Proof-Net Categories

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Preface

This study is the continuation of a project in categorial proof theory, which occupied us in the last few years and yielded the book [22]. This is a kind of appendix to that book, whose results are applied here. An acquaintance with that previous book is not absolutely necessary, provided the reader is prepared to trust the results on which we rely. It is, however, very desirable. Practically no other literature is presupposed, except for the sake of motivation. We rely only on very standard notions of logic and category theory, which, if by any chance they are not already known to the reader, may be found in [22].

The aim and context of our work are set forth in the introductory chapter. Our results should be of general interest to graduate students and researchers in general proof theory. They demonstrate how generality of proofs provides a criterion of identity for proofs. We believe these results bring something also to categorists interested in coherence questions, to whom they may illustrate the usefulness of syntactical methods in category theory. They should be of particular interest to investigators of linear logic, symmetric monoidal closed categories and star-autonomous categories. These or related matters seem to be interesting too in the borderline areas of theoretical computer science. This is, however, not a text belonging to that science. Our aims, our terminology and our style come from the related but, nevertheless, different and older field of logic.

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Chapter 1

Introduction

In this introductory chapter we state the aim of this work, and present the context, i.e. previous work related to the subject matter we treat. We also give a summary of the whole text.

§1.1. Aim and context

The aim of this work is to give a systematic account of the connection that exists between star-autonomous categories and the Kelly-Mac Lane graphs implicit in proof nets for the multiplicative fragment without propositional constants of linear logic. Star-autonomous categories are symmetric monoidal closed categories that have an object $\perp$ such that the canonical natural transformation from the identity functor to the functor $(\_ \rightarrow \perp) \rightarrow \perp$ is a natural isomorphism (see §§3.1-2 and §3.8; here $\_ \rightarrow \_$ is the internal hom-bifunctor).

For some results of this work it will perhaps be claimed that they are known—that they have already been established. We do not believe this claim is justified, and we have decided to present matters anew because we are not satisfied with the treatment they have received up to now. But even if it were true that these results are known, we think that they deserve a systematic and detailed presentation, following the canons of rigour that used to be the rule in logic. We feel there is a need for such a presentation, and we want to supply it.

There are in mathematics theorems that are more difficult to conjecture than to prove. Such is, for example, the Theorem of Pythagoras, or the theorem that $\sqrt{2}$ is not rational. There are, on the other hand, theorems that are more difficult to prove than to conjecture (and there are no doubt theorems were the conjecture and the proof are of equal difficulty).

The results we are going to present are of a kind called in logic *completeness theorems*. Such theorems are often not difficult to conjecture, but their proofs can be quite demanding. The prime example of such a result is the completeness theorem for first-order predicate logic. The axiomatization of this logic existed long before a precise completeness proof was given by Gödel, and throughout this period it was assumed the axiomatization is complete, with a more or less precise notion of completeness being envisaged. Nearer to our topic, we have the coherence theorem for symmetric monoidal closed categories proved by Kelly and Mac Lane in [32] (see the end of §3.1), where it also seems it was easier to conjecture the theorem than to prove it.

From the inception of proof nets in the late 1980s (see [26] and [13]), it could have been realized that they are connected with the graphs one finds in Kelly's and Mac Lane's coherence theorem. The earliest explicit reference for that we know about is [4] (see also [5]). It was also soon suggested that the multiplicative fragment of classical linear logic, which has an involutive negation that satisfies De Morgan laws, is closely related to Barr's star-autonomous categories, which stem from [1] (see [33], [42] and [2]). A number of results have been proposed since as completeness results connecting proof nets and particular categories (see the beginning of [28] for a recent survey). It seems to be an accepted opinion nowadays that, in the words of [28], "...the identifications [of proofs imposed by proof nets] correspond to coherences of free star-autonomous categories". The purpose of our work is to examine this opinion, and find out how much truth there is in it.

The problem with this opinion is that on the side of proof nets we do not have in the standard treatment the multiplicative propositional constants, while on the side of star-autonomous categories we have the corresponding unit objects. In the presence of these units, an unrestricted coherence theorem with respect to graphs of the Kelly-Mac Lane kind is not forthcoming. Kelly and Mac Lane had for their coherence theorem for symmetric monoidal closed categories of [32] a proviso concerning the unit object of the monoidal structure (see the end of §3.1, and see [43] for further work concerning this proviso), but we are not aware that a similar coherence involving a proviso for the units of star-autonomous categories has been proved up to now. (We provide such a result in Chapter 4 below.)

Two courses are open in this situation. The first course, which we will follow, is to reject the units on the side of star-autonomous categories, define precisely the resulting notion of category, and prove a standard, unrestricted, coherence result for it, akin to Kelly's and Mac Lane's coherence. (We do that in Chapter 2.) It is desirable to show here that the proposed notion of star-autonomous category without units catches exactly the corresponding fragment of star-autonomous categories, in a sense to be made precise in terms of category theory. (We do that in Chapters 3 and 4.) After that only, one can establish a match between the equations assumed for the categories and those imposed by the proof nets without the multiplicative propositional constants, and so vindicate the established opinion.

Relying on the unrestricted coherence for star-autonomous categories without units, one can obtain a restricted coherence theorem for standard star-autonomous categories. This coherence theorem has a proviso concerning the units: they are allowed to occur only in places such that the objects in which they are involved are isomorphic either to objects not involving the units or to one of the units. (This is the coherence result of Chapter 4 mentioned above, which will be phrased precisely in that chapter.) We will show that this proviso is of the same kind as the proviso that Kelly and Mac Lane had.

The second course is to add the multiplicative propositional constants without restriction on the side of proof nets, and claim that a completeness result connecting the modified proof nets and star-autonomous categories is the desired coherence result. This second course is more favoured in the existing literature, cited below and in Chapter 7. We should immediately notice that with this course coherence cannot be understood in the sense of Kelly and Mac Lane. Also, no precise notion of star-autonomous category without units arises.

As far as we know, the only coherence result in the style of Kelly and Mac Lane proved up to now for star-autonomous categories is still Kelly's and Mac Lane's own result of [32], which is, as we said above, about symmetric monoidal closed categories with a proviso concerning the unit object of the monoidal structure. Richard Blute in [5] purports to prove a general coherence result, which should yield coherence for star-autonomous categories without units with respect to Kelly-Mac Lane graphs. We find, however, this proof excessively wanting. The notion of star-autonomous category without units is not precisely defined. We do not know what is the "usual theory without units" of star-autonomous categories (mentioned in [5], pp. 9, 15), and one of our purposes in this work is to supply a language of arrow terms for that theory and the appropriate equations between these arrow terms. We could not find either of these in [5], or anywhere else. Recently, attempts have been made in [35] and [27] to define a notion of star-autonomous category without units, but the approach of these papers, different from ours, is not equational (at least not in our sense).

It is not, however, the case that once the notion of star-autonomous category without units is made precise, one obtains from the sketch in [5] (p. 23, right-to-left direction of Theorem 10.2) a recipe for proving coherence for this notion. A substantial part of the proof is covered by the sentence: "This amounts to a straightforward case analysis." This sentence occurs in a context where no specific equations are stated, and it is claimed that these equations cover a cut-elimination procedure. This is usually the most arduous part of a proof of coherence (see, for example, [32]).

Robert Seely and Robin Cockett in [12] (p. 104) consider coherence for star-autonomous categories without units to be "fairly straightforward, even trivial", and they refer to [5] and [6] for an exposition. We have already discussed [5], while in [6] we find neither a definition of star-autonomous category without units, nor a coherence result for them in the sense of Kelly and Mac Lane. Instead, the latter paper is about coherence for star-autonomous categories with units (and it is presumed that these are the weakly distributive categories of [11] with negation added) with respect to an extension of proof nets with multiplicative propositional constants. The subject of our work is to a great extent this matter previously dismissed as straightforward, or even trivial.

We will give an equational formulation of the notion of star-autonomous category without units, which we call *proof-net category*, and we will prove coherence for this notion with respect to Kelly-Mac Lane graphs, which means that there is a faithful functor from the proof-net category freely generated by a set of objects into the category whose arrows are these graphs. Another possibility would be to define the notion of proof-net category by coherence, i.e. by the existence of the faithful functor into the category whose arrows are graphs. This way, however, we would have no information about the axioms, which are the combinatorial building blocks of our notion.

The notion of monoidal category was introduced in such a nonaxiomatic way, via coherence, by Bénabou in [3], and in the axiomatic way, such as we favour, by Mac Lane in [37]. For Bénabou, coherence is built into the definition, and for Mac Lane it is a theorem. One could analogously define the theorems of classical propositional logic as being the tautologies (this is done, for example, in [9], Sections 1.2-3), in which case completeness would not be a theorem, but would be built into the definition.

To take another example, we could easily define nonaxiomatically a notion of Boolean category with respect to graphs of the Kelly-Mac Lane kind. (In this notion, conjunction would not be a product, because the diagonal arrows and the projections would not make natural transformations, and, analogously, disjunction would not be a coproduct; see [22], Section 14.3.) The resulting notion would not be trivial—the resulting freely generated categories would not be preorders—, but its nonaxiomatic definition would be trivial. We are looking for nontrivial axiomatic definitions. Such definitions give information about the combinatorial building blocks of our notions, as Reidemeister moves give information about the combinatorial building blocks of knot equivalence (see [8], Chapter 1). Our axiomatic equational definition of proof-net category is of this nontrivial, combinatorially informative, kind. Coherence of proof-net categories is for us a theorem, whose proof requires considerable effort.

§1.2. Summary

This study is a continuation of [22], whose ideas and style we have followed in general. Many notions we need are exposed more systematically in that book, which the reader may consult for more detailed explanations and definitions, and also for motivation from the perspective of general proof theory or categorial proof theory. At some key points, we rely on results proved before. In Chapter 2 we rely on matters proved in [20], [21] and [22], and in particular on a coherence result from [22] (Symmetric Net Coherence of Section 7.6). In Chapter 3 we rely on the coherence result of Kelly and Mac Lane for symmetric monoidal closed categories of [32]. We rely also on some well-known elementary notions of category theory, which may all be found in [38] or [22], and for the sake of motivation we rely on some acquaintance with linear logic and the proof nets of [26]. Except for that, we have strived to make our exposition self-contained to a great extent.

First, we give in Chapter 2 a precise definition of a notion that may be considered to correspond to star-autonomous categories without units. This notion, which we call proof-net category, is obtained by extending with an operation that corresponds to negation the notion of symmetric net category of [22] (Section 7.6); the notion of symmetric net category corresponds to the notion of linear (alias weakly) distributive category of [11] without units. For proof-net categories we prove in Chapter 2 a coherence result with respect to Kelly-Mac Lane graphs.

In Chapter 3, we prove precisely in categorial terms the equivalence of the notion of star-autonomous category with a notion amounting to the notion of linearly distributive category with negation of [11]. The latter notion is obtained by extending with the units our notion of proof net category. This categorial equivalence result was foreshadowed in [11] (Section 4, Theorem 4.5), but there its "straightforward" proof was said to depend on "pretty horrid" diagrams, and practically the whole of it was left "to the faith of the reader". The proof we supply is indeed pretty lengthy, though we have shortened it considerably by relying on Kelly's and Mac Lane's coherence for symmetric monoidal closed categories and on our coherence from Chapter 2 for proof-net categories. We can only imagine how "horrid" it would be without these tools, which are not mentioned in [11].

In Chapter 4, we prove that with the notion of proof-net category we have not only caught the notion of star-autonomous category without units, but with its help we can also obtain a coherence result for star-autonomous categories with respect to Kelly-Mac Lane graphs—a result of the same kind as Kelly's and Mac Lane's coherence result for symmetric monoidal closed categories. This result involves a proviso concerning the units, but does not exclude them completely (as we announced in the preceding section). This coherence of star-autonomous categories is a powerful tool for verifying whether a diagram of arrows commutes in star-autonomous categories.

After all that, the established opinion on the connection between proof nets and star-autonomous categories, which we mentioned in the preceding section, may be rephrased as the statement that the identifications of proofs imposed by proof nets correspond well to the equations of proof-net categories, and since proof-net categories catch the unit-free portion of star-autonomous categories, the opinion seems vindicated. We find that before it was accepted just on faith.

It is not true, however, that the identifications of proofs imposed by proof nets stem only from the cut-elimination procedure. There are also equations that serve to equate different cut-free proofs. These are equations similar to the so-called *permutative* reductions of natural deduction, which permute the order of rules in cut-free proofs, and also equations that atomize the identity axiomatic sequents. Such equations are indeed incorporated in the usual notion of proof net, and are there invisible, but in modifications of this notion they may reappear (cf. [24]). It is, in general, tricky to justify equations just by reference to cut elimination, because cut elimination tends to be sensitive to a particular syntax, and also to a particular procedure (cf. [14], Section 0.3.1 and passim). A justification independent of the vagaries of syntax is obtained by coherence theorems in the style of Mac Lane.

In Chapter 5, we consider how the assumptions concerning the involutive unary operation corresponding to negation, which we have in proof-net categories and star-autonomous categories, are tied to a particular kind of adjunction where an endofunctor is adjoint to itself.

In Chapter 6, we consider proof-net categories that have arrows corresponding to the mix principle of linear logic, and we prove coherence for the resulting notion by adapting the coherence proof for proof-net categories of Chapter 2.

In Chapter 7, the final chapter, we discuss the relationship between the Kelly-Mac Lane graphs and proof nets, which justifies the name we have given to proof-net categories. In general proof theory, one of the main problems is the investigation of identity of proofs (see [15] or [22], Sections 1.3-4), and it is desirable to find efficient means to check this identity. We approach coherence questions in that spirit, and we expect coherence theorems to yield a decision procedure (preferably easy) to answer the question whether a diagram of arrows commutes. From that standpoint, Kelly-Mac Lane graphs are the relevant core of proof nets, which we can use to answer efficiently the question whether two proofs are equal in the multiplicative fragment without propositional constants of linear logic, and also, according to the coherence theorem of Chapter 4, in a larger fragment of linear logic, where the multiplicative propositional constants occur at particular places. At the very end, we discuss further papers related to our work, and express some opinions on proof nets in the context of general proof theory.

Chapter 2

Coherence of Proof-Net Categories

In this chapter we define our notion of proof-net category. This notion is based on the notion of symmetric net category of [22] (Section 7.6); these are categories with two multiplications, $\wedge$ and $\vee$, associative and commutative up to isomorphism, which have moreover arrows of the *dissociativity* type $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$ (called *linear* or *weak* distribution by the authors of [11]). The symmetric net category freely generated by a set of objects is called **DS**. To symmetric net categories we add an operation on objects corresponding to negation, which is involutive up to isomorphism. With these operations come appropriate arrows. A number of equations between arrows, of the kind called *coherence conditions* in category theory, are satisfied in proof-net categories.

We introduce a category *Br* whose arrows are called *Brauerian split equivalences* of finite ordinals. These equivalence relations, which stem from results in representation theory from the 1930s, amount to the graphs used by Kelly and Mac Lane for their coherence theorem of symmetric monoidal categories. Brauerian split equivalences express generality of proofs in linear logic (see [20] and [21]).

The coherence theorem for proof-net categories says that there is a faithful functor from the proof-net category $\mathbf{PN}^{\neg}$ freely generated by a set of objects into *Br*. We call theorems of this kind *coherence theorems*. The coherence theorem for $\mathbf{PN}^{\neg}$ yields an elementary decision procedure for verifying whether a diagram of arrows commutes in $\mathbf{PN}^{\neg}$, and hence also in every proof-net category. This is a very useful tool, which will facilitate calculations later on.

The coherence theorem for $\mathbf{PN}^{\neg}$ is proved by finding a category $\mathbf{PN}$, equivalent to $\mathbf{PN}^{\neg}$, in which negation can be applied only to the generating objects, and coherence is first established for $\mathbf{PN}$ by relying on coherence for symmetric net categories, previously established in [22] (Chapter 7), and on an additional normalization procedure involving negation.

§2.1. The category DS

The objects of the category **DS** are the formulae of the propositional language $\mathcal{L}_{\wedge,\vee}$, generated from a set $\mathcal{P}$ of propositional letters, which we call simply *letters*, with the binary connectives $\wedge$ and $\vee$. We use $p, q, r, \dots,$ sometimes with indices, for letters, and $A, B, C, \dots,$ sometimes with indices, for formulae. As usual, we omit the outermost parentheses of formulae and other expressions later on.

To define the arrows of **DS**, we define first inductively a set of expressions called the *arrow terms* of **DS**. Every arrow term of **DS** will have a *type*, which is an ordered pair of formulae of $\mathcal{L}_{\wedge,\vee}$. We write $f\colon A \vdash B$ when the arrow term f is of type (A, B). (We use the turnstile $\vdash$ instead of the more usual $\rightarrow$, which we reserve for a connective and a bifunctor.) We use $f, g, h, \dots,$ sometimes with indices, for arrow terms.

For all formulae A, B and C of $\mathcal{L}_{\wedge,\vee}$ the following *primitive arrow terms*:

$$\mathbf{1}_A\colon A \vdash A,$$

$$\hat{b}^{\rightarrow}_{A,B,C}\colon A\wedge(B\wedge C)\vdash(A\wedge B)\wedge C,\quad \check{b}^{\rightarrow}_{A,B,C}\colon A\vee(B\vee C)\vdash(A\vee B)\vee C,$$

$$\hat{b}^{\leftarrow}_{A,B,C}\colon (A\wedge B)\wedge C\vdash A\wedge(B\wedge C),\quad \check{b}^{\leftarrow}_{A,B,C}\colon (A\vee B)\vee C\vdash A\vee(B\vee C),$$

$$\hat{c}_{A,B}\colon A\wedge B\vdash B\wedge A,\qquad \check{c}_{A,B}\colon B\vee A\vdash A\vee B,$$

$$d_{A,B,C}\colon A\wedge(B\vee C)\vdash(A\wedge B)\vee C$$

are arrow terms of **DS**. If $g\colon A \vdash B$ and $f\colon B \vdash C$ are arrow terms of **DS**, then $f \circ g\colon A \vdash C$ is an arrow term of **DS**; and if $f\colon A \vdash D$ and $g\colon B \vdash E$ are arrow terms of **DS**, then $f\,\xi\,g\colon A\,\xi\,B \vdash D\,\xi\,E$, for $\xi \in \{\wedge, \vee\}$, is an arrow term of **DS**. This concludes the definition of the arrow terms of **DS**.

Next we define inductively the set of *equations* of **DS**, which are expressions of the form $f = g$, where f and g are arrow terms of **DS** of the same type. We stipulate first that all instances of $f = f$ and of the following equations are equations of **DS**:

$(cat\ 1)$ $\quad f \circ \mathbf{1}_A = \mathbf{1}_B \circ f = f \colon A \vdash B,$

$(cat\ 2)$ $\quad h \circ (g \circ f) = (h \circ g) \circ f,$

for $\xi \in \{\wedge, \vee\}$,

$(\xi\ 1)$ $\quad \mathbf{1}_A \,\xi\, \mathbf{1}_B = \mathbf{1}_{A \xi B},$

$(\xi\ 2)$ $\quad (g_1 \circ f_1) \,\xi\, (g_2 \circ f_2) = (g_1 \,\xi\, g_2) \circ (f_1 \,\xi\, f_2),$

for $f \colon A \vdash D$, $g \colon B \vdash E$ and $h \colon C \vdash F$,

$(\overset{\xi}{b}{}^{\rightarrow}\ nat)$ $\quad ((f \,\xi\, g) \,\xi\, h) \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \overset{\xi}{b}{}^{\rightarrow}_{D,E,F} \circ (f \,\xi\, (g \,\xi\, h)),$

$(\hat{c}\ nat)$ $\quad (g \wedge f) \circ \hat{c}_{A,B} = \hat{c}_{D,E} \circ (f \wedge g),$

$(\check{c}\ nat)$ $\quad (g \vee f) \circ \check{c}_{B,A} = \check{c}_{E,D} \circ (f \vee g),$

$(d\ nat)$ $\quad ((f \wedge g) \vee h) \circ d_{A,B,C} = d_{D,E,F} \circ (f \wedge (g \vee h)),$

$(\overset{\xi}{b}\overset{\xi}{b})$ $\quad \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} = \mathbf{1}_{(A \xi B) \xi C}, \qquad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \mathbf{1}_{A \xi (B \xi C)},$

$(\overset{\xi}{b}5)$ $\quad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C \xi D} \circ \overset{\xi}{b}{}^{\leftarrow}_{A \xi B,C,D} = (\mathbf{1}_A \;\xi\; \overset{\xi}{b}{}^{\leftarrow}_{B,C,D}) \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B \xi C,D} \circ (\overset{\xi}{b}{}^{\leftarrow}_{A,B,C} \;\xi\; \mathbf{1}_D),$

$(\hat{c}\,\hat{c})$ $\quad \hat{c}_{B,A} \circ \hat{c}_{A,B} = \mathbf{1}_{A \wedge B},$

$(\check{c}\,\check{c})$ $\quad \check{c}_{A,B} \circ \check{c}_{B,A} = \mathbf{1}_{A \vee B},$

$(\hat{b}\,\hat{c})$ $\quad (\mathbf{1}_B \wedge \hat{c}_{C,A}) \circ \hat{b}^{\leftarrow}_{B,C,A} \circ \hat{c}_{A,B \wedge C} \circ \hat{b}^{\leftarrow}_{A,B,C} \circ (\hat{c}_{B,A} \wedge \mathbf{1}_C) = \hat{b}^{\leftarrow}_{B,A,C},$

$(\check{b}\,\check{c})$ $\quad (\mathbf{1}_B \vee \check{c}_{A,C}) \circ \check{b}^{\leftarrow}_{B,C,A} \circ \check{c}_{B \vee C,A} \circ \check{b}^{\leftarrow}_{A,B,C} \circ (\check{c}_{A,B} \vee \mathbf{1}_C) = \check{b}^{\leftarrow}_{B,A,C},$

$(d\wedge)$ $\quad (\hat{b}^{\leftarrow}_{A,B,C} \vee \mathbf{1}_D) \circ d_{A \wedge B,C,D} = d_{A,B \wedge C,D} \circ (\mathbf{1}_A \wedge d_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B,C \vee D},$

$(d\vee)$ $\quad d_{D,C,B \vee A} \circ (\mathbf{1}_D \wedge \check{b}^{\leftarrow}_{C,B,A}) = \check{b}^{\leftarrow}_{D \wedge C,B,A} \circ (d_{D,C,B} \vee \mathbf{1}_A) \circ d_{D,C \vee B,A},$

for $d^R_{C,B,A} =_{df} \check{c}_{C,B \wedge A} \circ (\hat{c}_{A,B} \vee \mathbf{1}_C) \circ d_{A,B,C} \circ (\mathbf{1}_A \wedge \check{c}_{B,C}) \circ \hat{c}_{C \vee B,A}\colon$
$$(C \vee B) \wedge A \vdash C \vee (B \wedge A),$$

$(d\hat{b})$ $\quad d^R_{A \wedge B,C,D} \circ (d_{A,B,C} \wedge \mathbf{1}_D) = d_{A,B,C \wedge D} \circ (\mathbf{1}_A \wedge d^R_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B \vee C,D},$

$(d\check{b})$ $\quad (\mathbf{1}_D \vee d_{C,B,A}) \circ d^R_{D,C,B \vee A} = \check{b}^{\leftarrow}_{D,C \wedge B,A} \circ (d^R_{D,C,B} \vee \mathbf{1}_A) \circ d_{D \vee C,B,A}.$

The set of equations of **DS** is closed under symmetry and transitivity of equality and under the rules

$$(cong\ \xi)\quad \frac{f = f_1 \qquad\qquad g = g_1}{f\ \xi\ g = f_1\ \xi\ g_1}$$

where $\xi \in \{\circ, \wedge, \vee\}$; if ξ is $\circ$, then $f \circ g$ is defined (namely, f and g have appropriate, composable, types), and analogously for $f_1 \circ g_1$.

On the arrow terms of **DS** we impose the equations of **DS**. This means that an arrow of **DS** is an equivalence class of arrow terms of **DS** defined with respect to the smallest equivalence relation such that the equations of **DS** are satisfied (see [22], Section 2.3, for details).

The equations (ξ 1) and (ξ 2) are called *bifunctorial* equations. They say that $\wedge$ and $\vee$ are biendofunctors (i.e. 2-endofunctors in the terminology of [22], Section 2.4).

It is easy to show that for **DS** we have the equations

$$(\overset{\xi}{b}{}^{\leftarrow}\ nat)\quad (f\ \xi\ (g\ \xi\ h)) \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} = \overset{\xi}{b}{}^{\leftarrow}_{D,E,F} \circ ((f\ \xi\ g)\ \xi\ h),$$

$$(d^R\ nat)\quad (h \vee (g \wedge f)) \circ d^R_{C,B,A} = d^R_{F,E,D} \circ ((h \vee g) \wedge f).$$

We call these equations and other equations with "*nat*" in their names, like those in the list above, *naturality* equations. Such equations say that $\hat{b}^{\rightarrow}$, $\hat{b}^{\leftarrow}$, $\hat{c}$, etc. are natural transformations.

The equations $(d\wedge)$, $(d\vee)$, $(d\hat{b})$ and $(d\check{b})$ stem from [11] (Section 2.1; see [10], Section 2.1, for an announcement). The equation $(d\check{b})$ of [22] (Section 7.2) amounts with $(\check{b}\check{b})$ to the present one.

§2.2. The category $\mathbf{PN}^{\neg}$

The category $\mathbf{PN}^{\neg}$ is defined as **DS** save that we make the following changes and additions. Instead of $\mathcal{L}_{\wedge,\vee}$, we have the propositional language $\mathcal{L}_{\neg,\wedge,\vee}$, which has in addition to what we have for $\mathcal{L}_{\wedge,\vee}$ the unary connective $\neg$.

To define the arrow terms of $\mathbf{PN}^{\neg}$, in the inductive definition we had for the arrow terms of **DS** we assume in addition that for all formulae A and B of $\mathcal{L}_{\neg,\wedge,\vee}$ the following *primitive arrow terms*:

$$\hat{\Delta}_{B,A}\colon A \vdash A \wedge (\neg B \vee B),$$
$$\check{\Sigma}_{B,A}\colon (B \wedge \neg B) \vee A \vdash A,$$

are arrow terms of $\mathbf{PN}^{\neg}$. We call the index B of $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$ the *crown* index, and A the *stem* index. The right conjunct $\neg B \vee B$ in the target of $\hat{\Delta}_{B,A}\colon A \vdash A \wedge (\neg B \vee B)$ is the *crown* of $\hat{\Delta}_{B,A}$, and the left disjunct $B \wedge \neg B$ in the source of $\check{\Sigma}_{B,A}\colon (B \wedge \neg B) \vee A \vdash A$ is the *crown* of $\check{\Sigma}_{B,A}$. We have analogous definitions of crown and stem indices, and crowns, for $\hat{\Sigma}$, $\hat{\Delta}'$, $\hat{\Sigma}'$, $\check{\Delta}$, $\check{\Sigma}'$ and $\check{\Delta}'$, which will be defined below. (The symbol Δ should be associated with the Latin *dexter*, because in $\hat{\Delta}_{B,A}$, $\hat{\Delta}'_{B,A}$, $\check{\Delta}_{B,A}$ and $\check{\Delta}'_{B,A}$ the crown is on the right-hand side of the stem; analogously, Σ should be associated with *sinister*.)

To define the arrows of $\mathbf{PN}^{\neg}$, we assume in the inductive definition we had for the equations of $\mathbf{DS}$ the following additional equations, which we call the $\mathbf{PN}$ *equations* (and *n*ot $\mathbf{PN}^{\neg}$ equations):

$(\hat{\Delta}\ nat)$ $\quad (f \wedge \mathbf{1}_{\neg B \vee B}) \circ \hat{\Delta}_{B,A} = \hat{\Delta}_{B,D} \circ f,$

$(\check{\Sigma}\ nat)$ $\quad f \circ \check{\Sigma}_{B,A} = \check{\Sigma}_{B,D} \circ (\mathbf{1}_{B \wedge \neg B} \vee f),$

$(\hat{b}\hat{\Delta})$ $\quad \hat{b}^{\leftarrow}_{A,B,\neg C \vee C} \circ \hat{\Delta}_{C,A \wedge B} = \mathbf{1}_A \wedge \hat{\Delta}_{C,B},$

$(\check{b}\check{\Sigma})$ $\quad \check{\Sigma}_{C,B \vee A} \circ \check{b}^{\leftarrow}_{C \wedge \neg C,B,A} = \check{\Sigma}_{C,B} \vee \mathbf{1}_A,$

for $\hat{\Sigma}_{B,A} =_{df} \hat{c}_{A,\neg B \vee B} \circ \hat{\Delta}_{B,A} : A \vdash (\neg B \vee B) \wedge A,$

$(d\hat{\Sigma})$ $\quad d_{\neg A \vee A,B,C} \circ \hat{\Sigma}_{A,B \vee C} = \hat{\Sigma}_{A,B} \vee \mathbf{1}_C,$

for $\check{\Delta}_{B,A} =_{df} \check{\Sigma}_{B,A} \circ \check{c}_{B \wedge \neg B,A} : A \vee (B \wedge \neg B) \vdash A,$

$(d\check{\Delta})$ $\quad \check{\Delta}_{A,C \wedge B} \circ d_{C,B,A \wedge \neg A} = \mathbf{1}_C \wedge \check{\Delta}_{A,B},$

$(\check{\Sigma}\hat{\Delta})$ $\quad \check{\Sigma}_{A,A} \circ d_{A,\neg A,A} \circ \hat{\Delta}_{A,A} = \mathbf{1}_A,$

for $\hat{\Delta}'_{B,A} =_{df} (\mathbf{1}_A \wedge \check{c}_{B,\neg B}) \circ \hat{\Delta}_{B,A} : A \vdash A \wedge (B \vee \neg B)$ and

$\check{\Sigma}'_{B,A} =_{df} \check{\Sigma}_{B,A} \circ (\hat{c}_{\neg B,B} \vee \mathbf{1}_A) : (\neg B \wedge B) \vee A \vdash A,$

$(\check{\Sigma}'\hat{\Delta}')$ $\quad \check{\Sigma}'_{A,\neg A} \circ d_{\neg A,A,\neg A} \circ \hat{\Delta}'_{A,\neg A} = \mathbf{1}_{\neg A}.$

It is easy to show that for $\mathbf{PN}^{\neg}$ we have the equations

$$(\hat{\Sigma}\ nat) \qquad (\mathbf{1}_{\neg B\vee B}\wedge f)\circ\hat{\Sigma}_{B,A}=\hat{\Sigma}_{B,D}\circ f,$$
$$(\check{\Delta}\ nat) \qquad f\circ\check{\Delta}_{B,A}=\check{\Delta}_{B,D}\circ(f\vee\mathbf{1}_{B\wedge\neg B}).$$

The naturality equations ($\hat{\Delta}$ *nat*) and ($\check{\Sigma}$ *nat*) together with these say that $\hat{\Delta}$, $\check{\Sigma}$, $\hat{\Sigma}$ and $\check{\Delta}$ are natural transformations in the stem index only, i.e. in the second index.

We also have the following abbreviations:

$$\hat{\Sigma}'_{B,A} =_{df} \hat{c}_{A,B\vee\neg B}\circ\hat{\Delta}'_{B,A} : A\vdash(B\vee\neg B)\wedge A,$$
$$\check{\Delta}'_{B,A} =_{df} \check{\Sigma}'_{B,A}\circ\check{c}_{\neg B\wedge B,A} : A\vee(\neg B\wedge B)\vdash A.$$

If Ξ stands for either Δ or Σ and $\xi\in\{\wedge,\vee\}$, then for every ($\overset{\xi}{\Xi}$ *nat*) equation we have in $\mathbf{PN}^{\neg}$ the equation ($\overset{\xi}{\Xi}{}'$ *nat*), which differs from ($\overset{\xi}{\Xi}$ *nat*) by replacing $\overset{\xi}{\Xi}$ by $\overset{\xi}{\Xi}{}'$, and the index of $\mathbf{1}$ by the appropriate index. For example, we have

$$(\hat{\Delta}'\ nat) \qquad (f\wedge\mathbf{1}_{B\vee\neg B})\circ\hat{\Delta}'_{B,A}=\hat{\Delta}'_{B,D}\circ f.$$

As alternative primitive arrow terms for defining $\mathbf{PN}^{\neg}$ we could take one of $\hat{\Xi}$ or $\hat{\Xi}'$ and one of $\check{\Xi}$ or $\check{\Xi}'$.

We can also derive for $\mathbf{PN}^{\neg}$ the following equations:

$$(\hat{b}\hat{\Delta}\hat{\Sigma}) \qquad \hat{b}^{\leftarrow}_{A,\neg B\vee B,C}\circ(\hat{\Delta}_{B,A}\wedge\mathbf{1}_C)=\mathbf{1}_A\wedge\hat{\Sigma}_{B,C},$$
$$(\hat{b}\hat{\Sigma}) \qquad \hat{b}^{\rightarrow}_{\neg C\vee C,B,A}\circ\hat{\Sigma}_{C,B\wedge A}=\hat{\Sigma}_{C,B}\wedge\mathbf{1}_A.$$

For the first equation, with indices omitted, we have

$$\begin{aligned}\hat{b}^{\leftarrow}\circ(\hat{\Delta}\wedge\mathbf{1}) &= \hat{b}^{\leftarrow}\circ\hat{c}\circ(\mathbf{1}\wedge\hat{\Delta})\circ\hat{c}, \quad \text{by } (\hat{c}\,\hat{c}) \text{ and } (\hat{c}\ nat),\\ &= \hat{b}^{\leftarrow}\circ\hat{c}\circ\hat{b}^{\leftarrow}\circ\hat{\Delta}\circ\hat{c}, \quad \text{by } (\hat{b}\hat{\Delta}),\\ &= (\mathbf{1}\wedge\hat{c})\circ\hat{b}^{\leftarrow}\circ\hat{\Delta}, \quad \text{with } (\hat{\Delta}\ nat) \text{ and } (\hat{b}\,\hat{c}),\\ &= \mathbf{1}\wedge\hat{\Sigma}, \quad \text{by } (\hat{b}\hat{\Delta}),\end{aligned}$$

and for the second equation we have

$$\begin{aligned}\hat{b}^{\rightarrow} \circ \hat{\Sigma} &= \hat{b}^{\rightarrow} \circ \hat{c} \circ \hat{b}^{\rightarrow} \circ (\mathbf{1} \wedge \hat{\Delta}), \quad \text{with } (\hat{b}\hat{\Delta}),\\ &= (\hat{c} \wedge \mathbf{1}) \circ \hat{b}^{\rightarrow} \circ (\mathbf{1} \wedge \hat{c}) \circ (\mathbf{1} \wedge \hat{\Delta}), \quad \text{by } (\hat{b}\,\hat{c}),\\ &= \hat{\Sigma} \wedge \mathbf{1}, \quad \text{with } (\hat{b}\hat{\Delta}\hat{\Sigma}).\end{aligned}$$

We derive analogously with the help of $(\check{b}\check{\Sigma})$ the equations

$$(\check{b}\check{\Delta}\check{\Sigma}) \qquad (\check{\Delta}_{B,A} \vee \mathbf{1}_C) \circ \check{b}^{\rightarrow}_{A,B\wedge\neg B,C} = \mathbf{1}_A \vee \check{\Sigma}_{B,C},$$

$$(\check{b}\check{\Delta}) \qquad \check{\Delta}_{C,A\vee B} \circ \check{b}^{\rightarrow}_{A,B,C\wedge\neg C} = \mathbf{1}_A \vee \check{\Delta}_{C,B}.$$

The arrows $\hat{\Delta}_{B,A}\colon A \vdash A \wedge (\neg B \vee B)$ and $\hat{\Sigma}_{B,A}\colon A \vdash (\neg B \vee B) \wedge A$ are analogous to the arrows of types $A \vdash A \wedge \top$ and $A \vdash \top \wedge A$ that one finds in monoidal categories. However, $\hat{\Delta}_{B,A}$ and $\hat{\Sigma}_{B,A}$ do not have inverses in $\mathbf{PN}^{\neg}$. The equations $(\hat{b}\hat{\Delta})$, $(\hat{b}\hat{\Delta}\hat{\Sigma})$, $(\hat{b}\hat{\Sigma})$ are analogous to equations that hold in monoidal categories (see [38], Section VII.1, [22], Section 4.6, and §3.1 below). An analogous remark can be made for $\check{\Sigma}_{B,A}$ and $\check{\Delta}_{B,A}$.

We can also derive for $\mathbf{PN}^{\neg}$ the following equations by using essentially $(d\hat{\Sigma})$ and $(d\check{\Delta})$:

$$(d^R\hat{\Delta}) \qquad d^R_{C,B,\neg A\vee A} \circ \hat{\Delta}_{A,C\vee B} = \mathbf{1}_C \vee \hat{\Delta}_{A,B},$$

$$(d^R\check{\Sigma}) \qquad \check{\Sigma}_{A,B\wedge C} \circ d^R_{A\wedge\neg A,B,C} = \check{\Sigma}_{A,B} \wedge \mathbf{1}_C.$$

These two equations could replace $(d\hat{\Sigma})$ and $(d\check{\Delta})$ for defining $\mathbf{PN}^{\neg}$. The analogues of the equations $(d\hat{\Sigma})$, $(d\check{\Delta})$, $(d^R\hat{\Delta})$ and $(d^R\check{\Sigma})$ may be found in [11] (Section 2.1), where they are assumed for linearly (alias weakly) distributive categories with negation (cf. [22], Section 7.9).

It is easy to infer that in $\mathbf{PN}^{\neg}$ we have analogues of the equations $(\hat{b}\hat{\Delta})$, $(\hat{b}\hat{\Delta}\hat{\Sigma})$, $(\hat{b}\hat{\Sigma})$, $(\check{b}\check{\Sigma})$, $(\check{b}\check{\Delta}\check{\Sigma})$, $(\check{b}\check{\Delta})$, $(d\hat{\Sigma})$, $(d\check{\Delta})$, $(d^R\hat{\Delta})$ and $(d^R\check{\Sigma})$ obtained by replacing $\overset{\xi}{\Xi}$ by $\overset{\xi}{\Xi}{}'$, and the indices of the form $\neg B \vee B$ and $B \wedge \neg B$ by $B \vee \neg B$ and $\neg B \wedge B$ respectively. For example, we have

$$(\hat{b}\hat{\Delta}') \qquad \hat{b}^{\leftarrow}_{A,B,C\vee\neg C} \circ \hat{\Delta}'_{C,A\wedge B} = \mathbf{1}_A \wedge \hat{\Delta}'_{C,B}.$$

We can also derive for $\mathbf{PN}^{\neg}$ the following equations by using essentially $(\check{\Sigma}\hat{\Delta})$ and $(\check{\Sigma}'\hat{\Delta}')$:

$(\check{\Delta}'\hat{\Sigma}')$ $\qquad \check{\Delta}'_{A,A} \circ d^R_{A,\neg A,A} \circ \hat{\Sigma}'_{A,A} = \mathbf{1}_A,$

$(\check{\Delta}\hat{\Sigma})$ $\qquad \check{\Delta}_{A,\neg A} \circ d^R_{\neg A,A,\neg A} \circ \hat{\Sigma}_{A,\neg A} = \mathbf{1}_{\neg A}.$

These two equations could replace $(\check{\Sigma}\hat{\Delta})$ and $(\check{\Sigma}'\hat{\Delta}')$ for defining $\mathbf{PN}^\neg$. The equations $(\check{\Sigma}\hat{\Delta})$, $(\check{\Sigma}'\hat{\Delta}')$, $(\check{\Delta}'\hat{\Sigma}')$ and $(\check{\Delta}\hat{\Sigma})$ are related to the triangular equations of an adjunction (see [38], Section IV.1, and §5.1 below; see also the next section). The analogues of these equations may be found in [11] (Section 4).

A *proof-net* category is a category with two biendofunctors $\wedge$ and $\vee$, a unary operation $\neg$ on objects, and the natural transformations $\hat{b}^{\rightarrow}$, $\hat{b}^{\leftarrow}$, $\check{b}^{\rightarrow}$, $\check{b}^{\leftarrow}$, $\hat{c}$, $\check{c}$, d, $\hat{\Delta}$ and $\check{\Sigma}$ that satisfy the equations $(\overset{\xi}{b}\,5)$, $(\overset{\xi}{b}\overset{\xi}{b})$, ... , $(\check{\Sigma}'\hat{\Delta}')$ of $\mathbf{PN}^\neg$. The category $\mathbf{PN}^\neg$ is up to isomorphism the free proof-net category generated by the set of letters $\mathcal{P}$ (the set $\mathcal{P}$ may be understood as a discrete category).

If β is a primitive arrow term of $\mathbf{PN}^\neg$ except $\mathbf{1}_B$, then we call β*-terms* of $\mathbf{PN}^\neg$ the set of arrow terms defined inductively as follows: β is a β-term; if f is a β-term, then for every A in $\mathcal{L}_{\wedge,\vee}$ we have that $\mathbf{1}_A \,\xi\, f$ and $f \,\xi\, \mathbf{1}_A$, where $\xi \in \{\wedge, \vee\}$, are β-terms.

In a β-term the subterm β is called the *head* of this β-term. For example, the head of the $\hat{b}^{\rightarrow}_{B,C,D}$-term $\mathbf{1}_A \wedge (\hat{b}^{\rightarrow}_{B,C,D} \vee \mathbf{1}_E)$ is $\hat{b}^{\rightarrow}_{B,C,D}$.

We define $\mathbf{1}$-*terms* as β-terms by replacing β in the definition above by $\mathbf{1}_B$. So $\mathbf{1}$-terms are headless.

An arrow term of the form $f_n \circ \ldots \circ f_1$, where $n \geq 1$, with parentheses tied to $\circ$ associated arbitrarily, such that for every $i \in \{1, \ldots, n\}$ we have that f_i is composition-free is called *factorized*. In a factorized arrow term $f_n \circ \ldots \circ f_1$ the arrow terms f_i are called *factors*. A factor that is a β-term for some β is called a *headed* factor. A factorized arrow term is called *headed* when each of its factors is either headed or a $\mathbf{1}$-term. A factorized arrow term $f_n \circ \ldots \circ f_1$ is called *developed* when f_1 is a $\mathbf{1}$-term and if $n > 1$, then every factor of $f_n \circ \ldots \circ f_2$ is headed. It is sometimes useful to write the factors of a headed arrow term one above the other, as it is done for example in Figure 1 at the end of §2.5.

By using the categorial equations $(cat\ 1)$ and $(cat\ 2)$ and bifunctorial equations we can easily prove by induction on the length of f the following lemma.

DEVELOPMENT LEMMA. *For every arrow term f there is a developed arrow term f' such that $f = f'$ in* $\mathbf{PN}^{\neg}$.

Analogous definitions of β-term and developed arrow term can be given for **DS**, and an analogous Development Lemma can be proved for **DS**.

§2.3. The category *Br*

We are now going to introduce a category called *Br*, which will serve to prove our main coherence result for proof-net categories. We will show that there is a faithful functor from $\mathbf{PN}^{\neg}$ to *Br*. The name of the category *Br* comes from "Brauerian". The arrows of this category correspond to graphs, or diagrams, that were introduced in [7] in connection with Brauer algebras (see [45]). Analogous graphs were investigated in [23], and in [32] Kelly and Mac Lane relied on them to prove their coherence result for symmetric monoidal closed categories (see §3.1).

Let $\mathcal{M}$ be a set whose subsets are denoted by X, Y, Z, ... For $i \in \{s, t\}$ (where s stands for "source" and t for "target"), let $\mathcal{M}^i$ be a set in one-to-one correspondence with $\mathcal{M}$, and let $i\colon \mathcal{M} \to \mathcal{M}^i$ be a bijection. Let X^i be the subset of $\mathcal{M}^i$ that is the image of the subset X of $\mathcal{M}$ under i. If $u \in \mathcal{M}$, then we use u_i as an abbreviation for $i(u)$. We assume also that $\mathcal{M}$, $\mathcal{M}^s$ and $\mathcal{M}^t$ are mutually disjoint.

For $X, Y \subseteq \mathcal{M}$, let a *split relation* of $\mathcal{M}$ be a triple $\langle R, X, Y\rangle$ such that $R \subseteq (X^s \cup Y^t)^2$. The set $X^s \cup Y^t$ may be conceived as the disjoint union of X and Y. We denote a split relation $\langle R, X, Y\rangle$ more suggestively by $R\colon X \vdash Y$.

A split relation $R\colon X \vdash Y$ is a *split equivalence* when R is an equivalence relation. We denote by $\mathrm{part}(R)$ the partition of $X^s \cup Y^t$ corresponding to the split equivalence $R\colon X \vdash Y$.

We say that a split equivalence $R\colon X \vdash Y$ is *Brauerian* when every member of $\mathrm{part}(R)$ is a two-element set. For $R\colon X \vdash Y$ a Brauerian split equivalence, every member of $\mathrm{part}(R)$ is either of the form $\{u_s, v_t\}$, in which case it is called a *transversal*, or of the form $\{u_s, v_s\}$, in which case it is called a *cup*, or, finally, of the form $\{u_t, v_t\}$, in which case it is called a *cap*.

For $X, Y, Z \subseteq \mathcal{M}$, we want to define the composition $P * R\colon X \vdash Z$ of the split relations $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ of $\mathcal{M}$. For that we need some auxiliary notions.

For $X, Y \subseteq \mathcal{M}$, let the function $\varphi^s\colon X \cup Y^t \to X^s \cup Y^t$ be defined by

$$\varphi^s(u) = \begin{cases} u_s & \text{if } u \in X \\ u & \text{if } u \in Y^t, \end{cases}$$

and let the function $\varphi^t\colon X^s \cup Y \to X^s \cup Y^t$ be defined by

$$\varphi^t(u) = \begin{cases} u & \text{if } u \in X^s \\ u_t & \text{if } u \in Y. \end{cases}$$

For a split relation $R\colon X \vdash Y$, let the relations $R^{-s} \subseteq (X \cup Y^t)^2$ and $R^{-t} \subseteq (X^s \cup Y)^2$ be defined by

$$(u, v) \in R^{-i} \quad \text{iff} \quad (\varphi^i(u), \varphi^i(v)) \in R$$

for $i \in \{s, t\}$. Finally, for an arbitrary binary relation R, let $\mathrm{Tr}(R)$ be the transitive closure of R.

Then we define $P * R$ by

$$P * R =_{df} \mathrm{Tr}(R^{-t} \cup P^{-s}) \cap (X^s \cup Z^t)^2.$$

It is easy to conclude that $P * R\colon X \vdash Z$ is a split relation of $\mathcal{M}$, and that if $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ are (Brauerian) split equivalences, then $P * R$ is a (Brauerian) split equivalence.

We now define the category *Br*. The objects of *Br* are the members of the set of finite ordinals $\boldsymbol{N}$. (We have $0 = \emptyset$ and $n{+}1 = n \cup \{n\}$, while $\boldsymbol{N}$ is the ordinal ω.) The arrows of *Br* are the Brauerian split equivalences $R\colon m \vdash n$ of $\boldsymbol{N}$. The identity arrow $\mathbf{1}_n\colon n \vdash n$ of *Br* is the Brauerian split equivalence such that

$$\mathrm{part}(\mathbf{1}_n) = \{\{m_s, m_t\} \mid m < n\}.$$

Composition in *Br* is the operation $*$ defined above.

That *Br* is indeed a category (i.e. that $*$ is associative and that $\mathbf{1}_n$ is an identity arrow) is proved in [20] and [21]. This proof is obtained via an isomorphic representation of *Br* in the category *Rel*, whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Rel* is the ordinary composition of relations. A direct formal proof would be more involved, though what we have to prove is rather clear if we represent Brauerian split equivalences geometrically (as this is done in [7], [23], and also in categories of tangles; see [31], Chapter 12, and references therein).

For example, for $R \subseteq (3^s \cup 9^t)^2$ and $P \subseteq (9^s \cup 1^t)^2$ such that

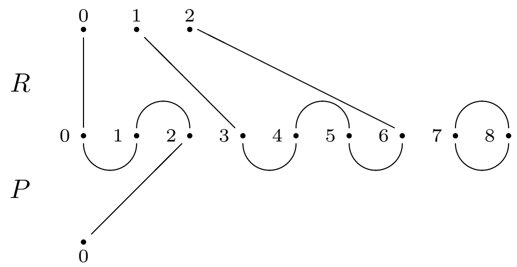
$$\mathrm{part}(R) = \{\{0_s, 0_t\}, \{1_s, 3_t\}, \{2_s, 6_t\}\} \cup \{\{n_t, (n{+}1)_t\} \mid n \in \{1, 4, 7\}\},$$

$$\mathrm{part}(P) = \{\{2_s, 0_t\}\} \cup \{\{n_s, (n{+}1)_s\} \mid n \in \{0, 3, 5, 7\}\},$$

the composition $P * R \subseteq (3^s \cup 1^t)^2$, for which we have

$$\mathrm{part}(P * R) = \{\{0_s, 0_t\}, \{1_s, 2_s\}\},$$

is obtained from the following diagram:



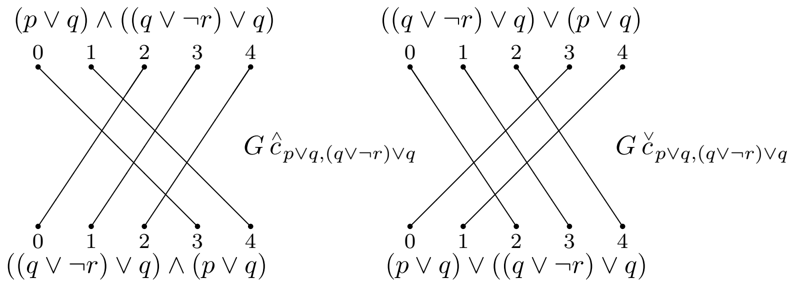
Every bijection f from X^s to Y^t corresponds to a Brauerian split equivalence $R\colon X \vdash Y$ such that the members of $\mathrm{part}(R)$ are of the form $\{u, f(u)\}$. The composition of such Brauerian split equivalences, which correspond to bijections, is then a simple matter: it amounts to composition of these bijections. If in *Br* we keep as arrows only such Brauerian split equivalences, then we obtain a subcategory of *Br* isomorphic to the category *Bij* whose objects are again the finite ordinals and whose arrows are the bijections between these objects. The category *Bij* is a subcategory of the category *Rel* (which played an important role in [22]), whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Bij* and *Rel* is the ordinary composition of relations. The category *Rel* is isomorphic to a subcategory of the category whose arrows are split relations of finite ordinals, of whom *Br* is also a subcategory.

We define a functor G from $\mathbf{PN}^{\neg}$ to Br in the following way. On objects, we stipulate that GA is the number of occurrences of letters in A. (If A has $n = \{0, 1, \dots, n{-}1\}$ occurrences of letters, then the first occurrence corresponds to 0, the second to 1, etc.)On arrows, we have first that $G\alpha$ is an identity arrow of Br for α being $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$ and $d_{A,B,C}$, where $\xi \in \{\wedge, \vee\}$.

Next, for $i, j \in \{s, t\}$, we have that $\{m_i, n_j\}$ belongs to part$(G\,\hat{c}_{A,B})$ iff $\{n_i, m_j\}$ belongs to part$(G\,\check{c}_{A,B})$, iff i is s and j is t, while $m, n < GA{+}GB$ and

$$(m{-}n{-}GA)(m{-}n{+}GB) = 0.$$

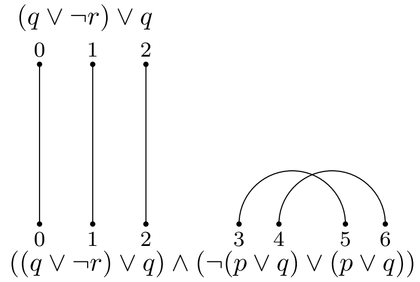
In the following example, we have $G(p \vee q) = 2 = \{0, 1\}$ and $G((q{\vee}\neg r){\vee}q) = 3 = \{0, 1, 2\}$, and we have the diagrams



We have that $\{m_i, n_j\}$ belongs to part$(G\,\hat{\Delta}_{B,A})$ iff either

> i is s and j is t, while $m, n < GA$ and $m = n$, or
>
> i and j are both t, while $m, n \in \{GA, \dots, GA{+}2GB{-}1\}$ and $|m{-}n| = GB$.

In the following example, for A being $(q \vee \neg r) \vee q$ and B being $p \vee q$, we have



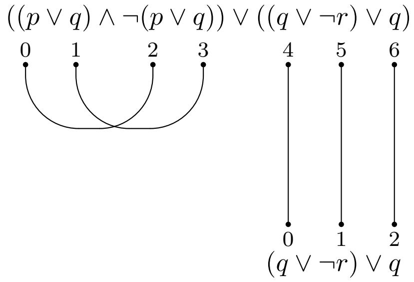
$G\hat{\Delta}_{p\vee q,(q\vee\neg r)\vee q}$

We have that $\{m_i, n_j\}$ belongs to part$(G\check{\Sigma}_{B,A})$ iff either

i is s and j is t, while $m \in \{2GB, \ldots, 2GB+GA-1\}$, $n < GA$ and $m-2GB = n$, or

i and j are both s, while $m, n < 2GB$ and $|m-n| = GB$.

For A and B being as in the previous example, we have



$G\check{\Sigma}_{p\vee q,(q\vee\neg r)\vee q}$

Let $G(f \circ g) = Gf * Gg$. To define $G(f\,\xi\,g)$, for $\xi \in \{\wedge, \vee\}$, we need an auxiliary notion.

Suppose b_X is a bijection from X to X_1 and b_Y a bijection from Y to Y_1. Then for $R \subseteq (X^s \cup Y^t)^2$ we define $R^{b_X}_{b_Y} \subseteq (X_1^s \cup Y_1^t)^2$ by

$$(u_i, v_j) \in R^{b_X}_{b_Y} \quad \text{iff} \quad (i(b_U^{-1}(u)), j(b_V^{-1}(v))) \in R,$$

where $(i, U), (j, V) \in \{(s, X), (t, Y)\}$.

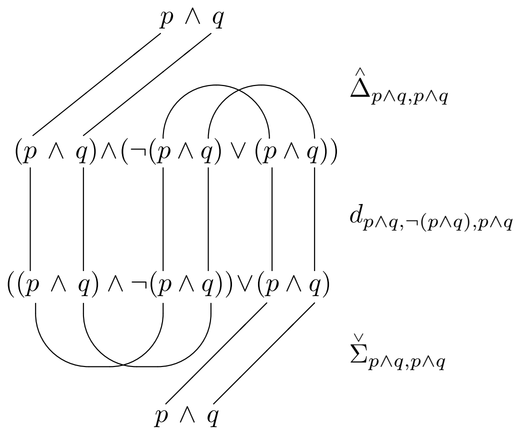
If $f : A \vdash D$ and $g : B \vdash E$, then for $\xi \in \{\wedge, \vee\}$ the set of ordered pairs $G(f\,\xi\,g)$ is

$$Gf \cup Gg^{+GA}_{+GD}$$

where $+GA$ is the bijection from GB to $\{n{+}GA \mid n \in GB\}$ that assigns $n{+}GA$ to n, and $+GD$ is the bijection from GE to $\{n{+}GD \mid n \in GE\}$ that assigns $n{+}GD$ to n.

It is not difficult to check that G so defined is indeed a functor from $\mathbf{PN}^{\neg}$ to Br. For that, we determine by induction on the length of derivation that for every equation $f = g$ of $\mathbf{PN}^{\neg}$ we have $Gf = Gg$ in Br.

Consider, for example, the following diagram, which illustrates an instance of $(\check{\Sigma}\hat{\Delta})$:



This diagram shows that the equation $(\check{\Sigma}\hat{\Delta})$, as well as the equation $(\check{\Sigma}'\hat{\Delta}')$, which is illustrated by analogous diagrams, is related to triangular equations of adjunctions (cf. [14], Section 4.10, and [16], Section 7). The triangular equations of adjunctions are essentially about "straightening a sinuosity", and this straightening is based on planar ambient isotopies of knot theory (cf. [8], Section 1.A).

We have shown by this induction that Br is a proof-net category, and the existence of a structure-preserving functor G from $\mathbf{PN}^{\neg}$ to Br follows from the freedom of $\mathbf{PN}^{\neg}$.

We can define analogously to G a functor, which we also call G, from the category $\mathbf{DS}$ to Br. We just omit from the definition of G above the clauses involving $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$. The image of $\mathbf{DS}$ by G in Br is the subcategory of Br isomorphic to Bij, which we mentioned above. The following is proved in [22] (Section 7.6).

DS Coherence. *The functor* G *from* **DS** *to Br is faithful.*

It follows immediately from this coherence result that **DS** is isomorphic to a subcategory of $\mathbf{PN}^{\neg}$ (cf. [22], Section 14.4).

Up to the end of §2.7 we will be occupied with proving the following.

$\mathbf{PN}^{\neg}$ Coherence. *The functor* G *from* $\mathbf{PN}^{\neg}$ *to Br is faithful.*

For this proof, we must deal first with some preliminary matters.

§2.4. Some properties of DS

In this section we will prove some results about the category **DS**, which we will use to ascertain that particular equations hold in $\mathbf{PN}^{\neg}$. We need these results also for the proof of $\mathbf{PN}^{\neg}$ Coherence.

First we introduce a definition. Suppose x is the n-th occurrence of a letter (counting from the left) in a formula A of $\mathcal{L}_{\neg,\wedge,\vee}$, and y is the m-th occurrence of the same letter in a formula B of $\mathcal{L}_{\neg,\wedge,\vee}$. Then we say that x and y are *tied* in an arrow $f\colon A \vdash B$ of $\mathbf{PN}^{\neg}$ when in the partition $\mathrm{part}(Gf)$ we have $\{(n-1)_s, (m-1)_t\}$ as a member. (Note that to find the n-th occurrence we count starting from 1, but the ordinal $n > 0$ is $\{0, \ldots, n-1\}$.) We have an analogous definition of tied occurrences of the same letter for **DS**: we just replace $\mathcal{L}_{\neg,\wedge,\vee}$ by $\mathcal{L}_{\wedge,\vee}$ and $\mathbf{PN}^{\neg}$ by **DS**.

It is easy to establish by induction on the complexity of f that for every arrow term $f\colon A \vdash B$ of **DS** we have $GA = GB$. Moreover, every occurrence of letter in A is tied to exactly one occurrence of the same letter in B, and vice versa. This is related to the fact that every arrow term $f\colon A \vdash B$ of **DS** may be obtained by substituting letters for letters out of an arrow term $f'\colon A' \vdash B'$ of **DS** such that every letter occurs in A' at most once, and the same for B' (see [22], Sections 3.3 and 7.6).

Suppose for Lemmata 1D and 2D below that $f\colon A \vdash B$ is an arrow term of **DS** such that A has a subformula D in which $\wedge$ does not occur and B has a subformula D' in which $\wedge$ does not occur, and suppose that every occurrence of a letter in D is tied to an occurrence of a letter in D' and vice versa. Then we can prove the following.

Lemma 1D. *The source* A *of* f *is* D *iff the target* B *of* f *is* D'.

This follows from the fact, noted above, that $GA = GB$. The arrow term f in this case can have as subterms that are primitive arrow terms only arrow terms of the forms $\mathbf{1}_E$, $\check{b}^{\rightarrow}_{E,F,G}$, $\check{b}^{\leftarrow}_{E,F,G}$ or $\check{c}_{E,F}$. We also have the following.

LEMMA 2D. *If $D \wedge A'$ or $A' \wedge D$ is a subformula of A, then $D' \wedge B'$ or $B' \wedge D'$ is a subformula of B for some B'.*

We will not go into the inductive proof of this lemma, in which we use Lemma 1D, because we need just a corollary of this lemma (Lemma 2 below), which is more easily proved directly.

Suppose for Lemmata 1C and 2C below that $f\colon A \vdash B$ is an arrow term of **DS** such that B has a subformula C in which $\vee$ does not occur and A has a subformula C' in which $\vee$ does not occur, and suppose that every occurrence of a letter in C is tied to an occurrence of a letter in C' and vice versa. Then we have the following duals of Lemmata 1D and 2D, proved in an analogous manner.

LEMMA 1C. *The target B of f is C iff the source A of f is C'.*

LEMMA 2C. *If $C \vee B'$ or $B' \vee C$ is a subformula of B, then $C' \vee A'$ or $A' \vee C'$ is a subformula of A for some A'.*

Suppose for the following lemma, which is a corollary of either Lemma 2D or Lemma 2C, that $f\colon A \vdash B$ is an arrow term of **DS** such that an occurrence x of a letter p in A is tied to an occurrence y of p in B. This lemma is easily proved by induction on the complexity of f.

LEMMA 2. *It is impossible that A has a subformula $x \wedge A'$ or $A' \wedge x$ and B has a subformula $y \vee B'$ or $B' \vee y$.*

Suppose for Lemmata 3D, 3C, 3 and 4 below that $f\colon A \vdash B$ is an arrow term of **DS**, and for $i \in \{1, 2\}$ let x_i in A and y_i in B be occurrences of the letter p_i tied in f (here p_1 and p_2 may also be the same letter).

LEMMA 3D. *If in A we have a subformula $A_1 \vee A_2$ such that x_i occurs in A_i, then in B we have a subformula $B_1 \vee B_2$ or $B_2 \vee B_1$ such that y_i occurs in B_i.*

This is easily proved by induction on the complexity of the arrow term f. We prove analogously the following.

LEMMA 3C. *If in B we have a subformula $B_1 \wedge B_2$ such that y_i occurs in B_i, then in A we have a subformula $A_1 \wedge A_2$ or $A_2 \wedge A_1$ such that x_i occurs in A_i.*

As a corollary of either Lemma 3D or Lemma 3C we have the following.

LEMMA 3. *It is impossible that A has a subformula $x_1 \vee x_2$ or $x_2 \vee x_1$ and B has a subformula $y_1 \wedge y_2$ or $y_2 \wedge y_1$.*

The following lemma, dual to Lemma 3, is a corollary of Lemma 2.

LEMMA 4. *It is impossible that A has a subformula $x_1 \wedge x_2$ or $x_2 \wedge x_1$ and B has a subformula $y_1 \vee y_2$ or $y_2 \vee y_1$.*

Lemma 3 is related to the acyclicity condition of proof nets, while Lemma 4 is related to the connectedness condition (see §7.1).

Next we can prove the following lemma.

p-q-r LEMMA. *Let $f\colon A \vdash B$ be an arrow of* **DS**, *let x_i for $i \in \{1,2,3\}$ be occurrences of the letters p, q and r, respectively, in A, and let y_i be occurrences of the letters p, q and r, respectively, in B, such that x_i and y_i are tied in f. Let, moreover, $x_2 \vee x_3$ be a subformula of A and $y_1 \wedge y_2$ a subformula of B. Then there is a $d_{p,q,r}$-term $h\colon A' \vdash B'$ such that x_i' are occurrences of the letters p, q and r, respectively, in the source $p \wedge (q \vee r)$ of the head of h and y_i' are occurrences of the letters p, q and r, respectively, in the target $(p \wedge q) \vee r$ of the head of h, such that for some arrows $f_x\colon A \vdash A'$ and $f_y\colon B' \vdash B$ of* **DS** *we have $f = f_y \circ h \circ f_x$ in* **DS**, *and x_i is tied to x_i' in f_x, while y_i' is tied to y_i in f_y.*

PROOF. The proof of this lemma, of which we give just a sketch, relies on a cut-elimination and related results of [22] (Sections 7.7-8). We first find in the category **GDS** introduced in [22] (Section 7.7) a cut-free Gentzen term $f'\colon X \vdash Y$, which corresponds to f, by the relationship that exists between **DS** and **GDS**. According to the equations at the beginning of Section 7.8 of [22], which are used for the proof of the Invertibility Lemmata in the same section, in **GDS** we have the equation $f' = f''$ for a Gentzen term f'' that has as a subterm either $\wedge_{p,q}(\mathbf{1}_p, \vee_{q,r}(\mathbf{1}_q, \mathbf{1}_r))$ or $\vee_{q,r}(\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r)$ both of type $p \wedge (q \vee r) \vdash (p \wedge q) \vee r$. By the relationship that exists between **DS** and **GDS**, we can find starting from f'' an arrow term $f_y \circ h \circ f_x$ equal to f in **DS**, which satisfies the conditions of the lemma. $\dashv$

The full force of the Cut-Elimination Theorem of Section 7.7 of [22] is not essential for this proof, but applying this theorem simplifies the proof.

§2.5. The category PN

We now introduce a category called **PN**, which is equivalent to $\mathbf{PN}^{\neg}$. In the objects of **PN**, the negation connective $\neg$ will be prefixed only to letters, and hence $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$ will be primitive only for the crown index B being a letter. Here is the formal definition of **PN**.

For $\mathcal{P}$ being the set of letters that we used to generate $\mathcal{L}_{\wedge,\vee}$ and $\mathcal{L}_{\neg,\wedge,\vee}$ in §§2.1-2, let $\mathcal{P}^{\neg}$ be the set $\{\neg p \mid p \in \mathcal{P}\}$. The objects of **PN** are the formulae of the propositional language $\mathcal{L}^{\neg p}_{\wedge,\vee}$ generated from $\mathcal{P} \cup \mathcal{P}^{\neg}$ with the binary connectives $\wedge$ and $\vee$. To define the arrow terms of **PN**, in the inductive definition we had for the arrow terms of **DS** we assume in addition that for every formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$ and every letter p

$$\hat{\Delta}_{p,A}\colon A \vdash A \wedge (\neg p \vee p),$$

$$\check{\Sigma}_{p,A}\colon (p \wedge \neg p) \vee A \vdash A$$

are primitive arrow terms of **PN**.

To define the arrows of **PN**, we assume as additional equations in the inductive definition we had for the equations of **DS** the **PN** equations of §2.2 restricted to the arrow terms $\hat{\Delta}_{p,A}$ and $\check{\Sigma}_{p,A}$. This means that in ($\hat{\Delta}$ *nat*) and ($\check{\Sigma}$ *nat*) the crown index B will be p, in $(\hat{b}\hat{\Delta})$ and $(\check{b}\check{\Sigma})$ the crown index C will be p, and in $(d\hat{\Sigma})$, $(d\check{\Delta})$, $(\check{\Sigma}\hat{\Delta})$ and $(\check{\Sigma}'\hat{\Delta}')$ the crown index A will be p. We define $\hat{\Sigma}_{p,A}$, $\check{\Delta}_{p,A}$, $\hat{\Delta}'_{p,A}$, $\check{\Sigma}'_{p,A}$, $\hat{\Sigma}'_{p,A}$ and $\check{\Delta}'_{p,A}$ for **PN** as they were defined in $\mathbf{PN}^{\neg}$ in terms of $\hat{\Delta}_{p,A}$ and $\check{\Sigma}_{p,A}$.

The following equations of **PN**, and hence also of $\mathbf{PN}^{\neg}$, which we call *stem-increasing* equations, enable us to have in developed arrow terms only $\hat{\Delta}_{A,B}$-terms and $\check{\Sigma}_{A,B}$-terms that coincide with their heads:

$(\mathbf{1}\wedge\hat{\Delta})\quad \mathbf{1}_A\wedge\hat{\Delta}_{p,B} = \hat{b}^{\leftarrow}_{A,B,\neg p\vee p}\circ\hat{\Delta}_{p,A\wedge B},\quad$ by $(\hat{b}\hat{\Delta})$,

$(\hat{\Delta}\wedge\mathbf{1})\quad \hat{\Delta}_{p,B}\wedge\mathbf{1}_A = \hat{c}_{A,B\wedge(\neg p\vee p)}\circ\hat{b}^{\leftarrow}_{A,B,\neg p\vee p}\circ(\hat{c}_{B,A}\wedge\mathbf{1}_{\neg p\vee p})\circ\hat{\Delta}_{p,B\wedge A}$,

by $(\hat{c}\,\hat{c})$, $(\hat{c}\ nat)$, $(\mathbf{1}\wedge\hat{\Delta})$ and $(\hat{\Delta}\ nat)$,

$(\mathbf{1}\vee\hat{\Delta})\quad \mathbf{1}_A\vee\hat{\Delta}_{p,B} = d^R_{A,B,\neg p\vee p}\circ\hat{\Delta}_{p,A\vee B},\quad$ by $(d^R\hat{\Delta})$,

$(\hat{\Delta}\vee\mathbf{1})\quad \hat{\Delta}_{p,B}\vee\mathbf{1}_A = \check{c}_{B\wedge(\neg p\vee p),A}\circ d^R_{A,B,\neg p\vee p}\circ(\check{c}_{A,B}\wedge\mathbf{1}_{\neg p\vee p})\circ\hat{\Delta}_{p,B\vee A}$,

by $(\check{c}\,\check{c})$, $(\check{c}\ nat)$, $(\mathbf{1}\vee\hat{\Delta})$ and $(\hat{\Delta}\ nat)$,

$(\check{\Sigma}\vee\mathbf{1})\quad \check{\Sigma}_{p,B}\vee\mathbf{1}_A = \check{\Sigma}_{p,B\vee A}\circ\check{b}^{\leftarrow}_{p\wedge\neg p,B,A},\quad$ by $(\check{b}\check{\Sigma})$,

$(\mathbf{1}\vee\check{\Sigma})\quad \mathbf{1}_A\vee\check{\Sigma}_{p,B} = \check{\Sigma}_{p,A\vee B}\circ(\mathbf{1}_{p\wedge\neg p}\vee\check{c}_{A,B})\circ\check{b}^{\leftarrow}_{p\wedge\neg p,B,A}\circ\check{c}_{(p\wedge\neg p)\vee B,A}$,

by $(\check{c}\,\check{c})$, $(\check{c}\ nat)$, $(\check{\Sigma}\vee\mathbf{1})$ and $(\check{\Sigma}\ nat)$,

$(\check{\Sigma}\wedge\mathbf{1})\quad \check{\Sigma}_{p,B}\wedge\mathbf{1}_A = \check{\Sigma}_{p,B\wedge A}\circ d^R_{p\wedge\neg p,B,A},\quad$ by $(d^R\check{\Sigma})$,

$(\mathbf{1}\wedge\check{\Sigma})\quad \mathbf{1}_A\wedge\check{\Sigma}_{p,B} = \check{\Sigma}_{p,A\wedge B}\circ(\mathbf{1}_{p\wedge\neg p}\vee\hat{c}_{B,A})\circ d^R_{p\wedge\neg p,B,A}\circ\hat{c}_{A,(p\wedge\neg p)\vee B}$,

by $(\hat{c}\,\hat{c})$, $(\hat{c}\ nat)$, $(\check{\Sigma}\wedge\mathbf{1})$ and $(\check{\Sigma}\ nat)$.

Note that in the stem-increasing equations the stem index B of $\hat{\Delta}$ and $\check{\Sigma}$ becomes more complex on the right-hand sides, whereas the crown index p does not change. We have analogous stem-increasing equations for $\hat{\Sigma}$, $\hat{\Delta}'$, $\hat{\Sigma}'$, $\check{\Delta}$, $\check{\Sigma}'$ and $\check{\Delta}'$.

We will next prove several lemmata concerning **PN**, which we will find useful for calculations later on. For these lemmata we need the following.

Let $\mathbf{DS}^{\neg p}$ be the category defined as **DS** save that it is generated not by $\mathcal{P}$, but by $\mathcal{P}\cup\mathcal{P}^{\neg}$. So the objects of $\mathbf{DS}^{\neg p}$ are formulae of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, i.e. the objects of **PN**. For A and B formulae of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, we define when an occurrence of p in A is tied to an occurrence of p in B in an arrow $f\colon A\vdash B$ of $\mathbf{DS}^{\neg p}$ analogously to what we had at the beginning of the preceding section.

Let $\overset{\xi}{\Xi}$ for $\xi\in\{\wedge,\vee\}$ stand for either $\overset{\xi}{\Delta}$, or $\overset{\xi}{\Delta}{}'$, or $\overset{\xi}{\Sigma}$, or $\overset{\xi}{\Sigma}{}'$, and let a $\overset{\xi}{\Xi}_{B,A}$-*term* be defined as a β-term in §2.2, save that β is replaced by $\overset{\xi}{\Xi}_{B,A}$. We use also Θ as a variable alternative to Ξ. Then we have the following.

$\hat{\Xi}$-PERMUTATION LEMMA. *Let $g\colon C \vdash D$ be a $\hat{\Xi}_{p,B}$-term of* **PN** *such that x_1 and $\neg x_2$ are respectively the occurrences within D of p and $\neg p$ in the crown of the head $\hat{\Xi}_{p,B}$ of g, and let $f\colon D \vdash E$ be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that we have an occurrence y_1 of p and an occurrence $\neg y_2$ of $\neg p$ within a subformula of E of the form $y_1 \vee \neg y_2$ or $\neg y_2 \vee y_1$, and x_i is tied to y_i for $i \in \{1,2\}$ in f. Then there is a $\hat{\Theta}_{p,B'}$-term $g'\colon D' \vdash E$ of* **PN** *the crown of whose head is $y_1 \vee \neg y_2$ or $\neg y_2 \vee y_1$, and there is an arrow term $f'\colon C \vdash D'$ of* $\mathbf{DS}^{\neg p}$ *such that in* **PN** *we have $f \circ g = g' \circ f'$.*

PROOF. By the Development Lemma we can assume that f is a developed arrow term, and then it is enough to consider the case when f is either a β-term for β a primitive arrow term of $\mathbf{DS}^{\neg p}$ or f is $\mathbf{1}_E$. Note that in the developed arrow term $f_n \circ \ldots \circ f_1$, which is equal to f, we have that f_1 is $\mathbf{1}_D$, and that f_2, if it exists, cannot be a $d_{B,p,\neg p}$-term or a $d_{B,\neg p,p}$-term such that x_1 and $\neg x_2$ are the occurrences of p and $\neg p$ in the right conjunct of the source $B \wedge (\neg p \vee p)$ or $B \wedge (p \vee \neg p)$ of the head of f_2. Otherwise, in the target of the head of f_2 we would obtain as the left disjunct $B \wedge \neg p$ or $B \wedge p$, which together with Lemma 2 would contradict the conditions put on f, and hence also on $f_n \circ \ldots \circ f_1$, in the formulation of the $\hat{\Xi}$-Permutation Lemma.

The case when f is $\mathbf{1}_E$ is trivial, and there are also many easy cases settled by bifunctorial and naturality equations. The remaining, more interesting, cases are settled by the following equations of **PN**:

$$\hat{b}^{\rightarrow}_{A,B,\neg p\vee p} \circ (\mathbf{1}_A \wedge \hat{\Delta}_{p,B}) = \hat{\Delta}_{p,A\wedge B}, \qquad \text{by } (\hat{b}\hat{\Delta}),$$
$$\hat{b}^{\leftarrow}_{B_1,B_2,\neg p\vee p} \circ \hat{\Delta}_{p,B_1\wedge B_2} = \mathbf{1}_{B_1} \wedge \hat{\Delta}_{p,B_2}, \qquad \text{by } (\hat{b}\hat{\Delta}),$$
$$\hat{b}^{\rightarrow}_{A,\neg p\vee p,B} \circ (\mathbf{1}_A \wedge \hat{\Sigma}_{p,B}) = \hat{\Delta}_{p,A} \wedge \mathbf{1}_B, \qquad \text{by } (\hat{b}\hat{\Delta}\hat{\Sigma}),$$
$$\hat{b}^{\leftarrow}_{B,\neg p\vee p,A} \circ (\hat{\Delta}_{p,B} \wedge \mathbf{1}_A) = \mathbf{1}_B \wedge \hat{\Sigma}_{p,A}, \qquad \text{by } (\hat{b}\hat{\Delta}\hat{\Sigma}),$$
$$\hat{b}^{\rightarrow}_{\neg p\vee p,B_1,B_2} \circ \hat{\Sigma}_{p,B_1\wedge B_2} = \hat{\Sigma}_{p,B_1} \wedge \mathbf{1}_{B_2}, \qquad \text{by } (\hat{b}\hat{\Sigma}),$$
$$\hat{b}^{\leftarrow}_{\neg p\vee p,B,A} \circ (\hat{\Sigma}_{p,B} \wedge \mathbf{1}_A) = \hat{\Sigma}_{p,B\wedge A}, \qquad \text{by } (\hat{b}\hat{\Sigma}),$$
$$\hat{c}_{B,\neg p\vee p} \circ \hat{\Delta}_{p,B} = \hat{\Sigma}_{p,B}, \qquad \text{by definition},$$
$$\hat{c}_{\neg p\vee p,B} \circ \hat{\Sigma}_{p,B} = \hat{\Delta}_{p,B}, \qquad \text{by definition and } (\hat{c}\,\hat{c}),$$

$$(\mathbf{1}_B \wedge \check{c}_{p,\neg p}) \circ \hat{\Delta}_{p,B} = \hat{\Delta}'_{p,B}, \qquad \text{by definition,}$$

$$(\check{c}_{p,\neg p} \wedge \mathbf{1}_B) \circ \hat{\Sigma}_{p,B} = \hat{\Sigma}'_{p,B}, \qquad \text{by definition and } (\hat{c}\ nat),$$

$$d_{\neg p\vee p,B_1,B_2} \circ \hat{\Sigma}_{p,B_1\vee B_2} = \hat{\Sigma}_{p,B_1} \vee \mathbf{1}_{B_2}, \qquad \text{by } (d\hat{\Sigma}).$$

Besides these equations, we have analogous equations where $\neg p \vee p$ is replaced by $p \vee \neg p$, while $\hat{\Delta}$ and $\hat{\Sigma}$ are replaced by $\hat{\Delta}'$ and $\hat{\Sigma}'$ respectively, and vice versa. ⊣

We prove analogously the following dual of the preceding lemma.

$\check{\Xi}$-Permutation Lemma. *Let $g\colon D \vdash C$ be a $\check{\Xi}_{p,B}$-term of* **PN** *such that x_1 and $\neg x_2$ are respectively the occurrences within D of p and $\neg p$ in the crown of the head $\check{\Xi}_{p,B}$ of g, and let $f\colon E \vdash D$ be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that we have an occurrence y_1 of p and an occurrence $\neg y_2$ of $\neg p$ within a subformula of E of the form $y_1 \wedge \neg y_2$ or $\neg y_2 \wedge y_1$, and y_i is tied to x_i for $i \in \{1,2\}$ in f. Then there is a $\check{\Theta}_{p,B'}$-term $g'\colon E \vdash D'$ of* **PN** *the crown of whose head is $y_1 \wedge \neg y_2$ or $\neg y_2 \wedge y_1$, and there is an arrow term $f'\colon D' \vdash C$ of* $\mathbf{DS}^{\neg p}$ *such that in* **PN** *we have $g \circ f = f' \circ g'$.*

Next we prove the following lemma, which involves the *p-q-r* Lemma of the preceding section.

p-¬*p*-*p* Lemma. *Let x_1, $\neg x_2$ and x_3 be occurrences of p, $\neg p$ and p, respectively, in a formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, and let y_1, $\neg y_2$ and y_3 be occurrences of p, $\neg p$ and p, respectively, in a formula B of $\mathcal{L}^{\neg p}_{\wedge,\vee}$. Let $\neg x_2 \vee x_3$ or $x_3 \vee \neg x_2$ be a subformula of A and $y_1 \wedge \neg y_2$ or $\neg y_2 \wedge y_1$ a subformula of B. Let $g_1\colon A' \vdash A$ be a $\hat{\Xi}_{p,C}$-term of* **PN** *such that $\neg x_2 \vee x_3$ or $x_3 \vee \neg x_2$ is the crown of the head of g_1, let $g_2\colon B \vdash B'$ be a $\check{\Theta}_{p,D}$-term of* **PN** *such that $y_1 \wedge \neg y_2$ or $\neg y_2 \wedge y_1$ is the crown of the head of g_2, and let $f\colon A \vdash B$ be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that x_i and y_i are tied in f for $i \in \{1,2,3\}$. Then $g_2 \circ f \circ g_1$ is equal in* **PN** *to an arrow term of* $\mathbf{DS}^{\neg p}$.

Proof. By the *p-q-r* Lemma, $f\colon A \vdash B$ is equal in $\mathbf{DS}^{\neg p}$, and hence also in **PN**, to an arrow term of the form $f_y \circ h \circ f_x$, where h is a $d_{p,\neg p,p}$-term, and the other conditions of the *p-q-r* Lemma are satisfied. So in **PN** we have

$$g_2 \circ f \circ g_1 = g_2 \circ f_y \circ h \circ f_x \circ g_1 = f'_y \circ g'_2 \circ h \circ g'_1 \circ f'_x,$$

by the $\overset{\xi}{\Xi}$-Permutation Lemmata above. Here the head of g'_1 must be $\hat{\Delta}_{p,p}$: $p \vdash p \wedge (\neg p \vee p)$, the head of h is $d_{p,\neg p,p} : p \wedge (\neg p \vee p) \vdash (p \wedge \neg p) \vee p$, and the head of g'_2 must be $\check{\Sigma}_{p,p}$: $(p \wedge \neg p) \vee p \vdash p$. By applying $(\check{\Sigma}\hat{\Delta})$, and perhaps bifunctorial equations, we obtain that $g'_2 \circ h \circ g'_1$ is equal in **PN** to an arrow term of the form $\mathbf{1}_A$, and hence we have $g_2 \circ f \circ g_1 = f'_y \circ f'_x$ in **PN**, which proves the lemma. ⊣

To give an example of the application of the p-$\neg p$-p Lemma, consider the diagram in Figure 1. This diagram corresponds to $G(\check{\Sigma}_{q,p\wedge q} \circ h \circ \hat{\Delta}_{q,p\wedge q})$ for an arrow term h of **PN**, which is of the form $g_2 \circ f \circ g_1$ for g_1 being $\mathbf{1}_{p\wedge q} \wedge (\mathbf{1}_{\neg q} \vee \hat{\Sigma}_{p,q})$, g_2 being $(\mathbf{1}_q \wedge \check{\Sigma}_{p,\neg q}) \vee \mathbf{1}_{p\wedge q}$ and f an arrow term of $\mathbf{DS}^{\neg p}$. Then by applying the p-$\neg p$-p Lemma we obtain an arrow term f' of $\mathbf{DS}^{\neg p}$ equal to $g_2 \circ f \circ g_1$ in **PN**, and next by applying the p-$\neg p$-p Lemma (as a matter of fact, the q-$\neg q$-q Lemma), we obtain an arrow term h' of $\mathbf{DS}^{\neg p}$ equal to $\check{\Sigma}_{q,p\wedge q} \circ f' \circ \hat{\Delta}_{q,p\wedge q}$ in **PN**. By **DS** Coherence of §2.3, we may conclude that h', and hence also $\check{\Sigma}_{q,p\wedge q} \circ h \circ \hat{\Delta}_{q,p\wedge q}$, is equal to $\mathbf{1}_{p\wedge q}$ in **PN**.

Here is a lemma analogous to the p-$\neg p$-p Lemma.

$\neg p$-p-$\neg p$ LEMMA. *Let $\neg x_1$, x_2 and $\neg x_3$ be occurrences of $\neg p$, p and $\neg p$, respectively, in a formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, and let $\neg y_1$, y_2 and $\neg y_3$ be occurrences of $\neg p$, p and $\neg p$, respectively, in a formula B of $\mathcal{L}^{\neg p}_{\wedge,\vee}$. Let $g_1 : A' \vdash A$ be a $\hat{\Xi}_{p,C}$-term of* **PN** *such that $x_2 \vee \neg x_3$ or $\neg x_3 \vee x_2$ is the crown of the head of g_1, let $g_2 : B \vdash B'$ be a $\check{\Theta}_{p,D}$-term of* **PN** *such that $\neg y_1 \wedge y_2$ or $y_2 \wedge \neg y_1$ is the crown of the head of g_2, and let $f : A \vdash B$ be an arrow term of $\mathbf{DS}^{\neg p}$ such that x_i and y_i are tied in f for $i \in \{1, 2, 3\}$. Then $g_2 \circ f \circ g_1$ is equal in* **PN** *to an arrow term of $\mathbf{DS}^{\neg p}$.*

To prove this lemma we proceed as for the p-$\neg p$-p Lemma, relying on the equation $(\check{\Sigma}'\hat{\Delta}')$ of **PN**.

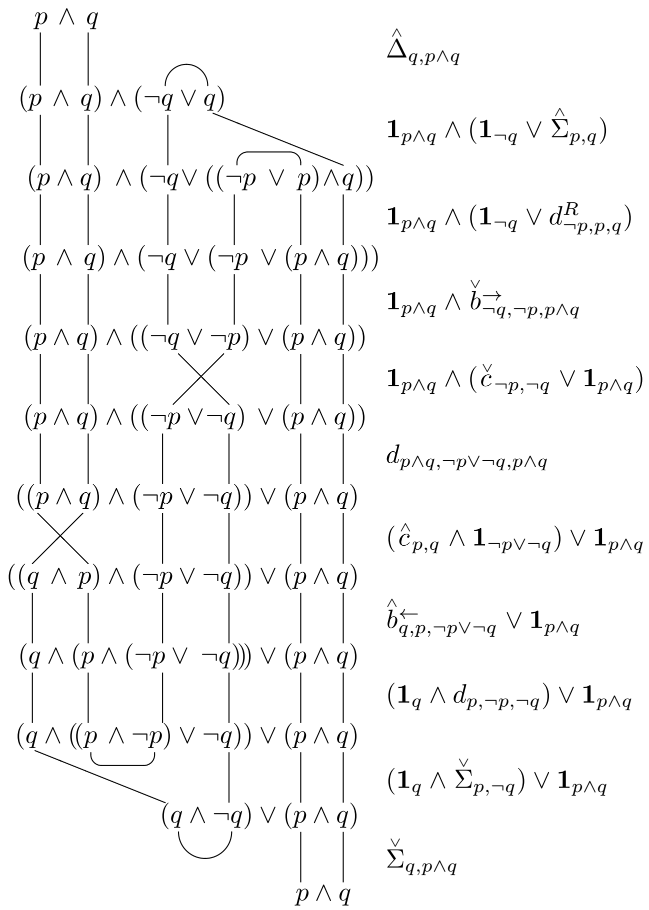

$p \wedge q$
$\hat{\Delta}_{q,p\wedge q}$
$(p \wedge q) \wedge (\neg q \vee q)$
$\mathbf{1}_{p\wedge q} \wedge (\mathbf{1}_{\neg q} \vee \hat{\Sigma}_{p,q})$
$(p \wedge q) \wedge (\neg q \vee ((\neg p \vee p) \wedge q))$
$\mathbf{1}_{p\wedge q} \wedge (\mathbf{1}_{\neg q} \vee d^{R}_{\neg p,p,q})$
$(p \wedge q) \wedge (\neg q \vee (\neg p \vee (p \wedge q)))$
$\mathbf{1}_{p\wedge q} \wedge \check{b}^{\rightarrow}_{\neg q,\neg p,p\wedge q}$
$(p \wedge q) \wedge ((\neg q \vee \neg p) \vee (p \wedge q))$
$\mathbf{1}_{p\wedge q} \wedge (\check{c}_{\neg p,\neg q} \vee \mathbf{1}_{p\wedge q})$
$(p \wedge q) \wedge ((\neg p \vee \neg q) \vee (p \wedge q))$
$d_{p\wedge q,\neg p\vee\neg q,p\wedge q}$
$((p \wedge q) \wedge (\neg p \vee \neg q)) \vee (p \wedge q)$
$(\hat{c}_{p,q} \wedge \mathbf{1}_{\neg p\vee\neg q}) \vee \mathbf{1}_{p\wedge q}$
$((q \wedge p) \wedge (\neg p \vee \neg q)) \vee (p \wedge q)$
$\hat{b}^{\leftarrow}_{q,p,\neg p\vee\neg q} \vee \mathbf{1}_{p\wedge q}$
$(q \wedge (p \wedge (\neg p \vee \neg q))) \vee (p \wedge q)$
$(\mathbf{1}_{q} \wedge d_{p,\neg p,\neg q}) \vee \mathbf{1}_{p\wedge q}$
$(q \wedge ((p \wedge \neg p) \vee \neg q)) \vee (p \wedge q)$
$(\mathbf{1}_{q} \wedge \check{\Sigma}_{p,\neg q}) \vee \mathbf{1}_{p\wedge q}$
$(q \wedge \neg q) \vee (p \wedge q)$
$\check{\Sigma}_{q,p\wedge q}$
$p \wedge q$

FIGURE 1

§2.6. The equivalence of $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$

In this section we show that the categories $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$ are equivalent categories. We define inductively a functor F from the category $\mathbf{PN}^{\neg}$ to $\mathbf{PN}$ in the following manner. On objects we have

$$\begin{aligned}
&Fp = p, \quad \text{for } p \text{ a letter},\\
&F(A\,\xi\,B) = FA\,\xi\,FB, \quad \text{for } \xi \in \{\wedge,\vee\},\\
&F\neg p = \neg p, \quad \text{for } p \text{ a letter},\\
&F\neg\neg A = FA,\\
&F\neg(A\wedge B) = F\neg A \vee F\neg B,\\
&F\neg(A\vee B) = F\neg A \wedge F\neg B.
\end{aligned}$$

On arrows we have

$$F\alpha_{A_1,\ldots,A_n} = \alpha_{FA_1,\ldots,FA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ being $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$, $\overset{\xi}{c}_{A,B}$ or $d_{A,B,C}$ where $\xi \in \{\wedge,\vee\}$,

$$\begin{aligned}
&F\hat{\Delta}_{p,A} = \hat{\Delta}_{p,FA}\colon FA \vdash FA \wedge (\neg p \vee p),\\
&F\check{\Sigma}_{p,A} = \check{\Sigma}_{p,FA}\colon (p\wedge\neg p)\vee FA \vdash FA,\\
&F\hat{\Delta}_{\neg B,A} = (\mathbf{1}_{FA} \wedge \check{c}_{FB,F\neg B}) \circ F\hat{\Delta}_{B,A}\colon FA \vdash FA \wedge (FB \vee F\neg B),\\
&F\check{\Sigma}_{\neg B,A} = F\check{\Sigma}_{B,A} \circ (\hat{c}_{F\neg B,FB} \vee \mathbf{1}_{FA})\colon (F\neg B \wedge FB)\vee FA \vdash FA,\\
&F\hat{\Delta}_{B\wedge C,A} = (\mathbf{1}_{FA} \wedge ((\check{c}_{F\neg B,F\neg C} \vee \mathbf{1}_{FB\wedge FC}) \circ \check{b}^{\rightarrow}_{F\neg C,F\neg B,FB\wedge FC} \circ\\
&\qquad \circ (\mathbf{1}_{F\neg C} \vee (d^{R}_{F\neg B,FB,FC} \circ \hat{c}_{FC,F\neg B\vee FB} \circ F\hat{\Delta}_{B,C})))) \circ F\hat{\Delta}_{C,A}\colon\\
&\qquad\qquad FA \vdash FA \wedge ((F\neg B \vee F\neg C) \vee (FB\wedge FC)),\\
&F\check{\Sigma}_{B\wedge C,A} = F\check{\Sigma}_{C,A} \circ ((\mathbf{1}_{FC} \wedge (F\check{\Sigma}_{B,\neg C} \circ d_{FB,F\neg B,F\neg C})) \circ\\
&\qquad \circ \hat{b}^{\leftarrow}_{FC,FB,F\neg B\vee F\neg C} \circ (\hat{c}_{FB,FC} \wedge \mathbf{1}_{F\neg B\vee F\neg C})) \vee \mathbf{1}_{FA})\colon\\
&\qquad\qquad ((FB\wedge FC)\wedge(F\neg B \vee F\neg C))\vee FA \vdash FA,
\end{aligned}$$

$$
\begin{aligned}
F\hat{\Delta}_{B\vee C,A} = {} & (\mathbf{1}_{FA}\wedge((\hat{c}_{F\neg C,F\neg B}\vee\mathbf{1}_{FB\vee FC})\circ\check{b}^{\leftarrow}_{F\neg C\wedge F\neg B,FB,FC}\circ \\
& \circ((d_{F\neg C,F\neg B,FB}\circ F\hat{\Delta}_{B,\neg C})\vee\mathbf{1}_{FC})))\circ F\hat{\Delta}_{C,A}\colon \\
& FA\vdash FA\wedge((F\neg B\wedge F\neg C)\vee(FB\vee FC)), \\
F\check{\Sigma}_{B\vee C,A} = {} & F\check{\Sigma}_{C,A}\circ((((F\check{\Sigma}_{B,C}\circ\check{c}_{FB\wedge F\neg B,FC}\circ d^{R}_{FC,FB,F\neg B})\wedge\mathbf{1}_{F\neg C})\circ \\
& \circ\hat{b}^{\rightarrow}_{FC\vee FB,F\neg B,F\neg C}\circ(\check{c}_{FC,FB}\wedge\mathbf{1}_{F\neg B\wedge F\neg C}))\vee\mathbf{1}_{FA})\colon \\
& ((FB\vee FC)\wedge(F\neg B\wedge F\neg C))\vee FA\vdash FA,
\end{aligned}
$$

$$
\begin{aligned}
& F(f\circ g)=Ff\circ Fg, \\
& F(f\,\xi\,g)=Ff\,\xi\,Fg, \quad \text{for } \xi\in\{\wedge,\vee\}.
\end{aligned}
$$

It is easy to infer

$$
\begin{aligned}
F\hat{\Delta}_{\neg B,A} &= F\hat{\Delta}'_{B,A}, & F\check{\Sigma}_{\neg B,A} &= F\check{\Sigma}'_{B,A}, \\
F\hat{\Delta}'_{\neg B,A} &= F\hat{\Delta}_{B,A}, & F\check{\Sigma}'_{\neg B,A} &= F\check{\Sigma}_{B,A}, \\
F\hat{\Delta}_{B,A} &= F\hat{\Delta}_{B,FA}, & F\check{\Sigma}_{B,A} &= F\check{\Sigma}_{B,FA}.
\end{aligned}
$$

To ascertain that F so defined is indeed a functor, we have to verify that if $f=g$ is an instance of one of the **PN** equations, then $Ff=Fg$ holds in **PN**. This is done by induction on the number od occurrences of connectives in the crown indices occurring in these equations.

For ($\hat{\Delta}$ *nat*) and ($\check{\Sigma}$ *nat*) this is a very easy matter. For ($\hat{b}\hat{\Delta}$), ($\check{b}\check{\Sigma}$), ($d\hat{\Sigma}$) and ($d\check{\Delta}$) we use essentially naturality equations. (In that context, it might be easier to rely on the equations ($d^{R}\hat{\Delta}$) and ($d^{R}\check{\Sigma}$), which are alternative to ($d\hat{\Sigma}$) and ($d\check{\Delta}$).)

To verify ($\check{\Sigma}\hat{\Delta}$) in cases where A is of the form $B\wedge C$ or $B\vee C$, we rely on the induction hypothesis that if $f=g$ is an instance of a **PN** equation such that the crown indices are B and C, then we have $Ff=Fg$ in **PN**. This induction hypothesis entails that we can proceed as in the proof of the p-$\neg p$-p Lemma in the preceding section, first for p replaced by B, and then for p replaced by C. Finally, we apply **DS** Coherence (see the example at the end of the preceding section). To verify ($\check{\Sigma}\hat{\Delta}$) in case A is of the form $\neg B$, we rely on the induction hypothesis for the equation ($\check{\Sigma}'\hat{\Delta}'$).

To verify ($\check{\Sigma}'\hat{\Delta}'$) we proceed analogously. In case A is $B\wedge C$ or $B\vee C$, we rely on the proof of the $\neg p$-p-$\neg p$ Lemma in the preceding section, and in case A is $\neg B$ we rely on the induction hypothesis for the equation $(\check{\Sigma}\hat{\Delta})$. This concludes the verification that F is a functor from $\mathbf{PN}^{\neg}$ to $\mathbf{PN}$.

(To verify that the functor F from $\mathbf{PN}^{\neg}$ to $\mathbf{PN}$ is a functor we could have proceeded by establishing $\mathbf{PN}$ Coherence first, before introducing the functor F. We do not need the functor F to prove $\mathbf{PN}$ Coherence in the next section. From $f = g$ in $\mathbf{PN}^{\neg}$ we pass to $Gf = Gg$, from which by relying on the first paragraph of §2.7 we pass to $GFf = GFg$, which by $\mathbf{PN}$ Coherence implies $Ff = Fg$.)

In the definition of F, there is some freedom in choosing the clauses for $F\overset{\xi}{\Xi}_{B\psi C,A}$, where $\Xi \in \{\Delta, \Sigma\}$ and $\xi, \psi \in \{\wedge, \vee\}$. We chose ours to be able to apply easily the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata in verifying that F is a functor.

We define a functor $F^{\neg}$ from $\mathbf{PN}$ to $\mathbf{PN}^{\neg}$ by stipulating that $F^{\neg}A = A$ and $F^{\neg}f = f$. It is clear that if $f = g$ in $\mathbf{PN}$, then $F^{\neg}f = F^{\neg}g$ in $\mathbf{PN}^{\neg}$; so $F^{\neg}$ is indeed a functor.

Our purpose is to show that $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$ are equivalent categories via the functors F and $F^{\neg}$. It is clear that $FF^{\neg}A = A$ and $FF^{\neg}f = f$. Since $F^{\neg}FA = FA$, we have to define in $\mathbf{PN}^{\neg}$ an isomorphism $i_A \colon A \vdash FA$. For that we need the following auxiliary definitions in $\mathbf{PN}^{\neg}$:

$$n^{\rightarrow}_{A} =_{df} \check{\Sigma}'_{\neg A,A} \circ d_{\neg\neg A,\neg A,A} \circ \hat{\Delta}_{A,\neg\neg A} : \neg\neg A \vdash A,$$

$$n^{\leftarrow}_{A} =_{df} \check{\Sigma}_{A,\neg\neg A} \circ d_{A,\neg A,\neg\neg A} \circ \hat{\Delta}'_{\neg A,A} : A \vdash \neg\neg A,$$

$$\begin{aligned}\hat{r}^{\rightarrow}_{A,B} =_{df} {} & \check{\Sigma}'_{A\wedge B,\neg A\vee\neg B} \circ d_{\neg(A\wedge B),A\wedge B,\neg A\vee\neg B} \circ (\mathbf{1}_{\neg(A\wedge B)} \wedge ((\mathbf{1}_{A\wedge B} \vee \check{c}_{\neg A,\neg B}) \circ \\ & \circ \check{b}^{\leftarrow}_{A\wedge B,\neg B,\neg A} \circ ((d_{A,B,\neg B} \circ \hat{\Delta}'_{B,A}) \vee \mathbf{1}_{\neg A}))) \circ \hat{\Delta}'_{A,\neg(A\wedge B)} : \\ & \neg(A\wedge B) \vdash \neg A \vee \neg B,\end{aligned}$$

$$\begin{aligned}\hat{r}^{\leftarrow}_{A,B} =_{df} {} & \check{\Sigma}'_{A,\neg(A\wedge B)} \circ ((((\check{\Delta}'_{B,\neg A} \circ d^{R}_{\neg A,\neg B,B}) \wedge \mathbf{1}_{A}) \circ \hat{b}^{\rightarrow}_{\neg A\vee\neg B,B,A} \circ \\ & \circ (\mathbf{1}_{\neg A\vee\neg B} \wedge \hat{c}_{A,B})) \vee \mathbf{1}_{\neg(A\wedge B)}) \circ d_{\neg A\vee\neg B,A\wedge B,\neg(A\wedge B)} \circ \hat{\Delta}'_{A\wedge B,\neg A\vee\neg B} : \\ & \neg A\vee\neg B \vdash \neg(A\wedge B),\end{aligned}$$

$$\overset{\vee}{r}{}^{\rightarrow}_{A,B} =_{df} \overset{\vee}{\Sigma}{}'_{A\vee B,\neg A\wedge\neg B} \circ d_{\neg(A\vee B),A\vee B,\neg A\wedge\neg B} \circ (\mathbf{1}_{\neg(A\vee B)} \wedge ((\overset{\vee}{c}_{A,B} \vee \mathbf{1}_{\neg A\wedge\neg B})\circ \\ \circ\, \overset{\vee}{b}{}^{\rightarrow}_{B,A,\neg A\wedge\neg B} \circ (\mathbf{1}_B \vee (d^R_{A,\neg A,\neg B} \circ \overset{\wedge}{\Sigma}{}'_{A,\neg B})))) \circ \overset{\wedge}{\Delta}{}'_{B,\neg(A\vee B)} : \\ \neg(A\vee B) \vdash \neg A \wedge \neg B,$$

$$\overset{\vee}{r}{}^{\leftarrow}_{A,B} =_{df} \overset{\vee}{\Sigma}{}'_{B,\neg(A\vee B)} \circ (((\mathbf{1}_{\neg B} \wedge (\overset{\vee}{\Sigma}{}'_{A,B} \circ d_{\neg A,A,B})) \circ \hat{b}{}^{\leftarrow}_{\neg B,\neg A,A\vee B} \circ \\ \circ\, (\hat{c}_{\neg A,\neg B} \wedge \mathbf{1}_{A\vee B})) \vee \mathbf{1}_{\neg(A\vee B)}) \circ d_{\neg A\wedge\neg B,A\vee B,\neg(A\vee B)} \circ \overset{\wedge}{\Delta}{}'_{A\vee B,\neg A\wedge\neg B} : \\ \neg A \wedge \neg B \vdash \neg(A\vee B).$$

It can be shown that in $\mathbf{PN}^{\neg}$ we have the following equations:

$$n^{\rightarrow}_A \circ n^{\leftarrow}_A = \mathbf{1}_A, \qquad n^{\leftarrow}_A \circ n^{\rightarrow}_A = \mathbf{1}_{\neg\neg A},$$
$$\hat{r}{}^{\rightarrow}_{A,B} \circ \hat{r}{}^{\leftarrow}_{A,B} = \mathbf{1}_{\neg A\vee\neg B}, \qquad \hat{r}{}^{\leftarrow}_{A,B} \circ \hat{r}{}^{\rightarrow}_{A,B} = \mathbf{1}_{\neg(A\wedge B)},$$
$$\overset{\vee}{r}{}^{\rightarrow}_{A,B} \circ \overset{\vee}{r}{}^{\leftarrow}_{A,B} = \mathbf{1}_{\neg A\wedge\neg B}, \qquad \overset{\vee}{r}{}^{\leftarrow}_{A,B} \circ \overset{\vee}{r}{}^{\rightarrow}_{A,B} = \mathbf{1}_{\neg(A\vee B)},$$

which means that $n^{\rightarrow}$ and $n^{\leftarrow}$, as well as $\overset{\xi}{r}{}^{\rightarrow}$ and $\overset{\xi}{r}{}^{\leftarrow}$ are inverses of each other. To derive these equations in $\mathbf{PN}^{\neg}$, we use essentially ($\overset{\wedge}{\Delta}$ *nat*), ($\overset{\vee}{\Sigma}$ *nat*), the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata, and **DS** Coherence. (If an equation holds in **PN**, then every substitution instance of it obtained by replacing letters uniformly by formulae of $\mathcal{L}_{\neg,\wedge,\vee}$ holds in $\mathbf{PN}^{\neg}$; this enables us to apply the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata.) The definitions of $n^{\rightarrow}$, $n^{\leftarrow}$, $\overset{\xi}{r}{}^{\rightarrow}$ and $\overset{\xi}{r}{}^{\leftarrow}$, for $\xi \in \{\wedge,\vee\}$, are such that they enable an easy application of the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata.

Then we define $i_A : A \vdash FA$ and its inverse $i^{-1}_A : FA \vdash A$ by induction on the complexity of the formula A of $\mathcal{L}_{\neg,\wedge,\vee}$ (cf. [22], Section 14.1):

$$i_A = i^{-1}_A = \mathbf{1}_A, \quad \text{if } A \text{ is } p \text{ or } \neg p, \text{ for } p \text{ a letter,}$$
$$i_{A_1 \xi A_2} = i_{A_1}\,\xi\, i_{A_2}, \qquad i^{-1}_{A_1\xi A_2} = i^{-1}_{A_1}\,\xi\, i^{-1}_{A_2}, \quad \text{for } \xi \in \{\wedge,\vee\},$$
$$i_{\neg\neg B} = i_B \circ n^{\rightarrow}_B, \qquad i^{-1}_{\neg\neg B} = n^{\leftarrow}_B \circ i^{-1}_B,$$
$$i_{\neg(A_1\wedge A_2)} = (i_{\neg A_1} \vee i_{\neg A_2}) \circ \hat{r}{}^{\rightarrow}_{A_1,A_2}, \qquad i^{-1}_{\neg(A_1\wedge A_2)} = \hat{r}{}^{\leftarrow}_{A_1,A_2} \circ (i^{-1}_{\neg A_1} \vee i^{-1}_{\neg A_2}),$$
$$i_{\neg(A_1\vee A_2)} = (i_{\neg A_1} \wedge i_{\neg A_2}) \circ \overset{\vee}{r}{}^{\rightarrow}_{A_1,A_2}, \qquad i^{-1}_{\neg(A_1\vee A_2)} = \overset{\vee}{r}{}^{\leftarrow}_{A_1,A_2} \circ (i^{-1}_{\neg A_1} \wedge i^{-1}_{\neg A_2}).$$

We can then prove the following (cf. [22], Section 14.1).

AUXILIARY LEMMA. *For every arrow term* $f : A \vdash B$ *of* $\mathbf{PN}^{\neg}$ *we have* $f = i^{-1}_B \circ Ff \circ i_A$ *in* $\mathbf{PN}^{\neg}$.

PROOF. We proceed by induction on the complexity of the arrow term f. If f is a primitive arrow term $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$, $\overset{\xi}{c}_{A,B}$ or $d_{A,B,C}$, for $\xi \in \{\wedge, \vee\}$, then we use naturality equations, and the fact that i_D is an isomorphism.

If f is $\hat{\Delta}_{D,A}$, then we proceed by induction on the complexity of D. (This is an auxiliary induction in the basis of the main induction.) If D is p, then we use ($\hat{\Delta}$ *nat*) and the fact that i_A is an isomorphism.

If D is $\neg B$, then we rely on the following equation of $\mathbf{PN}^{\neg}$, analogous to the clause defining $F\hat{\Delta}_{\neg B,A}$ above:

$$(\hat{\Delta} n) \qquad \hat{\Delta}_{\neg B,A} = (\mathbf{1}_A \wedge (n^{\leftarrow}_B \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A},$$

together with the induction hypothesis. To derive $(\hat{\Delta} n)$ we have

$$\begin{aligned}
&(\mathbf{1}_A \wedge (n^{\leftarrow}_B \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A} \\
&\quad = (\mathbf{1}_A \wedge (\check{\Sigma}_{B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ (\mathbf{1}_A \wedge (d_{B,\neg B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ \\
&\qquad\quad \circ (\mathbf{1}_A \wedge (\hat{\Delta}'_{\neg B,B} \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A}, \quad \text{by bifunctorial equations,} \\
&\quad = (\mathbf{1}_A \wedge (\check{\Sigma}_{B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ (\mathbf{1}_A \wedge ((d_{B,\neg B,\neg\neg B} \vee \mathbf{1}_{\neg B}) \circ \\
&\qquad\quad \circ \check{c}_{B\wedge(\neg B\vee\neg\neg B),\neg B} \circ d^R_{\neg B,B,\neg B\vee\neg\neg B} \circ (\check{c}_{\neg B,B} \wedge \mathbf{1}_{\neg B\vee\neg\neg B}))) \circ \\
&\qquad\quad \circ \hat{b}^{\leftarrow}_{A,\neg B,B\vee\neg B} \circ (\hat{\Delta}'_{B,A} \wedge \mathbf{1}_{\neg B\vee\neg\neg B}) \circ (\mathbf{1}_A \wedge \check{c}_{\neg B,\neg\neg B}) \circ \hat{\Delta}_{\neg B,A},
\end{aligned}$$

by stem-increasing equations involving $\hat{\Delta}'$ analogous to $(\mathbf{1} \vee \hat{\Delta})$ and $(\mathbf{1} \wedge \hat{\Delta})$ of the preceding section, and also ($\hat{\Delta}'$ *nat*). The equation $(\hat{\Delta} n)$ follows by applying the $\neg p$-p-$\neg p$ Lemma (with p replaced by B), and **DS** Coherence.

If D is $B \wedge C$, then we rely on the following equation of $\mathbf{PN}^{\neg}$, analogous to the clause defining $F\hat{\Delta}_{B\wedge C,A}$ above:

$$\begin{aligned}
(\hat{\Delta} r) \qquad \hat{\Delta}_{B\wedge C,A} = (\mathbf{1}_A \wedge (((\hat{r}^{\leftarrow}_{B,C} \circ \check{c}_{\neg B,\neg C}) \vee \mathbf{1}_{B\wedge C}) \circ \check{b}^{\rightarrow}_{\neg C,\neg B,B\wedge C} \circ \\
\circ (\mathbf{1}_{\neg C} \vee (d^R_{\neg B,B,C} \circ \hat{\Sigma}_{B,C})))) \circ \hat{\Delta}_{C,A},
\end{aligned}$$

together with the induction hypothesis. To show that $(\hat{\Delta} r)$ holds in $\mathbf{PN}^{\neg}$ we proceed as above, by applying essentially stem-increasing equations together with the p-$\neg p$-p Lemma. We proceed analogously when D is $B \vee C$.

The cases we have if f is $\check{\Sigma}_{D,A}$ are dual to those we had above for f being $\hat{\Delta}_{D,A}$. In all these cases we proceed in an analogous manner. This concludes the basis of the induction.

If f is $f_2 \circ f_1$, then by the induction hypothesis we have

$$f_2 \circ f_1 = i_B^{-1} \circ Ff_2 \circ i_C \circ i_C^{-1} \circ Ff_1 \circ i_A$$

which yields $f = i_B^{-1} \circ Ff \circ i_A$, by the fact that i_C is an isomorphism and by the functoriality of F.

If f is $f_1 \,\xi\, f_2$, for $\xi \in \{\wedge, \vee\}$, then $i_{A_1 \xi A_2}$ is $i_{A_1} \,\xi\, i_{A_2}$ and $i^{-1}_{B_1 \xi B_2}$ is $i^{-1}_{B_1} \,\xi\, i^{-1}_{B_2}$; we obtain $f = i_B^{-1} \circ Ff \circ i_A$ by using bifunctorial equations. ⊣

The Auxiliary Lemma shows that i_A is an isomorphism natural in A, and so we may conclude that $\mathbf{PN}^{\neg}$ and **PN** are equivalent categories.

§2.7. PN Coherence

We define a functor G from **PN** to Br as we defined it from $\mathbf{PN}^{\neg}$ to Br. In the clauses for $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$ we just restrict B to a letter p. For f an arrow term of $\mathbf{PN}^{\neg}$ we have that GFf coincides with Gf where F is the functor from $\mathbf{PN}^{\neg}$ to **PN** of the preceding section, G in GFf is the functor G from **PN** to Br and G in Gf is the functor G from $\mathbf{PN}^{\neg}$ to Br. To show that, it is essential to check that $GF\hat{\Delta}_{B,A}$ and $GF\check{\Sigma}_{B,A}$ coincide with $G\hat{\Delta}_{B,A}$ and $G\check{\Sigma}_{B,A}$ respectively.

In this section we will prove that G from **PN** to Br is faithful. This will imply that G from $\mathbf{PN}^{\neg}$ to Br is faithful too.

Analogously to what we had at the beginning of §2.4, we define when an occurrence x of a letter p in A is *tied* to an occurrence y of the same letter p in B in an arrow $f \colon A \vdash B$ of **PN**. We say that x and y are *directly tied* in a headed factorized arrow term $f_n \circ \ldots \circ f_1$ of **PN** when x and y are tied in the arrow $f_n \circ \ldots \circ f_1$, and for every $i \in \{2, \ldots, n\}$ if f_i is a $\check{\Sigma}_{p,C}$-term and z is one of the two occurrences of p in the crown $p \wedge \neg p$ of the head of f_i, then x and z are not tied in the arrow $f_{i-1} \circ \ldots \circ f_1$ (see the end of §2.2 for the definition of headed factorized arrow term).

An alternative definition of directly tied x and y in a headed factorized arrow term $f_1 \circ \ldots \circ f_n$ of **PN** is obtained by stipulating that x and y are tied in the arrow $f_1 \circ \ldots \circ f_n$, and for every $i \in \{2, \ldots, n\}$ if f_i is a $\hat{\Delta}_{p,D}$-term and z is one of the two occurrences of p in the crown $\neg p \vee p$ of the head of f_i, then z and y are not tied in the arrow $f_1 \circ \ldots \circ f_{i-1}$.

For example, the occurrence of q in the source $p \wedge q$ and the occurrence of q in the target $q \wedge p$ of

$$\hat{c}_{p,q} \circ (\check{\Sigma}_{p,p} \wedge \mathbf{1}_q) \circ (d_{p,\neg p,p} \wedge \mathbf{1}_q) \circ (\hat{\Delta}_{p,p} \wedge \mathbf{1}_q)$$

are directly tied in this headed factorized arrow term of **PN**, while the two occurrences of p in its source and target are not directly tied.

Take a headed factorized arrow term of **PN** of the form $g_2 \circ f \circ g_1$ where g_1 is a $\hat{\Delta}_{p,D}$-term and g_2 is $\check{\Sigma}_{p,C}$-term. Let $\neg x_1 \vee x_2$ be the crown of the head of g_1 (so x_1 and x_2 are both occurrences of p) and let $y_2 \wedge \neg y_1$ be the crown of the head of g_2 (so y_1 and y_2 are also occurrences of the same letter p). We say that g_1 and g_2 are *confronted* through f when x_i and y_i are directly tied for some $i \in \{1, 2\}$ in the arrow term f.

Let a $\hat{\Delta}_{p,A}$-term that is a factor of a factorized arrow term f be called a $\hat{\Delta}$-*factor*. We have an analogous definition of $\check{\Sigma}$-*factor* obtained by replacing $\hat{\Delta}$ by $\check{\Sigma}$. We can then prove the following lemma.

CONFRONTATION LEMMA. *For every headed factorized arrow term $g_2 \circ f \circ g_1$ of* **PN** *such that g_1 and g_2 are confronted through f there is a headed factorized arrow term h of* **PN** *with a subterm of the form $g'_2 \circ f' \circ g'_1$ such that g'_1 is a $\hat{\Delta}$-factor, g'_2 is a $\check{\Sigma}$-factor, g'_1 and g'_2 are confronted through f', and, moreover,*

(1) *f' is an arrow term of* $\mathbf{DS}^{\neg p}$,

(2) $g_2 \circ f \circ g_1 = h$ *in* **PN**,

(3) *the number of $\hat{\Delta}$-factors is equal in $g_2 \circ f \circ g_1$ and h, and the same for the number of $\check{\Sigma}$-factors.*

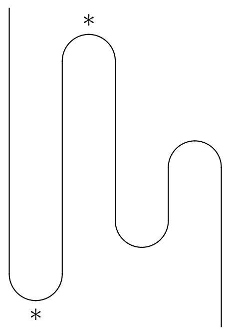
PROOF. We proceed by induction on the number n of factors of f that are $\hat{\Delta}$-factors or $\check{\Sigma}$-factors. If $n = 0$, then the arrow term f' coincides with the arrow term f.

If $n > 0$, then let $g_2 \circ f \circ g_1$ be of the form $f_2 \circ g \circ f_1$ for g a $\hat{\Delta}_{q,E}$-term (we proceed analogously when g is a $\check{\Sigma}_{q,E}$-term). According to the stem-increasing equations of §2.5, we may assume that g coincides with its head $\hat{\Delta}_{q,E}$. Then by ($\hat{\Delta}$ *nat*) we obtain in **PN**

$$g_2 \circ f \circ g_1 = f_2 \circ (f_1 \wedge \mathbf{1}_{\neg q \vee q}) \circ \hat{\Delta}_{q,E'}\,.$$

After $f_1 \wedge \mathbf{1}_{\neg q \vee q}$ in $f_2 \circ (f_1 \wedge \mathbf{1}_{\neg q \vee q})$ is replaced by a headed factorized arrow term $g_2 \circ f'' \circ (g_1 \wedge \mathbf{1}_{\neg q \vee q})$, we may apply the induction hypothesis to this arrow term, because it can easily be seen that $g_1 \wedge \mathbf{1}_{\neg q \vee q}$ and g_2 are confronted through f'', and f'' has one $\hat{\Delta}$-factor less than f. ⊣

A headed factorized arrow term of **PN** that has no subterm of the form $g_2 \circ f \circ g_1$ with g_1 and g_2 confronted through f is called *pure.* For a pure arrow term f there is a one-to-one correspondence, which we call the $\hat{\Delta}$-*cap bijection*, between the $\hat{\Delta}$-factors of f and the caps of the partition part(Gf). In this bijection, a cap ties, in an obvious sense, the occurrences of p in the crown $\neg p \vee p$ of the head of the corresponding $\hat{\Delta}$-factor. There is an analogous one-to-one correspondence, which we call the $\check{\Sigma}$-*cup bijection*, between the $\check{\Sigma}$-factors of f and the cups of part(Gf) (see §2.3 for the notions of cup and cap). Intuitively speaking, this follows from the fact that in a sequence of cups and caps tied to each other as in the following example:



cups and caps must alternate. For a pair made of a cap and a cup that is its immediate neighbour, like those marked with $*$ in the picture, we can find a subterm $g_2 \circ f \circ g_1$ such that g_1 and g_2 are confronted through f.

We can then prove the following.

Purification Lemma. *Every arrow term of* **PN** *is equal in* **PN** *to a pure arrow term of* **PN**.

Proof. We apply first the Development Lemma of §2.2. If in the resulting developed arrow term h we have a subterm $g_2 \circ f \circ g_1$ with g_1 and g_2 confronted through f, then we apply first the Confrontation Lemma to obtain a developed arrow term h' with a subterm of the form $g_2' \circ f' \circ g_1'$ where g_1' and g_2' are confronted through f', and f' is an arrow term of $\mathbf{DS}^{\neg p}$.

Suppose that $\neg x_2 \vee x_3$ is the crown of the head of g_1', and $y_1 \wedge \neg y_2$ is the crown of the head of g_2'. Suppose x_2 is tied to y_2 in f'. Then, by Lemma 3 of §2.4, it is impossible that x_3 is tied to y_1, and so there must be an occurrence x_1 of p different from x_3 in the source of f' such that x_1 is tied to y_1 in f', and there must be an occurrence y_3 of p different from y_1 in the target of f' such that x_3 is tied to y_3 in f'. Next we apply the p-$\neg p$-p Lemma of §2.5 to conclude that $g_2' \circ f' \circ g_1'$ is equal to an arrow term h'' of $\mathbf{DS}^{\neg p}$.

After replacing $g_2' \circ f' \circ g_1'$ in h' by h'', we obtain a headed factorized arrow term in which there is one $\hat{\Delta}$-factor and one $\check{\Sigma}$-factor less than in h', and hence also than in h, by clause (3) of the Confrontation Lemma.

If x_3 is tied to y_1, then we reason analogously by applying Lemma 3 of §2.4 and the $\neg p$-p-$\neg p$ Lemma of §2.5.

We can iterate this procedure, which must terminate, because the number of $\hat{\Delta}$-factors and $\check{\Sigma}$-factors in h is finite. ⊣

We can then prove the following.

PN COHERENCE. *The functor G from* **PN** *to Br is faithful.*

PROOF. Suppose for f and g arrow terms of **PN** of the same type $A \vdash B$ we have $Gf = Gg$. By the Purification Lemma, we can assume that f and g are pure arrow terms. Since $Gf = Gg$, by the $\hat{\Delta}$-cap and $\check{\Sigma}$-cup bijections we must have the same number $n \geq 0$ of $\hat{\Delta}$-factors in f and g and the same number $m \geq 0$ of $\check{\Sigma}$-factors in f and g. We proceed by induction on $n+m$.

If $n+m = 0$, then we just apply **DS** Coherence. Suppose now $n > 0$. So there is a $\hat{\Delta}$-factor in f and a $\hat{\Delta}$-factor in g that correspond by the $\hat{\Delta}$-cap bijections to the same cap of part(Gf), which is equal to part(Gg). By using the stem-increasing equations of §2.5, together with ($\hat{\Delta}$ *nat*), we obtain in **PN**

$$f = f' \circ \hat{\Delta}_{p,A}, \qquad g = g' \circ \hat{\Delta}_{p,A}$$

for f' and g' pure arrow terms of the same type $A \wedge (\neg p \vee p) \vdash B$, and such that the number of $\hat{\Delta}$-factors in f' and g' is $n-1$ in each, and the number of $\check{\Sigma}$-factors in f' and g' is m in each, the same number we had for the $\check{\Sigma}$-factors in f and g. So we have

$$G(f' \circ \hat{\Delta}_{p,A}) = Gf = Gg = G(g' \circ \hat{\Delta}_{p,A}).$$

We can show that $Gf' = Gg'$. This is because we obtain Gf' from $G(f' \circ \hat{\Delta}_{p,A})$ in the following manner. We first remove from the partition part$(G(f' \circ \hat{\Delta}_{p,A}))$ a cap $\{k_t, l_t\}$, where the $k{+}1$-th occurrence of letter in B is an occurrence of p in a subformula $\neg p$ of B, and the $l{+}1$-th occurrence of letter in B is an occurrence of p that is not in a subformula $\neg p$ of B (here we have either $k < l$ or $l < k$). After this removal, we add two new transversals:

$$\{GA_s, k_t\}, \quad \{(GA{+}1)_s, l_t\},$$

and this yields part(Gf'). Since Gg' is obtained from $G(g' \circ \hat{\Delta}_{p,A})$, which is equal to $G(f' \circ \hat{\Delta}_{p,A})$ in exactly the same manner, we obtain that $Gf' = Gg'$.

Then, by the induction hypothesis, we have that $f' = g'$ in **PN**, which implies that $f = g$ in **PN**. We proceed analogously in the induction step when $m > 0$, via $\check{\Sigma}$-factors. ⊣

From **PN** Coherence and the equivalence between the categories $\mathbf{PN}^{\neg}$ and **PN**, proved in the preceding section, we may conclude in the following manner that the functor G from $\mathbf{PN}^{\neg}$ to *Br* is faithful.

PROOF OF $\mathbf{PN}^{\neg}$ COHERENCE. Suppose that for f and g arrows of $\mathbf{PN}^{\neg}$ of the same type we have $Gf = Gg$. Then, as we noted at the beginning of this section, we have $GFf = GFg$, and hence $Ff = Fg$ in **PN** by **PN** Coherence. It follows that $f = g$ in $\mathbf{PN}^{\neg}$ by the equivalence of the categories $\mathbf{PN}^{\neg}$ and **PN**. ⊣

So we have proved $\mathbf{PN}^{\neg}$ Coherence, announced at the end of §2.3.

§2.8. The contravariant functor ¬

In every proof-net category $\mathcal{A}$ we can define a contravariant ¬ functor from $\mathcal{A}$ to $\mathcal{A}$ by relying on the following definition. For $f\colon A \vdash B$, we have

$$\neg f =_{df} \check{\Sigma}'_{B,\neg A} \circ d_{\neg B,B,A} \circ (\mathbf{1}_{\neg B} \wedge (f \vee \mathbf{1}_{\neg A})) \circ \hat{\Delta}'_{A,\neg B} \colon \neg B \vdash \neg A.$$

It is easy to check that we have

$$(\neg) \qquad \neg f = \check{\Delta}_{B,\neg A} \circ d^R_{\neg A,B,\neg B} \circ ((\mathbf{1}_{\neg A} \vee f) \wedge \mathbf{1}_{\neg B}) \circ \hat{\Sigma}_{A,\neg B},$$

which gives an alternative definition of $\neg f$.

To verify that $\neg$ is a contravariant functor we have first

$$(\neg 1) \qquad \neg \mathbf{1}_A = \mathbf{1}_{\neg A}$$

by $\mathbf{PN}^{\neg}$ Coherence (namely, we use here the equation $(\check{\Sigma}'\hat{\Delta}')$, or alternatively $(\check{\Delta}\hat{\Sigma})$, of §2.2).

Next, for $f \colon A \vdash B$ we have the following equation:

$$(\check{\Sigma}\ dinat) \qquad \check{\Sigma}_{A,C} \circ ((\mathbf{1}_A \wedge \neg f) \vee \mathbf{1}_C) = \check{\Sigma}_{B,C} \circ ((f \wedge \mathbf{1}_{\neg B}) \vee \mathbf{1}_C),$$

which together with $(\check{\Sigma}\ nat)$ says that $\check{\Sigma}$ is a *dinatural* transformation in the sense of [38] (Section IX.4). To verify $(\check{\Sigma}\ dinat)$ we have

$$\begin{aligned}
&\check{\Sigma}_{A,C} \circ ((\mathbf{1}_A \wedge \neg f) \vee \mathbf{1}_C) \\
&\quad = \check{\Sigma}_{A,C} \circ ((\mathbf{1}_A \wedge (\check{\Delta}_{B,\neg A} \circ d^R_{\neg A,B,\neg B})) \vee \mathbf{1}_C) \circ ((\mathbf{1}_A \wedge (((\mathbf{1}_{\neg A} \vee f) \wedge \mathbf{1}_{\neg B}) \circ \\
&\qquad\qquad \circ \hat{\Sigma}_{A,\neg B})) \vee \mathbf{1}_C), \quad \text{by } (\neg), \\
&\quad = \check{\Sigma}_{B,C} \circ ((((\check{\Sigma}_{A,B} \circ d_{A,\neg A,B}) \wedge \mathbf{1}_{\neg B}) \circ \hat{b}^{\rightarrow}_{A,\neg A \vee B,\neg B}) \vee \mathbf{1}_C) \circ \\
&\qquad\qquad \circ ((\mathbf{1}_A \wedge (((\mathbf{1}_{\neg A} \vee f) \wedge \mathbf{1}_{\neg B}) \circ \hat{\Sigma}_{A,\neg B})) \vee \mathbf{1}_C), \quad \text{by } \mathbf{PN}^{\neg} \text{ Coherence}, \\
&\quad = \check{\Sigma}_{B,C} \circ ((f \wedge \mathbf{1}_{\neg B} \vee \mathbf{1}_C),
\end{aligned}$$

by bifunctorial and naturality equations, and $\mathbf{PN}^{\neg}$ Coherence. We verify analogously the equation

$$(\hat{\Delta}\ dinat) \qquad (\mathbf{1}_C \wedge (\neg f \vee \mathbf{1}_B)) \circ \hat{\Delta}_{B,C} = (\mathbf{1}_C \wedge (\mathbf{1}_{\neg A} \vee f)) \circ \hat{\Delta}_{A,C},$$

From these equations we derive easily other analogous equations, which we call *dinaturality* equations, for $\hat{\Sigma}$, $\check{\Delta}$, $\hat{\Delta}'$, $\check{\Sigma}'$, $\hat{\Sigma}'$ and $\check{\Delta}'$. For example, we have

$$(\hat{\Sigma}\ dinat) \qquad ((\neg f \vee \mathbf{1}_B) \wedge \mathbf{1}_C) \circ \hat{\Sigma}_{B,C} = ((\mathbf{1}_{\neg A} \vee f) \wedge \mathbf{1}_C) \circ \hat{\Sigma}_{A,C},$$

$$(\check{\Sigma}'\ dinat) \qquad \check{\Sigma}'_{A,C} \circ ((\neg f \wedge \mathbf{1}_A) \vee \mathbf{1}_C) = \check{\Sigma}'_{B,C} \circ ((\mathbf{1}_{\neg B} \wedge f) \vee \mathbf{1}_C),$$

We can derive now the following equation:

$$(\neg 2) \qquad \neg(f_1 \circ f_2) = \neg f_2 \circ \neg f_1$$

for $f_1 \colon A \vdash B$ and $f_2 \colon C \vdash A$. We have

$$\begin{aligned}
&\neg f_2 \circ \neg f_1\\
&\quad = \check{\Sigma}'_{A,\neg C} \circ ((\neg f_1 \wedge \mathbf{1}_A) \vee \mathbf{1}_{\neg C}) \circ d_{\neg B,A,\neg C} \circ (\mathbf{1}_{\neg B} \wedge (f_2 \vee \mathbf{1}_{\neg C})) \circ \hat{\Delta}'_{C,\neg B},\\
&\qquad\qquad \text{by } (\hat{\Delta}' \ nat), (\wedge 2) \text{ and } (d \ nat),\\
&\quad = \check{\Sigma}'_{B,\neg C} \circ ((\mathbf{1}_{\neg B} \wedge f_1) \vee \mathbf{1}_{\neg C}) \circ d_{\neg B,A,\neg C} \circ (\mathbf{1}_{\neg B} \wedge (f_2 \vee \mathbf{1}_{\neg C})) \circ \hat{\Delta}'_{C,\neg B},\\
&\qquad\qquad \text{by } (\check{\Sigma}' \ dinat),\\
&\quad = \neg(f_1 \circ f_2), \quad \text{by } (d \ nat) \text{ and } (\wedge 2).
\end{aligned}$$

This proves that for every proof-net category $\mathcal{A}$ we have that ¬ is a contravariant functor from $\mathcal{A}$ to $\mathcal{A}$.

In every proof-net category $\mathcal{A}$, for every object A and for $\xi \in \{\wedge, \vee\}$, we have a functor $A\,\xi$ from $\mathcal{A}$ to $\mathcal{A}$, i.e. an endofunctor (1-endofunctor in the terminology of [22], Section 2.4), which on arrows is the operation $\mathbf{1}_A\,\xi$. It can easily be shown with the help of $\mathbf{PN}^{\neg}$ Coherence that in every proof-net category $A\wedge$ is left adjoint to $\neg A\vee$, and $\neg A\wedge$ is left adjoint to $A\vee$ (cf. §3.6 below; see [38], Section IV.1, or §5.1 below, for the notion of adjunction).

Chapter 3

Star-Autonomous Categories

The goal of this chapter is to prove the categorial equivalence between the star-autonomous category $\mathbf{SA}_s$ freely generated by a set of objects and the proof-net category with units $\mathbf{SA}'_s$ freely generated by the same set of objects. Our notion of proof-net category with units, which is obtained by extending the notion of proof-net category with unit objects and appropriate arrows and equations between these arrows, amounts to the notion of linearly distributive category with negation introduced in [11].

This chapter is rather technical. It demonstrates the equivalence of two notions of category formulated in two different languages, which happens to involve some pretty complex syntactical matters.

Since the language for star-autonomous categories we rely on only overlaps with the language for proof-net categories with units, we introduce two auxiliary freely generated categories called $\mathbf{SA}$ and $\mathbf{SA}'$, for which we have the union of these two different languages. The proof of equivalence of $\mathbf{SA}_s$ with $\mathbf{SA}'_s$ is broken into proofs of equivalence of $\mathbf{SA}_s$ with $\mathbf{SA}$ and $\mathbf{SA}'_s$ with $\mathbf{SA}'$, and a proof of isomorphism of $\mathbf{SA}$ with $\mathbf{SA}'$.

In the proofs of these equivalences and of this isomorphism, we rely heavily on Kelly's and Mac Lane's coherence theorem for symmetric monoidal closed categories, on our coherence theorem for $\mathbf{PN}^{\neg}$ from the preceding chapter, and on a coherence theorem for a freely generated category $\mathbf{PN}^{\neg}_{\to\bot}$, intermediary between $\mathbf{PN}^{\neg}$ and $\mathbf{SA}'$, which is equivalent with $\mathbf{PN}^{\neg}$. Without these tools, the computations needed would be excessively difficult, as it was prefigured in the literature (see [11], Section 4, Theorem 4.5).

In the course of these proofs, we define in §3.4 a nonsymmetric net structure in the sense of [22] (Section 7.2) in the category *Set* of sets with functions, and a nonassociative semiassociative structure in the sense of [22] (Section 4.2) in the same category.

§3.1. The category SMC

First we define the category **SMC**, which is the symmetric monoidal closed category (see [38], Section VII.7) freely generated by the set of letters $\mathcal{P}$. The objects of the category **SMC** are the formulae of the propositional language $\mathcal{L}_{\top,\wedge,\rightarrow}$ generated by $\mathcal{P}$ with the nullary connective (i.e. propositional constant) $\top$ and the binary connectives $\wedge$ and $\rightarrow$.

To define the arrows of **SMC**, we define inductively the arrow terms of **SMC** by assuming as primitive arrow terms $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$ (see §2.1), plus

$$\hat{\delta}^{\rightarrow}_{A} \colon A\wedge\top \vdash A, \qquad \hat{\delta}^{\leftarrow}_{A} \colon A \vdash A\wedge\top,$$
$$\varepsilon_{A,B} \colon A\wedge(A\rightarrow B) \vdash B, \qquad \eta_{A,B} \colon B \vdash A\rightarrow(A\wedge B);$$

as operations on arrow terms we have $\circ$ and $\wedge$ (which we know from **DS**; see §2.1) and the unary operations $A\rightarrow$, for every object A, such that for $f \colon B\vdash C$ we have the arrow term $A\rightarrow f \colon A\rightarrow B\vdash A\rightarrow C$. This concludes the definition of the arrow terms of **SMC**.

The equations of **SMC** are obtained by assuming besides $f=f$ the following equations: (*cat* 1), (*cat* 2), ($\wedge$ 1), ($\wedge$ 2) (see §2.1), plus

$$(A\rightarrow 1) \qquad A\rightarrow \mathbf{1}_B = \mathbf{1}_{A\rightarrow B},$$
$$(A\rightarrow 2) \qquad A\rightarrow (f\circ g) = (A\rightarrow f)\circ(A\rightarrow g),$$

($\hat{b}^{\rightarrow}$ *nat*), ($\hat{c}$ *nat*) (see §2.1), plus for $f \colon A\vdash B$

$$(\hat{\delta}^{\rightarrow}\ nat) \qquad f\circ\hat{\delta}^{\rightarrow}_{A} = \hat{\delta}^{\rightarrow}_{B}\circ(f\wedge\mathbf{1}_\top),$$
$$(\varepsilon\ nat) \qquad f\circ\varepsilon_{C,A} = \varepsilon_{C,B}\circ(\mathbf{1}_C\wedge(C\rightarrow f)),$$
$$(\eta\ nat) \qquad (C\rightarrow(\mathbf{1}_C\wedge f))\circ\eta_{C,A} = \eta_{C,B}\circ f,$$

($\hat{b}\hat{b}$), ($\hat{b}$ 5), ($\hat{c}\,\hat{c}$), ($\hat{b}\,\hat{c}$) (see §2.1), plus

$$(\hat{\delta}\,\hat{\delta})\qquad \hat{\delta}^{\rightarrow}_{A}\circ\hat{\delta}^{\leftarrow}_{A}=\mathbf{1}_{A},\qquad \hat{\delta}^{\leftarrow}_{A}\circ\hat{\delta}^{\rightarrow}_{A}=\mathbf{1}_{A\wedge\top},$$

$$(\hat{b}\,\hat{\delta})\qquad \hat{b}^{\leftarrow}_{A,B,\top}\circ\hat{\delta}^{\leftarrow}_{A\wedge B}=\mathbf{1}_{A}\wedge\hat{\delta}^{\leftarrow}_{B},$$

$$(\varepsilon\eta\wedge)\qquad \varepsilon_{A,A\wedge B}\circ(\mathbf{1}_{A}\wedge\eta_{A,B})=\mathbf{1}_{A\wedge B},$$

$$(\varepsilon\eta\rightarrow)\qquad (A\rightarrow\varepsilon_{A,B})\circ\eta_{A,A\rightarrow B}=\mathbf{1}_{A\rightarrow B}.$$

The equations $(A\rightarrow 1)$ and $(A\rightarrow 2)$ say that $A\rightarrow$ is a functor, while the last two equations are the triangular equations of an adjunction (see [38], Section IV.1, or §5.1 below).

The set of equations of **SMC** is closed under symmetry and transitivity of equality, under the rules $(cong\ \ \xi)$ for $\xi\in\{\circ,\wedge\}$ (see §2.1), and also under the rules

$$\frac{f=g}{A\rightarrow f=A\rightarrow g}$$

This defines the equations of **SMC**.

It is easy to see that for **SMC** we have the naturality equation

$$(\hat{\delta}^{\leftarrow}\ nat)\quad (f\wedge\mathbf{1}_{\top})\circ\hat{\delta}^{\leftarrow}_{A}=\hat{\delta}^{\leftarrow}_{B}\circ f.$$

With the definitions

$$\hat{\sigma}^{\rightarrow}_{A}=_{df}\hat{\delta}^{\rightarrow}_{A}\circ\hat{c}_{\top,A},\qquad \hat{\sigma}^{\leftarrow}_{A}=_{df}\hat{c}_{A,\top}\circ\hat{\delta}^{\leftarrow}_{A},$$

we obtain that $\hat{\sigma}^{\rightarrow}_{A}$ and $\hat{\sigma}^{\leftarrow}_{A}$ are inverse to each other. Note that there is an analogy between $\hat{\Delta}_{B,A}\colon A\vdash A\wedge(\neg B\vee B)$ and $\hat{\delta}^{\leftarrow}\colon A\vdash A\wedge\top$, and between $\hat{\Sigma}_{B,A}\colon A\vdash(\neg B\vee B)\wedge A$ and $\hat{\sigma}^{\leftarrow}\colon A\vdash\top\wedge A$, though $\hat{\Delta}_{B,A}$ and $\hat{\Sigma}_{B,A}$ are not isomorphisms. This analogy, which is the reason for our notation, is manifested by comparing the equation $(\hat{b}\hat{\Delta})$ of §2.2 and $(\hat{b}\,\hat{\delta})$ above. (Note that the equation $(\hat{b}\,\hat{\delta})$ above is not exactly the equation $(\hat{b}\,\hat{\delta})$ of Section 4.6 of [22], but the two equations follow from each other in the presence of isomorphism equations.)

For $g\colon A\vdash D$ and $f\colon B\vdash C$, we introduce the following definition in **SMC**:

$$g\rightarrow f=_{df}(A\rightarrow(\varepsilon_{D,C}\circ(g\wedge(D\rightarrow f))))\circ\eta_{A,D\rightarrow B}:D\rightarrow B\vdash A\rightarrow C.$$

With the help of (ε *nat*), ($A \rightarrow 2$) and ($\varepsilon\eta \rightarrow$) we obtain that

$$\mathbf{1}_A \rightarrow f = A \rightarrow f.$$

(For the sake of uniformity, we will later prefer to write $\mathbf{1}_A \rightarrow f$, rather than $A \rightarrow f$.) We then obtain

$$(\rightarrow 1) \qquad \mathbf{1}_A \rightarrow \mathbf{1}_B = \mathbf{1}_{A\rightarrow B},$$
$$(\rightarrow 2) \qquad (f_1 \circ g_1) \rightarrow (g_2 \circ f_2) = (g_1 \rightarrow g_2) \circ (f_1 \rightarrow f_2);$$

for ($\rightarrow 1$) we use ($A \rightarrow 1$), while for ($\rightarrow 2$) we use essentially ($A \rightarrow 2$), ($\wedge 2$), ($\varepsilon\eta \wedge$), (ε *nat*) and (η *nat*). So $\rightarrow$ is a bifunctor from $\mathbf{SMC}^{op} \times \mathbf{SMC}$ to $\mathbf{SMC}$ (see [38], Section IV.7, Theorem 3).

For $f\colon A \vdash B$, we derive in $\mathbf{SMC}$

$$(\varepsilon\ \mathit{dinat}) \qquad \varepsilon_{A,C} \circ (\mathbf{1}_A \wedge (f \rightarrow \mathbf{1}_C)) = \varepsilon_{B,C} \circ (f \wedge \mathbf{1}_{B\rightarrow C}),$$
$$(\eta\ \mathit{dinat}) \qquad (\mathbf{1}_A \rightarrow (f \wedge \mathbf{1}_C)) \circ \eta_{A,C} = (f \rightarrow \mathbf{1}_{B\wedge C}) \circ \eta_{B,C};$$

for (ε *dinat*) we use (ε *nat*) and ($\varepsilon\eta \wedge$), while for (η *dinat*) we use (η *nat*), ($\varepsilon\eta \wedge$) and bifunctorial equations. We call these two equations *dinaturality* equations for ε and η. Together with (ε *nat*) and (η *nat*), these dinaturality equations show that ε and η are dinatural transformations in the sense of [38] (Section IX.4).

We define a functor G from $\mathbf{SMC}$ to Br by using the appropriate clauses we had for the functor G from $\mathbf{PN}^{\neg}$ to Br in §2.3, to which we add that $G\alpha$ is an identity arrow of Br also when α is $\hat{\delta}^{\rightarrow}_{A}$ and $\hat{\delta}^{\leftarrow}_{A}$, and the following clauses:

$$G\varepsilon_{A,B} = G\check{\Sigma}_{A,B},$$
$$G\eta_{A,B} = G\hat{\Sigma}_{A,B}$$

(see §2.3). We define $G(f \circ g)$ and $G(f_1 \wedge f_2)$ as we did in §2.3. For $f\colon B \vdash C$, the set of ordered pairs of $G(A \rightarrow f)$ is

$$G\mathbf{1}_A \cup Gf^{+GA}_{+GA}$$

(see §2.3 for Gf^{+GA}_{+GA}).

For a Brauerian split equivalence $R \subseteq (X^s \cup Y^t)^2$, we define the Brauerian split equivalence $R^{op} \subseteq (Y^s \cup X^t)^2$ by replacing every ordered pair

(u_i, v_j) of R by $(u_{i'}, v_{j'})$ where $i, j \in \{s, t\}$, while $s' = t$ and $t' = s$. With that, for $g\colon A \vdash D$ and $f\colon B \vdash C$ it can be checked that the set of ordered pairs of $G(g \to f)$ is

$$Gg^{op} \cup Gf^{+GD}_{+GA}.$$

We call a formula A of $\mathcal{L}_{\top,\wedge,\to}$ *consequential* when for every subformula $B \to C$ of A we have that either B is letterless or C has letters occurring in it. An alternative way to characterize consequential formulae is to say that these are formulae A of $\mathcal{L}_{\top,\wedge,\to}$ for which there is an isomorphism of type $A \vdash A'$ of **SMC** such that either $\top$ does not occur in A' or A' is $\top$. (To establish the equivalence of these two characterizations, one may rely on the results of [17].)

Let $\mathbf{SMC}^c$ be the full subcategory of **SMC** whose objects are consequential formulae. (The category $\mathbf{SMC}^c$ is a replete subcategory of **SMC** in the sense of [29], Section A1.1; namely, every object of **SMC** isomorphic to an object of $\mathbf{SMC}^c$ is in $\mathbf{SMC}^c$.) The functor G from **SMC** to *Br* may be restricted to a functor G from $\mathbf{SMC}^c$ to *Br*. The following result is proved by Kelly and Mac Lane in [32].

$\mathbf{SMC}^c$ Coherence. *The functor G from $\mathbf{SMC}^c$ to Br is faithful.*

§3.2. The category SA

The objects of the category **SA** are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\to}$ generated by $\mathcal{P}$ with the nullary connectives (i.e. propositional constants) $\top$ and $\bot$, the unary connective $\neg$ and the binary connectives $\wedge$, $\vee$ and $\to$.

To define the arrows of **SA**, we define inductively the arrow terms of **SA** by assuming as primitive arrow terms all the primitive arrow terms we had for **SMC** (with A, B and C ranging over the formulae of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\to}$) plus

$\nu^{\to}_A\colon (A \to \bot) \to \bot \vdash A$,

$\lambda^{\to}_A\colon \neg A \vdash A \to \bot$, $\qquad\lambda^{\leftarrow}_A\colon A \to \bot \vdash \neg A$,

$\upsilon^{\to}_{A,B}\colon A \vee B \vdash (A \to \bot) \to B$, $\qquad\upsilon^{\leftarrow}_{A,B}\colon (A \to \bot) \to B \vdash A \vee B$;

the operations on these arrow terms are as for **SMC**.

The equations of **SA** are obtained by assuming all the equations we have assumed for **SMC**, plus for

$$\nu^{\leftarrow}_{A,B} =_{df} (\mathbf{1}_{A\rightarrow B} \rightarrow (\varepsilon_{A,B} \circ \hat{c}_{A\rightarrow B,A})) \circ \eta_{A\rightarrow B,A} \colon A \vdash (A \rightarrow B) \rightarrow B$$

the equations

$$
\begin{array}{lll}
(\nu\nu) & \nu^{\rightarrow}_{A} \circ \nu^{\leftarrow}_{A,\bot} = \mathbf{1}_A, & \nu^{\leftarrow}_{A,\bot} \circ \nu^{\rightarrow}_{A} = \mathbf{1}_{(A\rightarrow\bot)\rightarrow\bot}, \\
(\lambda\lambda) & \lambda^{\rightarrow}_{A} \circ \lambda^{\leftarrow}_{A} = \mathbf{1}_{A\rightarrow\bot}, & \lambda^{\leftarrow}_{A} \circ \lambda^{\rightarrow}_{A} = \mathbf{1}_{\neg A}, \\
(\upsilon\upsilon) & \upsilon^{\rightarrow}_{A,B} \circ \upsilon^{\leftarrow}_{A,B} = \mathbf{1}_{(A\rightarrow\bot)\rightarrow B}, & \upsilon^{\leftarrow}_{A,B} \circ \upsilon^{\rightarrow}_{A,B} = \mathbf{1}_{A\vee B}.
\end{array}
$$

The set of equations of **SA** is closed under the same rules as the set of equations of **SMC**.

The equations $(\nu\nu)$ assert that $\nu^{\leftarrow}_{A,\bot}$ is an isomorphism. This is the assumption used by Barr in [2] (Section 2) to define *star-autonomous* categories. The isomorphism equations $(\lambda\lambda)$ and $(\upsilon\upsilon)$ are auxiliary, and will be discarded in a language where $\neg$ and $\vee$ are not primitive (see §3.8).

§3.3. The category SA′

The objects of the category **SA**′ are as for **SA** the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$ generated by $\mathcal{P}$. As primitive arrow terms we have $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$, $\check{b}^{\rightarrow}_{A,B,C}$, $\check{b}^{\leftarrow}_{A,B,C}$, $\check{c}_{A,B}$, $d_{A,B,C}$ (see §2.1), $\hat{\Delta}_{B,A}$, $\check{\Sigma}_{B,A}$ (see §2.2), $\hat{\delta}^{\rightarrow}_{A}$, $\hat{\delta}^{\leftarrow}_{A}$ (see §3.1), plus

$$
\begin{array}{ll}
\check{\delta}^{\rightarrow}_{A} \colon A \vee \bot \vdash A, & \check{\delta}^{\leftarrow}_{A} \colon A \vdash A \vee \bot, \\
\pi^{\rightarrow}_{A,B} \colon A \rightarrow B \vdash \neg A \vee B, & \pi^{\leftarrow}_{A,B} \colon \neg A \vee B \vdash A \rightarrow B.
\end{array}
$$

These primitive arrow terms together with the operations on arrow terms $\circ$, $\wedge$ and $\vee$ (the same we had for **DS** and **PN**$^{\neg}$ in §§2.1-2) define the arrow terms of **SA**′.

The equations of **SA**′ are obtained by assuming all the equations we have assumed for **PN**$^{\neg}$ (which are the equations of **DS** of §2.1 plus the **PN** equations of §2.2), plus ($\hat{\delta}^{\rightarrow}$ *nat*), ($\hat{\delta}\,\hat{\delta}$), ($\hat{b}\,\hat{\delta}$) (see §3.1), with the dual equations

$$(\check{\delta}^{\rightarrow}\ nat)\qquad f\circ\check{\delta}^{\rightarrow}_{A}=\check{\delta}^{\rightarrow}_{B}\circ(f\vee\mathbf{1}_{\bot}),$$

$$(\check{\delta}\,\check{\delta})\qquad \check{\delta}^{\rightarrow}_{A}\circ\check{\delta}^{\leftarrow}_{A}=\mathbf{1}_{A},\qquad \check{\delta}^{\leftarrow}_{A}\circ\check{\delta}^{\rightarrow}_{A}=\mathbf{1}_{A\vee\bot},$$

$$(\check{b}\,\check{\delta})\qquad \check{b}^{\leftarrow}_{A,B,\bot}\circ\check{\delta}^{\leftarrow}_{A\vee B}=\mathbf{1}_{A}\vee\check{\delta}^{\leftarrow}_{B},$$

and, finally, with $\hat{\sigma}^{\leftarrow}_{A}$ defined as in §3.1, the following equations:

$$(d\,\hat{\sigma})\qquad d_{\top,B,C}\circ\hat{\sigma}^{\leftarrow}_{B\vee C}=\hat{\sigma}^{\leftarrow}_{B}\vee\mathbf{1}_{C},$$

$$(d\,\check{\delta})\qquad \check{\delta}^{\rightarrow}_{C\wedge B}\circ d_{C,B,\bot}=\mathbf{1}_{C}\wedge\check{\delta}^{\rightarrow}_{B},$$

$$(\pi\pi)\qquad \pi^{\rightarrow}_{A,B}\circ\pi^{\leftarrow}_{A,B}=\mathbf{1}_{\neg A\vee B},\qquad \pi^{\leftarrow}_{A,B}\circ\pi^{\rightarrow}_{A,B}=\mathbf{1}_{A\rightarrow B}.$$

The set of equations of $\mathbf{SA}'$ is closed under symmetry and transitivity of equality and under the rules $(cong\ \xi)$ for $\xi\in\{\circ,\wedge,\vee\}$ (see §2.1). This defines the equations of $\mathbf{SA}'$.

It is clear that in $\mathbf{SA}'$ we have the naturality equations $(\hat{\delta}^{\leftarrow}\ nat)$ (see §3.1) and

$$(\check{\delta}^{\leftarrow}\ nat)\qquad (f\vee\mathbf{1}_{\bot})\circ\check{\delta}^{\leftarrow}_{A}=\check{\delta}^{\leftarrow}_{B}\circ f.$$

Analogously to what we had in §3.1, we define $\hat{\sigma}^{\rightarrow}_{A}$ and

$$\check{\sigma}^{\rightarrow}_{A}=_{df}\check{\delta}^{\rightarrow}_{A}\circ\check{c}_{A,\bot},\qquad \check{\sigma}^{\leftarrow}_{A}=_{df}\check{c}_{\bot,A}\circ\check{\delta}^{\leftarrow}_{A},$$

which give isomorphisms in $\mathbf{SA}'$. Note that $\check{\sigma}^{\rightarrow}_{A}\colon\bot\vee A\vdash A$ is analogous to $\check{\Sigma}_{B,A}\colon(B\wedge\neg B)\vee A\vdash A$, though $\check{\Sigma}_{B,A}$ is not an isomorphism. The equation $(\hat{b}\check{\Sigma})$ of §2.2 is analogous to the following equation of $\mathbf{SA}'$ (an equation of monoidal categories):

$$\check{\sigma}^{\rightarrow}_{B\vee A}\circ\check{b}^{\leftarrow}_{\bot,B,A}=\check{\sigma}^{\rightarrow}_{B}\vee\mathbf{1}_{A}.$$

The equations $(d\,\hat{\sigma})$ and $(d\,\check{\delta})$, which amount to the equations $(\hat{\sigma}\,d^{L})$ and $(\check{\delta}\,d^{L})$ of Section 7.9 of [22] (these equations stem from [11], Section 2.1), are analogous to the equations $(d\,\hat{\Sigma})$ and $(d\,\check{\Delta})$ of §2.2. The equations $(\pi\pi)$ are auxiliary, and will be discarded in a language where $\rightarrow$ is not primitive (see §3.8).

§3.4. SA′ in SA

Our purpose now is to define the **SA′** structure in **SA**, and then show that the equations of **SA′** hold in **SA** for this defined structure.

To define $\check{b}^{\rightarrow}_{A,B,C}$ and $\check{b}^{\leftarrow}_{A,B,C}$ in **SA** we need some preliminary definitions. We note first that in **SMC**, and hence also in **SA**, we can introduce the following definitions:

$$i^{\rightarrow}_{A,B,C} =_{df} (\mathbf{1}_A \rightarrow ((\mathbf{1}_B \rightarrow (\varepsilon_{A\wedge B,C} \circ (\hat{c}_{B,A} \wedge \mathbf{1}_{(A\wedge B)\rightarrow C}) \circ \hat{b}^{\rightarrow}_{B,A,(A\wedge B)\rightarrow C})) \circ \\ \circ\, \eta_{B,A\wedge((A\wedge B)\rightarrow C)})) \circ \eta_{A,(A\wedge B)\rightarrow C} : (A\wedge B)\rightarrow C \vdash A \rightarrow (B \rightarrow C),$$

$$i^{\leftarrow}_{A,B,C} =_{df} (\mathbf{1}_{A\wedge B} \rightarrow (\varepsilon_{B,C} \circ (\mathbf{1}_B \wedge \varepsilon_{A,B\rightarrow C}) \circ \hat{b}^{\leftarrow}_{B,A,A\rightarrow(B\rightarrow C)} \circ \\ \circ\, (\hat{c}_{A,B} \wedge \mathbf{1}_C))) \circ \eta_{A\wedge B,A\rightarrow(B\rightarrow C)} : A \rightarrow (B \rightarrow C) \vdash (A\wedge B) \rightarrow C.$$

By $\mathbf{SMC}^c$ Coherence of §3.1, we can immediately conclude that in **SMC**, and hence also in **SA**, the arrows $i^{\rightarrow}_{A,B,C}$ and $i^{\leftarrow}_{A,B,C}$ are isomorphisms inverse to each other. By applying naturality and dinaturality equations, we can also conclude that $i^{\rightarrow}$ and $i^{\leftarrow}$ are natural transformations of **SMC** and **SA** in all their three indices A, B and C.

In **SA** we have the following definitions:

$$j^{\rightarrow}_{A,B} =_{df} \nu^{\rightarrow}_{A\wedge(B\rightarrow\bot)} \circ (i^{\rightarrow}_{A,B\rightarrow\bot,\bot} \rightarrow \mathbf{1}_\bot) \circ ((\mathbf{1}_A \rightarrow \nu^{\rightarrow}_B) \rightarrow \mathbf{1}_\bot) : \\ (A \rightarrow B) \rightarrow \bot \vdash A \wedge (B \rightarrow \bot),$$

$$j^{\leftarrow}_{A,B,C} =_{df} ((\mathbf{1}_A \rightarrow \nu^{\leftarrow}_{B,C}) \rightarrow \mathbf{1}_C) \circ (i^{\leftarrow}_{A,B\rightarrow C,C} \rightarrow \mathbf{1}_C) \circ \nu^{\leftarrow}_{A\wedge(B\rightarrow C),C} : \\ A \wedge (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow C.$$

The definition of $j^{\leftarrow}_{A,B,C}$ can be given already in **SMC**, but not the definition of $j^{\rightarrow}_{A,B}$. It is easy to see that in **SA** we have that $j^{\rightarrow}_{A,B}$ and $j^{\leftarrow}_{A,B,\bot}$ are isomorphisms inverse to each other.

We also have the following definitions in **SA**:

$$\check{\mathbf{b}}^{\rightarrow}_{A,B,C} =_{df} (j^{\rightarrow}_{A,B} \rightarrow \mathbf{1}_C) \circ i^{\leftarrow}_{A,B\rightarrow\bot,C} : \\ A \rightarrow ((B \rightarrow \bot) \rightarrow C) \vdash ((A \rightarrow B) \rightarrow \bot) \rightarrow C,$$

$$\check{\mathbf{b}}^{\leftarrow}_{A,B,C,D} =_{df} i^{\rightarrow}_{A,B\rightarrow D,C} \circ (j^{\leftarrow}_{A,B,D} \rightarrow \mathbf{1}_C) : \\ ((A \rightarrow B) \rightarrow D) \rightarrow C \vdash A \rightarrow ((B \rightarrow D) \rightarrow C).$$

The second of these definitions can be given already in **SMC**. It is easy to see that in **SA** we have that $\check{\mathbf{b}}^{\rightarrow}_{A,B,C}$ and $\check{\mathbf{b}}^{\leftarrow}_{A,B,C,\bot}$ are isomorphisms inverse to each other.

If $\bot$ is an arbitrary object, and we define $A\vee B$ as $(A\rightarrow\bot)\rightarrow B$ and $f\vee g$ as $(f\rightarrow\mathbf{1}_{\bot})\rightarrow g$, then we can check that in **SMC**, and hence also in every symmetric monoidal closed category, we have for this defined $\vee$ the bifunctorial equations $(\vee 1)$ and $(\vee 2)$ (see §2.1), while for $\check{b}^{\leftarrow}_{A,B,C}$ replaced by $\check{\mathbf{b}}^{\leftarrow}_{A\rightarrow\bot,B,C,\bot}$ we have ($\check{b}^{\leftarrow}$ *nat*) and ($\check{b}\,5$) of §2.1. So every symmetric monoidal closed category is a semiassociative category in the sense of Section 4.2 of [22]. Therefore, every cartesian closed category, and in particular the category *Set* of sets with functions, is a semiassociative category with $A\vee B$ being $(A\rightarrow\bot)\rightarrow B$, commonly written $B^{(\bot^{A})}$, where $\bot$ is an arbitrary set, not necessarily the initial object $\emptyset$ of *Set*. For $\bot$ distinct from $\emptyset$ and from a singleton, we have that $A\vee(B\vee C)$ need not be isomorphic to $(A\vee B)\vee C$.

With $A\vee B$ being $(A\rightarrow\emptyset)\rightarrow B$, the category *Set* is an associative category in the sense of Section 4.3 of [22]. We can check that in *Set* the arrow $\check{b}^{\leftarrow}_{A,B,C}$, defined as $\check{\mathbf{b}}^{\leftarrow}_{A\rightarrow\emptyset,B,C,\emptyset}$, is an isomorphism, and hence in *Set* we have a natural transformation whose members are of type

$$(A\rightarrow\emptyset)\rightarrow((B\rightarrow\emptyset)\rightarrow C)\vdash(((A\rightarrow\emptyset)\rightarrow B)\rightarrow\emptyset)\rightarrow C;$$

this defines the inverse $\check{b}^{\rightarrow}_{A,B,C}$ of $\check{b}^{\leftarrow}_{A,B,C}$, i.e. of $\check{\mathbf{b}}^{\leftarrow}_{A\rightarrow\emptyset,B,C,\emptyset}$.

Then we have the following definitions in **SA**:

$$\check{b}^{\rightarrow}_{A,B,C} =_{df} \upsilon^{\leftarrow}_{A\vee B,C}\circ((\upsilon^{\leftarrow}_{A,B}\rightarrow\mathbf{1}_{\bot})\rightarrow\mathbf{1}_C)\circ\check{\mathbf{b}}^{\rightarrow}_{A\rightarrow\bot,B,C}\circ(\mathbf{1}_{A\rightarrow\bot}\rightarrow\upsilon^{\rightarrow}_{B,C})\circ$$
$$\circ\,\upsilon^{\rightarrow}_{A,B\vee C} : A\vee(B\vee C)\vdash(A\vee B)\vee C,$$

$$\check{b}^{\leftarrow}_{A,B,C} =_{df} \upsilon^{\leftarrow}_{A,B\vee C}\circ(\mathbf{1}_{A\rightarrow\bot}\rightarrow\upsilon^{\leftarrow}_{B,C})\circ\check{\mathbf{b}}^{\leftarrow}_{A\rightarrow\bot,B,C,\bot}\circ((\upsilon^{\rightarrow}_{A,B}\rightarrow\mathbf{1}_{\bot})\rightarrow\mathbf{1}_C)\circ$$
$$\circ\,\upsilon^{\rightarrow}_{A\vee B,C} : (A\vee B)\vee C\vdash A\vee(B\vee C).$$

To define $\check{c}_{A,B}\colon B\vee A\vdash A\vee B$ in **SA**, we need some further preliminary definitions. In **SMC** we have

$$s_{A,B,C} =_{df} (\mathbf{1}_{B\rightarrow C}\rightarrow(\mathbf{1}_A\rightarrow(\varepsilon_{B,C}\circ(\varepsilon_{A,B}\wedge\mathbf{1}_{B\rightarrow C})\circ\hat{b}^{\rightarrow}_{A,A\rightarrow B,B\rightarrow C}))\circ$$
$$\circ\,\eta_{A,(A\rightarrow B)\wedge(B\rightarrow C)}\circ\hat{c}_{B\rightarrow C,A\rightarrow B}))\circ\eta_{B\rightarrow C,A\rightarrow B} :$$
$$A\rightarrow B\vdash(B\rightarrow C)\rightarrow(A\rightarrow C).$$

By applying naturality and dinaturality equations, we can verify that s is a natural transformation of **SA** in its first two indices A and B, and a dinatural transformation in its third index C.

We have in **SA** the following definitions based on s:

$$\check{\mathbf{c}}_{A,B} =_{df} (\mathbf{1}_{A\to\bot} \to \nu^{\rightarrow}_{B}) \circ s_{B\to\bot,A,\bot} : (B \to \bot) \to A \vdash (A \to \bot) \to B,$$

$$\check{c}_{A,B} =_{df} \upsilon^{\leftarrow}_{A,B} \circ \check{\mathbf{c}}_{A,B} \circ \upsilon^{\rightarrow}_{B,A} : B \vee A \vdash A \vee B.$$

Next we have in **SMC**:

$$\mathbf{d}_{A,B,C,D} =_{df} (i^{\rightarrow}_{A,B,D} \to \mathbf{1}_C) \circ j^{\leftarrow}_{A,B\to D,C} : \quad A \wedge ((B \to D) \to C) \vdash ((A \wedge B) \to D) \to C,$$

$$\mathbf{d}^R_{C,B,A,D} =_{df} (\mathbf{1}_{C\to D} \to ((\varepsilon_{C\to D,B} \wedge \mathbf{1}_A) \circ \hat{b}^{\rightarrow}_{C\to D,(C\to D)\to B,A})) \circ \circ\, \eta_{C\to D,((C\to D)\to B)\wedge A} : \quad ((C \to D) \to B) \wedge A \vdash (C \to D) \to (B \wedge A).$$

Note that both $\mathbf{d}$ and $\mathbf{d}^R$ are natural transformations in all their four indices.

If, as above, we define $A \vee B$ in *Set* as $(A \to \emptyset) \to B$, and $\check{b}^{\leftarrow}_{A,B,C}$ as $\check{\mathbf{b}}^{\leftarrow}_{A\to\emptyset,B,C,\emptyset}$, while $\check{b}^{\rightarrow}_{A,B,C}$ is its inverse, and if, moreover, $\wedge$ is cartesian product, while $d^L_{A,B,C}$ is $\mathbf{d}_{A,B,C,\emptyset}$ and $d^R_{C,B,A}$ is $\mathbf{d}^R_{C,B,A,\emptyset}$, then we can check that *Set* with this structure is a net category in the sense of Section 7.2 of [22]. To verify the equations of net categories (which stem from [11]), it here helps a lot to apply $\mathbf{SMC}^c$ Coherence. In this net structure of *Set* all arrows with the same source and target are equal, i.e. all diagrams commute; this follows from the Net Coherence of [22] (Section 7.3). This net structure of *Set* is not symmetric, because $(A \to \emptyset) \to B$ need not be isomorphic to $(B \to \emptyset) \to A$. Note that *Set* with $\wedge$ being cartesian product and $\vee$ being disjoint union cannot be a net category for any definition of d^L and d^R (see [11], Section 3, and [22], Section 11.3).

With the help of $\mathbf{d}_{A,B,C,\bot}$, we have the following definition in **SA**:

$$d_{A,B,C} =_{df} \upsilon^{\leftarrow}_{A\wedge B,C} \circ \mathbf{d}_{A,B,C,\bot} \circ (\mathbf{1}_A \wedge \upsilon^{\rightarrow}_{B,C}) : A \wedge (B \vee C) \vdash (A \wedge B) \vee C,$$

and for $f : A \vdash D$ and $g : B \vdash E$ also the following:

$$f \vee g =_{df} \upsilon^{\leftarrow}_{D,E} \circ ((f \rightarrow \mathbf{1}_\bot) \rightarrow g) \circ \upsilon^{\rightarrow}_{A,B} : A \vee B \vdash D \vee E.$$

With that we have finished defining what was missing to obtain the **DS** structure in **SA**. (We already have in **SA** the arrows $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$ and $\hat{c}_{A,B}$, and the operations on arrows $\circ$ and $\wedge$.)

To define $\hat{\Delta}_{B,A}$: $A \vdash A \wedge (\neg B \vee B)$ and $\check{\Sigma}_{B,A}$: $(B \wedge \neg B) \vee A \vdash A$ in **SA** we introduce first the following preliminary definitions in **SMC**:

$$\hat{\mathbf{\Delta}}_{B,A} =_{df} (\mathbf{1}_A \wedge ((\mathbf{1}_B \rightarrow \hat{\delta}^{\rightarrow}_B) \circ \eta_{B,\top})) \circ \hat{\delta}^{\leftarrow}_A : A \vdash A \wedge (B \rightarrow B),$$

for E being $B \wedge (B \rightarrow C)$,

$$\check{\mathbf{\Sigma}}_{B,A,C} =_{df} \varepsilon_{E\rightarrow E,A} \circ \hat{c}_{(E\rightarrow E)\rightarrow A,E\rightarrow E} \circ \hat{\mathbf{\Delta}}_{E,(E\rightarrow E)\rightarrow A} \circ$$
$$\circ ((\mathbf{1}_E \rightarrow \varepsilon_{B,C}) \rightarrow \mathbf{1}_A) : ((B \wedge (B \rightarrow C)) \rightarrow C) \rightarrow A \vdash A.$$

These definitions are not the only possible. It is easy to see with the help of $\mathbf{SMC}^c$ Coherence that many other definitions would do, and, in particular, shorter definitions of $\check{\mathbf{\Sigma}}_{B,A,C}$ are possible. (The present one was chosen to facilitate calculation in §3.7 below.)

Then we have the following definitions in **SA**:

$$\hat{\Delta}_{B,A} =_{df} (\mathbf{1}_A \wedge (\upsilon^{\leftarrow}_{\neg B,B} \circ ((\nu^{\rightarrow}_B \circ (\lambda^{\leftarrow}_B \rightarrow \mathbf{1}_\bot)) \rightarrow \mathbf{1}_B))) \circ \hat{\mathbf{\Delta}}_{B,A} :$$
$$A \vdash A \wedge (\neg B \vee B),$$

$$\check{\Sigma}_{B,A} =_{df} \check{\mathbf{\Sigma}}_{B,A,\bot} \circ (((\mathbf{1}_B \wedge \lambda^{\rightarrow}_B) \rightarrow \mathbf{1}_\bot) \rightarrow \mathbf{1}_A) \circ \upsilon^{\rightarrow}_{B\wedge\neg B,A} :$$
$$(B \wedge \neg B) \vee A \vdash A.$$

For the remainder of the **SA′** structure we have the following definitions in **SA**:

$$\check{\delta}^{\rightarrow}_A =_{df} \nu^{\rightarrow}_A \circ \upsilon^{\rightarrow}_{A,\bot} : A \vee \bot \vdash A,$$

$$\check{\delta}^{\leftarrow}_A =_{df} \upsilon^{\leftarrow}_{A,\bot} \circ \nu^{\leftarrow}_{A,\bot} : A \vdash A \vee \bot,$$

$$\pi^{\rightarrow}_{A,B} =_{df} \upsilon^{\leftarrow}_{\neg A,B} \circ ((\nu^{\rightarrow}_A \circ (\lambda^{\leftarrow}_A \rightarrow \mathbf{1}_\bot)) \rightarrow \mathbf{1}_B) : A \rightarrow B \vdash \neg A \vee B,$$

$$\pi^{\leftarrow}_{A,B} =_{df} (((\lambda^{\rightarrow}_A \rightarrow \mathbf{1}_\bot) \circ \nu^{\leftarrow}_{A,\bot}) \rightarrow \mathbf{1}_B) \circ \upsilon^{\rightarrow}_{\neg A,B} : \neg A \vee B \vdash A \rightarrow B.$$

Note that with $\pi^{\rightarrow}_{A,B}$ defined as above we have

$$\hat{\Delta}_{B,A} =_{df} (\mathbf{1}_A \wedge \pi^{\rightarrow}_{B,B}) \circ \hat{\boldsymbol{\Delta}}_{B,A}\,.$$

With that we have finished defining what was missing to obtain the $\mathbf{SA}'$ structure in $\mathbf{SA}$.

It remains now to verify that the equations of $\mathbf{SA}'$ hold for this defined structure in $\mathbf{SA}$. The equations $(\vee 1)$, $(\vee 2)$, $(\check{b}^{\rightarrow}\ nat)$, $(\check{c}\ nat)$ and $(\check{b}\check{b})$ of §2.1 are trivial to check. (We rely here and later on isomorphism equations without mentioning that explicitly.) For $(\check{b}5)$ we appeal to $\mathbf{SMC}^c$ Coherence, while for $(\check{c}\check{c})$ we need some preparation.

We have the following equations in $\mathbf{SA}$:

$$(\nu^{\leftarrow}_{A\rightarrow\bot,\bot}) \qquad \nu^{\leftarrow}_{A\rightarrow\bot,\bot} = \nu^{\rightarrow}_{A} \rightarrow \mathbf{1}_{\bot},$$

$$(\nu^{\rightarrow}_{A\rightarrow\bot}) \qquad \nu^{\rightarrow}_{A\rightarrow\bot} = \nu^{\leftarrow}_{A,\bot} \rightarrow \mathbf{1}_{\bot}.$$

To prove these equations it is enough to establish

$$(\nu^{\leftarrow}_{A,B} \rightarrow \mathbf{1}_B) \circ \nu^{\leftarrow}_{A\rightarrow B,B} = \mathbf{1}_{A\rightarrow B},$$

which follows from $\mathbf{SMC}^c$ Coherence, and then use isomorphism equations.

To verify $(\check{c}\check{c})$ we use the following:

$$\check{\mathbf{c}}_{A,B} \circ \check{\mathbf{c}}_{B,A} = (\mathbf{1}_{A\rightarrow\bot} \rightarrow \nu^{\rightarrow}_{B}) \circ s_{B\rightarrow\bot,A,\bot} \circ (\mathbf{1}_{B\rightarrow\bot} \rightarrow \nu^{\rightarrow}_{A}) \circ s_{A\rightarrow\bot,B,\bot}$$

$$= (\mathbf{1}_{A\rightarrow\bot} \rightarrow \nu^{\rightarrow}_{B}) \circ ((\nu^{\rightarrow}_{A} \rightarrow \mathbf{1}_{\bot}) \rightarrow \mathbf{1}_{(B\rightarrow\bot)\rightarrow\bot}) \circ s_{B\rightarrow\bot,(A\rightarrow\bot)\rightarrow\bot,\bot} \circ$$
$$\circ\, s_{A\rightarrow\bot,B,\bot}, \quad \text{by the naturality of } s,$$

$$= (\mathbf{1}_{A\rightarrow\bot} \rightarrow \nu^{\rightarrow}_{B}) \circ (\nu^{\leftarrow}_{A\rightarrow\bot,\bot} \rightarrow \mathbf{1}_{(B\rightarrow\bot)\rightarrow\bot}) \circ s_{B\rightarrow\bot,(A\rightarrow\bot)\rightarrow\bot,\bot} \circ$$
$$\circ\, s_{A\rightarrow\bot,B,\bot}, \text{ by } (\nu^{\leftarrow}_{A\rightarrow\bot,\bot}),$$

$$= (\mathbf{1}_{A\rightarrow\bot} \rightarrow \nu^{\rightarrow}_{B}) \circ (\mathbf{1}_{A\rightarrow\bot} \rightarrow \nu^{\leftarrow}_{B,\bot}), \quad \text{by } \mathbf{SMC}^c \text{ Coherence},$$

$$= \mathbf{1}_{(A\rightarrow\bot)\rightarrow B}.$$

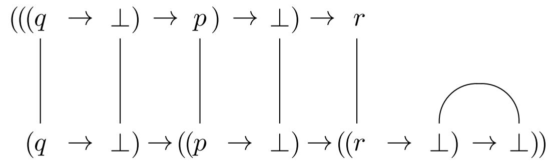
To verify $(\check{b}\check{c})$ we use the following equation of $\mathbf{SA}$:

$$(\mathbf{1}_{B\rightarrow\bot} \rightarrow \check{\mathbf{c}}_{A,C}) \circ \check{\mathbf{b}}^{\leftarrow}_{B\rightarrow\bot,C,A,\bot} \circ \check{\mathbf{c}}_{(B\rightarrow\bot)\rightarrow C,A} \circ \check{\mathbf{b}}^{\leftarrow}_{A\rightarrow\bot,B,C,\bot} \circ$$
$$\circ\, ((\check{\mathbf{c}}_{A,B} \rightarrow \mathbf{1}_{\bot}) \rightarrow \mathbf{1}_C) = \check{\mathbf{b}}^{\leftarrow}_{B\rightarrow\bot,A,C,\bot}.$$

To verify this equation we use the naturality of $\check{\mathbf{b}}^{\leftarrow}$ and s, and the equation $(\nu^{\leftarrow}_{A\rightarrow\bot,\bot})$ where A is instantiated by A and B. In this verification, for the arrow $g(p,q,r)$ of **SMC** defined as

$$\begin{aligned}(\mathbf{1}_{q\rightarrow\bot}\rightarrow((\nu^{\leftarrow}_{p\rightarrow\bot,\bot}\rightarrow\mathbf{1}_{(r\rightarrow\bot)\rightarrow\bot})\circ s_{r\rightarrow\bot,(p\rightarrow\bot)\rightarrow\bot,\bot}))\circ\check{\mathbf{b}}^{\leftarrow}_{q\rightarrow\bot,r,(p\rightarrow\bot)\rightarrow\bot,\bot}\circ\\ \circ\, s_{p\rightarrow\bot,(q\rightarrow\bot)\rightarrow r,\bot}\circ(\mathbf{1}_{p\rightarrow\bot}\rightarrow(\nu^{\leftarrow}_{q\rightarrow\bot,\bot}\rightarrow\mathbf{1}_r))\circ\check{\mathbf{b}}^{\leftarrow}_{p\rightarrow\bot,(q\rightarrow\bot)\rightarrow\bot,r,\bot}\circ\\ \circ((s_{q\rightarrow\bot,p,\bot}\rightarrow\mathbf{1}_\bot)\rightarrow\mathbf{1}_r),\end{aligned}$$

where $\bot$ is an arbitrary letter, we have that $Gg(p,q,r)$ corresponds to the diagram



and so, by $\mathbf{SMC}^c$ Coherence, the following holds:

$$g(A,B,C)=(\mathbf{1}_{B\rightarrow\bot}\rightarrow(\mathbf{1}_{A\rightarrow\bot}\rightarrow\nu^{\leftarrow}_{C,\bot}))\circ\check{\mathbf{b}}^{\leftarrow}_{B\rightarrow\bot,A,C,\bot}\,.$$

The equation $(d\ nat)$ is easily inferred from the naturality of $\mathbf{d}$ in all its four indices. To verify the equations $(d\wedge)$ and $(d\vee)$ we apply $\mathbf{SMC}^c$ Coherence.

For the equations $(d\hat{b})$ and $(d\check{b})$ we verify first that for

$$d^R_{C,B,A} =_{df} \check{c}_{C,B\wedge A}\circ(\hat{c}_{A,B}\vee\mathbf{1}_C)\circ d_{A,B,C}\circ(\mathbf{1}_A\wedge\check{c}_{B,C})\circ\hat{c}_{C\vee B,A}$$

of type $(C\vee B)\wedge A\vdash C\vee(B\wedge A)$ (see §2.1) we have in **SA**

$$d^R_{C,B,A}=\upsilon^{\leftarrow}_{C,B\wedge A}\circ\mathbf{d}^R_{C,B,A,\bot}\circ(\upsilon^{\rightarrow}_{C,B}\wedge\mathbf{1}_A).$$

For that we use the naturality of s and $\mathbf{d}$, the equation $(\nu^{\leftarrow}_{A\rightarrow\bot,\bot})$, and $\mathbf{SMC}^c$ Coherence. After this verification, we use essentially $\mathbf{SMC}^c$ Coherence to obtain $(d\hat{b})$ and $(d\check{b})$ in **SA**. With that we have all the equations of **DS** in **SA**.

We pass now to the **PN** equations of §2.2. It is trivial to check in **SA** the equations $(\hat{\Delta}\ nat)$ and $(\check{\Sigma}\ nat)$. For the equations $(\hat{b}\,\hat{\Delta})$, $(\check{b}\,\check{\Sigma})$ and $(d\hat{\Sigma})$ we apply various naturality equations and $\mathbf{SMC}^c$ Coherence.

For the equation $(d\overset{\vee}{\Delta})$ we verify first that we have

$$\nu^{\leftarrow}_{C\wedge B,\bot} = \mathbf{d}_{C,B,\bot,\bot}\circ(\mathbf{1}_C\wedge\nu^{\leftarrow}_{B,\bot}),$$

by $\mathbf{SMC}^c$ Coherence; so in **SA** we have

$$\nu^{\leftarrow}_{C\wedge B,\bot}\circ(\mathbf{1}_C\wedge\nu^{\rightarrow}_{B}) = \mathbf{d}_{C,B,\bot,\bot}.$$

We use this equation, together with the naturality of $\overset{\vee}{\Sigma}$ and $\mathbf{SMC}^c$ Coherence, to verify for E being $A\wedge(A\rightarrow\bot)$

$$\overset{\vee}{\mathbf{\Sigma}}_{A,C\wedge B,\bot}\circ(\mathbf{1}_{E\rightarrow\bot}\rightarrow\nu^{\rightarrow}_{C\wedge B})\circ s_{(C\wedge B)\rightarrow\bot,E,\bot}\circ\mathbf{d}_{C,B,E,\bot} = \\ \mathbf{1}_C\wedge(\overset{\vee}{\mathbf{\Sigma}}_{A,B,\bot}\circ(\mathbf{1}_{E\rightarrow\bot}\rightarrow\nu^{\rightarrow}_{B})\circ s_{B\rightarrow\bot,E,\bot}),$$

from which $(d\overset{\vee}{\Delta})$ follows.

For $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$ it is enough to verify the following:

$$\begin{aligned}\overset{\vee}{\mathbf{\Sigma}}_{A,A,\bot}\circ\mathbf{d}_{A,A\rightarrow\bot,A,\bot}\circ(\mathbf{1}_A\wedge(\nu^{\rightarrow}_{A}\rightarrow\mathbf{1}_A))\circ\overset{\wedge}{\mathbf{\Delta}}_{A,A}& \\ = \varepsilon_{A,A}\circ(\mathbf{1}_A\wedge(\nu^{\leftarrow}_{A,\bot}\rightarrow\mathbf{1}_A))\circ(\mathbf{1}_A\wedge(\nu^{\rightarrow}_{A}\rightarrow\mathbf{1}_A))\circ\overset{\wedge}{\mathbf{\Delta}}_{A,A},& \\ \text{by } \mathbf{SMC}^c \text{ Coherence,}& \\ = \mathbf{1}_A,\quad \text{by } (\nu\nu) \text{ and } \mathbf{SMC}^c \text{ Coherence.}&\end{aligned}$$

For $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$ it is enough to verify the following:

$$\overset{\vee}{\mathbf{\Sigma}}_{A,A\rightarrow\bot,\bot}\circ((\hat{c}_{A\rightarrow\bot,A}\rightarrow\mathbf{1}_\bot)\rightarrow\mathbf{1}_{A\rightarrow\bot})\circ\mathbf{d}_{A\rightarrow\bot,A,A\rightarrow\bot,\bot}\circ \\ \circ(\mathbf{1}_{A\rightarrow\bot}\wedge((\mathbf{1}_{A\rightarrow\bot}\rightarrow\nu^{\rightarrow}_{A\rightarrow\bot})\circ s_{(A\rightarrow\bot)\rightarrow\bot,A,\bot}\circ(\nu^{\rightarrow}_{A}\rightarrow\mathbf{1}_A)))\circ\overset{\wedge}{\mathbf{\Delta}}_{A,A\rightarrow\bot} = \\ \mathbf{1}_{A\rightarrow\bot},$$

which is done by using the equation $(\nu^{\rightarrow}_{A\rightarrow\bot})$, the naturality of s, the first $(\nu\nu)$ equation and $\mathbf{SMC}^c$ Coherence. With that we have finished verifying the **PN** equations in **SA**.

It remains to deal with the equations introduced in the preceding section. It is trivial to verify in **SA** the equations $(\overset{\vee}{\delta}{}^{\rightarrow}\ nat)$ and $(\overset{\vee}{\delta}\,\overset{\vee}{\delta})$, while for $(\overset{\vee}{b}\,\overset{\vee}{\delta})$ we apply the naturality of $\nu^{\leftarrow}$ and $\mathbf{SMC}^c$ Coherence. For $(d\overset{\wedge}{\sigma})$ we rely again on $\mathbf{SMC}^c$ Coherence, which also delivers readily

$$\overset{\vee}{\delta}{}^{\leftarrow}_{C\wedge B} = d_{C,B,\bot}\circ(\mathbf{1}_C\wedge\overset{\vee}{\delta}{}^{\leftarrow}_{B})$$

—an equation that, in the presence of $(\check{\delta}\,\check{\delta})$, amounts to $(d\,\check{\delta})$. It is trivial to verify the equations $(\pi\pi)$, and with that we have finished verifying all the equations of $\mathbf{SA}'$ in $\mathbf{SA}$.

§3.5. The category $\mathbf{PN}^{\neg}_{\rightarrow\bot}$

As an auxiliary for the proof of the isomorphism of the categories $\mathbf{SA}$ and $\mathbf{SA}'$, which will be completed in §3.7, we introduce a category intermediary between $\mathbf{PN}^{\neg}$ and $\mathbf{SA}'$ equivalent with $\mathbf{PN}^{\neg}$, which we call $\mathbf{PN}^{\neg}_{\rightarrow\bot}$. As a consequence of the equivalence of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ with $\mathbf{PN}^{\neg}$, we will obtain a coherence result for $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ with respect to *Br*, and this will enable us to shorten very considerably calculations in $\mathbf{SA}'$ in the next two sections.

The objects of the category $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ are all the formulae of the language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$ generated by $\mathcal{P}$ in which $\top$ does not occur and in which $\bot$ occurs only in subformulae of the form $A\rightarrow\bot$. The arrow terms of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ are defined as those of $\mathbf{PN}^{\neg}$ (in the extended language of formulae), save that among the primitive arrow terms we also have

$$\pi^{\rightarrow}_{A,B}\colon A\rightarrow B\vdash\neg A\vee B,\qquad \pi^{\leftarrow}_{A,B}\colon\neg A\vee B\vdash A\rightarrow B,$$

where B cannot be $\bot$, since $\neg A\vee\bot$ is not an object of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$,

$$\lambda^{\rightarrow}_{A}\colon\neg A\vdash A\rightarrow\bot,\qquad\qquad \lambda^{\leftarrow}_{A}\colon A\rightarrow\bot\vdash\neg A.$$

The equations of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ are defined as those of $\mathbf{PN}^{\neg}$ plus the equations $(\pi\pi)$ of §3.3 and $(\lambda\lambda)$ of §3.2. All the equations of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ will hold in $\mathbf{SA}'$ once $\lambda^{\rightarrow}_{A}$ and $\lambda^{\leftarrow}_{A}$ are defined in $\mathbf{SA}'$ as in the next section. This is an important fact for the applications we will make of the coherence of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ in the next two sections.

We will now show that $\mathbf{PN}^{\neg}$ and $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ are equivalent categories. From $\mathbf{PN}^{\neg}$ to $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ we have a functor I such that IA is A for every object A of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$, and If is f for every arrow term f of $\mathbf{PN}^{\neg}$. From $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ to $\mathbf{PN}^{\neg}$ we have a functor H defined inductively as follows. On objects we have

$$Hp=p,\quad\text{for } p \text{ a letter},$$
$$H\neg A=\neg HA,$$

$$H(A\,\xi\,B) = HA\,\xi\,HB, \quad \text{for } \xi \in \{\wedge, \vee\},$$
$$H(A \rightarrow B) = \neg HA \vee HB, \quad \text{if } B \text{ is not } \bot,$$
$$H(A \rightarrow \bot) = \neg HA.$$

On arrow terms we have

$$H\alpha_{A_1,\ldots,A_n} = \alpha_{HA_1,\ldots,HA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ a primitive arrow term different from $\pi^{\rightarrow}_{A,B}$, $\pi^{\leftarrow}_{A,B}$, $\lambda^{\rightarrow}_{A}$ and $\lambda^{\leftarrow}_{A}$,

$$H\pi^{\rightarrow}_{A,B} = H\pi^{\leftarrow}_{A,B} = \mathbf{1}_{\neg HA \vee HB},$$
$$H\lambda^{\rightarrow}_{A} = H\lambda^{\leftarrow}_{A} = \mathbf{1}_{\neg HA},$$
$$H(f\,\xi\,g) = Hf\,\xi\,Hg, \quad \text{for } \xi \in \{\circ, \wedge, \vee\}.$$

It is clear that for every object A of $\mathbf{PN}^{\neg}$ we have that $HIA = HA = A$. For every object A of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ we have $IHA = HA$, and we define an arrow $h_A \colon HA \vdash A$ of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$, which is a member of a natural isomorphism of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ (natural in A), with inverse $h^{-1}_A \colon A \vdash HA$. The arrows h_A and h^{-1}_A are defined inductively as follows:

$$h_p = h^{-1}_p = \mathbf{1}_p, \quad \text{for } p \text{ a letter},$$

$$h_{A\xi B} = h_A\,\xi\,h_B, \qquad h^{-1}_{A\xi B} = h^{-1}_A\,\xi\,h^{-1}_B,$$

for $\xi \in \{\wedge, \vee\}$,

$$h_{\neg A} = \neg h^{-1}_A, \qquad h^{-1}_{\neg A} = \neg h_A,$$

where the operation $\neg$ on arrows is defined as in §2.8,

$$h_{A\rightarrow B} = \pi^{\leftarrow}_{A,B} \circ (\neg h^{-1}_A \vee h_B), \qquad h^{-1}_{A\rightarrow B} = (\neg h_A \vee h^{-1}_B) \circ \pi^{\rightarrow}_{A,B},$$

if B is not $\bot$,

$$h_{A\rightarrow\bot} = \lambda^{\rightarrow}_{A} \circ \neg h^{-1}_A, \qquad h^{-1}_{A\rightarrow\bot} = \neg h_A \circ \lambda^{\leftarrow}_{A}.$$

For $f \colon A \vdash B$ we prove that we have

$$f \circ h_A = h_B \circ Hf$$

in $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ by induction on the complexity of the arrow term f of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$. In this induction we rely on various bifunctorial, naturality and isomorphism equations. We rely on ($\hat{\Delta}$ *dinat*) and ($\check{\Sigma}$ *dinat*) of §2.8, in addition to ($\hat{\Delta}$ *nat*) and ($\check{\Sigma}$ *nat*) of §2.2, when f is $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$. This establishes that h is a natural isomorphism, and it follows that the categories $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ and $\mathbf{PN}^{\neg}$ are equivalent via the functors H and I.

Let $G\alpha$ be an identity arrow of Br when α is $\pi^{\rightarrow}_{A,B}$, $\pi^{\leftarrow}_{A,B}$, $\lambda^{\rightarrow}_{A}$ or $\lambda^{\leftarrow}_{A}$. With other clauses for G being as for $\mathbf{PN}^{\neg}$ we obtain a functor G from $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ to Br. Note that $GA = GHA$. For every arrow f of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ we have that $Gf = GHf$. Then we can prove the following (cf. the proof at the end of §2.7).

$\mathbf{PN}^{\neg}_{\rightarrow\bot}$ COHERENCE. *The functor G from $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ to Br is faithful.*

PROOF. Suppose that for f and g arrows of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ of the same type we have $Gf = Gg$. Then $GHf = GHg$, and $Hf = Hg$ in $\mathbf{PN}^{\neg}$ by $\mathbf{PN}^{\neg}$ Coherence. It follows that $f = g$ in $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ by the equivalence of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ with $\mathbf{PN}^{\neg}$. ⊣

Note that in $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ we can define $\upsilon^{\rightarrow}_{A,B} : A \vee B \vdash (A \rightarrow \bot) \rightarrow B$ and $\upsilon^{\leftarrow}_{A,B} : (A \rightarrow \bot) \rightarrow B \vdash A \vee B$ as in $\mathbf{SA}'$ in the next section, and it is easy to see that the equations $(\upsilon\upsilon)$ of §3.2 hold for these defined arrows.

§3.6. SA in SA′

This section is opposite to §3.4. We will define in it the **SA** structure in **SA′**, and then we will show that the equations of **SA** hold in **SA′** for this defined structure.

First, we have the following definitions in **SA′**:

$$\varepsilon_{A,B} =_{df} \check{\Sigma}_{A,B} \circ d_{A,\neg A,B} \circ (\mathbf{1}_A \wedge \pi^{\rightarrow}_{A,B}) : A \wedge (A \rightarrow B) \vdash B,$$

$$\eta_{A,B} =_{df} \pi^{\leftarrow}_{A,A\wedge B} \circ d^{R}_{\neg A,A,B} \circ \hat{\Sigma}_{A,B} : B \vdash A \rightarrow (A \wedge B),$$

$$A \rightarrow f =_{df} \pi^{\leftarrow}_{A,C} \circ (\mathbf{1}_{\neg A} \vee f) \circ \pi^{\rightarrow}_{A,B} : A \rightarrow B \vdash A \rightarrow C, \quad \text{for } f : B \vdash C.$$

With that we have defined what was missing to obtain the **SMC** structure in **SA′**. (We already have in **SA′** the arrows $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$, $\hat{\delta}^{\rightarrow}_{A}$ and $\hat{\delta}^{\leftarrow}_{A}$, and the operations on arrows $\circ$ and $\wedge$.)

For the remainder of the **SA** structure we have the following definitions in **SA**$'$:

$$\lambda^{\rightarrow}_{A} =_{df} \pi^{\leftarrow}_{A,\bot} \circ \check{\delta}^{\leftarrow}_{\neg A} : \neg A \vdash A \rightarrow \bot,$$

$$\lambda^{\leftarrow}_{A} =_{df} \check{\delta}^{\rightarrow}_{\neg A} \circ \pi^{\rightarrow}_{A,\bot} : A \rightarrow \bot \vdash \neg A,$$

$$\nu^{\rightarrow}_{A} =_{df} n^{\rightarrow}_{A} \circ \neg\lambda^{\rightarrow}_{A} \circ \lambda^{\leftarrow}_{A\rightarrow\bot} : (A \rightarrow \bot) \rightarrow \bot \vdash A,$$

$$\upsilon^{\rightarrow}_{A,B} =_{df} \pi^{\leftarrow}_{A\rightarrow\bot,B} \circ ((\neg\lambda^{\leftarrow}_{A} \circ n^{\leftarrow}_{A}) \vee \mathbf{1}_B) : A \vee B \vdash (A \rightarrow \bot) \rightarrow B,$$

$$\upsilon^{\leftarrow}_{A,B} =_{df} ((n^{\rightarrow}_{A} \circ \neg\lambda^{\rightarrow}_{A}) \vee \mathbf{1}_B) \circ \pi^{\rightarrow}_{A\rightarrow\bot,B} : (A \rightarrow \bot) \rightarrow B \vdash A \vee B,$$

where the operation $\neg$ on arrows is defined as in §2.8, the arrows $\lambda^{\rightarrow}_{A}$, $\lambda^{\leftarrow}_{A}$ and $\lambda^{\leftarrow}_{A\rightarrow\bot}$ on the right-hand sides are defined as above, while $n^{\rightarrow}_{A} : \neg\neg A \vdash A$ and $n^{\leftarrow}_{A} : A \vdash \neg\neg A$ are defined as in §2.6. With that we have finished defining what was missing to obtain the **SA** structure in **SA**$'$.

It is not difficult to show with the help of ($\hat{\Sigma}$ *dinat*) and $\mathbf{PN}^{\neg}$ Coherence that for $g\colon A \vdash D$ and $f\colon B \vdash C$ in **SA**$'$ we have the equation

$$g \rightarrow f = \pi^{\leftarrow}_{A,C} \circ (\neg g \vee f) \circ \pi^{\rightarrow}_{D,B}$$

where $g \rightarrow f$ on the left-hand side is defined in **SA**$'$ as in §3.1 in terms of $\varepsilon_{D,C}$, $\eta_{A,D\rightarrow B}$ and the operations $A \rightarrow$ and $D \rightarrow$, which are themselves defined in **SA**$'$.

We verify next that the equations of **SA** hold for the defined **SA** structure in **SA**$'$. For the equations of **SMC** of §3.1 we have that $(A \rightarrow 1)$ and $(A \rightarrow 2)$ are trivial to check, (ε *nat*) and (η *nat*) follow from various naturality equations, while $(\varepsilon\eta\ \wedge)$ and $(\varepsilon\eta \rightarrow)$ follow from $\mathbf{PN}^{\neg}$ Coherence. For the equations of §3.2 we have that $(\lambda\lambda)$ follow from $(\check{\delta}\,\check{\delta})$ and $(\pi\pi)$ of §3.3, while for $(\upsilon\upsilon)$ we use $(\pi\pi)$, $(\lambda\lambda)$, the isomorphism of $n^{\rightarrow}_{A}$ (see §2.6) and $(\neg 2)$ of §2.8.

It remains only to verify $(\nu\nu)$ of §3.2. We will do that after some preparation, which will enable us to apply the $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ Coherence of the preceding section.

We have the following equation in **SA**$'$:

$$(\mathbf{1} \rightarrow \varepsilon) \qquad \mathbf{1}_A \rightarrow \varepsilon_{B,\bot} = \lambda^{\rightarrow}_{A} \circ \check{\Delta}_{B,\neg A} \circ (\mathbf{1}_{\neg A} \vee (\mathbf{1}_B \wedge \lambda^{\leftarrow}_{B})) \circ \pi^{\rightarrow}_{A,B\wedge(B\rightarrow\bot)}$$

where $\varepsilon_{B,\bot}$ and $\rightarrow$ on the left-hand side $\mathbf{1}_A \rightarrow \varepsilon_{B,\bot}$, which is equal to $A \rightarrow \varepsilon_{B,\bot}$, are defined in **SA**′, and so are $\lambda^{\rightarrow}_A$ and $\lambda^{\leftarrow}_B$ on the right-hand side. By definitions and isomorphism equations, for g being

$$\check{\delta}^{\rightarrow}_{\neg A} \circ (\mathbf{1}_{\neg A} \vee \check{\Sigma}_{B,\bot}) \circ (\mathbf{1}_{\neg A} \vee (d_{B,\neg B,\bot} \circ (\mathbf{1}_B \wedge \check{\delta}^{\leftarrow}_{\neg B}))),$$

in **SA**′ we have

$$\mathbf{1}_A \rightarrow \varepsilon_{B,\bot} = \lambda^{\rightarrow}_A \circ g \circ (\mathbf{1}_{\neg A} \vee (\mathbf{1}_B \wedge \lambda^{\leftarrow}_B)) \circ \pi^{\rightarrow}_{A,B\wedge(B\rightarrow\bot)}.$$

Next we have

$$g = \check{\delta}^{\rightarrow}_{\neg A} \circ (\check{\Delta}_{B,\neg A} \vee \mathbf{1}_\bot) \circ \check{b}^{\rightarrow}_{\neg A,B\wedge\neg B,\bot} \circ (\mathbf{1}_{\neg A} \vee \check{\delta}^{\leftarrow}_{B\wedge\neg B}),$$
$$\text{by } (\check{b}\check{\Delta}\check{\Sigma}) \text{ and } (d\,\check{\delta}),$$
$$= \check{\Delta}_{B,\neg A}, \quad \text{by } (\check{\delta}^{\rightarrow}\ nat),\ (\check{b}\,\check{\delta}) \text{ and } (\check{\delta}\,\check{\delta}),$$

which establishes the equation $(\mathbf{1} \rightarrow \varepsilon)$.

We can now verify $(\nu\nu)$ by establishing in **SA**′ the equation

$$\nu^{\leftarrow}_{A,\bot} = \lambda^{\rightarrow}_{A\rightarrow\bot} \circ \neg\lambda^{\leftarrow}_A \circ n^{\leftarrow}_A.$$

The left-hand side $\nu^{\leftarrow}_{A,\bot}$ of this equation is equal to

$$(\mathbf{1}_{A\rightarrow\bot} \rightarrow \varepsilon_{A,\bot}) \circ (\mathbf{1}_{A\rightarrow\bot} \rightarrow \hat{c}_{A\rightarrow\bot,A}) \circ \eta_{A\rightarrow\bot,A}$$

(see §3.2), where we can replace $\mathbf{1}_{A\rightarrow\bot} \rightarrow \varepsilon_{A,\bot}$ according to the equation $(\mathbf{1} \rightarrow \varepsilon)$, and then apply $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ Coherence. With that we have finished verifying all the equations of **SA** in **SA**′.

§3.7. The isomorphism of SA and SA′

In this section we will show that **SA** and **SA**′ are isomorphic categories. We have a functor F from **SA**′ to **SA** that is identity on objects and that maps every arrow of **SA**′ to the homonymous arrow in the defined **SA**′ structure of **SA**. For example,

$$F\,\check{b}^{\rightarrow}_{A,B,C} = \check{b}^{\rightarrow}_{A,B,C},$$

where the $\check{b}^{\rightarrow}_{A,B,C}$ on the right-hand side is defined as in §3.4. We define analogously a functor F' from **SA** to **SA′** which is identity on objects and which maps every arrow of **SA** to the homonymous arrow in the defined **SA** structure of **SA′**. That F and F' are indeed functors follows from what was established in §3.4 concerning the equations of **SA′** in **SA**, and in the preceding section concerning the equations of **SA** in **SA′**.

It is trivial that $F'FA$ and $FF'A$ are both A. We will show next that $F'Ff = f$ in **SA′**, and $FF'f = f$ in **SA**, from which it will follow that **SA** and **SA′** are isomorphic categories.

To verify $F'Ff = f$ in **SA′**, we have to verify this equation for f being $\check{b}^{\rightarrow}_{A,B,C}$, $\check{b}^{\leftarrow}_{A,B,C}$, $\check{c}_{A,B}$, $d_{A,B,C}$, $\hat{\Delta}_{B,A}$, $\check{\Sigma}_{B,A}$, $\check{\delta}^{\rightarrow}_{A}$, $\check{\delta}^{\leftarrow}_{A}$, $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$, and we also have to verify that in **SA′** we have

$$(F'F\,\xi) \qquad F'F(f \,\xi\, g) = F'Ff \,\xi\, F'Fg$$

for ξ being $\vee$. It is trivial that in **SA′** the equation $F'Ff = f$ holds for f being $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$, $\hat{\delta}^{\rightarrow}_{A}$ and $\hat{\delta}^{\leftarrow}_{A}$; the equations $(F'F\,\xi)$ for $\xi \in \{\circ, \wedge\}$ hold trivially too.

To verify $F'Ff = f$ in **SA′** for f being $\check{b}^{\rightarrow}_{A,B,C}$, $\check{b}^{\leftarrow}_{A,B,C}$, etc. we need some preparation. We have for $f\colon A \vdash B$ the following equation in **SA′**:

$$(f \rightarrow \mathbf{1}) \qquad f \rightarrow \mathbf{1}_{\perp} = \lambda^{\rightarrow}_{A} \circ \neg f \circ \lambda^{\leftarrow}_{B},$$

with the help of the equation $(g \rightarrow f)$ of the preceding section, together with the definitions of $\lambda^{\rightarrow}_{A}$ and $\lambda^{\leftarrow}_{B}$, and the equations $(\check{\delta}\!\rightarrow\ nat)$ and $(\check{\delta}\,\check{\delta})$. The equation $(f \rightarrow \mathbf{1})$, together with the equation $(\mathbf{1} \rightarrow \varepsilon)$ established for **SA′** in the preceding section, will enable us to apply below $\mathbf{PN}^{\neg}_{\rightarrow\perp}$ Coherence of §3.5.

By $\mathbf{PN}^{\neg}_{\rightarrow\perp}$ Coherence we have that $F'Ff = f$ holds in **SA′** for f being $\check{b}^{\rightarrow}_{A,B,C}$, $\check{c}_{A,B}$, $d_{A,B,C}$, $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$. We only have to check by using $(\mathbf{1} \rightarrow \varepsilon)$ and $(f \rightarrow \mathbf{1})$ that the arrow term $F'Ff$ is equal in **SA′** to an arrow term of $\mathbf{PN}^{\neg}_{\rightarrow\perp}$. This is a lengthy, but straightforward, exercise. We also need to verify that $GF'Ff = Gf$.

Once we have shown that $F'F\check{b}^{\rightarrow}_{A,B,C} = \check{b}^{\rightarrow}_{A,B,C}$ in **SA′**, we can use that to obtain the following:

$$\begin{aligned} F'F\check{b}^{\leftarrow}_{A,B,C} &= F'F\check{b}^{\leftarrow}_{A,B,C} \circ \check{b}^{\rightarrow}_{A,B,C} \circ \check{b}^{\leftarrow}_{A,B,C} \\ &= F'F\check{b}^{\leftarrow}_{A,B,C} \circ F'F\check{b}^{\rightarrow}_{A,B,C} \circ \check{b}^{\leftarrow}_{A,B,C} \\ &= \check{b}^{\leftarrow}_{A,B,C}, \end{aligned}$$

by the functoriality of F' and F, though $F'F\check{b}^{\leftarrow}_{A,B,C} = \check{b}^{\leftarrow}_{A,B,C}$ can also be verified directly with the help of $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ Coherence (this is not such a short verification).

For $F'F\hat{\Delta}_{B,A} = \hat{\Delta}_{B,A}$ we have a verification very much analogous to the verification of $(\mathbf{1} \rightarrow \varepsilon)$ in the preceding section. In this verification we establish that in **SA′** we have

$$(\hat{\mathbf{\Delta}}) \qquad \hat{\mathbf{\Delta}}_{B,A} = (\mathbf{1}_A \wedge \pi^{\leftarrow}_{B,B}) \circ \hat{\Delta}_{B,A},$$

and then, by using the equations $(f \rightarrow \mathbf{1})$ and $(\hat{\mathbf{\Delta}})$, together with $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ Coherence, we obtain $F'F\hat{\Delta}_{B,A} = \hat{\Delta}_{B,A}$. For $F'F\check{\Sigma}_{B,A} = \check{\Sigma}_{B,A}$ we use $(\mathbf{1} \rightarrow \varepsilon)$, $(f \rightarrow \mathbf{1})$ and $(\hat{\mathbf{\Delta}})$, together with $\mathbf{PN}^{\neg}_{\rightarrow\bot}$ Coherence.

For $F'F\check{\delta}^{\rightarrow}_{A} = \check{\delta}^{\rightarrow}_{A}$ we use

$$\lambda^{\leftarrow}_{A\rightarrow\bot} = \check{\delta}^{\rightarrow}_{\neg(A\rightarrow\bot)} \circ \pi^{\rightarrow}_{A\rightarrow\bot,\bot},$$

which holds in **SA′** by definition (see the preceding section), and then we apply ($\check{\delta}^{\rightarrow}$ *nat*), $(\neg 2)$ of §2.8, and isomorphism equations. We obtain $F'F\check{\delta}^{\leftarrow}_{A} = \check{\delta}^{\leftarrow}_{A}$ from $F'F\check{\delta}^{\rightarrow}_{A} = \check{\delta}^{\rightarrow}_{A}$ (see the verification of $F'F\check{b}^{\leftarrow}_{A,B,C} = \check{b}^{\leftarrow}_{A,B,C}$ above).

It remains to derive $(F'F\vee)$ in **SA′**. For this rather straightforward derivation we use $(\neg 2)$ and ($\check{\delta}^{\leftarrow}$ *nat*), together with isomorphism and bifunctorial equations.

To verify $FF'f = f$ in **SA**, we have to verify this equation for f being $\varepsilon_{A,B}$, $\eta_{A,B}$, $\nu^{\rightarrow}_{A}$, $\lambda^{\rightarrow}_{A}$, $\lambda^{\leftarrow}_{A}$, $\upsilon^{\rightarrow}_{A,B}$ and $\upsilon^{\leftarrow}_{A,B}$, and we also have to verify that in **SA** we have

$$FF'(A \rightarrow g) = FF'A \rightarrow FF'g.$$

For that we rely on lengthy, but also rather straightforward, derivations, in which we apply various bifunctorial, naturality, dinaturality and isomorphism equations. We also use the equations $(\nu^{\leftarrow}_{A\rightarrow\bot,\bot})$ and $(\nu^{\rightarrow}_{A\rightarrow\bot})$ of §3.4, and we apply $\mathbf{SMC}^c$ Coherence of §3.1. It is trivial that in **SA** the equation $FF'f = f$ holds for f being $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$, $\hat{\delta}^{\rightarrow}_{A}$ and $\hat{\delta}^{\leftarrow}_{A}$; the equations obtained from $(F'F\,\xi)$ for $\xi \in \{\circ, \wedge\}$ by replacing $F'F$ by FF' hold trivially too. With that we have finished establishing that **SA** and $\mathbf{SA}'$ are isomorphic categories.

§3.8. The categories $\mathbf{SA}_s$ and $\mathbf{SA}'_s$

The objects of the category $\mathbf{SA}_s$ are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\wedge,\rightarrow}$ generated by $\mathcal{P}$, which are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$ of §3.2 in which $\neg$ and $\vee$ do not occur. The arrow terms of $\mathbf{SA}_s$ are defined as those of **SA** save that we omit in the definition the primitive arrow terms $\lambda^{\rightarrow}_{A}$, $\lambda^{\leftarrow}_{A}$, $\upsilon^{\rightarrow}_{A,B}$ and $\upsilon^{\leftarrow}_{A,B}$. The equations of $\mathbf{SA}_s$ are defined as those of **SA** save that we omit the equations $(\lambda\lambda)$ and $(\upsilon\upsilon)$. This means that to the equations assumed for **SMC** we add only the equation $(\nu\nu)$. The category $\mathbf{SA}_s$ is the free *star-autonomous* category in the sense of [2] (Section 2) generated by $\mathcal{P}$.

We will establish that **SA** and $\mathbf{SA}_s$ are equivalent categories. From $\mathbf{SA}_s$ to **SA** we have a functor I such that IA is A for every object A of $\mathbf{SA}_s$, and If is f for every arrow term f of $\mathbf{SA}_s$.

From **SA** to $\mathbf{SA}_s$ we have a functor H defined inductively as follows. On objects we have

$$\begin{aligned}
HA &= A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot,\\
H\neg A &= HA \rightarrow \bot,\\
H(A\,\xi\,B) &= HA\,\xi\,HB, \quad \text{for } \xi \in \{\wedge, \rightarrow\},\\
H(A \vee B) &= (HA \rightarrow \bot) \rightarrow HB.
\end{aligned}$$

On arrow terms we have

$$H\alpha_{A_1,\ldots,A_n} = \alpha_{HA_1,\ldots,HA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ a primitive arrow term different from $\lambda^{\rightarrow}_{A}$, $\lambda^{\leftarrow}_{A}$, $\upsilon^{\rightarrow}_{A,B}$ and $\upsilon^{\leftarrow}_{A,B}$,

$$\begin{aligned}
H\lambda^{\rightarrow}_{A} &= H\lambda^{\leftarrow}_{A} = \mathbf{1}_{HA\rightarrow\bot},\\
H\upsilon^{\rightarrow}_{A,B} &= H\upsilon^{\leftarrow}_{A,B} = \mathbf{1}_{(HA\rightarrow\bot)\rightarrow HB},
\end{aligned}$$

$$H(f\,\xi\,g) = Hf\,\xi\,Hg, \quad \text{for } \xi \in \{\circ, \wedge\},$$
$$H(A \rightarrow f) = HA \rightarrow Hf.$$

It is clear that for every object A of $\mathbf{SA}_s$ we have that $HIA = HA = A$. For every object A of $\mathbf{SA}$ we have $IHA = HA$, and we define an arrow $h_A\colon HA \vdash A$ of $\mathbf{SA}$, which is a member of a natural isomorphism of $\mathbf{SA}$ (natural in A), with inverse $h_A^{-1}\colon A \vdash HA$. The arrows h_A and h_A^{-1} are defined inductively as follows:

$$h_A = h_A^{-1} = \mathbf{1}_A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot$$

$$h_{\neg A} = \lambda^{\leftarrow}_A \circ (h_A^{-1} \rightarrow \mathbf{1}_\bot), \qquad h^{-1}_{\neg A} = (h_A \rightarrow \mathbf{1}_\bot) \circ \lambda^{\rightarrow}_A,$$
$$h_{A\wedge B} = h_A \wedge h_B, \qquad h^{-1}_{A\wedge B} = h_A^{-1} \wedge h_B^{-1},$$
$$h_{A\vee B} = \upsilon^{\leftarrow}_{A,B} \circ ((h_A \rightarrow \mathbf{1}_\bot) \rightarrow h_B), \qquad h^{-1}_{A\vee B} = ((h_A^{-1} \rightarrow \mathbf{1}_\bot) \rightarrow h_B^{-1}) \circ \upsilon^{\rightarrow}_{A,B},$$
$$h_{A\rightarrow B} = h_A^{-1} \rightarrow h_B, \qquad h^{-1}_{A\rightarrow B} = h_A \rightarrow h_B^{-1}.$$

For $f\colon A \vdash B$ we prove that we have

$$f \circ h_A = h_B \circ Hf$$

in $\mathbf{SA}$ by induction on the complexity of the arrow term f of $\mathbf{SA}$. In this induction we rely on various bifunctorial, naturality, dinaturality and isomorphism equations. This establishes that h is a natural isomorphism, and it follows that the categories $\mathbf{SA}$ and $\mathbf{SA}_s$ are equivalent via the functors H and I.

From this equivalence we can deduce that $\mathbf{SA}_s$ is isomorphic to a full subcategory of $\mathbf{SA}$. For every object A of $\mathbf{SA}_s$ we have that $HA = A$ and $h_A = \mathbf{1}_A$. So, for A and B objects of $\mathbf{SA}_s$ and $f\colon A \vdash B$ an arrow term of $\mathbf{SA}$, there is an arrow term $Hf\colon A \vdash B$ of $\mathbf{SA}_s$ such that in $\mathbf{SA}$ we have $f = Hf$, because $h_A = \mathbf{1}_A$ and $h_B = \mathbf{1}_B$.

The objects of the category $\mathbf{SA}'_s$ are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$ generated by $\mathcal{P}$, which are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$ of §3.2 in which $\rightarrow$ does not occur. The arrow terms of $\mathbf{SA}'_s$ are defined as those of $\mathbf{SA}'$ save that we omit in the definition the primitive arrow terms $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$. The equations of $\mathbf{SA}'_s$ are defined as those of $\mathbf{SA}'$ save that we omit the equations $(\pi\pi)$.

With the definitions

$$\tau^L_B =_{df} \;\hat{\sigma}^{\rightarrow}_{\neg B\vee B} \circ \hat{\Delta}_{B,\top} : \top \vdash \neg B \vee B,$$

$$\gamma^R_B =_{df} \;\check{\Sigma}_{B,\bot} \circ \check{\delta}^{\leftarrow}_{B\wedge\neg B} : B \wedge \neg B \vdash \bot,$$

in $\mathbf{SA}'_s$, on the one hand, and

$$\hat{\Delta}_{B,A} =_{df} \;(\mathbf{1}_A \wedge \tau^L_B) \circ \hat{\delta}^{\leftarrow}_A : A \vdash A \wedge (\neg B \vee B),$$

$$\check{\Sigma}_{B,A} =_{df} \;\check{\sigma}^{\rightarrow}_A \circ (\gamma^R_B \vee \mathbf{1}_A) \colon (B \wedge \neg B) \vee A \vdash A,$$

on the other hand, it can easily be established that $\mathbf{SA}'_s$ is isomorphic to the free *symmetric linearly* (alias *weakly*) *distributive category with negation* in the sense of [11] (Section 4, Definition 4.3) generated by $\mathcal{P}$.

We will establish that $\mathbf{SA}'$ and $\mathbf{SA}'_s$ are equivalent categories. From $\mathbf{SA}'_s$ to $\mathbf{SA}'$ we have a functor I such that IA is A for every object A of $\mathbf{SA}'_s$, and If is f for every arrow term f of $\mathbf{SA}'_s$.

From $\mathbf{SA}'$ to $\mathbf{SA}'_s$ we have a functor H defined inductively as follows. On objects we have

$$HA = A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot,$$
$$H\neg A = \neg HA,$$
$$H(A\,\xi\,B) = HA\,\xi\,HB, \quad \text{for } \xi \in \{\wedge, \vee\}$$
$$H(A \rightarrow B) = \neg HA \vee HB.$$

On arrow terms we have

$$H\alpha_{A_1,\ldots,A_n} = \alpha_{HA_1,\ldots,HA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ a primitive arrow term different from $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$,

$$H\pi^{\rightarrow}_{A,B} = H\pi^{\leftarrow}_{A,B} = \mathbf{1}_{\neg HA\vee HB},$$

$$H(f\,\xi\,g) = Hf\,\xi\,Hg, \quad \text{for } \xi \in \{\circ, \wedge, \vee\}.$$

It is clear that for every object A of $\mathbf{SA}'_s$ we have that $HIA = HA = A$. For every object A of $\mathbf{SA}'$ we have $IHA = HA$, and we define an arrow $h_A \colon HA \vdash A$ of $\mathbf{SA}'$, which is a member of a natural isomorphism of $\mathbf{SA}'$ (natural in A), with inverse $h^{-1}_A \colon A \vdash HA$. The arrows h_A and h^{-1}_A are defined inductively as follows:

$$h_A = h_A^{-1} = \mathbf{1}_A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot$$

$$h_{\neg A} = \neg h_A^{-1}, \qquad h_{\neg A}^{-1} = \neg h_A,$$

$$h_{A\,\xi\,B} = h_A\,\xi\,h_B, \qquad h_{A\,\xi\,B}^{-1} = h_A^{-1}\,\xi\,h_B^{-1}, \quad \text{for } \xi \in \{\wedge, \vee\},$$

$$h_{A\to B} = \pi^{\leftarrow}_{A,B} \circ (\neg h_A^{-1} \vee h_B), \qquad h_{A\to B}^{-1} = (\neg h_A \vee h_B^{-1}) \circ \pi^{\rightarrow}_{A,B}.$$

We check as before that h is indeed a natural isomorphism, which establishes that the categories $\mathbf{SA}'$ and $\mathbf{SA}'_s$ are equivalent via the functors H and I. As we established that $\mathbf{SA}_s$ is isomorphic to a full subcategory of $\mathbf{SA}$, so we establish that $\mathbf{SA}'_s$ is isomorphic to a full subcategory of $\mathbf{SA}'$.

By combining the equivalences of $\mathbf{SA}$ with $\mathbf{SA}_s$ and of $\mathbf{SA}'$ with $\mathbf{SA}'_s$ and the isomorphism of $\mathbf{SA}$ with $\mathbf{SA}'$, established in the preceding section, we obtain that $\mathbf{SA}_s$ and $\mathbf{SA}'_s$ are equivalent categories. This is presumably what was meant in [11] (Section 4, Theorem 4.5) by saying that the notion of symmetric linearly distributive category with negation and the notion of star-autonomous category "coincide".

To establish the equivalence of $\mathbf{SA}_s$ and $\mathbf{SA}'_s$ directly, without proceeding via $\mathbf{SA}$ and $\mathbf{SA}'$ as we did, is possible, but this cannot be easier than what we did (as a matter of fact, this seems to us much more tangled). One cannot escape that way all the calculations we made in verifying the isomorphism of $\mathbf{SA}$ and $\mathbf{SA}'$. These calculations must be made at least implicitly. We were able to shorten them via $\mathbf{SMC}^c$ Coherence of §3.1 and our $\mathbf{PN}^{\neg}_{\to\bot}$ Coherence of §3.5, which is based on $\mathbf{PN}^{\neg}$ Coherence.

Chapter 4

Proof-Net and Star-Autonomous Categories

In this chapter we prove that the free proof-net category $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of the free proof-net category with units $\mathbf{SA}'_s$, and hence also to full subcategories of the categories $\mathbf{SA}'$ and $\mathbf{SA}$ of the previous chapter. All these categories are freely generated by the same set of objects. The proof is based on a Gentzen sequent formulation of $\mathbf{SA}'_s$, a cut-elimination theorem for this formulation, and a key technical lemma (Lemma 3 of §4.3). The proof of the cut-elimination is facilitated very much by relying on coherence for proof-net categories.

As a corollary, we obtain a coherence theorem with respect to *Br* for the category $\mathbf{SA}^c$, which is the full subcategory of $\mathbf{SA}$ whose objects are isomorphic either to objects in which units do not occur, i.e. to objects of $\mathbf{PN}^{\neg}$, or to one of the units. The restriction on the objects of $\mathbf{SA}$ brought by this coherence theorem for $\mathbf{SA}^c$ is of the same kind as the proviso concerning the unit object that Kelly and Mac Lane had in their coherence theorem for symmetric monoidal closed categories of [32] (see the end of §3.1). This restricted coherence of star-autonomous categories is a very useful tool for deciding whether a diagram of arrows commutes in these categories.

§4.1. The Gentzenization of $\mathbf{SA}'_s$

We will now define a new language of arrow terms to denote the arrows of the category $\mathbf{SA}'_s$ of §3.8. We call these arrow terms *Gentzen terms*, and we prove for Gentzen terms a result analogous to Gentzen's cut-elimination theorem, which we will use to prove that the category $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of $\mathbf{SA}'_s$.

As the arrow terms of $\mathbf{SA}'_s$, Gentzen terms will be defined inductively starting from primitive Gentzen terms. As *primitive* Gentzen terms we have $\mathbf{1}_A : A \vdash A$, for A being a letter, or $\top$, or $\bot$. To define the operations on Gentzen terms, called *Gentzen operations*, which are mostly partial operations, we need some preparation.

We define inductively a notion that for $\xi \in \{\wedge, \vee\}$ we call a ξ-*context*:

> $\Box$ is a ξ-context;
>
> if Z is a ξ-context and A an object of $\mathbf{SA}'_s$, then $Z\,\xi\,A$ and $A\,\xi\,Z$ are ξ-contexts.

A ξ-context is called *proper* when it is not $\Box$.

Next we define inductively what it means for a ξ-context Z to be applied to an object B of $\mathbf{SA}'_s$, which we write $Z(B)$, or to an arrow term f of $\mathbf{SA}'_s$, which we write $Z(f)$:

$$\begin{aligned} \Box(B) &= B, & \Box(f) &= f,\\ (Z\,\xi\,A)(B) &= Z(B)\,\xi\,A, & (Z\,\xi\,A)(f) &= Z(f)\,\xi\,\mathbf{1}_A,\\ (A\,\xi\,Z)(B) &= A\,\xi\,Z(B); & (A\,\xi\,Z)(f) &= \mathbf{1}_A\,\xi\,Z(f). \end{aligned}$$

We use X, perhaps with indices, as a variable for $\wedge$-contexts, and Y, perhaps with indices, as a variable for $\vee$-contexts.

Then we have the Gentzen operation $\hat{B}^{\leftarrow}_X$, which involves types specified by

$$\frac{f : X(A \wedge (B \wedge C)) \vdash D}{\hat{B}^{\leftarrow}_X f : X((A \wedge B) \wedge C) \vdash D}$$

This is read "if f is a Gentzen term, then $\hat{B}^{\leftarrow}_X f$ is a Gentzen term", all that of the required types. We use this rule notation for operations also in the future. The Gentzen term $\hat{B}^{\leftarrow}_X f$ denotes the arrow of $\mathbf{SA}'_s$ named on the right-hand side of the $=_{dn}$ sign below:

$$\hat{B}^{\leftarrow}_X f =_{dn} f \circ X(\hat{b}^{\leftarrow}_{A,B,C}).$$

We also have the following Gentzen operation:

$$\frac{f\colon D \vdash Y(A \vee (B \vee C))}{\check{B}^{\rightarrow}_Y f =_{dn} Y(\check{b}^{\rightarrow}_{A,B,C}) \circ f\colon D \vdash Y((A \vee B) \vee C)}$$

and the following four analogous Gentzen operations, where the types can be easily guessed:

$$\begin{array}{ll} \hat{B}^{\rightarrow}_X f =_{dn} f \circ X(\hat{b}^{\rightarrow}_{A,B,C}), & \check{B}^{\leftarrow}_Y f =_{dn} Y(\check{b}^{\leftarrow}_{A,B,C}) \circ f, \\ \hat{C}_X f =_{dn} f \circ X(\hat{c}_{A,B}), & \check{C}_Y f =_{dn} Y(\check{c}_{A,B}) \circ f. \end{array}$$

We also have the Gentzen operations in the following list:

$$\frac{f\colon A \vdash B}{\top^{\rightarrow} f =_{dn} f \circ \hat{\sigma}^{\rightarrow}_A\colon \top \wedge A \vdash B} \qquad \frac{f\colon B \vdash A}{\bot^{\leftarrow} f =_{dn} \check{\delta}^{\leftarrow}_A \circ f\colon B \vdash A \vee \bot}$$

$$\frac{g\colon \top \wedge A \vdash B}{\top^{\leftarrow} g =_{dn} g \circ \hat{\sigma}^{\leftarrow}_A\colon A \vdash B} \qquad \frac{g\colon B \vdash A \vee \bot}{\bot^{\rightarrow} g =_{dn} \check{\delta}^{\rightarrow}_A \circ g\colon B \vdash A}$$

for $\check{e}'_{D,C,B,A} =_{df} (\hat{c}_{C,D} \vee \mathbf{1}_{B\vee A}) \circ \check{b}^{\leftarrow}_{C\wedge D,B,A} \circ ((d_{C,D,B} \circ \hat{c}_{D\vee B,C}) \vee \mathbf{1}_A) \circ$
$\circ\, d_{D\vee B,C,A}\colon (D \vee B) \wedge (C \vee A) \vdash (D \wedge C) \vee (B \vee A)$,

$$\frac{f_1\colon B_1 \vdash A_1 \vee C_1 \qquad\qquad f_2\colon B_2 \vdash A_2 \vee C_2}{\wedge(f_1, f_2) =_{dn} \check{e}'_{A_1,A_2,C_1,C_2} \circ (f_1 \wedge f_2)\colon B_1 \wedge B_2 \vdash (A_1 \wedge A_2) \vee (C_1 \vee C_2)}$$

for $\hat{e}'_{A,B,C,D} =_{df} d_{A,C,B\wedge D} \circ (\mathbf{1}_A \wedge (\check{c}_{C,B\wedge D} \circ d_{B,D,C})) \circ \hat{b}^{\leftarrow}_{A,B,D\vee C} \circ$
$\circ\, (\mathbf{1}_{A\wedge B} \wedge \check{c}_{D,C})\colon (A \wedge B) \wedge (C \vee D) \vdash (A \wedge C) \vee (B \wedge D)$,

$$\frac{f_1\colon C_1 \wedge A_1 \vdash B_1 \qquad\qquad f_2\colon C_2 \wedge A_2 \vdash B_2}{\vee(f_1, f_2) =_{dn} (f_1 \vee f_2) \circ \hat{e}'_{C_1,C_2,A_1,A_2}\colon (C_1 \wedge C_2) \wedge (A_1 \vee A_2) \vdash B_1 \vee B_2}$$

(see [22], Section 7.6, for $\check{e}'$ and $\hat{e}'$),

$$\frac{f\colon B \vdash A \vee C}{\neg^{\mathrm{L}} f =_{dn} \check{\Sigma}'_{A,C} \circ d_{\neg A,A,C} \circ \hat{c}_{A\vee C,\neg A} \circ (f \wedge \mathbf{1}_{\neg A})\colon B \wedge \neg A \vdash C}$$

$$\frac{f\colon C \wedge A \vdash B}{\neg^{\mathrm{R}} f =_{dn} (\mathbf{1}_{\neg A} \vee f) \circ \check{c}_{\neg A,C\wedge A} \circ d_{C,A,\neg A} \circ \hat{\Delta}'_{A,C} : C \vdash \neg A \vee B}$$

To define the remaining Gentzen operations, we need some preparation. For every proper $\wedge$-context X we define inductively as follows an object E_X of $\mathbf{SA}'_s$:

$$E_{\Box\wedge B} = E_{B\wedge\Box} = B,$$
$$E_{X\wedge B} = E_X \wedge B, \quad \text{for } X \text{ proper},$$
$$E_{B\wedge X} = B \wedge E_X, \quad \text{for } X \text{ proper}.$$

For every proper $\wedge$-context X and every object A of $\mathbf{SA}'_s$ we define inductively as follows an arrow term $\hat{\tau}_{X,A}\colon E_X \wedge A \vdash X(A)$ of $\mathbf{SA}'_s$:

$$\hat{\tau}_{B\wedge\Box,A} =_{df} \mathbf{1}_{B\wedge A} : B \wedge A \vdash B \wedge A,$$
$$\hat{\tau}_{B\wedge X,A} =_{df} (\mathbf{1}_B \wedge \hat{\tau}_{X,A}) \circ \hat{b}^{\leftarrow}_{B,E_X,A} : (B \wedge E_X) \wedge A \vdash B \wedge X(A),$$
$$\text{for } X \text{ proper},$$
$$\hat{\tau}_{\Box\wedge B,A} =_{df} \hat{c}_{B,A} : B \wedge A \vdash A \wedge B,$$
$$\hat{\tau}_{X\wedge B,A} =_{df} (\hat{\tau}_{X,A} \wedge \mathbf{1}_B) \circ \hat{b}^{\rightarrow}_{E_X,A,B} \circ (\mathbf{1}_{E_X} \wedge \hat{c}_{B,A}) \circ \hat{b}^{\leftarrow}_{E_X,B,A}\colon$$
$$(E_X \wedge B) \wedge A \vdash X(A) \wedge B, \quad \text{for } X \text{ proper}.$$

For every proper $\vee$-context Y we define inductively as follows an object D_Y of $\mathbf{SA}'_s$:

$$D_{\Box\vee B} = D_{B\vee\Box} = B,$$
$$D_{Y\vee B} = D_Y \vee B, \quad \text{for } Y \text{ proper},$$
$$D_{B\vee Y} = B \vee D_Y, \quad \text{for } Y \text{ proper}.$$

For every proper $\vee$-context Y and every object A of $\mathbf{SA}'_s$ we define inductively as follows an arrow term $\check{\tau}_{Y,A}\colon Y(A) \vdash A \vee D_Y$ of $\mathbf{SA}'_s$:

$$\check{\tau}_{\Box\vee B,A} =_{df} \mathbf{1}_{A\vee B} : A\vee B\vdash A\vee B,$$

$$\check{\tau}_{Y\vee B,A} =_{df} \check{b}^{\leftarrow}_{A,D_Y,B}\circ(\check{\tau}_{Y,A}\vee\mathbf{1}_B) : Y(A)\vee B\vdash A\vee(D_Y\vee B),$$

for Y proper,

$$\check{\tau}_{B\vee\Box,A} =_{df} \check{c}_{A,B} : B\vee A\vdash A\vee B,$$

$$\check{\tau}_{B\vee Y,A} =_{df} \check{b}^{\leftarrow}_{A,B,D_Y}\circ(\check{c}_{A,B}\vee\mathbf{1}_{D_Y})\circ\check{b}^{\rightarrow}_{B,A,D_Y}\circ(\mathbf{1}_B\vee\check{\tau}_{Y,A}) : B\vee Y(A)\vdash A\vee(B\vee D_Y),$$

for Y proper.

For $f\colon A\vdash B$, the following equations hold in $\mathbf{SA}'_s$:

$$(\hat{\tau}\ nat)\qquad X(f)\circ\hat{\tau}_{X,A} = \hat{\tau}_{X,B}\circ(\mathbf{1}_{E_X}\wedge f),$$

$$(\check{\tau}\ nat)\qquad (f\vee\mathbf{1}_{D_Y})\circ\check{\tau}_{Y,A} = \check{\tau}_{Y,B}\circ Y(f);$$

they are proved by applying naturality equations.

It is clear that for $\xi\in\{\wedge,\vee\}$ and $\overset{\xi}{\tau}_{X,A}\colon A_1\vdash A_2$ there is an arrow term $\overset{\xi}{\tau}{}^{-1}_{X,A}\colon A_2\vdash A_1$ of $\mathbf{SA}'_s$, which is a "mirror image" of $\overset{\xi}{\tau}_{X,A}$, such that in $\mathbf{SA}'_s$ we have

$$\overset{\xi}{\tau}{}^{-1}_{X,A}\circ\overset{\xi}{\tau}_{X,A} = \mathbf{1}_{A_1},\qquad \overset{\xi}{\tau}_{X,A}\circ\overset{\xi}{\tau}{}^{-1}_{X,A} = \mathbf{1}_{A_2}.$$

For example, with

$$\hat{\tau}_{F\wedge((C\wedge\Box)\wedge B),A} = (\mathbf{1}_F\wedge(\hat{b}^{\rightarrow}_{C,A,B}\circ(\mathbf{1}_C\wedge\hat{c}_{B,A})\circ\hat{b}^{\leftarrow}_{C,B,A}))\circ\hat{b}^{\leftarrow}_{F,C\wedge B,A}$$

we have

$$\hat{\tau}^{-1}_{F\wedge((C\wedge\Box)\wedge B),A} = \hat{b}^{\rightarrow}_{F,C\wedge B,A}\circ(\mathbf{1}_F\wedge(\hat{b}^{\rightarrow}_{C,B,A}\circ(\mathbf{1}_C\wedge\hat{c}_{A,B})\circ\hat{b}^{\leftarrow}_{C,A,B})).$$

Officially, $\overset{\xi}{\tau}{}^{-1}_{X,A}$ is defined inductively as $\overset{\xi}{\tau}_{X,A}$, in a dual manner.

Next, we introduce the following abbreviation:

$$d_{X,A,Y} =_{df} \check{\tau}^{-1}_{Y,X(A)}\circ(\hat{\tau}_{X,A}\vee\mathbf{1}_{D_Y})\circ d_{E_X,A,D_Y}\circ(\mathbf{1}_{E_X}\wedge\check{\tau}_{Y,A})\circ\hat{\tau}^{-1}_{X,Y(A)} : X(Y(A))\vdash Y(X(A)).$$

When X or Y is $\Box$, then we assume that $d_{X,A,Y}$ stands for $\mathbf{1}_{X(Y(A))}$, which is of type $X(Y(A))\vdash Y(X(A))$, i.e. $Y(A)\vdash Y(A)$ or $X(A)\vdash X(A)$.

We can finally define the remaining Gentzen operations, which are all of the following form:

$$\frac{g\colon B \vdash Y(A) \qquad\qquad f\colon X(A) \vdash C}{cut_{X,Y}(f,g) =_{dn} Y(f) \circ d_{X,A,Y} \circ X(g) : X(B) \vdash Y(C)}$$

This concludes the definition of Gentzen operations. The set of Gentzen terms is the smallest set containing primitive Gentzen terms and closed under the Gentzen operations above.

It is easy to infer from **DS** Coherence of §2.3 that the following equations hold in $\mathbf{SA}'_s$:

$$\begin{array}{ll}
(d\wedge X) & d_{A\wedge X,C,Y} = d_{A\wedge\Box,X(C),Y} \circ (\mathbf{1}_A \wedge d_{X,C,Y}),\\
(dX\wedge) & d_{X\wedge A,C,Y} = d_{\Box\wedge A,X(C),Y} \circ (d_{X,C,Y} \wedge \mathbf{1}_A),\\
(d\vee Y) & d_{X,C,A\vee Y} = (\mathbf{1}_A \vee d_{X,C,Y}) \circ d_{X,Y(C),A\vee\Box},\\
(dY\vee) & d_{X,C,Y\vee A} = (d_{X,C,Y} \vee \mathbf{1}_A) \circ d_{X,Y(C),\Box\vee A}.
\end{array}$$

The equation $(d\wedge X)$ is analogous to the equation $(d\wedge)$ of §2.1, while $(d\vee Y)$ is analogous to $(d\vee)$ of §2.1.

We can then prove the following.

Gentzenization Lemma. *Every arrow of* $\mathbf{SA}'_s$ *is denoted by a Gentzen term.*

Proof. We first show by induction on the complexity of A that for every A the arrow $\mathbf{1}_A\colon A \vdash A$ is denoted by a Gentzen term. For A being a letter, or $\top$, or $\bot$, this is trivial. For the induction step we use the following equations of $\mathbf{SA}'_s$:

$$\begin{array}{ll}
(\wedge) & \bot^{\rightarrow}\bot^{\rightarrow}\check{B}^{\rightarrow}_{\Box}\wedge(\bot^{\leftarrow}f_1, \bot^{\leftarrow}f_2) = f_1 \wedge f_2,\\
(\vee) & \top^{\leftarrow}\top^{\leftarrow}\hat{B}^{\rightarrow}_{\Box}\vee(\top^{\rightarrow}f_1, \top^{\rightarrow}f_2) = f_1 \vee f_2.
\end{array}$$

For $(\wedge)$ we use

$$\check{e}'_{A_1,A_2,\bot,\bot} = (\mathbf{1}_{A_1\wedge A_2} \vee \check{\delta}^{\leftarrow}_{\bot}) \circ \check{\delta}^{\leftarrow}_{A_1\wedge A_2} \circ (\check{\delta}^{\rightarrow}_{A_1} \wedge \check{\delta}^{\rightarrow}_{A_2}),$$

which follows essentially from $(\check{b}\check{\delta})$ and $(d\check{\delta})$ of §3.3 (we may apply here the Symmetric Bimonoidal Coherence of [22], Section 6.4, which reduces to

Mac Lane's symmetric monoidal coherence of [37]; see [38], Section VII.7, and [22], Section 5.3). We proceed analogously for $(\vee)$.

We also have for the induction step the following equations of $\mathbf{SA}'_s$:

$$\bot^{\rightarrow}\neg^{\mathrm{R}}\,\hat{C}_{\Box}\,\neg^{\mathrm{L}}\bot^{\leftarrow}\mathbf{1}_A = \top^{\leftarrow}\neg^{\mathrm{L}}\,\check{C}_{\Box}\,\neg^{\mathrm{R}}\top^{\rightarrow}\mathbf{1}_A = \mathbf{1}_{\neg A},$$

for which we use $(d\,\check{\delta})$, $(\check{b}\check{\Sigma}')$ and $(\check{\Sigma}'\,\hat{\Delta}')$, among other obvious equations. The Gentzen term that denotes $\mathbf{1}_A$ is written $\mathbf{1}_A$.

Next we have the following in $\mathbf{SA}'_s$:

$$\hat{B}^{\rightarrow}_{\Box}\mathbf{1}_{(A\wedge B)\wedge C} =_{dn} \hat{b}^{\rightarrow}_{A,B,C}, \qquad \check{B}^{\rightarrow}_{\Box}\mathbf{1}_{A\vee(B\vee C)} =_{dn} \check{b}^{\rightarrow}_{A,B,C},$$

$$\hat{B}^{\leftarrow}_{\Box}\mathbf{1}_{A\wedge(B\wedge C)} =_{dn} \hat{b}^{\leftarrow}_{A,B,C}, \qquad \check{B}^{\leftarrow}_{\Box}\mathbf{1}_{(A\vee B)\vee C} =_{dn} \check{b}^{\leftarrow}_{A,B,C},$$

$$\hat{C}_{\Box}\mathbf{1}_{B\wedge A} =_{dn} \hat{c}_{A,B}, \qquad \check{C}_{\Box}\mathbf{1}_{B\vee A} =_{dn} \check{c}_{A,B},$$

$$cut_{A\wedge\Box,\Box\vee C}(\mathbf{1}_{A\wedge B},\mathbf{1}_{B\vee C}) =_{dn} d_{A,B,C};$$

by using abbreviations according to $(\wedge)$ and $(\vee)$ above,

$$\top^{\leftarrow}\,\hat{C}_{\Box}\,(\mathbf{1}_A\wedge\neg^{\mathrm{R}}\top^{\rightarrow}\mathbf{1}_B) =_{dn} \hat{\Delta}_{B,A},$$

$$\bot^{\rightarrow}\,\check{C}_{\Box}\,(\neg^{\mathrm{L}}\bot^{\leftarrow}\mathbf{1}_B\vee\mathbf{1}_A) =_{dn} \check{\Sigma}_{B,A},$$

$$\hat{C}_{\Box}\,\top^{\rightarrow}\mathbf{1}_A =_{dn} \hat{\delta}^{\rightarrow}_A, \qquad \bot^{\rightarrow}\mathbf{1}_{A\vee\bot} =_{dn} \check{\delta}^{\rightarrow}_A,$$

$$\top^{\leftarrow}\,\hat{C}_{\Box}\,\mathbf{1}_{A\wedge\top} =_{dn} \hat{\delta}^{\leftarrow}_A, \qquad \bot^{\leftarrow}\mathbf{1}_A =_{dn} \check{\delta}^{\leftarrow}_A.$$

(For the equations involving $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$ we rely on $(d\,\hat{\sigma})$ and $(d\,\check{\delta})$ of §3.3, and on the stem-increasing equations of §2.5.)

For composition we have the following equation of $\mathbf{SA}'_s$:

$$cut_{\Box,\Box}(f,g) = f\circ g,$$

and for the operations $\wedge$ and $\vee$ on arrows we have the equations $(\wedge)$ and $(\vee)$ above. ⊣

§4.2. Cut elimination in $\mathbf{SA}'_s$

For the proof of the Cut-Elimination Theorem below we will introduce analogues of Gentzen's notions of rank and degree. We need some preliminary definitions to define these notions.

For $\xi \in \{\wedge, \vee\}$, we define first by induction the notion of ξ-*superficial* subformula of a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$:

> if A is of the form p, $\bot$, $A_1 \vee A_2$, or $\neg A'$, then A is a $\wedge$-superficial subformula of A;
>
> if A is of the form p, $\top$, $A_1 \wedge A_2$, or $\neg A'$, then A is a $\vee$-superficial subformula of A;
>
> if A is a ξ-superficial subformula of B, then A is a ξ-superficial subformula of $B\,\xi\,C$ and $C\,\xi\,B$.

Consider a Gentzen term f of the form

$$\wedge(f_1, f_2) \colon B_1 \wedge B_2 \vdash (A_1 \wedge A_2) \vee (C_1 \vee C_2).$$

The $\vee$-superficial subformula $A_1 \wedge A_2$ that is the left disjunct of the target of f is called the *leaf* of f. All the other $\vee$-superficial subformulae of the target of f, which are subformulae of C_1 or C_2, and all the $\wedge$-superficial subformulae of the source of f, which are subformulae of B_1 or B_2, are called *lower parameters* of f.

To every lower parameter x of f, there corresponds unambiguously a subformula y in the target or the source of either $f_1 \colon B_1 \vdash A_1 \vee C_1$ or $f_2 \colon B_2 \vdash A_2 \vee C_2$, which we call the *upper parameter of f corresponding to x*. The lower parameter x is a $\wedge$-superficial subformula of the source of f iff the corresponding upper parameter y is a $\wedge$-superficial subformula of the source of either f_1 or f_2 (it cannot be in both), and analogously for parameters that are $\vee$-superficial subformulae of targets. If y is in the type of f_1, then f_1 is called the *subterm of f for the upper parameter y*, and analogously for f_2.

For example, if f is

$$\wedge(\mathbf{1}_{p\vee q}, \bot^{\leftarrow}\mathbf{1}_r) \colon (p \vee q) \wedge r \vdash (p \wedge r) \vee (q \vee \bot),$$

then $p \wedge r$ in the target is the leaf of f, while q in the target of f and $p \vee q$ and r in the source of f are lower parameters of f. To the lower parameter q of f corresponds the upper parameter of f that is the occurrence of q in the target of the subterm $\mathbf{1}_{p\vee q} \colon p \vee q \vdash p \vee q$ for this upper parameter; to the lower parameter $p \vee q$ of f corresponds the upper parameter of f that is the source of the subterm $\mathbf{1}_{p\vee q}$ for this upper parameter; and to the lower parameter r of f corresponds the upper parameter of f that is the source of the subterm $\bot^{\leftarrow}\mathbf{1}_r : r \vdash r \vee \bot$ for this upper parameter. Note that the subformula $\bot$ in the target of f is not a $\vee$-superficial subformula of this target, and hence is not a lower parameter of f.

If the Gentzen term f is of the form

$$\vee(f_1, f_2) : (C_1 \wedge C_2) \wedge (A_1 \vee A_2) \vdash B_1 \vee B_2,$$

then the $\wedge$-superficial subformula $A_1 \vee A_2$ that is the right conjunct of the source of f is the leaf of f, while all the other $\wedge$-superficial subformulae of the source of f and the $\vee$-superficial subformulae of the target of f are the lower parameters of f. The upper parameters of f corresponding to these lower parameters, and the subterms of f for these upper parameters, are defined analogously to what we had in the previous case.

The leaf of $\neg^{\mathrm{L}} f : B \wedge \neg A \vdash C$ is the $\wedge$-superficial subformula $\neg A$ that is the right conjunct of its source, while the leaf of $\neg^{\mathrm{R}} f : C \vdash \neg A \vee B$ is the $\vee$-superficial subformula $\neg A$ that is the left disjunct of its target. In both cases, the remaining $\wedge$-superficial subformulae of the source or the remaining $\vee$-superficial subformulae of the target are lower parameters, to whom correspond, analogously to what we had before, upper parameters in the source or target of the subterm f for these upper parameters.

If our Gentzen term is of the form

$$\hat{B}^{\leftarrow}_X f, \hat{B}^{\rightarrow}_X f, \check{B}^{\rightarrow}_Y f, \check{B}^{\leftarrow}_Y f, \hat{C}_X f, \check{C}_Y f, \top^{\rightarrow} f, \top^{\leftarrow} f, \bot^{\leftarrow} f, \bot^{\rightarrow} f, \text{ or } cut_{X,Y}(f, g),$$

then it has no leaves, and all the $\wedge$-superficial subformulae of its source and all the $\vee$-superficial subformulae of its target are lower parameters, to which upper parameters correspond in an obvious manner.

Finally, the Gentzen term $\mathbf{1}_p : p \vdash p$ has two leaves, which are its source p and its target p. There are no parameters of $\mathbf{1}_p$, neither lower nor upper. The Gentzen term $\mathbf{1}_\top : \top \vdash \top$ has as its leaf the target $\top$, and no parameters (the source $\top$ of $\mathbf{1}_\top$ is not a $\wedge$-superficial subformula of itself). The Gentzen term $\mathbf{1}_\bot : \bot \vdash \bot$ has as its leaf the source $\bot$, and no parameters (the target $\bot$ of $\mathbf{1}_\bot$ is not a $\vee$-superficial subformula of itself).

Let x be a $\wedge$-superficial subformula of the source of a Gentzen term f or a $\vee$-superficial subformula of the target of f. Then the *cluster* of x in f is a sequence of occurrences of formulae defined inductively as follows:

> if x is a leaf of f, then the cluster of x in f is x,
>
> if x is not a leaf of f, then x is a lower parameter of f, and for y_1 being the upper parameter of f corresponding to x, take the cluster $y_1 \dots y_n$, where $n \geq 1$, of y_1 in the proper subterm f' of f that is the subterm of f for the upper parameter y_1 (the sequence $y_1 \dots y_n$ is already defined, by the induction hypothesis); the cluster of x in f is the sequence $xy_1 \dots y_n$.

All occurrences of formulae in a cluster are ξ-superficial subformulae for ξ being one of $\wedge$ and $\vee$. If ξ is $\wedge$, then the cluster is a *source* cluster, and if ξ is $\vee$, then it is a *target* cluster.

A *cut* is a Gentzen term of the form $cut_{X,Y}(f,g)$. For $g\colon B \vdash Y(A)$ and $f\colon X(A) \vdash C$ let the formula A be called the *cut formula* of the cut $cut_{X,Y}(f,g)$. Let x be the displayed occurrence of A in the source $X(A)$ of f, and let s be the length of the cluster of x in f (we write s because we have here a source cluster). Let y be the displayed occurrence of A in the target $Y(A)$ of g, and let t be the length of the cluster of y in g (we write t because we have here a target cluster).

Depending on the form of A, we define a number r, which we call the *rank* of the cut $cut_{X,Y}(f,g)$. If the cut formula A is of the form p or $\neg A'$, then

$$\begin{array}{ll} r = \min(s,t)-1, & \text{if } A \text{ is } p,\\ r = s+t-2, & \text{if } A \text{ is } \neg A'. \end{array}$$

(As a matter of fact, when A is p, we could stipulate that r is either $s+t-2$, as when it is $\neg A'$, or $s-1$, or $t-1$, but the computation of rank we have introduced makes the cut-elimination procedure run faster, and does not complicate the proof.)

If the cut formula A is of the form $\top$ or $A_1 \wedge A_2$, then $r = t-1$. If, finally, the cut formula A is of the form $\bot$ or $A_1 \vee A_2$, then $r = s-1$.

We define the *degree* d of a cut as the number of occurrences of $\wedge$, $\vee$ and $\neg$ in its cut formula. The *complexity* of a cut is the ordered pair (d, r), where

d is its degree and r its rank. The complexities of cuts are lexicographically ordered (i.e., $(d_1, r_1) < (d_2, r_2)$ iff $d_1 < d_2$, or $d_1 = d_2$ and $r_1 < r_2$).

A Gentzen term is called *cut-free* when no subterm of it is a cut. A cut $cut_{X,Y}(f, g)$ is *topmost* when f and g are cut-free. (Since in the proof below, we compute the rank only for topmost cuts, our definition of cluster can be shortened a little bit by not considering the parameters of cuts; but this is not a substantial shortening.)

We can then prove the following.

CUT-ELIMINATION THEOREM. *For every Gentzen term h there is a cut-free Gentzen term h' such that $h = h'$ in* $\mathbf{SA}'_s$.

PROOF. It suffices to prove the theorem when h is a topmost cut. We proceed by induction on the complexity (d, r) of this topmost cut.

Suppose $r = 0$ and $d = 0$. Then h can be of one of the following forms:

$$cut_{X,\Box}(f, \mathbf{1}_A) \quad \text{for } A \text{ being } p \text{ or } \top,$$

$$cut_{\Box,Y}(\mathbf{1}_A, g) \quad \text{for } A \text{ being } p \text{ or } \bot,$$

and we have in $\mathbf{SA}'_s$

$$cut_{X,\Box}(f, \mathbf{1}_A) = f,$$

$$cut_{\Box,Y}(\mathbf{1}_A, g) = g.$$

This settles the basis of the induction.

Suppose $r = 0$ and $d > 0$. Then the cut formula must be of the form $A_1 \wedge A_2$ or $A_1 \vee A_2$ or $\neg A'$. In the first case, for $f\colon X(A_1 \wedge A_2) \vdash D$, $g_1\colon B_1 \vdash A_1 \vee C_1$ and $g_2\colon B_2 \vdash A_2 \vee C_2$ we have the equation

$$cut_{X,\Box\vee(C_1\vee C_2)}(f, \wedge(g_1, g_2)) = \check{\overleftarrow{B}}_{\Box}\, cut_{X'',\Box\vee C_2}(cut_{X',\Box\vee C_1}(f, g_1), g_2)$$

where $X'(C)$ is $X(C \wedge A_2)$ and $X''(C)$ is $X(B_1 \wedge C)$. To prove this equation we apply naturality equations and **DS** Coherence.

The complexity of the topmost cut $cut_{X',\Box\vee C_1}(f, g_1)$ is (d', r') with $d' < d$, and we can apply the induction hypothesis to obtain a cut-free Gentzen term f' equal to it in $\mathbf{SA}'_s$. The complexity of the topmost cut $cut_{X'',\Box\vee C_2}(f', g_2)$ is (d'', r'') with $d'' < d$, and we can again apply the induction hypothesis.

In case the cut formula is $A_1 \vee A_2$, we have an analogous equation, for which we use again **DS** Coherence, and we reason analogously, applying the induction hypothesis twice.

In case the cut formula is $\neg A'$, for $f\colon D \wedge A' \vdash E$ and $g\colon B \vdash A' \vee C$ we have the equation

$$cut_{B\wedge\Box,\Box\vee E}(\neg^{\mathrm{L}} g, \neg^{\mathrm{R}} f) = \check{C}_{\Box}\hat{C}_{\Box}\, cut_{D\wedge\Box,\Box\vee C}(f,g),$$

which holds by naturality equations and $\mathbf{PN}^{\neg}$ Coherence. Then we apply the induction hypothesis to the topmost cut on the right-hand side, which has a smaller degree.

Suppose now $r > 0$. If r was computed as $s-1$, or as $s+t-2$, where $s > 1$, then we may apply equations of $\mathbf{SA}'_s$ of the following form

$$(*) \quad cut_{X,Y}(\gamma f', g) = \gamma_1 \ldots \gamma_n\, cut_{X',Y}(f', g)$$

for γ, $\gamma_1, \ldots, \gamma_n$ unary Gentzen operations. If (d, r) is the complexity of the topmost cut $cut_{X,Y}(\gamma f', g)$, then the complexity of the topmost cut $cut_{X',Y}(f', g)$ is $(d, r-1)$, and so we may apply to it the induction hypothesis.

If γ is a unary Gentzen operation different from $\top^{\rightarrow}$, $\top^{\leftarrow}$, $\bot^{\leftarrow}$ and $\bot^{\rightarrow}$, then so are $\gamma_1, \ldots, \gamma_n$, and to prove $(*)$ we apply naturality equations and $\mathbf{PN}^{\neg}$ Coherence (sometimes **DS** Coherence suffices, depending on γ). We have analogous equations involving binary Gentzen operations, which are proved analogously, relying on **DS** Coherence (cf. [22], Section 11.2, Case (6), where on p. 251, in the second line $\wedge^R(f, cut(g, h))$ should be replaced by $\wedge^R(g, cut(f, h))$, and in the third line $cut(g, h)$ should be replaced by $cut(f, h)$).

If γ in $(*)$ is $\top^{\rightarrow}$, then $n = 1$ and γ_1 is $\top^{\rightarrow}$. To prove $(*)$, we then apply essentially the equation

$$Y(\hat{\sigma}^{\rightarrow}_{X(A)}) \circ d_{T\wedge X,A,Y} = d_{X,A,Y} \circ \hat{\sigma}^{\rightarrow}_{X(Y(A))},$$

which we obtain with the help of $(d\wedge X)$ of the preceding section, $(d\hat{\sigma})$ of §3.3, and $(\check{\tau}\ nat)$ of the preceding section (as a matter of fact, we may apply here the Symmetric Bimonoidal Coherence of [22], Section 6.4). We proceed analogously if γ is $\top^{\leftarrow}$.

If γ in $(*)$ is $\bot^{\leftarrow}$ or $\bot^{\rightarrow}$, then we apply essentially Mac Lane's symmetric monoidal coherence of [37] (see also [38], Section VII.7, and [22], Section 5.3).

If r was computed as $t-1$, or as $s+t-2$, where $t > 1$, then we proceed in a dual manner. Instead of $(*)$, we have equations of $\mathbf{SA}'_s$ of the following form:

$$cut_{X,Y}(f, \gamma g') = \gamma_1 \ldots \gamma_n \, cut_{X,Y'}(f, g').$$

This concludes the proof of the theorem. ⊣

§4.3. $\mathbf{SA}^c$ Coherence

There is a functor G from the category $\mathbf{SA}'$ of §3.3 to Br, which is defined as the functor G from $\mathbf{PN}^{\neg}$ to Br (see §2.3) with the additional clauses that say that $G\alpha$ is an identity arrow of Br for α being $\overset{\xi}{\delta}{}^{\rightarrow}_{A}$, $\overset{\xi}{\delta}{}^{\leftarrow}_{A}$, $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$, where $\xi \in \{\wedge, \vee\}$. There is analogously a functor G from $\mathbf{SA}'_s$ to Br, which is defined as G from $\mathbf{SA}'$ to Br save that we do not have the clauses for $\pi^{\rightarrow}_{A,B}$ and $\pi^{\leftarrow}_{A,B}$. It follows from the existence of these functors and $\mathbf{PN}^{\neg}$ Coherence that $\mathbf{PN}^{\neg}$ is isomorphic to subcategories of $\mathbf{SA}'$ and $\mathbf{SA}'_s$ (cf. [22], Section 14.4).

Our purpose in this section is to prove the following theorem.

CONSERVATIVENESS THEOREM. *If A and B are objects of $\mathbf{PN}^{\neg}$, then for every arrow $f\colon A \vdash B$ of $\mathbf{SA}'_s$ there is an arrow term $f'\colon A \vdash B$ of $\mathbf{PN}^{\neg}$ such that $f = f'$ in $\mathbf{SA}'_s$.*

This theorem implies that $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of $\mathbf{SA}'_s$, from which, according to what we established in §§3.7-8, we can conclude that $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of $\mathbf{SA}'$, and of $\mathbf{SA}$ too. In these isomorphisms every object of $\mathbf{PN}^{\neg}$ is mapped to itself, and so every object of $\mathbf{PN}^{\neg}$ in $\mathbf{SA}'_s$, $\mathbf{SA}'$ or $\mathbf{SA}$ is in the image of $\mathbf{PN}^{\neg}$.

Let the functor G from $\mathbf{SA}$ to Br be defined as G from $\mathbf{SMC}$ to Br (see §3.1) with the additional clauses that say that $G\alpha$ is an identity arrow of Br for α being $\nu^{\rightarrow}_{A}$, $\lambda^{\rightarrow}_{A}$, $\lambda^{\leftarrow}_{A}$, $\upsilon^{\rightarrow}_{A,B}$ and $\upsilon^{\leftarrow}_{A,B}$. One can easily check that this functor G restricted to the subcategory of $\mathbf{SA}$ isomorphic to $\mathbf{PN}^{\neg}$ satisfies all the clauses of the definition of the functor G from $\mathbf{PN}^{\neg}$ to Br (see §2.3).

Let $\mathbf{SA}''$ be the full subcategory of $\mathbf{SA}$ whose objects are all the objects A of $\mathbf{SA}$ such that there is an isomorphism of type $A \vdash A'$ of $\mathbf{SA}$ for A' an object of $\mathbf{PN}^{\neg}$. (The category $\mathbf{SA}''$ is a replete subcategory of $\mathbf{SA}$; cf. the end of §3.1.) Then we can restrict the functor G from $\mathbf{SA}$ to Br to a functor G from $\mathbf{SA}''$ to Br, for which we can prove the following, relying on the Conservativeness Theorem.

$\mathbf{SA}''$ COHERENCE. *The functor G from $\mathbf{SA}''$ to Br is faithful.*

PROOF. Suppose A and B are objects of $\mathbf{SA}''$, and let $j_A : A \vdash A'$ and $j_B : B \vdash B'$ be isomorphisms of $\mathbf{SA}$ for A' and B' objects of $\mathbf{PN}^{\neg}$. Suppose that $f_1, f_2 : A \vdash B$ are arrows of $\mathbf{SA}$, i.e. of $\mathbf{SA}''$, such that $Gf_1 = Gf_2$.

Since $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of $\mathbf{SA}$ such that every object of $\mathbf{PN}^{\neg}$ in $\mathbf{SA}$ is in the image of $\mathbf{PN}^{\neg}$, we have in $\mathbf{SA}$ that

$$j_B \circ f_i \circ j_A^{-1} = f_i'$$

for $i \in \{1, 2\}$ and f_i' an arrow term of $\mathbf{PN}^{\neg}$. It follows that $Gf_1' = Gf_2'$, and, according to what we said immediately after the definition of the functor G from $\mathbf{SA}$ to Br, by $\mathbf{PN}^{\neg}$ Coherence we have that $f_1' = f_2'$ in $\mathbf{PN}^{\neg}$, and hence also in $\mathbf{SA}$. So $f_1 = f_2$ in $\mathbf{SA}$. ⊣

The category $\mathbf{SA}''$ is a category equivalent to $\mathbf{PN}^{\neg}$, and its coherence is a consequence of $\mathbf{PN}^{\neg}$ Coherence. We can find full subcategories of $\mathbf{SA}''$, some of which are full subcategories of $\mathbf{SA}_s$ too, that are not only equivalent, but also isomorphic to $\mathbf{PN}^{\neg}$.

Let $\mathbf{SA}^c$ be the full subcategory of $\mathbf{SA}$ whose objects are all the objects A of $\mathbf{SA}$ such that there is an isomorphism of type $A \vdash A'$ of $\mathbf{SA}$ for A' being either an object of $\mathbf{PN}^{\neg}$, or $\top$, or $\bot$. (The category $\mathbf{SA}^c$ is as $\mathbf{SA}''$ a replete subcategory of $\mathbf{SA}$.) Then we can restrict the functor G from $\mathbf{SA}$ to Br to a functor G from $\mathbf{SA}^c$ to Br, for which we can prove the following, relying on the Conservativeness Theorem and on $\mathbf{SA}''$ Coherence.

$\mathbf{SA}^c$ COHERENCE. *The functor G from $\mathbf{SA}^c$ to Br is faithful.*

PROOF. There is no arrow of type $\top \vdash \bot$ in $\mathbf{SA}$. (Otherwise, classical propositional logic would be inconsistent.) There is also no arrow of type

$\bot \vdash \top$ in $\mathbf{SA}$. If $f\colon \bot \vdash \top$ were such an arrow, then we would have in $\mathbf{SA}$ the arrow

$$((\hat{\delta}^{\rightarrow}_{p} \circ (\mathbf{1}_p \wedge f)) \vee \mathbf{1}_q) \circ d_{p,\bot,q} \circ (\mathbf{1}_p \wedge \check{\sigma}^{\leftarrow}_{q})\colon p \wedge q \vdash p \vee q.$$

Hence, by the Conservativeness Theorem, there would be an arrow term $f'\colon p \wedge q \vdash p \vee q$ of $\mathbf{PN}^{\neg}$, and that such an f' does not exist can be shown by appealing to the connectedness condition of proof nets (see §7.1).

Suppose A and B are objects of $\mathbf{SA}^c$; so A and B are isomorphic in $\mathbf{SA}$ to respectively A' and B', each of which is either an object of $\mathbf{PN}^{\neg}$, or $\top$, or $\bot$. Suppose that $f_1, f_2\colon A \vdash B$ are arrows of $\mathbf{SA}$, i.e. of $\mathbf{SA}^c$, such that $Gf_1 = Gf_2$.

As we have seen above, it is excluded that one of A' and B' is $\top$ while the other is $\bot$. If A' and B' are objects of $\mathbf{PN}^{\neg}$, then we apply $\mathbf{SA}''$ Coherence.

Let $\mathbf{SA}_{+p}$ be $\mathbf{SA}$ generated by $\mathcal{P} \cup \{p\}$ for a letter p foreign to $\mathcal{P}$, and hence also to A and B. Let $\mathbf{SA}''_{+p}$ be the $\mathbf{SA}''$ subcategory of $\mathbf{SA}_{+p}$. In the remaining cases, if either A' or B' is $\top$, then $G(f_1 \wedge \mathbf{1}_p) = G(f_2 \wedge \mathbf{1}_p)$. It is easy to see that $f_1 \wedge \mathbf{1}_p, f_2 \wedge \mathbf{1}_p\colon A \wedge p \vdash B \wedge p$ are arrows of $\mathbf{SA}''_{+p}$, and so $f_1 \wedge \mathbf{1}_p = f_2 \wedge \mathbf{1}_p$ in $\mathbf{SA}_{+p}$ by $\mathbf{SA}''$ Coherence applied to $\mathbf{SA}''_{+p}$. Then in $\mathbf{SA}$ generated by $\mathcal{P}$ we have $f_1 \wedge \mathbf{1}_\top = f_2 \wedge \mathbf{1}_\top$ (we just substitute $\top$ for p in the derivation of $f_1 \wedge \mathbf{1}_p = f_2 \wedge \mathbf{1}_p$ in $\mathbf{SA}_{+p}$), and so we have in $\mathbf{SA}$

$$\begin{aligned} f_1 &= f_1 \circ \hat{\delta}^{\rightarrow}_{A} \circ \hat{\delta}^{\leftarrow}_{A}, \quad \text{by } (\hat{\delta}\,\hat{\delta}),\\ &= \hat{\delta}^{\rightarrow}_{B} \circ (f_1 \wedge \mathbf{1}_\top) \circ \hat{\delta}^{\leftarrow}_{A}, \quad \text{by } (\hat{\delta}^{\rightarrow}\ nat),\\ &= \hat{\delta}^{\rightarrow}_{B} \circ (f_2 \wedge \mathbf{1}_\top) \circ \hat{\delta}^{\leftarrow}_{A}\\ &= f_2. \end{aligned}$$

If either A' or B' in the remaining cases is $\bot$, then $G(f_1 \vee \mathbf{1}_p) = G(f_2 \vee \mathbf{1}_p)$, and we proceed analogously. ⊣

Both $\mathbf{SA}''$ Coherence and $\mathbf{SA}^c$ Coherence are analogous to Kelly's and Mac Lane's $\mathbf{SMC}^c$ Coherence (see the end of §3.1); for $\mathbf{SA}^c$ Coherence the analogy is complete.

Note that many computations of equality of arrows in Chapter 3, which we could not settle previously by $\mathbf{SMC}^c$ Coherence or $\mathbf{PN}^{\neg}$ Coherence alone, are now settled by simple applications of $\mathbf{SA}^c$ Coherence. As a matter of fact, $\mathbf{SA}''$ Coherence suffices. (Of course, we used these computations to establish $\mathbf{SA}^c$ Coherence, and we can judge now only retrospectively that they are dispensable in the presence of this coherence.) With $\mathbf{SA}^c$ Coherence we have found a powerful tool to establish equality of arrows in a considerable fragment of $\mathbf{SA}$, and also of $\mathbf{SA}_s$. This covers the $\{\wedge, \rightarrow\}$ fragment, the $\{\neg, \wedge, \vee\}$ fragment, and also other fragments of star-autonomous categories involving $\top$ and $\bot$ at some particular places.

Coherence with respect to Br for the whole of $\mathbf{SA}$ or $\mathbf{SA}_s$ presumably does not hold. According to [6] (Sections 4.2, 2.3), in $\mathbf{SA}_s$ we do not have

$$\nu^{\leftarrow}_{A\rightarrow\top,\top} \circ (\nu^{\leftarrow}_{A,\top} \rightarrow \mathbf{1}_\top) = \mathbf{1}_{((A\rightarrow\top)\rightarrow\top)\rightarrow\top}$$

(cf. [32], [43]), nor

$$\hat{c}_{\bot,\bot} = \mathbf{1}_{\bot\wedge\bot}, \qquad\qquad \check{c}_{\top,\top} = \mathbf{1}_{\top\vee\top},$$

while if $f = g$ is one of these equations we have $Gf = Gg$. (The claim made in [6], Section 2.3, that the category of sets with functions is a linearly distributive category is not correct.)

The remainder of this section is devoted to the proof of the Conservativeness Theorem. This will be accomplished with the help of several lemmata, for whose formulation we introduce the following terminology.

An object of $\mathbf{SA}'_s$, i.e. a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$, is *constant-free* when neither $\top$ nor $\bot$ occurs in it. In other words, the constant-free objects of $\mathbf{SA}'_s$ are the objects of $\mathbf{PN}^\neg$.

An object of $\mathbf{SA}'_s$ is called *literate* when at least one letter occurs in it; otherwise, it is *letterless*. Every constant-free formula is literate (but not conversely).

For $\xi \in \{\wedge, \vee\}$, we define inductively when a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$ is ξ-*nice*:

> $\top$ is $\wedge$-nice and $\bot$ is $\vee$-nice;
> constant-free objects of $\mathbf{SA}'_s$ are ξ-nice;
> if A and B are ξ-nice, then $A\,\xi\,B$ is ξ-nice.

For a ξ-nice formula A we define inductively an arrow term $\overset{\xi}{\rho}_A \colon A \vdash A^r$ of $\mathbf{SA}'_s$ such that A^r is constant-free if A is literate, A^r is $\top$ if A is letterless and $\wedge$-nice, and A^r is $\bot$ if A is letterless and $\vee$-nice:

$$\hat{\rho}_\top = \mathbf{1}_\top, \qquad \check{\rho}_\bot = \mathbf{1}_\bot, \qquad \overset{\xi}{\rho}_A = \mathbf{1}_A, \qquad \text{for } A \text{ constant-free,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\rho}_A \;\xi\; \overset{\xi}{\rho}_B, \qquad \text{for } A \text{ and } B \text{ literate,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\delta}{}^{\rightarrow}_A \circ (\overset{\xi}{\rho}_A \;\xi\; \overset{\xi}{\rho}_B), \qquad \text{for } B \text{ letterless,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\sigma}{}^{\rightarrow}_B \circ (\overset{\xi}{\rho}_A \;\xi\; \overset{\xi}{\rho}_B), \qquad \text{for } A \text{ letterless.}$$

It is clear that $\overset{\xi}{\rho}_A$ is an isomorphism of $\mathbf{SA}'_s$, with inverse $\overset{\xi}{\rho}{}^{-1}_A\colon A^r \vdash A$.

We can then prove the following lemma.

LEMMA 1. *Let* $f\colon A \vdash B$ *be a* $\overset{\xi}{b}{}^{\rightarrow}_{C,D,E}$*-term for* C, D *and* E *literate* ξ*-nice formulae. Then there is* $\overset{\xi}{b}{}^{\rightarrow}_{C^r,D^r,E^r}$*-term* $f^r\colon A^r \vdash B^r$ *for* C^r, D^r *and* E^r *constant-free such that*

$$\overset{\xi}{\rho}_B \circ f = f^r \circ \overset{\xi}{\rho}_A\,.$$

PROOF. We proceed by induction on the complexity of f. If f is $\overset{\xi}{b}{}^{\rightarrow}_{C,D,E}$, then we have that

$$\overset{\xi}{\rho}_{(C\xi D)\xi E} = (\overset{\xi}{\rho}_C \;\xi\; \overset{\xi}{\rho}_D)\,\xi\; \overset{\xi}{\rho}_E,$$

and we apply $(\overset{\xi}{b}{}^{\rightarrow}\ nat)$. For the induction step, suppose f is $g\,\xi\,\mathbf{1}_F : G\,\xi\,F \vdash H\,\xi\,F$ (we proceed analogously when f is $\mathbf{1}_F\,\xi\,g$). Then we have two cases.

If F is literate, then $\overset{\xi}{\rho}_{H\xi F} = \overset{\xi}{\rho}_H \;\xi\; \overset{\xi}{\rho}_F$, and we just apply bifunctorial equations and the induction hypothesis.

If F is letterless, then for $\zeta \in \{\top, \bot\}$ we have

$$\begin{aligned}\overset{\xi}{\rho}_{H\xi F} \circ (g\,\xi\,\mathbf{1}_F) &= \overset{\xi}{\delta}{}^{\rightarrow}_H \circ (\overset{\xi}{\rho}_H \;\xi\; \overset{\xi}{\rho}_F) \circ (g\,\xi\,\mathbf{1}_F)\\ &= \overset{\xi}{\delta}{}^{\rightarrow}_H \circ (g^r\,\xi\,\mathbf{1}_\zeta) \circ (\overset{\xi}{\rho}_G \;\xi\; \overset{\xi}{\rho}_F),\end{aligned}$$

by bifunctorial equations and the induction hypothesis. Then we apply $(\overset{\xi}{\delta}{}^{\rightarrow}\ nat)$ to obtain $g^r \circ \overset{\xi}{\rho}_{G\xi F}$. ⊣

We have analogous lemmata, which we call also *Lemma 1*, when f is a $\overset{\xi}{b}{}^{\leftarrow}_{C,D,E}$-term or a $\overset{\xi}{c}_{C,D}$-term. We also have the following.

LEMMA 2. *Let $f\colon A \vdash B$ be a $\stackrel{\xi}{b}{}^{\rightarrow}_{C,D,E}$-term, $\stackrel{\xi}{b}{}^{\leftarrow}_{C,D,E}$-term or $\stackrel{\xi}{c}_{C,D}$-term for C or D or E being a letterless ξ-nice formula, or let f be a $\stackrel{\xi}{\sigma}{}^{\rightarrow}_{F}$-term, $\stackrel{\xi}{\sigma}{}^{\leftarrow}_{F}$-term, $\stackrel{\xi}{\delta}{}^{\leftarrow}_{F}$-term or $\stackrel{\xi}{\delta}{}^{\rightarrow}_{F}$-term for F being ξ-nice. Then*

$$\stackrel{\xi}{\rho}_B \circ f = \stackrel{\xi}{\rho}_A .$$

PROOF. We proceed either by induction, applying essentially the following equations of monoidal categories:

$$\begin{aligned}
\stackrel{\xi}{b}{}^{\rightarrow}_{C,D,\zeta} &= \stackrel{\xi}{\delta}{}^{\leftarrow}_{C\xi D} \circ (\mathbf{1}_C \;\xi\; \stackrel{\xi}{\delta}{}^{\rightarrow}_{D}),\\
\stackrel{\xi}{b}{}^{\rightarrow}_{C,\zeta,E} &= \stackrel{\xi}{\delta}{}^{\leftarrow}_{C} \;\xi\; \stackrel{\xi}{\sigma}{}^{\rightarrow}_{E},\\
\stackrel{\xi}{b}{}^{\rightarrow}_{\zeta,D,E} &= (\stackrel{\xi}{\sigma}{}^{\leftarrow}_{D} \;\xi\; \mathbf{1}_E) \circ \stackrel{\xi}{\sigma}{}^{\rightarrow}_{D\xi E},
\end{aligned}$$

where ζ is $\top$ if ξ is $\wedge$, and $\bot$ if ξ is $\vee$,

$$\hat{c}_{\top,A} = \hat{\delta}{}^{\leftarrow}_{A} \circ \hat{\sigma}{}^{\rightarrow}_{A}, \qquad\qquad \check{c}_{\bot,A} = \check{\sigma}{}^{\leftarrow}_{A} \circ \check{\delta}{}^{\rightarrow}_{A},$$

or we infer that under the conditions of the lemma A^r and B^r must be equal, and then we apply essentially Mac Lane's symmetric monoidal coherence of [37] (see also [38], Section VII.7, and [22], Section 5.3). ⊣

We prove next the key lemma of this section, whose corollary is the Conservativeness Theorem (we just instantiate statement (1) of this lemma).

LEMMA 3. *Let $f\colon A \vdash B$ be an arrow of $\mathbf{SA}'_s$ such that A is $\wedge$-nice and B is $\vee$-nice.*

(1) *If both A and B are literate, then there is an arrow term $f^r : A^r \vdash B^r$ of $\mathbf{PN}^{\neg}$ such that in $\mathbf{SA}'_s$ we have*

$$\check{\rho}_B \circ f \circ \hat{\rho}_A^{-1} = f^r.$$

(2) *If A is letterless and B is literate, then for every constant-free C there is an arrow term $f^r\colon C \vdash C \wedge B^r$ of $\mathbf{PN}^{\neg}$ such that in $\mathbf{SA}'_s$ we have*

$$(\mathbf{1}_C \wedge (\check{\rho}_B \circ f \circ \hat{\rho}_A^{-1})) \circ \hat{\delta}{}^{\leftarrow}_{C} = f^r.$$

(3) *If A is literate and B is letterless, then for every constant-free C there is an arrow term $f^r : A^r \vee C \vdash C$ of* **PN**$^\neg$ *such that in* $\mathbf{SA}'_s$ *we have*

$$\check{\sigma}^{\rightarrow}_{C} \circ ((\check{\rho}_B \circ f \circ \hat{\rho}^{-1}_A) \vee \mathbf{1}_C) = f^r.$$

Before we start the proof of this lemma, note that it is impossible that A and B be both letterless. Otherwise, we would have an arrow of the type $\top \vdash \bot$ in $\mathbf{SA}'_s$.

Proof of Lemma 3. By the Gentzenization Lemma and the Cut-Elimination Theorem of the preceding two sections, we may suppose that f is a cut-free Gentzen term. Then we proceed by induction on the complexity of f.

In the basis, we have that f can only be $\mathbf{1}_p : p \vdash p$. It cannot be $\mathbf{1}_\top : \top \vdash \top$ or $\mathbf{1}_\bot : \bot \vdash \bot$, because $\top$ is not $\vee$-nice and $\bot$ is not $\wedge$-nice. Then f^r is also $\mathbf{1}_p$, and statement (1) of the lemma is satisfied, since $\check{\rho}_p = \hat{\rho}^{-1}_p = \mathbf{1}_p$.

Suppose f is of the form Sf_1 for $f_1 : A_1 \vdash B_1$ and S being $\hat{B}^{\leftarrow}_X$, $\hat{B}^{\rightarrow}_X$, $\check{B}^{\rightarrow}_Y$, $\check{B}^{\leftarrow}_Y$, $\hat{C}_X$, $\check{C}_Y$, $\top^{\rightarrow}$, $\top^{\leftarrow}$, $\bot^{\leftarrow}$ or $\bot^{\rightarrow}$. Then, by the induction hypothesis, we have either statement (1), or (2), or (3), of the lemma for f replaced by f_1, and A and B replaced by A_1 and B_1.

Suppose (1) is the case for f_1. Then by Lemmata 1 and 2 above we have that

$$\check{\rho}_B \circ Sf_1 \circ \hat{\rho}^{-1}_A$$

is equal in $\mathbf{SA}'_s$ to one of the following arrow terms of $\mathbf{SA}'_s$:

$$\begin{aligned} &\check{\rho}_B \circ f_1 \circ \hat{\rho}^{-1}_{A_1} \circ g^r, && \text{for } B \text{ being } B_1,\\ &g^r \circ \check{\rho}_{B_1} \circ f_1 \circ \hat{\rho}^{-1}_A, && \text{for } A \text{ being } A_1,\\ &\check{\rho}_{B_1} \circ f_1 \circ \hat{\rho}^{-1}_{A_1}, \end{aligned}$$

where g^r is a $\overset{\xi}{b}{}^{\rightarrow}_{C,D,E}$-term, or $\overset{\xi}{b}{}^{\leftarrow}_{C,D,E}$-term, or $\overset{\xi}{c}_{C,D}$-term, with ξ being $\wedge$ in the first arrow term and $\vee$ in the second arrow term. In either case, by the induction hypothesis, we infer (1) for f.

If (2) is the case for f_1, then by Lemmata 1 and 2 we have that

$$(\mathbf{1}_C \wedge (\check{\rho}_B \circ Sf_1 \circ \hat{\rho}_A^{-1})) \circ \hat{\delta}^{\leftarrow}_C$$

is equal in $\mathbf{SA}'_s$ to one of the following arrow terms of $\mathbf{SA}'_s$:

$$(\mathbf{1}_C \wedge g^r) \circ (\mathbf{1}_C \wedge (\check{\rho}_{B_1} \circ f_1 \circ \hat{\rho}_A^{-1})) \circ \hat{\delta}^{\leftarrow}_C, \quad \text{for } A \text{ being } A_1,$$
$$(\mathbf{1}_C \wedge (\check{\rho}_{B_1} \circ f_1 \circ \hat{\rho}_{A_1}^{-1})) \circ \hat{\delta}^{\leftarrow}_C,$$

where g^r is as above with ξ being $\vee$. In either case, we apply the induction hypothesis, and infer (2) for f. We proceed analogously if (3) is the case for f_1.

Suppose that for $f_i \colon B_i \vdash A_i \vee C_i$, where $i \in \{1, 2\}$, we have that f is

$$\wedge(f_1, f_2) \colon B_1 \wedge B_2 \vdash (A_1 \wedge A_2) \vee (C_1 \vee C_2).$$

Here A_1 and A_2 must be constant-free; otherwise, the target of f would not be $\vee$-nice. Let g be

$$\check{\rho}_{(A_1 \wedge A_2) \vee (C_1 \vee C_2)} \circ \check{e}'_{A_1, A_2, C_1, C_2}.$$

Depending on whether C_i is literate or letterless, we have the following equations in $\mathbf{SA}'_s$:

if C_1 and C_2 are both literate, then

(I) $\qquad g = \check{e}'_{A_1, A_2, C_1^r, C_2^r} \circ (\check{\rho}_{A_1 \vee C_1} \wedge \check{\rho}_{A_2 \vee C_2});$

if C_1 is literate and C_2 is letterless, then

(II) $\qquad g = (\hat{c}_{A_2, A_1} \vee \mathbf{1}_{C_1^r}) \circ d_{A_2, A_1, C_1^r} \circ \hat{c}_{A_1 \vee C_1^r, A_2} \circ (\check{\rho}_{A_1 \vee C_1} \wedge \check{\rho}_{A_2 \vee C_2}),$
by applying essentially $(\check{b}\check{\delta})$ and $(d\check{\delta})$;

if C_1 is letterless and C_2 is literate, then

(III) $\qquad g = d_{A_1, A_2, C_2^r} \circ (\check{\rho}_{A_1 \vee C_1} \wedge \check{\rho}_{A_2 \vee C_2}),$

by applying essentially

$$(\mathbf{1}_{A_2 \wedge A_1} \vee \check{\sigma}^{\rightarrow}_{C_2}) \circ \check{b}^{\leftarrow}_{A_2 \wedge A_1, \bot, C_2} = \check{\delta}^{\rightarrow}_{A_2 \wedge A_1} \vee \mathbf{1}_{C_2}$$

(cf. the second equation displayed in the proof of Lemma 2) and $(d\check{\delta})$;

if C_1 and C_2 are both letterless, then

(IV) $\quad g = \check{\rho}_{A_1 \vee C_1} \wedge \check{\rho}_{A_2 \vee C_2}$, by applying essentially $(\check{b}\check{\delta})$ and $(d\check{\delta})$.

Suppose statement (1) of the lemma holds for both f_1 and f_2. Then B_1 and B_2 are literate, and we have

$$\begin{aligned}\check{\rho}_{(A_1 \wedge A_2) \vee (C_1 \vee C_2)} \circ \wedge(f_1, f_2) \circ \hat{\rho}^{-1}_{B_1 \wedge B_2} &= g \circ (f_1 \wedge f_2) \circ (\hat{\rho}^{-1}_{B_1} \wedge \hat{\rho}^{-1}_{B_2}) \\ &= h \circ (f_1^r \wedge f_2^r)\end{aligned}$$

for h a $\mathbf{PN}^{\neg}$-term; here we apply one of (I)-(IV) and the induction hypothesis. So (1) holds for f.

Suppose (1) holds for f_1 and (2) holds for f_2. Then B_1 is literate and B_2 is letterless, and we have

$$\begin{aligned}\check{\rho}_{(A_1 \wedge A_2) \vee (C_1 \vee C_2)} \circ \wedge(f_1, f_2) \circ \hat{\rho}^{-1}_{B_1 \wedge B_2} &= g \circ (f_1 \wedge f_2) \circ (\hat{\rho}^{-1}_{B_1} \wedge \hat{\rho}^{-1}_{B_2}) \circ \hat{\delta}^{\leftarrow}_{B_1^r} \\ &= h \circ (f_1^r \wedge \mathbf{1}_{(A_2 \vee C_2)^r}) \circ f_2^r\end{aligned}$$

for h a $\mathbf{PN}^{\neg}$-term; here we apply again one of (I)-(IV) and the induction hypothesis, which for f_2 yields

$$(\mathbf{1}_{B_1^r} \wedge (\check{\rho}_{A_2 \vee C_2} \circ f_2 \circ \hat{\rho}^{-1}_{B_2})) \circ \hat{\delta}^{\leftarrow}_{B_1^r} = f_2^r.$$

So (1) holds for f.

If (2) holds for f_1 and (1) holds for f_2, then we proceed analogously to what we had in the previous case. Here, the induction hypothesis for f_1 yields

$$((\check{\rho}_{A_1 \vee C_1} \circ f_1 \circ \hat{\rho}^{-1}_{B_1}) \wedge \mathbf{1}_{B_2^r}) \circ \hat{\sigma}^{\leftarrow}_{B_2^r} = \hat{c}_{B_2^r, (A_1 \vee C_1)^r} \circ f_1^r.$$

Suppose (2) holds for both f_1 and f_2. Then both B_1 and B_2 are letterless, and we have

$$\begin{aligned}&(\mathbf{1}_C \wedge (\check{\rho}_{(A_1 \wedge A_2) \vee (C_1 \vee C_2)} \circ \wedge(f_1, f_2) \circ \hat{\rho}^{-1}_{B_1 \wedge B_2})) \circ \hat{\delta}^{\leftarrow}_C \\ &\quad = (\mathbf{1}_C \wedge (g \circ (f_1 \wedge f_2) \circ (\hat{\rho}^{-1}_{B_1} \wedge \hat{\rho}^{-1}_{B_2}) \circ \hat{\delta}^{\leftarrow}_{\top})) \circ \hat{\delta}^{\leftarrow}_C \\ &\quad = (\mathbf{1}_C \wedge h) \circ (\mathbf{1}_C \wedge ((\mathbf{1}_{(A_1 \vee C_1)^r} \wedge (\check{\rho}_{A_2 \vee C_2} \circ f_2 \circ \hat{\rho}^{-1}_{B_2})) \circ \hat{\delta}^{\leftarrow}_{(A_1 \vee C_1)^r})) \circ \\ &\qquad\qquad \circ (\mathbf{1}_C \wedge (\check{\rho}_{A_1 \vee C_1} \circ f_1 \circ \hat{\rho}^{-1}_{B_1})) \circ \hat{\delta}^{\leftarrow}_C \\ &\quad = (\mathbf{1}_C \wedge h) \circ (\mathbf{1}_C \wedge f_2^r) \circ f_1^r,\end{aligned}$$

for h a $\mathbf{PN}^{\neg}$-term; here we apply again one of (I)-(IV) and the induction hypothesis. So (2) holds for f. Since statement (3) of the lemma cannot hold for f_1 and f_2, this exhausts all possible cases when f is $\wedge(f_1, f_2)$.

If f is $\vee(f_1, f_2)$, then we proceed in a manner dual to the case when f is $\wedge(f_1, f_2)$, relying on statement (3) of the lemma in places where previously we relied on statement (2).

Suppose that for $f_1 : C \wedge A \vdash B$ we have that f is $\neg^{\mathrm{R}} f_1 : C \vdash \neg A \vee B$. Here A must be constant-free; otherwise the target of f would not be $\vee$-nice. We derive first the following equations of $\mathbf{SA}'_s$:

$$\text{(V)} \qquad \check{c}_{\neg A, C\wedge A} \circ d_{C,A,\neg A} \circ \hat{\Delta}'_{A,C} \circ \hat{\rho}^{-1}_{C} = (\mathbf{1}_{\neg A} \vee (\hat{\rho}^{-1}_{C} \wedge \mathbf{1}_A)) \circ \check{c}_{\neg A, C^r\wedge A} \circ d_{C^r,A,\neg A} \circ \hat{\Delta}'_{A,C^r},$$

$$\text{(VI)} \qquad (\mathbf{1}_D \wedge (\check{c}_{\neg A, \top\wedge A} \circ d_{\top,A,\neg A} \circ \hat{\Delta}'_{A,\top})) \circ \hat{\delta}^{\leftarrow}_{D} = (\mathbf{1}_D \wedge (\mathbf{1}_{\neg A} \vee \hat{\sigma}^{\leftarrow}_{A})) \circ \hat{\Delta}_{A,D},$$

by applying naturality equations for (V), and essentially $(\hat{b}\hat{\Delta}')$ and $(d\,\hat{\sigma})$ for (VI).

We have to consider four cases:

(i) both B and C are literate,
(ii) B is literate and C is letterless,
(iii) B is letterless and C is literate,
(iv) both B and C are letterless.

In case (i), we obtain easily by using (V) and the induction hypothesis that statement (1) of the lemma holds for f.

In case (ii), for D constant-free we have

$$(\mathbf{1}_D \wedge (\check{\rho}_{\neg A\vee B} \circ \neg^{\mathrm{R}} f_1 \circ \hat{\rho}^{-1}_{C})) \circ \hat{\delta}^{\leftarrow}_{D} = (\mathbf{1}_D \wedge (\mathbf{1}_{\neg A} \vee (\check{\rho}_B \circ f_1 \circ (\hat{\rho}^{-1}_{C} \wedge \mathbf{1}_A) \circ \hat{\sigma}^{\leftarrow}_{A})) \circ \hat{\Delta}_{A,D},$$

by using (V) and (VI), and from that, by applying

$$(\hat{\rho}^{-1}_{C} \wedge \mathbf{1}_A) \circ \hat{\sigma}^{\leftarrow}_{A} = \hat{\rho}^{-1}_{C\wedge A}$$

and the induction hypothesis, we obtain that statement (2) of the lemma holds for f.

In case (iii), we obtain that statement (1) of the lemma holds for f by applying (V) and the induction hypothesis, which yields

$$\check{\sigma}^{\rightarrow}_{\neg A} \circ ((\check{\rho}_B \circ f_1 \circ \hat{\rho}^{-1}_{C\wedge A}) \vee \mathbf{1}_{\neg A}) = f_1^r.$$

In case (iv), we obtain that statement (2) of the lemma holds for f by applying (V), (VI) and the induction hypothesis.

The only remaining case is when f is $\neg^{\mathrm{L}} f_1$, and this is settled dually to the previous case, where f is $\neg^{\mathrm{R}} f_1$. ⊣

Chapter 5

Involutive Adjunctions and Proof-Net Categories

One finds in the notion of star-autonomous category the well-known adjunction of symmetric monoidal closed categories involving the tensor (multiplicative conjunction), denoted by $\wedge$ in §3.1, and exponentiation (linear implication), denoted by $\rightarrow$ in §3.1. The functor $A\,\wedge$ is left adjoint to the functor $A \rightarrow$ (see §3.1). In proof-net categories, the functor $A\,\wedge$ is left adjoint to the functor $\neg A\,\vee$, and the functor $\neg A\,\wedge$ is left adjoint to the functor $A\,\vee$ (see the end of §2.8).

There is also in proof-net categories, and hence also implicitly in star-autonomous categories, something generalizing the notion of adjunction, which involves dissociativity $d_{A,B,C} : A \wedge (B \vee C) \vdash (A \wedge B) \vee C$, and is perceived on another level, where the objects A and $\neg A$ are conceived as functors. We have remarked in §§2.2-3 that the equations $(\check{\Sigma}\hat{\Delta})$, $(\check{\Sigma}'\hat{\Delta}')$, $(\check{\Delta}'\hat{\Sigma}')$ and $(\check{\Delta}\hat{\Sigma})$ are related to the triangular equations of an adjunction.

The goal of this chapter is to show that there is in proof-net categories yet another phenomenon of adjunction. The assumptions of proof-net categories involving only negation are a particular, trivial, case of an adjoint situation that we call an *involutive adjunction*. The notion of involutive adjunction amounts, in a sense to be made precise, to adjunction where an endofunctor is adjoint to itself, which in [18] is called *self-adjunction*.

§5.1. Self-adjunctions

To fix notation and terminology, we will rely on the following definition of the notion of adjunction (cf. [38], Section IV.1, and [14], Section 4.1.3).

An *adjunction* is a sextuple $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ where

$\mathcal{A}$ and $\mathcal{B}$ are categories,

F from $\mathcal{B}$ to $\mathcal{A}$ and G from $\mathcal{A}$ to $\mathcal{B}$ are functors,

φ is a natural transformation of $\mathcal{A}$ from the composite functor FG to the identity functor of $\mathcal{A}$, which means that the following equation holds in $\mathcal{A}$ for every arrow $f\colon A_1 \vdash A_2$ of $\mathcal{A}$:

$$(\varphi\ nat) \qquad f \circ \varphi_{A_1} = \varphi_{A_2} \circ FGf,$$

γ is a natural transformation of $\mathcal{B}$ from the identity functor of $\mathcal{B}$ to the composite functor GF, which means that the following equation holds in $\mathcal{B}$ for every arrow $g\colon B_1 \vdash B_2$ of $\mathcal{B}$:

$$(\gamma\ nat) \qquad GFg \circ \gamma_{B_1} = \gamma_{B_2} \circ g,$$

the following *triangular* equations hold in $\mathcal{A}$ and $\mathcal{B}$ respectively:

$$(\varphi\gamma F) \qquad \varphi_{FB} \circ F\gamma_B = \mathbf{1}_{FB},$$

$$(\varphi\gamma G) \qquad G\varphi_A \circ \gamma_{GA} = \mathbf{1}_{GA}.$$

A *self-adjunction* is a quadruple $\langle \mathcal{S}, L, \varphi, \gamma \rangle$ where $\langle \mathcal{S}, \mathcal{S}, L, L, \varphi, \gamma \rangle$ is an adjunction (this notion is taken over from [18], Section 10). So, in a self-adjunction, L is an endofunctor, and the equations (φ *nat*) and (γ *nat*) become

$$f \circ \varphi_{A_1} = \varphi_{A_2} \circ LLf,$$

$$LLf \circ \gamma_{A_1} = \gamma_{A_2} \circ f,$$

while the triangular equations become

$$(\varphi\gamma L) \qquad \varphi_{LA} \circ L\gamma_A = L\varphi_A \circ \gamma_{LA} = \mathbf{1}_{LA}.$$

A $\mathcal{K}$-*self-adjunction* is a self-adjunction that satisfies the additional equation

$$(\varphi\gamma\mathcal{K}) \qquad L(\varphi_A \circ \gamma_A) = \varphi_{LA} \circ \gamma_{LA},$$

and a $\mathcal{J}$*-self-adjunction* is a self-adjunction that satisfies the additional equation

$$(\varphi\gamma\mathcal{J}) \qquad \varphi_A \circ \gamma_A = \mathbf{1}_A$$

(these notions are also from [18], Section 10). It is easy to see that every $\mathcal{J}$-self-adjunction is a $\mathcal{K}$-self-adjunction (the converse need not hold).

A $\mathcal{J}$-self-adjunction that satisfies

$$(\gamma\varphi) \qquad \gamma_A \circ \varphi_A = \mathbf{1}_{LLA}$$

is called a *trivial* self-adjunction. Note that for trivial self-adjunctions it is superfluous to assume the equations (γ *nat*) and ($\varphi\gamma G$), or alternatively (φ *nat*) and ($\varphi\gamma F$); these equations can be derived from the remaining ones.

The *free self-adjunction* $\langle \mathcal{S}, L, \varphi, \gamma\rangle$ generated by $\{p\}$ (we call p a *letter*, as before) is defined as follows. The category $\mathcal{S}$ has as objects the formulae of the propositional language generated by $\{p\}$ with a unary connective L. We may identify the formulae p, Lp, LLp,... of this language with the natural numbers 0, 1, 2,...

The arrow terms of $\mathcal{S}$ are defined inductively out of the primitive arrow terms

$$\mathbf{1}_A : A \vdash A, \qquad \varphi_A : LLA \vdash A, \qquad \gamma_A : A \vdash LLA,$$

for every object A of $\mathcal{S}$, with the help of the operations of composition $\circ$ and the unary operation that assigns to the arrow term $f : A \vdash B$ the arrow term $Lf : LA \vdash LB$. On these arrow terms we impose the equations of self-adjunctions (cf. §2.1). In the set of these equations we have of course all the equations $f = f$, and this set is closed under symmetry and transitivity of equality, under the rule (*cong* ξ) for ξ being $\circ$ (see §2.1), and also under the rule

$$(cong\ L) \qquad \frac{f = g}{Lf = Lg}$$

We define analogously the free $\mathcal{K}$-self-adjunction, the free $\mathcal{J}$-self-adjunction and the free trivial self-adjunction generated by $\{p\}$, just by imposing additional equations.

§5.2. Involutive adjunctions

Consider a category $\mathcal{A}$ and a contravariant functor $\neg$ from $\mathcal{A}$ to $\mathcal{A}$, which means that for $f\colon A \vdash B$ in $\mathcal{A}$ we have $\neg f\colon \neg B \vdash \neg A$ in $\mathcal{A}$, and the equations $(\neg 1)$ and $(\neg 2)$ of §2.8 are satisfied. The contravariant functor $\neg$ may be conceived either as a functor from the category $\mathcal{A}^{op}$ to $\mathcal{A}$, which we denote by $\neg$ too, or as a functor from $\mathcal{A}$ to $\mathcal{A}^{op}$, which we denote by $\neg^{op}$.

Suppose that for every object A of $\mathcal{A}$ we have an arrow $n^{\rightarrow}_A\colon \neg\neg A \vdash A$ of $\mathcal{A}$. The arrow $n^{\rightarrow}_A$ becomes the arrow $n^{\rightarrow\, op}_A\colon A \vdash \neg\neg A$ in $\mathcal{A}^{op}$.

We say that $\langle \mathcal{A}, \neg, n^{\rightarrow}\rangle$ is an $n^{\rightarrow}$-*adjunction* when

$$\langle \mathcal{A}, \mathcal{A}^{op}, \neg, \neg^{op}, n^{\rightarrow}, n^{\rightarrow\, op}\rangle$$

is an adjunction. This means that in $\mathcal{A}$ we have for every $f\colon A_1 \vdash A_2$ the equation

$$(n^{\rightarrow}\ nat) \qquad f \circ n^{\rightarrow}_{A_1} = n^{\rightarrow}_{A_2} \circ \neg\neg f,$$

alternatively written $f \circ n^{\rightarrow}_{A_1} = n^{\rightarrow}_{A_2} \circ \neg\neg^{op} f$, which also delivers $(n^{\rightarrow\, op}\ nat)$ in $\mathcal{A}^{op}$, and the equation

$$(n^{\rightarrow}\ triang) \qquad n^{\rightarrow}_{\neg A} \circ \neg n^{\rightarrow}_A = \mathbf{1}_{\neg A},$$

which delivers both the equation $(\varphi\gamma F)$, i.e. $(n^{\rightarrow} n^{\rightarrow\, op}\ \neg)$, in $\mathcal{A}$, and the equation $(\varphi\gamma G)$, i.e. $(n^{\rightarrow} n^{\rightarrow\, op}\ \neg^{op})$, in $\mathcal{A}^{op}$.

Suppose now that we have as before a category $\mathcal{A}$ and a contravariant functor $\neg$ from $\mathcal{A}$ to $\mathcal{A}$, and that for every object A of $\mathcal{A}$ we have an arrow $n^{\leftarrow}_A\colon A \vdash \neg\neg A$ of $\mathcal{A}$. The arrow $n^{\leftarrow}_A$ becomes the arrow $n^{\leftarrow\, op}_A\colon \neg\neg A \vdash A$ in $\mathcal{A}^{op}$.

We say that $\langle \mathcal{A}, \neg, n^{\leftarrow}\rangle$ is an $n^{\leftarrow}$-*adjunction* when

$$\langle \mathcal{A}^{op}, \mathcal{A}, \neg^{op}, \neg, n^{\leftarrow\, op}, n^{\leftarrow}\rangle$$

is an adjunction. This means that in $\mathcal{A}$ we have for every $f\colon A_1 \vdash A_2$ the equation

$$(n^{\leftarrow}\ nat) \qquad \neg\neg f \circ n^{\leftarrow}_{A_1} = n^{\leftarrow}_{A_2} \circ f,$$

which also delivers $(n^{\leftarrow\, op}\ nat)$ in $\mathcal{A}^{op}$, and the equation

$$(n^{\leftarrow}\ \mathit{triang}) \qquad \neg n^{\leftarrow}_{A} \circ n^{\leftarrow}_{\neg A} = \mathbf{1}_{\neg A},$$

which delivers both the equation $(\varphi\gamma F)$, i.e. $(n^{\leftarrow\, op} n^{\leftarrow}\, \neg^{op})$, in $\mathcal{A}^{op}$, and the equation $(\varphi\gamma G)$, i.e. $(n^{\leftarrow\, op} n^{\leftarrow}\, \neg)$, in $\mathcal{A}$. Note that what we call $n^{\leftarrow}$-adjunction is called *self-adjunction* in [40] (Section 3.1; cf. also [39], Section I.8), which should not be confused with our notion of self-adjunction in the preceding section.

We say that $\langle \mathcal{A}, \neg, n^{\rightarrow}, n^{\leftarrow}\rangle$ is an *involutive* adjunction when $\langle \mathcal{A}, \neg, n^{\rightarrow}\rangle$ is an $n^{\rightarrow}$-adjunction and $\langle \mathcal{A}, \neg, n^{\leftarrow}\rangle$ is an $n^{\leftarrow}$-adjunction.

A $\mathcal{K}$-*involutive* adjunction is an involutive adjunction that satisfies the additional equation

$$(n^{\rightarrow} n^{\leftarrow} \mathcal{K}) \qquad \neg(n^{\rightarrow}_{A} \circ n^{\leftarrow}_{A}) = n^{\rightarrow}_{\neg A} \circ n^{\leftarrow}_{\neg A},$$

and a $\mathcal{J}$-*involutive* adjunction is an involutive adjunction that satisfies the additional equation

$$(n^{\rightarrow} n^{\leftarrow} \mathcal{J}) \qquad n^{\rightarrow}_{A} \circ n^{\leftarrow}_{A} = \mathbf{1}_{A}.$$

It is easy to see that every $\mathcal{J}$-involutive adjunction is a $\mathcal{K}$-involutive adjunction (the converse need not hold).

A $\mathcal{J}$-involutive adjunction that satisfies

$$(n^{\leftarrow} n^{\rightarrow}) \qquad n^{\leftarrow}_{A} \circ n^{\rightarrow}_{A} = \mathbf{1}_{\neg\neg A}$$

is called a *trivial* involutive adjunction.

Note that for trivial involutive adjunctions it is superfluous to assume the equations $(n^{\leftarrow}\ \mathit{nat})$ and $(n^{\leftarrow}\ \mathit{triang})$, or alternatively $(n^{\rightarrow}\ \mathit{nat})$ and $(n^{\rightarrow}\ \mathit{triang})$; these equations can be derived from the remaining ones. In trivial involutive adjunctions we have the equations

$$n^{\leftarrow}_{\neg A} = \neg n^{\rightarrow}_{A},$$
$$n^{\rightarrow}_{\neg A} = \neg n^{\leftarrow}_{A},$$

which should be compared with the equations $(\nu^{\leftarrow}_{A\to\bot,\bot})$ and $(\nu^{\rightarrow}_{A\to\bot})$ of §3.4.

The *free involutive adjunction* $\langle \mathcal{A}, \neg, n^{\rightarrow}, n^{\leftarrow} \rangle$ generated by $\{p\}$ is defined as follows. The category $\mathcal{A}$ has as objects the formulae of the propositional language generated by $\{p\}$ with a unary connective $\neg$. We may identify these formulae with the natural numbers.

The arrow terms of $\mathcal{A}$ are defined inductively out of the primitive arrow terms

$$\mathbf{1}_A : A \vdash A, \qquad n^{\rightarrow}_A : \neg\neg A \vdash A, \qquad n^{\leftarrow}_A : A \vdash \neg\neg A,$$

for every object A of $\mathcal{A}$, with the help of the operations of composition $\circ$ and the unary operation that assigns to the arrow term $f : A \vdash B$ the arrow term $\neg f : \neg B \vdash \neg A$. On these arrow terms we impose the equations of involutive adjunctions (cf. §2.1 and the preceding section). In the set of these equations we have of course all the equations $f = f$, and this set is closed under symmetry and transitivity of equality, under the rule (*cong* ξ) for ξ being $\circ$ (see §2.1), and also under the rule

$$(cong\ \neg) \qquad \frac{f = g}{\neg f = \neg g}$$

We define analogously the free $\mathcal{K}$-involutive adjunction, the free $\mathcal{J}$-involutive adjunction and the free trivial involutive adjunction generated by $\{p\}$, just by imposing additional equations.

Note that the category of the free involutive adjunction generated by an arbitrary set having more than one letter would be the disjoint union of isomorphic copies of the category $\mathcal{A}$ of the free involutive adjunction generated by $\{p\}$. An analogous remark applies to the category of the free self-adjunction generated by an arbitrary set having more than one member: it would be the disjoint union of isomorphic copies of the category $\mathcal{S}$ of the free self-adjunction generated by $\{p\}$.

§5.3. Self-adjunctions and involutive adjunctions

We are now going to prove that in the free self-adjunction $\langle \mathcal{S}, L, \varphi, \gamma \rangle$ and the free involutive adjunction $\langle \mathcal{A}, \neg, n^{\rightarrow}, n^{\leftarrow} \rangle$, both generated by $\{p\}$, the categories $\mathcal{S}$ and $\mathcal{A}$ are isomorphic categories.

First, we define $\neg$, $n^{\rightarrow}$ and $n^{\leftarrow}$ in $\mathcal{S}$ in the following manner. On objects we have that $\neg$ is L, while for the arrow term $f\colon A \vdash B$ of $\mathcal{S}$ we define the arrow term $\neg f\colon \neg B \vdash \neg A$ of $\mathcal{S}$ inductively as follows:

$$\begin{aligned}
&\neg \mathbf{1}_A = L\mathbf{1}_A = \mathbf{1}_{LA} = \mathbf{1}_{\neg A},\\
&\neg \varphi_A = L\gamma_A,\\
&\neg \gamma_A = L\varphi_A,\\
&\neg (f \circ g) = \neg g \circ \neg f,\\
&\neg Lf = L\neg f.
\end{aligned}$$

That this defines an operation $\neg$ on the arrows of $\mathcal{S}$ is shown by verifying that if $f = g$ in $\mathcal{S}$, then $\neg f = \neg g$ in $\mathcal{S}$; we verify, namely, that the equations of $\mathcal{S}$ are closed under the rule $(cong\ \neg)$ of the preceding section. This is done by a straightforward induction on the length of the derivation of $f = g$ in $\mathcal{S}$. For that we use the fact that for every arrow term f of $\mathcal{S}$ the arrow term $\neg f$ is equal in $\mathcal{S}$ to an arrow term of the form Lf'.

Finally, we have

$$n^{\rightarrow}_A =_{df} \varphi_A, \qquad\qquad n^{\leftarrow}_A =_{df} \gamma_A.$$

Next, we define L, φ and γ in $\mathcal{A}$ in the following manner. On objects we have that L is $\neg$, while for the arrow term $f\colon A \vdash B$ of $\mathcal{A}$ we define the arrow term $Lf\colon LA \vdash LB$ of $\mathcal{A}$ inductively as follows:

$$\begin{aligned}
&L\mathbf{1}_A = \neg \mathbf{1}_A = \mathbf{1}_{\neg A} = \mathbf{1}_{LA},\\
&Ln^{\rightarrow}_A = \neg n^{\leftarrow}_A,\\
&Ln^{\leftarrow}_A = \neg n^{\rightarrow}_A,\\
&L(f \circ g) = Lf \circ Lg,\\
&L\neg f = \neg Lf.
\end{aligned}$$

That this defines an operation L on the arrows of $\mathcal{A}$ is shown by verifying that if $f = g$ in $\mathcal{A}$, then $Lf = Lg$ in $\mathcal{A}$; we verify, namely, that the equations of $\mathcal{A}$ are closed under the rule $(cong\ L)$ of §5.1. This is done by a straightforward induction on the length of the derivation of $f = g$ in $\mathcal{A}$. For that we use the fact that for every arrow term f of $\mathcal{A}$ the arrow term Lf is equal in $\mathcal{A}$ to an arrow term of the form $\neg f'$.

Finally, we have

$$\varphi_A =_{df} n^{\rightarrow}_A, \qquad\qquad \gamma_A =_{df} n^{\leftarrow}_A.$$

We verify easily by induction on the complexity of the arrow term f that both in $\mathcal{S}$ and in $\mathcal{A}$ we have the equation

$$(LL\neg\neg) \qquad LLf = \neg\neg f.$$

Next we verify that the equations of involutive adjunctions hold for the defined $\neg$, $n^{\rightarrow}$ and $n^{\leftarrow}$ in $\mathcal{S}$. This is done in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use $(LL\neg\neg)$, $(\varphi\ nat)$ and $(\gamma\ nat)$ to verify $(n^{\rightarrow}\ nat)$ and $(n^{\leftarrow}\ nat)$, while the equations $(n^{\rightarrow}\ triang)$ and $(n^{\leftarrow}\ triang)$ reduce to $(\varphi\gamma L)$. In the induction step, we rely on the closure of $\mathcal{S}$ under $(cong\ \neg)$, which we established above.

We verify also that the equations of self-adjunctions hold for the defined L, φ and γ in $\mathcal{A}$. This is done again in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use $(LL\neg\neg)$, $(n^{\rightarrow}\ nat)$ and $(n^{\leftarrow}\ nat)$ to verify $(\varphi\ nat)$ and $(\gamma\ nat)$, while the equations $(\varphi\gamma L)$ reduce to $(n^{\rightarrow}\ triang)$ and $(n^{\leftarrow}\ triang)$. In the induction step, we rely on the closure of $\mathcal{A}$ under $(cong\ L)$, which we established above.

We have a functor $F_{\mathcal{A}}$ from $\mathcal{S}$ to $\mathcal{A}$ that maps the object of $\mathcal{S}$ corresponding to the natural number n to the object of $\mathcal{A}$ corresponding to n, and that maps every arrow of $\mathcal{S}$ to the homonymous arrow in the defined $\mathcal{S}$ structure of $\mathcal{A}$. For example,

$$F_{\mathcal{A}}\,\varphi_{LLp} = \varphi_{\neg\neg p} = n^{\rightarrow}_{\neg\neg p}.$$

We define analogously a functor $F_{\mathcal{S}}$ from $\mathcal{A}$ to $\mathcal{S}$ (cf. the functors F and F' in §3.7). That $F_{\mathcal{A}}$ and $F_{\mathcal{S}}$ are indeed functors follows from what we established above.

It is trivial that on objects we have that $F_{\mathcal{S}}F_{\mathcal{A}}A$ is A, and that $F_{\mathcal{A}}F_{\mathcal{S}}B$ is B. We show next by induction on the complexity of f that in $\mathcal{S}$ we have

$$F_{\mathcal{S}}F_{\mathcal{A}}\,f = f.$$

When f is of the form Lf', we make an auxiliary induction on the complexity of f', in which we use $(LL\neg\neg)$. We show analogously that in $\mathcal{A}$ we have

$$F_{\mathcal{A}}F_{\mathcal{S}}\, g = g.$$

This concludes the proof that $\mathcal{S}$ and $\mathcal{A}$ are isomorphic categories.

We demonstrate analogously that the categories of, respectively,

> the free $\mathcal{K}$-self-adjunction and the free $\mathcal{K}$-involutive adjunction,
>
> the free $\mathcal{J}$-self-adjunction and the free $\mathcal{J}$-involutive adjunction,
>
> the free trivial self-adjunction and the free trivial involutive adjunction,

all generated by $\{p\}$, are isomorphic categories.

The interest of considering $\mathcal{K}$ and $\mathcal{J}$ versions of self-adjunctions and involutive adjunctions comes from connections with Temperley-Lieb algebras and the associated geometrical interpretation (see [18] and references therein). Roughly speaking, $\mathcal{K}$ is what we find in Temperley-Lieb algebras, where only the number of circles (which correspond to $\varphi_A \circ \gamma_A$ or $n^{\rightarrow}_A \circ n^{\leftarrow}_A$) counts, while in $\mathcal{J}$ circles are disregarded. (How these circles arise may be grasped from the first diagram in §2.3, where there is a circle involving 7 and 8.)

The free trivial self-adjunction, and hence also the free trivial involutive adjunction, are preorders; namely, all arrows with the same source and target are equal. This follows from the results of [18] (unabridged version) or [19].

§5.4. Trivial involutive adjunctions and proof-net categories

In every proof-net category we encounter a trivial involutive adjunction, where $\neg$ is defined as in §2.8, while $n^{\rightarrow}$ and $n^{\leftarrow}$ are defined as in §2.6. That all the equations of trivial involutive adjunctions are satisfied with these definitions in proof-net categories is easily verified with what we have in §2.8, naturality equations and $\mathbf{PN}^{\neg}$ Coherence. According to what we established in the preceding section, in every proof-net category we have a subcategory that is a trivial self-adjunction. This does not mean, however, that in every proof-net category, and in $\mathbf{PN}^{\neg}$ in particular, we can define the endofunctor L of the trivial self-adjunction.

The notion of star-autonomous category arises out of the notion of symmetric monoidal closed category by assuming in addition the arrows $\nu^{\rightarrow}_{A} : (A \rightarrow \bot) \rightarrow \bot \vdash A$ and the isomorphism equations $(\nu\nu)$ that tie these arrows to the arrows $\nu^{\leftarrow}_{A,\bot} : A \vdash (A \rightarrow \bot) \rightarrow \bot$ of the symmetric monoidal closed structure (see §3.2). For every symmetric monoidal closed category $\mathcal{A}$ we have that $\langle \mathcal{A}, \_\!\_ \rightarrow \bot, \nu^{\leftarrow}_{\_\_,\bot} \rangle$ is an $n^{\leftarrow}$-adjunction. With $\nu^{\rightarrow}$ added together with the equations $(\nu\nu)$, we obtain a trivial involutive adjunction (see §5.2).

A non-equational definition of star-autonomous category is obtained by assuming instead of the arrows $\nu^{\rightarrow}_{A}$ and the equations $(\nu\nu)$ just that A and $(A \rightarrow \bot) \rightarrow \bot$ are naturally isomorphic. That $\nu^{\leftarrow}_{A,\bot}$ is an isomorphism follows then from a lemma in [30] (Lemma 1.3; see also [29], Section A1.1, Lemma 1.1.1) and the fact that $\langle \mathcal{A}, \_\!\_ \rightarrow \bot, \nu^{\leftarrow}_{\_\_,\bot} \rangle$ is an $n^{\leftarrow}$-adjunction. If $i_A : (A \rightarrow \bot) \rightarrow \bot \vdash A$ is a member of a natural isomorphism, then the inverse of $\nu^{\leftarrow}_{A,\bot} : A \vdash (A \rightarrow \bot) \rightarrow \bot$ is

$$i_A \circ (((i_A^{-1} \rightarrow \mathbf{1}_{\bot}) \circ \nu^{\leftarrow}_{A\rightarrow\bot,\bot}) \rightarrow \mathbf{1}_{\bot}) : (A \rightarrow \bot) \rightarrow \bot \vdash A.$$

Since i is a natural isomorphism, we have $i_{(A\rightarrow\bot)\rightarrow\bot} = (i_A \rightarrow \mathbf{1}_{\bot}) \rightarrow \mathbf{1}_{\bot}$.

Chapter 6

Coherence of Mix-Proof-Net Categories

In this chapter we add *mix* arrows of the type $A \wedge B \vdash A \vee B$ to proof-net categories, with appropriate conditions that will enable us to prove coherence with respect to *Br* for the resulting categories, which we call mix-proof-net categories. The mix arrows, which underly the mix principle of linear logic, were treated extensively in [22] (Chapters 8, 10, 11, 13). The proof of coherence for mix-proof-net categories is an adaptation of the proof of coherence for proof-net categories given in Chapter 2.

§6.1. The category MDS

The category **MDS** is defined as the category **DS** in §2.1 save that we have the additional primitive arrow terms

$$m_{A,B} \colon A \wedge B \vdash A \vee B$$

for all objects, i.e. for all formulae, A and B of $\mathcal{L}_{\wedge,\vee}$, and we assume the following additional equations:

$(m\ nat)$ $\quad (f \vee g) \circ m_{A,B} = m_{D,E} \circ (f \wedge g), \quad$ for $f \colon A \vdash D$ and $g \colon B \vdash E$,

$(\hat{b}\, m)$ $\quad m_{A\wedge B,C} \circ \hat{b}^{\rightarrow}_{A,B,C} = d_{A,B,C} \circ (\mathbf{1}_A \wedge m_{B,C}),$

$(\check{b}\, m)$ $\quad \check{b}^{\rightarrow}_{C,B,A} \circ m_{C,B\vee A} = (m_{C,B} \vee \mathbf{1}_A) \circ d_{C,B,A},$

(cm) $\quad m_{B,A} \circ \hat{c}_{A,B} = \check{c}_{B,A} \circ m_{A,B}.$

The proof-theoretical principle underlying $m_{A,B}$ is called *mix* (see [22], Section 8.1, and references therein).

To obtain the functor G from **MDS** to *Br*, we extend the definition of the functor G from **DS** to *Br* (see §2.3) by adding the clause that says that $Gm_{A,B}$ is the identity arrow $\mathbf{1}_{GA+GB}$ of *Br*. Then we have the following result of [22] (Section 8.4).

MDS COHERENCE. *The functor G from* **MDS** *to Br is faithful.*

In the remainder of this section we will prove some lemmata concerning **MDS**, which we will use for the proof of coherence in the next section. For that we need some preliminaries.

For x a particular proper subformula of a formula A of $\mathcal{L}_{\wedge,\vee}$, and $\xi \in \{\wedge,\vee\}$, we define A^{-x} inductively as follows:

$$(B\,\xi\,x)^{-x} = (x\,\xi\,B)^{-x} = B,$$

for x a proper subformula of C,

$$(B\,\xi\,C)^{-x} = B\,\xi\,C^{-x},$$
$$(C\,\xi\,B)^{-x} = C^{-x}\,\xi\,B.$$

For $i \in \{1,2\}$, let A_i be a formula of $\mathcal{L}_{\wedge,\vee}$ with a proper subformula x_i, which is an occurrence of a letter q, and let x_i be the n_i-th occurrence of letter counting from the left. We define the following functions $\mu_i\colon \boldsymbol{N} - \{n_i - 1\} \rightarrow \boldsymbol{N}$:

$$\mu_i(n) =_{df} \begin{cases} n & \text{if } n < n_i - 1 \\ n-1 & \text{if } n > n_i - 1. \end{cases}$$

The definition of *tied* occurrence of a letter in an arrow of **MDS** is analogous to what we had in §2.4. Then we can prove the following.

LEMMA 1. *For every arrow term $f\colon A_1 \vdash A_2$ of* **MDS** *such that x_1 and x_2 are tied in the arrow f, there is an arrow term $f^{-q}\colon A_1^{-x_1} \vdash A_2^{-x_2}$ of* **MDS** *such that the members of* $\mathrm{part}(Gf^{-q})$ *are $\{s(\mu_1(m_1)), t(\mu_2(m_2))\}$ for each $\{s(m_1), t(m_2)\}$ in* $\mathrm{part}(Gf)$*, provided $m_i \neq n_i - 1$.*

PROOF. We proceed by induction on the complexity of the arrow term f. If f is a primitive arrow term $\alpha_{B_1,\ldots,B_m}$, then for some $j \in \{1,\ldots,m\}$ we have that x_i occurs in a subformula B_j of A_i. If x_i is a proper subformula of this subformula B_j, then $B_j^{-x_i}$ is defined, and f^{-q} is

$$\alpha_{B_1,\dots,B_{j-1},B_j^{-x_i},B_{j+1},\dots,B_m}$$

(note that $B_j^{-x_1}$ and $B_j^{-x_2}$ are the same formula). If x_i is not a proper subformula of the subformula B_j, then $d^{-q}_{B_1,q,B_3}$ is m_{B_1,B_3} or f^{-q} is $\mathbf{1}_{A_i^{-x_i}}$.

If f is $g \circ h$, then f^{-q} is $g^{-q} \circ h^{-q}$, and if f is $g \,\xi\, h$ for $\xi \in \{\wedge, \vee\}$, then f^{-q} is either $g^{-q} \,\xi\, h$, or $g \,\xi\, h^{-q}$, or g when $h = \mathbf{1}_{x_1}$, or h when $g = \mathbf{1}_{x_1}$. ⊣

Note that this lemma does not hold for **DS**, because we cannot cover $d^{-q}_{B_1,q,B_3}$.

Here is an example of the application of Lemma 1. If $f\colon A_1 \vdash A_2$ is

$$((m_{q,p\wedge q} \circ (\mathbf{1}_q \wedge \hat{c}_{q,p}) \circ \hat{c}_{q\wedge p,q}) \vee \mathbf{1}_p) \circ d_{q\wedge p,q,p} \circ \hat{b}^{\rightarrow}_{q,p,q\vee p} \circ \hat{c}_{p\wedge(q\vee p),q}\colon$$
$$(p \wedge (q \vee p)) \wedge q \vdash (q \vee (p \wedge q)) \vee p,$$

where x_1 is the second (rightmost) occurrence of q in $(p\wedge(q\vee p))\wedge q$, while x_2 is the second occurrence of q in $(q\vee(p\wedge q))\vee p$, then $f^{-q}\colon A_1^{-q} \vdash A_2^{-q}$ is

$$((m_{q,p} \circ (\mathbf{1}_q \wedge \mathbf{1}_p) \circ \hat{c}_{p,q}) \vee \mathbf{1}_p) \circ d_{p,q,p} \circ \mathbf{1}_{p\wedge(q\vee p)} \circ \mathbf{1}_{p\wedge(q\vee p)} :$$
$$p \wedge (q \vee p) \vdash (q \vee p) \vee p,$$

which is equal to $((m_{q,p} \circ \hat{c}_{p,q}) \vee \mathbf{1}_p) \circ d_{p,q,p}$. As another example, we have that $(((m_{q,p} \circ \hat{c}_{p,q}) \vee \mathbf{1}_p) \circ d_{p,q,p})^{-q}$ is equal to $m_{p,p}$.

We define inductively a notion we call a *context* (analogous up to point to notions introduced in §4.1):

> $\Box$ is a context;
>
> if Z is a context and A a formula of $\mathcal{L}_{\wedge,\vee}$, then $Z \,\xi\, A$ and $A \,\xi\, Z$ are contexts for $\xi \in \{\wedge, \vee\}$.

Note that now we have contexts like $p \wedge (q \vee \Box)$, which are neither $\wedge$-contexts nor $\vee$-contexts in the sense of §4.1. We define $Z(B)$ and $Z(f)$ as in §4.1, and we use $X, Y, Z, \dots$ for contexts.

For $f\colon A \vdash C$ an arrow of **MDS**, we say that an occurrence x of a formula B as a subformula of A and an occurrence y of the same formula

B as a subformula of C are *tied* in f when the n-th letter in x is tied in f to the n-th letter in y.

Let $f\colon X(p)\wedge B\vdash Y(p\wedge B)$ be an arrow term of **MDS** such that the displayed occurrences of p in the source and target, and also the displayed occurrences of B, are tied in the arrow f. Then, by successive applications of Lemma 1, for each occurrence of a letter in B, we obtain the arrow term $f^{-B}\colon X(p)\vdash Y(p)$ of **MDS**, and the displayed occurrences of p in $X(p)$ and $Y(p)$ are tied in the arrow f^{-B}.

Let $f^{\dagger}\colon X(p\wedge B)\vdash Y(p\wedge B)$ be the arrow term of **MDS** obtained from f^{-B} by replacing the occurrences of p that correspond to those displayed in $X(p)$ and $Y(p)$ by occurrences of $p\wedge B$. This replacement is made in the indices of primitive arrow terms that occur in f^{-B}, and it need not involve all the occurrences of p in these indices. For example, if X is $\Box\wedge(q\vee p)$ and Y is $(q\vee\Box)\vee p$, while f^{-B} is

$$((m_{q,p}\circ \hat{c}_{p,q})\vee \mathbf{1}_p)\circ d_{p,q,p}\colon p\wedge(q\vee p)\vdash (q\vee p)\vee p,$$

then $f^{\dagger}$ is

$$((m_{q,p\wedge B}\circ \hat{c}_{p\wedge B,q})\vee \mathbf{1}_p)\circ d_{p\wedge B,q,p}\colon (p\wedge B)\wedge(q\vee p)\vdash (q\vee (p\wedge B))\vee p.$$

Then we can prove the following.

LEMMA 2$\wedge$. *Let $f\colon X(p)\wedge B\vdash Y(p\wedge B)$ and $f^{\dagger}\colon X(p\wedge B)\vdash Y(p\wedge B)$ be as above. Then there is an arrow term $h_X\colon X(p)\wedge B\vdash X(p\wedge B)$ of* **DS** *such that $f=f^{\dagger}\circ h_X$ in* **MDS**.

PROOF. We construct the arrow term h_X of **DS** by induction on the complexity of the context X. For the basis we have that $h_{\Box}$ is $\mathbf{1}_{p\wedge B}$. In the induction step we have

$$\begin{aligned}
h_{Z\wedge A} &= (h_Z\wedge \mathbf{1}_A)\circ \hat{c}_{A,Z(p)\wedge B}\circ \hat{b}^{\leftarrow}_{A,Z(p),B}\circ(\hat{c}_{Z(p),A}\wedge \mathbf{1}_B),\\
h_{Z\vee A} &= (h_Z\vee \mathbf{1}_A)\circ \check{c}_{Z(p)\wedge B,A}\circ d^{R}_{A,Z(p),B}\circ(\check{c}_{A,Z(p)}\wedge \mathbf{1}_B),\\
h_{A\wedge Z} &= (\mathbf{1}_A\wedge h_Z)\circ \hat{b}^{\leftarrow}_{A,Z(p),B},\\
h_{A\vee Z} &= (\mathbf{1}_A\vee h_Z)\circ d^{R}_{A,Z(p),B}.
\end{aligned}$$

It is easy to see that $Gf=G(f^{\dagger}\circ h_X)$, and then the lemma follows by applying **MDS** Coherence. $\dashv$

Let $f\colon Y(B\vee p)\vdash B\vee X(p)$ be an arrow term of **MDS** such that the displayed occurrences of p in the source and target, and also the displayed occurrences of B, are tied in the arrow f. Then, as above by Lemma 1, we obtain the arrow term $f^{-B}\colon Y(p)\vdash X(p)$ of **MDS**, and the displayed occurrences of p in $Y(p)$ and $X(p)$ are tied in the arrow f^{-B}.

Let $f^\dagger\colon Y(B\vee p)\vdash X(B\vee p)$ be the arrow term of **MDS** obtained from f^{-B} by replacing the occurrences of p that correspond to those displayed in $Y(p)$ and $X(p)$ by occurrences of $B\vee p$ (cf. the example above). Then we can prove the following, analogously to Lemma 2$\wedge$.

LEMMA 2$\vee$. *Let $f\colon Y(B\vee p)\vdash B\vee X(p)$ and $f^\dagger\colon Y(B\vee p)\vdash X(B\vee p)$ be as above. Then there is an arrow term $h_X\colon X(B\vee p)\vdash B\vee X(p)$ of* **DS** *such that $f=h_X\circ f^\dagger$ in* **MDS**.

§6.2. MPN$^\neg$ Coherence

The category **MPN**$^\neg$ is defined as the category **PN**$^\neg$ in §2.2 save that we have the additional primitive arrow terms $m_{A,B}\colon A\wedge B\vdash A\vee B$ for all objects A and B of **PN**$^\neg$, and we assume as additional equations (m *nat*), ($\hat{b}\,m$), ($\check{b}\,m$) and (cm) of the preceding section. To obtain the functor G from **MPN**$^\neg$ to *Br*, we extend the definition of the functor G from **PN**$^\neg$ to *Br* by adding the clause that says that $Gm_{A,B}$ is the identity arrow $\mathbf{1}_{GA+GB}$ of *Br*.

A *mix-proof-net* category is defined as a proof-net category (see §2.2) that has in addition a natural transformation m satisfying the equations ($\hat{b}\,m$), ($\check{b}\,m$) and (cm). The category **MPN**$^\neg$ is up to isomorphism the free mix-proof-net category generated by $\mathcal{P}$.

The category **MPN** is defined as the category **PN** in §2.5 save that we have the additional primitive arrow terms $m_{A,B}$ for all objects of **PN**, and we assume as additional equations (m *nat*), ($\hat{b}\,m$), ($\check{b}\,m$) and (cm). We can prove that **MPN**$^\neg$ and **MPN** are equivalent categories as in §2.6. (We have an additional case involving $m_{A,B}$ in the proof of the analogue of the Auxiliary Lemma of §2.6, and similar trivial additions elsewhere; otherwise the proof is quite analogous.)

We have a functor G from **MPN** to *Br* defined by restricting the definition of the functor G from $\mathbf{MPN}^{\neg}$ to Br (cf. the beginning of §2.7), and we will prove the following.

MPN Coherence. *The functor G from* **MPN** *to Br is faithful.*

The proof of this coherence proceeds as the proof of **PN** Coherence in §2.7. The only difference is in the $\hat{\Xi}$-Permutation and $\check{\Xi}$-Permutation Lemmata of §2.5.

The formulation of the $\hat{\Xi}$-Permutation Lemma is modified by replacing **PN** and $\mathbf{DS}^{\neg p}$ by respectively **MPN** and $\mathbf{MDS}^{\neg p}$, where the category $\mathbf{MDS}^{\neg p}$ is defined as **MDS** save that it is generated not by $\mathcal{P}$, but by $\mathcal{P} \cup \mathcal{P}^{\neg}$ (cf. §2.5); moreover, we assume that y_1 and $\neg y_2$ occur in E within a subformula of the form $p \wedge (\neg y_2 \vee y_1)$ or $\neg p \wedge (y_1 \vee \neg y_2)$. We modify the proof of this lemma as follows.

If in E we have $p \wedge (\neg y_2 \vee y_1)$, then by the stem-increasing equations of §2.5 we have that the $\hat{\Xi}_{p,B}$-term $g\colon C \vdash D$ is equal to $f'' \circ \hat{\Delta}_{p,C}$ for $f''\colon C \wedge (\neg p \vee p) \vdash D$ an arrow term of $\mathbf{DS}^{\neg p}$, and so for $f\colon D \vdash E$ an arrow term of $\mathbf{MDS}^{\neg p}$ satisfying the conditions of the lemma we have in **MPN**

$$f \circ g = f \circ f'' \circ \hat{\Delta}_{p,C}\,.$$

Then we apply Lemma 2$\wedge$ of the preceding section to

$$f \circ f''\colon C \wedge (\neg p \vee p) \vdash E,$$

where C is $X(p)$, $\neg p \vee p$ is B and E is $Y(p \wedge (\neg p \vee p))$. So for

$$h_X : X(p) \wedge (\neg p \vee p) \vdash X(p \wedge (\neg p \vee p))$$

an arrow term of $\mathbf{DS}^{\neg p}$, and

$$(f \circ f'')^{\dagger} : X(p \wedge (\neg p \vee p)) \vdash Y(p \wedge (\neg p \vee p))$$

we have

$$f \circ f'' = (f \circ f'')^{\dagger} \circ h_X.$$

By the $\hat{\Xi}$-Permutation Lemma of §2.5 we have

$$h_X \circ \hat{\Delta}_{p,C} = g' \circ f'$$

where g' is the $\hat{\Delta}_{p,p}$-term $X(\hat{\Delta}_{p,p})$, and by bifunctorial and naturality equations we have

$$(f \circ f'')^{\dagger} \circ X(\hat{\Delta}_{p,p}) = Y(\hat{\Delta}_{p,p}) \circ (f \circ f'')^{-(\neg p \vee p)}.$$

Note that $(f \circ f'')^{\dagger}$ is obtained from $(f \circ f'')^{-(\neg p \vee p)} \colon X(p) \vdash Y(p)$ by replacement of p.

So we have in **MPN**

$$\begin{aligned} f \circ g &= f \circ f'' \circ \hat{\Delta}_{p,C} \\ &= (f \circ f'')^{\dagger} \circ h_X \circ \hat{\Delta}_{p,C} \\ &= (f \circ f'')^{\dagger} \circ X(\hat{\Delta}_{p,p}) \circ f' \\ &= Y(\hat{\Delta}_{p,p}) \circ f''' \end{aligned}$$

for f''', which is $(f \circ f'')^{-(\neg p \vee p)} \circ f'$, an arrow term of $\mathbf{MDS}^{\neg p}$.

We proceed analogously if in E we have $\neg p \wedge (y_1 \vee \neg y_2)$; instead of $\hat{\Delta}_{p,p}$ we then have $\hat{\Delta}'_{p,p}$. We have an analogous reformulation of the $\check{\Xi}$-Permutation Lemma of §2.5, with a proof based on Lemma 2$\vee$ of the preceding section.

Instead of Lemma 2$\wedge$ of the preceding section, we could have proved, with more difficulty, an analogous lemma where f is of type

$$Z(X_1(p) \wedge X_2(B)) \vdash Y(p \wedge B),$$

and $f^{\dagger}$ is of one of the following types:

$$Z(X_1(p \wedge B) \wedge (X_2(B))^{-B}) \vdash Y(p \wedge B),$$
$$Z(X_1(p \wedge B)) \vdash Y(p \wedge B).$$

Then in the proof of the $\hat{\Xi}$-Permutation Lemma modified for **MPN** we would not need to pass from g to $f'' \circ \hat{\Delta}_{p,C}$ via stem-increasing equations, but this alternative approach is altogether less clear.

Note that we have no analogue of Lemma 2 of §2.4 for **MDS**. The lack of this lemma, on which we relied in §2.5 for the proof of the $\hat{\Xi}$-Permutation and $\check{\Xi}$-Permutation Lemmata, is tied to the modifications we made for these lemmata with **MPN**. We have also no analogue of Lemma 4 of §2.4, but the analogue of Lemma 3 of §2.4 does hold.

From **MPN** Coherence and the equivalence of the categories $\mathbf{MPN}^{\neg}$ and **MPN** we can then infer the following.

$\mathbf{MPN}^{\neg}$ COHERENCE. *The functor G from $\mathbf{MPN}^{\neg}$ to Br is faithful.*

If we extend the definition of the category $\mathbf{SA}'$ with the primitive arrow terms $m_{A,B} : A \wedge B \vdash A \vee B$, together with the equations (m nat), ($\hat{b}\, m$), ($\check{b}\, m$) and (cm), we obtain a star-autonomous category of the mix kind. In this category we have arrows of the types $\bot \wedge A \vdash A$ and $A \vdash A \vee \top$, and also $\bot \vdash \top$. (Arrows of type $\bot \vdash \top$ may be used to define arrows of the type of $m_{A,B}$; see the proof of $\mathbf{SA}^c$ Coherence in §4.3.)

Chapter 7

Proof Nets

In this, final, chapter we justify the name we have given to proof-net categories. We show how they are related to a two-sided version of the proof nets of [26], such as have already been considered in the literature. Roughly speaking, the Brauerian split equivalences of §2.3 are the *graph core* of proof nets. As we have shown previously, we need just this core to prove coherence for proof-net categories (see Chapter 2) and restricted coherence for star-autonomous categories (see §4.3).

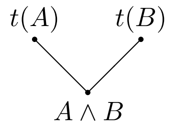
We discuss next the usefulness of proof nets in general proof theory. While they may be useful to decide the question whether there is an arrow of a particular kind (of a given type, or of a given type with a given graph), we do not find proof nets very useful to answer the question whether a diagram of arrows commutes. We find that this question, which is one of the central questions of general proof theory, is answered more efficiently by the graph core of proof nets.

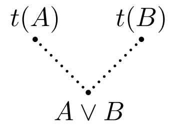
§7.1. Proof nets and proof-net categories

The connection between proof-net categories and the proof nets of [26] is the following.

For an object C of the category **PN** of §2.5, we define inductively the *target tree* $t(C)$ of C in the following manner:

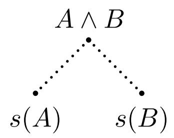
$t(p)$ and $t(\neg p)$ are the one-node tree labelled by respectively p and $\neg p$;

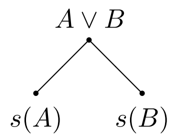
$t(A \wedge B)$ is 

and $t(A \vee B)$ is 

We define the *source tree* $s(C)$ of C inductively in a dual manner:

$s(p)$ and $s(\neg p)$ are the one-node tree labelled by respectively p and $\neg p$;

$s(A \wedge B)$ is 

and $s(A \vee B)$ is 

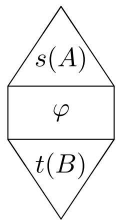
We have in target trees and source trees edges of two kinds: *solid* edges, like those in the clauses for $t(A \wedge B)$ and $s(A \vee B)$, and *dotted* edges, like those in the clauses for $t(A \vee B)$ and $s(A \wedge B)$. Nodes are labelled by subformulae of C.

An occurrence x of a letter in an object A of the category **PN** is called *negative* when $\neg x$ is a subformula of A; otherwise, the occurrence is *positive*.

An arrow $\varphi\colon GA \vdash GB$ of the category Br of §2.3 is said to *respect* A and B when every member of the partition $\mathrm{part}(\varphi)$ satisfies the following:

> if it is of the form $\{m_s, n_t\}$, then the $m+1$-th occurrence of letter in A (counting from the left) and the $n+1$-th occurrence of letter in B are occurrences of the same letter, and they are either both positive or both negative;
>
> if it is of the form $\{m_i, n_i\}$ for $i \in \{s, t\}$, then the $m+1$-th and the $n+1$-th occurrences of letter in A, when i is s, or in B, when i is t, are occurrences of the same letter, and one of them is positive while the other is negative.

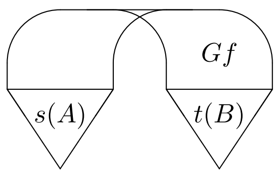
We call *proof structures* graphs of the form



where $\varphi\colon GA \vdash GB$ is an arrow of Br that respects A and B. The leaf of $t(B)$ labelled by the n-th occurrence of letter in B is identified with $n-1$ in the target GB of φ (more precisely, with $(n-1)_t$ in GB_t), and analogously with $s(A)$ and the source GA of φ. So all the nodes of this graph are nodes of the trees $s(A)$ and $t(B)$, while φ provides only edges, which are solid.

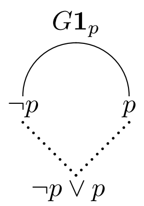
A proof structure where φ is Gf for some arrow $f\colon A \vdash B$ of **PN** is called a *proof net*. This notion of proof net is a *two-sided* version of the notion, like notions that may be found in [6], [11] and [41].

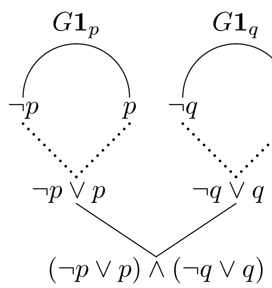
A two-sided proof net, such as we have introduced above, is transformed into a *one-sided* proof net, such as those of [26], in the following manner:



The source tree $s(A)$ is now conceived as being tied to $F\neg A$ (see §2.6), and the semicircles in Gf are called *axiom links*.

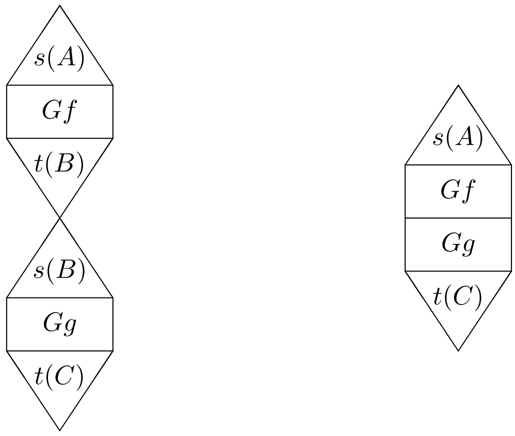
There are one-sided proof nets in [26] which are not obtained in this manner from our two-sided proof nets. For example, proof nets like



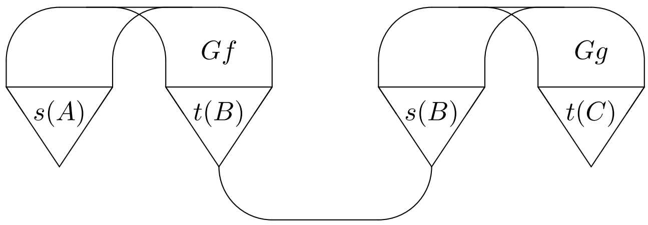


These proof nets are not tied to the category **PN**, but to the arrows of $\mathbf{SA}'_s$ or $\mathbf{SA}'$ whose source is $\top$ and whose target is an object of **PN**. We cover only one-sided proof nets corresponding to the sequents $\vdash \neg A, B$, but this is not an essential departure from the format of [26].

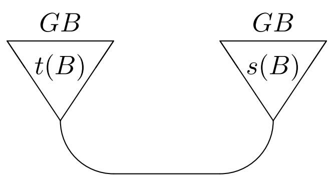
Composition, i.e. cut, of our two-sided proof nets is reduced to composition in Br in the following manner:



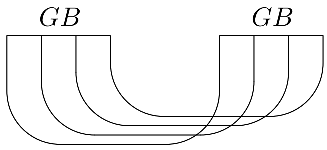
With one-sided proof nets one has instead



where



is transformed into a strip of semicircles



called *cut links*. So applying cut to one-sided proof nets also reduces to composition in *Br*.

One-sided proof nets serve to answer the question whether a given formula is provable in the multiplicative fragment of linear logic without propositional constants. (This question can also be answered, with apparently not more difficulty, by using standard sequent tools.) Formulated in terms of two-sided proof nets and categories, this is the question whether for a given type $A \vdash B$ there is an arrow of this type. We call this the *theoremhood problem*. A variant of the theoremhood problem, which we call the *graph-theoremhood problem*, is the problem whether for a given type $A \vdash B$ *and* a given arrow $\varphi\colon GA \vdash GB$ of *Br* there is an arrow $f\colon A \vdash B$ such that $Gf = \varphi$. Proof nets may serve to solve also this problem (which is of lesser complexity than the general theoremhood problem) for the categories **PN** and $\mathbf{PN}^{\neg}$. Both the general and the graph-theoremhood problem can be understood either constructively or nonconstructively, depending on the reading of the quantifier "there is" in the formulation of these problems. When only one φ respects A and B (and this is the case when $A \vdash B$ is *diversified*; i.e., when each letter occurs in it exactly twice), then solving the general theoremhood problem for $A \vdash B$ reduces to solving the graph-theoremhood problem.

Proof nets are connected more remotely with the question whether two arrow terms f and g of the same type $A \vdash B$ stand for the same arrow. To answer this latter question, which makes what we call the *commuting problem*, we do not need $s(A)$ and $t(B)$. One could say that in that context $s(A)$ and $t(B)$ are irrelevant material. As our **PN** Coherence shows, it is enough to check whether Gf is equal to Gg to answer this question: **PN** Coherence solves the commuting problem for the category **PN**.

The theoremhood problem for the category **PN** is solved by the acyclicity and connectedness conditions of proof nets (see [13]; another condition, equivalent to these two, may be found in [26]). A *switching* is a graph obtained from a proof structure by erasing for each pair of dotted edges growing in the same direction out of a common node one of these edges. A proof structure is called *acyclic* when each of its switchings is an acyclic graph, and it is called *connected* when each of its switchings is a connected graph.

It follows from [13] that a proof structure is acyclic and connected iff it is a proof net. Acyclicity implies Lemma 3 of §2.4, while connectedness implies Lemma 4 of §2.4. When we pass from **PN** to the category **MPN** of §6.2, so that in proof nets Gf arises out of an arrow f of **MPN** instead of **PN**, then connectedness is rejected and acyclicity is kept only. A solution of the theoremhood problem for **PN** and **MPN** yields a solution of this problem for the categories $\mathbf{PN}^{\neg}$ and $\mathbf{MPN}^{\neg}$.

As far as we know, a definition of the category $\mathbf{PN}^{\neg}$ of §2.2, and of the general notion of proof-net category, has first been given in this study. This is an equational definition. The same applies to the category **PN**. (It is not clear whether the non-equationally defined star-autonomous categories without units of [35] amount to our proof-net categories; the authors of [27] conjecture that their definition is equivalent to ours.) Related categories with the units $\top$ and $\bot$ have, however, been defined previously. These are either the symmetric linearly distributive categories with negation of [11], or the star-autonomous categories of [1] or [2] (see §3.8 for these two notions). Results that might be interpreted as coherence theorems for these categories with respect to proof nets, instead of the category *Br*, are stated in [11] and [6]. To these papers should be added as the most recent [24], [25] and [34], which are contemporaneous with our work.

In all these papers the categories envisaged have the units, and coherence in our sense with the units is not forthcoming (cf. §4.3). It is not clear how the results of [24], which are about proof nets with $\top$ and $\bot$, overcome the difficulties brought in by adding $\top$ and $\bot$ to $\mathbf{PN}^{\neg}$, of which the authors of [6] are aware. Another approach for overcoming these difficulties may be found in [34].

These papers do not state that proof nets bring in irrelevant material for the study of the commuting problem, though this may be gathered from [5] (which we have considered in §1.1). It is not even clear whether these papers are oriented towards solving the commuting problem, rather than some form of the theoremhood problem, or perhaps another problem.

We were clearly oriented towards solving the commuting problem, and our coherence results with respect to *Br* do that. Equality of arrows in *Br* is decidable in an elementary way, and the commuting problem is hence decidable in an elementary way in every category for which we have coherence with respect to *Br*.

Our approach differs also in style from these other papers. We have strived to present proofs as complete as possible. We do rely on previous results, but they may all be found exposed in detail in [22]. We find that proofs in the papers cited above can hardly qualify as complete. Sometimes, as in [5], the equations for the categories are not even stated, and have to be guessed. In [24] and [34], the previous results of [5] and [6] are not taken for granted, but other proofs are supplied.

If it is claimed that the category $\mathbf{PN}^{\neg}$, though it has not been previously defined equationally, has been defined by coherence, then we are in the situation that we have described at the end of §1.1. For us, as for Mac Lane, coherence is not built into the definition, but it is a theorem.

We would also not be satisfied with defining the category $\mathbf{PN}^{\neg}$ as the full subcategory on the objects of $\mathbf{PN}^{\neg}$ of the free symmetric linearly distributive category with negation (i.e. of the category $\mathbf{SA}'_s$ of §3.8). That $\mathbf{PN}^{\neg}$ is such a subcategory is for us a theorem, which we prove in §4.3, and is not built into the definition.

§7.2. Proof nets in general proof theory

Natural deduction is sometimes presented as being more practical than sequent systems because it involves less writing, less copying. Sequent systems note explicitly undischarged hypotheses, and keep copying them. Here is, for example, a proof of the dissociativity principle, corresponding to the type of $d_{p,q,r}$, in natural deduction format and in sequent format:

$$\dfrac{q\vee r \qquad \dfrac{\dfrac{p \quad \not{q}^{\,1}}{p\wedge q}}{(p\wedge q)\vee r} \qquad \dfrac{\not{r}^{\,1}}{(p\wedge q)\vee r}}{(p\wedge q)\vee r}\;{\scriptstyle 1}$$

$$\dfrac{\dfrac{\dfrac{p \vdash p \quad q \vdash q}{p, q \vdash p \wedge q}}{p, q \vdash (p \wedge q) \vee r} \qquad \dfrac{r \vdash r}{r \vdash (p \wedge q) \vee r}}{p, q \vee r \vdash (p \wedge q) \vee r}$$

The natural deduction proof involves 8 formulae, while the sequent proof involves 17 of them.

This advantage of natural deduction over sequents vanishes when we reach the standpoint of general proof theory, where we are interested in *proofs* and not in *provability* (see [15]). From that standpoint, it is not correct to say that a proof in natural deduction is a tree whose nodes are formulae. This is not precise enough. One should not forget about the rules used for building the trees, and in particular about the very important rules for discharging hypotheses in the leaves of the trees. We had such a rule in our example with disjunction elimination, and we noted the discharging with the label 1. It is more correct to say that a proof in natural deduction is the *building* of a tree. The tree itself provides just an incomplete record of this building, part of the information. Rules, i.e. operations (usually partial), for making proofs are not explicit in the tree, and we are very much interested in these operations. We want to see the operations, and do not want to keep guessing about them. We want to see how proofs are inductively built.

In general proof theory one studies inference rules, i.e. operations for building proofs, as in arithmetic one studies operations on natural numbers. A language for arithmetic in which operations would not be explicitly noted could hardly be suitable.

Complete information about proofs in natural deduction is obtained by introducing codes for proofs, in a notation usually inspired by the lambda calculus (following ideas of Curry and Howard). In our example we have

$$\dfrac{u\colon q \vee r \qquad \dfrac{\dfrac{x\colon p \quad y\colon q}{\langle x, y\rangle\colon p \wedge q}}{\iota^1\langle x, y\rangle\colon (p \wedge q) \vee r} \qquad \dfrac{z\colon r}{\iota^2 z\colon (p \wedge q) \vee r}}{\delta_{y,z}(u, \iota^1\langle x, y\rangle, \iota^2 z)\colon (p \wedge q) \vee r}$$

where $\delta_{y,z}$ is a ternary partial operation binding y and z. The tree of formulae contains now just the record of the types of the subterms of the term that codes the proof. This term, rather than the tree of formulae, stands for the proof.

We can incorporate the information about types in the term itself so that the tree of formulae disappears—or, rather, becomes implicit in the tree of the term:

$$\delta_{y_q,z_r}(u_{q\vee r},\iota^1_r\langle x_p,y_q\rangle,\iota^2_{p\wedge q}z_r).$$

In the same way, complete information about the sequent proof is obtained by coding. Here is a coding of the sequent proof above:

$$\dfrac{\dfrac{\dfrac{\mathbf{1}_p : p \vdash p \qquad \mathbf{1}_q : q \vdash q}{\wedge^R(\mathbf{1}_p,\mathbf{1}_q) : p,q \vdash p\wedge q}}{\iota^1_r \wedge^R(\mathbf{1}_p,\mathbf{1}_q) : p,q \vdash (p\wedge q)\vee r} \qquad \dfrac{\mathbf{1}_r : r\vdash r}{\iota^2_{p\wedge q}\mathbf{1}_r : r \vdash (p\wedge q)\vee r}}{\vee^L(\iota^1_r\wedge^R(\mathbf{1}_p,\mathbf{1}_q),\iota^2_{p\wedge q}\mathbf{1}_r) : p, q\vee r \vdash (p\wedge q)\vee r}$$

and the Gentzen term

$$\vee^L(\iota^1_r\wedge^R(\mathbf{1}_p,\mathbf{1}_q),\iota^2_{p\wedge q}\mathbf{1}_r),$$

in which the tree of sequents is implicit, is not more complicated than the term

$$\delta_{y_q,z_r}(u_{q\vee r},\iota^1_r\langle x_p,y_q\rangle,\iota^2_{p\wedge q}z_r)$$

above. (Actually, it is slightly shorter.)

The sequent proof becomes recorded with another Gentzen term when it is modified in the style of linear logic, so that the rules for $\vee$ are "multiplicative", as was the rule for introducing $\wedge$ on the right-hand side:

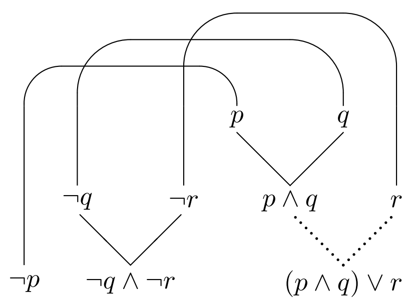
$$\dfrac{\dfrac{\dfrac{\mathbf{1}_p : p\vdash p \qquad \mathbf{1}_q : q\vdash q}{\wedge^R(\mathbf{1}_p,\mathbf{1}_q): p,q\vdash p\wedge q} \qquad \mathbf{1}_r : r\vdash r}{\vee^L(\wedge^R(\mathbf{1}_p,\mathbf{1}_q),\mathbf{1}_r) : p, q\vee r \vdash p\wedge q, r}}{\vee^R\vee^L(\wedge^R(\mathbf{1}_p,\mathbf{1}_q),\mathbf{1}_r) : p, q\vee r\vdash (p\wedge q)\vee r}$$

After introducing, multiplicatively, a conjunction on the left-hand side, we obtain a Gentzen term that should be equal to the **DS** arrow term

$$d_{p,q,r} : p \wedge (q \vee r) \vdash (p \wedge q) \vee r.$$

When they appeared nearly twenty years ago, proof nets were advertized as a new syntax bringing an economy over the sequent calculus, similar to the economy natural deduction brings. Proof nets were said to involve even less copying—even less "bureaucracy". As for natural deduction, this advantage vanishes from the standpoint of general proof theory. Unfortunately, proof nets never quite reached that standpoint. For the time being, they are approximately where natural deduction would be if typed lambda terms were not introduced to code derivations in natural deduction. Lambek has established in [36] a clear connection between cartesian closed categories and natural-deduction proofs in the conjunction-implication fragment of intuitionistic logic by proving an equivalence between the category of cartesian closed categories and the category of typed lambda calculuses. Could one obtain such a result without introducing typed lambda terms?

We would know that proof nets had reached the standpoint of general proof theory if codes were introduced to record the *building* of proof nets. Because what corresponds to a proof is not a proof net, but rather the building of a proof net. We want to see the operations for building proofs. If such codes were introduced, then we would see that the advantage over sequents vanishes. For example, here is a one-sided proof net corresponding to the proof of $p, q \vee r \vdash (p \wedge q) \vee r$:



and here is a code recording its building:

$$\cfrac{\cfrac{\cfrac{\mathbf{1}_p : \neg p, p \qquad \mathbf{1}_q : \neg q, q}{\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q) : \neg p, \neg q, p \wedge q} \qquad \mathbf{1}_r : \neg r, r}{\wedge_{\neg q, \neg r}(\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r) : \neg p, \neg q \wedge \neg r, p \wedge q, r}}{\vee_{p \wedge q, r} \wedge_{\neg q, \neg r} (\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r) : \neg p, \neg q \wedge \neg r, (p \wedge q) \vee r}$$

As far as length is concerned, the term

$$\vee_{p \wedge q, r} \wedge_{\neg q, \neg r} (\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r)$$

bears no advantage over the Gentzen term

$$\vee^R \vee^L (\wedge^R(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r),$$

which we had above. The proof net is a substitute for the tree of sequents, which is of secondary importance in general proof theory. The term coding the building of the proof net is not shorter than the term coding the building of the tree of sequents. In general proof theory, these terms occupy the centre of the stage, and not their types, which are implicit in the terms.

Proof nets do bring something more than just the types. It is as if besides $A \vdash B$ we were also given an arrow $\varphi \colon GA \vdash GB$ of Br that respects A and B. From our point of view, proof nets are not really syntax. In addition to the type, they bring something that belongs to the category Br, which is for us a model of our syntactical categories, with respect to which we prove completeness with our coherence theorems. Still, with a proof net we do not yet have an analogue of an arrow term f of type $A \vdash B$. Such arrow terms, like our arrow terms of **PN**, are syntax for us.

In some of the early papers of categorial proof theory, and in particular in the book [44], the types of arrow terms were more prominent than the arrow terms, which were often not mentioned. The readers were left to guess the arrow terms out of the types. This has serious disadvantages if the theory is concerned with commuting diagrams of arrows, i.e. equations between arrow terms that stand for proofs. The theory of proof nets suffered up to now from a similar disadvantage when it was proposed as a tool for recording equations between proofs. Again, we have to guess the syntax.

We have to guess the terms from the proof nets, which now replace the types.

Proponents of proof nets claim that theirs is a syntax with an advantage over ordinary syntax. Terms introduce an order on the application of operations, which is deemed unimportant. The term

$$\vee_{p\wedge q,r}\wedge_{\neg q,\neg r}(\wedge_{p,q}(\mathbf{1}_p,\mathbf{1}_q),\mathbf{1}_r),$$

which we had above, and the term

$$\wedge_{\neg q,\neg r}\vee_{p\wedge q,r}(\wedge_{p,q}(\mathbf{1}_p,\mathbf{1}_q),\mathbf{1}_r)$$

stand for the same proof net, drawn above, granted that the order of application of operations is unimportant. We can, however, introduce an equivalence relation on terms, which would make equivalent these two terms, and deal with equivalence classes of terms. This seems wiser than to relinquish completely the use of terms, and keep guessing what the terms are. Equations between terms, or equations between equivalence classes of terms, are more easily written down, and better understood, than equations between objects that are not syntactical in the ordinary sense, but are some sort of complicated graphs.

Here we touch upon deep questions. Why is it that we prefer languages made of *sequences* of symbols, rather than other structures of symbols, such as trees, or graphs of another sort (or simpler collections of symbols like multisets or sets)? This may have to do with the fact that we speak in time, which is one-dimensional, and not in something two-dimensional or three-dimensional. We do write in space, but nevertheless writing keeps to a great extent the one-dimensional organization of speech. Without that one-dimensional organization, written language is more difficult to understand. It is usually easier for pupils to absorb matters written on a blackboard by following them as they are being written down, than to face a blackboard fully filled before the lesson. This is so even when the matter on the blackboard is not one-dimensional, as in geometrical drawings. It is probably not fortuitous that Frege's two-dimensional notation for formulae, where formulae are drawn as trees, has not become standard in logic. (Although the tree structure implicit in formulae is very important, and Frege was not mistaken to stress it.)

We do not think, however, this is only a psychological or typographical problem. There may be mathematical reasons to prefer ordinary syntax. Variables are essential for the language of mathematics, and there is something in the ease with which we introduce variables for ordinary syntactical objects, and perform substitution for these variables, in a handy and very precise way, which seems to be lost by passing to a nonstandard syntax of more complicated graphs. What are variables for proof nets? As an analogue for that, one finds in the literature some sort of empty boxes with wires going out of them, which are not particularly easy to draw or handle, and for which the rules of substitution are not entirely explicit and precise.

A related matter is that we want syntactical notions to be decidable in an elementary way. The notions of term, formula and derivation in a formal system are decidable in this way in standard logic. The notion of proof net is not decidable in an elementary way. It may require considerable effort to recognize that a proof structure is a proof net. This is yet another disadvantage of proof nets taken as syntax.

For these reasons, we do not believe that proofs, as syntactical objects, should be identified with graphs like proof nets. They could perhaps be identified with the building of these graphs, in which building the order of application of operations would be unimportant, but we want to have these operations recorded nevertheless, and this recording is still achieved in the most secure way by sequences of symbols, i.e. terms.

A disadvantage of proof nets from the point of view of category theory is that with one-sided proof nets we do not have a clear information about the source and the target, but this can be remedied with two-sided proof nets. Still another disadvantage is excessive flexibility, verging on imprecision, when the order of formulae in proof nets is in question. If this order is disregarded, then we lose information about the arrows of Br implicit in proof nets, and lose a tool for solving the commuting problem. (This order can be disregarded sometimes, with diversified types, but this has to be justified; cf. [22], Section 3.3.) Sometimes the disadvantage of proof nets consists in bringing in irrelevant material, as we have indicated in the previous section, but this is when proof nets are taken as belonging to the model rather than to syntax.

This does not mean that proof nets cannot be useful for some purposes in general proof theory. They may serve to solve the graph-theoremhood problem for proof-net categories (see the preceding section). They have merit too for having attracted attention to coherence questions in logic, though they were not alone in doing that (interest in the generality of proofs has as much, if not more, merit; see [15] or [22], Sections 1.3-4), and though coherence may be, and usually is, proved without appealing to them, as this was done in [22] and in this work. (We appealed to proof nets only once, to decide a question of theoremhood in the proof of $\mathbf{SA}^c$ Coherence in §4.3, but this was not indispensable; we could have used a sequent system, or ordinary model-theoretical tools.)

In some circles at the border of logic and theoretical computer science, the belief is still spread with much enthusiasm that proof nets are an indispensable tool. We do not call into question here the role proof nets may play outside general proof theory. We have examined only to a certain extent what they brought up to now to this particular region of logic.

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