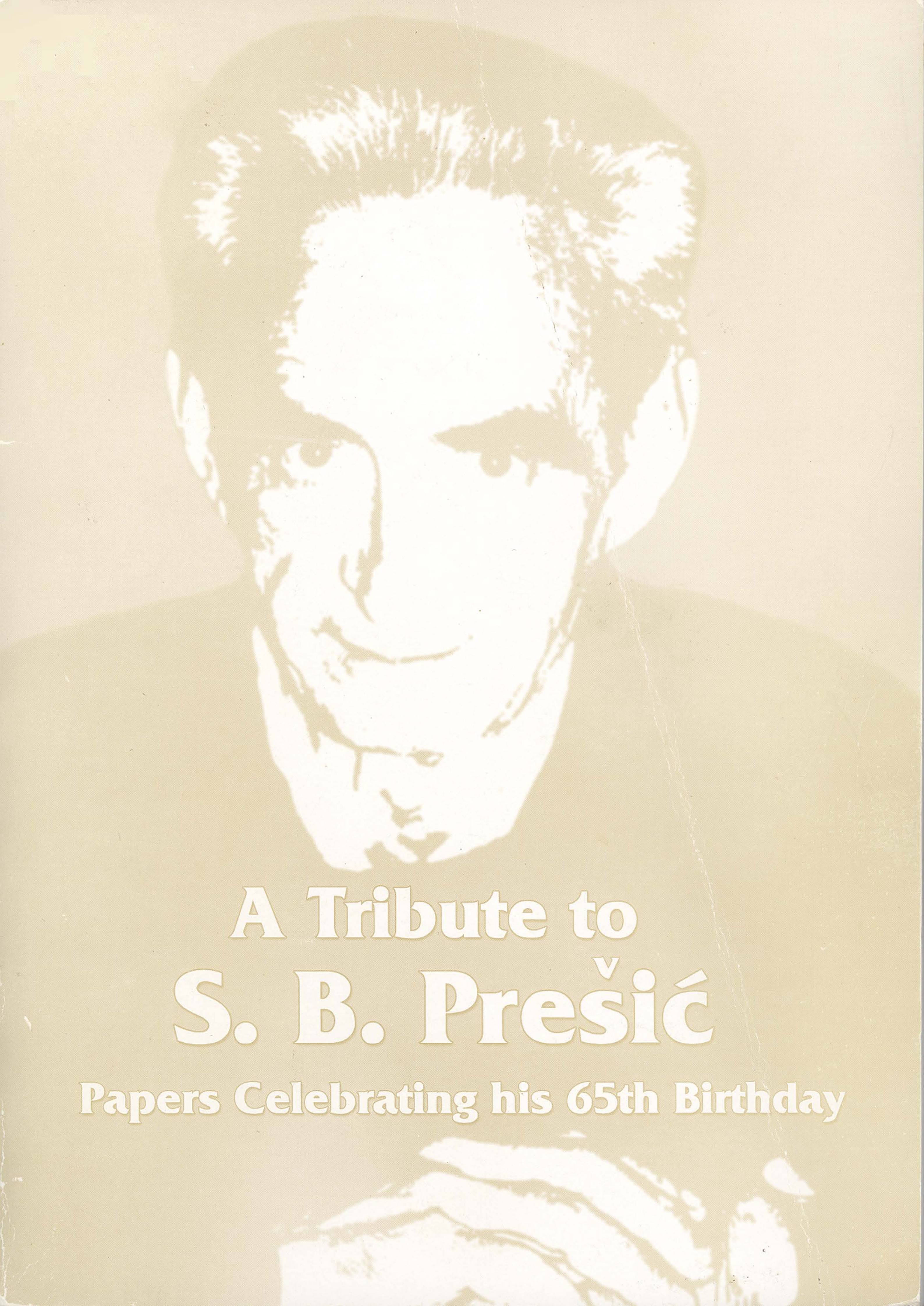

A Tribute to S. B. Prešić

Papers Celebrating his 65th Birthday

A. Krapež, ed.

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Matematički institut SANU

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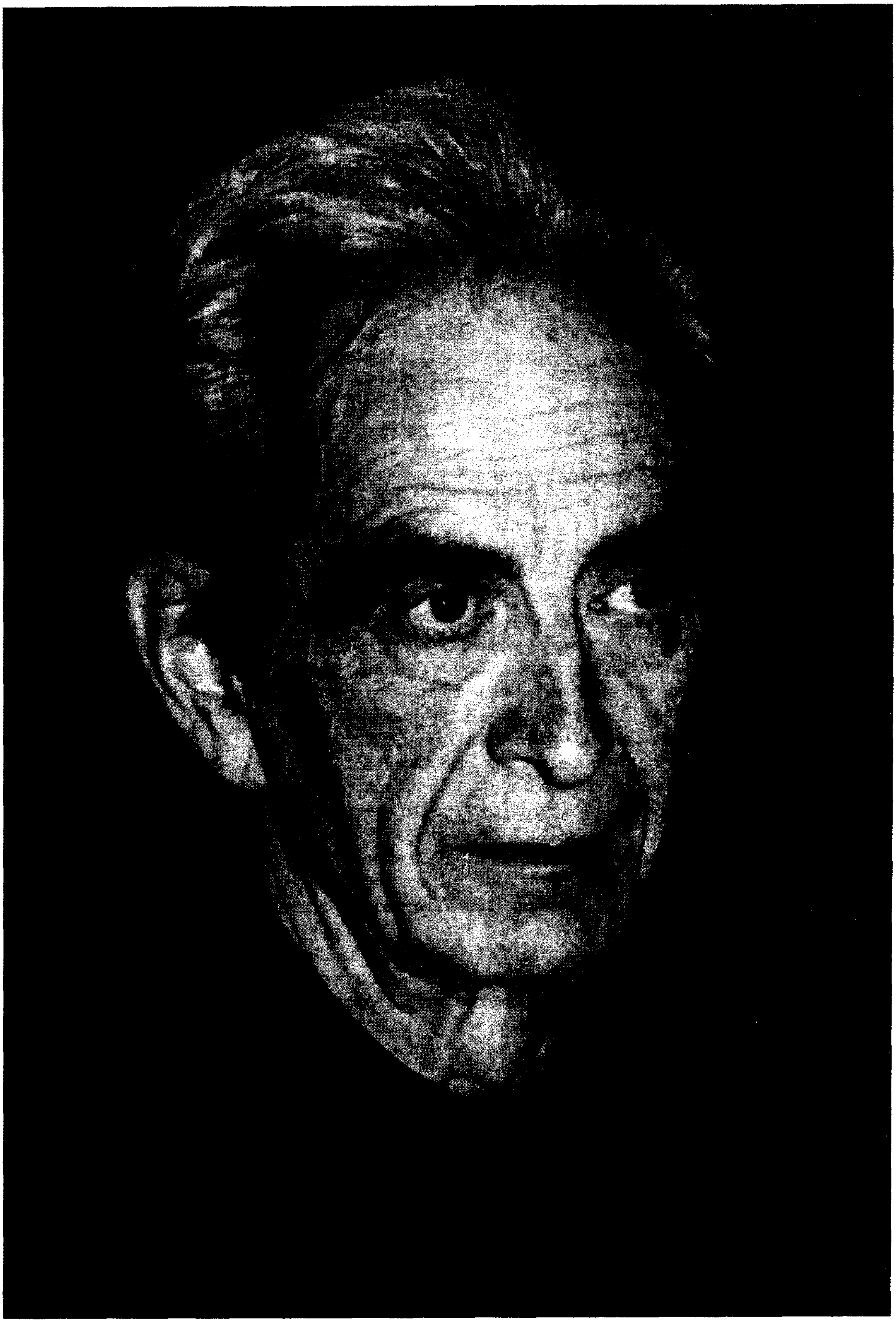

FOREWORD

I do not intend to write here of S. Prešić's results in mathematics. There is a glimpse of his achievments in the papers that follow. I would like to present a personal side of my relationship with him.

I met him for the first time in 1966 when I was a pupil of the Belgrade Mathematical Gimnasium. He was teaching to some other class but the news about him came to me quickly. 'He is different' was the unanimous opinion. And indeed he was. It was not just his lectures (although the best by far in our Gimnasium) that made him different. He also knew how to listen to us – 17 years old novices in mathematics. He had a way of treating us as his younger colleagues without intimidating us with a depth of his knowledge or his quick mind.

It was the same during my studies at the University of Belgrade. He was also the one to review my first research papers, to support my promotions, to supervise my masters studies. If ever I had a mathematical father – he is the one. With everything that this implies: adoration and admiration, quarels and cooperation, learning and maturing, growing up and getting free. Finally, there is a respect and a recognition that it is a privilege to be his student and a coauthor of joint papers.

There is a similar story that any author of a paper from this booklet can tell. You can find a hint of it in the papers by Ž. Mijajlović and I. Stavrev. So it was only natural that the idea of a Conference celebrating his 65th birthday met with unanimous agreement. Official support, by the Mathematical Faculty of the University of Belgrade and the Mathematical Institute of the Serbian Academy of Sciences and Arts, was assured as he has a life-long successfull involvement with both institutions.

Although the present booklet is not coextensive with the talks delivered during the Conference, it is certainly based on them.

The booklet consists of three parts. First, there are invited papers giving an overview of S. Prešić's results in general or in some particular mathematical discipline. Ž. Mijajlović in his article writes about S. Prešić's life and work in mathematics, particularly in mathematical logic. M. Kapetanović presents his results in computer science and artificial inteligence, emphasizing his extension of logic programming to any set of clauses, which need not be Horn clauses. reviewes his early work on functional equations including the profound idea of reproductive solutions of equations. The reproductivity is also the main theme of D. Banković's paper which deals with systems of equations on both Boolean algebras and finite sets.

The subject of the paper by G. Milovanović is S. Prešić's contribution to numerical factorization of polynomials. A striking new theory, numerical problem expressed in a first order formula over a $\{=,<\}$ with equalities restricted to segments, so called m–M calculus, is presented by V. Kovačević-Vujčić. There are many applications of m–M calculus: solving systems of equations and inequalities, finding n-dimensional integrals, solving problems of constrained and unconstrained optimization, *min–max* problems, problems from interval mathematics, finding functions satisfying a given m–M condition (functional condition, difference or differential equation) and many others. Lj. Ćirić writes on one of the first generalizations of the fixed point theorem of S. Banach, found by S. Prešić. This part concludes with the article by S. Vujić on the role of S. Prešić in education – at all levels – from the primary to the university. Conclusion is that he is the most deserving for the introduction and development of several modern mathematical disciplines in Serbia.

Next, there are research papers, not necessarily connected to his work, but in the fields he was familiar with.

Finally, there are several appendices:

1. a list of his books and papers (up to 2001),
2. an impression of him as a teacher – given by the one of his students,
3. the list of PhD's supervised by him and
4. the list of reviewers of papers submitted for this booklet.

Professor S. Prešić retired in 2000 but works furiously as ever. He is giving talks at various occasions and writing books on different (mostly mathematical) subjects. Let us mention just a series of books on the borderline of mathematics and methodology of scientific research under a common title: 'Misaona vidjenja' ('Mind frames').

Editor

Invited papers

ON THE SCIENTIFIC WORK OF SLAVIŠA B. PREŠIĆ

Žarko Mijajlović

Slaviša B. Prešić was born in 1933 in Kragujevac. In early childhood he experienced the hard days of the German occupation. S. Prešić once said about that period of his life:

I remember the horrible time of the Second World War, but even today the behaviour of my teacher Mateja Veljković has the effect of spiritual light. When it was the hardest, when the German troops were in front of Stalingrad, he encouraged us and cheered us up, and at the end of school lessons we would sing led by him "Let's sing with love to St. Sava".

He finished grammar school in Kragujevac in 1952, as a distinguished pupil. Of schoolteachers the greatest influence on him had Vitaly Hvorostansky, a strict, but very good teacher of mathematics. In 1957 S. Prešić graduated the Mathematics faculty of Belgrade University. Immediately after the graduation, he became the assistant of Professor Dragoljub Marković who had great merits for introducing modern algebra at the Mathematical Department. Very soon Professor Marković allowed his young and talented assistant to read lectures on algebra. After Professor Marković's death in 1965, the Chair of Algebra was founded and S. Prešić became the head of the Chair continuing to be in charge of ever since that time. At the beginning of seventies Prof. Prešić introduced in undergraduate studies a course of mathematical logic, and since then the Chair carries the name "the Chair of Algebra and Logic".

S. Prešić has never paid too much attention to institutional and authoritative approach to science. Let us grasp Professor S. Prešić's approach to science by putting it into the following context. Today we understand that "Scientific Revolution" took place in Europe when free thinkers began to doubt the then existing doctrines of natural philosophy, which the Church strictly adhered to. The doubt found its reflection in the statement that the accepted dogmas did not have a rational confirmation, and that the belief in them was mainly based on the authority of the Fathers of the Church. At the same time free thinkers challenged and questioned these doctrines. These challenges resulted in the traditional dichotomy between the rational knowledge, on the one hand, and religious belief as an irrational dogma, on the other.

As a response to this, free thinkers were punished. Giordano Bruno was burnt, and Galilei was taken to the Inquisition court. Fortunately, things like that no longer happen in science, though we are witness to the fact that people are burnt or punished for other, equally irrational, reasons.

About sixty years ago Bertrand Russell observed that there appeared indications that natural sciences were becoming a religion based on belief without reservation, which is incompatible with a rational standpoint. Really, four hundred years after the beginning of the Scientific Revolution the belief in scientific doctrines was still based, in many cases, on the authority of distinguished scientists. It is a fact that today scientific professionalism, solidarity, institution and rules are given priority over independent thinking, confirmation of truth and critical attitude. Expert evaluation and thinking is dominant on every level, from the local to the global, from elementary school to the world famous research institutions in establishing the value and the truth. It probably must be so, but mathematicians are really fortunate, because mathematics is, by its very nature, the purest science in this respect. However, the history of mathematics teaches us that evaluation of mathematical results may be a relative matter. A healthy doubt and a reasonable amount of dissatisfaction with the present state of affairs are the main types of motive power for all changes, including those in science.

I have every right to say that these particular qualities distinguish professor S. Prešić, which he himself emphasized on several occasions. He wrote once:

> *Ever since I was a little boy I have been a "doubting Thomas", that is, I have always been inclined to discover a shadow of prejudice, too much habit of thought, a routine in almost every thing, which, when eliminated, made my mind move farther, see better, and sometimes discover something altogether new.*

These distinctive features form the basis of Slaviša Prešić's entire work: his research, his books, monographs, his public speeches, no matter where they were made — at scientific conferences, expert meetings or popular talks. In his scientific papers and books we often come across completely new ideas and original ways of solving mathematical problems. Mathematics, as well as other modern sciences, is characterized by a complicated and complex language. Even a good mathematician finds it difficult to adequately present his results to the general mathematical public, as the main ideas often remain hidden under the heavy garment of technicalities and formalities. S. Prešić's language is clear and direct. His public speeches and discussions are remembered not only thanks to the picturesque and vital language with the help of which he immediately presents to the audience the essence of some mathematical problem or theory, but also thanks to his open criticism, a criticism without a compromise, of something that is not valid in mathematics. On such occasions the audience never remained indifferent. The discussion would become heated, with sharp remarks and plenty of enthusiasm. His opponents used to stay after the lecture "to see the matter through", and they often left with their opinion changed or at least less convinced that their point of view was right.

Professor S. Prešić's mathematical interests and research are universal and very extensive. His 50 articles and 10 books deal with a large number of mathematical disciplines. They are algebra, logic, numerical analysis, the theory of functional equations, and the theory of equations in general. In the past 10 years he has been doing intensive research in the field of computer science, where he is especially interested in the subject of artificial intelligence. His papers are written in French, English and German.

The first decade of his research is remarkable for a great number of published scientific papers, five of which were printed by C.R. Acad. Sci.. Later, besides scientific papers, he published several books: three university textbooks, one of which was written together with Marica Prešić, and five monographs, one of which was written with Marica Prešić, and two — with a group of authors. From the very start of his research work he was co-operative and ready to collaborate. Thus, about one fourth of his papers was written together with other authors, our well-known mathematicians. They are D. Đoković, D. Mitrinović, P. Vasić, M. Marjanović, Z. Ivković, B. Zarić, J. Petrić and M. Prešić.

The main distinctive feature of his works is their interdisciplinary character and close connection with other branches of mathematics. Thus, for example, if the work deals with the solution of functional equations, the methods are not simple manipulations with formulas, but include the application of group theory, linear algebra, Boolean algebra and mathematical logic. If a certain problem from the theory of algebraic equations is analyzed, like, for example, the evaluation of a polynomial root or factorization problem, here the methods from analysis, numerical analysis, the number theory are likely to be used. If a logic problem is considered, it is never studied in isolation, but in the light of the possible applications. Sometimes the application refers to the equation theory, on other occasions — to computer science or algebra.

Another specific feature of S. Prešić's works is originality and witnesses. The originality has in several cases resulted in completely new methods and theories. One of the main examples of this kind is the reproductivity theory as one of the rare universal methods in solving general equations. The reproductivity method showed its fruitfulness in the theory of functional equations, Boolean algebras and finite structures. The method is included into world famous monographs dealing with functional equations theory and Boolean algebras (M. Kuczma, S. Rudeanu). Another example is the method of solving a system of real equations, known as M-m calculus. S. Prešić developed this theory at the end of the eighties and at the beginning of the nineties. The method is applicable to a wide class of equation systems, including all equations of the algebraic type and many classes of transcendental equations. In principle, the solving procedure is based on exhaustion method, and uses specific features of functions of bounded variation. The idea of the method consists in exhausting subsets of the domains which do not contain a solution, so that what is left over in the limit is the solutions of the system. The main advantage of the method is not only its universal character, but also the fact that with the help of the method it is possible to determine all the roots of the given system in the assigned domain. The domain can be described with the complicated conditions such as formulas of the first order predicate calculus. The third example, taken from the field of artificial intelligence, is a PL-prover — an algorithm for an automated proving of theorems. Here we deal with a case of extension of Prolog algorithm. Namely, a standard procedure in Prolog, which is based on the properties of the Horn formulas, is extended on arbitrary formulas of the propositional type.

This does not exhaust all the themes of S. Prešić's scientific opus. Among his works we shall find papers on subjects dealing with universal algebra, fixed point theory, theory of quasigroups and their foundations. The results of his research were cited by the world leading authorities in various mathematical disciplines, such as P.M. Cohn in universal algebra, S. Rudeanu in Boolean algebra, and M. Kuczma in functional equations theory. In our country a large number of mathematicians study and develop S. Prešić's results in their research papers, books, MSc and PhD theses. We have every right to say that professor S. Prešić belongs to the narrow circle of our mathematicians who influenced considerably the development of modern mathematics in this country.

A greater part of Slavisa Prešić's scientific achievement was made in the field of algebra and logic. Let us present some of his most important works in this field.

One part of them deals with the polynomial theory. Thus his work [25] is concerned with the PhD thesis (defended in Paris) of S. Zervos, a Greek mathematician. In the thesis Zervos proves a general and significant inequality of the upper limit of a real polynomial root. The results of A.L. Cauchy, P. Montel, E. Landau, J.L. Jensen, D. Marković could be made for the chosen values of the inequality parameter. In his paper S. Prešić derived Zervos's result from one natural lemma on one page only, thus shortening the original proof many times. The work can be said to belong to the field of polynomial geometry, which concerns itself with the arrangement of the zero polynomial in a complex-valued plane as well as on a real line. It should be mentioned that Mihailo Petrović had significant results in this field. Academician Miodrag Tomić said once that the polynomial geometry is probably Petrović's most significant field in which his greatest achievement was made. It is probably under his influence that several outstanding Serbian mathematicians (D. Marković, M. Tomić, J. Karamata, B. Baišanski, S. Raljević) did research in this field and had significant and valuable results. S. Prešić's work in some way completes the long-standing tradition of this discipline in this country, as there are practically no more papers published on this theme. S. Prešić's several works on polynomial theory deal with the factorization of polynomials [18, 22, 42]. The first two papers give a procedure of a polynomial factorization on the polynomial of a given degree, so that in case of linear factors an iterative model for a simultaneous determination of all polynomial roots could be obtained. In the third paper the following very interesting idea is introduced. For a given polynomial $P(x)$ with integer coefficients, a good natural number M is found, so that the problem of factorization of polynomial $P(x)$ is reduced to the problem of factorization of the number $p(M)$.

The paper [33] belongs to the field of universal algebras. Namely, the discussion here is concerned with the so-called quasi-algebras, which, thanks to their properties, become a natural instrument in implementing various significant constructions, such as free algebras constructions, in solving word problems, as well as different problems connected with embedding of structures. By the use of quasi-algebra all these constructions obtain the same form: they are reduced to the solution of appropriate term systems of equations so that positive diagrams would naturally appear at the end as a solution. It should be mentioned here that S. Prešić is the first mathematician in this country to occupy himself with universal algebra. His results had a response in the world, he was cited, for example, by P. M. Cohn. Moreover, he introduced this and some other related fields such as model theory, for example, as part of the curriculum for full-time and post-graduate studies of algebra at the faculty of Natural Science and Mathematics of Belgrade University, which brought the curriculum up-to-date.

The third part of algebra papers refers to the application of the reproductivity idea to solving equations in algebraic structures. S. Prešić has published about 15 articles on this theme. As professor Banković and professor Rudeanu will dwell on these results, I shall only say that the most fruitful domain for the application of this idea is represented by Boolean algebras. Namely, S. Prešić observed that the property of reproductivity could be expressed in the language of Boolean algebras, and that thanks to it the conditions of existence and the description of general solutions of equations over these structures are easily established. Unexpectedly a unique theory was discovered, which united a great number of separate results obtained by Schröder, Lowenheim and others. S. Prešić uses the same method successfully and elegantly in the case of other algebraic structures such as a matrix ring, with the application in determining a generalized inverse; semigroups; in solving functional equations, etc. This idea is developed by the great number of authors: D. Banković, S. Rudeanu, J. Kečkić, M. Prešić, M. Božić, S. Milić, B. Alimpić, A. Krapež, Z. Mijajlović and others.

Professor S. Prešić has concerned himself with mathematical logic and its applications for already 35 years. It is rather difficult and risky to interpret a person's work in the presence of the person, especially if that person is your professor.

To avoid this I shall use a roundabout way and say a few words about our old mathematician Bogdan Gavrilović. In his academic talk on the problem of infinity in mathematics in 1926 he says: "Mathematics cannot tell us: space is infinite; it cannot tell us that space is finite, either." Gavrilović actually thinks that in mathematics the most important thing is *demonstration* or *proof*, and that thus obtained truths do not in the least predict the nature of space, in spite of the fact that they are starting assumptions. In that sense Gavrilović's point of view is close to Hilbert's formalistic approach, according to which infinity is a useful fiction that can be easily eliminated. This approach is also apparent when Gavrilović says on some other occasion: "We can think whatever we like about axioms; we can say that they are just a convention, *a priori* judgements; we can accept some of them, and reject others, but when they are accepted, what develops from them must be logically correct."

One part of S. Prešić's papers on mathematical logic is of a theoretical character. They mainly deal with algebraization of logical theories. Thus in work [30] it is proved that any formal theory can be in a definite way reformulated into an equational theory. In another paper [32], for one class of propositional calculus a corresponding class of algebras is introduced, and then a necessary and sufficient condition is found for the appropriate algebra to be adequate for the given propositional calculus. By adequacy we mean that for the considered pair 'algebra — propositional calculus' a completeness theorem holds.

A certain part of other works refers to the application of logic in other branches of mathematics. An example of that kind is the cited M-m calculus. It is interesting to note that besides the author himself, a Berkeley University publication [46] classifies the monograph as a paper on logic.

S. Prešic also wrote several books on logic. The best-known and highly influential book of that kind in our country is probably "Mathematical logic" from the "Mathematical library" edition, which has been used at our universities for already 30 years. Another reason that makes the work remarkable is the fact that for the first time in our country logic is presented as a mathematical discipline. Soon after the publication of the book and thanks to S. Prešić's personal engagement and influence logic was introduced as a special discipline into mathematics curriculum at our universities, while at secondary schools logic as a subject was transferred from the philosophy subject group to the group of mathematical subjects. This was the actual recognition of logic as a mathematical discipline in our country.

In some of his books S. Prešić intensively studied the problem of foundation. Let us mention one work of that kind, a voluminous monograph "Real numbers" published in 1985. In it S. Prešić creates the structure of real numbers in an original way, as well as discusses many other aspects connected with this, probably the most important, mathematical structure, like, for example, algebraic properties and the problem of solving algebraic equations.

This does not complete S. Prešić's activities in logic. We must first of all mention a seminar on mathematical logic, which is regularly held by the Mathematical Institute of the Serbian Academy of Sciences and Arts, and which was founded by S. Prešić 35 years ago. This is the oldest seminar of the Mathematical Institute, and dozens of mathematicians from the former and present Yugoslavia took part in its work. It is worth mentioning that Professor Đuro Kurepa was a constant and active member of the seminar up to the end of his life. The chronicles of the Mathematical Institute say that among the guests of this seminar were the most prominent logicians and mathematicians of other fields of specialization from all over the world: L. Henkin, J. Keisler, A.V. Arhangelski, M. Magidor, T. Jech, K. Devlin, Van Benthem, S. Negrepontis, A. Dragalin, … . Within the limits of the seminar program numerous specialized courses were held, which embraced almost all disciplines of mathematical logic. Every MSc or PhD candidate that specialized in logic and algebra had to report his main results at the seminar before the formal defense of a MSc or PhD thesis. About ten members of the seminar, the former students of professor S. Prešić or students of his students, are now respectable professors at well-known universities in the USA, Canada, France. Let us also mention that more than ten PhD theses and about 20 MSc theses were defended under professor S. Prešić's supervision. The Pure and Applied Logic Society was formed as a division of the seminar. Today professor S. Prešić is one of the most active members of the seminar and its spiritual leader.

S.B. Prešić fathered three sons and has six grandchildren.

WORK OF SLAVIŠA PREŠIĆ IN ARTIFICIAL INTELLIGENCE

M. Kapetanović

Slaviša Prešić got interested in computers mainly for practical reasons. In 1985 he began to work on his $m - M$-calculus (it took him 11 years to finish it!). He himself describes it as 'logical numerics' and in any case it contains, among other things, a large number of numerical algorithms. It happened so that the same year he visited New York. There he stayed for a year using the opportunity to work in the Courant Institute where he had an access to computers of the kind existing at the time. Then he used a little Commodore as well as the university mainframe and got acquainted with most of the well known programming languages. Besides practical programming techniques S. Prešić learned LISP and Prolog, and immediately realized the significance of symbolic languages for artificial intelligence (though he appreciates qualities of C as well). All this brought a considerable change in his views on mathematics: from those days on the art of design and analysis of algorithms, *algorithmics* as he calls it, has been constantly a subject of his interest. So he studied and discovered some fundamental algorithms (not usually found in the standard literature) concerning programming languages, trees and databases. In that context his interest in Prolog came as no surprise. First, problems can be expressed in Prolog in elegant and relatively short way and second, more important, being a logician, S. Prešić must have liked the fact that the theoretical basis of Prolog is a fragment of the first order logic, known as Horn logic.

As an introduction to S. Prešić's presentation and use of Prolog let us give a very short overview of the main ideas and concepts underlying the development of Prolog. One of the original motivations was to build a *declarative* programming language, i.e. a user friendly language requiring from a programmer only to define the problem correctly stating the conditions, while Prolog would take the whole burden of solving, using its own algorithm. This turned out to be too optimistic, "too good to be true", but nevertheless a new discipline was born and that was *logic programming.* Although the field expanded very fast, Prolog remains at its heart.

As already mentioned, the syntax of Prolog is based on the so called *Horn clauses.* These are universal closures of the first order formulas of the form $A_1 \wedge$

$\ldots \wedge A_n \to B$, where all A_i as well as B are atomic. Special cases are also important: when all A_i are missing, logic programmers talk of B as *facts*, while the clauses with $\perp$ in place of B are *goals.* On the dynamic side there is a built-in algorithm consisting of *unification, resolution* and *backtracking.*

Unlike some other mathematicians Slaviša Prešić enjoys talking about problems he is currently working on and enjoys teaching in general. When he got involved with Prolog it was natural that he should share his enthusiasm with his students and colleagues. As a result a book appeared under the name PROLOG, Relational language, quite a long text of more than 300 pages. The book represents a careful introduction to essential features of Prolog, with many details and worked examples. In order to stress some syntactical aspects S. Prešić in fact compared three Prologs: Micro Prolog (whose syntax owes much to LISP), LPA Prolog and Arity Prolog with its, now widely accepted, Edinburgh syntax. We are not going to review the whole book, but only to illustrate S. Prešić's own approach to the subject. The book comprises some standard parts (usually found in textbooks on Prolog), such as description of syntax and Prolog predicates (even some most practical advice: how to make exe-files, for instance!), as well as some more subtle things. Thus there is an important separate chapter on the Prolog algorithm, as S. Prešić names it, and a detailed disccussion of the important, though somewhat problematic *cut* operator. There is also a chapter on databases and their treatment in Prolog, but we choose to say few more words about Chapter 7, *Horn formulas; deductive models.* The reason is not only the inclination of the author of this text towards logic, but also the fact that these concepts do not get proper attention in books on Prolog. Their fundamental importance was certainly appreciated by S. Prešić and in the case of propositional calculus the exposititon should not be too dificult even for a newcomer to the subject. After a brief on truth tables and formal theories he introduces propositional Horn formulas and defines the central notion of *deductive model.* The point is that although any consistent theory has a model, the more is true about Horn theories: they have the least model! More precisely, if $\mathcal{H}$ is a consistent Horn theory (meaning that all of its axioms are Horn formulas, modus ponens is the only rule of inference and $\perp$ is not deducible), set $v(p) = \top$ exactly for those propositional variables p for which $\mathcal{H} \vdash p$. Then not only $v \models \mathcal{H}$, but all these p remain true in any model of $\mathcal{H}$. S. Prešić proves this and supplies examples to clarify the proof. The whole idea can be presented in a more general setting and this is also done in the book. The idea can be explained by examining the nature of Prolog itself. Although it is seen as a relational programming language (as the subtitle of the book suggests), what we have is in fact *computation done by term manipulation.* This brings us to *term models*, structures whose universe, also known as *Herbrand universe,* consists of all constant terms of a given first order language. For Horn clause theories we can now repeat the above definition of deductive model, where instead of propositional variables we take all atomic sentences θ such that $\mathcal{H} \vdash \theta$ and they induce the relations among terms in the intended term model. This may be beyond comprehension of a general reader and to overcome that S. Prešić not only offers examples of model construction but also finds a close link with Prolog. Namely if a Prolog computation (displayed as a tree) terminates with an answer θ, we can erase all unsuccessful branches and get what S. Prešić calls 'shortened Prolog proof' of θ. This in turn can be transformed into a Horn logic proof of θ. Let us end with one thing noticeable throughout the book and typical for S. Prešić: his relentless efforts to establish original Serbian terminology in the field.

This overview of S. Prešić's book should help us understand his later concern with logic programming. With all its depth and elegance Prolog could not satisfy S. Prešić completely. Although accclaimed as based on logic, Prolog programs may well be "illogical", such as the following one:

$$\begin{array}{l} a :- b, c. \\ b :- c. \\ c :- a. \\ a :- d. \\ d :- e. \\ e. \end{array}$$

Now the question $? - d$ leads to the positive answer, due to the presence of the clauses $d : - e$ and e in the database, but the question $? - a$ forces Prolog into an infinite loop, although these two clauses together with $a : - d$ suggest that a holds. Of course it is the ordering of clauses that causes trouble in this case. Dissatisfied with these peculiarities S. Prešić decided to prevent such things happen by setting up a *new formal system* which we here reproduce from [2]. It consists of the following four rules:

(R1) $\mathcal{F}, \bot \vdash \bot \longleftarrow \vdash \top$
(R2) $\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \longleftarrow \mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot$
$\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash \neg p \longleftarrow \mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \vdash \bot$
(R3) $\mathcal{F}, p_1 \vee \ldots p_k \vdash \bot \longleftarrow \mathcal{F} \vdash \neg p_1, \ldots, \mathcal{F} \vdash \neg p_k$
(R4) $\mathcal{F}, \top \vdash A \longleftarrow \mathcal{F} \vdash A$

Here $\mathcal{F}$ is an arbitrary set of clauses, p_i are literals, $\neg p_i$ their duals ($\neg\neg p_i$ are identified with p_i) and A is a literal or $\bot$. For this system S. Prešić proves *completeness*: $\mathcal{F} \vdash \psi$ iff $\mathcal{F} \models \psi$, where ψ is a literal or $\bot$. Then he goes on to describe a Prolog-like algorithm (extracted from the proof), named by him PL, with the following main features:

- *PL applies to arbitrary clauses, not just Horn clauses;*
- *the procedure always terminates and if it is applied to a Horn clause program, the usual Prolog algorithm appears as a special case;*
- *the priority is given to so called relevant clauses.*

Rather than going into details, we shall follow S. Prešić's style and motivate the procedure using the above program as an example, asking again if a is a consequence of the program. Translated into the clause form the problem reads:

$$a \vee \neg b \vee \neg c,\ b \vee \neg c,\ c \vee \neg a,\ a \vee \neg d,\ d \vee \neg e,\ e \vdash a.$$

Now we could replace a by $\bot$, and use (R2) and (R4), but we will turn that into the following *computation rule*: erase all ocurrences of the literal a and erase all clauses in which $\neg a$ appears. Moreover, call the clauses in which a occurred *relevant* and push them to the left (or to the top if we look at this sequence as a stack). This results in the following seqent:

(1) $$\underline{\neg b \vee \neg c},\ \underline{\neg d},\ b \vee \neg c,\ d \vee \neg e, e \ \vdash\ \bot\,.$$

Here relevant clauses are underlined and the priority is given to the leftmost clause. By (R3) our task is reduced to the following two:

(2) $$\neg d, b \vee \neg c,\ d \vee \neg e,\ e \ \vdash\ b$$

(3) $$\neg d, b \vee \neg c,\ d \vee \neg e,\ e \ \vdash\ c$$

Then by the above rule (2) is transformed into

$$\neg c,\ \neg d,\ d \vee \neg e,\ e \vdash \bot\,.$$

Notice that $\neg c$ is now given higher priority than $\neg d$. By (R3) we have

$$\neg d,\ d \vee \neg e,\ e \vdash c.$$

We could now follow the same procedure but observing that c has no occurrence in the hypotheses, we decide to give up proving b (as well as the other conjunct c) and backtrack to the next relevant clause $\neg d$ instead. This time we succeed as the following sequence shows.

(4) $$\neg d, b \vee \neg c,\ d \vee \neg e,\ e \ \vdash\ \bot$$

(5) $$b \vee \neg c, d \vee \neg e,\ e \ \vdash\ d$$

(6) $$\neg e,\ b \vee \neg c,\ e \ \vdash\ \bot$$

(7) $$b \vee \neg c,\ e \ \vdash\ e$$

(8) $$b \vee \neg c,\ \bot \ \vdash\ \bot$$

(9) $$\vdash\ \top.$$

The clause (4) is obtained from (1) by erasing $\neg b \vee \neg c$, (5) and (7) follow by (R3) and (6) and (8) by the computation rule formulated above. Finally (9) follows from (8) by (R1) proving that a indeed is a consequence of the program.

It is natural to try to extend this approach to the more general case of predicate formulas and S. Prešić showed how this could be carried out. As an illustration consider the program

$$\begin{aligned} \beta(f(X)) \ &:-\ \alpha(X) \\ \alpha(a) \ &:-\ \beta(a) \\ \alpha(b)& \end{aligned}$$

and ask whether $?-\beta(Y)$ can be satisfied. Here α, β are unary predicate symbols, f is a unary operational symbol, a, b are constants and X, Y free individual variables. Our aim is to find a "solution" for Y, i.e. a term t from Herbrand universe $\{a, b, f(a), f(b), \dots\}$ such that $\beta(t)$ follows from the program. To achieve that we need a new ingredient, *unification of terms*. Transforming our problem into the clause form

$$\beta(f(X)) \vee \neg\alpha(X),\ \alpha(a) \vee \neg\beta(a),\ \alpha(b) \vdash \beta(Y)$$

we observe that the only chance to apply the computation rule is to make $\beta(f(X))$ and $\beta(Y)$ equal. This is done by unifying $f(X)$ with Y, in this case simply by assigning $Y := f(X)$, Doing that we have

$$\neg\alpha(X),\ \alpha(a) \vee \neg\beta(a),\ \alpha(b),\ \beta(f(Z)) \vee \neg\alpha(Z) \vdash \bot\ .$$

Notice another novelty: since $\beta(f(X)) \vee \neg\alpha(X)$ is a universal formula, it should be saved, so we made a fresh copy of it, renaming its free variable at the same time. By (R3) we get

$$\alpha(a) \vee \neg\beta(a),\ \alpha(b),\ \beta(f(Z)) \vee \neg\alpha(Z) \vdash \alpha(X)$$

which calls for another unification. We first try to unify $\alpha(X)$ with $\alpha(a)$ putting $X := a$ which gives us

$$\neg\beta(a),\ \alpha(b),\ \beta(f(Z)) \vee \neg\alpha(Z) \vdash \bot\ .$$

By (R3) again we have

$$\alpha(b),\ \beta(f(Z)) \vee \neg\alpha(Z) \vdash \beta(a).$$

We realize now that $\beta(a)$ cannot unify with $\beta(f(Z))$, so we are forced, and this is an important point, *to backtrack to the place where X got its value,* annul that assignment and try the next clause which is $\alpha(b)$. The new value $X := b$ gives us

$$\alpha(a) \vee \neg\beta(a),\ \bot,\ \beta(f(Z)) \vee \neg\alpha(Z) \vdash \bot,$$

and by (R1) this is a success. Moreover, combining the substitutions $Y := f(X)$ and $X := b$ we get the required solution $Y = f(b)$.

With this sketch we end our survey of S. Prešiś's contributions to the field and we would like to do it with a personal impression: S. Prešić's work is far from finished. His abilities and wide interests leave no doubt that he will pursue his research with as much enthusiasm.

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CONTRIBUTION OF PROFESSOR S.B. PREŠIĆ TO FUNCTIONAL EQUATIONS

Jovan D. Kečkić

0. Introduction

Functional equations were the main field of interest of Professor S.B. Prešić at the beginning of his career. His work on functional equations led him to the discovery of what might be called "the theory of reproductivity" (see, e.g. [22]). In this survey we shall be chiefly concerned with the results obtained in papers [1]–[**16**] which are listed chronologically.

These papers can be divided into two distinct groups according to the two types of considered equations. Namely, suppose that m variables appear in a functional equation for unknown functions $f_1, \dots, f_k$ in $n_1, \dots, n_k$ variables, respectively, and let $n = \min(n_1, \dots, n_k)$. If $n < m$, we say that the equation is of type 1; such equations are treated extensively in monograph [23]. If $n = m$ we say that the equation is of type 2; some such equations are treated in monograph [24]. It can be argued that it is "easier" to solve an equation of type 1 than an equation of type 2.

Papers [1], [3], [4], [5], [6], [**10**], [11] comprise the first group and are devoted to some equations of type 1 for functions in two variables. The seven papers from this group were published in a short interval, from 1960 till 1963; with the exception of [1], they were all written jointly with other authors and the methods used were more or less standard: giving certain variables fixed values.

On the other hand, papers from the second group, that is to say papers [2], [7], [8], [9], [12], [13], [14], [15], [16] were devoted to rather general classes of equations of type 2. With the exception of [16], these papers were not co-authored and the basic method used was highly original.

1. Equations of type 1

The so-called translation equation

$$f(f(x,y),z) = f(x,y+z) \tag{1.1}$$

was solved in [1] on the set S of those functions $f: C^2 \to C$ which have the property: for any $x, t \in C$ the equation $f(x,y) = t$ has unique solution for y. It was proved that the general solution of (1.1) on the set S is given by

$$f(x,y) = g^{-1}(g(x) + y),$$

where g is an arbitrary complex function having its inverse function g^{-1}.

The equation

$$\sum_{i=1}^{m+n} C^{i-1} F_i(x_1 + x_2 + \cdots + x_m, x_{m+1} + x_{m+2} + \cdots + x_{m+n}) = 0 \tag{1.2}$$

where $F_i: R^2 \to R$ and where C is the cyclic operator defined by

$$Cf(x_1, x_2, \dots, x_k) = f(x_2, x_3, \dots, x_k, x_1)$$

is the subject of paper [3]. It was proved that the general continuous solution of (1.2) is given by:

$$F_i(x,y) = (nx - my)f(x+y) + g_i(x+y) \qquad (i = 1, 2, \dots, m+n-1)$$
$$F_{m+n}(x,y) = (nx - my)f(x+y) - \sum_{i=1}^{m+n-1} g_i(x+y),$$

where $f: R \to R$ and $g_i: R \to R$ are arbitrary continuous functions.

The remaining five papers from this group were concerned with the real equation

$$\begin{aligned} F(x_1, x_2, x_3, \dots, x_{m-1}, x_m) + F(x_1, x_3, x_4, \dots, x_m, x_2) + \cdots \\ + F(x_1, x_m, x_2, \dots, x_{m-2}, x_{m-1}) = 0. \end{aligned} \tag{1.3}$$

The equation (1.3) with $m = 2n$ was considered in [4], [5] and [10]. In those papers F is made to depend upon the unknown function f in two variables. We give the forms of the function F and the general solutions of the corresponding equations, where $g, h: R \to R$ always denote arbitrary functions.

Paper [4]:

$$F(x_1, x_2, \dots, x_{2n}) = \left(\sum_{i=1}^{k} f(x_{2i-1}, x_{2i})\right)\left(\sum_{i=k+1}^{n} f(x_{2i-1}, x_{2i})\right),$$

where $n \geq 2$, $1 \leq k \leq n-1$. The general solution is:

$$f(u,v) = \begin{cases} g(u)h(v) - g(v)h(u) & \text{for } n = 2 \\ h(v) - h(u) & \text{for } n > 2 \end{cases} \tag{1.4}$$

Paper [5]:

$$F(x_1,x_2,\dots,x_{2n})=\sum_{k=1}^{n} f(x_{2k-1},x_{2k}),$$

where $n \geq 2$. The general solution is:

$$f(u,v)=\begin{cases} g(u)h(v)-g(v)h(u) & \text{for } n=2\\ 0 & \text{for } n>2.\end{cases}$$

Paper [10]: Two forms of F were considered, namely

$$F(x_1,x_2,\dots,x_{2n})=\left(\sum_{i=1}^{k} f(x_{2i-1},x_{2i})\right)\left(\sum_{i=k+1}^{n} f(x_{k+i},x_{k+2n+1-i})\right),$$

where $n \geq 2$; $1 \leq k \leq n-1$ and

$$\begin{aligned} F(x_1,x_2,\dots x_{2n}) &= A_{n-2}f(x_1,x_2)f(x_{n+1},x_{n+2})\\ &\quad + f(x_1,x_2)\sum_{i=0}^{n-3} A_i(f(x_{i+3},x_{i+4})+f(x_{2n-1-i},x_{2n-i})), \end{aligned}$$

where $\sum_{i=0}^{n-3}=0$ for $n<3$.

It was proved that in both cases the general solution of the corresponding equation (1.3) is given by (1.4).

For $n=2$ all the equations considered in [4], [5] and [10] reduce to

$$f(x_1,x_2)f(x_3,x_4)+f(x_1,x_3)f(x_4,x_2)+f(x_1,x_4)f(x_2,x_3)=0 \tag{1.5}$$

and hence they can be taken to be various generalizations of the equation (1.5). Some more equations which have the same general solutions as the equation (1.5), that is to say

$$f(u,v)=g(u)h(v)-g(y)h(u) \qquad (g,h \text{ arbitrary}) \tag{1.6}$$

were constructed in [6]. First, it was shown that (1.6) is the general solution of the equation

(1.7)

$$\begin{aligned} &f(x_1,x_2+x_3)f(x_4+x_5,x_6+x_7)+f(x_1,x_3+x_4)f(x_5+x_6,x_7+x_2)\\ &+f(x_1,x_4+x_5)f(x_6+x_7,x_2+x_3)+f(x_1,x_5+x_6)f(x_7+x_2,x_3+x_4)\\ &+f(x_1,x_6+x_7)f(x_2+x_3,x_4+x_5)+f(x_1,x_7+x_2)f(x_3+x_4,x_5+x_6)=0. \end{aligned}$$

When the equation (1.7) is generalized to

$$\begin{aligned} &f(x_1,\varphi(x_2,x_3))f(\varphi(x_4,x_5),\varphi(x_6,x_7))\\ &\quad+f(x_1,\varphi(x_3,x_4))f(\varphi(x_5,x_6),\varphi(x_7,x_2))\\ &\quad+f(x_1,\varphi(x_4,x_5))f(\varphi(x_6,x_7),\varphi(x_2,x_3))\\ &\quad+f(x_1,\varphi(x_5,x_6))f(\varphi(x_7,x_2),\varphi(x_3,x_4))\\ &\quad+f(x_1,\varphi(x_6,x_7))f(\varphi(x_2,x_3),\varphi(x_4,x_5))\\ &\quad+f(x_1,\varphi(x_7,x_2))f(\varphi(x_3,x_4),\varphi(x_5,x_6))=0, \end{aligned} \tag{1.8}$$

where $\varphi: R^2 \to R$ is a given function, then (1.6) is again its solution, but it need not be general, as shown by the example when $\varphi(u,v) = u - v$, in which case the particular solution $f(u,v) = u$ cannot be obtained from (1.6). However, if there exists a real number a such that

$$\varphi(x,a) = \varphi(a,x) = x \quad \text{for all} \quad x \in R, \tag{1.9}$$

then the general solution of (1.8) is given by (1.6). Besides $\varphi(u,v) = u + v$, there are other examples of the function φ satisfying (1.9); e.g.

$$\varphi(u,v) = uv, \quad \varphi(u,v) = u + v + uv, \quad \varphi(u,v) = \frac{u+v}{1+uv^2}, \quad \text{etc.}$$

A further extension from [6] is provided by the equation (1.3) where $m = 3k+1$ and

$$\begin{aligned} &f(u_1, u_2, u_3, \dots, u_{3k}, u_{3k+1}) = \\ &\qquad f(u_1, \varphi(u_2, u_3, \dots, u_{k+1}))f(\varphi(u_{k+2}, \dots, u_{2k+1}), \varphi(u_{2k+2}, \dots, u_{3k+1})), \end{aligned}$$

which for $k = 1$, $\varphi(u) = u$ reduces to (1.5). Again (1.6) is a solution of the corresponding equation, and it is general if there exists a real number a such that

$$\varphi(u,a,a,\dots,a) = \varphi(a,u,a,\dots,a) = \cdots = \varphi(a,a,a,\dots,u) = u$$

for all $u \in R$.

Finally, the equation (1.2) with

$$\begin{aligned} &F(u_1, u_2, u_3, \dots, u_{n-1}, u_n) \\ &f(u_1, g_1(u_2, u_3, \dots, u_{k+1}))\{f(g_2(u_{k+2}, u_{k+3}, \dots, u_{k+l+1}), g_3(u_{k+i+2}, \dots, u_n)) \\ &\qquad\qquad + f(g_3, (u_{k+2}, u_{k+3}, \dots, u_{k+m+1}), g_2(u_{k+m+2}, \dots, u_n))\} \\ &f(u_1, g_2(u_2, u_3, \dots, u_{l+1}))\{f(g_1(u_{l+2}, u_{l+3}, \dots, u_{k+l+1}), g_3(u_{k+l+2}, \dots, u_n)) \\ &\qquad\qquad + f(g_3, (u_{l+2}, u_{l+3}, \dots, u_{l+m+1}), g_2(u_{l+m+2}, \dots, u_n))\} \\ &f(u_1, g_3(u_2, u_3, \dots, u_{m+1}))\{f(g_1(u_{m+2}, u_{m+3}, \dots, u_{k+m+1}), g_2(u_{k+m+2}, \dots, u_n)) \\ &\qquad\qquad + f(g_3, (u_{m+2}, u_{m+3}, \dots, u_{l+m+1}), g_1(u_{l+m+2}, \dots, u_n))\} \\ &\qquad\qquad\qquad\qquad (k + l + m + 1 = n) \end{aligned}$$

was considered in [11]. It was shown that if there exists an $a \in R$ such that

$$\varphi_i(u,a,\dots,a) = \varphi_i(a,u,a,\dots,a) = \cdots = \varphi_i(a,a,\dots,a,u) = u,$$

for $i = 1,2,3$ and all $u \in R$ then (1.6) is the general solution of the equation in question.

All the papers from this group are mentioned in Aczél's monograph [23].

2. Equations of type 2

Let E and S be nonempty sets and suppose that g is a bijection of E. The equation

$$f(x) = f(g(x)) \tag{2.1}$$

for the unknown function $f: S \to E$ was considered in [2].

The basic idea underlying the original method applied in this paper led its author, S.B. Prešić, to other important results and so we shall devote somewhat more space to it.

In order to fix ideas, let $E = R^3$, $S = R$, $x = (x_1, x_2, x_3)$, $g(x_1, x_2, x_3) = (x_2, x_3, x_1)$. Then

$$g^2(x_1, x_2, x_3) = g(g(x_1, x_2, x_3)) = g(x_2, x_3, x_1) = (x_3, x_1, x_2),$$
$$g^3(x_1, x_2, x_3) = g(g^2(x_1, x_2, x_3)) = g(x_3, x_1, x_2) = (x_1, x_2, x_3),$$

i.e. $g^3 = i$, the identity mapping. The equation (2.1) becomes

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_1) \tag{2.2}$$

and from (2.2) we obtain

$$f(x_2, x_3, x_1) = f(x_3, x_1, x_2), \qquad f(x_3, x_1, x_2) = f(x_1, x_2, x_3)$$

which, together with (2.2) implies

$$\begin{aligned} f(x_1, x_2, x_3) &= f(x_1, x_2, x_3), \\ f(x_1, x_2, x_3) &= f(x_2, x_3, x_1), \\ f(x_1, x_2, x_3) &= f(x_3, x_1, x_2). \end{aligned} \tag{2.3}$$

Adding up the equations (2.3) we get

$$f(x_1, x_2, x_3) = \frac{1}{3}\big(f(x_1, x_2, x_3) + f(x_2, x_3, x_1) + f(x_3, x_1, x_2)\big). \tag{2.4}$$

Clearly, the equations (2.2) and (2.4) are equivalent, but (2.4) has a special property. Namely, if we replace the unknown function $f : R^3 \to R$ on the right hand side of (2.4) by an arbitrary function $\Pi : R^3 \to R$ we obtain the formula

$$f(x_1, x_2, x_3) = \frac{1}{3}\big(\Pi(x_1, x_2, x_3) + \Pi(x_2, x_3, x_1) + \Pi(x_3, x_1, x_2)\big) \tag{2.5}$$

which represents the general solution of the equation (2.2).

Indeed, it is easily verified that the function f defined by (2.5) satisfies (2.2) for arbitrary Π. Conversely, if f_0 is a solution of (2.2) then f_0 satisfies the equations (2.3) and setting $\Pi = f_0$ into (2.5) we get: $f(x_1, x_2, x_3) = f_0(x_1, x_2, x_3)$.

Consider now the equation (2.2) where the unknown function f maps S^3 into S, where S is any nonempty set. As before, from (2.2) follows (2.3), but now we cannot simply "add up" those equations. We are now working on arbitrary sets and the only tools we have are those of set theory.

So, from (2.3) we derive

$$\{f(x_1,x_2,x_3),\ f(x_1,x_2,x_3),\ f(x_1,x_2,x_3)\} \\ = \{f(x_1,x_2,x_3),\ f(x_2,x_3,x_1),\ f(x_3,x_1,x_2)\},$$

that is to say

$$\{f(x_1,x_2,x_3)\} = \{f(x_1,x_2,x_3),\ f(x_2,x_3,x_1),\ f(x_3,x_1,x_2)\} \tag{2.6}$$

By the Axiom of Choice, there exists a mapping $M : P(S) \to S$, such that a singleton is mapped into its element, i.e. $M\{u\} = u$, for all $u \in S$. Applying M to (2.6) we get

$$f(x_1,x_2,x_3) = M\{f(x_1,x_2,x_3),\ f(x_2,x_3,x_1),\ f(x_3,x_1,x_2)\} \tag{2.7}$$

The equations (2.2) and (2.7) are equivalent, but the equation (2.7) has the same special property as the equation (2.5). Namely when we replace the unknown function f on the right hand side of (2.7) by an arbitrary function $\Pi : S^3 \to S$, we obtain the general solution

$$f(x_1,x_2,x_3) = M\{\Pi(x_1,x_2,x_3),\ \Pi(x_2,x_3,x_1),\ \Pi(x_3,x_1,x_2)\}$$

of the equation (2.2).

We now return to the equation (2.1), where $g : E \to E$ is a given bijection and $f : E \to S$ is the unknown function. From (2.1) we obtain as consequences, the following equations

$$f(x) = f(g^\nu(x)) \qquad (\nu \in Z)$$

which imply

$$\{f(x)\} = \bigcup_{\nu \in Z} \{f(g^\nu(x))\}. \tag{2.8}$$

Let $M : P(S) \to S$ have the property: $M\{u\} = u$ for $u \in S$. Applying M to (2.8) we get

$$f(x) = M \bigcup_{\nu \in Z} \{f(g^\nu(x))\} \tag{2.9}$$

and the equation (2.9), which is equivalent to (2.1), shares the same special property with the equations (2.4) and (2.7). Indeed, the general solution of (2.1) is given by

$$f(x) = M \bigcup_{\nu \in Z} \{\Pi(g^\nu(x))\} \tag{2.10}$$

where $\Pi : E \to S$ is arbitrary.

The general solution (2.10) of (2.1) was obtained in [2] and various particular examples were also displayed.

The idea which led to the solution of (2.1) was applied in [7] to a very general equation, namely

$$f(x) = H(x, f(x), f(\Theta_2 x)), \dots, f(\Theta_n(x)) \tag{2.11}$$

where $\Theta_1, \Theta_2, \dots, \Theta_n$ are given mappings of a nonempty set E into itself (Θ_1 is the identity mapping), the function $H : E \times S^n \to S$, where S is another nonempty set, is also given, whereas $f : E \to S$ is unknown.

Denote by G the semigroup generated by $\Theta_1, \Theta_2, \dots, \Theta_n$. The general solution of the equation (2.11) can be obtained if one of its consequences is an equation of the form

$$f(x) = \Phi(x, f(x), f(\sigma_1(x)), \dots, f(\sigma_k(x)))$$

where $\sigma_1, \dots, \sigma_k \in G$ and the function f determined by

$$f(x) = \Phi(x, \Pi(x), \Pi(\sigma_1(x)), \dots, \Pi(\sigma_k(x))) \tag{2.12}$$

satisfies (2.11) for any function $\Pi : E \to S$. In that case the general solution of (2.11) is given by (2.12).

The equation (2.11) is so general that we cannot expect to be able to write down its general solution effectively. In connection with this equation Kuczma wrote in his survey paper [25]:

> *"A way to obtain the general solution of equation (2.11) has been indicated in S. Prešić* [7]"

and in his later monograph [24, p. 244]:

> *"We shall not endeavour to give a construction of the general solution of equation (2.11); an attempt to do this may be found in S. Prešić* [7]"

However, if the equation (2.11) is linear and if G is a finite group, the general solution can effectively be obtained. This was done by S. Prešić in papers [8], [9] and [14]. We shall describe here how S. Prešić in [8] and [9] solved the equation

$$a_1(x)f(\Theta_1 x) + \cdots + a_n(x)f(\Theta_n x) = 0 \qquad (\Theta_i x = \Theta_i(x)), \tag{2.13}$$

where a_i are given functions mapping a nonempty set G into a field F of characteristic 0, the given functions $\Theta_i : S \to S$ form a group G of order n (Θ_1 is the identity mapping) and the function $f : S \to F$ is unknown. The method of solving (2.13) which follows was called by Kuczma [24, p. 268] "an elegant method given by S. Prešić" and he devoted some four pages of his monograph to it.

Since $\Theta_1, \dots, \Theta_n$ form a group of order n, then for all $i, j = 1, \dots, n$ we have

$$\Theta_i(\Theta_j x) = \Theta_p x \qquad (1 \le p \le n) \tag{2.14}$$

and the index $p = p_{ij}$ is unambiguously defined by (2.14). If we replace x by $\Theta_i x$ $(i = 1, \ldots, n)$ in (2.13) we obtain the system of equations which has the matrix form

$$A(x)F(x) = 0 \tag{2.15}$$

where $A(x) = \|a_{ij}(x)\|_{n\times n}$, $a_{ij}(x) = a_k(\Theta_i x)$ with $p_{ki} = j$, and $F(x)$ is the one-column matrix

$$F(x) = \|f(\Theta_1 x) \ldots f(\Theta_n x)\|^T.$$

S. Prešić looked for the general solution of (2.13) in the form

$$F(x) = B(x)\Phi(x), \tag{2.16}$$

where $B(x) = \|b_{ij}(x)\|_{n\times n}$, $\Phi(x) = \|\Pi(\Theta_i x), \ldots, \Pi(\Theta_n x)\|^T$ and $\Pi : S \to F$ is arbitrary.

In general, the expressions obtained from (2.16) for $f(\Theta_i x)$ are contradictory. If they are not, the matrix $B(x)$ is said to be compatible with the group G. S. Prešić first proved the following lemma.

Let the matrices $M_k = \|a^k_{ij}\|$, where $1 \leq i, j, k \leq n$ be defined by $a^k_{ij} = 1$ if $j = p_{ik}$ and $a^k_{ij} = 0$ if $j \neq p_{ik}$. Then a sufficient condition for the compatibility of $B(x)$ with the group G is given by

$$B(\Theta_i x) = M_i B(x) M_i^{-1} \qquad (x \in S; \quad i = 1, \ldots, n).$$

He then constructed a matrix $B(x)$ which has the following two properties:

(P_1) $\quad A(x)B(x)A(x) + A(x) = 0$ for all $x \in S$;

(P_2) $\quad B(x)$ is compatible with the group G.

The construction runs as follows. Let $r(x)$ be the rank of $A(x)$. The matrix $A(x)$ can be written in the form $A(x) = P(x)D(x)Q(x)$, where $P(x)$ and $Q(x)$ are regular for all $x \in S$, while $D(x)$ is a diagonal matrix with 1's and 0's on the diagonal, the number of 1's being equal to $r(x)$. Now, if we put

$$B_0(x) = -Q(x)^{-1}D(x)P(x)^{-1} \tag{2.17}$$

$$B(x) = \frac{1}{n}\sum_{k=1}^{n} M_k^{-1} B_0(\Theta_k x) M_k \quad (x \in S), \tag{2.18}$$

then it can be verified that the matrix $B(x)$ defined by (2.17) and (2.18) has the properties (P_1) and (P_2).

Now the general solution of the equation (2.13) is given by

$$\left\|\begin{matrix} f(x) \\ f(\Theta_2 x) \\ \vdots \\ f(\Theta_n x)\end{matrix}\right\| = (B(x)A(x) + I)\left\|\begin{matrix} \Pi(x) \\ \Pi(\Theta_2 x) \\ \vdots \\ \Pi(\Theta_n x)\end{matrix}\right\|,$$

where $B(x)$ has properties (P_1) and (P_2), I is the unit matrix, and $\Pi: S \to F$ is arbitrary.

This means that the general solution of (2.13) has the form

$$f(x) = c_1(x)\Pi(x) + c_2(x)\Pi(\Theta_2 x) + \cdots + c_n(x)\Pi(\Theta_n x)$$

where the coefficients $c_i(x)$ are determined by the given functions $a_i(x)$ and Π is arbitrary.

S. Prešić later generalized this result to nonhomogeneous linear equation

$$a_1(x)f(\Theta_1 x) + \cdots + a_n(x)f(\Theta_n x) = g(x), \tag{2.19}$$

where a_i, Θ_i and f are as before and $g: S \to F$ is a given function. Namely, in [14] he proved that:

(i) If $B(x)$ is a matrix with the properties (P_1) and (P_2), the equation (2.19) has a solution if and only if

$$(A(x)B(x) + I)\|g(x)\; g(\Theta_2 x) \cdots g(\Theta_n x)\|^T = 0. \tag{2.20}$$

(ii) If the equality (2.20) is true, the general solution of (2.19) is given by

$$\begin{Vmatrix} f(x) \\ f(\Theta_2 x) \\ \vdots \\ f(\Theta_n x) \end{Vmatrix} = -B(x) \begin{Vmatrix} g(x) \\ g(\Theta_2 x) \\ \vdots \\ g(\Theta_n x) \end{Vmatrix} + (B(x)A(x) + I) \begin{Vmatrix} \Pi(x) \\ \Pi(\Theta_2 x) \\ \vdots \\ \Pi(\Theta_n x) \end{Vmatrix}$$

where $\Pi: S \to F$ is an arbitrary function.

We now return to the equation (2.2), where $f: R^3 \to R$ is the unknown function and to the equation (2.4) which is equivalent to it. Denote, as usual, the set of all functions which map R^3 into R by $R^{(R^3)}$, and let $\mathcal{F}: R^{(R^3)} \to R^{(R^3)}$ be defined by

$$\mathcal{F}f(x_1, x_2, x_3) = \frac{1}{3}\big(f(x_1, x_2, x_3) + f(x_2, x_3, x_1) + f(x_3, x_1, x_2)\big).$$

Then it is easily verified that

$$\mathcal{F}^2 f(x_1, x_2, x_3) = \mathcal{F}f(x_1, x_2, x_3),$$

i.e. that $\mathcal{F}^2 = \mathcal{F}$ for all $f \in R^{(R^3)}$.

In other words, the equation (2.2) is equivalent to the equation

$$f(x_1, x_2, x_3) = \mathcal{F}f(x_1, x_2, x_3),$$

where $\mathcal{F}^2 = \mathcal{F}$, and its general solution is given by

$$f(x_1, x_2, x_3) = \mathcal{F}\Pi(x_1, x_2, x_3),$$

where $\Pi: R^3 \to R$ (or $\Pi \in R^{(R^3)}$) is arbitrary.

Similarly, it can be shown that the matrix equation (2.15) is equivalent to the equation

$$F(x) = \mathcal{F}F(x),$$

where the operator $\mathcal{F}$ which maps the set of $n \times 1$ matrices into itself is defined by

$$\mathcal{F}F(x) = (B(x)A(x) + I)F(x)$$

and $B(x)$ is an $n \times n$ matrix with the properties (P_1) and (P_2). It is easily shown that $\mathcal{F}^2 = \mathcal{F}$ and the general solution of (2.15) is given by

$$F(x) = \mathcal{F}\Pi(x),$$

where Π is an arbitrary $n \times 1$ matrix.

On the basis of the above facts S. Prešić proved in [12] the following simple, but far-reaching theorem:

Suppose that f maps a nonempty set S into itself and that $f^2 = f$, i.e. $f(f(x)) = f(x)$ for all $x \in S$. The general solution of the equation in x:

$$x = f(x) \tag{2.21}$$

is given by $x = f(\Pi)$, where $\Pi \in S$ is arbitrary.

The equation (2.21), with $f^2 = f$, is called a reproductive equation. S. Prešić also proved in [12] that for any equation in x

$$x = g(x),$$

which has at least one solution, there exists a reproductive equation which is equivalent to it.

The method of reproductivity can be applied not only to functional equations (as S. Prešić essentially did in [2], [8], [9] and [14]) but to equations in general. Indeed, in **[12]** S. Prešić used reproductive equations to solve a class of linear matrix equations and a system of Boolean equations. Various authors followed S. Prešić and applied the reproductivity method to various kinds of equations. For instance, the present author applied reproductivity to some matrix, integral and differential equations. The work of S.B. Prešić on reproductive equations will be surveyed in a separate paper. Nevertheless, we shall briefly comment the papers [13] and [15] where the idea of reproductivity is implicitly used.

Let E_1 and E_2 be nonempty sets and let $E = E_1 \times E_2^n$ where $n \in N$. Suppose that the mappings $\Theta_1, \ldots, \Theta_n : E_1 \to E_1$ form a group G of order n (Θ_1 is the identity mapping) and suppose that J maps the set E into the set $\{\top, \bot\}$. The so-called general group equation

$$J(x, f(x), f(\Theta_2 x), \ldots, f(\Theta_n x)) = \top \tag{2.22}$$

where $f: E_1 \to E_2$ is the unknown function is solved in [15]. The special case when G is a cyclic group is treated in the previous paper [13]. We describe the construction of the general solution of (2.22).

The equation (2.22) is clearly equivalent to the system (conjunction)

$$\bigwedge_{k=1}^{n} J(\Theta_k x, f(\Theta_k x), f(\Theta_2\Theta_k x), \dots, f(\Theta_n\Theta_k x)) = \top$$

which will simply be denoted by

$$C(x, f(x), f(\Theta_2 x), \dots, f(\Theta_n x)) \tag{2.23}$$

S. Prešić first showed in [15] that (2.22) has a solution if and only if for any $x \in E_1$ there exist $u_1^0, \dots, u_n^0 \in E_2$ such that

$$C(x, u_1^0, \dots, u_n^0).$$

If $x \in E_1$, denote $\Theta_i x$ by $x_i \in E_1$ and $f(\Theta_i\Theta_j x)$ by $u_{ij} \in E_2$ $(i,j = 1,\dots,n)$. Then it is easily seen that the conjunction (2.23) has the following property

$$C(x_i, u_{1i}, u_{2i}, \dots, u_{ni}) \Leftrightarrow C(x_j, u_{1j}, u_{2j}, \dots, u_{nj}) \quad (i,j = 1,\dots,n) \tag{2.24}$$

which follows from the fact that G is a group and hence that $Gg_i = Gg_j$ $(i,j = 1,\dots,n)$.

Introduce the set S by the following definition

$$(x, u_1, u_2, \dots, u_n) \in S \Leftrightarrow C(x, u_1, u_2, \dots, u_n). \tag{2.25}$$

From the equivalences (2.24) and (2.25) it easily follows that

$$(x_i, u_{1i}, \dots, u_{ni}) \in S \Leftrightarrow (x_j, u_{1j}, \dots, u_{nj}) \in S \quad (i,j = 1,\dots,n)$$

Define the function F in the following way:

(i) If $(x_i, u_{1i}, u_{2i}, \dots, u_{ni}) \in S$ then

$$F(x_i, u_{1i}, u_{2i}, \dots, u_{ni}) = u_{1i} \quad (i = 1,2,\dots,n).$$

(ii) If $(x_i, u_{1i}, u_{2i}, \dots, u_{ni}) \notin S$, there exists an n-tuple $(u_1^0, u_2^0, \dots, u_n^0) \in E_2^n$ such that $C(x, u_1^0, u_2^0, \dots, u_n^0)$. Then

$$F(x_i, u_{1i}, u_{2i}, \dots, u_{ni}) = u_i^0 \quad (i = 1,2,\dots,n).$$

The general solution of the group equation (2.22) is given by

$$f(x) = F(x, \Pi(x), \Pi(\Theta_2 x), \dots, \Pi(\Theta_n x))$$

where the function F is defined by (i) and (ii) and $\Pi: E_1 \to E_2$ is arbitrary.

All the papers from this group considered up to now are in a way linked by the implicitely present idea of reproductivity. The remaining paper [16] is different. It contains a proof of a theorem regarding the cyclic equation

$$a_0 f(x) + a_1 f(\Theta x) + \dots + a_{n-1} f(\Theta^{n-1} x) = 0$$

where Θ maps a nonemtpy set S into itself and $\Theta^n(x) = x$ for some $n \in N$, a_i are real (or complex) numbers and $f: S \to R$ is unknown, originally proved in [26, p. 369]. The proof given in [16] is much simpler than the proof given in [26].

3. Some related results

Five more papers [17]–[21] of S.B. Prešić are related to the 16 papers reviewed above. Some simple matrix equations were solved in [17] by a method which anticipates the general reproductive method discovered later. Papers [18] and [21] are devoted to equations on finite sets, paper [19] is a set-theoretic discussion of reproductive solutions and paper [20] is concerned with the so-called algebraic functional equations. Those five papers will be analysed elsewhere.

4. Concluding remarks

D.S. Mitrinović (1908–1995) initiated the work on functional equations in Serbia, he gave full support to younger mathematicians who wished to join him, and it can be said that he founded his "school" of functional equations. A substantial number of papers on functional equations were published in Belgrade by the members of this "school", particularly in the period 1961–1964.

As a young man, in 1959, S.B. Prešić got in touch with D.S. Mitrinović; the result of their cooperation are jointly written papers surveyed above in Section 1. But there was a distinct difference between S. Prešić and other associates of Mitrinović.

Some 20 years ago I was asked by the Mathematical Faculty of Skoplje to write about Mitrinović's contribution to functional equations. I finished my article [27] by the following text:

"As Professor Mitrinović turned over to inequalities, the Belgrade production in functional equations began to decrease, so that nowadays only isolated results are published from time to time. This comment cannot be applied to S.B. Prešić, whose approach to functional equations is essentially different, so that he is not really a member of "Mitrinović's school".

Indeed, S. Prešić not only introduced new methods, but he was chiefly interested in general classes of functional as well as other equations. His ideas were, and still are, used and developed by many mathematicians from Serbia and abroad, so that it can be said that he started his own "school" of equations.

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S. B. PREŠIĆ'S WORK IN REPRODUCTIVE EQUATIONS

Dragić Banković

ABSTRACT. We present three S. Prešić's papers, related to the reproductive solutions of equations, and their influence to the other authors in this subject. In the paper [18] S. Prešić initiated the study of general and reproductive general solutions for the most general concept of equation. He also described all reproductive general solutions of such equation.

In the paper [17] S. Prešić considered the case of equation over a finite set, on which he introduced a certain algebraic structure. S. Prešić gave the formula of the reproductive general solution of this equation.

In the paper [19] S. Prešić considered the equation over a finite set, where the equation was given by a term. Namely, he introduced a kind of generalization of boolean and postian equations in one unknown.

All theorems in this article are either S. Prešić's or written under the immediate influence of S. Prešić's papers.

Introduction

The concept of general solution of an equation was known in various fields of mathematics. Schröder [25] introduced reproductive general solutions of Boolean equations. The term "reproductive" was introduced by Löwenheim [11]. The general solutions were very extensively studied in boolean algebras. The first result within a set-theoretical framework was obtained by S. Prešić [16]. S. Prešić introduced the notion of reproductive equation [16] and he proved that for every equation there exists a reproductive equation equivalent to it. The reproductive equations are of the form $x = f(x)$, where the function f satisfies the condition $f^2 = f$. In that case the formula $x = f(t)$ represents a reproductive general solution of the equation $x = f(x)$. S. Prešić [18] initiated the study of general and reproductive general solutions for the most general concept of equation. S. Prešić

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[17], [19] also considered finite equations and he described all reproductive general solutions of such equations.

General equations

S. Prešić studied general solutions and reproductive general solutions for the most general equations. Namely, let r be a given unary relation of T i.e. $r : T \to \{0,1\}$. S. Prešić considered the equation $r(x) = 1$. The set S of the all elements satisfying the equation $r(x) = 1$ is called the solution set of $r(x) = 1$. The elements of S are called the solutions of $r(x) = 1$. The equation $r(x) = 1$ is consistent if and only if S is not empty. We state S. Prešić's definitions of the general solution and reproductive general solution.

DEFINITION 1. A general solution of a consistent equation $r(x) = 1$ is a function $\phi : T \to T$ which satisfies

$$(\forall t) r(\phi(t)) = 1 \wedge (\forall x)(r(x) = 1 \Rightarrow (\exists t)(x = \phi(t))). \tag{1}$$

DEFINITION 2. A reproductive general solution of a consistent equation $r(x) = 1$ is a function $\psi : T \to T$ if and only if

$$(\forall t) r(\psi(t)) = 1 \wedge (\forall t)(r(t) = 1 \Rightarrow t = \psi(t)). \tag{2}$$

If r and ϕ are determined by formulas, we say that a formula $x = \phi(t)$ represents a general solution of a consistent equation $r(x) = 1$ if and only if the condition (1) is fulfilled. Similarly, if r and ψ are determined by formulas, we say that a formula $x = \psi(t)$ represents a reproductive general solution of a consistent equation $r(x) = 1$ if and only if the condition (2) is fulfilled.

LEMMA. (S. Prešić [18]) *A formula $x = \psi(t)$ represents a reproductive general solution of the equation $r(x) = 1$ if and only if the following conditions hold:*

$$(\forall t \in S)(r(x) = 1 \Rightarrow x = \psi(t))$$
$$(\forall t \in S)(r(x) = 0 \Rightarrow r(\psi(t)) = 1)$$

Supposing that a general solution of the equation $r(x) = 1$ is known, S. Prešić gave, in the next theorem, the formulas of all reproductive general solutions of this equation.

THEOREM 1. (S. Prešić [18]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of $r(x) = 1$. A formula $x = f(t)$, where $f : T \to T$, represents a reproductive general solution of $r(x) = 1$ if and only if there exists a function $h : T \to T$ such that $f(t) = r(t) \cdot t + r'(t) \cdot g(h(t))$.*

Proof. $x = f(t)$ is a reproductive general solution of $r(x) = 1$
$\Leftrightarrow (\forall x \in S)(f(x) = x) \wedge (\forall x \in T \setminus S)(f(x) \in S)$
(Lemma)

$\Leftrightarrow (\forall x \in S)\,(f(x) = x) \wedge (\forall x \in T \setminus S)(\exists t \in T)(f(x) = g(t))$
($x = g(t)$ is a general solution of $r(x) = 1$)
$\Leftrightarrow (\forall x \in S)\,(f(x) = x) \wedge (\exists \overline{h} : T \setminus S \to T)(\forall x \in T \setminus S)(f(x) = g(\overline{h}(x))$
(Axiom of choice)
$\Leftrightarrow (\exists h : T \to T)\,(\forall x \in T)(f(x) = r(x) \cdot x + r'(x) \cdot g(h(x))$
(h is an extension of $\overline{h}$). □

Using Theorem 1 S. Prešić described all reproductive general solutions of boolean equation in the set $\{0, 1\}$. Once S. Prešić applied Horn formulas in order to describe solutions of arbitrary boolean equation. This idea for the use of Horn sentences in boolean algebras was developed in [13] by Ž. Mijajlović. S. Prešić's followers described the formulas of all general reproductive and all general solutions of the equation $r(x) = 1$ in certain ways (Theorems 2–8).

THEOREM 2. (Božić [4]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of $r(x) = 1$. A formula $x = f(t)$, , where $f : T \to T$, represents a general solution of $r(x) = 1$ if and only if there exist functions $h, k : T \to T$ such that $f = g\,h$ and $g = g\,h\,k\,g$.*

THEOREM 3. (Božić [4]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of $r(x) = 1$. A formula $x = f(t)$, where $f : T \to T$, represents a reproductive general solution of $r(x) = 1$ if and only if there exist function $h : T \to T$ such that $g = g\,h\,g$ and $f = g\,h$.*

Božić [4] also solved the functional equation $g = g\,h\,g$.

THEOREM 4. (Rudeanu [22]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of $r(x) = 1$. A formula $x = f(t)$ represents a reproductive general solution of $r(x) = 1$ if and only if there exists a function $h : T \to T$ such that $f(T) \subset S$ and $f = g\,h$.*

THEOREM 5. (Rudeanu [22]) *Let $f, g : T \to T$ be functions such that the formula $x = g(t)$ represents a general solution and the formula $x = f(t)$ represents a reproductive general solution of $r(x) = 1$. Then $g = g\,f$.*

THEOREM 6. (Rudeanu [22]) *Let $f, g : T \to T$ be functions such that the formula $x = g(t)$ represents a reproductive general solution of $r(x) = 1$, $f(T) \subset S$ and $f = g\,f$. Then the formula $x = f(t)$ represents a reproductive general solution of $r(x) = 1$.*

In view of Theorems 4-6, the determination of all general solutions and of all reproductive solutions is reduced to the solution of the functional equations $f = g\,h$ and $f = g\,f$, respectively. This is done in [22].

THEOREM 7. (Banković [1]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of the equation $r(x) = 1$. The formula $x = f(t)$, where $f : T \to T$, represents a general solution of $r(x) = 1$ if and only if there exists a function $h : T \to T$, such that $f = g\,h$ and*

$$(\forall s \in f(T))(\exists x \in T)\, f(h(x)) = s.$$

THEOREM 8. (Banković [1]) *Let $g : T \to T$ be a function such that the formula $x = g(t)$ represents a general solution of the equation $r(x) = 1$. The formula $x = f(t)$, where $f : T \to T$, represents a reproductive general solution of $r(x) = 1$ if and only if there exists a function $h : T \to T$ such that $f = g\,h$ and*

$$(\forall x \in S)\, f(h(x)) = x.$$

Using theorems 7 and 8, P. Smith [24] got three propositions equivalent to the axiom of choice. Note that the use of the axiom of choice in this context was initiated by S. Prešić [18].

Finite equations

S. Prešić considered the case of equation over finite set, on which he introduced a certain algebraic structure. S. Prešić gave the formula of the reproductive general solution of this equation. Let $Q = \{q_1, \dots, q_m\}$ be a finite set of m elements and let E ($|E| > 1$) be the set containing element 0. S. Prešić considered the equation $J(x) = 0$, where $J : S \to E$. Let, for every $q \in Q$, $C_q : Q \to Q$ be a cycle, depending on q. The following notation will be used: $C_q^2(q) = C_q(C_q(q))$, $C_q^3(q) = C_q(C_q^2(q)), \dots$.

Let $A : S \times E^{m-1} \to S$ be the function defined in the following way:

$$\begin{aligned}
&A(q, 0, U_2, \dots, U_{m-1}) = q\\
&A(q, u_1, 0, U_3, \dots, U_{m-1}) = C_q(q)\\
&A(q, u_1, \dots, u_i, 0, U_{i+2}, \dots, U_{m-1}) = C_q^i(q)\\
&\dots\\
&A(q, u_1, \dots, u_{m-2}, 0) = C_q^{m-2}(q)\\
&A(q, u_1, \dots, u_{m-1}) = C_q^{m-1}(q)
\end{aligned}$$

where $q \in S, u_1, \dots, u_{m-1}, U_2, \dots, U_{m-1} \in E$ and $u_1 \neq 0, \dots, u_{m-1} \neq 0$.

THEOREM 9. (S. Prešić [17]) *If the equation $J(x) = 0$ is consistent, then the formula*

$$x = A(t, J(t), J(C_t(t)), \dots, J(C_t^{m-2}(t))) \tag{1}$$

represents the reproductive general solution of the equation $J(x) = 0$.

Proof. Let t be an arbitrary element of Q. If the equation $J(x) = 0$ is consistent then at least one of the elements $J(t), J(C_t(t)), \dots, J(C_t^{m-1}(t))$ is equal to 0. Let $J(C_t^i(t))$ be the first term, in the previous sequence, which is equal 0. Then $x = A(t, J(t), J(C_t(t)), \dots, J(C_t^{m-2}(t)))$ is a solution of $J(x) = 0$, by the definition of the function A.

Let x be a solution of the equation $J(x) = 0$. If we replace, in (1), t by x, we get $A(x, 0, \dots) = x$. □

The function A is called "a solving function". It can be determined, for instance, in the following structure: let $+$ and $\cdot$ be a binary operations on the set $Q \cup E$ satisfying

$$0 \cdot e = e \cdot 0 = 0 \cdot 0 = 0, \quad e \cdot e = e, \quad 0 \cdot q = 0,$$
$$e \cdot q = q, \quad q + 0 = 0 + q = q, \quad 0 + 0 = 0$$

($q \in Q$, e is the fixed element from E, $e \neq 0$). Let $* : E \to E$ and $- : E \to E$ be the functions defined by

$$x^* = \begin{cases} e, & \text{for } x = 0 \\ 0, & \text{for } x \neq 0 \end{cases} \qquad \overline{x} = \begin{cases} 0, & \text{for } x = 0 \\ e, & \text{for } x \neq 0. \end{cases}$$

The function A can be determined by

$$\begin{aligned} A(q, U_1, \dots, U_{m-1}) = {} & \overline{U}_1 \cdot q + U_1^* \cdot \overline{U}_2 \cdot C_q(q) + \cdots \\ & + U_1^* \cdot U_2^* \cdots U_{m-2}^* \cdot \overline{U}_{m-1} \cdot C_q^{m-2}(q) + U_1^* \cdot U_2^* \cdots U_{m-1}^* \cdot C_q^{m-1}(q) \end{aligned}$$

where

$$U + V + W + \cdots + R = (\cdots((U + V) + W) + \cdots + R)$$

and

$$U \cdot V \cdot W \cdots R = (\cdots((U \cdot V) \cdot W) \cdots R).$$

EXAMPLE. Let $Q = E = \{0, 1\}$ and $+$ and $\cdot$ are *max* and *min*, respectively. Boolean equation

$$ax + bx' = 0, \qquad (a, b, x \in \{0, 1\})$$

is consistent if and only if $ab = 0$. Let $e = 1$, $x^* = x$, $\overline{x} = x'$, $C_0(x) = C_1(x) = x'$. The reproductive general solution of $ax + bx' = 0$ is

$$x = \overline{J(t)}\, t + J(t)\, \overline{t} \qquad (J(t) = a\,t + b\,t')$$

i.e.

$$x = (a\,t + b\,t')' + (a\,t + b\,t')\, t'$$

i.e.

$$x = a'\,t + b\,t'.$$

As the previous Example shows, Theorem 9 gives the formula of the reproductive general solution of a given equation.

Using the function A, C. Ghilezan [6] solved the relation $J(x) \in D$, where $D \subset E$. The function A was also used in [7]. C. Ghilezan considered the equation $J(x_1, \dots, x_n) = 0$, where $J : Q^n \to E$.

THEOREM 10. (Ghilezan [6]) *A function $J : Q^n \to E$ can be written in the form*

$$J(x_1,\dots,x_n) = \sum_{a\in S\setminus\{b\}} x_i^a\, g_{ai}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) + h_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

where b is a fixed element from Q and $g_{ai}, h : Q^{n-1} \to E$.

In accordance with Theorem 9, the equation $J(x_1,\dots,x_n) = 0$ can be written in the form

$$\sum_{i=1}^{m} x_i^{q_i} g_{i1}(x_2,\dots,\dots,x_n) = 0.$$

The latter equation is consistent if and only if

$$\prod_{i=0}^{m} g_{i1}(x_2,\dots,x_n) = 0.$$

In this way, the unknown x_1 is eliminated. In the similar way, using successive elimination, the equation $J(x_1,\dots,x_n) = 0$ can be solved.

S. Prešić's Theorem 9 was the motivation for the following Theorem.

THEOREM 11. (M. Prešić [15]) *Let $J(x) = 0$ be equation on Galois field $GP(p^n)$ and let α be the generic element of the cyclic group of that field. If equation $J(x) = 0$ is consistent then the formula*

$$\begin{aligned} x = t &+ (J(t))^{p^n-1} + \alpha\,(J(t)J(t+1))^{p^n-1} + \alpha^2\,(J(t)J(t+1)J(t+1+\alpha))^{p^n-1} + \\ &\cdots + \alpha^{p^n-3}(J(t)\,J(t+1)\cdots J(t+1+\alpha+\cdots+\alpha^{p^n-4}))^{p^n-1} \\ &+ (\alpha^{p^n-2}+s)\,((J(t)\,J(t+1)+\cdots+J(t+1+\alpha+\cdots+\alpha^{p^n-3}))^{p^n-1} \end{aligned}$$

represents the reproductive general solution of the equation $J(x) = 0$, where $s = 2 + 3e + \cdots + (p^n-1)\,\alpha^{p^n-3}$.

Further study of finite equations

S. Prešić considered the equation over finite set, where the equation was given by the term. Namely, he introduced a kind of generalization of boolean and postian equations in one unknown. Let $Q = \{q_0, q_1, \dots, q_m\}$ be a given set of $m+1$ elements. Define the operations $+$, $\cdot$ and x^y in the following way:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

$$x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise } (x, y \in Q \cup \{0,1\}). \end{cases}$$

Assuming that

$$(\forall x \in \{0,1\} \cup Q)(x+0=x \wedge 0+x=x \wedge x\cdot 0=0 \wedge x\cdot 1=x \wedge 1\cdot x=x)$$

S. Prešić considered the following x-equation

$$s_0 \cdot x^{q_0} + s_1 \cdot x^{q_1} + \cdots + s_m \cdot x^{q_m} = 0 \tag{3}$$

where $s_i \in \{0,1\}$, $x \in Q$. The latter equation is consistent (has a solution) if and only if $s_0 \cdot s_1 \cdots s_m = 0$. Denote by S the solution set of the equation (3).

In the sequel $\cdot$ will be omitted.

One can prove that every equation over Q is equivalent to an equation of the form $s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0$. Note that $h(x) = s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m}$ is the function which maps the set Q in the set $\{0,1\}$ i.e. $h : Q \to \{0,1\}$.

S. Prešić introduced "the zero-set" $Z(a_0, \dots, a_m)$ of $(a_0, \dots, a_m)$ in the following way

$$q_i \in Z(_0, \dots, a_m) \Leftrightarrow a_i = 0 \quad (i = 0, 1, \dots, m).$$

THEOREM 12. (S. Prešić [19]) *Let the equation*

$$s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0 \tag{3}$$

be consistent (i.e. $s_0\, s_1 \cdots s_m = 0$). A formula $x = A(t)$ represents a reproductive general solution of the equation (3) if and only if $A(t)$ is of the form

$$A(t) = \sum_{k=0}^{m} (q_k s_k^0 + \sum_{a_k \neq 0, a_0 \cdots a_m = 0} F_k(a_0, \dots, a_m) s_0^{a_0} \dots s_n^{a_n}) t^{b_k}. \tag{4}$$

where $F_k(a_0, \dots, a_m) \in Z(a_0, \dots, a_m)$.

Proof. Let $A(t)$ be of the form (4). For arbitrary $t \in Q$ there exists $k \in \{0, 1, \dots, m\}$ such that $t = q_k$. If $q_k \in Z(s_0, s_1, \dots, s_m)$, then the formula $x = A(t)$ gives $t = q_k$, because $q_i \in Z(s_0, \dots, s_m) \Leftrightarrow s_i = 0$. If $q_k \notin Z(s_0, s_1, \dots, s_m)$, then the formula $x = A(t)$ becomes $x = F_k(s_0, \dots, s_m)$. We also have $F_k(s_0, \dots, s_m) \in Z_k(s_0, \dots, s_m)$. Therefore the formula $x = A(t)$ represents a reproductive general solution of (3).

Let the formula $x = A(t)$ represents a reproductive general solution of (3). If we write the function A as

$$A = \begin{pmatrix} q_0 & q_1 & \cdots & q_m \\ q_{i_0} & q_{i_1} & \cdots & q_{i_m} \end{pmatrix}$$

then $\{q_{i_0}, q_{i_1}, \dots, q_{i_m}\} = Z(s_0, s_1, \dots, s_m)$. If $q_k \notin Z(s_0, s_1, \dots, s_m)$, then we determine $F_k(s_0, s_1, \dots, s_m)$ in the following way:

$$F_k(s_0, s_1, \dots, s_m) = q_{i_k} \qquad (k = 0, 1, \dots, m).$$

Since A represents a reproductive general solution of (3) we have

$$q_k \in F_k(s_0, s_1, \ldots, s_m) \Rightarrow A(q_k) = q_k.$$

Since

$$q_k \in Z(s_0, s_1, \ldots, s_m) \Rightarrow s_k = 0 \Rightarrow q_k\, s_k^0 = q_k$$

and the sum

$$\sum_{a_k=0, a_0\cdots a_m=0} F_k(s_0, s_1, \ldots, s_m)\, s_0^{a_0} \cdots s_m^{a_m}$$

reduces to $F_k(s_0, s_1, \ldots, s_m)$, the function A can be written as

$$A(t) = \sum_{k=0}^{m} (q_k s_k^0 + \sum_{a_k \neq 0, a_0\cdots a_m=0} F_k(a_0, \ldots, a_m) s_0^{a_0} \ldots s_n^{a_n}) t^{b_k}. \quad \square$$

Let $M = \{0, 1, \ldots, m\}$).

THEOREM 13. (Banković [2]) *Let the equation*

$$(3) \qquad s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0$$

be consistent (i.e. $s_0\, s_1 \cdots s_m = 0$*). A formula* $x = A(t)$ *represents a general solution of the equation (3) if an only if there exists a permutation* $h : M \to M$ *such that*

$$A(t) = \sum_{k=0}^{m} (q_{h(k)}\, s_{h(k)}^0 + \sum_{a_{h(k)} \neq 0, a_0\cdots a_m=0} F_{h(k)}(a_0, \ldots, a_m) s_0^{a_0} \ldots s_n^{a_n}) t^{b_k}.$$

where, for every $k \in M$ *and every* $(m+1)$*-tuple* $(a_0, \ldots, a_m) \in \{0, 1\}$*, holds*

$$a_0 \cdots a_m = 0 \Rightarrow F_{h(k)}(a_0, \ldots, a_m) \in Z(a_0, \ldots, a_m).$$

THEOREM 14. (Rudeanu [23]) *Let the equation*

$$(3) \qquad s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0$$

be consistent (i.e. $s_0\, s_1 \cdots s_m = 0$*). A formula* $x = A(t)$ *represents a general solution of the equation (3) if an only if* $A(t)$ *is of the form*

$$A(t) = \sum_{k=0}^{m} (q_{h(k)}\, s_{h(k)}^0 + s_{h(k)}\, r_k)\, t^{b_k}$$

where $h : M \to M$ *is a permutation of* M *and* $r_k \in S$ $(k = 0, 1, \ldots, m)$.

THEOREM 15. (Rudeanu [23]) *Let the equation*

$$(3) \qquad s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0$$

be consistent (i.e. $s_0\, s_1 \cdots s_m = 0$*). A formula* $x = A(t)$ *represents a reproductive general solution of the equation (3) if an only if* $A(t)$ *is of the form*

$$A(t) = \sum_{k=0}^{m} (q_k\, s_k^0 + s_k\, r_k)\, t^{b_k}$$

where $r_k \in S$ $(k = 0, 1, \dots, m)$.

The previous four theorems describe all reproductive general solutions and all general solutions of a finite equation, supposing that particular solutions are known. Next theorem, where the idea of S. Prešić's solving function from Theorem 9 is present, describes all general solutions (including all reproductive general solutions) of a finite equation without the above supposition.

THEOREM 16. (Banković [3]) *Let the equation*

$$(3) \qquad s_0\, x^{q_0} + s_1\, x^{q_1} + \cdots + s_m\, x^{q_m} = 0$$

be consistent (i.e. $s_0\, s_1 \cdots s_m = 0$*). A formula* $x = A(t)$ *represents a (reproductive) general solution of the equation (3) if an only if*

$$A(t) = \sum_{k=0}^{m} \big(s_{i_{k,0}}^0\, q_{i_{k,0}} + s_{i_{k,0}}\, s_{i_{k,1}}^0\, q_{i_{k,1}} + s_{i_{k,0}}\, s_{i_{k,1}}\, s_{i_{k,2}}^0\, q_{i_{k,1}} + s_{i_{k,0}}\, s_{i_{k,1}} \cdots s_{i_{k,m-1}}^0\, q_{i_{k,m-1}} + s_{i_{k,0}}\, s_{i_{k,1}} \cdots s_{i_{k,m-1}}\, q_{i_{k,m}}\big) t^{q_k}$$

under the following conditions:

$(i_{k,0}, i_{k,1}, \dots, i_{k,m})$ *are permutations of* $\{0, 1, \dots, m\}$,
$(i_{k,0}, i_{k,1}, \dots, i_{k,m})$ *are permutations of* $\{0, 1, \dots, m\}$,
(under the conditions
$(i_{k,0}, i_{k,1}, \dots, i_{k,m})$ *are permutations of* $\{0, 1, \dots, m\}$,
$(i_{0,0}, i_{1,0}, \dots, i_{m,0}) = (0, 1, \dots, m)$*).*

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ON S.B. PREŠIĆ'S TYPE GENERALIZATION OF BANACH CONTRACTION MAPPING PRINCIPLE

Ljubomir Ćirić

Many problems in pure and applied mathematics reduce to convergence problems of corresponding sequences. For this reason the studying of convergence problems, first in the structure $\mathcal{R}$ of reals and later in more general spaces, is very actual for decades. In 1965 young mathematician Slaviša Prešić [3], [4] gave his significant contribution to that area. He devised a very elegant and a new way by which in complete metric spaces he investigated the convergence problem of sequences defined recursively. In such a way he also generalized in a natural way the well known result of S. Banach (1922).

THEOREM 1. [1] *Let (E, d) be a complete metric space and let $f : E \to E$ be a mapping which satisfies the following contraction condition*

$$d(f(x), f(y)) \leq qd(x, y) \qquad \text{for all } x, y \in E \tag{1}$$

where $q \in (0, 1)$ is a constant. Then for each $x_0 \in E$, the sequence $\langle x_n \rangle$ defined by

$$x_{n+1} = f(x_n) \qquad (n = 0, 1, \dots) \tag{2}$$

converges to a point $\xi \in E$ and ξ is the unique fixed point of f.

Briefly said, a proof of this theorem can flow as follows.
Starting with (2) we have the following equalities

$$x_{n+2} = f(x_{n+1}), \qquad x_{n+1} = f(x_n) \tag{3}$$

hence we obtain

$$d(x_{n+2}, x_{n+1}) = d(f(x_{n+1}), f(x_n)) \tag{4}$$

Using (1) and the denotation $\Delta_n = d(x_{n+1}, x_n)$ we have the following inequality

$$\Delta_{n+1} \leq q\Delta_n \tag{5}$$

In a trivial way from (5) we obtain the following inequality

$$\Delta_n \leq q^{n-1}\Delta_1 \qquad (n = 1, 2, \dots) \tag{6}$$

As it is well known, having this estimation for Δ_n one can easily complete the proof.

Having in mind a personal communication by S. Prešić we shall describe the methodological idea which he used.

First, he considered a mapping $f : E^k \to E$, where $k = 1, 2, \dots$ is a constant, and as before E is a complete metric space. Related to this f he defined a sequence $\langle x_n \rangle$ like (2)

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \qquad (n = 1, 2, \dots) \tag{2'}$$

Of course, in general the convergence problem for such a sequence $\langle x_n \rangle$ is very difficult. Trying to follow steps similar to (3), (4), (5), (6) S. Prešić did the following

First, similarly to (3) he considered the equalities

$$x_{n+k+1} = f(x_{n+1}, x_{n+2}, \dots, x_{n+k}), \quad x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \tag{3'}$$

Second, similarly to (4) he formed the equality

$$d(x_{n+k+1}, x_{n+k}) = d(f(x_{n+1}, x_{n+2}, \dots, x_{n+k}), f(x_n, x_{n+1}, \dots, x_{n+k-1})) \tag{4'}$$

But now we do not have a condition like (1). At this step S. Prešić generalized (1) by the following condition

(1′)

$$d(f(t_1, t_2, \dots, t_k), f(t_2, t_3, \dots, t_{k+1})) \leq a_1 d(t_1, t_2) + a_2 d(t_2, t_3) + \cdots + a_k d(t_k, t_{k+1})$$

where $t_1, \dots, t_k, t_{k+1}$ are any elements of E and $a_1, \dots, a_k$ are non-negative constants whose sum $a_1 + \cdots + a_k$ is less then 1. If $k = 1$ the condition (1′) reduces to (1).

Now supposing the condition (1′), from (4′) we obtain

$$\Delta_{n+k} \leq a_1\Delta_n + a_2\Delta_{n+1} + \cdots + a_k\Delta_{n+k-1} \tag{5'}$$

where Δ_n is an abbreviation for $d(x_{n+1}, x_n)$. Unlike (5) now we have a non-trivial inequality. One way to derive some estimation for Δ_n is to involve the algebraic equation $x^k = a_1 + a_2 x^2 + \cdots + a_k x^{k-1}$, the spectrum of all its solutions, and so on. Obviously, such a way is rather complicated.

S. Prešić devised a very nice way to overcome these difficulties. Namely, he first proved the following assertion.

ASSERTION. *It the sequence Δ_n of non-negative reals satisfies (5′), where $a_i \geq 0$, $a_1 + a_2 + \cdots + a_k < 1$, then there exist two positive constants $K4$ and θ (with $\theta \in (0,1)$) such that the following inequality holds for every n*

$$\Delta_n \leq K\theta^n \tag{7}$$

The main problem is how to find K and θ? Suppose that somewhere we have seen the inequality (7) but now we cannot remember the values for K and θ. Despite this, we are going to prove (7). According to the structure of (5′) we use the induction of the following type $m, m+1, \cdots, m+k-1 \vdash m+k$. Hypothetical inductive proof reads

Base of induction. The inequalities

$$\Delta_1 \leq K\theta, \quad \Delta_2 \leq K\theta^2, \quad \cdots \quad, \Delta_k \leq K\theta^k \tag{8}$$

are true[1]

Induction step. Suppose that (7) is true when n is m, $m+1, \ldots m+k-1$, i.e., that the following inequalities

$$\Delta_m \leq K\theta^m, \quad \Delta_{m+1} \leq K\theta^{m+1}, \quad \ldots, \quad \Delta_{m+k-1} \leq K\theta^{m+k-1}$$

hold. From these inequalities we easily infer the following one

$$a_1\Delta_m + a_2\Delta_{m+1} + \cdots + a_k\Delta_{m+k-1} \leq K\theta^m(a_1 + a_2\theta + \cdots + a_k\theta^{k-1})$$

Using (5′) we obtain the following inequality

$$\Delta_{m+k} \leq K\theta^m(a_1 + a_2\theta + \cdots + a_k\theta^{k-1})$$

Obviously the inductive proof will be completed if the following inequalities

$$a_1 + a_2\theta + \cdots + a_k\theta^{k-1} \leq \theta^k, \qquad 0 < \theta < 1 \tag{9}$$

hold. So, the hypothetical proof will become a real proof if we can find K and θ satisfying the inequalities (8) and (9). This can be done as follows:

First, consider the function $g : R \to R$ defined by $g(x) = x^k - (a_1 + a_2x + \cdots + a_kx^{k-1})$. Since $g(1) > 0$, there exists $\theta \in (0,1)$ such that (9) is satisfied.

Second, using such a θ we define K in the following way

$$K = \max\left(\frac{\Delta_1}{\theta}, \frac{\Delta_2}{\theta^2}, \ldots, \frac{\Delta_k}{\theta^k}\right)$$

Obviously with such K and θ both (8) and (9) are satisfied. In such a way we have composed the S. Prešić's proof mentioned before.

By means of the assertion (7) S. Prešić proved the following theorem, a generalization of Theorem 1

[1] As a matter of fact, we now see that K and θ must satisfy the conditions (8)

THEOREM 2. [3], [4] *Let* (E,d) *be a complete metric space and let* $f: E^k \to E$ *be a mapping satisfying the contraction condition* (1′). *Then there exists the unique* $\xi \in E$ *such that* $f(\xi,\xi,\dots,\xi) = \xi$. *This* ξ *is the limit value of the sequence* $\langle x_n \rangle$ *defined by*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \qquad (n = 1, 2, \dots)$$

where $x_1, \dots, x_k$ *are arbitrarily chosen elements of* E.

Notice that in the case $k = 1$ this theorem reduces to Banach's Theorem 1. As I know, Theorem 2 is one of the first generalizations of Banach's theorem. Also many mathematicians were inspired by that S. Prešić's result; for instance M. Marjanović, M. Tasković, D. Arandjelović, Lj. Ćirić, B. C. Dhage.

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CONTRIBUTION AND INFLUENCE OF S.B. PREŠIĆ TO NUMERICAL FACTORIZATION OF POLYNOMIALS

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ABSTRACT. This paper is devoted to contributions of S.B. Prešić in numerical factorization of algebraic polynomials, as well as to influence of his work in this subject. Beside a general factorization of polynomials, we consider some important special cases and point out some accelerated iterative formulas.

1. Introduction

The numerical factorization of algebraic polynomials is a very important mathematical subject. There are several methods for it in the literature, beginning with the well-known methods of Bairstow [2] and of Lin [19–20]. Many of them are quadratically convergent, but most require a sufficiently close starting values for factorization. In their survey paper, Householder and Stewart [14] mentioned also the method of Graeffe and the *qd* algorithm, though they are not primarily for this assignment. A number of these methods can be related to an algorithm proposed by Sebastião e Silva [38]. Some generalizations of this algorithm were given by Householder [11] in 1971 (see also [12], [41], [6]). In addition we mention also a method of Samelson [36] from 1959, which generalizes the Bauer-Samelson iteration [3]. In his paper Samelson noted that his method is related to Bairstow's method. Taking a monic algebraic polynomial over the field of complex numbers, with zeros $z_1, z_2, \ldots, z_n$, i.e.,

$$P(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n = \prod_{k=1}^{n} (z - z_k), \tag{1.1}$$

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Samelson [36] seeks its factorization by two factors

$$u(z) = (z - z_1)(z - z_2) \cdots (z - z_m)$$

and

$$v(z) = (z - z_{m+1})(z - z_{m+2}) \cdots (z - z_n).$$

Let p and q be monic polynomials of degree m and $n - m$ approximating u and v repectively. Then his quadratically convergent iterative procedure defines improved approximations p^* and q^* by the formula

$$p^* q + q^* p = P + pq. \tag{1.2}$$

If p and q are relatively prim, then p^* and q^* are uniquely defined by (1.2) Samelson's iteration was discovered independently by Stewart [39], who characterized pq^* as the linear combination of $P, q, zq, \ldots, z^{m-1}q$ that is divisible by p. Householder and Stewart [13] gave the exact connection between these characterizations (see also [14] and [40] for another derivation of Samelson's method and the corresponding error bounds for the iteration, as well as the paper of Schröder [37] for a connection with Newton's method).

In 1966 and 1968 S.B. Prešić [34–35], inspired only by some results of D. Marković [21], gave an iterative method for numerical factorization of algebraic polynomials by s $(2 \leq s \leq n)$ factors. The purpose of the present paper is to show contributions of S. Prešić, as well as to point out an influence of S. Prešić's work to this subject. The paper is organized as follows. In Section 2 we explain S. Prešić's approach to numerical factorization of polynomials and give an example on $2 - 2$ factorization of a polynomial of fourth degree. Sections 3 and 4 are dedicated to an $1 - 1 - \cdots - 1$ factorization and some accelerated iterative formulas, respectively.

Later, in 1969 Dvorčuk [9] considered a factorization into quadratic factors, and in 1971 Grau [10] used a Newton-type of approximation for simultaneously improving a complete set of approximate factors for a given polynomial. Recently, Carstensen [4] and Carstensen and Sakurai [5] gave some generalizations of this method.

Here, we mention also that in the last period many papers have been published on factorization of polynomials over finite fields, on factorization methods for multivariable polynomials, as well as on factorization of matrix polynomials.

2. S. Prešić's approach to numerical factorization

Let P be a monic algebraic polynomial over the field of complex numbers given by (1.1) and let it be expressed in a factorized form

$$P(z) = A_1(z)A_2(z) \cdots A_s(z) \qquad (2 \leq s \leq n), \tag{2.1}$$

where $A_\nu(z)$ are monic polynomials of degree n_ν, i.e.,

$$A_\nu(z) = \sum_{i=0}^{n_\nu} a_{\nu i} z^{n_\nu - i}, \qquad a_{\nu 0} = 1 \quad (\nu = 1, 2, \ldots, s), \tag{2.2}$$

and $\sum\limits_{\nu=1}^{s} n_\nu = n$. The case $s = 2$ is mentioned in the previous section.

Assuming that zeros of (1.1) are simple, S.B. Prešić gave an iterative method for numerical determination such a factorization, so-called $n_1 - n_2 - \cdots - n_s$ factorization, in which successive iterated monic factors

$$A_\nu^{(k)}(z) = \sum_{i=0}^{n_\nu} a_{\nu i}^{(k)} z^{n_\nu - i}, \quad a_{\nu 0}^{(k)} = 1 \qquad (\nu = 1, 2, \ldots, s) \tag{2.3}$$

are determined from the relation

$$A_1^{(k+1)} A_2^{(k)} \cdots A_s^{(k)} + A_1^{(k)} A_2^{(k+1)} \cdots A_s^{(k)} + \cdots + A_1^{(k)} A_2^{(k)} \cdots A_s^{(k+1)} - (s-1) A_1^{(k)} A_2^{(k)} \cdots A_s^{(k)} = P,$$

i.e.,

$$A_1^{(k)}(z) A_2^{(k)}(z) \cdots A_s^{(k)}(z) \left(\sum_{\nu=1}^{s} \frac{A_\nu^{(k+1)}(z)}{A_\nu^{(k)}(z)} - s + 1 \right) \equiv P(z). \tag{2.4}$$

Taking the coefficients $a_{\nu i}$ of polynomials (2.2) as coordinates of an n-dimensional vector

$$\boldsymbol{a} = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n_1} \quad a_{21} \quad a_{2n_2} \quad \cdots \quad a_{22} \quad \cdots \quad a_{s1} \quad a_{22} \quad \cdots \quad a_{sn_s}]^T$$

and $a_{\nu i}^{(k)}$ (coefficients of iterated factors (2.3)) as coordinates of the corresponding also n-dimensional vector $\boldsymbol{a}^{(k)}$, S. Prešić observed that (2.4) implies a system of linear equations of the form

$$A_n(\boldsymbol{a}^{(k)}) \boldsymbol{a}^{(k+1)} = \boldsymbol{b}_n(\boldsymbol{a}^{(k)}, \boldsymbol{p}) \tag{2.5}$$

where A_n is an $n \times n$ matrix depending only on $\boldsymbol{a}^{(k)}$, and $\boldsymbol{b}_n$ is an n-dimensional vector depending also on $\boldsymbol{a}^{(k)}$ and on coefficients of the polynomial (1.1), $\boldsymbol{p} = [p_1 \quad p_2 \quad \cdots \quad p_n]^T$. Further, he concluded that there exists a neighbourhood V of $\boldsymbol{a} \in \mathbb{C}^n$, such that (2.5) can be expressed in the following form

$$\boldsymbol{a}^{(k+1)} = F(\boldsymbol{a}^{(k)}) \qquad (k = 0, 1, \ldots;\ \boldsymbol{a}^{(k)} \in V), \tag{2.6}$$

where $F\colon V \to V$ is an enough times differentiable operator (in Fréchet sense). Practically, S. Prešić proved that $F(\boldsymbol{a}) = \boldsymbol{a}$ and $F'_{(\boldsymbol{a})}$ is a zero operator, so that

$$\|\boldsymbol{a}^{(k+1)} - \boldsymbol{a}\| = O(\|\boldsymbol{a}^{(k)} - \boldsymbol{a}\|^2) \qquad \Big(\boldsymbol{a} = \lim_{k \to +\infty} \boldsymbol{a}^{(k)}\Big).$$

Thus, S. Prešić's result can be sumarized as:

THEOREM 1.1. *There is an neighbourhood V of $\boldsymbol{a} \in \mathbb{C}^n$ so that for an arbitrary $\boldsymbol{a}^{(0)} \in V$, the iterative process (2.6) quadratically converges to $\boldsymbol{a}$.*

Thus,

$$\lim_{k\to+\infty} A_\nu^{(k)}(z) = A_\nu(z) \qquad (\nu = 1,2,\dots,s),$$

give the factorization (2.1).

In his paper [35], S. Prešić derived formulas for a $2-2-2$ factorization of a polynomial of degree 6. Here, as an illustration, we give a simpler case when $P(z) = z^4 + p_1z^3 + p_2z^2 + p_3z + p_4$ and when we seek its $2-2$ factorization, with

$$A_1(z) = z^2 + a_{11}z + a_{12}, \quad A_2(z) = z^2 + a_{21}z + a_{22}.$$

In that case the system (2.5) becomes

$$\begin{aligned}
a_{11}^{(k+1)} \qquad\qquad + \qquad a_{21}^{(k+1)} \qquad\qquad\qquad &= b_1^{(k)},\\
a_{21}^{(k)}a_{11}^{(k+1)} + \quad a_{12}^{(k+1)} + a_{11}^{(k)}a_{21}^{(k+1)} + \quad a_{22}^{(k+1)} &= b_2^{(k)},\\
a_{22}^{(k)}a_{11}^{(k+1)} + a_{21}^{(k)}a_{12}^{(k+1)} + a_{12}^{(k)}a_{21}^{(k+1)} + a_{11}^{(k)}a_{22}^{(k+1)} &= b_3^{(k)},\\
a_{22}^{(k)}a_{12}^{(k+1)} \qquad\qquad + a_{12}^{(k)}a_{22}^{(k+1)} &= b_4^{(k)},
\end{aligned}$$

where

$$\begin{aligned}
b_1^{(k)} &= p_1, & b_2^{(k)} &= p_2 + a_{11}^{(k)}a_{21}^{(k)},\\
b_3^{(k)} &= p_3 + a_{11}^{(k)}a_{22}^{(k)} + a_{12}^{(k)}a_{21}^{(k)}, & b_4^{(k)} &= p_4 + a_{12}^{(k)}a_{22}^{(k)}.
\end{aligned}$$

Solving this system we obtain an iterative procedure of the form (2.6). This case $(s = 2)$ reduces to Samelson's iteration.

Using the previous idea on polynomial factorization, J.J. Petrić and S.B. Prešić [32] treated a problem of simultaneous determination of all solutions of the system of algebraic equations

$$\begin{aligned}
J_1(x,y) &\equiv A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0,\\
J_2(x,y) &\equiv A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0.
\end{aligned}$$

3. Factorization $1-1-\cdots-1$

In the case $s = n$, i.e., $n_\nu = 1$ $(\nu = 1,2,\dots,n)$, the factors are linear

$$A_\nu(z) = z + a_{\nu 0} = z - z_\nu \qquad (\nu = 1,2,\dots,n),$$

and (2.4) reduces to

$$(z - z_1^{(k)})(z - z_2^{(k)})\cdots(z - z_n^{(k)})\left(\sum_{\nu=1}^{n} \frac{z - z_\nu^{(k+1)}}{z - z_\nu^{(k)}} - n + 1\right) \equiv P(z).$$

Then, the scalar form of (2.6) can be obtained easily as

$$(3.1)\qquad z_\nu^{(k+1)} = z_\nu^{(k)} - \frac{P(z_\nu^{(k)})}{\prod\limits_{\substack{j=1\\ j\neq\nu}}^{n}\left(z_\nu^{(k)} - z_j^{(k)}\right)} \qquad (\nu = 1,2,\dots,n;\ k = 0,1,\dots).$$

Thus, in this important case, S. Prešić's factorization approach leads to the Weierstrass' formulas (3.1) (see [44]), which were not well-known in that period. These formulas were obtained several times in various ways by many authors. Weierstrass used them in a new constructive proof of fundamental theorem of algebra. In a book on numerical solution of algebraic equations from 1960, written by French mathematician E. Durand [8], one chapter was dedicated to iterative methods for simultaneous finding polynomial zeros, where the author obtained formulas (3.1) in an implicit form. It seems that Bulgarian mathematician K. Dočev [7] was the first who used these formulas in their original form for numerical calculation and who proved their quadratic converegence.

Introducing $Q(z) = \prod\limits_{j=1}^{n}(z - z_j^{(k)})$, formulas (3.1) can be represented in the form (Newtonian type)

$$(3.2)\qquad z_\nu^{(k+1)} = z_\nu^{(k)} - \frac{P(z_\nu^{(k)})}{Q'(z_\nu^{(k)})} \qquad (\nu = 1,2,\dots,n;\ k = 0,1,\dots).$$

Beside the polynomial $Q(z)$ we consider also polynomials $R_\nu(z)$ defined by

$$R_\nu(z) = \frac{Q(z)}{z - z_\nu^{(k)}} = \prod_{\substack{j=1\\ j\neq\nu}}^{n}\left(z - z_j^{(k)}\right) \qquad (\nu = 1,2,\dots,n).$$

Their expanded forms are

$$Q(z) = z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \dots + (-1)^n\sigma_n,$$
$$R_\nu(z) = z^{n-1} - \sigma_1^{(\nu)} z^{n-2} + \sigma_2^{(\nu)} z^{n-3} - \dots + (-1)^{n-1}\sigma_{n-1}^{(\nu)},$$

where $\sigma_1, \sigma_2, \dots, \sigma_n$ are elementary symmetric functions of $z_1, z_2, \dots, z_n$ (see [23, Section 1.3.1]). For the sake of simplicity, we omit the upper index in $z_\nu^{(k)}$, and for $z_\nu^{(k+1)}$ we use the notation $\hat{z}_\nu$. Similarly, $\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \dots, \sigma_{n-1}^{(\nu)}$ are also such functions that do not involve z_ν. It is easy to see that $Q'(z_\nu) = R_\nu(z_\nu)$ ($\nu \in I = \{1,2,\dots,n\}$). In the note [43], which was our first paper in mathematics inspired only by the S. Prešić paper [35], we showed: *If all zeros of $Q(z)$ are simple, then the inverse matrix of*

$$(3.3)\qquad W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_1^{(2)} & & \sigma_1^{(n)} \\ \vdots & & & \\ \sigma_{n-1}^{(1)} & \sigma_{n-1}^{(2)} & & \sigma_{n-1}^{(n)} \end{bmatrix}$$

is given by

$$(3.4)\qquad W^{-1}=\begin{bmatrix} D_1z_1^{n-1} & -D_1z_1^{n-2} & \cdots & (-1)^{n-1}D_1 \\ D_2z_2^{n-1} & -D_2z_2^{n-2} & \cdots & (-1)^{n-1}D_2 \\ \vdots & & & \\ D_nz_n^{n-1} & -D_nz_n^{n-2} & \cdots & (-1)^{n-1}D_n \end{bmatrix},$$

with $D_\nu = 1/Q'(z_\nu)$ $(\nu \in I)$.

The corresponding S. Prešić's form (2.6), i.e., a vector form of (3.2) can be written as

$$(3.5)\qquad \boldsymbol{z}^{(k+1)} = T(\boldsymbol{z}^{(k)}) \qquad (k=0,1,\ldots),$$

where $T(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{e}(\boldsymbol{z})$ and

$$\boldsymbol{z}=\begin{bmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{bmatrix},\quad \boldsymbol{e}(\boldsymbol{z})=\begin{bmatrix} e_1\\ e_2\\ \vdots\\ e_n\end{bmatrix},\quad e_\nu=\frac{P(z_\nu)}{Q'(z_\nu)},\quad Q(z)=\prod_{j=1}^{n}(z-z_j).$$

Taking the system of Viète's formulas for polynomial $P(z)$, given by (1.1), $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$, where the i-th coordinate in the vector $\boldsymbol{f}(\boldsymbol{z})$ is equal to $\sigma_i + (-1)^{i-1}p_i$ $(i = 1, 2, \ldots, n)$, and applying the known iterative procedure of Newton-Kantorovič,

$$(3.5)\qquad \boldsymbol{z}^{(k+1)} = \boldsymbol{z}^{(k)} - W^{-1}(\boldsymbol{z}^{(k)})\boldsymbol{f}(\boldsymbol{z}^{(k)}) \qquad (k=0,1,\ldots),$$

in order to solve the previous system of nonlinear equations, we obtain (3.5). Here, the Jacobi matrix is exactly given by (3.3) and its inverse by (3.4). It seems that Kerner [16] was the first who observed this fact. His proof was slightly different from ours.

Regarding to the iterative method (3.2), in 1980 Dirk P. Laurie [18] stated the following problem: *If* $\sum_{\nu=1}^{n} z_\nu = -p_1$, *prove that*

$$(3.6)\qquad \sum_{\nu=1}^{n} \hat{z}_\nu = -p_1.$$

It is a nice property of the method (3.2) and it was known earlier (see Dočev [7]).

Relation (3.6) holds regardless of the value of $\sum_{\nu=1}^{n} z_\nu$. We gave now a proof of that as an application of the Cauchy residue method and it was published in the book [26, pp. 347–348]. Indeed, since $\hat{z}_\nu = z_\nu - P(z_\nu)/Q'(z_\nu)$, $\nu = 1, 2, \ldots, n$, we have

$$(3.7)\qquad \sum_{\nu=1}^{n} \hat{z}_\nu = \sum_{\nu=1}^{n} z_\nu - \sum_{\nu=1}^{n} \frac{P(z_\nu)}{Q'(z_\nu)}.$$

No doubt that the S. Prešić's work on this area is very important and that it has a great influence on the development of this field in our country. In the last thirty years several mathematicians in Serbia, especially those from the University of Niš and University of Novi Sad, have been very active in this field. For the references see, for instance, [28] and [29].

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CONTRIBUTIONS OF S. B. PREŠIĆ TO THE FIELD OF APPLIED MATHEMATICS

Vera V. Kovačević-Vujčić

ABSTRACT. We give an overview of S. Prešić's results related to problems in different disciplines of applied mathematics such as numerical analysis, optimization, interval mathematics, etc. In particular, we discuss results on factorization of polynomials and the theory of m–M calculus.

1. Introduction

During his long and fruitful mathematical career Prof. Slaviša Prešić was attracted with many areas in mathematics such as algebra, functional equations, foundations, logic, computer science and applied mathematics, including numerical mathematics and optimization. Our aim is to give a brief survey of Presic's results in applied mathematics. The paper is organized as follows:

In Section 2 we present S. Prešić's results on factorization of polynomials and solution of systems of nonlinear algebraic equations published in [3], [4], [5]. Section 3 is devoted to the m–M calculus, the theory proposed by S. Prešić in [6], [7], [8]. This theory combines in an original way the ideas of mathematical logic and numerical analysis and has various applications in numerical analysis, interval mathematics, optimization, etc.

2. Factorization of polynomials and related results

During the sixties Slaviša Prešić was an assistant of Prof. D. Marković who encouraged him to study problems of solving algebraic equations. The work in this direction has resulted in three papers which will be outlined here.

In [3] S. Prešić proposes an iterative method for factorization of polynomials of degree n with complex coefficients. Let

$$P = P(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0$$

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be a given polynomial and assume that all its roots have multiplicity one. Let n be decomposed into $s+1$ summands, i.e. $n = a + b + \cdots + l$ and let $A, B, \ldots, L$ be polynomials of degree $a, b, \ldots, l$, respectively:

$$\begin{aligned} A &= x^a + \alpha_{a-1}x^{a-1} + \cdots + \alpha_0 \\ B &= x^b + \beta_{b-1}x^{b-1} + \cdots + \beta_0 \\ &\vdots \\ L &= x^l + \lambda_{l-1}x^{l-1} + \cdots + \lambda_0 \end{aligned}$$

Now the factorization problem can be formulated as follows:

(1) Determine coefficients of $A, B, \ldots, L$ such that $P = AB \cdots L$.

S. Prešić proposes the following iterative procedure for obtaining A, $B, \ldots,$ L. Let for $k = 1, 2, \ldots$

$$\begin{aligned} A(k) &= x^a + \alpha_{a-1}(k)x^{a-1} + \cdots + \alpha_0(k) \\ B(k) &= x^b + \beta_{b-1}(k)x^{b-1} + \cdots + \beta_0(k) \\ &\vdots \\ L(k) &= x^l + \lambda_{l-1}(k)x^{l-1} + \cdots + \lambda_0(k) \end{aligned}$$

be the sequences of polynomials defined by conditions

$$\begin{aligned} (2)\quad & A(k+1)B(k)\cdots L(k) + A(k)B(k+1)\cdots L(k) + \cdots \\ & + A(k)B(k)\cdots L(k+1) - sA(k)B(k)\cdots L(k) = P, \quad k = 1, 2, \ldots \end{aligned}$$

If $p(k) = (\alpha_{a-1}(k), \ldots, \alpha_0(k), \beta_{b-1}(k), \ldots, \beta_0(k), \ldots, \lambda_{l-1}(k), \ldots, \lambda_0(k))$, then using (2) it is possible to express $p(k+1)$ as a function of $p(k)$, i.e., it is possible to determine function F such that $p(k+1) = F(p(k))$.

The following theorems are proved in [3]:

THEOREM 1. *If the sequence $(p(k))$ converges, then $\lim_{k\to\infty} A(k)$, $\lim_{k\to\infty} B(k)$, $\ldots, \lim_{k\to\infty} L(k)$ are factors of P, i.e.,*

$$P = \lim_{k\to\infty} A(k) \lim_{k\to\infty} B(k) \cdots \lim_{k\to\infty} L(k).$$

THEOREM 2. *There exists an open set V such that if $p(1) \in V$, then $p(k) \to p$, $k \to \infty$ and moreover $\|p(k+1) - p\| = O(\|p(k) - p\|^2)$.*

In the case of $n = 1 + \cdots + 1$ decomposition, the factors are linear. If we introduce notation $A(k) = x - a_1(k)$, $B(k) = x - a_2(k), \ldots,$ $L(k) = x - a_n(k)$, then the iterative procedure can be described by simple formulas:

$$(3)\qquad a_i(k+1) = a_i(k) - \frac{P(a_i(k))}{Q'(a_i(k))}, \quad i = 1, \ldots, n, \ k = 1, 2, \ldots$$

where $Q(x) = (x - a_1(k)) \cdots (x - a_n(k))$. It is interesting to note that (3) is related to the Newton method for solving the system of nonlinear equations

$$\begin{aligned} a_1^n + p_{n-1}a_1^{n-1} + \cdots + p_0 &= 0 \\ &\vdots \\ a_n^n + p_{n-1}a_n^{n-1} + \cdots + p_0 &= 0 \end{aligned}$$

S. Prešić's results on factorization of polynomials have been extended by M. Prešić who has formulated a quadratically convergent method for finding k roots of a polynomial P of degree n $(1 \leq k \leq n)$ [2]. It has been proved by M. Ašić and V. Kovačević that M. Prešić's method belongs to the class of quasi-Newton methods [1].

In [5] Petrić and S. Prešić generalize the method proposed in [3] to systems of nonlinear algebraic equations of the following type:

$$\begin{aligned} J_1(x,y) &= A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0 \\ J_2(x,y) &= A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0 \end{aligned} \tag{4}$$

It is assumed that system (4) has four different solutions (a, α), (b, β), (c, γ), (d, δ). Then it is equivalent to the system

$$\begin{aligned} AB \cdot CD &= 0 \\ AC \cdot BD &= 0 \end{aligned}$$

where

$$\begin{aligned} AB &= (\beta - \alpha)(x - a) - (b - a)(y - \alpha) \\ &\vdots \\ BD &= (\delta - \beta)(x - b) - (d - b)(y - \beta) \end{aligned}$$

Moreover, there exist constants λ, μ, ρ, φ such that

$$\begin{aligned} AB \cdot CD + \lambda J_1(x,y) + \mu J_2(x,y) &= 0 \\ AC \cdot BD + \rho J_1(x,y) + \varphi J_2(x,y) &= 0 \end{aligned} \tag{5}$$

The iterative procedure defined in [3] can now be applied to (5) yielding equations

$$\begin{aligned} R_n + \Delta R_n &= 0 \\ S_n + \Delta S_n &= 0 \end{aligned} \tag{6}$$

where

$$R_n = A_nB_n \cdot C_nD_n + \lambda_n J_1(x,y) + \mu_n J_2(x,y)$$

$$\begin{aligned}
S_n &= A_nC_n \cdot B_nD_n + \rho_n J_1(x,y) + \varphi_n J_2(x,y)\\
A_nB_n &= (\beta_n - \alpha_n)(x - a_n) - (b_n - a_n)(y - \alpha_n)\\
C_nD_n &= (\delta_n - \gamma_n)(x - c_n) - (d_n - c_n)(y - \gamma_n)\\
A_nC_n &= (\gamma_n - \alpha_n)(x - a_n) - (c_n - a_n)(y - \alpha_n)\\
B_nD_n &= (\delta_n - \beta_n)(x - b_n) - (d_n - b_n)(y - \beta_n)
\end{aligned}$$

and operator Δ refers to polynomial expressions for a_n, α_n, b_n, $\beta_n, \dots, \rho_n$, φ_n, a_{n+1}, α_{n+1}, b_{n+1}, $\beta_{n+1}, \dots, \rho_{n+1}$, φ_{n+1}. Using polynomial identities (6) it is possible to determine a_{n+1}, α_{n+1}, b_{n+1}, $\beta_{n+1}, \dots, \rho_{n+1}$, φ_{n+1} as the functions of a_n, α_n, b_n, β_n, c_n, γ_n, d_n, δ_n.

The following theorem is proved in [5]:

THEOREM 3. *If the sequences* (a_n), (α_n), (b_n), (β_n), (c_n), (γ_n), (d_n), (δ_n), (λ_n), (μ_n), (ρ_n), (φ_n), *converge to* a, α, b, β, c, γ, d, δ, λ, μ, ρ, φ, , *respectively, and* $\lambda\varphi - \rho\mu \neq 0$, *then* (a,α), (b,β), (c,γ), (d,δ), *are solutions of system (4).*

Numerical evidence reported at the end of [5] shows very good performance of the proposed method in practice.

The paper [4] is related to the following result by S. Zervos:

THEOREM 4. (Zervos, 1960) *Let* $I_1, \dots, I_m$ *be sets of indices and* $\theta_{i_j} \geq 0$ *be such that*

$$\sum_{i_j \in I_j} \theta_{i_j} = j - t \quad j = 1, \dots, n$$

where $t \in (0,1]$ *is fixed. Then any positive root* ξ *of the equation*

$$x^n = a_1x^{n-1} + \dots + a_n \quad a_i \geq 0, \ \sum_{i=1}^{n} a_i > 0$$

satisfies

$$\xi \leq \max\left\{M, \left(\sum_{j=1}^{n} a_j \left(\prod_{i_j \in I_j} M_{i_j}^{\theta_{I_j}}\right)^{-1}\right)^{1/t}\right\}$$

where $M = \max(M_{i_j})$ *and* M_{i_j} *are arbitrary positive numbers.*

It should be pointed out that this theorem generalizes results of many mathematicians (Cauchy, Landau, Montel, Jensen, Birkhoff, D. Marković, Carmichael, Walsh, Kojima, etc.), which follow for particular choices of parameters. In [4] S. Prešić proposes a short (one page) and elegant proof of this deep result.

3. The m–M calculus

The m–M calculus is an original theory proposed by S. Prešić in the monograph published in 1996 [6], which has already had the second edition [5]. A brief version of the monograph was published in 1998 [8]. In this section we shall outline the main ideas of this widely applicable theory.

The m–M calculus deals with the so-called m–M functions, i.e., functions $f: D \to R$ ($D = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset R^n$) for which on each n-dimensional segment $\Delta = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset D$ generalized minimum $m(f)(\Delta)$ and generalized maximum $M(f)(\Delta)$ are effectively given. The definition of the m–M function and some applications of that concept will be given in 3.1. Suppose now that

$$\text{for}(x_1, \ldots, x_n, f, g, \ldots, <, \leq, \wedge, \vee, \neg, \forall, \exists) \tag{7}$$

is a formula built up from variables $x_1, \ldots, x_n$, the symbols of real numbers, symbols of m–M functions $f, g, \ldots$, relational symbols $<, \leq$ and the logical symbols $\wedge, \vee, \neg, \forall, \exists$. The m–M calculus considers problems of the following type:

(8) Find all $(x_1, \ldots, x_n) \in D$ for which formula (7) is satisfied.

The very general formulation (8) includes as special cases problems of solving systems of equations and inequalities, problems of unconstrained, constrained and disjunctive optimization, problems of interval mathematics, computation of n-dimensional integrals and solutions of differential equations, etc. In 3.2 we shall explain the methodology used in the m–M calculus for solving (8) and point out to the most important applications.

3.1. The notion of the m–M pair of a given function. Applications. The key notion of the m–M calculus is introduced by the following definition:

DEFINITION 1. The function $f : D \to R$ is an m–M function if for each n-dimensional segment $\Delta = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset D$ a pair of real numbers $m(f)(\Delta)$, $M(f)(\Delta)$ satisfying conditions

$$m(f)(\Delta) \leq f(x) \leq M(f)(\Delta) \quad \text{for all } x \in \Delta \tag{9}$$

$$M(f)(\Delta) - m(f)(\Delta) \to 0, \quad \operatorname{diam} \Delta = \left(\sum_{i=1}^{n} (\beta_i - \alpha_i)^2 \right)^{1/2} \to 0 \tag{10}$$

is effectively given.

It is easy to see that f is an m–M function if and only if it is continuous on D. The crucial issue in the m–M calculus is how to effectively compute an m–M pair of a given function. To this end the following rules are introduced:

(i) $m(C)(\Delta) = C$, $M(C)(\Delta) = C$, (C is a constant),
$m(x_i)(\Delta) = \alpha_i$, $M(x_i)(\Delta) = \beta_i$, $(i = 1, \ldots, n)$

(ii) $m(f+g)(\Delta) = m(f)(\Delta) + m(g)(\Delta)$, $M(f+g)(\Delta) = M(f)(\Delta) + M(g)(\Delta)$,

(iii) $m(-f)(\Delta) = -M(f)(\Delta)$, $M(-f)(\Delta) = -m(f)(\Delta)$,

(iv) $m(fg)(\Delta) = \min(m(f)(\Delta)m(g)(\Delta), m(f)(\Delta)M(g)(\Delta), M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$
$M(fg)(\Delta) = \max(m(f)(\Delta)m(g)(\Delta), m(f)(\Delta)M(g)(\Delta), M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$

(v) $m(\min(f,g))(\Delta) = \min(m(f)(\Delta), m(g)(\Delta))$,
$M(\min(f,g))(\Delta) = \min(M(f)(\Delta), M(g)(\Delta))$

(vi) $m(\max(f,g))(\Delta) = \max(m(f)(\Delta), m(g)(\Delta))$,
$M(\max(f,g))(\Delta) = \max(M(f)(\Delta), M(g)(\Delta))$

(vii) $m(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{m(f)(\Delta)}$, $M(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{M(f)(\Delta)}$,

(viii) $m(\exp f)(\Delta) = \exp m(f)(\Delta)$, $M(\exp f)(\Delta) = \exp M(f)(\Delta)$,

(ix) $m(\sin f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \sin m(f)(\Delta)$,
$M(\sin f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \sin M(f)(\Delta)$,

(x) $m(\cos f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \cos m(f)(\Delta)$,
$M(\cos f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \cos M(f)(\Delta)$,

(xi) $m(1/f)(\Delta) = 1/M(f)(\Delta)$,
$M(1/f)(\Delta) = 1/m(f)(\Delta)$, if $0 \notin [m(f)(\Delta), M(f)(\Delta)]$

(xii) $m(\arcsin f)(\Delta) = \arcsin m(f)(\Delta)$,
$M(\arcsin f)(\Delta) = \arcsin M(f)(\Delta)$, if $-1 \leq m(f)(\Delta)$ and $M(f)(\Delta) \leq 1$

(xiii) $m(\ln f)(\Delta) = \ln m(f)(\Delta)$, $M(\ln f)(\Delta) = \ln M(f)(\Delta)$, if $m(f) > 0$.

(xiv) $m(\sqrt[2k]{f})(\Delta) = \sqrt[2k]{m(f)(\Delta)}$, $M(\sqrt[2k]{f})(\Delta) = \sqrt[2k]{M(f)(\Delta)}$, $k > 0$, $k \in N$, if $m(f) \geq 0$.

Using these rules we can obtain m–M pairs for various elementary functions. It is easy to show that in the case of differentiable functions m–M pairs can be computed using the corresponding Taylor expansion (Theorem 1.2 in [7]).

The other important notion in the m–M calculus is the so-called cell-decomposition of n-dimensional segments. We shall first define a cell-decomposition of an interval $[a,b] \subset R$. Any such decomposition $\mathcal{D}$ is an infinite set of segments $[a',b'] \subset [a,b]$, the so-called cells, where to each cell one of the numbers 0, 1, 2,... (the so-called order of the decomposition) is assigned. In addition, the following holds:

(i) $[a,b] \subset \mathcal{D}$.

(ii) For each $r \in N$ there exists a finite number of cells in $\mathcal{D}$ of order r. The set of all cells of order r is denoted by $\mathcal{D}_r$. The segment $[a,b]$ is the unique cell of order 0.

(iii) The union of all cells of order r, is equal to $[a,b]$.

(iv) The interiors of two different cells of the same order r are disjoint.

(v) If $d(r)$ denotes the maximum length of all cells of order r, then $\lim_{t\to\infty} d(r) = 0$.

Notice that, by the definition of the cell-decomposition, for each decomposition $\mathcal{D}$ of the segment $[a,b]$ the following fact holds:

To each point $x \in [a,b]$ at least one sequence $(C_r(x))$ of r-cells is related such that the condition $(\forall r \in N)\ x \in C_r(x)$ is satisfied.

Consider now n-dimensional segment $D = [a_1,b_1] \times \cdots \times [a_n,b_n]$. Let $\mathcal{D}[a_i,b_i]$, $i = 1,\ldots,n$ be some cell-decompositions of segments $[a_i,b_i]$, $i = 1,\ldots,n$, and let $\mathcal{D}_r(D) = \{P_1 \times \cdots \times P_n \mid P_i \in \mathcal{D}_r[a_i,b_i],\ i = 1,\ldots,n\}$. Now cell-decomposition of D is defined as $\mathcal{D}(D) = \bigcup_{r\in N} \mathcal{D}_r(D)$.

One of the main methodological ideas of the m–M calculus can now be described as follows:

(a) Sufficient conditions that an arbitrary n-dimensional segment Δ does not contain a solution of the given problem are formulated in terms of m–M pairs.

(b) A cell-decomposition of the initial segment is chosen and cells which do not contain a solution are eliminated, i.e., only feasible cells are being further refined.

There are two possible outcomes of such a procedure. Either in the limit all solutions are obtained or the conclusion that the solution does not exist is reached in finitely many steps. As an illustration, consider the problem of solving the system

$$\begin{aligned} &f_i(x_1,\dots,x_n) \geq 0, \quad i=1,\dots,n \\ &(x_1,\dots,x_n) \in [a_1,b_1] \times \cdots \times [a_n,b_n] \end{aligned} \tag{11}$$

Sufficient condition that a segment Δ does not contain a solution in this case is given by

$$(\exists i) M(f_i)(\Delta) < 0 \tag{12}$$

and it is now easy to formulate a cell-decomposition-based algorithm for solving (11).

From the computational point of view the crucial issue is the number of cells which is being generated by such an algorithm. Various examples presented in [7] illustrate that this number need not grow exponentially. Consider the following:

EXAMPLE 1. Equation $\sin x = 1/x$, $x \in [0,20]$. Then

$$\begin{aligned} m(f)[\alpha,\beta] &= \alpha + \sin\alpha - \beta - 1/\alpha \\ M(f)[\alpha,\beta] &= \beta + \sin\beta - \alpha - 1/\beta \end{aligned}$$

The number of cells which are generated at steps 1,...,25 is the following: (1,1), (2,2), (3,4), (4,8), (5,15), (6,16), (7,16), (8,15), (9,16), (10,14), (11,14), (12,16), (13,16), (14,15), (15,15), (16,15), (17,17), (18,16), (19,15), (20,15), (21,15), (22,15), (23,15), (24,15), (25,15). At the 25th step all 7 solutions of the given equation are obtained with 6 significant digits.

Under suitable assumptions it is possible to prove that the behavior observed in the case of Example 1 holds in the general case. Namely, the following theorem can be proved (Theorem 2.4 in [7]):

THEOREM 5. *Let* $f_1(x_1,\dots,x_n) = 0,\dots, f_n(x_1,\dots,x_n) = 0$, $(x_1,\dots,x_n) \in D$ *be a system of equations where* $f_1,\dots,f_n : D \to R$ *are m–M functions for which* $(m(f_i), M(f_i))$ *are* σ_i*-Lipschitz m–M pairs.*[1] *Let* $c = (c_1,\dots,c_n) \in D$ *be a solution of the given system and suppose that the following condition holds:*

In some neighborhood $\Delta \ni (c_1,\dots,c_n)$ *the functions* $f_1,\dots,f_n$ *have continuous partial derivatives of the first order and their Jacobian at* $(c_1,\dots,c_n)$ *is different from zero.*

Let $\mathrm{Fis}(c,r)$ *be the maximal set such that its elements are feasible r-cells,* $C_r(c) \in \mathrm{Fis}(c,r)$ *and* $\mathrm{Fis}(c,r)$ *is cell-connected (each 2 elements can be connected by a*

[1]An m–M pair of f is σ-Lipschitz if there exist positive numbers ε, K, σ such that $|M(f)(\Delta) - m(f)(\Delta)| \leq K(\operatorname{diam}\Delta)^\sigma$ whenever $\operatorname{diam}\Delta \leq \varepsilon$, $\Delta \subset D$.

chain of neighboring elements). Then it is possible to formulate a cell-elimination procedure such that there exist positive constants L, σ and $r_0 \in N$ such that for every $r \geq r_0$ the inequality $\mathrm{diam}(\bigcup \mathrm{Fis}(c,r)) \leq L(d(r))^\sigma$ *holds.*

The following example illustrates the applications of m–M calculus to systems of equations.

EXAMPLE 2. Consider the system in $(x,y,z) \in D \subset R^3$

$$\begin{aligned} e^x + x + \sin y + \cos z &= p \\ x^3 + e^{\sin y} - z - e^z &= q \\ \sin(x-z) + (x+y)^5 - x - y - z &= r \end{aligned}$$

where p, q, r are given real parameters.

Case 1. $p = 2$, $q = 0$, $r = 0$, $D = [1,2] \times [-2,1] \times [-3,2]$. There is exactly one solution $(x,y,z) = (0,0,0)$. Starting with the 6th step the number of feasible cells is between 40 and 50. At the 24th step the following result is obtained:

$$\begin{aligned} -0.0000152587891 \leq x \leq 0.0000247955322 \\ -0.0000247955322 \leq y \leq 0.0000324249268 \\ -0.00000762939453 \leq z \leq 0.0000114440918 \end{aligned}$$

Case 2. $p = 2$, $q = 0$, $r = 0$, $D = [-5,5] \times [1,5]$. Step-by-step the number of feasible cells is 1, 8, 21, 32, 24, 0. Hence in 6 steps it is concluded that the system has no solution.

The use of the m–M calculus in solving complex equations is illustrated by the following:

EXAMPLE 3. Complex equation $e^z = z$, where $z = x + iy$. In the domain $[-20,20] \times [-20,20]$ this equation has 6 solutions $x_j + iy_j$, $j = 1,\ldots,6$ described as follows

$$\begin{array}{ll} 2.65319109 \leq x_1 \leq 2.65319228 & -13.94920826 \leq y_1 \leq -13.94920731 \\ 2.06227660 \leq x_2 \leq 2.06227899 & -7.58863215 \leq y_2 \leq -7.58863020 \\ 0.31813025 \leq x_3 \leq 0.31813264 & -1.33723736 \leq y_3 \leq -1.33723497 \\ x_4 = x_3 & y_4 = -y_3 \\ x_5 = x_2 & y_5 = -y_2 \\ x_6 = x_1 & y_6 = -y_1 \end{array}$$

The calculations up to the 25th step show that starting with the 6th step the number of feasible cells is about 16. For instance at steps 24 and 25 these numbers are 15 and 16, respectively.

The m–M calculus can also be applied to evaluation of n-dimensional integrals, as well as to functions given by means of infinite sums or integrals, which is illustrated by the following examples:

EXAMPLE 4. Evaluate $\iint_C xy\,dx\,dy$, where $C = \{(x,y) \in R^2 \mid 0 \le x \le 2,\ 0 \le y \le 2,\ 2 + e(x+y+z) \ge e^{x+1} + e^{y+1}\}$

Using a suitable cell-decomposition the following estimates can be obtained:

Step 1:	$0.0000000000000000 \le I \le 0.5625000000000000$
Step 2:	$0.0278320312500000 \le I \le 0.2424316406250000$
Step 3:	$0.0950307846069336 \le I \le 0.1581497192382812$
Step 4:	$0.1179245151579380 \le I \le 0.1341890022158623$
Step 5:	$0.1239582093403442 \le I \le 0.1281216432544170$
Step 6:	$0.1255188694198068 \le I \le 0.1265613023800256$
Step 7:	$0.1259090350460332 \le I \le 0.1261702301487544$
Step 8:	$0.1259090350460332 \le I \le 0.1259090350460332$
Step 9:	$0.1259090350460332 \le I \le 0.1259090350460332$

EXAMPLE 5. Let f be a function defined by the equality $f(x) = \sum_{i=0}^{\infty} \frac{1}{x+2^i}$ Consider the equation $f(x) = c$, $a \le x \le b$, where a, b, c are given parameters.

Case 1. $c = 1.5$, $a = 0$, $b = 1$. At the 20th step the double inequality $0.54416 \le x \le 0.54417$ is obtained. The number of feasible cells at steps $1, \ldots, 20$ is $(1,1)$, $(2,2)$, $(3,3)$, $(4,3)$, $(5,2)$, $(6,3)$, $(7,2)$, $(8,2)$, $(9,3)$, $(10,2)$, $(11,3)$, $(12,3)$, $(13,2)$, $(14,2)$, $(15,2)$, $(16,3)$, $(17,2)$, $(18,3)$, $(19,2)$, $(20,3)$.

Case 2. $c = 1.5$, $a = 0.6$, $b = 1$. At the 3rd step the number of feasible cells is 0 so that the given equation has no solutions.

EXAMPLE 6. Let f be defined by $f(x) = \int_0^x \frac{e^t}{1+t}\,dt$. Consider the equation $f(x) = c$, $a \le x \le b$, where a, b, c are given parameters.

Case 1. $c = 1$, $[a,b] = [0,1]$. At the 13th step the double inequality $0.905029297 \le x \le 0.905761719$ is obtained, while the number of feasible cells is: $(1,1)$, $(2,1)$, $(3,2)$, $(4,2)$, $(5,4)$, $(6,3)$, $(7,3)$, $(8,3)$, $(9,1)$, $(10,3)$, $(11,3)$, $(12,4)$, $(13,4)$.

Case 2. $c = 1$, $[a,b] = [3, 100]$. The number of feasible cells is $(1,1)$, $(2,2)$, $(3,2)$, $(4,1)$, $(5,0)$, so that the given equation has no solutions.

3.2. The notion of m–M pairs of the first order $<$, $\le$ formulas. Applications. The most important extension of the concept of an m–M function is that of an m–M pair of the first order formula. Briefly speaking, the first order $<$, $\le$ formulas are built from some variables, the symbols of real numbers, symbols of some m–M functions, relational symbols $<$, $\le$ and logical symbols $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$. A variable v is free in some first order formula φ if v does not occur in some part of φ which has the form $(\forall v)(\cdots)$ or $(\exists v)(\cdots)$, where $(\cdots)$ denotes the scope of the quantifier; otherwise v is a bounded variable in φ. The formula φ is called $\le$-positive ($<$-positive) if it is built using the relational symbols $\le$ ($<$) and the logical symbols $\wedge$, $\vee$, $\forall$, $\exists$ (without the negation symbol).

Let now φ be a given $<$, $\le$ formula whose all variables are among $x_1, \ldots, x_n$. Suppose that to each segment $I(x_i)$ a cell-decomposition $\mathcal{D}(I(x_i))$ is assigned, $i =$

$1,\dots,m$. The m–M pair of φ with respect to the cell-decompositions $\mathcal{D}(I(x_i))$, $i=1,\dots,m$ is a sequence of ordered pairs $(m_0(\varphi), M_0(\varphi)),\dots,(m_r(\varphi), M_r(\varphi)),\dots$ obtained according to the following rules which are applied recursively. We point out that during the process each bounded variable x_i is replaced by a new variable X_i.

(i) $m_r(f(y_1,\dots,y_p)\,\rho\, g(z_1,\dots,z_q)) = m_r(f)(y_1'\times\cdots\times y_p')\,\rho\, M_r(g)(z_1'\times\cdots\times z_q')$
$M_r(f(y_1,\dots,y_p)\,\rho\, g(z_1,\dots,z_q)) = M_r(f)(y_1'\times\cdots\times y_p')\,\rho\, m_r(g)(z_1'\times\cdots\times z_q')$
(ρ may be $<$ or $\le$)
Variables $y_1,\dots,y_p,z_1,\dots,z_q$ are some of $x_1,\dots,x_m,X_1,\dots,X_m$. Denotation $'$ means the following: x_i' denotes $C_r(x_i)$, while X_i' denotes X_i

(ii) $m_r(\alpha\wedge\beta) = m_r(\alpha)\wedge m_r(\beta)$, $M_r(\alpha\wedge\beta) = M_r(\alpha)\wedge M_r(\beta)$,

(iii) $m_r(\alpha\vee\beta) = m_r(\alpha)\vee m_r(\beta)$, $M_r(\alpha\vee\beta) = M_r(\alpha)\vee M_r(\beta)$,

(iv) $m_r(\neg\alpha) = \neg M_r(\alpha)$, $M_r(\neg\alpha) = \neg M_r(\alpha)$,

(v) Let $\alpha(x_i)$ be a formula having x_i as a free variable and q be quantifier $\forall$ or $\exists$. Then we have the following equalities
$m_r((qx_i\in I(x_i))\alpha(x_i)) = (qX_i\in\mathcal{D}_r(I(x_i)))m_r(\alpha(X_i))$
$M_r((qx_i\in I(x_i))\alpha(x_i)) = (qX_i\in\mathcal{D}_r(I(x_i)))M_r(\alpha(X_i))$

For instance, if φ is the formula $(\forall x_2)\, f(x_1,x_2) < g(x_2,x_3)$, then $m_r(\varphi)$, $(r = 0,1,\dots)$ can be constructed as follows

$$\begin{aligned} m_r((\forall x_2)\, f(x_1,x_2) < g(x_2,x_3)) &= (\forall X_2\in\mathcal{D}_r(I(x_2)))m_r(f(x_1,X_2) < g(X_2,x_3)) \\ &= (\forall X_2\in\mathcal{D}_r(I(x_2)))m(f)(C_r(x_1)\times X_2) < M(g)(X_2\times C_r(x_3)) \end{aligned}$$

Hence, $m_r(\varphi)$ can also be treated as some first-order formula. Then X_2 is a bounded variable and the symbols $C_r(x_1)$, $C_r(x_3)$ should be taken as its free variables. Accordingly, if we denote φ by $\varphi(x_1,x_3)$, emphasizing that x_1 and x_3 are free variables of φ, then it is natural that $m_r(\varphi)$ is denoted by $m_r(\varphi)(C_r(x_1),C_r(x_3))$. Such notation will also be used in the general case.

The following theorem has been proved in [7] (Theorem 4.1).

THEOREM 6. *Let $\varphi(x_1,\dots,x_m)$ (with $m\ge 0$) be a $<$, $\le$ – formula whose all free variables are among $x_1,\dots,x_m$. Then for every $r\in N$ the following double implication is true*

$$M_r(\varphi)(C_r(x_1),\dots,C_r(x_m)) \Rightarrow \varphi(x_1,\dots,x_m) \Rightarrow m_r(\varphi)(C_r(x_1),\dots,C_r(x_m))$$

provided that the variables $x_1,\dots,x_m$ have any values from their segments $I(x_1)$, $\dots, I(x_m)$, respectively.

Intuitively speaking, $m_r(\varphi)$ and $M_r(\varphi)$ are "logical" minorant and majorant of formula φ, respectively.

The next theorem is important for the application of the m–M calculus:

THEOREM 7. (Theorem 4.3 in [7]) *Let all free variables in formula φ be among the variables $x_1,\dots,x_m$ (with $m\ge 0$). Then:*

(i) if φ is a $<$-positive formula, then the following equivalence holds:
$\varphi(x_1,\ldots,x_m) \Leftrightarrow (\exists r \in N)M_r(\varphi)(C_1(x_1).\ldots,C_r(x_m))$
(ii) if φ is a $\leq$- positive formula, then the following equivalence holds:
$\varphi(x_1,\ldots,x_m) \Leftrightarrow (\forall r \in N)\, m_r(\varphi)(C_r(x_1),\ldots,C_r(x_m))$

In both cases it is supposed that the variables $x_1,\ldots,x_m$ have any values from their segments $I(x_1),\ldots,I(x_m)$, respectively.

Consider now an application of the stated results on the following class of problems:

(i) If $x_1,\ldots,x_m$ are all free variables of formula φ, find all values of $x_i \in I(x_i)$, $i = 1,\ldots,m$ for which formula φ is satisfied.
(ii) If formula φ has no free variables establish whether φ is true or false.

In the case (i) we can apply a procedure similar to that outlined in 3.1. In the case (ii) the following procedure may be used:

PROCEDURE 1.
(i) Set $r = 0$
(ii) Calculate $m_r(\varphi)$. If $m_r(\varphi)$ is false the procedure halts and the answer is: φ is false. Otherwise, go to (iii).
(iii) Calculate $M_r(\varphi)$. If $M_r(\varphi)$ is true the procedure halts and the answer is: φ is true. Otherwise, go to (iv).
(iv) Replace r by $r + 1$ and go to (ii).

Many different problems have equivalent reformulations which have the form (i) or (ii), which is illustrated by the following examples:

smallskip Example 7 Examine the truth of the formula $(\forall x \in [1.4, 1.5])x^2 \geq 1.8\ldots$ where $1.8\ldots$ is a constant satisfying $1.8 \leq 1.8\ldots \leq 1.9$. Obviously, the problem is logically equivalent to the problem:

Is the formula $(\forall c \in [1.8, 1.9])(\forall x \in [1.4, 1.5])x^2 \geq c$ true or false.

EXAMPLE 8. Find $x \in [0, 1]$ such that $x^2 = c$ where c is a constant satisfying $1.69 \leq c \leq 1.96$. It has been shown in [7] (Example 5.3) that the problem is logically equivalent to the problem of the type (i):

Find $x \in [1, 2]$ such that the formula $(\exists c \in [1.69, 1.96])x^2 = c$ is true.

In the conclusion let us point out that the ideas outlined in 3.1 and 3.2 can be used for constructing various algorithms in different areas of applied mathematics, such as:

- Solving systems of equations and inequalities
- Finding n-dimensional integrals
- Solving problems expressed by positive $\leq$ - formulas. Among others
  Problems of unconstrained optimization
  Problems of constrained optimization
  Problems of disjunctive programming
  Problems of interval mathematics.

In Chapter 6 of [7] it has been shown how approximately to determine functions satisfying a given m–M condition, which is some functional condition, or some difference condition, or a differential equation. This enables extensions of the m–M calculus to the initial value problems for differential equations and other types of functional equations. More details on the m–M calculus and its applications can be found in [6], [7], [8].

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EDUCATIONAL AND TUTORIAL WORK OF PROFESSOR SLAVIŠA B. PREŠIĆ

Slobodan Vujić

We shall speak here today about the work of Professor Slaviša Prešić, particularly about segment of his work being performed for decades for the benefit of our junior students of mathematics. These efforts of his include, besides teaching, long and frequent professional discussions with students of the Faculty of Mathematics which is today, generally speaking, one of the best schools of its kind. Together with his prolific scientific work, which in principle represents a difficult road paved with many unsuccessful trials and with only a few achievements, Prof. S. Prešić is always ready to find time for bringing culture and enlightenment to his public in various ways: one cannot avoid spending one's life, but one can dedicate parts of it to others.

Enlightening others - words of our language quite clear in their meaning, today sound a little obsolescent. Those activities of Prof. S. Prešić include the lectures he gave throughout this country for teachers of mathematics, writing of teachers' handbooks, starting and managing professional journal of mathematics methodology and pedagogy and particularly, writing textbooks for secondary schools and university. All this was done in an effort to prove in practice advantages of a new approach to studies of mathematics.

The beginnings of such work of Prof. S. Prešić belong to rather distant past, if passage of time is measured by human life-span: it was the end of the sixties and the beginning of the seventies. A fight for new study programs was begun at different professional commissions and panels of politicians; how those proceedings looked like one can see from the minutes of one such meeting, where, after hours of discussion an almost incredible question was asked by the defenders of the "old and proven programs" : "What's the use of implication for children?" Such old programs and textbooks had to be strongly criticized. Even today we remember the eloquent, decisive and rational criticism of Prof. S. Prešić during those long sessions. After the victory, which duly came, it was necessary to create excellent mathematical textbooks on the spirit of the new professional approach. The decisive factor in this was the appearance of the "red-and-black book" by Prof. S. Prešić, which represented a cannon of good mathematics and good teaching of this subject, although it was only a modest textbook for the first grade of secondary specialized schools.

Some more good book followed: Modern Approach to Teaching (1975), Textbook by authors S. Prešić, B. Alimpic (1977), journal "Mathematics", the Textbook of Mathematical Logic (1974), etc.

The acceptance of these textbooks was not instantaneous, and some other half-baked products were still being published, specially among so-called "collections" of mathematical problems. This situation, unfortunately, still exists to a degree.

The "red-and-black book", important for being the first of the kind, and published without a single printing error in its first edition, is still fresh and "modern", which is invariably the case with all worthy work. Many teachers and their students have found in it a new much more beautiful world. We can find there fundamental concepts of mathematical logic, set theory, notions of the function, relations, operations. An edifice called "real numbers field" was erected with rigorous attention to detail. (here we could stop listing further items) The exposition is done with utmost clarity and care. Some new thoughts are planed in a form adapted to the users' age; for example, ostensibly non-definable concept of a set he explained by using the right and cautious words: he went from individual object to a set of such objects by using the image of sticks and thread binding them together. We remember sardonic smiles with which "triviality"of such an approach was greeted. This correct idea has, after right considerations, brought about a beautiful miniature. recently presented by Prof. S. Prešić, and called Algebraic Definition Of Finite Sets. This happened to be a small subsidiary result.

Pedagogical work of Prof. S. Prešić is intertwined whit deep insight into Mathematics: the language used at this level of thought called syntax is different from higher levels of meaning called semantics. As for the syntax side, almost nowhere in the world can we find mathematical texts with such usage and with such effects of written logic. In this limelight many concepts are recreated, so that we are compelled to comprehend and view some seemingly known fields and segments of mathematics in a new light. For example, "Varia I, II", his newest work is full of true Mathematics and full of true enlightenment. Ideas of great thinkers are respected without futile reverence. Mathematics is not a finite notion, it is not "geography-like", which is well-known and favorite expression of Prof. S. Prešić. This attitude goes against such presentation of mathematics in which students are instructed that "this belongs here" and "that belongs there", and everything is finite and beyond any doubt.

Mathematics should be discovered, but it also should be made-teaches us Prof. S. Prešić very convincingly; in his books which bring enlightenment to us he shows how mathematical is "made". What is the meaning of being "modern" in teaching mathematics? Works of Prof. S. Prešić bestow an honorable meaning upon that word; it means that while teaching others one should sow ideas of true Mathematics. in adapted form, without petrifying them. In that way we can look up from the foot of the edifices forming Mathematics and see, beyond their tops, sky free for new enterprise. In such intellectual environment the works we tried to speak about are being made.

Research papers

CUT ELIMINATION IN A CATEGORY-LIKE SEQUENT SYSTEM

Zoran Petrić

Dedicated to Professor Slaviša Prešić

ABSTRACT. A sequent system $\mathcal{L}$ for the conjunction-implication fragment of intuitionistic propositional logic is introduced. Sequents of $\mathcal{L}$ are of the form $A \vdash B$, where A and B are formulae, i.e. sequences of formulae with exactly one member. With a modification of Gentzen's procedure a cut elimination theorem for $\mathcal{L}$ is proved. Some categorial consequences of this result are pointed out.

Introduction

The work on this note was inspired by the paper of Kelly and MacLane [1971] where the cut elimination procedure was used to prove two facts connected with symmetric monoidal closed categories, namely the naturality of its canonical transformations and the property of coherence. The authors were inspired by Lambek (see [1968]) who was the first who has used a cut-elimination technique in category theory. However, we stay here in the logical framework and try to clarify the process of preparation of a logical system for further categorial purposes.

System $\mathcal{L}$. The sequent system $\mathcal{L}$ for the conjunction-implication fragment of intuitionistic propositional logic is introduced as follows. *Formulae* of the logic are built from an infinite set of *propositional letters* and the constant $\top$, by the logical connectives $\wedge$ and $\rightarrow$. The set of all formulae is denoted by $\mathcal{F}$. *Sequents* of $\mathcal{L}$ are of the form $A \vdash B$ for A and B in $\mathcal{F}$. We call A in $A \vdash B$ the *antecedent*, and B the *consequent* of the sequent. In order to introduce the rules of inference of $\mathcal{L}$ we need the following auxiliary notion of $\wedge$*-context*, which corresponds to the notion of (poly)functor in categories. A $\wedge$-context is defined inductively as follows:

1° The symbol $\Box$ is a $\wedge$-context.

2° If F is a $\wedge$-context and $A \in \mathcal{F}$, then $(F \wedge A)$ and $(A \wedge F)$ are $\wedge$-contexts.

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3° If F and G are $\wedge$-contexts, then $(F \wedge G)$ is a $\wedge$-context.

For a $\wedge$-context F we say that it is a $\wedge_1$-context if the symbol $\Box$ occurs in F exactly once. For F a $\wedge$-context and $A \in \mathcal{F}$, we obtain $F(A)$ by substituting A for $\Box$ in F, e.g. if $F \equiv (B \wedge \Box) \wedge C$, then $F(A) = (B \wedge A) \wedge C$.

The *axioms* of $\mathcal{L}$ are

$$a_A : A \vdash A, \quad \text{for every } A \in \mathcal{F},$$

The *structural rules* of $\mathcal{L}$ are

$$(\beta_F^{\leftarrow})\ \frac{F(A \wedge (B \wedge C)) \vdash D}{F((A \wedge B) \wedge C) \vdash D} \qquad (\beta_F^{\rightarrow})\ \frac{F((A \wedge B) \wedge C) \vdash D}{F(A \wedge (B \wedge C)) \vdash D}$$

$$(\gamma_F)\ \frac{F(A \wedge B) \vdash C}{F(B \wedge A) \vdash C}$$

$$(\omega_F)\ \frac{F(A \wedge A) \vdash B}{F(A) \vdash B} \qquad (\theta_F^A)\ \frac{F(\top) \vdash B}{F(A) \vdash B}$$

$$(\tau_F)\ \frac{F(A) \vdash B}{F(A \wedge \top) \vdash B} \qquad (\tau_F')\ \frac{F(A \wedge \top) \vdash B}{F(A) \vdash B}$$

$$(\mu_G)\ \frac{A \vdash B \qquad G(B) \vdash C}{G(A) \vdash C},$$

where F is a $\wedge_1$ context and G is a $\wedge$ context.

The *rules for connectives* are

$$(\wedge)\ \frac{A \vdash C \qquad B \vdash D}{A \wedge B \vdash C \wedge D}$$

$$(*)\ \frac{A \wedge B \vdash C}{B \vdash A \to C} \qquad (\Delta)\ \frac{A \vdash B \qquad C \wedge D \vdash E}{(A \wedge (B \to C)) \wedge D \vdash E}$$

A *proof* of a sequent $A \vdash B$ in $\mathcal{L}$ is a binary tree with sequents in its nodes, such that $A \vdash B$ is in the root, axioms are in the leaves and consecutive nodes are connected by some of the inference rules above.

What are the differences between $\mathcal{L}$ and the corresponding fragment of Gentzen's system LJ (see [1935])? In $\mathcal{L}$ we have just one meta-logical symbol $\vdash$ in a sequent and we omit Gentzen's commas in the antecedents, whose role is now covered by the logical connective $\wedge$. Also, we can't have empty either the antecedent or the consequent of a sequent in $\mathcal{L}$. The logical constant $\top$ serves to fill gaps in antecedents. These discrepancies between $\mathcal{L}$ and LJ arise because in $\mathcal{L}$ we want antecedents and consequents of sequents to be of the same sort (namely members of $\mathcal{F}$) and this enables us to look at an $\mathcal{L}$ sequent as an arrow with the source being the antecedent and the target the consequent of the sequent.

Also, the rule $(\wedge)$ is a rule of simultaneous introduction of the connective $\wedge$ on the both sides of a sequent: there is no counterpart for this rule in LJ. This difference is not categorially motivated. We believe that $\mathcal{L}$ completely separates structural rules from the rules for connectives. On the other hand, the LJ rules &-IS and&-IA (see 1.22 of [1935]) have hidden interchanges, contractions and thinnings.

Since we prove the cut-elimination theorem through elimination of *mix*, as Gentzen did too, we have postulated mix rule (μ) as primitive. However this mix is something different from Gentzen's mix. It is liberal in the sense that the $\wedge$-context G in (μ_G) need not to capture all factors B (see the definition below) as arguments in $G(B)$. The formula B may also be used in Step 2° of the construction of the $\wedge$-context G, i.e. mix need not to "swallow" all the occurrences of B in $G(B)$. Also, there are no categorial reasons to prefer cut to such a mix. In both cases, we do not have categorial composition of arrows corresponding to both premises of the rule, but a more involved composition of the right premise with an image of the left premise under the functor corresponding to a $\wedge$-context. The only difference is that in the case of cut this is always a $\wedge_1$-context.

An advantage of $\mathcal{L}$ is that its proofs can be easily coded. For example the proof

$$\cfrac{\cfrac{p \vdash p \qquad \cfrac{q \vdash q}{q \wedge \top \vdash q}}{(p \wedge (p \to q)) \wedge \top \vdash q}}{p \wedge (p \to q) \vdash q}$$

is coded by

$$\tau'_{\Box}(a_p \Delta \tau_{\Box} a_q)$$

This fact helps when we want to postulate equalities that should hold between the proofs of $\mathcal{L}$.

For the proof of our main result we need the following notions of *degree* and *rank*. The degree of a formula is the number of logical connectives in it. However, because of the categorially motivated elimination of the comma, the symbol $\wedge$ plays a double role and in order to define rank, we define as follows a set of *factors* of A, for every $A \in \mathcal{F}$:

1° A is a factor of A,

2° if A is of the form $A_1 \wedge A_2$ then every factor of A_1 or A_2 is a factor of A.

Now, we introduce (in the style of Došen) an auxiliary indexing of consequents and factors of antecedents in a mixless proof of $\mathcal{L}$, which will help us in defining the rank of an occurrence of a formula in such a proof. First we index all the consequents and all the factors of antecedents of axioms by 1 and inductively proceed as follows. In all the structural rules and the rule (Δ) the index of the consequent in the conclusion is increased by 1. In ($\wedge$) and ($*$) the index of the consequent in the conclusion is 1. Every factor of the antecedent preserved by a rule has the index increased by 1, and all the factors introduced or modified by the rule (take care that we always speak about occurrences of formulae and not just about formulae) have index 1 in the conclusion. In (ω_F) the occurence of A in the conclusion is indexed by the maximum of indices of distinguished A's in the premise, increased by 1. In the example of the proof given above this indexing looks like

$$\dfrac{p^1 \vdash p^1 \qquad \dfrac{q^1 \vdash q^1}{(q^2 \wedge \top^1)^1 \vdash q^2}}{\dfrac{((p^2 \wedge (p \to q)^1)^1 \wedge \top^2)^1 \vdash q^3}{(p^3 \wedge (p \to q)^2)^2 \vdash q^4}}$$

Then the rank of an occurrence of a formula in a proof is given by its index.

Cut-elimination theorem and consequences

Our main result is the following.

THEOREM. *Every proof in $\mathcal{L}$ can be transformed into a proof of the same root-sequent with no applications of the rule (μ).*

Proof. As in the standard cut-elimination procedure it is enough to consider a proof whose last rule is (μ) and there is no more applications of (μ) in the proof. So let our proof be of the form

$$\dfrac{\begin{matrix}\pi_1\\ A \vdash B\end{matrix} \qquad \begin{matrix}\pi_2\\ G(B) \vdash C\end{matrix}}{G(A) \vdash C}$$

with π_1 and π_2 mixless. Then we define the *degree of this proof* as the degree of B and the *rank of this proof* as the sum of the *left rank*, i.e. the rank of the occurrence of B in the left premise of (μ), in the subproof π_1, and the *right rank*, i.e. the maximum of all ranks of distinguished factors B in the right premise of (μ) in the subproof π_2. Then we prove our theorem by induction on the lexicographically ordered pairs $\langle d, r\rangle$ for the degree d and the rank r of the proof.

1. $r = 2$ The following situations should be considered: 1.1. π_1 or π_2 are axioms; 1.2. π_1 ends with $(\wedge)$; 1.3.1. π_1 ends with $(*)$ and π_2 ends with $(\triangle)$; 1.3.2. π_1 ends with $(*)$ and π_2 ends with (θ^B). We illustrate here just Case 1.2.

Suppose our proof is of the form

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ A_1 \vdash B_1\end{matrix} \qquad \begin{matrix}\pi_1''\\ A_2 \vdash B_2\end{matrix}}{A_1 \wedge A_2 \vdash B_1 \wedge B_2}\,\wedge \qquad \begin{matrix}\pi_2\\ G(B_1 \wedge B_2) \vdash C\end{matrix}}{G(A_1 \wedge A_2) \vdash C}\,\mu$$

Then this proof is transformed into the proof

$$\dfrac{\begin{matrix}\pi_1'\\ A_1 \vdash B_1\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1''\\ A_2 \vdash B_2\end{matrix} \qquad \begin{matrix}\pi_2\\ G(B_1 \wedge B_2) \vdash C\end{matrix}}{G(B_1 \wedge A_2) \vdash C}\,\mu}{G(A_1 \wedge A_2) \vdash C}\,\mu$$

where both applications of (μ) have lower degree.

2. $r > 2$ The following situations should be considered: 2.1. π_2 ends with a structural rule; 2.2. π_2 ends with $(\wedge)$; 2.3. π_2 ends with $(*)$; 2.4. π_2 ends with $(\triangle)$; 2.5. π_1 ends with a structural rule; 2.6. π_1 ends with $(\triangle)$. Cases 2.1–2.4 are considered under the assumption that the right rank is greater than 1, while 2.5 and 2.6 are connected with the assumption that the left rank is greater than 1. Case 2.1 includes a lot of subcases and we illustrate one of them here.

Suppose our proof is of the form

$$\dfrac{\begin{array}{c}\pi_1\\ A \vdash B\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ G_1(B) \vdash C\end{array}}{G(B) \vdash C}\,\beta^{\rightarrow}}{G(A) \vdash C}\,\mu$$

where G is obtained from G_1 by substituting $H \wedge \square$ for a subcontext $(H \wedge B_1) \wedge (B_2 \wedge B_3)$ of G_1, and H is a $\wedge$-context and $B \equiv B_1 \wedge (B_2 \wedge B_3)$. We call this new box of G the *principal box*. Then this proof is transformed into the proof

$$\dfrac{\begin{array}{c}\pi_1\\ A \vdash B\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ A \vdash B\end{array} \qquad \begin{array}{c}\pi_2\\ G_1(B) \vdash C\end{array}}{G_1(A) \vdash C}\,\mu}{G_2(B) \vdash C}\,\beta^{\rightarrow}}{G(A) \vdash C}\,\mu$$

where G_2 is obtained from G by substituting A for all boxes except the principal one which remains the unique box in G_2. Then the upper application of (μ) has its rank decreased by one and the right rank of the lower application of (μ) is 1.

It is possible to check that all the reduction steps of our cut-elimination procedure are covered by the equalities of cartesian closed categories which can be naturally defined in the language of $\mathcal{L}$. These equations are sufficient for cut elimination, but they need not all be necessary. This is an argument for the justification of these categories. However, the main consequence of our Theorem is another proof of the result from [1992], which claims that all canonical transformations from cartesian closed categories are natural in the extended sense. The fact that all proofs of $\mathcal{L}$ can be reduced to a cut-free form, directly eliminates all obstacles in the way of naturality. This result was originally proved by the apparatus of natural deduction, and this is an alternative, sequent system, approach.

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AN AUTOMATED THEOREM PROVER FOR THE PROBABILITY LOGIC *LPP*

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Dedicated to Professor Slaviša Prešić

ABSTRACT. We consider a propositional probability logic denoted LPP. LPP is a conservative extension of the classical propositional logic. The language of LPP contains probability operators of the form $P_{\geq s}$ for every real number $s \in [0, 1]$. The intended meaning of a formula of the form $P_{\geq s}\alpha$ is 'α holds with the probability at least s'. We obtain a decision procedure for LPP by reducing probability formulas to systems of linear equalities and inequalities. We describe an automated theorem prover based on this procedure.

1. Introduction

Probabilistic reasoning has become a subject of increased interest in theoretical computer sciences, artificial intelligence, analyzing distributed systems, cryptography, etc. Since in [**4**] a method for probabilistic deduction was proposed, various attempts to deal with such problems appeared in the literature. Some of them concerned probability logics as a suitable framework for uncertainty reasoning [**1, 2, 5, 6, 7, 8, 9, 10**]. Formulas from these logics speak about probabilities, but they remain either true or false. Thus, the probability logics are not fuzzy logics. The propositional probability languages are obtained by adding probability operators to the propositional language. The probability operators have (in our notation) the form $P_{\geq s}$, with the intended meaning that $P_{\geq s}\alpha$ holds if the probability of α is greater or equal to s.

In this paper we consider a probability logic denoted LPP. Axiomatizations for some variants of the logic and the corresponding completeness and decidability theorems are given in [**1, 7, 8, 9**]. The language of LPP contains probability operators of the form $P_{\geq s}$ for every real number $s \in [0, 1]$. This logic allows

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statements like 'if α holds with the probability s, and β follows from α with the probability r, then the probability of β is t', where α and β are events described by classical propositional formula. We obtain a decision procedure for LPP by reducing probability formulas to systems of linear equalities and inequalities and describe an automated theorem prover based on this procedure.

2. Probability logic LPP

The LPP-language is obtained by adding a list of probability operators of the form $P_{\geq s}$, for every real number $s \in [0,1]$, to the classical propositional language. Starting from a set of propositional letters $\phi = \{p, q, r, \dots\}$ and the classical operators $\neg$ and $\wedge$, the set of classical propositional formulas For_C is defined in the usual way. Let us denote formulas from For_C by α, β, ... The set For_P of all probability formulas is defined as follows. If $\alpha \in \mathrm{For}_C$, then $P_{\geq s}\alpha$ is a basic probability formula. The set of all probability formulas is the least set For_P containing all basic probability formulas, and closed under formation rules: if $A, B \in \mathrm{For}_P$, then $\neg A$, $A \wedge B \in \mathrm{For}_P$. Let formulas from For_P be denoted by A, B, ... , $\mathrm{For}_C \cup \mathrm{For}_P$ by For, and formulas from For by Φ, Ψ, ... For example, $\neg P_{\geq s}\alpha \wedge P_{\geq r}(\alpha \rightarrow \beta)$ is a syntactically correct formula, while $(P_{\geq r}\alpha) \rightarrow \beta$ and $P_{\geq s}P_{\geq r}\alpha$ are not. In other words, combinations of classical propositional and probability formulas and iterations of probability operators are not allowed. We use the usual abbreviation for the other classical connectives ($\vee$, $\rightarrow$, $\leftrightarrow$), and also denote $\neg P_{\geq s}(\alpha)$ by $P_{<s}(\alpha)$, $P_{\geq 1-s}(\neg\alpha)$ by $P_{\leq s}(\alpha)$, $\neg P_{\leq s}(\alpha)$ by $P_{>s}(\alpha)$, and $P_{\geq s}(\alpha) \wedge \neg P_{>s}(\alpha)$ by $P_{=s}(\alpha)$.

Note that there are uncountably many formulas. This does not make any problem, since we only consider decidability of the logic. On the other hand, the completeness problem for LPP is not so straightforward. See **[1]** for more discussion on this subject.

In order to give semantics to formulas from the set For, we use some notions from the measure theory. We suppose that the reader is familiar with them.

Definition 1. An LPP_{Meas}-model is a structure $\langle W, H, \mu, v \rangle$ where:

- W is a set of elements called worlds,
- H is a σ-algebra of subsets of W,
- $\mu : H \rightarrow [0,1]$ is a σ-additive probability measure,
- $v : W \times \phi \rightarrow \{\top, \bot\}$ is a valuation which associated with every world $w \in W$ a truth assignment $v(w)$ on the propositional letters, and
- for every propositional letter $p \in \phi$ the set $[p]_W = \{w \in W : v(w)(p) = \top\}$ is measurable (i.e. $[p]_W \in H$).

The valuation v is extended to a truth assignment on all classical propositional formulas in a usual way. Note that we insist that every set of the form $[p]_W$ is measurable. It is easy to see that for every formula $\alpha \in \mathrm{For}_C$, the set $[\alpha]_W$ is also measurable, i.e. that in every model every event described by a classical propositional formula is associated with a measurable set of worlds. We call such models - *measurable models.* The subscript Meas in LPP_{Meas} denotes that we work with the class of all measurable models.

DEFINITION 2. The satisfiability relation $\models \subset LPP_{\text{Meas}} \times \text{For}$ fulfills the following conditions for every LPP_{Meas}-model $M = \langle W, H, \mu, v\rangle$:

1. for any $\alpha \in \text{For}_C$, $M \models \alpha$ iff for every world $w \in W$, $v(w)(\alpha) = \top$,
2. for any $\alpha \in \text{For}_C$, $M \models P_{\geq s}\alpha$ iff $\mu([\alpha]_M) \geq s$,
3. for any $A \in \text{For}_P$, $M \models \neg A$ iff $M \not\models A$, and
4. for all $A, B \in \text{For}_P$, $M \models A \wedge B$ iff $M \models A$ and $M \models B$.

A formula $\Phi \in \text{For}$ is *satisfiable* if there is an LPP_{Meas}-model M such that $M \models \Phi$; Φ is *valid* ($\models \Phi$) if for every LPP_{Meas}-model M, $M \models \Phi$. Note that classical propositional formulas do not behave in the usual way. For example, it follows from Definition 2, that for some $\alpha, \beta \in \text{For}_C$ and some model M it can be $M \models \alpha \vee \beta$, but that neither $M \models \alpha$, nor $M \models \beta$. Similarly, it can be that $M \not\models \alpha$ and $M \not\models \neg\alpha$.

3. Decidability

In the sequel we will use $\pm\Phi$ to denote either Φ or $\neg\Phi$. Let $A \in \text{For}_P$, and $p_1, \ldots, p_n$ be a list of all propositional letters that appear in A. A basic conjunction a of A is a formula of the form $\pm p_1 \wedge \ldots \wedge \pm p_n$. If A contains n propositional letters, there are 2^n basic conjunctions of A. For different basic conjunctions a_i and a_j we have $\vdash a_i \to \neg a_j$. Thus, in every LPP_{Meas}-model $\mu(a_i \vee a_j) = \mu(a_i) + \mu(a_j)$. It is easy, using propositional reasoning and the theorem

THEOREM 1. *Let* $\alpha, \beta \in \text{For}_C$. *If* $\models \alpha \leftrightarrow \beta$, *then* $\models P_{\geq s}\alpha \leftrightarrow P_{\geq s}\beta$, *for every* $s \in [0, 1]$.

to show that every probability formula $A \in \text{For}_P$ is equivalent to a formula

$$\text{DNF}(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} \pm P_{\geq s_{ij}}(p_1, \cdots, p_n) \tag{1}$$

called a disjunctive normal form of A, where $P_{\geq s_{i,j}}(p_1, \cdots, p_n)$ denotes that the propositional formula which is in the scope of the probability operator $P_{\geq s_{i,j}}$ is in the complete disjunctive normal form, i.e. that the propositional formula is a disjunction of the basic conjunctions from A. The next theorem is proved in [**9**]. We give it here since our prover relies on the described decision procedure.

THEOREM 2. *The logic LPP is decidable.*

PROOF. There is a procedure for deciding satisfiability and validity for classical propositional formulas. Hence, we consider probability formulas only. A probability formula A is equivalent to $\text{DNF}(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} \pm P_{\geq s_{ij}}(p_1, \cdots, p_n)$. A is satisfiable iff at least one disjunct from $\text{DNF}(A)$ is satisfiable. Let the probability of the basic conjunction a_i be denoted by y_i. We use an expression of the form $a_i \in \pm P_{\geq s}(p_1, \cdots, p_n)$ to denote that a_i appears in the classical propositional part of $\pm P_{\geq s}(p_1, \cdots, p_n)$. A formula of the form $P_{\geq s}(p_1, \cdots, p_n)$ is satisfiable in a model $M = \langle W, H, \mu, v\rangle$ iff the probability of $[\bigvee_{a_{t_i} \in P_{\geq s}(p_1, \ldots, p_n)} a_{t_i}]_M$ is at least s. And, since basic conjunctions are mutually exclusive, $\mu([\bigvee_{a_{t_i} \in P_{\geq s}(p_1, \ldots, p_n)} a_{t_i}]_M) \geq$

s holds iff $\sum_{a_{t_i} \in P_{\geq s}(p_1,\ldots,p_n)} \mu([a_{t_i}]_M) \geq s$. Similarly, a formula of the form $\neg P_{\geq s}(p_1,\ldots,p_n)$ holds in a model M iff the probability of $[\bigvee_{a_{t_i} \in P_{\geq s}(p_1,\ldots,p_n)} a_{t_i}]_M$ is less than s iff $\sum_{a_{t_i} \in P_{\geq s}(p_1,\ldots,p_n)} \mu([a_{t_i}]_M) < s$.

Thus, a disjunct $D = \bigwedge_{j=1}^{k} \pm P_{\geq s_j}(p_1,\ldots,p_n)$ from DNF(A) is satisfiable iff the following system of linear equalities and inequalities is consistent:

$$
(2) \qquad \begin{array}{rl}
\sum_{i=1}^{2^n} y_i & = 1 \\
y_i & \geq 0, \text{ for } i = 1,\ldots,2^n \\
\sum_{a_t \in P_{\geq s_1}(p_1,\ldots,p_n) \in D} y_t & \begin{cases} \geq s_1 & \text{if } \pm P_{\geq s_1} = P_{\geq s_1} \\ < s_1 & \text{if } \pm P_{\geq s_1} = P_{< s_1} \end{cases} \\
\ldots & \\
\sum_{a_t \in P_{\geq s_k}(p_1,\ldots,p_n) \in D} y_t & \begin{cases} \geq s_k & \text{if } \pm P_{\geq s_k} = P_{\geq s_k} \\ < s_k & \text{if } \pm P_{\geq s_k} = P_{< s_k} \end{cases}
\end{array}
$$

The first equation corresponds to the fact that the measure of the set of all worlds in a model is 1, while the set of inequalities $y_i \geq 0$ corresponds to the nonnegativity of the probability. The other inequalities correspond to formulas from D. Now, the problem of satisfiability of an arbitrary formula A is reduced to the linear systems solving problem, and the satisfiability problem for the LPP-logic is decidable. Since a formula is valid iff its negation is not satisfiable, the validity problem for the LPP-logic is decidable, too. □ □

4. An automated theorem prover

Our LPP-theorem prover is, in fact, a satisfiability checker for probability formulas. Here is a high level description of the procedure in a Pascal-like language:

```
procedure CheckSatisfiability (Formula A)
begin
        DNF(A) := disjunctive_normal_form(A);
        for every disjunct D from DNF(A) do
        begin
                SYSTEM(D) := generate_system(D);
                solution := solve(SYSTEM(D));
                if solution is not empty then
                begin
                        write ( 'A is satisfiable' );
                        exit;
                end;
        end;
        write ( 'A is not satisfiable' );
end;
```

For example, if the input of the prover is the formula $P_{\geq 0.8}p \wedge P_{<0.9}p \wedge P_{\geq 0.7}q \wedge P_{<0.8}q \wedge P_{\geq 0.9}(p \wedge q)$, the obtained result is that the formula is not satisfiable. On the other hand, $P_{\geq 0.5}p$ is satisfiable, as well as $\neg P_{\geq 0.5}p$. Thus, neither $P_{\geq 0.5}p$ nor $\neg P_{\geq 0.5}p$ are valid.

The above procedure allows the following modification. The probability language can be extended by a probability operator of the form $P_{\geq x}$, where x denotes unknown probability. Then, we can solve the corresponding linear systems (2) and find for what x they are (un)satisfiable. For example, the formula $(P_{\geq 0.8}p \wedge P_{\geq 0.9}(p \rightarrow q) \wedge P_{<0.95}(p \rightarrow q)) \rightarrow P_{<x}q$ is valid for every $x \in [0.95, 1]$. The formula $\neg P_{\geq x}p$ is satisfiable for every $x \in (0, 1]$ which means that $\neg P_{\geq 0}p$ is a contradiction, i.e. that $P_{\geq 0}p$ is valid, while $P_{\geq x}p$ is not valid for any $x > 0$.

Finally, it is easy to detect inherent parallelism in the procedure CheckSatisfiability: disjuncts from DNF(A) can be processed independently by individual processes. Afterwards, their results can be combined to form the solution.

5. Conclusion

There are many places in artificial intelligence (expert systems, decision making systems, fault tree analysis, ...) where knowledge is not crisp. If we are able to attach probabilities to uncertain information, it would be useful to have an effective formal procedure to infer conclusions. Presented LPP-logic and the corresponding decision procedure offer a suitable way to reason in such situations. For example, using probability formulas we can describe some events and check whether some events with attached probabilities are consequences of some other events. Also, we can compute what is the probability of a consequence of some premises, and find the most promising consequence among a set of events.

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ON THE FUNCTIONAL EQUATION $A(x, B(x,y)) = y$ IN THE VARIETY OF GROUPOIDS

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Dedicated to Professor Slaviša B. Prešić in occasion of his 65^{th} anniversary

ABSTRACT. We focus on finding general solutions of the functional equation $A(x, B(x,y)) = y$ in the class of groupoids where A, B are unknown groupoid operations over the same set. We also consider functional equations symmetric to the mentioned one, as well as systems of such functional equations.

1. Preliminaries

In what follows we present description of the general solutions of the functional equation $A(x, B(x,y)) = y$, as well as symmetric ones: $A(x, B(y,x)) = y$, $A(B(y,x),x) = y$, $A(B(x,y),x) = y$, where A and B are unknown groupoid operations over a same set. We characterize the solutions in the class of finite groupoids and in the class of all groupoids. The main motivation to consider such functional equations arose from the groupoid identity $A(x, A(x,y)) = y$ [5]. Clearly, if this identity holds in a groupoid, then the pair (A, A) is a solution to our equation, where A denotes the operation of the groupoid. Further back, consideration of such an identity was motivated by the fact that groupoids that satisfy it have orthogonal complements which are right zero (or left unit) groupoids. So, we thought that finding the general solutions of these functional equations might be of interest. In the sequel we frequently use the following notions. A right zero groupoid is a groupoid satisfying the law $xy = y$. Every right zero groupoid is a semigroup where each element is a right zero and a left unit. A groupoid $(G; A)$ is said to be left (right) cancellative groupoid if

$$A(x,y) = A(x,z) \implies y = z \qquad (A(y,x) = A(z,x) \implies y = z)$$

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for all $x, y, z \in G$. A groupoid that is both left and right cancellative will be called cancellative. $(G; A)$ is called left (right) solvable groupoid if the equation $A(x, a) = b$ ($A(a, y) = b$) has a solution x (y), for every $a, b \in G$. If $(G; A)$ is both left and right solvable then it is solvable. $(G; A)$ is said to be left (right) quasigroup if it is left (right) solvable and right (left) cancellative. The solution x (y) of the equation $A(x, a) = b$ ($A(a, y) = b$) in a left (right) quasigroup is unique, and vice versa. $(G; A)$ is a quasigroup if it is left and right quasigroup. Sometimes we shall call the operation A (left, right) cancellative, solvable, quasigroup if the groupoid $(G; A)$ has the mentioned property. The following simple facts hold.

PROPOSITION 1.

(i) *Every finite left (right) cancellative groupoid is a right (left) quasigroup.*
(ii) *Every finite left (right) solvable groupoid is left (right) quasigroup.*

If $(G; A)$ is a quasigroup, then the quasigroup operations A^{-1}, ${}^{-1}A$, A^*, defined on G by $A^{-1}(x, z) = y$, ${}^{-1}A(z, y) = x$, $A^*(y, x) = z$ if $A(x, y) = z$, are called parastrophes (or conjugates) of A. If $(G; A)$ is left (right) quasigroup, then $(G; A^*)$ is right (left) quasigroup, the left parastrophe ${}^{-1}A$ (the right parastrophe A^{-1}) is defined and $(G; {}^{-1}A)$ ($(G; A^{-1})$) is left (right) quasigroup as well.

2. Functional equation $A(x, B(x, y)) = y$ in the variety of groupoids

We focus our attention on the functional equation $A(x, B(x, y)) = y$ where A and B are unknown groupoid operations on a same set.

PROPOSITION 2. *If $A(x, B(x, y)) = y$ is satisfied by some groupoid operations A and B, then:*

(i) *B is left cancellative.*
(ii) $B(x, y) = z \implies A(x, z) = y$.
(iii) *A is right solvable.*
(iv) $A(x, z) = y \wedge B(x, y) = t \implies A(x, z) = A(x, t) = y$.
(v) *If B is right solvable, then A is right quasigroup.*
(vi) *If A is left cancellative, then B is right quasigroup.*
(vii) *A is right quasigroup if and only if B is right quasigroup, and when A and B are right quasigroups then $B(x, y) = z \iff A(x, z) = y$ i.e. $A = B^{-1}$ ($B = A^{-1}$).*

PROOF. (i), (ii) and (iii) are obvious.

(iv) When $A(x, z) = y$ and $B(x, y) = t$ we have $A(x, t) = A(x, B(x, y)) = y = A(x, z)$.
(v) Let $A(x, z) = A(x, t)$. There exist y_1, y_2 such that $B(x, y_1) = z$, $B(x, y_2) = t$ and we have $y_1 = A(x, B(x, y_1)) = A(x, z) = A(x, t) = A(x, B(x, y_2)) = y_2$, hence $z = t$. By *(iii)*, A is right quasigroup.
(vi) Let A be left cancellative. By *(i)*, it is enough to prove that B is right solvable. Let x, z be given and $A(x, z) = y$, $B(x, y) = t$. Then by *(iv)* we have $A(x, z) = A(x, t) = y$ and hence $z = t$. So, $B(x, y) = z$.

(vii) The first part is a consequence of (*i*), (*iii*), (*v*) and (*vi*). Let A and B be right quasigroups. $B(x,y) = z \implies A(x,z) = y$ holds by (ii). Let $A(x,z) = y$ and $B(x,y) = t$. Then $A(x,t) = y = A(x,z)$ by (iv), hence $z = t$.

□

THEOREM 1. *The solution of the equation* $A(x, B(x,y)) = y$ *over the class of all finite groupoids consists of arbitrary mutually right-inverse right quasigroups i.e., right quasigroups satisfying* $A = B^{-1}$.

PROOF. If A and B form a solution of the equation and are over a finite set then, by Proposition 2(i), B is left cancellative, and the statement follows by Proposition 1(i) and Proposition 2(vii). On the other hand, if A and B are mutually right-inverse right quasigroups and if $B(x,y) = z$, then $A(x,z) = y$ i.e. A and B form a solution of the equation. □

EXAMPLE 2.1. The following operations A and B defined on the set $\mathbb{N}$ of positive integers by:

$$A(x,y) = div(x,y) = \left[\frac{y}{x}\right], \ B(x,y) = xy$$

are solutions of the functional equation. Namely, $A(x, B(x,y)) = div(x,xy) = y$. Note that A is neither left nor right cancellative and B is neither left nor right solvable.

THEOREM 2. *The solution of the functional equation*

$$A(x, B(x,y)) = y$$

over the class of all groupoid operations consists of any left cancellative operation B *and a corresponding right solvable operation* A *that satisfies the condition* $B(x,y) = z \implies A(x,z) = y$.

PROOF. If A and B form a solution then it satisfies the conditions by Propositions 2 (i), (ii), (iii). The other direction of the statement follows directly from the assumptions. □

In the special case of the equation $A(x, A(x,y)) = y$ we get that the solution is a right quasigroup that is self-right-inverse. Out of symmetry similar results hold for the functional equations

$$A(x, B(y,x)) = y, \quad A(B(y,x),x) = y, \quad A(B(x,y),x) = y.$$

So, we have the following properties.

THEOREM 3. *The solution of the functional equation* $A(x, B(y,x)) = y$ *over the class of all groupoids consists of a right cancellative operation* B *and a right solvable operation* A *satisfying the condition* $B(y,x) = z \implies A(x,z) = y$. *In the finite case, the solution consists of a right quasigroup* A *and a left quasigroup* B *satisfying* $A = ({}^{-1}B)^*$.

THEOREM 4. *The solution of the functional equation $A(B(y,x),x) = y$ over the class of all groupoids consists of a right cancellative operation B and a left solvable operation A satisfying the condition $B(y,x) = z \implies A(z,x) = y$. In the finite case, the solution consists of arbitrary finite mutually left-inverse left quasigroups i.e., left quasigroups satisfying $A = {}^{-1}B$.*

THEOREM 5. *The solution of the functional equation $A(B(x,y),x) = y$ over the class of all groupoids consists of a left cancellative operation B and a left solvable operation A satisfying the condition $B(x,y) = z \implies A(z,x) = y$. In the finite case, the solution consists of a right quasigroup B and a left quasigroup A satisfying $A = (B^{-1})^*$.*

3. Systems of equations

Here we note some consequences of the results in previous section, considering systems of functional equations consisting of pairs of equations of the mentioned types. By Theorem 2 we get the following theorem.

THEOREM 6. *The solution of the system of functional equations*

$$A(x, B(x,y)) = y, \quad B(x, A(x,y)) = y$$

over the class of all groupoids consists of right quasigroups A and B satisfying the condition $A = B^{-1}$.

THEOREM 7. *The solution of the system of functional equations*

$$A(x, B(y,x)) = y, \quad B(x, A(y,x)) = y$$

over the class of all groupoids consists of quasigroups A and B such that $B = A = {}^{-1}(A^{-1}) = ({}^{-1}A)^{-1}$, $A^* = A^{-1} = {}^{-1}A$.

PROOF. If A and B are solutions then, by *Theorem 3*, we have that they are both right solvable and right cancellative. Then also $B(y,x) = z \implies A(x,z) = y \implies B(z,y) = x \implies A(y,x) = z \implies B(x,z) = y$. Hence, $A = B$ and $A(x,y_1) = A(x,y_2) = z \implies y_1 = A(x,z) = y_2$, i.e. A is right quasigroup. Moreover, $x = A(a,b)$ is the solution of the equation $A(x,a) = b$, therefore A is left quasigroup, i.e. A is quasigroup. By $A(z,y) = x \iff A(y,x) = z$ we have $A^{-1} = {}^{-1}A$. Hence, A and B are quasigroups such that $B = A$, $A^{-1} = {}^{-1}A$, and we only have to note that $A = (A^{-1})^{-1} = {}^{-1}({}^{-1}A)$, $A^* = {}^{-1}(({}^{-1}A)^{-1})$. □

The next two theorems are obtained in the same manner as above.

THEOREM 8. *The solution of the system of functional equations*

$$A(B(y,x),x) = y, \quad B(A(y,x),x) = y$$

over the class of all groupoids consists of left quasigroups A and B satisfying $A = {}^{-1}B$.

THEOREM 9. *The solution of the system of functional equations*

$$A(B(x,y),x) = y, \quad B(A(x,y),x) = y$$

over the class of all groupoids consists of quasigroups A and B satisfying the conditions $B = A = {}^{-1}(A^{-1}) = ({}^{-1}A)^{-1}$, $A^* = A^{-1} = {}^{-1}A$.

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A GENERALIZATION OF RECTANGULAR LOOPS

A. Krapež

Dedicated to professor Slaviša B. Prešić on the occasion of his 65th birthday

ABSTRACT. It is proved that the class of groupoids which are isotopes of rectangular loops can be axiomatized by 18 universal equations in a language with three binary and six unary operations. The problem of the existence of some simpler (in particular independent) axiom system for this class is posed.

Introduction

A *left zero semigroup* is a semigroup in which every element is a left zero and which therefore may be defined by the universal equation $xy = x$. The dual notion of a *right zero semigroup* is defined by the universal equation $xy = y$. A *rectangular band* is the direct product of a left zero semigroup and a right zero semigroup.

All these are important but fairly trivial types of semigroups. Not so trivial and even more important is a *rectangular group* which is the direct product of a group and a rectangular band (see for example M. Petrich [5]).

A generalization of rectangular group called *rectangular loop* is defined in [4] as the direct product of a left zero semigroup, a loop and a right zero semigroup. The following theorem from [4] gives us an axiomatization of rectangular loops:

THEOREM 1. *A groupoid* $(S;\cdot)$ *is a rectangular loop iff it is a reduct of an algebra* $(S;\cdot,/,\backslash)$, *satisfying the axioms:*

(Q1) $$x\backslash(x\cdot y) = (x\backslash x)\cdot y$$

(Q2) $$x\cdot(x\backslash y) = (x\backslash x)\cdot y$$

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Key words and phrases. loop, rectangular loop, rectangular loop isotope, isotopy, axiomatization.

(Q3) $$(x \cdot y)/y = x \cdot (y/y)$$

(Q4) $$(x/y) \cdot y = x \cdot (y/y)$$

(E1) $$(x/x) \cdot x = x$$

(E2) $$x\backslash x = x/x$$

(A1) $$(x/x) \cdot (y \cdot z) = ((x/x) \cdot y) \cdot z$$

(A2) $$x \cdot ((y/y) \cdot z) = x \cdot z$$

(A3) $$(x \cdot y) \cdot (z/z) = x \cdot (y \cdot (z/z))$$

(U1) $$(x \cdot y)/(x \cdot y) = (x/x) \cdot (y/y)$$

(U2) $$(x\backslash y)/(x\backslash y) = (x/x) \cdot (y/y)$$

(U3) $$(x/y)/(x/y) = (x/x) \cdot (y/y)$$

This axiom system will be denoted by ($\Box\Lambda$) and the class of all rectangular loops by [$\Box\Lambda$]. Algebras satisfying ($\Box\Lambda$) will be called equational rectangular loops.

Rectangular loop isotopes

EXAMPLE 1. Let $(G; \cdot, e, ^{-1})$ be a group (equationally defined), which is not boolean (i.e. $xx \neq e$ for some x) and let $x/y = x \cdot y^{-1}$. Then the groupoid $(G; /)$ is a unipotent ($x/x = e$) quasigroup with right unit e. But e is not a left unit and therefore $(G; /)$ is not a loop.

Let L, R be nonempty sets, $S = L \times G \times R$ and $\circ, //, I$ operations on S defined by: $(a, x, p) \circ (b, y, q) = (a, xy, q), (a, x, p)//(b, y, q) = (a, x/y, q)$ and $I(a, x, p) = (a, x^{-1}, p)$ (for all appropriate a, b, x, y, p, q). Then $(S; \circ)$ is a rectangular group, I is bijection and $(S; //)$ is an isotope of $(S; \circ)$ ($u//v = u \circ I(v)$) which is not a rectangular loop.

EXAMPLE 2. Let $(L; \cdot)$ be a left zero semigroup, $f \neq id$ a permutation of L and $x \circ y = f(x)$. Then $(L; \circ)$ is so called left groupoid (see [3]) which is obviously isotopic to $(L; \cdot)$ but is not a rectangular loop.

Such examples inspired us to make a generalization of rectangular groups which we call *rectangular loop isotopes*. The key requirement for the class of all rectangular loop isotopes is that it should be closed under taking isotopies, the property not satisfied by either of the classes of all rectangular groups, rectangular loops. For the definition and properties of the notion of *isotopy* of groupoids, in particular of quasigroups, the reader may consult [1] or [2].

DEFINITION 1. The class $[RLI]$ of all rectangular loop isotopes is the smallest class of groupoids containing the classes $[Q]$ of all quasigroups, $[L]$ of all left zero semigroups, $[R]$ of all right zero semigroups and closed under taking direct products and isotopies.

LEMMA. *The class* [$\Box\Lambda$] *is properly contained in the class* $[RLI]$.

Proof. Trivially, $[\Box\Lambda] \subseteq [RLI]$. The following example shows that the inclusion is strict. Let $S = \{0, 1\}, f(x) = 1 - x$ and $x \cdot y = f(x)$. Then $(S; \cdot)$ is a left groupoid and consequently an isotope of the left zero semigroup $(S;\circ), x \circ y = x$. Hence $(S;\cdot) \in [RLI]$.

Assume that $(S;\cdot)$ is a rectangular loop. Being of prime order, it should be either a left or right zero semigroup or else a loop. But it obviously is neither. This contradiction proves that $(S;\cdot) \notin [\square\Lambda]$. $\square$

Further, the usual rules for omitting parentheses apply. All binary operations bind terms with equal strength, except juxtaposition which binds them stronger (juxtaposition replaces the multiplication $\cdot$ in some cases).

For unary operations (and there are only six of them: f, f^-, g, g^-, i and j) fx stands for $f(x)$ and fgx for $f(g(x))$ and similarly in other cases. Unary operations always bind terms stronger then binary ones. However, we shall never write potentially ambiguous expressions like fxy but use parentheses $(f(xy))$ or multiplication symbol $(fx \cdot y)$ to enhance readability.

DEFINITION 2. $\mathcal{A}$ is the class of all algebras $(S; \cdot, /, \backslash, f, f^-, g, g^-, i, j)$ satisfying the system (RLI) of axioms:

(q1) $$x\backslash xy = g(i(x) \cdot y)$$
(q2) $$x(x\backslash y) = i(x) \cdot g(y)$$
(q3) $$xy/y = f(x \cdot j(y))$$
(q4) $$(x/y)y = f(x) \cdot j(y)$$
(e1) $$if(x) \cdot g(x) = x$$
(e2) $$f(x)\backslash x = jg(x)$$
(e3) $$x/g(x) = if(x)$$
(e4) $$f^-if(x) = g^-jg(x)$$
(a1) $$i(x) \cdot g(f(y) \cdot z) = f(i(x) \cdot g(y)) \cdot z$$
(a2) $$x \cdot g(i(y) \cdot z) = xz$$
(a3) $$f(x \cdot g(y)) \cdot j(z) = x \cdot g(f(y) \cdot j(z))$$
(u1) $$if(xy) = f(i(x) \cdot j(y))$$
(u2) $$j(x\backslash y) = g(i(x) \cdot jg(y))$$
(u3) $$i(x/y) = f(if(x) \cdot j(y))$$
(i1) $$ff^-(x) = x$$
(i2) $$f^-f(x) = x$$
(i3) $$gg^-(x) = x$$
(i4) $$g^-g(x) = x$$

By $\mathcal{A}_{\{\cdot\}}$ we denote the class of all $\{\cdot\}$–reducts of algebras from $\mathcal{A}$. The class of all isotopes of rectangular loops is denoted by $\mathcal{B}$.

A more detailed study of rectangular loop isotopes will be published elsewhere. Here we prove only the main result justifying the name 'rectangular loop isotopes' for the groupoids from $[RLI]$ and giving an equational axiomatization for them.

THEOREM 2. $\mathcal{A}_{\{\cdot\}} = \mathcal{B} = [RLI]$.

Proof. a) Let $(S;\cdot,/,\backslash,f,f^-,g,g^-,i,j)$ be an algebra from $\mathcal{A}$ and let $x\circ y = fx\cdot gy, x//y = f^-(x/gy)$ and $x\backslash\backslash y = g^-(fx\backslash y)$. Then the algebra $(S;\circ,//,\backslash\backslash)$ is an equational rectangular loop. The proof requires checking axioms (Q1)–(U3). As an example we prove just $(A1)$:

$$\begin{aligned}(x//x)\circ(y\circ z) &= f(x//x)\cdot g(fy\cdot gz) = ff^-(x/gx)\cdot g(fy\cdot gz) = ifx\cdot g(fy\cdot gz)\\ &= f(ifx\cdot gy)\cdot gz = (ff^-ifx\cdot gy)\circ z = (f^-ifx\circ y)\circ z\\ &= (f^-(x/gx)\circ y)\circ z = ((x//x)\circ y)\circ z.\end{aligned}$$

Therefore $(S;\cdot)$ is an isotope of the rectangular loop $(S;\circ)$ i.e. $(S;\cdot)\in\mathcal{B}$. It follows that $\mathcal{A}_{\{\cdot\}}\subseteq\mathcal{B}$.

b) Let $(S;\cdot)$ be an algebra from $\mathcal{B}$. As every isotope is isomorphic to a principal isotope, we may assume that $x\circ y = fx\cdot gy$, where $(S;\circ)$ is a rectangular loop. By theorem 1, there is an equivalent equational rectangular loop $(S;\circ,//,\backslash\backslash)$. Operations f and g are bijections and therefore $ff^- = f^-f = gg^- = g^-g = id$ for f^-,g^- inverse mappings of f,g respectivelly (and id identity mapping of S). If we define

$$\begin{aligned}x/y &= f(x//g^-(y)),\\ x\backslash y &= g(f^-(x)\backslash\backslash y),\\ i(x) &= f^-(x)/gf^-(x),\\ j(x) &= gf^-ifg^-(x),\end{aligned}$$

then we can easily verify that the algebra $(S;\cdot,/,\backslash,f,f^-,g,g^-,i,j)$ satisfies all axioms (q1)–(i4) and consequently $(S;\cdot)\in\mathcal{A}_{\{\cdot\}}$. Therefore $\mathcal{B}\subseteq\mathcal{A}_{\{\cdot\}}$.

c) Every quasigroup is an isotope of some loop which is a special rectangular loop. Therefore an algebra obtained from a quasigroup by expanding its language using division operations / and \, $i(x) = x\backslash x, j(x) = x/x$ and $f = f^- = g = g^- = id$, belongs to $\mathcal{A}$. Similarly, the expanded versions of left (right) zero semigroups (assuming $x/y = x\backslash y = xy$) belong to $\mathcal{A}$. Consequently, the class $\mathcal{A}_{\{\cdot\}}$ contains classes $[Q],[L]$ and $[R]$. Being a variety, $\mathcal{A}$ is closed under taking direct products and so is $\mathcal{A}_{\{\cdot\}}$. If $(S;\cdot)$ is an isotope of some groupoid from $\mathcal{A}_{\{\cdot\}} = \mathcal{B}$, then $(S;\cdot)$ is an isotope of an isotope of a rectangular loop and consequently is itself an isotope of a rectangular loop i.e. $(S;\cdot)\in\mathcal{B} = \mathcal{A}_{\{\cdot\}}$. It follows that $[RLI]\subseteq\mathcal{A}_{\{\cdot\}}$.

d) We have already noted that $[\Box\Lambda]\subset[RLI]$. As $[RLI]$ is closed under isotopies, it follows that all groupoids from $\mathcal{B} = \mathcal{A}_{\{\cdot\}}$ also belong to $[RLI]$. □

The independence of the axioms (q1)–(i4) for the rectangular loop isotopes remains an open problem (see [4] for the related ir ɔendence problem of $(\Box\Lambda)$). The system (RLI) can be reduced trivially, replacing function j by gf^-ifg^- and eliminating axiom $(e4)$ (and similarly for i), but we have a more substantial reduction in mind.

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SEMIGROUPS IN WHICH ANY PROPER IDEAL IS SEMILATTICE INDECOMPOSABLE

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Dedicated to Professor S. B. Prešić on the occasion of his 65th birthday

ABSTRACT. The main purpose of this note is to study semigroups all of whose proper ideals from an arbitrary nontrivial complete 1-sublattice of the lattice of ideals are semilattice indecomposable or archimedean semigroups. We also determine some conditions under which there exists the largest proper ideal in this lattice.

A significant problem of Semigroup theory is to study semigroups all of whose proper subsemigroups or ideals have certain properties. Semigroups whose proper ideals are groups, commutative or archimedean semigroups were investigated by Schwarz in [**11**], Tamura in [**12**], Bogdanović in [**2, 3, 4, 5**], Bogdanović and Ćirić in [**6**] and others. In the present paper we consider any complete 1-sublattice $\mathrm{Id}^\pi(S)$ of the lattice $\mathrm{Id}(S)$ of ideals of a semigroup S. As was shown in [**8**], it is uniquely determined by some positive quasi-order π on S. We study the set $M_\pi(S)$ of all elements of S that generate a proper ideal from $\mathrm{Id}^\pi(S)$, for which we show that it is an ideal and the union of all proper ideals from $\mathrm{Id}^\pi(S)$, and we find some conditions under which $M_\pi(S)$ is also a proper ideal of S. Finally, we determine the conditions under which any proper ideal from $\mathrm{Id}^\pi(S)$ is a semilattice indecomposable or archimedean semigroup.

By a *complete 1-sublattice* of a complete lattice L we mean any complete sublattice of L containing the unity of L. For a semigroup S, S^1 denotes the semigroup obtained from S by adjoining the unity. The *division relation* $\mid$ on S is defined by: $a \mid b$ if and only if $b = xay$, for some $x, y \in S^1$, the relation $\longrightarrow$ on S is defined by: $a \longrightarrow b$ if and only if $a \mid b^n$, for some natural number n, and $\longrightarrow^\infty$ denotes the transitive closure of $\longrightarrow$. The lattice of all ideals of S is denoted by $\mathrm{Id}(S)$. Any ideal of S different than S is called a *proper ideal* of S. An ideal I of S is called *completely semiprime* if for any $a \in S$, $a^2 \in I$ implies $a \in I$, and it

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is called *completely prime* if for any $a, b \in S$, $ab \in I$ implies that $a \in I$ or $b \in I$. The set of all completely semiprime ideals of S, which is a complete 1-sublattice of $\mathrm{Id}(S)$, is denoted by $\mathrm{Id}^{cs}(S)$.

By a *quasi-order* on a set A we mean any reflexive and transitive binary relation π on A, and the pair (A, π) is called a *quasi-ordered set*. For $a \in A$ we set $a\pi = \{x \in A \mid a \,\pi\, x\}$ and for $X \subseteq A$ we set $X\pi = \bigcup_{x \in X} x\pi$. In other words, $a\pi$ and $X\pi$ are the filters (dual ideals) of a quasi-ordered set (A, π) generated by $\{a\}$ and X, respectively. By π^{-1} we denote a relation on A defined by: $a \,\pi^{-1}\, b$ if and only if $b \,\pi\, a$. The relation $\widetilde{\pi} = \pi \cap \pi^{-1}$ is the greatest equivalence relation contained in π and it is called the *natural equivalence* of π. As is well known, $a \,\widetilde{\pi}\, b$ if and only if $a\pi = b\pi$, and the set of all $\widetilde{\pi}$-classes is partially ordered, where the partial order $\leq$ s defined by: $a\widetilde{\pi} \leq b\widetilde{\pi}$ if and only if $a \,\pi\, b$ (cf. **[1]** and **[8]**). The partially ordered set of all quasi-orders on A is a complete lattice and it is denoted by $\mathcal{Q}(S)$.

Let π be a quasi-order on a semigroup S. We say that π is *positive* if $a \,\pi\, ab$ and $b \,\pi\, ab$, for all $a, b \in S$, that it is *lower-potent* if $a^2 \,\pi\, a$, for any $a \in S$, and that it satisfies the *cm-property* (*common multiple property*, **[13, 14]**) if for any $a, b, c \in S$, $a \,\pi\, c$ and $b \,\pi\, c$ implies $ab \,\pi\, c$. The set of all positive quasi-orders on S s the principal filter of the lattice $\mathcal{Q}(S)$ generated by the division relation on S, whereas the set of all positive lower-potent quasi-orders on S is the principal filter of $\mathcal{Q}(S)$ generated by the quasi-order $\longrightarrow^{\infty}$. If π is the division relation on S, then $\widetilde{\pi}$ is the well known Green's $\mathcal{J}$-relation, and if $\pi = \longrightarrow^{\infty}$, then $\widetilde{\pi}$ is the smallest semilattice congruence on S, and the partially ordered set of all $\widetilde{\pi}$-classes is the greatest semilattice homomorphic image of S (cf. **[13]**, **[14]**, **[7]** and **[8]**). As is well known, a semigroup S is *semilattice indecomposable*, i.e. the universal relation on S is the only semilattice congruence on S, if and only if $a \longrightarrow^{\infty} b$, for all $a, b \in S$ (cf. **[13]**). If $a \longrightarrow b$ for all $a, b \in S$, then S is called an *archimedean semigroup*.

Let π be a positive quasi-order on a semigroup S. An ideal I of S is called a π*-ideal* if $I\pi = I$, i.e. if $a \in I$ implies $a\pi \subseteq I$, for any $a \in S$. In other words, I is a π-ideal of S if and only if it is an ideal of S and a filter of the quasi-ordered set (S, π). The set of all π-ideals of S is denoted by $\mathrm{Id}^{\pi}(S)$. As was proved in [8], $\mathrm{Id}^{\pi}(S)$ is a complete 1-sublattice of $\mathrm{Id}(S)$ and the mapping $\pi \mapsto \mathrm{Id}^{\pi}(S)$ is a dual isomorphism of the lattice of all positive quasi-orders on S onto the lattice of all complete 1-sublattices of $\mathrm{Id}(S)$. The same mapping also determines a dual isomorphism of the lattice of positive lower-potent quasi-orders on S onto the lattice of all complete 1-sublattices of $\mathrm{Id}^{cs}(S)$. Hence, if π is the division relation on S, then $\mathrm{Id}^{\pi}(S) = \mathrm{Id}(S)$, and if $\pi = \longrightarrow^{\infty}$, then $\mathrm{Id}^{\pi}(S) = \mathrm{Id}^{cs}(S)$.

For undefined notions and notations we refer to **[1]**, **[5]**, **[6]** and **[10]**.

Let π be a positive quasi-order on a semigroup S. Then we set

$$M_{\pi}(S) = \{a \in S \mid a\pi \subset S\}.$$

The proof of the first lemma is immediate and it will be omitted.

LEMMA 1. *Let π be a positive quasi-order on a semigroup S. Then $M_{\pi}(S) = \varnothing$ and only if S does not have proper π-ideals.*

Next we consider the case when $M_{\pi}(S)$ is nonempty.

THEOREM 1. *Let π be any positive quasi-order π on a semigroup S such that S has a proper π-ideal. Then $M_\pi(S)$ is the union of all proper π-ideals of S.*

If, in addition, π satisfies the cm-property, then $M_\pi(S)$ is a completely prime ideal of S.

PROOF. Let U denote the union of all proper π-ideals of S and let I be any proper π-ideal of S. For any $a \in I$ we have that $a\pi \subseteq I \subset S$, so $a \in M_\pi(S)$. Thus, $I \subseteq M_\pi(S)$ and we conclude that $U \subseteq M_\pi(S)$. On the other hand, for any $a \in M_\pi(S)$ we have that $a\pi$ is a proper ideal of S, that is $a\pi \subseteq U$, and hence $a \in U$. Therefore, $M_\pi(S) \subseteq U$, and we have proved that $M_\pi(S) = U$.

Suppose now that π satisfies the cm-property, and let $a, b \in S$ such that $ab \in M_\pi(S)$. If $a \notin M_\pi(S)$ and $b \notin M_\pi(S)$, i.e. if $a\pi = S$ and $b\pi = S$, then by Lemma 2 of [8] we have that $(ab)\pi = a\pi \cap b\pi = S$, which contradicts the hypothesis $ab \in M_\pi(S)$. Thus, we conclude that $a \in M_\pi(S)$ or $b \in M_\pi(S)$, so we have proved that $M_\pi(S)$ is a completely prime ideal of S. □

Now we determine some conditions under which $M_\pi(S)$ is a proper ideal.

THEOREM 2. *Let π be any positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then the following conditions are equivalent:*

(i) *$M_\pi(S)$ is a proper ideal of S;*
(ii) *S has a largest proper π-ideal;*
(iii) *The partially oredered set of $\widetilde{\pi}$-classes has a least element.*

PROOF. (i)⇒(ii). Since $\mathrm{Id}^\pi(S)$ is a complete 1-sublattice of $\mathrm{Id}(S)$, then by Theorem 1 we have that $M_\pi(S) \in \mathrm{Id}^\pi(S)$. Therefore, if $M_\pi(S)$ is a proper ideal of S, then it is the largest proper π-ideal of S, again by Theorem 1.

(ii)⇒(i). Let S has a largest π-ideal U. Then U is the union of all proper π-ideals of S, and by Theorem 1 it follows that $U = M_\pi(S)$. Therefore, $M_\pi(S)$ is a proper ideal of S.

(i)⇒(iii). Let $X = S \setminus M_\pi(S)$. If $a, b \in X$. then $a\pi = S = b\pi$, so by Proposition 1 of [8] it follows that $(a, b) \in \widetilde{\pi}$. Thus, X is contained in some $\widetilde{\pi}$-class C of S. On the other hand, for any $c \in C$ and $a \in X$ we have that $(c, a) \in \widetilde{\pi}$, whence $c\pi = a\pi = S$, so $c \in X$. Therefore $X = C$, i.e. X is a $\widetilde{\pi}$-class of S. Let Y be any $\widetilde{\pi}$-class of S and let $a \in X$ and $b \in Y$ be arbitrary elements. Then $a\pi = S$ and $b \in S$, whence $a\,\pi\, c$. This means that $X \leq Y$ in the partially ordered set of all $\widetilde{\pi}$-classes of S, so we have proved that X is the least element in this partially ordered set.

(iii)⇒(i). Let X be the least element in the partially ordered set of $\widetilde{\pi}$-classes of S. First we prove that $X = \{a \in S \,|\, a\pi = S\}$. Let $a \in X$ and $b \in S$. Then $a\widetilde{\pi} \leq b\widetilde{\pi}$ implies $a\,\pi\, b$, so $b \in a\pi$. This means that $a\pi = S$. Conversely, let $a \in S$ such that $a\pi = S$ and let $b \in X$ be an arbitrary element. Then $a\pi = S$ yields $a\,\pi\, b$, whereas by $b\widetilde{\pi} \leq a\widetilde{\pi}$ it follows $b\,\pi\, a$. Thus $(a, b) \in \pi \cap \pi^{-1} = \widetilde{\pi}$, so we have that $a \in X$. Now we have that $\varnothing \neq X \neq S$, since S has at least one proper π-ideal, whence it follows that $M_\pi(S) = S \setminus X$ is a proper ideal of S. □

Note that the condition (ii) of the above theorem means that the lattice $\mathrm{Id}^\pi(S)$ has a unique dual atom.

Let $M_{cs}(S)$ denote the union of all proper completely semiprime ideals of a semigroup S. By Theorem 2 we obtain the following consequence.

COROLLARY 1. *The following conditions on a semigroup S are equivalent:*
(i) *$M_{cs}(S)$ is a proper ideal of S;*
(ii) *S has a largest proper completely semiprime ideal;*
(iii) *S has a largest proper completely prime ideal;*
(iii) *The greatest semilattice homomorphic image of S has a unity.*

By the previous corollary it follows that the largest proper completely semiprime ideal of a semigroup, if it exists, is completely prime.

The following theorem describes semigroups whose proper π-ideals are semilattice indecomposable semigroups.

THEOREM 3. *Let π be a positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then any proper π-ideal is a semilattice indecomposable semigroup if and only if $M_\pi(S)$ is a semilattice indecomposable semigroup.*

PROOF. Let any proper π-ideal of S be a semilattice indecomposable semigroup and let $a, b \in M_\pi(S)$. Then $a\pi$ and $b\pi$ are proper π-ideals of S, and they are semilattice indecomposable. Moreover, $a, ab \in a\pi$ and $b, ab \in b\pi$, whence $a \longrightarrow^\infty ab$ and $ab \longrightarrow^\infty a$ in $a\pi$ and $b \longrightarrow^\infty ab$ and $ab \longrightarrow^\infty b$ in $b\pi$. Since the ideals $a\pi$ and $b\pi$ are contained in $M_\pi(S)$, by Theorem 1, then we have that $a \longrightarrow^\infty ab \longrightarrow^\infty b$ and $b \longrightarrow^\infty ab \longrightarrow^\infty a$ in $M_\pi(S)$. Therefore, $M_\pi(S)$ is a semilattice indecomposable semigroup.

Conversely, let $M_\pi(S)$ be a semilattice indecomposable semigroup. For any proper π-ideal I of S, by Theorem 1 we have that I is an ideal of $M_\pi(S)$, and by Theorem 3.4 and Corollary 3.9 of **[9]** we have that any ideal of a semilattice indecomposable semigroup is also semilattice indecomposable. □

In a similar way we prove the next corollary which generalizes some results from **[3]**, **[5]** and **[6]**.

COROLLARY 2. *Let π be a positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then any proper π-ideal is an archimedean semigroup if and only if $M_\pi(S)$ is an archimedean semigroup.*

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A NOTE ON THE SET-THEORETIC REPRESENTATION OF ARBITRARY LATTICES

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Dedicated to Professor Slaviša B. Prešić

ABSTRACT. Every lattice is isomorphic to a lattice whose elements are sets of sets and whose operations are intersection and the operation $\vee^*$ defined by $A \vee^* B = A \cup B \cup \{Z : (\exists X \in A)(\exists Y \in B) X \cap Y \subseteq Z\}$. This representation spells out precisely Birkhoff's and Frink's representation of arbitrary lattices, which is related to Stone's set-theoretic representation of distributive lattices.

As a generalization of his representation theory for Boolean algebras, Stone has developed in [4] a representation theory for distributive lattices. This representation theory has set-theoretic and topological aspects. Set-theoretically, every distributive lattice L is isomorphic to a set lattice L^*, i.e. a lattice whose elements are sets and whose operations are intersection and union. In Stone's representation, the elements of L^* are certain subsets of the set $F(L)$ of prime filters of L. Topologically, $F(L)$ can be viewed as a T_0-space with the elements of L^* constituting a subbasis.

Following ideas of Priestley's [3], Urquhart has developed in [5] the topological aspects of this representation theory to cover arbitrary bounded lattices. However, Birkhoff and Frink had already in [1, section 6] a simple set-theoretic representation for arbitrary lattices, also inspired by Stone, but different from Urquhart's representation.

In the Birkhoff-Frink representation, every lattice L is isomorphic to a lattice L^* whose elements are sets of sets, whose meet operation is intersection and whose join operation is a set-theoretic operation $\vee^*$ unspecified by Birkhoff and Frink. The elements of L^* are certain subsets of a set $F(L)$, which may be either the set of all filters of L, or the set of all principal filters of L, or any set of filters of L that for every pair of distinct elements of L has a filter containing one element of the pair but not the other. Stone's set-theoretic representation for distributive lattices may be viewed as a special case of the Birkhoff-Frink representation: if for

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a distributive lattice L we take $F(L)$ to be the set of all prime filters of L, then $\vee^*$ collapses into set-theoretic union.

The aim of this note is to make precise some details of the Birkhoff-Frink representation, which doesn't seem to be very well known. We shall explicitly characterize the operation $\vee^*$ when $F(L)$ is the set of all filters of L, or of all principal filters of L. The interest of this exercise is in applications that may be found in the models of nondistributive nonclassical logics, where the semantic clause for disjunction may be derived from the operation $\vee^*$.

Let $L = \langle D, \wedge, \vee\rangle$ be an arbitrary lattice, and let $F(L) = \{X : X \text{ is a filter of } L\}$. For every $a \in D$, let $f(a) = \{X \in F(L) : a \in X\}$. Let now $D^* = \{f(a) : a \in D\}$, and let

$$f(a) \wedge^* f(b) = f(a) \cap f(b),$$
$$f(a) \vee^* f(b) = f(a) \cup f(b) \cup \{Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b)) X \cap Y \subseteq Z\}.$$

The second of these equalities corresponds to the semantic clause for disjunction introduced in [2, section 3.2], which has since found its way into a number of papers on models of substructural logics.

In the proof of the following proposition we assume for $a \in D$ that $[a) = \{b \in D : a \leq b\}$; that is, $[a)$ is the principal filter generated by a.

PROPOSITION 1. *The following equalities hold:*
(1.1) $f(a) \wedge^* f(b) = f(a \wedge b)$,
(1.2) $f(a) \vee^* f(b) = f(a \vee b)$.

PROOF. The proof of (1.1) is quite straightforward, and we only need to consider the proof of (1.2). So suppose $Z \in f(a) \vee^* f(b)$. If $a \in Z$ or $b \in Z$, then, since Z is a filter, $a \vee b \in Z$. If, on the other hand, for some X and Y we have that $a \in X$, $b \in Y$ and $X \cap Y \subseteq Z$, then, since X and Y are filters, $a \vee b \in X \cap Y$, and so $a \vee b \in Z$. For the converse, suppose $Z \in f(a \vee b)$, that is $a \vee b \in Z$. If $c \in [a \vee b)$, then $a \vee b \leq c$, and, since Z is a filter, $c \in Z$. So $[a \vee b) \subseteq Z$, but, since $[a) \cap [b) = [a \vee b)$, we have that $[a) \cap [b) \subseteq Z$. Hence for some X, namely $[a)$, and some Y, namely $[b)$, we have that $a \in X$, $b \in Y$ and $X \cap Y \subseteq Z$, and so we have proved (1.2). □

Since it is quite easy to see that $f : D \to D^*$ is one-one and onto, we obtain that $L = \langle D, \wedge, \vee\rangle$ is isomorphic to $L^* = \langle D^*, \wedge^*, \vee^*\rangle$.

Note that we obtain the isomorphism of L with L^* also when $F(L)$ is taken to be the set of all principal filters of L, and not the set of all filters of L. Another alternative, yielding again the isomorphism of L with L^*, is to replace $\vee^*$ by the operation $\vee^{**}$ defined by

$$A \vee^{**} B = \{Z : (\exists X \in A)(\exists Y \in B) X \cap Y \subseteq Z\}.$$

We have preferred to work with $\vee^*$, rather than with the more simply defined operation $\vee^{**}$, which coincides with $\vee^*$ on D^* as it was defined up to now, in order to be able to connect smoothly the isomorphism of L and L^* with Stone's representation theory. This connection is made by the following proposition.

PROPOSITION 2. *If L is a distributive lattice and $F(L)$ is the set of all prime filters of L, then $f(a) \vee^* f(b) = f(a) \cup f(b)$.*

PROOF. Suppose $Z \in f(a) \vee^* f(b)$. As in the proof of the previous proposition, it follows that $a \vee b \in Z$. Since Z is prime, $a \in Z$ or $b \in Z$, that is $Z \in f(a) \cup f(b)$. The converse, namely, $f(a) \cup f(b) \subseteq f(a) \vee^* f(b)$, is trivial. □

This trivial converse can, however, be blocked if $\vee^*$ is replaced by $\vee^{**}$. Indeed, suppose $a \in Z$; then we must show that for some prime filters X and Y we have that $a \in X$, $b \in Y$ and $X \cap Y \subseteq Z$. The prime filter X can be Z, but, since b may be the least element of L, there is no guarantee that there is a prime, i.e. proper, filter Y such that $b \in Y$.

To conclude, we note that for the sake of symmetry we can define $f(a) \wedge^* f(b)$ either as $f(a) \cap f(b) \cap \{Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b)) X \cup Y \subseteq Z\}$, or as $\{Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b)) X \cup Y \subseteq Z\}$; both of these sets are equal to $f(a) \cap f(b)$. In these new definitions of $\wedge^*$, unions of filters occur where in the definitions of $\vee^*$ and $\vee^{**}$ we had intersections. Then remark that the set of filters $F(L)$, which is a semilattice with $\cap$, is not necessarily closed under $\cup$.

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NOTE ON THE METHODS FOR NUMERICAL SOLUTION OF EQUATIONS

Boško S. Jovanović

Dedicated to Prof. Slaviša B. Prešić on the occasion of his 65^{th} birthday

ABSTRACT. We consider iterative methods for simultaneous determination of all roots of a given polynomial. We discuss the connection between such methods and standard iterative methods for numerical solution of equations, generalizations on other types of equations, and construction of higher order iterative methods.

1. Introduction

In [16] and [17] S. B. Prešić proposed an iterative method for factorization of polynomials. Let $P_n(x)$ be a monic polynomial of n–th degree on the field of complex numbers, with different roots $x_1, x_2, \ldots, x_n$. Let $P_n(x)$ allow a factorization of the form

$$P_n(x) = \prod_{i=1}^{m} P_{n_i}(x)\,, \tag{1}$$

where $P_{n_i}(x)$ is monic polynomial of degree n_i and $n_1 + n_2 + \cdots + n_m = n$. One constructs sequences of polynomials $P_{n_i,k}(x)$ $(k = 0, 1, 2, \ldots)$, where $P_{n_i,k}(x)$ converges to $P_{n_i}(x)$ when $k \to \infty$.

In the case of linear factors $P_{n_i}(x)$ one obtains

$$P_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n)\,, \tag{2}$$

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while formulae for determining of roots are

$$(3)\qquad x_{i,k+1} = x_{i,k} - \frac{P_n(x_{i,k})}{P'_{n,k}(x_{i,k})}, \qquad i = 1, 2, \ldots, n; \quad k = 0, 1, \ldots$$

where $P_{n,k}(x) = (x - x_{1,k})(x - x_{2,k}) \cdots (x - x_{n,k})$.

Iterative method (3) was widely used and investigated. It is known as Weierstrass method, or Durand–Dochev–Kerner method (see [2], [3], [4], [11]). Method (3) also may be considered as a modification of Newton method [13]

$$(4)\qquad x_{k+1} = \frac{F(x_k)}{F'(x_k)}, \qquad k = 0, 1, \ldots$$

for numerical solution of equation

$$(5)\qquad F(x) = 0.$$

Possible generalizations of method (3) on more general equations are considered. For example, factorization of a trigonometric polynomial of the form

$$T_n(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx) = A \prod_{j=1}^{n} \left[\sin(x + \alpha_j) + c_j \right]$$

is considered in [10] and iterative formulae for simultaneous approximation of parameters A, α_j and c_j are obtained. An iterative method for simultaneous finding of k roots of equation (5), based on substitution of function $F(x)$ with interpolation polynomial, is proposed in [15].

2. Iterative methods of higher order

Let us consider iterative method

$$(6)\qquad x_{k+1} = \varphi(x_k), \qquad k = 0, 1, \ldots$$

for numerical solution of equation (5). Suppose that the sequence x_k converges to the root x_* of equation. One says that (6) has order of convergence p if the following condition is satisfied

$$|x_{k+1} - x_*| = O(|x_k - x_*|^p), \quad k \to \infty.$$

The following assertion holds.

THEOREM 1. (see [8]) *Let (6) be an iterative method of order p. Let the function $\varphi(x)$ be $p+1$ times differentiable in the neighborhood of the limit point x_* and let $\varphi(x_*) \neq p$. Then*

$$(7)\qquad x_{k+1} = x_k - \frac{x_k - \varphi(x_k)}{1 - \dfrac{\varphi'(x_k)}{p}}, \qquad k = 0, 1, \ldots$$

is at least an iterative method of order $p+1$.

Starting from Newton method (4) and repeating the procedure given in Theorem 1 one obtains a family of iterative formulae

$$x_{k+1} = x_k - \frac{\Delta_{p-2}(x_k)\,F(x_k)}{\Delta_{p-1}(x_k)}, \qquad k = 0,\,1,\,2,\,\ldots; \quad p = 2,\,3,\,\ldots \tag{8}$$

where $\Delta_0(x) = 1$ and

(9)

$$\Delta_j(x) = \begin{vmatrix} F'(x) & F(x) & 0 & \cdots & 0 \\ \dfrac{F''(x)}{2!} & F'(x) & F(x) & \cdots & 0 \\ \dfrac{F'''(x)}{3!} & \dfrac{F''(x)}{2!} & F'(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{F^{(j)}(x)}{j!} & \dfrac{F^{(j-1)}(x)}{(j-1)!} & \dfrac{F^{(j-2)}(x)}{(j-2)!} & \cdots & F'(x) \end{vmatrix}, \qquad j = 1,\,2,\,3,\,\ldots$$

Method (8) is an iterative method of order p. For $p = 2$ from (8) one obtains Newton method (4). For $p = 3$ one obtains Halley iterative formula [5]

$$x_{k+1} = x_k - \frac{F(x_k)\,F'(x_k)}{\left(F'(x_k)\right)^2 - 0.5\,F(x_k)\,F''(x_k)}, \qquad k = 0,\,1,\,\ldots \tag{10}$$

For $p = 4$ the corresponding formula is

$$x_{k+1} = x_k - \frac{F(x_k)\left[\left(F'(x_k)\right)^2 - 0.5\,F(x_k)\,F''(x_k)\right]}{\left(F'(x_k)\right)^3 - F(x_k)\,F'(x_k)\,F''(x_k) + \frac{1}{6}\left(F(x_k)\right)^2 F'''(x_k)}$$

etc.

Iterative methods (8) were investigated by many mathematicians. For example an application of Halley method for determination of matrix square root is considered in [12]. An interval version of Halley method is investigated in [14]. In [1] formula (8) is used in the framework of Padé approximation. Notice that formula (8) can be obtained by application of Newton method to equation

$$\frac{F(x)}{\sqrt[p-1]{\Delta_{p-2}(x)}} = 0$$

(see [9]).

3. Higher order methods for simultaneous approximation of all roots of polynomial

Combining results from two previous chapters we can construct iterative methods of higher order for simultaneous approximation of all roots of polynomial $P_n(x)$. Since $P_{n,k}(x)$ contains k^{th} iterations of all roots of polynomial $P_n(x)$, it is sufficient to substitute $F(x)$ with $P_n(x)$ and $F'(x)$ with $P_{n,k}(x)$ in (8) and (9). Derivatives of higher order $F^{(j)}(x)$ can be substituted with $P_n^{(j)}(x)$ or with $P_{n,k}^{(j)}(x)$. In such a way one obtains the formula:

$$x_{i,k+1} = x_{i,k} - \frac{\overline{\Delta}_{p-2}(x_{i,k})\, P_n(x_{i,k})}{\overline{\Delta}_{p-1}(x_{i,k})}, \tag{11}$$
$$i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots; \quad p = 2, 3, \ldots$$

where $\overline{\Delta}_0(x) = 1$ and

$$\overline{\Delta}_j(x) = \begin{vmatrix} P'_{n,k}(x) & P_n(x) & 0 & \ldots & 0 \\ \frac{P''_n(x)}{2!} & P'_{n,k}(x) & P_n(x) & \ldots & 0 \\ \frac{P'''_n(x)}{3!} & \frac{P''_n(x)}{2!} & P'_{n,k}(x) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{P_n^{(j)}(x)}{j!} & \frac{P_n^{(j-1)}(x)}{(j-1)!} & \frac{P_n^{(j-2)}(x)}{(j-2)!} & \ldots & P'_{n,k}(x) \end{vmatrix}, \qquad j = 1, 2, 3, \ldots$$

or

$$\overline{\Delta}_j(x) = \begin{vmatrix} P'_{n,k}(x) & P_n(x) & 0 & \ldots & 0 \\ \frac{P''_{n,k}(x)}{2!} & P'_{n,k}(x) & P_n(x) & \ldots & 0 \\ \frac{P'''_{n,k}(x)}{3!} & \frac{P''_{n,k}(x)}{2!} & P'_{n,k}(x) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{P_{n,k}^{(j)}(x)}{j!} & \frac{P_{n,k}^{(j-1)}(x)}{(j-1)!} & \frac{P_{n,k}^{(j-2)}(x)}{(j-2)!} & \ldots & P'_{n,k}(x) \end{vmatrix}, \qquad j = 1, 2, 3, \ldots$$

For $p = 2$ from (11) one obtains Weierstrass method (3). For $p = 3$ one obtains imultaneous versions of Halley method:

$$x_{i,k+1} = x_{i,k} - \frac{P_n(x_{i,k})\, P'_{n,k}(x_{i,k})}{\left(P'_{n,k}(x_{i,k})\right)^2 - 0.5\, P_n(x_{i,k})\, P''_n(x_{i,k})},$$

and

$$x_{i,k+1} = x_{i,k} - \frac{P_n(x_{i,k})\,P'_{n,k}(x_{i,k})}{\left(P'_{n,k}(x_{i,k})\right)^2 - 0.5\,P_n(x_{i,k})\,P''_{n,k}(x_{i,k})},$$
$$i = 1, 2, \ldots, n; \quad k = 0, 1, \ldots$$

etc. Formulae of this kind were also frequently used and investigated. For example, safe convergence of certain modifications of simultaneous version of Halley method is investigated in [6] and [7].

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Appendices

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RAZNICE

Raznica: putnici misaonog voza

(Nakon što se upoznaju Zadatak i Rešenje.)

Zadatak: Izgledam ponešto bedasto pored tebe.

Rešenje: Pa čekaj malo, ko zna čega će ovo predstavljati početak. Što sad tako reaguješ? Zar ti se ne čini da se ni moje ni tvoje kolege baš ne razbacuju velikim idejama?

Z: Mislim da ti imaš priliku da na sve to gledaš s boljeg mesta.

R: Može biti, ali zar ti ja nisam dao moje uporišnice. I ti si valjda putnik misaonog voza. Nema veze što si skoro ušao.

Z: Ali pitanje je da li je ta tvoja sposobnost talenat ili se stiče. Jer ako je talenat, ostaje mi samo da se prisećam ovih trenutaka s tobom i da se molim da je talenat prelazan.

R: Naravno da ni ja ne znam gde ide ovaj voz niti iko može to da zna. Ali to nije nikakav razlog da ti imaš ovakav stav prema putovanju.

Z: Znaš li šta je problem s ovim putovanjem? Al' nema ljutiš. Što ćeš ti ne samo meni nego i mojim kolegama biti samo onoliko dobar koliko ti to nešto u nama dopusti. Pa evo ti primer: jedan moj stariji kolega, da sad ne kažem ko je, na rečima ima priličnu averziju prema ovom obliku putovanja. Ne treba ni da ti napominjem da njegova putovanja sve više liče na ovo.

R: Pa dobro, ali...

VARIA

Problem and Solution are both passengers in the train of thoughts. They've just met.

Problem: I look a bit foolish next to you.

Solution: Hold on a second. Who knows where this may lead? And why do you react like this? No colleague of mine or yours really throw around big ideas, right?

P: I think you have a better perspective on that from your position.

S: Maybe, but didn't I tell you what grounds me ? And you are also a passenger in this train, aren't you? It doesn't matter that you've come aboard only recently.

P: The question is whether your ability is a talent or it can be acquired. For if it's a talent I can only remember these moments with you and pray that talent is contagious.

S: Nobody, myself included, can tell where this train is heading to. But that is no reason for having such an attitude towards this journey.

P: Do you know what's wrong with this journey? I'll tell you, but no hard feelings here, OK? The problem is that you could help me and my colleagues up to the extent we ourselves let you help us. And here is the example: one senior colleague of mine (no names mentioned) would verbally express quite an aversion towards this type of journey. Needless to say, his own journey resembles this one more and more.

Z: Znam! Ja znam – ja to mislim – da su sve ideje koje si mi rekao suho zlato, ali... Ja to moram da osetim, da osetim kako teče kroz mene. Bojim se da ti tu više ne možeš ništa. To je pitanje mog metabolizma, i ja ne znam, zaista ne znam, hoće li i kad će to da se slegne. Ne znam.

R: Kako hoćeš. Moram da idem i kod drugih zadataka. Ako hoćeš sa mnom...

Z: Daj mi vremena da razmislim.

Napomena uz raznicu:

Prvi čas – programiranje na I godini – ulazi profesor; predstavlja se i, naravno, odmah traži da se otvore prozori. Piše po tabli: "Algoritam uopšte". Reakcije studenata su razne – od nekritičkog ispisivanja "Algoritmika" do "silna mi čuda!" na profesorovo: "Algoritam – ja vam to radije ne bih definisao." što se ovih drugih tiče, ubrzo ipak shvataju da su za njihovog profesora te standardne programerske priče samo pričice za malu decu ili, što bi on rekao, "za domaćice". Bilo kako bilo, većina studenata će pamtiti taj prvi čas po tome da im je nešto bilo čudno, i da su se smijali. Profesor se tu malo postavlja kao zabavljač. Jedan dio studenata ne zna da li on tako samo sad priča ili će to tako stalno da bude. Neki ne znaju ni kako da vode zabeleške – neki ih ni ne vode. U najboljem slučaju zapisuju: "Algoritam $\approx$ konstruktivna funkcija" i prepisuju algoritme faktorijela s table, pri čemu – uzgred budi rečeno – profesor funkciju $n!$ zove Fak(n), na šta se cijela grupa studenata smije kao dečica u pubertetu, a profesor zadovoljan što mu je dosetka uspela komentariše: "što mi znate engleski". Studenti odlaze s

S: Well OK, but...

P: I know! I am aware (I think) that all of your ideas are truly priceless, but... It's not enough to know that, I have to feel it, feel it become part of me. I am afraid you can't do much more about it. That's for me to process and I don't know, really don't know if and when that would settle in me. I don't know.

S: Well, do as you please. I have to go now to other problems. If you want to join me...

P: Let me have some time to think this over.

A remark to varia:

It's the first class of the programming course for the freshmen. Professor enters the classroom, introduces himself and immediately asks windows to be opened. He writes on the board: "Algorithms in general". Students react differently: from blindly copying down what's on the board to simply "Yeah, right!" as the reaction to professor's: "I would rather not define an algorithm". Soon, though, the students realize that the professor considers these standard programming terms just kids stuff—"stories for housewives" as he would say. In any case, for the majority of students this first class will be somewhat of a strange experience. They will remember that they laughed a lot. The professor entertains more than he teaches. Some students wonder whether that's just for the first class or for the whole course. Some also wonder how to take notes—some don't take them at all! At best they write: "Algorithm $\approx$ constructive function" and copy down the factorial function algorithm. By the way, professor's name for the $n!$ is Fak(n), which sends the whole class to adolescent giggling. Professor, pleased with his witty časa s rečima "dobar je, dobar je".

Sledeći čas počinje sa $*$-termovima, završava u smehu sa "jesi drvce" algoritmom. Neki studenti se samo kikoću i ne primećuju kako im se kroz primere i smeh poturaju vrlo suptilne ideje. Oni, pak, koji bi da ostanu ozbiljni vremenom počinju da kukaju kako ga ne razumiju. Na časove dolaze stariji studenti koji sad polažu ispit. Profesor nije nikakvo zakeralo i ispite drži svakog petka – po principu "kad naučiš javi se, ja ionako hoću da ocenim tvoje znanje, a ne neznanje".

Mali studenti se smiju dok gledaju starije kako prepisuju. Posle se kod njih raspituju o ispitu: gledaju pitanja – λ-operator, "Šta da mu pričamo o λ-operatoru kad ništa nismo ni shvatili".

"Nema šta da se brinete", kažu stariji studenti, "objasniće vam on to. Videćete, kad on završi predavanje sve će vam biti super." Mali studenti se mole da to bude tako. čitaju dalje pitanja... "Deduktivni množač", "dodelnik", "razreč", "liste u paskalu – krpelj", "sortiranje nizova – mališa na početak". Profesor je pun živopisnih izraza koji su tu da ti pomognu u mišljenju i uvek je tu njegova reč, samo će jednom čitati s papira, primer, naravno. Sažetak čuvenog pitanja "penjanje na brdo", to jest, "planinarski metod" studenti pronalaze u jednom članku iz novina koje su izašle tog dana kad su imali predavanje. Ako ništa drugo, barem ga nije čitao s nekog požutelog papira na pragu raspadanja. Nekad, kad treba da napravi pauze između misli, dopušta sebi da sa studentima priča o vezi između srpskog jezika i sanskrta, što se ne retko znalo i odužiti. Studenti uživaju slušajući profesora. Kako čovjek da se

notation, comments: "My, my, so many experts in English in this room!". Students leave the class saying: "He is cool, real cool."

Next class starts with $*$-terms and ends in laughter over "You-are-a-tree algorithm". Some students just giggle, not realizing to what extent some very subtle ideas are introduced through examples and jokes.

Some students who would like to remain serious start moaning that they don't understand the professor. This course is not just for freshmen-this professor had already taught some of the students in the class. And as far as the exam goes, it is held each Friday under slogan "Come when you know something—I grade what you know, not what you don't know".

Freshmen look at the list of questions for the exam and lament to their upperclassmen: "What can we say about λ-operator if we didn't understand a thing about it!?"

"Nothing to worry about" , they say "professor will explain it. You'll see, when he finishes the course everything will be just fine". The freshmen pray for that. They read on through the list: "Deductive multiplier", "lists in Pascal—tisks", "array sorting—put the tiny one first"... Professor makes his own vivid words, so full of meaning, in order to help students think. And he rarely reads in the class, only examples. Once after the class, students find his abstract of the famous "climb-the-hill" method in the article of that same day newspaper. If anything, professor didn't read it to them from some aged-about-to-fall-apart notes.

Sometimes, as a break between thoughts, professor wanders into the relationship between Serbian and Sanskrit. And that can go for quite some time! Students enjoy listening to the professor.

ne smije kad on pita: "Ovo se rešava preko radikala. Znate li vi koja je to reč radikal?" Malo posle eto odgovora: "radikal–root–rotkva–rotkvica". Studenti se smiju. Ili: "Znate li kako se kaže cipela na sanskrtu? Upanak." Ipak, neki studenti prestaju da idu na predavanja – ostavljaju svojim najboljim drugovima indigo da svoje zabeleške pišu preko njega. U isto vreme, oni drugi studenti prvi put čuju za intuicionizam i logicizam, Markovljeve algoritme i Tjuringove mašine. Uče o nizovima, listama, drvetima – i uče razne algoritme na jeziku nizova, lista, drveta. Profesor pokušava da nagovori studente da prave šume jer Matematika se i pravi – ali ne uspeva u tome, "tja, to je samo programiranje" misle studenti u sebi.

Treća godina – Algebra 2 – studenti kao malo matematički porasli – položili Analizu 2 – misle ko zna šta su sve naučili... U sebi misle: "O, ne opet!" kad vide "jednakosna logika" na tabli. U stvari, zima im je jer profesoru treba vazduha. Intimna atmosfera ovog puta je neizbežna jer na predavanje dolazi polovina studenata, znači pet–šest njih. Studenti se pitaju kakve sve to veze ima s algebrom – Čerčova teza – to je kako mali student zamišlja logiku. Oni jedva čekaju da slušaju teoriju Galoa. Na prvoj zgodnoj pauzi profesor pokušava da s njima priča o tome – oni se boje da kažu bilo šta određenije jer nisu sigurni da li postoji nešto što ne bi trebalo da kažu. Profesor studentima: "Vi se ne usuđujete raspravljati sa mnom." Studenti: "Mi? Profesore, to Vam se samo čini." Studenti su tu ispali donekle smešni. Sedam dana kasnije, studenti ne znaju kako da se postave u raspravi dvaju

It's hard not to laugh when he asks: "We can solve this problem using radicals. Do you know what kind of word is that?" And the answer comes: "Radical–root–radish". Laughter. Or this one: "What is the Sanskrit word for shoe? Upanak[1]." Still, some students stop attending lectures and only get Xeroxed notes. At the same time, other discover intuitionism, logicism, Markov algorithms, Turing machine. They learn about arrays, lists, trees and pertinent algorithms. Professor makes futile attempts to persuade students into making "forests" because Mathematics is not something finished, fully defined. On the contrary, he believes that mathematics can be made, created, that it is "alive". "It's only programming" students think.

Junior year and Algebra 2. Students, having passed Multivariable Calculus, think they have mathematically matured... They cringe when they see "Equational logic" on the board. Actually, they are freezing since professor needs fresh air and windows are wide open. Only half of the class attends lectures, thus creating an intimate atmosphere. About half a dozen students wonder how Church's thesis is connected to algebra—they are impatiently waiting for the Galois theory. Professor, sensing that, tries to engage them in the conversation about it. Students are afraid to volunteer all of their thoughts since they want to be completely "PC". Professor tries to provoke them: "I dare you discuss this with me!". Students make some lame attempts to show they are not afraid, a bit silly of them. A week later, they still don't know how to take part in the discussion between two other mathematicians. As they leave the classroom they listen to the professor explaining that there are hens and eggdrugih matematičara. Izlaze iz sale zajedno s profesorom i slušaju ga dok im objašnjava kako postoje kokoške koje nose i koje ne nose jaja – i kako se on na njihovom mestu ne bi dvoumio.

Sledeći mesec dana studenti se gube među pretpričama i nekim drugim pretpričama i nadaju se da će im se na kraju konačno ukazati šta je profesor htio reći. "Ko vam je predavao Algebru 1?" Studenti odgovore. "Pa, dobro, on više vuče na semantiku. Nemojte vi mene pogrešno da shvatite, nemam ja ništa protiv semantike, ali sintaksa..." Sintaksa! Tih mesec dana im stalno priča o misaonim svetovima, misaonim spratovima, vinuću, dunuću i vrnuću. Studenti nisu sigurni da razumiju - pokušavaju da se vinu, ali kad treba da ih ponese, oduva ih ono što duva kroz prozore. Zato kad treba da se vrnu na sintaksu, ne samo da se vrnu, nego i tresnu o nju. Termi? Sve su termi. Kako su sve termi? Kako je termovska algebra pra-algebra?

No, eto studentima utehe – kaže profesor: "Samo budali je sve jasno." Dešava se da se profesor i studenti zapričaju i da ih profesorka engleskog prekine u 4,15. Poneki put bi se samo preselili u profesorov kabinet i raspravljali bi o tome da li ima još koji kurs koji na ovaj način pristupa odnosu sintaksa-semantika. Nekad se desi da u tim pričama doguraju i do toga kako naći burek u Njujorku ili kako ukiseliti kupus za jedan dan. Vremenom su studenti zamenili čas Algebra 2 za pravi ručak i dobru (protestnu) šetnju.

Četiri meseca posle eto opet studenata i profesora. Ovog puta mora da im se pohvali šta je radio prethodna četiri meseca, i priča im P-L algoritmu. Naprosto želi da podeli odu-

laying hens and that he wouldn't think twice if he were in their shoes.

Months go by and students lose themselves in previews of some other previews, hoping that one day they will finally grasp what professor is saying. "Who was your teacher in Algebra 1?" Students respond. "Well then, his approach emphasizes semantics. Don't get me wrong, I am all for semantics but syntax..." Syntax! During that month professor constantly talks about realms of thoughts, thought levels, Soaratia, Inspiratia and Comprehentia. Students are not quite sure they get it—they try to soar, but windows are still open and before they can get swept away, the wind blows them off. For that reason, when the time comes to return to syntax, not only they return, they are knocked down. What is a term? Everything is a term- how on Earth can that be ?! How can algebra of terms be pre-algebra?

Professor consoles students: "Only a fool understands everything". It is not unusual that professor and students get carried away in their conversation and go well beyond the class time, only to be interrupted by the English professor at 4:15 pm. Sometimes they just move to professor's cabinet and continue their discussion. Students wonder whether any other course treats relationship between syntax and semantics in the similar way. At times, these conversations answer such questions like how one can find burek[2] in New York or how one can pickle cabbage in only one day! In time, real lunch and a good walk (in protest) take place of Algebra 2.

After four months students and professor are again in the classroom. This time he wants to share his enthusiasm about what he worked on in the meantime. He talks about P-L algorithm. Stuševljenje s njima. Studenti se čude kako to sad sve izgleda tako prosto. Nakon dva meseca raznih pretpriča i priča, dolazi jedan čudesan momenat na koji su studenti prilično ponosni. Profesor im objašnjava slobodne algebre, ali pre nego što i spomene sečenje po kongruenciji studenti shvataju koliko je sati i između sebe se kikoću i govore šta će sve sledeće profesor uraditi. Kao da se sve najednom složilo u sliku, i sve je ličilo na profesorovu priču o tome kako postoje ljudi koji toliko umeju da primaju da posle nekog vremena i sami počinju da zrače. Studenti misle da znaju šta je profesor hteo da kaže kad im je pričao da nisu u stanju svi oni koji se nazivaju matematičarima da osećaju Matematiku. Profesor pokazuje kako se ovim novim naočarima mogu gledati i korenska polja i algebarska zatvorenja i sve. Sve te ideje studenti doživljavaju kao jako dragocene i trude se da ih sve pohvataju, dok još imaju prilike. I uspevaju neke da uhvate: naučili su da se Matematika i pravi, naučili su da naprave što kad im zatreba,nešto za šta će zaraditi komplimente od asistenata, tipa: "čoveče koji ste vi monstrumi."

Sada kada su studenti položili Algebru 2 i bliže se završetku studija, sretnu ih neki drugi studenti i pitaju iz kojih knjiga da uče Algebru 2. Dobijaju odgovor da im nijedna knjiga neće ni značiti ni odrediti toliko koliko profesorova predavanja. Jednog dana, ako studenti matematike porastu, sigurno će o profesoru pričati kao o nekom ko je bio uz njih dok su oni rasli.

dents are amazed at how simple all that feels right now. After months of previews and conversations the proud moment has come for students. Professor starts explaining free algebras and before he even mentions congruences and factor systems, students anticipate correctly what he plans to say. They giggle. It is as if all pieces came together into one mosaic and everything is as if it came straight from the professor's story about people who can accept so much that after a while they start radiating. Students now think they know what professor meant when he said that not all who call themselves mathematicians can truly feel mathematics. Professor illustrates how one can use this new outlook to examine the field of roots and algebraic closure, in fact everything. Students find these ideas very valuable and try to seize them while they still can. They succeed with some: they know now that Math is "alive" , that it is made and they have learned to make what they need when they need it—something that can make TA's say: "Man, you are monsters!"

After Algebra 2 students move through other courses and graduation is at sight. Some new students ask them which book to use for Algebra 2. They answer that no book would come close to professor's lectures. One day, when students become grown-up mathematically, they will no doubt talk of the professor as of somebody who has been there for them when they were growing up.

1. *opanak* is the traditional Serbian peasant shoe—*upanak* differs by just one letter.

2. *Burek* is a traditional Balkan pastry usually filled with meat or cheese.

Translated from Serbian by Barbara Blažek.

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Many textbooks, handbooks, as well as numerous articles dedicated to pedagogical questions and popularizations of mathematics.

A long series of public lectures and appearances in various media on the topics of mathematics, philosophy of mathematics and metodology of scientific research.

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LIST OF REVIEWERS

The following is a list of all reviewers of the papers submitted for publication in the present issue (including the papers which were not published).

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Contents