MODELS OF DEDUCTION

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ABSTRACT

In standard model theory, deductions are not the things one models. But in general proof theory, in particular in categorial proof theory, one finds models of deductions, and the purpose here is to motivate a simple example of such models. This will be a model of deductions performed within an abstract context, where we don't have any particular logical constant, but something underlying all logical constants. In this context, deductions are represented by arrows in categories involved in a general *adjoint situation*.

To motivate the notion of adjointness, one of the central notions of category theory, and of mathematics in general, it is first considered how some features of it occur in set-theoretical axioms and in the axioms of the lambda calculus. Next, it is explained how this notion arises in the context of deduction, where it characterizes logical constants. It is shown also how the categorial point of view suggests an analysis of propositional identity. The problem of propositional identity, i.e. the problem of identity of meaning for propositions, is no doubt a philosophical problem, but the spirit of the analysis proposed here will be rather mathematical. Finally, it is considered whether models of deductions can pretend to be a semantics. This question, which as so many questions having to do with meaning brings us to that wall that blocked linguists and philosophers during the whole of the twentieth century, is merely posed. At the very end, there is the example of a geometrical model of adjunction. Without pretending that it is a semantics, it is hoped that this model may prove illuminating and useful.

Since the text of this talk was written in 1999, the author has published several papers about related matters (see ‘Identity of proofs based on normalization and generality’, *The Bulletin of Symbolic Logic* 9(2003), pp. 477-503, corrected version available at: http:// arXiv. org/ math. LO/ 0208094; other titles are available in the same archive).

1. INTRODUCTION

According to the traditional vocation of logic to study deductive reasoning, deductions should indeed be of central concern to logicians. However, as an object of study, deductions have really a central place in a rather restricted area of logic called *general proof theory*—namely, proof theory done in the tradition of Gentzen. There, by studying normalization of logical deductions, one is led to consider criteria of identity of deductions. The goal of this brand of proof theory might be to find a mathematical answer to the philosophical question “What is deduction?”, as recursion theory has found, with much success, a mathematical answer to the question “What is computation?”.

In proof theory done in the tradition of Hilbert’s program, where one is concerned with consistency proofs for fragments of mathematics, deducing is less central. The goal there is not to answer the question “What is deduction?”, but to prove consistency by some particular means. Hilbertian proof theory, incorporating the fundamental lessons of Gödel, was for a long time dominant in proof theory, but nowadays it seems it may be yielding ground. It has become a rather secluded branch of mathematics, where one studies intricate problems about ordinals, not particularly appealing to other logicians, let alone other mathematicians.

However, following a general trend, seclusion has become the norm in logic, as well as elsewhere in mathematics. The trend is quite conspicuous in model theory, which was no doubt the dominant branch of logic during a long period in the second half of the twentieth century. As the century was drawing to its end, so model theory, which had drifted to some particular branches of algebra, appeared more and more esoteric. The remaining two great branches of logic, recursion theory and set theory, leave the same impression nowadays. Logicians of these various branches meet at congresses, and politely listen to each other’s talks, but don’t seem much moved by them.

Although we are speaking here of the dominant branches of *logic*, deductive reasoning hardly makes their subject matter. The study of deduction was for a long time confined to rather marginal fields of nonclassical logics. Perhaps the growth of general proof theory, and its connection with category theory and computer science, might bring deduction to the fore. (In that, the role of category theory and computer science would presumably not be the same, the former being otherworldly and the latter mundane, but—who knows—the two ways might end up by being in harmony.)

In standard model theory, deductions are not the things one models. But in general proof theory, in particular in categorial proof theory, one finds models of deductions, and my purpose in this talk is to motivate a simple example of such models. This will be a model of deductions performed within an abstract context, where we don't have any particular logical constant, but something underlying all logical constants. In this context, deductions are represented by arrows in categories involved in a general *adjoint situation*.

To motivate the notion of adjointness, one of the central notions of category theory, and of mathematics in general, we shall first consider how some features of it occur in set-theoretical axioms and in the axioms of the lambda calculus. Next, it will be explained how this notion arises in the context of deduction, where it characterizes logical constants. We shall see also how the categorial point of view suggests an analysis of propositional identity. The problem of propositional identity, i.e. the problem of identity of meaning for propositions, is no doubt a philosophical problem, but the spirit of the analysis proposed here will be rather mathematical. Finally, we shall consider whether models of deductions can pretend to be a semantics. I merely ask this question, which as so many questions having to do with meaning, brings us to that wall, which blocked linguists and philosophers during the whole of the twentieth century. At the very end, we reach our example of a geometrical model of adjunction, for which I don't pretend that it is a semantics. Nevertheless, I hope that this model may prove illuminating and useful.

2. TERMS, PROPOSITIONS AND INVERSION

In logic, as well as in the philosophy of language, we are especially interested in two kinds of linguistic activity: referring and asserting. We engage in the first kind of activity with the help of *terms* (which abbreviates *individual terms*), while for the second we use *propositions* (or *formulae*). The two grammatical categories of terms and propositions are basic grammatical categories, with whose help other grammatical categories can be defined as functional categories: predicates map terms into propositions, functional expressions map terms into terms, and connectives and quantifiers map propositions into propositions.

The set-abstracting expression $\{x : \dots\}$ maps a proposition A into the term $\{x : A\}$. This term is significant in particular when x is free in A, but it makes sense for any A, too. The expression $x \in \dots$ is a unary predicate: it maps a term a into the proposition $x \in a$. The ideal set theory would just assume that $\{x : \dots\}$ and $x \in \dots$ are in some sense inverse to each other. Namely, we would have the following postulates:

$$\begin{array}{ll} \textit{Comprehension}: & x \in \{x : A\} \leftrightarrow A, \\ \textit{Extensionality}: & \{x : x \in a\} = a, \end{array}$$

provided x is not free in a. In the presence of replacement of equivalents and of Comprehension, Extensionality is equivalent to the more usual extensionality principle

$$\forall x(x \in a_1 \leftrightarrow x \in a_2) \ \rightarrow a_1 = a_2,$$

provided x is not free in a_1 and a_2 (see [D. 2001], Section 2). We know that ideal set theory is inconsistent if in propositions we find negation, or at least implication. To get consistency, either $\{x : A\}$ will not always be defined, and we replace Comprehension by a number of restricted postulates, or we introduce types for terms.

Instead of $\{x : \dots\}$ let us now write $(\lambda_x \dots)$, and instead of $x \in \dots$ let us write $(\dots x)$. Then Comprehension and Extensionality become respectively

$$((\lambda_x A)x) \ \leftrightarrow A,$$
$$(\lambda_x (ax)) = a.$$

If we take that $(\lambda_x \dots)$ maps a term a into the term $(\lambda_x a)$, while $(\dots x)$ maps a term a into the term (ax), and if, furthermore, we replace equivalence by equality, and omit outermost parentheses, our two postulates become the following postulates of the lambda calculus:

$$\begin{array}{ll} \beta\textit{-equality}: & (\lambda_x a)x = a, \\ \eta\textit{-equality}: & \lambda_x (ax) = a, \end{array}$$

provided x is not free in a in η-equality. The present form of β-equality yields the usual form in the presence of substitution for free variables.

(The usual form of β-equality and η-equality imply α-equality.) The fact that the lambda calculus based on β-equality and η-equality is consistent is due to the fact that the language has been restricted, either by preventing anything like negation or implication to occur in terms, or by introducing types. Without restrictions, in type-free *illative* theories, we regain inconsistency.

So the general pattern of Comprehension and Extensionality, on the one hand, and of β and η-equality, on the other, is remarkably analogous. These postulates assert that a variable-binding expression Γ_X and application to a variable Φ_X are inverse to each other, in the sense that $\Phi_X\Gamma_X\,\alpha$ and $\Gamma_X\Phi_X\,\alpha$ are either equivalent or equal to α, depending on the grammatical category of α. It is even more remarkable that theories so rich and important as set theory and the lambda calculus are based on such a simple inversion principle.

3. DEDUCTIONS AND INVERSION

Besides referring and asserting, there is a third kind of activity of particular interest to logic: deducing, which is also linguistic, as far as it consists in passing from propositions to propositions.

To speak about deductions we may use *labelled sequents* of the form $f : \Gamma \vdash B$, where Γ is a collection of propositions making the premises, the proposition B is the conclusion, and the term f records the rules justifying the deduction. If the premises can be collected into a single proposition, and this is indeed the case if Γ is finite and we have a connective like conjunction, then we can restrict our attention to simple sequents of the form $f : A \vdash B$, where both A and B are propositions. We can take that $f : A \vdash B$ is an arrow in a category in which A and B are objects. When we don't need it, we omit mentioning the *type* $A \vdash B$ of $f : A \vdash B$, and write just the *arrow term* f.

Special arrows in a category are axioms, and operations on arrows are rules of inference. Equalities of arrows are equalities of deductions. For that, categorial equalities between arrows have to make proof-theoretical sense, as indeed they do, by following closely reductions in a normalization or cut-elimination procedure in intuitionistic and substructural logics.

In categorial proof theory we are not concerned with a consequence *relation*, but with a consequence *graph*, where more than one arrow, i.e. deduction, can join the same pair of objects, i.e. propositions. This should be the watershed between proof theory and the rest of logic. It is indeed a defect of traditional general proof theory, unaware of categories, that it is still very much under the spell of sequents understood in terms of consequence relations—as if all deductions with the same premises and conclusions were equal. The traditional theory has trouble in representing deductions and in coding them. It draws trees and has no clear criteria of identity of deductions. (Applying the typed lambda calculus in general proof theory usually brings awareness of categories.)

We shall now inquire whether there is something in the context of deductions, as they are understood in categories, which would be analogous to the inversion principle we encountered before in set theory and the lambda calculus.

Take a category $\mathcal{K}$ with a terminal object T (this object behaves like the constant true proposition), and take the *polynomial category* $\mathcal{K}[x]$ obtained by extending $\mathcal{K}$ with an *indeterminate arrow* $x : \mathrm{T} \vdash A$ (see [Lambek & Scott 1986], Part I, Chapters 4-5, and [D. 2001]). We obtain $\mathcal{K}[x]$ by adding to the graph of arrows of $\mathcal{K}$ a new arrow $x : \mathrm{T} \vdash A$, and then by imposing on the new graph equalities required by the particular sort of category to which $\mathcal{K}$ belongs. Note that $\mathcal{K}[x]$ is not simply the free category of the required sort generated by the new graph, because the operations on arrows of $\mathcal{K}[x]$ should coincide with those of $\mathcal{K}$ on the arrows inherited from $\mathcal{K}$ (see [D. 2001], Section 5). We can conceive of $\mathcal{K}[x]$ as the extension of a deductive system $\mathcal{K}$ with a new axiom A.

Now consider the variable-binding expression Γ_x that assigns to every arrow term $f : C \vdash B$ of $\mathcal{K}[x]$ the arrow term $\Gamma_x f : C \vdash A \to B$ of $\mathcal{K}$, where $\to$, which corresponds to implication, is an operation on the objects of $\mathcal{K}$ (in categories, $A \to B$ is more often written B^A). Passing from f to $\Gamma_x f$ corresponds to the deduction theorem. Conversely, we have application to x, denoted by Φ_x, which assigns to an arrow term $g : C \vdash A \to B$ of $\mathcal{K}$ the arrow term $\Phi_x g : C \vdash B$ of $\mathcal{K}[x]$. Now, passing from g to $\Phi_x g$ corresponds to modus ponens. If we require that

$$(\beta) \qquad \Phi_x \Gamma_x f = f,$$
$$(\eta) \qquad \Gamma_x \Phi_x g = g,$$

we obtain a bijection between the hom-sets $\mathcal{K}(C, A \to B)$ and $\mathcal{K}[x](C, B)$. If, moreover, we require that this bijection be natural in the arguments B

and C, we obtain an *adjunction*. The left-adjoint functor in this adjunction is the *heritage* functor from $\mathcal{K}$ to $\mathcal{K}[x]$, which assigns to objects and arrows of $\mathcal{K}$ their heirs in $\mathcal{K}[x]$, while the right-adjoint functor is a functor from $\mathcal{K}[x]$ to $\mathcal{K}$ that assigns to an object B the object $A \rightarrow B$. We find such an adjunction in cartesian closed categories, whose arrows correspond to deductions of the implication-conjunction fragment of intuitionistic logic, and also in bicartesian closed categories, whose arrows correspond to deductions of the whole of intuitionistic propositional logic.

In cartesian closed and bicartesian closed categories, as well as in cartesian categories tout court, we also have the adjunction given by the bijection between the hom-sets $\mathcal{K}(A \times C, B)$ and $\mathcal{K}[x](C, B)$. Here the heritage functor is right adjoint, and a functor from $\mathcal{K}[x]$ to $\mathcal{K}$ that assigns to an object C the object $A \times C$ is left adjoint. The binary product operation on objects $\times$ corresponds to conjunction, both intuitionistic and classical, as $\rightarrow$ corresponds to intuitionistic implication.

These adjunctions, which were first considered by Lambek, and which he called *functional completeness*, are a refinement of the deduction theorem (see [Lambek 1974], [Lambek & Scott 1986], Part I, [D. 1996] and [D. 2001]). Through the categorial equivalence of the typed lambda calculus with cartesian closed categories, which was also discovered by Lambek, they are closely related to the so-called Curry-Howard correspondence between typed lambda terms and natural-deduction proofs. They shed much light on this correspondence. The adjunctions of functional completeness may serve to characterize conjunction and intuitionistic implication.

4. LOGICAL CONSTANTS AND ADJUNCTION

Adjointness phenomena pervade logic, as well as much of mathematics. An essential ingredient of the spirit of logic is to investigate inductively defined notions, and inductive definitions engender free structures, which are tied to adjointness. We find also in logic the important model-theoretical adjointness between syntax and semantics, behind theorems of the "if and only if" type called semantical completeness theorems. However, adjunction is present in logic most specifically through its connection with logical constants.

Lawvere put forward the remarkable thesis that all logical constants are characterized by adjoint functors (see [Lawvere 1969]). Lawvere's thesis about logical constants is just one part of what he claimed for adjunction, but it is a significant part.

Actually, Lawvere didn't characterize conjunction and intuitionistic implication through the adjunctions of functional completeness we mentioned in the preceding section. Instead, there is for conjunction, i.e. binary product in cartesian categories, the adjunction between the diagonal functor $\mathrm{D}: \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ as left adjoint and the internal product bifunctor $\times: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ as right adjoint. Coproduct, i.e. disjunction, is analogously left adjoint to the diagonal functor. The terminal and initial objects, which correspond respectively to the constant true proposition and the constant absurd proposition, may be conceived as empty product and empty coproduct. They are characterized by functors right and left-adjoint, respectively, to the constant functor into the trivial category with a single object and a single identity arrow. Functors tied to the universal and existential quantifiers are, respectively, right adjoint and left adjoint to the substitution functor, which we find in hyperdoctrines, or fibered categories.

In all that, one of the adjoint functors carries the logical constant to be characterized, i.e., it involves the corresponding operation on objects and depends on the inner constitution of the category, while the other adjoint functor is a *structural* functor, which does not involve the inner operations of the category ("structural" is here used as in the "structural rules" of Gentzen's proof theory). The diagonal functor and the constant functor are clearly structural: they make sense for any sort of category. The substitution functor may also be conceived as structural, and such is the heritage functor too. (We relied above on the presence of the terminal object to characterize implication through functional completeness, but this was done only to simplify the exposition, and is not essential.)

Lawvere's way to characterize intuitionistic implication through adjunction is by relying on the bijection between $\mathcal{K}(A \times C, B)$ and $\mathcal{K}(C, A \to B)$, which can be obtained by composing the two adjunctions with the heritage functor mentioned in the preceding section. The disadvantage of this characterization is that none of the adjoint functors $A\times$ and $A\to$ is structural (though the former resembles such a functor more than the latter).

In the late seventies (see [D. 1989], which summarizes the results of my doctoral thesis written ten years before), I was engaged in characterizing logical constants of classical, intuitionistic and substructural logics through equivalences between a sequent involving the logical constant in question at a particular place and a *structural*, purely schematic, sequent, not involving any logical constant. A typical such equivalence is

$$\Gamma \vdash A \rightarrow B \;\; iff \;\; A, \Gamma \vdash B.$$

I called such equivalences *analyses*, and not *definitions*, because they may lack some essential traits of definitions, like conservativeness and replaceability by the defining expression in every context.

I realized more recently that my analyses were just superficial aspects of adjunctions. They pointed to the inversion principle, but didn't mention the naturalness condition of adjunctions. This last condition may perhaps be taken as implicit, but I lacked a clear idea of identity of deductions. However, this idea is also unclear in all of traditional general proof theory untouched by categorial proof theory. Gentzen's and Prawitz's inversion principle for natural deduction, which says that the elimination rules can be recovered from the introduction rules, amounts to analytical equivalence, and is in the same way a superficial aspect of adjointness (see Gentzen's *Untersuchungen über das logische Schließen*, II, § 5.13, and [Prawitz 1965], Chapter II).

However, what I did brings something which I think should be added to Lawvere's thesis: namely, the functor carrying the logical constant should be adjoint to a *structural* functor recording some features of deduction. With this amendment the thesis might serve to separate logical constants from other expressions.

I suppose that my notion of analysis corresponds to an adjoint situation that does not amount to an adjoint equivalence of categories, while ordinary definitions are based on an equivalence of categories.

In some poignant passages of his book on Frege, Dummett has argued very convincingly that the inversion principle of natural deduction, discovered by Gentzen and studied by Prawitz, operates in ordinary language too (see [Dummett 1973], pp. 396-397, 454-455). With pejorative expressions this principle is broken so that sufficient conditions for an assertion are weaker than the conclusions we may draw from the assertion. "Long-winded" may be taken as a pejorative expression because conclusions one can infer from the assertion that somebody's performance is such, like the conclusion that the matter should be ignored, need not be warranted by a sufficient condition for the assertion, which can be merely that the thing is long. Actually, the point of using pejoratives is to licence some otherwise unwarranted inferences. (Unwarranted conclusions in the case of pejoratives are condemning, whereas the point of using flattery terms is to licence commending conclusions, which may also be unwarranted.)

To complete what Dummett is saying, one could add that with euphemisms, dually to what one has with pejoratives, the sufficient conditions for an assertion are stronger than the conclusions we are expected to draw from the assertion. A sufficient condition for asserting that some text is "not concise" might be that it is unbearably long, and the conclusion that it should be ignored, which could be drawn from this sufficient condition, is meant to be blocked by using the euphemism. The point of using euphemisms is to block unwanted inferences (or, at least, the speaker pretends he means to block them).

5. Identity of Deductions and Propositional Identity

Many successful philosophical analyses are achieved by a shift in grammatical categories. Such is Frege's analysis of the predicate "exists" in terms of the existential quantifier, or Russell's analysis of definite descriptions. We find this shift in grammatical form in the analyses of logical constants mentioned in the preceding section. In

$$\Gamma \vdash A \rightarrow B \;\; iff \;\; A, \Gamma \vdash B$$

the connective of implication is analyzed in terms of the turnstile, which stands for deducibility.

A simple example of a good analysis with shift in grammatical form, mentioned by Frege in *Die Grundlagen der Arithmetik* (§§ 64-68), is the analysis of the notion of the direction of a line *a* as the equivalence class of lines parallel to *a*. This amounts to the analytical equivalence

The direction of a is equal to the direction of b iff a is parallel to b.

What is achieved in passing from the left-hand side of this equivalence to the right-hand side is that we have eliminated a spurious individual term "the direction of *a*" and used instead the uncontroversial binary predicate "is parallel to".

Leibniz's analysis of identity, given by the equivalence

a is identical to b iff "a" can always be replaced by "b" salva veritate,

achieves a fundamental grammatical shift. It assumes as given and uncontroversial propositional equivalence, i.e. identity of truth value, and analyzes in terms of it identity of individuals. It is because of this shift that

Hide Ishiguro could find behind Leibniz's analysis a form of Frege's context principle, which says that we should explain the sense of a word in terms of the truth and falsity of propositions in which it may occur (see [Ishiguro 1990], Chapter II). To put it in a nutshell, Frege's principle says that when it comes to explaining how language functions, asserting is more basic than referring (see [Dummett 1973], pp. 3-7).

What about deducing? Is it less or more basic than asserting or referring? If we surmise that it is more basic than asserting, in the order of explaining how language functions, we have opened the way to analyze propositional identity in terms of an equivalence relation between deductions, much as Leibniz analyzed identity of individuals in terms of propositional equivalence. The most plausible candidate for an equivalence relation that would do the job is identity of deductions as codified in categories. We have said that this equivalence of deductions is motivated by normalization in natural deduction or by cut elimination.

Propositional equivalence, which in classical logic is defined by identity of truth value, is understood as follows in a proof-theoretical context:

A is equivalent to B iff there is a deduction $f : A \vdash B$ *and a deduction* $g : B \vdash A$.

This relation between the propositions *A* and *B*, which certainly doesn't amount to the stricter relation of propositional identity, does not rely on a criterion of identity of deduction.

By relying on such a criterion, we could analyze propositional identity as follows, quite in tune with how category theory understands identity of objects:

A is the same propositions as B iff A and B are isomorphic.

Isomorphism is here understood in the precise way how category theory understands isomorphism of objects: namely, there is a deduction, i.e. arrow, $f : A \vdash B$ and a deduction $g : B \vdash A$ such that *g* composed with *f* and *f* composed with *g* are equal respectively to the identity deductions from *A* to *A* and from *B* to *B*. That two objects are isomorphic means that they behave exactly in the same manner in deductions: by composing, we can always extend deductions involving one of them, either as premise or as conclusion, to deductions involving the other, so that nothing is lost, nor gained. There is always a way back. By composing further with the inverses, we return to the original deductions.

6. IS THERE A SEMANTICS OF DEDUCTION?

In theoretical linguistics syntactical theory was much more prosperous than semantical theory. Not so in logic, where semantics, i.e. model theory, has for a long time been preponderant over syntax.

Logicians are concerned with language much more than other mathematicians, and the old name of mathematical logic, *symbolic logic*, rightly stressed that. It is true that linguistic preoccupations are not foreign to some other branches of mathematics—in particular, algebra—but their involvement with language rarely matches that of logic.

Proof theory is entirely within the sphere of language, and, with many good reasons, *syntactical* is usually taken as synonymous with *proof-theoretical*. It is also pretty secure to consider that the set-theoretic models of classical model theory give the *semantics* of mathematical theories based on classical logic. But to call “semantics” the production of any kind of models for other sorts of systems, like the lambda calculus, or systems of nonclassical logics, may well be abusive, if we understand “semantics” *à la lettre*, as the theory giving an explanation of meaning.

Did the untyped lambda calculus really acquire meaning only when, at a rather late date, some sorts of models were found for it? Do the extant models of intuitionistic logic, or various substructural logics, give meaning to these logics, which are otherwise motivated mostly by proof theory? And what to say about various uses of the word “semantics” in theoretical computer science, or its borderlines, where some rather syntactical activities, like coding natural-deduction proofs with typed lambda terms, or just translating one formal language into another, are deemed a matter of semantics?

Completeness proofs are the glory of logic (though incompleteness proofs are even more glorious), but they should not serve as an excuse for the cheap question, often posed irresponsibly after colloquium talks, or in referee’s reports: “What can you tell us about the semantics of your system? What about its models?” And this question should not receive a cheap answer, which consists in producing anything resembling models, or even not resembling them, as a semantics.

Classical model-theoretical semantics gives meaning to referential expressions like terms through models, and propositions acquire truth values through these models, but these models can hardly serve to give meaning to deductions. A consequence relation may be defined with respect to models, but we said that we need rather a consequence graph, where between the same premise and conclusion there may be several deductions. From the point of view of classical model theory, deductions are not bound to the models, but only to the language.

The fact that there is no room for deductions in the classical semantical framework, whose spirit is Platonistic, should be significant for the philosophy of mathematics. That part of mathematics which is bound to deduction—namely, logic—could be understood in a formalistic vein, whereas in the rest of mathematics we would have Platonism. This sort of formalistic conception would resemble Hilbert's formalism in so far as it is not purely formalistic—it understands formalistically just one part of mathematics. However, it differs very much from Hilbert's conception by finding formalism in logic, whereas Hilbert looked for it in those parts of mathematics transcending the finite. Moreover, Hilbert understood the foundational, finitistic, part of mathematics in a constructivist vein. Logic need not coincide with the finitistic part of mathematics, but it should presumably be found in the foundations. So Hilbert's formalism would indeed be turned upside down: formalism is in the foundations, and Platonism above, whereas with Hilbert, constructivism is in the foundations, and formalism above.

The fact that we are not prone to speak about models of deduction, and that this topic has not received much attention up to now, is in accordance with a formalistic understanding of deduction. When we encounter different reconstructions of the same deductions, as happens when we have sequents on the one hand and natural deduction on the other hand, the usual inclination is not to speak about one reconstruction being a model of the other, but both are taken as alternative syntaxes.

Still, isn't there something model-theoretical in passing from the calculus of sequents to natural deduction? Couldn't one take natural deduction not just as an alternative syntax, but as a model giving meaning to the sequent calculus?

And what about modelling natural deduction itself? We can code natural-deduction proofs by typed lambda terms, according to the Curry-Howard correspondence, but this seems to be rather a matter of finding a suitable syntax to describe natural-deduction proofs, though there are authors who speak about the typed lambda calculus as providing a semantics of deductions.

Another kind of coding of natural deduction is obtained in categories, by proceeding as Lambek (see [Lambek & Scott 1986] and references therein). The possibility of this coding is not fortuitous: one can prove rigorously that if we want to represent deductive systems set-theoretically by identifying, in the style of intuitionism, a proposition with deductions leading to it, or deductions starting from it, we must end up with categories. In this set-theoretical representation, one can also exhibit effectively the duality between composition of deductions, i.e. cut, and the identity deduction. Composition leads us from the deductive system to the representing category, and the identity deduction brings us back. This representation, which is summarized in the theorem that every small category is isomorphic to a concrete category, i.e. a subcategory of the category of sets with functions, is an elementary aspect of the Yoneda representation, and is related to some aspects of Stone's representation of lattice-orders and to Cayley's representation of monoids (see [D. 1998] or [D. 1999], § 1.9).

We can always take as a model of a category the skeleton of this category, i.e. the category obtained by identifying isomorphic objects, but there are also more "dynamic" kinds of models. ("Dynamic" is often used in theoretical computer science and borderline areas just as a commending expression. It means roughly "okay", while "static" is a pejorative expression.)

Lambek, who first realized in the sixties that categorial equality of arrows coincides with proof-theoretical equivalence induced by normalization, conjectured also that the same equivalence relation between deductions could be characterized by saying that the deductions have the same *generality*, by which he meant that by generalizing the deductions, by diversifying schematic letters as much as possible, while keeping the same rules, we shall end up with the same thing. Lambek's way of making precise the notion of generality of deductions was not successful. At roughly the same time, under the influence of Gentzen and Lambek, the matter was approached with more success in a geometrical vein by Eilenberg, Kelly and Mac Lane, in connection with so-called *coherence* problems in category theory (see [Eilenberg & Kelly 1966] and [Kelly & Mac Lane 1971]). These are roughly decidability problems for the commuting of various classes of diagrams, i.e. decidability problems in an equational calculus of algebraic partial operations.

Lambek's conjecture that generality characterizes Gentzenian equivalence of deductions is not true in general; in particular, it is not true for intuitionistic implication (as it was conclusively shown in [Petrić 1997]). But what appears from these studies of coherence problems is that for categories interesting for logic, which codify deductions in various fragments of intuitionistic and substructural logics, we can find interesting geometrical models. That is, these categories can be faithfully embedded in some categories of geometrical morphisms. The matter was rediscovered two decades latter with the proof nets of linear logic, and there is also a more recent rediscovery in [Buss 1991] and [Carbone 1997]. However, proof nets are officially presented as a new kind of syntax, while Buss and Carbone disregard categories and don't deal explicitly with identity of deductions.

Lambek remarks in [1999] that this geometrization of algebraic matters goes against the direction given to mathematics by Descartes, but, concerning the matter at hand, this may nevertheless be the right direction.

Do these geometrical models give a semantics of deduction? I would refrain from answering the question. What is certain is that they give models of deduction, and these models are appealing and useful. To corroborate that, I shall present in the last section of this talk a geometrical model for the general notion of adjunction. (This model is explored in detail in [D. 1999].)

7. A Geometrical Model of Adjunction

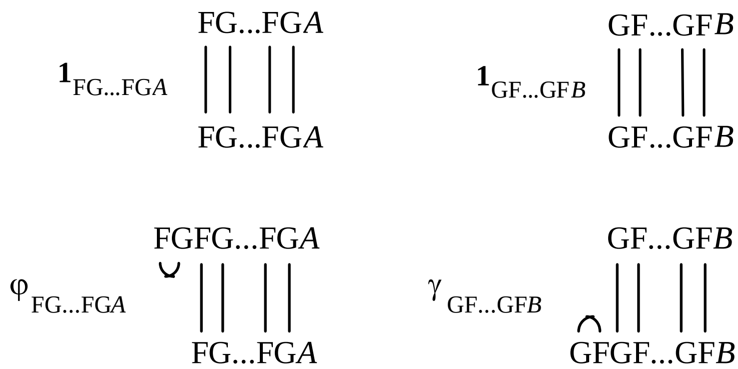
Let us first briefly review one of the standard equational definitions of adjunction. We have two categories $\mathcal{A}$ and $\mathcal{B}$, and two functors, F from $\mathcal{B}$ to $\mathcal{A}$ and G from $\mathcal{A}$ to $\mathcal{B}$. The former functor is called *left adjoint* and the latter *right adjoint*. Next, we have a natural transformation φ in $\mathcal{A}$, called the *counit* of the adjunction, whose components are $\varphi_A : \mathrm{FG}A \vdash A$ for every object A of $\mathcal{A}$, and a natural transformation γ in $\mathcal{B}$, called the *unit* of the adjunction, whose components are $\gamma_B : B \vdash \mathrm{GF}B$ for every object B of $\mathcal{B}$. Finally, the following *triangular equalities* must be satisfied:

$$\varphi_{\mathrm{F}B} \circ \mathrm{F}\gamma_B = \mathbf{1}_{\mathrm{F}B},$$
$$\mathrm{G}\varphi_A \circ \gamma_{\mathrm{G}A} = \mathbf{1}_{\mathrm{G}A}.$$

In logical situations we should imagine that one of the adjoint functors F and G is structural, and hence "invisible". Then the unit and counit correspond to rules for introducing and eliminating a connective.

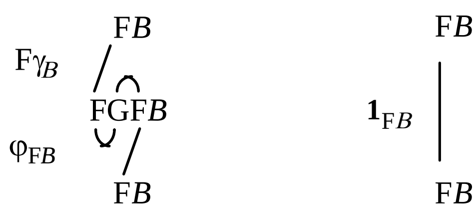
Since all the assumptions in this definition are equational (the equalities in question are the categorial axioms of composing with identity arrows and the associativity of composition, the equalities of the functoriality of F and G, the equalities of naturalness of φ and γ, and the triangular equalities), we can take that we have here an equational calculus. Out of the linguistic material of this calculus we can build the *free adjunction* generated by a set of objects. The details of this construction, as well as other technical details concerning matters in this section, are exposed in [D. 1999].

To every arrow term in the free adjunction we assign a *graph*, which is made of links between occurrences of F and G in the source and target of the arrow term (these graphs should not be confused with the graphs of arrows underlying a category). Identity arrows and the components of the counit and unit have graphs like the following:



while given the graphs for $g : B_1 \vdash B_2$ and $f : A_1 \vdash A_2$ we obtain the graphs of Fg and Gf as follows:

Composition of graphs is defined in the obvious manner; for example, for the first triangular equality we have



It is clear that the composed graph on the left-hand side is equal to the graph on the right-hand side.

One can give a reformulation of the notion of adjunction where composition can be eliminated, in the style of cut elimination. For this reformulation one should replace the families of arrows, i.e. families of components, making the counit and unit of the adjunction by operations on arrows, as Gentzen replaced axioms like $A \wedge B \vdash A$ by rules like

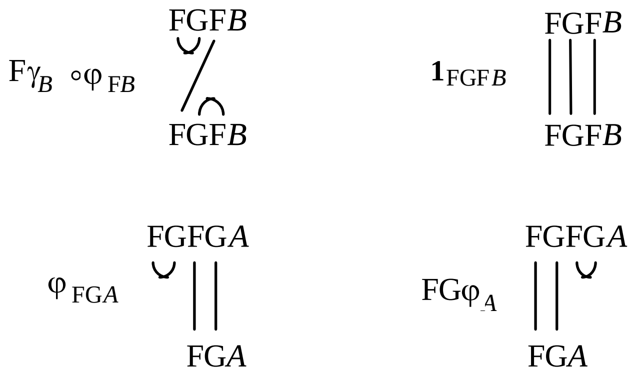
$$\frac{A, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}$$

One can then obtain a composition-free normal form for arrow terms, which is unique for every arrow.

With the help of this composition-free formulation of adjunction it can be proved that for all arrow terms h_1 and h_2 we have $h_1 = h_2$ in the free adjunction iff the graphs of h_1 and h_2 are equal. This result is of the kind called "coherence theorems" in category theory.

So graphs yield a very simple decision procedure for commuting of diagrams in free adjunctions. They enable us also to reduce to normal form arrow terms in the composition-free formulation without syntactical reduction steps. Uniqueness of normal form can also be demonstrated with the help of these graphs without involving anything like the Church-Rosser property of some reduction steps.

Consider, in the free adjunction, the following two pairs of arrow terms of the same type with different graphs:



There are infinitely many such pairs. It can be shown that if we extend the notion of adjunction by equating *any* such pair, we would trivialize the notion. Namely, the resulting free adjunction would be a preorder: any two arrow terms of the same type would be equal. This means that all these equalities of arrows with different graphs are equivalent with each other.

So the notion of adjunction is Post complete in some sense. Graphs are not absolutely needed to demonstrate this result, but they help to shorten calculations of a rather lengthy inductive argument.

Our coherence result for graphs in free adjunctions guarantees that there are faithful functors from the categories involved in the free adjunction to categories whose objects are finite sequences of alternating F’s and G’s, and whose arrows are the graphs. The faithfulness of these functors guarantees that we can speak of completeness with respect to the graph models (soundness amounts to functoriality). Our coherence result is exactly like a completeness theorem.

These categories of graphs are subcategories of categories of *tangles*, which have played recently a prominent role in the theory of quantum groups, in low-dimensional topology and in knot theory (see [Kassel 1995], Chapter XII, [Kauffman & Lins 1994], and references therein). Equality between our graphs covers planar ambient isotopies of tangles without crossings.

Since every logical constant is characterized by an adjunction, we can expect to find in the geometrical models of deductions involving these constants various avatars of our graphs of adjunction.

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