdash \Delta\} \vdash^2 \{\emptyset \vdash \Delta\}}.$$

20.3

$$\frac{A^2}{C^2} \quad \frac{\{A \to B, A\} \vdash \{B\} \quad \underline{h}^0((m\varphi))}{\varnothing \vdash^2 \{\{A \to B, A\} \vdash \{B\}\}} \qquad\qquad \underline{h}^1(\underline{C}^1)$$

$$\frac{\{\Gamma \vdash \Delta \cup \{A \to B\}, \{A \to B, A\} \vdash \{B\}\} \vdash \{\Gamma \cup \{A\} \vdash \Delta \cup \{B\}\}}{\{\Gamma \vdash \Delta \cup \{A \to B\}\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta \cup \{B\}\}}.$$

20.4

$$\frac{A^1}{C^2} \quad \frac{A \to (A \vee B) \quad \underline{a}_1}{\varnothing \vdash^1 \{A \to (A \vee B)\}} \qquad \{A \to (A \vee B), A\} \vdash \{A \vee B\} \quad \underline{h}^0((m\varphi))$$

$$\frac{A^2}{C^2} \quad \frac{\{A\} \vdash^1 \{A \vee B\}}{\varnothing \vdash^2 \{\{A\} \vdash \{A \vee B\}\}} \qquad\qquad \underline{h}^1(\underline{C}^1)$$

$$\frac{\{\{A\} \vdash \{A \vee B\}, \Gamma \cup \{A \vee B\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta\}}{\{\Gamma \cup \{A \vee B\} \vdash \Delta\} \vdash \{\Gamma \cup \{A\} \vdash \Delta\}};$$

and homologously with $\underline{a}_2$.

20.5 $\quad \{\Gamma \vdash \varnothing\} \vdash^2 \{\Gamma \vdash \{\bot\}\}$ $\quad \underline{h}^1(T_{e\ell}^1)$ or $\underline{h}^1(T_{\nwarrow r}^1)$.

Q.E.D.

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Lemma 21   Let S be the extension of <u>hADICT</u> with the postulates

of CP/O.

21.1 $(\rightarrow)\!\downarrow$ is not admissible in S.

21.2 $(\vee)\!\downarrow$ is not admissible in S.

21.3 $(\perp)\!\uparrow$ is not admissible in S.

21.41 $(\neg)_1\!\downarrow$ is not admissible in S.

21.42 $(\neg)_1\!\uparrow$ is not admissible in S.

<u>Demonstration</u>:   We have

$$21.1 \quad \underline{T}\!\uparrow \frac{\quad}{(\rightarrow)\!\downarrow\dfrac{\{A\}\vdash\{A\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\dfrac{\{A\}\vdash\{A,B\}}{\varnothing\vdash\{A,A\rightarrow B\}}}} \; ;$$

$$21.2 \quad \underline{T}\!\uparrow\; (\vee)\!\downarrow \dfrac{\dfrac{\{A\}\vdash\{A\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\}\vdash\{A,B\}} \qquad \underline{T}\!\uparrow\dfrac{\{B\}\vdash\{B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{B\}\vdash\{A,B\}}}{\{A\vee B\}\vdash\{A,B\}} \; ;$$

$$21.3 \quad (\perp)\!\uparrow\dfrac{\{\perp\}\vdash\{\perp\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\perp\}\vdash\varnothing} \; ;$$

$$21.41 \quad (\neg)_1\!\downarrow\dfrac{\{A\}\vdash\{A\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\varnothing\vdash\{A,\neg A\}} \; ;$$

$$21.42 \quad (\neg)_1\!\uparrow\dfrac{\{\neg A\}\vdash\{\neg A\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A,\neg A\}\vdash\varnothing} \; .$$

Then we can easily show that if $\Gamma \vdash^1 \Delta$ is provable in S,

it is admissible in S and $\Delta \neq \emptyset$ [cf. Lemma 10] . But the

formulae proved for 21.1, 21.2 and 21.41 are not admissible, and

in those for 21.3 and 21.42, $\Delta = \emptyset$. Hence these formulae are not

provable in S, though their premises are.

<div align="right">Q.E.D.</div>

From Lemmata 20 and 21 we can conclude that only (&) and (T)

are replaceable by postulates of CP/O, whereas (→), (v) and

(⊥) are not even weakly replaceable by these postulates. Moreover,

this will still hold if we restrict ourselves only to sentences of

levels 0 and 1.

But we can show that a certain restricted form of some of these

analytic rules is at least weakly replaceable by postulates of

CP/O if we restrict ourselves to sentences of levels 0 and 1.

First we show

Lemma 22   If $\Gamma \vdash^1 \{A\}$ is provable CP/D, it is provable in the

extension of $\underline{h^0 A^1 c^1}$ with the postulates of CP/O.

Demonstration: Suppose $\Gamma \vdash^1 \{A\}$ is provable in CP/D. If

$\Gamma = \{B_1, \ldots, B_k\}$, $k \geqslant 1$, we have

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$$\frac{\Gamma \vdash^1 \{A\}}{k \text{ applications of } (\rightarrow)\!\!\downarrow}$$

$$\underline{D}^1 \quad \frac{\emptyset \vdash^1 \{B_1 \rightarrow (B_2 \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)\}}{B_1 \rightarrow (B_2 \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)} \quad .$$

But then by Lemma 19 this formula is provable in CP/0, and hence also in the extension S of the Lemma. Then we have

$$\underline{A}^1 \quad \frac{B_1 \rightarrow (B_2 \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)}{\emptyset \vdash^1 \{B_1 \rightarrow (B_2 \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)\}} \quad ,$$

and we also have for every i, $1 \leqslant i \leqslant k$,

$$\{B_i, B_i \rightarrow (B_{i+1} \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)\} \vdash \{B_{i+1} \rightarrow (B_{i+2} \rightarrow \ldots \rightarrow (B_k \rightarrow A) \ldots)\} \; \underline{h}^0(\underline{mp}).$$

By k applications of $\underline{C}^1$ we get that $\Gamma \vdash^1 \{A\}$ is provable in S. If $\Gamma = \emptyset$, we have that A is provable in S and we apply $\underline{A}^1$.

Q.E.D.

Then we have

Lemma 23   Let S be the extension of hADICT with the postulates of CP/0.

23.1 $(\rightarrow)\!\!\downarrow$ is admissible in S provided $\Delta = \emptyset$.

23.2 $(\vee)\!\!\downarrow$ is admissible in S provided $\Delta = \{C\}$, for some C.

23.3 $(\neg)_1\!\!\downarrow$ is admissible in S provided $\Delta = \emptyset$.

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<u>Demonstration</u>: 23.1 Suppose $\Gamma \cup \{A\} \vdash^1 \{B\}$ is provable in S .

Then by Lemma 18, it is provable in CP/D, and so $\Gamma \vdash^1 \{A \to B\}$

is provable in CP/D. Then we use Lemma 22 to show that $\Gamma \vdash^1 \{A \to B\}$

is provable in S.

23.2 Suppose $\Gamma \cup \{A\} \vdash^1 \{C\}$ and $\Gamma \cup \{B\} \vdash^1 \{C\}$ are provable in S .

Then by Lemma 18, they are provable in CP/D, and so $\Gamma \cup \{A \lor B\} \vdash^1 \{C\}$

is provable in CP/D. Then we use Lemma 22 to show that $\Gamma \cup \{A \lor B\} \vdash^1 \{C\}$

is provable in S.

23.3 If $\Delta = \emptyset$, $(\neg)_1 \downarrow$ is vacuously admissible since the premise

cannot be provable in S [cf. the demonstration of Lemma 21].

<div align="right">Q.E.D.</div>

The rule $(\to)\downarrow$ with the proviso $\Delta = \emptyset$ corresponds to the

Deduction Theorem of CP/0. Later we shall see that though this

rule with this proviso is admissible, it is not deducible, and,

<u>a fortiori</u>, not derivable or horizontalizable, in $\underline{hADICT^{(mp)}c_{1-3}}$ [v. §80]. If

$(\to)\downarrow$ were horizontalizable in S, $(\lor)\downarrow$ would be horizontalizable too.

We can conclude that the restricted analytic rules $(\to)$ and $(\lor)$

are weakly replaceable if we restrict ourselves to sentences of

levels 0 and 1; i.e. unrestricted $(\to)$ and $(\lor)$ are weakly replaceable

if we consider only sentences of level 0 and sequents of level 1 with

the right set a singleton.

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[§28]   That parentheses can be conceived as belonging to

U rather than O, connectives being infixes, is explained in

Smullyan 1968 (p.7).   If connectives are prefixes, parentheses

are anyway superfluous.   The antecendents of our canonical

set of analytic rules for propositional constants can be found in

the references of §25.   We shall make a brief remark on the

explicit definition of ⌐ .   The definition

$$\neg A \; =_{df} \quad A \leftrightarrow \bot$$

would give the same effect as the well known definition we have used,

in classical and Heyting   logic;   i.e. for systems where $\bot \rightarrow A$

is provable.   But it would not be acceptable for systems lacking

$\bot \rightarrow A$, like KP/O and subsystems of it.   For example, none of the

laws of contraposition would be provable in KP/O with this other

definition.

[§29]   Analytic rules like $(\&)_{1.1}$ and $(\vee)_{1.1}$ would seem more natural

than (&) and (∨) in a context like D3 (probably because they have a single

formula above the lines).   But we shall see that even in such a

context, these analytic rules could not be the canonical rules for

& and ∨.   In the absence of T these analytic rules would analyze

constants different from conjunction and disjunction   [v.§§45,49].

[§31]   Axiomatizations in O are also often called "Hilbert-

-style formulations of systems".

Except for $\mathcal{L}$ and $\mathcal{V}$, all the axioms for CP/O, HP/O and KP/O,

together with n_{1-3} for the first two systems, and n_{1-2} for the third,

are given in Kanger 1955, where separativeness is also demonstrated

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(cf. also Prior 1962, p.303). It is well known that b is

sufficient for negation in CP/O and HP/O when $\neg$ is explicitly

defined, and that δ and υ preserve separativeness (v. Curry 1963,

pp.283-288, 306-307, where enough information on these matters can

be found).

The propositional calculus corresponding to KP/O is more

frequently called "the minimal propositional calculus". We

think that this name is unfortunate, and we prefer to name it

by its authors (Kolmogorov 1925 gave its implication-negation

fragment, and Johansson 1936 the whole calculus).

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Chapter 5

NON-CLASSICAL PROPOSITIONAL CONSTANTS

§35   This chapter is divided into three parts. In the first part

we investigate analytic extensions of structural systems with

$\underline{T}_{sr}$ and $\underline{T}_{er}$, in the second analytic extensions of structural

systems with $\underline{T}_{sl}$ and $\underline{T}_{el}$, and in the third analytic extensions

of the structural system for which $\underline{T}$ is rejected completely. In

the first part we shall obtain systematizations of intuitionistic

propositional logics, in the second systematizations of propositional

logics dual to the intuitionistic ones, and in the third a

systematization of a relevant propositional logic related to the

logic of the system R. This last logic we propose to call

"intuitionistic relevant logic".

In all these parts we investigate the replaceability of analytic

rules by other analytic rules, the O-equivalence of systems in Dl

with axiomatizations in O, and in the first and third, the

replaceability of the set of analytic rules for propositional constants

by postulates in O, analogously to what we had in the previous chapter.

We also make some remarks on the intuitive interpretation of

the systems considered.

§36

INTUITIONISTIC PROPOSITIONAL CONSTANTS

§36   First, we shall consider the analytic extensions of $\underline{hADICT}_{\delta\hbar}$

and $\underline{hADICT}_{e\hbar}$ with $(\rightarrow)$, $(\&)$, (v), $(\bot)$ and (T).

The system

$$\underline{hADICT}_{\delta\hbar}\,(\rightarrow)\,(\&)\,(v)\,(\bot)\,(T)$$

will be called "HP/D", and the system

$$\underline{hADICT}_{e\hbar}\,(\rightarrow)\,(\&)\,(v)\,(\bot)\,(T)$$

will be called "KP/D".

Concerning these systems we immediately note the following

<u>Lemma 24</u>   24.1   If $\Gamma\vdash^{n+1}\Delta$, $n \geqslant 0$, is provable in HP/D, Δ is

either a singleton or empty.

24.2   A sentence is provable in HP/D without $(\bot)$ iff it is

provable in KP/D without $(\bot)$.

The demonstration of this lemma is made by inspection of the

postulates given above.   For 24.2 we show that if $\Gamma\vdash^{n+1}\Delta$ is

provable in the subsystems mentioned, Δ is never empty (hence it

must be a singleton).

The property which HP/D, and hence also its subsystems, have

by 24.1 will be called "the <u>single-conclusion property</u>".

Next we shall consider the replaceability of $(\rightarrow)$, $(\&)$, (v), $(\bot)$

and (T) in the context with $\underline{T}_{\delta\hbar}$ and $\underline{T}_{e\hbar}$.

It can easily be shown that Lemmata 12 and 15 are still

demonstrable when instead of $\underline{T}$ we have $\underline{T}_{\delta\hbar}$ or $\underline{T}_{e\hbar}$. Also the remarks

after these lemmata still hold.   That $(\rightarrow)_1, (v)_{1.1}$ and $(\bot)_1$ cannot be

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·obtained in this context follows from the fact that they would

not preserve the single-conclusion property.   Indeed we can show

Lemma 25 $\underline{T}^1$ is horizontalizable in the system $\underline{hADICT}_{er}(\to)_1(\bot)$.

Demonstration: We have

$$(\to)_1\uparrow \frac{\{B\to A\}\vdash^1\{B\to A\}\;\underline{h}^\circ(\underline{I}^\circ)}{\underline{C}^1 \quad \varnothing\vdash^1\{B\to A,B\}} \quad \underline{T}er\uparrow\frac{\{A\}\vdash^1\{A\}\;\underline{h}^\circ(\underline{I}^\circ)}{\{B\to A,A\}\vdash^1\{A\}}$$

$$\underline{A}^2\frac{\{A\}\vdash^1\{A,B\}}{\underbrace{\varnothing\vdash^2\{\{A\}\vdash^1\{A,B\}\}}_{①}}\;,\;\text{where } A\neq B$$

$$\underline{C}^1\frac{① \quad \{\Gamma\vdash^1\Delta_1\cup\{A\},\{A\}\vdash^1\{A,B\}\}\vdash^2\{\Gamma\vdash^1\Delta_1\cup\{A,B\}\}\;\underline{h}^1(\underline{C}^1)}{\{\Gamma\vdash^1\Delta_1\cup\{A\}\}\vdash^2\{\Gamma\vdash^1\Delta_1\cup\{A,B\}\}}.$$

Then we can easily show $\{\Gamma\vdash^1\Delta\}\vdash^2\{\Gamma\vdash^1\Delta\cup\Theta\}$ by repeating this

proof for every B∈Θ, such that B∉Δ, and applying $\underline{C}^2$.   If $\Delta=\varnothing$, we

have $\{\Gamma\vdash^1\varnothing\}\vdash^2\{\Gamma\vdash^1\{\bot\}\}\;\underline{h}^1((\bot)\downarrow)$ and

$$(\bot)\uparrow\frac{\{\bot\}\vdash^1\{\bot\}\;\underline{h}^\circ(\underline{I}^\circ)}{\underline{A}^2\frac{\{\bot\}\vdash^1\varnothing}{\underline{C}^2\;\varnothing\vdash^2\{\{\bot\}\vdash^1\varnothing\}}}\quad \frac{\{\Gamma\vdash^1\Theta\cup\{\bot\},\{\bot\}\vdash^1\varnothing\}\vdash^2\{\Gamma\vdash^1\Theta\}\;\underline{h}^1(\underline{C}^1)}{\{\Gamma\vdash^1\Theta\cup\{\bot\}\}\vdash^2\{\Gamma\vdash^1\Theta\}},$$

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reducing this case, with the help of $\underline{C}^2$, to the case where $\Delta \neq \emptyset$.

Q.E.D.

It follows that if we further extended the system of this lemma with (&), (∨) and (T) , we would get a system 0 and 1--equivalent with CP/D. This shows that if CP/D were formulated with $(\rightarrow)_1$ instead of $(\rightarrow)$, Thinning on the right of level 1, and its horizontalization, would be eliminable.

7   If to the analytic rule $(\neg)_1$ we add the proviso $\Delta = \emptyset$, we obtain the analytic rule

$$(\neg)_3 \quad \frac{\Gamma \cup \{A\} \vdash^1 \emptyset}{\Gamma \vdash^1 \{\neg A\}} \quad .$$

We state the following omitting the demonstration

<u>Lemma 26</u>   If $A =_{df} A \rightarrow \perp$, $(\neg)_3$ is horizontalizable in <u>hADIC</u>$(\rightarrow)(\perp)$.

In at least some contexts $(\neg)_3$ could be used to analyze classical negation too, since we have

<u>Lemma 27</u> $(\neg)_1$ is horizontalizable in <u>hADICT</u>$(\rightarrow)(\neg)_3$.

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Demonstration:

$$\frac{\frac{\{A\}\vdash\{A\}\ \underline{h}^\circ(I^\circ)}{\{A\}\vdash\{A,\neg A\}}(\to)\downarrow}{\varnothing\vdash\{A,A\to\neg A\}}\underline{C}^\wedge \qquad \frac{\frac{\frac{\{A\to\neg A\}\vdash\{A\to\neg A\}\ \underline{h}^\circ(I^\circ)}{\{A,A\to\neg A\}\vdash\{\neg A\}}(\to)\uparrow \quad \frac{\{\neg A\}\vdash\{\neg A\}\ \underline{h}^\circ(I^\circ)}{\{\neg A,A\}\vdash\varnothing}(\neg)\uparrow_3}{\{A,A\to\neg A\}\vdash\varnothing}\underline{C}^\wedge}{\{A\to\neg A\}\vdash\{\neg A\}}(\neg)\downarrow_3}{\varnothing\vdash\{A,\neg A\}}\ ;$$

and we have also shown in this proof $\{A,\neg A\}\vdash^1 \varnothing$.

Q.E.D.

But in general this is not the case, since we have

__Lemma 28__ $A\vee\neg A$ is not provable in $\underline{hADICT}(\vee)(\neg)_3$.

Demonstration: Take the o_1-translation from the demonstration of Lemma 19. We can show by an induction on the length of proof that if A^n, $0\leqslant n\leqslant 1$, is provable in the system S of the Lemma, $o_1(A^n)$ is provable in HP/O, where $\neg$ is explicitly defined in this latter system. So $A\vee\neg A$ is not provable in S.

Q.E.D.

As an immediate corollary we get that $\varnothing\vdash^1\{A,\neg A\}$ is not provable in $\underline{hADICT}(\neg)_3$.

§38

Clause (O) of the definition of structurally analyzed

constants [v.§24] requires that a constant be analyzed by a

certain analytic rule relative to any language L in which it might

occur. Now suppose L has only two constants: classical disjunction

and classical negation; then by clause (O), O also has only these

two constants. Then Lemma 28 shows that classical negation cannot be

structurally analyzed with $(\neg)_3$.

§38   We shall now demonstrate the O-equivalence of HP/D and HP/O, and of

KP/D and KP/O.

Lemma 29   29.1 (mp) is horizontalizable and c_{1-2} are provable

in $\underline{hADICT}_{\varrho\hbar}(\rightarrow)$.

29.2 k_{1-3} are provable in $\underline{hADICT}_{\varrho\hbar}(\rightarrow)(\&)$.

29.3 a_{1-3} are provable in $\underline{hADICT}_{\varrho\hbar}(\rightarrow)(\lor)$.

29.4 δ is provable in $\underline{hADICT}_{\delta\hbar}(\rightarrow)(\bot)$.

29.5 ν is provable in $\underline{hADICT}_{\varrho\hbar}(T)$.

29.6 n_{1-2} are provable in $\underline{hADICT}_{\varrho\hbar}(\rightarrow)(\neg)_3$.

29.7 n_3 is provable in $\underline{hADICT}_{\delta\hbar}(\rightarrow)(\neg)_3$.

To demonstrate this lemma we proceed exactly as in the demonstration

of Lemma 18, replacing occasionally applications of $\underline{T}$ by

applications of $\underline{T}_{\varrho\hbar}$ or $\underline{T}_{\delta\hbar}$, and applications of $(\neg)_1$ by applications

of $(\neg)_3$.

Lemma 30   30.1   If a sentence of level 0 is provable in HP/D, it is provable in HP/O.

    30.2 If a sentence of level 0 is provable in KP/D, it is provable in KP/O.

Demonstration: We proceed as in the demonstration of Lemma 19. In the basis in case 2) only the second two formulae are relevant; in case 3) with T_{er} only the second; in case 4) only the second and the fourth; and in case 5) only the second. The rest remains unchanged.

<div align="right">Q.E.D.</div>

    From Lemmata 29 and 30 we get immediately

Theorem 2   2.1 HP/D and HP/O are 0-equivalent.

    2.2 KP/D and KP/O are 0-equivalent.

§39   We shall now consider whether the analytic rules of HP/D or KP/D are replaceable by postulates of HP/O or KP/O.

    Lemma 20 applies also to this context, except for 20.5 which applies only to the system with T_{sr}. But now we have also

Lemma 31   Let S_1 be the extension of hADICT$_{sr}$ with the postulates of HP/O, and S_2 the extension of hADICT$_{er}$ with the postulates of KP/O.

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31.1 $(\rightarrow)\downarrow$ is admissible in S_1 and S_2.

31.2 $(\vee)\downarrow$ is admissible in S_1 and S_2.

31.3 $(\neg)_3\downarrow$ is admissible in S_1 and S_2.

31.4 $(\bot)\downarrow$ is admissible in S_2.

<u>Demonstration</u>: We proceed in principle as for Lemma 23, using

a lemma homologous to Lemma 22. For 31.2-31.4 we have, by inspecting

the postulates of S, that every provable sequent has a singleton

on the right.

<div align="right">Q.E.D.</div>

Lemma 31.4   <u>a fortiori</u> holds for S_1 by Lemma 20.   The rules $(\bot)\uparrow$

and $(\neg)_3\uparrow$ are not admissible in S_1 and S_2 for the reasons given

for 21.3 and 21.42 of Lemma 21.

From this we can conclude that (&) and (T) are replacable

by postulates of HP/O or KP/O, and that $(\bot)$ is not weakly

replacable even if we restrict ourselves to sentences of levels 0

and 1. With the restriction to these sentences, $(\rightarrow)$ and $(\vee)$ are

weakly replaceable by the postulates of HP/O and KP/O. The analytic

rule $(\vee)$ would be replaceable in general if we had $(\rightarrow)$.

[For the question whether the weak replaceability of $(\rightarrow)$ can be strengthened

to replaceability cf. §80.]

We can see that as far as sentences of level 0 are concerned

$(\bot)$ is eliminable from KP/D.

Because of the single-conclusion property, the systems HP/D

and KP/D can be seen as systematizations of propositional <u>logics of</u>

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proof, where "proof" stands for a somewhat more general notion
of deduction than the one exemplified by what we have called
"proofs in a system". This notion would allow for an empty
set of premises and an empty set of conclusions. If we reject
($\perp$), these proofs would practically coincide with proofs in a
system, since we can count the premises of the directives
corresponding to axioms as also being part of a proof. The logics
of proof can be conceived as proper parts of classical logic.

PROPOSITIONAL CONSTANTS OF THE LOGIC OF REFUTATION

§40   In this part of the present chapter we shall consider
the analytic extensions of $\underline{hADICT}_{\delta\ell}$ an $\underline{hADICT}_{e\ell}$ with ($\rightarrow$), (&),
(v), ($\perp$) and (T), and a new analytic rule. We assume that O has
the following propositional constants

converse non-implication: $\nleftarrow$,

which is of the category $\dfrac{\delta}{\delta\delta}$, and

rejection: ⅂ ,

which is of the category $\dfrac{\delta}{\delta}$.

For $\nleftarrow$ we give the following analytic rule

$$(\nleftarrow) \quad \dfrac{\Gamma \cup \{A\} \vdash^1 \Delta \cup \{B\}}{\Gamma \cup \{B \nleftarrow A\} \vdash^1 \Delta} \quad ,$$

and for ⅂ the following explicit definition

$$⅂A \ =_{df} \ A \nleftarrow T \ .$$

Now we introduce two systems which will be dual to HP/D and
KP/D : the system

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$$\underline{\text{hADICT}}_{\delta\ell}(\nrightarrow)(\&)(\vee)(\bot)(\top)$$

called "RHP/D" ("R" stands for "refutation"), and the system

$$\underline{\text{hADICT}}_{e\ell}(\nrightarrow)(\&)(\dot{\vee})(\bot)(\top)$$

called "RKP/D".

A number of results concerning these systems can be obtained in virtue of the duality with HP/D and KP/D, but this duality is not always exact. For example, we have no general single-premise property: we can prove for sentences $\Gamma \vdash^{1} \Delta$ that if they are provable in RHP/D, Γ is a singleton or empty, but not so for sentences of higher levels.

The analytic rule ($\rightarrow$) is not appropriate in the context with $\underline{T}_{\delta\ell}$ or $\underline{T}_{e\ell}$, for we have

Lemma 32 $\underline{T}^{1}$ is horizontalizable in the system $\underline{\text{hADICT}}_{e\ell}(\rightarrow)(\bot)$.

Demonstration: If $\Delta \neq \emptyset$, $B \in \Delta$ and $A \notin \Gamma$, we have

$$(\rightarrow)\uparrow \frac{\{A \rightarrow B\} \vdash^{1} \{A \rightarrow B\}}{\{A \rightarrow B, A\} \vdash^{1} \{B\}} \underline{h}^{\circ}(\underline{I}^{\circ})$$

$$\frac{A^{2}}{\underline{C}^{2}} \frac{}{\emptyset \vdash^{2} \{\{A \rightarrow B, A\} \vdash^{1} \{B\}\}}$$

$$\frac{\{\Gamma \vdash^{1} \Delta \cup \{A \rightarrow B\}, \{A \rightarrow B, A\} \vdash^{1} \{B\}\} \vdash^{2} \{\Gamma \cup \{A\} \vdash^{1} \Delta\}}{\{\Gamma \vdash^{1} \Delta \cup \{A \rightarrow B\}\} \vdash^{2} \{\Gamma \cup \{A\} \vdash^{1} \Delta\}} \underline{h}^{1}(\underline{C}^{1})$$

$$\underline{C}^{2} \frac{\{\Gamma \vdash^{1} \Delta\} \vdash^{2} \{\Gamma \vdash^{1} \Delta \cup \{A \rightarrow B\}\}}{\{\Gamma \vdash^{1} \Delta\} \vdash^{2} \{\Gamma \cup \{A\} \vdash^{1} \Delta\}} \underline{h}^{1}(\underline{T}_{e\ell}^{1}) .$$

If $\Delta = \Delta_{1} \cup \{A \rightarrow B\}$, we obtain $\{\Gamma \vdash^{1} \Delta\} \vdash^{2} \{\Gamma \cup \{A\} \vdash^{1} \Delta_{1}\}$ as above, and then use $\underline{h}^{1}(\underline{T}_{e\ell}^{1})$ and $\underline{C}^{2}$.

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Then we can easily show $\{\Gamma \vdash^1 \Delta\} \vdash^2 \{\Gamma \cup \Theta \vdash^1 \Delta\}$ by repeating this

proof for every $A \in \Theta$, and applying $\underline{c}^2$. If $\Delta = \emptyset$, we have

$$\{\Gamma \vdash^1 \Delta\} \vdash^2 \{\Gamma \vdash^1 \{\bot\}\} \quad \underline{h}^1((\bot)\not\uparrow) \text{ and}$$

$$\{\Gamma \cup \Theta \vdash^1 \{\bot\}\} \vdash^2 \{\Gamma \cup \Theta \vdash^1 \Delta\} \quad \underline{h}^1((\bot)\uparrow),$$

reducing this case, with the help of $\underline{c}^2$, to the case where
$\Delta \neq \emptyset$.

<div align="right">Q.E.D.</div>

It follows that the extension of RKP/D with $(\rightarrow)$ would be
0 and 1-equivalent with CP/D. This shows that Thinning on the
left of level 1, and its horizontalization, are eliminable from
CP/D.

Adding $(\rightarrow)$ to RKP/D would not give rise to a conservative
extension of RKP/D [v.§84 for the notion of conservative extension].

Because of the single-premise property of sequents of level
1, we propose to interpret the systems RHP/D and RKP/D as
systematizations of propositional logics of refutation. A
refutation would be a deduction where we have at most one premise;
from this premise we try to deduce a number of conclusions, with
the intent to show that all these conclusions are refutable, so
that the premise must be refutable too. Sequents are read backwards:
if all sentences on the right are refutable, a sentence on the
left is refutable. The present interest is to see what sentences
of the form $\{A\} \vdash^1 \emptyset$ are provable in this logic. But first we
shall make some remarks on the replaceability of analytic rules
in this context.

§41   Dually to what we had with $\underline{T}_{\delta\hbar}$ and $\underline{T}_{\varrho\hbar}$, with $\underline{T}_{\delta\ell}$ and $\underline{T}_{\varrho\ell}$ the

analytic rules (v) and (⊥) are replaceable by $(v)_{1.1}$ and $(v)_{1.2}$,

and $(⊥)_1$; whereas $(\&)_0$, $(\&)_{1.1}$ and $(T)_1$ cannot be obtained

in this context, for they would not preserve the single-premise

property of sentences of level 1. And dually to Lemma 25, we could

show that the analytic rule

$$(\not\vdash)_1 \quad \dfrac{\Gamma\vdash^1 \Delta \cup \{A\} \qquad \Gamma \cup \{B\}\vdash^1 \Delta}{\Gamma\vdash^1 \Delta \cup \{B \not\vdash A\}}$$

would give us $\underline{T}^1$ in the presence of (T).

        With $\underline{T}$ unrestricted, i.e. in classical logic, we could give

$$B \not\vdash A =_{df} \ (A \to B) \to \ ⊥,$$

and both $(\not\vdash)$ and $(\not\vdash)_1$ would be horizontalizable and the following

formulae provable

$$\{A\}\vdash^1 \{B, \ B \not\vdash A\} \ ; \ \{B \not\vdash A\}\vdash^1 \{A\} \ ; \ \{B \not\vdash A, B\}\vdash^1 \emptyset \ .$$

        With $\underline{T}_{\delta\hbar}$ or $\underline{T}_{\varrho\hbar}$, in the presnece of (T), $(\not\vdash)$ would give us $\underline{T}^1$,

dually to what we had in Lemma 32. But extending HP/D or KP/D

with $(\not\vdash)_1$ we would not "fall" into classical logic. Then $B \not\vdash A$

would <u>not</u> be explicitly definable as above, for with $(\not\vdash)_1$ and $(\neg)_3$

we would get $(\to)_1$, but as follows

$$B \not\vdash A =_{df} \ A\&(B \to ⊥),$$

for we have that the following double-line rules are horizontalizable

$$(\neg)_3 \ \dfrac{\Gamma \cup \{B\}\vdash^1 \emptyset}{\Gamma\vdash^1 \{B \to ⊥\}} \qquad ; \qquad (\&) \ \dfrac{\Gamma\vdash^1 \{A\} \qquad \Gamma\vdash^1 \{B \to ⊥\}}{\Gamma\vdash^1 \{A\&(B \to ⊥)\}} \ .$$

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Dually, we can extend RHP/D and RKP/D with $(\rightarrow)_1$ without

"falling" into classical logic. Then we could explicitly define $\rightarrow$

as follows

$$A \rightarrow B =_{df} \quad (A \not\vdash T) \vee B, \quad \text{i.e.}$$

$$A \rightarrow B =_{df} \quad \daleth A \vee B \; ,$$

for we have that the following double-line rules are horizontalizable

$$(\daleth) \; \frac{\emptyset \vdash^1 \Delta \cup \{A\}}{\{\daleth A\} \vdash^1 \Delta} \quad ; \; (\vee) \; \frac{\{\daleth A\} \vdash^1 \Delta \quad \{B\} \vdash^1 \Delta}{\{\daleth A \vee B\} \vdash^1 \Delta} \; .$$

The explicit definiton

$$A \rightarrow B =_{df} \quad \daleth (B \not\vdash A)$$

would not do, for with $(\rightarrow)_1$ and $(\daleth)$, we would get $(\not\vdash)_1$.

By duality we also have that $\not\vdash$ in RHP/D and RKP/D is not

explicitly definable in terms of $\&, \vee, \perp$ and T; otherwise $\rightarrow$

would be explicitly definable in HP/D and KP/D in terms of these

constants, and familiar arguments could show that it isn't.

We have given above the analytic rule for $\daleth$, horizontalizable

in RHP/D and RKP/D. This analytic rule is dual to $(\daleth)_3$.

§42   We shall now give axiomatizations in O which correspond to

RHP/D and RKP/D in such a way that whenever A is provable in one

of these axiomatizations $\{A\} \vdash^1 \emptyset$ is provable in the corresponding

system in D1, and vice versa. Intuitively, if A is provable in one

of these axiomatizations, this would mean that A is refutable.

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Consider the following rule (corresponding to <u>modus tollens</u>)

$$(\underline{mt}) \quad \frac{A \qquad A \supset B}{B} \quad ,$$

and the following formulae

rc_1 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

rc_2 $A \supset (B \supset A)$;

rk_1 $A \supset (A\&B)$;

rk_2 $B \supset (A\&B)$;

rk_3 $(A \supset C) \supset ((B \supset C) \supset ((A\&B) \supset C))$;

ra_1 $(A \vee B) \supset A$;

ra_2 $(A \vee B) \supset B$;

ra_3 $(C \supset A) \supset ((C \supset B) \supset (C \supset (A \vee B)))$;

$r\emptyset$ $\bot$;

$r\upsilon$ $\top \supset A$;

<div align="center">*</div>

rn_1 $(A \supset B) \supset (\neg B \supset \neg A)$;

rn_2 $A \supset \neg\neg A$;

rn_3 $\neg\neg A \supset (\neg A \supset A)$

(substitution is everywhere free).

The system

$$(\underline{mt}) \;\; \underline{rc_{1\text{-}2} \;\; rk_{1\text{-}3} \;\; ra_{1\text{-}3} \;\; r\emptyset \;\; r\upsilon}$$

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will be called "RHP/O", and the system

$$(mt) \qquad \ulcorner c_{1-2} \; \ulcorner k_{1-3} \; \ulcorner a_{1-3} \; \ulcorner b$$

will be called "RKP/O".

Next we give the following translation of sentences of O into sentences of O

$t_1(A) = A$, if $\neq$, &,∨,⊥,T [and ¬] do not occur in A,

$t_1 (A \neq B) = t_1(A) \rightarrow t_1(B)$,

$t_1 (A\&B) = t_1(A) \vee t_1(B)$,

$t_1 (A \vee B) = t_1(A) \& t_1(B)$,

$t_1 (\perp) = T$,

$t_1 (T) = \perp$

$[t_1(\neg A) = \neg t_1(A)]$,

$$t_1(\Gamma) = \begin{cases} \{t_1(A_1), \; \dots, \; t_1(A_k)\}, \text{ if } \Gamma = \{A_1, \; \dots, A_k\}, \; k \geqslant 1; \\ \Gamma, \text{ if } \Gamma = \emptyset \; . \end{cases}$$

It can be shown easily by inspection of postulates

<u>Lemma 33</u>   33.1   A is provable in RHP/O iff $t_1(A)$ is provable in HP/O.

33.2   A is provable in RKP/O iff $t_1(A)$ is provable in KP/O.

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Next we can show

Lemma 34   34.1 $\Gamma \vdash^1 \Delta$ is provable in RHP/D iff $t_1(\Delta) \vdash^1 t_1(\Gamma)$
is provable in HP/D.

    34.2 $\Gamma \vdash^1 \Delta$ is provable in RKP/D iff $t_1(\Delta) \vdash^1 t_1(\Gamma)$ is provable
in KP/D.

Demonstration: We first make an induction on the length of the proof
of $\Gamma \vdash^1 \Delta$ to show the Lemma from left to right.

    For the basis we have only $\underline{h}^0(\underline{I}^0)$.

    For the induction step suppose that if $\Gamma \vdash^1 \Delta$ is proved
in $\leqslant k$ steps, the Lemma holds from left to right. Let it be
proved in $k + 1$ steps. $\underline{D}^2$ is eliminable; so the last step
can be an instance of:

1) $\underline{A}^1 \dfrac{A}{\emptyset \vdash^1 \{A\}}$, $\Gamma = \emptyset$, $\Delta = \{A\}$; then the only

    way to prove A is to use $\underline{D}^1$ ($\underline{I}$ being eliminable);

2) $\underline{C}^1$; then we use the induction hypothesis and $\underline{C}^1$;

3) $\underline{T}_{el}^1$ (or $\underline{T}_{sl}^1$); then we use the induction hypothesis and
    $\underline{T}_{er}^1$ (or $\underline{T}_{sr}^1$);

4) $(\nrightarrow)$; then we use the induction hypothesis and $(\rightarrow)$;

5) $(\&)$; then we use the induction hypothesis and $(\vee)$;

6) $(\vee)$; then we use the induction hypothesis and $(\&)$;

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7)   (⊥); then we use the induction hypothesis and (T);

8)   (T); then we use the induction hypothesis and (⊥).

For the other direction we proceed homologously.

<div align="right">Q.E.D.</div>

Theorem 3   3.1   A is provable in RHP/O iff $\{A\} \vdash^1 \emptyset$ is provable in RHP/D.

   3.2   A is provable in RKP/O iff $\{A\} \vdash^1 \emptyset$ is provable in RKP/D.

Demonstration: 3.1   By Lemma 33, A is provable in RHP/O iff $t_1(A)$ is provable in HP/O. By Theorem 2, $t_1(A)$ is provable in HP/O iff $t_1(A)$ is provable in HP/D. $t_1(A)$ is provable in HP/D iff $\emptyset \vdash^1 \{t_1(A)\}$ is.   By Lemma 34, $\emptyset \vdash^1 \{t_1(A)\}$ is provable in HP/D iff $\{A\} \vdash^1 \emptyset$ is provable in RHP/D. For 3.2 we proceed homologously.

<div align="right">Q.E.D.</div>

§43   The translation t_1 serves well to find the characteristics of RHP/O and RKP/O by connecting them with HP/O and KP/O, but it does not give the intuitive interpretation of the provable sentences.

     One intuitive interpretation would be given by the translation t_2 in the language of intuitionistic logic, where

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$t_2(A) = \neg A$, if $\nrightarrow, \&, \vee \perp$ and T do not occur in A,

and the other clauses are obtained by subsituting "t_2" for

"t_1" in all the clauses for t_1 except the first. It can easily

be seen that if A is provable in RHP/O (or RKP/O), $t_2(A)$ is

provable in HP/O (or KP/O); but the converse does not hold

necessarily. The formula $t_2 \; (((A \nrightarrow B) \nrightarrow A) \nrightarrow A)$ is

provable in HP/O and KP/O when $\nrightarrow, \&, \vee, \perp$ and T do not

occur in A and B, since $((\neg A \rightarrow \neg B) \rightarrow \neg A) \rightarrow \neg A$ is provable in

both these systems. On the other hand, $((A \nrightarrow B) \nrightarrow A) \nrightarrow A$ is

not provable in RHP/O or RKP/O.

That the understanding of provable sentences of RHP/O and

RKP/O suggested by t_2 should square with our expectations, can

be shown by the following considerations. Suppose A is provable

in one of these systems. This means that A is refutable. If

$\nrightarrow, \&, \vee, \perp$ and T do not occur in A, this means that $\neg A$, where $\neg$

is understood intuitionistically; if A is $B \nrightarrow C$, this means

that from a refutation of B we can construct a refutation of C;

if A is B&C, this means that either B is refutable or C is

refutable; if A is BvC, this means that both B and C are refutable;

if A is $\perp$, this means that something refutable is refutable, i.e.

A is provable; if A is T, this means that something unrefutable

is refutable, i.e. A is refutable (and if A is $\overline{\neg} B$, this means

that from a refutation of B we can construct a refutation of

something unrefutable, i.e. this means that it is refutable that

B is refutable).

It should also square with our expectations that the

t_2-translation goes only one way: interpreted intuitionistically,

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the logic of refutation is only a proper part of the logic of proof.

By using the following translation t_3, homologous to t_2, it is possible to give a reading of intuitionistic propositional logic in the logic of refutation

$$t_3(A) = \daleth A, \text{ if } \to, \&, \vee, \bot, T \text{ [and } \neg \text{] do not occur in A,}$$

$$t_3(A \to B) = t_3(A) \not\to t_3(B),$$

$$t_3(A\&B) = t_3(A) \vee t_3(B),$$

$$t_3(A \vee B) = t_3(A) \& t_3(B),$$

$$t_3(\bot) = T,$$

$$t_3(T) = \bot$$

$$[t_3(\neg A) = \daleth t_3(A)] \quad .$$

Now if A is provable in HP/O (or KP/O), $t_3(A)$ is provable RHP/O (or RKP/O), but the converse does not hold necessarily. Essentially, this amounts to interpreting "it is provable that A" as "it is refutable that it is refutable that A". This, of course, preserves the validity of intuitionistic principles, but does not square with an intuitive understanding of them. This asymmetry, the possibility to interpret intuitively the logic of refutation in intuitionistic logic but not conversely, might be due to the fact that we are interested primarily in sentences that can be affirmed in some sense. If, as it seems per impossibile, we were interested primarily in sentences that can be denied, intuitionistic logic could be taken to be only a part of the logic of refutation.

The logic of refutation is a proper part of classical

logic, but in this section we shall try to show that it is

possible to interpret classical logic at level 0 as a logic

of refutation.

RHP/0 and RKP/0 are certainly awkward systems. We are

usually not interested in systems of provable sentences which

cannot be affirmed in some sense, but must be denied. So we

can consider what sentences of level 0, or what sentences of

the form $\emptyset \vdash^1 \{A\}$,   can be proved in RHP/D and RKP/D. Then

the constants $\not\leftarrow$, &,v,⊥ and T(and $\daleth$) don't get their

dual reading, but correspond to their intuitive reading ($\not\leftarrow$ being

the negation of converse implication, and $\daleth$, rejection, being

a kind of negation). A constant of implication can then be

explicitly defined as follows

$$A \Rightarrow B =_{df} \daleth (B \not\leftarrow A) \ .$$

Then we introduce the following translation for sentences which

are such that if a propositional constant occurs in them, it is

among: $\Rightarrow$,&,v,⊥ and T :

$t_4(A) = A$, if no propositional constant occurs in A,

$t_4(A \Rightarrow B) = t_4(A) \to t_4(B)$,

$t_4(A\&B)\quad = t_4(A)\&t_4(B)$,

$t_4(A v B)\ = t_4(A)v t_4(B)$,

$t_4(\bot) = \bot$,

$t_4(T) = T.$

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We can show

Theorem 4   Let A be a sentence of level 0 such that if a propositional constant occurs in A, it is among : $\Rightarrow$,&,$\vee$,$\perp$ and T; then A is provable in RHP/D iff $t_4(A)$ is provable in CP/0.

Demonstration: A is provable in RHP/D iff $\emptyset \vdash^1 \{A\}$ is, and $\emptyset \vdash^1 \{A\}$ is provable in RHP/D if $\{\neg A\} \vdash^1 \emptyset$ is. By Lemma 34.1, $\{\neg A\} \vdash^1 \emptyset$ is provable in RHP/D iff $\emptyset \vdash^1 \{\neg t_1(A)\}$ is provable in HP/D; and $\emptyset \vdash^1 \{\neg t_1(A)\}$ is provable in HP/D iff $\neg t_1(A)$ is. By Theorem 2.1, $\neg t_1(A)$ is provable in HP/D iff $\neg t_1(A)$ is provable in HP/0. Then by Glivenko's Theorem, $\neg t_1(A)$ is provable in HP/0 iff $\neg t_1(A)$ is provable in CP/0.

Let us consider the $\neg t_1$-translation in this context:

$\neg t_1(B) = \neg B$, if no propositional constant occurs in B,

$\neg t_1(B \Rightarrow C) = \neg t_1(\neg(C \not\equiv B)) = \neg\neg(t_1(C) \rightarrow t_1(B))$,

$\neg t_1(B\&C) = \neg(t_1(B) \vee t_1(C))$,

$\neg t_1(B \vee C) = \neg(t_1(B) \& t_1(C))$,

$\neg t_1(\perp) = \neg T$,

$\neg t_1(T) = \neg \perp$.

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But we have in CP/O

$$\neg\neg(t_1(C) \to t_1(B)) \leftrightarrow (\neg t_1(B) \to \neg t_1(C)),$$

$$\neg(t_1(B) \vee t_1(C)) \leftrightarrow (\neg t_1(B) \& \neg t_1(C)),$$

$$\neg(t_1(B) \& t_1(C)) \leftrightarrow (\neg t_1(B) \vee \neg t_1(C)),$$

$$\neg T \quad \leftrightarrow \quad \perp,$$

$$\neg\perp \quad \leftrightarrow \quad T.$$

So, we can define a new translation t_5 which is like t_4 except that it prefixes $\neg$ to every sentence A_i occurring in A such that no propositional constant occurs in A_i; then we get that $\neg t_1(A)$ is provable in CP/O iff $t_5(A)$ is. By substituting $\neg A_i$ for every A_i and using $A_i \leftrightarrow \neg\neg A_i$, we get that $t_5(A)$ is provable in CP/O iff $t_4(A)$ is provable in CP/O.

Q.E.D.

This result is dual to the result that $\{A\} \vdash^1 \emptyset$ is provable in HP/D iff $\{A\} \vdash^1 \emptyset$ is provable in CP/D, which is another consequence of Glivenko's Theorem.

We leave open the question what sentences of level 0 are provable in RKP/D. (Glivenko's Theorem does not hold between KP/O and CP/O: in KP/O, e.g., $\neg\neg(\neg\neg A \to A)$ is not provable; but it holds between KP/O and Johansson-Curry's system D [v. §50].)

If we don't introduce $\Rightarrow$, we can modify t_4 by giving

$$t_4(A \not\to B) = \neg(t_4(B) \to t_4(A)),$$

and an appropriate form of Theorem 4 will still hold.

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PROPOSITIONAL CONSTANTS OF INTUITIONISTIC RELEVANT LOGIC

§45   In this part of the present chapter we shall consider the
analytic extensions of <u>hADIC</u> with (→), (&), (v), (⊥) and (T).

The system

$$\underline{hADIC}(\rightarrow)\,(\&)\,(v)\,(\bot)\,(T)$$

will be called "IRP/D" ("IR" stands for "intuitionistic relevant").

By Lemma 24.1, IRP/D has the single-conclusion property,
but contrary to RHP/D or RKP/D it does not have the single-
-premise property even for sequents of level 1. Concerning these
latter systems we had, by Lemma 32, that (→) was not appropriate ,
since it would give us $\underline{T}^1$. On the other hand, in the present context we
shall see that $\underline{T}_{\varrho\hbar}^{1}$ is not admissible in IRP/D [v.§48].

Without $\underline{T}$ none of (→), (&), (v), (⊥) and (T) are replaceable
by the alternative analytic rules we have considered. Only in a
particular case we have that (T) is replaceable by $(T)_1$ in the
presence of (→).   Indeed, analytic rules exactly corresponding to
$(\&)_{1.1}$ and $(\&)_{1.2}$, and to $(v)_{1.1}$ and $(v)_{1.2}$, could be used in this
context to analyze constants <u>different</u> from conjunction and
disjunction.

We assume that we have in O the propositional constants

intensional conjunction: o , and

intensional disjunction: + ,

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which are of the category $\frac{\delta}{\delta\delta}$. Then we give the analytic

rules

$(\circ)_1$ $\dfrac{\Gamma \cup \{A,B\} \vdash^1 \Delta}{\Gamma \cup \{A\circ B\} \vdash^1 \Delta}$; $(\circ)_2$ $\dfrac{\Gamma \cup \{A\} \vdash^1 \Delta}{\Gamma \cup \{A\circ A\} \vdash^1 \Delta}$;

$(+)_1$ $\dfrac{\Gamma \vdash^1 \Delta \cup \{A,B\}}{\Gamma \vdash^1 \Delta \cup \{A+B\}}$; $(+)_2$ $\dfrac{\Gamma \vdash^1 \Delta \cup \{A\}}{\Gamma \vdash^1 \Delta \cup \{A+A\}}$.

In a language like D3, with rules of Repetition [v.§16], analytic

rules corresponding to $(\circ)_1$ and $(+)_1$ would be sufficient [cf. §29] .

The analytic rule $(+)_1$ is not appropriate in the context of

IRP/D, for it would not preserve the single-conclusion property;

but we shall consider a closely related system to which it could

be added.

Canonically, we have given IRP/D without $(\circ)_{1-2}$, but

we shall also consider extensions with these analytic rules. Some

properties of $\circ$ and + will become apparent later [v. also §50

for historical remarks].

Concerning the analytic rule for $\lnot$ we can appeal to Lemma

26.

Systems without T are about a concept of deduction in which

no additional irrelevant premises and no alternative irrelevant

conclusions can be found. Every premise must be used in the

deduction of the conclusion from the premises. This concept

of deduction is made intuitively clear by natural deduction calculi

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of relevant logic using the <u>subscri b ing</u> technique *[v.§50]*,

and a system like IRP/D could in principle be shown sound and

complete in a certain sense with respect to such a natural deduction

calculus. This concept of deduction is not an uncommon one:

it is appealed to when we imagine proofs as trees of occurrences

of sentences in which every possible premise is connected with the

sentence to be proved by a chain of deductive links. It is also

connected with our notion of a derivable rule, which is a standard

logical notion. We also think that the natural way in which we

get IRP/D by gradually restricting <u>T</u>, lends support to the

claim that this system is important. Because of the single-

-conclusion property, and because it is a subsystem of HP/D and

KP/D, we propose to interpret IRP/D as a systematization of a

propositional <u>logic of relevant proof</u> .

§46   We shall now give an axiomatization in O which will be

shown to be O-equivalent with IRP/D. As we have done before

with negation, we also list some axiom-postulates which are not

in the canonical axiomatization, but   will be used when we

want to show the separativeness of analytic rules, and when we

consider extensions of IRP/D.

Consider the following rules

$$(\underline{mp}) \quad \frac{A \quad A \to B}{B} \quad ; \quad (\underline{adj}) \quad \frac{A \quad B}{A\&B} \quad ;$$

and the following formulae

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$\underline{c_1}$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$;

$\underline{c_4}$ $A \to A$;

$\underline{c_5}$ $(A \to (B \to C)) \to (B \to (A \to C))$;

$\underline{c_6}$ $(A \to B) \to ((B \to C) \to (A \to C))$;

$\underline{k_1}$ $(A\&B) \to A$;

$\underline{k_2}$ $(A\&B) \to B$;

$\underline{k_4}$ $((C \to A)\&(C \to B)) \to (C \to (A\&B))$;

$\underline{a_1}$ $A \to (A \lor B)$;

$\underline{a_2}$ $B \to (A \lor B)$;

$\underline{a_4}$ $((A \to C)\&(B \to C)) \to ((A \lor B) \to C)$;

$\underline{\upsilon}$ T ;

$\underline{\upsilon_1}$ $T \to (A \to A)$;

<div align="center">*</div>

$\underline{n_1}$ $(A \to B) \to (\neg B \to \neg A)$;

$\underline{n_2}$ $A \to \neg\neg A$;

$\underline{n_4}$ $\neg\neg A \to A$;

$\underline{ik_1}$ $A \to (B \to (A \circ B))$,   provided $A \neq B$;

$\underline{ik_2}$ $(A \to (B \to C)) \to ((A \circ B) \to C)$;

$\underline{ik_3}$ $A \to (A \circ A)$;

$\underline{ik_4}$ $(A \circ A) \to A$;

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$$\underline{ia_1} \quad (\neg A \to B) \to (A + B) \quad ;$$

$$\underline{ia_2} \quad (A + B) \to (\neg A \to B), \text{ provided } A \neq B ;$$

$$\underline{ia_3} \quad A \to (A + A);$$

$$\underline{ia_4} \quad (A + A) \to A$$

(substitution is everywhere free except for $\underline{ik_1}$ and $\underline{ia_2}$).

The system

$$(\text{mp})(\text{adj}) \quad c_1 c_{4-6} k_{1-2} k_4 a_{1-2} a_4 v v_1$$

will be called "IRP/0".

The system

$$(\text{mp}) \quad c_1 c_{4-6}$$

is an axiomatization of the implicational fragment of the system of relevant implication R. IRP/0 is a proper subsystem of R. The positive fragment of R would be obtained by extending IRP/0 without $\underline{v \, v_1}$ with

$$\underline{\text{distr}} \quad (A\&(B \lor C)) \to ((A\&B) \lor C) \quad .$$

It is known that $\underline{\text{distr}}$ is independent from the postulates of IRP/0. The lack of $\underline{\text{distr}}$ in IRP/0 implies the lack of

$$(A\&(B \lor C)) \to ((A\&B) \lor (A\&C)) \quad .$$

(The converse of this last formula and also $(A \lor (B\&C)) \to ((A \lor B)\&(A \lor C))$ are provable in IRP/0. The converse of $\underline{\text{distr}}$ is not provable in CP/0.)

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The whole system R would be obtained by extending the positive fragment of R with $n_{1-2}n_4$. We shall consider extending IRP/O, which is a subsystem of KP/O, only with n_{1-2}. We shall also consider the extension of IRP/O with $n_{1-2}n_4$, which will be called "CRP/O" ("C" stands for "classical"), and the extensions of IRP/O with ik_{1-4}, and CRP/O with ik_{1-4} and ia_{1-4}.

It is known that the positive fragment of R is a conservative extension of $(mp)c_1c_{4-6}$, and that the addition of vv_1 would also give a conservative extension of this last system. A fortiori, IRP/O without vv_1 and IRP/O are conservative extensions of this system. Some other facts on the separativeness of the axiomatization of IRP/O and the extensions we have mentioned can be inferred from some known facts on the axiomatization of R. [For references on all these systems and the assertions we have made here v. §50]

§47   . We shall now demonstrate the O-equivalence of IRP/D and IRP/O.

Lemma 35   35.1 (mp) is horizontalizable and c_1c_{4-6} are provable in hADIC($\rightarrow$).

35.2 (adj) is derivable and $k_{1-2}k_4$ are provable in hADIC($\rightarrow$)(&).

35.3 $a_{1-2}a_4$ are provable in hADIC($\rightarrow$)(&)(v).

35.4 vv_1 are provable in hADIC($\rightarrow$)(T).

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35.5 n_{1-2} are provable in $\underline{\text{hADIC}}(\rightarrow)(\neg)_3$.

35.6 $n_{1-2}n_4$ are provable in $\underline{\text{hADIC}}(\rightarrow)(\neg)_1$.

35.7 ik_{1-4} are provable in $\underline{\text{hADIC}}(\rightarrow)(\circ)_{1-2}$.

35.8 ia_{1-4} are provable in $\underline{\text{hADIC}}(\rightarrow)(\neg)_1(+)_{1-2}$.

<u>Demonstration</u>: 35.1 For $\underline{(mp)}$ and c_1 we proceed as for Lemma 18.1;

$$(\rightarrow)\downarrow \frac{\{A\} \vdash \{\overline{A}\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underset{\underline{D}^1}{\frac{\varnothing \vdash \{A \rightarrow A\}}{A \rightarrow A}};}$$

$$\underset{\underline{C}^1}{(\rightarrow)\uparrow} \frac{\{A\rightarrow(B\rightarrow C)\} \vdash \{A\rightarrow(B\rightarrow C)\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{A\rightarrow(B\rightarrow C),A\} \vdash \{B\rightarrow C\}} \qquad (\rightarrow)\uparrow \frac{\{B\rightarrow C\} \vdash \{B\rightarrow C\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{B\rightarrow C,B\} \vdash \{C\}}$$
$$\frac{\{A\rightarrow(B\rightarrow C),B,A\} \vdash \{C\}}{\frac{3 \text{ applications of } (\rightarrow)\downarrow \text{ and } \underline{D}^1}{(A\rightarrow(B\rightarrow C))\rightarrow(B\rightarrow(A\rightarrow C))}}$$

if $B=A$, $\underline{c_5}$ is an instance of $\underline{c_4}$;

$$\underset{\underline{C}^1}{(\rightarrow)\uparrow} \frac{\{A\rightarrow B\} \vdash \{A\rightarrow B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{A\rightarrow B,A\} \vdash \{B\}} \qquad (\rightarrow)\uparrow \frac{\{B\rightarrow C\} \vdash \{B\rightarrow C\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{B\rightarrow C,B\} \vdash \{C\}}$$
$$\frac{\{A\rightarrow B, B\rightarrow C, A\} \vdash \{C\}}{\frac{3 \text{ applications of } (\rightarrow)\downarrow \text{ and } \underline{D}^1}{(A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C))}},$$

if $A = B\rightarrow C$:

$$\underset{\underline{C}^1}{(\rightarrow)\uparrow} \frac{\{A\rightarrow B\} \vdash \{A\rightarrow B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{A\rightarrow B,A\} \vdash \{B\}} \qquad \begin{array}{c}(\rightarrow)\uparrow \frac{\{A\} \vdash \{A\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{A,B\} \vdash \{C\}} \\ (\rightarrow)\downarrow \frac{}{\{B\} \vdash \{A\rightarrow C\}}\end{array}$$
$$\frac{}{\{A\rightarrow B, B\rightarrow C\} \vdash \{A\rightarrow C\}} \text{ etc.}$$

if $A \to B = B \to C$:

$$(\to)\!\uparrow \frac{\{B \to B\} \vdash \{B \to B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{B \to B, B\} \vdash \{B\}} \underline{C}\!\uparrow$$

$$(\to)\!\uparrow \frac{\{B \to B\} \vdash \{B \to B\} \quad \underline{h}^\circ(\underline{I}^\circ)}{(\to)\!\downarrow \frac{\{B, B \to B\} \vdash \{B\}}{\{B\} \vdash \{(B \to B) \to B\}}}$$

$$(\to)\!\downarrow \frac{\{B \to B, B\} \vdash \{(B \to B) \to B\}}{\underbrace{\{B \to B\} \vdash \{B \to ((B \to B) \to B)\}}_{①}}$$

$$\frac{(B \to ((B \to B) \to B)) \to ((B \to B) \to (B \to B))}{\underline{A}\!\uparrow \text{ and } (\to)\!\uparrow} \underline{c}_s$$

$$\underline{C}\!\uparrow \quad ① \quad \frac{\{B \to ((B \to B) \to B)\} \vdash \{(B \to B) \to (B \to B)\}}{\{B \to B\} \vdash \{(B \to B) \to (B \to B)\} \text{ etc.}}$$

35.2

$$(\&)\!\downarrow \frac{\underline{A}\!\uparrow \dfrac{A}{\varnothing \vdash \{A\}} \qquad \underline{A}\!\uparrow \dfrac{B}{\varnothing \vdash \{B\}}}{\underline{D}\!\uparrow \dfrac{\varnothing \vdash \{A \& B\}}{A \& B} \; ;}$$

for $\underline{k}_{1-2}$ we proceed as for Lemma 18.2;

$$(\&)\!\uparrow \frac{\{(c \to A)\&(c \to B)\} \vdash \{(c \to A)\&(c \to B)\} \quad \underline{h}^\circ(\underline{I}^\circ)}{(\to)\!\uparrow \frac{\{(c \to A)\&(c \to B)\} \vdash \{c \to A\}}{(\&)\!\downarrow \{(c \to A)\&(c \to B), c\} \vdash \{A\}}}$$

$$(\&)\!\uparrow \frac{\{(c \to A)\&(c \to B)\} \vdash \{(c \to A)\&(c \to B)\} \quad \underline{h}^\circ(\underline{I}^\circ)}{(\to)\!\uparrow \frac{\{(c \to A)\&(c \to B)\} \vdash \{c \to B\}}{\{(c \to A)\&(c \to B), c\} \vdash \{B\}}}$$

$$\frac{\{(c \to A)\&(c \to B), c\} \vdash \{A \& B\}}{2 \text{ applications of } (\to)\!\downarrow, \text{ and } \underline{D}\!\uparrow}$$

$$((c \to A)\&(c \to B)) \to (c \to (A \& B)) .$$

35.3   For $\underline{a_{1-2}}$ we proceed as for Lemma 18.3;

$$
\text{(\&)}\uparrow \frac{\{(A\to C)\&(B\to C)\}\vdash\{(A\to C)\&(B\to C)\}\overset{h^\circ(\underline{I}^\circ)}{}}{\{(A\to C)\&(B\to C)\}\vdash\{A\to C\}}
$$
$$
\text{($\to$)}\uparrow \frac{}{\{(A\to C)\&(B\to C)\}\vdash\{A\to C\}}
$$
$$
\text{(v)}\downarrow \frac{\{(A\to C)\&(B\to C),A\}\vdash\{C\}}{}
$$

$$
\text{(\&)}\uparrow \frac{\{(A\to C)\&(B\to C)\}\vdash\{(A\to C)\&(B\to C)\}\overset{h^\circ(\underline{I}^\circ)}{}}{\{(A\to C)\&(B\to C)\}\vdash\{B\to C\}}
$$
$$
\text{($\to$)}\uparrow \frac{}{\{(A\to C)\&(B\to C),B\}\vdash\{C\}}
$$

$$
\frac{\{(A\to C)\&(B\to C),A\vee B\}\vdash\{C\}}{}
$$
$$
\frac{\text{2 applications of ($\to$)$\downarrow$, and }\underline{D}^1}{((A\to C)\&(B\to C))\to((A\vee B)\to C).}
$$

35.4 For $\underline{v}$ we proceed as for Lemma 18.5;

$$
\text{($\to$)}\downarrow \frac{\{A\}\vdash\{A\}\ \overset{h^\circ(\underline{I}^\circ)}{}}{\emptyset\vdash\{A\to A\}}
$$
$$
\text{(T)}\downarrow \frac{}{\{T\}\vdash\{A\to A\}}
$$
$$
\frac{\text{($\to$)$\downarrow$ and }\underline{D}^1}{T\to(A\to A).}
$$

For 35.5 we proceed as for Lemma 18.6 replacing $(\neg)_1$ by $(\neg)_3$.

35.6
$$
\text{$(\neg)_1\downarrow$}\ \underline{C}^1 \frac{\{A\}\vdash\{A\}\ \overset{h^\circ(\underline{I}^\circ)}{}}{\emptyset\vdash\{A,\neg A\}}
$$
$$
\text{$(\neg)_1\uparrow$} \frac{\{\neg\neg A\}\vdash\{\neg\neg A\}\ \overset{h^\circ(\underline{I}^\circ)}{}}{\{\neg\neg A,\neg A\}\vdash\emptyset}
$$
$$
\frac{\{\neg\neg A\}\vdash\{A\}}{}
$$
$$
\frac{\text{($\to$)$\downarrow$ and }\underline{D}^1}{\neg\neg A\to A.}
$$

35.7

$(\circ)_1\uparrow$
$$\frac{\{A\circ B\}\vdash\{A\circ B\}}{\{A,B\}\vdash\{A\circ B\}}$$

2 applications of $(\rightarrow)\downarrow$, and $\underline{D}^1$

$$A\rightarrow(B\rightarrow(A\circ B))\;;$$

$(\rightarrow)\uparrow$
$\underline{C}^1$
$$\frac{\{A\rightarrow(B\rightarrow C)\}\vdash\{A\rightarrow(B\rightarrow C)\}\quad\underline{h}^\circ(\underline{I}^\circ)}{\{A\rightarrow(B\rightarrow C),A\}\vdash\{B\rightarrow C\}}$$

$(\rightarrow)\uparrow$
$$\frac{\{B\rightarrow C\}\vdash\{B\rightarrow C\}\quad\underline{h}^\circ(\underline{I}^\circ)}{\{B\rightarrow C,B\}\vdash\{C\}}$$

$$\underbrace{\{A\rightarrow(B\rightarrow C),A,B\}\vdash\{C\}}_{①}$$

if $A=B$, instead of ① we get ⟨1.1⟩ :
$$\{A\rightarrow(A\rightarrow C),A\}\vdash\{C\}$$

$(\circ)_1\downarrow$ or $(\circ)_2\downarrow$
$$\frac{①\qquad or\qquad ⟨1.1⟩}{\{A\rightarrow(B\rightarrow C),A\circ B\}\vdash\{C\}}$$

2 applications of $(\rightarrow)\downarrow$, and $\underline{D}^1$

$$(A\rightarrow(B\rightarrow C))\rightarrow((A\circ B)\rightarrow C)\;;$$

$(\circ)_2\uparrow$
$$\frac{\{A\circ A\}\vdash\{A\circ A\}\quad\underline{h}^\circ(\underline{I}^\circ)}{\{A\}\vdash\{A\circ A\}}$$
$(\rightarrow)\downarrow$ and $\underline{D}^1$
$$A\rightarrow(A\circ A)\qquad;$$

$(\circ)_2\downarrow$
$$\frac{\{A\}\vdash\{A\}\quad\underline{h}^\circ(\underline{I}^\circ)}{\{A\circ A\}\vdash\{A\}}$$
$(\rightarrow)\downarrow$ and $\underline{D}^1$
$$(A\circ A)\rightarrow A\;.$$

35.8

$(\neg)_1\downarrow$ $\dfrac{\{A\}\vdash^\mu\{A\}\ \underline{h}^\circ(\underline{I}^\circ)}{\emptyset\vdash^\mu\{A,\neg A\}}$ $\underline{C}^\mu$ $\qquad\qquad (\to)\uparrow\ \dfrac{\{\neg A\to B\}\vdash^\mu\{\neg A\to B\}\ \underline{h}^\circ(\underline{I}^\circ)}{\{\neg A\to B,\neg A\}\vdash^\mu\{B\}}$

$$\underbrace{\{\neg A\to B\}\vdash^\mu\{A,B\}}_{①}$$

if $A = B$, instead of $①$ we get $\boxed{1.1}$:
$$\{\neg A\to A\}\vdash^\mu\{A\}.$$

$(+)_1\downarrow\ or\ (+)_2\downarrow$ $\dfrac{①\qquad or\qquad \boxed{1.1}}{\{\neg A\to B\}\vdash^\mu\{A+B\}}$

$$\dfrac{}{\{\neg A\to B\}\vdash^\mu\{A+B\}}$$
$$\dfrac{(\to)\downarrow\ and\ \underline{D}^\mu}{(\neg A\to B)\to(A+B)\ ;}$$

$(+)_1\uparrow$ $\dfrac{\{A+B\}\vdash^\mu\{A+B\}\ \underline{h}^\circ(\underline{I}^\circ)}{\{A+B\}\vdash^\mu\{A,B\}}$ $\underline{C}^\mu$ $\qquad\qquad (\neg)_1\uparrow\ \dfrac{\{\neg A\}\vdash^\mu\{\neg A\}\ \underline{h}^\circ(\underline{I}^\circ)}{\{A,\neg A\}\vdash^\mu\emptyset}$

$$\dfrac{\{A+B,\neg A\}\vdash^\mu\{B\}}{}$$
$$\dfrac{2\ applications\ of\ (\to)\downarrow,\ and\ \underline{D}^\mu}{(A+B)\to(\neg A\to B)\ ;}$$

$(+)_2\downarrow$ $\dfrac{\{A\}\vdash^\mu\{A\}\ \underline{h}^\circ(\underline{I}^\circ)}{\{A\}\vdash^\mu\{A+A\}}$ $\qquad$ $(+)_2\uparrow$ $\dfrac{\{A+A\}\vdash^\mu\{A+A\}\ \underline{h}^\circ(\underline{I}^\circ)}{\{A+A\}\vdash^\mu\{A\}}$

$\dfrac{(\to)\downarrow\ and\ \underline{D}^\mu}{A\to(A+A)}\quad;$ $\qquad\qquad$ $\dfrac{(\to)\downarrow\ and\ \underline{D}^\mu}{(A+A)\to A}\quad.$

Q.E.D.

§47

Lemma 36   If a sentence of level 0 is provable in IRP/D, it is provable in IRP/0.

Demonstration: Consider the following o_2-translation of sentences of levels 0 and 1

$$o_2\left(A^0\right) = A^0 \ ;$$

$$o_2\left(\{A_1,\ldots,A_k\}\vdash\{B\}\right) = \begin{cases} A_1 \to (A_2 \to \cdots \to (A_k \to B)\cdots) \ , & \text{if } k \geq 1, \\ B & , \text{if } k = 0; \end{cases}$$

$$o_2\left(\{A_1,\ldots,A_k\}\vdash\varnothing\right) = \begin{cases} A_1 \to (A_2 \to \cdots \to (A_k \to \bot)\cdots) \ , & \text{if } k \geq 1, \\ \bot & , \text{if } k = 0. \end{cases}$$

We shall show that if A^n, $0 \leqslant n \leqslant 1$, is provable in IRP/D, $o_2(A^n)$ is provable in IRP/0. We make an induction on the length of the proof of A^n.

For the basis we have only $\underline{h}^0(\underline{I}^0)$, for which we use $A \to A$. Now suppose for the induction step that if A^n is proved in $\leqslant k$ steps, $o_2(A^n)$ is provable in IRP/0. Let it be proved in $k + 1$ steps. The last step can be :

1) $\underline{A}^1$; then we use $A \to A$;

2) $\underline{D}^1$; then we use $A \to A$; $\underline{D}^2$ is eliminable by Lemma 10;

3) $\underline{C}^1$; then we have

$$\left(C_1 \to (C_2 \to \cdots \to (C_k \to A)\cdots)\right) \to \left(\left(D_1 \to (D_2 \to \cdots \to (D_\ell \to (A \to B))\cdots)\right) \to \right.$$
$$\left. \to (E_1 \to (E_2 \to \cdots \to (E_m \to B)\cdots))\right),$$

$$\{C_1,\ldots,C_k\} \cup \{D_1,\ldots,D_\ell\} = \{E_1,\ldots,E_m\} \ , \ m \geq 0$$
$$\left(\{C_1,\ldots,C_k\} \text{ and } \{D_1,\ldots,D_\ell\} \text{ are not necessarily disjoint}\right);$$

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to show this we make an induction on m:

3.1)   basis: $A \rightarrow ((A \rightarrow B) \rightarrow B)$ is provable in (mp) $\underline{c_{4-5}}$;

3.2)   induction step:

$(A_1 \rightarrow (A_2 \rightarrow A_3)) \rightarrow ((F \rightarrow A_1) \rightarrow ((F \rightarrow A_2) \rightarrow (F \rightarrow A_3)))$,

$(A_1 \rightarrow (A_2 \rightarrow A_3)) \rightarrow ((F \rightarrow A_1) \rightarrow (A_2 \rightarrow (F \rightarrow A_3)))$,

and $(A_1 \rightarrow (A_2 \rightarrow A_3)) \rightarrow (A_1 \rightarrow ((F \rightarrow A_2) \rightarrow (F \rightarrow A_3)))$

are provable in $\text{(mp)} c_1 \underline{c_{4-6}}$;

4) $(\rightarrow)$; then we use $A \rightarrow A$; the order of the sentences in the left set is without importance due to $\underline{c_5}$;

5.1) (&)$\downarrow$; then we use (adj), $\underline{k_4}$ and

$((A\&B) \rightarrow C) \rightarrow (((D \rightarrow A)\&(D \rightarrow B)) \rightarrow (D \rightarrow C))$;

5.2) (&)$\uparrow$; then we use $\underline{k_{1-2}}$ and

$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$;

6.1) (v)$\downarrow$; then we use (adj), $\underline{a_4}$ and

$((A\&B) \rightarrow C) \rightarrow (((D \rightarrow A)\&(D \rightarrow B)) \rightarrow (D \rightarrow C))$;

6.2) (v)$\uparrow$; then we use $\underline{a_{1-2}}$, $\underline{c_6}$, and

$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$;

7)   ($\perp$) ; then we use $A \rightarrow A$;

8.1) (T)$\downarrow$; then we use $A \rightarrow (T \rightarrow A)$ which is provable in $\text{(mp)} v_1 \underline{c_5}$;

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8.2) $(T)^{\uparrow}$; then we use $(T \to A) \to A$ which is provable in

$\underline{(mp)vc}_{4-5}$.

This concludes the induction.

If $n = 0$, $\sigma_2(A^n) = A^n$, and the Lemma follows.

Q.E.D.

From Lemmata 35 and 36 we get immediately

Theorem 5   IRP/D and IRP/O are 0-equivalent.

As a corollary of Theorem 5 we have that $T^1_{\mathcal{er}}$ is not admissible in IRP/D. For suppose it is; then $\{A,B\}\vdash^1\{A\}$ would be provable, and $A \to (B \to A)$ would be provable in IRP/O. But it is known that it isn't. As another corollary we have that (adj) is not horizontalizable in IRP/D.

§48   We shall now consider the replaceability of analytic rules of IRP/D, and some of its extensions, by postulates in 0.

Lemma 20.12, 20.22, 20.3 and 20.4 can be applied in the present context too. We also have

Lemma 37   Let S be the extension of hADIC with the postulates of IRP/O, where (adj) is not in the scope of h.

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37.1 $(\to)\downarrow$ is admissible in S.

37.2 $(\&)\downarrow$ is admissible in S.

37.3 $(\vee)\downarrow$ is admissible in S.

37.4 $(\bot)\downarrow$ is admissible in S.

37.5 $(\top)\downarrow$ is horizontalizable in S .

37.6 $(\neg)_3\downarrow$ is admissible in S.

Demonstration: Except for 37.5, we proceed in principle as for Lemma 23 using a lemma homologous to Lemma 22.

For 37.3 - 37.4 and 37.6 we have that if $\Gamma\vdash^1 \Delta$ is provable in S, Δ must be a singleton.

37.5

$$\dfrac{A^1}{\underline{C^1}} \quad \dfrac{c \to (\top \to c) \quad \text{provable in } (mp)\, c_5 \upsilon_1}{\varnothing \vdash \{c \to (\top \to c)\} \qquad\qquad \underline{h}^0((\underline{mp}))}$$

$$\dfrac{A^2}{\underline{C^2}} \quad \dfrac{\{c\} \vdash \{\top \to c\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{h}(\underline{c}^1)}{\varnothing \vdash^2 \{\{c\} \vdash \{\top \to c\}\} \quad \{\varnothing \vdash \{c\}, \{c\} \vdash \{\top \to c\}\} \vdash^2 \{\varnothing \vdash \{\top \to c\}\}}$$

$$\underbrace{\{\varnothing \vdash \{c\}\} \vdash^2 \{\varnothing \vdash \{\top \to c\}\}}_{①}$$

② : $\{\varnothing \vdash \{\top \to c\}\} \vdash^2 \{\{\top\} \vdash \{c\}\}$ by Lemma 20.3

$$\underline{C^2} \quad \dfrac{① \qquad\qquad ②}{\{\varnothing \vdash \{c\}\} \vdash^2 \{\{\top\} \vdash \{c\}\}.}$$

Q.E.D.

§49

Note that we cannot give $\underline{h}^0\!\!\left(\underline{\text{adj}}\right)$ for S, for otherwise $\{A,B\} \vdash^1 \{A\}$ would be provable. But we could give the following form of $(\&){\downarrow}$

$$\underline{(\text{adj}}^1) \quad \frac{\emptyset \vdash^1 \{A\} \qquad \emptyset \vdash^1 \{B\}}{\emptyset \vdash^1 \{A\&B\}} \quad,$$

and this rule can be in the scope of $\underline{h}$. The rule (adj) is only derivable, and is $\underline{\text{not}}$ horizontalizable in IRP/D *[cf.§47]*.

The rules $(\bot){\uparrow}$ and $(\neg)_3{\uparrow}$ are not admissible in S for the reasons given for 21.3 and 21.42 of Lemma 21.

We can conclude that (T) is replaceable by postulates of IRP/O. If we restrict ourselves to sentences of levels O and 1, $(\rightarrow)$, (&) and (v) are weakly replaceable by postulates of IRP/O.

We also state without demonstration

$\underline{\text{Lemma 38}}$ 38.1 $(\circ)_{1\text{-}2}$ are horizontalizable in $\underline{\text{hADIC}}(\rightarrow)\mathit{ik}_{\underline{1\text{-}4}}$.

38.2 $(+)_{1\text{-}2}$ are horizontalizable in $\underline{\text{hADIC}}(\rightarrow)(\neg)_1\mathit{ia}_{\underline{1\text{-}4}}$.

Hence $(\circ)_{1\text{-}2}$ and $(+)_{1\text{-}2}$ are replaceable by postulates in O.

§49 Consider the system

$$\underline{\text{hADIC}}(\rightarrow)(\&)(\text{v})(\bot)(\text{T})(\neg)_1 \quad,$$

which we shall call "CRP/D".

§49

By Lemma 35, we know that CRP/0 is contained in this system. We can also show

Lemma 39   If a sentence of level 0 is provable in CRP/D, it is provable in CRP/0.

Demonstration: We use the following o_3-translation of sentences of levels 0 and 1

$$o_3(A^0) = A^0 \; ;$$
$$o_3(\{A_1,\dots,A_k\} \vdash \{B_1,\dots,B_\ell\}) = (A_1 \to \cdots \to (A_k \to (\neg B_1 \to \cdots \to (\neg B_\ell \to \bot)\cdots))\cdots),$$
$$k \geq 0, \ell \geq 0;$$

and we make an induction on the length of the proof of A^n, $0 \leq n \leq 1$, in CRP/D, to show that $o_3(A)$ is provable in CRP/0.

<div align="right">Q.E.D.</div>

distr is not provable in CRP/D, since it is known that distr is not provable in CRP/0. The lack of distr is the only significant respect in which CRP/0 differs from R, since we can conservatively extend R with vv_1.

CRP/D cannot be a systematization of a logic of relevant proof , but it could be a systematization of a logic of relevant deduction in a broader sense, if indeed it is not a hybrid system. CRP/D provides a context in which + can be introduced.

The constants ∘ and + given with $(∘)_{1\text{-}2}$ and $(+)_{1.2}$ are not the usual constants ∘ and + treated in connection with R.

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Usually $(A \circ A) \to A$ and $A \to (A + A)$ are not given, and ik_1 and ia_2 are given without proviso. ik_3 and ia_4 are then superfluous, and we can explicitly define $+$ as follows

$$A + B =_{df} \neg A \to B \ .$$

$(A \circ A) \to A$ and $A \to (A + A)$ are usually given in a context where it is possible to formulate a rule of Repetition. And indeed, we have seen that in general rules like $(\circ)_{1-2}$ are better suited for a language like D3. Our only aim in treating $\circ$ and $+$ here was to show that $(\circ)_{1-2}$ and $(+)_{I-2}$ characterize constants different from & and $\vee$. That this is the case is shown by the fact that $(A \circ B) \to A$ and $(A \circ B) \to B$ are not provable in IRP/0 with ik_{1-4} : in this system we have $(A\&B) \to (A \circ B)$ but not the converse. Analogously, $A \to (A + B)$ and $B \to (A + B)$ are not provable in CRP/0 with ia_{1-4}: in this system we have $(A + B) \to (A \vee B)$ but not the converse [v.§50 for references].

In a language like D3 $\circ$ and $+$ could appropriately be analyzed with single analytic rules. There could analyze the usual $\circ$ and $+$ connected with R, with rules corresponding to $(\circ)_1$ and $(+)_1$, provided we did not assume rules of Thinning and Repetition.

In classical logic $\circ$ and $+$ are synonymous with & and $\vee$ respectively, and in intuitionistic logic $\circ$ is synonymous with &. This explains why in the o_1-translation we can use & to perform the function of $\circ$, and $\vee$ to perform the function of $+$. Canonically, when we are picturing sequents in the language O, the constants we need are $\to$, $\circ$, T, $+$ and $\bot$:

§50

§50   [§§36-39]   The single-conclusion property as a distinguishing

mark of intuitionistic sequent systems was discovered by Gentzen

1934.

[§§40-43]   Czermak 1977 has shown some interest in

investigating a sequent calculus of level 1 with the dual,

single-premise property.   Shoesmith & Smiley 1978 (pp.204-206)

consider briefly single-premise single-conclusion natural

deduction calculi.   Converse non-implication in classical logic

is studied in Church 1956.

[§44]   Glivenko's Theorem is the second theorem of Glivenko

1929.   Glivenko's Theorem in connection with KP/O, and Johansson-

-Curry's system D, are treated in Johansson 1936 (pp.124, 129)

and Curry 1950, 1952 (pp.104ff.), 1963 (pp.279, 288).

[§45]   The subscribing technique in natural deduction calculi,

and relevant logic in general, are treated in Anderson & Belnap

1975 (referred to in this section by "op.cit!").

[§46]   The implicational fragment of R was given by Moh Shaw-

-Kwei 1950 and Church 1951 (v. also Church 1951a).   Church 1951

doubts that this system can be extended with conjunction and

disjunction.   The axiomatization $(mp)c_1c_{4-6}$ is in op.cit. (§3);

for R v. op.cit.(Ch.5); for vv_1 v.op.cit.(§27.1.2).   Prawitz 1965

(Ch.7) has proposed a natural deduction calculus for "relevant

implication extended with minimal logic", where "relevant implication"

is the implication of R (Urquhart 1972 has a similar proposal

with different motives).

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The calculus corresponding to IRP/O without $\upsilon\upsilon_1$ is O-
-equivalent with the propositional part of the <u>absolute predicate
calculus</u> of Smirnov 1972 (rechristened "RA" in Smirnov 1979).
Smirnov has also investigated a corresponding natural deduction
calculus and corresponding sequent calculi   of level 1 with a
motivation which seems to be close to ours.   In particular,
Smirnov's sequent rules in one of these calculi are very similar
to our analytic rules.   Smirnov has also investigated a calculus
O-equivalent with CRP/O without $\upsilon\upsilon_1$ (v. Smirnov 1979).   These
two propositional calculi of Smirnov have been shown decidable
by sequent calculi methods (v. Popov 1977; Smirnov 1979).   The
decidability of R is one of the outstanding open questions
of relevant logic (v. Meyer 1979).

The independence of <u>distr</u> in the positive fragment of R
is shown in op.cit. (§29.9).   By trying to prove <u>distr</u> in IRP/D,
it can also be intuitively realized that T_{ex} is needed for the
proof.   It seems that the strongest reason given by Anderson
and Belnap for taking <u>distr</u> in R is their wish to make this system
as close an approximation to classical logic as possible, short
of the "paradoxes of implication" (v. op.cit. §23.3).   But <u>distr</u>
stands apart: for example, in the natural deduction formulation
of R, the rule corresponding to it has to be assumed in addition
to the introduction-elimination rules for & and v (v. op.cit.
§27.2, and Anderson 1963).   (However, cf. op.cit. §25.2.3,where
Meyer and Dunn assert that <u>distr</u> is necessary for showing that
$\dfrac{A \quad \neg A\vee B}{B}$ is an admissible rule in R and related systems. This rule
is admissible also in IRP/O, as well as $\dfrac{A\&(B\vee C)}{(A\&B)\vee C}$. This is
a consequence of the intuitionistic property that A∨B is
provable in IRP/O iff either A or B is, which can be shown
with the help of Smirnov's sequent techniques.)   The

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option to take classical negation for R seems to stem from the
same encompassing tendency, and has not remained unchallenged
(v. remarks in op.cit. and in Meyer 1974).

Remarks on the conservativeness of extensions of fragments
of R can be found in op.cit.   (§28.3.2).

[§49]   Remarks on the constants ∘ and + in connection with
R can be found in op.cit. (§§27.1.4, 29.3.1).   (A∘A) → A and
A → (A + A) are assumed in connection with RM (R-mingle).   If
(A∘A) → A were assumed in connection with R, with ik_1 without
proviso and with contraction, we would get the mingle axiom
A → (A → A).   Our proviso for ik_1 serves to prevent this.

§51

Chapter 6

MODAL CONSTANTS

1   After introducing the propositional modal constants □, ◇ and ⊣ ,

we propose a general definition of modal constants, and we give

analytic rules of level 2 for the constants □, ◇ and ⊣ .

Next we consider two analytic extensions with these rules,

one systematizing the S5 propositional modal logic, and the other

systematizing the S4 propositional modal logic. The only difference

between these two systems is that $\underline{T}^2$ is restricted in the second.

We demonstrate the O-equivalence of these systems with

axiomatizations of propositional S5 and S4, and we consider the

replaceability of the analytic rule for the necessity operator

by other postulates, in particular sequent rules of level 1.

We conclude this chapter with some remarks on the use of

systems with sequents of levels higher than 1 and on the choice

we can make of $\underline{T}$-rules of higher levels.

§52

§52   For this chapter we assume that we have in O, in addition
to what we have assumed for Chapter 4, the following propositional
constants

          the necessity operator : $\Box$,

          the possibility operator: $\Diamond$,

which are of the category $\frac{\delta}{\delta}$, and

          strict implication: $\dashv$,

which is of the category $\frac{\delta}{\delta\delta}$.

      These constants will be called "modal constants". As a
general definition of modal constants we propose the following

Definition of modal constants

      A structural constant of O is modal iff it is explicitly
definable in terms of structurally analyzed constants of O
at least one of which is structurally analyzed with an analytic
rule of level 2.

      As a consequence of this definition we have that constants
which are structurally analyzed with analytic rules of level 2 are
modal. We shall see that $\Box$, $\Diamond$ and $\dashv$ can all be considered to
be either primary structural modal constants or secondary structural
modal constants.

      We give the following analytic rules in D1 for the constants
above

$$(\Box) \quad \frac{\Pi \cup \{\emptyset \vdash \{A\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}}{\Pi \qquad \vdash^2 \Sigma \cup \{\Theta \cup \{\Box A\} \vdash \Xi\}} \quad ;$$

$$(\Diamond) \quad \frac{\Pi \cup \{\{A\} \vdash \emptyset\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}}{\Pi \qquad \vdash^2 \Sigma \cup \{\Theta \vdash \Xi \cup \{\Diamond A\}\}} \quad ;$$

$$(\to) \quad \frac{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}}{\Pi \qquad \vdash^2 \Sigma \cup \{\Theta \cup \{A \to B\} \vdash \Xi\}}$$

(substitution is free for "A" and "B" in (→)).

We can show the following

Lemma 40   If $A \to B =_{df} \Box\,'(A \to B)$, (→) is horizontalizable in

hADIC(→)(□).

Demonstration:

$\overbrace{\qquad ① \qquad}$

$\dfrac{A^3}{C^3} \dfrac{\{\emptyset \vdash \{A \to B\}\} \vdash \{\{A\} \vdash \{B\}\} \quad \underline{h}((\to)\uparrow)}{\emptyset \vdash^3 \{①\} \quad \{①, \Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\emptyset \vdash \{A \to B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \quad \underline{h}(C^2)}$

$\{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\emptyset \vdash \{A \to B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\}$

$\underbrace{\qquad\qquad\qquad ② \qquad\qquad\qquad}$

$$\underline{C}^3 \, \textcircled{2} \, \frac{\{\Pi \cup \{\emptyset \vdash \{A \to B\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \vdash^2 \Sigma \cup \{\Theta \cup \{\Box(A \to B)\} \vdash \Xi\}\} \, \underline{h}^2((\Box)\downarrow)}{\{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \vdash^2 \Sigma \cup \{\Theta \cup \{\Box(A \to B)\} \vdash \Xi\}\}} \, ;$$

for the converse we have

$$\frac{A^3}{C^3} \, \frac{\overbrace{\{\{A\} \vdash \{B\}\} \vdash^2 \{\emptyset \vdash \{A \to B\}\}}^{\textcircled{3}} \, \underline{h}^2((\to)\downarrow)}{\emptyset \vdash^3 \{\textcircled{3}\} \quad \{\textcircled{3}, \Pi \cup \{\emptyset \vdash \{A \to B\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \, \underline{h}^2(\underline{C}^2)}$$

$$\underbrace{\{\Pi \cup \{\emptyset \vdash \{A \to B\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\}}_{\textcircled{4}}$$

$$\underline{C}^3 \, \frac{\{\Pi \vdash^2 \Sigma \cup \{\Theta \cup \{\Box(A \to B)\} \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\emptyset \vdash \{A \to B\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\} \, \underline{h}^2((\Box)\downarrow) \quad \textcircled{4}}{\{\Pi \vdash^2 \Sigma \cup \{\Theta \cup \{\Box(A \to B)\} \vdash \Xi\}\} \vdash^3 \{\Pi \cup \{\{A\} \vdash \{B\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\}} \, .$$

Q.E.D.

§52

And, without demonstration, we state the following

<u>Lemma 41</u>   If □A $=_{df}$ T → A, (□) is horizontalizable in <u>hADIC</u>(T)(→).

We can also show

<u>Lemma 42</u>   If ◇A $=_{df}$ ¬□¬A, (◇) is horizontalizable in

<u>hADICT</u>[1](→)(⊥)(□).

<u>Demonstration:</u>   We first show that (¬)$_1$ is horizontalizable

/cf. Lemma 16/

$$
\underline{C}^2 \;\cfrac{\{\Gamma \cup \{A\} \vdash \Delta\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta \cup \{\bot\}\}\;\underline{h}(\underline{T}^\circ) \quad \{\Gamma \cup \{A\} \vdash \Delta \cup \{\bot\}\} \vdash^2 \{\Gamma \vdash \Delta \cup \{\neg A\}\} \;\overset{\wedge}{\underline{h}}((\to)\wedge)}{\{\Gamma \cup \{A\} \vdash \Delta\} \vdash^2 \{\Gamma \vdash \Delta \cup \{\neg A\}\}}\;;
$$

$$
\begin{array}{l}
(\bot)\uparrow \\
\underline{A}^2 \quad \cfrac{\{\bot\} \vdash^2 \{\bot\}\;\overset{0}{\underline{h}}(\underline{I}^\circ)}{\{\bot\} \vdash \varnothing} \\
\underline{C}^2 \;\cfrac{\varnothing \vdash^2 \{\{\bot\} \vdash \varnothing\} \quad \{\Gamma \cup \{A\} \vdash \Delta \cup \{\bot\}\},\{\bot\} \vdash \varnothing\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta\}\;\underline{h}(\underline{C}^\circ)}{\{\Gamma \cup \{A\} \vdash \Delta \cup \{\bot\}\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta\}}
\end{array}
$$

$$
\underline{C}^2\;\cfrac{\{\Gamma \vdash \Delta \cup \{\neg A\}\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta \cup \{\bot\}\}\; \overset{\wedge}{\underline{h}}((\to)\wedge)}{\{\Gamma \vdash \Delta \cup \{\neg A\}\} \vdash^2 \{\Gamma \cup \{A\} \vdash \Delta\}}\;.
$$

Then we have

①

$$\dfrac{A^3\ \{\emptyset\vdash\{\neg A\}\vdash\{\{A\}\vdash\emptyset\}\ \underline{h}^1((\neg)_\wedge\uparrow)}{C^3\ \emptyset\vdash^3\{①\}\ \{①,\Pi\cup\{\{A\}\vdash\emptyset\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\cup\{\emptyset\vdash\{\neg A\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\}}\ \ \underline{h}^2(\underline{C}^2)$$

$$\{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\cup\{\emptyset\vdash\{\neg A\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\}$$

③   ②

$$\dfrac{A^3\ \{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\vdash^2\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\ \underline{h}^1((\neg)_\wedge\downarrow)}{C^3\ \emptyset\vdash^3\{③\}\ \{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\},③\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\}}\ \ \underline{h}^2(\underline{C}^2)$$

$$\{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\}$$

④

$$\dfrac{C^3\ ②\ \ \{\Pi\cup\{\emptyset\vdash\{\neg A\}\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}\ \underline{h}^2((\Box)\downarrow)}{C^3\ \{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}\ \ ④}$$

$$\{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\};$$

⑤

$$\dfrac{A^3\ \{\{A\}\vdash\emptyset\}\vdash^2\{\emptyset\vdash\{\neg A\}\}\ \underline{h}^1((\neg)\downarrow)}{C^3\ \emptyset\vdash^3\{⑤\}\ \{⑤,\Pi\cup\{\emptyset\vdash\{\neg A\}\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\vdash^3\{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}}\ \ \underline{h}^2(\underline{C}^2)$$

$$\{\Pi\cup\{\emptyset\vdash\{\neg A\}\}\vdash^2\Sigma\cup\{\Theta\vdash\Xi\}\}\vdash^3\{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}$$

⑦   ⑥

$$\dfrac{A^3\ \{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\vdash^2\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\ \underline{h}^1((\neg)_\wedge\uparrow)}{C^3\ \emptyset\vdash^3\{⑦\}\ \{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\},⑦\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}}\ \ \underline{h}^2(\underline{C}^2)$$

$$\{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\}\vdash^3\{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}$$

⑧

$$\dfrac{C^3\ ⑧\ \ \{\Pi\vdash^2\Sigma\cup\{\Theta\cup\{\Box\neg A\}\vdash\Xi\}\}\vdash^3\{\Pi\cup\{\emptyset\vdash\{\neg A\}\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\ \underline{h}^2((\Box)\uparrow)}{C^3\ \{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\}\vdash^3\{\Pi\cup\{\emptyset\vdash\{\neg A\}\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}\ \ ⑥}$$

$$\{\Pi\vdash^2\Sigma\cup\{\Theta\vdash^2\Xi\cup\{\neg\Box\neg A\}\}\}\vdash^3\{\Pi\cup\{\{A\}\vdash\emptyset\}\vdash^2\Sigma\cup\Theta\vdash\Xi\}.$$

Q.E.D.

And, without demonstration, we state the following

<u>Lemma 43</u>   If $\square A =_{df} \neg \lozenge \neg A$, $(\square)$ is horizontalizable in $\underline{hADICT}^1 (\rightarrow) (\bot) (\lozenge)$.

By Lemma 40, in all the contexts concerning $\underline{T}$ we have considered we can assume that $\dashv$ is explicitly defined in terms of $\rightarrow$ and $\square$. Since in this chapter we shall be concerned with contexts which have $\underline{T}^1$, we can assume, in virtue of Lemma 42, that $\lozenge$ is explicitly defined in terms of $\rightarrow$, $\bot$ and $\square$. So, in the demonstrations below it is enough if we consider the analytic rule $(\square)$.   But we shall mention also contexts in which the explicit definitions of Lemmata 42 and 43 are not available.

§53   The system

$\underline{hADICT} (\rightarrow) (\&) (\lor) (\bot) (T) (\square) (\lozenge) (\dashv)$,

i.e. CP/D extended with $(\square)$, $(\lozenge)$ and $(\dashv)$, will be called "S5P/D".

The subsystem of S5P/D where $\underline{T}^2$ and $\underline{h}^2 (\underline{T}^2)$ are replaced by $\underline{T}^2_{\mathcal{S}\mathcal{r}}$ and $\underline{h}^2 (\underline{T}^2_{\mathcal{S}\mathcal{r}})$ will be called "S4P/D".

Note that in S4P/D the rule $\underline{T}^2_{\mathcal{R}}$ would have the same effect as $\underline{T}^2_{\mathcal{S}\mathcal{r}}$; only if sentences of level 1 were taken to constitute an object language and $\bot$ or $\neg$ of level 1 were analyzed by analytic rules of level 2, a difference would arise.   (As a matter of fact, we shall see that rejecting $\underline{T}^2$ completely from S4P/D would not alter the provable sentences of level 0.)

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It can easily be shown that in S4P/D, sequents of level 2 have the single-conclusion property, or more precisely, that they always have a singleton right set. So, in applying (□), (◊) and (→), Σ will always be empty.

In virtue of Lemma 10, $\underline{D}$ of all levels $\geqslant 3$ is eliminable from S5P/D and S4P/D.

Consider now the following rule

$$(\underline{nec}) \quad \frac{A}{\Box A} \,,$$

and the following formulae

$\underline{\ell_1}$ $\Box(A \to B) \to (\Box A \to \Box B)$;

$\underline{\ell_2}$ $\Box A \to A$;

$\underline{\ell_3}$ $\neg \Box A \to \Box \neg \Box A$;

$\underline{\ell_4}$ $\Box A \to \Box \Box A$.

The system CP/0 extended with $(\underline{nec})$ and $\underline{\ell_{1-3}}$, together with the explicit definitions of Lemmata 40 and 42, will be called "S5P/0". The system got from "S5P/0" by replacing $\underline{\ell_3}$ by $\underline{\ell_4}$ will be called "S4P/0". It is well known that $\underline{\ell_4}$ is provable in S5P/0, but that $\underline{\ell_3}$ is not provable in S4P/0.

We shall now demonstrate that S5P/D and S5P/0, and S4P/D and S4P/0 are 0-equivalent. In this demonstration we shall also mention the rule

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$$(\underline{nec}^1) \quad \frac{\emptyset \vdash^1 \{A\}}{\emptyset \vdash^1 \{\Box A\}}$$

[cf. with the rule $(\underline{adj}^1)$ in §48].

Lemma 44   44.1 $(\underline{nec}^1)$ is horizontalizable, $(\underline{nec})$ is

derivable and $\ell_{1-2}\ell_4$ are provable in $\underline{hADIC}(\rightarrow)(\Box)$.

  44.2 $\underline{\ell_3}$ is provable in $\underline{hADICT}^{\leqslant 2}(\rightarrow)(\bot)(\Box)$.

Demonstration:

44.1 $\underline{\underset{(\Box)\uparrow}{A^2}} \dfrac{\{\Box A\} \vdash \{\Box A\} \quad \underline{h}^0(\underline{I}^0)}{\dfrac{\emptyset \vdash^2 \{\{\Box A\} \vdash \{\Box A\}\}}{\{\emptyset \vdash \{A\}\} \vdash \{\emptyset \vdash \{\Box A\}\}}} \; ; \qquad \underset{\underline{D}^1}{\overset{A^1}{(\underline{nec}^1)}} \dfrac{\dfrac{\dfrac{A}{\emptyset \vdash \{A\}}}{\emptyset \vdash \{\Box A\}}}{\Box A} \; ;$

$\underline{C}^2 \dfrac{\{\emptyset \vdash \{A \rightarrow B\}\} \vdash^2 \{\{A\} \vdash \{B\}\} \quad \underline{h}^1((\rightarrow)^1) \quad \{\emptyset \vdash \{A\}, \{A\} \vdash \{B\}\} \vdash^2 \{\emptyset \vdash \{B\}\} \quad \overset{\underline{h}^2(\underline{C}^1)}{}}{\underbrace{\{\emptyset \vdash \{A \rightarrow B\}, \emptyset \vdash \{A\}\} \vdash^2 \{\emptyset \vdash \{B\}\}}_{\textcircled{1}}}$

$\underline{C}^2 \dfrac{\textcircled{1} \qquad\qquad \{\emptyset \vdash \{B\}\} \vdash^2 \{\emptyset \vdash \{\Box B\}\} \quad \underline{h}^1((\underline{nec}^1))}{\{\emptyset \vdash \{A \rightarrow B\}, \emptyset \vdash \{A\}\} \vdash^2 \{\emptyset \vdash \{\Box B\}\}}$

 2 applications of $(\Box)\downarrow$

$\underline{D}^2 \dfrac{\emptyset \vdash^2 \{\{\Box(A \rightarrow B), \Box A\} \vdash \{\Box B\}\}}{\{\Box(A \rightarrow B), \Box A\} \vdash \{\Box B\}}$

 2 applications of $(\rightarrow)\downarrow$, and $\underline{D}^1$

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \; ;$$

$$(\square)\!\downarrow \quad \underline{D^2} \quad \cfrac{\cfrac{\cfrac{\cfrac{\{\emptyset \vdash \{A\}\} \vdash^2 \{\emptyset \vdash \{A\}\} \quad \underline{h}^{\backprime}(\underline{I}^{\backprime})}{\emptyset \vdash^2 \{\{\square A\} \vdash \{A\}\}}}{\{\square A\} \vdash \{A\}}}{(\rightarrow)\!\downarrow \text{ and } \underline{D}^{\backprime}}}{\square A \rightarrow A} \; ;$$

$$\underline{C}^2 \quad \cfrac{\cfrac{\{\emptyset \vdash \{A\}\} \vdash^2 \{\emptyset \vdash \{\square A\}\} \quad \underline{h}^{\backprime}((\underline{nec}^{\backprime})) \qquad \{\emptyset \vdash \{\square A\}\} \vdash^2 \{\emptyset \vdash \{\square\square A\}\} \quad \underline{h}^{\backprime}((\underline{nec}^{\backprime}))}{(\square)\!\downarrow \quad \underline{D}^2 \quad \cfrac{\cfrac{\cfrac{\{\emptyset \vdash \{A\}\} \vdash^2 \{\emptyset \vdash \{\square\square A\}\}}{\emptyset \vdash^2 \{\{\square A\} \vdash \{\square\square A\}\}}}{\{\square A\} \vdash \{\square\square A\}}}{(\rightarrow)\!\downarrow \text{ and } \underline{D}^{\backprime}}}}{\square A \rightarrow \square\square A}.$$

44.2   For the demonstration that $(\neg)_1$ is horizontalizable

γ. the demonstration of Lemma 42 (in fact we need only $(\neg)_3$).

We have

①: $\{\{\square A\} \vdash \emptyset\} \vdash^2 \{\emptyset \vdash \{\neg\square A\}\} \quad \underline{h}^{\backprime}((\neg)_1\!\downarrow)$

②: $\{\emptyset \vdash \{\neg\square A\}\} \vdash^2 \{\emptyset \vdash \{\square\neg\square A\}\} \quad \underline{h}^{\backprime}((\underline{nec}^{\backprime}))$

③: $\{\emptyset \vdash \{\square A\}\} \vdash^2 \{\emptyset \vdash \{\square A, \square\neg\square A\}\} \quad \underline{h}^{\backprime}(\underline{T}^{\backprime})$

④: $\{\emptyset \vdash \{\square\neg\square A\}\} \vdash^2 \{\emptyset \vdash \{\square A, \square\neg\square A\}\} \quad \underline{h}^{\backprime}(\underline{T}^{\backprime})$

$(¬)_4\uparrow$ $\dfrac{\{¬□A\} \vdash \{¬□A\}}{}$ $\underline{h}^0(I^0)$

$\underline{A}^2$ $\dfrac{\{□A, ¬□A\} \vdash \varnothing}{}$

$\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\{□A, ¬□A\} \vdash^2 \varnothing\}}{}$ $\underline{h}^1(C^1)$

$\dfrac{\{\varnothing \vdash \{□A, □¬□A\}, \{□A, ¬□A\} \vdash \varnothing\} \vdash^2 \{\{¬□A\} \vdash \{□¬□A\}\}}{\{\varnothing \vdash \{□A, □¬□A\}\} \vdash^2 \{\{¬□A\} \vdash \{□¬□A\}\}}$

⑤

I^2 $\dfrac{\{\varnothing \vdash \{A\}\} \vdash^2 \{\varnothing \vdash \{□A\}\}}{}$ $\underline{h}^4((\underline{nec}^1))$

$(□)\downarrow$ $\dfrac{\{\varnothing \vdash \{A\}\} \vdash^2 \{\varnothing \vdash \{□A\}, \varnothing \vdash \varnothing\}}{}$

$\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\varnothing \vdash \{□A\}, \{□A\} \vdash \varnothing\}}{}$ ①

$|\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\varnothing \vdash \{□A\}, \varnothing \vdash \{¬□A\}\}}{}$ ②

$|\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\varnothing \vdash \{□A\}, \varnothing \vdash \{□¬□A\}\}}{}$ ③

$|\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\varnothing \vdash \{□A, □¬□A\}, \varnothing \vdash \{□¬□A\}\}}{}$ ④

$|\underline{C}^2$ $\dfrac{\varnothing \vdash^2 \{\varnothing \vdash \{□A, □¬□A\}\}}{}$ ⑤

$|\underline{D}^2$ $\dfrac{\varnothing \vdash^2 \{\{¬□A\} \vdash \{□¬□A\}\}}{}$

$\dfrac{\{¬□A\} \vdash \{□¬□A\}}{}$

$\dfrac{(\rightarrow)\downarrow \text{ and } \underline{D}^1}{}$

$¬□A \rightarrow □¬□A.$

Q.E.D.

<u>Lemma 45</u>   45.1   If a sentence of level 0 is provable in S5P/D ,
it is provable in S5P/0.

45.2   If a sentence of level 0 is provable in S4P/D, it
is provable in S4P/0.

<u>Demonstration</u> :   Consider the following σ_4-translation

$$\sigma_4(A^0) = A^0 ;$$

$\hat{\sigma}_4(\Gamma)$ and $\check{\sigma}_4(\Gamma)$ are obtained by substituting everywhere
"σ_4" for "σ" in the clauses for $\hat{\sigma}(\Gamma)$ and $\check{\sigma}(\Gamma)$ [v§15];

$$\sigma_4(\Gamma \vdash^1 \Delta) = \square(\hat{\sigma}_4(\Gamma) \rightarrow \check{\sigma}_4(\Delta)) ;$$
$$\sigma_4(\Gamma \vdash^{n+2} \Delta) = \hat{\sigma}_4(\Gamma) \rightarrow \check{\sigma}_4(\Delta) , \; n \geqslant 0.$$

45.1 Suppose now that a sentence A^n, $n \geqslant 0$, is provable in
S5P/D. We can show by an induction on the length of
the proof of A^n that $\sigma_4(A^n)$ is provable in S5P/0.

For the basis we have for cases 1) - 8) all the formulae of
cases 1) - 8) of the basis in the induction of the demonstration
of Lemma 19, and moreover

$$\frac{A \rightarrow B}{\square A \rightarrow \square B} \quad , \text{ derivable in (mp)(nec) } \ell_1$$

and $\square(A\&B) \leftrightarrow (\square A \& \square B)$, provable in S4P/0.   The remaining cases
are :

9.1) $\underline{h}^2((\square)\dashv);$ then we use

$$((B \,\&\, \Box(T \to A)) \to (C \lor \Box(D \to E))) \to (B \to (C \lor \Box((D \,\&\, \Box A) \to E))) \;,$$
$$(\Box(T \to A) \to (C \lor \Box(D \to E))) \to (T \to (C \lor \Box((D \,\&\, \Box A) \to E))) \;,$$
$$((B \,\&\, \Box(T \to A)) \to \Box(D \to E)) \to (B \to \Box((D \,\&\, \Box A) \to E)) \;,$$
$$(\Box(T \to A) \to \Box(D \to E)) \to (T \to \Box((D \,\&\, \Box A) \to E)) \;,$$

which are all provable in S5P/O [but not in S4P/O:

the S5-diagram in the style of Hughes & Cresswell 1968

showing the S5-validity of the first formula is

w_1 | $((B \,\&\, \overset{*}{\Box}(T \to A)) \to (C \lor \Box(D \to E))) \to (B \to (C \lor \Box((D \,\&\, \overset{*}{\Box}A) \to E)))$
below: 1 1 1 1 1 1 1 0 1 1 1 0 1 0 0 0 0 1 1

w_2 | $(D \,\&\, \overset{*}{\Box}A) \to E \;;\; D \to E$
below: 1 1 1 1 0 0 1 1 0 ;
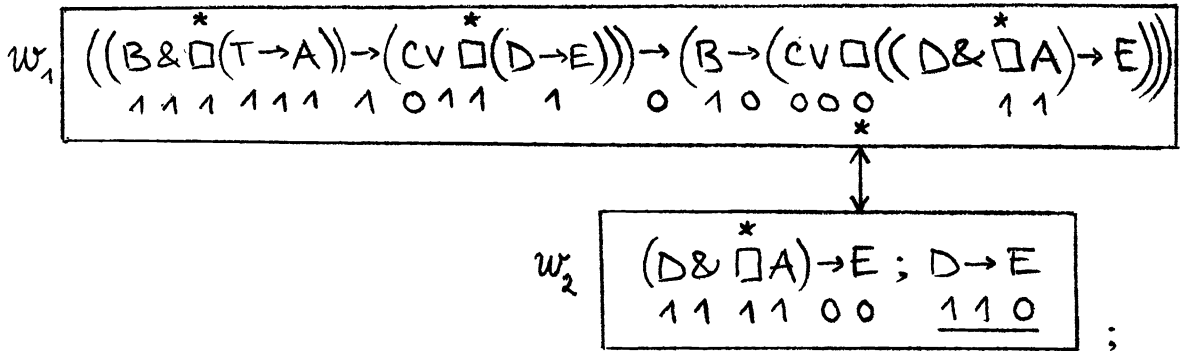
the S4-diagram showing the S4-invalidity of the third formula is

w_1 | $((B \,\&\, \Box(T \to A)) \to \Box(D \to E)) \to (B \to \Box((D \,\&\, \Box A) \to E))$
below: 1 0 0 1 0 0 1 0 1 0 0 0 1 0 0 1 0 0 0 1 0

w_2 | $(D \,\&\, \overset{*}{\Box}A) \to E$
below: 1 1 1 1 0 0

J.
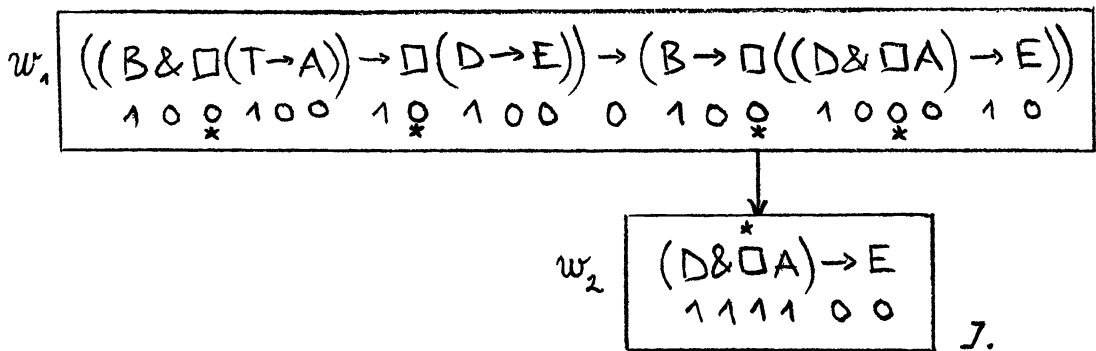
9.2) $\underline{h}^2((\square)\uparrow)$; then we use the converses of the formulae of $9.1)$

which are all provable in S4P/O.

For the induction step suppose that if A^n is proved in

$\leqslant k$ steps, $o_4(A^n)$ is provable in S5P/O. Let it be proved in

$k + 1$ steps. The last step can be:

1) $\underline{A}^n$; then we use A → (T → A), (mp) and (nec);

2) $\underline{D}^{n+1}$; then we use (T → A) → A, $\square$A → A and (mp);

for the other cases we use the formulae of the basis, $\underline{k}_{1-3}$

and (mp). This concludes the induction. If $n = 0$, $o_4(A^n) = A^n$,

and 45.1 follows.

45.2 Suppose that A^n, $0 \leqslant n \leqslant 2$, is provable in S4P/D.

We can show by an induction on the length of the proof of A^n

that $o_4(A^n)$ is provable in S4P/O.

For the basis we proceed as for 45.1 omitting cases 9.1) and

9.2).

In the induction step we prodeed as for 45.1 for all the

cases where the last step is not $(\square)\downarrow$. If the last step is $(\square)\downarrow$,

we know that Σ must be empty, and we have by the induction

hypothesis that $(\hat{o}_4(\Pi)\&\square(T \rightarrow A)) \rightarrow \square(\hat{o}_4(\Theta) \rightarrow \check{o}_4(\Xi))$ is provable

in S4P/O. It follows that $(\hat{o}_4(\Pi)\&\square A) \rightarrow (\hat{o}_4(\Theta) \rightarrow \check{o}_4(\Xi))$ is provable

too, and using CP/O we get that $\hat{o}_4(\Pi) \rightarrow ((\hat{o}_4(\Theta)\&\square A) \rightarrow \check{o}_4(\Xi))$ is

provable. If $\Pi = \{B_1^1, \ldots, B_k^1\}$, $k \geqslant 1$, then $\hat{o}_4(\Pi) = \square C_1 \&\ldots\& \square C_k$,

where $\square C_i = o_4(B_i^1)$, $1 \leqslant i \leqslant k$. Then let $D = ((\hat{o}_4(\Theta)\&\square A) \rightarrow \check{o}(\Xi))$;

we have

$$(\text{nec}) \quad \frac{(\Box C_1 \,\&...\&\, \Box C_k) \to D}{\Box((\Box C_1 \,\&...\&\, \Box C_k) \to D)}$$

$$\frac{\ell_1 \text{ and } (\text{mp})}{\Box(\Box C_1 \,\&...\&\, \Box C_k) \to \Box D}$$

we use $(\Box C_1 \,\&...\&\, \Box C_k) \to \Box(\Box C_1 \,\&...\&\, \Box C_k)$
which is provable in S4P/O

$$\overline{(\Box C_1 \,\&\,...\&\Box C_k) \to \Box D.}$$

If $\Pi = \emptyset$, we have:

$$(\text{nec}) \quad \frac{\dfrac{T \to D}{D}}{\Box D}$$

$$\overline{T \to \Box D \; .}$$

Note that $\underline{D}^3$ is eliminable from S4P/D. This concludes the induction.

If $n = 0$, $o_4(A^n) = A^n$, and 45.2 follows.

$$\text{Q.E.D.}$$

We could have used for the demonstration of Lemma 45 the o_5-translation which is obtained by substituting "o_5" for "o_4" in all the clauses for o_4 except the last and replacing this last clause by

$$o_5(\Gamma \vdash^{n+2} \Delta) = \Box(\hat{o}_5(\Gamma) \to \check{o}_5(\Delta)), \; n \geqslant 0,$$

in order to show that if A^n, $n \geqslant 0$, is provable in S5P/D (or S4P/D), $o_5(A^n)$ is provable in S5P/O (or S4P/O). But the o_4--translation is somewhat more economical, and it serves to mark a

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difference in the way how S5P/O and S4P/O can represent their deductive meta-logic.

From Lemmata 44 and 45, Theorem 1, and the fact that we can demonstrate Theorem 1 when we replace CP/D with the subsystem of CP/D without $\underline{T}^2$ and $\underline{h}^2(\underline{T}^2)$, we get immediately

Theorem 6   6.1 S5P/D and S5P/O are O-equivalent.

6.2 S4P/D and S4P/O are O-equivalent.

As an immediate corollary of Theorem 6 we have that (nec) is not horizontalizable in S5P/D and S4P/D.

§54   We shall now consider whether the analytic rule (□) is replaceable by other postulates, and in particular other analytic rules. In order to prevent the non-replaceability of analytic rules of level 1 to influence our treatment, we shall consider only the replaceability of (□), with analytic rules for the non--modal propositional constants given.

First we show the following

Lemma 46   (□)↑ is horizontalizable in hAC(nec^1).

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Demonstration: ① : $\{\varnothing \vdash \{A\}\} \vdash^2 \{\varnothing \vdash \{\square A\}\}$ h((nec))

$$A^3 \frac{①}{\varnothing \vdash \{①\}}$$

$$C^3 \qquad \frac{\{①, \{\varnothing \vdash \{\square A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}\} \vdash^3 \{\varnothing \vdash \{A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}}{\{\{\varnothing \vdash \{\square A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}\} \vdash^3 \{\varnothing \vdash \{A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}}}$$ h(C²)

②

$$A^3 \frac{\{\varnothing \vdash \{\square A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}}{\varnothing \vdash^3 \{\{\varnothing \vdash \{\square A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}\}}}$$ h(C¹) ②

$$C^3 \qquad \frac{}{\varnothing \vdash^3 \{\{\varnothing \vdash \{A\}, \Theta \cup \{\square A\} \vdash \Xi\} \vdash^2 \{\Theta \vdash \Xi\}\}}$$

③

$$\varnothing \vdash^3 \{③\} \quad \{\sqcap \vdash^2 \Sigma \cup \{\Theta \cup \{\square A\} \vdash \Xi\}, ③\} \vdash^3 \{\sqcap \cup \{\varnothing \vdash \{A\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\}$$ h(C²)

$$C^3 \frac{}{\{\sqcap \vdash^2 \Sigma \cup \{\Theta \cup \{\square A\} \vdash \Xi\}\} \vdash^3 \{\sqcap \cup \{\varnothing \vdash \{A\}\} \vdash^2 \Sigma \cup \{\Theta \vdash \Xi\}\}}.}$$

Q.E.D.

So we can replace (□)↑ by (nec¹); but not by (nec) (which cannot be horizontalizable).

Next consider the following double-line rules

$$(\square_{S5}) \quad \frac{\Gamma \vdash^1 \Delta \cup \{A\}}{\Gamma \vdash^1 \Delta \cup \{\square A\}} \quad ,$$

provided the following is satisfied: if G∈Γ, G = □B, for some B, and if D∈Δ, D = □C, for some C ;

§54

$$(\square_{S4}) \quad \dfrac{\Gamma \vdash^1 \{A\}}{\Gamma \vdash^1 \{\square A\}} \qquad ,$$

provided the following is satisfied: if $G \in \Gamma$, $G = \square B$, for some

B.

Since we have made no requirement concerning the occurrence

of a constant analyzed, or concerning the referring to such a

constant, in the proviso of an analytic rule, $(\square_{S5})$ and $(\square_{S4})$ could

be taken as analytic rules. Note however that they differ in

this respect from the other analytic rules used up to now. We

shall also see that $(\square_{S5})$ and $(\square_{S4})$ cannot serve for the structural

analysis of $\square$ since they don't guarantee the uniqueness of $\square$

[v.§82].

We can show the following

Lemma 47   47.1 $(\square_{S5})$ is horizontalizable in hADICT$(\rightarrow)$ $(\bot)$ $(\text{nec}^1)\ell_{1\text{-}3}$

47.2 (nec^1) is horizontalizable and $\ell_{1\text{-}3}$ are provable in

hADICT$(\rightarrow)$ $(\bot)$ $(\square_{S5})$.

Demonstration: 47.1   We shall only sketch this demonstration

for the case $\Gamma = \{\square B\}$ and $\Delta = \{\square C\}$. The general case can be

obtained by repeated appeals to the same principles.

For $(\square_{S5})\!\downarrow$ we have:

① : $\{\{\square B\} \vdash \{\square C, A\}\} \vdash^2 \{\{\square B, \neg\square C\} \vdash \{A\}\}$ $\underline{h}^1((\neg)_2\!\downarrow)$;

② : $\{\{\square B, \neg\square C\} \vdash \{A\}\} \vdash^2 \{\varnothing \vdash \{\square B \to (\neg\square C \to A)\}\}$
    provable with $\underline{h}^1$, $(\to)\!\downarrow$ and $\underline{C}^2$;

③ : $\{\varnothing \vdash \{\square B \to (\neg\square C \to A)\}\} \vdash^2 \{\varnothing \vdash \{\square(\square B \to (\neg\square C \to A))\}\}$
    $\underline{h}^1((\text{nec}^1)$;

next we can prove in $(\underline{mp})(\underline{nec})\,\underline{c}_{1\text{-}3}\,\&\,\underline{l}_{1\text{-}3}$

$$\underbrace{\square(\square B \to (\neg\square C \to A))}_{\mathcal{A}} \to \underbrace{(\square B \to (\neg\square C \to \square A))}_{\mathcal{B}} ,$$

and we have:

$$\underline{C}^2 \; \dfrac{\dfrac{\dfrac{\mathcal{A} \to \mathcal{B}}{\underline{A}^1,\,(\to)\!\uparrow\text{ and }\underline{A}^2}}{\varnothing \vdash^2 \{\{\mathcal{A}\} \vdash \{\mathcal{B}\}\}} \qquad \{\varnothing \vdash \{\mathcal{A}\},\{\mathcal{A}\} \vdash \{\mathcal{B}\}\} \vdash^2 \{\varnothing \vdash \{\mathcal{B}\}\}\;\underline{h}^1(\underline{C}^1)}{\{\varnothing \vdash \{\mathcal{A}\}\} \vdash^2 \{\varnothing \vdash \{\mathcal{B}\}\}} \text{ , i.e.}$$

④ : $\{\varnothing \vdash \{\square(\square B \to (\neg\square C \to A))\}\} \vdash^2 \{\varnothing \vdash \{\square B \to (\neg\square C \to \square A)\}\}$;

⑤ : $\{\varnothing \vdash \{\square B \to (\neg\square C \to \square A)\}\} \vdash^2 \{\{\square B, \neg\square C\} \vdash \{\square A\}\}$
    provable with $\underline{h}^1$, $(\to)\!\uparrow$ and $\underline{C}^2$;

⑥ : $\{\{\square B, \neg\square C\} \vdash \{\square A\}\} \vdash^2 \{\{\square B\} \vdash \{\square C, \square A\}\}\,\underline{h}^1((\neg)_2\!\uparrow)$.
    Then we use ① - ⑥ and $\underline{C}^2$.

For $(\square_{S5})\!\uparrow$ we have: $\dfrac{\dfrac{\square A \to A\;\;\underline{l}_2}{\underline{A}^1,\,(\to)\!\uparrow\text{ and }\underline{A}^2}}{\underbrace{\varnothing \vdash^2 \{\{\square A\} \vdash \{A\}\}}_{⑦}}$

$$\underline{C}^2 \; \dfrac{⑦ \qquad \{\{\square B\} \vdash \{\square C, \square A\}, ⑦\} \vdash^2 \{\{\square B\} \vdash \{\square C, A\}\}\;\underline{h}^1(\underline{C}^1)}{\{\{\square B\} \vdash \{\square C, \square A\}\} \vdash^2 \{\{\square B\} \vdash \{\square C, A\}\}}.$$

47.2 (<u>nec⁴</u>) is an instance of $(\square_{SS})\!\downarrow$;

$$(\square_{SS})\!\uparrow \;\;\underline{C^4} \;\; \frac{\{\square A\} \vdash^4 \{\square A\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\square A\} \vdash^4 \{A\}} \qquad (\square_{SS})\!\uparrow \;\; (\rightarrow)\!\uparrow \;\; \frac{\dfrac{\{\square(A\rightarrow B)\} \vdash^4 \{\square(A\rightarrow B)\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\square(A\rightarrow B)\} \vdash^4 \{A\rightarrow B\}}}{\{\square(A\rightarrow B), A\} \vdash^4 \{B\}}$$

$$(\square_{SS})\!\downarrow \;\; \frac{\{\square(A\rightarrow B), \square A\} \vdash^4 \{\underline{B}\}}{\{\square(A\rightarrow B), \square A\} \vdash^4 \{\square B\}}$$

2 applications of $(\rightarrow)\!\downarrow$, and $\underline{D}^4$

$$\square(A\rightarrow B) \rightarrow (\square A \rightarrow \square B) \; ;$$

$$(\square_{SS})\!\uparrow \;\; \frac{\{\square A\} \vdash^4 \{\square A\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\square A\} \vdash^4 \{A\}}$$

$(\rightarrow)\!\downarrow$ and $\underline{D}^4$

$$\square A \rightarrow A \; ;$$

$$\underline{T}^4 \;\; (\rightarrow)\!\downarrow \;\; (\square_{SS})\!\downarrow \;\; \underline{C}^4 \;\; \frac{\dfrac{\dfrac{\{\square A\} \vdash^4 \{\square A\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\square A\} \vdash^4 \{\square A, \bot\}}}{\varnothing \vdash^4 \{\square A, \neg\square A\}}}{\varnothing \vdash^4 \{\square A, \square\neg\square A\}} \qquad (\rightarrow)\!\uparrow \;\; (\bot)\!\uparrow \;\; \frac{\dfrac{\{\neg\square A\} \vdash^4 \{\neg\square A\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\neg\square A, \square A\} \vdash^4 \{\bot\}}}{\{\neg\square A, \square A\} \vdash^4 \varnothing}$$

$$\{\neg\square A\} \vdash^4 \{\square\neg\square A\}$$

$(\rightarrow)\!\downarrow$ and $\underline{D}^4$

$$\neg\square A \rightarrow \square\neg\square A \; .$$

Q.E.D.

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<u>Lemma 48</u>   48.1 ($\square_{S4}$) is horizontalizable in <u>hADIC</u>($\rightarrow$) (nec^1)$\ell_{1-2}\ell_4$.

48.2 (nec^1) is horizontalizable and $\ell_{1-2}\ell_4$ are provable in <u>hADIC</u>($\rightarrow$) ($\square_{S4}$).

<u>Demonstration:</u>   We shall only consider the case $\Gamma = \{\square B\}$. The general case is obtained by repeated appeals to the same principles.

For ($\square_{S4}$)$\downarrow$ we have:

① $\{\{\square B\} \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\square B \rightarrow A\}\}$ $\underline{h}^1((\rightarrow)\downarrow)$;

② $\{\emptyset \vdash^1 \{\square B \rightarrow A\}\} \vdash^2 \{\emptyset \vdash^1 \{\square(\square B \rightarrow A)\}\}$ $\underline{h}^1((\text{nec}^1))$;

next we can prove in (mp) (nec)$c_1 c_{4-6}\ell_{1-2}\ell_4$

$$\square(\square B \rightarrow A) \rightarrow (\square B \rightarrow \square A),$$

and we have as in the demonstration of Lemma 47.1

③ $\{\emptyset \vdash^1 \{\square(\square B \rightarrow A)\}\} \vdash^2 \{\emptyset \vdash^1 \{\square B \rightarrow \square A\}\}$;

④ $\{\emptyset \vdash^1 \{\square B \rightarrow \square A\}\} \vdash^2 \{\{\square B\} \vdash^1 \{\square A\}\}$ $\underline{h}^1((\rightarrow)\uparrow)$.

Then we use ① - ④ and $\underline{c}^2$.

For ($\square_{S4}$)$\uparrow$ we proceed as in the demonstration of Lemma 47.1.

48.2   In addition to what can be found in the demonstration of Lemma 47.2, we have

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$$(\square_{S4})\downarrow \frac{\dfrac{\{\square A\}\vdash^1 \{\square A\} \qquad \underline{h}^0(\underline{I}^0)}{\{\square A\}\vdash^1 \{\square\square A\}}}{\dfrac{(\rightarrow)\downarrow \quad \text{and} \quad \underline{D}^1}{\square A \rightarrow \square\square A \; .}}$$

<div align="right">Q.E.D.</div>

It follows from Lemmata 44 and 47 that $(\square_{S5})$ is horizontalizable in hADICT$(\rightarrow)(\perp)(\square)$ (in fact, it is horizontalizable even if we omit $(\rightarrow)$ and $(\perp)$), and from Lemmata 44 and 48 that $(\square_{S4})$ is horizontalizable in hADIC$(\rightarrow)(\square)$ (in fact, it is horizontalizable even if we omit $(\rightarrow)$). We also have by Lemma 46 that $(\square)\uparrow$ can be replaced by $(\square_{S5})\downarrow$, or $(\square_{S4})\downarrow$, in the context of hADICT, or hADIC. But we can show the following

Lemma 49 $(\square)\downarrow$ is not admissible in the extension of CP/O with $(\square_{S5})$.

Demonstration: We show first that every sequent of level 2 provable in this extension is admissible (cf. the demonstration of Lemma 10). In this extension we can prove $\{\emptyset\vdash^1 \{A\}\}\vdash^2 \{\emptyset\vdash^1\{A\},\emptyset\vdash^1\emptyset\}$, but not $\emptyset\vdash^2 \{\emptyset\vdash^1\{A\},\{\square A\}\vdash^1\emptyset\}$, since this second sequent is not admissible.

<div align="right">Q.E.D.</div>

On the other hand, if we add to $(\square)\downarrow$ the proviso $\Sigma = \emptyset$, we can show that $(\square)\downarrow$ is admissible in the extension of the Lemma above. We can also show that $(\square)\downarrow$ is admissible in the extension with $(\square_{S4})$

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of the system called "$C^{S4}P/D$", which is the subsystem of CP/D

where $\underline{T}^2_{-\Delta\hbar}$ and $\underline{h}^2(\underline{T}^2_{-\Delta\hbar}$) replace $\underline{T}^2$ and $\underline{h}^2$ ($\underline{T}^2$). We shall first

demonstrate a more general result.

Consider the rules which are of the following form and

which we shall call "rules of expansion" (E-rules)

$$\underline{E}^{n+2} \quad \frac{\Pi \cup \{\Gamma_1 \vdash^{n+1} \Delta_1, \ldots, \Gamma_k \vdash^{n+1} \Delta_k\} \vdash^{n+2} \{\Theta_1 \vdash^{n+1} \Xi_1, \ldots, \Theta_\ell \vdash^{n+1} \Xi_\ell\}}{\Pi \cup \{\Gamma_1 \cup \Phi_1 \vdash^{n+1} \Delta_1 \cup \Psi_1, \ldots, \Gamma_k \cup \Phi_k \vdash^{n+1} \Delta_k \cup \Psi_k\} \vdash^{n+2} \{\Theta_1 \cup \Phi \vdash^{n+1} \Xi_1 \cup \Psi, \ldots, \Theta_\ell \cup \Phi \vdash^{n+1} \Xi_\ell \cup \Psi\}},$$

$$\Phi = \Phi_1 \cup \cdots \cup \Phi_k, \quad \Psi = \Psi_1 \cup \cdots \cup \Psi_k, \quad n \geqslant 0, \quad k \geqslant 0, \quad \ell \geqslant 0.$$

Lemma 50 E-rules are admissible in CP/D, but are not always

deducible.

Demonstration: If $\ell = 0$, $\underline{E}^{n+2}$ is admissible vacuously, since

no sequent of levels $\geqslant 2$ with an empty right set is provable

in CP/D. If $k = 0$, $\Phi = \Psi = \emptyset$.

If $\ell > 0$ and $k > 0$, we proceed as follows. Let (p) be

the premise and (c) the conclusion of an instance

of $\underline{E}^{n+2}$. We shall show by an induction on the length of the

proof of (p) that (c) is provable if (p) is.

For the basis we have that (p) can be an instance of

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1) $\underline{h}^{n+1}(\underline{I}^{n+1})$;   then (c) is too;

2) $\underline{h}^{n+1}(\underline{C}^{n+1})$; then (c) is too;

3) $\underline{h}^{n+1}(\underline{T}^{n+1})$; then (c) is too;

4) $\underline{h}^1((\rightarrow)\downarrow)$ and $\underline{h}^1((\rightarrow)\uparrow)$; then (c) is too;

5.1) $\underline{h}^1((\&)\downarrow)$; then we have

$$\{\Gamma \cup \Phi_1, \vdash \triangle \cup \Psi_1 \cup \{A\}\} \vdash^2 \{\Gamma \cup \Phi \vdash \triangle \cup \Psi \cup \{A\}\} \; \underline{\hat{h}}(\underline{T}^{\wedge}) \; ;$$
$$\{\Gamma \cup \Phi_2, \vdash \triangle \cup \Psi_2 \cup \{B\}\} \vdash^2 \{\Gamma \cup \Phi \vdash \triangle \cup \Psi \cup \{B\}\} \; \underline{\hat{h}}(\underline{T}^{\wedge}) \; ;$$
$$\underline{\hat{h}}((\&)\downarrow)$$
$$\{\Gamma \cup \Phi \vdash \triangle \cup \Psi \cup \{A\}, \Gamma \cup \Phi \vdash \triangle \cup \Psi \cup \{B\}\} \vdash^2 \{\Gamma \cup \Phi \vdash \triangle \cup \Psi \cup \{A \& B\}\}$$

and we apply $\underline{C}^2$ (with obvious modifications if A = B);

5.2) $\underline{h}^1((\&)\uparrow)$; then (c) is too;

6.1) $\underline{h}^1((\vee)\downarrow)$; we proceed analogously to 5.1);

6.2) $\underline{h}^1((\vee)\uparrow)$; then (c) is too;

7) $\underline{h}^1((\perp)\downarrow)$ and $\underline{h}^1((\perp)\uparrow)$; then we use Lemma 14;

8) $\underline{h}^1((T)\downarrow)$ and $\underline{h}^1((T)\uparrow)$; then we use Lemma 15.

For the induction step suppose that if (p) is proved in
$\leqslant m$ steps, (c) is provable. Let (p) be proved in $m + 1$ steps.
The last step cannot be $\underline{A}^{n+2}$ since $k > 0$, $\underline{D}^{n+3}$ is eliminable,
and $(\rightarrow)$, $(\&)$,$(\vee)$,$(\perp)$ and (T) are excluded since (p) is at least
of level 2. So the last step can be:

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1) $\underline{C}^{n+2}$ $\dfrac{\Sigma_1 \vdash^{n+2} \Sigma_2 \cup \{A^{n+1}\} \qquad \Sigma_3 \cup \{A^{n+1}\} \vdash^{n+2} \Sigma_4}{\Sigma_5 \vdash^{n+1} \Sigma_6}$

$\Sigma_5 \vdash^{n+2} \Sigma_6 = (p)$, $\Sigma_1 \cup \Sigma_3 = \Sigma_5$, $\Sigma_2 \cup \Sigma_4 = \Sigma_6$, Σ_4 cannot be empty; then we apply the induction hypothesis adding Φ_i and Ψ_i to every sequent $\Gamma_i \vdash^{n+1} \Delta_i$, $1 \leqslant i \leqslant k$ which is a member of Σ_1 or Σ_3, and Φ and Ψ to A^{n+1} and to every sequent which is a member of Σ_2 or Σ_4, and we apply $\underline{C}^{n+2}$ to get (c);

2) $\underline{T}^{n+2}$ $\dfrac{\Sigma_1 \vdash^{n+2} \Sigma_2}{\Sigma_3 \vdash^{n+2} \Sigma_4}$, $\Sigma_3 \vdash^{n+2} \Sigma_4 = (p)$, $\Sigma_1 \subseteq \Sigma_3$,

$\Sigma_2 \subseteq \Sigma_4$, Σ_2 cannot be empty; then we apply the induction hypothesis adding Φ_i and Ψ_i to every sequent $\Gamma_i \vdash^{n+1} \Delta_i$, $1 \leqslant i \leqslant k$, which is a member of Σ_1 (if none are, we leave Σ_1 unchanged), and Φ and Ψ to every sequent which is a member of Σ_2, and we apply $\underline{T}^{n+2}$ to get (c).

This concludes the induction and the demonstration that $\underline{E}$-rules are admissible in CP/D.

To show that they are not always deducible, let $n = 0$, and let CP/D be extended with $\underline{h}^1((\underline{nec}^1))$.

Then $\{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}\}$ is provable, but

$\{\{A\} \vdash^1 \{A\}\} \vdash^2 \{\{A\} \vdash^1 \{\Box A\}\}$ is not; otherwise $A \to \Box A$ would be provable in S5P/D, and by Theorem 6 we know that it is not. So, this $\underline{E}$-rule is not admissible in this extension, and hence it is not deducible in it. It follows that it is not deducible in

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CP/D [v. §11].

<div align="right">Q.E.D.</div>

[A demonstration for our examples of §§14 and 16 can be extracted from the preceding demonstration. Since D is admissible in hAICT, every horizontalizable rule is deducible; so, the formula in §14 is not provable.]

E-rules in general are not admissible in the extension of $c^{S4}P/D$ with $(□_{S4})$. This follows from the end of the demonstration of Lemma 50. But the E-rules where $n = 0$, $k \geq 0$, $\ell = 1$, $\Psi = \emptyset$, and where for every i, $1 \leq i \leq k$, if $F \in \Phi_i$, then $F = □A$, for some A, can be shown admissible in this extension. To demonstrate this we proceed in principle as for Lemma 50, with two additional cases in the basis of the induction. Now suppose that

$$\Pi \cup \{\emptyset \vdash^1 \{A\}\} \vdash^2 \Sigma \cup \{\Theta \vdash^1 \Xi\}$$

is provable in this extension. Σ must be empty, and we have that

$$\Pi \cup \{\{□A\} \vdash^1 \{A\}\} \vdash^2 \Sigma \cup \{\Theta \cup \{□A\} \vdash^1 \Xi\}$$

is provable too. Then we use

$$(□_{S4})^{+} \quad \underline{A}^2 \quad \cfrac{\cfrac{\{□A\} \vdash^1 \{□A\} \quad \underline{h}^0(\underline{I}^0)}{\{□A\} \vdash^1 \{A\}}}{\emptyset \vdash^2 \{\{□A\} \vdash^1 \{A\}\}}$$

and apply $\underline{c}^2$ to get that $\Pi \vdash^2 \Sigma \cup \{\Theta \cup \{□A\} \vdash^1 \Xi\}$ is provable. Hence we have

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Lemma 51 $(\Box)\!\downarrow$ is admissible in the extension of $C^{S4}P/D$ with $(\Box_{S4})$.

In the same way we can show that the E-rules with the proviso above would be admissible in the extension of CP/D with $(\Box_{S5})$ and its horizontalizations. Hence we have

Lemma 52 $(\Box)\!\downarrow$ is admissible in the extension of CP/D with $(\Box_{S5})$, provided $\Sigma = \emptyset$.

The situation with the admissibility of $(\Box)\!\downarrow$ in the system with $(\Box_{S5})$ and the system with $(\Box_{S4})$ is analogous to the situation with the admissibility of $(\rightarrow)\!\downarrow$ in the extension of hADICT with the postulates of CP/O, and the admissibility of $(\rightarrow)\!\downarrow$ in the extension of hADICT$_{\Delta\hbar}$ with the postulates of HP/O. There $(\rightarrow)\!\downarrow$ was not admissible in the first system due to the presence of unrestricted $\underline{T}^1$, which gave rise to sequents of level 1 without the single-
-conclusion property [v. Lemma 21.1]. On the other hand, it was shown that $(\rightarrow)\!\downarrow$ is admissible in the second system, and this was due to the single-conclusion property of sequents of level 1 [v. Lemma 31.1]. We have also shown that $(\rightarrow)\!\downarrow$ with the proviso $\Delta = \emptyset$ is admissible in the first system [v. Lemma 23.1].

With $(\Box)\!\downarrow$ we have the following situation. In the system with $(\Box_{S5})$ it is not admissible due to the lack of the single-
-conclusion property of sequents of level 2 [v. Lemma 49].   On the other hand, it is admissible in the system with $(\Box_{S4})$ due to the single-conclusion property of sequents of level 2 [v. Lemma 51].

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It is admissible also in the system with $(\square_{S5})$, provided $\Sigma = \emptyset$ [v. Lemma 52].

We can conclude that $(\square)$ is not weakly replaceable in S5P/D by $(\square_{S5})$ or $(\text{nec}^1)\underline{\ell_{1\text{-}3}}$ even if we restrict ourselves to sentences of levels 0, 1 and 2, and that with a restriction to these sentences, it is weakly replaceable in S4P/D by $(\square_{S4})$ or $(\text{nec}^1)\underline{\ell_{1\text{-}2}\ell_4}$.

5   With modal constants we can see clearly why it is necessary to take into account sequents of levels higher than 1. With non-modal constants we have used sequents of level 2 to distinguish level-preserving deducible rules which are horizontalizable from those which are not, but the utility of doing that might have been questioned.

It can still be questioned why we had assumed from the beginning a structural system with postulates of all levels. An alternative would have been to assume for the context in which we treat non-modal constants only postulates of levels 0, 1 and eventually 2, and then eventually to enlarge this system when we came to deal with modal constants with postulates of level 3. But our approach, which does not differ substantially, has the advantage of showing that it is possible to take the same structural systems in the background, and add analytic rules for modal constants without changing this background.

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The choice to present structural systems from the outset with sentences and postulates of all levels has obliged us however to make some assumptions concerning various postulates which do not affect the provability of sentences of the levels in which we are interested. Levels irrelevant in this sense are shown by the levels of eliminable D-rules. We have made such assumptions, in particular, concerning T-rules and their horizontalizations. When dealing with non-modal constants, all T-rules of levels $\geqslant 2$ were irrelevant, and with modal constants all T-rules of levels $\geqslant 3$ were in the same position.

We could have followed either of the following policies concerning assumptions about T-rules

(1) make <u>uniform restrictions</u> for all levels;

(2) make <u>minimal restrictions</u>, i.e. make only the restrictions needed to obtain the desired effect at those levels in which we are interested, and at all other levels assume T-rules without restriction;

(3) make <u>maximal restrictions</u>, i.e. assume only the T-rules needed to obtain the desired effect at those levels in which we are interested, and at all other levels assume no T-rule.

Everywhere, except with S4P/D, we have followed the uniform restrictions policy (this policy can also coincide with another policy). With S4P/D it was necessary to abandon this policy, since T-rules of levels 1 and 2 must differ, and we have chosen to make minimal restrictions in this case. This is why we have

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chosen $T^2_{-\delta\hbar}$ rather than $T^2_{-\ell\hbar}$ or no $\underline{T}$ of level 2 at all.

According to Lemma 32 and the demonstration of Lemma 44, the maximal restrictions policy for S5P/D would be to assume only $T^{\leqslant 2}_{-\ell\ell}$, and for S4P/D only $T^1_{-\ell\ell}$.

By restricting $\underline{T}^1$ in some way, making some assumptions concerning $\underline{T}$-rules of other levels, and extending with analytic rules for modal constants, it is possible to give systems of various non-classical modal logics.

Two such systems with uniform restrictions on $\underline{T}$-rules are

$$\underline{hADICT}_{\delta\hbar}(\rightarrow)\,(\&)\,(\vee)\,(\perp)\,(T)\,(\square)\,(\dashv)\quad,$$

which is a modal Heyting propositonal system, and

$$\underline{hADIC}(\rightarrow)\,(\&)\,(\vee)\,(\perp)\,(T)\,(\square)\quad(\dashv)\quad,$$

which is an intutionistic relevant modal propositional system. The demonstration of Lemma 44.1 shows that both of these systems will have a certain S4 character. For these two systems the explicit definition of Lemma 42 fails, and extending them with (◊) would destroy the single-conclusion property of sequents of level 1. It is questionable whether in the second of the systems above we should take (&) and (∨) in the scope of $\underline{h}$. For if we do, we have

$$\underline{c}^2 \frac{\{\varnothing\vdash\{A\},\varnothing\vdash\{B\}\}\vdash^2\{\varnothing\vdash\{A\&B\}\}\ \underline{h}((\&)\!\!\downarrow)\quad \{\varnothing\vdash\{A\&B\}\}\vdash^2\{\varnothing\vdash\{A\}\}\ \underline{h}((\&)\!\!\uparrow)}{(\square)\!\!\downarrow\frac{\{\varnothing\vdash\{A\},\varnothing\vdash\{B\}\}\vdash^2\{\varnothing\vdash\{A\}\}}{\{\varnothing\vdash\{A\}\}\vdash^2\{\{\square B\}\vdash\{A\}\}}}\,;$$

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now suppose A is provable; then $\Box B \to A$, will be provable too, though $\Box B$ need not be used in any proof of A. With $\underline{h}^1((v)\downarrow)$ and $\underline{h}^1((v)\uparrow)$ we would get $\Box\neg B \to \neg A$, where $\neg A$ is provable. Note that if we don't assume (&) and (v) in the scope of $\underline{h}$, this has no effect on the provable sentences of level 0 of the non-
-modal system, and would leave (&)$\uparrow$ and (v)$\uparrow$ horizontalizable [cf. §§60,69].

It is clear that we could sustain various ideas about the logic of different levels, and accordingly have systems, like S4P/D, with $\underline{T}$-assumptions different than those dictated by the uniform restrictions policy. This policy presupposes that our logic at all levels is of the same kind. The minimal restrictions policy would presuppose that our logic is classical wherever there are no express indications that it isn't.

$\underline{T}$-rules are in this work in a special position when compared with other structural rules: no restrictions for these other rules were envisaged, and this is why we could safely assume them for all levels.

§56   [§§52-53] The idea to treat modality with sequents of higher levels is perhaps suggested by Kneale & Kneale 1962 (pp.559ff), but the connection with our treatment is not clear.

A more direct connection is with Scott 1971 and 1974. In these papers can be found the double-line rule

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$$\frac{\{A\} \vdash^1 \{B\}}{\emptyset \vdash^1 \{A \to B\}}$$

and a rule for transforming deductions of the form

$$\frac{\{A_1\} \vdash^1 \{B_1\} \ \ldots \ \{A_{n-1}\} \vdash^1 \{B_{n-1}\}}{\{A_n\} \vdash^1 \{B_n\}}$$

into sentences of the form

$$\{A_1 \to B_1, \ \ldots \ , A_{n-1} \to B_{n-1}\} \vdash^1 \{A_n \to B_n\} \quad .$$

It is by combining these two ideas, and defining a formal

context for sequents of levels higher than 1, that one could

be led to the double-line rules

$$\frac{\{A\} \vdash^1 \{B\}}{\emptyset \vdash^1 \{A \to B\}} \quad ; \quad \frac{\{\{A_1\} \vdash^1 \{B_1\}, \ldots, \{A_{n-1}\} \vdash^1 \{B_{n-1}\}\} \vdash^2 \{\{A_n\} \vdash^1 \{B_n\}\}}{\emptyset \vdash^2 \{\{A_1 \to B_1, \ldots, A_{n-1} \to B_{n-1}\} \vdash^1 \{A_n \to B_n\}\}}$$

or

$$\frac{\emptyset \vdash^1 \{A\}}{\emptyset \vdash^1 \{\Box A\}} \quad ; \quad \frac{\{\emptyset \vdash^1 \{A_1\}, \ldots, \emptyset \vdash^1 \{A_{n-1}\}\} \vdash^2 \{\emptyset \vdash^1 \{A_n\}\}}{\emptyset \vdash^2 \{\{\Box A_1, \ldots, \Box A_{n-1}\} \vdash^1 \{\Box A_n\}\}} \quad .$$

The modal system in 0 to which these two pairs of double-line rules give

rise is not S4, but a weaker system having as characteristic rules

and axiom-schemata to be added to the non-modal classical system

$$\frac{A}{\Box A} \quad ; \quad \Box(A \to B) \to (\Box A \to \Box B); \quad \Box A \leftrightarrow \Box\Box A \quad ,$$

and lacking $\Box A \to A$. However, this system - let us call it "S4⁻" -

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is stronger than Sobociński's S4° (v. Sobociński 1962 and

Zeman 1973; Hughes & Cresswell 1968, p.302, are misleading

as to the origin of S4°; §*68 of Feys 1965, which is not by

Feys but by Dopp, seems to contain an outright mistake concerning

S4° when it claims that <<Sobociński has shown that his system

S4° is deductively equivalent with the system which Feys has

called 'T' ... >>). S4° can be obtained by replacing the second

axiom-schema above by $\Box A \to \Box\Box A$. The formula $\Box\Box A \to \Box A$ is not

provable in S4° (enough information to infer that can be found

in Zeman 1973). That $\Box A \to A$, which added to S4° would give S4,

is not provable in S4⁻ is shown by the following matrices, derived

from Group IV of Lewis & Langford 1932

→	1 2 3 4		ר		□	
* 1	1 2 3 4	1	4	1	1	; $\Box A \to A$.
2	1 1 3 3	2	3	2	3	32 2 2
3	1 2 1 2	3	2	3	3	
4	1 1 1 1	4	1	4	3	

So, S4⁻ is between S4° and S4. It could be conjectured that there

is an infinity of systems between S4° and S4, having as axiom-

-schemata added to S4°

$$\underbrace{\Box...\Box}_{n+k} A \to \underbrace{\Box ...\Box}_{n} A, \quad n \geqslant 1, \quad k \geqslant 1$$

(cf. Thomas 1964 for a result of Sobociński about an infinite

hierarchy between T and S4 obtained by adding successively to T

$$\underbrace{\Box...\Box}_{n} A \to \underbrace{\Box...\Box}_{n+1} A \quad) .$$

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The following double-line rule would give rise to S4

$$\frac{\{\emptyset \vdash^1 \{A_1\}, \ldots, \emptyset \vdash^1 \{A_{n-1}\}\} \vdash^2 \{\emptyset \vdash^1 \{A_n\}\}}{\emptyset \vdash^2 \{\{\square A_1, \ldots, \square A_{n-1}\} \vdash^1 \{A_n\}\}} \quad ,$$

and this naturally leads to a form of analytic rules for the necessity operator of S4

$$\frac{\Gamma \cup \{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{B\}\}}{\Gamma \qquad \vdash^2 \{\{\square A\} \vdash^1 \{B\}\}} \quad .$$

Of course, the double-line rules for S4⁻ can be obtained with the analytic rule for S4, but not the other way round.

These remarks are put loosely, but they could also be stated precisely in the more formal context of this work.

[§54]   Sequent calculi for S5 and S4 having rules closely connected with our ($\square_{S5}$) and ($\square_{S4}$) are studied in Ohnishi & Matsumoto 1957 and 1959 (v. also Curry 1950, 1963; Zeman 1973).

[§55]   The outcome of our approach is that a consistently intuitionistic standpoint, restricting $\underline{T}$ of all levels to $\underline{T}_{sh}$ or $\underline{T}_{eh}$, would give rise to a modal system with an S4 character, and not an S5 character like the calculus of Bull 1966. And with a consistent rejection of $\underline{T}$ at all levels, with which we have obtained the intuitionistic relevant system, the ensuing modal system is again of an S4 character. This perhaps justifies Anderson & Belnap 1975, who make a case for S4 principles, and against S5 principles, in relevant logic.

Chapter 7

FIRST-ORDER QUANTIFIERS

§57   We first make some preliminary remarks concerning the

language O and assumptions which should be made in order to

obtain a context in which first-order quantifiers can be analyzed.

Then we introduce first-order quantifiers and we give

analytic rules for them. Since these quantifiers are infinite

in number, the primary analytic rules we give involve, unlike

the preceding analytic rules, schemata for the constants

analyzed, and not the constants themselves. Analytic rules for the

constants are obtained from these by substitution.

Next, we demonstrate the O-equivalence of the analytic

extension of CP/D with these rules and an axiomatization in

O of classical first-order quantificational logic, and we

consider the replaceability of the analytic rules by postulates

in O.

We then make some remarks on non-classical and modal first-

-order quantificational systems. In connection with these last

systems we consider an approach which would make different

assumptions concerning the formal framework of D1 and would

apparently make our treatments of quantification and modality

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incompatible. We shall try to assess the merits of this
approach.

§58   When dealing with propositional constants, we assumed
about O only that it has expressions of the category s and the
constants for which we gave analytic rules. In this chapter we
shall make some additional assumptions concerning O.

O will be obtained in the following way. We first specify
a language called "O_{cl}" ("cl" stands for "closed"). For O_{cl}
we assume that we have in it at least some expressions of the category
t , and at least some expressions of the categories p_k, $k \geqslant 1$,
where "p_k" is an abbreviation for $\underbrace{\overset{"\ \ \ s\ \ \ "}{t...t}}_{k}$ (so that there will be
at least some expressions of the category s in O_{cl}). We also
assume that we have in it the constants for which we shall give
analytic rules. O is obtained by extending O_{cl} with schematic
letters for expressions of O_{cl} of at least some (possibly all)
categories represented in O_{cl}. These schematic letters of O will
be called "variables".   (We shall not specify what variables look
like, as we have not given such a specification for any expression
of O, save the constants analysed.)

So an expression of O is either an expression of O_{cl} or
a schema for expressions of O_{cl} . The language which contains all
the expressions of O which are not expressions of O_{cl} will be
called "O_{op}" ("op" stands for "open").

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A broader conception of O, which we shall not consider here, results when we allow O_{op} to have variables of a certain category even if there are no expressions of this category in O_{cl}. These variables can then be conceived as schematic letters for themselves. A language O which has such variables can be conceived as a kind of zero case of the language O presented above, and our structural analyses in principle can be compatible with it too.

In addition to the schemata of U used before, we shall use the following schematic letters of U for expressions of O of the category t

$$a, b, c, a_1, \ b_1, \ \ldots$$

These schematic letters are both for singular terms of O_{cl} and for variables of O_{op}. Next we shall use the following schematic letters of U for the variables of O_{op} of the category t, which are <u>denumerably infinite</u> in number,

$$x, y, z, x_1, y_1, \ \ldots$$

For this chapter we assume that we have only these variables in O_{op}.

Basic schemata of U of the form

$$A^n(x_1, \ \ldots, x_k)$$

$n \geqslant 0$, $k \geqslant 1$, will be used for sentences of D1 in which x_i, for every i, $1 \leqslant i \leqslant k$, <u>occurs at least once</u> (if $n = 0$, we omit the superscript as usual). x_i, for every i, $1 \leqslant i \leqslant k$, can occur more than once in sentences for which the schema is used, but it is

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listed only once. So a schema like "A(x,x)" will have no

sense in this notation. $x_1, \ldots, x_k$ are not necessarily all

the expressions, nor all the variables, of the cateogry t which

occur in $A^n(x_1, \ldots, x_k)$.

As before, we shall use these basic schemata also for formulae

of D1 which satsify the requirement concerning $x_1, \ldots, x_k$.

We shall distinguish in D1 the language D1 built on O_{cl} and

the language D1 built of O_{op} which are obtained by restricting O

to O_{cl} and O_{op} respectively.

When constructing systems of provable sentences of D1 we must

in the presence of the variables of O take into account some

additional structural rules. Namely, we must explicitly give some

form of the rules of substitution which will permit us to pass

from sentences of D1 to their instances, salva provability. For

instances we shall use basic schemata of the form

$$S_a^{x_i} A^n(x_1, \ldots, x_k)|$$

$n \geqslant 0$, $k \geqslant 1$, $1 \leqslant i \leqslant k$. $S_a^{x_i} A^n(x_1, \ldots, x_k)|$ is the result

of substituting a at the place of every occurrence of x_i in $A^n(x_1, \ldots, x_k)$.

Note that the notation "$S_a^x A^n|$" will be used only if A^n is $B^n(x)$,

for some $B^n(x)$.

Then we give the following primitive rule for structural

systems in D1

t-substitution $(\underline{S}_t)$: $\underline{S}_t^n \dfrac{A^n(x)}{S_a^x \, A^n(x)|}$, $n \geqslant 0$.

Note that no empty application of this rule is possible *[cf. §4]*.

This rule will be subject to some provisions in some contexts *[v.§59]*.

In general, it is subject to the proviso that $S_a^x \, A^n(x)|$ is a sentence

of Dl *[cf.§9]*. As before, we shall envisage restricting this

rule only to some levels.

In this chapter we shall consider the structural system

$$\underline{S}_t \text{ hADICT} \quad .$$

In spite of the fact that $\underline{S}_t$ is level-preserving, it cannot

be in the scope of $\underline{h}$. Otherwise we would have the following

$$\underline{S}_t^1 \frac{\{A(x)\} \vdash^1 \{S_a^x A(x)|\}}{\{S_b^x A(x)|\} \vdash^1 \{S_a^x A(x)|\}} \qquad \underline{h}^0(\underline{S}_t^0)$$

where b can be any expression of the category t of $O_{c\ell}$.

Extending hADICT with $\underline{S}_t$ should not be understood as giving

rise to a structural system essentially different from the one used

previously.   If O had t-variables, $\underline{S}_t$ would be admissible in hADICT;

and even if O did not have these variables, $\underline{S}_t$ would be admissible

vacuously.   It would remain admissible in the analytic extensions

considered thus far.   Thus we could have assumed $\underline{S}_t$ from the

beginning.   The same holds for other rules of substitution, involving

other variables of O than t-variables, which we shall consider later

[v.§§68,74].   To assume $\underline{S}_t$ when dealing with propositional constants

would be superfluous, in the same way that assuming other rules of

substitution would be superfluous in this chapter.   But assuming $\underline{S}_t$

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for this chapter is not superfluous. $\underline{S}_{t}$ is admissible in hADICT,

but it is not admissible in every extension of hADICT (hence

it is not deducible in hADICT).   In particular, it would not

be admissible in the analytic extensions with analytic rules we

shall consider in this chapter.

§59   We shall now introduce the constants of O_{cl} called "first-

-order quantifiers" (until further notice, by "quantifier" we shall

understand "first-order quantifier"). Quantifiers are not

elementary but complex constants. A quantifier is constructed

from a quantifier prefix which is of the form

$$\forall x \quad \text{or} \quad \exists x$$

and all the k, $k \geqslant 1$, occurrences of the variable x, identical

with the variable which occurs in the quantifier prefix, in a

sentence $A(x)$ of O_{op} to which the quantifier prefix is prefixed. When

this prefixing is effected the result is a sentence of the form

$$\forall x A(x) \quad \text{or} \quad \exists x A(x) .$$

A quantifier is underline universal if it has a prefix of the first form, and

existential if it has a prefix of the second form.

A variable which occurs in a quantifer is called "bound".

We can obtain an infinity of quantifiers by varying the bound variable

x and the number k of occurrences of the variable bound by the

quantifier.

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A variable which is not bound is called "free". Only free

variables are schematic letters of O_{op}: bound variables are not

schematic letters of O_{op} but parts of constants of O_{cl}. Until

now, when we have spoken of a variable occurring in a sentence,

we have had in mind only variables which occur qua schematic letters,

i.e. free. Accordingly, "$A^n(x_1, \ldots, x_k)$" will be used only when x_i

for every i, $1 \leqslant i \leqslant k$, occurs free. Also, $S_a^x A^n(x)|$ will be the

result of substituting a for every occurrence of the free variable x

in $A^n(x)$, and the free variable x only. Moreover, if in a there occurs

the free variable y, this y must not become bound in $S_a^x A^n(x)|$, i.e.

x must not occur free in a formula $\forall y B^m(y)$ or $\exists y C^\ell(y)$ which occurs

in $A^n(x)$.

N.B. The notation " $S_a^x A^n(x)|$ " and the rule $S_{\to t}$ will be applied

only if the provisions we have now stated are satisfied.

§60   Since the number of quantifiers is infinite, we shall give

analytic rules not for quantifiers but for quantifier-schemata.

Analytic rules for quantifiers are obtained from these analytic rules

by substitution. In these instances the schematic letter of U "x"

is replaced by a variable of O_{op} and the number of occurrences of

the bound variable is fixed.

We give the following analytic rules

$$(\forall) \quad \frac{\Gamma \vdash^1 \Delta \cup \{A(x)\}}{\Gamma \vdash^1 \Delta \cup \{\forall x A(x)\}} \quad ,$$

provided x does not occur free in Γ or Δ;

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$$(\exists) \quad \frac{\Gamma \cup \{A(x)\} \vdash^1 \Delta}{\Gamma \cup \{\exists x A(x)\} \vdash^1 \Delta} \quad ,$$

provided x does not occur free in Γ or Δ.

An obvious modification of our definition of analytic rules is

needed to justify our calling ($\forall$) and ($\exists$) "analytic rules",

instead of "schemata of analytic rules". (The notation "($\forall$)" and "($\exists$)"

need not differ from the standard, since "$\forall$" and "$\exists$" can be taken as

names for quantifier-schemata /cf. § 20/.)

We note immediately that the proviso holds not only for ($\forall$)↓

and ($\exists$)↓, but also for ($\forall$)↑ and ($\exists$)↑, so that our analytic rules

are completely symmetrical. But in ($\forall$)($\exists$)<u>hADIC</u> , the rules ($\forall$)↑ and ($\exists$)↑

without the proviso are horizontalizable, and hence derivable :

and homologously with ($\exists$)↑.

That ($\forall$)↑ and ($\exists$)↑ without the proviso are derivable can be

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demonstrated with the help of $(\forall)\uparrow, (\exists)\uparrow$, $\underline{h}^0(\underline{I}^0)$ and $\underline{c}^1$ alone,
or $(\forall)\uparrow, (\exists)\uparrow$ and $\underline{S}^1_{\rightarrow t}$ alone.

Analogously to what we had with $\underline{S}_{\rightarrow t}$, the rules $(\forall)\downarrow$ and $(\exists)\downarrow$
cannot be in the scope of $\underline{h}$. Otherwise we would have the following

$$\underline{S}^2_{\rightarrow t}\frac{\{\emptyset\vdash^1\{A(x)\}\}\vdash^2\{\emptyset\vdash^1\{\forall xA(x)\}\}}{\{\emptyset\vdash^1\{S^x_a A(x)|\}\}\vdash^2\{\emptyset\vdash^1\{\forall xA(x)\}\}}\qquad \underline{h}^0((\forall)\downarrow)$$

where a can be any expression of the category t of 0_{cl}.

The explanation for the impossibility of horizontalizing
$\underline{S}_{\rightarrow t}$, $(\forall)\downarrow$ and $(\exists)\downarrow$ seems to lie in the fact that, in contradistinction
to the other analytic rules we have considered, they involve
essentially transitions from sentences in which occur expressions
of 0_{op} to sentences in which occur expressions of 0_{cl}, or, as it
is the case with $\underline{S}_{\rightarrow t}$, they at least involve an occurrence of
expressions of 0_{op} in the premises. To horizontalize these rules
would reduce them to schemata for sentences of D1 built on 0_{cl},
and would abolish this essential characteristic. We shall return
to these matters later [v.§65].

So, when we extend $\underline{S}_{\rightarrow t}\underline{hADICT}$, the analytic rules $(\forall)$ and $(\exists)$
will not be in the scope of $\underline{h}$. This will not prevent $(\forall)\uparrow$ and $(\exists)\uparrow$
from being horizontalizable, as we have seen above. In contra-
distinction to $(\forall)\downarrow$ and $(\exists)\downarrow$, these rules do not involve essentially
transitions from sentences in which occur expressions of one language
to sentences in which occur expressions of another language, which
in this case would refer to transitions from sentences in which occur
expressions of 0_{cl} to sentences in which occur expressions of 0_{op}.

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The universal and existential first-order quantifiers in

our treatment belong to categories of the form

$$\frac{\delta}{p_k} \; , \; k \geqslant 1.$$

So, the universal quantifier in

Vx∃y (x is divisible by y)

is of the category $\frac{\delta}{p_1}$, whereas the universal quantifier in

Vx (x is divisible by x)

is of the category $\frac{\delta}{p_2}$.   The expression " _ is divisible by _"

in both these cases is of the category p_2. The universal quantifier

in

∃yVx (x is divisible by y)

is also of the category $\frac{\delta}{p_1}$. When this quantifier is applied to

"_ is divisible by _", the value of the ensuing expression is p_1,

as we have explained before [v. §3].

To understand how we construct VxA(x) we need not invoke any

notion of substitution. We are given A(x) which is of the category

δ, and we apply to it the quantifier prefix Vx which is of the

category $\frac{\delta}{\delta}$. This gives the same result as applying the quantifier,

which is of the category $\frac{\delta}{p_k}$, for some $k \geqslant 1$, to the expression

obtained from A(x) by deleting all the k free occurrences of x in

this expression, and which is of the category p_k.

Step-by-step, "∃yVx (x is divisible by y)" is constructed by

prefixing Vx to "x is divisible by y", and then prefixing ∃y to

"$\forall x(x$ is divisible by $y)$". This is why our ascription of

categories to quantifiers still holds even if the expression to

which the quantifier prefix is prefixed is not a sentence.

The quantifier of $\forall x A(x)$ could be counted as being always

of the category $\dfrac{\delta}{p_1}$ if all the free occurrences of x in $A(x)$,

irrespectively of their number, were counted as filling only <u>one</u>

<u>kind of free places</u> in the expression obtained by deleting every

free occurrence of x from $A(x)$. This expression would then be

of the category p_1. However, we shall not follow this approach.

It would require us to introduce into our categorial apparatus

a device for indentifying kinds of free places, and the notion

of a scope of a quantifier. The scope of a quantifier in our

approach is exactly determined by the number of occurrences of the

variable bound by it.

It is also possible to introduce generalized first-order

quantifiers which would belong to categories of the form

$$\dfrac{\delta}{p_{k_1}\ldots\; p_{k_m}} \; , \; m \geqslant 1, \; k_i \geqslant 1, \; 1 \leqslant i \leqslant m.$$

For some quantifiers of this kind, which it would be natural to

consider, we can give the following analytic rule

$$(\forall)^2 \quad \dfrac{\Gamma \cup \{A(x)\} \vdash^1 \Delta \cup \{B(x)\}}{\Gamma \vdash^1 \Delta \cup \{\forall x(A(x),B(x))\}^2} \quad ,$$

provided x does not occur free in Γ or Δ. These quantifiers would

belong to the categories

$$\dfrac{\delta}{p_{k_1} p_{k_2}} \; , \quad k_1 \geqslant 1, \; k_2 \geqslant 1.$$

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The expression "$\overset{2}{\forall x}(A(x),B(x))$" can be explicitly defined as

"$\forall x(A(x) \to B(x))$".

There is a certain correspondence between the categories
of generalized first-order quantifiers and the categories of
propositional constants. The categories of propositional constants
are obtained when in

$$\frac{\delta}{p_{k_1} \cdots p_{k_m}}$$

we take $m \geqslant 0$ and $k_i = 0$, for every i, $1 \leqslant i \leqslant m$.

§62   Consider the following rules

(∀gen) $\dfrac{B \to A(x)}{B \to \forall x A(x)}$, provided x does not occur free in B;

(∃gen) $\dfrac{A(x) \to B}{\exists x A(x) \to B}$, provided x does not occur free in B;

and the following formulae

$\underline{u}$ $\forall x A(x) \to S_a^x A(x) |$;

$\underline{e}$ $S_a^x A(x) | \to \exists x A(x)$.

The extension of CP/O with these postulates will be called
"CQ/O" ("Q" stands for "quantificational").

In this section we shall demonstrate the O-equivalence of
CQ/O and the system

$\underline{S}_t(\forall)(\exists)\underline{hADICT}$ $(\to)(\&)(\lor)(\bot)(T)$

called "CQ/D".

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Lemma 53   (∀gen) and (∃gen) are derivable, and u and e are

provable in $\underline{S}_t(\forall)(\exists)\underline{hADIC}(\rightarrow)$.

Demonstration:

$$\underline{A}^{\wedge}\frac{\underline{B}\rightarrow A(x)}{(\rightarrow)\uparrow\frac{\varnothing\vdash\{B\rightarrow A(x)\}}{(\forall)\downarrow\frac{\{B\}\vdash\{A(x)\}}{(\rightarrow)\downarrow\frac{\{B\}\vdash\{\forall x A(x)\}}{\underline{D}^{\wedge}\frac{\varnothing\vdash\{B\rightarrow\forall x A(x)\}}{\underline{}\ B\rightarrow\forall x A(x)}}}}}\ ;$$

$$(\forall)\uparrow\frac{\{\forall x A(x)\}\vdash\{\forall x A(x)\}\ \underline{h}^{\circ}(\underline{I}^{\circ})}{(\rightarrow)\downarrow\frac{\{\forall x A(x)\}\vdash\{A(x)\}}{\underline{D}^{\wedge}\frac{\varnothing\vdash\{\forall x A(x)\rightarrow A(x)\}}{\underline{S}^{\circ}_t\frac{\forall x A(x)\rightarrow A(x)}{\forall x A(x)\rightarrow S^x_a A(x)|}}}}\ ;$$

we proceed homologously for (∃gen) and e, using (∃).

                                          Q.E.D.

Lemma 54   If a sentence of level 0 is provable in CQ/D, it

is provable in CQ/0.

Demonstration: We proceed as for Lemma 19 using the o_1-translation.

The only addition is in the induction step, where the last step

can be

   1) $\underline{S}_t\dfrac{B^n}{A^n}$;   then we show that $\underline{S}^0_t$ is derivable in CQ/0;

let x be the variable for which we substitute, and let $o_1(B^n)=C(x)$

and $o_1(A^n)=S^x_a C(x)|$, then

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$$(\forall gen) \dfrac{\dfrac{\dfrac{C(x)}{T \to C(x)}}{T \to \forall x\, C(x)}}{\forall x\, C(x)} \qquad \forall x\, C(x) \to S^{x}_{a} C(x)| \quad \underline{u}$$

$$S^{x}_{a} C(x)| \; ;$$

2) $(\forall)\downarrow$ or $(\forall)\uparrow$; then we use $\dfrac{G \to (D \vee A(x))}{G \to (D \vee \forall x\, A(x))}$;

3) $(\exists)\downarrow$ or $(\exists)\uparrow$; then we use $\dfrac{(G \& A(x)) \to D}{(G \& \exists x\, A(x)) \to D}$.

Q.E.D.

From Lemmata 53 and 54 and Theorem 1 we get immediately

Theorem 7 CQ/D and CQ/O are O-equivalent.

§63   We shall now consider whether $(\forall)$ and $(\exists)$ are replaceable in CQ/D by postulates in O, in the presence of the analytic rules for propositional constants.

We have

Lemma 55   55.1 $(\forall)\uparrow$ and $(\exists)\uparrow$ are horizontalizable in hA C(mp)ue.

55.2 $(\forall)\downarrow$ and $(\exists)\downarrow$ are derivable in the extension of CP/D with ($\forall$gen) and ($\exists$gen), where ($\forall$gen) and ($\exists$gen) are not in the scope of h.

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Demonstration: For 55.1 we have

$$\dfrac{\underline{A^1}}{\underline{C^1}}\dfrac{\dfrac{\forall x\,A(x)\to A(x)\quad\underline{u}}{\emptyset \vdash \{\forall x\,A(x)\to A(x)\}}\quad\quad\underline{\overset{\circ}{h}}\,((mp))}{\{\forall x\,A(x)\}\vdash\{A(x)\}}$$

and then we proceed as in §60; and homologously for $(\exists)\!\uparrow$ using $\underline{e}$.

55.2   Let $\hat{\Gamma}=\begin{cases} A_1 \&\ldots\& A_k\,,\ \text{if } \Gamma=\{A_1,\ldots,A_k\}\,,\ k\geq 1,\\ \qquad\top\qquad,\ \text{if } \Gamma=\emptyset\ ;\end{cases}$

$\quad\quad\quad\quad\quad\quad\check{\Gamma}=\begin{cases} A_1\vee\ldots\vee A_k\,,\ \text{if } \Gamma=\{A_1,\ldots,A_k\}\,, k\geq 1,\\ \qquad\perp\qquad,\ \text{if } \Gamma=\emptyset\ ;\end{cases}$

then we have

$$\dfrac{\Gamma\vdash\Delta\cup\{A(x)\}}{\text{applications of }(\&)_{1\ldots 1}\!\downarrow\text{ or }(\top)\!\downarrow,}$$
and $(\vee)_{1\ldots 1}\!\downarrow$ or $(\perp)_1\!\downarrow$
$$\overline{\dfrac{\{\hat{\Gamma}\}\vdash\{\check{\Delta},A(x)\}}{(\neg)_1\!\downarrow,(\&)_{1\ldots 1}\!\downarrow,(\to)\!\downarrow\text{ and }\underline{D^1}}}$$
$(\forall\text{gen})\dfrac{(\hat{\Gamma}\&\neg\check{\Delta})\to A(x)}{(\hat{\Gamma}\&\neg\check{\Delta})\to\forall x\,A(x)}$
$$\dfrac{}{\underline{A^1},(\to)\!\uparrow,(\&)_{1\ldots 1}\!\uparrow\text{ and }(\neg)_1\!\uparrow}$$
$$\{\hat{\Gamma}\}\vdash\{\check{\Delta},\forall x\,A(x)\}$$
applications of $(\vee)_{1\ldots 1}\!\uparrow$ or $(\perp)_1\!\uparrow$,
and $(\&)_{1\ldots 1}\!\uparrow$ or $(\top)\!\uparrow$
$$\Gamma\vdash\Delta\cup\{\forall x\,A(x)\}\ ;$$

$$\dfrac{\Gamma\cup\{A(x)\}\vdash\Delta}{\text{applications of }(\&)_{1\ldots 1}\!\downarrow\text{ or }(\top)_1\!\downarrow,}$$
and $(\vee)_{1\ldots 1}\!\downarrow$ or $(\perp)\!\downarrow$
$$\dfrac{\{\hat{\Gamma},A(x)\}\vdash\{\check{\Delta}\}}{\text{2 applications of }(\to)\!\downarrow,}$$
and $\underline{D^1}$
$(\exists\text{gen})\dfrac{A(x)\to(\hat{\Gamma}\to\check{\Delta})}{\exists x\,A(x)\to(\hat{\Gamma}\to\check{\Delta})}$
$$\dfrac{}{\underline{A^1}\text{ and 2 applications}}$$
of $(\to)\!\uparrow$
$$\{\hat{\Gamma},\exists x\,A(x)\}\vdash\{\check{\Delta}\}$$
applications of $(\vee)_{1\ldots 1}\!\uparrow$ or $(\perp)\!\uparrow,$
and $(\&)_{1\ldots 1}\!\uparrow$ or $(\top)_1\!\uparrow$
$$\Gamma\cup\{\exists x\,A(x)\}\vdash\Delta\ .$$

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[For $(\&)_{1.1}$, $(\vee)_{1.1}$, $(\perp)_1$, $(T)_1$ and $(\neg)_1$ v.§§29,30*].*

Q.E.D.

Note that the demonstration of 55.2 depends on the presence

of $(\rightarrow)\downarrow$, so that if we replace $(\rightarrow)$ by some postulates such that

$(\rightarrow)\downarrow$ ceases to be derivable, or even admissible, this can

influence the derivability of $(\forall)\downarrow$ and $(\exists)\downarrow$. Note also that we

must take care in 55.2 to put (∀gen) and (∃gen) out of the scope

of h, for the reasons given above *[v. §60].* Even (∀gen) in

the form

$$\frac{\emptyset \vdash^1 \{B \rightarrow A(x)\}}{\emptyset \vdash^1 \{B \rightarrow \forall x A(x)\}} \quad , \text{ with the proviso,}$$

and the homologous form of (∃gen), could not be horizontalized. The

situation is unlike what we had with (adj) and (adj ^{1}), and (nec) and

(nec^1) *[v.§§48,53,54].*

We can conclude that $(\forall)$ and $(\exists)$ are replaceable in CQ/D by

(∀gen) (∃gen) ue, where (∀gen) and (∃gen) are not in the scope of h.

§64   The systems obtained by replacing "T" in the name of CQ/D by

(1) "T_{sh}", (2) "T_{eh}" and (3) by omitting "T", will be called

(1) "HQ/D", (2) "KQ/D" and (3) "IRQ/D". We shall not consider these

systems in detail. By Lemma 53 (∀gen),(∃gen), u and e can be obtained

in all these systems. This is all we need to add to HP/O, KP/O and IRP/O,

together with

$\underline{u}_1$ $\forall x (B \rightarrow A(x)) \rightarrow (B \rightarrow \forall x A(x))$, provided x does not
                                              occur free in B,

for IRP/O,   to get HQ/O, KQ/O and IRQ/O. To show that these

systems in O are O-equivalent with the corresponding systems in D1,

we can proceed by enlarging the demonstrations of lemmata we

already have, as we have done for Lemma 54. We can also use Lemma 55.1,

and a procedure involving lemmata homologous to Lemmata 22 and 23,

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to show that $(\forall)\downarrow$ and $(\exists)\downarrow$ are admissible in the extensions of

HP/O, KP/O and IRP/O with $(\forall gen)$,$(\exists gen)$, $\underline{u}$ and $\underline{e}$, and also $\underline{u}_1$ for IRP/O,

where $(\forall gen)$ and $(\exists gen)$ are not in the scope of $\underline{h}$

The analytic rules $(\forall)$ and $(\exists)$ are so formulated that the

explicit interdefinability of the two sorts of quantifiers in the

presence of $(\neg)_1$ and $(\neg)_2$, and their non-interdefinability in the

absence of $(\neg)_1$ and $(\neg)_2$, are obvious.

§65   The system

$\underline{S}_{\mathcal{t}}(\forall)(\exists)\underline{hADICT}(\rightarrow)(\&)(\vee)(\bot)(T)(\square)(\lozenge)(\dashv)$

will be called "S5Q/D"; the subsystem of S5Q/D where $\underline{T}^2$ and $\underline{h}^2(\underline{T}^2)$

are replaced by $\underline{T}^2_{\Delta\mathcal{h}}$ and $\underline{h}^2(\underline{T}^2_{\Delta\mathcal{h}})$ will be called "S4Q/D".

The extensions of S5P/O and S4P/O with $(\forall gen)$,$(\exists gen)$, $\underline{u}$ and $\underline{e}$

will be called "S5Q/O" and "S4Q/O". To show the O-equivalence of

S5Q/D and S5Q/O, and of S4Q/D and S4Q/O, we can again appeal to Lemma

53 and enlargements of demonstrations we already have (in the

demonstration of Lemma 45, apart from what we have in the three

additional cases of the demonstration of Lemma 54, we would have to

use $\square A \rightarrow A$ and (nec)).

In S5Q/D the Barcan formula would be provable and in S4Q/D not.

The whole of our approach to analytic rules for quantifiers

was influenced by our decision to ban the horizontalizing of $(\forall)$ and $(\exists)$.

The decision to ban the horizontalizing of $\underline{S}_{\mathcal{t}}$, $(\forall)$ and $(\exists)$ was

motivated by a certain understanding of the use of schemata of O.

Namely, we assumed that the schematic letters of these schemata, i.e.

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free variables, continued to function as schematic letters for

expressions of $0_{c\ell}$ at every level and in every context of D1.

This assumption was made by giving $\underline{S}_{\neq t}$ of every level $\geqslant 0$. When

we assume $\underline{S}_{\neq t}$, a sequent like

$$\{\emptyset \vdash^1 \{A(x)\}\} \vdash^2 \{\emptyset \vdash^1 \{\forall x A(x)\}\} \ ,$$

where x is the only free variable of $A(x)$, does $\underline{not}$ express that a

sentence of D1 built on $0_{c\ell}$, i.e. $\emptyset \vdash^1 \{\forall x A(x)\}$, can be deduced from

a sentence of D1 built on 0_{op}, i.e. $\emptyset \vdash^1 \{A(x)\}$, which would be

correct, but that $\emptyset \vdash^1 \{\forall x A(x)\}$ can be deduced from any sentence of

D1 built on $0_{c\ell}$ of the form $\emptyset \vdash^1 \{A(x)\}$, which is not correct. This

is because in the sequent $\{\emptyset \vdash^1 \{A(x)\}\} \vdash^2 \{\emptyset \vdash^1 \{\forall x A(x)\}\}$, x continues

to be $\underline{used}$ as a schematic letter for expressions of $0_{c\ell}$, according

to $\underline{S}_{\neq t}^2$. Sentences of D1 can be either sentences of D1 built on $0_{c\ell}$

or schemata for sentences of D1 built of $0_{c\ell}$, in the same way as

sentences of 0 can be either sentences of $0_{c\ell}$ or schemata for sentences

of $0_{c\ell}$. There is no way to represent with a sentence of D1 a

transition from a sentence of D1 built on 0_{op} to a sentence of D1

built on $0_{c\ell}$.

An alternative approach would be to permit D1 to express these

transitions. We can achieve this by making the schematic letters of

0 function as schematic letters for expressions of $0_{c\ell}$ only at some

levels or in some contexts of D1, and this can be made possible by

restricting $\underline{S}_{\neq t}$ only to those levels or those contexts. In all other

cases, these schematic letters would, so to speak, not be $\underline{used}$ as

schematic letters, but only $\underline{mentioned}$ as schematic letters. Certain

sentences of D1 would then cease to function as schemata for sentences

of D1 built on $0_{c\ell}$, and would be able to represent transitions from

§65.

sentences of Dl built on 0_{op} to sentences of Dl built on $0_{c\ell}$.

What would be the appropriate restriction on $\underline{S}_t$ in this approach? We can attempt an answer by assuming only the following

$$\underline{S}_t^* \quad \frac{\emptyset \vdash^1 \{A(x)\}}{S_a^x \emptyset \vdash^1 \{A(x)\}|}$$

The rule $\underline{S}_t^*$ can then be horizontalized, and also $(\forall)\downarrow$ and $(\exists)\downarrow$. A sequent like $\{\emptyset \vdash^1 \{A(x)\}\} \vdash^2 \{\emptyset \vdash^1 \{\forall x A(x)\}\}$ would then express something correct. So we could consider the system

$\underline{hADICTS}_t^*(\rightarrow)(\&)(\vee)(\bot)(T)(\forall)(\exists)$.

This system can be shown 0-equivalent with CQ/0 (also $\underline{S}_t^{\leqslant 1}$ would be derivable in it). However, this system has a serious drawback. Extending it with (□) would have unacceptable consequences. For example:

$$(\Box)\downarrow \frac{\{\emptyset \vdash \{A(x)\}\} \vdash^2 \{\emptyset \vdash \{\forall x\, A x\}\}}{\emptyset \vdash^2 \{\{\Box A(x)\} \vdash \{\forall x A(x)\}\}} \underline{h}((\forall)\downarrow)$$
$$\underline{D}^2 \frac{}{\{\Box A(x)\} \vdash \{\forall x\, A(x)\}}$$
$$(\rightarrow)\downarrow \frac{}{\emptyset \vdash \{\Box A(x) \rightarrow \forall x\, A(x)\}}$$
$$\underline{S}_t^* \frac{}{\emptyset \vdash \{\Box S_a^x A(x)| \rightarrow \forall x\, A(x)\}}$$
$$(\rightarrow)\uparrow \frac{}{\{\Box S_a^x A(x)|\} \vdash \{\forall x A(x)\}}$$
$$\underline{A}^2 \frac{}{\emptyset \vdash^2 \{\{\Box S_a^x A(x)|\} \vdash \{\forall x A(x)\}\}}$$
$$(\Box)\uparrow \frac{}{\{\emptyset \vdash \{S_a^x A(x)|\}\} \vdash^2 \{\emptyset \vdash \{\forall x A(x)\}\}} \quad ,$$

which is exactly what we wanted to prevent.

If instead of $\underline{S}^*_{\underline{t}}$ we assume in this system $\underline{S}^0_{\underline{t}}$ out of the scope of $\underline{h}$, this would not change the situation: in the proof above we would have in addition an application $\underline{D}^1$ before applying $\underline{S}^0_{\underline{t}}$, and then an application of $\underline{A}^1$. And at least $\underline{S}^0_{\underline{t}}$ is bound to be a derivable rule of CQ/O.

Our ban on horizontalizing would here be replaced by a ban on introducing ($\Box$). And intuitively it can be seen why this should be the case. Roughly speaking, ($\Box$) serves to reduce sequents of level 2 to sequents of level 1, and the analytic rules for the other propositional constants can always reduce sequents of level 1 to sentences of level 0. Now, at level 2 we should have sequents representing deductions from sentences built on O_{op} to sentences built on $O_{c\ell}$. But with ($\Box$), these decutions would be represented at level 1, or ultimately at level 0, and this is impossible. Sentences of level 0 are either sentences of $O_{c\ell}$ or schemata for sentences of $O_{c\ell}$: in them schematic letters of O_{op} cannot be mentioned, but only used.

Rather than concluding that there is a tension between quantification and modality in our treatment, we decided to put sentences of all levels in D1, hence also those of level 2, in the position which sentences of O occupy relative to $O_{c\ell}$. This leaves certain characteristic deductions of quantificational logic out of the expressive power of D1. On the other hand, this gives to the levels of systems in D1 a certain uniformity, and enables the other constants, including $\Box$, to perform their work of representing deductions of a higher level on a lower level.

§66   [§§58-61]   The spirit of our appraoch to the analytic rules

for quantifiers differs somewhat from the spirit of some usual

treatments of quantifiers in logical calculi. There rules for

quantifiers are made to perform the function of the rule of

substitution, whereas here we have tried to separate substitution

proper from the rules for quantifiers. In a certain sense, we don't

need the notion of substitution in order to understand the analytic

rules for the quantifiers; in a more indirect sense, we need this

notion because it is tied to the notion of a variable. But the

function of substitution is taken over by a separate rule.

It seems that there is no unanimity concerning the grammatical

category of expressions called "quantifiers". In addition to the approach

followed here (and which probably can be found in Frege 1879), it is

possible to find in the literature at least two other approaches. In one

they are of the category $\frac{\delta}{\delta}$, where by "a quantifier" is understood a

quantifier prefix; in the other they are of the category $\frac{\delta}{t\delta}$, where

by "a quantifier" is understood just "∀" or "∃" (v.e.g. Belnap 1975,

Borkowski 1958).

Also it seems that there is no unanimity concerning the question

whether or not the same letters should be used for free and bound

variables. Both orientations seem to be equally entrenched.

Distinguishing bound and free variables by different types of letters

may be recommended if we are afraid of confusing an expression of

0_{op} with parts of expressions of $0_{c\ell}$; but even then, free and bound

occurrences of variables of quantifiers can be distinguished. The

approach we have followed recommends itself because it makes possible

the giving of analytic rules without involving substitution, and

achieves a certain economy which does not seem more confusing than

it is illuminating.

§66

Another open choice in constructing quantification theory is whether or not to treat ∀xA as well formed even if no free x occurs in A. Our choice concerning the category of quantifiers dictates the second of these options.

Some of these alternative approaches to quantification theory may have advantages over our treatment, if some specific, often mathematical, purposes are to be achieved. The approach we have adopted is, of course, motivated by the general framework of this enquiry.

A study of generalized quantifiers like those mentioned in §61 can be found in Borkowski 1958 (cf. also Prior 1963 and Dummett 1973, p.162).

[§62] The characteristic rules and axiom-schemata of CQ/O are essentially those found e.g. in Kleene 1952. Considering Kleene's exact formulation we can illustrate some of the possibilities in formulating quantification theory mentioned above:

the rules and axiom-schemata do not separate principles of quantification from principles of substitution;

quantifiers are of the category $\frac{\delta}{\delta}$;

the same letters are used for free and bound variables;

∀xA is well formed even if x does not occur free in A.

("A(x)" in Kleene 1952 does not mean that x occurs in A(x): it is only part of the notation for substitution; v. Kleene 1952, pp.78 ff.

§66

Our notation concerning substitution, which is taken from Church 1956, differs from Kleene's.)

[§64] The system IRQ/O should be compared with the system RA of Smirnov *[v. §50]*.

Chapter 8

IDENTITY

§67   We first make some preliminary remarks concerning the language

O and assumptions which should be made in order to obtain a second-

-order context in which identity can be analyzed.

Then we consider an analytic rule for identify in this context,

and we demonstrate the O-equivalence of the extension of CQ/D with

this analytic rule and an axiomatization in O of classical first-

-order quantificational logic with identity. We also consider the

replaceability of this analytic rule by another analytic rule and by

formulae of O.

We then introduce non-classical and modal systems with identity.

Next, we consider an analytic rule which can serve for the

structural analysis of identity in a first-order context, like the

context of the previous chapter. In connection with this analytic

rule we also consider some single axiom-schemata for identity.

Finally, we make some remarks on second-order quantifiers, which

are placed in this chapter because of the connection these quantifiers

have with identity.

§68   In addition to the assumptions made about O in the last

chapter , we here assume that we have in O_{op} a <u>denumerably infinite</u>

number of variables of the category p_1. As schematic letters of U

for these variables we shall use

$$P, \; Q, \; R, \; P_1, \; Q_1, \; \ldots$$

Schemata of the form

$$P\alpha$$

will be used for sentences of O_{op} which are constructed from the

variable P and the singular term α.   (Note that the notation

"Px" differs from the notation "A(x)".)

    Basic schemata of U of the form

$$A^n(P_1, \; \ldots, \; P_k)$$

$n \geqslant 0$, $k \geqslant 1$, will be used for sentences of D1 in which P_i , for

every i, $1 \leqslant i \leqslant k$, <u>occurs at least once</u>, homologously to what we had

with "$A^n(x_1, \; \ldots, \; x_k)$".

    If an expression of O of the category p_k, $k \geqslant 1$, is applied to k

occurences of an expression α, we assume that we can replace the

resulting sentence by a synonymous sentence which is by explicit

definition an abbreviation of this sentence, and in which α occurs

only once.   This abbreviation can then be represented as constructed

from an expression of the category p_1 and α.   Since the schema $P\alpha$ of

O will be used for such abbreviations, it will also be used for the

original sentences. $P\alpha$ will then be a schema for all sentences

$S_\alpha^x A(x)|$ of O_{cl}, irrespectively of the number of occurrences of α in

$S_\alpha^x A(x)|$.   This qualifies in a certain sense the assumption that a

schematic letter is of the category c in O_{op} iff it is a

schematic letter for expressions of O_{cl} of the category c

[v.§4]. However, there is no contradiction: P is always a schema

of O_{op} for expressions of the category p_1, but sentences with these

expressions can be synonymous with some specific sentences in which

expressions of the category $p_k, k \geq 1$, occur.

Accordingly, in the presence of the variables P, Q, ..., for

structural systems we give the following rule

p_1-substitution $(S_{\to p_1})$: $\quad S_{\to p_1}^n \quad \dfrac{A^n(P)}{S_{S_a^x B(x)|}^{Pa} \; \left| A^n(P) \right|}$, $n \geq 0$,

where $S_{\;\; S_a^x B(x)|}^{\;Pa} \left| A^n(P) \right.$ is the result of substituting $S_a^x B(x)|$ at

the place of every occurrence of Pa in $A^n(P)$, for every a, such that Pa

occurs in $A^n(P)$. No empty applications of this rule are possible.

Also, the notation "$S_B^{Pa} A^n |$" will be used iff A^n is $C^n(P)$, for

some $C^n(P)$.

(In $S_a^x B(x)|, x$ occurs only as a place-marker, so that for $S_{\to p_1}$

we need not necessarily assume that we have the variables $x, y, \ldots$,

in O; they can be used only as an auxiliary of the notation for

instances. $S_a^x B(x)|$ can be either a sentence of O_{op} or of O_{cl} :)

If free and bound occurrences of P,Q, ..., are distinguished,

and also if we have the variables $x, y, \ldots$, in O, which can themselves

be free and bound, the rule $S_{\to p_1}$ would be subject to a number of provisions.

Additional provisions would be needed for the rule $S_{\to t}$ in this context.

We shall not try to spell out these provisions in detail [v. §75 for references on this topic].

The decision to give $S_{\to p_1}$ for all levels will have an influence on our treatment of the problems of this chapter homologous to the influence which the giving of $S_{\to t}$ for all levels had in the last chapter [cf. §65]. Also the rule $S_{\to p_1}$ will not be in the scope of h for similar reasons. As explained in the last chapter, the extension of $S_{\to t}$hADICT with $S_{\to p_1}$ does not give a structural system essentially different from the one used previously [v.§58].

§69   We assume that O has the following constant

identity: = ,

which is of the category p_2.

The analytic rule for identity is

$$(=) \quad \frac{\Gamma \cup \{Pa\} \vdash^1 \Delta \cup \{Pb\}}{\Gamma \vdash^1 \Delta \cup \{a=b\}} \quad ,$$

provided P does not occur free in Γ or Δ (substitution for "a" and "b" is free).

The proviso does not hold only for (=)↓, but also for (=)↑, so that our analytic rule is completely symmetric. But in (=)hADIC, the rule (=)↑ without the proviso is horizontalizable, and hence derivable:

$$(=)\!\uparrow \; \underline{A^2} \; \frac{\{a=b\} \vdash \{a=b\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\underbrace{\dfrac{\{a=b, Pa\} \vdash \{Pb\}}{\varnothing \vdash^2 \{\{a=b, Pa\} \vdash \{Pb\}\}}}_{①}}$$

$$\underline{h}^\wedge(\underline{C}^\wedge)$$

$$\underline{C^2} \; \frac{① \quad \{\Gamma \vdash \Delta \cup \{a=b\}, \{a=b, Pa\} \vdash \{Pb\}\} \vdash^2 \{\Gamma \cup \{Pa\} \vdash \Delta \cup \{Pb\}\}}{\{\Gamma \vdash \Delta \cup \{a=b\}\} \vdash^2 \{\Gamma \cup \{Pa\} \vdash \Delta \cup \{Pb\}\}.}$$

That $(=)\uparrow$ without the proviso is derivable can be demonstrated with the help of $(=)\uparrow$, $\underline{h}^0(\underline{I}^0)$ and $\underline{C}^1$ alone, or $(=)\uparrow$ and $\underline{S}^1_{\not{p}_1}$ alone.

Homologously to what we had with $(\forall)\downarrow$ and $(\exists)\downarrow$, the rule $(=)\downarrow$ cannot be in the scope of $\underline{h}$ [cf.§60]. So, when we extend $\underline{S}_{\not{\kappa}\not{p}_1}\underline{\text{hADICT}}$, the analytic rule $(=)$ will not be in the scope of $\underline{h}$. This will not prevent $(=)\uparrow$ from being horizontalizable, as we have seen above.

Consider the analytic rule

$$(=)_1 \quad \dfrac{\Gamma \cup \{P a\} \vdash^1 \Delta \cup \{P b\} \qquad \Gamma \cup \{P b\} \vdash^1 \Delta \cup \{P a\}}{\Gamma \vdash^1 \Delta \cup \{a=b\}} \quad,$$

provided P does not occur free in Γ or Δ (substitution for "a" and "b" is free).

We can show

<u>Lemma 56</u>   56.1 The rules given by $(=)_1$ are deducible in $(=)\underline{S}_{\not{p}_1}\underline{\text{hIC}}$.

56.2 The rules given by $(=)$ are deducible in $(=)_1\underline{S}_{\not{p}_1}\underline{\text{hIC}}$.

<u>Demonstration</u>: 56.1 We only need to show the following, where a is not b,

$$(=)\downarrow \frac{\{Pa\}\vdash\{Pa\}\quad \underline{h}°(\underline{I}°)}{\emptyset\vdash\{a=a\}} \qquad (=)\uparrow\ \underline{S}_{p_1}\frac{\dfrac{\{a=b\}\vdash\{a=b\}\quad \underline{h}°(\underline{I}°)}{\{a=b,Pa\}\vdash\{Pb\}}}{\{a=b,a=a\}\vdash\{b=a\}} \qquad \underline{h}°(\underline{I}°)$$

$$\underline{C}^1\ \frac{}{} \qquad (=)\uparrow\frac{\{b=a\}\vdash\{b=a\}}{\{b=a,Pb\}\vdash\{Pa\}}$$

$$\underline{C}^1\ \frac{\{a=b\}\vdash\{b=a\}}{}$$

$$\underline{C}^1\ \frac{\Gamma\vdash\Delta\cup\{a=b\}\qquad \{a=b,Pb\}\vdash\{Pa\}}{\Gamma\cup\{Pb\}\vdash\Delta\cup\{Pa\}}.$$

56.2 We only need to show the following

$$(=)_1\downarrow\ \underline{C}^1\frac{\dfrac{\{Pa\}\vdash\{Pa\}\ \underline{h}°(\underline{I}°)\quad \{Pa\}\vdash\{Pa\}\ \underline{h}°(\underline{I}°)}{\emptyset\vdash\{a=a\}}\qquad \underline{S}_{p_1}\dfrac{\Gamma\cup\{Pa\}\vdash\Delta\cup\{Pb\}}{\Gamma\cup\{a=a\}\vdash\Delta\cup\{a=b\}}}{\Gamma\vdash\Delta\cup\{a=b\}}\ ;$$

if $\Gamma=\Gamma_1\cup\{a=a\}$, then we have

$$\underline{C}^1\frac{\emptyset\vdash\{a=a\}\qquad \Gamma\cup\{Pa\}\vdash\Delta\cup\{Pb\}}{\underline{S}_{p_1}\dfrac{\Gamma_1\cup\{Pa\}\vdash\Delta\cup\{Pb\}}{\Gamma\ \ \vdash\Delta\cup\{a=b\}}}\ ;$$

for $(=)\uparrow$ if $Pb\in\Gamma$ or $Pa\in\Delta$, we have

$$(=)_1\uparrow\ \underline{S}_{p_1}\frac{\dfrac{\Gamma\vdash\Delta\cup\{a=b\}}{\Gamma\cup\{Qa\}\vdash\Delta\cup\{Qb\}}}{\Gamma\cup\{Pa\}\vdash\Delta\cup\{Pb\}}. \qquad Q\text{ does not occur free in }\Gamma\text{ or }\Delta$$

Q.E.D.

From this we easily get that (=) is replaceable by $(=)_1$ in the contexts we shall consider.

§70   Consider the following formulae

$$\underline{i_1} \quad a = a \quad ;$$

$$\underline{i_2} \quad a = b \rightarrow (S_a^x A(x)| \rightarrow S_b^x A(x)|).$$

The extension of CQ/O with $\underline{i_{1-2}}$ will be called "CQ$\underline{i}$/O".

In this section we shall demonstrate the O-equivalence of CQ$\underline{i}$/O and the system

$$\underline{S_t S_{p_1}} \text{ (∀) (∃) (=)} \underline{hADICT} (\rightarrow) \text{ (\&) (∨) (⊥) (T)}$$

called "CQ$\underline{i}$/D".

<u>Lemma 57</u> $\underline{i_{1-2}}$ are provable in $\underline{S_{p_1}}$ (=)$\underline{hADIC}(\rightarrow)$.

<u>Demonstration:</u>

$$(=)\!\!\downarrow \quad \dfrac{\dfrac{\{Pa\} \vdash \{Pa\} \quad \underline{h}^o(\underline{I}^o)}{\varnothing \vdash \{a=a\}}}{a=a} \quad ; \qquad (=)\!\!\uparrow \quad \dfrac{\{a=b\} \vdash \{a=b\} \quad \underline{h}^o(\underline{I}^o)}{\{a=b, Pa\} \vdash \{Pb\}}$$

$$2 \text{ applications of } (\rightarrow)\!\!\downarrow, \text{ and } \underline{D}'$$

$$\underline{S}_{p_1}^o \quad \dfrac{a=b \rightarrow (Pa \rightarrow Pb)}{a=b \rightarrow (S_a^x A(x)| \rightarrow S_b^x A(x)|).}$$

Q.E.D.

§70

Lemma 58 (=)↓ is derivable, and (=)↑ is horizontalizable in

$\underline{\underline{S}}_{p_1}$ hAC(mp)$\mathcal{L}_{1\text{-}2}$.

Demonstration: For (=)↓ we have $\mathcal{L}_1$ and $\underline{A}^1$, and then we proceed

as for Lemma 56.2.

$①: \{\Gamma \vdash^1 \Delta \cup \{a=b\}, \{a=b, Pa\} \vdash \{Pb\}\} \vdash^2 \{\Gamma \cup \{Pa\} \vdash \Delta \cup \{Pb\}\}$ $\underline{h}^1(\underline{C}^1)$

$\underline{A}^1 \dfrac{a=b \rightarrow (Pa \rightarrow Pb) \quad \mathcal{L}_2}{}$

$\underline{C}^1 \dfrac{\varnothing \vdash^1 \{a=b \rightarrow (Pa \rightarrow Pb)\} \qquad\qquad \underline{h}^0((\underline{m\!b}))}{}$

$\underline{C}^1 \dfrac{\{a=b\} \vdash^1 \{Pa \rightarrow Pb\} \qquad\qquad \underline{h}^0((\underline{m\!b}))}{}$

$\underline{A}^2 \dfrac{\{a=b, Pa\} \vdash^1 \{Pb\}}{}$

$\underline{C}^2 \dfrac{\varnothing \vdash^2 \{\{a=b, Pa\} \vdash^1 \{Pb\}\} \qquad\qquad ①}{}$

$\dfrac{\{\Gamma \vdash^1 \Delta \cup \{a=b\}\} \vdash^2 \{\Gamma \cup \{Pa\} \vdash^1 \Delta \cup \{Pb\}\}.}{}$

Q.E.D.

So we have that (=) is replaceable in CQ$_1$/D by $\mathcal{L}_{1\text{-}2}$. Note

that this replaceability does not depend on the presence of (→)↓.

This replaceability will facilitate our demonstration of

Lemma 59   If a sentence of level 0 is provable in CQ$_1$/D, it is

provable in CQ$_1$/0.

Demonstration: We replace (=) in CQ$_1$/D by $\mathcal{L}_{1\text{-}2}$. Then we proceed

§71

as for Lemma 54. The only addition is in the induction step when $\underline{S}_{\mathcal{P}_1}$ is used. Then we show that $\underline{S}^0_{\mathcal{P}_1}$ is admissible in CQ_i/O.

Q.E.D.

From Lemmata 57 and 59 and Theorem 7 we get immediately

<u>Theorem 8</u> CQ_i/D and CQ_i/O are 0-equivalent.

§71   The systems obtained by replacing "<u>T</u>" in the name of CQ_i/D by (1) "<u>T</u>$_{\delta h}$", (2) "<u>T</u>$_{eh}$" and (3) by omitting "<u>T</u>", will be called (1) "HQ_i/D", (2)"KQ_i/D" and (3) "IRQ_i/D". The extension of CQ_i/D with (□), (◊) and (→) will be called "$S5Q_i/D$", and the subsystem of $S5Q_i/D$ where $\underline{T}^2$ and $\underline{h}^2(\underline{T}^2)$ are replaced by $\underline{T}^2_{\delta h}$ and $\underline{h}^2(\underline{T}^2_{\delta h})$ will be called "$S4Q_i/D$".

These systems could be shown 0-equivalent with HQ_i/O, KQ_i/O, IRQ_i/O, $S5Q_i/O$ and $S4Q_i/O$,respectively. These latter systems are obtained by extending HQ/O, KQ/O, IRQ/O, $S5Q/O$ and $S4Q/O$ with $\underline{\dot{\iota}}_{1-2}$.

Remarks homologous to those we have made in the previous chapter [v. §65] could be made concerning an alternative treatment in which $\underline{S}_{\mathcal{P}_1}$ would be restricted and (=)↓ horizontalizable. In this alternative treatment it would appear that there is a certain tension between identity and modality.

§72   We shall now consider an analytic rule which can serve to analyze identity in a first-order context. We make the same assumptions concerning 0 as in the chapter on first-order quantifiers, and we

0

§72

assume further that identity is in O.   Then we give the following

analytic rule

$$(=)_2 \quad \frac{S_a^x \; \Gamma\vdash^1 \Delta\,|}{\Gamma \cup \{x=a\}\vdash^1 \Delta}\quad.$$

The analytic rule $(=)_2$ is subject only to the provisions for the

notation "$S_a^x A^n|$"   [v. §59].   Hence, x must occur free in $\Gamma\vdash^1\Delta$

[cf. §73].

To show that $(=)_2$ can serve for the structural analysis of identity

we shall show that $(=)$ in CQi/D, HQi/D, KQi/D, IRQi/D, S5Qi/D

and S4Qi/D is replaceable by $(=)_2$.   This can be shown directly by

demonstrating that i_{1-2} are replaceable by $(=)_2$, but it is more

profitable to show first that i_{1-2} are replacable by a single axiom-

-schema, and then to show that this axiom-schema is replaceable by $(=)_2$.

Consider the formulae

$\underline{w_1}$ $S_a^x A(x)| \leftrightarrow \forall x(x = a \rightarrow A(x));$

$\underline{w_2}$ $S_a^x A(x)| \leftrightarrow \exists x(x = a \circ A(x));$

$\underline{w_3}$ $S_a^x A(x)| \leftrightarrow \exists x(x = a \; \& \; A(x)).$

We can demonstrate the following

<u>Lemma 60</u>   Let S be the extension of IRQ/O with ik_1 without proviso

and ik_2 [v. §§46,64].

60.1 $\underline{w_1}$ is provable in IRQ/O with i_{1-2} .

60.2 $\underline{w_2}$ is provable in S with $\underline{w_1}$.

§72

60.3 $\underline{i}_{1-2}$ are provable in S with $\underline{w}_2$.

60.4 $\underline{i}_{1-2}$ are provable in IRQ/0 with $\underline{w}_1$.

<u>Demonstration:</u> We shall only sketch the proofs needed, in general

without justifying the steps.

60.1

$$\cfrac{x=x \;\; \underline{i}_1 \qquad \cfrac{x=a \to (x=x \to a=x) \;\; \underline{i}_2}{x=x \to (x=a \to a=x)}}{x=a \to a=x}$$

$$\cfrac{a=x \to (S^x_a A(x)| \to A(x)) \;\; \underline{i}_2}{\cfrac{x=a \to (S^x_a A(x)| \to A(x))}{\cfrac{S^x_a A(x)| \to (x=a \to A(x))}{S^x_a A(x)| \to \forall x (x=a \to A(x))}}} \; ;$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x=a \to (A(x) \to S^x_a A(x)|) \;\; \underline{i}_2}{A(x) \to (x=a \to S^x_a A(x)|)}}{(x=a \to A(x)) \to (x=a \to (x=a \to S^x_a A(x)|))}}{(x=a \to A(x)) \to (x=a \to S^x_a A(x)|)}}{x=a \to ((x=a \to A(x)) \to S^x_a A(x)|)}}{\cfrac{a=a \;\; \underline{i}_1}{\exists x (x=a)} \quad \exists x (x=a) \to \exists x ((x=a \to A(x)) \to S^x_a A(x)|)}}{\cfrac{\exists x ((x=a \to A(x)) \to S^x_a A(x)|)}{\forall x (x=a \to A(x)) \to S^x_a A(x)|}} \; .$$

60.2

$$\cfrac{S^x_a A(x)| \to \forall x (x=a \to A(x)) \;\; \underline{w}_1}{\cfrac{S^x_a A(x)| \to (x=a \to A(x))}{\underbrace{x=a \to (S^x_a A(x)| \to A(x))}_{\textcircled{1}}}}$$

$$(x=a \circ A(x)) \rightarrow (x=a \circ A(x))$$

$$\overline{\quad x=a \rightarrow (A(x) \rightarrow (x=a \circ A(x)))\quad}$$

$$\overline{x=a \rightarrow ((S_a^x A(x)| \rightarrow A(x)) \rightarrow (S_a^x A(x)| \rightarrow (x=a \circ A(x))))}$$

$$① \quad \overline{(S_a^x A(x)| \rightarrow A(x)) \rightarrow (x=a \rightarrow (S_a^x A(x)| \rightarrow (x=a \circ A(x))))}$$

$$\overline{x=a \rightarrow (x=a \rightarrow (S_a^x A(x)| \rightarrow (x=a \circ A(x))))}$$

$$\overline{x=a \rightarrow (S_a^x A(x)| \rightarrow (x=a \circ A(x)))}$$

$$\overline{\exists x (x=a) \rightarrow \exists x (S_a^x A(x)| \rightarrow (x=a \circ A(x)))}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{②}$$

$$\dfrac{\forall x (x=a \rightarrow x=a) \rightarrow a=a \quad \underline{w_1}}{\dfrac{a=a}{\dfrac{\exists x (x=a) \qquad\qquad ②}{\dfrac{\exists x (S_a^x A(x)| \rightarrow (x=a \circ A(x)))}{S_a^x A(x)| \rightarrow \exists x (x=a \circ A(x)) \quad ;}}}}$$

$$\dfrac{S_x^y S_y^x A(x)|| \rightarrow \forall y (y=x \rightarrow S_y^x A(x)|) \quad \underline{w_1} \quad \begin{array}{l}\text{y does not}\\\text{occur free in } A(x)\end{array}}{\dfrac{A(x) \rightarrow \forall y (y=x \rightarrow S_y^x A(x)|)}{\dfrac{A(x) \rightarrow (y=x \rightarrow S_y^x A(x)|)}{\underbrace{y=x \rightarrow (A(x) \rightarrow S_y^x A(x)|)}_{③}}}}$$

$$\dfrac{\forall x (x=y \rightarrow x=y) \rightarrow y=y \quad \underline{w_1}}{\dfrac{y=y}{}}$$

$$\dfrac{\qquad\qquad\qquad y=y \rightarrow \forall x (x=y \rightarrow y=x) \quad \underline{w_1}}{\dfrac{\forall x (x=y \rightarrow y=x)}{\dfrac{x=y \rightarrow y=x \qquad\qquad ③}{S_t^o \dfrac{x=y \rightarrow (A(x) \rightarrow S_y^x A(x)|)}{\dfrac{x=a \rightarrow (A(x) \rightarrow S_a^x A(x)|)}{\dfrac{(x=a \circ A(x)) \rightarrow S_a^x A(x)|}{\dfrac{\forall x ((x=a \circ A(x)) \rightarrow S_a^x A(x)|)}{\exists x (x=a \circ A(x)) \rightarrow S_a^x A(x)| \quad .}}}}}}$$

60.3
$$\frac{S_a^x(x=a \to x=a)| \to \exists x(x=a \circ (x=a \to x=a)) \; \underline{w_2}}{\exists x(x=a \circ (x=a \to x=a))} \quad ④$$

$$\frac{\dfrac{(x=a \circ x=a) \to (x=a \circ x=a)}{\dfrac{x=a \to (x=a \to (x=a \circ x=a))}{\dfrac{x=a \to (x=a \circ x=a)}{\exists x(x=a) \to \exists x(x=a \circ x=a)}}}}{\;}$$

$$\frac{\dfrac{x=a \to ((x=a \to x=a) \to x=a)}{\dfrac{(x=a \circ(x=a \to x=a)) \to x=a}{④ \quad \dfrac{\exists x(x=a \circ(x=a \to x=a)) \to \exists x(x=a)}{\exists x(x=a)}}}}{\;}$$

$$\frac{\exists x(x=a \circ x=a) \qquad \exists x(x=a \circ x=a) \to a=a \; \underline{w_2}}{a=a \quad ;}$$

$$\frac{\exists x(x=b \circ A(x)) \to S_b^x A(x)| \quad \underline{w_2}}{\dfrac{\forall x((x=b \circ A(x)) \to S_b^x A(x)|)}{\dfrac{(x=b \circ A(x)) \to S_b^x A(x)|}{\underline{S}_t^o \dfrac{(a=b \circ S_a^x A(x)|) \to S_b^x A(x)|}{a=b \to (S_a^x A(x)| \to S_b^x A(x)|).}}}}$$

$$\frac{S_a^x(x=a \to a=a)| \to \exists x(x=a \circ(x=a \to a=a))}{\dfrac{\exists x(x=a \circ(x=a \to a=a))}{a=a}}$$

60.4
$$\frac{\forall x(x=a \to x=a) \to a=a \; \underline{w_1}}{a=a \quad ;}$$

$$\frac{S_a^y S_y^x A(x)|| \to \forall y(y=a \to S_y^x A(x)|) \quad \underline{w_1}, \; y \text{ does not}}{\dfrac{S_a^x A(x)| \to (y=a \to S_y^x A(x)|)}{\underline{S}_t^o \dfrac{y=a \to (S_a^x A(x)| \to S_y^x A(x)|)}{b=a \to (S_a^x A(x)| \to S_b^x A(x)|)}}} \quad \text{occur free in } A(x)$$

$$⑤$$

$$\frac{b=b \to \forall x(x=b \to b=x) \; \underline{w_1}}{\dfrac{\forall x(x=b \to b=x)}{a=b \to b=a}} \qquad ⑤$$

$$a=b \to (S_a^x A(x)| \to S_b^x A(x)|).$$

$$Q.E.D.$$

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(If S is a conservative extension of IRQ/0, the demonstration of 60.4 is superfluous.)

Since IRQi/0 is a subsystem of the systems CQi/0, HQi/0, KQi/0, S5Qi/0 and S4Qi/0, we have that in all these systems i_{1-2} is replaceable by w_1. Moreover, since ik_1 without proviso and ik_2 hold in the latter systems when we replace "∘" by "&", we have that in them i_{1-2} is replaceable by w_3. From this we get that (=) is replaceable by w_1 in CQi/D, HQi/D, KQi/D, IRQi/D, S5Qi/D and S4Qi/D, and by w_3 in all these systems except IRQi/D.

Next we show the following

<u>Lemma 61</u>   61.1 w_1 is provable in $(\forall)(=)_2 \underline{hDI}(\rightarrow)$.

61.2 $(=)_2 \downarrow$ is horizontalizable, and $(=)_2 \uparrow$ derivable in CQ/D, HQD, KQ/D, IRQ/D, S5Q/D and S4Q/D with $\dot{w}_1$.

<u>Demonstration:</u>

61.1

$$(\forall)\uparrow \frac{\{\forall x\,(x=a \to A(x))\} \vdash \{\forall x\,(x=a \to A(x))\}\,\underline{h}(\underline{I}^0)}{(\to)\uparrow\frac{\{\forall x\,(x=a \to A(x))\}\vdash\{x=a \to A(x)\}}{(=)_2\uparrow\frac{\{\forall x(x=a \to A(x)),x=a\}\vdash\{A(x)\}}{\{\forall x(x=a \to A(x))\}\vdash\{S_a^x A(x)\}}}}$$

$$\frac{}{\forall x\,(x=a \to A(x)) \to S_a^x A(x)\vert}\ (\to)\downarrow\ \text{and}\ \underline{D}^1\ ;$$

$$(=)_2\downarrow\frac{S_a^x\{S_a^x A(x)\}\vdash\{A(x)\}\vert\quad \underline{h}(\underline{I}^0)}{(\to)\downarrow\frac{\{S_a^x A(x)\vert, x=a\}\vdash\{A(x)\}}{(\forall)\downarrow\frac{\{S_a^x A(x)\vert\}\vdash\{x=a \to A(x)\}}{\{S_a^x A(x)\vert\}\vdash\{\forall x(x=a\to A(x))\}}}}$$

$$\frac{}{S_a^x A(x)\vert \to \forall x(x=a \to A(x)).}\ (\to)\downarrow\ \text{and}\ \underline{D}^1$$

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61.2   In all these systems we have that

①: $\{S_a^x \Gamma \vdash^1 \Delta |\} \vdash^2 \{\emptyset \vdash^1 \{S_a^x A(x)|\}\}$ and

②: $\{\emptyset \vdash^1 \{S_a^x A(x)|\}\} \vdash^2 \{S_a^x \Gamma \vdash^1 \Delta|\}$

are provable, where A(x) is obtained as follows

$$\frac{\Gamma \vdash^1 \Delta}{\text{eventual applications of } (v)_1 \text{ or } (\bot)}$$

$$\frac{}{\Gamma \vdash^1 \{D\}}$$

eventual applications of ($\rightarrow$)

$$\frac{}{\emptyset \vdash^1 \{A(x)\};}$$

we also have that

③: $\{\emptyset \vdash^1 \{\forall x(x = a \rightarrow A(x))\}\} \vdash^2 \{\Gamma \cup \{x = a\} \vdash^1 \Delta\}$ is provable,

and

④: $\dfrac{\Gamma \cup \{x = a\} \vdash^1 \Delta}{\emptyset \vdash^1 \{\forall x(x = a \rightarrow A(x))\}}$

derivable. Then we can easily show with $\underline{W}_1$ that

⑤ : $\{\emptyset \vdash^1 \{S_a^x A(x)|\}\} \vdash^2 \{\emptyset \vdash^1 \{\forall x(x = a \rightarrow A(x))\}\}$

and ⑥ : $\{\emptyset \vdash^1 \{\forall x(x = a \rightarrow A(x))\}\} \vdash^2 \{\emptyset \vdash^1 \{S_a^x A(x)|\}\}$

are provable.

From ①, ⑤, ③ and $\underline{C}^2$ it follows that $(=)_2\downarrow$ is horizontal-izable, and from ④, ⑥ and ② that $(=)_2\uparrow$ is derivable.

Q.E.D.

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We can conclude that (=) is replaceable by $(=)_2$ in the systems with (=) we have considered. The analytic rule $(=)_2$ can also be shown sufficient for $\underline{\iota_{1-2}}$ in the absence of ($\forall$).

The analytic rule $(=)_2$ suggests analytic rules for the constant

$$\text{non-equality: } \neq \quad,$$

which is of the category p_2. These are the following analytic rules

$$(\neq)_2 \quad \frac{S^x_a \; \Gamma \vdash^A \Delta \;|}{\Gamma \vdash^A \Delta \cup \{x \neq a\}} \quad ;$$

$$(\neq)_3 \quad \frac{S^x_a \; \Gamma \vdash^A \emptyset \;|}{\Gamma \vdash^A \{x \neq a\}} \quad .$$

The analytic rule $(\neq)_2$ would be appropriate for classical logic, but only $(\neq)_3$ would be appropriate for intuitionistic logics. In classical logic $x \neq a$ is explicitly definable as $\neg\, x = a$, and $x = a$ as $\neg\, x \neq a$. In intuitionistic logics we could have only the first of these explicit definitions.

A single axiom-schema for $\neq$ in both classical and intuitionistic logics would be

$$\neg S^x_a A(x) \;|\; \leftrightarrow \; \forall x(A(x) \rightarrow x \neq a).$$

In classical logic we could have also with the same effect

$$S^x_a A(x) \;|\; \leftrightarrow \; \forall x(x \neq a \lor A(x)) .$$

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§73   The rule

$$\frac{\Gamma \cup \{x = a\} \vdash^1 \Delta}{\Gamma \vdash^1 \Delta}$$

is derivable in $\underline{S_{\neq t}}AC\underline{\iota_1}$ if x does not occur free in $\Gamma \vdash^1 \Delta$, and

with $\underline{T_{e\hbar}}$ we have that the following

$$\frac{\Gamma \vdash^1 \Delta}{\Gamma \cup \{x = a\} \vdash^1 \Delta}$$

is derivable.

Let us assume for the moment that the notation $"S_a^x A|"$ can

be used even if x does not occur free in A^u. Then, for all the

logics we have considered, with the exception of intuitionistic relevant

logic, we could give the analytic rule $(=)_2$ even if x does not

occur free in $\Gamma \vdash^1 \Delta$. Also, with this same assumption about $"S_a^x A|"$,

the following forms of $\underline{w_1}$ and $\underline{w_3}$ could be used as single axiom-schemata

for identity in all these logics, with the exception of the relevant

one,

$$\underline{w_1'} \quad S_a^x A| \leftrightarrow \forall x \; (x = a \to A);$$

$$\underline{w_3'} \quad S_a^x A| \leftrightarrow \exists x \; (x = a \; \& \; A).$$

That $(=)_2$ and $\underline{w_1'}$ could not be used in this form for intuitionistic

relevant logic follows from the fact that $A \to \forall x \; (x = a \to A)$ cannot

be provable for any A in this logic.

The analytic rule $(=)_2$ involves essentially substitution by $\underline{S_{\neq t}}$,

and whereas the logics with $\underline{T_{e\hbar}}$ seem to be compatible with empty

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applications of $\underline{S}_{\tau}$, intuitionistic relevant logic in a certain

sense is not. This holds if we want the analytic rule $(=)_2$ and

take it that $\underline{i}_{1-2}$ are the appropriate forms of axiom-schemata

for identity. If we took $\underline{i}_2$ in the form

$$\underline{i'_2} \quad a = b \ \rightarrow \ (S^x_a A| \ \rightarrow \ S^x_b A|),$$

where "$S^x_a A|$" is used even if x does not occur free in A, then it

would seem that even intuitionistic relevant logic is compatible

with empty applications of substitution (this would also require a

change in our formulation of $\underline{S}_{\tau}$ and $\underline{S}_{p_1}$). But a sentence of the form

$$a = b \ \rightarrow (A \ \rightarrow \ A),$$

where a and b do not occur in A, does not seem to be based on some

valid principle concerning identity, and seems to be based on the

acceptabilility of $\underline{T}_{er}$.

Of course, in a certain sense nothing is lost by banning empty

applications of substitution: these are instances of $\underline{I}$. Something

is lost in a context where principles justified by $\underline{T}_{er}$ are "smuggled in"

as valid principles about identity, and $\underline{T}_{er}$ is not acceptable in this

context.

4   In this section we shall make some remarks on second-order

quantifiers. Suppose that we have in O_{op} variables for expressions of

O_{cl} of all the categories p_k, $k \geqslant 0$. As ambiguous schematic letters

of U for all these variables we shall use

$$X, Y, Z, X_1, Y_1, \ \ldots$$

Schemata of the form

$$X a_1 \ \ldots \ a_k \quad ,$$

$k \geqslant 0$, will be used for sentences of O_{op} which are constructed from the

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variable X and the expressions $a_1,\ldots,a_k$, if any. We have remarked

before that "A(a,a)" has no sense, but "Xaa" has a sense: i.e.

at least some of the expressions $a_1,\ldots,a_k$ can be identical.

Basic schemata of the form

$$A^n(X_1, \ldots, X_k) \quad ,$$

$k \geqslant 1$, will be used for sentences of Dl in which X_i, for every i,

$1 \leqslant i \leqslant k$, occurs at least once.

We must in the presence of these schemata give corresponding

provability-preserving rules of substitution. In general, these

rules of p_k-substitution will be of the following form

$$\underset{\substack{S^{Xa_1}\ldots a_k \\ S^{x_1}_{a_1} \ldots S^{x_k}_{a_k} B(x_1, \ldots, x_k)|\ldots|}}{\underline{S^n_{p_k} \quad \dfrac{A^n(X)}{} \quad A^n(X)|}} \quad ,$$

$n \geqslant 0$, $k \geqslant 0$, where the notation under the line is interpreted

homologously to the notation

$$S^{Pa}_{\substack{x}} \; S^x_a B(x)| \; A^n(P)|$$

[v.§68];   if $k = 0$, the schema "B($x_1, \ldots,x_k$)" is the schematic

letter "B".

When free and bound occurrences of X, Y,..., are distinguished,

and also if we have the variables $x,y, \ldots,$ in O which can themselves

be free and bound, the rules $\underline{S}^n_{\neg p_k}$ are subject to a number of provisions.
Additional provisions are needed for $\underline{S}_t$ in this context.

Special provisions pertaining to the occurrence of quantifiers
in $B(x_1, \ldots, x_k)$ can be envisaged which would substantially influence
the ensuing logic. We shall not consider these matters here.

Our aim is to introduce the expressions of $O_{c\ell}$ called "second-
-order quantifiers". A second-order quantifier is constructed
homologously to a first-order quantifier, where instead of the variables
$x, y, \ldots$, we use the variables $X, Y, \ldots$

The analytic rules for second-order quantifier-schemata would be
of the following form

$$(\forall_{p_k}) \quad \frac{\Gamma \vdash^1 \Delta \cup \{A(X)\}}{\Gamma \vdash^1 \Delta \cup \{\forall X A(X)\}} \quad ,$$

provided X does not occur free in Γ or Δ;

$$(\exists_{p_k}) \quad \frac{\Gamma \cup \{A(X)\} \vdash^1 \Delta}{\Gamma \cup \{\exists X A(X)\} \vdash^1 \Delta} \quad ,$$

provided X does not occur free in Γ or Δ.

Considerations of analytic extensions with $(\forall_{p_k})$ and $(\exists_{p_k})$ would
in many ways parallel our treatment of first-order quantifiers.

Second-order quantifiers can be considered to belong to the
categories of the forms

$$p_k \quad \text{and} \quad \underbrace{\frac{\dfrac{\delta}{\delta}}{p_k \cdots p_k}}_{\ell} \quad ,$$

$k \geqslant 0, \ell \geqslant 1$.

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   If only bound variables X, Y, ..., of a particular category

p_m, $m \geqslant 0$, are envisaged, then all the quantifiers binding these

variables will be of the categories obtained when in the forms above

we take $k = m$. Let "$q_{k\ell}$" be an ambiguous abbreviation which stands

for either "$\dfrac{s}{\underbrace{p_k \cdots p_k}_{\ell}}$" or "$\underbrace{t \cdots t}_{k}$" .

Then the categories of second-order quantifiers are of the form

$$\frac{s}{q_{k\ell}} \quad , \quad k \geqslant 0 \ , \ \ell \geqslant 1 \ .$$

It is possible to introduce generalized second-order quantifiers

which would belong to the categories of the form

$$\frac{s}{q_{k\ell_1} \cdots q_{k\ell_m}} \quad ,$$

$k \geqslant 0$, $m \geqslant 1$, $\ell_i \geqslant 1$, $1 \leqslant i \leqslant m$.   For example, some quantifiers

of this kind would have analytic rules of the following form, where X

is of the category p_1,

$$(\overset{2}{\forall}_{p_1}) \quad \frac{\Gamma \cup \{A(X)\} \vdash^1 \Delta \cup \{B(X)\}}{\Gamma \vdash^1 \Delta \cup \{\overset{2}{\forall}X(A(X),B(X))\}} \quad ,$$

provided X does not occur free in Γ or Δ. These quantifiers would

belong to the categories

$$\frac{s}{q_{1\ell_1} q_{1\ell_2}} \quad , \quad \ell_1 \geqslant 1, \ \ell_2 \geqslant 1.$$

"$\overset{2}{\forall}X(A(X),B(X))$" can be explicitly defined as "$\overset{2}{\forall}X(A(X) \to B(X))$"

[cf. §61].

The variables P,Q,..., are just these particular cases of

X,Y,..., of the category p_1, and the rule $\underset{\to p_1}{S}$ a particular case

of the rules $\underset{\to p_k}{S^n}$. Also, in the presence of $(\forall_{p_1})$ and $(\to)$, identity

is explicitly definable as follows

$$a = b \quad =_{df} \quad \forall P(Pa \to Pb)$$

(or $\qquad a = b \quad =_{df} \quad \forall P(Pa \leftrightarrow Pb) \quad$).

Accordingly, identity could be conceived as a kind of generalized

second-order quantifier, "$\overset{2}{\forall}P(Pa,Pb)$" being a particular case of

"$\overset{2}{\forall}X(A(X),B(X))$". The generalized form of categories to which

identity would belong would be that of second-order quantifiers.

It is well known that the classical second-order quantificational

calculus, to which a system with analytic rules of the form $(\forall_{p_k})$

and $(\exists_{p_k})$ would correspond, is not complete with respect to a certain

standard interpretation of its expressions. The only conclusion we

can draw is that our analytic rules do not analyze the second-order

quantifiers which can be interpreted in that way. These expressions

are grammatically identical with the expressions we can analyze and

might have other close similarities, but they are not the same

expressions. We can conclude that the constants we can structurally

analyze are structural, whereas for the second-order quantifiers

interpreted in the standard way the question remains open whether in

some context with a certain notion of structural analysis they can

be analyzed [cf.§§88-89].

If in O_{op} we have variables of any other category not considered

up to now, we can have corresponding quantifiers in O_{cl}, which would

have analytic rules homologous to those we have given for first-order

and second-order quantifiers.

§75   [§§68,74]   The exact formulation of the rules of p_k-substitution,

of which $\underline{S}_{\overline{-p}_1}$ is a particular case, with all the provisions needed

in the presence of free and bound variables, was a vexing subject

in logic, judging by its relatively late appearance, the number of

unsuccessful attempts, and the names of those who made them (v.

Church 1956, §49). A precise treatment of this topic and of second-

-order quantification can be found in Church 1956. A more precise

treatment was evaded here in order not to lose sight of the main

subject of this work.

[§72]   A form of the single axiom-schema for identity $\underline{w}_3$

was discovered by Wang (v. Quine 1963, p.13, where it is shown

that it is necessary and sufficient for a form of $\underline{\iota}_1$ and $\underline{\iota}_2$, but

where the proof of sufficiency given is not valid intuitionistically).

As a single axiom-schema for identity, $\underline{w}_1$ seems to be more satisfactory

than $\underline{w}_3$: it involves constants which are somehow more basic, its

connection with the analytic rule $(=)_2$ is clearer, it can be

used without change of constants in intuitionistic relevant logic too,

and its intuitive understanding is certainly as easy as that of $\underline{w}_3$.

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Chapter 9

UNIQUENESS

§76   In this chapter we shall first formulate practical criteria

for uniqueness and non-uniqueness of expressions in systems

in D1 and systems in O.

Then using these criteria we show that, with certain assumptions

concerning O, the constants analyzed are unique in the analytic

extensions we have considered. On the other hand, we show that not

all constants in axiomatizations in O are unique: → is not

unique in the implicational fragments of CP/O, HP/O,

KP/O and IRP/O, and □ is not unique in S5P/O and S4P/O.

The uniqueness of → can be secured by rules of level 1, but we

show that some customary rules of level 1 which are given for □

cannot guarantee the uniqueness of □ . (In this context, we

suggest a form of natural deduction rules for □, derived from

our analytic rules, which would make □ unique.) Thus, even

disregarding the form we require for structural analyses, the

corresponding axiomatizations in O cannot serve for the analysis of

→, and neither the corresponding axiomatizations in O nor the

corresponding rules of level 1 can serve for the analysis of □.

Next, we consider a result connected with uniqueness.

According to this result, in the context of CP/D we can find

for every sentence of D1 a set of occurrences of sentences of D1

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which is in a certain sense equivalent to this sentence and which, under
some assumptions, satisfies the condition that no non-modal propositional
constant occurs in the sentences occurring in this set.

Finally, we make some remarks on the question of conservative-
ness of our analytic extensions.

§77   In this section we shall demonstrate a result which will
provide us with practical criteria, for determining that an
expression is unique or non-unique in a system in D1.

First, we give the following definitions

Definition of deductively equivalent formulae

The formulae of D1, A^n and B^n, not necessarily distinct,
are deductively equivalent in a system S in D1 iff $\{A^n\} \vdash^{n+1} \{B^n\}$
and $\{B^n\} \vdash^{n+1} \{A^n\}$ are provable in S.

Definition of D-subformulae

Let A^n be a formula of D1. Then

(1) A^n is a D-subformula of A^n;

(2)   if $\Gamma \cup \{B^{n-m}\} \vdash^{n-m+1} \Delta$ or $\Gamma \vdash^{n-m+1} \Delta \cup \{B^{n-m}\}$ is
a D-subformula of A^n , B^{n-m} is a D-subformula of A^n;

(3) nothing else is a D-subformula of A^n.

Let "X^n" be a basic schema for the basic schemata
$$A^n, B^n, \ldots ,$$

and let "$A^{n+m}[X^n]$" be a basic schema for formulae of Dl in which

X^n occurs as a D-subformula. We shall use schemata of the

form

$$S^{X^n}_{B^n} A^{n+m} [X^n] |$$

for formulae of Dl obtained by uniform substitution of B^n for

X^n in $A^{n+m}[X^n]$, and also the schematic and conceptual apparatus

of $\mho$ introduced previously [v.§22].

Then we can show the following

Lemma 62   The expressions α and β of O are synonymous in an

extension S of hADIC iff, for every $A^n(\xi)$ such that $S^\xi_\alpha A^n(\xi)|$

and $S^\xi_\beta A^n(\xi)|$ are formulae of Dl, $S^\xi_\alpha A^n(\xi)|$ and $S^\xi_\beta A^n(\xi)|$ are

deductively equivalent in S.

Demonstration: From right to left we use Lemma 1 to show that

$S^\xi_\alpha A^n(\xi)|$ and $S^\xi_\beta A^n(\xi)|$ are inter-derivable, and hence also inter-

-deducible and inter-admissible. For the other direction we have

$$\{ S^\xi_\alpha A^n(\xi) | \} \vdash^{n+1} \{ S^\xi_\alpha A^n(\xi) | \} \; \underline{h}^n(\underline{I}^n) \; ,$$

and by adm. — synonymity, and hence also by ded.-synonymity

and der.-synonymity, we get that

$$\{ S^\xi_\alpha A^n(\xi) | \} \vdash^{n+1} \{ S^\xi_\beta A^n(\xi) | \}$$

is provable in S . We proceed homologously for

$$\{ S^\xi_\beta A^n(\xi) | \} \vdash^{n+1} \{ S^\xi_\alpha A^n(\xi) | \} \; .$$

Q.E.D.

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This demonstration shows that in the extensions of <u>hADIC</u>
the three notions of synonymity coincide. According to our
convention [v.§22], "synonymous" in Lemma 62 means "<u>adm</u>.-
-synonymous", but we could appeal to one of the other notions
of synonymity as well.

<u>Lemma 63</u> A_1^n and A_2^n are deductively equivalent in an extension
S of <u>hAC</u> iff, for every $B^{n+m}[X^n]$, $m \geqslant 0$, such that $S_{A_1^n}^{X^n} B^{n+m}[X^n]|$
and $S_{A_2^n}^{X^n} B^{n+m}[X^n]|$ are formulae of Dl, $S_{A_1^n}^{X^n} B^{n+m}[X^n]|$ and
$S_{A_2^n}^{X^n} B^{n+m}[X^n]|$ are deductively equivalent in S.

<u>Demonstration</u>:   From right to left we just take that $B^{n+m}[X^n]$ is X^n.
From left to right we shall make an induction on the complexity
of $B^{n+m}[X^n]$. For the basis we have that if $B^{n+m}[X^n]$ is X^n, the
demonstration is trivial. For the induction step we only need to
show the following

$$\underline{A}^{k+2} \frac{\{C_1^k\} \vdash^{k+1}\{C_2^k\}}{\underline{C}^{k+2} \varnothing \vdash^{k+2}\{\{C_1^k\}\vdash^{k+1}\{C_2^k\}\}} \quad \frac{\{\{C_1^k\}\vdash^{k+1}\{C_2^k\}, \Gamma\cup\{C_2^k\}\vdash^{k+1}\Delta\}\vdash^{k+2}\{\Gamma\cup\{C_1^k\}\vdash^{k+1}\Delta\}}{\{\Gamma\cup\{C_2^k\}\vdash^{k+1}\Delta\}\vdash^{k+2}\{\Gamma\cup\{C_1^k\}\vdash^{k+1}\Delta\}} ; \underline{h}^{k+1}(\underline{C}^{k+1})$$

C_1^k cannot be a member of Γ, for otherwise the condition that both
$S_{A_1^n}^{X^n} B^{n+m}[X^n]|$ and $S_{A_2^n}^{X^n} B^{n+m}[X^n]|$ are formulae of Dl would not be
satisfied, and homologously below;

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$$\frac{A^{k+2}\{C_1^k\}\vdash^{k+1}\{C_2^k\}}{C^{k+2}\varnothing\vdash^{k+2}\{\{C_1^k\}\vdash^{k+1}\{C_2^k\}\}} \qquad \{\Gamma\vdash^{k+1}\Delta\cup\{C_1^k\}\},\{C_1^k\}\vdash^{k+1}\{C_2^k\}\}\vdash^{k+2}\{\Gamma\vdash^{k+1}\Delta\cup\{C_2^k\}\} \qquad h^{k+1}(C^{k+1})$$

$$\{\Gamma\vdash^{k+1}\Delta\cup\{C_1^k\}\}\vdash^{k+2}\{\Gamma\vdash^{k+1}\Delta\cup\{C_2^k\}\};$$

①: $\{\{\Gamma\cup\{C_2^k\}\}\vdash^{k+1}\Delta\cup\{C_2^k\}\}\vdash^{k+2}\{\{\Gamma\cup\{C_1^k\}\}\vdash^{k+1}\Delta\cup\{C_2^k\}\}$

②: $\{\{\Gamma\cup\{C_1^k\}\}\vdash^{k+1}\Delta\cup\{C_2^k\}\}\vdash^{k+2}\{\{\Gamma\cup\{C_1^k\}\}\vdash^{k+1}\Delta\cup\{C_1^k\}\}$

$$\frac{C^{k+2} \qquad ① \qquad ②}{\{\{\Gamma\cup\{C_2^k\}\}\vdash^{k+1}\Delta\cup\{C_2^k\}\}\vdash^{k+2}\{\{\Gamma\cup\{C_1^k\}\}\vdash^{k+1}\Delta\cup\{C_1^k\}\}};$$

the other cases needed are obtained by interchanging C_1^k and C_2^k above.

Q.E.D.

Next we give the following definitions

<u>Definition of formulae typical for an expression</u>

A formula of O is typical for an expression α of O iff α is either a sentence which is identical with this formula or is a sentence-forming functor last applied in the construction of this formula.

If α is of the categories

$$s, \quad \frac{s}{s}, \quad \frac{s}{ss}, \quad \frac{s}{p_{k+1}}, \quad p_2,$$

the formulae typical for α will be respectively of the forms

$$\alpha \, , \alpha A, \; A\alpha B, \; \alpha^{\prime} A(\varkappa), \; a \alpha b \; ,$$

where the $\varkappa$'s in $A(\varkappa)$ and $\alpha^{\prime}$ taken together constitute α.

We shall use basic schemata of the form

$$A(X^0)$$

for formulae of O in which X^0 occurs at least once, and

$$S_B^{X^0} A(X^0)|$$

for formulae of O obtained by uniform substitution of B for X^0 in $A(X^0)$.

Definition of deductively monotonic languages O

A language O is deductively monotonic with respect to a system S in D1 built on O iff [A and B are deductively equivalent in S iff, for every $C(X^0)$, $S_A^{X^0} C(X^0)|$ and $S_B^{X^0} C(X^0)|$ are deductively equivalent in S].

For this definition, and in the discussion which follows, we assume that substitution is everywhere free for formulae of O. Otherwise we would have to say above "for every $C(X^0)$ such that $S_A^{X^0} C(X^0)|$ and $S_B^{X^0} C(X^0)|$ are formulae of O", analogously to what we had with formulae of D1, for which substitution was not assumed to be everywhere free.

We have

Lemma 64   If O is deductively monotonic with respect to an extension S of hADIC, and $A(\xi)$ is a formula typical for ξ, then

§77

α and β are synonymous in S iff, for every $A(\xi)$, $S_\alpha^\xi A(\xi)|$ and $S_\beta^\xi A(\xi)|$ are deductively equivalent in S.

Demonstration: From left to right we use Lemma 62. From right to left we have, since $S_\alpha^\xi A(\xi)|$ and $S_\beta^\xi A(\xi)|$ are deductively equivalent for every $A(\xi)$, and since 0 is deductively monotonic, that $S_\alpha^\xi B(\xi)|$ and $S_\beta^\xi B(\xi)|$ are deductively equivalent for every $B(\xi)$, where $B(\xi)$ is not not necessarily typical for ξ. Then we use Lemma 63 to show that $S_\alpha^\xi C^n(\xi)|$ and $S_\beta^\xi C^n(\xi)|$ are deductively equivalent for every $C^n(\xi)$, such that $S_\alpha^\xi C^n(\xi)|$ and $S_\beta^\xi C^n(\xi)|$ are formulae of D1. By Lemma 62 it follows that α and β are synonymous.

Q.E.D.

From Lemma 64 we can easily obtain a practical criterion for uniqueness, and from Lemma 62 a practical criterion for non-uniqueness. These criteria are given in the following

Lemma 65 65.1 If 0 is deductively monotonic with respect to an extension $SS^\#$ of hADIC, and $A(\xi)$ is a formula typical for ξ, then α is unique in S iff, for every $A(\xi)$, $S_\alpha^\xi A(\xi)|$ and $S_{\alpha\#}^\xi A(\xi)|$ are deductively equivalent in $SS^\#$.

65.2 For any 0 and any extension S of hADIC, if for some $B(\xi)$, $S_\alpha^\xi B(\xi)|$ and $S_{\alpha\#}^\xi B(\xi)|$ are not deductively equivalent in $SS^\#$, α is not unique in S.

§78   The criteria for uniqueness and non-uniqueness of Lemma

65 can serve only for systems which are properly in D1 and not

only in O. In this section we shall formulate criteria for

uniqueness and non-uniqueness which can serve for systems in O

which satisfy certain conditions. One of these conditions is that

O has the constant $\rightarrow$, and we shall presuppose in this chapter that

O satisfies this condition. First we give the following definition

Definition of implicationally equivalent formulae

The formulae of O, A and B, not necessarily distinct, are

implicationally equivalent in a system S in O iff A $\rightarrow$ B and B $\rightarrow$ A

are provable in S.

Then we show a lemma corresponding to Lemma 62:

Lemma 66   If S is a system in O in which (mp) is derivable

and A $\rightarrow$ A provable, then α and β are synonymous in S iff for every

$B(\xi)$, $S_\alpha^\xi B(\xi)|$ and $S_\beta^\xi B(\xi)|$ are implicationally equivalent.

Demonstration: From right to left we use (mp). From left to right

we have

$$S_\alpha^\xi B(\xi)| \;\rightarrow\; S_\alpha^\xi B(\xi)|$$

and by synonymity we obtain that

$$S_\alpha^\xi B(\xi)| \;\rightarrow\; S_\beta^\xi B(\xi)|$$

is provable in S. We proceed homologously for the converse implication.

Q.E.D.

§78

This demonstration can show, analogously to what we had

with Lemma 62, that the three notions of synonymity coincide

for systems which satisfy the conditions of Lemma 66.

Next, we give the following definition

Definition of implicationally monotonic languages O

A language O is implicationally monotonic with respect to

a system S in O iff [A and B are implicationally equivalent in S

iff, for every $C(X^0)$, $S_A^{X^0}C(X^0)|$ and $S_B^{X^0}C(X^0)|$ are implicationally

equivalent in S].

Then we have corresponding to Lemma 64

Lemma 67   If O is implicationally monotonic with respect to

a system S in O in which (mp) is derivable and A → A provable, and

$A(\xi)$ is a formula typical for ξ, then α and β are synonymous

in S iff, for every $A(\xi)$, $S_\alpha^\xi A(\xi)|$ and $S_\beta^\xi A(\xi)|$ are implicationally

equivalent in S .

Demonstration: From left to right we use Lemma 66. From right

to left we have, in virtue of the implicational monotonicity of O,

that $S_\alpha^\xi B(\xi)|$ and $S_\beta^\xi B(\xi)|$ are implicationally equivalent for every

$B(\xi)$, where $B(\xi)$ is not necessarily typical for ξ. Then by Lemma

66 we get that α and β are synonymous.

Q.E.D.

§79

From Lemma 67 we can easily obtain a practical criterion
for uniqueness and from the demonstration of Lemma 66 a practical
criterion for non-uniqueness relative to some systems in O. These
criteria are given in the following

<u>Lemma 68</u>   68.1   If O is implicationally monotonic with
respect to a system $SS^{\#}$ in O in which (mp) is derivable and
A → A provable, and A (ξ) is a formula typical for ξ , then α
is unique in S iff, for every A(ξ), $S_{\alpha}^{\xi}A(\xi)|$ and $S_{\alpha^{\#}}^{\xi}A(\xi)|$ are
implicationally equivalent in $SS^{\#}$.

   68.2   For any O, and any system S in O in which A → A is
provable, if for some B(ξ), $S_{\alpha}^{\xi}B(\xi)|$ and $S_{\alpha^{\#}}^{\xi}B(\xi)|$ are not
implicationally equivalent in $SS^{\#}$, α is not unique in S .

We need both the criteria of Lemma 65 and those of Lemma
68 since extensions of <u>hADIC</u> need not be based on a language O
with →, and for axiomatizations in O the criteria of Lemma 65
are not applicable.

§79   In this section we shall use the criterion of Lemma 65.1
to show that the constants for which we have given analytic rules
are unique in the analytic extensions S of structural systems with
these analytic rules. In order to do this we must assume that O
is deductively monotonic with respect to the systems $SS^{\#}$. We
cannot show that O is such in general, since we have left undetermined
what specific language O is. But we must at least show that if we
start with a language O_1 without the constants →,&,∨,↔,¬,□,◊ and —3,

which is deductively monotonic with respect to an extension of S_1

of hADIC, adding the constants above, and $\alpha^{\#}$, where α is one

of the constants above, and also the corresponding analytic

rules to S_1, will preserve deductive monotonicity. Without some

specific assumptions about O_1, for example, that O_1 has functors

of the categories $\dfrac{p_k}{\delta}$, $k \geqslant 1$, or $\dfrac{t}{\delta}$, it is impossible to study

first-order quantifiers and identity with respect to this

monotonicity-preserving. Assuming that $\leftrightarrow, \neg, \Diamond$ and $\rightarrow$ are

explicitly defined, we have

Lemma 69   Let O_1 be the fragment of O without $\rightarrow, \&, \vee$ and $\square$,

and let O_1 be deductively monotonic with respect to an extension

S_1 of hADIC; next, let S be the system in D1 built on O obtained

by extending S_1 with $(\rightarrow)$, or $(\&)$, or $(\vee)$, or $(\square)$. Then O is

deductively monotonic with respect to $SS^{\#}$, where $\alpha^{\#}$ is $\rightarrow^{\#}$, or $\&^{\#}$,

or $\vee^{\#}$, or $\square^{\#}$.

Demonstration: It is enough to show that the following rules

are derivable

$$\dfrac{\{A\} \vdash \{B\}}{\{C \rightarrow A\} \vdash \{C \rightarrow B\}} \quad ; \qquad \dfrac{\{A\} \vdash \{B\}}{\{B \rightarrow C\} \vdash \{A \rightarrow C\}} \quad ;$$

$$\dfrac{\{A\} \vdash \{B\}}{\{A \& C\} \vdash \{B \& C\}} \quad ; \qquad \dfrac{\{A\} \vdash \{B\}}{\{C \& A\} \vdash \{C \& B\}} \quad ;$$

$$\dfrac{\{A\} \vdash \{B\}}{\{A \vee C\} \vdash \{B \vee C\}} \quad ; \qquad \dfrac{\{A\} \vdash \{B\}}{\{C \vee A\} \vdash \{C \vee B\}} \quad ;$$

$$\dfrac{\{A\} \vdash \{B\}}{\{\square A\} \vdash \{\square B\}} \quad .$$

§79

As an example, we show only that the last rule is derivable:

$$
\begin{array}{c}
(\square)\!\downarrow \dfrac{\{\emptyset \vdash^{1} \{A\}\} \vdash^{2} \{\emptyset \vdash^{1} \{A\}\} \quad \underline{h}^{1}(\underline{I}^{1})}{\emptyset \vdash^{2} \{\{\square A\} \vdash^{1} \{A\}\}} \\[4pt]
\underline{D}^{2} \\[2pt]
\underline{C}^{1} \dfrac{\{\square A\} \vdash^{1} \{A\} \qquad \{A\} \vdash^{1} \{B\}}{}
\end{array}
$$

$$
\begin{array}{cc}
\underline{A}^{2} \dfrac{\{\square A\} \vdash^{1} \{B\}}{\emptyset \vdash^{2} \{\{\square A\} \vdash^{1} \{B\}\}} &
\underline{A}^{2} \dfrac{\{\square B\} \vdash^{1} \{\square B\} \quad \underline{h}^{0}(\underline{I}^{0})}{\emptyset \vdash^{2} \{\{\square B\} \vdash^{1} \{\square B\}\}} \\[6pt]
(\square)\!\uparrow \quad \underline{C}^{2} \dfrac{\{\emptyset \vdash^{1}\{A\}\} \vdash^{2} \{\emptyset \vdash^{1}\{B\}\}}{} &
(\square)\!\uparrow \dfrac{\{\emptyset \vdash^{1}\{B\}\} \vdash^{2} \{\emptyset \vdash^{1}\{\square B\}\}}{}
\end{array}
$$

$$
(\square)\!\downarrow \dfrac{\{\emptyset \vdash^{1} \{A\}\} \vdash^{2} \{\emptyset \vdash^{1} \{\square B\}\}}{\emptyset \vdash^{2} \{\{\square A\} \vdash^{1} \{\square B\}\}}
$$

$$
\underline{D}^{2} \quad \{\square A\} \vdash^{1} \{\square B\} .
$$

For the constants $\alpha^{\#}$ we proceed homologously.

Q.E.D.

Before stating the following theorem we remark that when α is a first-order quantifier, $\alpha^{\#}$ must have the same variables as α.

<u>Theorem 9</u> $\to,\&,\vee,\bot,\top,\square$, the universal and existential first--order quantifiers, and $=$, are unique in any extension S of <u>hADIC</u> with some structural postulates and the analytic rules $(\to)$, $(\&)$, $(\vee)$, $(\bot)$, $(\top)$, $(\square)$, $(\forall)$, $(\exists)$ and $(=)$ (these last three analytic rules being out of the scope of <u>h</u>), provided O is deductively monotonic with respect to $SS^{\#}$.

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Demonstration: For each constant α, above, we have $\alpha^{\#}$, and we show in the systems $SS^{\#}$:

$$(\rightarrow)\!\uparrow \frac{\{A\rightarrow B\}\vdash\{A\rightarrow B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\rightarrow B, A\}\vdash\{B\}}$$
$$(\rightarrow^{\#})\!\downarrow \frac{}{\{A\rightarrow B\}\vdash\{A\rightarrow^{\#}B\}}\;;$$

$$(\&)\!\uparrow \frac{\{A\&B\}\vdash\{A\&B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\&B\}\vdash\{A\}} \qquad (\&)\!\uparrow \frac{\{A\&B\}\vdash\{A\&B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\&B\}\vdash\{B\}}$$
$$(\&^{\#})\!\downarrow \frac{}{\{A\&B\}\vdash\{A\&^{\#}B\}}\;;$$

$$(\vee)\!\uparrow \frac{\{A\vee B\}\vdash\{A\vee B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A\}\vdash\{A\vee B\}} \qquad (\vee)\!\uparrow \frac{\{A\vee B\}\vdash\{A\vee B\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{B\}\vdash\{A\vee B\}}$$
$$(\vee^{\#})\!\downarrow \frac{}{\{A\vee^{\#}B\}\vdash\{A\vee B\}}\;;$$

$$(\bot)\!\uparrow \frac{\{\bot\}\vdash\{\bot\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\bot\}\vdash\varnothing} \qquad\qquad (\top)\!\uparrow \frac{\{\top\}\vdash\{\top\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\varnothing\vdash\{\top\}}$$
$$(\bot^{\#})\!\downarrow \frac{}{\{\bot\}\vdash\{\bot^{\#}\}}\;; \qquad (\top^{\#})\!\uparrow \frac{}{\{\top^{\#}\}\vdash\{\top\}}\;;$$

$$A^{2}\;\frac{\{\Box A\}\vdash\{\Box A\}\quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\varnothing\vdash^{2}\{\{\Box A\}\vdash\{\Box A\}\}}$$
$$(\Box)\!\uparrow \frac{}{}$$
$$(\Box^{\#})\!\downarrow \frac{\{\varnothing\vdash\{A\}\}\vdash\{\varnothing\vdash\{\Box A\}\}}{}$$
$$D^{2}\;\frac{\varnothing\vdash^{2}\{\{\Box^{\#}A\}\vdash\{\Box A\}\}}{\{\Box^{\#}A\}\vdash\{\Box A\}}\;;$$

$$(\forall)\uparrow \quad \dfrac{\{\forall x A(x)\} \vdash \{\forall x A(x)\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{\forall x A(x)\} \vdash \{A(x)\}}$$
$$(\forall^{\#})\downarrow \quad \dfrac{}{\{\forall x A(x)\} \vdash \{\forall^{\#} x A(x)\}} \;\; ;$$

$$(\exists)\uparrow \quad \dfrac{\{\exists x A(x)\} \vdash \{\exists x A(x)\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{A(x)\} \vdash \{\exists x A(x)\}}$$
$$(\exists^{\#})\uparrow \quad \dfrac{}{\{\exists^{\#} x A(x)\} \vdash \{\exists x A(x)\}} \;\; ;$$

$$(=)\uparrow \quad \dfrac{\{a=b\} \vdash \{a=b\} \;\; \underline{h}^{\circ}(\underline{I}^{\circ})}{\{a=b, Pa\} \vdash \{Pb\}}$$
$$(=^{\#})\downarrow \quad \dfrac{}{\{a=b\} \vdash \{a=^{\#}b\}} \;\; .$$

The remaining cases are obtained by interchanging α and $\alpha^{\#}$ everywhere. Then we apply Lemma 65.1.

Q.E.D.

It can easily be shown for a number of other analytic rules we have considered, for example $(\neg)_3$ or $(=)_2$, that they would guarantee uniqueness.

§80   In this section we shall use the criteria of Lemma 68 to investigate the uniqueness of constants in the axiomatizations CQi/O, HQi/O, KQi/O, IRQi/O, S5Qi/O and S4Qi/O. Concerning the implicational monotonicity of O we could make some remarks analogous to those on deductive monotonicity. We can easily show that in all the systems above the following rules are derivable

$$\dfrac{A \to B}{(C \to A) \to (C \to B)} \;\; ; \quad \dfrac{A \to B}{(B \to C) \to (A \to C)} \;\; ; \quad \dfrac{A \to B}{(A \& C) \to (B \& C)} \;\; ; \quad \dfrac{A \to B}{(C \& A) \to (C \& B)} \;\; ;$$

$$\dfrac{A \to B}{(A \lor C) \to (B \lor C)} \;\; ; \quad \dfrac{A \to B}{(C \lor A) \to (C \lor B)} \;\; ; \quad \dfrac{A \to B}{\Box A \to \Box B} \;\; .$$

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We also have that if S is one of the systems above, the following would be derivable in $SS^{\#}$

$$\frac{A \to B}{(A \&^{\#} C) \to (B \&^{\#} C)} \; ; \; \frac{A \to B}{(C \&^{\#} A) \to (C \&^{\#} B)} \; ; \; \frac{A \to B}{(A \lor^{\#} C) \to (B \lor^{\#} C)} \; ;$$

$$\frac{A \to B}{(C \lor^{\#} A) \to (C \lor^{\#} B)} \; .$$

Remark that $SS^{\#}$ systems are obtained by adding only one $\alpha^{\#}$.

Next, we shall demonstrate

Lemma 70   70.1 $\&, \lor, T$, the universal and existential first-
-order quantifiers, and $=$, are unique in CQi/O, HQi/O, KQi/O,
IRQi/O, S5Qi/O and S4Qi/O, provided O is implicationally monotonic
with respect to $SS^{\#}$, where S is one of the systems above.

70.2 $\bot$ is unique in CQi/O, HQi/O, S5Qi/O and S4Qi/O,
provided O is implicationally monotonic with respect to $SS^{\#}$,
where S is one of the systems above.

Demonstration: 70.1   We have in $SS^{\#}$

1) $(A \& B) \to A \; \underline{\ell_1}$ and $(A \& B) \to B \; \underline{\ell_2}$ and either
   $(C \to A) \to ((C \to B) \to (C \to (A \&^{\#} B))) \; \overline{\underline{\ell_3^{\#}}}$ or
   $((C \to A) \&^{\#} (C \to B)) \to (C \to (A \&^{\#} B)) \; \overline{\underline{\ell_4^{\#}}}$; then we
   use (mp), or $(\underline{adj^{\#}})$ and (mp) to get $(A \& B) \to (A \&^{\#} B)$;

2) $A \to (A \lor B) \; \underline{a_1}$ and $B \to (A \lor B) \; \underline{a_2}$ and either
   $(A \to C) \to ((B \to C) \to ((A \lor^{\#} B) \to C)) \; \underline{a_3^{\#}}$ or
   $((A \to C) \& (B \to C)) \to ((A \lor^{\#} B) \to C) \; \underline{a_4^{\#}}$; then we
   use (mp), or $(\underline{adj})$ and (mp) to get $(A \lor^{\#} B) \to (A \lor B)$;

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3) we use $T \underline{\nu}$ and $T \to (T^{\#} \to T) \underline{c_2}$ or $T \underline{\nu}$, $T^{\#} \to (T \to T) \underline{\nu_1^{\#}}$ and $(A \to (B \to C)) \to (B \to (A \to C)) \underline{c_5}$ with $\underline{(mp)}$ to get $T^{\#} \to T$;

4) we have $(\forall^{\#} \underline{gen})$ $\dfrac{\forall x A(x) \to A(x) \; \underline{u}}{\forall x A(x) \to \forall^{\#} x A(x)}$;

5) we have $(\exists^{\#} \underline{gen})$ $\dfrac{A(x) \to \exists x A(x) \; \underline{e}}{\exists^{\#} x A(x) \to \exists x A(x)}$;

6) $\dfrac{a = {}^{\#} a \; \underline{i_1^{\#}} \qquad \dfrac{a = b \to (a = {}^{\#} a \to a = {}^{\#} b) \; \underline{i_2}}{a = {}^{\#} a \to (a = b \to a = {}^{\#} b)}}{a = b \to a = {}^{\#} b .}$

70.2 We have $\perp \to \perp^{\#} \; \underline{\delta}$.

The remaining cases are obtained by interchanging α and $\alpha^{\#}$
everywhere. Then we apply Lemma 68.1.

<div align="right">Q.E.D.</div>

The constant $\perp$ is not unique in KQi/0 and IRQi/0 since no
postulate is given for it: it is given as non-unique for these
systems. Accordingly, negation is non-unique in these systems,
whereas it is unique in the systems of Lemma 70.2, for we have

$$\dfrac{\neg A \to (A \to (B \& \neg^{\#} B)) \qquad (A \to (B \& \neg^{\#} B)) \to \neg^{\#} A}{\neg A \to \neg^{\#} A .}$$

On the other hand, we have

Lemma 71 $\to$ is not unique in $(mp) c_{1-3}$, $(mp) c_{1-2}$ and $(mp) c_1 c_{4-6}$.

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Demonstration: We show first that if S is one of the systems

of the Lemma ,

$$(A \to B) \to (A \to^{\#} B)$$

is not provable in $SS^{\#}$.   It is enough to show

that when S is $(mp)c_{1-3}$.   A fortiori, $(A \to B) \to (A \to^{\#} B)$ cannot

then be provable in any weaker system.

Take the matrices

$\to$	1 2 3 4
* 1	1 2 3 4
2	1 1 1 1
3	1 1 1 1
4	1 1 1 1

$\to^{\#}$	1 2 3 4
1	1 2 3 4
2	1 1 3 3
3	1 2 1 2
4	1 1 1 1

which give the value 1 for any assignment of values to

$$\underline{c_1} \ \left(A \to (B \to C)\right) \to \left(\left(A \to B\right) \to \left(A \to C\right)\right); \quad \underline{c_1^{\#}} \left(A \to^{\#} (B \to^{\#} C)\right) \to^{\#} \left(\left(A \to^{\#} B\right) \to^{\#} \left(A \to^{\#} C\right)\right);$$

$$\underline{c_2} \ A \to (B \to A); \qquad\qquad \underline{c_2^{\#}} \ A \to^{\#} (B \to^{\#} A);$$

$$\underline{c_3} \ \left((A \to B) \to A\right) \to A ; \qquad \underline{c_3^{\#}} \ \left((A \to^{\#} B) \to^{\#} A\right) \to^{\#} A ,$$

and satisfy

$$(mp) \ \frac{A \qquad A \to B}{B} \quad ; \quad (mp^{\#}) \ \frac{A \qquad A \to^{\#} B}{B} .$$

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Then we have

$$\left(A \to B\right) \to \left(A \to^{\#} B\right) \quad .$$

$$\begin{array}{ccccccc} 3 & 1 & 2 & 2 & 3 & 2 & 2 \end{array}$$

Hence, $(A \to B) \to (A \to^{\#} B)$ is not provable. Then we apply Lemma 68.2 and get that $\to$ is not unique.

Q.E.D.

Using the demonstration of this lemma we can show that $(\to)\downarrow$, with the proviso $\Delta = \emptyset$, is not deducible, and, a fortiori, not derivable or horizontalizable, in   extensions of structural systems with postulates in O [cf. §33]:

<u>Lemma 72</u>   Let S be <u>hADIC</u> extended with either $(\text{mp})c_{1-3}$ and $\underline{T}$; or $(\text{mp})c_{1-2}$ and $\underline{T}_{\mathit{sh}}$, or $\underline{T}_{\mathit{eh}}$; or $(\text{mp})c_1 c_{4-6}$. Then $(\to)\downarrow$, where $\Delta = \emptyset$, is not deducible in S.

<u>Demonstration:</u>   If $(\to)\downarrow$, $\Delta = \emptyset$, is deducible in S, $(\to^{\#})\downarrow, \Delta = \emptyset$, is deducible in $S^{\#}$, and both of these rules are deducible in $SS^{\#}$ [v. §11]. Then we should get

$$(\to^{\#})\downarrow \frac{\dfrac{\{A \to B, A\} \vdash \{B\} \quad \overset{\circ}{h}((\text{mp}))}{\{A \to B\} \vdash \{A \to^{\#} B\}}}{(\to)\downarrow \text{ and } \underline{D}} \\ (A \to B) \to (A \to^{\#} B) \quad .$$

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But if a sentence of level 0 is provable in $SS^\#$, it is provable

in $S_0S_0^\#$, where S_0 is $(mp)c_{1-3}$, $(mp)c_{1-2}$ or $(mp)c_1 c_{4-6}$, and

$\alpha^\#$ is $\rightarrow^\#$. This follows from the fact that $\underline{D}$ is eliminable

from $SS^\#$ [v.§21]. But $(A \rightarrow B) \rightarrow (A \rightarrow^\# B)$ is not provable in

$S_0S_0^\#$, as we have shown for Lemma 72. Hence, $(\rightarrow)\downarrow$, $\Delta = \emptyset$, is

not admissible in $SS^\#$, and hence, it is not deducible in S.

Q.E.D.

In other words, it would be possible to show that the Deduction

Theorem fails for the systems $S_0S_0^\#$ of the demonstration above.

(In an attempted demonstration of the Deduction Theorem the step that

would be blocked would be the one from $\Gamma \vdash A \rightarrow (B \rightarrow^\# C)$ and $\Gamma \vdash A \rightarrow B$ to

$\Gamma \vdash A \rightarrow C$.) The failure of the Deduction Theorem in $S_0S_0^\#$, and the

non-deducibility of $(\rightarrow)\downarrow$, $\Delta = \emptyset$, in S, underlie the non-uniqueness

of $\rightarrow$ in S_0. It is instructive to compare our results on the

replaceability of analytic rules by postulates in 0 with these

results connected with the uniqueness of $\rightarrow$.

We can also show

Lemma 73 $\square$ is not unique in S5 P /0 and S4 P /0.

Demonstration: We show first that if S is S5 P /0 or S4 P /0,

$$\square^\# A \rightarrow \square A$$

is not provable in $SS^\#$. It is enough to show that when S is S5P/0,

since S4P/0 is contained in S5P/0.

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Take the matrices

→	1 2 3 4		□			□$^{\#}$		¬
* 1	1 2 3 4	1	1	1	1	1	4
2	1 1 3 3	2	4	2	2	2	3
3	1 2 1 2	3	4	3	3	3	2
4	1 1 1 1	4	4	4	4	4	1 ; i.e. ⊥ takes

the value 4 (the matrices for the other constants, if they were

wanted, could be obtained from these by explicitly defining them

in terms of → and ⊥). These matrices give the value 1 for every

assignment of values to the axiom-postulates of CP/0 and

$$\underline{\ell_1} \quad \Box(A \to B) \to (\Box A \to \Box B) ;$$
$$\underline{\ell_2} \quad \Box A \to A ;$$
$$\underline{\ell_3} \quad \neg\Box A \to \Box\neg\Box A ;$$

$$\underline{\ell_1^{\#}} \quad \Box^{\#}(A \to B) \to (\Box^{\#}A \to \Box^{\#}B) ;$$
$$\underline{\ell_2^{\#}} \quad \Box^{\#}A \to A ;$$
$$\underline{\ell_3^{\#}} \quad \neg\Box^{\#}A \to \Box^{\#}\neg\Box^{\#}A ,$$

and they also satisfy (mp);

$$(\underline{nec}) \; \frac{A}{\Box A} \qquad ; \qquad (\underline{nec^{\#}}) \; \frac{A}{\Box^{\#}A} \; .$$

Then we have

$$\Box^{\#}A \to \Box A \; .$$
$$2 \quad 2 \quad 3 \quad 4 \; 2$$

Hence, $\Box^{\#}A \to \Box A$ is not provable. Then we apply Lemma 68.2

and get that □ is not unique.

Q.E.D.

The postulates which when added to the systems $SS^{\#}$ of the

demonstration above would enable us to prove $\Box^{\#}A \to \Box A$ and

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$\square A \rightarrow \square^{\#}A$ are

$$\square A \rightarrow \square^{\#}\square A \qquad \text{and}$$

$$\square^{\#}A \rightarrow \square\square^{\#}A,$$

for with $\ell_1, \ell_1^{\#}, \ell_2, \ell_2^{\#}$, (nec) and (nec$^{\#}$) we can prove $\square^{\#}\square A \rightarrow \square^{\#}A$ and $\square\square^{\#}A \rightarrow \square A$.

Let S be CQi/O, HQi/O, KQi/O, IRQi/O, S5Qi/O or S4Qi/O. It is known that all the systems S are conservative extensions of their implicational fragments S_1 [cf. §§31,34,46,50]. If it is also the case that $SS^{\#}$ are conservative extensions of $S_1 S_1^{\#}$, where $\alpha^{\#}$ is $\rightarrow^{\#}$, it would follow from the demonstration of Lemma 71 that

$\rightarrow$ is not unique in CQi/O, HQi/O, KQi/O, IRQi/O, S5Qi/O and S4Qi/O.

Then we could also show the following, corresponding to Lemma 72,

Let S be $\underline{S\text{-}S_{p_1}}$ $\underline{\text{hADIC}}$ extended with either the postulates of CQi/O and $\underline{T}$; HQi/O and $\underline{T_{\delta\hbar}}$; KQi/O and $\underline{T_{\varrho\hbar}}$; IRQi/O; S5Qi/O and $\underline{T}$; S4Qi/O and $\underline{T}$, where $T^2_{\delta\hbar}$ and $\underline{h}^2(T^2_{\delta\hbar})$ replace $\underline{T}^2$ and $\underline{h}^2(T^2)$ ((∀gen), (∃gen), (adj) and (nec) are not in the scope of $\underline{h}$). Then $(\rightarrow)↓$, where $\Delta = \emptyset$, is not deducible in S .

By giving the value 4 to $\perp$, the demonstrations of Lemmata 71 and 72 can easily be extended to show that $\rightarrow$ is not unique in $\underline{(mp)c_{1-3}ठ}$ and that $(\rightarrow)↓$, where $\Delta = \emptyset$, is not deducible in $\underline{\text{hADICT}(mp)c_{1-3}ठ}$. The other constants being explicitly defined in terms of $\rightarrow$ and $\perp$, $\underline{(mp)c_{1-3}ठ}$ gives a complete axiomatization of classical propositional logic.

Now let S be S5Qi/0 or S4Qi/0. It is known that S is a conservative extension of S_1, where S_1 is S5P/0 or S4P/0. If it is also the case that $SS^{\#}$ are conservative extensions of $S_1 S_1^{\#}$, where $\alpha^{\#}$ is $\square^{\#}$, it would follow from the demonstration of Lemma 73 that

$\square$ is not unique in S5Qi/0 and S4Qi/0.

1   In this section we shall consider an alternative notion of synonymity, and a corresponding notion of uniqueness, to show that with them we would obtain the same results as in the previous two sections.

Definition of derivably uniform (d.u.) synonymous expressions

α and β are d.u. synonymous in S iff, for every $A^n(\xi)$ such that $S_{\alpha}^{\xi} A^n(\xi)|$ and $S_{\beta}^{\xi} A^n(\xi)|$ are formulae of D1 and such that neither α nor β occurs in $A^n(\xi)$, $S_{\alpha}^{\xi} A^n(\xi)|$ and $S_{\beta}^{\xi} A^n(\xi)|$ are inter-derivable in S.

This notion of synonymity permits only uniform replacements of α by β and _vice versa_.

Definition of d.u. uniqueness

α is d.u. unique in S iff α and $\alpha^{\#}$ are d.u. synonymous in $SS^{\#}$.

Lemmata 62 and 66 guarantee that in the systems of these lemmata uniqueness entails d.u. uniqueness. Let us now take $\rightarrow$ and $\square$, for which we have shown that they are not unique in the systems of Lemmata 72 and 73. These constants are also not d.u. unique in these systems. For $\rightarrow$ we use the fact that $\dfrac{A \rightarrow B}{A \rightarrow^{\#} B}$ is not

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derivable. With the matrices of the demonstration of Lemma 72

there is an assignment of values which makes $A \rightarrow B$ designated

and $A \rightarrow^{\#} B$ not, so that the rule above is not designation-

-preserving, as a derivable rule would be.   For $\square$ we use the

fact that $\dfrac{A \rightarrow \square^{\#} A}{A \rightarrow \square A}$ is not derivable. With the matrices of the

demonstration of Lemma 73 we have that $A \rightarrow \square^{\#} A$ is always designated ,

whereas $A \rightarrow \square A$ is not.

We shall not investigate further the connection between

uniqueness and d.u. uniqueness. We only wanted to show that

with this second notion, which might also seem natural, and

need not coincide in general with our notion of uniqueness, we would

obtain the same results. Note that in the definition of d.u.

synonymity we could replace "inter-derivable" by "inter-deducible"

with the same results; but we could not replace it by "inter-

-admissible", for then it would follow that α is trivially unique

in the corresponding sense in any system.

§82   By, so to speak, "climbing" one level, and giving postulates

for $\rightarrow$ with rules which would correspond to the rules usually given

in natural deduction calculi

$(\underline{\rightarrow \text{intr}})$ $\dfrac{\Gamma \cup \{A\} \vdash^{1} \{B\}}{\Gamma \vdash^{1} \{A \rightarrow B\}}$; $(\underline{\rightarrow \text{elim}})$ $\dfrac{A \quad A \rightarrow B}{B}$,

we can obtain the uniqueness of $\rightarrow$, for we would have

$(\underline{\rightarrow^{\#}\text{intr}})$ $\dfrac{\{A \rightarrow B, A\} \vdash^{1} \{B\} \quad h^{0}((\underline{\rightarrow \text{elim}}))}{\{A \rightarrow B\} \vdash^{1} \{A \rightarrow^{\#} B\}}$.

We could also show in this context that adding $\rightarrow^{\#}$ would preserve

the deductive monotonicity of O.

§82

However, the rules corresponding to the usual natural

deduction rules for □ in S5 and S4, or the double-line rules

$(\Box_{S5})$ and $(\Box_{S4})$, cannot guarantee uniqueness for □ . The rules

$(\Box_{S5})$, or $(\Box_{S4})$, are equivalent to $(\Box_{S5})\!\downarrow$, or $(\Box_{S4})\!\downarrow$, and

$$\{\Box A\} \vdash^1 \{A\} \ ,$$

in the presence of $\underline{C}^1$.

We can show

Lemma 74   Let S be hADICT$(\rightarrow)$ $(\bot)$ extended with either $(\Box_{S5})$

or $(\Box_{S4})$. Then □ is not unique in S.

Demonstration : By Lemmata 47 and 48 we have that $(\Box_{S5})$, or $(\Box_{S4})$,

is replaceable in S by $\underline{(\mathrm{nec}^1)\ell_{1-3}}$, or $\underline{(\mathrm{nec}^1)\ell_{1-2}\ell_4}$. Consequently, we

need only to show that □ is not unique in the system S_1 obtained

by making this replacement. It can be shown that if a sentence

of level 0 is provable in $S_1 S_1^{\#}$, it is provable in $S_0 S_0^{\#}$,

where $\alpha^{\#}$ is $\Box^{\#}$, and S_0 is S5P/0, or S4P/0.   For this we can use

the o_1-translation to show that if A^n, $0 \leqslant n \leqslant 1$, is provable in $S_1 S_1^{\#}$,

$o_1(A^n)$ is provable in $S_0 S_0^{\#}$. $\underline{D}^2$ is eliminable from $S_1 S_1^{\#}$ [cf. §21]. But

$\Box^{\#}A \rightarrow \Box A$ is not provable in $S_0 S_0^{\#}$, as we have shown for Lemma 73; hence,

it is not provable in $S_1 S_1^{\#}$, and also $\{\Box^{\#}A\} \vdash^1 \{\Box A\}$ is not provable

in $S_1 S_1^{\#}$, since the previous formula can be deduced from this one

with $(\rightarrow)\!\downarrow$ and $\underline{D}^1$.   Then by Lemma 65.2, it follows that □ is not

unique in S_1. Hence, it is not unique in S.

Q.E.D.

§82

We can also replace "hADICT(→)(⊥)" in Lemma 74 by "hADIC" and obtain the same result.

To obtain uniqueness we would need the rules $(\square_{S5})$ and $(\square_{S4})$ with the proviso "if G∈Γ, then G = □B or G = $\square^{\#}$B for some B, and if D∈Δ, then D = □C or D = $\square^{\#}$C for some C", and a homologous form of the rules $(\square^{\#}_{S5})$ and $(\square^{\#}_{S4})$, where "$\square^{\#}$A" replaces "□A" in $(\square_{S5})$ and $(\square_{S4})$. These rules are horizontalizable in hADICT(□)($\square^{\#}$) and the system where $\underline{T}^2_{\Delta\hbar}$ and $\underline{h}^2(\underline{T}^2_{\Delta\hbar})$ replace $\underline{T}^2$ and $\underline{h}^2(\underline{T}^2)$, respectively. In these two systems $\{\square^{\#}A\}\vdash^1\{\square A\}$ is provable [v. Theorem 9].

Thus, we have to "climb" a further level, and assume the rules (□), in order to obtain uniqueness for □. Using the same metaphor, we can say that the non-uniqueness of □ at level 0 is "deeper" than the non-uniqueness of → .

Usually deductions in natural deduction calculi do not have deductions of levels ⩾ 1 or sentences of levels ⩾ 2 as premises or conclusions. A tentative formulation of something like natural deduction rules for □ in S4 which would give uniqueness is

$$(\underline{\square intr}) \quad \frac{\emptyset\vdash^1\{A\}}{\emptyset\vdash^1\{\square A\}} \quad ; \quad (\underline{\square elim}) \quad \frac{\Gamma \cup \{\emptyset\vdash^1\{A\}\}\vdash^2\{\emptyset\vdash^1\{B\}\}}{\Gamma\vdash^2\{\{\square A\}\vdash^1\{B\}\}} \quad ;$$

or to be more in the spirit of natural deduction notation, (□elim) would be written

$$\frac{\square A \qquad \overset{\displaystyle [\emptyset\vdash^1\{A\}]}{\emptyset\vdash^1\{B\}}}{B} \quad .$$

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For $\Diamond$ in S4 we could have

$(\Diamond\underline{intr})$ $\dfrac{\Gamma \cup \{\{A\}\vdash^1\emptyset\} \vdash^2 \{\{B\}\vdash^1\emptyset\}}{\Gamma\vdash^2\{\{B\}\vdash^1\{\Diamond A\}\}}$; $(\Diamond\underline{elim})$ $\dfrac{\{A\}\vdash^1\emptyset}{\{\Diamond A\}\vdash^1\emptyset}$;

or to be more in the spirit of natural deduction notation,

$(\Diamond\underline{intr})$ would be written

$$\dfrac{\begin{matrix} & [\{A\}\vdash^1\{\bot\}] \\ B & \{B\}\vdash^1\{\bot\} \end{matrix}}{\Diamond A} \quad ,$$

and $(\Diamond\underline{elim})$

$$\dfrac{\begin{matrix} & [A] \\ \Diamond A & \bot \end{matrix}}{\bot}$$

(The rule $(\Diamond\underline{intr})$ could perhaps be reduced to $\dfrac{\begin{matrix}[\{A\}\vdash^1\{\bot\}]\\ \bot\end{matrix}}{\Diamond A}$.)

It can easily be shown that ($\Box\underline{intr}$) and ($\Box\underline{elim}$) are horizontalizable in the presence of ($\Box$) and that they are sufficient for

$(nec)\ell_{1-2}\ell_4$. With these rules the uniqueness of $\Box$ can be

shown as follows

$$(\Box^{\#}\underline{elim})\ \underline{D^2\ \dfrac{\{\emptyset\vdash^1\{A\}\}\vdash^2\{\emptyset\vdash\{\Box A\}\}\ \underline{h}^1((\Box\underline{intr}))}{\emptyset\vdash^2\{\{\Box^{\#}A\}\vdash^1\{\Box A\}\}}}_{\{\Box^{\#}A\}\vdash^1\{\Box A\}} .$$

It is also possible to show the uniqueness of $\Diamond$ with $(\Diamond\underline{intr})$ and

$(\Diamond\underline{elim})$.

The language D1 is not designed to cover the mixing of

levels typical for some natural deduction rules, like the

introduction of $\rightarrow$, or the elimination of $\vee$. But a language like

DC [v. §16] could be better suited in this respect: in it we could

have a provable sequent of the form $\{\{A\}\vdash^1\{B\}\}\vdash^2\{A\to B\}$,

presumably deduced from $\{\{A\}\vdash^1\{B\}\}\vdash^2\{\emptyset\vdash^1\{A\to B\}\}$ and

$\{\emptyset\vdash^1\{A\to B\}\}\vdash^2\{A\to B\}$ with Cut of level 2. Thus, our

systems suggest natural deduction calculi in which we could

find deductions having as premises or conclusions deductions of

levels $\geqslant 1$ or sentences of levels $\geqslant 2$, i.e. sentences expressing

that deductions or levels $\geqslant 1$ can be made.

3   In this section we shall demonstrate a result connected

with uniqueness, in the context of CP/D. For this demonstration

we assume that D1 has an additional constant

                 D-conjunction: $\&_D$,

which is of the category $\dfrac{\delta}{\delta\delta}$ and is <u>not</u> a constant of O.

If A^n and B^n are sentences of D1 of level n, $A^n\&_D B^n$ is a sentence

of level n, where $n\geqslant 1$. For this constant we give the following

double-line rule, homologous to $(\&)_O$

$$(\&_D)\ \dfrac{A^n\qquad B^n}{A^n\&_D B^n}\quad,\quad n\geqslant 1\ .$$

Next we shall say that a set of occurrences of formulae

$\{B_1^n,\ \ldots,B_k^n\}, k\geqslant 1$, is inter-derivable with a formula A^n iff all

the rules given by the double-line rule

$$\dfrac{B_1^n\ \ldots\ B_k^n}{A^n}$$

are derivable.

Definition of propositionally decomposed sentences of D1

A sentence of D1 is propositionally decomposed iff none of
the constants $\rightarrow,\&,\vee,\perp,\top,\leftrightarrow,\neg,\&_D$ occurs in it out of the scope of a
functor. of O which is not in this list.

Then we can show

Lemma 75   Every sentence of D1 is inter-derivable in CP/D
extended with ($\&_D$) with a finite set of occurrences of
propositionally decomposed sentences of D1.

Demonstration: We have that the following formulae are
deductively equivalent with each other in CP/D extended with
($\&_D$)

$$\Gamma \vdash^A \Delta \cup \{A \rightarrow B\} \quad \text{and} \quad \Gamma \cup \{A\} \vdash \Delta \cup \{B\} \; ;$$

$$\Gamma \cup \{A \rightarrow B\} \vdash \Delta \quad \text{and} \quad (\Gamma \vdash \Delta \cup \{A\}) \&_D (\Gamma \cup \{B\} \vdash \Delta) \; ;$$

$$\Gamma \vdash^A \Delta \cup \{A \& B\} \quad \text{and} \quad (\Gamma^A \vdash \Delta \cup \{A\}) \&_D (\Gamma \vdash \Delta \cup \{B\}) \; ;$$

$$\Gamma \cup \{A \& B\} \vdash \Delta \quad \text{and} \quad \Gamma \cup \{A, B\} \vdash \Delta \text{ , or } \Gamma \cup \{A\} \vdash^A \Delta \; ;$$

$$\Gamma \cup \{A \vee B\} \vdash^A \Delta \quad \text{and} \quad (\Gamma \cup \{A\} \vdash \Delta) \&_D (\Gamma \cup \{B\} \vdash \Delta) \; ;$$

$$\Gamma \vdash^A \Delta \cup \{A \vee B\} \quad \text{and} \quad \Gamma \vdash^A \Delta \cup \{A, B\}, \text{ or } \Gamma \vdash^A \Delta \cup \{A\} \; ;$$

$$\Gamma \vdash^A \Delta \cup \{\perp\} \quad \text{and} \quad \Gamma \vdash^A \Delta \; ;$$

$$\Gamma \cup \{\perp\} \vdash \Delta \quad \text{and} \quad \{A\} \vdash \{A\} \; ;$$

$$\Gamma \cup \{\top\} \vdash \Delta \quad \text{and} \quad \Gamma \vdash \Delta \; ;$$

$$\Gamma \vdash^A \Delta \cup \{\top\} \quad \text{and} \quad \{A\} \vdash \{A\} \; .$$

To show this, we use Lemmata 11-15 and the following proofs

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$$(\bot)\uparrow \quad \frac{\dfrac{\dfrac{\dfrac{\{\bot\}\vdash\{\bot\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\{\bot\}\vdash\varnothing}\,T^1}{\Gamma\cup\{\bot\}\vdash\Delta}\,A^2}{\varnothing\vdash^2\{\Gamma\cup\{\bot\}\vdash\Delta\}}\,T^2}{\{\{A\}\vdash\{A\}\}\vdash\{\Gamma\cup\{\bot\}\vdash\Delta\}} \ ; \qquad (\top)\uparrow \quad \frac{\dfrac{\dfrac{\dfrac{\{\top\}\vdash\{\top\} \quad \underline{h}^\circ(\underline{I}^\circ)}{\varnothing\vdash\{\top\}}\,T^1}{\Gamma\vdash\Delta\cup\{\top\}}\,A^2}{\varnothing\vdash^2\{\Gamma\vdash\Delta\cup\{\top\}\}}\,T^2}{\{\{A\}\vdash\{A\}\}\vdash^2\{\Gamma\vdash\Delta\cup\{\top\}\}} \ ;$$

for the converses of the sequents proved we use $\{A\}\vdash^1\{A\}$, $\underline{A}^2$ and

$\underline{T}^2$. Negation can either be taken as explicitly defined or we

can use Lemma 16. We can also show that the following formulae

are deductively equivalent.

$$\Gamma\vdash^{n+1}\Delta\cup\{A^n\&_D B^n\} \quad \text{and} \quad (\Gamma\vdash^{n+1}\Delta\cup\{A^n\})\&_D(\Gamma\vdash^{n+1}\Delta\cup\{B^n\}) \ ;$$

$$\Gamma\cup\{A^n\&_D B^n\}\vdash^{n+1}\Delta \quad \text{and} \quad \Gamma\cup\{A^n,B^n\}\vdash^{n+1}\Delta, \text{ or } \Gamma\cup\{A^n\}\vdash^{n+1}\Delta,$$

using a result homologous to Lemma 12. Then we extend the

definition of D-subformulae with the clause

if $B^{n-m}\&_D C^{n-m}$ is a D-subformula of A^n, B^{n-m}

and C^{n-m} are D-subformulae of A^n ,

and we show that Lemma 63 still holds. For that we only need

$$\underline{C}^{k+1}\frac{\dfrac{\{B^k\&_D C_1^k\}\vdash^{k+1}\{C_1^k\} \quad \underline{h}^k((\&_D)\uparrow)}{\{B^k\&_D C_1^k\}\vdash^{k+1}\{C_2^k\}} \qquad \dfrac{\{C_1^k\}\vdash^{k+1}\{C_2^k\} \quad \underline{h}^k((\&_D)\downarrow)}{\{B^k,C_2^k\}\vdash^{k+1}\{B^k\&_D C_2^k\}}}{\underline{C}^{k+1}\dfrac{\{B^k\&_D C_1^k\}\vdash^{k+1}\{B^k\} \quad \underline{h}^k((\&_D)\uparrow) \qquad \{B^k\&C_1^k,B^k\}\vdash^{k+1}\{B^k\&_D C_2^k\}}{\{B^k\&_D C_1^k\}\vdash^{k+1}\{B^k\&_D C_2^k\}}} \ ;$$

if $B^k = C_2^k$, we omit the last application of $\underline{C}^{k+1}$; for

$\{C_1^k \&_D B^k\} \vdash^{k+1} \{C_2^k \&_D B^k\}$ we proceed homologously.

Then, using Lemma 63, the pairs of deductively equivalent

formulae above, and the fact that deductive equivalence is

transitive due to $\underline{C}$, we can show for every A^n, $n \geqslant 1$, that A^n

is deductively equivalent with $B_1^n \&_D \dots \&_D B_k^n$, $k \geqslant 1$,

where for every i, $1 \leqslant i \leqslant k$, B_i^n is propositionally decomposed.

Hence, using $(\&_D)$, A^n is inter-derivable with the set $\{B_1^n, \dots, B_k^n\}$.

If we have A^n, $n = 0$, A^n is inter-derivable with $\emptyset \vdash^{n+1} \{A^n\}$.

Q.E.D.

To demonstrate a form of this lemma for sentences of levels

0 and 1, we need not assume that we have $\&_D$ in D1.

The uniqueness of the constants $\rightarrow, \&, \vee, \perp, T, \leftrightarrow$ and $\neg$

of classical logic follows immediately from this Decomposition

Lemma (provided we have deductive monotonicity),

for we can decompose a sentence in which α occurs , where α

is one of these constants, and recompose it using $\alpha^{\#}$, the sentence

with α and the sentence with $\alpha^{\#}$ being synonymous. Roughly speaking,

if we have analytic rules which in a certain context guarantee

decomposition, they will also guarantee uniqueness.

However, a Decomposition Lemma need not be demonstrable in

every context we have considered, and we leave open the question

of its status in systems other than CP/D.

§84   Definition of conservative extensions

Let S_1 be an extension of a system S, such that the

expressions $\alpha_1, \ldots, \alpha_k$, $k \geqslant 1$, are essential for S_1, and that

the expression α_i, for every i, $1 \leqslant i \leqslant k$, is not essential for S;

and let A^n be a sentence in which α_i, for every i, $1 \leqslant i \leqslant k$, does

not occur. Then S_1 is a conservative extension of S iff

[for every A^n, A^n is provable in S_1 iff A^n is provable in S].

We have already had occasion to use this notion of conservative-

ness in the course of our work.

Whether an analytic extension will be conservative does not

depend only on the form of the analytic rules which are used for the

extension. As an example we can take Lemma 32, where ($\rightarrow$) and ($\perp$)

do not give rise to a conservative extension, whereas in other

contexts they do. So, to determine whether the analytic extensions

we have considered are conservative we should work from case to case.

If we restrict ourselves to sentences of level 0, we can take

advantage of the 0-equivalence theorems and of some results already

known regarding the separativeness of our axiomatizations in 0,

to demonstrate some results relating to conservativeness. We can

in some cases also appeal to the fact that a sentence A^1 is

provable iff a certain sentence of level 0 is provable, to extend

these results relating to conservativeness to sentences of level 1.

For sentences of level 1 we could also in some cases appeal

to the replaceability of our analytic rules by more conventional

sequent rules for introducing constants on the left and on the right.

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Then we would use results on the eliminability of $\underline{C}^1$ and on

the subformula property. But it is not clear how these methods

could be used also for higher levels.

In some cases we could also take advantage of a Decomposition

Lemma to demonstrate some results relating to conservativeness.

We can also show the following. If S_1 is a conservative extension

of S with respect to sentences of level $n + 1$, it is a conservative

extension with respect to sentences of level n, provided we have

$\underline{A}^{n+1}$ and $\underline{D}^{n+1}$. For suppose S_1 is not a conservative extension

at level n; then some A^n, in which none of $\alpha_1, \ldots, \alpha_k$ occur, is

provable in S_1 and not provable in S. But then $\emptyset \vdash^{n+1} \{A^n\}$ is

provable in S_1 and not provable in S; for it it were provable in S,

A^n would be too. Hence, S_1 is not a conservative extension at level

$n + 1$. But it is not clear that we could also show the converse.

We shall not treat questions concerning the conservativeness

of our analytic extensions in more detail. Conservativeness can

be important for enquiries closely related to ours, and it is

also possible that the main analytic extensions we have considered

are in fact conservative extensions of their subsystems. However,

in the next chapter we shall try to justify the position that

conservativeness does not play an essential role in structural

analyses.

§85   [§§22,77-81]   The notion of uniqueness as applied to

logical constants can be found in a certain form in Belnap 1962

(Belnap acknowledges his debt to a lecture of Hiż for this notion).

However, our understanding of this notion is nearer to a notion of

Smiley 1962.   There Smiley does not use the term "uniqueness" but

"functional dependence", and shows that & is functionally

dependent, whereas → is functionally independent in the Heyting

propositional calculus.   Smiley does not seem to take uniqueness

as a desirable property for a constant in a system.   He proposes

uniqueness as a counterpart to Padoa's criterion for dependence.

From Padoa 1900 one can get necessary, but not sufficient, conditions

for _definitional_ dependence (i.e. sufficient, but not necessary,

conditions for definitional independence), and necessary

and sufficient conditions for _functional_ dependence (or functional

independence).   This view is also expressed in McKinsey 1939.   To

show that a constant is unique means to show that it is somehow

dependent on the rest of the system.   Smiley shows that this does

not mean that it is explicitly definable in terms of other

expressions of the system.   Roughly speaking, if α is synonymous

with some β already in the system S, then α and $\alpha^{\#}$ will be

synonymous in $SS^{\#}$, provided α remains synonymous with β in $SS^{\#}$.

So, explicit definability gives uniqueness.   But if α and $\alpha^{\#}$

are synonymous in $SS^{\#}$, this does not mean that there is a β in

S synonymous with α (as a superficial analogy with Beth's Definability

Theorem might induce us to think).   Smiley discusses these questions

in detail.

Natural deduction rules do not guarantee uniqueness by their

very form, and this is why Belnap 1962 suggests that we state the

requirement of uniqueness when we want these rules to _define_ a

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Chapter 10

CONCLUSION

§86   This chapter is divided into two parts. In the first we

summarize the results of our analyses and then consider the relation of these

analyses to Thesis [I]. We shall also try to provide some grounds for this

thesis and to consider the general notion of analysis of which

structural analysis can be an instance.

In the second part we introduce the notion of structurally

alternative systems, and consider the relation of this notion,

together with those of our results which can be expressed with the

help of this notion, to Thesis [II]. We shall also try to provide some

grounds for this thesis.

In the historical remarks we shall try to compare our enquiry to

some other enquiries which have a similar goal or are in general

similar in spirit.

§87

THESIS [I]

§87   In the preceding chapters we have tried to show that the constants which are essential for the systems CQi/O, HQi/O, KQi/O, IRQi/O, S5Qi/O and S4Qi/O are structural.

We have shown that →,& ,∨,⊥,T, the universal and existential first-order quantifiers, and =, are primary structural constants. For any language L in which we have these constants we have shown first how to obtain the language O of clause (O) of the definition of structurally analyzed constants [v. §24]. L can either satisfy the conditions set for O, or O can be obtained by extending L. If the latter is the case, we can assume that L has only the constants analyzed, and the following situations are possible: (i) to L which has propositional constants we add some schematic letters of the category ⩔; (ii) to L which has first-order quantifiers we add some schematic letters of the categories t and p_k, $k \geqslant 1$, to obtain O_{cl}, and then add denumberably   many t-variables to obtain O; (iii) to L which has = we either add some schematic letters of the category t to obtain O_{cl}, and then add denumerably many p_1-variables to obtain O, or we proceed as for the first-order quantifiers (when we are using $(=)_2$). In cases (ii) and (iii), L will be included in O_{cl}. The schematic letters added to obtain O or O_{cl} will be schemata relative to O or O_{cl}, and appropriate rules of substitution should be assumed for them. Note that the schematic letters of O_{cl} would differ from the schematic letters of O_{op}, i.e. variables of O_{op}, of the same category.

The cases listed above do not exhaust all the possibilities in which O can be obtained by extending L. For example, in a case corresponding to (ii), L can have first-order quantifiers and constant

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singular terms, so that t-schemata of $O_{c\ell}$ need not be added; or

L can have first-order quantifiers and t-variables, so that

t-variables of O_{op} need not be added. Of course, we can also add to

L more than is strictly needed to obtain a language O sufficient

for structural analyses.

A language O obtained by extending L⁻ in which we have only the

constants analyzed will be called "schematic O", and we shall

later comment on it   [v. §93].

Next   we have given the analytic rules $(\rightarrow),(\&),(v),(\bot),(T)$,

the analytic rules which are instances of $(\forall)$ and $(\exists)$, and $(=)$(or

$(=)_2$), for clause (1) of the definition of structurally analyzed

constants.

For clause (2) of that definition we can appeal to our results

on the O-equivalence of systems in D1 and systems in O. That in

these systems in O are provable all the correct sentences involving

the constants analyzed is established either by fiat or by

independent semantical considerations.

For clause (3) we have the result of Theorem 9. Concerning

the proviso about the deductive monotonicity of O, we can make

a comment analogous to the comment on the proviso about the completeness

mentioned in clause (2) [v. §24]. If the language L we start with is

such that O cannot be deductively monotonic with respect to S , we

assume that there is a fragment of L in which we have α for which

a language O deductively monotonic with respect to a system with (α)

can be given. That the constant α of this fragment is the same as

the constant α of L would be shown by semantical considerations.

§88

We have shown that the constants ↔, ◇ and —⫟ are secondary structural constants, but it would also be possible to show that they are primary. The constant ¬ has also been shown a secondary structural constant. It is possible to show that it is primary as well, but then we would not be able to use the same analytic rule for all the contexts we have considered, as we have done with the other constants, and we should better distinguish symbolically the constant analyzed with $(¬)_1$ from the constant analyzed with $(¬)_3$. On the other hand, the same explicit definition of ¬ can be used in all the contexts we have considered.

§88   Let us now consider

Thesis [I]   A constant is logical iff it can be ultimately analyzed in purely structural terms.

When the intuitive notions used for stating Thesis [I] are interpreted in the more precise framework set up by our enquiry, this thesis entails

Thesis [Ia]
If a constant is structural, it is logical.

Later [§91] we shall argue that the inference of Thesis [Ia] from Thesis [I] is justified.

What "ultimately analyzed in purely structural terms" can mean besides "explicitly definable in terms of structurally analyzed constants", and consequently,what other constants besides structural constants could be called "logical" according to Thesis [I], is suggested by the following definitions

§88

  (1) a double-line rule is <u>analytic of the first degree</u>
    iff it is analytic;

  (2) a constant is a <u>structural constant of the first degree</u>
    iff it is structural;

and the following schemata of definitions

  (m_1) a double-line rule is <u>analytic of the mth degree</u> iff
  it differs from an analytic rule of the m-1th degree only
  by having structural constants of the m-1th degree occurring in the
  formulae B_1^n, ..., B_k^n ;

  (m_2) a constant is <u>structurally analyzed at the mth degree</u> by
  the double line rule (α) iff (α) is an analytic rule of the
  mth degree in which α is the only constant of O occurring in
  A^n , and clauses (O), (2) and (3) of the definition of
  structurally analyzed constants are satisfied

(as a consequence of (m_2) we get that a constant is structurally

analyzed at the first degree iff it is structurally analyzed);

  (m_3) a constant is a <u>structural constant of the mth degree</u> iff
  it is explicitly definable in terms of some constants structurally
  analyzed at degrees ℓ_1, ...,ℓ_k, $k \geqslant 1$, $\ell_i \leqslant m$, for every
  i, $1 \leqslant i \leqslant k$, $\ell_j = m$, for some j, $1 \leqslant j \leqslant k$.

Definition of general structural constants

  A constant of O is general structural iff it is a structural

constant of the mth degree for some m.

  (In a language deductively monotonic with respect to any system

considered, an explicit definition of a sentence-forming functor α,

or sentence α, can be given by a horizontalizable double-line rule

§88

$\dfrac{B}{A}$, where A is a formula typical for α and B is a formula in which, besides all the schematic letters of A, there occur only the expressions in terms of which α is defined. An example is

$$\dfrac{A \rightarrow \perp}{\neg A} \; ;$$

this double-line rule can be counted as analytic of the second degree.)

We assume that general structural constants could be said to be "ultimately analyzed in purely structural terms", and hence according to Thesis [I] they could be called "logical". We remark that in some cases where a Decomposition Lemma can be demonstrated, general structural constants could coincide with structural constants. Since we conjecture that there are contexts in which such a lemma is not available, we must leave open the possibility of there being general structural constants which are not structural.

The definitions of structural constants and general structural constants do not presuppose any particular language D like D1: they would in principle be compatible with a language D different from D1, though in this work it is only with D1 that these definitions have been given a precise sense. To get a plausible reconstruction of the full import of Thesis [I] we should also consider alternative languages D, and some of the notions defined only with reference to D1 would have to be redefined. In particular, various other conceptions on the form of our analyses could be envisaged.

§89   Thesis [Ia] can only serve to show that a constant <u>is</u> logical

and, of course, it cannot serve to show that a constant <u>is not</u>

logical. On the other hand, Thesis [I] which apparently also provides

a criterion for the second of these properties, need not guarantee

that the property of being logical is effectively decidable.

To take a disputed case: it can be shown that in the standard

interpretation the universal and existential second-order quantifiers

are not structural, nor general structural, if we understand these

notions on the background of D1 and our notion of analytic rule.

But this does not mean that with a certain conception of the

deductive meta-language and a certain conception of the form of

our analyses, we could not show that they are "ultimately analyzable

in purely structural terms ".

Another disputed case is the constant $\in$ of set-membership.

Concerning it we remark only that a double-line rule of the form

$$\frac{S^x_a A(x) \mid}{a \in \{x \mid A(x)\}} \quad ,$$

with or without proviso, would not be an analytic rule ($\in$ is of the

category p_2 and $\{x \mid A(x)\}$ of the category t; the notation for

set-abstraction is a constant of the category $\dfrac{t}{p_k}$, $k \geqslant 1$).   The

constant $\in$ and the constant of set-abstraction are not to be considered

as parts of one constant, since from $a \in \{x \mid A(x)\}$ we can deduce

$\exists y (a \in y)$. Otherwise , $a \in \{x \mid A(x)\}$ would be just a notational variant

of $S^x_a A(x) \mid$.

In general, on the background surveyed in this work we are

prepared to appeal only to Thesis [Ia].

§89

With this thesis we can e.g. answer the question whether the turnstiles of D1 are logical constants. Let us take that D1 is an object language and let us construct the deductive meta-language of D1 homologous to a language D3. We can assume that we have in D1 the constants $\to_D$, $\circ_D$, T_D, $+_D$ and $\perp_D$ for which we give analytic rules homologous to $(\to), (\circ)_{1.1}$, $(T), (+)_{1.1}$ and $(\perp)$. Then we can explicitly define $\Gamma \vdash^{n+1} \Delta$ in the following way

$$\Gamma \vdash^{n+1} \Delta \ =_{df} \ \hat{\Gamma} \to_D \check{\Delta} \ , \quad \text{where}$$

$$\hat{\Gamma} = \begin{cases} A_1^u \circ_D \cdots \circ_D A_k^u & , \text{ if } \Gamma = \{A_1^u, \ldots, A_k^u\}, k \geqslant 1, \\ T_D & , \text{ if } \Gamma = \emptyset ; \end{cases}$$

$$\check{\Delta} = \begin{cases} A_1^u +_D \cdots +_D A_k^u & , \text{ if } \Delta = \{A_1^u, \ldots, A_k^u\}, k \geqslant 1, \\ \perp_D & , \text{ if } \Delta = \emptyset . \end{cases}$$

Not all expressions of $\Gamma \vdash^{n+1} \Delta$ excluding $\vdash^{n+1}$ occur in $\hat{\Gamma} \to_D \check{\Delta}$, i.e. $\emptyset$ and the curly brackets are omitted. But these constants have, so to speak, only a grammatical function (in contradistinction to the notation for set-abstraction we mentioned in connection with $\in$). They transform $k \geqslant 0$ expressions of the category s into an expression of the category t, where $\vdash^{n+1}$ is of the category p_2; but alternatively we could take that $\vdash^{n+1}$ is of the categories

§90

$$
\cfrac{\quad \overbrace{\underline{\Delta \ldots \Delta}}_{k_1}^{\underline{\Delta}} \quad}{\underbrace{\Delta \ldots \Delta}_{k_2}} \quad , \; k_1 \geqslant 0 \;, \; k_2 \;\geqslant\; 0 \;,
$$

and then ∅ and the curly brackets would not stand for genuine

expressions of D1, but will only be an auxiliary notation of U

for showing the arguments of $\vdash^{n+1}$. This holds if we do not envisage

first-order quantification over singular terms of D1.

Hence, the turnstiles of D1 are structural constants, and by

Thesis [Ia] they are logical.

§90   If we have the constants $\bullet_D$ and $+_D$ in D1 we can see that an

analytic rule $\dfrac{B_1^n \; \ldots \; B_k^n}{A^n}$ could be given in the form $\dfrac{B_1^n \bullet_D \; \ldots \; \bullet_D B_k^n}{A^n}$

with the same effect. Can we also envisage an analytic rule of the

form $\dfrac{B_1^n +_D \; \ldots \; +_D B_k^n}{A^n}$, which would suggest a broadening of our definition

of analytic rules? So far as we are concerned with constructing

systems of <u>provable</u> sentences of D1, it seems we cannot, but we could ,

if we envisaged systems in D1 based on a kind of multiple-conclusion

"proofs".

On the other hand, the definition we have given of analytic

rules could possibly be strengthened. By considering the analytic

rules (→), (&),(∨),(⊥),(T),(□),(◇),(⊣),(∀),(∃) and (=), we shall

try to see what more stringent necessary and sufficient conditions

an analytic rule can be expected to satisfy.

§90

Take the analytic rule

$$(\alpha) \quad \frac{B^n_1 \ \ldots \ B^n_k}{A^n} \quad .$$

Let us call the D-subformula of A^n which is a formula of O in

which α occurs "the <u>constant part</u> of A^n". For example, "A → B"

is the constant part of "$\Gamma \vdash^1 \Delta \cup \{A \to B\}$" in (→). Let C be the

constant part of (α). If α is deleted from C, we are left either

(1) with some schemata for sentences of O; this is

e.g. the case with (→) or (□); or

(2) with some incomplete schemata for sentences of O; this

is e.g. the case with (∀); or

(3) with some schemata for singular terms of O; this is e.g.

the case with (=); or

(4) with nothing; this is e.g. the case with (⊥).

(These cases do not exclude each other: e.g. we could be left with

what is in both (1) and (2), though we have not considered such cases

in this work.)

The <u>components</u> of C will be either what we are left with in (1)

or (4), or some schemata for sentences of O constructed with what

we are left with in (2) or (3), using eventually additional

schemata of O_{op}. So, for example, the components

§90

of "A → B"   are   "A" and "B" ;

of "□A"   ,   "A" ;

of "∀xA(x)" ,   "A(x)" ;

of "$a = b$"   ,   "Pa" and "Pb" ;

of "⊥"   ,   none .

This is not an exhaustive characterization of components, but only an account which should cover the analytic rules we have treated.

The basic schemata for sentences or singular terms of D1 which occur in A^n and in which the constant part of A^n does not occur will be called "the <u>parametric parts</u> of A^n" (a basic schema can occur in another basic schema).

We can now state the following clauses which could be added to the right-hand side of the definition of analytic rules

(3) all the components of the constant part of the formula A^n occur in the formulae B^n_1, ..., B^n_k, and

(4) apart from these components, the basic schemata which occur in the formulae B^n_1, ..., B^n_k are identical with the parametric parts of the formula A^n .

All the analytic rules we have considered would satisfy (3) and (4). This new definition would reject as analytic the following double-line rule, which would be analytic according to   the old definition,

$$(\mathfrak{N}) \quad \frac{\Gamma \vdash^1 \Delta \cup \{A\}}{\Gamma \vdash^1 \Delta \cup \{\mathfrak{N}\}} \quad ,$$

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where $\mathcal{X}$ is of the category δ. $(\mathcal{X})$ would give rise to inconsistent

extensions when added to consistent systems; for in $\underline{hDI}(T)\,(\mathcal{X})$

we have

$$
(T)\uparrow \cfrac{\{T\}\vdash^{\prime}\{T\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\cfrac{\emptyset \vdash^{\prime} \{T\}}{\cfrac{\emptyset \vdash^{\prime}\{\mathcal{X}\}}{\cfrac{\emptyset \vdash^{\prime}\{A\}}{A}}}}
$$

$(\mathcal{X})\downarrow$, $(\mathcal{X})\uparrow$, $\underline{D}'$ appear as labels on the lines. A .

But even the definition with (3) and (4) would permit us to have

analytic rules like

$$
(\mathcal{X}_1)\;\frac{\Gamma \vdash^1 \Delta \qquad \emptyset \vdash^1 \{A\}}{\Gamma \vdash^1 \Delta \cup \{\mathcal{X}_1 A\}}\;,
$$

where $\mathcal{X}_1$ is of the category $\frac{\delta}{\delta}$. $(\mathcal{X}_1)$ would also give rise to

inconsistent extensions when added to consistent systems; for in

$\underline{hDI}(\mathcal{X}_1)$ we have

$$
(\mathcal{X}_1)\uparrow \cfrac{\{\mathcal{X}_1 A\}\vdash^{\prime}\{\mathcal{X}_n A\} \quad \underline{h}^{\circ}(\underline{I}^{\circ})}{\cfrac{\emptyset \vdash^{\prime}\{A\}}{A}}
$$

$\underline{D}'$.

We can envisage restricting our definition of analytic rules

still further in order to exclude double-line rules like $(\mathcal{X}_1)$. It

would not be enough to require only that all the parametric parts

of A^n occur in B^n_i, for $\underline{every}$ i, $1 \leqslant i \leqslant k$, for we could have

$$
(\mathcal{X}_2)\;\frac{\Gamma \vdash^1 \Delta \qquad \Delta \vdash^1 \Gamma \cup \{A\}}{\Gamma \vdash^1 \Delta \cup \{\mathcal{X}_2 A\}}\;,
$$

§90

where $\mathcal{K}_2$ is of the category $\frac{\delta}{\delta}$; in $\underline{hDIC}(\mathcal{K}_2)$ we have

$$(\mathcal{K}_2)\uparrow \underline{c}^1 \frac{\{\mathcal{K}_2 A\} \vdash \{\mathcal{K}_2 A\} \quad \underline{h}^0(\underline{I}^0)}{\varnothing \vdash \{\mathcal{K}_2 A, A\}} \qquad (\mathcal{K}_2)\uparrow \frac{\{\mathcal{K}_2 A\} \vdash \{\mathcal{K}_2 A\} \quad \underline{h}^0(\underline{I}^0)}{\{\mathcal{K}_2 A\} \vdash \varnothing}$$

$$\underline{D}^1 \frac{\varnothing \vdash \{A\}}{A} \ .$$

To exclude both $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ we could require that

(5) all the parametric parts of A^n occur in B^n_i, for every i,

$1 \leqslant i \leqslant k$, on the same side of the turnstile on which

they occurred in A^n;

or perhaps also that

(6) in B^n_i, for every i, $1 \leqslant i \leqslant k$, there occurs at least

one component of the constant part of A^n.

The analytic rules we have considered in this work would

satisfy these more stringent requirements. We shall leave open

the question whether analytic rules for an inconsistent constant

like $\mathcal{K}$, $\mathcal{K}_1$, or $\mathcal{K}_2$ could even then be given.   [In §92 we shall

make some other remarks on such inconsistent constants.]

The analytic rules we have considered in this work are for

constants which are sentence-forming functors or sentences;

hence they would satisfy an even more stringent definition of

analytic rules which requires that the constant part of A^n be a

formula typical for the constant analyzed.

§90

As an example of analytic rules for a constant which is neither a sentence-forming functor nor a sentence we give

$$(\lambda) \quad \frac{S_a^x A(x)\,|}{\overline{\overline{\qquad\qquad}}} \quad , \\ (\lambda x A(x))a$$

where the λ-functor is of the category $\frac{p_1}{p_k}$, $k \geqslant 1$. The schema of analytic rules (λ) gives a form of the rules of conversion.

Because of the uncertainties arising in the attempt to strengthen our original definition of analytic rules, and the imprecision of our notion of components, we shall continue to rely on this definition as our basic definition of analytic rules. Any definition we could subsequently adopt can be seen as a development of something implicit in the understanding of this definition. *[In §92 we shall consider whether we are justified in calling the double-line rules which satisfy this definition "analytic", and why a strengthening of this definition of the kind we have envisaged can be required.]*

The inconsistent constants $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ would also be logical according to Thesis [Ia], if we have assumed that there are correct sentences in which these constants occur and that these are the sentences provable in the analytic extensions with analytic rules for these constants (which is something which, of course, we shouldn't do). If such an outcome is thought unacceptable, we can envisage strengthening our definition of analytic rules as above. However, we shall try to show later *[v. §92]* that this difficulty for Thesis [Ia] is not real and that this thesis can still

be assumed to hold with our original definition of analytic

rules in the background.

It is also possible to envisage requirements pertaining to

the occurrence of the constant analyzed,or to the referring to such

a constant, in the proviso of an analytic rule [cf. §54]. The

strongest of these requirements would be to banish such an

occurrence, or referring, altogether.

§91   The results summarized in §87 provide some <u>inductive grounds</u>

for Thesis [Ia], and hence also for Thesis [I]. In this section

and the next, we shall present some considerations on the nature

of logic and on the significance of our analyses which will provide

some <u>deductive grounds</u> for Thesis [I] and the inference of Thesis [Ia]

from Thesis [I].

We state the following hypothesis

[A] Logic is the science of formal deductions.

This hypothesis is not generally acknowledged: there is a strong

tradition in modern logic which assumes that logic is the

science of a certain kind of truths of the object language. It

seems also that a large number of mathematical subjects studied

by contemporary logicians are not in the domain of logic as specified

by [A], so that [A] cannot be justified by usage. However, we

don't think that [A] needs some special justification: formal

deductions were for a long time the only subject of logic, and

still represent one of the main subjects of the science. The

import of hypothesis [A] is terminological. [For more on this topic

v. §99]

When [A] is interpreted in the framework set up by our enquiry, it entails the hypothesis

[Aa]   Logical systems in a language D are systems
for which only formal postulates are given.

If the inference of [Aa] from [A] is to be justified, the language D must be appropriate for speaking about deductions. We shall now consider the question what deductions are _formal_, or what postulates assumed for systems in D are _formal_. We state the following hypothesis

[B]   Basic formal deductions are structural deductions.

When [B] is interpreted in the framework set up by our enquiry, it entails the hypothesis

[Ba]   Basic formal postulates of systems in a language
D are structural postulates.

The assumption that structural deductions are formal, or that structural postulates, i.e. postulates given for structural systems, are formal, does not seem to need special arguing. But in what sense are these deductions and postulates _basic_? In one sense this is obvious from the construction of systems in D: in order to systematize the principles concerning a more or less natural notion of deduction , these systems must be based on structural systems. Systems in D which do not assume any background structural system are either not properly systems in D (like the axiomatizations in O) or are not natural. But there is another, more important, sense in which structural deductions and postulates are basic, and this will be shown by the explanation of the following hypothesis

§91

[C] Formal deductions which are not structural are
determined by structural deductions. This means that
any constant on the presence of which a non-structural
formal deduction depends can be ultimately analyzed
in purely structural terms.

When [C] is interpreted in the framework set up by our enquiry,
it entails the hypothesis

[Ca] The structural postulates of a system in a
language D with only formal postulates determine
the whole system. This is the case if formal postulates
which are not structural are such that any constant of O
which occurs in them, or occurs, or is referred to,
in a proviso of such a postulate, is (general)
structural.

For the defence of [C] we shall say that it enables us to get
a uniform picture of logical form. Logical form is primarily
exhibited by structural deductions, and when logical constants
are introduced they   serve, so to speak, as punctuation marks
for some structural features of deductions. The logical form of
sentences with logical constants then mirrors in some way the form
of structural deductions. Other reasons for accepting [C] can be
inductive, and our aim in this work was to find such reasons.

If we accept that the form studied by logic which is mentioned
in [A] is specified by [B] and [C], we can conclude that the constants
mentioned in the second sentence of [C] can rightly be called "logical",
and that Thesis [I] is justified. If the inference of [Aa], [Ba]
and [Ca] from [A], [B] and [C] is justified, the inference of Thesis
[Ia] from Thesis [I] should also be justified.

§92   A notion which still needs some explanation is the notion

of analysis used in [C]. The notion of analysis presupposed by

[Ca] was defined in this work, but does it correspond to the intuitive

notion which we appealed to in [C]?

We shall try to sketch a general account of analysis of which

the notion of analysis in [C] can be an instance. We suppose that a

certain notion of analysis may reasonably be assumed to satisfy the

following conditions

(1) An analysis of an expression α of a language L
establishes that a sentence A in which α occurs
only once, and in which there can also occur only
expressions of a language M generally understood to
which α does not belong, is equivalent with a
sentence B in which only expressions of M can occur.

(2) This equivalence must be such that it is sufficient
to assume it together with an understanding of
expressions of M and expressions of L without α,
in order to deduce every sentence which is assumed
to be correct in virtue of the meaning of expressions
of L and no incorrect one.

(3) Two expressions have the same analysis iff they have
the same meaning.

In general, M will have schemata for expressions of L, and at least

in this respect it will be a meta-language of L. However, it is not

absolutely necessary that M be such a language.

Condition (1) states the <u>form of an analysis</u>, condition (2)

requires that an analysis characterizes <u>soundly</u> and <u>completely</u>

the expression analyzed, and condition (3) requires that this characterization be unique.

An explicit definition can satisfy the three conditions for analysis, but these conditions do not require that α be explicitly definable in terms of some other expressions, or that an analysis should satisfy Pascal's Condition. Replacing this weaker notion of analysis by explicit definability of some sort threatens to exclude practically all philosophically interesting analyses and can at most hope to establish only more or less lexicographical facts, like the fact that "a is a brother" is equivalent with "a is a male sibling".

Let us call an expression α "strongly inconsistent" iff from the meaning of α and some consistent assumptions in the language L we can deduce inconsistent conclusions. Some expressions, like the predicate " is a round square", which it would be natural to call "inconsistent", are not strongly inconsistent. An example of a strongly inconsistent expression is Peano's μ ,the analysis of which is given by the equivalence of $\frac{a}{b} \mu \frac{c}{d}$ with $\frac{a+c}{b+d}$, where $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{a+c}{b+d}$ are rational numbers. Introducing μ into arithmetic would give rise to inconsistencies.

Can a strongly inconsistent expression be analyzed? As
we have just seen above, the answer is: yes, provided we have
assumed that we have such an expression (which, of course, we
shouldn't do). The sentences correct in virtue of the meaning of
expressions of L mentioned in condition (2) were left completely
undetermined, so that if it happens that α is strongly inconsistent,
every sentence of L will be correct, and an analysis of α must
enable us to deduce every sentence of L.

No doubt, strongly inconsistent expressions are to be avoided,
and languages containing them should not be constructed for actual
use. But constructing a language is something we do before trying
to analyze the expressions of this language, and if we have been
so unreasonable, or unfortunate, as to construct a language with
strongly inconsistent expressions, the fact that these too can be
analyzed is not a defect in analysis. If we take the term
"correct sentence" in a reasonable way, and in constructing a
language do not introduce strongly inconsistent expressions, then
there will be no strongly inconsistent expression which can be
analyzed, because there will be no strongly inconsistent expressions.
But there is no reason to require that the conditions for analysis
should single out these expressions as unanalyzable, so that analysis
should be impracticable with unreasonably constructed languages.
On the contrary, analysis can sometimes help us in investigating a
language which is unreasonably constructed and in locating the
source of the trouble.

§92

were consistent, then the addition of α would give rise to a

conservative extension, and hence α could not be strongly inconsistent.

We can consider the prospect of using analytic equivalences in

this way, but our main concern will be with analysis as it is specified

by conditions (1) - (3), and which does not assume that α is

introduced and has been given meaning by the analytic equivalence.

Some other remarks should be made concerning condition (1).

In it we require that α occurs only once in A. We think that this

requirement is justified, since otherwise it could not be said

that we are analyzing α, and not a series of occurrences of α. However,

a more relaxed view of analysis can perhaps be envisaged in which

this restriction is lifted. In another respect condition (1) could

be strengthened, viz. we could require that the same basic schemata

of M should occur in A and B, or something more elaborate of the

same kind. (There is no necessity that these schemata occur in

A and B, but in general they will.) This requirement would serve

to exclude would-be analyses in which there is an element in A or B

which is irrelevant to what is conveyed by the other sentence of the

analytic equivalence. It would also prevent at least some strongly

inconsistent expressions from being analyzed. This point is connected

with the explanation of the nature of the analytic equivalence.

It seems that this mutual entailment should be based on a relation

stronger than both material or strict entailment. Otherwise, it could

not be said that an analysis stems directly and exclusively from the

meanings of the terms involved. So the required entailment should

presumably be a kind of relevant entailment. But this mutual

entailment need not yield an equivalence as strong as propositional

§92

identity: otherwise we would be in the danger of excluding
practically all philosophically interesting analyses. However,
it is not clear what exactly this entailment relation is and what
demands it makes on the sharing of basic schemata. Accordingly,
we leave open the question whether an analysis must comply with
requirements about the sharing of basic schemata.

It seems that (1)-(3) cannot represent all the necessary
conditions for a philosophically acceptable notion of analysis.
In particular, they say nothing about the <u>clarificatory value</u> of
an analysis. It is not enough merely to assume that the language M
is generally understood. It is usually, and rightly, assumed
further that M is of a different order than L, in that it is more
<u>basic</u>, in the sense that it makes less assumptions and that the
understanding of L is somehow dependent on a previous understanding
of M, and not the other way round. If we are allowed to use the
term "basic" without further explanation, a fourth condition for
analysis can be stated as follows

(4) The language M should be more basic than the language
L.

Even if we cannot pretend that (1)-(4) are <u>all</u> the necessary
conditions that a certain notion of analysis should satisfy, it
can still be argued plausibly that they are neither too weak nor
too strong, and that they could be strengthened to yield a sufficient
condition by developing something which they already contain in an
implicit form. It should be noted that many philosophically interesting
analyses are not given exactly in the form of a single equivalence.

§92

Indeed, they need not be given in the form of equivalences at all.
Hence, what we have been seeking are not conditions which every possible
notion of analysis must satisfy.

This concludes our consideration of the intuitive notion of
analysis appealed to in [C].

It can be seen that our notion of structural analysis is
parallel to this notion. (The numbering of clauses in the definition
of structurally analyzed constants enhances this parallelism.)
Structural analysis satisfies condition (1), where it is understood
that the language M is that fragment of U in which we speak about
structural deductions and B can also be an implicit conjunction.
Next, we presume that clauses (2) and (3) match conditions (2)
and (3). This presupposes that the notion of synonymity involved in
clause (3) is a natural one.

An expression explicitly definable in terms of some other
expressions will be unique, but we have seen that a uniquely
characterized expression need not have a proper explicit definition.
The Decomposition Lemma shows that in some contexts the analyses
provided by analytic rules satisfy a form of Pascal's Condition, but
such a lemma need not be demonstrable in every context. We have
also seen that strongly inconsistent expressions like $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$
can be structurally analyzed, but just as in the discussion above,
this is only the case if we assume that we have these constants and
correct sentences in which they occur, which of course, we should
not do. A fortiori, expressions which would give rise to non-conservative
extensions can be structurally analyzed. If we want analytic rules

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to serve as a kind of definition, not necessarily satisfying
Pascal's Condition, so that correct sentences of O will coresspond
trivially to those provable in a system S for which (α) is given,
we should require further that analytic extensions be conservative.
In principle, we could not allow that analytic extensions be
conservative only with respect to sentences of level
O, for analytic rules will be constitutive of the language at all
other levels too, and the whole language D should be reasonable.
So, in this case, e.g., we could not use ($\rightarrow$) to define $\rightarrow$ in a
context where we have only $T_{-\varrho\ell}$ [v. Lemma 32].

If condition (1) is strengthened as envisaged above, this would
require a strengthening of our definition of analytic rules in the
way we have suggested [v.§90]. We have seen that this need not
be simply a matter of sharing all the basic schemata. With such
a strengthening of this definition, it would be plausible to argue
that the inter-derivability in an analytic rule is <u>directly</u> and
<u>exclusively</u> based on the meaning of the expressions involved, but we
leave open the question exactly what form this strengthening should
take.

Finally, we suppose that our notion of structural analysis
satisfies condition (4), since we suppose that the fragment of U
in which we speak about structural deductions is more basic than O
or a language included in O. So our analyses of logical constants
should have a clarificatory value. In the next section we shall
make some remarks on the clarificatory value of these analyses.

§93   Logical systems as specified by hypothesis [Aa] can be in a

language D built on any language O, provided only formal

postulates are given for them. However, it is systems in a

language D built on a schematic language O [v. §87] which are

considered logical par excellence.   This is connected with the

following hypothesis

[D] Logic is independent of subject matter.

The following hypothesis will be a paraphrase of at least a

part of the import of [D]

[Da] No constant of O save the logical constants is
essential for a logical system.

So for any logical system in D built on a particular language

O there are homologous logical systems in languages D built on

any language O which has the required constants and expressions which

can be instances of the basic schemata in the formal postulates

[cf.§98]. By appealing to schemata of U, which we use for giving

postulate-schemata, we can dispense with an appeal to a schematic

language O. (Even when we have a schematic language O, a certain

use of schemata of U to formulate transition-schemata is in

general not dispensed with.)

Hypothesis [Da] is a consequence of hypotheses [Aa], [Ba] and

[Ca], and is relatively uncontroversial.   On the other hand,

hypothesis [D] can be taken to express more than is conveyed by

[Da], and it is questionable to what extent it can be sustained

[v. §98].

Next we state the following hypothesis

[E]   The level of discourse of logic is higher than
the level of discourse in which we treat of a particular
subject matter relying on logical principles.

Part of what [E] can mean could be given by [D], i.e.
[Da]. But another ingredient of [E] can be paraphrased as follows
in the framework set up by our enquiry

[Ea]   A logical system is formulated in a language D
built on O, whereas we treat of a particular subject matter
in a language L included in O.

The sense in which the level of discourse of D, and eventually
that fragment of O which is not included in L, is <u>higher</u> than
the level of discourse of L is only metaphorical, but it is a useful
metaphor. To simplify matters we shall call D, together with that
fragment of O which is not included in L, "the higher language",
and L "the lower language".

By hypothesis [Ea], the systems mentioned by [Aa], [Ba] and
[Ca] are all the logical systems which can be found, so that in a
certain sense [Ea] makes explicit something implicit in [Aa].

With the help of [E] and [Ea] we can account for the clarificatory
value of structural analyses of logical constants. Logical constants
are expressions of the lower language which have their <u>raison d'être</u>
in some features of the higher language. When we treat of a
particular subject matter, some features of the higher language are
only <u>implicit</u> in the <u>activity</u> of making deductions. Logical constants

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serve to make <u>explicit</u> these features in the lower language:
they help us to reduce truths of the higher language to truths
of the lower language. To use a metaphor we have already mentioned
in connection with [Ca]: <u>logical constants are punctuation marks</u>
<u>for some structural features of deductions.</u> This metaphor is at
the back of Thesis [I].

We shall now try to survey briefly the punctuation functions which
belong to the logical constants we have structurally analyzed. This
survey will not be precise, but will only try to find a suggestive
reading of something exhibited more precisely in the analytic rules.
Our structural analyses depend on the context of D1, which, as we have
already said, is in a certain sense a minimal context among the contexts
of the various languages D we have mentioned. D1 pictures fewer
features of deductions than these other languages. Some constants,
for example, ∘ , cannot be analyzed exactly in a language like D1,
but the constants which can will in principle have the same
punctuation function in other kinds of language D too.

In a certain sense, → is the main logical constant. It is,
up to a point, a subsitute for the turnstile at level 0: it can
reduce a sentence of level 1 to a sentence of level 0. The constants
& and ∨ serve to economize: they reduce to one deduction two deductions
which differ at only one place in the conclusions or in the
premises. The constant ⊥ is a substitute for the empty set of
conclusions, and T a substitute for the empty set of premises.

The modal constant □ differs from the constants mentioned
above by having analytic rules of level 2. It can reduce a sentence

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of level 2 to a sentence of level 1. Modal logic arises when

at level 0 we also want to represent some features of the higher language

of level 2.

It would be interesting to show that the well known translations

from classical and Heyting logics into S5 and S4 respectively, and

vice versa, are possible for the following reason. Let us take the

sentences of level 1 of D to constitute an object language O_1, and

then let us introduce constants into O_1. With $\underline{T}^2$ unrestricted, the

logic obtained will be classical; with $\underline{T}^2$ restricted to $\underline{T}^2_{\to sh}$, it

will be intuitionistic. We know that at level 0 with these forms

of $\underline{T}^2$ we should obtain S5 and S4 respectively (where $\underline{T}^1$ is

unrestricted, making the logic of level 0 classical). We could

then say that S5 represents at level 0 a classical logic of level

1, and S4 represents at level 0 an intuitionistic logic of level 1.

That is, with S5 we represent in classical logic the principles of

a classical meta-logic, and with S4 we represent in classical logic

the principles of an intuitionistic meta-logic. The translations

would work because by them we connect the provable sentences of S5

and S4 with what they represent.

The first-order quantifiers and identity serve to represent

in the lower language some features of the higher language involving

the presence of an arbitrary expression of a certain category in deductions.

The provisos for the analytic rules (∀), (∃) and (=) serve to guarantee

that the expression in question is arbitrary. Roughly speaking, all

quantifiers and identity express something about "any".

Our treatment of the quantifiers favours the substitutional

interpretation, but it should be noted that this is not due to any

inherent feature of the analytic rules. It is due to our understanding

of variables, and had we presented a different understanding of

variables with the same analytic rules, we should have obtained

a different understanding of quantifiers. We reduce the question

which interpretation of the quantifiers is right, the substitutional

or the referential, to the question how we should understand sentences

of O_{op}. If we think that these sentences are schemata relative to

$O_{c\ell}$, we shall favour the substitutional interpretation; and if

we understand these sentences in the way specified by the standard

interpretation, which appeals to domains and individuals, we shall

favour the referential interpretation. Thesis [I] is compatible

with both interpretations. Our treatment suggests a substitutional

interpretation for all quantifiers, and not only first-order ones.

The analytic rule $(=)_2$ exhibits the function $=$ can be

expected to play in a first-order context, as distinguished from the

second-order context of $(=)$. The constant $=$ serves to indicate

substitution possibilities: it serves to express that something

which holds for a holds also for anything assumed to be identical with

a.

A notable feature exhibited by analytic rules is that some

logical constants are structurally analyzed (which also means

completely characterized) when formulae typical for them are

in the left set, whereas others when formulae typical for them are

in the right set. (The situation is somewhat more complicated in

the case of modal constants.) A grouping of logical constants would

also be possible along these lines, and it might be important if we

considered that analytic rules serve as definitions of the constants

analyzed. The primary meaning of a constant would then be tied to

a certain position in deductions of the formulae typical for this

constant.

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We conclude this section with some remarks on the prospects opened up by our work. Treating logical constants with analytic rules in the framework we have set could enable us to predict what other constants, perhaps still unknown, could be structurally analyzed. The possibilities of analytic rules of level 2 do not seem to be exhausted by (□), (◊) and (→). We could also ask if it is possible to give an analytic rule of level n, $n \geqslant 3$ for a constant of O. Also, we have only occasionally mentioned the possibility of introducing logical constants at higher levels of D simultaneously with their introduction in O, taking each of these higher levels as an object language. It might also be interesting to try to find a connection between analytic extensions in propositional logic and decidability. Finally, we could try to treat within the framework of our enquiry the problem of functional completeness: roughly speaking, the question whether we have all the logical constants that we need.

THESIS [II]

§94   In this section we shall define the notion of structurally alternative systems. First we give the following

Definition of analytically identical primary structural constants

The primary structural constants α and β are analytically identical iff α and β are structurally analyzed by using analytic rules which can be obtained from each other by substituting α for β, and <u>vice versa.</u>

This definition requires that homologous analytic rules be used for the structural analysis of α and β, but not necessarily the same system in D without (α) and (β). Hence, classical $\rightarrow$ and intuitionistic $\rightarrow$ are analytically identical. Our general policy in this work has been to identify analytically identical constants with the same expression of O.

Definition of analytically identical secondary structural constants

The secondary structural constants α and β are analytically identical iff from the explicit definition of α in terms of the primary structural constants α_1, ..., $\alpha_k, k \geqslant 1$, we can obtain the explicit definition of β by substituting β for α and β_i for α_i, for every i, $1 \leqslant i \leqslant k$, where β_i is analytically identical with α_i, and _vice versa_.

Definition of directly structurally alternative systems

The structural systems or analytic extensions S_1 and S_2 are directly structurally alternative iff

(1) for every constant α_1 of O which occurs in the analytic rule (α_1) of S_1, there is a constant α_2 of O which occurs in the analytic rule (α_2) of S_2 such that (α_2) is obtained from (α_1) by substituting α_2 for α_1, and _vice versa_,

(2) (α_1) is in the scope of _h_ iff (α_2) is,   and

(3) if the same expression is used for every pair α_1, α_2 in S_1 and S_2, it is not the case that both S_1 is contained in S_2, and S_2 in S_1.

Definition of structurally alternative systems

The systems S_1 and S_2 in D or O for which only structural constants are essential are structurally alternative iff

(1) for every structural constant α_1 which is essential for S_1, there is a structural constant α_2 which is essential for S_2, and <u>vice versa</u>,

(2) α_1 and α_2 are analytically identical, and

(3) if the same expression is used for analytically identical constants in S_1 and S_2, it is not the case that both S_1 is contained in S_2, and S_2 in S_1.

As a consequence of these definitions we have that two structural systems are directly structurally alternative iff they are structurally alternative, and that they are structurally alternative iff it is not the case that both S_1 is contained in S_2, and S_2 in S_1. With structural systems, clauses (1) and (2), and the antecedent of (3), in the last two definitions, are satisfied vacuously.

Another consequence of these definitions is that if two systems are directly structurally alternative, they are structurally alternative. The converse is not necessarily the case, but we can show that if two systems S_1 and S_2 are structurally alternative, there are two systems S_1' and S_2' such that S_1 is contained in S_1', S_2 is contained in S_2', and S_1' and S_2' are directly structurally alternative.

§95   The main results which we have tried to establish in this
work can be summed up as follows.

The systems CQi/O, HQi/O, KQi/O and IRQi/O are structurally
alternative to each other. The analytic extensions in which these
systems are contained, and which are directly structurally
alternative to each other, are, respectively, CQi/D, HQi/D, KQi/D
and IRQi/D. These latter systems differ only with respect to their
assumptions on T-rules and horizontalizations of T-rules.

We have also shown that the systems S5Qi/O and S4Qi/O are
structurally alternative. The corresponding directly structurally
alternative systems are S5Qi/D and S4Qi/D, which differ only with
respect to their assumptions on T-rules and horizontalizations of
T-rules.

The modal systems above are not structurally alternative with
the non-modal systems, since these systems are not based on
analytically identical constants. Hence, these modal systems should
be treated as supplementary to CQi/O or CQi/D, or the systems
obtained from CQi/O or CQi/D by restricting $\underline{T}^2$.

We leave open the question whether by changing the deductive
meta-language in some way, a modal constant like $\dashv$ could be shown
analytically identical to the non-modal $\rightarrow$.

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Let us now consider

> Thesis [II]   Two logical systems are alternative iff
> they differ only in their assumptions on structural
> deductions.

When the intuitive notions used for stating Thesis [II] are
interpreted in the more precise framework set up by our enquiry,
this thesis entails

> Thesis [IIa]   If two systems are structurally alternative,
> they are alternative logical systems.

In what sense two systems can "differ only in their assumptions
on structural deductions" besides being structurally alternative,
and consequently what other pairs of systems besides structurally
alternative ones could be considered alternative logical systems
according to Thesis [II], is suggested by the following considerations.

First, a definition of analytically identical general structural
constants can be envisaged, upon which a definition of general
structurally alternative systems can be based.

Second, we could perhaps form a notion of analytically identical
general structural constants by relying on a notion of analytic
rules which is not tied to a particular language D. Then, upon this
notion, we could base a notion of general structurally alternative
systems which would differ not only in some structural assumptions
within the context of a particular language D, but would differ with
respect to the assumptions on which are based the languages D in
which these systems are given.

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With this we could possibly obtain a plausible reconstruction of the import of Thesis [II].

Thesis [IIa] can only serve to show that two systems <u>are</u> alternative logical systems and, of course, it cannot serve to show that two logical systems <u>are not</u> alternative. On the other hand, Thesis [II], which apparently also provides a criterion for the second of these relations, need not guarantee that the question whether two logical systems are alternative is effectively decidable. In general, on the background surveyed in this work we are prepared to appeal only to Thesis [IIa].

However, it is probable that some logics proposed as alternatives to classical logic would necessarily fail the test of Thesis [II], and the result that some allegedly alternative logics are not genuine alternatives would not be undesirable.

§97   The results summarized in §95 provide some <u>inductive grounds</u> for Thesis [IIa], and hence also for Thesis [II].   In this section we shall try to show what <u>deductive grounds</u> could be given for the justification of Thesis [II] and the inference of Thesis [IIa] from Thesis [II].

Thesis [II] is a consequence of hypotheses [A],[B] and [C]. According to these hypotheses different formal deductions studied by logic can arise either

(1) by having different structural deductions and logical constants which are ultimately analyzed <u>in the same way</u>, or

(2) by having the same structural deductions and logical constants which are ultimately analyzed in different ways, or

(3) by having both different structural deductions and logical constants which are ultimately analyzed in different ways.

Then we suppose that the situation is best described by saying that in case (1) we are confronted with alternative formal deductions, in case (2) with formal deductions which supplement each other, and in case (3) with formal deductions which are both alternative and supplement each other. A justification for this description can be found in the following.

Condition (3) for analysis stated that two expressions have the same analysis iff they have the same meaning. If two expressions have the same analysis, then the language M used for giving this analysis must always be understood in the same way. If, on the other hand, this language can be understood in two different ways, so that we have in fact two languages M_1 and M_2, then the analysis based on M_1 will not be the same as the analysis based on M_2, but we could still say that these two analyses are given in the same way. The expression α_1 with the analysis based on M_1 will not have the same meaning as the expression α_2 with the analysis based on M_2, but we could say that α_1 and α_2 are analytically identical. This analytic identity is a kind of identity of function. Analytic identity can be defined more precisely in the particular case of structural constants, as we have done above. In general, we have that two constants are analytically identical iff they can be ultimately analyzed in the same way. Analytic identity is not identity of meaning, but it is akin to it.

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Using this terminology we can say that in case (1) above we have alternative understandings of analytically identical constants, whereas in case (2) we have analytically different constants, and no alternatives. In case (3) we also have analytically different constants, although in principle it is possible to introduce in this context analytically identical constants which will be understood in alternative ways. Accordingly, we say that formal deductions are alternative if they are based on constants which can be understood in alternative ways; and if these deductions are structural we say that they are alternative because they provide contexts for understanding analytically identical constants in alternative ways. (However, it is also possible to envisage an argument which would show that the turnstiles of various structural systems are analytically identical,where D is taken as an object language.) Thesis [II] is then obtained by describing case (1) as we have suggested.

To infer Thesis [IIa] from Thesis [II] we assume that analytic identity for structural constants is a particular case of analytic identity, and that two systems can "differ only in their assumptions on structural deductions" not only in cases where these assumptions are explicitly given for these systems. They may also differ in this respect when contained in two analytic extensions in which this difference is explicit.

Our argument for Thesis [II] could also be summarized as follows. The import of Thesis [I] is that a logical system is completely determined by its assumptions on structural deductions. Hence, a logical system which differs only in its assumptions on non-structural

deductions will not represent an alternative logical system, but
a possible supplement: an alternative logical system can be
obtained only by changing assumptions on structural deductions.

Our use of the term "alternative" is in accordance with the
assumption that wherever there are alternatives, as opposed to
simply differentthings, there must be a certain common purpose for
which both alternatives can be used, a "common denominator" of
these alternatives. Analytically identical logical constants, which
are either already present in alternative logical systems, or which
it is possible to introduce, are the "common denominator" of these
systems. This is probably only one ingredient in the ordinary use
of the expression "alternative logic", where a logic would be
considered an alternative to classical logic when it resembles
classical logic in a certain sense, and hence can be assumed to
serve the same purpose. In this more or less vague sense, we could say
that two logical systems which are not contained in each other are
alternative merely because they both serve to codify principles
about deductions, provided they resemble each other enough. When
these two systems are structural, the deductions in question will be
structural.

Our explanation of why logical systems are alternative is
somewhat different. In particular, two structural systems are not
taken as alternative for the above reasons, but because they provide
two different contexts for understanding analytically, i.e. functionally,
identical constants. However, the ordinary use of "alternative" is
not incompatible with that. Two structural systems can also be
considered alternative because they both serve to codify principles
about structural deductions. This codification is implied by the

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attempt to provide a context for understanding logical constants.

We conjecture that our use of the term "alternative" will coincide

in application with the ordinary use in all important cases.

§98   If Thesis [II] is justified, in order to understand why

alternative logics arise, we should understand why alternative

structural deductions can be assumed to hold, i.e. why we have

alternative <u>valid</u> structural deductions, or alternative sets of

<u>correct</u> sentences provable in structural systems in D.

We have made some comments in this work on some notions of

deductions which can be taken into account and which give rise to

classical logic and various logics of proof. We have also

considered a logic in which sequents could correspond to refutations

of a certain form. Finally, we have seen that it is possible that

sequents of various levels within the same system correspond to

various notions of deductions.

According to hypothesis [D], logic is independent of subject

matter, but we have remarked that it is questionable to what extent

this hypothesis can be sustained. It might happen that in order

that the expressions of a language 0 <u>can be</u> instances of the schemata

in the structural postulates assumed for a logical system, it is not

enough that 0 has the expressions of the grammatical categories required

by the basic schemata of these postulates, but also our understanding

of 0 should not preclude using this particular logic for it. For

example, we might find that with a certain language 0, sentences of

0 can be asserted iff they are <u>proved</u>, so that only a logic of

proof can be used for 0. And it could be argued that this depends

on the subject matter of 0.

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Of course, [D] cannot mean that logic is independent of
the subject matter of logic, which in the framework of our enquiry
means that logical systems depend on the subject matter of D.

§99   [§88]   Our attempt to analyze logical constants and Thesis [I]
should not be confused with the programme to <u>define</u> logical constants
with syntactical principles. First, the main goal of this programme
is to show that syntactical principles give the <u>meaning</u> of logical
constants, whereas our analyses are neutral with respect to this claim
and are equally compatible with the view that the meaning of logical
constants is to be given in a more conventional semantical framework.
Second, the search for a criterion for what is a logical constant
does not always have a very important place in this programme.
Either this problem is not considered, and it is taken that the only
problem is to show how the meaning of expressions usually called
logical constants is given by syntactical principles, or it is taken
that logical constants will be those expressions whose meaning can
be given by some syntactical principles, making the search for this
criterion dependent on the main goal of the programme. Thesis [I]
is an attempt to formulate such a criterion without tying it to a
thesis on the meaning of logical constants. On the other hand,it
is clear that our analyses and Thesis [I] are congenial to this programme:
they are certainly compatible with its main goal, and if the
term "meaning" is taken in a loose sense, our analyses are also
concerned with giving a certain account of the meaning of the
logical constants.

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The origins of this programme can be traced to Gentzen 1934,

who remarked concerning the introduction and elimination rules of

his natural deduction calculi: <<Die Einführungen stellen sozusagen

die „Definitionen" der betreffenden Zeichen dar, und die

Beseitigungen sind letzten Endes nur Konsequenzen hiervon, ... >>

(p.189).

Popper 1947, 1947a-b, 1948, 1948a-b, attempted to use a certain

sort of sequent systems to define logical constants and to find a

criterion for the demarcation of logic. These papers were rather

confusing (v. the reviews by McKinsey 1948; Kleene 1948, 1949; and

Curry 1948), and Popper 1974 calls them "bad and ill-fated". The

attempt by Lejewski 1974 to reconstruct Popper's programme does

not seem to correspond to Popper's original intentions. However,

one of Popper's critics, Curry   1950 (p.25), endorses a position

akin to Gentzen's. A programme similar to Popper's is formulated

by Kneale 1956 (v. also Kneale & Kneale 1962), who tries to

formulate for this purpose a natural deduction multiple-conclusion

calculus. The programme has been recently pursued by Hacking 1977

and 1979. Hacking is concerned in principle with standard Gentzen-

-style sequent calculi with rules for introducing constants on the

left and on the right, but he also tries to introduce a certain

notion pertaining to the parametric parts of these rules which

apparently makes it impossible for him to deal with modal constants.

One of Hacking's aims is to show that quantum logic is logic.

That part of the proof-theoretical programme of Prawitz 1971,

1973, 1974, 1977, 1978, which is relevant to our discussion is not

so much concerned with the demarcation of logic as with pursuing

the idea expressed in Gentzen's remark on natural deduction introduction

rules. This also holds for the relevant views of Dummett 1975 and 1977 (pp.389-403). Some other recent papers which are in the tradition of Gentzen's remark are Bendall 1978, Zucker & Tragesser 1978 and Zucker 1978.

Prior 1960 criticizes this general programme by giving natural deduction rules for what we would call "a strongly inconsistent constant". In the reply by Belnap 1962 the requirements of conservativeness and uniqueness are stated for rules which pretend to define a constant. The framework of Belnap's discussion is a sequent calculus with the single conclusion property, where sequents are called "deducibility statements". Belnap does not mention any requirement concerning what we would call "components", which would also disqualify Prior's constant. It is not clear what the reply of Prior 1964 achieves.

[§91] We have remarked that today the conception of logic expressed by hypothesis [A] is neither the only possible nor the dominant conception of logic. It is, however, congenial to the programme mentioned above. A clear assessment of the role that this and the alternative conception (that logic is the science of logical truths of the object language) have played in modern logic can be found in Dummett 1973 (pp.432-435; cf. also Kneale 1946).

The problem "Where are the limits of logic?" was, obviously, of crucial importance to the logicists, with whose work modern logic started, but no definite criterion for setting these limits seems to

have come out of their work. (What they needed was, at least,

a criterion for being a logical constant, and not necessarily

also a criterion for not being a logical constant.) On the other

hand, one of the main reasons why logicism was abandoned was that at

some point it was felt that the limits of logic must have been

transgressed in the logicist reconstruction of mathematics. For

the logicists logic was the science of logical truths of the object

language.

The question what are the formal deductions studied by logic,

as specified by hypothesis [A], is answered by hypotheses [B] and

[C], where in hypothesis [C] we appeal to the notion of a logical

constant as specified by Thesis [I]. Likewise, with the view that

logic is the science of logical truths of the object language, we

can determine what the logical truths are, once we know what the

logical constants are: v. e.g. Quine 1970(Chapter 4) and Wang 1974

(pp.143-165). Quine proposes to draw a distinction between logical

constants and other expressions (his terms are respectively

"particles" and "lexicon") by saying that the grammatical categories

of the latter are "infinite and indefinite" (pp.28-30, v. also p.59).

(It is not clear whether it is possible to infer from our work

restrictions on the kind or multiplicity of the grammatical

categories of logical constants.) Whatever the merits of Quine's

proposal, a much more characteristic attitude in modern logic is

a certain scepticism as to whether this distinction can be clearly

drawn. Wang, who examines some proposals, including a grammatical

one linked with Quine's, is in such a sceptical mood. But Wang is

not on the side of those logicians who, like so many followers of

Protagoras, just list what they take as logical constants.

A clear exponent of this scepticism, and probably one of those
who made of it the accepted position, is Tarski 1936. It is
interesting that Bolzano 1837 anticipated Tarski not only in his
definition of the semantical notion of logical consequence, but
also in the belief that it is doubtful that a distinction can be
drawn between logical and non-logical expressions (v. Bolzano 1837,
§148, and also Kneale & Kneale 1962, p.366). However, in a lecture
in 1966, Tarski seems to have abandoned this scepticism up to
a point, and to have found a criterion for the demarcation of
logic by elaborating some results of Lindenbaum & Tarski 1935.
The main of these results seems to be that <<... every relation
between objects (individuals, classes, relations, etc.) which can
be expressed by purely logical means is invariant with respect to
every one-one mapping of the 'world' (i.e. the class of all
individuals) onto itself ...>> (v. Tarski 1956, p.385; a reference
to Tarski's lecture of 1966 and a critique of his views can be found
in Mikeladze 1979). The proposal for the demarcation of
logical constants of Dummett 1973 (pp.21-22) which is in principle
grammatical, also has a footnote dealing with identity in which a
view very similar to Tarski's conception of 1966 is propounded.
This conception is probably connected with hypothesis [Da].

For some other recent discussions on the demarcation of logic
v. Lindström 1969, 1974; Tharp 1975; and Boolos 1975.

We have said that our views are in principle compatible with
the view that the meaning of logical constants is given in a standard
semantical framework. They are also in principle compatible with a
semantical account of the notion of logical consequence, which can be

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used to explain the notion of deduction to which we have

appealed. Finally, our analyses and Thesis [I] would in principle

be compatible with the conception of logic as the science of logical

truths of the object language, if only it could be seen on what

grounds one might then accept Thesis [I] or show an interest in

the deductive meta-language.

[§92] In the philosophical tradition which continues to be

called "analytical", but which has gone through many significant

changes in this century, it seems that the most precise account

of analysis which can be found, and to which many accepted opinions

can still be traced, is in the writings of Moore. On the other hand,

this account which ultimately treats analysis as a kind of explicit

definition cannot be supported by many examples of plausible

philosophical analyses which conform to its standards. A detailed

critique of Moore's account of analysis, and of those accepted

opinions which can be traced to this account, can be found in Kojen

1977.

Our replacement of the term "definition" by the term "analysis"

in the programme mentioned at the beginning of this section is not

a merely verbal move. One substantial difference between this

programme and our enquiry is that Belnap's requirement of conservative-

ness ceases to play with us the role it has to play in the former

programme. (For a related discussion of conservativeness in the

theory of meaning v. Dummett 1973, pp.453-455, 396-397.)   The

characterization of constants by analytic rules, or an analogous

syntactical or functional characterization, is certainaly closely

tied with the meaning of the constants characterized; but, on the

other hand, it can achieve something that is independent of

the question whether by this the whole meaning of these constants is

given. We think that it can achieve a philosophically significant

analysis. It is possible that something not very far from our

notion of analysis was on Gentzen's mind when he made his brief

remark on the sozusagen Definitionen.

Peano's strongly inconsistent operational sign μ is considered in Peano 1921 (v. also Belnap 1962). Some other examples of strongly inconsistent mathematical expressions can be found in Suppes 1957 (Chapter 8).

[§93] The distinction between logical systems in a schematic

O and logical systems in a non-schematic O, and the corresponding

distinction for systems in D built on O, is relevant to the distinction

between what Kneale 1946 calls "truths of logic" and "formal truisms".

Hypothesis [E] should be regularly invoked in the exegesis of

Carroll 1895 (the Tortoise is stuck at level O). It can be argued

that one of the misleading aspects of the picture of logic given

by Quine 1951, and of the general conception that logic is the

science of logical truths of the object language, is that it does

not take into account hypothesis [E] (cf. Dummett 1973, p.596,

and Dummett 1974, sp.p.353). On the other hand, hypothesis [E]

seems to be presupposed in the critique of conventionalism of Quine

1936.

The metaphor which we have used to describe the function of

logical constants is reminiscent of the following remark of

Wittgenstein 1921: <<Die logischen Operationszeichen sind Interpunktionen.>>

(5.4611), and also : <<...; in der Logik ist jeder Satz die Form

eines Beweises.>> (6.1264) But one should probably separate these

remarks from their immediate context in order to connect them with

the views we are expressing here.

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The translations from classical and Heyting logic into S5 and S4, and vice versa, are treated in: Gödel 1933; McKinsey & Tarski 1948; Prawitz & Malmnäs 1968; Schütte 1968; Fitting 1970, Czermak 1975. It would be interesting to connect our treatment of modality with the provability interpretation of modal logic as expounded by Lemmon 1959 (cf. also Skyrms 1978 and the references therein).

[§§94-98] Our views on alternative logics are inspired by the well known treatment of classical and Heyting sequent calculi in Gentzen 1934.

BIBLIOGRAPHY

After each unit can be found a reference to the sections in which
it is mentioned. We shall use the following abbreviations

JPh   : The Journal of Philosophy ;

JPhL   : Journal of Philosophical Logic ;

JSL   : The Journal of Symbolic Logic ;

NDJFL   : Notre Dame Journal of Formal Logic ;

SL   : Studia Logica .

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right set : §9
RKP/D : §40
RKP/O : §42
rule : §6

S4P/D : §53
S4P/O : §53
S4Q/D : §65
S4Q/O : §65
S4Qi/D : §71
S4Qi/O : §71
S5P/D : §53
S5P/O : §53
S5Q/D : §65
S5Q/O : §65
S5Qi/D : §71
S5Qi/O : §71

INDEX OF SYMBOLS

For symbols for languages and systems consult also the preceding index. Symbols used only for the sake of examples are not listed.

ADDENDA

On p.46 after line 7 bottom insert:

Since the order of sentences in Γ is not unique, there are in fact many translations $\hat{\sigma}(\Gamma), \breve{\sigma}(\Gamma)$ and $\sigma(A^{n+1})$. When we consider the translation $\sigma(A^{n+1})$, we consider any of the relevant translations.

On p.286 after line 4 bottom insert:

If "any" is prefixed to a consequent, or an asserted proposition, it becomes "every", and if it is prefixed to an antecedent, it becomes "some".

On p.297 after line 1 bottom insert:

We have seen that first-order quantificational logic can be understood roughly as the logic of arbitrary singular terms, or, if we accept the referential interpretation, the logic of arbitrary individuals. This pressuposes that the language to which we apply this logic is about some individuals. [D] is not warranted if from it we can conclude that quantificational logic should be applicable even if there are no individuals we are speaking about, as has been done in connection with logics "free of existential assumptions".

On p.306 insert:

Aristotle

Topica et Sophistici Elenchi. W.D. Ross ed., Oxford (Clarendon Press) 1958 [Engl. transl. by W.D. Ross in The Works of Aristotle, vol. 1. Oxford (University Press) 1928] [§25].

On p.313 insert:

Peano, G.

1921. Le definizioni in matematica. Periodico di matematiche (4)1, pp. 175-189 [Engl. transl. in Selected Works of Giuseppe Peano. London (Allen & Unwin) 1973] [§99].

On p.313 insert:

Pogorzelski, W.A.

1968. Kilka uwag o pojęciu zupełności rachunku zdań, I. SL 23, pp. 43-50 [Engl. summary: Some remarks on the concept of completeness of the propositional calculus. ibid.,pp.55-58] [§18].

On p. 313 insert:

Pogorzelski, W.A.

1975. On the notion of the rule of inference and completeness
of systems, Some comments on H.C. Wasserman's remarks.
Reports on Mathematical Logic 5, pp. 73-75 [§18] .

On p. 316 insert:

Wasserman, H.C.

1974. Admissible rules, derivable rules, and extendible logistic
systems. NDJFL 15, pp. 265-278 [§18] .

On p. 107 insert after line 5:

, save that we use the eliminability of $\underline{D}^2$ (v. Lemma 10).

Armand & Nicole : for Pascal's condition
Leibniz : for Ramsey's analysis of truth
            also Tarski