Graphs for Juncture

Kosta Došen and Zoran Petrić

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Abstract

An alternative foundation for 2-categories is explored by studying graph-theoretically a partial operation on 2-cells named juncture, which can replace vertical and horizontal composition. Juncture is a generalized vertical composition of 2-cells that need not involve the whole target and the whole source; it may involve them only partly, provided the result is again a 2-cell. Since commuting diagrams of arrows of ordinary categories may be conceived as invertible 2-cells, this study concerns ordinary category theory too. The operation of juncture has a connection with proof theory, where it corresponds to a kind of cut rule on sequents, and it is related also to an operation on which the notion of operad can be based. The main achievement of the work is a detailed description of the specific planarity involved in juncture and graphs of 2-cells, comparable to the usual combinatorial characterizations of planarity in graph theory. This work points out to an alternative foundation for bicategories, i.e. weak 2-categories, and more generally weak n-categories.

Subject Classification (MSC2010)

Graph Theory:

05C10 (Planar graphs), 05C20 (Directed graphs), 05C62 (Graph representations), 05C76 (Graph operations)

Category Theory:

18A10 (Graphs, diagram schemes), 18D05 (2-categories)

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Chapter 1

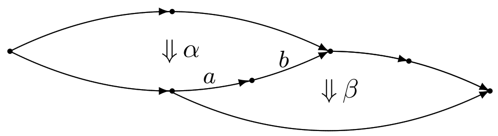
Introduction

§1.1. Aim and scope

Our aim is to explore a matter related to an alternative foundation for 2-categories (see [10] and [11], Section XII.3, for the standard notion of 2-category). We study graph-theoretically a partial operation on 2-cells we call *juncture*, which can replace vertical and horizontal composition. Juncture corresponds to the gluing of diagrams of 2-cells along their borders so that the result is again a diagram of 2-cells. We do not study everything needed for a theory of 2-categories, but only these matters related to horizontal and vertical compositions.

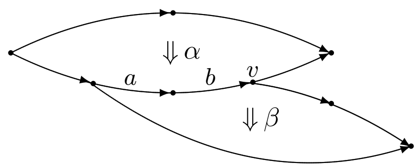
Commuting diagrams of arrows of ordinary categories may be conceived as invertible 2-cells, and the gluing of such commuting diagrams along their borders so as to make other such commuting diagrams is what juncture is about. So our study of juncture concerns ordinary category theory too. It is a contribution to the theory of diagrams of ordinary categories (see the end of §7.5 for some further comments concerning that matter). The operation of juncture, for which we will try to show that it is worth studying from the point of view of graph theory, has also a connection with proof theory, where it corresponds to a kind of cut rule on sequents (see the last paragraph of this section).

Juncture, which is definable in terms of vertical composition, horizontal composition and identity 2-cells, permits us to define vertical composition, but with its help we can define also horizontal composition only in the presence of identity 2-cells (see Chapter 6). Juncture is a generalized vertical composition of 2-cells, where the target of the first 2-cell may coincide only partly with the source of the second 2-cell, provided the result is again a 2-cell, as, for example, in



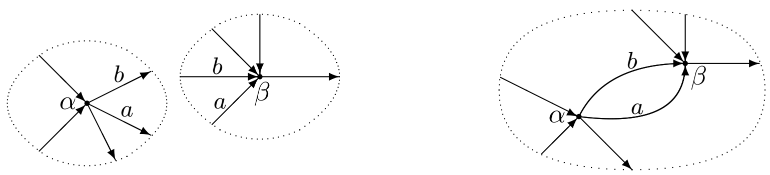
The associativity of vertical and horizontal composition and the *intermuting* of these two operations (see the equation $(\otimes\circ)$ in §6.3) are now replaced by two kinds of associativity of juncture (see the equations of $S\Box$ in §1.5), for which we will prove completeness (see §1.6 and Chapter 6).

An essential ingredient of juncture is that its correct application, where the result is a 2-cell, is based on conditions that are respected in the example above, but are violated in, for example,

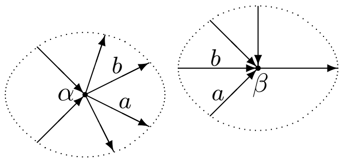


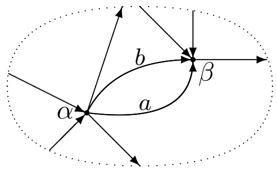
In this last picture the bifurcation at the point v makes it impossible to say what 1-cells are sources and targets of the result, and so the result is not a 2-cell.

We will formulate these conditions by treating diagrams of 2-cells as planar graphs of a specific sort, and then by considering dual graphs of these graphs. By this, and by further modifying the dual graphs (see §7.6 for details), the juncture in the first example above becomes an operation, which we call juncture too, that transforms the two graphs on the left into the graph on the right



while in the second example, which violates conditions for correctness, we have





For the graph on the right in this last picture, the dotted circle surrounding it cannot be divided into two semicircles, one with outgoing arrows and the other with incoming arrows. This division of the surrounding circle is what the exclusion of the bifurcation mentioned above corresponds to.

The modified dual graphs we have just introduced are analogous to the *string diagrams* of [9] (Chapter 1), [14] (Sections 4-5) and [15] (Section 4), while graphs that correspond to the diagrams of 2-cells like that in the first picture are the *pasting schemes* of [13] (see §7.3 for the definition of this notion; see also §6.6 and §6.7). In the definition of string diagram, as in the definition of pasting scheme, planarity is assumed.

Juncture for our modified dual graphs is applicable to a wider class of graphs than these modified dual graphs; we call the graphs in this wider class D1-graphs (see §6.5). We do not assume for D1-graphs the special kind of planarity, which consists in these graphs being realizable within a disk as in all the pictures above except the last (where the dotted circle could not be divided in an appropriate manner into two semicircles). Let us call this special planarity *disk planarity*.

Planarity need not be taken as a difficult notion from a geometrical point of view, but from a combinatorial, i.e. properly graph-theoretical, point of view, it is not simple, and our goal is to replace the assumption of disk planarity by purely combinatorial assumptions. In other words, our goal is to characterize disk planarity, i.e. the disk planar realizability of D1-graphs, in combinatorial terms. This is a goal analogous to that achieved by Kuratowski's and other characterizations of planarity in graph theory (see [8], Chapter 11; as a byproduct of our characterization of planarity in this work, we obtained in [6] another characterization of planarity for ordinary graphs, akin to Kuratowski's). Our reason for dualizing the graphs of 2-cells is this characterization of disk planarity, which otherwise we could not give.

We will find it more practical for our characterization of disk planarity of D1-graphs to concentrate on juncture in the absence of identity 2-cells, which yields the notion of *D-graph*. The D-graphs that have a disk planar realization will be called *P-graphs*. This notion is extended later (in

Chapter 6) to the notion of *P1-graph* (see §6.5), a disk realizable D1-graph, which has what is needed for identity 2-cells, and where what corresponds to horizontal composition is definable.

We define P-graphs in an inductive manner involving juncture with the notion of P′-graph (see §1.8), and non-inductively, again involving juncture, with the notion of P‴-graph (see §1.10). The notion of P‴-graph gives in the most accomplished form our combinatorial characterization of disk planarity for D-graphs, which provides the gist of what we need for D1-graphs. The inductively defined notion of P″-graph is intermediary, and serves as a tool to prove the equivalence of the notions of P′-graph and P‴-graph.

The proof of this equivalence, which will occupy us in most of our work (see Chapters 2-5), is interesting not only because of the final result it yields, but because of the light it sheds on the articulation of the notion of P-graph. We believe that the notions and techniques this proof relies on are of an intrinsic interest too. The length and the difficulties of this proof may come as a surprise, because our three definitions of P-graph are not that different. If however there is no proof much simpler than the one we found, then, judging by the distance our proof covers, these notions are indeed wide apart.

The last chapter of our work (Chapter 7) is about geometrical realizations of P-graphs and P1-graphs. Having both the P′ and P‴ version of the notion of P-graph will help us for that matter.

Another result of our work is a criterion for a graph to be realizable as a graph associated with a diagram of 2-cells, a criterion not based on dualizing (see §7.5). For that we rely on our combinatorial characterization of disk planarity.

Juncture is related to the operation of cut on sequents that one encounters in proof theory. The aspects of juncture as they occur in proof theory were treated in [5], the results of which are related to the definition of planar polycategory—a notion that generalizes the notions of multicategory and operad. As the notions of polycategory and multicategory, the notion of operad may be based on an operation like cut (see [4]). The operation of juncture treated in the present work is more general than all these operations related to cut. Proof-theoretically, it allows for cuts via finite non-empty sequences of formulae, and with the commas on the left and right of the turnstile being of the same nature; i.e., they are both understood conjunctively, or both disjunctively. Moreover, what we have in this work is about sequents where we do not assume the structural rule of permutation; i.e., neither of the commas corresponds to a commutative operation. Gentzen's cut, the plural (multiple-conclusion) form, which is treated in [5], proceeds via sequences that have just a single formula, and the two commas are of different nature. Graph-theoretically, matters are more complicated with this more general cut, i.e. with juncture, than with Gentzen's cut.

Our work points out to an alternative foundation for bicategories, i.e. weak 2-categories, and, further, to an alternative foundation for weak n-categories, a matter much debated these days. We hope that our approach may shed new light on this matter. Our equations for juncture would be replaced by cells of a higher level, cells that are isomorphisms.

Although the motivation for this work comes from category theory, we do not deal much with this theory, and do not presuppose the reader has any extended knowledge of it, except for the sake of motivation. We deal in this work with matters of graph theory, but in that theory we define everything we need, and do not presuppose anything in particular. For the remaining mathematical disciplines touched in our work, like geometry, topology and logic, we presuppose only elementary matters.

§1.2. D-graphs

We introduce first the notion of graph that is common in category theory (see [11], Sections I.2 and II.7). This notion, under the label *graph*, tout court, may be found in [1] (Section 1.1), and, under the labels *directed graph* and *digraph*, in [2] (Section 10.1), [17] (Section 1.4) and [3] (Section 1.10). What we call graphs are not the pseudographs of [8] (Chapter 2), which are not directed. In the style of [8], we could call the graphs of this work *directed pseudographs*.

So a *graph* in this work is given by two functions $W, E\colon A \to V$, where the elements of the set A are called *edges* and those of the set V *vertices* (in category theory, they would be respectively *arrows*, or *morphisms*, and *objects*). The names of the functions W and E come from *West* and *East*, which accords with how we will draw pictures for particular kinds of these graphs, from left to right (in category theory, W and E would be respectively the *source* and *target*, or *domain* and *codomain*, functions). In these pictures, an edge a is represented by an arrow going from the point representing the vertex $W(a)$ to the point representing the vertex $E(a)$. The names we use for W and E (rather than something derived from *left* and *right*, or the categorial terminology) become natural when we deal with geometrical realizations in Chapter 7, where *North* and *South* appear too (see also §6.6; for such reasons we used already an analogous terminology in [5]).

We use X as a variable for W and E. We write $\bar{W}$ and $\bar{E}$ for E and W respectively.

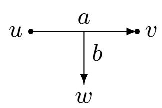
We say sometimes that an edge a of a graph is an edge *from* $W(a)$ *to* $E(a)$, and we say that $W(a)$ and $E(a)$ are *incident* with a.

A *graph morphism* from the graph $W_1, E_1 : A_1 \to V_1$ to the graph $W_2, E_2 \colon A_2 \to V_2$ (which is analogous to a *functor* between categories) is a pair of functions $F_A \colon A_1 \to A_2$ and $F_V : V_1 \to V_2$ such that for every edge a in A_1 and every X in $\{W, E\}$ we have $F_V(X_1(a)) = X_2(F_A(a))$. This means that $F_A(a)$ is an edge from $F_V(W_1(a))$ to $F_V(E_1(a))$.

A graph morphism is an *isomorphism* when both F_A and F_V are bijections.

Let a graph $W, E \colon A \to V$ be *distinguished* when A and V are disjoint. (This condition of disjointness for graphs does not seem to be often mentioned in textbooks of graph theory—exceptions are [2] and [3]—but it is presumably tacitly assumed by many authors.) A non-distinguished graph is given, for example, by

$$\begin{array}{ll} A = \{a, b\}, & V = \{u, v, w, a\}, \\ W(a) = u, & E(a) = v, \\ W(b) = a, & E(b) = w. \end{array}$$



More natural examples of non-distinguished graphs are found in category theory, where sometimes objects, i.e. vertices, are identified with identity arrows on these objects; so all vertices are edges.

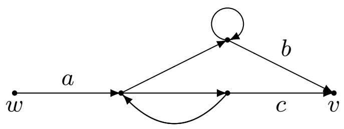
It is trivial to show that every graph is isomorphic to a distinguished graph. Just replace either the set of edges or the set of vertices by a new set in one-to-one correspondence with the original one. Every plane graph (see §7.1) is distinguished, and because of that the picture of the non-distinguished graph we had above as an example is not very natural.

From now on we assume that *graph* means *distinguished graph*, though this assumption is not always essential.

Note that a graph can have V empty, in which case A must be empty too, and for both W and E we have the empty set of ordered pairs. The graph that has V empty is the *empty graph*. A graph that is not the empty graph is said to be *non-empty*.

A graph is *finite* when A and V are finite. In this work we shall be concerned only with finite graphs.

A vertex v of a graph is an X-*vertex* of that graph when there is no edge a of that graph such that $\bar{X}(a) = v$. For example, in the graph of the following picture



the vertex w is a W-vertex, while v is an E-vertex; the other vertices are neither W-vertices nor E-vertices. (In the style of [8], Chapter 16, we could say that W-vertices have indegree 0, while E-vertices have outdegree 0.)

The notion of X-vertex is given with respect to a particular graph, and we mentioned that explicitly in the definition. We shall next define a series of notions that should likewise be understood as given with respect to a particular graph, but we will take this for granted, and will not mention it explicitly.

A vertex is an *inner* vertex when it is neither a W-vertex nor an E-vertex; W-vertices and E-vertices are accordingly called *outer*. All the vertices in our example above except the outer vertices w and v are inner vertices.

An edge a is an X-*edge* when $X(a)$ is an X-vertex. An edge is *inner* when it is neither a W-edge nor an E-edge. Alternatively, a is an inner edge when $W(a)$ and $E(a)$ are inner vertices. In our example, a is a W-edge, while b and c are E-edges; the remaining edges are inner.

An X-edge a is X-*functional* when for every edge b of our graph different from a the vertices $X(a)$ and $X(b)$ are different. In our example, the W-edge a is W-functional, while the E-edges b and c are not E-functional.

A graph is W-E-*functional* when all its W-edges are W-functional and all its E-edges are E-functional.

We give next the definitions of a number of notions analogous to those that may be found in [8], and for which accordingly we use the same terms. The reader should however keep in mind that these are not exactly the same notions, but analogous notions adapted to our context; the *graphs* of [8] are ordinary graphs, and not our graphs.

A *semiwalk* is either a vertex v_0, in which case the semiwalk is *trivial*, or for $n \geq 1$ this is a sequence $v_0 a_1 v_1 \ldots v_{n-1} a_n v_n$ such that for every i in $\{1, \ldots, n\}$

(1) $W(a_i) = v_{i-1}$ and $E(a_i) = v_i$, or
(2) $W(a_i) = v_i$ and $E(a_i) = v_{i-1}$.

A trivial semiwalk v_0 is a semiwalk *from* v_0 *to* v_0, while a non-trivial one is a semiwalk *from* v_0 *to* v_n. (Semiwalks from v_0 to v_n correspond bijectively to semiwalks from v_n to v_0.) We also say that a semiwalk from v_0 to v_n *connects* v_0 *with* v_n. For σ a semiwalk and x a vertex or edge, we write $x \rhd \sigma$ when x occurs in σ.

By omitting (2) from the definition of semiwalk we obtain the definition of *walk*.

A *semipath* is a semiwalk such that

($*$) no vertex in it occurs more than once.

Hence all the edges in a semipath are also mutually distinct. (Examples of semipaths may be found in §1.9.) A *path* is a walk such that ($*$) holds.

A graph is *weakly connected* when for every two distinct vertices v_0 and v_n there is a semipath from v_0 to v_n (which must be non-trivial).

A *semicycle* is a non-trivial semiwalk from v_0 to v_n such that

($**$) no vertex in it occurs more than once, except that v_0 is v_n.

So, in the limit case, $v_0 a v_0$ may be a semicycle based on a non-trivial semiwalk. A *cycle* is a non-trivial walk such that ($**$) holds.

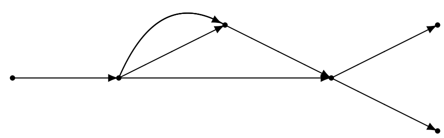
A graph is *acyclic* when it has no cycles.

Now we have all we need to define one of the main kinds of graph with which we deal in this work, and which we call *D-graph*. A D-graph is a graph that is finite, acyclic, W-E-functional, weakly connected and with an inner vertex.

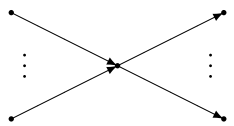
A graph is *incidented* when for each of its vertices v there is an edge a such that $W(a) = v$ or $E(a) = v$; i.e., v is incident with a. It is easy to infer that every D-graph is incidented.

A *loop* is an edge a such that $W(a) = E(a)$. The acyclicity condition excludes loops in D-graphs.

We will draw D-graphs from left to right, and here is a picture of one of them:



A *basic D-graph* is a D-graph with a single inner vertex. Basic D-graphs are all of the form



§1.3. Cocycles and juncture

We say that the graph G_1, which is $W_1, E_1 \colon A_1 \to V_1$, is a *subgraph* of the graph G_2, which is $W_2, E_2 : A_2 \to V_2$, when for Z being one of A, V, W and E we have $Z_1 \subseteq Z_2$. (It is clear that the relation of being a subgraph is a partial order.)

A *component* of a graph G is a weakly connected non-empty subgraph G' of G such that for every weakly connected subgraph G'' of G, if G' is a subgraph of G'', then G' is G''.

Consider a set S of inner edges of a D-graph D. The *removal* of S from D leaves a new graph with the same vertices and with the edges from S missing; the W and E functions of the new graph are obtained by restricting those of D to the new set of edges. The new graph is made of a family of components $D_1, \ldots, D_n$ of this graph, for $n \geq 1$. Note that the graphs in this family are not necessarily D-graphs.

When $n \geq 2$, in which case S must be non-empty, we say that S is a *cutset* (which is a term standing for an analogous notion of [8], Chapter 4).

A *directed graph*, in the sense of [8] (Chapter 2; also called *digraph*) is an irreflexive binary relation on a finite set of vertices; the ordered pairs of the binary relation are the edges. Such a graph may be identified with a finite graph in our sense where there are no multiple edges with the same vertices incident with them, and no loops (see the end of §1.2). Various notions concerning directed graphs, like weak connectedness and acyclicity, which we will rely on in a moment (and other notions we have in §1.6), may either be given definitions analogous to those we gave for graphs (see [8], Chapter 16, for weak connectedness and acyclicity of directed graphs), or having in mind the identification of directed graphs with a special kind of graph, we may apply the definitions we gave for graphs.

Let $C_S(D)$, the *componential graph* of D with respect to S, be the directed graph in the sense of [8] whose vertices are $D_1, \ldots, D_n$, and such that for some distinct i and j in $\{1, \ldots, n\}$ we have that the ordered pair

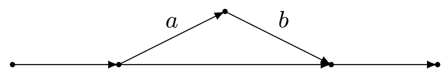
(D_i, D_j) is an edge of $C_S(D)$ iff there is an edge in S from a vertex of D_i to a vertex of D_j, i.e. an edge a in S such that

(‡) $W(a)$ is a vertex of D_i and $E(a)$ is a vertex of D_j.

Less formally, we may say that the edge a connects D_i with D_j.

It is easy to see that since the D-graph D is weakly connected the directed graph $C_S(D)$ is weakly connected.

We call a cutset S of D *strict* when for every a in S there are distinct i and j in $\{1, \ldots, n\}$ such that (‡), and $C_S(D)$ is acyclic. The acyclicity condition for the componential graph precludes $\{a, b\}$ from being a strict cutset in the D-graph of the following picture



A strict cutset where $n = 2$ will be called a *cocycle* (which is a term standing for an analogous notion of [8], Chapter 4).

Cocycles are related to a binary partial operation on D-graphs, which we will call *juncture*, and which now we proceed to define.

For X being W or E, let D_X be $W_X, E_X\colon A_X \to V_X$, and assume that the two graphs D_W and D_E are D-graphs. Assume moreover that

$C =_{df} A_W \cap A_E \neq \emptyset$,
$(\forall a \in C)\ E_W(a) = W_E(a)$,
$(\forall a \in C)\ a$ is an $\bar{X}$-edge of D_X.

Let $V_C = \{v \mid (\exists a \in C)\ v = E_W(a)\} = \{v \mid (\exists a \in C)\ v = W_E(a)\}$, and assume that

$V_W \cap V_E = V_C$.

It can be inferred that every vertex in V_C is an $\bar{X}$-vertex of D_X.

Then we define the D-graph $D_W \Box D_E$, which is $W, E\colon A \to V$, in the following way:

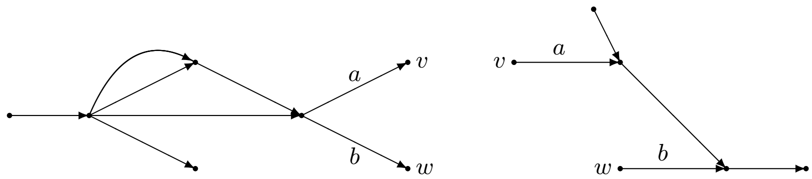
$A = A_W \cup A_E$,
$V = (V_W \cup V_E) - V_C$,

for a in A,

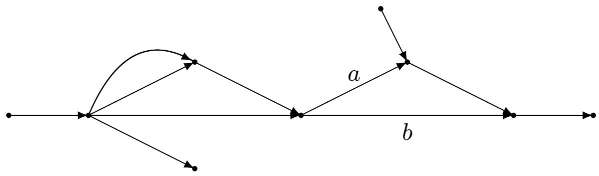
$$X(a) = \begin{cases} X_X(a) & \text{if } a \in A_X, \\ X_{\bar{X}}(a) & \text{if } a \in A_{\bar{X}} - C. \end{cases}$$

This concludes the definition of the operation of juncture □.

For example, consider the D-graphs in the following picture:



The D-graph $D_W \Box D_E$ is in the picture



It is easy to check that $D_W \Box D_E$ is always a D-graph.

Note that in the resulting D-graph $D_W \Box D_E$ the set of edges C is a cocycle. By removing C from $D_W \Box D_E$ we obtain the graphs D_W and D_E with the edges of C removed and the isolated vertices of V_C omitted.

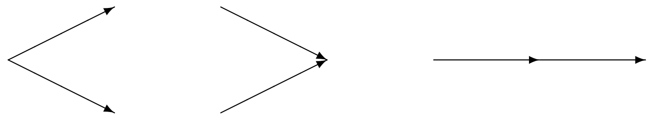
Conversely, if we start from a D-graph D and an arbitrary cocycle C of D, then we can construct two D-graphs D_W and D_E such that D is $D_W \Box D_E$ and C is $A_W \cap A_E$ (see §1.10 for details).

§1.4. Edge-graphs

In this section we consider a notion of graph without vertices, called *edge-graph*, which is equivalent to the notion of incidented graph (see the end of §1.2). Among edge-graphs, those that correspond to D-graphs will enable us to reformulate juncture in a particularly simple way. It will boil down to union.

The notion of edge-graph shows that vertices are in principle dispensable in our exposition, but we keep them because sometimes it is more convenient to rely on them, and also because we do not want to depart too far from established terminology. We will however rely on edge-graphs in §6.5. Matters exposed in this section are not essential for our results later on, and this is why here we will not dwell on the details of the proofs.

The definition of edge-graph does not mention vertices, but instead it mentions the relations on edges of having a common vertex. There are three such binary relations, because the common vertex may be on the west in both edges, or on the east in both edges, or on the east in one edge and on the west in the other, in which case the first edge precedes the second; here are the three relations in pictures:



An *edge-graph* is a set A, whose element are called *edges*, together with three binary relations $\mathbf{W}, \mathbf{E}, \mathbf{P} \subseteq A^2$, such that $\mathbf{W}$ and $\mathbf{E}$ are equivalence relations, and for every a, b and c in A we have

$$\begin{array}{ll} a\mathbf{W}b \Rightarrow (c\mathbf{P}a \Rightarrow c\mathbf{P}b), & a\mathbf{E}b \Rightarrow (a\mathbf{P}c \Rightarrow b\mathbf{P}c), \\ (c\mathbf{P}a \;\&\; c\mathbf{P}b) \Rightarrow a\mathbf{W}b, & (a\mathbf{P}c \;\&\; b\mathbf{P}c) \Rightarrow a\mathbf{E}b. \end{array}$$

For $\mathbf{X}$ being $\mathbf{W}$ or $\mathbf{E}$, we read intuitively $a\mathbf{X}b$ as *a and b have the same $\mathbf{X}$-end*, while $a\mathbf{P}b$ is read as *a precedes b*.

The equivalence of the notion of edge-graph with the notion of incidented graph is a result about equivalence of categories. We define first the category $\mathcal{E}$, where the objects are edge-graphs and the arrows are edge-graph morphisms, which we are now going to define.

An *edge-graph morphism* from the edge-graph $\langle A_1, \mathbf{W}_1, \mathbf{E}_1, \mathbf{P}_1\rangle$ to the edge-graph $\langle A_2, \mathbf{W}_2, \mathbf{E}_2, \mathbf{P}_2\rangle$ is a function $F_A : A_1 \to A_2$ such that for $\mathbf{Z}$ being $\mathbf{W}$, $\mathbf{E}$ or $\mathbf{P}$, if $a\mathbf{Z}_1 b$, then $F_A(a)\mathbf{Z}_2 F_A(b)$. The identity functions on edges serve to define the identity edge-graph morphisms, and composition in $\mathcal{E}$ is given by composition of functions.

The category $\mathcal{I}$, with which $\mathcal{E}$ is equivalent, has as its objects incidented graphs and as arrows graph morphisms (see §1.2). The identity graph morphisms of $\mathcal{I}$ are based on the identity functions on edges and vertices, and composition in $\mathcal{I}$ is based on composition of functions.

For a graph G, which is $W, E\colon A \to V$, let the edge-graph $\mathcal{H}(G)$, which is $\langle A, \mathbf{W}, \mathbf{E}, \mathbf{P}\rangle$, be obtained by stipulating that for every a and b in A we have

$$\begin{array}{rcl} a\mathbf{W}b & \Leftrightarrow & W(a) = W(b), \\ a\mathbf{E}b & \Leftrightarrow & E(a) = E(b), \\ a\mathbf{P}b & \Leftrightarrow & E(a) = W(b). \end{array}$$

It is clear that $\mathbf{W}$ and $\mathbf{E}$ are equivalence relations on A; we also have

$$W(a)=W(b)\Rightarrow(E(c)=W(a)\Rightarrow E(c)=W(b)),$$
$$(E(c)=W(a)\;\&\;E(c)=W(b))\Rightarrow W(a)=W(b),$$

and analogously with $E(a)=E(b)$ instead of $W(a)=W(b)$, so that we may conclude that $\langle A,\mathbf{W},\mathbf{E},\mathbf{P}\rangle$ is an edge-graph.

The empty graph $\emptyset,\emptyset\colon\emptyset\to\emptyset$ (see §1.2) is mapped by $\mathcal{H}$ to the *empty edge-graph* $\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$. A *single-vertex graph*, which is $\emptyset,\emptyset:\emptyset\to\{v\}$, is mapped by $\mathcal{H}$ to the empty edge-graph too. Note that a single-vertex graph is not an incidented graph.

For an edge-graph H, which is $\langle A,\mathbf{W},\mathbf{E},\mathbf{P}\rangle$, we obtain as follows the incidented graph $\mathcal{G}(H)$, which is $W,E\colon A\to V_H$. For a in A let

$$\begin{aligned}
{}_W[a]&=\{b\in A\mid a\mathbf{W}b\}, & [a]_E&=\{b\in A\mid a\mathbf{E}b\},\\
{}_P[a]&=\{b\in A\mid b\mathbf{P}a\}, & [a]_P&=\{b\in A\mid a\mathbf{P}b\},\\
W(a)&=({}_P[a],{}_W[a]), & E(a)&=([a]_E,[a]_P),
\end{aligned}$$
$$V_H=\{(A',A'')\mid(\exists a\in A)(W(a)=(A',A'')\ or\ E(a)=(A',A''))\}.$$

If H is the empty edge-graph, then $\mathcal{G}(H)$ is the empty graph.

It is straightforward to prove the following proposition.

PROPOSITION 1.4.1. *For every incidented graph G, the graph $\mathcal{G}(\mathcal{H}(G))$ is isomorphic to G.*

The graph isomorphism of this proposition is identity on edges, and, for X being W or E, it maps the vertex $X(a)$ of $\mathcal{G}(\mathcal{H}(G))$ to the vertex $X(a)$ of G. (One must verify that this function on vertices is well defined; part of that consists in verifying that if $({}_P[a_1],{}_W[a_1])=({}_P[a_2],{}_W[a_2])$, then $W(a_1)=W(a_2)$ in G.)

The following proposition is also straightforward to prove.

PROPOSITION 1.4.2. *For every edge-graph H, the edge-graph $\mathcal{H}(\mathcal{G}(H))$ is H.*

It is straightforward to extend $\mathcal{H}$ to a functor from the category $\mathcal{I}$ to the category $\mathcal{E}$; we just forget the F_V part of a graph morphism. It is also straightforward to extend $\mathcal{G}$ to a functor from $\mathcal{E}$ to $\mathcal{I}$. (A vertex $X(a)$ of $\mathcal{G}(H_1)$ is mapped by F_V to the vertex $X(F_A(a))$ of $\mathcal{G}(H_2)$.) Starting from Propositions 1.4.1 and 1.4.2, we then obtain that the categories $\mathcal{E}$ and $\mathcal{I}$ are equivalent.

Let a *D-edge-graph* be an edge-graph H such that $\mathcal{G}(H)$ is a D-graph. A *W-edge* of an edge-graph H is an edge a such that there is no edge b of

H with $b\mathbf{P}a$. An E-*edge* is defined analogously with $a\mathbf{P}b$ replacing $b\mathbf{P}a$.

Let H_1, which is $\langle A_1, \mathbf{W}_1, \mathbf{E}_1, \mathbf{P}_1\rangle$, and H_2, which is $\langle A_2, \mathbf{W}_2, \mathbf{E}_2, \mathbf{P}_2\rangle$, be two D-edge-graphs such that C, which is $A_1 \cap A_2$, is non-empty, and for every edge a in C we have that a is an E-edge of H_1 and a W-edge of H_2. Then let $H_1 \cup H_2$ be $\langle A_1 \cup A_2, \mathbf{W}_1 \cup \mathbf{W}_2, \mathbf{E}_1 \cup \mathbf{E}_2, \mathbf{P}_1 \cup \mathbf{P}_2\rangle$. It is straightforward to verify that $H_1 \cup H_2$ is a D-edge-graph.

It is also straightforward to verify that there are graphs G_1 and G_2 isomorphic to $\mathcal{G}(H_1)$ and $\mathcal{G}(H_2)$, respectively, such that $\mathcal{G}(H_1 \cup H_2) = G_1 \Box G_2$. (All that is involved in passing from $\mathcal{G}(H_1)$ and $\mathcal{G}(H_2)$ to G_1 and G_2 is the renaming of vertices incident with edges that will be in the cocycle of the juncture, in order to ensure the sharing of these vertices for G_1 and G_2.) Finally, it is straightforward to verify that $\mathcal{H}(D_W \Box D_E) = \mathcal{H}(D_W) \cup \mathcal{H}(D_E)$.

§1.5. The system S$\Box$

We will introduce now an equational system for juncture, called S$\Box$, which will have various associativity axioms, and which in §1.6 we will show sound and complete with respect to an interpretation in D-graphs.

The equations of S$\Box$ will have on their two sides terms that we will call D-terms. There will be three functions, W, E and A, mapping the set of D-terms to the power set of an arbitrary infinite set so that for a D-term δ neither of $W(\delta)$ and $E(\delta)$ is empty. As before, we write X for W or E. Intuitively, $X(\delta)$ is the set of X-edges of a D-graph for which δ stands, while $A(\delta)$ is the set of all edges of that D-graph. Hence the sets $W(\delta)$ and $E(\delta)$ will be disjoint, and we will have $W(\delta) \cup E(\delta) \subseteq A(\delta)$.

We define *D-terms* inductively by starting from *basic D-terms*, which are atomic symbols. To each such symbol β we assign three sets $W(\beta)$, $E(\beta)$ and $A(\beta)$ such that $W(\beta)$ and $E(\beta)$ are non-empty, finite and disjoint, while $A(\beta) = W(\beta) \cup E(\beta)$.

The inductive clause of our definition of D-term says that if δ_W and δ_E are D-terms such that

$$C =_{df} A(\delta_W) \cap A(\delta_E) = E(\delta_W) \cap W(\delta_E) \neq \emptyset,$$

then $(\delta_W \Box \delta_E)$ is a D-term. As usual, we omit the outermost parentheses of D-terms, and take them for granted.

We define as follows the values of W, E and A for the argument $\delta_W \Box \delta_E$:

$$X(\delta_W \Box \delta_E) = X(\delta_X) \cup (X(\delta_{\bar{X}}) - C),$$

$A(\delta_W \Box \delta_E) \; = A(\delta_W) \cup A(\delta_E).$

This concludes our definition of D-term.

The triple $(W(\delta), E(\delta), A(\delta))$ will be called the *edge type* of the D-term δ. In another notation, we could have written this edge type together with δ in our language.

Note that if $(\delta_1 \Box \delta_2) \Box \delta_3$ is defined—i.e., it is a D-term—then $\delta_2 \Box \delta_3$ may be defined or not, but $\delta_3 \Box \delta_2$ is never defined, because $E(\delta_3) \cap W(\delta_2)$ must be empty. Otherwise, $A(\delta_1 \Box \delta_2) \cap A(\delta_3)$ would not be equal to $E(\delta_1 \Box \delta_2) \cap W(\delta_3)$.

The equations of our system, which we call S$\Box$, will be of the form $\delta = \delta'$ for δ and δ' being D-terms of the same edge type. The rules of S$\Box$ are symmetry and transitivity of $=$, and congruence with $\Box$:

if $\delta_1 = \delta_2$ and $\delta_3 = \delta_4$, then $\delta_1 \Box \delta_3 = \delta_2 \Box \delta_4$,

provided that $\delta_1 \Box \delta_3$ and $\delta_2 \Box \delta_4$ are defined.

The axiomatic equations of S$\Box$ are $\delta = \delta$ and the following equations:

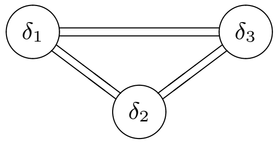
(Ass 1) $\quad (\delta_1 \Box \delta_2) \Box \delta_3 = \delta_1 \Box (\delta_2 \Box \delta_3)$,

(Ass 2.1) $\quad (\delta_1 \Box \delta_2) \Box \delta_3 = (\delta_1 \Box \delta_3) \Box \delta_2$,

(Ass 2.2) $\quad \delta_1 \Box (\delta_2 \Box \delta_3) = \delta_2 \Box (\delta_1 \Box \delta_3)$,

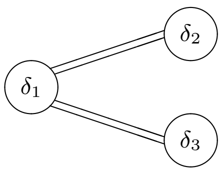
provided that for each of these equations both sides are defined. It is straightforward to verify that in all of these equations the two sides are D-terms of the same edge type.

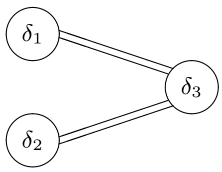
To help intuition, for (Ass 1) we have the picture



with the direct link between δ_1 and δ_3 at the top of the triangle perhaps missing. On the left-hand side of (Ass 1) we have joined first δ_1 and δ_2, and then joined the result with δ_3; on the right-hand side of (Ass 1) we have joined first δ_2 and δ_3, and then joined δ_1 with the result.

With (Ass 2.1) and (Ass 2.2) we have the pictures



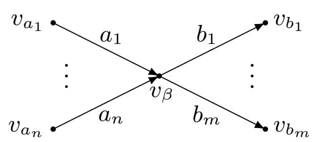


Note that for (Ass 1) we may have $\delta_1 \Box \delta_3$ defined, but this is not necessary. For (Ass 2.1) we must have that neither $\delta_2 \Box \delta_3$ nor $\delta_3 \Box \delta_2$ is defined, and for (Ass 2.2) we must have that neither $\delta_1 \Box \delta_2$ nor $\delta_2 \Box \delta_1$ is defined.

§1.6. The completeness of S$\Box$

We will now interpret the system S$\Box$ in D-graphs, and prove the completeness of S$\Box$ with respect to this interpretation. We introduce an interpretation function ι that assigns to a D-term a D-graph, and is defined inductively as follows.

For a basic D-term β such that $W(\beta)$ is $\{a_1, \ldots, a_n\}$, for $n \geq 1$, and $E(\beta)$ is $\{b_1, \ldots, b_m\}$, for $m \geq 1$, let $\iota(\beta)$ be the basic D-graph of the following picture



We assume that an edge e corresponds bijectively to the vertex v_e, and that this bijection is the same for all basic D-graphs; it will not vary from one basic D-graph to another. We also have a bijection assigning the vertex v_β to β.

To conclude our definition of the function ι we have the inductive clause

$$\iota(\delta_W \Box \delta_E) = \iota(\delta_W) \Box \iota(\delta_E),$$

where $\Box$ on the right-hand side is juncture. As this function, analogous interpretation functions, introduced later, will be homomorphic.

It is straightforward to verify by induction on the number of occurrences of $\Box$ in the D-term δ that $A(\delta)$ is the set of edges of the D-graph $\iota(\delta)$. For $X_e(D)$ being the set of X-edges of the D-graph D, we may also verify, by the same kind of induction, that $X(\delta) = X_e(\iota(\delta))$. (We put the subscript e in X_e because later we use $X(D)$ for the set of X-vertices of D.)

With the help of that it becomes straightforward to verify that if $\delta_W \Box \delta_E$ is defined, i.e., it is a D-term, then $\iota(\delta_W) \Box \iota(\delta_E)$ is defined, i.e., it is a D-graph.

The conditions for juncture $\Box$ in D-graphs and for the operation $\Box$ on D-terms are very similar, but not exactly the same, because D-graphs make a somewhat more general class than the D-graphs that are images of D-terms under ι. In defining the latter, we have introduced, for β a basic D-term, a bijection between the edges and the outer vertices of $\iota(\beta)$. This bijection, together with the bijection between basic D-terms and inner vertices, enables us not to mention vertices when we speak of syntax, i.e. when we speak of D-terms, before introducing their interpretation. This is more economical, but it imposes a restriction on our general notion of D-graph. (A similar restriction would be obtained with an interpretation in edge-graphs; see §1.4.)

We can establish easily the following soundness proposition by induction on the length of derivation in the system S□.

PROPOSITION 1.6.1. *If in* S□ *we can derive* $\delta = \delta'$*, then the D-graphs* $\iota(\delta)$ *and* $\iota(\delta')$ *are the same.*

Our purpose next is to establish also the converse implication, i.e. the completeness of S□ with respect to ι. For that we need a number of preliminary results, and to state them we need to introduce some terminology.

For a D-graph D and a strict cutset S of D, a vertex D_i of the directed graph $C_S(D)$ (see §1.3) is *inner* when, in accordance with our terminology of §1.2, there are two edges of $C_S(D)$ of the form (D_j, D_i) and (D_i, D_k). Otherwise, the vertex is *outer*.

The *removal* of a vertex D_i of $C_S(D)$ leaves a directed graph with the vertices of $C_S(D)$ without D_i and the edges of $C_S(D)$ without the edges in which D_i occurs (cf. [8], Chapter 2).

The vertex D_i is a *cutvertex* of $C_S(D)$ if the removal of D_i increases the number of weakly connected components of $C_S(D)$. (The notion of weak connectedness for directed graphs, which is analogous to our notion of weak connectedness for graphs of §1.2, may be found in [8], Chapter 16, as well as the notion of weakly connected component, called there *weak* component; cf. also with our notion of component of §1.3.)

Let a *componential extreme* of $C_S(D)$ be an outer vertex of $C_S(D)$ that is not a cutvertex. For example, in

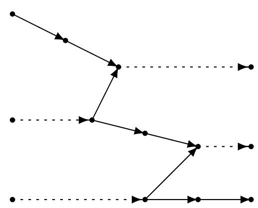
$$\begin{tikzcd} & D_1 \arrow[dr] & \\ & & D_2 \\ D_3 \arrow[r] & D_4 \arrow[ur] & \end{tikzcd}$$

D_1 and D_3 are componential extremes, while D_2, though it is an outer vertex, is not a componential extreme. The notion of componential extreme, and the notions it relies on, are not peculiar to $C_S(D)$. They could be given for arbitrary directed graphs, and the proposition on componential extremes that follows could be established for arbitrary acyclic directed graphs (which are finite by definition). We need it however for $C_S(D)$, and we formulate it accordingly.

PROPOSITION 1.6.2. *If S is a strict cutset of the D-graph D, then there are at least two componential extremes in $C_S(D)$.*

PROOF. As a consequence of the acyclicity and finiteness of $C_S(D)$, there must be at least two outer vertices in $C_S(D)$—at least one W-vertex and at least one E-vertex. We take the outer vertices of $C_S(D)$ to be the vertices of an ordinary graph in the sense of [8] (Chapter 2; the edges of ordinary graphs are unordered pairs of distinct vertices), which we call $G_S(D)$; in $G_S(D)$ we have an edge $\{D_i, D_j\}$ when $i \neq j$ and there is a path from D_i to D_j in $C_S(D)$. The definition of path for directed graphs is analogous to the definition of path we gave for graphs in §1.2 (see [8], Chapter 16). For notions concerning ordinary graphs, like the notions of path and connectedness, we rely on the definitions in [8] (Chapter 2) (but the definitions of path and connectedness for ordinary graphs is analogous to the definitions of semipath and weak connectedness we gave for graphs in §1.2).

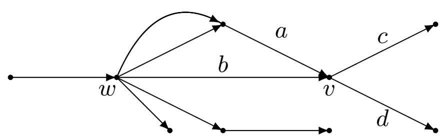
That the ordinary graph $G_S(D)$ is connected is shown as follows. Since $C_S(D)$ is weakly connected, we have for every pair of distinct vertices of $G_S(D)$ a semipath of $C_S(D)$ connecting them. It is easy to pass from this semipath to a path of $G_S(D)$ connecting these two vertices, as in the following picture:



Since $G_S(D)$ is connected and has at least two vertices, there are in $G_S(D)$ two distinct vertices connected by a path of $G_S(D)$. Take two such vertices D_i and D_j at the greatest possible distance from each other (this distance is the length of the shortest path of $G_S(D)$ from D_i to D_j; see [8], Chapter 2.) If D_j is a cutvertex of $G_S(D)$ (this notion of cutvertex is analogous to the notion given above for directed graphs, and may be found, under the name *cutpoint*, in [8], Chapter 3), then its removal leaves a connected ordinary graph G' (a connected subgraph of $G_S(D)$) in which D_i is a vertex, and another connected ordinary graph G'' in which we have a vertex D_k different from D_i and D_j, such that there is a path of $G_S(D)$ connecting D_i with D_k. Since D_j must occur in every such path, the distance between D_i and D_k must be greater than the distance between D_i and D_j, which contradicts our assumption that D_i and D_j are at the greatest possible distance. So D_j is not a cutvertex of $G_S(D)$, and we conclude analogously that D_i is not such (cf. [8], Theorem 3.4, Chapter 3).

To conclude that D_i and D_j are not cutvertices of $C_S(D)$, we have the following. Suppose D_j is a cutvertex of $C_S(D)$. Then the removal of D_j from $C_S(D)$ would leave two weakly connected components H' and H'' of $C_S(D)$ such that in one of them—let that be H'—we have the vertex D_i. Since $C_S(D)$ is acyclic and finite, there must be an outer vertex D_k of $C_S(D)$ in H'', which is different from D_i and D_j. In every semipath of $C_S(D)$ connecting D_i with D_k we find D_j. From that it is easy to conclude that in every path of $G_S(D)$ connecting D_i with D_k we find D_j. Since $G_S(D)$ is connected, we obtain that D_j is a cutvertex of $G_S(D)$, which contradicts what we have established above. We conclude analogously that D_i is not a cutvertex of $C_S(D)$. ⊣

An inner vertex v is an X-*border* vertex when for every edge a such that $\bar{X}(a) = v$ we have that $X(a)$ is an X-vertex. For example, in the D-graph of the following picture:



we have that v is an E-border vertex and w is a W-border vertex.

Let $X(v)$ be the set of all edges such that $\bar{X}(a) = v$. In the example above, $W(v)$ is $\{a, b\}$ and $E(v)$ is $\{c, d\}$.

We say for a non-basic D-graph D that it is n-*valent*, for $n \geq 1$, with respect to an X-border vertex v when for the set S of all the inner edges in $\bar{X}(v)$, which is a strict cutset, we have that $C_S(D)$ has $n{+}1$ vertices. The D-graph in our example above is 1-valent with respect to v, with the strict cutset S having two edges, and it is 2-valent with respect to w, with S now having four edges.

As usual, a subterm of a D-term is a D-term that occurs in it as a part, not necessarily proper. We have the following.

PROPOSITION 1.6.3.1. *Suppose the basic D-term β is a subterm of the D-term δ, and v_β is a W-border vertex of $\iota(\delta)$. In* S$\Box$ *we have an equation of the form*

$$\delta = (\ldots(\beta\Box\sigma_1)\Box\ldots)\Box\sigma_n,$$

for $n \geq 0$, where for distinct i and j in $\{1,\ldots,n\}$ we have that $\sigma_i\Box\sigma_j$ is not defined. (If $n = 0$, then our equation is $\delta = \beta$.)

PROOF. We proceed by induction on the number k of occurrences of $\Box$ in δ. If $k = 0$, then δ is β.

If $k > 0$, then δ is of the form $\delta_1\Box\delta_2$. If β is in δ_2, then v_β is a W-border vertex in $\iota(\delta_2)$, as well as in $\iota(\delta)$, and by the induction hypothesis we have in S$\Box$

$$\delta_2 = (\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_m,$$

for $m \geq 1$. We cannot have $\delta_2 = \beta$; otherwise, v_β would not be a W-border vertex. So we have in S$\Box$ the equation $\delta_2 = \tau\Box\tau_m$, with β in τ, and hence also the equation

$$\delta = \delta_1\Box(\tau\Box\tau_m).$$

If $\delta_1\Box\tau$ is defined, then, by (Ass 1), in S$\Box$ we have

$$\delta = (\delta_1\Box\tau)\Box\tau_m,$$

and if $\delta_1\Box\tau$ is not defined, then, by (Ass 2.2), in S$\Box$ we have

$$\delta = \tau\Box(\delta_1\Box\tau_m).$$

So it is enough to consider the case when δ is of the form $\delta_1\Box\delta_2$ with β in δ_1.

Then, by the induction hypothesis, in S$\Box$ we have

$$\delta_1 = (\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_m,$$

for $m \geq 0$. We will show that in S$\Box$ we have

$$(*) \qquad \delta = ((\ldots(\beta\Box\delta_2^*)\Box\tau_{i_1})\Box\ldots)\Box\tau_{i_l},$$

for some l in $\{0,\ldots,m\}$ and $i_1,\ldots,i_l$ in $\{1,\ldots,m\}$, so that for every j in $\{1,\ldots,l\}$ we have that $\delta_2^*\Box\tau_{i_j}$ is not defined. If $l=0$, then $(*)$ is $\delta=\beta\Box\delta_2^*$.

We prove the equation $(*)$, which suffices for our proposition, by an auxiliary induction on m. If $m=0$, then δ is $\beta\Box\delta_2$, and we are done. Suppose $m>0$. Then in S$\Box$ we have

$$\delta=((\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_m)\Box\delta_2.$$

If $\tau_m\Box\delta_2$ is defined, then, by (Ass 1), in S$\Box$ we have

$$\delta=((\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_{m-1})\Box\delta_2'$$

for δ_2' being $\tau_m\Box\delta_2$. We may then apply the induction hypothesis of the auxiliary induction.

If $\tau_m\Box\delta_2$ is not defined, then, by (Ass 2.1), in S$\Box$ we have

$$\delta=(((\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_{m-1})\Box\delta_2)\Box\tau_m,$$

and we apply the induction hypothesis of the auxiliary induction to

$$((\ldots(\beta\Box\tau_1)\Box\ldots)\Box\tau_{m-1})\Box\delta_2. \qquad \dashv$$

We prove analogously the following dual of Proposition 1.6.3.1.

PROPOSITION 1.6.3.2. *Suppose the basic D-term β is a subterm of the D-term δ, and v_β is an E-border vertex of $\iota(\delta)$. In* S$\Box$ *we have an equation of the form*

$$\delta=\sigma_n\Box(\ldots\Box(\sigma_1\Box\beta)\ldots),$$

for $n\geq 0$, where for distinct i and j in $\{1,\ldots,n\}$ we have that $\sigma_i\Box\sigma_j$ is not defined. (If $n=0$, then our equation is $\delta=\beta$.)

An inner vertex v of a non-basic D-graph D is an X-*extreme* when it is an X-border vertex and D is 1-valent with respect to v (see the example before Proposition 1.6.1, where v is E-extreme). An *extreme* of D is a W-extreme or an E-extreme.

Proposition 1.6.2 implies that there are at least two extremes in every non-basic D-graph. For that take as S in Proposition 1.6.2 the set of all inner edges of D. The unique inner vertex of D in a componential extreme of $C_S(D)$ is an extreme of D. Note that when v_β is an extreme, in Propositions 1.6.3.1 and 1.6.3.2 we have $n=1$.

We can now prove the completeness of S$\Box$ with respect to ι.

THEOREM 1.6.4. *In* S$\Box$ *we can derive* $\delta = \delta'$ *iff the D-graphs* $\iota(\delta)$ *and* $\iota(\delta')$ *are the same.*

PROOF. For the direction from left to right we have Proposition 1.6.1. For the direction from right to left we proceed by induction on the number k of inner vertices in $\iota(\delta)$. If $k = 1$, then δ and δ' are the same basic D-term.

If $k > 1$, then $\iota(\delta)$ is not basic. Take an extreme v of $\iota(\delta)$, and find the basic D-term β that is a subterm of δ and δ' such that v is v_β. Suppose v_β is W-extreme. Then, by Proposition 1.6.3.1, in S$\Box$ we have $\delta = \beta\Box\sigma_1$ and $\delta' = \beta\Box\sigma_1'$. Since, by Proposition 1.6.1, we have that $\iota(\delta)$ is $\iota(\beta)\Box\iota(\sigma_1)$ and $\iota(\delta')$ is $\iota(\beta)\Box\iota(\sigma_1')$, and since $\iota(\delta)$ is $\iota(\delta')$, we must have that $\iota(\sigma_1)$ is $\iota(\sigma_1')$, and, by the induction hypothesis, in S$\Box$ we have $\sigma_1 = \sigma_1'$, and hence also $\delta = \delta'$.

We proceed analogously when v_β is an E-extreme, in which case we apply Proposition 1.6.3.2. $\dashv$

Note that not every D-graph is $\iota(\delta)$ for some D-term δ, but every D-graph is isomorphic to $\iota(\delta)$ for some δ. This may be demonstrated by an easy argument concerning strict cutsets. A strict cutset for a D-graph that is not basic always exists (take, if nothing else, the set of all inner edges, as we did above). An arbitrary strict cutset can easily be reduced to a cocycle. (As a matter of fact, this cocycle may be made to contain an arbitrarily chosen edge of our initial cutset, but we don't need this for our results later on.) Formally, we then make an induction on the number of inner edges of our D-graph.

§1.7. Compatible lists

Let us consider sequences of distinct elements of an arbitrary non-empty set (which later on will be mostly vertices, and sometimes edges), and let such a finite (possibly empty) sequence be called a *list*.

For Γ a list, let Γ^s be the set of members of Γ. We say, as expected, that Γ is a *list of* Γ^s. The lists Γ and Δ are *disjoint* when Γ^s and Δ^s are disjoint, and the list Γ is *empty* when $\Gamma^s = \emptyset$.

Two non-empty lists are said to be *compatible* when they are either of the forms $\Phi\Xi$ and $\Xi\Psi$ or $\Phi\Xi\Psi$ and Ξ, for Φ, Ξ and Ψ mutually disjoint lists, and Ξ a non-empty list. As a particular case, we have that Ξ is compatible with Ξ. (Compatibility is, of course, a symmetric relation.)

An alternative definition of compatibility is given as follows. For Ξ a non-empty list, and Φ_1, Φ_2, Ψ_1, Ψ_2 and Ξ mutually disjoint lists, the lists $\Phi_1\Xi\Psi_1$ and $\Phi_2\Xi\Psi_2$ are *compatible* when at least one Φ_1 and Φ_2, and at least one Ψ_1 and Ψ_2, are empty lists.

The conditions we have in these definitions, and in particular the disjointness conditions, ensure that with compatible lists we have a *unified* list $\Phi\Xi\Psi$ with the first definition, and $\Phi_1\Phi_2\Xi\Psi_1\Psi_2$ with the second definition.

For every non-empty list Γ of the form $x_1 \dots x_n$, with $n \geq 1$, consider the set of ordered pairs defined for $n > 1$ by

$$R_\Gamma = \{(x_i, x_{i+1}) \mid 1 \leq i \leq n{-}1\},$$

while for $n = 1$ we have $R_\Gamma = \emptyset$. If Γ is the empty list, then R_Γ is again $\emptyset$.

For Γ a list, let $\langle \Gamma^s, R_\Gamma\rangle$, i.e. the binary relation R_Γ on Γ^s, be called a *chain*. Lists correspond bijectively to chains. (The one member list x corresponds to the chain $\langle\{x\}, \emptyset\rangle$, and the empty list corresponds to the chain $\langle\emptyset, \emptyset\rangle$.) Chains will serve to give another, quite natural, definition of compatibility.

We say that the chains $\langle \Gamma_1^s, R_{\Gamma_1}\rangle$ and $\langle \Gamma_2^s, R_{\Gamma_2}\rangle$ are *compatible* when there is a list Γ such that

$$\langle \Gamma^s, R_\Gamma\rangle = \langle \Gamma_1^s \cup \Gamma_2^s, R_{\Gamma_1} \cup R_{\Gamma_2}\rangle.$$

We will not go into the rather straightforward proofs of the following propositions, which show that the compatibility of lists and the compatibility of the corresponding chains are in complete agreement. These propositions are not essential for our results later on.

PROPOSITION 1.7.1. *If the lists* Γ_1 *and* Γ_2 *are compatible, then the chains* $\langle \Gamma_1^s, R_{\Gamma_1}\rangle$ *and* $\langle \Gamma_2^s, R_{\Gamma_2}\rangle$ *are compatible.*

PROPOSITION 1.7.2. *If the chains* $\langle \Gamma_1^s, R_{\Gamma_1}\rangle$ *and* $\langle \Gamma_2^s, R_{\Gamma_2}\rangle$ *are compatible, then*

(1) $C =_{df} \Gamma_1^s \cap \Gamma_2^s \neq \emptyset$,

(2) $\langle C, R_{\Gamma_1} \cap C^2\rangle$ *and* $\langle C, R_{\Gamma_2} \cap C^2\rangle$ *are chains*,

(3) $R_{\Gamma_1} \cap C^2 = R_{\Gamma_2} \cap C^2$,

(4) *there are no* $x, y \notin C$ *and a* $z \in C$ *such that* $(xR_{\Gamma_1}z$ *and* $yR_{\Gamma_2}z)$ *or* $(zR_{\Gamma_1}x$ *and* $zR_{\Gamma_2}y)$.

PROPOSITION 1.7.3. *If for the chains* $\langle \Gamma_1^s, R_{\Gamma_1}\rangle$ *and* $\langle \Gamma_2^s, R_{\Gamma_2}\rangle$ *we have* (1), (2), (3) *and* (4) *of Proposition 1.7.2, then the lists* Γ_1 *and* Γ_2 *are compatible.*

§1.8. P′-graphs

Our purpose now is to define a kind of D-graph realizable in a particular manner in the plane (see Chapter 7). First, in this section and in §1.9, we will have two inductive definitions, which will yield the notions of P′-graph and P″-graph. Then in §1.10 we will have a non-inductive definition, which will yield the notion of P‴-graph. All these definitions are based on juncture. In Chapters 2-5 we will show that these three notions cover the same graphs, which we will call *P-graphs*.

A *construction of a* P'*-graph* (for short, *construction*) is a finite binary tree such that in each node we have a triple (D, L_W, L_E) where D is a D-graph and L_X, for X being W or E, is a list of all the X-vertices of D, the set of which is designated by $X(D)$. For the triple (D, L_W, L_E) at the root of a construction K we call D the *root graph* of K, while L_W and L_E are the *root lists* of K. We say that a construction is a construction of its root graph.

Here are the two inductive clauses of our definition of construction:

(1) The single-node tree in whose single node we have a basic D-graph (see the end of §1.2) together with an arbitrary list of all of its W-vertices and an arbitrary list of all of its E-vertices is a construction;

(2) For X being W or E, let K_X be a construction that in its root has (D_X, L^X_W, L^X_E) so that the lists L^W_E and L^E_W are compatible (see §1.7). Out of K_W and K_E we obtain a new construction $K_W \Box K_E$ by adding a new node to serve as its root, whose successors are the roots of K_W and K_E; in the new root we have $(D_W \Box D_E, L_W, L_E)$, where if L^W_E is $\Phi_E \Xi \Psi_E$ and L^E_W is $\Phi_W \Xi \Psi_W$, then L_X is $\Phi_X L^X_X \Psi_X$.

The compatibility of L^W_E and L^E_W in clause (2) implies that at least one of Φ_E and Φ_W, and at least one of Ψ_E and Ψ_W, are empty lists (see §1.7).

A P'*-graph* is the root graph of a construction.

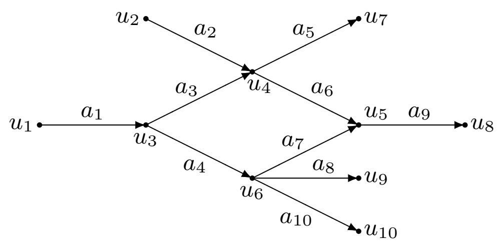
Note that this definition could have relied on lists of the X-edges instead of the X-vertices of a D-graph D, because the X-edges and the X-vertices of D correspond bijectively to each other. For some of our purposes concentrating on the vertices seems better, and more natural, while for other purposes it is easier to concentrate on the edges. On a few occasions (see, for example, the proof of Proposition 2.2.1), it may seem unnecessarily tedious to the reader to pass from one point of view to the other, but we believe that any exposition of our subject matter would have if not this some other kind of shortcoming.

§1.9. P″-graphs

For u and v vertices of a D-Graph D, let $[u, v]$ be the set of all semipaths from u to v. (This set is, of course, in a bijection with $[v, u]$.) Let

$[u]_X = \bigcup\{[u, v] \mid v \in X(D)\}$.

For example, in



we have

$$[u_7, u_9] = \{u_7a_5u_4a_6u_5a_7u_6a_8u_9, u_7a_5u_4a_3u_3a_4u_6a_8u_9\},$$
$$[u_8]_W = \{u_8a_9u_5a_6u_4a_3u_3a_1u_1, u_8a_9u_5a_7u_6a_4u_3a_1u_1, u_8a_9u_5a_6u_4a_2u_2, u_8a_9u_5a_7u_6a_4u_3a_3u_4a_2u_2\}.$$

We say that two semipaths *intersect* when they have a common vertex.

Let u, v and w be distinct X-vertices of a D-graph. We write $\psi_X(v, u, w)$ when every semipath in $[v, w]$ and every semipath in $[u]_{\bar{X}}$ intersect. It is clear that $\psi_X(v, u, w)$ implies $\psi_X(w, u, v)$.

In the example above, we have $\psi_E(u_7, u_8, u_9)$ and not $\psi_E(u_7, u_9, u_8)$, because we have $u_7a_5u_4a_6u_5a_9u_8$ in $[u_7, u_8]$ and $u_9a_8u_6a_4u_3a_1u_1$ in $[u_9]_W$.

For $n \geq 3$, we write $\Gamma: x_1 - x_2 - x_3 - \ldots - x_n$ to assert that in the list Γ the distinct members $x_1, x_2, x_3, \ldots, x_n$ occur either in that order or in the order $x_n, \ldots, x_3, x_2, x_1$, where for i in $\{1, 2, \ldots, n-1\}$ the members x_i and x_{i+1} are not necessarily immediate neighbours. For example, if Γ is 75465983, then we have $\Gamma: 7-6-9-8$ and $\Gamma: 8-9-6-7$.

For D a D-graph, we say that a list Λ of $X(D)$ is *grounded* in D when for every v, u and w in Λ^s if $\Lambda: v-u-w$, then $\psi_X(v, u, w)$.

In our example, we have that $u_7u_8u_9u_{10}$ and $u_7u_8u_{10}u_9$ are grounded, while $u_7u_9u_8u_{10}$ and $u_7u_{10}u_9u_8$ are not grounded.

For two D-graphs D_W and D_E such that $D_W \Box D_E$ is defined, we say that they are *P-compatible* when a list of $E(D_W)$ grounded in D_W and a list of $W(D_E)$ grounded in D_E are compatible.

A *P″-graph* is defined inductively by the following two clauses:

(1) basic D-graphs are P″-graphs;

(2) if D_W and D_E are P-compatible P″-graphs, then $D_W \Box D_E$ is a P″-graph.

The remainder of this section is an appendix, which is not essential for the later exposition, and can hence be skipped. We defined in §1.6 an X-border vertex as an inner vertex x such that for *every* edge a where $\bar{X}(a) = v$ we have that $X(a)$ is an X-vertex. Let an X-*peripheral* vertex be an inner vertex x such that for *some* edge a where $\bar{X}(a) = v$ we have that $X(a)$ is an X-vertex. Every X-border vertex is an X-peripheral vertex, but not necessarily vice versa.

We show in this appendix that our notion of grounding, based on the ternary relation ψ_X on X-vertices, could be replaced by an equivalent notion based on a ternary relation on X-peripheral vertices. The interest of this is that it contributes to showing that in D-graphs inner vertices are essential. Vertices that are not inner play a secondary role.

Let u, v and w be X-peripheral vertices of a D-graph, not necessarily distinct. We write $\psi^b_X(v, u, w)$ when every semipath in $[v, w]$ and every semipath in $[u]_{\bar{X}}$ intersect.

Take an X-vertex x, and consider the edge a such that $X(a) = x$. Then we say that the X-peripheral vertex $\bar{X}(a)$ is the *mate* of x, which we designate by $m(x)$. The function m from X-vertices to X-peripheral vertices is onto, but not one-one. We can prove the following for every D-graph D and every distinct vertices v, u and w in $X(D)$.

PROPOSITION 1.9.1. *We have $\psi_X(v, u, w)$ iff $\psi^b_X(m(v), m(u), m(w))$.*

PROOF. For the proof from left to right, suppose we have a semipath σ in $[m(v), m(w)]$ and a semipath τ in $[m(u)]_{\bar{X}}$. We extend σ to σ^+ in $[v, w]$ just by adding two edges at the ends and the vertices v and w, and we extend τ to τ^+ in $[u]_{\bar{X}}$ just by adding one edge and the vertex u. Since $\psi_X(v, u, w)$, we have that σ^+ and τ^+ intersect, but since v, u and w are distinct vertices, we obtain that σ and τ intersect.

For the proof from right to left it is enough to remark that for every semipath ρ in $[v, w]$ we have that $m(v)$ and $m(w)$ occur in ρ, and for every semipath π in $[u]_{\bar{X}}$ we have that $m(u)$ occurs in π. Let ρ^- in $[m(v), m(w)]$ be obtained from ρ by rejecting the vertices v and w and two edges at the ends incident with v and w respectively, and let π^- in $[m(u)]_{\bar{X}}$ be obtained for π by rejecting u and the edge incident with u.

Since $\psi^b_X(m(v), m(u), m(w))$, we have that ρ^- and π^- intersect, and hence ρ and π intersect. $\dashv$

§1.10. P$'''$-graphs

Consider a cocycle C of a D-graph D. Let the removal (see the beginning of §1.3) of C from D leave the graphs D_1 and D_2 such that for an edge a in C we have $W(a)$ in D_1 and $E(a)$ in D_2. Out of D_1 we build a D-graph D_W by adding to the edges of D_1 all the edges in C, and by stipulating that for every a in C we have $W(a)$ equal to what it was in D , while $E(a)$ is a new vertex v_a, which we add to the vertices of D_1 for every edge a in C. We build the D-graph D_E analogously out of D_2, by adding again the edges of C, and all the new vertices v_a we have added to D_1 to obtain D_W; now we have $E(a)$ as in D, while $W(a)$ is v_a.

We say that D_W and D_E are obtained by *cutting D through C*. It is obvious that D is $D_W \Box D_E$.

A D-graph D is a *P'''-graph* when for every cocycle C of D the D-graphs D_W and D_E obtained by cutting D through C are P-compatible (see §1.9 for P-compatibility).

Chapter 2

P′-Graphs and P‴-Graphs

§2.1. Interlacing and parallelism

In this chapter our goal is to prove that every P′-graph (as defined in §1.8) is a P‴-graph (as defined in §1.10). Before achieving that in §2.3, we deal with preliminary matters. In this section we start with combinatorial matters concerning lists (see §1.7).

When in a list L we have that x and y are immediate neighbours—i.e., x is the immediate predecessor or the immediate successor of y—we say that x and y are *L-neighbours*.

Let A be the list $a_{k+1} \dots a_{k+n}$, for $k \geq 0$ and $n \geq 1$, and let B be the list $b_{l+1} \dots b_{l+m}$, for $l \geq 0$ and $m \geq 1$. (We need k and l because we will have lists where indexing does not start from 1; see A' in the lemmata below.) Assume the sets $\{a_{k+1}, \dots, a_{k+n}\}$ and $\{b_{l+1}, \dots, b_{l+m}\}$ are disjoint, and let M be a list of the union of these two sets.

We have that in A and B respectively a_{k+1} and b_{l+1} are *initial*, while a_{k+n} and b_{l+m} are *final*. For F being A or B, let u^p be the immediate predecessor of u in F, provided this predecessor exists, i.e., u is not initial in F, and let u^s be the immediate successor of u in F, provided this successor exists, i.e., u is not final in F.

We say that two members u and v of M are of the same *parity*, and write $u \equiv_\Pi v$, when their indices are either both even or both odd. For example, we have $a_3 \equiv_\Pi b_7$, as well as $a_3 \equiv_\Pi a_{17}$.

Take two distinct members u and v of a list M, which are either one in A and other in B, or both in A, or both in B, and assume that v^s exists. We say that u is *interlaced* in M with v and v^s, and write $M[v, u, v^s]$, when

$M: v - u - v^s$ and

(1) if $u \equiv_\Pi v^s$ and u^p exists, then not $M: v - u^p - v^s$, and

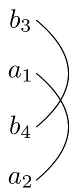
(2) if $u \equiv_\Pi v$ and u^s exists, then not $M: v - u^s - v^s$.

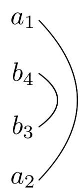
When M is clear from the context, we may omit "in M" from "interlaced in M".

Note that clause (1) is trivially satisfied when u^p does not exist, and (2) is trivially satisfied when u^s does not exist. For example, suppose we have $M: a_1 - b_4 - a_2$; then we have b_4 interlaced with a_1 and a_2 if either $M: b_3 - a_1 - a_2$, or $M: a_1 - a_2 - b_3$, or b_4 is initial in B. We do not have b_4 interlaced with a_1 and a_2 if $M: a_1 - b_3 - a_2$.

To help the intuition, let us draw the list M vertically. On the right of the line of M let us draw lines connecting the successive members u and u^s of A and B where u has an odd index, and on the left let us draw lines connecting u and u^s where u has an even index. These lines make A and B.

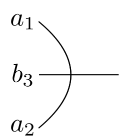
For example, in the first of these two pictures, with intersecting lines on the right, b_4 is interlaced with a_1 and a_2, while in the second it is not, and lines on the right do not intersect:





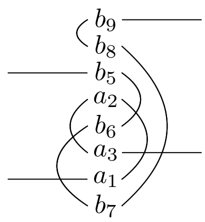
In the first picture, we have also a_1 interlaced with b_3 and b_4.

Nothing changes when we replace b_3 and b_4 by a_3 and a_4 respectively. When b_3 is final we may draw a horizontal line from it to the right to ensure intersection, which indicates interlacing:



For final members with an even index, the horizontal line would go to the left, and for initial members we have dual conventions (see the example below).

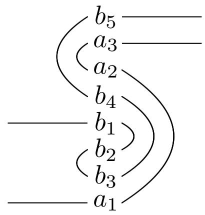
For a more involved example, let A be $a_1a_2a_3$, let B be $b_5b_6b_7b_8b_9$, and let M be $b_9b_8b_5a_2b_6a_3a_1b_7$:



As before, the intersections of the A and B lines indicate interlacing. For example, a_3 is interlaced with b_6 and b_7, as well as with a_1 and a_2, and with b_7 and b_8.

Let $F, G \in \{A, B\}$ (so F may be either different from or equal to G). We say that F and G are *parallel* in M, and write $F \parallel_M G$, when no member of F is interlaced in M with two successive members of G and no member of G is interlaced in M with two successive members of F. (One of the two conjuncts in this definition does not entail the other when initial and final members of A and B are involved; for example, if A is a_1a_2 and B is b_2b_3, with M being $b_3a_1a_2b_2$, we have that both a_1 and a_2 are interlaced with b_2 and b_3, but neither b_2 nor b_3 is interlaced with a_1 and a_2.)

To obtain an example of parallelism, let A be $a_1a_2a_3$, let B be $b_5b_6b_7b_8b_9$, and let M be $b_5a_3a_2b_4b_1b_2b_3a_1$:



It is straightforward to check that we have $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$.

For A having at least three members, assume two A-neighbours a_i and a_{i+1} are also M-neighbours, and let the lists A' and M' be obtained by omitting a_i and a_{i+1} from A and M respectively. We can prove the following.

LEMMA 2.1.1.1. *If $A \parallel_M B$, then $A' \parallel_{M'} B$.*

PROOF. Suppose we have a_{i-1} and a_{i+2} in A, i.e., a_i is not initial and a_{i+1} is not final in A. For w being a_i and a_{i+1}, we must have one of the following:

(I) $\quad M: a_{i-1} - w - a_{i+2}$,

(II) $M: w - a_{i-1} - a_{i+2}$,

(III) $M: a_{i-1} - a_{i+2} - w$.

Suppose not $A' \parallel_{M'} B$. We will infer that not $A \parallel_M B$.

(B in A') We consider first that $A' \parallel_{M'} B$ fails because for some member u of B and some members v and v^s of A' we have that $M'[v, u, v^s]$. If v is different from a_{i-1}, then we obtain easily that $M[v, u, v^s]$. If v is a_{i-1}, then v^s is a_{i+2}, and we have to consider separately the three cases (I)-(III) above.

In case (I) we can infer easily that either $M[a_{i-1}, u, a_i]$ or $M[a_{i+1}, u, a_{i+2}]$. In case (II) we have several subcases to consider.

(II.1) Suppose $u \equiv_\Pi a_{i+2}$ and u^p exists. We may have

$M: u^p - w - a_{i-1} - u - a_{i+2}$, or

$M: w - a_{i-1} - u - a_{i+2} - u^p$, or

$M: w - u^p - a_{i-1} - u - a_{i+2}$.

In the first two cases we conclude that $M[a_{i+1}, u, a_{i+2}]$, while in the third case we obtain that $M[a_{i-1}, u^p, a_i]$.

If $u \equiv_\Pi a_{i+2}$ and u^p does not exist, then $M[a_{i+1}, u, a_{i+2}]$.

(II.2) Suppose $u \equiv_\Pi a_{i-1}$ and u^s exists. Then we have again three cases as in (II.1), obtained by substituting u^s for u^p, and we continue reasoning analogously to what we had in (II.1). In case we have (III), we reason analogously to what we had for (II) above.

(A' in B) We consider now that $A' \parallel_{M'} B$ fails because for some member z of A' and some members u and u^s of B we have that $M'[u, z, u^s]$. Suppose z is a_{i-1}.

If $a_{i-1} \equiv_\Pi u^s$, then we conclude easily that $M[u, a_{i-1}, u^s]$. If $a_{i-1} \equiv_\Pi u$, then either we have that $M[u, a_{i-1}, u^s]$ or, in case that we have $M: u–w–u^s$, we have that $M[u, a_{i+1}, u^s]$.

We reason analogously if we suppose that z is a_{i+2}. This concludes the proof of $A' \parallel_{M'} B$, under the assumption in the first sentence of the proof.

In case in A we have a_{i-1} but not a_{i+2}, or a_{i+2} but not a_{i-1}, we reason by simplifying the reasoning we had above. From (B in A') we keep just the easy case when v is different from a_{i-1}, while from (A' in B) we keep just the case when z is a_{i-1}, or just the case when z is a_{i+2}. Since A has at least three members, one of a_{i-1} and a_{i+2} must exist. $\dashv$

LEMMA 2.1.1.2. *If $A \parallel_M A$, then $A' \parallel_{M'} A'$.*

PROOF. The proof is analogous to the proof of Lemma 2.1.1.1; we make only some obvious adaptations. Note that in (I) we may replace w by $a_i - a_{i+1}$, but not by $a_{i+1} - a_1$, because $A \parallel_M A$; analogously, in (II) and (III) we may replace w by $a_{i+1} - a_i$, but not by $a_i - a_{i+1}$. This does not however influence essentially the exposition of the proof. $\dashv$

The following holds for A' being any subset of A.

LEMMA 2.1.1.3. *If $B \parallel_M B$, then $B \parallel_{M'} B$.*

For the lemmata that follow we assume that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$.

LEMMA 2.1.2. *If A or B has at least two members, then two A-neighbours or two B-neighbours are M-neighbours.*

PROOF. For v and w distinct members of M, let $d_M(v, w)$ be the number of members of M between v and w and let k be the minimal number in the set

$$S = \{d_M(v, w) \mid v \text{ and } w \text{ are } A\text{-neighbours or } B\text{-neighbours}\}.$$

This set is non-empty because either A or B has at least two members.

When $k = 0$, it is clear that the lemma holds. Next we show that the assumption that $k \neq 0$ leads to a contradiction.

Suppose $k > 0$, and suppose v and w are A-neighbours or B-neighbours such that $d_M(v, w) = k$, and suppose $M : v - u - w$ for u a member of F, which is either A or B. If u has no F-neighbours (so it is both initial and final), then it is interlaced with v and w, which contradicts our assumptions about parallelism for A and B. So u has at least one F-neighbour.

If u has an F-neighbour u', which is either v or w, or $M : v - u' - w$, then $d_M(u, u') < d_M(v, w)$, which contradicts the assumption that k is minimal in the set S. So for every F-neighbour u' of u we have that u' is neither v nor w, nor $M : v - u' - w$. Then we have two cases.

One case is that u has two F-neighbours (so it is neither initial nor final in F), in which case we easily obtain that u is interlaced with v and w. The other case is that u has only one F-neighbour; so u is either initial or final in F without being both. Suppose u is initial in F, and w is the immediate successor of v in A or B. If $u \equiv_\Pi w$, then (1) of the definition of interlacing is trivially satisfied, and if $u \equiv_\Pi v$, then (2) of this definition is satisfied. So u is interlaced with v and w. The cases when v is the immediate successor of w, and when u is final in F are treated analogously. In any case, we contradict our assumptions about parallelism for A and B. ⊣

Let us write $x <_L y$ if x precedes y in the list L, not necessarily as an immediate predecessor.

LEMMA 2.1.3.1. *If $n \geq 3$, and a_{k+1} and a_{k+2} are M-neighbours, and $a_{k+1} \equiv_\Pi b_{l+1}$, then*

$$a_{k+1} <_M b_{l+1} \quad \textit{iff} \quad a_{k+3} <_M b_{l+1}.$$

PROOF. It is enough to see that $M\colon a_{k+1}-b_{l+1}-a_{k+3}$ entails that $M[a_{k+2}, b_{l+1}, a_{k+3}]$. ⊣

We prove analogously the following

LEMMA 2.1.3.2. *If $n \geq 3$, and a_{k+n} and a_{k+n-1} are M-neighbours, and $a_{k+n} \equiv_\Pi b_{l+m}$, then*

$$a_{k+n} <_M b_{l+m} \quad \textit{iff} \quad a_{k+n-2} <_M b_{l+m}.$$

LEMMA 2.1.3.3. *If $n \geq 3$, and a_{k+1} and a_{k+2} are M-neighbours, and $a_{k+2} \equiv_\Pi b_{l+m}$, then*

$$a_{k+1} <_M b_{l+m} \quad \textit{iff} \quad a_{k+3} <_M b_{l+m}.$$

PROOF. It is enough to see that $M\colon a_{k+1}-b_{l+m}-a_{k+3}$ entails that $M[a_{k+2}, b_{l+m}, a_{k+3}]$. ⊣

We prove analogously the following.

LEMMA 2.1.3.4. *If $n \geq 3$, and a_{k+n} and a_{k+n-1} are M-neighbours, and $a_{k+n-1} \equiv_\Pi b_{l+1}$, then*

$$a_{k+n} <_M b_{l+1} \quad \textit{iff} \quad a_{k+n-2} <_M b_{l+1}.$$

LEMMA 2.1.4. *If $a_{k+1} \equiv_\Pi b_{l+1}$ and n and m are both odd, then*

$$a_{k+1} <_M b_{l+1} \quad \textit{iff} \quad a_{k+n} <_M b_{l+m}.$$

PROOF. We make an induction on $n+m$. The basis, when $n = m = 1$, is trivial. In the induction step, apply Lemma 2.1.2, and suppose the members a_i and a_{i+1} of A are M-neighbours. By our assumptions on parallelism and

Lemmata 2.1.1.1, 2.1.1.2 and 2.1.1.3, we have $A' \parallel_{M'} B$, $A' \parallel_{M'} A'$ and $B \parallel_{M'} B$; so we will be able to apply the induction hypothesis. If $i \neq k+1$ and $i+1 \neq k+n$, then we are done. If $i = k+1$ or $i+1 = k+n$, then we use Lemmata 2.1.3.1 and 2.1.3.2 respectively. $\dashv$

LEMMA 2.1.5. *If $a_{k+1} \equiv_\Pi b_{l+1}$ and n is even and m odd, then*

$$a_{k+1} <_M b_{l+1} \quad \textit{iff} \quad a_{k+n} <_M b_{l+1}.$$

PROOF. We make an induction on $n+m$. In the basis we have $n = 2$ and $m = 1$. If our equivalence did not hold, then we would have that $M[a_{k+1}, b_{l+1}, a_{k+2}]$. In the induction step we proceed as in the induction step of the proof of Lemma 2.1.4, by using Lemmata 2.1.3.1 and 2.1.3.4 when we pass to A', and by using appropriately renamed variants of Lemmata 2.1.3.1 and 2.1.3.3 when we pass to B'. $\dashv$

LEMMA 2.1.6. *If $a_{k+1} \equiv_\Pi b_{l+1}$ and n and m are both even, then*

$$M : a_{k+1} - b_{l+1} - a_{k+n} \quad \textit{iff} \quad M : a_{k+1} - b_{l+m} - a_{k+n}.$$

PROOF. Note first that the equivalence of this lemma can be stated equivalently as follows:

$$(*) \qquad M : b_{l+1} - a_{k+1} - b_{l+m} \quad \textit{iff} \quad M : b_{l+1} - a_{k+n} - b_{l+m}.$$

To prove that, we proceed by induction on $n+m$. In the basis, we have $n = m = 2$, since n and m are both even, but are different from 0. If our equivalence did not hold, then we would contradict our assumptions on parallelism. For the induction step, apply first Lemma 2.1.2, and suppose the members a_i and a_{i+1} of A are M-neighbours. If $n = 2$, then $(*)$ holds trivially.

If $n \geq 3$, and $k+1 < i$ and $i+1 < k+n$, then we just apply Lemmata 2.1.1.1, 2.1.1.2 and 2.1.1.3 and the induction hypothesis to obtain $(*)$. Suppose $n \geq 3$ and $k+1 = i$, and suppose $b_{l+1} <_M a_{k+1}$ and $a_{k+1} <_M b_{l+m}$. Then, since for u being a_{k+1} and a_{k+3} we have $b_{l+1} <_M u$ iff not $u <_M b_{l+1}$, by Lemmata 2.1.3.1 and 2.1.3.3, we infer $b_{l+1} <_M a_{k+3}$ and $a_{k+3} <_M b_{l+m}$. We can also make the converse inference, and conclude that

$$(b_{l+1} <_M a_{k+1} \text{ and } a_{k+1} <_M b_{l+m}) \text{ iff } (b_{l+1} <_M a_{k+3} \text{ and } a_{k+3} <_M b_{l+m}).$$

Since we can prove analogously the equivalence obtained from this one by interchanging b_{l+1} and b_{l+m}, we obtain

$$(**)\qquad M: b_{l+1} - a_{k+1} - b_{l+m} \quad \textit{iff} \quad M: b_{l+1} - a_{k+3} - b_{l+m}.$$

Since for $3 \leq j \leq n$ we have

$$(***)\qquad M: b_{l+1} - a_{k+j} - b_{l+m} \quad \textit{iff} \quad M': b_{l+1} - a_{k+j} - b_{l+m},$$

and since by Lemmata 2.1.1.1, 2.1.1.2 and 2.1.1.3 and the induction hypothesis we have

$$(****)\qquad M': b_{l+1} - a_{k+3} - b_{l+m} \quad \textit{iff} \quad M': b_{l+1} - a_{k+n} - b_{l+m},$$

we derive $(*)$ as follows:

$$\begin{array}{lll} M: b_{l+1} - a_{k+1} - b_{l+m} & \text{iff} \;\; M: b_{l+1} - a_{k+3} - b_{l+m}, & \text{by } (**), \\ & \text{iff} \;\; M': b_{l+1} - a_{k+3} - b_{l+m}, & \text{by } (***), \\ & \text{iff} \;\; M': b_{l+1} - a_{k+n} - b_{l+m}, & \text{by } (****), \\ & \text{iff} \;\; M: b_{l+1} - a_{k+n} - b_{l+m}, & \text{by } (***). \end{array}$$

We reason analogously when $n \geq 3$ and $i + 1 = k + n$. ⊣

§2.2. P′-graphs and grounding

Suppose we have a construction K of the P′-graph D, and suppose that in the root of K we have the triple (D, L_W, L_E). Our purpose now is to prove that L_W and L_E are grounded in D. This is (2) of the proposition we are going to prove. That proposition asserts also something else concerning related matters, which are involved in the proof of (2).

PROPOSITION 2.2.1. (1) *If $x <_{L_W} y$ and $z <_{L_E} u$, then every semipath in $[x, u]$ and every semipath in $[y, z]$ intersect.*

(2) *If $L_X : x - y - z$, then every semipath in $[x, z]$ and every semipath in $[y]_{\bar{X}}$ intersect (i.e., we have $\psi_X(x, y, z)$).*

(3) *If $L_X : x - y - u - z$, then every semipath in $[x, u]$ and every semipath in $[y, z]$ intersect.*

PROOF. We proceed by induction on the number k of inner vertices of D. If $k = 1$, then D is a basic D-graph, and it is easy to convince oneself that the unique inner vertex of D serves for all the intersections we need in (1), (2) and (3).

Suppose now that $k > 1$. So we have $D = D_W \square D_E$. We prove first (1).

(1) Suppose $x <_{L_W} y$ and $z <_{L_E} u$. We have the following cases depending on where x, y, z and u are.

(1.1) Suppose $x, y \in W(D_W)$ and $z, u \in E(D_E)$. Out of a semipath σ in $[x, u]$ we construct the list A, which is $a_1 \dots a_n$, with $n \geq 1$ and odd, made of all the edges of σ that are elements of the set of edges C involved in $D_W \Box D_E$; the edges of this list are listed in the order in which they appear in σ. In the list A we could alternatively take that a_i for $i \in \{1, \dots, n\}$ instead of being an edge is a vertex v_{a_i} such that in D_E we have $W_E(a_i) = v_{a_i}$ and in D_W we have $E_W(a_i) = v_{a_i}$. Our lists A of edges and such lists of vertices correspond bijectively to each other. (The disadvantage of the list of vertices A is however that the vertex v_{a_i} is not in $D_W \Box D_E$, while the edge a_i is.)

We construct the list B, which is $b_1 \dots b_m$, with $m \geq 1$ and odd, out of a semipath τ in $[y, z]$ in an analogous manner. Since the lists L_E^W and L_W^E are compatible, there is a unified list L of these two lists (see §1.7). We construct a list M of the members of A and B where these members are listed in the order in which they occur in L.

The following part of our proof will be repeated in several analogous variants later on, and this is why we mark it with (†).

(†) Suppose for $F, G \in \{A, B\}$ we do not have $F \parallel_M G$. Suppose a member a of F is interlaced with two members b and b^s of G. If $a \equiv_\Pi b^s$ and a^p exists, then we have either $(L_X^{\bar{X}})^e : a^p$–b–a–b^s or $(L_X^{\bar{X}})^e : b$–a–b^s–a^p, where $(L_X^{\bar{X}})^e$ is the list of edges corresponding to the list of vertices $L_X^{\bar{X}}$. If v_e is the vertex of D_X corresponding bijectively to e (see the comment above after the definition of A), then we have $L_X^{\bar{X}} : v_{a^p} - v_b - v_a - v_{b^s}$ or $L_X^{\bar{X}} : v_b - v_a - v_{b^s} - v_{a^p}$.

If b has an odd index, then $\bar{X}$ is E, while if b has an even index, then $\bar{X}$ is W. Then we apply the induction hypothesis (3) to $D_{\bar{X}}$, and obtain an intersection of every semipath in $[v_a, v_{a^p}]$ with every semipath in $[v_b, v_{b^s}]$. We cannot have that F and G are both A or both B, because then σ or τ would not be semipaths (a vertex cannot occur twice in a semipath). So one of F and G is A, while the other is B, and we may conclude that σ and τ intersect, as required.

If a^p does not exist, i.e., a is initial in A, then we apply the induction hypothesis (2) to D_W, and obtain again that σ and τ intersect. If $a \equiv_\Pi b$, then again we have two cases, in one of which we apply the induction hypothesis (3) and in the other the induction hypothesis (2). This concludes the (†) part of the proof.

Suppose now that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$. If we have $a_1 <_M b_1$, then by Lemma 2.1.4 we have $a_n <_M b_m$, and hence $v_{a_n} <_{L^E_W} v_{b_m}$. We may apply the induction hypothesis (1) to D_E, to infer that σ and τ intersect. If we have $b_1 <_M a_1$, then we have $v_{b_1} <_{L^W_E} v_{a_1}$, and we apply the induction hypothesis (1) to D_W.

(1.2) Suppose $x \in W(D_E)$, $y \in W(D_W)$ and $z, u \in E(D_E)$. We construct A, B and M as in (1.1) except that a_1 is not an element of C, but it is the edge e such that $W(e) = x$. Then we reason as in (†) until we reach the supposition that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$. Now we know that we have $v_{a_1} <_{L_E} v_{b_1}$, and we apply again Lemma 2.1.4 and the induction hypothesis (1) to D_E.

(1.3) Suppose $x, y \in W(D_E)$ and $z, u \in E(D_E)$. We construct A, B and M as in (1.2) except that b_1 too is not an element of C, but it is the edge e such that $W(e) = y$. In the remainder of this case we reason as for (1.2).

(1.4) Suppose $x \in W(D_E)$, $y \in W(D_W)$, $z \in E(D_E)$ and $u \in E(D_W)$. We construct A, B and M as in (1.2) except that a_n too is not an element of C, but it is the edge e such that $E(e) = u$. Since we must have $a_1 <_M b_1$ and $b_m <_M a_n$, by Lemma 2.1.4 we can conclude that $A \parallel_M B$ fails, and we obtain that σ intersects τ as in (†).

All the other cases are treated analogously. Note that the case $x \in W(D_E)$, $y \in W(D_W)$, $z \in E(D_W)$ and $u \in E(D_E)$ is impossible, since it makes L^W_E, which is of the form $\Phi_E \Lambda \Psi_E$, and L^E_W, which is of the form $\Phi_W \Lambda \Psi_W$, not compatible; in Φ_E we have x and in Φ_W we have z. There are other such impossible cases, excluded by the compatibility of L^W_E and L^E_W.

(2) Suppose $x <_{L_X} y$ and $y <_{L_X} z$. (It will help the intuition to suppose that X is here W while $\bar{X}$ is E; or the other way round.) We have the following cases depending on where x, y and z are.

(2.1) Suppose $x, y, z \in X(D_X)$. Out of a semipath σ in $[x, z]$ we construct the list A, which is $a_1 \dots a_n$, with $n \geq 0$ and even, in the same manner as in (1.1). (If $n = 0$, then A i empty.) We construct out of a semipath τ in $[y]_{\bar{X}}$ the list B, which is $b_1 \dots b_m$, with $m \geq 1$ and odd, in an analogous manner. We construct M as in (1.1).

If $n = 0$, i.e., the list A is empty, then we apply the induction hypothesis (2) to D_X to obtain that σ and τ intersect. If $n \geq 2$, then we continue as in (†) until the supposition that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$.

If we have $a_1 <_M b_1$, then by Lemma 2.1.5 we have $a_n <_M b_1$, and hence $v_{a_n} <_{L^X_{\bar{X}}} v_{b_1}$. We may then apply the induction hypothesis (1) to D_X (with v_{a_n}, v_{b_1}, y and z standing respectively for x, y, z and u) to infer that σ and τ intersect.

If we have $b_1 <_M a_1$, then we apply immediately the induction hypothesis (1) to D_X (with v_{b_1}, v_{a_1}, x and y standing respectively for x, y, z and u).

(2.2) Suppose $x \in X(D_{\bar{X}})$ and $y, z \in X(D_X)$. We construct A, B and M as in (2.1) except that a_1 is not an element of C, but (as in (1.2)) it is the edge e such that $\bar{X}(e) = x$. We cannot have A empty now. We continue as in (†) until the supposition that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$.

If we have $a_1 <_M b_1$, then we continue reasoning as in (2.1) by relying on Lemma 2.1.5. It is now excluded that $b_1 <_M a_1$.

(2.3) Suppose $x, y \in X(D_{\bar{X}})$ and $Z \in X(D_X)$. We construct A, B and M as in (2.2) except that b_1 is not an element of C, but it is the edge e such that $\bar{X}(e) = y$. It is excluded that A is empty. Since we must have $a_1 <_M b_1$ and $b_1 <_M a_n$, because a_n is in C, by Lemma 2.1.5 we conclude that $A \parallel_M B$ fails, and we obtain that σ intersects τ as in (†).

In cases where $x, y, z \in X(D_{\bar{X}})$, or where $u, z \in X(D_{\bar{X}})$ and $y \in X(D_X)$, we reason analogously to what we had for case (2.3). We reason analogously when $z <_{L_X} y$ and $y <_{L_X} x$.

(3) Suppose $L_X : x{-}y{-}u{-}z$. We have the following cases depending on where x, y, u and z are.

(3.1) Suppose $x, y, u, z \in X(D_X)$. We construct the lists A, B and M as in (1.1) save that for A, which is $a_1 \ldots a_n$, and for B, which is $b_1 \ldots b_m$, we have that $n \geq 0$ and $m \geq 0$, and they are both even.

If $n = m = 0$, i.e., both A and B are empty, then we apply the induction hypothesis (3) to D_X to obtain that σ and τ intersect.

If $n = 0$ and $m > 0$, then we apply the induction hypothesis (2) to D_X (with x, y and u standing respectively for x, y and z), in order to infer that σ and τ intersect. We proceed analogously when $n > 0$ and $m = 0$.

If $n > 0$ and $m > 0$, then we continue as in (†) until the supposition that $A \parallel_M B$, $A \parallel_M A$ and $B \parallel_M B$. Then by applying Lemma 2.1.6 we obtain the possibility to apply the induction hypothesis (1) to D_X, in order to infer that σ and τ intersect.

The cases when

(3.2) $x \in X(D_{\bar{X}})$ and $y, u, z \in X(D_X)$,

(3.3) $x, y \in X(D_{\bar{X}})$ and $u, z \in X(D_X)$,

(3.4) $x, z \in X(D_{\bar{X}})$ and $y, u \in X(D_X)$,

are treated analogously to case (3.1). In (3.2) we may have $n > 0$ and $m =$

0, where we apply the induction hypothesis (2). If $n > 0$ and $m > 0$, which we may have with (3.2), and which must be the case with (3.3) and (3.4), we have (†) and we apply Lemma 2.1.6 and the induction hypothesis (1).

In the cases when

(3.5) $x, y, u \in X(D_{\bar{X}})$ and $z \in X(D_X)$,

(3.6) $x, y, z \in X(D_{\bar{X}})$ and $u \in X(D_X)$,

(3.7) $x, y, z, u \in X(D_{\bar{X}})$,

we conclude by Lemma 2.1.6 that $A \parallel_M B$ fails, and we obtain that σ intersects τ as in (†). For example, in (3.5) we must have $L_X^{\bar{X}} : x{-}y{-}u{-}v_{b_m}$, with x, y and u being respectively v_{a_1}, v_{b_1} and v_{a_n}.

All the other cases are treated analogously to these. ⊣

§2.3. P′-graphs are P‴-graphs

We may enter now into the proof that every P′-graph is a P‴-graph. The essential ingredient of that proof will be (2) of Proposition 2.2.1, together with the following lemmata, for which we assume that K_1, K_2 and K_3 are constructions of the P′-graphs D_1, D_2 and D_3 respectively. In these lemmata "construction" is short for "construction of a P′-graph" (see §1.8). We have first two lemmata corresponding to (Ass 1) (see §1.5).

LEMMA 2.3.1.1. *If $K_1 \Box K_2$ and $(K_1 \Box K_2) \Box K_3$ are constructions, and $D_2 \Box D_3$ is a D-graph, then $K_2 \Box K_3$ and $K_1 \Box (K_2 \Box K_3)$ are constructions.*

PROOF. Let us write $K : L_W \vdash L_E$ to indicate that in its root the construction K has (D, L_W, L_E).

Suppose that $D_1 \Box D_3$ is not defined. Then we have $K_1 : \Gamma \vdash \Delta_1^1 \Theta \Delta_2^1$, $K_2 : \Gamma_1^2 \Theta \Gamma_2^2 \vdash \Delta_1^2 \Xi \Delta_2^2$ and $K_3 : \Gamma_1^3 \Xi \Gamma_2^3 \vdash \Delta$, and hence

$$\dfrac{\dfrac{K_1 : \Gamma \vdash \Delta_1^1 \Theta \Delta_2^1 \quad K_2 : \Gamma_1^2 \Theta \Gamma_2^2 \vdash \Delta_1^2 \Xi \Delta_2^2}{K_1 \Box K_2 : \Gamma_1^2 \Gamma \Gamma_2^2 \vdash \Delta_1^1 \Delta_1^2 \Xi \Delta_2^2 \Delta_2^1} \qquad K_3 : \Gamma_1^3 \Xi \Gamma_2^3 \vdash \Delta}{(K_1 \Box K_2) \Box K_3 : \Gamma_1^3 \Gamma_1^2 \Gamma \Gamma_2^2 \Gamma_2^3 \vdash \Delta_1^1 \Delta_1^2 \Delta \Delta_2^2 \Delta_2^1}$$

with Θ and Ξ non-empty, and with some requirements concerning the emptiness of Δ_1^1, Δ_2^1, etc., so as to ensure compatibility.

If we cannot obtain $K_2 \Box K_3 : \Gamma_1^3 \Gamma_1^2 \Theta \Gamma_2^2 \Gamma_2^3 \vdash \Delta_1^2 \Delta \Delta_2^2$ because Δ_1^2 and Γ_1^3 are both non-empty, then we could not obtain $(K_1 \Box K_2) \Box K_3$, and if we cannot obtain $K_2 \Box K_3$ because Δ_2^2 and Γ_2^2 are both non-empty, then again we could not obtain $(K_1 \Box K_2) \Box K_3$. So $K_2 \Box K_3$ is a construction.

If we cannot obtain $K_1\Box(K_2\Box K_3)\colon \Gamma_1^3\Gamma_1^2\Gamma\Gamma_2^2\Gamma_2^3 \vdash \Delta_1^1\Delta_1^2\Delta\Delta_2^2\Delta_2^1$ because Δ_1^1 and $\Gamma_1^3\Gamma_1^2$ are both non-empty, then Γ_1^3 is non-empty or Γ_1^2 is non-empty. If Δ_1^1 and Γ_1^3 are non-empty, then we could not obtain $(K_1\Box K_2)\Box K_3$, and if Δ_1^1 and Γ_1^2 are non-empty, then we could not obtain $K_1\Box K_2$.

If we cannot obtain $K_1\Box(K_2\Box K_3)$ because Δ_2^1 and $\Gamma_2^2\Gamma_2^3$ are both non-empty, then Γ_2^2 is non-empty or Γ_2^3 is non-empty. If Δ_2^1 and Γ_2^2 are non-empty, then we could not obtain $K_1\Box K_2$, and if Δ_2^1 and Γ_2^3 are non-empty, then we could not obtain $(K_1\Box K_2)\Box K_3$. So $K_1\Box(K_2\Box K_3)$ is a construction.

It remains to consider the case when $D_1\Box D_3$ is defined. Then we may have

$$\dfrac{\dfrac{K_1\colon \Gamma \vdash \Delta_1^1\Phi\Theta\Delta_2^1 \quad K_2\colon \Theta\Gamma_2^2 \vdash \Xi\Delta_2^2}{K_1\Box K_2\colon \Gamma\Gamma_2^2 \vdash \Delta_1^1\Phi\Xi\Delta_2^2\Delta_2^1} \qquad K_3\colon \Gamma_1^3\Phi\Xi\Gamma_2^3 \vdash \Delta}{(K_1\Box K_2)\Box K_3\colon \Gamma_1^3\Gamma\Gamma_2^2\Gamma_2^3 \vdash \Delta_1^1\Delta\Delta_2^2\Delta_2^1}$$

with Θ, Ξ and Φ non-empty, and with some requirements concerning the emptiness of Δ_1^1, Δ_2^1, etc., so as to ensure compatibility. There is a mirror case where $\Phi\Theta$ and $\Phi\Xi$ are replaced by $\Theta\Phi$ and $\Xi\Phi$ (which requires that Γ_2^2 and Δ_2^2 be empty instead of Γ_1^2 and Δ_1^2); we deal with that case analogously.

If we cannot obtain $K_2\Box K_3\colon \Gamma_1^3\Phi\Theta\Gamma_2^2\Gamma_2^3 \vdash \Delta\Delta_2^2$ because Δ_2^2 and Γ_2^3 are both non-empty, then we could not obtain $(K_1\Box K_2)\Box K_3$. So $K_2\Box K_3$ is a construction.

If we cannot obtain $K_1\Box(K_2\Box K_3)\colon \Gamma_1^3\Gamma\Gamma_2^2\Gamma_2^3 \vdash \Delta_1^1\Delta\Delta_2^2\Delta_2^1$ because Δ_1^1 and Γ_1^3 are both non-empty, then we could not obtain $(K_1\Box K_2)\Box K_3$. If we cannot obtain $K_1\Box(K_2\Box K_3)$ because Δ_2^1 and $\Gamma_2^2\Gamma_2^3$ are both non-empty, then we continue reasoning as in the analogous case we had above. $\dashv$

We can prove analogously the following kind of converse of Lemma 2.3.1.1.

LEMMA 2.3.1.2. *If $K_2\Box K_3$ and $K_1\Box(K_2\Box K_3)$ are constructions, and $D_1\Box D_2$ is a D-graph, then $K_1\Box K_2$ and $(K_1\Box K_2)\Box K_3$ are constructions.*

We also have the following two lemmata corresponding to (Ass 2.1) and (Ass 2.2), of which we prove only the first (the proof of the second is analogous).

LEMMA 2.3.2.1. *If $K_1\Box K_2$ and $(K_1\Box K_2)\Box K_3$ are constructions, and $D_2\Box D_3$ is not defined, then $K_1\Box K_3$ and $(K_1\Box K_3)\Box K_2$ are constructions.*

PROOF. We have

$$\cfrac{\cfrac{K_1\colon \Gamma\vdash\Delta^1_1\Theta\Delta^1_2\Xi\Delta^1_3 \quad K_2\colon\Gamma^2_1\Theta\vdash\Delta^2}{K_1\square K_2\colon\Gamma^2_1\Gamma\vdash\Delta^1_1\Delta^2\Delta^1_2\Xi\Delta^1_3} \qquad K_3\colon\Xi\Gamma^3_2\vdash\Delta^3}{(K_1\square K_2)\square K_3\colon\Gamma^2_1\Gamma\Gamma^3_2\vdash\Delta^1_1\Delta^2\Delta^1_2\Delta^3\Delta^1_3}$$

with Θ, Ξ and Δ^2 non-empty, and with some requirements concerning the emptiness of Δ^1_1, Δ^1_2, etc., so as to ensure compatibility.

If we cannot obtain $K_1\square K_3\colon\Gamma\Gamma^3_2\vdash\Delta^1_1\Theta\Delta^1_2\Delta^3\Delta^1_3$ because Δ^1_3 and Γ^3_2 are both non-empty, then we could not obtain $(K_1\square K_2)\square K_3$. So $K_1\square K_3$ is a construction.

If we cannot obtain $(K_1\square K_3)\square K_2 : \Gamma^2_1\Gamma\Gamma^3_2\vdash\Delta^1_1\Delta^2\Delta^1_2\Delta^3\Delta^1_3$ because Δ^1_1 and Γ^2_1 are both non-empty, then we could not obtain $K_1\square K_2$. So $(K_1\square K_3)\square K_2$ is a construction. ⊣

LEMMA 2.3.2.2. *If $K_2\square K_3$ and $K_1\square(K_2\square K_3)$ are constructions, and $D_1\square D_2$ is not defined, then $K_1\square K_3$ and $K_2\square(K_1\square K_3)$ are constructions.*

We can prove the following concerning D-terms (see §1.5) and their interpretations with ι (see §1.6) .

LEMMA 2.3.3. *For every D-graph $\iota(\delta)$ for a D-term δ and for every cocycle C of $\iota(\delta)$ there are two D-terms δ_W and δ_E such that $\iota(\delta_W)$ and $\iota(\delta_E)$ are the D-graphs obtained by cutting $\iota(\delta)$ through C.*

PROOF. Let D_W and D_E be the D-graphs obtained by cutting $\iota(\delta)$ through C. Let δ'_W and δ'_E be D-terms such that $\iota(\delta'_W)$ and $\iota(\delta'_E)$ are isomorphic respectively to D_W and D_E (such D-terms exist, as noted at the end of §1.6). By renaming, if need there is, the edges assigned to δ'_W and δ'_E so that they accord with the edges assigned to δ we pass to δ_W and δ_E. ⊣

Consider a map φ that assigns to a construction K of a P$'$-graph D a D-term $\varphi(K)$ such that $\iota(\varphi(K))$ is isomorphic to D, and which satisfies that $\varphi(K_W\square K_E)$ is $\varphi(K_W)\square\varphi(K_E)$. It is clear that such maps exist. Relying on Lemmata 2.3.1.1, 2.3.1.2, 2.3.2.1 and 2.3.2.2 it is easy to prove the following by induction on the length of derivation.

LEMMA 2.3.4. *If $\delta=\delta'$ is derivable in S$\square$, then there is a construction K of the P'-graph $\iota(\delta)$ such that $\varphi(K)$ is δ iff there is a construction K' of $\iota(\delta')$ (which is equal to $\iota(\delta)$) such that $\varphi(K')$ is δ'.*

We can then prove the following.

LEMMA 2.3.5. *Every P'-graph $\iota(\delta)$ for which there is a construction K such that $\varphi(K)$ is δ is a P'''-graph.*

PROOF. If $\iota(\delta)$ is a basic D-graph, then it has no cocycles, and it is hence trivially a P‴-graph. Assume $\iota(\delta)$ is not basic, and take an arbitrary cocycle C of $\iota(\delta)$. Let $\iota(\delta_W)$ and $\iota(\delta_E)$ be the D-graphs obtained by cutting $\iota(\delta)$ through C, which we have by Lemma 2.3.3. Hence we have

$$\iota(\delta) = \iota(\delta_W)\Box\iota(\delta_E) = \iota(\delta_W\Box\delta_E),$$

and, by the completeness of S$\Box$ (see §1.6), we obtain that $\delta = \delta_W\Box\delta_E$ is derivable in S$\Box$. Since there is a construction K of $\iota(\delta)$ such that $\varphi(K)$ is δ, there is, by Lemma 2.3.4, a construction K' of $\iota(\delta_W\Box\delta_E)$ such that $\varphi(K')$ is $\delta_W\Box\delta_E$.

Since $\varphi(K_W\Box K_E)$ is $\varphi(K_W)\Box\varphi(K_E)$, from $\varphi(K')$ being $\delta_W\Box\delta_E$ we may conclude that δ_W and δ_E are $\varphi(K_W)$ and $\varphi(K_E)$ respectively, and that K' is $K_W\Box K_E$, for K_X a construction of $\iota(\delta_X)$. So we have that the D-graph $\iota(\delta_X)$ is a P′-graph.

Take the list of $E(\iota(\delta_W))$ from the root of K_W and the list of $W(\iota(\delta_E))$ from the root of K_E. By (2) of Proposition 2.2.1 these lists are grounded in $\iota(\delta_W)$ and $\iota(\delta_E)$ respectively. They are compatible because K' is a construction. So $\iota(\delta)$ is a P‴-graph. ⊣

Every P′-graph is isomorphic to a P′-graph $\iota(\delta)$ for which there is a construction K such that $\varphi(K)$ is δ. Just take an arbitrary construction K of our P′-graph, and take δ to be $\varphi(K)$.

It is also clear that every graph isomorphic to a P‴-graph is a P‴-graph. So from Lemma 2.3.5 we may conclude the following.

THEOREM 2.3.6. *Every P'-graph is a P'''-graph.*

Chapter 3

Grounding and Pivots

§3.1. Grounding and juncture

This chapter contains preliminary results that will help us to establish in §4.4 and §5.3 that P″-graphs (as defined in §1.9) are P′-graphs (as defined in §1.8) and that P‴-graphs (as defined in §1.10) are P″-graphs. The main of these results is the Pivot Theorem of §3.4.

In this section we prove results that will be used to establish that P‴-graphs are P″-graphs. These results are about the relationship between groundedness (see §1.9) in $D_W \Box D_E$ and groundedness in D_W and D_E.

Let Γ and Δ be lists such that x, which is either the initial or the final member of Γ, is either not a member of Δ or it is not the only member that Γ and Δ share, and let Γ' be obtained from Γ by removing x from Γ. Then we can easily establish the following.

LEMMA 3.1.1. *If Γ and Δ are compatible, then Γ' and Δ are compatible.*

We establish easily the following for D-graph $D_W \Box D_E$.

LEMMA 3.1.2. *If $x, y, z \in X(D_X)$ and $\psi_X(x, y, z)$ in $D_W \Box D_E$, then $\psi_X(x, y, z)$ in D_X.*

For Γ a list and S a subset of the set of members of Γ, let $\Gamma|_S$ be the list obtained from Γ by keeping the elements of S and omitting the others. We have the following as a consequence of Lemma 3.1.2.

LEMMA 3.1.3. *If Λ is a list of $X(D_W \Box D_E)$ grounded in $D_W \Box D_E$, then $\Lambda|_{X(D_X)}$ is grounded in D_X.*

We can prove also the following for V_C being the set of vertices defined as in §1.3, which is equal to $E(D_W) \cap W(D_E)$.

LEMMA 3.1.4. *If $y \in \bar{X}(D_{\bar{X}})$, while $x, z \in \bar{X}(D_X)$, $u \in V_C$ and $\psi_{\bar{X}}(y, x, z)$ in $D_W \Box D_E$, then $\psi_{\bar{X}}(u, x, z)$ in D_X.*

PROOF. To help the intuition, suppose X is W, and take a semipath σ of D_W in $[u, z]$ and a semipath τ of D_W in $[x]_W$. Since D_E is weakly connected, there is a semipath σ' of D_E in $[y, u]$. Out of σ' and σ we build a semipath σ'' of $D_W \Box D_E$ in $[y, z]$ obtained by omitting the vertex u and one copy of the edge a such that $E_W(a) = u$ in D_W and $W_E(a) = u$ in D_E. Since σ'' intersects τ in a vertex of D_W, we infer that σ intersects τ in the same vertex. We proceed analogously when X is E. $\dashv$

We prove the following in an analogous manner.

LEMMA 3.1.5. *If $y \in \bar{X}(D_{\bar{X}})$, while $x, z \in \bar{X}(D_X)$, $u \in V_C$ and $\psi_{\bar{X}}(x, y, z)$ in $D_W \Box D_E$, then $\psi_{\bar{X}}(x, u, z)$ in D_X.*

The difference with the preceding proof is that we enlarge a semipath τ of D_W in $[u]_W$, instead of a semipath σ of D_W in $[x, z]$, with a semipath τ' of D_E in $[y, u]$ so as to obtain a semipath τ'' of $D_W \Box D_E$ in $[y]_W$.

The following lemmata are established immediately.

LEMMA 3.1.6. *If $x, y, z \in \bar{X}(D_X)$ and $\psi_{\bar{X}}(x, y, z)$ in $D_W \Box D_E$, then $\psi_{\bar{X}}(x, y, z)$ in D_X.*

LEMMA 3.1.7. *If $x, z \in X(D_X)$ and $y \in X(D_{\bar{X}})$, then we do not have $\psi_{\bar{X}}(x, y, z)$ in $D_W \Box D_E$.*

The following is an immediate consequence of Lemma 3.1.7.

LEMMA 3.1.8. *If Λ is a list of $X(D_W \Box D_E)$ grounded in $D_W \Box D_E$, while $x, z \in X(D_X)$ and $y \in X(D_{\bar{X}})$, then we do not have $\Lambda\colon x{-}y{-}z$.*

§3.2. Pivots and their ordering

The preliminary results of this section are about a particular ordering of some vertices of D-graphs that we call pivots, which we are now going to define.

For x and y vertices of a D-graph, we write $y \prec_\exists x$ when $x \neq y$ and y occurs in some σ in $[x]_W$, which using the abbreviated notation introduced in §1.2 may be written $y \rhd \sigma$.

We write $y \prec x$, and say that y is a *pivot* of x, when $y \neq x$ and y occurs in every σ in $[x]_W$. We have analogous notions with W replaced by E, but for the sake of definiteness we concentrate on W.

For a semipath σ and any vertices x and y in σ consider the sequence σ' of vertices and edges of σ that make a semipath from x to y. We call σ' a *subsemipath* of σ from x to y, and write $\sigma_{[x,y]}$ for σ'. (Note that in this definition the order of vertices and edges in σ' may either coincide or be converse to the order of σ.)

For every semipath σ in $[x, y]$ and every semipath τ in $[y, z]$ there is a vertex v common to σ and τ such that the subsemipath $\sigma_{[x,v]}$ and the subsemipath $\tau_{[v,z]}$ have no vertex in common except v. (Formally, this is shown by an induction in which the basis is the case where v is y; in the induction step, where y' is a vertex common to σ and τ that differs from y, we apply the induction hypothesis to $\sigma_{[x,y']}$ and $\tau_{[y',z]}$.) We designate by $\sigma * \tau$ the semipath in $[x, z]$ obtained by concatenating $\sigma_{[x,v]}$ and $\tau_{[v,z]}$, with one of the two occurrences of v deleted.

Note that $*$ is not an operation because $\sigma * \tau$ is not uniquely determined by σ and τ. For example, in the D-graph

$$\begin{tikzcd}
 & & w \arrow[dr, "c"] & & z & & \\
x \arrow[r, "a"] & u \arrow[ur, "b"] \arrow[rr, "d"] \arrow[rrrr, bend right, "h"'] & & v \arrow[ur, "f"] \arrow[rr, "e"] & & y \arrow[r, "g"] & t
\end{tikzcd}$$

if σ is $xaubwcvey$ and τ is $yhudvfz$, then $\sigma * \tau$ can be either $xaubwcvfz$, which is $\sigma_{[x,v]}$ concatenated with $\tau_{[v,z]}$, or $xaudvfz$, which is $\sigma_{[x,u]}$ concatenated with $\tau_{[u,z]}$.

We can prove the following.

LEMMA 3.2.1. *If* $v_1 \prec x$, $v_2 \prec x$ *and* $v_1 \neq v_2$, *then* $v_2 \prec v_1$ *or* $v_1 \prec v_2$.

PROOF. Suppose $v_1 \prec x$, $v_2 \prec x$ and $v_1 \neq v_2$, and suppose there is a σ_1 in $[v_1]_W$ in which v_2 does not occur and a σ_2 in $[v_2]_W$ in which v_1 does not occur. Let $\sigma \in [x]_W$, and suppose v_1 is between x and v_2 in σ. Then $\sigma_{[x,v_1]} * \sigma_1$ is a semipath in $[x]_W$ in which v_2 does not occur, which contradicts $v_2 \prec x$. We reason analogously when v_2 is between x and v_1 in σ. $\dashv$

For $\sigma \in [x]_W$ and an arbitrary vertex y in σ, let $\sigma_{[y]_W}$ be the subsemipath of σ obtained by rejecting from σ everything that occurs in the subsemipath of σ from x to y except y.

LEMMA 3.2.2. *If $v_2 \prec v_1$, then not $v_1 \prec_\exists v_2$.*

PROOF. Suppose $v_2 \prec v_1$, and suppose there is a σ in $[v_2]_W$ in which v_1 occurs. Take the semipath $\sigma_{[v_1]_W}$. Since $v_2 \prec v_1$, we have that $v_2 \rhd \sigma_{[v_1]_W}$, which implies that v_2 occurs twice in σ. This contradicts the assumption that σ is a semipath. $\dashv$

As a consequence of Lemma 3.2.2, and of the nonemptiness of $[x]_W$ for every x, we obtain the following.

LEMMA 3.2.3. *If $v_2 \prec v_1$, then not $v_1 \prec v_2$.*

This means that we could replace "$v_2 \prec v_1$ or $v_1 \prec v_2$" in Lemma 3.2.1 by "$v_2 \prec v_1$ or $v_1 \prec v_2$, but not both". We also have that $\prec$ is transitive.

LEMA 3.2.4. *If $x \prec y$ and $y \prec z$, then $x \prec z$.*

PROOF. Suppose $x \prec y$ and $y \prec z$. By Lemma 3.2.3 we have that $x \neq z$, and then it is clear that $x \prec z$. $\dashv$

Let $y \rhd [x, z]$ mean that y occurs in every semipath σ in $[x, z]$. It is clear that $y \rhd [x, z]$ iff $y \rhd [z, x]$. Then we can prove the following.

LEMMA 3.2.5. *If $v_2 \prec_\exists v_1$ and $v_1 \prec x$, then $v_1 \rhd [x, v_2]$.*

PROOF. Suppose $\sigma \in [x, v_2]$ and not $v_1 \rhd \sigma$. Let $\tau \in [v_1]_W$ and $v_2 \rhd \tau$. There must be such a semipath because $v_2 \prec_\exists v_1$. Then $\sigma * \tau_{[v_2]_W} \in [x]_W$, but not $v_1 \rhd \sigma * \tau_{[v_2]_W}$, which contradicts $v_1 \prec x$. $\dashv$

LEMMA 3.2.6. *If $v_2 \rhd [x, z]$ and not $v_2 \rhd [x, v_1]$, then $v_2 \rhd [v_1, z]$.*

PROOF. Suppose $\sigma \in [v_1, z]$ and not $v_2 \rhd \sigma$. Because not $v_2 \rhd [x, v_1]$, for some τ in $[x, v_1]$ we have that not $v_2 \rhd \tau$. Then $\tau * \sigma \in [x, z]$ and not $v_2 \rhd \tau * \sigma$, contradicting $v_2 \rhd [x, z]$. $\dashv$

LEMMA 3.2.7. *If $v_1 \rhd [z, y]$ and $v_2 \rhd [z, v_1]$, then $v_2 \rhd [z, y]$.*

PROOF. Let $\sigma \in [z, y]$. Since $v_1 \rhd [z, y]$, we have $v_1 \rhd \sigma$. Then since $v_2 \rhd [z, v_1]$, we have $v_2 \rhd \sigma_{[z, v_1]}$, and hence $v_2 \rhd \sigma$. $\dashv$

As a corollary of Lemma 3.2.5, we have the following.

LEMMA 3.2.8.1. *If* $v_1 \prec v_2$ *and* $v_2 \prec x$*, then* $v_2 \rhd [x, v_1]$.

LEMMA 3.2.8.2. *If* $v_1 \prec v_2$ *and* $v_2 \prec x$ *and* $v_1 \rhd [z, x]$*, then* $v_2 \rhd [z, x]$.

PROOF. In Lemma 3.2.7 put x for z and z for y, and then use Lemma 3.2.8.1. ⊣

LEMMA 3.2.8.3. *If* $v_1 \prec v_2$*,* $v_2 \prec x$ *and* $\sigma \in [x, v_2]$*, then not* $v_1 \rhd \sigma$.

PROOF. Suppose $v_1 \prec v_2$, $v_2 \prec x$ and for some σ in $[x, v_2]$ we have $v_1 \rhd \sigma$. From $v_1 \prec v_2$ we have that $v_1 \neq v_2$, and, by Lemma 3.2.8.1, we have $v_2 \rhd \sigma_{[x,v_1]}$. But then we would have two occurrences of v_2 in σ, which is impossible, because σ is a semipath. ⊣

As a corollary of Lemma 3.2.8.3, we have the following.

LEMMA 3.2.8.4. *If* $v_1 \prec v_2$ *and* $v_2 \prec x$*, then not* $v_1 \rhd [x, v_2]$.

§3.3. Further results for the Pivot Theorem

The lemmata of this section give further preliminary results for the Pivot Theorem of §3.4. They are grouped together because of their common combinatorial inspiration.

Let x, $y_1, \ldots, y_n$, z and $v_1, \ldots, v_n$ be vertices of a D-graph. Consider the following condition:

$$(yji) \qquad v_j \rhd [z, x] \quad \& \quad v_j \rhd [z, y_i] \quad \& \quad v_j \prec x \quad \& \quad v_j \prec y_i.$$

Then we can prove the following for $n \geq 1$.

LEMMA 3.3.1. *If for every* $i \in \{1, \ldots, n\}$ *we have* (yii)*, then for some* $j \in \{1, \ldots, n\}$ *for every* $i \in \{1, \ldots, n\}$ *we have* (yji).

PROOF. We proceed by induction on n. If $n = 1$, then the lemma holds trivially. For the induction step, when $n > 1$, suppose that

$(*)$ for every $i \in \{1, \ldots, n\}$ we have (yii).

Hence for every $k \in \{1, \ldots, n-1\}$ we have (ykk), and by the induction hypothesis there is an element of $\{1, \ldots, n-1\}$, which we call m, such that

$(**)$ for every $k \in \{1, \ldots, n-1\}$ we have (ymk).

If $v_m = v_n$, then from (ynn), which follows from $(*)$, and from $(**)$, we obtain for $j = m = n$ that

$(***)$ for every $i \in \{1, \ldots, n\}$ we have (yji).

Suppose $v_m \neq v_n$. From $(*)$ we conclude that we have $v_m \prec x$ and $v_n \prec x$. Then by Lemma 3.2.1 we have $v_n \prec v_m$ or $v_m \prec v_n$.

Suppose that

(1) $v_n \prec v_m$.

We want to show $(***)$ for j being n, and to achieve that, since we have (ynn), it suffices to show (ynk) for every $k \in \{1, \ldots, n-1\}$. We have

(1.1) $v_n \rhd [z, x]$, by (ynn),

(1.2) $v_m \prec x$, by (ymm), which follows from $(*)$,

(1.3) not $v_n \rhd [x, v_m]$, from (1) and (1.2), by Lemma 3.2.8.4,

(1.4) $v_n \rhd [v_m, z]$, from (1.1) and (1.3), by Lemma 3.2.6,

(1.5) $v_m \rhd [z, y_k]$, by (ymk), which follows from $(**)$,

(1.6) $v_n \rhd [z, y_k]$, from (1.5) and (1.4), by Lemma 3.2.7,

(1.7) $v_n \prec x$, by (ynn),

(1.8) $v_m \prec y_k$, by (ymk),

(1.9) $v_n \prec y_k$, from (1) and (1.8), by Lemma 3.2.4.

With (1.1), (1.6), (1.7) and (1.9) we have the four conjuncts of (ynk), for every $k \in \{1, \ldots, n-1\}$.

Suppose now that

(2) $v_m \prec v_n$.

We want to show $(***)$ for j being m, and to achieve that, since we have $(**)$, it suffices to show (ymn). We have a derivation of (ymn) obtained from the derivation of (ynk) above by putting m for n and n for m and k.

So, in any case, from $(*)$ we have inferred $(***)$ for some j, which yields our lemma. $\dashv$

Let x, y, $z_1, \ldots, z_n$ and $v_1, \ldots, v_n$ be vertices of a D-graph. Consider the following condition:

(zji) $\quad v_j \rhd [z_i, x] \quad \& \quad v_j \rhd [z_i, y] \quad \& \quad v_j \prec x \quad \& \quad v_j \prec y.$

Then we can prove the following for $n \geq 1$.

LEMMA 3.3.2. *If for every $i \in \{1, \ldots, n\}$ we have (zii), then for some $j \in \{1, \ldots, n\}$ for every $i \in \{1, \ldots, n\}$ we have (zji).*

PROOF. We proceed by induction on n. If $n = 1$, then the lemma holds trivially. For the induction step, when $n > 1$, suppose that

$(*)$ for every $i \in \{1, \dots, n\}$ we have (zii).

Hence for every $k \in \{1, \dots, n-1\}$ we have (zkk), and by the induction hypothesis there is an element of $\{1, \dots, n-1\}$, which we call m, such that

$(**)$ for every $k \in \{1, \dots, n-1\}$ we have (zmk).

If $v_m = v_n$, then from (znn), which follows from $(*)$, and from $(**)$, we obtain for $j = m = n$ that

$(***)$ for every $i \in \{1, \dots, n\}$ we have (zji).

Suppose $v_m \neq v_n$. From $(*)$ we conclude that we have $v_m \prec x$ and $v_n \prec x$. Then by Lemma 3.2.1 we have $v_m \prec v_n$ or $v_n \prec v_m$.

Suppose that

(1) $v_m \prec v_n$.

We want to show $(***)$ for j being n, and to achieve that, since we have (znn), it suffices to show (znk) for every $k \in \{1, \dots, n-1\}$.

Take a semipath σ in $[z_k, x]$. Then we have

(1.1) $v_m \rhd [z_k, x]$, by (zmk), which follows from $(**)$,

(1.2) $v_n \prec x$, by (znn),

(1.3) $v_n \rhd [v_m, x]$, from (1) and (1.2), by Lemma 3.2.8.1,

(1.4) $v_n \rhd \sigma_{[v_m,x]}$, from (1.1) and (1.3),

(1.5) $v_n \rhd \sigma$, from (1.4).

So we have $v_n \rhd [z_k, x]$. We derive analogously $v_n \rhd [z_k, y]$, the second conjunct of (znk), and we have the third and fourth conjunct by (znn).

Suppose now that

(2) $v_n \prec v_m$.

We want to show $(***)$ for j being m, and to achieve that, since we have $(**)$, it suffices to show (zmn). We have a derivation of (zmn) obtained from the derivation of (znk) above by putting m for n and n for m and k.

So, in any case, from $(*)$ we have inferred $(***)$ for some j, which yields our Lemma. ⊣

We can also prove the following two lemmata.

LEMMA 3.3.3.1. *Let $\sigma \in [x, r]$ and $\tau \in [y, p]$. If $v \rhd \sigma$, $v \rhd [y, r]$ and not $v \rhd \tau$, then $\sigma_{[v,r]}$ and τ do not intersect.*

PROOF. If $\sigma_{[v,r]}$ and τ intersect in s, then not $v \rhd \tau_{[y,s]} * \sigma_{[s,r]}$, since not $v \rhd \tau$; this contradicts $v \rhd [y, r]$. ⊣

LEMMA 3.3.3.2. *Let $\sigma \in [x, r]$ and $\tau \in [q, o]$. If $v \rhd \sigma$, $v \rhd [x, o]$, and not $v \rhd \tau$, then $\sigma_{[x,v]}$ and τ do not intersect.*

PROOF. If $\sigma_{[x,v]}$ and τ intersect in s, then not $v \rhd \sigma_{[x,s]} * \tau_{[s,o]}$, since not $v \rhd \tau$; this contradicts $v \rhd [x, o]$. ⊣

As a matter of fact, one of these lemmata can be derived from the other, but it is easier to give independent proofs than to make these derivations by appropriate substitutions.

§3.4. The Pivot Theorem

In this section we give the proof of the following theorem, which is the Pivot Theorem that we have announced.

THEOREM 3.4.1. *If $\psi_E(x, y, z)$ and $\psi_E(y, x, z)$, then there is a vertex v such that*

$$(*) \qquad v \rhd [z, x] \quad \& \quad v \rhd [z, y] \quad \& \quad v \prec x \quad \& \quad v \prec y.$$

PROOF. Assume $\psi_E(x, y, z)$ and $\psi_E(y, x, z)$. We proceed by induction on the number of inner vertices in our D-graph. The basis, where it has just one inner vertex, is trivial, because v is that vertex.

For the induction step, suppose our D-graph is $D_W \square D_E$. With respect to the distribution of x, y and z in D_W and D_E we have the following cases:

	$E(D_W)$	$E(D_E)$
(I)		x, y, z
(IIx)	x, z	y
(IIy)	y, z	x
(III)	x, y	z
(IVx)	x	y, z
(IVy)	y	x, z
(V)	x, y, z	
(VI)	z	x, y

The order of these cases is dictated by the structure of our proof.

Note first that the cases (IVx) and (IVy) are impossible because of Lemma 3.1.7. The other cases are possible and will now be treated.

(I) Let x, y and z be distinct vertices of $E(D_E)$. (Their distinctness is a consequence of the assumptions $\psi_E(x,y,z)$ and $\psi_E(y,x,z)$.) By Lemma 3.1.2, we have $\psi_E(x,y,z)$ and $\psi_E(y,x,z)$ in D_E. So by the induction hypothesis we have a v such that $(*)$ for D_E. We will show that for that v we have $(*)$ for $D_W \Box D_E$.

For the first conjunct of $(*)$ for $D_W \Box D_E$, take a semipath σ of $D_W \Box D_E$ in $[z,x]$. If σ is in D_E, then $v \rhd \sigma$ by the induction hypothesis. If σ is not in D_E, then by replacing in a subsemipath of σ a single vertex of D_W by a vertex in V_C we obtain a semipath σ' of D_E in $[x]_W$. By the induction hypothesis we have $v \rhd \sigma'$, because $v \prec x$ in D_E; hence $v \rhd \sigma$. So we have the first conjunct of $(*)$ for $D_W \Box D_E$.

For the third conjunct of $(*)$ for $D_W \Box D_E$, take a semipath σ of $D_W \Box D_E$ in $[x]_W$. Then by replacing in a subsemipath of σ a single vertex of D_W by a vertex in V_C we obtain a semipath σ' of D_E in $[x]_W$. By the induction hypothesis, as above, we have $v \rhd \sigma'$; hence $v \rhd \sigma$. So we have the third conjunct of $(*)$ for $D_W \Box D_E$.

For the second and fourth conjunct we proceed analogously by replacing x by y. So we have $(*)$ for $D_W \Box D_E$.

(IIx) Let x and z be distinct vertices of $E(D_W)$, and let y be a vertex of $E(D_E)$. Let $V_C = \{y_1, \ldots, y_n\}$; here $n \geq 1$. By Lemmata 3.1.4 and 3.1.5 for every $i \in \{1, \ldots, n\}$ we obtain $\psi_E(y_i, x, z)$ and $\psi_E(x, y_i, z)$ in D_W, and then by applying the induction hypothesis n times to D_W we obtain (yii) for every $i \in \{1, \ldots, n\}$. By Lemma 3.3.1, for some $j \in \{1, \ldots, n\}$ for every $i \in \{1, \ldots, n\}$ we have (yji). We will show $(*)$ for $D_W \Box D_E$ with v being v_j.

For the first conjunct, $v_j \rhd [z,x]$, suppose σ is a semipath of $D_W \Box D_E$ in $[z,x]$. If σ is a semipath of D_W, then we use the first conjunct of (yji). If σ passes through D_E, then by replacing in a subsemipath of σ a single vertex of D_E by a vertex y_k for some $k \in \{1, \ldots, n\}$, we obtain a semipath σ' of D_W in $[z, y_k]$. Then we use the second conjunct of (yjk), and we have that $v_j \rhd \sigma'$; hence $v_j \rhd \sigma$.

We proceed analogously for the remaining conjuncts $v_j \rhd [z,y]$, $v_j \prec x$ and $v_j \prec y$. We look for a subsemipath which is either a semipath of D_W or which after replacement of a single vertex by y_k becomes a semipath of D_W. Then we apply (yjk). So we have $(*)$ for $D_W \Box D_E$. We proceed analogously for (IIy).

(III) Let x and y be distinct vertices of $E(D_W)$, and let z be a vertex of $E(D_E)$. Let $V_C = \{z_1, \ldots, z_n\}$; here $n \geq 1$. By Lemma 3.1.4 where $\psi_E(y, x, z)$ is replaced by $\psi_E(z, x, y)$ and $\psi_E(z, y, x)$, for every $i \in \{1, \ldots, n\}$ we obtain $\psi_E(z_i, x, y)$ and $\psi_E(z_i, y, x)$ in D_W, and then by applying the induction hypothesis n-times to D_W we obtain (zii) for every $i \in \{1, \ldots, n\}$. By Lemma 3.3.2 , for some $j \in \{1, \ldots, n\}$ for every $i \in \{1, \ldots, n\}$ we have (zji). We will show $(*)$ for $D_W \Box D_E$ with v being v_j. We proceed as in case (IIx), with the help of vertex z_k from V_C instead of y_k, if need there is.

(V) Let x, y and z be distinct vertices of $E(D_W)$. By Lemma 3.1.6, we have that $\psi_E(x, y, z)$ and $\psi_E(y, x, z)$ in D_W. Consider the set of vertices v of D_W such that $(*)$ holds in D_W. By the induction hypothesis, this set is non-empty, and let $v_1, \ldots, v_n$, for $n \geq 1$, be all its elements. By Lemmata 3.2.1 and 3.2.4, we have that this set is linearly ordered by the relation $\prec$. Let us assume that we have $v_1 \prec \ldots \prec v_n$.

We have two cases to consider. Assume first that in D_W we have that

(V.1) $\quad (\forall u \in V_C)(v_n \rhd [u, x] \quad \& \quad v_n \rhd [u, y])$.

We will show $(*)$ for $D_W \Box D_E$ with v being v_n.

For the first conjunct, $v_n \rhd [z, x]$, suppose σ is a semipath of $D_W \Box D_E$ in $[z, x]$. If σ is a semipath of D_W, then we are done. If σ passes through D_E, then by replacing in a subsemipath of σ a single vertex of D_E by a vertex u of V_C, we obtain a semipath σ' of D_W in $[u, x]$. Then, by (V.1) we conclude that $v_n \rhd \sigma'$; hence $v_n \rhd \sigma$. We prove the second conjunct analogously.

For the third conjunct, $v_n \prec x$, we take a semipath σ of $D_W \Box D_E$ in $[x]_W$. If σ is a semipath of D_W, then we are done. If σ passes through D_E, then as above we obtain a semipath σ' of D_W in $[x, u]$ for $u \in V_C$, and reason as above with (V.1). For the fourth conjunct we proceed analogously, and hence we have $(*)$ for $D_W \Box D_E$.

If for some u in V_C and for t being x or y we have in D_W

(V.2) $\quad$ not $v_n \rhd [u, t]$,

then we show $(*)$ for $D_W \Box D_E$ with v being v_1.

We prove first that in D_W we have

(†) $\quad$ not $(\psi_E(y, x, u) \quad \& \quad \psi_E(x, y, u))$.

Suppose not (†). Then, by the induction hypothesis applied to D_W, we have a vertex w such that

(††) $\quad w \rhd [u,x] \quad \& \quad w \rhd [u,y] \quad \& \quad w \prec x \quad \& \quad w \prec y.$

Since $w \rhd [u,t]$, by (V.2), we obtain $w \neq v_n$. Since $v_n \prec t$, $w \prec t$ and $v_n \neq w$, we obtain $v_n \prec w$ or $w \prec v_n$ by Lemma 3.2.1. We will show that not $w \prec v_n$.

By (V.2) we have a semipath σ of D_W in $[u,t]$ such that not $v_n \rhd \sigma$. From the conjunct $w \rhd [u,t]$ of (††), we have $w \rhd \sigma$. Then we have not $v_n \rhd \sigma_{[w,t]}$. By putting v_n, t and w for v_2, x and v_1 respectively in Lemma 3.2.8.1, we obtain not $w \prec v_n$. Hence we have $v_n \prec w$.

We show $(*)$ for D_W with v being w. The third and fourth conjunct are given by the third and fourth conjunct of (††). For the first conjunct, $w \rhd [z,x]$, we apply Lemma 3.2.8.2 with v_1 and v_2 being v_n and w respectively. For the second conjunct, $w \rhd [z,y]$, we proceed analogously, and hence we have $(*)$ for D_W with v being w. So $w \in \{v_1, \ldots, v_n\}$, but this is in contradiction with $v_1 \prec \ldots \prec v_n$ and $v_n \prec w$. So we can infer (†).

Suppose not $\psi_E(y,x,u)$. So there is a semipath ρ of D_W in $[y,u]$, and a semipath π of D_W in $[x,r]$, for r in $W(D_W)$, which do not intersect. We can show first that

(1) $\quad$ for every v in $W(D_E)$ we have $v \in V_C$.

Suppose $v \in W(D_E)$ and $v \notin V_C$. Take a semipath σ of D_W in $[x,z]$, and take an $i \in \{1,\ldots,n\}$; we have $v_i \rhd \sigma$, by $(*)$ for D_W. By putting v_i, z, ρ and u for respectively v, r, τ and p in Lemma 3.3.3.1 we obtain that $\sigma_{[v_i,z]}$ and ρ do not intersect. Let $u = W_E(a_u)$ in D_E; so $a_u \in C$. Let φ be a semipath of D_E in $[E_E(a_u), v]$, and let ρ' be obtained from ρ by replacing u by $E_E(a_u)$. Then $\rho' * \varphi$, a semipath of $D_W \Box D_E$ in $[y]_W$, and $\pi_{[x,v_i]} * \sigma_{[v_i,z]}$, a semipath of D_W, and hence of $D_W \Box D_E$, in $[x,z]$, do not intersect, which contradicts the assumption that $\psi_E(x,y,z)$ in $D_W \Box D_E$. This proves (1).

For every v in V_C, we can show that $\psi_E(x,v,z)$ and $\psi_E(v,x,z)$. We prove first that $\psi_E(x,v,z)$, which is similar to the proof of (1) we have just given. Suppose not $\psi_E(x,v,z)$ in D_W. Then there is a semipath σ of D_W in $[x,z]$ and a semipath τ of D_W in $[v]_W$ that do not intersect. Take an $i \in \{1,\ldots,n\}$; we have $v_i \rhd \sigma$. By putting v_i, z, ρ and u for v, r, τ and p in Lemma 3.3.3.1 we obtain that $\sigma_{[v_i,z]}$ and ρ do not intersect. By putting v_i, π and τ for v, σ and τ in Lemma 3.3.3.2, we obtain that $\pi_{[x,v_i]}$ and τ do not intersect.

Let $u = W_E(a_u)$ and $v = W_E(a_v)$ in D_E; so $a_u, a_v \in C$. Let φ be a semipath of D_E in $[E_E(a_u), E_E(a_v)]$, and let ρ' and τ' be obtained from

ρ and τ by replacing u and v by $E_E(a_u)$ and $E_E(a_v)$ respectively. Then $\rho' * \varphi * \tau'$, a semipath of $D_W \Box D_E$ in $[y]_W$, and $\pi_{[x,v_i]} * \sigma_{[v_i,z]}$, a semipath of $D_W \Box D_E$ in $[x,z]$, do not intersect, which contradicts the assumption that $\psi_E(x,y,z)$ in $D_W \Box D_E$. So we have $\psi_E(x,v,z)$.

Suppose not $\psi_E(v,x,z)$ in D_W. Then there is a semipath σ of D_W in $[v,z]$ and a semipath τ of D_W in $[x]_W$ that do not intersect. Take an $i \in \{1,\dots,n\}$; we have $v_i \rhd \tau$. By putting v_i, τ, ρ and u for v, σ, τ and p in Lemma 3.3.3.1, we obtain that $\tau_{[v_i,r]}$ and ρ do not intersect. By putting v_i, π and σ for v, σ and τ in Lemma 3.3.3.2, we obtain that $\pi_{[x,v_i]}$ and σ do not intersect. Let φ and ρ' be defined as above, and let σ' be obtained from σ by replacing v by the vertex $E_E(a_v)$ of D_E. Then $\rho' * \varphi * \sigma'$, a semipath of $D_W \Box D_E$ in $[y,z]$, and $\pi_{[x,v_i]} * \tau_{[v_i,r]}$, a semipath of $D_W \Box D_E$ in $[x]_W$, do not intersect, which contradicts the assumption that $\psi_E(y,x,z)$ in $D_W \Box D_E$. So we have $\psi_E(v,x,z)$, and hence we have shown that $\psi_E(x,v,z)$ and $\psi_E(v,x,z)$.

We apply then the induction hypothesis for D_W and obtain for every v in V_C a vertex w of D_W such that

$$(*vw) \qquad w \rhd [z,x] \quad \& \quad w \rhd [z,v] \quad \& \quad w \prec x \quad \& \quad w \prec v.$$

We can then prove

$$(2) \qquad \text{for every } v \text{ in } V_C \text{ we have } v_1 \prec v \text{ in } D_W.$$

Suppose for some v in V_C we do not have $v_1 \prec v$ in D_W. We can then infer that $w \prec v_1$. Since by $(*vw)$ we have $w \prec v$, we also have $w \neq v_1$, because not $v_1 \prec v$. Since by $(*vw)$ we have $w \prec x$, and we have $v_1 \prec x$, we obtain that $w \prec v_1$ or $v_1 \prec w$ by Lemma 3.2.1. If we had $v_1 \prec w$, with $w \prec v$, we would have $v_1 \prec v$, which contradicts our assumption. Hence we have $w \prec v_1$.

We will then prove that in D_W we have $(*yw)$, which is $(*vw)$ with v replaced by y, or $(*)$ with v replaced by w. The first and third conjunct of $(*yw)$ are obtained from the respective conjuncts of $(*vw)$. The fourth conjunct of $(*yw)$, namely $w \prec y$, follows immediately from $w \prec v_1$ and $v_1 \prec y$.

For the only remaining conjunct, $w \rhd [z,y]$, we show first that for every $i \in \{1,\dots,n\}$ we do not have $w \rhd \pi_{[x,v_i]}$, by Lemma 3.2.8.3; we put w, v_i and $\pi_{[x,v_i]}$ for v_1, v_2 and σ, and we use $w \prec v_i$, which follows from $w \prec v_1$.

Let $\sigma \in [z,y]$. Since by $(*vw)$ we have $w \rhd [x,z]$, we obtain $w \rhd \pi_{[x,v_i]} * \sigma_{[v_i,z]}$. Since not $w \rhd \pi_{[x,v_i]}$, we have $w \rhd \sigma_{[v_i,z]}$, and hence $w \rhd \sigma$. So we have $w \rhd [z,y]$, and hence $(*yw)$ holds.

So $w \in \{v_1, \dots, v_n\}$, but $w \prec v_1$ contradicts the assumption that $v_1 \prec \dots \prec v_n$. So (2) holds. We prove also the following

(3) for every v in V_C we have $v_1 \rhd [v, z]$ in D_W.

Suppose not $v_1 \rhd [v, z]$. Then for some σ in $[v, z]$ we have not $v_1 \rhd \sigma$. By putting v_1, π, σ, v and z for v, σ, τ, y and p in Lemma 3.3.3.1, we obtain that $\pi_{[v_1,r]}$ and σ do not intersect; for the assumption $v_1 \rhd [v, r]$ of Lemma 3.3.3.1 after the replacement, we use $v_1 \prec v$. Let π^{-1} be the semipath $\pi_{[r,x]}$ in $[r, x]$, which is obtained by taking π in reverse order, and let σ^{-1} be $\sigma_{[z,v]}$.

By putting v_1, π^{-1}, r, x, σ^{-1}, z and v for v, σ, x, r, τ, y and p in Lemma 3.3.3.1, we obtain that $\pi^{-1}_{[v_1,x]}$ and σ^{-1} do not intersect. Hence $\pi_{[x,v_1]}$ and σ do not intersect. Since π is $\pi_{[x,v_1]} * \pi_{[v_1,r]}$, we conclude that π and σ do not intersect.

By defining ρ', φ and σ' as before (see the proofs of $\psi_E(x, v, z)$ and $\psi_E(v, x, z)$ in D_W), we have that π and $\rho' * \varphi * \sigma'$ do not intersect, which contradicts the assumption that $\psi_E(y, x, z)$ in $D_W \Box D_E$. So (3) holds.

Now we can show $(*)$ for $D_W \Box D_E$ with v being v_1. For $v_1 \rhd [z, x]$, the first conjunct, take a semipath σ of $D_W \Box D_E$ in $[z, x]$. If σ is a semipath of D_W, we are done, by the induction hypothesis. If σ passes through D_E, then let s be the first vertex in σ not in D_W. Consider $\sigma_{[z,s]}$, and let σ^* be obtained from it by replacing s by the corresponding $v \in V_C$. (If $s = E_E(a)$ in D_E, then $v = W_E(a)$ in D_E; alternatively $v = E_W(a)$ in D_W.) By (3) we have that $v_1 \rhd \sigma^*$, and since $v_1 \neq v$, we have $v_1 \rhd \sigma$. For $v_1 \rhd [z, y]$, the second conjunct, we proceed analogously.

For $v_1 \prec x$, the third conjunct, take a semipath σ of $D_W \Box D_E$ in $[x]_W$. If σ is a semipath of D_W, we are done. If σ passes through D_E, we proceed in principle as above in order to obtain a semipath σ^* of D_W. The vertex s of D_E is now the last vertex of D_E in σ. We use (1) to guarantee that s, which is not in $W(D_E)$, has a corresponding vertex v in V_C, as above. By (2) we have that $v_1 \rhd \sigma^*$, and hence $v_1 \rhd \sigma$. For $v_1 \prec y$, the last conjunct, we proceed analogously. This concludes the proof of (V).

(VI) Let z be a vertex of $E(D_W)$, and let x and y be distinct vertices of $E(D_E)$. We proceed by an auxiliary induction on the number n of inner vertices of D_E. If $n = 1$, then D_E is a basic D-graph, and its unique inner vertex is the v required by $(*)$. If $n > 1$, then let D_E be $D'_E \Box D''_E$. We have the following cases.

(1) Suppose $x, y \in E(D'_E)$.

(1.1) If $D_W \Box D'_E$ is defined, then $D_W \Box (D'_E \Box D''_E)$ is equal to $(D_W \Box D'_E) \Box D''_E$, and we are in case (V) for $(D_W \Box D'_E) \Box D''_E$, for which we have $x, y, z \in E(D_W \Box D'_E)$. We continue reasoning as for (V) above.

(1.2) If $D_W \Box D'_E$ is not defined, then $D_W \Box (D'_E \Box D''_E)$ is equal to $D'_E \Box (D_W \Box D''_E)$ and we are in case (III).

(2) Suppose $x, y \in E(D''_E)$.

(2.1) If $D_W \Box D'_E$ is defined, then we may apply the hypothesis of the auxiliary induction to $(D_W \Box D'_E) \Box D''_E$, since the number of inner vertices of D''_E is lesser than n.

(2.2) If $D_W \Box D'_E$ is not defined, then we are in case (I) for $D'_E \Box (D_W \Box D''_E)$.

(3) Suppose $x \in E(D'_E)$ and $y \in E(D''_E)$.

(3.1) If $D_W \Box D'_E$ is defined, then we are in case (IIx) for $(D_W \Box D'_E) \Box D''_E$.

(3.2) If $D_W \Box D'_E$ is not defined, then we are in case (IVx) for $D'_E \Box (D_W \Box D''_E)$, which is impossible. This concludes our proof of the theorem. $\dashv$

Let us write $y \rhd [x]_X$ when for every σ in $[x]_X$ we have that $y \rhd \sigma$, and let $y \prec_X x$ stand for $y \rhd [x]_X$ and $y \neq x$. So $y \prec_W x$ iff $y \prec x$. It is clear that for all we have proven since §3.2 about pivots, $\prec_W$ and ψ_E there are dual results about $\prec_E$ and ψ_W.

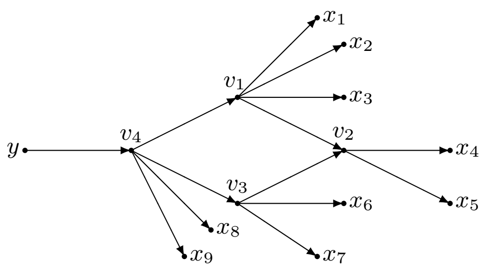
Chapter 4

P$''$-Graphs and P$'$-Graphs

§4.1. Petals

In this chapter the goal is to prove that every P$''$-graph (as defined in §1.9) is a P$'$-graph (as defined in §1.8). For that we must first deal with some preliminary matters in this and in the next two sections.

For a vertex v of a D-graph D let $\mathcal{C}(v)$, the *corolla* of v, be the set of all vertices x of D such that $v \prec x$. For example, in



$\mathcal{C}(v_1) = \{x_1, x_2, x_3\}$, while $\mathcal{C}(v_4)$ is made of all the vertices except v_4 and y.

The binary relation that holds between the elements x and y of $\mathcal{C}(v)$ whenever *not* $v \rhd [x, y]$ is an equivalence relation on $\mathcal{C}(v)$. For reflexivity, we have that *not* $v \rhd [x, x]$ because $v \neq x$ (which is assumed with $v \prec x$), and v does not belong to the one-vertex trivial semipath from x to x. Symmetry is trivial, because we can always read a semipath in reverse order. For transitivity, assume we have the semipaths σ in $[x, y]$ and τ in $[y, z]$ such that not $v \rhd \sigma$ and not $v \rhd \tau$. Then we have that $\sigma * \tau \in [x, z]$ and not $v \rhd \sigma * \tau$.

For x in $\mathcal{C}(v)$, let the equivalence class

$$[\![x]\!]_v =_{df} \{y \in \mathcal{C}(v) \mid not\ v \rhd [x,y]\}$$

be called a *petal*. In the example above we have $[\![x_1]\!]_{v_1} = \{x_1\}$, while $[\![x_1]\!]_{v_4}$ is made of all the vertices except x_8, x_9, v_4 and y.

LEMMA 4.1.1. *If $v \prec x$ and not $v \rhd [x,u]$, then $v \prec u$.*

PROOF. Suppose $v \prec x$ and not $v \prec u$. Hence for some $\sigma \in [u]_W$ we have not $v \rhd \sigma$. Take a τ in $[x,u]$. Then for $\tau * \sigma \in [x]_W$ we have $v \rhd \tau * \sigma$. Since not $v \rhd \sigma$, we must have $v \rhd \tau$. ⊣

As a corollary we have the following.

LEMMA 4.1.2. *If $v \prec x$ and not $v \rhd [x,u]$, then $u \in [\![x]\!]_v$.*

LEMMA 4.1.3. *If $v \prec x$, $v \rhd [x,y]$, $\tau \in [y]_W$ and $u \rhd \tau$, then $v \rhd [x,u]$.*

PROOF. Suppose $v \prec x$, $v \rhd [x,y]$, $\tau \in [y,z]$, $z \in W(D)$, and $u \rhd \tau$. Suppose not $v \rhd [x,u]$. Hence for some σ in $[x,u]$ we have not $v \rhd \sigma$. Then $\sigma * \tau_{[u,z]} \in [x]_W$, and since $v \prec x$, we have $v \rhd \sigma * \tau_{[u,z]}$. Since not $v \rhd \sigma$, we have $v \rhd \tau_{[u,z]}$. Hence not $v \rhd \tau_{[u,y]}$; otherwise, τ would not be a semipath. So not $v \rhd \sigma * \tau_{[u,y]}$, which together with $\sigma * \tau_{[u,y]} \in [x,y]$ contradicts our assumption that $v \rhd [x,y]$. ⊣

LEMMA 4.1.4. *Let x, x' and y be distinct vertices of $E(D)$. If $v \prec x$, $v \prec x'$, $v \rhd [x,y]$ and not $v \rhd [x,x']$, then not $\psi_E(x,y,x')$.*

PROOF. Suppose $v \prec x$, $v \prec x'$, $v \rhd [x,y]$, $\sigma \in [x,x']$ and not $v \rhd \sigma$. Suppose for τ in $[y]_W$ we have a vertex u such that $u \rhd \sigma$ and $u \rhd \tau$. By Lemma 4.1.3, we have $v \rhd [x,u]$, and hence $v \rhd \sigma_{[x,u]}$, which contradicts not $v \rhd \sigma$. Hence not $\psi_E(x,y,x')$. ⊣

As a corollary we have the following.

LEMMA 4.1.5. *Let x_1, x_2 and y be distinct vertices of $E(D)$. If $v \prec x$, $y \notin [\![x]\!]_v$ and $x_1, x_2 \in [\![x]\!]_v$, then not $\psi_E(x_1,y,x_2)$.*

PROOF. From $v \prec x$, $y \notin [\![x]\!]_v$ and $x_1, x_2 \in [\![x]\!]_v$, we conclude $v \prec x_1$, $v \prec x_2$, $v \rhd [x_1,y]$ and not $v \rhd [x_1,x_2]$, with the help of Lemma 4.1.2; then we apply Lemma 4.1.4. ⊣

§4.2. P-moves

For any list A, let $\bar{A}$ be the converse list, i.e. A read in reverse order. For the lists A and A', let $P_{A,A'}$ be the set of all ordered pairs (a,b) such that a precedes b in A and b precedes a in A'.

Let $[\![x]\!]_v^E$ be $[\![x]\!]_v \cap E(D)$, and let $\Lambda_v(x)$ stand for a list of $[\![x]\!]_v^E$. Since the equivalence class $[\![x]\!]_v$ is non-empty, we conclude that $[\![x]\!]_v^E$ and hence also $\Lambda_v(x)$ are always non-empty.

For x and y distinct E-vertices, let $V(x,y)$ be $\{v \mid v \prec x \quad \& \quad v \prec y\}$, i.e. the set of common pivots of x and y. We say that v is the *closest common pivot* of x and y, and write $v\mathrm{CCP}(x,y)$, when $v \in V(x,y)$ and for every w in $V(x,y)$ either $w \prec v$ or $w = v$.

Let Π and Θ be two lists of $E(D)$. Then we call $\mathrm{P}_{\Pi,\Theta}$-*moves*, or sometimes P-*moves* for short, the following rewrite rules for lists of $E(D)$; we read these rules as stating that we can pass from the list of $E(D)$ above the horizontal line, which we call Π, to the list of $E(D)$ below, which we call Π', provided $(x,y) \in P_{\Pi,\Theta}$:

$$\text{Tr-}(x,y) \qquad \frac{\Gamma\Lambda_v(x)\Lambda_v(y)\Delta}{\Gamma\Lambda_v(y)\Lambda_v(x)\Delta}$$

$$\text{Sf-}(x,y) \qquad \frac{\Gamma\Lambda_v(z)\Delta}{\Gamma\overline{\Lambda_v(z)}\Delta}$$

provided that in Sf-(x,y) we have that x precedes y in $\Lambda_v(z)$ and $v\mathrm{CCP}(x,y)$,

$$\text{Bf-}(x,y) \qquad \frac{\Pi}{\overline{\Pi}}$$

provided that in Bf-(x,y) we have that x precedes y in Π and $V(x,y) = \emptyset$. In the names of these rules, Tr stands for *transposition*, Sf stands for *small flip*, and Bf stands for *big flip*.

Let D be the D-graph form the beginning of §4.1. As an example, consider the following lists of $E(D)$:

$$\Pi\colon x_1x_2x_3x_4x_5x_6x_7x_8x_9$$

$$\Theta\colon x_8x_6x_7x_5x_4x_1x_2x_3x_9$$

The P-move Tr-(x_5,x_8) is

$$\frac{x_1x_2x_3x_4x_5x_6x_7x_8x_9}{x_8x_1x_2x_3x_4x_5x_6x_7x_9}$$

with Γ being empty, $\Lambda_{v_4}(x_5)\colon x_1x_2x_3x_4x_5x_6x_7$, $\Lambda_{v_4}(x_8)\colon x_8$ and $\Delta\colon x_9$. The P-move Sf-(x_5, x_6) is

$$\frac{x_1x_2x_3x_4x_5x_6x_7x_8x_9}{x_7x_6x_5x_4x_3x_2x_1x_8x_9}$$

with Γ being empty, $\Lambda_{v_4}(x_5)$ as above and Δ being x_8x_9.

Note that for Tr-(x, y) we can infer that $v\mathrm{CCP}(x, y)$, as in the proviso for Sf-(x, y). Otherwise, for some w we would have $v \prec w$, $w \prec x$ and $w \prec y$. From that by Lemma 3.2.8.4 we would obtain that $w \in [\![x]\!]_v$ and $w \in [\![y]\!]_v$, and hence $[\![x]\!]_v = [\![y]\!]_v$, which is contradictory to our assumptions (lists are without repetitions, and hence $\Lambda_v(x)$ and $\Lambda_v(y)$ are lists with different members).

LEMMA 4.2.1. *For every $P_{\Pi,\Theta}$-move, if Π is grounded in D, then Π' is grounded in D.*

PROOF. Suppose the vertices r,s and t occur in that order in Π, and we have $\psi_E(r, s, t)$. Suppose first our $\mathrm{P}_{\Pi,\Theta}$-move is Tr-(x, y). All the cases where not more than one of r, s and t occur in $\Lambda_v(x)\Lambda_v(y)$ are settled in the obvious manner, as well as the two cases where they are all in $\Lambda_v(x)$ or all in $\Lambda_v(y)$, and the cases where two of r, s and t are in one of $\Lambda_v(x)$ and $\Lambda_v(y)$, while the remaining vertex is in Γ or Δ. In all these cases, r, s and t occur in the same order in Π and Π'.

Note also that the case where r and t are in one of $\Lambda_v(x)$ and $\Lambda_v(y)$, while s is in the other, is impossible, because we have $\Pi : r - s - t$. As interesting cases, only the following remain.

(1) Suppose r and s are in $\Lambda_v(x)$, and t is in $\Lambda_v(y)$. We need to show that $\psi_E(t, r, s)$. Take a σ in $[t, s]$. We have $v \prec s$ and $t \notin [\![s]\!]_v$. By Lemma 4.1.2, we obtain $v \rhd \sigma$. Since $v \prec r$, we obtain $\psi_E(t, r, s)$. We reason analogously when r is in $\Lambda_v(x)$, while s and t are in $\Lambda_v(y)$.

(2) Suppose r is in Γ, s is in $\Lambda_v(x)$ and t is in $\Lambda_v(y)$. We need to show that $\psi_E(r, t, s)$. Take a σ in $[r, s]$. Then we have $v \prec s$ and $r \notin [\![s]\!]_v$. By Lemma 4.1.2, we obtain $v \rhd \sigma$. Since $v \prec t$, we obtain $\psi_E(r, t, s)$. We reason analogously when r is in $\Lambda_v(x)$, s is in $\Lambda_v(y)$ and t is in Δ.

Suppose next our $\mathrm{P}_{\Pi,\Theta}$-move is Sf-(x, y). Excluding obvious cases, like those mentioned above, we have as interesting cases only the following. Suppose r is in Γ, while s and t are in $\Lambda_v(z)$. Then we reason as in (2), by using Lemma 4.1.2. We reason analogously when r and s are in $\Lambda_v(z)$

and t is in Δ. The case where our $\mathrm{P}_{\Pi,\Theta}$-move is Bf-(x,y) is settled in the obvious way. ⊣

Note that this Lemma holds without taking account of the provisos for Sf-(x,y) and Bf-(x,y). As a matter of fact, our proof of the lemma shows that for every P-move, respecting the provisos or not, Π is grounded in D iff Π' is grounded in D.

The following proposition shows that making a Tr-(x,y) move brings us closer to Θ, "closer" in a sense that will be made precise later (see Proposition 4.3.2 below). For this proposition we assume that Π is $\Gamma\Lambda_v(x)\Lambda_v(y)\Delta$, as for Tr-$(x,y)$.

PROPOSITION 4.2.2. *Suppose* Π *and* Θ *are grounded in* D, *and* $r, r' \in [\![x]\!]_v^E$ *and* $s, s' \in [\![y]\!]_v^E$. *Then* $(r,s) \in P_{\Pi,\Theta}$ *implies* $(r',s') \in P_{\Pi,\Theta}$.

PROOF. Suppose $(r,s) \in P_{\Pi,\Theta}$ and $(r',s') \notin P_{\Pi,\Theta}$. Hence r precedes s in Π and s precedes r in Θ. Suppose s' precedes r' in both Π and Θ.

(1) Suppose r' precedes r in Π. Then we have $\Pi\colon s'{-}r'{-}r{-}s$, and by using Lemma 4.1.5 with x_1, x_2 and y replaced by s, s' and r or r' we obtain that not $\psi_E(s,r,s')$ or not $\psi_E(s,r',s')$, which contradicts our assumption.

(2) Suppose r precedes r' in Π.

(2.1) Suppose s precedes s' in Π. Then we have $\Pi\colon r{-}s{-}s'{-}r'$, and by using Lemma 4.1.5 with x_1, x_2 and y replaced by r, r' and s or s' we obtain that not $\psi_E(r,s,r')$ or not $\psi_E(r,s',r')$, which contradicts our assumption.

(2.2) Suppose s' precedes s in Π.

(2.21) Suppose r precedes s' in Π. Then we have $\Pi\colon r{-}s'{-}r'$, and by using Lemma 4.1.5 as in (2.1) we obtain a contradiction.

(2.22) Suppose s' precedes r in Π. Then we have $\Pi\colon s'{-}r{-}s$, and by using Lemma 4.1.5 as in (1) we obtain a contradiction.

When r' precedes s' in both Π and Θ we proceed in the same manner after replacing Π, r, r', s, and s' by respectively Θ, r', r, s' and s. We obtain in that case contradictions with our assumption that Θ is grounded in D. ⊣

We can prove the following.

LEMMA 4.2.3. *If* $u_1, u_2, u_3 \in [\![z]\!]_v^E$, $\psi_E(u_1,u_2,u_3)$ *and* $\psi_E(u_2,u_1,u_3)$, *then not* $v\mathrm{CCP}(u_1,u_2)$.

PROOF. Since $\psi_E(u_1, u_2, u_3)$ and $\psi_E(u_2, u_1, u_3)$, we have, by Theorem 3.4.1, a vertex w such that $w \rhd [u_3, u_1]$ and $w \in V(u_1, u_2)$. Since $u_1, u_3 \in [\![z]\!]^E_v$, there is a σ in $[u_3, u_1]$ such that not $v \rhd \sigma$. Since $w \rhd [u_3, u_1]$, we have $w \rhd \sigma$, and hence not $v \rhd \sigma_{[u_1,w]}$. So not $v \rhd [u_1, w]$, and since $v \prec u_1$, we obtain $v \prec w$ by Lemma 4.1.1. ⊣

For the following two lemmata we assume that vCCP(x, y), and that Π and Θ are grounded in D. First we have a lemma that is a direct corollary of Lemma 4.2.3.

LEMMA 4.2.4. *If $x, y, u \in [\![z]\!]^E_v$, then it is impossible that $\Pi\colon x{-}y{-}u$ and $\Theta\colon y{-}x{-}u$.*

Next we have the following.

LEMMA 4.2.5. *If $x, y, x', y' \in [\![z]\!]^E_v$, then it is impossible that*

(1) $\Pi\colon y'{-}x{-}y{-}x'$ *and* $\Theta\colon y{-}y'{-}x'{-}x$,

(2) $\Pi\colon x{-}y'{-}x'{-}y$ *and* $\Theta\colon y'{-}y{-}x{-}x'$.

PROOF. Suppose we have (1). Then since $\Pi\colon y'{-}x{-}y$ and $\Theta\colon x{-}y'{-}y$, we have, by Lemma 4.2.3, a vertex w_1 in $V(x, y')$ such that $v \prec w_1$. And since $\Pi\colon y'{-}y{-}x'$ and $\Theta\colon y{-}y'{-}x'$, we have, by Lemma 4.2.3, a vertex w_2 in $V(y', y)$ such that $v \prec w_2$.

It is impossible that $w_1 = w_2$, because otherwise not vCCP(x, y). Then since $w_1 \prec y'$ and $w_2 \prec y'$, we have, by Lemma 3.2.1 that either $w_1 \prec w_2$ or $w_2 \prec w_1$. If $w_1 \prec w_2$, then $w_1 \in V(x, y)$ and since $v \prec w_1$, we obtain a contradiction with vCCP(x, y), , and we reason analogously if $w_2 \prec w_1$. This proves that (1) is impossible.

An alternative proof that (1) is impossible is obtained by showing that we have a vertex w_1 in $V(y, x')$ such that $v \prec w_1$, and a vertex w_2 in $V(x', x)$ such that $v \prec w_2$. We prove (2) analogously with two applications of Lemma 4.2.3. ⊣

LEMMA 4.2.6. *If $V(x, y) \neq \emptyset$, then there is a w in $V(x, y)$ such that wCCP(x, y).*

PROOF. It follows from Lemmata 3.2.1 and 3.2.4 that $V(x, y)$ is linearly ordered by $\prec$. Then w is the greatest element of this linear order. ⊣

Besides the assumptions that vCCP(x, y), and that Π and Θ are grounded in D, which we made before Lemma 4.2.4, we assume also that $(x, y) \in$

$P_{\Pi,\Theta}$, that $x, y \in [\![z]\!]^E_v$, that Π is $\Gamma\Lambda_v(z)\Delta$, and that Π' is $\Gamma\overline{\Lambda_v(z)}\Delta$, as for Sf-$(x, y)$. Then we have the following.

PROPOSITION 4.2.7. *If* $(x', y') \in P_{\Pi',\Theta} - P_{\Pi,\Theta}$, *then not* vCCP(x', y').

PROOF. Let P_Π and $P_{\Pi'}$ abbreviate $P_{\Pi,\Theta}$ and $P_{\Pi',\Theta}$ respectively. Suppose $(x', y') \in P_{\Pi'} - P_\Pi$; i.e., $(x', y') \notin P_\Pi$ and $(x', y') \in P_{\Pi'}$. From that we derive that y' precedes x' in both Θ and Π, and that $x', y' \in [\![z]\!]^E_v$. It is excluded that $x = y$ because $(x, y) \in P_\Pi$, and that $x' = y'$ because $(x', y') \in P_{\Pi'}$. It is excluded that $x' = x$ and $y' = y$ because $(x, y) \in P_\Pi$ and $(x', y') \notin P_\Pi$, and that $x' = y$ and $y' = x$ because $(x', y') \in P_{\Pi'}$ and $(y, x) \notin P_{\Pi'}$.

(1) Suppose $x' = x$ and $y' \neq y$. Then, since y' precedes x' in Π, we have $\Pi\colon y'-x-y$.

It is excluded that $\Theta : y'-y-x$, because of Lemma 4.2.4, and that $\Theta\colon y–x–y'$, because y precedes x and y' precedes x' in Θ; hence $\Theta\colon y–y'–x$. From $\Pi : y'-x-y$ and $\Theta : y-y'-x$ we conclude that $\psi_E(y', x', y)$ and $\psi_E(x', y', y)$, and by, Lemma 4.2.3, not vCCP(x', y'). The case where $x' \neq x$ and $y' = y$ is treated analogously.

(2) Suppose $x' = y$ and $y' \neq x$. Then, since y' precedes x' in Θ, we have $\Theta\colon y'-y-x$.

It is excluded that $\Pi\colon y'-x-y$, because of Lemma 4.2.4, and that $\Pi\colon x–y–y'$, because x precedes y and y' precedes x' in Π. Hence $\Pi\colon x–y'–y$, and by reasoning as in (1), we obtain, by Lemma 4.2.3, not vCCP(x', y'). The case where $x' \neq y$ and $y' = x$ is treated analogously. The only remaining case is the following.

(3) Suppose x, y, x' and y' are all mutually distinct.

(3.1) Suppose $\Pi\colon y'-x-y$. It is excluded that $\Theta\colon y'-y-x$, because of Lemma 4.2.4.

(3.11) Suppose $\Theta\colon y–y'–x$. It is excluded that $\Pi\colon x'-y'-x-y$, because x precedes y and y' precedes x' in Π.

(3.111) Suppose $\Pi\colon y'-x'-x-y$ or $\Pi\colon y'-x-x'-y$. It is excluded that $\Theta\colon x'-y-y'-x$ and $\Theta\colon y-x'-y'-x$, because y precedes x and y' precedes x' in Θ. If $\Theta\colon y-y'-x'-x$ or $\Theta\colon y-y'-x-x'$, then since $\Pi\colon y-x'-y'$ and $\Theta\colon y-y'-x'$, we obtain, by Lemma 4.2.3, not vCCP(x', y').

(3.112) Suppose $\Pi\colon y'–x–y–x'$. It is excluded that $\Theta\colon x'–y–y'–x$ and $\Theta\colon y-x'-y'-x$, for the reasons given in (3.111). It is excluded also that $\Theta\colon y-y'-x'-x$, because of (1) of Lemma 4.2.5, and that $\Theta\colon y-y'-x-x'$, because of Lemma 4.2.4. So (3.112) is impossible.

(3.12) Suppose $\Theta\colon y{-}x{-}y'$. Hence $\Theta\colon y{-}x{-}y'{-}x'$, because y precedes x and y' precedes x' in Θ. From now on we will take for granted this sort of justification based on precedence. We may have that either $\Pi\colon y'{-}x'{-}x{-}y$ or $\Pi\colon y'{-}x{-}x{-}y$, as in (3.111), and we reason as for (3.111). If we suppose $\Pi\colon y'-x-y-x'$ (as in (3.112)), then, since $\Pi\colon x-y-x'$ and $\Theta\colon y-x-x'$, we obtain a contradiction with Lemma 4.2.4.

(3.2) Suppose $\Pi\colon x-y'-y$.

(3.21) Suppose $\Theta\colon y'-y-x$.

(3.211) Suppose $\Pi\colon x-y'-x'-y$. Then with either $\Theta\colon y'-x'-y-x$ or $\Theta : y'-y-x'-x$ we apply Lemma 4.2.3 to obtain not $v\mathrm{CCP}(x', y')$. It is excluded that $\Theta\colon y'-y-x-x'$, because of (2) of Lemma 4.2.5.

(3.212) Suppose $\Pi\colon x-y'-y-x'$. Then with either $\Theta\colon y'-x'-y-x$ or $\Theta : y'-y-x'-x$ we apply Lemma 4.2.3 to obtain not $v\mathrm{CCP}(x', y')$. It is excluded that $\Theta\colon y'-y-x-x'$, because of Lemma 4.2.4.

(3.22) Suppose $\Theta\colon y-y'-x$.

(3.221) Suppose $\Pi\colon x-y'-x'-y$. Then with either $\Theta\colon y-y'-x'-x$ or $\Theta\colon y-y'-x-x'$ we apply Lemma 4.2.3 to obtain not $v\mathrm{CCP}(x', y')$.

(3.222) Suppose $\Pi\colon x-y'-y-x'$. Then with $\Theta\colon y-y'-x'-x$ we apply Lemma 4.2.3 to obtain not $v\mathrm{CCP}(x', y')$. It is excluded that $\Theta\colon y{-}y'{-}x{-}x'$, because of Lemma 4.2.4.

(3.23) Suppose $\Theta : y-x-y'$. Hence $\Theta : y-x-y'-x'$. Then with $\Pi\colon x-y'-x'-y$ we apply Lemma 4.2.3 to obtain not $v\mathrm{CCP}(x', y')$. It is excluded that $\Pi\colon x-y'-y-x'$, because of Lemma 4.2.4.

(3.3) Suppose $\Pi\colon x-y-y'$. Hence $\Pi\colon x-y-y'-x'$.

(3.31) Suppose $\Theta\colon y'{-}y{-}x$. With either $\Theta\colon y'{-}x'{-}y{-}x$ or $\Theta\colon y'{-}y{-}x'{-}x$, we obtain, by Lemma 4.2.3, not $v\mathrm{CCP}(x', y')$. It is excluded that $\Theta\colon y'{-}y{-}x{-}x'$, because of Lemma 4.2.4.

(3.32) Suppose $\Theta\colon y{-}y'{-}x$. With $\Theta\colon y{-}y'{-}x'{-}x$, we obtain, by Lemma 4.2.3, not $v\mathrm{CCP}(x', y')$. It is excluded that $\Theta : y-y'-x-x'$, because of Lemma 4.2.4.

(3.33) Suppose $\Theta\colon y-x-y'$. Hence $\Theta\colon y-x-y'-x'$, which is excluded because of Lemma 4.2.4. $\dashv$

Under the assumptions that x precedes y in Π, that $V(x, y) = \emptyset$, and that Π' is $\overline{\Pi}$, as for Bf-(x, y), we can prove the following version of Proposition 4.2.7.

PROPOSITION 4.2.8. *If* $(x', y') \in P_{\Pi',\Theta} - P_{\Pi,\Theta}$, *then* $V(x', y') \neq \emptyset$.

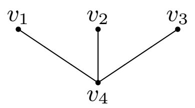
PROOF. Let the W-edges of D be $a_1, \ldots, a_n$ for $n \geq 2$. (If n where 1, then $V(x, y)$ would not be $\emptyset$.) Let B be the basic D-graph whose E-edges are $a_1, \ldots, a_n$, whose inner vertex is v, and whose W-edges are arbitrarily chosen so that $B \Box D$ is defined. Then a P-move Bf-(x, y) of D becomes Sf-(x, y) of $B \Box D$. Proposition 4.2.7 for that P-move Sf-(x, y) of $B \Box D$ yields Proposition 4.2.8 for D. ⊣

§4.3. Completeness of P-moves

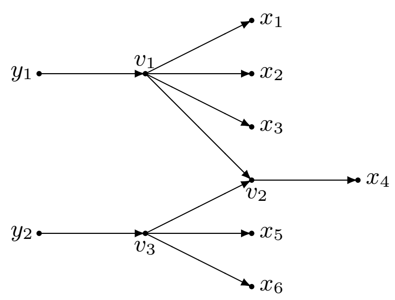
For a vertex v of D, let $\mathcal{C}_E(v) = \mathcal{C}(v) \cap E(D)$, where $\mathcal{C}(v)$ is the corolla of v defined at the beginning of §4.1. In the D-graph from the beginning of §4.1 we have, for example, $\mathcal{C}_E(v_1) = \mathcal{C}(v_1) = \{x_1, x_2, x_3\}$ and $\mathcal{C}_E(v_4) = \{x_1, \ldots, x_9\}$. Let U_E be the set of all the vertices v of D such that $\mathcal{C}_E(v)$ has at least two elements.

In the set-theoretic sense, a *tree* is a partially ordered set in which for every element the set of predecessors is well-ordered. (Such a tree need not have a single root.) It follows from Lemmata 3.2.1 and 3.2.4 that $\langle U_E, \prec \rangle$ is a finite tree. (This finite tree corresponds to a forest in the graph-theoretic sense; see [8], Chapter 4.)

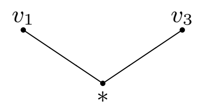
Let $\langle U_E^+, \prec \rangle$ be the single-rooted tree obtained by adding to U_E a new element $*$ and by assuming that for every v in U_E we have $* \prec v$. The new element $*$ has the same function as the inner vertex of B in the proof of Proposition 4.2.8. The tree $\langle U_E^+, \prec \rangle$ is interesting when $\langle U_E, \prec \rangle$ is not single-rooted. In our example above, the tree $\langle U_E, \prec \rangle$ is pictured by



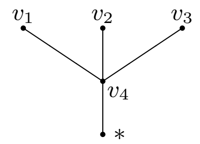
In the D-graph



the tree $\langle U_E, \prec\rangle$ would be made just of two roots v_1 and v_3, and $\langle U_E^+, \prec\rangle$ would be



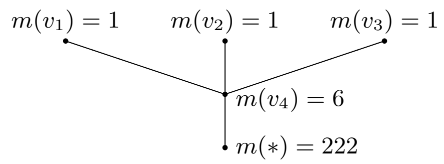
For every element v of U_E^+ let $S(v) = \{w \in U_E^+ \mid v \prec w\}$; i.e., $S(v)$ is the set of successors of v in the tree, not necessarily immediate successors; v is a *leaf* when $S(v)$ is empty. In the tree $\langle U_E^+, \prec\rangle$ obtained from the first $\langle U_E, \prec\rangle$ tree above



$S(*) = \{v_1, v_2, v_3, v_4\}$, $S(v_4) = \{v_1, v_2, v_3\}$, and v_1, v_2 and v_3 are leaves.

Let k_v be the cardinality of $\mathcal{C}_E(v)$. We assign inductively to every element v of U_E^+ a natural number $m(v) \geq 1$. For a leaf v, let $m(v)$ be 1, and, for other elements v of U_E^+, let $m(v)$ be $(\sum_{w\in S(v)} \binom{k_w}{2} \cdot m(w)) + 1$. Since $\sum_{w\in S(v)} \binom{k_w}{2} \cdot m(w)$ is 0 when $S(v) = \emptyset$, we compute $m(v)$ for a leaf v in the same way. The number $\binom{k_w}{2}$ is the number of pairs of distinct E-vertices in $\mathcal{C}(w)$.

For the last example for $\langle U_E^+, \prec\rangle$ above we have



$$\begin{aligned}
m(v_4) &= \tbinom{k_{v_1}}{2} \cdot m(v_1) + \tbinom{k_{v_2}}{2} \cdot m(v_2) + \tbinom{k_{v_3}}{2} \cdot m(v_3) + 1, \\
&= \tbinom{3}{2} \cdot 1 + \tbinom{2}{2} \cdot 1 + \tbinom{2}{2} \cdot 1 + 1 = 5 + 1 = 6, \\
m(*) &= \tbinom{k_{v_1}}{2} \cdot m(v_1) + \ldots + \tbinom{k_{v_4}}{2} \cdot m(v_4) + 1, \\
&= 5 + \tbinom{9}{2} \cdot 6 + 1 = 222.
\end{aligned}$$

For (x, y) a pair of distinct elements of $E(D)$, let

$$M(x,y) = \begin{cases} \min\{m(v) \mid v \in V(x,y)\}, & \text{when } V(x,y) \neq \emptyset \\ m(*), & \text{when } V(x,y) = \emptyset. \end{cases}$$

An alternative way to define $M(x, y)$ when $V(x, y) \neq \emptyset$ is to say that it is $m(v)$ for v such that $v\mathrm{CCP}(x, y)$; it yields the same number as the definition above.

Finally, we define $\mu_\Theta(\Pi)$ as $\sum_{(x,y)\in P_{\Pi,\Theta}} M(x, y)$. For the D-graph from the beginning of §4.1 and the lists Π and Θ given as an example in §4.2 after introducing the P-moves, we have

$$\begin{aligned} P_{\Pi,\Theta} = \{&(x_1, x_8), \dots, (x_7, x_8),\\ &(x_1, x_6), \dots, (x_5, x_6),\\ &(x_1, x_7), \dots, (x_5, x_7),\\ &(x_1, x_5), \dots, (x_4, x_5),\\ &(x_1, x_4), \dots, (x_3, x_4)\}. \end{aligned}$$

With Π' of Tr-(x_5, x_8), which we call Π'_{Tr}, we have

$$P_{\Pi'_{\mathrm{Tr}},\Theta} = P_{\Pi,\Theta} - \{(x_1, x_8), \dots, (x_7, x_8)\}.$$

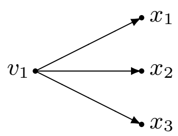
With Π' of Sf-(x_5, x_6), which we call Π'_{Sf}, we have

$$P_{\Pi'_{\mathrm{Sf}},\Theta} = \{(x_1, x_8), \dots, (x_7, x_8)\} \cup \{(x_7, x_6), (x_3, x_2), (x_3, x_1), (x_2, x_1)\}.$$

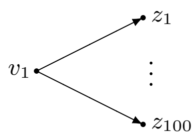
We then have

$$\begin{aligned} \mu_\Theta(\Pi) &= 7 \cdot 6 + 5 \cdot 6 + 5 \cdot 6 + (3 \cdot 6 + 1) + 3 \cdot 6 = 139,\\ \mu_\Theta(\Pi'_{\mathrm{Tr}}) &= 139 - 42 = 97,\\ \mu_\Theta(\Pi'_{\mathrm{Sf}}) &= 42 + 4 = 46. \end{aligned}$$

Note that if in our example we replace



by



and in Π and Θ we replace $x_1x_2x_3$ by $z_1 \dots z_{100}$, then $P_{\Pi,\Theta}$ has 509 elements, while $P_{\Pi'_{\mathrm{Sf}},\Theta}$ has 5055 elements, but $\mu_\Theta(\Pi) = 2\,516\,124$, while $\mu_\Theta(\Pi'_{\mathrm{Sf}}) = 520\,063$.

We make the same assumptions as for Proposition 4.2.7, and we prove the following.

PROPOSITION 4.3.1. *We have* $\sum_{(x',y')\in P_{\Pi',\Theta} - P_{\Pi,\Theta}} M(x', y') < M(x, y)$.

PROOF. We have that v, for which we have $v\mathrm{CCP}(x,y)$, is in U_E^+, because x and y are distinct members of a list. If $P_{\Pi',\Theta} - P_{\Pi,\Theta}$ is empty, then the sum on the left is 0; this is lesser than $M(x,y)$, which is $m(v)$, and is at least 1.

If $P_{\Pi',\Theta} - P_{\Pi,\Theta}$ is non-empty, then, by using Proposition 4.2.7 and Lemma 4.2.6, we conclude that for every pair (x',y') in it there is a w such that $w\mathrm{CCP}(x',y')$ and $w \in U_E^+$, because x' and y' are distinct members of a list, and $w \in S(v)$. We have

$$\begin{aligned}
&\textstyle\sum_{(x',y')\in P_{\Pi',\Theta}-P_{\Pi,\Theta}} M(x',y') \\
&\qquad = \textstyle\sum_{w\in S(v)}(\sum_{(x',y')\in P_{\Pi',\Theta}-P_{\Pi,\Theta}\ \&\ w\mathrm{CCP}(x',y')} M(x',y')) \\
&\qquad \leq \textstyle\sum_{w\in S(v)} \binom{k_w}{2} \cdot m(w), \text{ since the number of pairs } (x',y') \text{ in} \\
&\qquad\qquad\qquad\qquad P_{\Pi',\Theta} - P_{\Pi,\Theta} \text{ such that } w\mathrm{CCP}(x',y') \\
&\qquad\qquad\qquad\qquad \text{is lesser than or equal to } \binom{k_w}{2}, \\
&\qquad < (\textstyle\sum_{w\in S(v)} \binom{k_w}{2} \cdot m(w)) + 1 \\
&\qquad = M(x,y). \qquad \dashv
\end{aligned}$$

We also have Proposition 4.3.1 under the same assumptions as for Proposition 4.2.8. The proof is very much analogous, with $*$ functioning as v, and Proposition 4.2.8 replacing Proposition 4.2.7. If we have a P-move Tr-(x,y), then Proposition 4.2.2 guarantees that $P_{\Pi',\Theta}$ is a proper subset of $P_{\Pi,\Theta}$.

Then Proposition 4.2.2 and Proposition 4.3.1 in both versions, the Sf and Bf versions, yield the following proposition where Π and Π' are from any P-move.

PROPOSITION 4.3.2. *We have* $\mu_\Theta(\Pi') < \mu_\Theta(\Pi)$.

Then we can prove the following proposition, which says that P-moves are complete, in the sense that they enable us to pass from a grounded list to any other grounded list.

PROPOSITION 4.3.3. *If* Π *and* Θ *are grounded in* D*, then they are either the same or there is a finite sequence of P-moves* $P_1,\ldots,P_n$*, with* $n \geq 1$*, such that* Π *is the upper list of* P_1*, while* Θ *is the lower list of* P_n*, and for every* P_i *such that* $1 \leq i < n$ *we have that the lower list of* P_i *is the upper list of* P_{i+1}*.*

PROOF. If $\mu_\Theta(\Pi) = 0$, then $P_{\Pi,\Theta} = \emptyset$, and Π and Θ are the same. If $\mu_\Theta(\Pi) > 0$, then $P_{\Pi,\Theta} \neq \emptyset$, and Π and Θ are distinct.

The *distance* $d(a,b)$ between the distinct members a and b of a list A is the number of members of A between a and b. Among all the pairs in $P_{\Pi,\Theta}$ take a pair (x,y) with a minimal distance $d(x,y)$. We have the following possibilities.

Suppose $V(x,y) \neq \emptyset$ and let $v\mathrm{CCP}(x,y)$. This v exists by Lemma 4.2.6. We have two subcases. Suppose $[\![x]\!]_v \neq [\![y]\!]_v$. It follows from Lemma 4.1.5 that Π must be of the form $\Gamma\Lambda_v(x)\Xi\Lambda_v(y)\Delta$. That Ξ must be the empty list follows from our assumption about the minimality of $d(x,y)$. (If there were a member z in Ξ, then by the minimality of $d(x,y)$ we would have that (x,z) and (z,y) are not in $P_{\Pi,\Theta}$, from which it would follow that (x,y) is not in $P_{\Pi,\Theta}$.) Then we may apply Tr-(x,y).

If $[\![x]\!]_v = [\![y]\!]_v$, then we appeal again to Lemma 4.1.5, and we apply Sf-(x,y). If $V(x,y) = \emptyset$, then we apply Bf-(x,y). Our proof is formally an induction on $\mu_\Theta(\Pi)$, which relies on Proposition 4.3.2. $\dashv$

§4.4. P$''$-graphs are P$'$-graphs

In this section we will prove that every P$''$-graph is a P$'$-graph. For that we need some more preliminary results. (Here we use the notation V_W, V_E and V_C introduced in §1.3.)

LEMMA 4.4.1. *Suppose v and x are vertices of $D_W \Box D_E$ and $v \prec x$. If v is from D_W, then either* (1) $V_E \cap [\![x]\!]_v = \emptyset$ *or* (2) $V_E - V_C \subseteq [\![x]\!]_v$.

PROOF. Suppose $y \in V_E \cap [\![x]\!]_v$, and let $z \in V_E - V_C$. Then for some semipath σ of $D_W \Box D_E$ in $[y,z]$ we have not $v \rhd \sigma$, because D_E is weakly connected. Hence $z \in [\![x]\!]_v$. $\dashv$

LEMMA 4.4.2. *If there is a construction K of a P'-graph with the root list L_E, which is $\Gamma\Lambda_v(x)\Lambda_v(y)\Delta$, then there is a construction K' of the same P'-graph with the root list L'_E being $\Gamma\Lambda_v(y)\Lambda_v(x)\Delta$, while the root lists L_W and L'_W of K and K' respectively are the same.*

PROOF. We proceed by induction on the number of nodes in K. In the basis, when K has a single node, then in this node, which is the root of K, we have (B, L_W, L_E), and v is the inner vertex of the basic D-graph B. All petals with respect to v are singletons, and we pass from L_E to L'_E by one transposition of x and y. The construction K' has (B, L_W, L'_E) in its root.

For the induction step, we have that K is $K_W \Box K_E$, and the root graphs of K_W and K_E are D_W and D_E respectively.

(1) If the vertex v of the petals $[\![x]\!]_v$ and $[\![y]\!]_v$ is in D_E, then we just apply the induction hypothesis to K_E to obtain K'_E, and K' is $K_W \Box K'_E$. (Note that $[\![x]\!]_v$ in $D_W \Box D_E$ and $[\![x]\!]_v$ in D_E are here the same sets of vertices.)

If v is in D_W, then, according to Lemma 4.4.1, we have three possibilities.

(2) Suppose $V_E \cap [\![x]\!]_v = V_E \cap [\![y]\!]_v = \emptyset$. Then we apply the induction hypothesis to K_W to obtain K'_W, and K' is $K'_W \Box K_E$.

(3) Suppose $V_E - V_C \subseteq [\![x]\!]_v$. Then $V_E \cap [\![y]\!]_v = \emptyset$. Replace in $\Lambda_v(x)$ the subset L^E_E by the list Ξ, which is the common sublist of L^W_E and L^E_W, made of the vertices of V_C, and let the result of this replacement be $\Lambda_v(x_1)\dots\Lambda_v(x_k)$, where $k \geq 1$, with $\Lambda_v(x_i)$ for $i \in \{1,\dots,k\}$ being a list of an $[\![x_i]\!]^E_v$ for $[\![x_i]\!]_v$ a petal of D_W. The induction hypothesis allows us to make k applications of moves like P-moves of the Tr-(x_i, y) type to obtain K'_W, and K' will be $K'_W \Box K_E$. Note that Ξ coincides with L^E_W, which follows from $V_E - V_C \subseteq [\![x]\!]_v$.

This is important to ascertain that L'_W is L_W and that $(L^W_E)'$ and L^E_W are compatible. The case when $V_E - V_C \subseteq [\![y]\!]_v$ is treated analogously. $\dashv$

LEMMA 4.4.3. *If there is a construction K of a P'-graph with the root lists L_W and L_E, then there is a construction K' of the same P'-graph with the root lists L'_W and L'_E being respectively $\overline{L_W}$ and $\overline{L_E}$.*

PROOF. In every leaf of K replace (B, L_W, L_E) by $(B, \overline{L_W}, \overline{L_E})$. Formally, we have again an induction on the number of nodes in K, with the induction step trivial. $\dashv$

LEMMA 4.4.4. *If there is a construction K of a P'-graph with the root list L_E, which is $\Gamma\Lambda_v(z)\Delta$, then there is a construction K' of the same P'-graph with the root list L'_E being $\Gamma\overline{\Lambda_v(z)}\Delta$, while the root lists L_W and L'_W of K and K' respectively are the same.*

PROOF. We proceed by induction on the number of nodes in K. In the basis, when K has a single node, we take that K is just K'. This is because all petals are singletons, as in the proof of Lemma 4.4.2.

For the induction step, if v is in D_E, we proceed as in (1) of the proof of Lemma 4.4.2. Suppose v is in D_W. If we have $V_E \cap [\![x]\!]_v = \emptyset$, then we proceed as for (2) of the proof of Lemma 4.4.2.

Suppose $V_E - V_C \subseteq [\![x]\!]_v$. Replace in $\Lambda_v(x)$ the sublist L^E_E by the list Ξ, as in (3) of the proof of Lemma 4.4.2, and let the result be $\Lambda_v(x_1)\dots\Lambda_v(x_k)$, where $k \geq 1$. By the induction hypothesis, we obtain a construction with $\overline{\Lambda_v(x_1)}\dots\overline{\Lambda_v(x_k)}$, and then by Lemma 4.4.2 we have a construction K'_W with $\overline{\Lambda_v(x_k)}\dots\overline{\Lambda_v(x_1)}$ in its root.

By Lemma 4.4.3, we have a construction K'_E obtained by replacing the lists L^E_W and L^E_E of K_E by $\overline{L^E_W}$ and $\overline{L^E_E}$ respectively. Note that Ξ coincides with L^E_W, which follows from $V_E - V_C \subseteq [\![x]\!]_v$, and is important for the reasons mentioned in the proof of Lemma 4.4.2. Then the construction K' will be $K'_W \Box K'_E$. $\dashv$

It is clear that for all the results based on $\prec$, which is $\prec_W$, as at the end of §3.4, and obtained starting from §4.1 up to now, we have analogous results based on $\prec_E$, with completely analogous proofs. Then we can prove the following.

THEOREM 4.4.5. *For every P''-graph D there is a construction of a P'-graph with root graph D, which means that D is a P'-graph.*

PROOF. We proceed by induction on the number k of inner vertices of D. When k is 1, the theorem is trivial.

For the induction step, suppose D is $D_W \Box D_E$. So we have a list Θ_W of $E(D_W)$ grounded in D_W and a list Θ_E of $W(D_E)$ grounded in D_E, which are compatible. By the induction hypothesis we have the constructions K_W and K_E with root graphs D_W and D_E respectively.

Let the root lists L_W and L_E of K_X be respectively Π^X_W and Π^X_E. By Proposition 4.3.3 and Lemmata 4.4.2, 4.4.3 and 4.4.4, there is a finite sequence of constructions of the P′-graph D_W starting with K_W, for which L_E is Π^W_E, and ending with K'_W, for which the root list L'_E is Θ_W.

By analogous results, in an analogous manner, we obtain out of K_E a construction K'_E of the P′-graph D_E, for which the root list L'_W is Θ_E. Since Θ_W and Θ_E are compatible, we have that $K'_W \Box K'_E$ is a construction with root graph D. $\dashv$

Chapter 5

P$'''$-Graphs and P$''$-Graphs

§5.1. B$_m$-moves

In this chapter we will finish establishing that the three definitions of P-graph are equivalent by proving that every P$'''$-graph (as defined in §1.10) is a P$''$-graph (as defined in §1.9). For that we must first deal with some preliminary matters in this and in the next section. The present section is based on matters introduced in §4.2.

Let Π, which is $\Theta\Psi\Theta'$, be a list of $E(D)$. Let T be the set of members of Θ, while T' is the set of members of Θ'. Consider a set $F \subseteq T \cup T'$, and let F' be the relative complement of F with respect to $T \cup T'$. Let B be the set $(T \cap F') \cup (F \cap T')$, which amounts to the symmetric difference of the sets T and F.

For m a member of Ψ, we call B$_m$-moves the following rewrite rules from Π to Π', provided $x \in B$:

$$\begin{array}{ll}
\text{Tr-}(x,m) & \dfrac{\Gamma\Lambda_v(x)\Phi\Lambda_v(m)\Delta}{\Gamma\Phi\Lambda_v(m)\Lambda_v(x)\Delta} \\[2ex]
\text{Tr-}(m,x) & \dfrac{\Gamma\Lambda_v(m)\Phi\Lambda_v(x)\Delta}{\Gamma\Lambda_v(x)\Lambda_v(m)\Phi\Delta} \\[2ex]
\text{Sf-}(x,m) & \dfrac{\Gamma\Lambda_v(m)\Delta}{\Gamma\overline{\Lambda_v(m)}\Delta}
\end{array}$$

provided that in Sf-(x,m) we have that x is a member of $\Lambda_v(m)$ and

vCCP(x, m),

$$\text{Bf-}(x,m) \qquad \frac{\Pi}{\overline{\Pi}}$$

provided that in Bf-(x, m) we have that $V(x, m) = \emptyset$. Note that, as in §4.2, we can infer for Tr-(x, m) and Tr-(m, x) that vCCP(x, m).

From now on we assume that D is $D_W \Box D_E$, and let Ψ of $\Theta\Psi\Theta'$ be a list of $E(D_E)$. Hence x is an E-vertex of D_W. We may infer that v of Tr-(x, m), Tr-(m, x) and Sf-(x, m) is a vertex of D_W; otherwise, if v were a vertex of D_E, then, since D_W is weakly connected, for some σ in $[x]_W$ we would have not $v \rhd \sigma$, and so we would not have $v \prec x$, which is presupposed by $\Lambda_v(x)$. We may also infer that Ψ is a sublist of $\Lambda_v(m)$, by Lemma 4.4.1 for x being m. We may further infer that in Tr-(x, m) we have that $\Gamma\Lambda_v(x)\Phi$ is a sublist of Θ and Δ a sublist of Θ'; that in Tr-(m, x) we have that Γ is a sublist of Θ and $\Phi\Lambda_v(x)\Delta$ a sublist of Θ'; and that in Sf-(x, m) we have that Γ is a sublist of Θ and Δ a sublist of Θ'. With a proof analogous to the proof of Lemma 4.2.1, we establish that for every B$_m$-move, if Π is grounded in D, then Π' is grounded in D.

Let V_C be as in §1.3 for $D_W \Box D_E$, and let $k \in V_C$. Then we have the following lemma, which will help us to prove Proposition 5.1.4, a proposition that will play a similar role to Proposition 4.2.2.

Lemma 5.1.1. *For Π being the upper list of Tr-(x, m) or Tr-(m, x), let x' be a member of $\Lambda_v(x)$. Then not $\psi_E(x, k, x')$ in D_W.*

Proof. It is easy to infer that $v \prec x$ in D_W. Next we show that $k \notin [\![x]\!]^E_v$ in D_W. Otherwise, not $v \rhd [x, k]$ in D_W, which would yield not $v \rhd [x, m]$ in $D_W \Box D_E$, and this contradicts $m \notin [\![x]\!]^E_v$ in $D_W \Box D_E$. We show also that $x' \in [\![x]\!]^E_v$ in D_W. Otherwise, x and x' would be connected by a semipath with vertices of D_E in which v does not occur, and this would again contradict $m \notin [\![x]\!]^E_v$ in $D_W \Box D_E$. Hence by Lemma 4.1.5 we have not $\psi_E(x, k, x')$ in D_W. $\dashv$

Lemma 5.1.2. *For Π being the upper list of Sf-(x, m), let x' be, as x, an element of $T \cup T'$ and a member of $\Lambda_v(m)$, and assume that for every k in V_C we have $\psi_E(x, k, x')$ and $\psi_E(k, x, x')$ in D_W. Then not vCCP(x, m) in $D_W \Box D_E$.*

Proof. By Theorem 3.4.1 and Lemma 3.3.1 we obtain a vertex w of D_W

such that for every k in V_C

$$w \rhd [x', x] \quad \& \quad w \rhd [x', k] \quad \& \quad w \prec x \quad \& \quad w \prec k$$

in D_W. Since for every k in V_C we have $w \prec k$ in D_W, we infer $w \in V(x,m)$ in $D_W \Box D_E$. Since $v \in V(x,m)$ in $D_W \Box D_E$, we must have by Lemma 3.2.1 either $v = w$, or $w \prec v$, or $v \prec w$ in $D_W \Box D_E$. We show next that $v = w$ or $w \prec v$ implies a contradiction.

Since there is a semipath σ of $D_W \Box D_E$ in $[x', m]$ such that not $v \rhd \sigma$, because x' is a member of $\Lambda_v(m)$, there is a k' in V_C and a semipath σ' of D_W in $[x', k']$ such that not $v \rhd \sigma'$. Since for every k in V_C we have $w \rhd [x', k]$, we have that $w \rhd \sigma'$.

Since not $v \rhd \sigma'_{[x',w]}$, we have not $v \rhd [x', w]$, and since $v \prec x'$ in D_W, because $v \prec x'$ in $D_W \Box D_E$, which follows from x' being a member of $\Lambda_v(m)$, we conclude, by Lemma 4.1.1, that $v \prec w$ in D_W. This contradicts $v = w$ immediately, and it contradicts also $w \prec v$ in $D_W \Box D_E$, which implies $w \prec v$ in D_W; we rely on Lemma 3.2.2. Hence we must have $v \prec w$ in $D_W \Box D_E$, which implies not vCCP(x, m). $\dashv$

The following lemma is the analogue of Lemma 5.1.2 for Bf-(x, m).

Lemma 5.1.3. *For* Π *being the upper list of Bf-*(x, m)*, let* x' *be, as* x*, a member of* Π*, and assume that for every* k *in* V_C *we have* $\psi_E(x, k, x')$ *and* $\psi_E(k, x, x')$ *in* D_W*. Then* $V(x, m) \neq \emptyset$ *in* $D_W \Box D_E$*.*

The proof is as the proof of Lemma 5.1.2 until we reach the conclusion that $w \in V(x, m)$ in $D_W \Box D_E$.

Suppose we have the D-graphs D_1, D_2 and D_3 such that $D_1 \Box D_2$ and $D_1 \Box D_3$ are defined, but neither $D_2 \Box D_3$ nor $D_3 \Box D_2$ is defined. This is the situation analogous to what we had with (Ass 2.1) in §1.5. Alternatively, it is equivalent to suppose that $(D_1 \Box D_2) \Box D_3$ and $(D_1 \Box D_3) \Box D_2$ are defined.

Let Π, which is a list of $E(D_1 \Box D_2)$, be of the form $\Theta\Psi\Theta'$ for Ψ a list of $E(D_2)$; here D_1 and D_2 correspond respectively to what was above D_W and D_E. As before, the sets of vertices T and T' of $E(D_1)$ are respectively the sets of members of Θ and Θ'.

Let Σ be a list of $E(D_1 \Box D_3)$, which is of the form $\Omega\Xi\Omega'$ for Ξ a list of V_C, which is $E(D_1) \cap W(D_2)$, and for all the members of the list Ω being elements of $E(D_1)$. If the members of Ω are not all in $E(D_1)$, but some are elements of $E(D_3)$, then all the members of the list Ω' are elements of $E(D_1)$, and all that we do up to the end of §5.2 would be done in a dual manner, involving Θ' and Ω' instead of Θ and Ω. Let F be the set of members of Ω, and let T', F' and B be defined with respect to T and F as at the beginning of this section.

Our purpose now is to show that B_m-moves are complete, in the sense that they enable us to pass from any list Π of $E(D_1\Box D_2)$ grounded in $D_1\Box D_2$ to a list $\Theta\Psi\Theta'$ of $E(D_1\Box D_2)$ grounded in $D_1\Box D_2$ such that Θ is a list of F and Ψ is a list of $E(D_2)$. For that we assume that Σ is grounded in $D_1\Box D_3$.

For the propositions that follow we assume that Π is grounded in $D_1\Box D_2$ and that Σ is grounded in $D_1\Box D_3$. First, we have a proposition analogous to Proposition 4.2.2, which says that with B_m-moves of the kind Tr-(x,m) or Tr-(m,x) the set B diminishes, in a sense which will be made precise later (see Proposition 5.2.2).

PROPOSITION 5.1.4. *For Π being the upper list of Tr-(x,m) or Tr-(m,x), let x' be a member of $\Lambda_v(x)$. Then $x'\in B$.*

PROOF. By our assumption for B-moves, we have that $x\in B$. Suppose Π is the upper list of Tr-(x,m), and suppose $x'\notin B$. Suppose $x\in T\cap F'$. We have that x is a member of Θ, i.e., x precedes m in Π. Hence x' precedes m in Π by Lemma 4.1.5 and by the groundedness of Π. For every k in V_C we have $\Sigma\colon x'-k-x$, since $x\in B$ and $x'\notin B$. Since Σ is grounded in $D_1\Box D_3$, we have $\psi(x',k,x)$ in $D_1\Box D_3$. By Lemma 3.1.6 we have $\psi_E(x',k,x)$ in D_1, which contradicts Lemma 5.1.1.

If $x\in F\cap T'$, then we proceed analogously, and obtain again a contradiction with Lemma 5.1.1. Hence $x'\in B$. We proceed analogously when Π is the upper list of Tr-(m,x). ⊣

Next we have a proposition related to Proposition 4.2.7.

PROPOSITION 5.1.5. *For Π being the upper list of Sf-(x,m), let x' be, as x, an element of $T\cup T'$ and a member of $\Lambda_v(m)$, and suppose $x'\notin B$. If*

(1) $\Pi\colon x'-x-m$, *or*

(2) $\Pi\colon x-m-x'$,

then not vCCP(x,m) in $D_1\Box D_2$, and if

(3) $\Pi\colon x-x'-m$,

then not vCCP(x',m) in $D_1\Box D_2$.

PROOF. We prove first the implication from (1) or (2) to not vCCP(x,m) in $D_1\Box D_2$.

(I) Suppose $x \notin E(D_1) \cap W(D_3)$, and suppose we have (1). Then for every k in V_C we have $\Sigma : x' - k - x$, since $x \in B$ and $x' \notin B$. Since Σ is grounded in $D_1 \Box D_3$, we have $\psi_E(x', k, x)$ in $D_1 \Box D_3$. By Lemma 3.1.6 we obtain $\psi_E(x', k, x)$ in D_1. From (1), and the groundedness of Π in $D_1 \Box D_2$, we infer $\psi_E(x', x, k)$ in D_1 by Lemma 3.1.4. Then by Lemma 5.1.2 we obtain that not $v\mathrm{CCP}(x, m)$ in $D_1 \Box D_2$. If we have (2), then we proceed analogously, with Lemma 3.1.4 replaced by Lemma 3.1.5.

(II) Suppose $x' \in E(D_1) \cap W(D_3)$. Suppose we have (1). Then for every k in V_C and for some w in $E(D_3)$ we have $\Sigma : w - k - x$, since $x \in B$ and $x' \notin B$. Since Σ is grounded in $D_1 \Box D_3$, we have $\psi_E(w, k, x)$ in $D_1 \Box D_3$. By Lemma 3.1.4 we obtain $\psi_E(x', k, x)$ in D_1. After that we proceed as in (I) to show that not $v\mathrm{CCP}(x, m)$ in $D_1 \Box D_2$. If we have (2), then we proceed analogously, with Lemma 3.1.4 replaced by Lemma 3.1.5.

To prove the implication from (3) to not $v\mathrm{CCP}(x', m)$ in $D_1 \Box D_2$, we proceed analogously to what we had with (1) above. Instead of Lemma 5.1.2, we now apply the lemma obtained from Lemma 5.1.2 by interchanging x and x'. $\dashv$

Finally, we have a proposition related to Proposition 4.2.8.

PROPOSITION 5.1.6. *For Π being the upper list of Bf-(x, m), let x' be, as x, a member of Π, and suppose $x' \notin B$. If* (1) *or* (2) *of Proposition 5.1.5, then $V(x, m) \neq \emptyset$ in $D_1 \Box D_2$, and if* (3) *of Proposition 5.1.5, then $V(x', m) \neq \emptyset$ in $D_1 \Box D_2$.*

The proof is as for Proposition 5.1.5 by relying on Lemma 5.1.3 instead of Lemma 5.1.2.

§5.2. Completeness of B_m-moves

Let D be a D-graph $D_1 \Box D_2$. With the tree $\langle U_E^+, \prec \rangle$ of this D-graph, we define $m(v)$ and $M(x, y)$ as in §4.3. As a matter of fact, we could now modify the definition of $m(v)$ by replacing $\binom{k_w}{2}$ in it by k_w, or by k_w diminished by the number of vertices in $E(D_2)$. We write $M(x)$ as an abbreviation for $M(x, m)$, where m is the vertex involved in our B_m-moves. We define μ_B, which is analogous to $\mu_\Theta(\Pi)$, as $\sum_{x \in B} M(x)$.

With D being $D_1 \Box D_2$, for every B_m-move, Π' is the lower list, which, as Π, may be conceived as being of the form $\Theta\Psi\Theta'$ for Ψ a list of $E(D_2)$. This is because, as we remarked after introducing the B_m-moves, the sublist Ψ

of Π is a sublist of $\Lambda_v(m)$. Let B' be defined for this Π', as B was defined for Π, namely as the symmetric difference of T and F, with T being the set of members of the sublist Θ of Π', and F being as for B the set of members of Ω (see the assumptions concerning Σ before Proposition 5.1.4).

We can prove the following proposition analogous to Proposition 4.3.1. Assume for that proposition that Π and Π' are as for an Sf-(x, m) move.

PROPOSITION 5.2.1. *We have* $\sum_{x'\in B'-B} M(x') < M(x)$.

PROOF. As a corollary of Proposition 5.1.5 we may ascertain that if $x' \in B'-B$, then not $v\mathrm{CCP}(x', m)$, a proposition analogous to Proposition 4.2.7. Note first that $x' \in B' - B$ implies that x' is an element of $T \cup T'$ and a member of $\Lambda_v(m)$. We also have, of course, $x' \notin B$. Since we have a Sf-(x, m) move, we have $v\mathrm{CCP}(x, m)$, and hence (1) and (2) of Proposition 5.1.5 are impossible. The only remaining possibility is (3), which yields not $v\mathrm{CCP}(x', m)$.

We may then continue reasoning as in the proof of Proposition 4.2.7. (Now, y' is either omitted or replaced by m.) $\dashv$

By relying on Proposition 5.1.4 and Proposition 5.1.6, and by reasoning in a manner analogous to what we had before Proposition 4.3.2, we obtain the following for any B_m-move.

PROPOSITION 5.2.2. *We have* $\mu(\Pi') < \mu(\Pi)$.

Then we can prove that B_m-moves are complete as explained before Proposition 5.1.4. The proof of this completeness proceeds as the proof of Proposition 4.3.3. (We replace y by m, and disregard matters concerning the distance $d(x, m)$.) So we may assume that in the list Π, which is $\Theta\Psi\Theta'$, of $E(D_1\Box D_2)$ grounded in $D_1\Box D_2$, the members of Θ make F; i.e., the members of Θ and Ω are the same.

Out of this list Π we make the list Π_Ξ of $E(D_1)$ by replacing Ψ by the Ξ of Σ; the list Σ, which is $\Omega\Xi\Omega'$, is a list of $E(D_1\Box D_3)$ grounded in $D_1\Box D_3$. So Π_Ξ is $\Theta\Xi\Theta'$.

For Π_Ξ, with the assumptions that the members of Θ make F, as the members of Ω do, we can prove the following.

PROPOSITION 5.2.3. *The list* Π_Ξ *is grounded in* D_1.

PROOF. Suppose $\Pi_\Xi : x-y-z$. We have the following cases.

If $x, y, z \notin V_C$, where $V_C = E(D_1) \cap W(D_2)$, then we appeal to the groundedness of Π in $D_1\Box D_2$ and to Lemma 3.1.6 to obtain that $\psi_E(x,y,z)$ in D_1.

If $x \in V_C$ and $y,z \notin V_C$, then for any m in $E(D_E)$ we have that $\Pi: m{-}y{-}z$, since $\Pi_\Xi: x{-}y{-}z$. Hence $\psi_E(m,y,z)$ in $D_1\Box D_2$, and, by Lemma 3.1.4, we have that $\psi_E(x,y,z)$ in D_1.

If $y \in V_C$ and $x,z \notin V_C$, then we proceed analogously by applying Lemma 3.1.5 to obtain $\psi_E(x,y,z)$ in D_1.

If $z \in V_C$ and $x,y \notin V_C$, then we reason as when $x \in V_C$ and $y,z \notin V_C$.

If $y,z \in V_C$ and $x \notin V_C$, with $x \in F$, then $\Sigma: x{-}y{-}z$, since $\Pi_\Xi: x{-}y{-}z$. Since Σ is grounded in $D_1\Box D_3$, by Lemma 3.1.6, we obtain that $\psi_E(x,y,z)$ in D_1.

If $y,z \in V_C$ and $x \notin V_C$, with $x \notin F$, then z precedes y in Σ, because $\Pi_\Xi: z{-}y{-}x$. The vertex x, which is not in F, is a member of Θ'. We have two subcases.

If $x \notin E(D_1) \cap W(D_3)$, then x is a member of Ω', and then y precedes x in Σ. So we have $\Sigma: z{-}y{-}x$, and, by the groundedness of Σ in $D_1\Box D_3$ and by Lemma 3.1.6, we obtain $\psi_E(z,y,x)$ in D_1.

If $x \in E(D_1) \cap W(D_3)$, then there is a vertex w in $E(D_3)$ such that y precedes w in Σ; since Θ and Ω have the same members, w cannot be a member of Ω, and is hence a member of Ω'. So we have $\Sigma: z{-}y{-}w$ and, by the groundedness of Σ in $D_1\Box D_3$ and by Lemma 3.1.4, we obtain $\psi_E(z,y,x)$ in D_1. This concludes the case when $y,z \in V_C$ and $x \notin V_C$. The case when $x,y \in V_C$ and $z \notin V_C$ is treated analogously.

The final case is when $x,y,z \in V_C$. Then we rely on the groundedness of Σ in $D_1\Box D_3$ and on Lemma 3.1.6 to obtain that $\psi_E(x,y,z)$ in D_1. $\dashv$

§5.3. P$'''$-graphs are P$''$-graphs

In this section we will prove the assertion that is in its title.

Suppose both $D_1\Box(D_2\Box D_3)$ and $(D_1\Box D_2)\Box D_3$ are defined, i.e., stand for a D-graph. This is analogous to what we have with (Ass 1) (see §1.5). Then we can prove the following.

PROPOSITION 5.3.1.1. *If D_1 and $D_2\Box D_3$ are P-compatible, then D_1 and D_2 are P-compatible.*

PROOF. Suppose a list Λ_W of $E(D_1)$ grounded in D_1 and a list Λ_E of $W(D_2\Box D_3)$ grounded in $D_2\Box D_3$ are compatible. By Lemma 3.1.8, we conclude that it is impossible that for some x and z in $W(D_2)$ and some y in $W(D_3)$ we have $\Lambda_E : x - y - z$.

Remove from Λ_E all the W-vertices of $D_2 \Box D_3$ that belong to $W(D_3)$. The resulting list Λ'_E is a list of $W(D_2)$ compatible with Λ_W by Lemma 3.1.1. (Note that since $D_1 \Box D_2$ is defined, there must be in Λ'_E a member of Λ_W.) By Lemma 3.1.3, this list is grounded in D_2, because Λ_E was grounded in $D_2 \Box D_3$. $\dashv$

With the same assumptions as above Proposition 5.3.1.1, we establish the following in an analogous manner by using Lemmata 3.1.8, 3.1.1 and 3.1.3.

PROPOSITION 5.3.1.2. *If $D_1 \Box D_2$ and D_3 are P-compatible, then D_2 and D_3 are P-compatible.*

Suppose both $(D_1 \Box D_2) \Box D_3$ and $(D_1 \Box D_3) \Box D_2$ are defined. This is analogous to what we have with (Ass 2.1) (see §1.5). Then we can prove the following.

PROPOSITION 5.3.2.1. *If $D_1 \Box D_2$ and D_3 are P-compatible, and if $D_1 \Box D_3$ and D_2 are P-compatible, then D_1 and D_2, as well as D_1 and D_3, are P-compatible.*

PROOF. Assume for Π and Σ all that was assumed for them before Proposition 5.1.4. Assume moreover that there is a list Φ of $W(D_2)$ grounded in D_2 such that Σ and Φ are compatible. Because we have assumed that the members of Ω are elements of $E(D_1)$, we have that Φ is of the form $\Phi'\Xi$, since $E(D_3) \neq \emptyset$.

According to what we concluded after Proposition 5.2.2, we may assume that the members of Θ and Ω are the same. Then by applying Proposition 5.2.3 we obtain that the list Π_Ξ of $E(D_1)$ is grounded in D_1.

If Φ' is the empty list, then Π_Ξ, which is $\Theta\Xi\Theta'$, and Φ, which is Ξ, are compatible. If Φ' is not empty, then Ω must be empty, and hence Θ is empty. It follows that Π_Ξ, which is $\Xi\Theta'$, and Φ, which is $\Phi'\Xi$, are compatible. So Π_Ξ and Φ are compatible.

If we assume that the members of Ω', instead of those of Ω, are elements of $E(D_1)$, we proceed in a dual manner, making the members of Θ' and Ω' coincide, instead of those of Θ and Ω. So D_1 and D_2 are P-compatible. The proof that D_1 and D_3 are P-compatible is obtained by renaming. $\dashv$

By a general dualizing of all that we had done to prove Proposition 5.3.2.1 we may prove the following. Suppose $D_1\Box(D_2\Box D_3)$ and $D_2\Box(D_1\Box D_3)$ are defined. This is analogous to what we have with (Ass 2.2) (see §1.5).

PROPOSITION 5.3.2.2. *If D_1 and $D_2\Box D_3$ are P-compatible, and if D_2 and $D_1\Box D_3$ are P-compatible, then D_1 and D_3, as well as D_2 and D_3, are P-compatible.*

We can now prove the following.

THEOREM 5.3.3. *Every P'''-graph is a P''-graph.*

PROOF. We proceed by induction on the number of inner vertices of a P$'''$-graph D. In the basis, when D has a single inner vertex, it is a basic D-graph, and we are done. For the induction step, suppose D is $D_W\Box D_E$. We will prove that D_W and D_E are P$'''$-graphs.

Take D_W. If D_W has no cocycles, then it is trivially a P$'''$-graph. If it has a cocycle, then take an arbitrary cocycle of D_W, and assume D_W is $D'_W\Box D''_W$. Then we have two cases.

If $D''_W\Box D_E$ is defined, then, since D is a P$'''$-graph, D'_W and $D''_W\Box D_E$ are P-compatible. By Proposition 5.3.1.1 we obtain that D'_W and D''_W are P-compatible.

If $D''_W\Box D_E$ is not defined, then $D'_W\Box D_E$ is defined, and, since D is a P$'''$-graph, $D'_W\Box D_E$ and D''_W are P-compatible. Since $D'_W\Box D''_W$ and D_E are P-compatible, for the same reason, we conclude by Proposition 5.3.2.1 that D'_W and D''_W are P-compatible. So D_W is a P$'''$-graph, and by the induction hypothesis it is a P$''$-graph. We conclude analogously by relying on Propositions 5.3.1.2 and 5.3.2.2 that D_E is a P$''$-graph, and by the inductive clause of the definition of P$''$-graphs, since D_W and D_E are P-compatible by the assumption that D is a P$'''$-graph, we obtain that $D_W\Box D_E$ is a P$''$-graph. ⊣

By Theorem 2.3.6, Theorem 4.4.5 and the theorem we have just proven, we conclude that the notions of P$'$-graph, P$''$-graph and P$'''$-graph define the same class of D-graphs, which we call simply P-graphs.

Chapter 6

The Systems S1 and S2

§6.1. The system $\mathrm{S}\Box_P$

In this chapter we put juncture into a wider context, which from the point of view of 2-categories involves, besides vertical composition, the remaining operations on 2-cells—horizontal composition and identity 2-cells. As a preliminary, we show in this section that the equations of $\mathrm{S}\Box$ are complete not only with respect to D-graphs, as we proved in §1.6, but also with respect to P-graphs. We introduce next the system S1, which is an extension of $\mathrm{S}\Box$ with unit terms and appropriate axiomatic equations. This system is proven equivalent, i.e. homomorphically intertranslatable, with the system S2, which has operations corresponding to the standard operations on 2-cells—viz., vertical composition, horizontal composition and identity 2-cells—and as axiomatic equations the standard assumptions for 2-categories. The systems S1 and S2 are then proven complete with respect to interpretations in appropriate kinds of graphs. For S1 these are graphs based on P-graphs, while for S2 these are graphs, dual in a certain sense, which correspond to the usual diagrams of category theory, and which will be called M-graphs. The duality in question will be investigated in more detail in §7.6, but it is already described in this chapter by our completeness results for S1 and S2, and the equivalence of these two systems.

The system $\mathrm{S}\Box_P$ will not differ essentially from $\mathrm{S}\Box$. Its equations will be of the same form, but, instead of equations between D-terms, they will now be equations between what we will call P-terms. These P-terms are also of the same form as D-terms, but they will be interpretable in P-graphs, and not in any D-graph.

A P-term will be a D-term (see §1.5) δ for which, in addition to the functions W, E and A, we provide two functions $\mathcal{L}_W$ and $\mathcal{L}_E$ such that $\mathcal{L}_X$ is a list (see §1.7) of $X(\delta)$, for X being W or E. The ordered pair $(\mathcal{L}_W(\delta), \mathcal{L}_E(\delta))$ is the *sequential type* of the P-term δ. The equations of the system $\mathrm{S}\Box_P$ will be equations between P-terms of the same edge type (see §1.5) and the same sequential type.

We define *P-terms* inductively by starting from a set of *basic P-terms*, which are basic D-terms β (these are atomic symbols; see §1.5), and we have that $\mathcal{L}_X(\beta)$ is an arbitrary list of $X(\beta)$. Next we have the following inductive clause:

> if δ_W of sequential type $(\Lambda_W^W, \Phi_E\Xi\Psi_E)$ and δ_E of sequential type $(\Phi_W\Xi\Psi_W, \Lambda_E^E)$ are P-terms, then $\delta_W\Box\delta_E$ is a P-term of sequential type $(\Phi_W\Lambda_W^W\Psi_W, \Phi_E\Lambda_E^E\Psi_E)$, provided that $\Phi_E\Xi\Psi_E$ and $\Phi_W\Xi\Psi_W$ are compatible (see §1.7) and Ξ is a list of $C =_{df} A(\delta_W) \cap A(\delta_E) = E(\delta_W) \cap W(\delta_E) \neq \emptyset$.

The condition concerning C ensures that $\delta_W\Box\delta_E$ is a D-term. We define the values of W, E and A for the argument $\delta_W\Box\delta_E$ as we did for the definition of D-term in §1.5. (As before, we take the outermost parentheses of $\delta_W\Box\delta_E$ for granted.)

The system $\mathrm{S}\Box_P$ is defined as $\mathrm{S}\Box$ (see §1.5) with "P-term" substituted for "D-term". We can prove the following.

PROPOSITION 6.1.1. *If $\delta = \delta'$ is derivable in* $\mathrm{S}\Box$*, then δ is a P-term of sequential type (Γ, Δ) iff δ' is a P-term of sequential type (Γ, Δ).*

PROOF. We proceed by induction on the length of derivation of $\delta = \delta'$ in $\mathrm{S}\Box$. If δ' is δ, then we are done. If $\delta = \delta'$ is an instance of (Ass 1), (Ass 2.1) or (Ass 2.2), then we rely on lemmata analogous to Lemmata 2.3.1.1, 2.3.1.2, 2.3.2.1 and 2.3.2.2. For example, the lemma analogous to Lemma 2.3.1.1 says that if $\delta_1\Box\delta_2$ and $(\delta_1\Box\delta_2)\Box\delta_3$ are P-terms, and $\delta_2\Box\delta_3$ is a D-term, then $\delta_2\Box\delta_3$ and $\delta_1\Box(\delta_2\Box\delta_3)$ are P-terms. The proof of that is analogous to the proof of Lemma 2.3.1.1, and likewise for the analogues for the other lemmata.

With that we have proven the basis of the induction. The induction step, which involves the symmetry and transitivity of $=$, and congruence with $\Box$, is straightforward. $\dashv$

PROPOSITION 6.1.2. *For δ and δ' being P-terms, $\delta = \delta'$ is derivable in* $\mathrm{S}\Box$ *iff $\delta = \delta'$ is derivable in* $\mathrm{S}\Box_P$*.*

Proof. The implication from right to left is trivial. From left to right we proceed by induction on the length of derivation of $\delta = \delta'$ in S$\Box$. If $\delta = \delta'$ is an axiomatic equation of S$\Box$, then it is an axiomatic equation of S$\Box_P$ as well. If $\delta = \delta'$ is derived in S$\Box$ by the symmetry of $=$, or by the congruence with $\Box$, then we proceed easily by applying the induction hypothesis. The only more difficult case is when $\delta = \delta'$ is derived in S$\Box$ by the transitivity of $=$ from $\delta = \delta''$ and $\delta'' = \delta'$. Then, by Proposition 6.1.1, we have that δ'' is a P-term, and, by the induction hypothesis, we obtain that $\delta = \delta''$ and $\delta'' = \delta'$ are derivable in S$\Box_P$. Hence $\delta = \delta'$ is derivable in S$\Box$. ⊣

The following proposition is proven by a straightforward induction on the number of occurrences of $\Box$ in δ.

Proposition 6.1.3. *If δ is a P-term, then there is a construction with $(\iota(\delta), \mathcal{L}_W(\delta), \mathcal{L}_E(\delta))$ in its root.*

As a corollary of Proposition 6.1.3, and of the fact that P-graphs may be defined as P′-graphs, we have the following.

Proposition 6.1.4. *If δ is a P-term, then $\iota(\delta)$ is a P-graph.*

From Theorem 1.6.4, the completeness theorem for S$\Box$, with the help of Propositions 6.1.2 and 6.1.4, we obtain the following completeness theorem.

Theorem 6.1.5. *In S$\Box_P$ we can derive $\delta = \delta'$ iff the P-graphs $\iota(\delta)$ and $\iota(\delta')$ are the same.*

§6.2. The system S1

The functions W, E and A associated with D-terms, map D-terms into the power set of an infinite set, which we will now call $\mathcal{A}$. (Intuitively, $\mathcal{A}$ is the set of all possible edges.)

Let a *unit term* be $\mathbf{1}_\Gamma$ where Γ is a list (see §1.7) of some elements of $\mathcal{A}$. We stipulate that $W(\mathbf{1}_\Gamma) = E(\mathbf{1}_\Gamma) = A(\mathbf{1}_\Gamma) = \Gamma^s$, which is the finite (possibly empty) set of the members of Γ, and we stipulate that $\mathcal{L}_W(\mathbf{1}_\Gamma) = \mathcal{L}_E(\mathbf{1}_\Gamma) = \Gamma$.

The P1-terms we are now going to define have, as P-terms (see §6.1), the functions W, E, A, $\mathcal{L}_W$ and $\mathcal{L}_E$ associated with them, subject to the same conditions, and their sequential types are defined analogously. The equations of the system S1 will be equations between P1-terms of the same edge type (see §1.5) and the same sequential type (see §6.1).

We define *P1-terms* inductively by starting from a set of *atomic P1-terms*, which are either basic P-terms (see §6.1) or unit terms. We have an inductive clause for P1-terms involving $\Box$, which is obtained by substituting "P1-term" for "P-term" in the inductive clause of the definition of P-term (see §6.1), and with this clause we define the values of W, E and A for the argument $\delta_W \Box \delta_E$ as we did for the definition of D-term in §1.5. We also have one more inductive clause involving $\Box$:

> if δ_W of sequential type (Γ_W, Δ_W) and δ_E of sequential type (Γ_E, Δ_E) are P1-terms, and $A(\delta_W)$ or $A(\delta_E)$ is empty, then $\delta_W \Box \delta_E$ is a P1-term of sequential type $(\Gamma_W \Gamma_E, \Delta_W \Delta_E)$.

In this case, for Z being one of W, E and A, we define $Z(\delta_W \Box \delta_E)$ as $Z(\delta_W) \cup Z(\delta_E)$. This concludes the definition of P1-term.

Note that in the second inductive clause above we must have that $(\Gamma_W \Gamma_E, \Delta_W \Delta_E)$ is either (Γ_W, Δ_W) or (Γ_E, Δ_E).

The system S1 is defined as S$\Box$ (see §1.5) with "P1-term" substituted for "D-term" and with the following additional axiomatic equations for δ of sequential type $(\Gamma_1 \Phi \Gamma_2, \Delta_1 \Psi \Delta_2)$:

(**1**1) $\quad \mathbf{1}_\Phi \Box \delta = \delta = \delta \Box \mathbf{1}_\Psi,$

for Γ_1 and Δ_1 empty,

(**12**L) $\quad \mathbf{1}_{\Theta\Phi} \Box \delta = \delta \Box \mathbf{1}_{\Theta\Psi},$

for Γ_2 and Δ_2 empty,

(**12**R) $\quad \mathbf{1}_{\Phi\Theta} \Box \delta = \delta \Box \mathbf{1}_{\Psi\Theta}.$

This defines the system S1.

The equations (**12**L) and (**12**R) could be replaced by two of their instances: the equation (**12**LΦ), which is (**12**L) with Γ_2 empty, and the equation(**12**RΦ), which is (**12**R) with Γ_1 empty. We can write down these new equations as follows, for δ of sequential type $(\Gamma_1 \Phi \Gamma_2, \Delta_1 \Psi \Delta_2)$:

for Γ_1 and Δ_1 empty,

(**12**LΦ) $\quad \mathbf{1}_{\Theta\Phi\Gamma_2} \Box \delta = \delta \Box \mathbf{1}_{\Theta\Psi},$

for Γ_2 and Δ_2 empty,

(**12**RΦ) $\quad \mathbf{1}_{\Gamma_1\Phi\Theta} \Box \delta = \delta \Box \mathbf{1}_{\Psi\Theta}.$

By taking δ in (**12**LΦ) to be $\mathbf{1}_{\Phi\Gamma_2}$, and by using (**1**1), we obtain

$$\mathbf{1}_{\Theta\Phi\Gamma_2} = \mathbf{1}_{\Phi\Gamma_2} \Box \mathbf{1}_{\Theta\Phi}.$$

From this equation, by using ($\mathbf{1}$1) and (Ass 1), we obtain

$$\mathbf{1}_{\Theta\Phi\Gamma_2} = \mathbf{1}_{\Theta\Phi}\Box\mathbf{1}_{\Phi\Gamma_2}.$$

Hence, after renaming Γ_2 into Γ, we have

$$(\mathbf{1}\Box\mathbf{1}) \qquad \mathbf{1}_{\Theta\Phi}\Box\mathbf{1}_{\Phi\Gamma} = \mathbf{1}_{\Theta\Phi\Gamma} = \mathbf{1}_{\Phi\Gamma}\Box\mathbf{1}_{\Theta\Phi}.$$

Then we derive ($\mathbf{1}$2L) as follows:

$$\begin{aligned} \mathbf{1}_{\Theta\Phi}\Box\delta &= \mathbf{1}_{\Theta\Phi}\Box(\mathbf{1}_{\Phi\Gamma_2}\Box\delta), && \text{by } (\mathbf{1}1), \\ &= \mathbf{1}_{\Theta\Phi\Gamma_2}\Box\delta, && \text{by (Ass 1) and } (\mathbf{1}\Box\mathbf{1}), \\ &= \delta\Box\mathbf{1}_{\Theta\Psi}, && \text{by } (\mathbf{1}2\mathrm{L}\Phi). \end{aligned}$$

We derive analogously ($\mathbf{1}$2R) by using ($\mathbf{1}$2RΦ). Hence ($\mathbf{1}$2LΦ) and ($\mathbf{1}$2RΦ) can replace ($\mathbf{1}$2L) and ($\mathbf{1}$2R).

An alternative is to replace ($\mathbf{1}$2L) and ($\mathbf{1}$2R) by their instances ($\mathbf{1}$2LΨ) and ($\mathbf{1}$2RΨ), in which we have, respectively, Δ_2 and Δ_1 empty. We could write down these new equations as follows, for δ, as before, of sequential type $(\Gamma_1\Phi\Gamma_2, \Delta_1\Psi\Delta_2)$:

for Γ_1 and Δ_1 empty,

$$(\mathbf{1}2\mathrm{L}\Psi) \qquad \mathbf{1}_{\Theta\Phi}\Box\delta = \delta\Box\mathbf{1}_{\Theta\Psi\Delta_2},$$

for Γ_2 and Δ_2 empty,

$$(\mathbf{1}2\mathrm{R}\Psi) \qquad \mathbf{1}_{\Phi\Theta}\Box\delta = \delta\Box\mathbf{1}_{\Delta_1\Psi\Theta}.$$

Still other alternatives are to replace ($\mathbf{1}$2L) and ($\mathbf{1}$2R) by ($\mathbf{1}$2LΦ) and ($\mathbf{1}$2RΨ), or by ($\mathbf{1}$2LΨ) and ($\mathbf{1}$2RΦ).

In S1 one of (Ass 2.1) and (Ass 2.2) is superfluous as an axiom; it is derivable in the presence of the other. Here is a derivation of (Ass 2.2):

$$\begin{aligned} \delta_1\Box(\delta_2\Box\delta_3) &= ((\mathbf{1}_{\mathcal{L}_W(\delta_1\Box(\delta_2\Box\delta_3))}\Box\delta_1)\Box\delta_2)\Box\delta_3, \text{ by } (\mathbf{1}1) \text{ and (Ass 1)}, \\ &= ((\mathbf{1}_{\mathcal{L}_W(\delta_1\Box(\delta_2\Box\delta_3))}\Box\delta_2)\Box\delta_1)\Box\delta_3, \text{ by (Ass 2.1)}, \\ &= \delta_2\Box(\delta_1\Box\delta_3), \text{ by (Ass 1) and } (\mathbf{1}1). \end{aligned}$$

§6.3. The system S2

The P2-terms we are now going to define have, as P-terms and P1-terms (see §6.1 and §6.2), the functions W, E, A, $\mathcal{L}_W$ and $\mathcal{L}_E$ associated with them, subject to the same conditions, and their sequential types are defined analogously. The equations of the system S2 will be equations between

P2-terms of the same edge type (see §1.5) and the same sequential type (see §6.1).

We define *P2-terms* inductively by starting from a set of *atomic* P2-terms, which are the same as the atomic P1-terms; i.e., they are either basic P-terms (see §6.1) or unit terms (see §6.2). We have the following two inductive clauses:

> if δ_W of sequential type (Λ_W^W, Ξ) and δ_E of sequential type (Ξ, Λ_E^E) are P2-terms, then $\delta_W \circ \delta_E$ is a P2-term of sequential type $(\Lambda_W^W, \Lambda_E^E)$, provided Ξ is a list of $C =_{df} A(\delta_W) \cap A(\delta_E) = E(\delta_W) = W(\delta_E)$;

for X being W or E, we have $X(\delta_W \circ \delta_E) = X(\delta_X)$, and $A(\delta_W \circ \delta_E) = A(\delta_W) \cup A(\delta_E)$;

> if δ_N of sequential type (Γ_N, Δ_N) and δ_S of sequential type (Γ_S, Δ_S) are P2-terms, then $\delta_N \otimes \delta_S$ is a P2-term of sequential type $(\Gamma_N\Gamma_S, \Delta_N\Delta_S)$, provided $A(\delta_N)$ and $A(\delta_S)$ are disjoint;

for Z being one of W, E and A, we define $Z(\delta_N \otimes \delta_S)$ as $Z(\delta_N) \cup Z(\delta_S)$. This concludes the definition of P2-term. (The reason for using in the second clause of this definition the indices N and S, rather than 1 and 2, will become apparent in clause (2$\otimes$) of the definition of M-graph in §6.6, an later on.)

Note that in the first inductive clause above, for $\circ$, we may have C also empty, but, with the atomic P2-terms at our disposal, this will happen only if $A(\delta_W)$ and $A(\delta_E)$ are both empty. With that, we obtain P2-terms like $(\mathbf{1}_\Lambda \circ \mathbf{1}_\Lambda) \circ \mathbf{1}_\Lambda$ for Λ the empty list. With our atomic P2-terms, we cannot have $\delta_W \circ \delta_E$ defined when one of $A(\delta_W)$ and $A(\delta_E)$ is empty and the other is not.

It is easy to establish that for every P2-term δ we have $A(\delta)$ empty iff all the atomic P2-terms occurring in δ are $\mathbf{1}_\Lambda$ for Λ the empty list.

The rules of the system S2 are symmetry and transitivity of $=$, and congruence with $\circ$ and $\otimes$ (these two congruence rules are obtained from the congruence with $\Box$ of §1.5 by substituting $\circ$ and $\otimes$ respectively for $\Box$). The axiomatic equations of S2 are $\delta = \delta$ and the following equations:

(Ass $\circ$) $\quad (\delta_1 \circ \delta_2) \circ \delta_3 = \delta_1 \circ (\delta_2 \circ \delta_3)$,

($\mathbf{1}\circ$) $\quad \mathbf{1}_{\mathcal{L}_W(\delta)} \circ \delta = \delta = \delta \circ \mathbf{1}_{\mathcal{L}_E(\delta)}$,

(Ass $\otimes$) $\quad (\delta_1 \otimes \delta_2) \otimes \delta_3 = \delta_1 \otimes (\delta_2 \otimes \delta_3)$,

for Λ the empty list,

($\mathbf{1}\otimes$) $\quad \mathbf{1}_\Lambda \otimes \delta = \delta = \delta \otimes \mathbf{1}_\Lambda$,

$$(\otimes\circ)\qquad (\delta_1\circ\delta_2)\otimes(\delta_3\circ\delta_4)=(\delta_1\otimes\delta_3)\circ(\delta_2\otimes\delta_4),$$
$$(\otimes\mathbf{1})\qquad \mathbf{1}_\Gamma\otimes\mathbf{1}_\Delta=\mathbf{1}_{\Gamma\Delta},$$

provided that for each of these equations both sides are defined, i.e., they are P2-terms. It is straightforward to verify that in all of these equations the two sides are P2-terms of the same edge type and the same sequential type.

The axiomatic equations of S2 are like the assumptions for 2-cells in 2-categories, where $\circ$ is interpreted as vertical composition, $\otimes$ as horizontal composition and unit terms as identity 2-cells (see [10] and [11], Sections XII.3 and XII.6). Note however that in category theory the notation is usually different (and so it is in the references we gave), not only because it uses different symbols, but also because, contrary to what we do here, the terms composed are written from right to left. The P2-terms stand for 2-cells, while the elements of $\mathcal{A}$, i.e. the edges, stand for 1-cells. Nothing is provided in this syntax for 0-cells, i.e. vertices. It would be more in the spirit of this reading of S2, but not very perspicuous, to write $\circ_2$ for $\circ$ and $\circ_1$ for $\otimes$.

From the point of view of ordinary category theory, in S2 we assume that we have lists as objects and arrows between these lists. With the axiomatic equations (Ass $\circ$) and ($\mathbf{1}\circ$) of S2 we assume that we have a category with composition $\circ$ and identity arrows $\mathbf{1}_\Gamma$. We have moreover a strict monoidal structure with a bifunctor $\otimes$ and unit object $\mathbf{1}_\Lambda$ for Λ the empty list (see [11], Sections VII.1 and XI.3); on the objects, $\otimes$ is concatenation. The axiomatic equations (Ass $\otimes$) and ($\mathbf{1}\otimes$) tell that this monoidal structure is strict (associativity isomorphisms and isomorphisms involving the unit are identity arrows), while ($\otimes\circ$) and ($\otimes\mathbf{1}$) are the assumptions of bifunctoriality. This reading of the axiomatic equations of S2 explains our notation.

By the equation ($\mathbf{1}\otimes$), the P2-term $\mathbf{1}_\Lambda$ with the empty list Λ behaves like the unit for horizontal composition. Having this P1-term and P2-term is helpful, from a notational, computational and aesthetic point of view (like having zero), but it is not essential. Every P2-term that is not equal in S2 to $\mathbf{1}_\Lambda$ is equal to a P2-term in which $\mathbf{1}_\Lambda$ does not occur. In the graphs corresponding to diagrams of 2-cells, which we will call M-graphs (see §6.6), we have allowed the empty graph, because we will interpret $\mathbf{1}_\Lambda$ by the empty graph (see §6.7). Had we however omitted $\mathbf{1}_\Lambda$ from the language of S2, the empty graph would be excluded from M-graphs, and nothing would change essentially. In the notion of pasting scheme (see §7.3), which is a planar realization of an M-graph, the empty graph is not taken into account.

Omitting $\mathbf{1}_\Lambda$ from P2-terms would not make it difficult to axiomatize the remaining complete fragment of S2. From the axiomatic equations of S2 we would just omit $(\mathbf{1}\otimes)$. For the equivalent fragment of S1, in the axiomatic equations of S1 in §6.2 we would require that the lists Φ and Ψ, as well as Θ, are not empty.

§6.4. The equivalence of S1 and S2

We show in this section that there are two translations, i.e. homomorphic maps, one from P1-terms to P2-terms and the other from P2-terms to P1-terms, which are inverse to each other up to derivable equality in S2 and S1 (see Propositions 6.4.1 and 6.4.2). These translations preserve derivability of equality in S1 and S2 (see Propositions 6.4.3 and 6.4.4).

We define first inductively a map t_2 from P1-terms to P2-terms:

$$t_2(\delta) = \delta, \quad \text{when } \delta \text{ is atomic;}$$

for δ_W a P1-term of sequential type $(\Lambda^W_W, \Phi_E\Xi\Psi_E)$ and δ_E a P1-term of sequential type $(\Phi_W\Xi\Psi_W, \Lambda^E_E)$, where $\Phi_E\Xi\Psi_E$ and $\Phi_W\Xi\Psi_W$ are compatible lists (which means that at least one of Φ_W and Φ_E, and at least one of Ψ_W and Ψ_E, are the empty list; see §1.7),

$$t_2(\delta_W\Box\delta_E) = (\mathbf{1}_{\Phi_W} \otimes t_2(\delta_W) \otimes \mathbf{1}_{\Psi_W}) \circ (\mathbf{1}_{\Phi_E} \otimes t_2(\delta_E) \otimes \mathbf{1}_{\Psi_E}),$$

where the P2-term on the right-hand side is of sequential type $(\Phi_W\Lambda^W_W\Psi_W, \Phi_E\Lambda^E_E\Psi_E)$. Since we have (Ass $\otimes$) in S2, we may restore the missing parentheses involving $\otimes$ in this P2-term as we wish.

Next define inductively a map t_1 from P2-terms to P1-terms:

$$t_1(\delta) = \delta, \quad \text{when } \delta \text{ is atomic,}$$
$$t_1(\delta_W \circ \delta_E) = t_1(\delta_W)\Box t_1(\delta_E),$$
$$t_1(\delta_N \otimes \delta_S) = (\mathbf{1}_{\mathcal{L}_W(\delta_N)\mathcal{L}_W(\delta_S)}\Box t_1(\delta_N))\Box t_1(\delta_S).$$

We can prove the following.

Proposition 6.4.1. *In* S1 *we can derive* $t_1(t_2(\delta)) = \delta$.

Proof. We proceed by induction on the number k of occurrences of $\Box$ in the P1-term δ. If $k = 0$, then δ is atomic, and $t_1(t_2(\delta))$ is δ.

If δ is $\delta_W\Box\delta_E$, then in S1 we have

$$t_1(t_2(\delta_W \Box \delta_E)) = t_1(((\mathbf{1}_{\Phi_W} \otimes t_2(\delta_W)) \otimes \mathbf{1}_{\Psi_W}) \circ ((\mathbf{1}_{\Phi_E} \otimes t_2(\delta_E)) \otimes \mathbf{1}_{\Psi_E})),$$
$$= (\mathbf{1}_{\Phi_W \mathcal{L}_W(\delta_W)\Psi_W} \Box (\mathbf{1}_{\Phi_W \mathcal{L}_W(\delta_W)} \Box \delta_W)) \Box (\mathbf{1}_{\Phi_E \mathcal{L}_W(\delta_E)\Psi_E} \Box (\mathbf{1}_{\Phi_E \mathcal{L}_W(\delta_E)} \Box \delta_E)),$$
by the induction hypothesis and (**11**),
$$= (\mathbf{1}_{\Phi_W \Lambda_W^W \Psi_W} \Box \delta_W) \Box (\mathbf{1}_{\Phi_E \Phi_W \Xi \Psi_W \Psi_E} \Box \delta_E),$$
by (Ass 1) and (**11**),
$$= (\delta_W \Box \mathbf{1}_{\Phi_E \Phi_W \Xi \Psi_W \Psi_E}) \Box \delta_E, \quad \text{by (Ass 1) and } (\mathbf{11}),$$
$$= \delta_W \Box \delta_E, \qquad \text{by } (\mathbf{12}\text{L}) \text{ or } (\mathbf{12}\text{R}), \text{ (Ass 1) and } (\mathbf{11}). \quad \dashv$$

Proposition 6.4.2. *In* S2 *we can derive* $t_2(t_1(\delta)) = \delta$.

Proof. We proceed again by induction on the number k of occurrences of $\circ$ or $\otimes$ in the P2-term δ. If $k = 0$, then δ is atomic, and $t_2(t_1(\delta))$ is δ.

If δ is $\delta_W \circ \delta_E$, then in S2 we have

$$t_2(t_1(\delta_W \circ \delta_E)) = t_2(t_1(\delta_W) \Box t_1(\delta_E)),$$
$$= (\mathbf{1}_\Lambda \otimes \delta_W \otimes \mathbf{1}_\Lambda) \circ (\mathbf{1}_\Lambda \otimes \delta_E \otimes \mathbf{1}_\Lambda),$$
by the induction hypothesis, for Λ the empty list,
$$= \delta_W \circ \delta_E, \quad \text{by } (\mathbf{1}\otimes).$$

If δ is $\delta_N \otimes \delta_S$, then in S2 we have

$$t_2(t_1(\delta_N \otimes \delta_S)) = t_2((\mathbf{1}_{\mathcal{L}_W(\delta_N)\mathcal{L}_W(\delta_S)} \Box t_1(\delta_N)) \Box t_1(\delta_S)),$$
$$= (\delta_N \otimes \mathbf{1}_{\mathcal{L}_W(\delta_S)}) \circ (\mathbf{1}_{\mathcal{L}_E(\delta_N)} \otimes \delta_S),$$
by the induction hypothesis, (**1**$\otimes$) and (**1**$\circ$),
$$= \delta_N \otimes \delta_S, \quad \text{by } (\otimes\circ) \text{ and } (\mathbf{1}\circ). \quad \dashv$$

Proposition 6.4.3. *If in* S1 *we can derive* $\delta = \delta'$, *then in* S2 *we can derive* $t_2(\delta) = t_2(\delta')$.

To prove this proposition we proceed by induction on the length of the derivation of $\delta = \delta'$ in S1. Most of the work is in the basis, where for the axiomatic equations of S1 we have lengthy, but straightforward, derivations in S2 of their t_2 variants. Finally, we have the following proposition.

Proposition 6.4.4. *If in* S2 *we can derive* $\delta = \delta'$, *then in* S1 *we can derive* $t_1(\delta) = t_1(\delta')$.

To prove this proposition, we proceed as for Proposition 6.4.3. Note that we use (Ass 2.1) only to derive in S1 the t_1 variant of $(\otimes\circ)$.

§6.5. The completeness of S1

In this section we show that S1 is complete with respect to an interpretation in graphs of a particular kind, with a juncture operation and unit graphs. For convenience, we rely here on edge-graphs (see §1.4). By the equivalence of this notion with incidented graphs, we obtain a completeness result with respect to a notion of graph based on the notion of P-graph. This requires however a modification of our notion of juncture in the presence of units, a modification involving the vertices incident with the edges of the cocycle of the juncture. To disregard vertices, as we do by relying on edge-graphs, seemed to us the best way to get around the trivial, but annoying, difficulties involved in these modifications. (The kind of difficulty we avoid this way may be sensed in the definition of μ in §6.7.)

Let a *D1-graph* be a graph that is finite, acyclic, W-E-functional and incidented (see §1.2 for these notions, and for the related notion of D-graph). Every D-graph is a D1-graph. We have that D1-graphs differ from D-graphs by possibly not being weakly connected and by possibly lacking inner vertices. (Note that D-graphs are incidented.)

Let a *D1-edge-graph* be an edge-graph H such that $\mathcal{G}(H)$ is a D1-graph (see §1.4 for $\mathcal{G}$).

The empty graph (see §1.2), which is not a D-graph, is a D1-graph, but a single-vertex graph (see §1.4) is neither a D-graph nor a D1-graph, because it is not incidented.

In a *straight single-edge graph* $W, E : A \to V$ we have $A = \{a\}$ and $V = \{W(a), E(a)\}$ with $W(a) \neq E(a)$ (see also §6.6; a single-edge graph is not straight when $W(a) = E(a)$).

It is easy to infer that an equivalent alternative definition of D1-graph is that these are graphs where every component is either a D-graph or a straight single-edge graph (see the beginning of §1.3 for the notion of component). In the empty graph the set of components is empty, and hence every component is trivially what is required to make the empty graph a D1-graph.

Joining two components of a D1-graph into a single D1-graph could be conceived as the result of a new kind of juncture, via an empty set of edges C (see §1.3). Such a juncture, which we disallowed before, is an operation related to the $\otimes$ of §6.3.

A *basic D-edge-graph* is $\langle A, \mathbf{W}, \mathbf{E}, \mathbf{P} \rangle$ where $A = A_W \cup A_E$, $A_W \neq \emptyset$, $A_E \neq \emptyset$ and $A_W \cap A_E = \emptyset$, and, moreover,

$$a_1 \neq a_2 \Rightarrow (a_1\mathbf{E}a_2 \;\Leftrightarrow a_1, a_2 \in A_W),$$
$$a_1 \neq a_2 \Rightarrow (a_1\mathbf{W}a_2 \Leftrightarrow a_1, a_2 \in A_E),$$
$$a_1\mathbf{P}a_2 \;\Leftrightarrow (a_1 \in A_W \;\&\; a_2 \in A_E).$$

It is straightforward to verify that H is a basic D-edge-graph iff $\mathcal{G}(H)$ is a basic D-graph.

A *unit D1-edge-graph* is $\mathbf{1}_A = \langle A, I_A, I_A, \emptyset\rangle$, where $I_A = \{(a,a) \mid a \in A\}$ is the identity relation on A. Here A can be the empty set, in which case I_A is the empty set too. If H is a unit D1-edge-graph, then in $\mathcal{G}(H)$ every component is a straight single-edge graph.

We define now inductively for every P1-term (see §6.2) δ a D1-edge-graph $\eta(\delta)$.

If β is a basic P-term (see §6.1 and §1.5), then let $\eta(\beta)$ be the basic D-edge-graph such that $A = A(\beta)$ and $A_X = X(\beta)$.

If $\mathbf{1}_\Gamma$ is a unit term (see the beginning of §6.2), then let $\eta(\mathbf{1}_\Gamma)$ be the unit D1-edge graph $\mathbf{1}_{\Gamma^s}$, where Γ^s is the set, possibly empty, of members of the list Γ.

If $\delta_W \Box \delta_E$ is a P1-term, then let $\eta(\delta_W \Box \delta_E)$ be the D1-edge-graph $\eta(\delta_W) \cup \eta(\delta_E)$, where this union of D1-edge graphs is defined as the union of D-edge graphs that corresponds to juncture (we take the union of the two sets of edges, and the unions of the two functions $\mathbf{W}$, $\mathbf{E}$ and $\mathbf{P}$; see the end of §1.4).

We also have the following definitions for every P1-term δ:

$$\rho(\delta) = \{\beta \mid \beta \text{ is a basic P-term occurring in } \delta\},$$
$$\eta^*(\delta) = \langle \eta(\delta), \rho(\delta), \mathcal{L}_W(\delta), \mathcal{L}_E(\delta)\rangle.$$

We can prove the following lemma.

LEMMA 6.5.1. *For P-terms δ and δ' we have $\iota(\delta) = \iota(\delta')$ iff $\eta^*(\delta) = \eta^*(\delta')$.*

PROOF. From left to right we rely on the right-to-left direction of Theorem 6.1.5, and on the implication from $\delta = \delta'$ in $S\Box_P$ to $\eta^*(\delta) = \eta^*(\delta')$. This implication is easy to establish because the $\Box$ of $S\Box_P$ is interpreted in terms of union, and because in all the axiomatic equations the two sides have the same basic P-terms. We rely also on Proposition 6.1.1.

From right to left we pass from $\eta^*(\delta)$ to a unique $\iota(\delta)$ by taking for each basic P-term, i.e. basic D-term, β occurring in δ the inner vertex v_β from $\iota(\beta)$ (see the beginning of §1.6), and adding this vertex to $\eta(\delta)$. Remember that for every pair (A', A'') of sets of edges of $\eta(\delta)$ such that (A', A'') belongs to $V_{\eta(\delta)}$ (see the definition of V_H in §1.4) we have a unique β in $\rho(\delta)$ such that $W(\beta) = A'$ and $E(\beta) = A''$. The X-vertices of $\iota(\delta)$ are induced by the edges (see the beginning of §1.6). ⊣

The following proposition is analogous to Proposition 1.6.3.1.

PROPOSITION 6.5.2.1. *For every P1-term δ, in* S1 *we have an equation of the form*

$$\delta = (\ldots(\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \sigma_1) \Box \ldots) \Box \sigma_n,$$

for $n \geq 0$, where for distinct i and j in $\{1, \ldots, n\}$ we have that $\sigma_i \Box \sigma_j$ is not defined, and for every i in $\{1, \ldots, n\}$ we have that σ_i is a P-term. (If $n = 0$, then our equation is $\delta = \mathbf{1}_{\mathcal{L}_W(\delta)}$.)

PROOF. We proceed by induction on the number k of occurrences of $\Box$ in δ. If $k = 0$, then δ is either basic, in which case we have $\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \delta$ by ($\mathbf{1}1$), or δ is $\mathbf{1}_{\mathcal{L}_W(\delta)}$.

If $k > 0$, then δ is of the form $\delta_W \Box \delta_E$, and by the induction hypothesis, for X being W or E, in S1 we have

$$\delta_X = (\ldots(\mathbf{1}_{\Gamma_X} \Box \sigma_1^X) \Box \ldots) \Box \sigma_{n_X}^X,$$

where the right-hand side is abbreviated by $\mathbf{1}_{\Gamma_X} \Box \vec{\sigma}_{n_X}$. By applying (Ass 1), in S1 we obtain

$$\delta_W \Box \delta_E = (\delta_W \Box \mathbf{1}_{\Gamma_E}) \Box \vec{\sigma}_{n_E},$$

with an abbreviated notation of the same kind.

Then we make an auxiliary induction on n_W to prove that in S1 we have

$$\delta_W \Box \mathbf{1}_{\Gamma_E} = \mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W}.$$

If $n_W = 0$, then δ_W is $\mathbf{1}_{\Gamma_W}$, and in S1 we have $\mathbf{1}_{\Gamma_W} \Box \mathbf{1}_{\Gamma_E} = \mathbf{1}_{\mathcal{L}_W(\delta)}$ by ($\mathbf{1}\Box\mathbf{1}$) (see §6.2).

If $n_W > 0$, then δ_W is $\delta'_W \Box \sigma_{n_W}^W$. Then in S1 we have either

$$(\delta'_W \Box \sigma_{n_W}^W) \Box \mathbf{1}_{\Gamma_E} = \delta'_W \Box \sigma_{n_W}^W,$$

by ($\mathbf{1}1$), or we have

$$(\delta'_W \Box \sigma_{n_W}^W) \Box \mathbf{1}_{\Gamma_E} = (\delta'_W \Box \mathbf{1}_{\Gamma'_E}) \Box \sigma_{n_W}^W,$$

by using either (Ass 2.1), in which case Γ'_E is Γ_E, or (Ass 1) together with ($\mathbf{1}2\mathrm{L}\Phi$) or ($\mathbf{1}2\mathrm{R}\Phi$) (see §6.2). Then we apply the induction hypothesis to $\delta'_W \Box \mathbf{1}_{\Gamma'_E}$. This concludes the auxiliary induction. So in S1 we have

$$\delta = (\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W}) \Box \vec{\sigma}_{n_E}.$$

For the remainder of the proof we proceed as in the proof of Proposition 1.6.3.1, by applying only (Ass 1) and (Ass 2.1). Formally, we need auxiliary inductions on n_W and n_E to show that in S1 we have $\delta = \mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_n$, with the conditions of the proposition satisfied.

If $\sigma^W_{n_W} \Box \sigma^E_1$ is defined, then, by (Ass 1), in S1 we have

$$((\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W-1}) \Box \sigma^W_{n_W}) \Box \sigma^E_1 = (\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W-1}) \Box (\sigma^W_{n_W} \Box \sigma^E_1),$$

and if $\sigma^W_{n_W} \Box \sigma^E_1$ is not defined, then, by (Ass 2.1), in S1 we have

$$((\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W-1}) \Box \sigma^W_{n_W}) \Box \sigma^E_1 = ((\mathbf{1}_{\mathcal{L}_W(\delta)} \Box \vec{\sigma}_{n_W-1}) \Box \sigma^E_1) \Box \sigma^W_{n_W}. \qquad \dashv$$

An example of a P1-term in the form on the right-hand side of the equation of Proposition 6.5.2.1 may be found at the end of §6.7.

By relying on ($\mathbf{12}$LΨ) or ($\mathbf{12}$RΨ) (see §6.2) instead of ($\mathbf{12}$LΦ) or ($\mathbf{12}$RΦ), we prove analogously the following proposition, which corresponds to Proposition 1.6.3.2.

PROPOSITION 6.5.2.2. *For every P1-term δ, in* S1 *we have an equation of the form*

$$\delta = \sigma_n \Box (\ldots \Box (\sigma_1 \Box \mathbf{1}_{\mathcal{L}_E(\delta)}) \ldots),$$

for $n \geq 0$, where for distinct i and j in $\{1, \ldots, n\}$ we have that $\sigma_i \Box \sigma_j$ is not defined, and for every i in $\{1, \ldots, n\}$ we have that σ_i is a P-term. (If $n = 0$, then our equation is $\delta = \mathbf{1}_{\mathcal{L}_W(\delta)}$.)

Then we can prove the completeness of S1 with respect to η^*.

THEOREM 6.5.3. *In S1 we can derive $\delta = \delta'$ iff $\eta^*(\delta) = \eta^*(\delta')$.*

PROOF. From left to right we proceed by an easy induction on the length of the derivation of $\delta = \delta'$ in S1. For the axiomatic equations (Ass 1), (Ass 2.1) and (Ass 2.2) we rely on the interpretation of $\Box$ in terms of union in $\eta(\delta)$ (see the proof of the left-to-right direction of Lemma 6.5.1). We rely moreover on a variant of Proposition 6.1.1, which says that if $\delta = \delta'$ is derivable in S1, then the sequential types of δ and δ' are the same. The proof of that is straightforward.

For the proof from right to left, suppose $\eta^*(\delta) = \eta^*(\delta')$. By Proposition 6.5.2.1 we have that equations of the form $\delta = \mathbf{1}_\Gamma \Box \vec{\sigma}_n$ and $\delta' = \mathbf{1}_\Gamma \Box \vec{\tau}_{n'}$

are derivable in S1 (see the proof of that proposition for the vector abbreviations). By the direction from left to right, which we have just proven, $\eta^*(\mathbf{1}_\Gamma\Box\vec{\sigma}_n) = \eta^*(\mathbf{1}_\Gamma\Box\vec{\tau}_{n'})$. From that we infer that $n = n'$, because n is the number of components in $\eta(\delta)$ and $\eta(\delta')$, components for edge-graphs being defined as for the corresponding incidented graphs. We infer also that there is a bijection π of $\{1,\ldots,n\}$ to itself such that for every i in $\{1,\ldots,n\}$ we have $\eta^*(\sigma_i) = \eta^*(\tau_{\pi(i)})$. By Lemma 6.5.1, we infer that $\iota(\sigma_i) = \iota(\tau_{\pi(i)})$, which, by Theorem 6.1.5, yields that $\sigma_i = \tau_{\pi(i)}$ is derivable in $S\Box_P$, and hence also in S1. We apply then (Ass 2.1), if needed, to derive $\mathbf{1}_\Gamma\Box\vec{\sigma}_n = \mathbf{1}_\Gamma\Box\vec{\tau}_{n'}$ in S1. ⊣

Let a *P1-graph* be a D1-graph where every component is either a P-graph or a straight single-edge graph. Note that the empty graph is trivially a P1-graph.

Let a *P1-edge-graph* be a D1-edge-graph H such that $\mathcal{G}(H)$ is a P1-graph (see §1.4 for $\mathcal{G}$, and §6.7 for an example). Analogously, let a *P-edge-graph* be a D-edge-graph H such that $\mathcal{G}(H)$ is a P-graph.

The following proposition shows that the interpretation function η^* is based on P1-graphs.

Proposition 6.5.4. *For every P1-term δ we have that $\eta(\delta)$ is a P1-edge-graph.*

Proof. By Proposition 6.5.2.1, in S1 we have

$$\delta = (\ldots(\mathbf{1}_{\mathcal{L}_W(\delta)}\Box\sigma_1)\Box\ldots)\Box\sigma_n,$$

where for every i in $\{1,\ldots,n\}$ we have that σ_i is a P-term. By Proposition 6.1.4, we have that $\iota(\sigma_i)$ is a P-graph.

We can then verify easily by induction on the number of occurrences of $\Box$ in the P-term τ that $\mathcal{H}(\iota(\tau))$ is $\eta(\tau)$ (see §1.4 for $\mathcal{H}$). For that we need that $\mathcal{H}(D_W\Box D_E) = \mathcal{H}(D_W)\cup\mathcal{H}(D_E)$.

By Proposition 1.4.1, we know that $\mathcal{G}(\mathcal{H}(\iota(\sigma_i)))$ is isomorphic to $\iota(\sigma_i)$, and since $\mathcal{H}(\iota(\sigma_i))$ is $\eta(\sigma_i)$, we have that $\eta(\sigma_i)$ is a P-edge-graph. It is then easy to conclude that $\eta(\delta)$, which is

$$(\ldots(\eta(\mathbf{1}_{\mathcal{L}_W(\delta)})\Box\eta(\sigma_1))\Box\ldots)\Box\eta(\sigma_n),$$

is a P1-edge-graph. ⊣

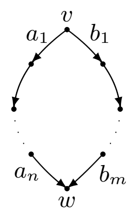
§6.6. M-graphs

The M-graphs (M comes from *mandorla*; see the picture below), which we will introduce in this section, correspond to diagrams of 2-cells in 2-categories. This correspondence will be made manifest in the pictures of §6.7, and in §7.3 through the notion of pasting scheme—a notion of plane graph from [13] (see §7.3). Every non-empty M-graph is realizable in the plane as a pasting scheme, and every pasting scheme is an M-graph. In §6.7 we will prove the completeness of S2 with respect to an interpretation in M-graphs.

Except for the empty graph (see §1.2), which is an M-graph too, every M-graph M will have two special distinct vertices $N(M)$ and $S(M)$ (which are respectively the source and sink of M; see §7.3). (The names of the functions N and S come from *North* and *South*.) When M is not the empty graph, we define also two distinct paths $W(M)$ and $E(M)$ from $N(M)$ to $S(M)$, which are the *domain* and *codomain* of M; since $N(M)$ and $S(M)$ are distinct, the paths $W(M)$ and $E(M)$ are non-trivial (see §1.2).

Note that diagrams of 2-cells in 2-categories are usually drawn so that the domain and codomain of a 2-cell are not in the West and in the East, respectively, but in the North and in the South, i.e. above and below, as in the most common maps. The terminology of *vertical* and *horizontal* composition is suggested by this way of drawing. The usual drawings are reflected with respect to the axis $y = -x$, as well as the dual diagrams of 2-cells (see the pictures of §6.7), to connect them with our way of drawing P-graphs and P1-graphs. These graphs are the main subject of our work, and to draw them as we did seems more practical. This is however done at the price of having vertical composition going from West to East, and horizontal composition from North to South.

For v distinct from w and $n, m \geq 1$, consider a graph B that corresponds to the following picture (in the shape of a mandorla)



Let $N(B)$ be the vertex v, let $S(B)$ be the vertex w, let $W(B)$ be the path from v to w with the edges $a_1, \dots, a_n$, and let $E(M)$ be the path from v to w with the edges $b_1, \dots, b_m$.

We say that B is a *basic M-graph* when $W(B)$ and $E(B)$ have no common edge and no common vertex except for v and w.

A straight single-edge graph (as defined in §6.5) is B where $n = m = 1$ and $a_1 = b_1$.

We can now give the clauses of our inductive definition of *M-graph*.

(1) Every basic M-graph, every straight single-edge graph and the empty graph are M-graphs.

($2\circ$) For X being W or E, let M_X, which is $W_X, E_X : A_X \to V_X$, be an M-graph. If M_W and M_E are not the empty graph, and they have in common as vertices and edges just the vertices and edges of $E(M_W)$, which is the same as $W(M_E)$, then $M_W \circ M_E$, which is the graph $W, E\colon A_W \cup A_E \to V_W \cup V_E$ such that for every a in $A_W \cup A_E$

$$X(a) = \begin{cases} X_W(a) & \text{if } a \in A_W, \\ X_E(a) & \text{if } a \in A_E \end{cases}$$

is an M-graph. (Note that for an edge a in $A_W \cap A_E$, i.e. an edge in the path $E(M_W)$, which coincides with the path $W(M_E)$, we have $X_W(a) = X_E(a)$.) For Y being N or S, let $Y(M_W \circ M_E) = Y(M_W) = Y(M_E)$, let $W(M_W \circ M_E) = W(M_W)$, and let $E(M_W \circ M_E) = E(M_E)$.

If one of M_W and M_E is the empty graph, then $M_W \circ M_E$ is defined only if the other is the empty graph too, and $M_W \circ M_E$ for both M_W and M_E being the empty graph is the empty graph.

($2\otimes$) For Y being N or S, let M_Y, which is $W_Y, E_Y : A_Y \to V_Y$, be an M-graph. If M_N and M_S are not the empty graph, and they have in common as vertices and edges just the vertex $S(M_N)$, which is the same vertex as $N(M_S)$, then $M_N \otimes M_S$, which is the graph $W, E : A_N \cup A_S \to V_N \cup V_S$ where $X(a)$ is defined as in clause ($2\circ$) above, with N and S substituted respectively for W and E, is an M-graph. For Y being N or S, let $Y(M_N \otimes M_S) = Y(M_Y)$, while $X(M_N \otimes M_S)$ is the path from $N(M_N)$ to $S(M_S)$ obtained by concatenating the paths $X(M_N)$ and $X(M_S)$ with one of the two occurrences of $S(M_N)$, which is equal to $N(M_S)$, deleted. (This joining of paths is analogous to what we had with $*$ and semipaths in §3.2.)

If one of M_N and M_S is the empty graph, then $M_N \otimes M_S$ is the other graph of these two graphs, from which, if this other graph is not the empty graph, it inherits the functions N, S, W and E.

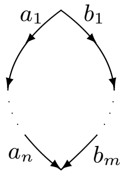
This concludes our definition of M-graph. Examples of M-graphs, with pictures, may be found in §6.7.

An *M-edge-graph* is an edge-graph H such that $\mathcal{G}(H)$ is an M-graph. When $\mathcal{G}(H)$ is a basic M-graph, H is a *basic M-edge-graph*, and when $\mathcal{G}(H)$ is a straight single-edge graph, H is the *straight single-edge edge-graph* $\langle\{a\}, I_{\{a\}}, I_{\{a\}}, \emptyset\rangle$, which is the unit D1-edge graph $\mathbf{1}_{\{a\}}$ (see §6.5).

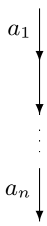
§6.7. The completeness of S2

We will now interpret the system S2 in M-graphs, and prove the completeness of S2 with respect to this interpretation. We introduce an interpretation function μ that assigns to a P2-term an M-graph, and is defined inductively as follows. As an auxiliary for this definition, we have a function α that assigns to an atomic P2-term an M-edge-graph.

For atomic P2-terms, which are P1-terms, we have first that for a basic P-term β such that $\mathcal{L}_W(\beta)$ is $a_1 \ldots a_n$, for $n \geq 1$, and $\mathcal{L}_E(\beta)$ is $b_1 \ldots b_m$, for $m \geq 1$, the M-edge-graph $\alpha(\beta)$ is the basic M-edge-graph that corresponds to the following picture:



For the unit term $\mathbf{1}_\Gamma$ where Γ is $a_1 \ldots a_n$ for $n \geq 1$ we have that $\alpha(\mathbf{1}_\Gamma)$ is the M-edge-graph that corresponds to the following picture:



If Λ is the empty list, then $\alpha(\mathbf{1}_\Lambda)$ is the empty edge-graph $\langle\emptyset, \emptyset, \emptyset, \emptyset\rangle$ (see §1.4).

For every atomic P1-term δ we have that $\mu(\delta)$ is $\mathcal{G}(\alpha(\delta))$. Note that $\mu(\mathbf{1}_\Lambda)$ for Λ the empty list is the empty graph.

Suppose we have the P2-term $\delta_W \circ \delta_E$, and we are given the M-graphs $\mu(\delta_W)$ and $\mu(\delta_E)$. Consider the paths $E(\mu(\delta_W))$ and $W(\mu(\delta_E))$, which are made of the same edges in the same order. Let the k-th vertex in the first path be (A'_W, A''_W), and let the k-th vertex in the second path be (A'_E, A''_E); here $k \geq 1$. Let the M-graph M_X isomorphic to $\mu(\delta_X)$ be obtained from $\mu(\delta_X)$ by replacing every such vertex (A'_X, A''_X) by $(A'_W \cup A'_E, A''_W \cup A''_E)$. We have that the paths $E(M_W)$ and $W(M_E)$ coincide. If $\mu(\delta_X)$ is the empty graph, then the M-graph M_X is also the empty graph. We take $\mu(\delta_W \circ \delta_E)$ to be $M_W \circ M_E$ (see $(2\circ)$ in §6.6).

Suppose we have the P2-term $\delta_N \otimes \delta_S$, and we are given the M-graphs $\mu(\delta_N)$ and $\mu(\delta_S)$. If neither $\mu(\delta_N)$ nor $\mu(\delta_S)$ is the empty graph, then we have that $S(\mu(\delta_N))$ is a vertex of the form $(A', \emptyset)$, and $N(\mu(\delta_S))$ is a vertex of the form $(\emptyset, A'')$. Let Y be N or S, and let $\bar{N}$ be S, and $\bar{S}$ be N. Let the M-graph M_Y isomorphic to $\mu(\delta_Y)$ be obtained from $\mu(\delta_Y)$ by replacing the vertex $\bar{Y}(\mu(\delta_Y))$ by (A', A''). If $\mu(\delta_Y)$ is the empty graph, then the M-graph M_Y is also the empty graph. We take $\mu(\delta_N \otimes \delta_S)$ to be $M_N \otimes M_S$ (see $(2\otimes)$ in §6.6), and this concludes our definition of μ.

For the clause concerning $\mu(\delta_W \circ \delta_E)$ in this definition to be correct, i.e. for $\mu(\delta_W \circ \delta_E)$ to be defined, we cannot have that one of M_W and M_E is the empty graph and the other is not, because this is required by the definition of $M_W \circ M_E$. Suppose $\mu(\delta_W)$ and $\mu(\delta_E)$ are both defined. It is easy to see that the edges of $\mu(\delta_X)$ are the elements of $A(\delta_X)$. Since the edges of $\mu(\delta_X)$ and M_X are the same, if we had that one of M_W and M_E is the empty graph and the other is not, then we would have that one of $A(\delta_W)$ and $A(\delta_E)$ is empty and the other is not. In that case, however, as we noted after the definition of P2-graph in §6.3, we would not have that $\delta_W \circ \delta_E$ is a P2-term.

Let $\mu^*(\delta) = \langle \mu(\delta), \rho(\delta) \rangle$, where ρ is defined as for η^* in §6.5. We can establish in a straightforward manner the following soundness proposition by induction on the length of derivation in the system S2.

PROPOSITION 6.7.1. *If in* S2 *we can derive* $\delta = \delta'$, *then* $\mu^*(\delta) = \mu^*(\delta')$.

To establish also the converse implication, i.e. the completeness of S2 with respect to μ^*, we consider first some preliminary matters.

We say that a P2-term is *developed* when it is of the form $\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n$, where $n \geq 0$, parentheses tied to $\circ$ are associated arbitrarily, δ_0 is a unit term $\mathbf{1}_\Gamma$, and if $n > 0$, then for each i in $\{1, \ldots, n\}$ we have that δ_i is of the form $(\mathbf{1}_{\Gamma'_i} \otimes \beta_i \otimes \mathbf{1}_{\Gamma''_i})$, with parentheses tied to the two occurrences of

$\otimes$ associated arbitrarily, and β_i a basic P-term. Here β is the *core* of δ_i. If $n = 0$, then $\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n$ is just δ_0, which is of the form $\mathbf{1}_\Gamma$. An example of a developed P2-term may be found in γ, at the end of this section.

We can prove the following development lemma.

LEMMA 6.7.2. *For every P2-term δ there is a developed P2-term $\delta^\dagger$ such that $\delta = \delta^\dagger$ is derivable in S2.*

PROOF. We proceed by induction on the complexity of δ. If δ is an atomic P2-term, the lemma is established easily by using, if need there is, $(\mathbf{1}\circ)$ or $(\mathbf{1}\otimes)$.

If δ is of the form $\delta' \circ \delta''$, then we apply the induction hypothesis to δ' and δ'', and in S2 we have

$$(\mathbf{1}_{\Gamma'} \circ \delta'_1 \circ \ldots \circ \delta'_{n'}) \circ (\mathbf{1}_{\Gamma''} \circ \delta''_1 \circ \ldots \circ \delta''_{n''}) = \mathbf{1}_{\Gamma'} \circ \delta'_1 \circ \ldots \circ \delta'_{n'} \circ \delta''_1 \circ \ldots \circ \delta''_{n''},$$

by $(\mathbf{1}\circ)$ and (Ass $\circ$).

If δ is of the form $\delta' \otimes \delta''$, then we apply again the induction hypothesis to δ' and δ'', and we make an auxiliary induction on $n'+n''$. If $n'+n'' = 0$, then, by $(\otimes\mathbf{1})$, in S2 we have

$$\mathbf{1}_{\Gamma'} \otimes \mathbf{1}_{\Gamma''} = \mathbf{1}_{\Gamma'\Gamma''}.$$

If $n' > 0$, then in S2 we have

$$\begin{aligned}
&(\mathbf{1}_{\Gamma'} \circ \delta'_1 \circ \ldots \circ \delta'_{n'}) \otimes (\mathbf{1}_{\Gamma''} \circ \delta''_1 \circ \ldots \circ \delta''_{n''})\\
&\quad = (\mathbf{1}_{\Gamma'} \circ \delta'_1 \circ \ldots \circ \delta'_{n'}) \otimes (\mathbf{1}_{\Gamma''} \circ \delta''_1 \circ \ldots \circ \delta''_{n''} \circ \mathbf{1}_\Delta), \quad \text{by } (\mathbf{1}\circ),\\
&\quad = ((\mathbf{1}_{\Gamma'} \circ \delta'_1 \circ \ldots \circ \delta'_{n'-1}) \otimes (\mathbf{1}_{\Gamma''} \circ \delta''_1 \circ \ldots \circ \delta''_{n''})) \circ (\delta'_{n'} \otimes \mathbf{1}_\Delta), \text{ by } (\otimes\circ).
\end{aligned}$$

Then we apply the induction hypothesis of the auxiliary induction and $(\otimes\mathbf{1})$. We proceed analogously if $n'' > 0$.

In all that we rely on (Ass $\circ$) and (Ass $\otimes$) to associate parentheses as we wish. $\dashv$

Then we can prove the completeness of S2 with respect to μ^*.

THEOREM 6.7.3. *In S2 we can derive $\delta = \delta'$ iff $\mu^*(\delta) = \mu^*(\delta')$.*

PROOF. From left to right we have Proposition 6.7.1. For the other direction we proceed as follows.

By Lemma 6.7.2, in S2 we have

$$\delta = \delta_0 \circ \delta_1 \circ \ldots \circ \delta_n \text{ and } \delta' = \delta'_0 \circ \delta'_1 \circ \ldots \circ \delta'_{n'},$$

for the right-hand sides developed. Since $\mu(\delta) = \mu(\delta')$, by Proposition 6.7.1, we infer that in S2 we have

$$\mu^*(\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n) = \mu^*(\delta'_0 \circ \delta'_1 \circ \ldots \circ \delta'_{n'}).$$

From $\rho(\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n) = \rho(\delta'_0 \circ \delta'_1 \circ \ldots \circ \delta'_{n'})$ we infer that $n = n'$. We proceed then by induction on n.

If $n = 0$, then δ and δ', which are respectively δ_0 and δ'_0, must both be the same unit term $\mathbf{1}_\Gamma$. If $n > 0$, then since

$$\rho(\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n) = \rho(\delta'_0 \circ \delta'_1 \circ \ldots \circ \delta'_{n'}),$$

there must be an i in $\{1, \ldots, n\}$ such that δ_n and δ'_i have the same core β. If $i \neq n$, then by using equations of the form

$$(\mathbf{1}_{\Gamma'} \otimes \beta_1 \otimes \mathbf{1}_{\Gamma''}) \circ (\mathbf{1}_{\Delta'} \otimes \beta_2 \otimes \mathbf{1}_{\Delta''}) = (\mathbf{1}_{\Xi'} \otimes \beta_2 \otimes \mathbf{1}_{\Xi''}) \circ (\mathbf{1}_{\Pi'} \otimes \beta_1 \otimes \mathbf{1}_{\Pi''}),$$

with the proviso that $E(\beta_1) \cap W(\beta_2)$ and $E(\beta_2) \cap W(\beta_1)$ are empty, which are derivable in S2 with the help of $(\otimes\, \mathbf{1})$ and (Ass $\otimes$), and the essential use of $(\mathbf{1}\, \circ)$ and two applications of $(\otimes\, \circ)$, we obtain that, for δ'' being

$$\delta_0 \circ \delta'_1 \circ \ldots \circ \delta'_{i-1} \circ \delta''_{i+1} \circ \ldots \circ \delta''_n,$$

in S2 we can derive $\delta' = \delta'' \circ \delta_n$. If $i = n$, then, for δ'' being $\delta_0 \circ \delta'_1 \circ \ldots \circ \delta'_{n-1}$, in S2 we can derive $\delta' = \delta'' \circ \delta_n$.

For δ'' being either of these two, we infer that $\mu(\delta_0 \circ \delta_1 \circ \ldots \circ \delta_{n-1}) = \mu(\delta'')$, and, by the induction hypothesis, we obtain that $\delta_0 \circ \delta_1 \circ \ldots \circ \delta_{n-1} = \delta''$ is derivable in S2. From that we infer that $\delta_0 \circ \delta_1 \circ \ldots \circ \delta_n = \delta'' \circ \delta_n$, and hence also $\delta = \delta'$, are derivable in S2. $\dashv$

Note that ρ plays an essential role in this completeness proof. Without involving ρ in μ^*, and by having just the interpretation function μ, completeness for S2 would fail for the simple reason that there may be two different basic P-terms β_1 and β_2 of the same sequential type (a, b); we have $\mu(\beta_1) = \mu(\beta_2)$, but $\beta_1 = \beta_2$ is not derivable in S2. However, even if we secured that there are no different basic P-terms of the same sequential type, we would still need ρ, as the following example shows. For

β_1 of sequential type (a, b),	β'_1 of sequential type (a, c),
β_2 of sequential type (b, c),	β'_2 of sequential type (c, b),
β_3 of sequential type (c, d),	β'_3 of sequential type (b, d)

we have $\mu(\beta_1 \circ \beta_2 \circ \beta_3) = \mu(\beta_1' \circ \beta_2' \circ \beta_3')$, but $\beta_1 \circ \beta_2 \circ \beta_3 = \beta_1' \circ \beta_2' \circ \beta_3'$ is not derivable in S2.

The completeness of S2 of Theorem 6.7.3 corresponds to the unicity part of Theorem 3.3 of [13], and the proof just given provides details for the sketch of the proof in the last paragraph of [13]. The remaining part of Theorem 3.3 of [13] is tied to our Lemma 6.7.2. One may understand P2-terms as formalizing what is called there "ways to obtain composites", and M-graphs correspond, as we already said at the beginning of §6.6, to what is called there "pasting schemes". Pasting schemes are not defined inductively as M-graphs are, and an essential ingredient of their definition, which may be found in §7.3, is planarity. One of the main purposes of this work is a combinatorial analysis of this planarity in terms of the notion of P-graph.

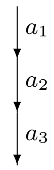
The connection between M-graphs and P-graphs may be derived from Theorem 6.5.3, the completeness of S1 with respect to η^*, which is based on P1-graphs and P-graphs, next from Theorem 6.7.3 above, the completeness of S2 with respect to μ^*, which is based on M-graphs, and finally from the translations that establish the equivalence of S1 and S2 in §6.4. In P1-graphs one forgets about the lists of edges, which are incorporated in M-graphs in the paths made of the duals of theses edges. This duality, which is treated more precisely for planar realizations in §7.6, will here be only illustrated by some pictures, and the accompanying comments.

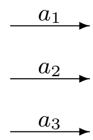
We have here on the left a picture of a basic M-edge-graph (see the end of §6.6) and on the right a picture of the corresponding D-edge-graph (see §1.4):

The region between the two paths in the left picture is replaced by a vertex in the right picture. The order of the edges in the paths in the left picture is replaced by their lists in the right picture. When in the right picture we forget about this order, and deal not with a given order, but with *orderability*, then we reach the level at which we have dealt with P-graphs.

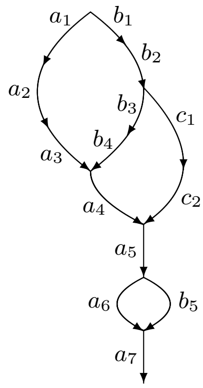
Here is next on the left a picture for the M-edge-graph (see the end of

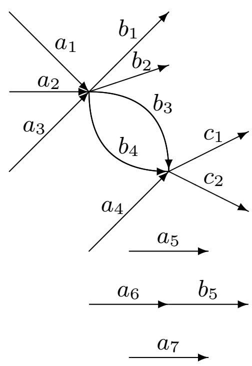
§6.6) corresponding to $\mu(\mathbf{1}_{a_1a_2a_3})$, and on the right a picture of the P1-edge-graph (see §6.5, before Proposition 6.5.4) $\eta(\mathbf{1}_{a_1a_2a_3})$:





Here is finally on the left a picture for a more complex M-edge-graph, and on the right a picture for the corresponding P1-edge-graph:





With β_1 of sequential type $(a_1a_2a_3,\ b_1b_2b_3b_4)$, β_2 of sequential type $(b_3b_4a_4,\ c_1c_2)$ and β_3 of sequential type (a_6, b_5), and with γ being

$$\mathbf{1}_{a_1a_2a_3a_4a_5a_6a_7} \circ (\mathbf{1} \otimes \beta_1 \otimes \mathbf{1}_{a_4a_5a_6a_7}) \circ (\mathbf{1}_{b_1b_2} \otimes \beta_2 \otimes \mathbf{1}_{a_5a_6a_7}) \\ \circ (\mathbf{1}_{b_1b_2c_1c_2a_5} \otimes \beta_3 \otimes \mathbf{1}_{a_7}),$$

for the picture on the left we have $\mu^*(\gamma) = \langle \mu(\gamma), \{\beta_1, \beta_2, \beta_3\}\rangle$, and with γ' being

$$(\mathbf{1}_{a_1a_2a_3a_4a_5a_6a_7} \Box (\beta_1 \Box \beta_2)) \Box \beta_3,$$

for the picture on the right we have

$$\eta^*(\gamma') = \langle \eta(\gamma'), \{\beta_1, \beta_2, \beta_3\},\ a_1a_2a_3a_4a_5a_6a_7,\ b_1b_2c_1c_2a_5b_5a_7\rangle.$$

The P-term γ is developed, while the P1-term γ' is in the form on the right-hand side of the equation of Proposition 6.5.2.1.

Chapter 7

Disk D-Graphs and P-Graphs

§7.1. Disk D-graphs

In this chapter we deal with geometrical matters concerning our graphs. We deal in particular with a special kind of realization of P-graphs in the plane. Such a realization is a plane graph situated within a disk with the boundary divided into two halves, one for the W-vertices and the other for the E-vertices. The plane graphs in question are called disk D-graphs, and the D1-graphs (see §6.5) based on disk D-graphs are called disk D1-graphs. We prove that every P-graph is isomorphic to a disk D-graph, and that, conversely, every disk D-graph is a P-graph. It follows that a graph is a P-graph iff it is isomorphic to a disk D-graph. This entails an analogous relationship between P1-graphs and disk D1-graphs.

We introduce in this chapter what we will call D1$'$-graphs, which are obtained from D1-graphs by adding a source and sink; namely, a single W-vertex and a single E-vertex. The disk D1$'$-graphs, i.e. the disk realizations of D1$'$-graphs, are known in the literature as pasting schemes, and we provide here two presumably new definitions of this notion. The equivalence of various definitions of P-graph, which we established in Chapters 2-5, enables us to obtain through the notion of P$'''$-graph a viable criterion for testing whether a D1$'$-graph is isomorphic to a pasting scheme. In the last section we state precisely the duality illustrated at the end §6.7. This is, namely, the particular relationship that exists between plane graphs that correspond to diagrams of 2-cells and disk D1-graphs, which we mentioned already in §1.1.

In this section, before defining disk D-graphs, to fix terminology, we introduce as preliminary notions the notion of plane graph and a few associated notions. Our terminology and these notions are pretty standard, but they should be adapted to the notion of graph of §1.2.

A *plane graph* is a graph $W, E\colon A \to V$ where A is a set of simple, open or closed, Jordan curves in $\mathbf{R}^2$ and V is a set of points in $\mathbf{R}^2$ such that

(1) for every open a in A the points $W(a)$ and $E(a)$ are the two distinct end points of a, and for every closed a in A we have that $W(a)$ and $E(a)$ are the same point of a,

(2) for every distinct a and b in A, if $v \in a \cap b$, then $v = W(a)$ or $v = E(a)$.

It follows immediately that we have also (2) with the consequent replaced by "$v = W(b)$ or $v = E(b)$".

An unessentially different notion of plane graph is obtained by requiring further that

for every a in A and every v in V, if $v \in a$, then $v = W(a)$ or $v = E(a)$

(cf. [16], Section 2.2, Definition 2.1). If the graph is incidented (see §1.2), then this additional requirement is met anyway.

For every plane graph G, which is $W, E\colon A \to V$, let the *point set* $\mathcal{U}(G)$ of G be the set of points of $\mathbf{R}^2$ that belong either to an edge in A or are elements of V.

When a graph G is isomorphic to a plane graph G' we say that G' is a *realization* of G. The graph G is *planar*, or *realizable in the plane*, when there is such a G'.

(We work all the time with the assumption that our graphs are distinguished; see §1.2. Relinquishing this assumption for a moment, note that non-distinguished plane graphs do not exist, although non-distinguished planar graphs would be possible.)

A *topological disk* in $\mathbf{R}^2$ is a closed subset of $\mathbf{R}^2$ homeomorphic to the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$. A *compass disk* κ is a topological disk in $\mathbf{R}^2$ with two distinct points on its boundary, called the *north pole* and the *south pole* of κ. The north pole and the south pole of κ determine within the boundary of κ two disjoint subsets not including the two poles, called the *W-meridian* and the *E-meridian*. Which of these two subsets is the W-meridian and which is the E-meridian is not arbitrary. We suppose from now on that we have fixed an orientation of $\mathbf{R}^2$, and it is with respect to this orientation that the sequence *north pole, E-meridian, south pole, W-meridian* proceeds clockwise.

Let a *disk D-graph* be a plane graph D that is a D-graph such that, for a compass disk κ, all the W-vertices of D are in the W-meridian of κ, all the E-vertices of D are in the E-meridian of κ, and the remaining points in $\mathcal{U}(D)$ are in the interior of κ.

We say that the compass disk κ of this definition is *associated* with the disk D-graph D. Although this associated compass disk is not uniquely determined, it is unique up to homeomorphisms that are identity maps on D. Examples of disk D-graphs with the associated compass disks may be found in the pictures of the proof of Proposition 7.2.1.

For a disk D-graph D, and for X being W or E, let $L_X(D)$ be the list of the X-vertices of D obtained by going along the X-meridian from the north pole to the south pole.

For every edge a of a plane graph, since a is a Jordan curve, we have a one-one continuous map f_a from the interval $[0,1]$ onto a. We say that a plane graph is *eastward-growing* when for r_W and r_E in $[0,1]$, and $f_a(r_X) = (x_X, y_X)$, if $r_W < r_E$, then $x_W < x_E$. (An analogous notion is called *upward planarity* in the literature; see [7].)

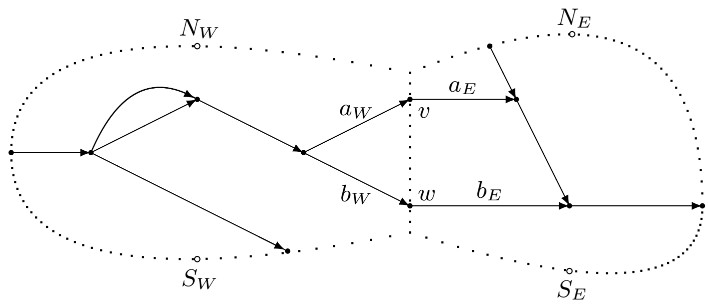
§7.2. P-graphs are realizable as disk D-graphs

In this section we prove what is announced in its title, which follows from the following proposition.

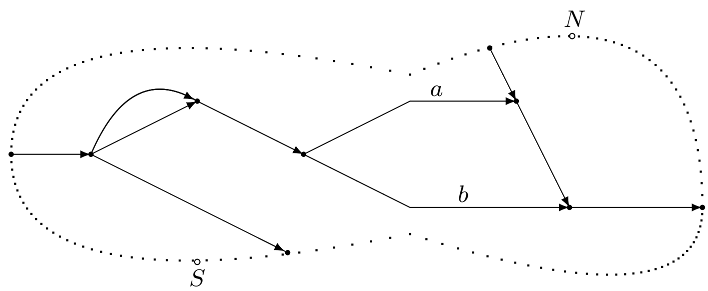
PROPOSITION 7.2.1. *Every P-graph is isomorphic to an eastward-growing disk D-graph.*

PROOF. Take a P-graph D conceived as a P$'$-graph (see §1.8). So there is a construction K with (D, L_W, L_E) in its root. We will show by induction on the number k of nodes in the tree of K that there is a graph isomorphism from D to an eastward-growing disk D-graph R such that L_X is $L_X(R)$. If $k = 1$, then D is a basic D-graph, for which the proposition is obvious (see the pictures at the end of §1.2 and at the beginning of §1.6).

Suppose K is $K_W \Box K_E$. For X being W or E, let D_X be the root graph of K_X. By applying the induction hypothesis to D_X, we obtain the eastward-growing disk D-graph R_X, which is a realization of D_X such that L^W_E is $L_E(R_W)$ and L^E_W is $L_W(R_E)$. If κ_X is a compass disk associated with R_X, then by appealing to the compatibility of the lists L^W_E and L^E_W, we may assume that part of the E-meridian of κ_W coincides with part of the W-meridian of κ_E, so that the vertices that D_W and D_E share are realized by the same points of these two meridians. For example, we may have R_W and R_E, with κ_W and κ_E drawn with dotted lines, as in the following pictures (which corresponds to the first two pictures in the example at the end of §1.3):



The eastward-growing disk D-graph R that is a realization of D such that L_X is $L_X(R)$ is obtained by removing the vertices v and w that were common to R_W and R_E, and by gluing into one edge a the two edges a_W and a_E, and into one edge b the two edges b_W and b_E, as in the following picture (which corresponds to the last pictures in §1.3):



The boundary of the compass disk κ associated here with R is drawn with dotted lines in this picture of R. This disk is obtained from κ_W and κ_E by omitting the part of their boundaries that they share, and by taking as the north pole N of κ the north pole N_E of κ_E, while the south pole S of κ will be the south pole S_W of κ_W.

The rules for choosing the poles of κ are the following. We have that the list L_E^W is $\Phi_E \Xi \Psi_E$, while L_W^E is $\Phi_W \Xi \Psi_W$. If Φ_E and Ψ_W are empty, as in our example above, then we take $N = N_E$ and $S = S_W$. If Φ_W and Ψ_E are empty, then we take $N = N_W$ and $S = S_E$, and, for X being W or E, if Φ_X and Ψ_X are empty, then we take $N = N_X$ and $S = S_X$. If more than two of the four lists Φ_W, Φ_E, Ψ_W and Ψ_E are empty, then we have more than one of these rules for poles applying. The results are not the same, but the differences are not important. $\dashv$

§7.3. Disk D-graphs are P-graphs

In this section we are going to prove that every disk D-graph is a P-graph. This implies the converse of the proposition that every P-graph is isomorphic to a disk D-graph, which follows from Proposition 7.2.1. More precisely, we will show that every disk D-graph is a P‴-graph. For Proposition 7.2.1 we relied, on the other hand, on the notion of P′-graph. Here is where our proof of the equivalence of the notions of P′-graph and P‴-graph helps us. We must first deal however with a number of preliminary matters.

The following lemma is a variant of Lemma 2 of [12], which is proven there with the help of Brouwer's Fixed Point Theorem.

LEMMA 7.3.1. *For four distinct points v_1, v_2, v_3 and v_4 occurring in that order on the boundary of a topological disk κ in $\mathbf{R}^2$, every open Jordan curve joining v_1 and v_3 and every open Jordan curve joining v_2 and v_4, which are both included in κ, must intersect.*

We can then prove the following, for X being W or E.

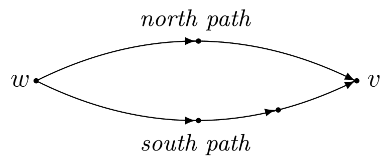
LEMMA 7.3.2. *For every disk D-graph D the list $L_X(D)$ is grounded in D.*

PROOF. In this proof we use the notation introduced in §1.9. Suppose we have $L_X(D)\colon v$–u–w. Take a semipath σ in $[v,w]$ and a semipath τ in $[u,t]$ such that t is an $\bar{X}$-vertex of D. So we have t, v, u and w occurring in that order on the boundary of a compass disk associated with D. It follows from Lemma 7.3.1 that σ and τ must intersect, and hence $\psi_X(v,u,w)$, since D is a plane graph. $\dashv$

Our purpose next is to show that the edges of every cocycle can be linearly ordered, so as to make a list. A theorem in [1] (Theorem 3, Section 2.2) asserts and proves that to a cycle of a plane pseudograph in the sense of [8] (Chapter 2, with multiple edges and loops) there corresponds a cocycle in the dual graph. This theorem asserts also the converse—namely, that to a cocycle there corresponds a cycle in the dual graph—but without proof. This converse assertion is close to what we need for the linear orderability of the edges of every cocycle, but since it is not exactly the same (we pass from a cocycle not to a cycle in the dual graph, but to a list, which is not cyclic), since the notions of graph in question are not exactly the same, and since [1] does not provide a proof, we give an independent proof of what we need.

For a plane graph G, a *face* of G is a connected component (in the topological sense) of $\mathbf{R}^2 - \mathcal{U}(G)$. A face of G is an open subset of $\mathbf{R}^2$, in which we do not find the edges and vertices of G. Assuming that $\mathcal{U}(G)$ is included in some sufficiently large disk $\mathcal{D}$, which we may do when G is finite, exactly one of the faces of G is unbounded—namely, the face in which $\mathbf{R}^2 - \mathcal{D}$ is included. This face is the *outer face* of G; the other faces are the *inner faces* of G. (The terminology of this paragraph agrees with that of [3], Section 4.2, and is close to that of [1], Section 2.2, and [8], Chapter 11.) The *boundary* of a face f is the closure of f minus f.

An inner face f in a plane graph will be called *bipolar* when its boundary is made of two paths from a vertex w to a vertex v. The two distinct paths must be non-trivial and w must be distinct from v. We call the path with the face on the right-hand side the *north path*, and the path with the face on the left-hand side the *south path*, as in the following picture:



The left and right position of the face is here determined by the orientation we have assumed for $\mathbf{R}^2$ (see §7.1, where we have decided upon the W-meridian and E-meridian). The notion of bipolar face (not under that name) may be found in Proposition 2.6 of [13], which we will state after introducing some other notions.

When a graph G has a single W-vertex that vertex is the *source* of G, and when G has a single E-vertex that vertex is the *sink* of G.

Let a *pasting scheme* be a finite plane graph with source and sink, which are distinct, which are both on the boundary of the outer face, and which are such that for every vertex v there is a path from the source to v and a path from v to the sink; moreover, every inner face is bipolar. In [13], where one may find this definition, it is shown in Proposition 2.6 that an equivalent alternative definition of pasting scheme is obtained by replacing the requirement of bipolarity for inner faces by the requirement of acyclicity for the graph.

It is not difficult to show by induction that every M-graph (see §6.6) that is not the empty graph is isomorphic to a pasting scheme. Conversely, one can show that every pasting scheme is an M-graph. The proof of that would proceed by induction on the number of inner faces in the pasting scheme (cf. Proposition 2.10 of [13]). It is easy to see that every graph isomorphic to an M-graph is an M-graph, and so we may conclude that a non-empty graph is an M-graph iff it is isomorphic to a pasting scheme. (We have found it more convenient in Chapter 6 to allow the empty graph as an M-graph for the reasons mentioned at the end of §6.3.)

For D a disk D-graph, consider a plane graph D' obtained by adding two new vertices s and t, and new edges from s to every W-vertex of D, and from every E-vertex of D to t, such that s and t are on the boundary of the outer face of D'. It is easy to conclude that D' is a pasting scheme by relying on the alternative definition mentioned above. We call D' a *source-sink closure* of D.

Then, by the right-to-left direction of Proposition 2.6 of [13], which says that acyclicity implies the bipolarity of inner faces, we have the following.

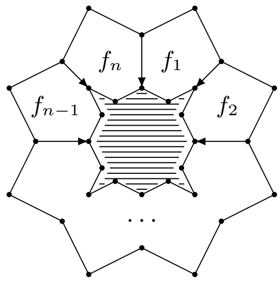
LEMMA 7.3.3. *Every inner face of a disk D-graph is bipolar.*

It is enough to note that, for a disk D-graph D, a source-sink closure D' of D is acyclic, and if every inner face of D' is bipolar, so is every inner face of D.

When two distinct bipolar inner faces f and g of a plane graph share an edge a so that a is in the south path of f and in the north path of g, we will say that f *precedes* g. We can prove the following.

LEMMA 7.3.4. *There is no sequence $f_1, \dots, f_n$, with $n \geq 2$, of inner faces of a disk D-graph such that for every i in $\{1, \dots, n-1\}$ we have that f precedes f_{i+1} and f_n precedes f_1.*

Otherwise, we would not have a disk D-graph, because we would have either something like



where at least one E-vertex would be in the shaded area, and hence it would not be on the boundary of a compass disk associated with our disk D-graph, or we would have something like the graph in the dual picture with inverted arrows, where the same thing holds for at least one W-vertex. The E-vertex or W-vertex in question must be in the shaded area because all the inner faces $f_1, \dots, f_n$ are bipolar.

For a cocycle C of a disk D-graph D, a face of D is *C-cocyclic* when an edge of C belongs to the boundary of that face. We can prove the following.

LEMMA 7.3.5. *Every C-cocyclic inner face of a disk D-graph D contains exactly two edges of the cocycle C of D, one of which is in the north path and the other in the south path.*

PROOF. If either in the north or in the south path we had more than one edge from C, then the componential graph $C_C(D)$ (see §1.3) would not be acyclic. The first edge from C in the north or south path would connect D_1 with D_2, for D_1 and D_2 vertices of $C_C(D)$, while from the second edge from C in that path we would have that it must connect D_2 with D_1.

There cannot be a single edge from C in a C-cocyclic face of D; otherwise, (‡) of §1.3 would not hold. ⊣

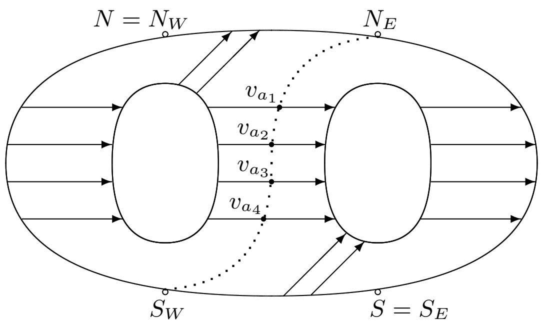
For a cocycle C of a disk D-graph D, and for f and g being C-cocyclic inner faces of D, let us write fP_Cg, and say that f is a *C-predecessor* of g, while g is a *C-successor* of f, when there is an edge in C that is in the south path of f and in the north path of g. It is clear that if fP_Cg, then f precedes g, according to the definition before Lemma 7.3.4. By Lemma 7.3.5, the relation P_C is linear in the following sense: if f_1P_Cg and f_2P_Cg, then $f_1 = f_2$, and if fP_Cg_1 and fP_Cg_2, then $g_1 = g_2$.

From Lemmata 7.3.4 and 7.3.5, and from $C_C(D)$ having exactly two vertices, we may conclude that if we have C-cocyclic inner faces in our disk

D-graph D, then they make a list $f_1 \dots f_n$, with $n \geq 1$, such that f_1 has no C-predecessor, f_n has no C-successor, and if $n \geq 2$, then for every i in $\{1, \dots, n-1\}$ we have $f P_C f_{i+1}$. The cocycle C has a single edge iff there are no C-cocyclic inner faces of D, and our list is empty.

Out of such a non-empty list we make a list $L(C)$ of the edges of C by starting with the edge in the north path of f_1, and by passing to the edge in the south path of f_1. If $n \geq 2$, and we have reached the edge in the south path of f_i for i in $\{1, \dots, n-1\}$, then that edge is the edge in the north path of f_{i+1}, and we pass to the edge in the south path of f_{i+1}. We proceed in that manner until we reach the edge in the south path of f_n (for an example, see the next picture). If C has a single edge, then the list made of that edge is $L(C)$.

We will show next how to make out of a disk D-graph D two disk D-graphs D'_W and D'_E closely related to the D-graphs D_W and D_E obtained by cutting D through a cocycle C (see §1.10). How we obtain D'_W and D'_E should be clear from the following picture, and the explanations we give:

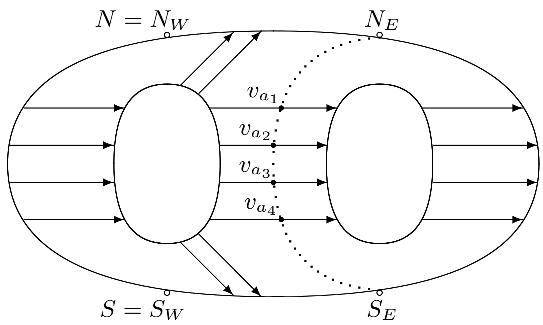


Let $C = \{a_1, a_2, a_3, a_4\}$ be our cocycle, which in the picture is represented by the four edges on which we have chosen the points v_{a_1}, v_{a_2}, v_{a_3} and v_{a_4}; these points are not end points. Here $L(C)$ is $a_1a_2a_3a_4$.

The points N and S are respectively the north pole and the south pole of the compass disk κ associated with D, whose boundary is the outermost circle in the picture. The point N becomes the north pole N_W of the compass disk κ_W associated with D'_W, and N_E is a point on the boundary of κ that may be joined with v_{a_1} by a Jordan curve—a dotted line in our picture—which besides v_{a_1} does not contain any point from $\mathcal{U}(D)$ (see §7.1). The point N_E is the north pole of the compass disk κ_E associated with D'_E. We connect analogously by Jordan curves, represented by dotted lines, v_{a_1} with v_{a_2}, the point v_{a_2} with v_{a_3}, the point v_{a_3} with v_{a_4}, and finally v_{a_4}

with a point S_W on the boundary of κ. This last point is the south pole of κ_W, while S, which is the south pole of κ, is also the south pole of κ_E. The boundary of κ_W is made of the west side of the boundary of κ from N_E to S_W together with the Jordan curve represented by the dotted line, which is the union of all the dotted lines introduced above. The boundary of κ_E is made analogously with the east side. For X being W or E, the disk D-graph D'_X is that part of D within κ_X, with v_{a_1}, v_{a_2}, v_{a_3} and v_{a_4} as new $\bar{X}$-vertices.

Another possible situation is



and other possibilities are treated analogously.

The disk D-graph D'_W differs from the D-graph D_W obtained by cutting D through C by having, for every i in $\{1, 2, 3, 4\}$, the edge a_i replaced by "the west half" of a_i; the vertices are the same. Note that with our choice of v_{a_i}, the D-graph D_W is not a plane graph: v_{a_i} is not an end point of a_i. Analogously, in D'_E we have "the east half" of a_i instead a_i, which is in D_E.

By Lemma 7.3.2, the list $L_{\bar{X}}(D'_X)$ is grounded in D'_X, and it is easy to conclude that the same list is grounded in D_X. It can be verified that the lists $L_E(D_W)$ and $L_W(D_E)$ are compatible. In our example, both lists have $v_{a_1}v_{a_2}v_{a_3}v_{a_4}$ as the common sublist, and it is easy to see that the requirements of compatibility are met. From that we may infer that D_W and D_E are P-compatible. Hence we have that every disk D-graph is a P$'''$-graph, and hence we have the following.

PROPOSITION 7.3.6. *Every disk D-graph is a P-graph.*

It is easy to see that every graph isomorphic to a P-graph is a P-graph (as we remarked already at the end of §2.3), and so this proposition and Proposition 7.2.1 yield the following.

PROPOSITION 7.3.7. *A graph is a P-graph iff it is isomorphic to a disk D-graph.*

As other corollaries, we have that a graph is a P-graph iff it is isomorphic to an eastward-growing disk D-graph, and that every disk D-graph is isomorphic to an eastward-growing disk D-graph. A survey of criteria for eastward growing in the plane (i.e. for upward planarity) and of related questions may be found in [7].

§7.4. D1′-graphs

Let a *disk D1-graph* be defined as a disk D-graph in §7.1 by substituting "D1-graph" for "D-graph" (for the notion of D1-graph see §6.5). (Note that the empty graph is trivially a disk D1-graph.)

It is easy to derive from Propositions 7.2.1, 7.3.6 and 7.3.7 that the assertions of these propositions hold when "D1-graph" is substituted for "D-graph" and "P1-graph" is substituted for "P-graph". (Related matters are considered in §7.5.)

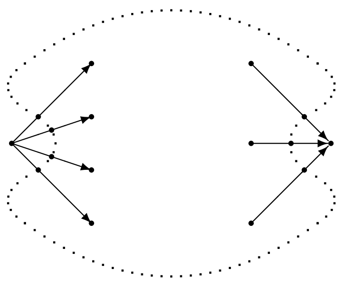
Let a *D1′-graph* be a graph that is finite, acyclic, incidented and has a source and sink (see §1.2 and §7.3 for these notions). This definition differs from the definition of D1-graph by replacing the requirement of W-E-functionality by the requirement of possessing a source and sink. While D1-graphs need not be weakly connected, D1′-graphs are always such.

Let a *disk D1′-graph* be defined as a disk D-graph in §7.1 by substituting "D1′-graph" for "D-graph", as we did for the notion of disk D1-graph above.

A *source-sink closure* of a non-empty disk D1-graph is defined in the same manner as a source-sink closure of a disk D-graph (see §7.3, before Lemma 7.3.3); just substitute "D1-graph" for "D-graph". We can prove the following.

PROPOSITION 7.4.1. *A graph is a source-sink closure of a disk D1-graph iff it is a disk D1′-graph.*

PROOF. The direction from left to right is obtained immediately by a stretching of the compass disk towards West and East. The other direction is also straightforward. After removing the source and sink together with their small neighbourhoods, and replacing them by new distinct end points on the remainder of the incident edges so as to ensure W-E-functionality, we shrink the compass disk as in the following picture:



⊣

Proposition 7.4.2. *A graph is a disk D1′-graph iff it is a pasting scheme.*

Proof. From left to right it is enough to remark that the source and sink of a D1′-graph must be on the boundary of the outer face because they are on the boundary of an associated compass disk. From finiteness, acyclicity and incidentedness we infer that for every vertex v there is a path from the source to v and a path from v to the sink. The other direction is even easier to prove. ⊣

With Propositions 7.4.1 and 7.4.2 we have obtained two other alternative definitions of the notion of pasting scheme.

§7.5. Realizing D1′-graphs

We will consider in this section the question when a D1′-graph is isomorphic to a disk D1′-graph, i.e., when it is realizable in the plane as a pasting scheme.

We define first for a D1-graph D the D1′-graph D', which is the *abstract source-sink closure* of D: the graph D' differs from D by replacing all its W-vertices by a new vertex u_W, all its E-vertices by a new vertex u_E, and by assuming for every X-edge a, where X is W or E, that $X(a) = u_X$.

Next we define for a D1′-graph D the D1-graph D^-, which is the *D1-interior* of D: the graph D^- is obtained from D by rejecting its source and sink, and by assuming that for every X-edge a of D the vertex $X(a)$ is a new vertex v^X_a. We have that if $a_1 \neq a_2$, then $v^X_{a_1} \neq v^X_{a_2}$.

It is clear that for every D1-graph D we have that D'^- is isomorphic to D, and that for every D1′-graph D we have that $D^{-'}$ is isomorphic to D.

We say that a D1′-graph D is *disk realizable* when there is a disk D1′-graph isomorphic to D. We have two more analogous definitions obtained by substituting "D1-graph" and "D-graph" respectively for "D1′-graph". It is clear that we have the following two lemmata.

LEMMA 7.5.1. *If the D1′-graph D is disk realizable, then the D1-graph D^- is disk realizable.*

LEMMA 7.5.2. *If the D1-graph D is disk realizable, then the D1′-graph D' is disk realizable.*

From these lemmata we infer the following proposition.

PROPOSITION 7.5.3. *For every D1′-graph D we have that D is disk realizable iff the D1-graph D^- is disk realizable.*

We can also prove the following.

PROPOSITION 7.5.4. *For every D1-graph D we have that D is disk realizable iff every component of D is disk realizable.*

Every component of a D1-graph D is either a D-graph or a straight single-edge graph (see §6.5). Straight single-edge graphs are of course always disk realizable (as disk D1-graphs of a particularly simple kind), and a D-graph is disk realizable iff it is a P-graph, by Proposition 7.3.7. So we have reduced the question of disk realizability of D1′-graphs to the question whether a graph is a P-graph, and to answer this last question the notion of P‴-graph suggests the most viable criterion, among those we have considered.

Another, easier, way of reducing the question of disk realizability of D1′-graphs to the notion of P-graph is to pass from a D1′-graph D to a D-graph $D^\dagger$, which is obtained from D by adding a new source $s^\dagger$ and a new sink $t^\dagger$, and two new edges, from $s^\dagger$ to the source s of D, and from the sink t of D to $t^\dagger$. It is easy to see that the D1′-graph D is disk realizable iff the D-graph $D^\dagger$ is disk realizable, i.e., iff $D^\dagger$ is a P-graph.

It is clear that the added edges from $s^\dagger$ to s and from t to $t^\dagger$ do not play an essential role here. We have them only to conform to our definition of D-graph. With a more general notion, we could dispense with this addition.

This more general notion could be based on a wider class of basic D-graphs, which could also look like

with the vertex v in both pictures considered as an *inner* vertex. This presupposes a new notion of inner vertex. Out of these enlarged basic graphs, we would obtain the other graphs in our enlarged family of P-graphs with the operation of juncture.

One could envisage further generalizations, and investigate juncture in these wider contexts as an operation for building graphs that correspond to diagrams of ordinary categories that are not only commuting diagrams of arrows (the diagrams of the last two pictures are not commuting diagrams). Juncture, which consists in identifying the tokens of the same edges in different diagrams, would in this perspective replace ordinary composition of arrows in categories, which consists in such an identifying of tokens of the same object.

This was not our point of view in this work. For us, juncture was an operation on graphs that correspond to diagrams of 2-cells, or from the point of view of ordinary categories, just to commuting diagrams of arrows. The operation of juncture was applied not to graphs that correspond directly to the diagrams of 2-cells, but to graphs that are a kind of dual of these graphs. In §7.6, the last section of this work, we deal with this duality.

§7.6. Duality

We need the following notions for the definition of dual of a disk D1′-graph. The edges of a disk D1′-graph D may be of four kinds.

(1) An edge a may separate two inner faces f and g of D, in which case when f precedes g (see §7.3 before Lemma 7.3.4) we say that a is an *interior edge on the way from f to g*.

(2) An edge a may separate the outer face of D from an inner face f so that a is in the north path of f; in that case we say that a is a *northern outer edge of f*.

(3) An edge a may separate an inner face f from the outer face of D so that a is in the south path of f; in that case we say that a is a *southern outer edge of f*.

(4) An edge a may be such that there is no inner face with a belonging to its boundary; in that case we say that a is a *totally outer edge*.

When from a compass disk κ associated with a disk D1′-graph D we reject all the points of $\mathcal{U}(D)$ (see §7.1) and all the inner faces of D, we are left with two disjoint sets of points of κ, which we call κ_N, in which we find the north pole of κ, and κ_S, in which we find the south pole of κ. We can then pass to our definition of dual.

A *dual* of a disk D1′-graph D is a plane graph D^* obtained as follows. For every inner face f of D a point f^* from f will be a vertex of D^*, and, moreover, we have as additional vertices of D^* the north pole N and the south pole S of a compass disk κ associated with D. For every interior edge a on the way from the inner face f to the inner face g we have as an edge of D^* a Jordan curve a^* joining f^* with g^*, such that $a^* \subseteq f \cup g \cup a$; we take that $W(a^*) = f^*$ and $E(a^*) = g^*$. For every northern outer edge a of an inner face f we have as an edge of D^* a Jordan curve a^* joining the vertex N with f^*, such that $a^* \subseteq \kappa_N \cup f \cup a$; we take that $W(a^*) = N$ and $E(a^*) = f^*$. For every southern outer edge a of an inner face f we have as an edge of D^* a Jordan curve a^* joining f^* with the vertex S, such that $a^* \subseteq f \cup \kappa_S \cup a$; we take that $W(a^*) = f^*$ and $E(a^*) = S$. For every totally outer edge a we have as an edge of D^* a Jordan curve joining the vertex N with the vertex S, such that $a^* \subseteq \kappa_N \cup \kappa_S \cup a$; we take that $W(a^*) = N$ and $E(a^*) = S$. This concludes our definition of D^*.

Note that we have required that D^* be a plane graph. So we must ensure that the Jordan curves that make its edges intersect only in the vertices of D^* that are the end points of these edges, as in condition (2) of the definition of plane graph (see §7.1).

We can now prove the following.

PROPOSITION 7.6.1. *For every disk D1′-graph D the graph D^* is a disk D1′-graph.*

PROOF. We check first that D^* is a D1′-graph. It is clear that it is finite, and, by a lemma for D1′-graphs analogous to Lemma 7.3.4, we obtain acyclicity. It is clear that D^* is incidented, and finally the vertex N is the source, while the vertex S is the sink of D^*. The compass disk associated with D will also be associated with D^*, with the new north pole being the sink of D, and the new south pole being the source of D. ⊣

For a graph G, which is $W, E\colon A \to V$, let G^{op} be the graph $W^{\mathrm{op}}, E^{\mathrm{op}}\colon A \to V$ such that $W^{\mathrm{op}} = E$ and $E^{\mathrm{op}} = W$. It is possible to show that for a D1′-graph D the graph D^{**} is isomorphic to D^{op}, but we will not go into the proof of that. (Analogous facts in graph theory are usually skipped over, as in [1], Section 2.2, or left as exercises, as in [2], Exercise 9.2.4, Section 9.2.)

Pasting schemes may be combined one with another with two operations that correspond to vertical and horizontal composition in 2-categories (cf. §6.6). Our goal was to study the operation definable in terms of these two operations that consists in gluing two pasting schemes along a common path on the boundaries, as in the first picture of §1.1. When applied to 2-cells, we called this operation *juncture* in §1.1, but when applied to pasting schemes, we better find now another name for it, not to create confusion. We could call it *gluing*.

Instead of dealing with pasting schemes, i.e. disk D1′-graphs, we pass to modified duals of these graphs. For every disk D1′-graph D we take the D1-interior D^{*-} of D^* (see §7.5), which is easily seen to be isomorphic to a disk D1-graph (by proceeding as in the proof of Proposition 7.4.1; for examples of passing from D to D^{*-} see the end of §6.7). For the modified duals D^{*-}, gluing becomes juncture, and the passing from D to D^{*-} was made to obtain the operation of juncture, more manageable in wider classes of graph, which need not be plane. We did not stop at the dual D^*, which is a D1′-graph, but passed further to the D1-graph D^{*-}, because the analogue of juncture for D1′-graphs would be less manageable. The analogue of juncture for two D1′-graphs D_W and D_E is best defined as corresponding to $(D_W^- \Box D_E^-)'$ (with $'$ being the abstract source-sink closure of §7.5).

We forgot about disk realizability, to obtain more general notions, and we ended up with the notion of D1-graph and the essential ingredient of that notion, which is the notion of D-graph. Neither of these two notions has a natural dual correlate in the world of pasting schemes. (These would be, roughly, pasting schemes with vertices removed.) For the notion of D1-graph we may then ask when it is disk realizable, and this disk realizability reduces to the disk realizability of D-graphs (see Proposition 7.5.4). This last question is answered by Proposition 7.3.7, and our work serves to explain the notion of P-graph of that proposition.

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