Coherent Bicartesian and Sesquicartesian Categories

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Abstract

Coherence is here demonstrated for sesquicartesian categories, which are categories with nonempty finite products and arbitrary finite sums, including the empty sum, where moreover the first and the second projection from the product of the initial object with itself are the same. (Every bicartesian closed category, and, in particular, the category **Set**, is such a category.) This coherence amounts to the existence of a faithful functor from categories of this sort freely generated by sets of objects to the category of relations on finite ordinals, and it yields a very easy decision procedure for equality of arrows. Restricted coherence holds also for bicartesian categories where, in addition to this equality for projections, we have that the first and the second injection to the sum of the terminal object with itself are the same.

The printed version of this paper (in: R. Kahle et al. eds, *Proof Theory in Computer Science*, Lecture Notes in Computer Science, vol. 2183, Springer, Berlin, 2001, pp. 78-92) and versions previously posted here purported to prove unrestricted coherence for the bicartesian categories mentioned above. Lemma 5.1 of these versions, on which the proof of coherence for sesquicartesian categories relied too, is however not correct. The present version of the paper differs from the previous ones also in terminology.

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1 Introduction

The connectives of conjunction and disjunction in classical and intuitionistic logic make a structure corresponding to a distributive lattice. Among nonclassical logics, and, in particular, among substructural logics, one finds also conjunction and disjunction that make a structure corresponding to a free lattice, which is not distributive.

In proof theory, however, we are not concerned only with a consequence *relation*, which in the case of conjunction and disjunction gives rise to a lattice order, but we may distinguish certain deductions with the same premise and the same conclusion. For example, there are two different deductions from $A \wedge A$ to A, one corresponding to the first projection and the other to the second projection, and two different deductions from A to $A \vee A$, one corresponding to the first injection and the other to the second injection.

If we identify deductions guided by normalization, or cut elimination, we will indeed distinguish the members in each of these two pairs of deductions, and we will end up with natural sorts of categories, whose arrows stand for deductions. Instead of nondistributive lattices, we obtain then categories with binary products and sums (i.e. coproducts), where the product $\times$ corresponds to conjunction and the sum $+$ to disjunction. If, in order to have all finite products and sums, we add also the empty product, i.e. the terminal object, which corresponds to the constant true proposition, and the empty sum, i.e. the initial object, which corresponds to the constant absurd proposition, we obtain *bicartesian* categories, namely categories that are at the same time *cartesian*, namely, with all finite products, and *cocartesian*, with all finite sums. Bicartesian categories need not be distributive.

One may then enquire how useful is the representation of deductions involving conjunction and disjunction in bicartesian categories for determining which deductions are equal and which are not. A drawback is that here cut-free, i.e. composition-free, form is not unique. For example, for $f : A \to C$, $g : A \to D$, $h : B \to C$ and $j : B \to D$ we have the following two composition-free arrow terms

$$\frac{\langle f, g\rangle : A \to C \times D \qquad \langle h, j\rangle : B \to C \times D}{[\langle f, g\rangle, \langle h, j\rangle] : A + B \to C \times D}$$

$$\frac{[f, h] : A + B \to C \qquad [g, j] : A + B \to D}{\langle [f, h], [g, j]\rangle : A + B \to C \times D}$$

which designate the same arrow, but it is not clear which one of them is to be considered in normal form. The same problem arises also for distributive bicartesian categories, i.e. for classical and intuitionistic conjunction and disjunction. (This problem is related to questions that arise in natural deduction with Prawitz's reductions tied to disjunction elimination, sometimes called "commuting conversions".)

Cut elimination may then be supplemented with a *coherence* result. The notion of coherence is understood here in the following sense. We say that coherence holds for some sort of category iff for a category of this sort freely generated by a set of objects there is a faithful functor from it to a *graphical* category, whose arrows are sets of links between the generating objects. These links can be drawn, and that is why we call the target category "graphical". Intuitively, these links connect the objects that must remain the same after "generalizing" the arrows. We shall define precisely below the graphical category we shall use for our coherence results, and the intuitive notion of generalizing (which originates in [6] and [7]) will become clear. This category will be the category of relations on finite ordinals. In other cases, however, we might have a different, but similar, category. It is desirable that the graphical category be such that through coherence we obtain a decision procedure for equality of arrows, and in good cases, such as those investigated in [3] and here, this happens indeed.

Although this understanding of coherence need not be the most standard one, the paradigmatic results on coherence of [10] and [5] can be understood as faithfulness results of the type mentioned above, and this is how we understand coherence here. We refer to [3], and papers cited therein, for further background motivation on coherence.

This paper is a companion to [3], where it was shown that coherence holds for categories with binary, i.e. nonempty finite, products and sums, but without the terminal and the initial object, and without distribution. One obtains thereby a very easy decision procedure for equality of arrows. With the help of this coherence, it was also demonstrated that the categories in question are maximal, in the sense that in any such category that is not a preorder all the equations between arrows inherited from the free categories with binary products and sums are the same. An analogous maximality can be established for cartesian categories (see [1]) and cartesian closed categories (see [11] and [2]).

In this paper we shall be concerned with categories with nonempty finite products and arbitrary finite sums, including the empty sum, i.e. the initial object. We call such categories *sesquicartesian*. As a matter of fact, *sesquicocartesian* would be a more appropriate label for these categories, since they are *cocartesian* categories—namely, categories with arbitrary finite sums—to which we add a part of the cartesian structure—namely, nonempty finite products. *Sesquicartesian* is more appropriate as a label for cartesian categories—namely, categories with arbitrary finite products—to which we add nonempty finite sums, which are a part of the cocartesian structure. However, sesquicocartesian and sesquicartesian categories so understood are dual to each other, and in a context where both types of categories are not considered, there is no necessity to burden oneself with the distinction, and make a strange new name even stranger. So we call here *sesquicartesian* categories with nonempty finite products and arbitrary finite sums.

We shall show that coherence holds for sesquicartesian categories in which the first and the second projection arrow from the product of the initial object with itself are equal. Such sesquicartesian categories were called *coherent* sesquicartesian categories in the printed version of this paper. Now we use just *sesquicartesian category* to designate what we used to call *coherent sesquicartesian category.* Every bicartesian closed category, and, in particular, the sesquicartesian category **Set** of sets with functions as arrows, is a sesquicartesian category in this new sense of the term. It is not true, however, that sesquicartesian categories are maximal, in the sense in which cartesian categories, cartesian closed categories and categories with binary products and sums are maximal.

Bicartesian categories are also not maximal. Coherence does not hold for bicartesian categories in general. We will prove however a restricted coherence result for bicartesian categories where besides the equality of the projection arrows mentioned above we also have that the first and the second injection arrow to the sum of the terminal object with itself are equal. We call such bicartesian categories *dicartesian* categories. (In the printed version of this paper we called them *coherent* bicartesian categories.) The bicartesian category **Set** is not such a category, but a few natural examples of such categories may be found in Section 7. Such categories are also not maximal.

As in [3], our coherence for sesquicartesian categories yields a very easy decision procedure for equality of arrows in the categories of this kind freely generated by sets of objects. For dicartesian categories this procedure is partial. Without maximality, however, the application of this decision procedure to an arbitrary category of the appropriate kind is limited. We can use this decision procedure only to show that two arrows inherited from the free category are equal, and we cannot use it to show that they are not equal.

We said that this paper is a companion to [3], but except for further motivation, for which we refer to [3], we shall strive to make the present paper self-contained. So we have included here definitions and proofs that are just versions of material that may be found also in [3], but which for the sake of clarity it is better to adapt to the new context.

2 Free Dicartesian and Sesquicartesian Categories

The propositional language $\mathcal{P}$ is generated from a set of *propositional letters* $\mathcal{L}$ with the nullary connectives, i.e. propositional constants, I and O, and the binary connectives $\times$ and $+$. The fragment $\mathcal{P}_{\times,+,\mathrm{O}}$ of $\mathcal{P}$ is obtained by omitting all formulae that contain I. For the propositional letters of $\mathcal{P}$, i.e. for the members of $\mathcal{L}$, we use the schematic letters $p, q, \ldots$, and for the formulae of $\mathcal{P}$, or of its fragments, we use the schematic letters $A, B, \ldots, A_1, \ldots$ The propositional letters and the constants I and O are *atomic* formulae. The formulae of $\mathcal{P}$ in which no propositional letter occurs will be called *constant objects*.

Next we define inductively the *terms* that will stand for the arrows of the free *dicartesian category* $\mathcal{D}$ generated by $\mathcal{L}$. Every term has a *type*, which is a pair (A, B) of formulae of $\mathcal{P}$. That a term f is of type (A, B) is written $f : A \to B$. The *atomic* terms are for every A and every B of $\mathcal{P}$

$$\mathbf{1}_A : A \to A,$$

$$\begin{array}{ll} k_A : A \to \mathrm{I}, & l_A : \mathrm{O} \to A, \\ k^1_{A,B} : A \times B \to A, & l^1_{A,B} : A \to A + B, \\ k^2_{A,B} : A \times B \to B, & l^2_{A,B} : B \to A + B, \\ w_A : A \to A \times A, & m_A : A + A \to A. \end{array}$$

The terms $\mathbf{1}_A$ are called *identities*. The other terms of $\mathcal{D}$ are generated with the following operations on terms, which we present by rules so that from the terms in the premises we obtain the terms in the conclusion:

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

$$\frac{f : A \to B \qquad g : C \to D}{f \times g : A \times C \to B \times D} \qquad \frac{f : A \to B \qquad g : C \to D}{f + g : A + C \to B + D}$$

We use $f, g, \dots, f_1, \dots$ as schematic letters for terms of $\mathcal{D}$.

The category $\mathcal{D}$ has as objects the formulae of $\mathcal{P}$ and as arrows equivalence classes of terms so that the following equations are satisfied for $i \in \{1, 2\}$:

$$\begin{array}{ll} (cat\,1) & \mathbf{1}_B \circ f = f \circ \mathbf{1}_A = f, \\ (cat\,2) & h \circ (g \circ f) = (h \circ g) \circ f, \end{array}$$

$(\times 1)$ $\mathbf{1}_A \times \mathbf{1}_B = \mathbf{1}_{A \times B}$,
$(\times 2)$ $(g_1 \circ g_2) \times (f_1 \circ f_2) = (g_1 \times f_1) \circ (g_2 \times f_2)$,
(k^i) $k^i_{B_1,B_2} \circ (f_1 \times f_2) = f_i \circ k^i_{A_1,A_2}$,
(w) $w_B \circ f = (f \times f) \circ w_A$,
$(kw1)$ $k^i_{A,A} \circ w_A = \mathbf{1}_A$,
$(kw2)$ $(k^1_{A,B} \times k^2_{A,B}) \circ w_{A \times B} = \mathbf{1}_{A \times B}$,
(k) for $f : A \to \mathrm{I}$, $f = k_A$,
$(k\mathrm{O})$ $k^1_{\mathrm{O},\mathrm{O}} = k^2_{\mathrm{O},\mathrm{O}}$,

$(+1)$ $\mathbf{1}_A + \mathbf{1}_B = \mathbf{1}_{A+B}$,
$(+2)$ $(g_1 \circ g_2) + (f_1 \circ f_2) = (g_1 + f_1) \circ (g_2 + f_2)$,
(l^i) $(f_1 + f_2) \circ l^i_{A_1,A_2} = l^i_{B_1,B_2} \circ f_i$,
(m) $f \circ m_A = m_B \circ (f + f)$,
$(lm1)$ $m_A \circ l^i_{A,A} = \mathbf{1}_A$,
$(lm2)$ $m_{A+B} \circ (l^1_{A,B} + l^2_{A,B}) = \mathbf{1}_{A+B}$,
(l) for $f : \mathrm{O} \to A$, $f = l_A$,
$(l\mathrm{I})$ $l^1_{\mathrm{I},\mathrm{I}} = l^2_{\mathrm{I},\mathrm{I}}$.

If we omit the equations $(k\mathrm{O})$ and $(l\mathrm{I})$, we obtain the free bicartesian category generated by $\mathcal{L}$.

The free *sesquicartesian category* $\mathcal{S}$ generated by $\mathcal{L}$ has as objects the formulae of $\mathcal{P}_{\times,+,\mathrm{O}}$. In that case, terms in which k_A occurs are absent, and the equations (k) and $(l\mathrm{I})$ are missing. The remaining terms and equations are as in $\mathcal{D}$.

We shall call terms of $\mathcal{D}$ in which the letters l and m don't occur *K-terms.* (That means there are no subterms of K-terms of the form l_A, $l^i_{A,B}$ and m_A.) Terms of $\mathcal{D}$ in which the letters k and w don't occur will be called L-terms.

3 Cut Elimination

For inductive proofs on the length of objects or terms, it is useful to be able to name the arrows of a category by terms that contain no composition. In this section we shall prove for $\mathcal{D}$ and $\mathcal{S}$ a theorem analogous to cut-elimination theorems of logic, which stem from Gentzen [4]. This theorem says that every term of these categories is equal to a term of a special sort, which in an appropriate language would be translated by a composition-free term. We shall call terms of this special sort *cut-free Gentzen terms*.

We define first the following operations on terms of $\mathcal{D}$, which we call *Gentzen operations*:

$$\begin{array}{ll} K^1_B f =_{def.} f \circ k^1_{A,B}, & L^1_B f =_{def.} l^1_{A,B} \circ f, \\ K^2_A f =_{def.} f \circ k^2_{A,B}, & L^2_A f =_{def.} l^2_{A,B} \circ f, \\ \langle f, g\rangle =_{def.} (f \times g) \circ w_C, & [f, g] =_{def.} m_C \circ (f + g). \end{array}$$

Starting from the identities and the terms k_A and l_A, for every A of $\mathcal{P}$, and closing under the Gentzen operations, we obtain the set of *cut-free Gentzen terms* of $\mathcal{D}$. The *Gentzen terms of* $\mathcal{D}$ are obtained by closing cut-free Gentzen terms under composition.

It is easy to show that every term of $\mathcal{D}$ is equal in $\mathcal{D}$ to a Gentzen term, since we have the following equations:

$$\begin{array}{ll} k^1_{A,B} = K^1_B \mathbf{1}_A, & l^1_{A,B} = L^1_B \mathbf{1}_A, \\ k^2_{A,B} = K^2_A \mathbf{1}_B, & l^2_{A,B} = L^2_A \mathbf{1}_B, \\ w_A = \langle \mathbf{1}_A, \mathbf{1}_A\rangle, & m_A = [\mathbf{1}_A, \mathbf{1}_A], \\ f \times g = \langle K^1_C f, K^2_A g\rangle, & f + g = [L^1_D f, L^2_B g]. \end{array}$$

We need the following equations of $\mathcal{D}$:

$$\begin{array}{llll} (K1) & g \circ K^i_A f = K^i_A (g \circ f), & (L1) & L^i_A g \circ f = L^i_A (g \circ f), \\ (K2) & K^i_A g \circ \langle f_1, f_2\rangle = g \circ f_i, & (L2) & [g_1, g_2] \circ L^i_A f = g_i \circ f, \\ (K3) & \langle g_1, g_2\rangle \circ f = \langle g_1 \circ f, g_2 \circ f\rangle, & (L3) & g \circ [f_1, f_2] = [g \circ f_1, g \circ f_2], \end{array}$$

in order to prove the following theorem for $\mathcal{D}$.

CUT ELIMINATION. *Every term is equal to a cut-free Gentzen term.*

Proof. We first find for an arbitrary term of $\mathcal{D}$ a Gentzen term h equal to it. Let the *degree* of a Gentzen term be the number of occurrences of Gentzen operations in this term. Take a subterm $g \circ f$ of h such that both f and g are cut-free Gentzen terms. We call such a term a *topmost cut*. We show that $g \circ f$ is either equal to a cut-free Gentzen term, or it is equal to a Gentzen term whose topmost cuts are of strictly smaller degree than the degree of $g \circ f$. The possibility of eliminating the main compositions of topmost cuts, and hence of finding for h a cut-free Gentzen term, follows by induction on degree.

The cases where f or g is $\mathbf{1}_A$, or f is l_A, or g is k_A, are taken care of by $(cat\,1)$, (l) and (k). The cases where f is $K^i_A f'$ or g is $L^i_A g'$ are taken care of by

$(K1)$ and $(L1)$. And the cases where f is $[f_1, f_2]$ or g is $\langle g_1, g_2\rangle$ are taken care of by $(L3)$ and $(K3)$.

The following cases remain. If f is k_A, then g is of a form covered by cases we dealt with above.

If f is $\langle f_1, f_2\rangle$, then g is either of a form covered by cases above, or g is $K^i_A g'$, in which case we apply $(K2)$.

If f is $L^i_A f'$, then g is either of a form covered by cases above, or g is $[g_1, g_2]$, in which case we apply $(L2)$. This covers all possible cases. ⊣

This proof, with the cases involving k_A omitted, suffices to demonstrate Cut Elimination for $\mathcal{S}$.

Let the *cut-free Gentzen K-terms* of $\mathcal{D}$ be obtained from the identities and the terms k_A by closing under $+$ and the Gentzen operations K^i and $\langle , \rangle$. The *Gentzen K-terms* of $\mathcal{D}$ are obtained by closing the cut-free Gentzen K-terms under composition. Let, dually, the *cut-free Gentzen L-terms* of $\mathcal{D}$ be obtained from the identities and the terms l_A by closing under $\times$ and the Gentzen operations L^i and $[,]$. The *Gentzen L-terms* of $\mathcal{D}$ are obtained by closing the cut-free Gentzen L-terms under composition.

Then we can prove the following version of Cut Elimination for the K-terms and the L-terms of $\mathcal{D}$.

CUT ELIMINATION FOR K-TERMS AND L-TERMS. *Every K-term is equal to a cut-free Gentzen K-term, and every L-term is equal to a cut-free Gentzen L-term.*

Proof. It is easy to see that every K-term is equal in $\mathcal{D}$ to a Gentzen K-term. That this Gentzen K-term is equal to a cut-free Gentzen K-term is demonstrated as in the proof of Cut Elimination above by induction on the degree of topmost cuts. We have to consider the following additional cases.

If f is k_A or $\langle f_1, f_2\rangle$, then g cannot be of the form $g_1 + g_2$. If f is $f_1 + f_2$, and g is not of a form already covered by cases in the proof above, then g is of the form $g_1 + g_2$, in which case we apply $(+2)$. This covers all possible cases.

Cut Elimination for L-terms follows by duality. ⊣

Let the *cut-free Gentzen K-terms* of $\mathcal{S}$ be terms of $\mathcal{S}$ obtained from the identities by closing under $+$ and the Gentzen operations K^i and $\langle , \rangle$. The *Gentzen K-terms* of $\mathcal{S}$ are obtained by closing the cut-free Gentzen K-terms of $\mathcal{S}$ under composition. Let the *cut-free Gentzen L-terms* of $\mathcal{S}$ be terms of $\mathcal{S}$ obtained from the identities and the terms l_A by closing under $\times$ and the Gentzen operations L^i and $[,]$. The *Gentzen L-terms* of $\mathcal{S}$ are obtained by closing the cut-free Gentzen L-terms of $\mathcal{S}$ under composition.

Then we can establish Cut Elimination for K-terms and L-terms of $\mathcal{S}$, where equality in $\mathcal{D}$ is replaced by equality in $\mathcal{S}$. We just proceed as in the proof above, with inapplicable cases involving k_A omitted.

4 The Graphical Category

We shall now define a graphical category $\mathcal{G}$ into which $\mathcal{D}$ and $\mathcal{S}$ can be mapped. The objects of $\mathcal{G}$ are finite ordinals. An arrow $f : n \to m$ of $\mathcal{G}$ will be a binary relation from n to m, i.e. a subset of $n \times m$ with domain n and codomain m. The identity $\mathbf{1}_n : n \to n$ of $\mathcal{G}$ is the identity relation on n, and composition of arrows is composition of relations.

For an object A of $\mathcal{D}$, let $|A|$ be the number of occurrences of propositional letters in A. For example, $|(p \times (q+p)) + (\mathrm{I} \times p)|$ is 4.

We now define a functor G from $\mathcal{D}$ to $\mathcal{G}$ such that $G(A) = |A|$. It is clear that $G(A \times B) = G(A + B) = |A| + |B|$. We define G on arrows inductively:

$$
\begin{array}{rcl}
G(\mathbf{1}_A) & = & \{(x,x) : x \in |A|\} = \mathbf{1}_{|A|}, \\[1ex]
G(k^1_{A,B}) & = & \{(x,x) : x \in |A|\}, \\
G(k^2_{A,B}) & = & \{(x+|A|,x) : x \in |B|\}, \\
G(w_A) & = & \{(x,x) : x \in |A|\} \cup \{(x,x+|A|) : x \in |A|\}, \\
G(k_A) & = & \emptyset, \\[1ex]
G(l^1_{A,B}) & = & \{(x,x) : x \in |A|\}, \\
G(l^2_{A,B}) & = & \{(x,x+|A|) : x \in |B|\}, \\
G(m_A) & = & \{(x,x) : x \in |A|\} \cup \{(x+|A|,x) : x \in |A|\}, \\
G(l_A) & = & \emptyset, \\[1ex]
G(g \circ f) & = & G(g) \circ G(f),
\end{array}
$$

and for $f : A \to B$ and $g : C \to D$,

$$G(f \times g) = G(f+g) = G(f) \cup \{(x+|A|, y+|B|) : (x,y) \in G(g)\}.$$

Though $G(\mathbf{1}_A)$, $G(k^1_{A,B})$ and $G(l^1_{A,B})$ are the same as sets of ordered pairs, in general they have different domains and codomains, the first being a subset of $|A| \times |A|$, the second a subset of $(|A| + |B|) \times |A|$, and the third a subset of $|A| \times (|A| + |B|)$. We have an analogous situation in some other cases.

The arrows $G(f)$ of $\mathcal{G}$ can easily be represented graphically, by drawings linking propositional letters, as it is illustrated in [3]. This is why we call this category "graphical".

It is easy to check that G is a functor from $\mathcal{D}$ to $\mathcal{G}$. We show by induction on the length of derivation that if $f = g$ in $\mathcal{D}$, then $G(f) = G(g)$ in $\mathcal{G}$. (Of course, G preserves identities and composition.)

For the bicartesian structure of $\mathcal{G}$ we have that the operations $\times$ and $+$ on objects are both addition of ordinals, the operations $\times$ and $+$ on arrows coincide and are defined by the clauses for $G(f \times g)$ and $G(f+g)$, and the terminal and the initial object also coincide: they are both the ordinal zero. The category $\mathcal{G}$ has zero arrows—namely, the empty relation. The bicartesian category $\mathcal{G}$ is a linear category in the sense of [9] (see p. 279). The functor G from $\mathcal{D}$ to $\mathcal{G}$ is not just a functor, but a bicartesian functor; namely, a functor that preserves the bicartesian structure of $\mathcal{D}$.

We also have a functor defined analogously to G, which we call G too, from $\mathcal{S}$ to $\mathcal{G}$. It is obtained from the definition of G above by just rejecting clauses that are no longer applicable.

Our aim is to show that the functor G from $\mathcal{S}$ is faithful.

5 *K*-*L* Normalization

We shall say for a term of $\mathcal{D}$ of the form $f_n \circ \ldots \circ f_1$, for some $n \geq 1$, where f_i is composition-free, that it is *factorized*. By using $(\times 2)$, $(+2)$ and $(cat\,1)$ it is easy to show that every term of $\mathcal{D}$ is equal to a factorized term of $\mathcal{D}$. A subterm f_i in a factorized term $f_n \circ \ldots \circ f_1$ is called a *factor*.

A term of $\mathcal{D}$ where all the atomic terms are identities will be called a *complex identity*. According to $(\times 1)$, $(+1)$ and $(cat\,1)$, every complex identity is equal to an identity. A factor which is a complex identity will be called an *identity factor*. It is clear that if $n > 1$, we can omit in a factorized term every identity factor, and obtain a factorized term equal to the original one.

A term of $\mathcal{D}$ is said to be in *K-L normal form* iff it is of the form $g \circ f : A \rightarrow B$ for f a K-term and g an L-term. Note that K-L normal forms are not unique, since $(m_A \times m_A) \circ w_{A+A}$ and $m_{A\times A} \circ (w_A + w_A)$, which are both equal to $w_A \circ m_A$, are both in K-L normal form.

We can prove the following proposition for $\mathcal{D}$.

K-*L* Normalization. *Every term is equal to a term in K-L normal form.*

Proof. Suppose $f : B \rightarrow C$ is a composition-free K-term that is not a complex identity, and $g : A \rightarrow B$ is a composition-free L-term that is not a complex identity. We show by induction on the length of $f \circ g$ that

$$(*) \quad f \circ g = g' \circ f' \text{ or } f \circ g = f' \text{ or } f \circ g = g'$$

for f' a composition-free K-term and g' a composition-free L-term.

We shall not consider below cases where g is m_B, which are easily taken care of by (m). Cases where f is k_B or g is l_B are easily taken care of by (k) and (l). The following cases remain.

If f is $k^i_{C,E}$ and g is $g_1 \times g_2$, then we use (k^i). If f is w_B, then we use (w). If f is $f_1 \times f_2$ and g is $g_1 \times g_2$, then we use $(\times 2)$, the induction hypothesis, and perhaps $(cat\,1)$.

Finally, if f is $f_1 + f_2$, then we have the following cases. If g is $l^i_{B_1,B_2}$, then we use (l^i). If g is $g_1 + g_2$, then we use $(+2)$, the induction hypothesis, and perhaps $(cat\,1)$. This proves $(*)$.

Every term of $\mathcal{D}$ is equal to an identity or to a factorized term $f_n \circ \ldots \circ f_1$ without identity factors. Every factor f_i of $f_n \circ \ldots \circ f_1$ is either a K-term or an L-term or, by $(cat\,1)$, $(\times 2)$ and $(+2)$, it is equal to $f''_i \circ f'_i$ where f'_i is a K-term and f''_i is an L-term. For example, $(k^1_{A,B} \times l^1_{C,D}) + (w_E + l_F)$ is equal to

$$((\mathbf{1}_A \times l^1_{C,D}) + (\mathbf{1}_{E\times E}) + l_F) \circ ((k^1_{A,B} \times \mathbf{1}_C) + (w_E + \mathbf{1}_{\mathrm{O}})).$$

Then it is clear that by applying $(*)$ repeatedly, and by applying perhaps $(cat1)$, we obtain a term in K-L normal form. $\dashv$

Note that to reduce a term of $\mathcal{D}$ to K-L normal form we have used in this proof all the equations of $\mathcal{D}$ except $(kw1)$, $(kw2)$, $(lm1)$, $(lm2)$, $(k\mathrm{O})$ and $(l\mathrm{I})$.

The definition of K-L normal form for $\mathcal{S}$ is the same. Then the proof above, with some parts omitted, establishes K-L Normalization also for $\mathcal{S}$.

6 Coherence for Sesquicartesian Categories

We shall prove in this section that the functor G from $\mathcal{S}$ to $\mathcal{G}$ is faithful, i.e. we shall show that we have coherence for sesquicartesian categories. These categories are interesting because the category **Set** of sets with functions, where cartesian product is $\times$, disjoint union is $+$, and the empty set is O, is such a category. As a matter of fact, every bicartesian closed category is a sesquicartesian category. A bicartesian closed category is a bicartesian category that is cartesian closed, i.e., every functor $A \times ...$ has a right adjoint $...^{A}$. And in every cartesian closed category with an initial object O we have $(k\mathrm{O})$, because $Hom(\mathrm{O}\times\mathrm{O},\mathrm{O}) \cong Hom(\mathrm{O},\mathrm{O}^{\mathrm{O}})$. In every cartesian category in which $(k\mathrm{O})$ holds we have that $Hom(A,\mathrm{O})$ is a singleton or empty, because for $f,g : A \to \mathrm{O}$ we have $k^1_{\mathrm{O},\mathrm{O}} \circ \langle f,g\rangle = k^2_{\mathrm{O},\mathrm{O}} \circ \langle f,g\rangle$ (cf. [8], Proposition 8.3, p. 67).

The category **Set** shows that sesquicartesian categories are not maximal in the following sense. In the category $\mathcal{S}$ the equations $l_{\mathrm{O}\times A} \circ k^1_{\mathrm{O},A} = \mathbf{1}_{\mathrm{O}\times A}$ and $l_{A\times\mathrm{O}} \circ k^2_{A,\mathrm{O}} = \mathbf{1}_{A\times\mathrm{O}}$ (in which only terms of $\mathcal{S}$ occur) don't hold, but they hold in **Set**, and **Set** is not a preorder. That these two equations don't hold in $\mathcal{S}$ follows from the fact that G is a functor from $\mathcal{S}$ to $\mathcal{G}$, but $G(l_{\mathrm{O}\times p} \circ k^1_{\mathrm{O},p})$ and $G(l_{p\times\mathrm{O}} \circ k^2_{p,\mathrm{O}})$ are empty, whereas $G(\mathbf{1}_{\mathrm{O}\times p})$ and $G(\mathbf{1}_{p\times\mathrm{O}})$ contain $(0,0)$. In the case of cartesian categories and categories with binary products and sums, we had coherence, and used this coherence to prove maximality in [1] and [3]. With sesquicartesian categories, however, coherence and maximality don't go hand in hand any more.

First we prove the following lemmata.

LEMMA 6.1. *A constant object of $\mathcal{P}_{\times,+,\mathrm{O}}$ is isomorphic in $\mathcal{S}$ to* O.

Proof. We have in $\mathcal{S}$ the isomorphisms

$$\begin{array}{ll} k^1_{\mathrm{O},\mathrm{O}} = k^2_{\mathrm{O},\mathrm{O}} : \mathrm{O}\times\mathrm{O} \to \mathrm{O}, & l_{\mathrm{O}\times\mathrm{O}} = w_{\mathrm{O}} : \mathrm{O} \to \mathrm{O}\times\mathrm{O}, \\ m_A \circ (\mathbf{1}_A + l_A) : A + \mathrm{O} \to A, & l^1_{A,\mathrm{O}} : A \to A + \mathrm{O}, \\ m_A \circ (l_A + \mathbf{1}_A) : \mathrm{O} + A \to A, & l^2_{\mathrm{O},A} : A \to \mathrm{O} + A. \quad \dashv \end{array}$$

LEMMA 6.2. *If $f,g : A \to B$ are terms of $\mathcal{S}$ and either A or B is isomorphic in $\mathcal{S}$ to* O*, then $f = g$ in $\mathcal{S}$.*

Proof. Suppose $i : A \to \mathrm{O}$ is an isomorphism in $\mathcal{S}$. Then from

$$f \circ i^{-1} = g \circ i^{-1} = l_B$$

we obtain $f = g$.

Suppose $i : B \to \mathrm{O}$ is an isomorphism in $\mathcal{S}$. Then from

$$k^1_{\mathrm{O},\mathrm{O}} \circ \langle i \circ f, i \circ g\rangle = k^2_{\mathrm{O},\mathrm{O}} \circ \langle i \circ f, i \circ g\rangle$$

we obtain $i \circ f = i \circ g$, which yields $f = g$. ⊣

LEMMA 6.3. *If $f, g : A \to B$ are terms of $\mathcal{S}$ and $G(f) = G(g) = \emptyset$ in $\mathcal{G}$, then $f = g$ in $\mathcal{S}$.*

Proof. By K-L Normalization, $f = f_2 \circ f_1$ for $f_1 : A \to C$ a K-term and $f_2 : C \to B$ an L-term. Since for every $z \in |C|$ there is a $y \in |B|$ such that $(z, y) \in G(f_2)$, we must have $G(f_1)$ empty; otherwise, $G(f)$ would not be empty. On the other hand, if there is a propositional letter in C at the z-th place, there must be for some $x \in |A|$ a pair (x, z) in $G(f_1)$. So C is a constant object of $\mathcal{P}_{\times,+,\mathrm{O}}$.

Analogously, $g = g_2 \circ g_1$ for $g_1 : A \to D$ a K-term, and $g_2 : D \to B$ an L-term. As before, D is a constant object of $\mathcal{P}_{\times,+,\mathrm{O}}$. By Lemma 6.1, there is an isomorphism $i : C \to D$ of $\mathcal{S}$, and $f = f_2 \circ i^{-1} \circ i \circ f_1$. By Lemmata 6.1 and 6.2, we obtain $i \circ f_1 = g_1$ and $f_2 \circ i^{-1} = g_2$, from which $f = g$ follows. ⊣

We shall next prove the following coherence proposition.

SESQUICARTESIAN COHERENCE. *If $f, g : A \to B$ are terms of $\mathcal{S}$ and $G(f) = G(g)$ in $\mathcal{G}$, then $f = g$ in $\mathcal{S}$.*

Proof. Lemma 6.3 covers the case when $G(f) = G(g) = \emptyset$. So we assume $G(f) = G(g) \neq \emptyset$, and proceed by induction on the sum of the lengths of A and B. In this induction we need not consider the cases when either A or B is a constant object; otherwise, $G(f)$ and $G(g)$ would be empty. So in the basis of the induction, when both of A and B are atomic, we consider only the case when both of A and B are propositional letters. In this case we conclude by Cut Elimination that f and g exist iff A and B are the same propositional letter p, and $f = g = \mathbf{1}_p$ in $\mathcal{S}$. (We could conclude the same thing by interpreting $\mathcal{S}$ in conjunctive-disjunctive logic.) Note that we didn't need here the assumption $G(f) = G(g)$.

If A is $A_1 + A_2$, then $f \circ l^1_{A_1,A_2}$ and $g \circ l^1_{A_1,A_2}$ are of type $A_1 \to B$, while $f \circ l^2_{A_1,A_2}$ and $g \circ l^2_{A_1,A_2}$ are of type $A_2 \to B$. We also have

$$\begin{array}{rcl} G(f \circ l^i_{A_1,A_2}) & = & G(f) \circ G(l^i_{A_1,A_2}) \\ & = & G(g) \circ G(l^i_{A_1,A_2}) \\ & = & G(g \circ l^i_{A_1,A_2}), \end{array}$$

whence, by the induction hypothesis, or Lemma 6.3,

$$f \circ l^i_{A_1,A_2} = g \circ l^i_{A_1,A_2}$$

in $\mathcal{S}$. Then we infer that

$$[f \circ l^1_{A_1,A_2}, f \circ l^2_{A_1,A_2}] = [g \circ l^1_{A_1,A_2}, g \circ l^2_{A_1,A_2}],$$

from which it follows that $f = g$ in $\mathcal{S}$. We proceed analogously if B is $B_1 \times B_2$.

Suppose now A is $A_1 \times A_2$ or a propositional letter, and B is $B_1 + B_2$ or a propositional letter, but A and B are not both propositional letters. Then, by Cut Elimination, f is equal in $\mathcal{S}$ either to a term of the form $f' \circ k^i_{A_1,A_2}$, or to a term of the form $l^i_{B_1,B_2} \circ f'$. Suppose $f = f' \circ k^1_{A_1,A_2}$. Then for every $(x, y) \in G(f)$ we have $x \in |A_1|$. (We reason analogously when $f = f' \circ k^2_{A_1,A_2}$.)

By Cut Elimination too, g is equal in $\mathcal{S}$ either to a term of the form $g' \circ k^i_{A_1,A_2}$, or to a term of the form $l^i_{B_1,B_2} \circ g'$. In the first case we must have $g = g' \circ k^1_{A_1,A_2}$, because $G(g) = G(f' \circ k^1_{A_1,A_2}) \neq \emptyset$, and then we apply the induction hypothesis to derive $f' = g'$ from $G(f') = G(g')$. Hence $f = g$ in $\mathcal{S}$.

Suppose $g = l^1_{B_1,B_2} \circ g'$. (We reason analogously when $g = l^2_{B_1,B_2} \circ g'$.) Let $f'' : A_1 \to B_1 + B_2''$ be the substitution instance of $f' : A_1 \to B_1 + B_2$ obtained by replacing every occurrence of propositional letter in B_2 by O. There is an isomorphism $i : B_2'' \to \mathrm{O}$ by Lemma 6.1, and f'' exists because in $G(f)$, which is equal to $G(l^1_{B_1,B_2} \circ g')$, there is no pair (x, y) with $y \geq |B_1|$. So we have an arrow $f''' : A_1 \to B_1$, which we define as $[\mathbf{1}_{B_1}, l_{B_1}] \circ (\mathbf{1}_{B_1} + i) \circ f''$. It is easy to verify that $G(l^1_{B_1,B_2} \circ f''') = G(f')$, and that $G(f''' \circ k^1_{A_1,A_2}) = G(g')$. By the induction hypothesis, we obtain $l^1_{B_1,B_2} \circ f''' = f'$ and $f''' \circ k^1_{A_1,A_2} = g'$, from which we derive $f = g$.

We reason analogously when $f = l^i_{B_1,B_2} \circ f'$. ⊣

To verify whether for $f, g : A \to B$ in the language of $\mathcal{S}$ we have $f = g$ in $\mathcal{S}$ it is enough to draw $G(f)$ and $G(g)$, and check whether they are equal, which is clearly a finite task. So we have here an easy decision procedure for the equations of $\mathcal{S}$.

It is clear that we also have coherence for coherent dual sesquicartesian categories, which are categories with arbitrary finite products and nonempty finite sums where (lI) holds.

As a consequence of Cut Elimination, of the functoriality of G from $\mathcal{D}$, and of Sesquicartesian Coherence, we obtain that $\mathcal{S}$ is a full subcategory of $\mathcal{D}$.

7 Restricted Coherence for Dicartesian Categories

First we give a few examples of dicartesian categories. Note that the bicartesian category **Set**, where product, sum and the initial object are taken as usual (see the beginning of the previous section), and a singleton is the terminal object I, is not a dicartesian category. The equation (lI) does not hold in **Set**.

Every bicartesian category in which the terminal and the initial object are isomorphic is a dicartesian category. Such is, for instance, the category **Set*** of pointed sets, i.e. sets with a distinguished element $*$ and $*$-preserving maps, which is isomorphic to the category of sets with partial maps. In **Set*** the objects I and O are both $\{*\}$, the product $\times$ is cartesian product, the sum $A + B$ of the objects A and B is $\{(a, *) : a \in A\} \cup \{(*, b) : b \in B\}$, with $(*, *)$

being the $*$ of the product and of the sum, while

$$l^1_{A,B}(a) = (a,*), \quad l^2_{B,A}(a) = (*,a),$$

$$m_A(a,b) = \begin{cases} a & \text{if } a \neq * \\ b & \text{if } b \neq * \\ * & \text{if } a = b = * \end{cases}$$

$$(f+g)(a,b) = (f(a), g(b)).$$

In **Set*** we have that I+I is isomorphic to I.

A fortiori, every bicartesian category in which all finite products and sums are isomorphic (i.e. every linear category in the sense of [9], p. 279), is a dicartesian category. Such are, for example, the category of commutative monoids with monoid homomorphisms, and its subcategory of vector spaces over a fixed field with linear transformations. We shall next present a concrete dicartesian category in which the terminal and the initial object, as well as finite products and sums in general, are not isomorphic.

As sesquicartesian categories were not maximal, so dicartesian categories are not maximal either. This is shown by the category **Set***+$\emptyset$, which is the category of pointed sets extended with the empty set $\emptyset$ as an additional object. The arrows of **Set***+$\emptyset$ are the $*$-preserving maps plus the empty maps with domain $\emptyset$. Let I be $\{*\}$, let O be $\emptyset$, let $\times$ be cartesian product, and let the sum $A+B$ be defined as in **Set***, with the same clause. (If both A and B are $\emptyset$, then $A+B$ is $\emptyset$.) If $A \times B$ and $A+B$ are not $\emptyset$, then their $*$ is $(*,*)$, as in **Set***. Next, let l_B be $\emptyset : \emptyset \to B$, and if A is not $\emptyset$, let $l^1_{A,B}$, $l^2_{B,A}$ and m_A be defined as in **Set***. If A is $\emptyset$, then $l^1_{A,B}$ is $\emptyset : \emptyset \to \emptyset + B$, $l^2_{B,A}$ is $\emptyset : \emptyset \to B + \emptyset$ and m_A is $\emptyset : \emptyset \to \emptyset$. We also have

$$(f+g)(a,b) = \begin{cases} (f(a), g(b)) & \text{if neither } f \text{ nor } g \text{ is an empty map} \\ (f(a), *) & \text{if } f \text{ is not an empty map and } g \text{ is} \\ (*, g(b)) & \text{if } g \text{ is not an empty map and } f \text{ is.} \end{cases}$$

If f and g are both empty maps, then $f+g$ is the empty map with appropriate codomain.

With other arrows and operations on arrows defined in the obvious way, we can check that **Set***+$\emptyset$ is a dicartesian category. In this category we have that $\emptyset \times A = A \times \emptyset = \emptyset$. In the category $\mathcal{D}$ the equations $l_{\mathrm{O}\times A} \circ k^1_{\mathrm{O},A} = \mathbf{1}_{\mathrm{O}\times A}$ and $l_{A\times\mathrm{O}} \circ k^2_{A,\mathrm{O}} = \mathbf{1}_{A\times\mathrm{O}}$ (in which only terms of $\mathcal{D}$ occur) don't hold, as we explained at the beginning of the previous section, but they hold in **Set***+$\emptyset$, and **Set***+$\emptyset$ is not a preorder.

Note that a dicartesian category $\mathcal{C}$ is cartesian closed only if $\mathcal{C}$ is a preorder. We have $Hom_{\mathcal{C}}(A,B) \cong Hom_{\mathcal{C}}(\mathrm{I}, B^A)$, and for $f,g : \mathrm{I} \to D$ with $(l\mathrm{I})$ we obtain $[f,g] \circ l^1_{\mathrm{I,I}} = [f,g] \circ l^2_{\mathrm{I,I}}$, which gives $f = g$. The category **Set** is cartesian closed, whereas **Set*** and **Set***+$\emptyset$ are not.

We can prove the following statements, which extend Lemmata 6.1-6.3.

LEMMA 7.1. *A constant object of* $\mathcal{P}$ *is isomorphic in* $\mathcal{D}$ *to either* O *or* I.

Proof. In addition to the isomorphisms of the proof of Lemma 6.1, we have in $\mathcal{D}$ the isomorphisms

$$\begin{array}{ll} k_{\rm I} = m_{\rm I} : {\rm I}+{\rm I} \rightarrow {\rm I}, & l^1_{\rm I,I} = l^2_{\rm I,I} : {\rm I} \rightarrow {\rm I}+{\rm I}, \\ k^1_{A,{\rm I}} : A \times {\rm I} \rightarrow A, & (\mathbf{1}_A \times k_A) \circ w_A : A \rightarrow A \times {\rm I}, \\ k^2_{{\rm I},A} : {\rm I} \times A \rightarrow A, & (k_A \times \mathbf{1}_A) \circ w_A : A \rightarrow {\rm I} \times A. \quad \dashv \end{array}$$

LEMMA 7.2. *If $f, g : A \rightarrow B$ are terms of $\mathcal{D}$ and either A or B is isomorphic in $\mathcal{D}$ to* O *or* I, *then $f = g$ in $\mathcal{D}$.*

Proof. We repeat what we had in the proof of Lemma 6.2, and reason dually when A or B is isomorphic to I. ⊣

RESTRICTED DICARTESIAN COHERENCE I. *If $f, g : A \rightarrow B$ are terms of $\mathcal{D}$ and $G(f) = G(g) = \emptyset$ in $\mathcal{G}$, then $f = g$ in $\mathcal{D}$.*

Proof. As in the proof of Lemma 6.3, by *K*-*L* Normalization, we have $f = f_2 \circ f_1$ for $f_1 : A \rightarrow C$ a K-term, $f_2 : C \rightarrow B$ an L-term, and C a constant object of $\mathcal{P}$; we also have $g = g_2 \circ g_1$ for $g_1 : A \rightarrow D$ a K-term, $g_2 : D \rightarrow B$ an L-term, and D a constant object of $\mathcal{P}$. Next we apply Lemma 7.1. If C and D are both isomorphic to O, we reason as in the proof of Lemma 6.3, and we reason analogously when they are both isomorphic to I. If $i : C \rightarrow {\rm O}$ and $j : {\rm I} \rightarrow D$ are isomorphisms of $\mathcal{D}$, then we have

$$\begin{array}{rcl} f_2 \circ f_1 & = & g_2 \circ j \circ k_{\rm O} \circ i \circ f_1, \text{ by Lemma 7.2,} \\ & = & g_2 \circ g_1, \text{ by Lemma 7.2,} \end{array}$$

and so $f = g$ in $\mathcal{D}$. (Note that $k_{\rm O} = l_{\rm I}$ in $\mathcal{D}$.) ⊣

Besides Restricted Dicartesian Coherence I, we can prove another partial coherence result for $\mathcal{D}$. For that result we need the following lemma, and the definitions that follow.

LEMMA 7.3 *If $u : A \rightarrow B_1 + B_2$ is such that $G(u) \neq \emptyset$ and there is no (x, y) in $G(u)$ such that $y \geq |B_1|$ and $+$ does not occur in A, then there is an arrow $v : A \rightarrow B_1$ such that $G(u) = G(l^1_{B_1,B_2} \circ v)$.*

Proof. By induction on the length of A. Suppose u is a cut-free term. If A is a propositional letter, then by the assumption on $G(u)$ we have $u = L^1_{B_2}u'$ and we can take $v = u'$.

If A is not a propositional letter and u is not of the form $L^1_{B_2}u'$ (by the assumption on $G(u)$, the term u cannot be of the form $L^2_{B_1}u'$), then since $+$ does not occur in A we have that u is of the form $K^i_{A''}u'$ for $u' : A' \rightarrow B_1 + B_2$. Since $G(u') \neq \emptyset$ and there is no (x, y) in $G(u')$ such that $y \geq |B_1|$ and $+$ does not occur in A' we may apply the induction hypothesis to u' and obtain v' such that $G(u') = G(l^1_{B_1,B_2} \circ v')$ and hence we can take $v = v' \circ k^i$. ⊣

A formula C of $\mathcal{P}$ is called a *contradiction* when there is in $\mathcal{D}$ an arrow of the type $C \rightarrow {\rm O}$. For every formula that is not a contradiction there is a substitution instance isomorphic to I. Suppose C is not a contradiction, and let C^{I} be obtained from C by substituting I for every propositional letter. If C^{I} were not isomorphic to I, then by Lemma 7.1 we would have an isomorphism $i : C^{\mathrm{I}} \rightarrow \mathrm{O}$. Since there is obviously an arrow $u : C \rightarrow C^{\mathrm{I}}$ formed by using k_p, we would have $i \circ u : C \rightarrow \mathrm{O}$, and C would be a contradiction.

A formula C of $\mathcal{P}$ is called a *tautology* when there is in $\mathcal{D}$ an arrow of the type $\mathrm{I} \rightarrow C$. For every formula that is not a tautology there is a substitution instance isomorphic to O. (This is shown analogously to what we had in the preceding paragraph.)

A formula of $\mathcal{P}$ is called O-*normal* when for every subformula $D \times C$ or $C \times D$ of it with C a contradiction, there is no occurrence of $+$ in D. A formula of $\mathcal{P}$ is called I-*normal* when for every subformula $D + C$ or $C + D$ of it with C a tautology, there is no occurrence of $\times$ in D.

We can now formulate our second partial coherence result for dicartesian categories.

Restricted Dicartesian Coherence II. *If $f, g : A \rightarrow B$ are terms of $\mathcal{D}$ such that $G(f) = G(g)$ and either A is O-normal or B is I-normal, then $f = g$ in $\mathcal{D}$.*

Proof. Suppose A is O-normal. Restricted Dicartesian Coherence I covers the case when $G(f) = G(g) = \emptyset$. So we assume $G(f) = G(g) \neq \emptyset$, and proceed as in the proof of Sesquicartesian Coherence by induction on the sum of the lengths of A and B. The basis of this induction and the cases when A is of the form $A_1 + A_2$ or B is of the form $B_1 \times B_2$ are settled as in the proof of Sesquicartesian Coherence.

Suppose A is $A_1 \times A_2$ or a propositional letter and B is $B_1 + B_2$ or a propositional letter, but A and B are not both propositional letters. (The cases when A or B is a constant object are excluded by the assumption that $G(f) = G(g) \neq \emptyset$.) We proceed then as in the proof of Sesquicartesian Coherence until we reach the case when $f = f' \circ k^1_{A_1,A_2}$ and $g = l^1_{B_1,B_2} \circ g'$.

Suppose A_2 is not a contradiction. Then there is an instance A_2^{I} of A_2 and an isomorphism $i : \mathrm{I} \rightarrow A_2^{\mathrm{I}}$. (To obtain A_2^{I} we substitute I for every letter in A_2.) Let $g'' : A_1 \times A_2^{\mathrm{I}} \rightarrow B_1$ be the substitution instance of $g' : A_1 \times A_2 \rightarrow B_1$ obtained by replacing every occurrence of propositional letter in A_2 by I. Such a term exists because in $G(g)$, which is equal to $G(f' \circ k^1_{A_1,A_2})$, there is no pair (x, y) with $x \geq |A_1|$.

So we have an arrow $g''' = g'' \circ (\mathbf{1}_{A_1} \times i) \circ \langle \mathbf{1}_{A_1}, k_{A_1} \rangle : A_1 \rightarrow B_1$. It is easy to verify that $G(l^1_{B_1,B_2} \circ g''') = G(f')$ and that $G(g''' \circ k^1_{A_1,A_2}) = G(g')$. By the induction hypothesis we obtain $l^1_{B_1,B_2} \circ g''' = f'$ and $g''' \circ k^1_{A_1,A_2} = g'$, from which we derive $f = g$.

Suppose A_2 is a contradiction. Then by the assumption that A is O-normal we have that $+$ does not occur in A_1. We may apply Lemma 7.3 to $f' : A_1 \rightarrow B_1 + B_2$ to obtain $f''' : A_1 \rightarrow B_1$ such that $G(f') = G(l^1_{B_1,B_2} \circ f''')$. It is easy to verify that then $G(g') = G(f''' \circ k^1_{A_1,A_2})$, and we may proceed as in the proof of Sesquicartesian Coherence.

We proceed analogously when B is I-normal, relying on a lemma dual to Lemma 7.3. $\dashv$

Let $A^0_{\rm O}$ be $A\times{\rm O}$, and let $A^{n+1}_{\rm O}$ be $(A^n_{\rm O}+{\rm I})\times{\rm O}$. Let $f^0_{\rm O}$ be $f\times\mathbf{1}_{\rm O}$, and let $f^{n+1}_{\rm O}$ be $(f^n_{\rm O}+\mathbf{1}_{\rm I})\times\mathbf{1}_{\rm O}$. Let, dually, $A^0_{\rm I}$ be $A+{\rm I}$, and let $A^{n+1}_{\rm I}$ be $(A^n_{\rm I}\times{\rm O})+{\rm I}$. Let $f^0_{\rm I}$ be $f+\mathbf{1}_{\rm I}$, and let $f^{n+1}_{\rm I}$ be $(f^n_{\rm I}\times\mathbf{1}_{\rm O})+\mathbf{1}_{\rm I}$. Then for f^n being

$$(l^1_{A,{\rm I}}\times\mathbf{1}_{\rm O})^n_{\rm I}\circ k^1_{(A\times{\rm O})^n_{\rm I},{\rm O}}:A^{n+1}_{\rm O}\vdash A^{n+1}_{\rm I}$$

and g^n being

$$l^1_{(A+{\rm I})^n_{\rm O},{\rm I}}\circ(k^1_{A,{\rm O}}+\mathbf{1}_{\rm I})^n_{\rm O}:A^{n+1}_{\rm O}\vdash A^{n+1}_{\rm I}$$

we have $G(f^n)=G(g^n)$, but we suppose that $f^n=g^n$ does not hold in $\mathcal{D}$. The equation $f^0=g^0$ is

$$((l^1_{A,{\rm I}}\times\mathbf{1}_{\rm O})+\mathbf{1}_{\rm I})\circ k^1_{(A\times{\rm O})+{\rm I},{\rm O}}=l^1_{(A+{\rm I})\times{\rm O},{\rm I}}\circ(k^1_{A,{\rm O}}+\mathbf{1}_{\rm I})\times\mathbf{1}_{\rm O}:$$
$$((A\times{\rm O})+{\rm I})\times{\rm O}\vdash((A+{\rm I})\times{\rm O})+{\rm I}.$$

Note that $A^{n+1}_{\rm O}$ is not O-normal, and $A^{n+1}_{\rm I}$ is not I-normal.

We don't know whether it is sufficient to add to $\mathcal{D}$ the equations $f^n=g^n$ for every $n\geq 0$ in order to obtain full coherence for the resulting category.

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