TORTUOUS APPLICATION

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Professor S.L. Achilles works in the Department of Computer Science of Dodgson University. Although his education is in practical computational matters, he has developed an interest in theory. Lately, he has been much influenced by a newcomer to the department, Professor L.C. Tortoise, a logician who has turned to applying theory.* Tortoise gave a course on the lambda calculus, which Achilles tried to follow, and there Achilles learned how to pursue an idea that has haunted him for some time - namely, how to improve the notation for 2+2.

The ordinary theory of formal languages teaches us that 2 and + are symbols of equal stature and that 2+2 is just a finite sequence of such symbols - or rather an element of a free semigroup made of the generators 2 and + with the help of the binary operation of concatenation. So we can take that 2+2 is formed by first applying the operation of concatenation to 2 and +, which yields 2+, and then applying concatenation to 2+ and 2 to get 2+2. (Of course, we may as well obtain +2 first, and then apply concatenation to 2 and +2 to get 2+2.)

Achilles was bothered by the fact that 2 and + are not exactly on an equal footing. The symbol + stands for an operation, while 2 stands for an argument of this operation. Hence, wouldn't it be better not to introduce the operation of concatenation (which is anyway not usually represented by a symbol) and take rather that in 2+2 the operation + is applied to two occurrences of 2? This becomes clearer if we write 2+2 as +(2,2).

Then Achilles learned in Tortoise's course how in the lambda calculus and combinatory logic we get rid of functions of two arguments by replacing them with functions of one argument. So, instead of having the function + of two arguments, we can introduce a function + of one argument, which applied to 2 yields another function of one argument, +(2). In (+(2))(2) the latter function is applied to 2 to yield an expression with the same meaning as 2+2. Achilles also learned to write +(2) as (+2), in the style of the lambda calculus; then (+(2))(2) becomes ((+2)2). And if, still in the style of the lambda calculus, we omit parentheses by associating to the left, we obtain +22. Achilles found this Polish-looking expression, +22, rather satisfying and quite a

* Tortoise's achievements in logic should be quite well known - so let's skip them. The more interesting part of his vita, which he is fond of stressing in book blurbs, notes in *The Mathematical Intelligencer*, or similar edifying material, is his culinary ability. He is now perfecting a lamb curry sauce Howard. For soup he is dependent on a recipe he got from his cousin the linguist M. Turtle, a pupil of the distinguished French chef Lucien Tesnière.

bit better theoretically grounded than 2+2. Practical offshoots were bound to come, and he immediately sat to write a paper oriented towards such matters.

But then torture started for Achilles. Because Tortoise, commenting on the draft of the paper, told him that in +22 we have omitted writing something that might be important. When we restore parentheses in +22, so that we have ((+2)2), we have still not written quite explicitly the two-argument function of application. True, parentheses in (+2) remind us that + has been applied to 2, but wouldn't it be wiser if we introduced explicitly a symbol α for the function of application that has + and 2 as arguments; so we could write (+α2), which means the same as α(+,2), instead of (+2), leaving to parentheses just the auxiliary role they usually have? Instead of ((+2)2) we would then write ((+α2)α2).

Tortoise also taught Achilles how to frame α in the style of the lambda calculus. Remember how we replaced a binary + by a unary +. We shall do something analogous with α: we shall replace the binary α by a unary one. Instead of (+α2) we shall have ((α+)2), constructed by first applying the one-argument function α to + and then applying the one-argument function (α+) to 2. So ((+α2)α2) becomes ((α((α+)2))2). As before, some parentheses can be omitted by assuming we always associate to the left. So we get the expression α(α+2)2.

Achilles was quite happy with that one, and was in general impressed by all that. So he immediately started writing a new paper, in which the advantages of the notation α(α+2)2 over 2+2 were to become apparent.

When Tortoise read the draft of the new paper he remarked that as far as ((α((α+)2))2) is concerned, the application of α to + in (α+), the application of (α+) to 2, etc. are not written explicitly. Shouldn't we write down these applications explicitly, let us say by β, so that ((α((α+)2))2) becomes ((αβ((αβ+)β2))β2)? And by replacing the binary β by a unary one we would get ((β((βα)((β((βα)+))2)))2). After omitting superfluous parentheses, as before, we get β(βα(β(βα+)2))2.

Achilles wrote a paper on the advantages of β(βα(β(βα+)2))2 over 2+2, which was duly criticized by Tortoise, and after that a whole series of papers with other, longer and longer expressions, involving γ, δ, ε, ... He has now reached ξ and wonders what will happen if he has to go all the way up to ω. Should he switch to another font?

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