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Marko Kostić

Abstract Degenerate Volterra Integro-Differential Equations

Mathematical Institute
of the Serbian Academy
of Sciences and Arts

Marko Kostić
marco.s@verat.net, markokostic121@yahoo.com

ABSTRACT DEGENERATE VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Contents

PREFACE

The theory of linear degenerate Volterra integro-differential equations is in its very early stage. The present monograph is probably the first one that is specifically dedicated to abstract degenerate Volterra integro-differential equations and abstract degenerate (multi-term) fractional differential equations. It is a logical continuation of my previously published books entitled "Generalized Semigroups and Cosine Funtions" (Mathematical Institute SANU, Belgrade, 2011; [**291**]) and "Abstract Volterra Integro-Differential Equations" (CRC Press, Boca Raton, FL, 2015; [**292**]). Decomposition of material into chapters, sections and subsections, as well as the style used for presenting bibliographic information are almost the same as those already employed in [**292**].

Fractional calculus and fractional differential equations are rapidly growing fields of research, having invaluable importance in modeling of various problems appearing in physics, engineering, chemistry, biology, medicine and many other sciences. Degenerate fractional differential equations have not attracted the attention of a large number of authors working in the field of applied science; fractional models of some well known degenerate equations of mathematical physics considered in this monograph still do not have reasonable interpretations in the world of real phenomena. Although formulated in the general setting of (infinite-dimensional) sequentially complete locally convex spaces, a great part of our results on degenerate Volterra equations and degenerate fractional differential equations seems to be new even in the setting of Banach spaces. Throughout the book, we use four different kinds of fractional derivatives: the Caputo derivative, which is the most commonly used, the Riemann–Liouville derivative, the modified Liouville right-sided fractional derivative and the Weyl fractional derivative. The (multivalued) linear operators employed in our analyses need not be densely defined, in general.

Although the often large, we are sure that list of references is not complete. The book is not compulsively detailed and there is a large number of important topics that will not be analyzed here. Selection is based on the personal views and aspirations of the author, who pays special attention on examining possibilities to apply various types of convoluted or C-regularized families of solution operators in the analysis of abstract degenerate equations under consideration.

The target audiences are researchers and experts interested in getting to know the basic methods of linear theory of abstract degenerate Volterra integro-differential equations; the book can be also useful for graduate students in mathematics, physics or engineering science. The required mathematical preparation is no higher than basic functional analysis, complex analysis in one variable, integration theory and elementary partial differential equations.

There is a long list of people deserving my thanks. First of all, I would like to express sincere thanks to my family, closest friends and colleagues for permanent support of my work. I would also like to extend my sincere appreciation to Prof. V. Fedorov (Chelyabinsk, Russia), Prof. D. Velinov (Skopje, Macedonia), Prof. P. J. Miana, L. Abadias (Zaragoza, Spain), Prof. M. Murillo-Arcila, J. A. Conejero, A. Peris, J. Bonet (Valencia, Spain), Prof. D. Sidorov (Moscow State University, Russia), Prof A. Favaron (Milano, Italy), Prof. B. Chaouchi (Khemis Miliana, Algeria), Prof. M. Li, C. Chen, C.-G. Li (Chengdu, China), Prof. R. Ponce (Talca, Chile), Prof. C. Bianca (Paris, France), Prof. V. Keyantuo (Rio Piedras Campus, Puerto Rico, USA), Prof. T. Diagana (Huntsville, USA) and G. M. N'Guérékata (Baltimor, USA), Prof. E. M. A. El-Sayed (Alexandria, Egypt), Prof. M. S. Moslehian (Mashhad, Iran), Prof. C.-C. Kuo (New Taipei City, Taiwan) and C.-C. Chen (Taichung, Taiwan), for many stimulating discussions during the research. Special thanks go to Prof. S. Pilipović (Novi Sad, Serbia).

Loznica/Novi Sad
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NOTATION

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the natural numbers, integers, rationals, reals, complexes.
For any $s \in \mathbb{R}$, we denote $\lfloor s \rfloor = \sup\{l \in \mathbb{Z} : s \geqslant l\}$ and $\lceil s \rceil = \inf\{l \in \mathbb{Z} : s \leqslant l\}$.
$\operatorname{Re} z$, $\operatorname{Im} z$: the real and imaginary part of a complex number $z \in \mathbb{C}$; $|z|$: the modul of z, $\arg(z)$: the argument of a complex number $z \in \mathbb{C} \smallsetminus \{0\}$.
$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.
$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$ $(z_0 \in \mathbb{C},\ r > 0)$.
$\Sigma_\alpha = \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$.
$\operatorname{card}(G)$: the cardinality of G.
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\mathbb{N}_n = \{1, \cdots, n\}$.
$\mathbb{N}_n^0 = \{0, 1, \cdots, n\}$.
$\mathbb{R}^n$: the real Euclidean space, $n \geqslant 2$.
The Euclidean norm of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is denoted by $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.
If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index, then we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$.
$f^{(\alpha)} := \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$; $D^\alpha f := (-i)^{|\alpha|} f^{(\alpha)}$.
In (X, τ) is a topological space and $F \subseteq X$, then the interior, the closure, the boundary, and the complement of F with respect to X are denoted by $\operatorname{int}(F)$ (or F°), $\bar{F}$, ∂F and F^c, respectively.
Let Γ be a Jordan curve in the Euclidean plane $\mathbb{R}^2$. Then we denote by $int(\Gamma)$ $(ext(\Gamma))$ the bounded (unbounded) connected component of $\mathbb{R}^2 \smallsetminus \Gamma$.
If X is a vector space over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$, then for each non-empty subset F of X by $span(F)$ we denote the smallest linear subspace of X which contains F.
$\circledast$: the abbreviation for the fundamental system of seminorms which defines the topology of a sequentially complete locally convex space E.
SCLCS: shorthand used to denote a sequentially complete locally convex space.
$L(E, X)$: the space of all continuous linear mappings from E into another SCLCS X, $L(E) = L(E, E)$.
$\mathcal{B}$: the family of bounded subsets of E.
E^*: the dual space of E.
E^{**}: the bidual of E.
A: a linear operator on E.

$\mathcal{A}$: a multivalued linear operator on E (MLO).
C: a continuous linear operator on E.
If F is a subspace of E, then we denote by $\mathcal{A}_{|F}$ the part of $\mathcal{A}$ in F.
$\mathcal{A}^*$: the adjoint operator of $\mathcal{A}$.
$D(\mathcal{A})$, $R(\mathcal{A})$, $\rho(\mathcal{A})$, $\sigma(\mathcal{A})$: the domain, range, resolvent set and spectrum of $\mathcal{A}$.
$N(\mathcal{A})$ or Kern($\mathcal{A}$): the null space of $\mathcal{A}$.
$\bar{\mathcal{A}}$: the closure of $\mathcal{A}$.
$n(\mathcal{A})$: the stationarity of $\mathcal{A}$.
$\sigma_p(\mathcal{A})$: the point spectrum of $\mathcal{A}$.
$\rho_C(\mathcal{A})$: the C-resolvent set of $\mathcal{A}$.
Let A be a closed linear operator on E. Then $[D(A)]$ denotes the sequentially complete locally convex space $D(A)$ equipped with the following system of seminorms $p_A(x) = p(x) + p(Ax)$, $x \in D(A)$, $p \in \circledast$.
$D_\infty(A) = \bigcap_{n \geqslant 1} D(A^n)$.
$\chi_\Omega(\cdot)$: the characteristic function, defined to be identically one on Ω and zero elsewhere.
$\Gamma(\cdot)$: the Gamma function.
If $\alpha > 0$, then $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$; $g_0(t) \equiv$ the Dirac delta distribution.
$\mathcal{D} = C_0^\infty(\mathbb{R})$, $\mathcal{E} = C^\infty(\mathbb{R})$: the Schwartz spaces of test functions.
$\mathcal{S}(\mathbb{R}^n)$: the Schwartz space of rapidly decreasing functions ($n \in \mathbb{N}$); $\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$.
$\mathcal{D}_0$: the subspace of $\mathcal{D}$ which consists of those functions whose support is contained in $[0, \infty)$.
$\mathcal{D}'(E) = L(\mathcal{D}, E)$, $\mathcal{E}'(E) = L(\mathcal{E}, E)$, $\mathcal{S}'(E) := L(\mathcal{S}, E)$: the spaces of continuous linear functions $\mathcal{D} \to E$, $\mathcal{E} \to E$ and $\mathcal{S} \to E$, respectively.
$\mathcal{D}'_0(E)$, $\mathcal{E}'_0(E)$, $\mathcal{S}'_0(E)$: the subspaces of $\mathcal{D}'(E)$, $\mathcal{E}'(E)$ and $\mathcal{S}'(E)$, respectively, containing the elements whose support is contained in $[0, \infty)$.
If $1 \leqslant p < \infty$, $(X, \|\cdot\|)$ is a complex Banach space, and $(\Omega, \mathcal{R}, \mu)$ is a measure space, then $L^p(\Omega, X, \mu)$ denotes the space which consists of those strongly μ-measurable functions $f \colon \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p d\mu)^{1/p}$ is finite; $L^p(\Omega, \mu) \equiv L^p(\Omega, \mathbb{C}, \mu)$.
$L^\infty(\Omega, X, \mu)$: the space which consists of all strongly μ-measurable, essentially bounded functions.
$\|f\|_\infty = ess\sup_{t \in \Omega} \|f(t)\|$, the norm of a function $f \in L^\infty(\Omega, X, \mu)$.
$L^p(\Omega : X) \equiv L^p(\Omega, X) \equiv L^p(\Omega, X, \mu)$, if $p \in [1, \infty]$ and $\mu = m$ is the Lebesgue measure; $L^p(\Omega) \equiv L^p(\Omega : \mathbb{C})$.
$L^p_{loc}(\Omega : X)$: the space which consists of those Lebesgue measurable functions $u(\cdot)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in L^p(\Omega' : X)$; $L^p_{loc}(\Omega) \equiv L^p_{loc}(\Omega : \mathbb{C})$ $(1 \leqslant p \leqslant \infty)$.
$C_0(\mathbb{R}^n)$: the space consisted of those functions $f \in C(\mathbb{R}^n)$ for which $\lim_{|x| \to \infty} |f(x)| = 0$, topologized by the norm $|f| := \sup_{x \in \mathbb{R}^n} |f(x)|$.
$C_b(\mathbb{R}^n)$ $(BUC(\mathbb{R}^n))$: the space of bounded continuous functions (bounded uniformly continuous functions) on $\mathbb{R}^n$, topologized by the norm $|f| := \sup_{x \in \mathbb{R}^n} |f(x)|$.

$C^{\sigma}(\mathbb{R}^n)$: the space of bounded Hölder continuous functions on $\mathbb{R}^n$, topologized by the norm $|f|_{\sigma} := \sup_{x\in\mathbb{R}^n}|f(x)| + \sup_{x,y\in\mathbb{R}^n, x\neq y}\frac{|f(x)-f(y)|}{|x-y|^{\sigma}}$ $(0<\sigma<1)$.
If X is a Banach space, then the abbreviation $AC_{loc}([0,\infty):X)$ $(BV_{loc}([0,\infty):X))$ stands for the space of all X-valued functions that are absolutely continuous (of bounded variation) on any closed subinterval of $[0,\infty)$.
$AC_{loc}([0,\infty)) \equiv AC_{loc}([0,\infty):\mathbb{C})$, $BV_{loc}([0,\infty)) \equiv BV_{loc}([0,\infty):\mathbb{C})$.
$BV[0,T]$, $BV_{loc}([0,\tau))$, $BV_{loc}([0,\tau):X)$: the spaces of functions of bounded variation.
$C^k(\Omega : E)$: the space of k-times continuously differentiable functions ($k\in\mathbb{N}_0$) from a non-empty subset $\Omega\subseteq\mathbb{C}$ into E; $C(\Omega:E)\equiv C^0(\Omega:E)$.
If $k\in\mathbb{N}$, $p\in[1,\infty]$ and Ω is an open non-empty subset of $\mathbb{R}^n$, then we denote by $W^{k,p}(\Omega:X)$ the Sobolev space which consists of those X-valued distributions $u\in\mathcal{D}'(\Omega:X)$ such that, for every $i\in\mathbb{N}_k^0$ and for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\leqslant k$, one has $D^{\alpha}u\in L^p(\Omega:X)$; $H^k(\mathbb{R}^n:X)\equiv W^{k,2}(\mathbb{R}^n:X)$.
$W^{k,p}_{loc}(\Omega:X)$: the space of those X-valued distributions $u\in\mathcal{D}'(\Omega:X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'}\in W^{k,p}(\Omega':X)$.
$\mathrm{S}^{\alpha,p}(\mathbb{R}^n)$: the fractional Sobolev space of order $\alpha>0$ ($p\in[1,\infty]$).
$\mathcal{L}$, $\mathcal{L}^{-1}$: the Laplace transform and its inverse transform; $\tilde{f}(\lambda)\equiv\mathcal{L}f(\lambda)$.
$\mathcal{F}$, $\mathcal{F}^{-1}$: the Fourier transform and its inverse transform.
$LT-E$: we say that a function $h(\cdot)$ belongs to the class $LT-E$ if there exists a function $f\in C([0,\infty):E)$ such that for each $p\in\circledast$ there exists $M_p>0$ satisfying $p(f(t))\leqslant M_pe^{at}$, $t\geqslant 0$ and $h(\lambda)=(\mathcal{L}f)(\lambda)$, $\lambda>a$.
If a function $K(t)$ satisfies the condition (P1) stated in Section 1.2, then we denote $\mathrm{abs}(K)=\inf\{\operatorname{Re}\lambda : \tilde{K}(\lambda) \text{ exists}\}$.
$L^1_{loc}([0,\infty))$, resp. $L^1_{loc}([0,\tau))$: the space of scalar valued locally integrable functions on $[0,\infty)$, resp. $[0,\tau)$.
J_t^{α}: the Riemann–Liouville fractional integral of order $\alpha>0$.
D_t^{α}: the Riemann–Liouville fractional derivative of order $\alpha>0$.
$\mathbf{D}_t^{\alpha}$: the Caputo fractional derivative of order $\alpha>0$.
W_+^{α}: the Weyl fractional derivative of order α.
$D_-^{\beta}u(s)$: the modified Liouville right-sided fractional derivative of order $\beta>0$.
$E_{\alpha,\beta}(z)$: the Mittag-Leffler function ($\alpha>0$, $\beta\in\mathbb{R}$); $E_{\alpha}(z)\equiv E_{\alpha,1}(z)$.
$\Psi_{\gamma}(t)$: the Wright function $(0<\gamma<1)$.
$\wp(R)$: the set which consists of all subgenerators of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$.
$a^{*,n}(t)$: the n-th convolution power of function $a(t)$.
$\delta_{j,l}$: Kronecker's delta.
$\mathrm{supp}(f)$: the support of function $f(t)$.

The notation of various spaces of generalized (asymptotically) almost periodic functions will be explained in Section 3.11 and Section 3.12.

INTRODUCTION

The main purpose of this monograph is to provide an overview of the recent research on abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations in sequentially complete locally convex spaces. Considering only linear equations, we also contribute to the theories of abstract degenerate differential equations of first and second order. Some results of ours seem to be new even for abstract non-degenerate differential equations.

The organization and main ideas of this monograph, which is composed of three chapters, are given as follows. In order to make it useful for self-study, in Chapter 1 we have collected together in one place the mathematical preliminaries and diverse tools required for reading the material from Chapter 2 and Chapter 3. In Section 1.1, we remind ourselves of basic things concerning sequentially complete locally convex spaces (SCLCSs), closed linear operators, analytical functions with values in SCLCSs, integration of functions with values in SCLCSs, function spaces used, and complex powers of almost C-nonnegative operators. We present the most important definitions and results from the theory of multivalued linear operators (MLOs) in Section 1.2; in a separate subsection, we give some new results about hypercyclic and disjoint hypercyclic classes of multivalued linear operators. In Section 1.3, we explore a new theoretical approach to the Laplace transform of functions with values in SCLCSs **[312]** and collect various properties of vector-valued Laplace transform needed for our further work. The operators of fractional differentiation, Mittag-Leffler and Wright functions are investigated in Section 1.4.

The second chapter is consisted from twelve sections. One of the main subjects considered in this chapter is the following abstract degenerate Cauchy problem:

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \in [0,\tau), \tag{1}$$

where $0 < \tau \leqslant \infty$, $t \mapsto f(t)$, $t \in [0,\tau)$ is a continuous mapping with values in a Hausdorff sequentially complete locally convex space E over the field of complex numbers, $a \in L^1_{loc}([0,\tau))$ and A, B are closed linear operators with domain and range contained in E. The reader may consult the monograph **[459]** by J. Prüss, the author's one **[292]** and the references cited there for the general theory of abstract non-degenerate Volterra equations in Banach and sequentially complete locally convex spaces, i.e., the theory of various types of resolvent (sometimes also called solution) families for (1), with $B = I$. Compared with non-degenerate case, increasingly less has been said about the well-posedness of abstract degenerate

Cauchy problem (1). The study of (1) starts presumably with the papers [**195**,**196**] by A. Favini and H. Tanabe (for some other references on degenerate integro-differential equations, one may refer e.g. to [**91**, **92**, **173**, **175**, **182**, **183**, **188**, **189**, **207**,**280**,**305**,**309**,**311**,**319**,**321**,**327**,**397**] and [**524**]), who have analyzed the well-posedness of equation (1) in the setting of Banach spaces, considering separately the so-called hyperbolic case

$$\sup_{s>0,k\in\mathbb{N}} \|(B(sB+A)^{-1})^k\| < \infty \tag{2}$$

and the parabolic case

$$\sup_{\operatorname{Re}\lambda\geqslant 0} (1+|\lambda|)^{-1}\|B(\lambda B+A)^{-1}\| < \infty. \tag{3}$$

Blank hypothesis in [**195**] is that the operator $T = BA^{-1}$ is a bounded linear operator on E, as well as that the space E has a direct decomposition representation $E = N(T) \oplus \overline{R(T)}$ (similar assumptions have been used in [**424**, Sections 1.1.5–1.1.6], where the authors have investigated degenerate integrated semigroups, as well as in many other research papers concerning abstract degenerate differential equations).

Example 0.0.1. [**195**]

(i) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and let $m \in C^2(\bar{\Omega})$ be a nonnegative superharmonic function. Set $E := L^2(\Omega)$, $A := -\Delta$ with the Dirichlet boundary conditions, $D(B) := \{f; mf \in E\}$ and $Bf := mf$ $(f \in D(B))$. Then (2) holds, with the corresponding supremum being less than or equal to 1.

(ii) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, let $m \in C(\bar{\Omega})$ be a nonnegative function, and let B be a multiplication operator by $m(x)$, acting in $E := H^{-1}(\Omega)$. Set $D(A) := H_0^1(\Omega)$ and $A := -\Delta$. Then (3) holds.

Section 2.1 is devoted to the study of (C, B)-resolvents of closed linear operators. The material from Section 2.2 is taken from [**306**] and [**319**]. Following the approach of T.-J. Xiao and J. Liang (cf. [**542**, Definition 1.4] for the case $a(t) = k(t) = 1$, and [**543**, Definition 2.3] for the case $a(t) = g_n(t)$, $k(t) = 1$), we introduce the notion of an exponentially equicontinuous (a, k)-regularized C-resolvent family for (1). Generally, in our approach, the resolvent set of A does not contain 0 and can be even the empty set, which clearly implies that the operator T need not be defined. Although providing only partial information about the C-wellposedness of the problem (1), it is worth noting that our method has some advantages compared with other existing because we do not use any assumption on the decomposition of the state space E. For the introduced class, we analyze Hille–Yosida type theorems, subordination principles and perturbations in Subsection 2.2.1, as well as differential and analytical properties in Subsection 2.2.2. Abstract degenerate time-fractional differential equations of Caputo order $\alpha \in (0, 2]$ are investigated in Subsection 2.2.3–Subsection 2.2.4, while semilinear degenerate relaxation equations associated with abstract differential operators are investigated in Subsection 2.2.5.

Unless specified otherwise, by E we denote a Hausdorff sequentially complete locally convex space over the field of complex numbers and by $A := A_0, A_1, \dots, A_{n-1}$, $A_n := B$ we denote closed linear operators acting on E. Let $n \in \mathbb{N} \setminus \{1\}$, $0 \leqslant \alpha_1 < \dots < \alpha_n$, $f \colon [0,\infty) \to E$ be a continuous function, and let $\mathbf{D}_t^{\alpha}$ denote the Caputo fractional derivative of order α [**61**, **292**]. The well-posedness of the following multi-term fractional differential equation has been analyzed in a series of recent papers (cf. [**292**, Section 2.10] for an extensive survey of results on abstract multi-term fractional differential equations with Caputo fractional derivatives):

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = f(t),\ t \geqslant 0; \quad u^{(j)}(0) = u_j,\ j = 0, \dots, \lceil \alpha_n \rceil - 1.$$

Define $\mathbb{N}_{n-1} := \{1, \dots, n-1\}$, $\mathbb{N}_{n-1}^0 := \{0, 1, \dots, n-1\}$, $m_i := \lceil \alpha_i \rceil$, $i \in \mathbb{N}_{n-1}$, $T_{i,L}u(t) := A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geqslant 0$, $i \in \mathbb{N}_{n-1}$ and $\alpha_i > 0$, and $T_{i,R}u(t) := \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$. For every $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$, we denote by $T_i u(t)$ either $T_{i,L}u(t)$ or $T_{i,R}u(t)$. In Section 2.3, the principal object of investigation is the following abstract degenerate multi-term problem:

$$\sum_{i=1}^{n-1} T_i u(t) = f(t), \quad t \geqslant 0. \tag{4}$$

Set $\mathcal{I} := \{i \in \mathbb{N}_{n-1} : \alpha_i > 0$ and $T_{i,L}u(t)$ appears on the left hand side of (4)$\}$, $Q := \max \mathcal{I}$, if $\mathcal{I} \neq \emptyset$ and $Q := m_Q := 0$, if $\mathcal{I} = \emptyset$. We subject the following initial conditions to the equation (4); cf. [**308**] for more details:

$$u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant m_Q - 1 \text{ and } (A_i u)^{(j)}(0) = u_{i,j} \text{ if } m_i - 1 \geqslant j \geqslant m_Q. \tag{5}$$

If $T_{n-1}u(t) = T_{n-1,L}u(t)$, then (5) reads as follows:

$$u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant m_{n-1} - 1.$$

From a theoretical point of view, the problem [(4)–(5)] is not most general and later we will see how it can be further generalized. The analysis of problem [(4)–(5)] occupies a great deal of our attention and, for the sake of better exposition and understanding, we will be forced sometimes to repeat the basic things from its definition (sometimes the summation index on the left hand side of (4) runs differently and sometimes the time-variable will be denoted by s). It is our strong belief that this will not cause any form of plagiarism and inconsistency.

The most important subcases of problem [(4)–(5)] are the following fractional Sobolev equations:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^{\alpha} Bu(t) = Au(t) + f(t), \quad t \geqslant 0, \\ Bu(0) = Bx; \quad (Bu)^{(j)}(0) = 0,\ 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B\mathbf{D}_t^{\alpha} u(t) = Au(t) + f(t), \quad t \geqslant 0, \\ u(0) = x; \quad u^{(j)}(0) = 0,\ 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

where $\alpha > 0$. These problems are generalizations of the usual Sobolev linear degenerate equations of first order:

$$B\frac{d}{dt}u(t) = Au(t) + f(t),\ u(0) = u_0\ (t \geqslant 0)$$

and

$$\frac{d}{dt}Bu(t) = Au(t) + f(t),\ Bu(0) = Bu_0\ (t \geqslant 0).$$

For further information concerning the wellposedness of Sobolev first order degenerate equations, the reader may consult the monographs by G. V. Demidenko, S. V. Uspenskii [**140**], A. Favini, A. Yagi [**199**], S. G. Krein [**361**], R. W. Carroll, R. E. Showalter [**95**], I. V. Melnikova, A. Filinkov [**424**], A. G. Sveshnikov, A. B. Al'shin, M. O. Korpusov, Yu. D. Pletner [**503**] and G. A. Sviridyuk, V. Fedorov [**210**], as well as the papers [**6**, **25**, **49**, **50**, **90**, **180**, **197**, **202**, **219**, **247**, **258**, **396**, **420**, **453**, **454**, **507**, **517**, **518**] and [**542**]. The well-posedness of various types of degenerate second-order Sobolev equations has been analyzed in [**14**, **95**, **199**, **203**, **306**, **317**, **421**, **431**, **490**, **525**, **543**] and [**555**]. The corresponding results on degenerate Sobolev equations with integer higher-order derivatives can be found in [**16**, **18**, **43**, **210**, **305**, **515**, **543**] and [**199**, Section 5.7]; concerning abstract degenerate fractional differential equations, we may refer to [**42**, **210**, **214**, **271**, **306**, **317**, **321**, **325**, **379**] and [**558**].

We continue by explaining the organization of material in Section 2.3; cf. [**307**, **311**, **314**] and [**327**]. Various types of solutions of abstract Cauchy problem [(4)–(5)] and its integral analogues are presented in Subsection 2.3.1. For the purpose of study of abstract multi-term problem [(4)–(5)], we introduce the classes of exponentially equicontinuous k-regularized C-resolvent (i, j)-propagation families (Subsection 2.3.2) and exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation families (Subsection 2.3.3). We investigate subordination principles, regularity properties, existence and uniqueness of solutions of problem [(4)–(5)] and related Cauchy problems obtained by integration; following the approach of N. H. Abdelaziz and F. Neubrander [**6**] for abstract degenerate equations of first order, in Subsection 2.3.3 we introduce and analyze the classes of exponentially equicontinuous (a, k)-regularized C-resolvent families generated by a pair of operators and exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation families for [(4)–(5)], clarifying also some new results on the C-wellposedness of the equation $(\mathrm{DFP})_L$ with abstract differential operators. In Theorem 2.3.20, we reconsider the assertion of [**292**, Theorem 2.3.3] for systems of abstract degenerate differential equations involving polynomial matrices of abstract differential operators. Later on, we explain how we can use this result in the analysis of existence and uniqueness of entire solutions of some well known degenerate differential equations of mathematical physics, like the Rossby wave equation, the Sobolev equation, the internal wave equation in the Boussinesq approximation, the gravity-gyroscopic wave equation, and the equation of small amplitude oscillations of a rotating viscous fluid. In Subsection 2.3.4–Subsection 2.3.5, we take up the study of the following special case of problem [(4)–(5)]:

$$(6)\qquad \begin{aligned} &B\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t),\quad t\geqslant 0;\\ &u^{(j)}(0)=u_j,\quad j=0,\dots,\lceil\alpha_n\rceil-1,\end{aligned}$$

where $0\leqslant\alpha<\alpha_n$. Following the method employed in our joint papers with C.-G. Li and M. Li [**346**, **347**], we introduce the class of degenerate k-regularized (C_1,C_2)-existence and uniqueness propagation families for (6), the class of degenerate k-regularized (C_1,C_2)-existence and uniqueness families, and the class of (k,C_2)-uniqueness families for (6). We present some applications to abstract degenerate equations analyzed by M. V. Falaleev and S. S. Orlov in [**174**], as well as an illustrative example of abstract degenerate Cauchy problem (6) whose solution can be defined only locally. We investigate subordination principles and perturbation results for the introduced classes. In the remainder of Subsection 2.3.5, we analyze entire and analytical solutions of (6), and apply our theoretical results in the study of abstract Boussinesq-Love equation, which is important in the modeling the longitudinal waves in an elastic bar with the transverse inertia, and the abstract Barenblatt–Zheltov–Kochina equation, which is important in the study of fluid filtration in fissured rocks, as well as in the studies of moisture transfer in soil and the process of two-temperature heat conductivity. We also analyze some fractional analogues of these equations.

Denote by D_t^{α} the Riemann–Liouville fractional derivative of order α [**61**, **315**, **345**]. In Section 2.4, we analyze the following abstract multi-term fractional differential equation:

$$(7)\qquad BD_t^{\alpha_n}u(t)+\sum_{j=1}^{n-1}A_jD_t^{\alpha_j}u(t)=AD_t^{\alpha}u(t)+f(t),\quad t\in(0,\tau),$$

in a complex Banach space E. If $\alpha_n>1$ ($\alpha_n=1$, $\alpha_n<1$), then we say that (7) is of type (SC1) ((SC2), (SC3)). Depending on type of (7), we subject initial conditions to (7) and carry out further analyses. The main result of Section 2.4 is Theorem 2.4.9, in which we analyze some applications of certain subclasses of k-regularized (C_1,C_2)-existence and uniqueness families in the study of unique solvability of (7).

Section 2.5 conveys results of a joint research study [**213**] with V. E. Fedorov. Suppose that $n\in\mathbb{N}$, $\zeta\in(0,1]$, $p_0,p_1,\dots,p_n$ and $q_0,q_1,\dots,q_n$ are given nonnegative integers satisfying $p_0=q_0=0$ and $0<p_1+q_1\leqslant p_2+q_2\leqslant\dots\leqslant p_n+q_n$. Set $T_iu(s):=(\mathbf{D}_s^{\zeta})^{p_i}A_i(\mathbf{D}_s^{\zeta})^{q_i}u(s)$, $s\geqslant 0$, $i\in\mathbb{N}_n^0$, $S_l:=\{i\in\mathbb{N}_n : q_i\geqslant 1\}$, $S_r:=\{i\in\mathbb{N}_n : p_i\geqslant 1\}$, and conventionally, $\max(\emptyset):=\emptyset$, $\mathbb{N}_{\emptyset}^0:=\emptyset$. We analyze the following abstract degenerate Cauchy problem:

$$(8)\qquad \sum_{i=0}^{n}T_iu(s)=0,\quad s\geqslant 0,$$

accompanied with the following initial conditions:

$$(9)\qquad ((\mathbf{D}_s^{\zeta})^{j}u(s))_{s=0}=u_j,\quad j\in\mathbb{N}^0_{\max\{q_i-1:i\in S_l\}},\ \text{and}$$

$$((\mathbf{D}_s^\zeta)^j A_i(\mathbf{D}_s^\zeta)^{q_i} u(s))_{s=0} = u_{i,j} \quad (i \in S_r,\ j \in \mathbb{N}^0_{p_i-1}).$$

Although not visible at first glance, the study of problem [(8)–(9)] taken up in this section leans heavily on the method used in our previous paper [**303**] on regularization of ultradistribution semigroups and ultradistribution sines of Beurling class. The crucial thing is the existence and polynomial boundedness of the operators P_λ^{-1}, where

$$P_\lambda := \lambda^{(p_n+q_n)\zeta} B + \sum_{i=0}^{n-1} \lambda^{(p_i+q_i)\zeta} A_i, \quad \lambda \in \mathbb{C} \smallsetminus \{0\},$$

on an appropriate ultra-logarithmic region $\Lambda_{\alpha,\beta,l}$ (cf. J. Chazarain [**99**], 1971).

Assume now that $n \in \mathbb{N}$, $0 < \zeta \leqslant 2$, $q_0, q_1, \ldots, q_n$ are given non-negative integers satisfying $q_0 = 0$ and $0 < q_1 \leqslant q_2 \leqslant \ldots \leqslant q_n$, as well as that $p_i = 0$ for all $i \in \mathbb{N}^0_n$. Hence, $T_i u(s) = A_i(\mathbf{D}_s^\zeta)^{q_i} u(s)$, $s \geqslant 0$, $i \in \mathbb{N}^0_n$ and $P_\lambda = \lambda^{q_n\zeta} B + \sum_{i=0}^{n-1} \lambda^{q_i\zeta} A_i$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$. In Section 2.6, we investigate the abstract degenerate multi-term Cauchy problem (8) accompanied with the initial conditions of the following form:

$$((\mathbf{D}_s^\zeta)^j u(s))_{s=0} = u_j,\ j \in \mathbb{N}^0_{q_n-1},\ \text{if } \zeta \in (0,1],\ \text{resp.},$$

$$((\mathbf{D}_s^\zeta)^j u(s))_{s=0} = u_j,\ j \in \mathbb{N}^0_{q_n-1};\ \Big(\frac{d}{ds}(\mathbf{D}_s^\zeta)^j u(s)\Big)_{s=0} = v_j,\ j \in \mathbb{N}^0_{q_n-1},\ \text{if } \zeta \in (1,2]. \tag{10}$$

In Definition 2.6.1, we introduce the notion of an entire (analytic) solution of problem [(8), (10)]. Denote by $\mathfrak{W}$ ($\mathfrak{W}_e$) the subspace of E^{q_n}, resp. E^{2q_n}, consisting of all initial values $(u_0, \ldots, u_{q_n-1}) \in E^{q_n}$, resp. $(u_0, \ldots, u_{q_n-1}; v_0, \ldots, v_{q_n-1}) \in E^{2q_n}$, subjected to some analytical solution $u(\cdot)$ of problem (8) on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ (entire solution $u(\cdot)$ of problem (8)). The main result of Section 2.6 is Theorem 2.6.2, which asserts that the set $\mathfrak{W}$ is dense in $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$ for the topology of E^{q_n}, provided that $0 < \zeta < 1$, resp. $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$ for the topology of E^{2q_n}, provided that $1 < \zeta < 2$. Similar statements hold for the equations with integer order derivatives, when we have that the set $\mathfrak{W}_e$ is dense in $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$ for the topology of E^{q_n}, provided that $\zeta = 1$, resp. $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$ for the topology of E^{2q_n}, provided that $\zeta = 2$. The class of C-regularized semigroups of growth order $r > 0$ has recently been introduced in [**103**] following the ideas of G. Da Prato [**126**] (1966). In Section 2.7, we continue our previous research studies [**101**, **103**] by investigating abstract incomplete Cauchy problems degenerate in time. For that purpose, we introduce the concept of degenerate (B,C)-regularized semigroups of growth order $r > 0$ and clarify their most important structural properties. In Theorem 2.7.4, we deal with the following abstract incomplete degenerate differential equations with modified Liouville right-sided fractional derivatives:

$$(FP_{\alpha_1,\beta_1,\theta}) : \begin{cases} u \in C^\infty((0,\infty) : E), \\ D_-^{\alpha_1} B D_-^{\beta_1} u(s) = e^{i\theta/\gamma} A u(s), \quad s > 0, \\ \lim_{s\to 0+} Bu(s) = Cx, \\ \text{the set } \{Bu(s) : s > 0\} \text{ is bounded in } E \end{cases}$$

and

$$(FP_{\alpha_1,\beta_1,\theta})' : \begin{cases} u \in C^{\infty}((0,\infty) : E), \\ D_-^{\alpha_1} B D_-^{\beta_1} u(s) = e^{i\theta/\gamma} Au(s), \quad s > 0, \\ \lim_{s\to 0+} Bu(s) = Cx, \\ \text{the sets } \{(1+s^{-(q+1)/\gamma})^{-1}u(s) : s > 0\} \\ \text{ and } \{(1+s^{-(q+1)/\gamma})^{-1}Bu(s) : s > 0\} \text{ are bounded in } E; \end{cases}$$

here, $0 < \gamma < 1/2$, $\alpha_1 \geqslant 0$, $\beta_1 \geqslant 0$, $\alpha_1 + \beta_1 = 1/\gamma$, $\theta \in (\gamma\pi - (\pi/2), (\pi/2) - \gamma\pi)$, $C \in L(E)$ is injective, A belongs to the class $\mathcal{M}_{B,C,q}$ defined later, and $q > -1$, resp. $-1 - \gamma < q \leqslant -1$, in the case of consideration of problem $(FP_{\alpha_1,\beta_1,\theta})$, resp. $(FP_{\alpha_1,\beta_1,\theta})'$. In Theorem 2.7.5, we treat the following abstract incomplete degenerate Cauchy problem of second order

$$(P_{2,q,B}) : \begin{cases} u \in \mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)} : E),\ \ Bu \in \mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)} : E), \\ Bu''(z) = \frac{d^2}{dz^2}Bu(z) = Au(z),\ z \in \Sigma_{(\pi/2)-(\omega/2)}, \\ \lim_{z\to 0, z\in\Sigma_\delta} Bu(z) = Cx,\ \text{for every } \delta \in (0,(\pi/2)-(\omega/2)), \\ \text{the sets } \{(1+|z|^{-(2q+2)})^{-1}u(z) : z \in \Sigma_\delta\} \text{ and} \\ \{(1+|z|^{-(2q+2)})^{-1}Bu(z) : z \in \Sigma_\delta\} \text{ are bounded in } E \\ \text{for every } \delta \in (0,(\pi/2)-(\omega/2)), \end{cases}$$

where $(-3)/2 < q < (-1)/2$, $\mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)} : E)$ denotes the set consisting of all analytic functions from the sector $\Sigma_{(\pi/2)-(\omega/2)} = \{\lambda \in \mathbb{C} \smallsetminus \{0\} : |\arg(\lambda)| < (\pi/2)-(\omega/2)\}$ into E, and the condition (H) specified later holds with some number $\omega \in [0,\pi)$. We raise a question of finding some sufficient conditions guaranteeing the uniqueness of solutions of problem $(P_{2,q,B})$. The main aim of Section 2.8 is to present the most important results from our second joint research study with V. E. Fedorov [**214**]. In this section, we follow the methods employed in the monograph G. A. Sviridyk–V. E. Fedorov [**509**], which are maybe insufficiently reconsidered for abstract degenerate Volterra integro-differential equations so far (cf. Section 2.12 for more details). The notion of a regular resolvent set of a closed linear operator A acting on E plays an important role in this section: let us recall that the regular resolvent set of A, $\rho^r(A)$ shortly, is defined as the union of those complex numbers $\lambda \in \rho(A)$ for which $(\lambda - A)^{-1} \in R(E)$, where $R(E)$ denotes the set consisting of all regular bounded linear operators $D \in L(E)$, i.e., the operators $D \in L(E)$ for which there exists a positive constant $c > 0$ such that for each seminorm $p \in ⊛$ there exists another seminorm $q \in ⊛$ such that $p(D^n x) \leqslant c^n q(x)$, $x \in E$, $n \in \mathbb{N}$ (infimum of such constants is said to be constant of regularity of D). Here, ⊛ is the abbreviation for the fundamental system of seminorms which defines the topology of E. Assume now that X, Y are two SCLCSs as well as that A and B are two closed linear operators acting between the spaces X and Y. Set $R_\lambda^B(A) := (\lambda B - A)^{-1}B$ and $L_\lambda^B(A) := B(\lambda B - A)^{-1}$. By a regular B-resolvent set of the operator A, $\rho_r^B(A)$ for short, we mean the set

$$\rho_r^B(A) := \{\lambda \in \mathbb{C} : (\lambda B - A)^{-1} \in L(Y,X),\ R_\lambda^B(A) \in L(X),\ L_\lambda^B(A) \in L(Y)\}.$$

It is said that the operator A is (B,σ)-regular, or equivalently, that a pair of operators (A,B) is σ-regular, iff there exists a finite number $a>0$ such that, for every $\lambda\in\mathbb{C}$ with $|\lambda|>a$, we have $\lambda\in\rho_r^B(A)$. We also define the notion of a (B,p)-regular operator A (a p-regular pair of operators (A,B)), where $p\in\mathbb{N}_0$. The main result of Section 2.8 is Theorem 2.8.6, where we give some sufficient conditions for the existence of a unique strong solution of the following inhomogeneous degenerate Cauchy problem

$$(\mathrm{DF})_f:\begin{cases}\mathbf{D}_t^{\alpha}Bu(t)=Au(t)+f(t), & t\geqslant 0,\\ u^{(k)}(0)=x_k, & 0\leqslant k\leqslant\lceil\alpha\rceil-1,\end{cases}$$

under the assumption on (B,p)-regularity of operator A. In [**459**, Chapter II], J. Prüss has studied abstract non-scalar Volterra equations. Applications have been given in the analysis of viscoelastic Timoshenko beam model, Midlin–Timoshenko plate model and viscoelastic Kirchhoff plate model, with the corresponding materials being non-synchronous, as well as in the analysis of some problems appearing in the theories of linear thermoviscoelasticity and electrodynamics. Section 2.9 is written in an expository manner (we do have obligation to say that Subsection 2.3.5, Section 2.4, Section 2.8 and Subsection 3.6.4 are also written without giving the proofs of structural results; the only exception is Theorem 2.4.6) and its aim is to show how the techniques established in [**459**] and [**299**] can be helpful in the analysis of a substantially large class of abstract degenerate Volterra integral equations of non-scalar type [**318**]. More precisely, we treat the following linear degenerate Volterra equation:

$$Bu(t)=f(t)+\int_0^t A(t-s)u(s)\,ds,\quad t\in[0,\tau), \tag{11}$$

where X and Y are two complex Banach spaces satisfying that Y is continuously embedded in X, B is a closed linear operator with domain and range contained in X, $\tau\in(0,\infty]$, $f\in C([0,\tau):X)$ and $A\in L^1_{loc}([0,\tau):L(Y,X))$. In Definition 2.9.1, we introduce the notion of a strong (mild) solution of problem (11) as well as the notion of (kC)-wellposedness of (11); here, $k(t)$ is a scalar-valued kernel on $[0,\tau)$ and the operator $C\in L(X)$ is injective. For the purpose of research of (11), we introduce the classes of (weak) (A,k,B)-regularized C-pseudoresolvent families and (A,k,B)-regularized C-resolvent families in Definition 2.9.2. After that, we investigate the generation of (A,k,B)-regularized C-pseudoresolvent families, the main solution concept considered, as well as their analytical properties and hyperbolic perturbation results.

The main aim of Section 2.10 is to continue our previous researches of hypercyclic and topologically mixing properties of abstract non-degenerate (multi-term) fractional differential equations with Caputo derivatives (cf. [**292**, Chapter 3] for a comprehensive survey of results). It is our strong belief that our study of (disjoint) hypercyclic and (disjoint) topologically mixing extensions of multivalued linear operators (binary relations) will receive some attention of the authors working in the field of linear topological dynamics. Concerning the other subjects, we would like to note that we are not primarily concerned with studying new concepts in the theory of hypercyclicity and that some of our results have origins in the theory of hypercyclic single valued operators. The most important assertions, from the viewpoint of possible applications, will be those in which we reconsider the well known Desch–Schappacher–Webb and Banasiak–Moszyński criteria for chaos of strongly continuous semigroups.

Suppose, for the time being, that E is a separable infinite-dimensional Fréchet space over the field $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C}\}$, whose topology is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms. Then the translation invariant metric $d\colon E \times E \to [0,\infty)$, defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x, y \in E,$$

satisfies, among many other properties, the following:

$$d(x+u, y+v) \leqslant d(x,y) + d(u,v) \text{ and } d(cx, cy) \leqslant (|c|+1)d(x,y),$$

for any $c \in \mathbb{K}$ and $x, y, u, v \in E$. A linear mapping $T\colon E \to E$ is said to be hypercyclic, resp. cyclic, iff there exists an element $x \in E$ whose orbit $\operatorname{Orb}(x,T) := \{T^n x : n \in \mathbb{N}_0\}$ is dense in E, resp. iff there exists an element $x \in E$ such that the linear span of $\operatorname{Orb}(x,T)$ is dense in E, while T is said to be topologically transitive iff for for any pair of open non-empty subsets U, V of E there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Using a simple Baire Category argument, it readily follows that T is hypercyclic iff T is topologically transitive (G. D. Birkhoff [**77**], 1929). Furthermore, a linear mapping $T\colon E \to E$ is said to be chaotic (supercyclic, resp. positively supercyclic) iff T is hypercyclic and the set of periodic points of T, defined by $\{x \in E :$ there exists $n \in \mathbb{N}$ such that $T^n x = x\}$, is dense in E (iff there exists an element $x \in E$ whose projective orbit $\{cT^n x : n \in \mathbb{N}_0,\ c \in \mathbb{K}\}$, resp. positive projective orbit $\{cT^n x : n \in \mathbb{N}_0,\ c \geqslant 0\}$, is dense in E). Chronologically, the first example of a hypercyclic operator was constructed by G. D. Birkhoff in the afore-mentioned paper [**77**]. In actual fact, there were proved that the translation operator $f \mapsto f(\cdot + a)$, $f \in \mathrm{H}(\mathbb{C})$, $a \in \mathbb{C} \smallsetminus \{0\}$ is hypercyclic in $\mathrm{H}(\mathbb{C})$; the hypercyclicity of the derivative operator $f \mapsto f'$, $f \in \mathrm{H}(\mathbb{C})$ was proved by G. R. MacLane [**403**] in 1952. The first example of a hypercyclic operator acting on a Banach space was constructed by S. Rolewicz [**464**] in 1969; the state space in his analysis is chosen to be $l^2(\mathbb{N})$. The most commonly used criterion for proving hypercyclicity of single operators, the so-called Hypercyclicity Criterion, was discovered independently by C. Kitai [**284**] (1982) and R. M. Gethner, J. Shapiro [**224**] (1987). From the period 1987 onwards, the study of hypercyclicity of single valued operators and various generalizations of this concept have experimented a great development. For further information concerning dynamical properties of single operators, we refer the reader to the monographs [**238**] by K.-G. Grosse-Erdmann, A. Peris and [**60**] by F. Bayart, E. Matheron.

Although the first examples of chaotic semigroups were given by C. R. MacCluer [**402**] and V. Protopopescu, Y. Azmy [**458**] already in 1992, the hypercyclic and chaotic properties of strongly continuous semigroups were studied in a systematic way for the first time in the paper [**143**] by W. Desch, W. Schappacher and G. F.

Webb (1997). The study of hypercyclicity of second order non-degenerate equations starts with the paper [**85**] by A. Bonilla and P. J. Miana (2008), while the study of hypercylicity of non-degenerate fractional differential equations goes back to author's paper [**300**] (2012).

The non-existence of an appropriate reference which treats the hypercyclicity of abstract degenerate PDEs may has been a strong motivational factor that influenced us to write the papers [**308, 309**], from which the material of Section 2.10 is taken; our results seem to be new even for abstract degenerate differential equations of first and second order. The analysis we have carried out indicates that the class of hypercyclic abstract degenerate equations is, by all means, substantially larger than the corresponding class of non-degenerate equations. Hypercyclic and topologically mixing properties of solutions of abstract (multi-term) fractional differential equations with Riemann–Liouville derivatives can be analyzed similarly, and we will only refer the reader to the papers [**345**] and [**382**] for more details about this topic.

Distributional chaos is a very popular field of research in the theory of topological dynamics of linear operators. Let us recall that the notion of distributional chaos for interval maps was introduced by B. Schweizer and J. Smítal [**481**] in 1994. Distributional chaos was firstly considered in the setting of linear operators when studying a quantum harmonic oscillator [**159**] (1999) and [**441**] (2006). A systematic study of distributional chaos for backward shifts operators was initiated in [**414**], while an example of a backward shift operator with a full scrambled set appeared in [**415**]. For some other relevant references on distributional chaos, one may refer e.g. to [**46,66,68,440**] and [**499**]. A linear continuous operator T acting on a Fréchet space E is said to be distributionally chaotic iff there exist an uncountable set $S \subseteq E$ (scrambled set) and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x, y \in S$ of distinct points we have that

$$\overline{dens}(\{k \in \mathbb{N} : d(T^k x, T^k y) \geqslant \sigma\}) = 1 \text{ and}$$
$$\overline{dens}(\{k \in \mathbb{N} : d(T^k x, T^k y) < \varepsilon\}) = 1,$$

where $d(\cdot, \cdot)$ denotes the metric on E and the upper density of a set $D \subseteq \mathbb{N}$ is defined by

$$\overline{dens}(D) := \limsup_{n \to +\infty} \frac{\operatorname{card}(D \cap [1, n])}{n}.$$

If we can choose S to be dense in E, then we say that T is densely distributionally chaotic. The notion of a (densely) distributionally chaotic strongly continuous semigroup on Fréchet space has recently been introduced in [**112**] (joint work with J. A. Conejero, P. J. Miana and M. Murillo-Arcila; cf. also [**13,55,56**] and [**116**] for further information concerning distributionally chaotic strongly continuous semigroups on Banach spaces) as follows: A strongly continuous semigroup $(T(t))_{t \geqslant 0} \subseteq L(E)$ is said to be distributionally chaotic iff there are an uncountable set $S \subseteq E$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x, y \in S$ of distinct points we have that

$$\overline{Dens}(\{t \geqslant 0 : d(T(t)x, T(t)y) \geqslant \sigma\}) = 1 \text{ and}$$

$$\overline{Dens}(\{t \geqslant 0 : d(T(t)x, T(t)y) < \varepsilon\}) = 1,$$

where the upper density of a set $D \subseteq [0, \infty)$ is defined now by

$$\overline{Dens}(D) := \limsup_{t \to +\infty} \frac{m(D \cap [0, t])}{t},$$

with $m(\cdot)$ being the Lebesgue measure on $[0, \infty)$. If, moreover, we can choose S to be dense in E, then $(T(t))_{t \geqslant 0}$ is said to be densely distributionally chaotic. The question of whether an operator $T \in L(X)$ or a strongly continuous semigroup $(T(t))_{t \geqslant 0} \subseteq L(X)$ is distributionally chaotic or not is closely connected with the existence of distributionally irregular vectors, i.e., those elements $x \in X$ such that for each $\sigma > 0$

$$\overline{dens}(\{k \in \mathbb{N} : d(T^k x, 0) > \sigma\}) = 1 \text{ and}$$
$$\overline{dens}(\{k \in \mathbb{N} : d(T^k x, 0) < \sigma\}) = 1,$$

respectively,

$$\overline{Dens}(\{t \geqslant 0 : d(T(t)x, 0) > \sigma\}) = 1 \text{ and}$$
$$\overline{Dens}(\{t \geqslant 0 : d(T(t)x, 0) < \sigma\}) = 1.$$

For the basic information concerning distributionally chaotic properties of ill-posed abstract non-degenerate equations of first order, we refer the reader to [**112**]. Distributionally chaotic properties of abstract non-degenerate fractional differential equations in Banach spaces has recently been analyzed in [**320**], where it has been pointed out that the notion of distributional chaos is much more appropriate for dealing with fractional equations than that of the usually considered Devaney chaos. The main purpose of Section 2.11 is to analyze a class of distributionally chaotic abstract degenerate (multi-term) fractional differential equations. For further information about (distributional) chaos of general binary relations, multivalued linear operators and abstract integro-differential equations, we refer the reader to the recent monograph of M. Kostić [**294**].

In this monograph, we will not discuss linear topological dynamics of abstract degenerate differential equations associated with the use of backward shift operators. For more details concerning discrete case, we refer the reader to the doctoral dissertation of Ö. Martin [**409**] and references cited therein.

We believe that our researches will enjoy reading Section 2.10 and Section 2.11. In order to make our monograph a convenient reference, we have concluded the second and third chapter with a concise summary and further guidance notes. Appendices to the second chapter are given in Section 2.12.

A large number of research papers, starting presumably with that of A. Yagi [**549**], written over the last twenty five years, have concerned applications of multivalued linear operators to abstract degenerate differential equations (cf. [**199**] for the basic source of information on this subject). In the third chapter of monograph, we investigate the abstract degenerate Volterra integro-differential equations in sequentially complete locally convex spaces by using multivalued linear operators and vector-valued Laplace transform. We follow the method which is based on the use of (a,k)-regularized C-resolvent families generated by multivalued linear operators (cf. [**366**, **367**, **421**, **423**, **424**] and [**482**] for some special cases of this notion) and which suggests a very general way of approaching abstract Volterra equations. We also introduce and analyze the class of (a,k)-regularized (C_1,C_2)-existence and uniqueness families. The results presented in Chapter 3, which is composed of thirteen sections, are completely new even for abstract degenerate Caputo fractional inclusions in Banach spaces.

In Section 3.1, we analyze the following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A}\int_0^t a(t-s)u(s)\,ds + \mathcal{F}(t), \quad t \in [0,\tau), \tag{12}$$

where $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A}\colon X \to P(Y)$ and $\mathcal{B}\colon X \to P(Y)$ are given multivalued linear operators acting between sequentially complete locally convex spaces X and Y, and $\mathcal{F}\colon X \to P(Y)$ is a given mutivalued mapping, as well as the following fractional Sobolev inclusions:

$$(\mathrm{DFP})_{\mathbf{R}} : \begin{cases} \mathbf{D}_t^{\alpha} Bu(t) \in \mathcal{A}u(t) + \mathcal{F}(t), & t \geqslant 0, \\ (Bu)^{(j)}(0) = Bx_j, & 0 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

where we assume that $B = \mathcal{B}$ is single-valued, and

$$(\mathrm{DFP})_{\mathbf{L}} : \begin{cases} \mathcal{B}\mathbf{D}_t^{\alpha} u(t) \subseteq \mathcal{A}u(t) + \mathcal{F}(t), & t \geqslant 0, \\ u^{(j)}(0) = x_j, & 0 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

We define various types of solutions of problems (12), $(\mathrm{DFP})_{\mathbf{R}}$ and $(\mathrm{DFP})_{\mathbf{L}}$. In Theorem 3.1.3 and Theorem 3.1.5, we reconsider the main results of research of M. Kim [**280**], while in Theorem 3.1.6 we prove an extension of [**285**, Theorem 3.5] for abstract degenerate fractional differential inclusions. Subordination principles are clarified in Theorem 3.1.8 and Theorem 3.1.9 following the methods proposed by J. Prüss [**459**, Section 4] and E. Bazhlekova [**61**, Section 3] (cf. [**210**, **314**] and [**327**] for similar results known in the degenerate case).

Following the ideas of R. deLaubenfels [**132**], in Section 3.2 we introduce and analyze the class of (a,k)-regularized (C_1,C_2)-existence and uniqueness families (see [**292**, Section 2.8] for non-degenerate case). Later on, we single out the class of (a,k)-regularized C-resolvent families for special considerations. We focus our attention on the analysis of Hille–Yosida's type theorems for (a,k)-regularized C-resolvent families generated by multivalued linear operators (as in all previous researches of non-degenerate case, we introduce the notion of a subgenerator of an (a,k)-regularized C-resolvent family and investigate the most important properties of subgenerators). It is well known (see e.g. [**199**, Theorem 2.4], [**285**, Theorem 3.6] and [**280**, p. 169]) that Hille–Yosida's type estimates for the resolvent of a multivalued operator $\mathcal{A}$ immediately implies that $\mathcal{A}$ is single-valued in a certain sense. In part (ii) of Theorem 3.2.12, we will prove a similar assertion provided that the Hille–Yosida condition (288) below holds. For the validity of Theorem 3.2.12(ii), we have found the condition $k(0) \neq 0$ very important to be satisfied; in other words, the existence of above-mentioned single-valued branch of $\mathcal{A}$ can be proved exactly in non-convoluted or non-integrated case, so that we have arrived to a diametrically opposite conclusion to that stated on l. 7-13, p. 169 of [**280**]. Nevertheless, the existence or non-existence of such a single-valued branch of $\mathcal{A}$ is not sufficient for obtaining a fairly complete information on the well-posednesss of inclusion (12) with $\mathcal{B} = I$ (the reading of papers [**280**] and [**285**] has strongly influenced us to write [**312**], which contains a great part of our results presented in Section 3.1–Section 3.2; compared with the results of [**280**], here we do not need the assumption that $a(t)$ is a normalized function of local bounded variation). In the remainder of Section 3.2, we enquire into the possibility to extend the most important results from [**292**, Section 2.1, Section 2.2] to (a, k)-regularized C-resolvent families generated by multivalued linear operators, and present several examples and possible applications of our abstract theoretical results. We clarify the complex characterization theorem for the generation of exponentially equicontunuous (a, k)-regularized C-resolvent families, the generalized variation of parameters formula, and subordination principles; in a separate subsection, we analyze differential and analytical properties of (a, k)-regularized C-resolvent families. Furthermore, we investigate the following degenerate Volterra integral inclusion:

$$0 \in \mathcal{B}u(t) + \sum_{j=0}^{n-1} \mathcal{A}_j(a_j * u)(t) + \mathcal{F}(t), \quad t \in [0, \tau),$$

where $n \in \mathbb{N}$, $0 < \tau \leqslant \infty$, $\mathcal{F}\colon [0, \tau) \to P(Y)$, $a_0, \cdots, a_{n-1} \in L^1_{loc}([0, \tau))$, and $\mathcal{A} \equiv \mathcal{A}_0, \cdots, \mathcal{A}_{n-1}, \mathcal{B} \equiv \mathcal{A}_n$ are multivalued linear operators acting between the sequentially complete locally convex spaces X and Y ([**327**]). In Subsection 3.2.2, we analyze the situation in which some of the regularizing operators C_2 or C is not injective, and introduce the notion of an (a, k, C)-subgenerator of arbitrary strongly continuous operator family $(Z(t))_{t \in [0,\tau)} \subseteq L(X)$, where $0 < \tau \leqslant \infty$. The main aim of Subsection 3.2.3 is to clarify the most important results about degenerate K-convoluted C-semigroups and degenerate K-convoluted C-cosine functions in locally convex spaces. In this subsection, degenerate operator families under examination are defined locally or globally and their subgenerators are allowed to be multivalued linear operators. We thus provide a new unification concept in the theory of abstract degenerate differential equations of first and second order, paying special attention to explain, in a brief and concise manner, how we can improve our structural results from the second chapter of monograph [**291**] to degenerate operator families. We analyze the basic properties of subgenerators, extension and adjoint type theorems, real and complex characterization theorems, as well as generation of local degenerate K-convoluted C-semigroups (K-convoluted C-cosine functions). Following P. C. Kunstmann [**363**], we introduce the notion of stationarity of a multivalued linear operator and give some estimates on the upper bounds for stationarity of generators of degenerate fractionally integrated semigroups and cosine functions. The study of degenerate K-convoluted C-groups is without the scope of this monograph.

It is well known that the class of distribution semigroups in Banach spaces was introduced by J. L. Lions [**391**] in 1960 as an attempt to seek for the solutions of abstract first order differential equations that are not well-posed in the usual sense, i.e., whose solutions are not governed by strongly continuous semigroups of linear operators. From then on, distribution semigroups have attracted the attention of a large number of mathematicians. The class of distribution semigroups with not necessarily densely defined generators has been introduced independently by P. C. Kunstmann [**364**] and S. W. Wang [**530**], while the class of C-distribution semigroups has been introduced by the author in [**332**]. Ultradistribution semigroups in Banach spaces, with densely or non-densely defined generators, and abstract Beurling spaces have been analyzed in the papers of R. Beals [**64**, **65**], J. Chazarain [**99**], I. Ciorănescu [**109**], H. A. Emami-Rad [**167**] and H. Komatsu [**289**] (cf. also [**291**,**351**,**365**] and [**424**]). On the other hand, the study of distribution semigroups in locally convex spaces has been initiated by R. Shiraishi, Y. Hirata [**488**], T. Ushijima [**522**] and M. Ju Vuvunikjan [**528**]. In our recent joint research study with S. Pilipović and D. Velinov [**354**], we have introduced and systematically analyzed the classes of C-distribution semigroups and C-ultradistribution semigroups in locally convex spaces. The main aim of Section 3.3 is to explore the classes of degenerate C-distribution semigroups and degenerate C-ultradistribution semigroups in locally convex spaces [**352**,**353**]. Section 3.4 is devoted to the study of degenerate C-distribution cosine functions and degenerate C-ultradistribution cosine functions in locally convex spaces (cf. [**331**,**333**,**348**] and [**428**] for single-valued case). Even in non-degenerate case, with C being the identity operator and the pivot space being one of Banach's, the results presented in Section 3.4 are completely new in ultradistributional case [**523**].

In Section 3.5, we continue the analysis of A. Favini, A. Yagi [**199**, Chapter III] and numerous other authors by investigating subordinated fractional resolvent families with removable singularities at zero and semilinear degenerate fractional differential inclusions [**324**, **325**]. Our main contributions are contained in Subsection 3.5.2, where we also analyze a new class of abstract relaxation differential equations that are not degenerate in time.

The main ideas and organization of Section 3.6, where we reconsider our previously established results on hypercyclic and topologically mixing properties of abstract degenerate (multi-term) time-fractional equations by using the multivalued linear operators approach [**321**], will be explained within themselves. Let us only note, for now, that this section is consisted from four subsections and that an interesting application has been made in the study of topologically mixing properties of the Poisson heat equation in $C^2(\mathbb{R})$. Chronologically, G. A. Sviridyuk and N. A. Manakova were the first to investigate perturbations of a class of abstract degenerate differential equations of first order [**510**], 2003. Using the perturbation theory for strongly continuous semigroups and the theory developed by G. A. Sviridyuk [**507**], V. E. Fedorov and O. A. Ruzakova have analyzed in [**219**] the unique solvability for the Cauchy problem and Showalter problem for a class of perturbations of abstract degenerate differential equations of first order. The paper [**219**] contains a great number of applications to initial boundary value problems and we can freely say that this is the first systematic study of perturbations of abstract degenerate differential equations. Recently, A. Favini [**186**] has looked into inverse problems of degenerate differential equations by using perturbation results for linear relations (cf. also M. A. Horani, A. Favini [**15**] and A. Favaron, A. Favini, H. Tanabe [**184**]). The main aim of Section 3.7 is to reconsider perturbation results for abstract non-degenerate Volterra integro-differential equations [**292**, Section 2.6] from the point of view of the theory of multivalued linear operators [**323**]. We provide several illustrative applications of our results, primarily to degenerate fractional differential equations with Caputo derivatives. We also provide a few instructive examples emphasizing that certain perturbation properties of abstract degenerate Volterra integro-differential equations can be analyzed by using the results from the perturbation theory for non-degenerate equations.

Approximation theory is an established field of mathematics whose focus is primarily on the approximation of real-valued continuous functions by some simpler class of functions. Concerning the theory of strongly continuous semigroups, of particular interest is the well-known Trotter–Kato theorem, which makes a relationship between the convergence of a sequence of infinitesimal generators (their resolvents) to the convergence of associated semigroups of operators. Briefly told, the main aim of Section 3.8 [**316**] is to prove some Trotter–Kato type formulae for degenerate (a,k)-regularized C-resolvent families in locally convex spaces (cf. [**25**,**50**,**79**,**95**,**187**,**199**,**370**,**509**] and [**519**] for the basic references on approximation of abstract degenerate differential equations), as well as to investigate the Laguerre expansions of degenerate (a,k)-regularized C-resolvent families (cf. [**2**] for the study of Laguerre expansions of non-degenerate strongly continuous semigroups in Banach spaces).

The first results about fractional powers of non-negative multivalued linear operators was given by El H. Alaarabiou [**11**,**12**] in 1991. In these papers, he extended the well known Hirsch functional calculus to the class $\mathcal{M}$ of non-negative multivalued linear operators in a complex Banach space. Unfortunately, the method proposed in [**11**,**12**] had not allowed one to consider the product formula and the spectral mapping theorem for powers. Nine years later, in 2000, C. Martínez, M. Sanz and J. Pastor [**411**] improved a functional calculus established in [**11**,**12**], providing a new definition of fractional powers. A very stable and consistent theory of fractional powers of the operators belonging to the class $\mathcal{M}$ has been constructed, including within itself the above-mentioned product formula, spectral mapping theorem, as well as almost all other fundamental properties of fractional powers of non-negative single-valued linear operators. Some later contributions have been given by J. Pastor [**444**], who considered relations between the multiplicativity and uniqueness of fractional powers of non-negative multivalued linear operators.

In order to motivate our research in Section 3.9, let us first look into the class consisting of multivalued linear operators $\mathcal{A}$, acting on a complex Banach space $(X, \|\cdot\|)$, for which $(-\infty, 0] \subseteq \rho\mathcal{A}$ and there exist finite numbers $M_1 \geqslant 1$, $\beta \in (0,1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leqslant M_1(1+|\lambda|)^{-\beta}, \quad \lambda \leqslant 0. \tag{13}$$

Assuming that (13) is true, we can apply the usual von Neumann's expansion in order to see that there exist positive real constants $c > 0$ and $M > 0$ such that the resolvent set of $\mathcal{A}$ contains an open region $\Omega_{c,M} \supseteq \Omega'_{c,M} := \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant (2M)^{-1}(c - \operatorname{Re}\lambda)^{\beta},\ \operatorname{Re}\lambda \leqslant c\}$, where we have the estimate $\|R(\lambda : \mathcal{A})\| = O((1 + |\lambda|)^{-\beta})$, $\lambda \in \Omega_{c,M}$. Let Γ' be the upwards oriented curve $\{\xi \pm i(2M)^{-1}(c-\xi)^{\beta} : -\infty < \xi \leqslant c\}$. In [**199**], cf. also Subsection 1.2.1, A. Favini and A. Yagi define the fractional power $\mathcal{A}^{-\theta}$, for $\operatorname{Re}\theta > 1 - \beta$, by

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi i}\int_{\Gamma'} \lambda^{-\theta}(\lambda - \mathcal{A})^{-1} d\lambda,$$

$\mathcal{A}^{\theta} := (\mathcal{A}^{-\theta})^{-1}$ $(\operatorname{Re}\theta > 1-\beta)$; then $\mathcal{A}^{-\theta} \in L(E)$ for $\operatorname{Re}\theta > 1 - \beta$, and the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1+\theta_2)}$, $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1+\theta_2}$ of powers hold for $\operatorname{Re}\theta_1, \operatorname{Re}\theta_2 > 1-\beta$. The case $\beta \in (0,1)$ occurs in many applications and then we cannot define satisfactorily the power $\mathcal{A}^{\theta}$ for $|\operatorname{Re}\theta| \leqslant 1-\beta$. As explained in the introductory part of paper [**184**] by A. Favaron and A. Favini, the method of closed extensions used in the pioneering works [**44**] by A. Balakrishnan and [**290**] by H. Komatsu cannot be used here for construction of power $\mathcal{A}^{\theta}$ $(\operatorname{Re}\theta \in (0, 1-\beta))$. In this place, we would like to observe that the method proposed by F. Periago, B. Straub [**447**] and C. Martinez, M. Sanz, A. Redondo [**413**] (cf. also [**101**]) cannot be of any help for construction of power $\mathcal{A}^{\theta}$ $(\operatorname{Re}\theta \in (0, 1-\beta))$, as well. In [**184**, Section 9], the fractional power $\mathcal{A}^{\theta}$ has been constructed for $|\operatorname{Re}\theta| \leqslant 1-\beta$, provided the validity of condition [**184**, (H3)]. In general case $\beta \in (0,1)$, the condition (H3) does not hold.

Assume now that $\alpha \geqslant -1$ and a closed multivalued linear operator $\mathcal{A}$ satisfies:

($\Diamond$) $(0,\infty) \subseteq \rho(\mathcal{A})$ and

($\Diamond\Diamond$) $\sup_{\lambda>0}(1+|\lambda|)^{-\alpha}||R(\lambda : \mathcal{A})|| < \infty$.

Given $\beta \geqslant -1$, $\varepsilon \in (0,1]$, $d \in (0,1]$, $c' \in (0,1)$ and $\theta \in (0,\pi]$, put $B_d := \{z \in \mathbb{C} : |z| \leqslant d\}$, $\Sigma_{\theta} := \{z \in \mathbb{C} : z \neq 0,\ \arg(z) \in (-\theta,\theta)\}$ and $P_{\beta,\varepsilon,c'} := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta \in \mathbb{R},\ |\eta| \leqslant c'(1+\xi)^{-\beta}\}$. Then is checked at once that the hypotheses ($\Diamond$)–($\Diamond\Diamond$) imply the existence of numbers $d \in (0,1]$, $c \in (0,1)$, $\varepsilon \in (0,1]$ and $M > 0$ such that:

(§) $P_{\alpha,\varepsilon,c} \cup B_d \subseteq \rho(\mathcal{A})$, $(\varepsilon, c(1+\varepsilon)^{-\alpha}) \in \partial B_d$ and

(§§) $\|R(\lambda : \mathcal{A})\| \leqslant M(1+|\lambda|)^{\alpha}$, $\lambda \in P_{\alpha,\varepsilon,c} \cup B_d$.

Suppose now that X is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. Keeping in mind the above analysis (the notion will be explained later), it seems reasonable to introduce the following condition:

$(\mathrm{H})_0$: Let $C \in L(X)$ be not necessarily injective, let $\mathcal{A}$ be closed, and let $C\mathcal{A} \subseteq \mathcal{A}C$. There exist real numbers $d \in (0,1]$, $c \in (0,1)$, $\varepsilon \in (0,1]$ and $\alpha \geqslant -1$ such that $P_{\alpha,\varepsilon,c} \cup B_d \subseteq \rho_C(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-\alpha}(\lambda - \mathcal{A})^{-1}C : \lambda \in P_{\alpha,\varepsilon,c} \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is strongly analytic on $\operatorname{int}(P_{\alpha,\varepsilon,c} \cup B_d)$ and strongly continuous on $\partial(P_{\alpha,\varepsilon,c} \cup B_d)$.

The first aim of Subsection 3.9.1 is to construct the complex power $(-\mathcal{A})_b$, $b \in \mathbb{C}$ of a multivalued linear operator $\mathcal{A}$ satisfying the condition $(\mathrm{H})_0$. Although very elegant and elementary, our construction has some serious disadvantages because the introduced powers behave very badly (for example, we cannot expect the additivity property of powers clarified in [**103**, Remark 2.11]) in the case that the regularizing operator C_1, defined in this subsection, is not injective (since the resolvents and C-resolvents of a really multivalued linear operator are not injective, this is the main case in our examinations). The method proposed for construction of power $(-\mathcal{A})_b$ is different from that already employed in the single-valued linear case [**334**]; in Subsection 3.9.1, we first apply regularization with the operator C_1 and follow after that the approach from our joint research paper with C. Chen, M. Li and M. Žigić [**103**]. In particular, we define any complex power of a multivalued linear operator satisfying (13) and not the above-mentioned condition (H3).

The following sectorial analogue of (H) is most important in applications:

$(\mathrm{HS})_0$: Let $C \in L(X)$ be not necessarily injective, let $\mathcal{A}$ be closed, and let $C\mathcal{A} \subseteq \mathcal{A}C$. There exist real numbers $d \in (0,1]$, $\vartheta \in (0,\pi/2)$ and $\alpha \geqslant -1$ such that $\Sigma_\vartheta \cup B_d \subseteq \rho_C(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-\alpha}(\lambda-\mathcal{A})^{-1}C : \lambda \in \Sigma_\vartheta \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$ is strongly analytic on $\mathrm{int}(\Sigma_\vartheta \cup B_d)$ and strongly continuous on $\partial(\Sigma_\vartheta \cup B_d)$.

The construction of power $(-\mathcal{A})_b$, $b \in \mathbb{C}$ of a multivalued linear, non-sectorial, operator $\mathcal{A}$ for which $0 \notin \mathrm{int}(\rho_C(\mathcal{A}))$ is not trivial and we will not discuss this theme here. For some other approaches concerning the construction of fractional powers, the reader may consult [**101**, **138**] and [**502**].

In Section 3.9, we will also see that a great number of resolvent equations and generalized resolvent equations clarified in Section 1.2 holds for C-resolvents of multivalued linear operators, where C is non-injective, in general. The third and, probably, the main aim of Section 3.9 is to continue our researches of abstract incomplete fractional degenerate differential equations with modified Liouville right-sided fractional derivatives [**292**] and abstract incomplete degenerate differential equations of second order (cf. Section 2.7). We investigate fractionally integrated C_1-regularized semigroups generated by the negatives of introduced powers, and provide a few relevant applications of our theoretical results to abstract incomplete degenerate PDEs.

Chronologically speaking, R. deLaubenfels proved in 1988 that any injective infinitesimal generator A of a bounded analytic C_0-semigroup in Banach space E has the property that the inverse operator A^{-1} also generates a bounded analytic C_0-semigroup of the same angle [**133**]. In this paper, the author asked whether any injective infinitesimal generator A of a bounded C_0-semigroup in E has the property that the inverse operator A^{-1} generates a C_0-semigroup. As we know today, the answer is negative in general: a simple counterexample is given by H. Komatsu already in 1966, who constructed an injective infinitesimal generator of a contraction semigroup on the Banach space c_0, for which the inverse operator is not an infinitesimal generator of a C_0-semigroup [**233**, **290**]. Concerning the inverse generator problem, it should be noted that A. Gomilko, H. Zwart and Y. Tomilov proved in 2007 that the answer to R. deLaubenfels's question is negative in the state space l^p, where $1 < p < \infty$ and $p \neq 2$ (see [**233**]), as well as that S. Fackler proved in 2016 that the answer to this question is negative in the state space $L^p(\mathbb{R})$, where $1 < p < \infty$ and $p \neq 2$ (see [**172**]). We do not yet know whether there exists an injective infinitesimal generator A of a bounded C_0-semigroup in a Hilbert space H such that A^{-1} does not generate a C_0-semigroup in H (it is well known that for any injective infinitesimal generator A of a contraction C_0-semigroup in H, the inverse operator A^{-1} likewise generates a contraction C_0-semigroup in H by the Lumer-Phillips theorem; see the paper [**394**] by R. Liu for the fractional analogue of this result). For further information about the inverse generator problem, we refer the reader to the papers [**135**] by R. deLaubenfels, [**162**] by T. Eisner, H. Zwart, [**563, 564**] by H. Zwart and the recent survey [**232**] by A. Gomilko.

We would like to note that the complexity of inverse generator problem lies also in the fact that the use of real or complex representation theorems for the Laplace transform does not take a satisfactory effect. To explain this in more detail, assume that A is injective and generates a bounded C_0-semigroup in the Banach space E equipped with the norm $\|\cdot\|$. Then a simple calculation involving the Hille–Yosida theorem yields that, for every $\lambda > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{d^n}{d\lambda^n}[(\lambda - A^{-1})^{-1}] &= (-1)^n n![\lambda^{-1} - \lambda^{-2}(\lambda^{-1} - A)^{-1}]^{n+1} \\ &= (-1)^n n! \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \lambda^{-(n+1+k)} (\lambda^{-1} - A)^{-k},\end{aligned}$$

so that

$$\left\| \frac{d^n}{d\lambda^n}[(\lambda - A^{-1})^{-1}] \right\| \leqslant \frac{Mn!}{\lambda^{n+1}} \cdot 2^{n+1}.$$

Since the multiplication with number 2^{n+1} has appeared above, this estimate is completely useless if one wants to prove that the operator A^{-1} generates an exponentially bounded r-times integrated semigroup for some real number $r \geqslant 0$ (see also [**232**, Proposition 3.1, Theorem 3.2, Theorem 3.4]). On the other hand, a simple computation shows that the resolvent of A^{-1} is bounded in norm by Const $\cdot|\lambda|$ on any right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > a\}$, where $a > 0$, so that the complex characterization theorem for the Laplace transform immediately yields that the operator A^{-1} generates an exponentially bounded r-times integrated semigroup for any real number $r > 2$. In 2007, S. Piskarev and H. Zwart proved that the operator A^{-1} generates an exponentially bounded once integrated semigroup [**449**], while M. Li, J. Pastor and S. Piskarev improved this result in 2018 by showing that the operator A^{-1} generates a tempered r-times integrated semigroup for any real number $r > 1/2$. Moreover, they formulated a corresponding result for tempered fractional resolvent operator families of order $\alpha \in (0, 2]$; see [**383**] for more details.

Up to now, we do not have any relevant reference which treats the inverse generator problem for abstract degenerate Volterra integro-differential equations, even for abstract degenerate differential equations of first order. In contrast to non-degenerate differential equations, we have found the inverse generator problem much more important from the point of view of possible applications. The main aim of Section 3.10 is to analyze the inverse generator problem for the classes of mild (a, k)-regularized (C_1, C_2)-existence and uniqueness families, where the operator C_2 is not necessarily injective, and (a, k)-regularized C-resolvent families, where the operator C is not necessarily injective [**343**]; see, especially, Subsection 3.10.1, where we investigate degenerate time-fractional equations with abstract differential operators. We reconsider some results from **[134]** and **[383]** for abstract degenerate Volterra integro-differential equations in Theorem 3.10.9 and Theorem 3.10.11. We also investigate the situation in which we do not assume the existence of C-resolvent of a corresponding multivalued linear operator $\mathcal{A}$ on some right half plane. Moreover, we observe that the existence of C-resolvent set of $\mathcal{A}$ at infinity does not play any role for the generation of certain classes of (a,k)-regularized C-resolvent operator families, C-(ultra)distribution semigroups and C-(ultra)distribution cosine functions by the inverse of a closed multivalued linear operator $\mathcal{A}$; sometimes it is crucial to investigate the behaviour of C-resolvent set of $\mathcal{A}$ around zero, only (see e.g. Proposition 3.10.5 and Example 3.10.8).

The theory of almost periodic and almost automorphic type functions is unavoidable nowdays. The existing literature on various types of almost periodic (automorphic) properties and asymptotically almost periodic (automorphic) properties of abstract non-degenerate Volterra integro-differential equations in Banach spaces is enormous. For a basic source of information in this direction, we refer the reader to **[20, 87, 93, 119, 256, 274, 434, 435, 478]** and **[501]**. Concerning degenerate case, mention should be made of papers by V. Barbu-A. Favini **[49]**, A. Favini-G. Marinoschi **[191]**, S. Q. Bu **[91, 92]**, N. T. Lan **[371]**, C. Lizama-R. Ponce **[396, 397]** and R. Ponce **[455]**.

The most relevant details of research studies **[49, 191]** and **[396]** are described as follows. In **[49]**, V. Barbu and A. Favini have considered 1-periodic solutions of abstract degenerate differential equation $(d/dt)(Bu(t)) = Au(t)$, $t \geqslant 0$, subjected with inital condition $(Bu)(0) = (Bu)(1)$, by using P. Grisvard's sum of operators method and some results from investigation of J. Prüss **[461]** in non-degenerate case (here, A and B are closed linear operators acting on a Banach space X). The authors have used the multivalued linear approach to degenerate differential equations (cf. the next chapter for further information) and reduce the above problem to $v'(t) \in \mathcal{A}v(t)$, $t \geqslant 0$, $v(0) = v(1)$, where the multivalued linear operator $\mathcal{A}$ is defined by $\mathcal{A} = AB^{-1}$. The main problem in the whole analysis is the question of whether the inclusion $1 \in \rho(\mathcal{A})$ holds or not; in connection with this, we would like to remind ourselves **[461]** that $1 \in \rho(A)$ iff $2\pi i\mathbb{Z} \subseteq \rho(A)$ and $\sup(\{\|(2\pi in - A)^{-1}\| : n \in \mathbb{Z}\}) < \infty$, provided that A generates a non-degenerate strongly continuous semigroup. Applications are given to the Poisson heat equation in $H^{-1}(\Omega)$ and $L^2(\Omega)$, as well as to some systems of ordinary differential equations. In **[191]**, A. Favini and G. Marinoschi have continued the analysis from **[49]** by assuming the possible non-linearity of (multi-valued) operator A. C. Lizama and R. Ponce **[396]** have applied the Fourier multipliers techniques for establishment the necessary and sufficient conditions for the existence of 2π-periodic solutions to the following abstract inhomogeneous linear equation

$$\frac{d}{dt}(Bu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t \geqslant 0, \tag{14}$$

subjected with the initial condition $(Bu)(0) = (Bu)(2\pi)$. Here, $f\colon [0,\infty) \to X$ is a 2π-periodic function and $a \in L^1([0,\infty))$ is a scalar-valued kernel on $[0,\infty)$.

The authors also showed some results on the maximal regularity of (14) in periodic Besov, Triebel–Lizorkin and Lebesgues vector-valued function spaces.

For a basic source of information about almost automorphic solutions of abstract non-degenerate integro-differential equations, the reader may consult the monograph [**293**], as well as the papers [**7, 9, 10, 124, 125, 146, 147**] and [**239**]. In [**293**], I have investigated the almost periodic and almost automorphic type solutions of various classes of abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations in Banach spaces, as well. This is probably the first research monograph regarding this problematic. After the final release of the monograph [**293**], I have written several new research papers in this field. Two of them, [**148**] and [**149**], have been completed in co-autorship with Professor T. Diagana. The material of [**148**] is presented in Section 3.12, while the material of [**149**] will appear as a chapter in the forthcoming edited book "Recent Studies in Differential Equations" by Nova Science Publishers, Inc. (New York). It is worth noting here that the concept of almost periodicity (respectively, almost automorphy, pseudo-almost periodicity, and pseudo-almost automorphy) in the Lebesgue space with variable exponent $L^{p(x)}(I,X)$ was first introduced and studied by T. Diagana and M. Zitane [**150, 151**]. However, the translation-invariance of these newly introduced spaces depends heavily upon the function $p \in C([0,\infty))$. To remove such a restriction, in Section 3.12 we introduce some new concepts so that the obtained almost periodic (respectively, asymptotically almost periodic) in $L^{p(x)}(I,X)$ are automatically translation-invariant. Among other things, it will be shown that these new functions generalize in a natural fashion the classical notion of almost periodicity (respectively, asymptotic almost periodicity). Many properties of the new functions are analyzed including their compositions. Further, we will make extensive use of these new functions to study some abstract Volterra integro-differential equations in Banach spaces including multi-valued ones.

In Section 3.11, we present the material from my recent paper [**344**] concerning quasi-asymptotically almost periodic functions in Banach spaces and related applications. The organization and main ideas of this section are given as follows. The concept introduced by H. Weyl [**534**] suggests a very general way of approaching almost periodicity. To the best knowledge of the author, the question whether the class of asymptotically Stepanov p-almost periodic functions, introduced by H. R. Henríquez [**249**], is contained in the class of Weyl-p-almost periodic functions taken without any ergodic components, has not been examined elsewhere by now. In this section, we introduce the class of Stepanov p-quasi-asymptotically almost periodic functions and prove later that this class contains all asymptotically Stepanov p-almost periodic functions and make a subclass of the class consisting of all Weyl p-almost periodic functions (taken in the sense of A. S. Kovanko's approach [**356**], which is also followed in the definition of a quasi-asymptotically almost periodic function [**336**]). In such a way, we initiate the study of generalized (asymptotical) almost periodicity that intermediate Stepanov and Weyl concept. Further on, in Subsection 3.11.1 and Subsection 3.11.2, we recall the basic definitions and results about asymptotically almost periodic type functions, asymptotically almost automorphic type functions and evolution systems, Green's functions, respectively. In

Definition 3.11.9, we recall the notion of a quasi-asymptotically almost periodic (q-aap., for short) function, defined on the interval I, where $I = \mathbb{R}$ or $I = [0, \infty)$. After providing some observations and illustrative examples, in Theorem 3.11.13 we prove that any asymptotically almost automorphic (aaa.) function which is also q-aap. needs to be asymptotically almost periodic (aap.). We present a simple example of a q-aap. function that is uniformly continuous and whose range is not relatively compact in X (this is a simple modification of [**251**, Example 3.1]); we also show that there exists a q-aap. function that is uniformly continuous and not aap.. The notion of a Stepanov p-q-aap. function is introduced in Definition 3.11.17, while an analogue of Theorem 3.11.18 for Stepanov class has been proved in Theorem 3.11.18. The (Stepanov) class of S-asymptotically ω-periodic functions, introduced by H. R. Henríquez, M. Pierri and P. Táboas in [**251**], is a subclass of the class consisting of the (Stepanov) class of q-aap. functions (see Proposition 3.11.15). The main structural properties of (Stepanov) q-aap. functions are proved in Theorem 3.11.21 and Proposition 3.11.23. In Example 3.11.24 and Example 3.11.25, we verify that the class of (Stepanov) q-aap. functions is not closed under pointwise products with bounded scalar-valued (Stepanov) q-aap. functions, while in Example 3.11.26 we show that (Stepanov) q-aap. functions do not form vector spaces equipped with the usual operations of addition and multiplication with scalars, unfortunately. Subsection 3.11.4 is devoted to the analysis of (Stepanov) q-aap. functions depending on two parameters and related composition principles. In Theorem 3.11.28 and Theorem 3.11.29, we analyze the composition principles for q-aap. functions depending on two parameters following the approach presented in the monograph of T. Diagana [**144**] for aap. functions. The main objective in Theorem 3.11.31 and Theorem 3.11.32 is to prove corresponding results for Stepanov q-aap. functions; at these places, we follow the approach of W. Long and S.-H. Ding from [**399**].

Concerning applications, our main results are given in Subsection 3.11.5, where we analyze the invariance of quasi-asymptotical almost periodicity under the action of convolution products. We do not require the non-degeneracy of solution operator families in Subsection 3.11.5, so that the results established in this subsection can be simply incorporated in the analysis of certain classes of abstract degenerate inhomogeneous fractional inclusions and abstract degenerate inhomogeneous Volterra integro-differential inclusions in Banach spaces. With a view to motivate our researchers for the analyses of evolution systems and abstract quasi-linear differential equations of first order, in Subection 3.11.6 we investigate the existence and uniqueness of q-aap. solutions of abstract (semilinear) nonautonomous differential equations of first order. In Subection 3.11.6, we analyze the abstract nonautonomous differential equations

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R},$$
$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x$$

and their semilinear analogues; here, the operator family $A(\cdot)$ is consisted of closed linear operators with domain and range contained in X, the condition (H1) clarified below holds and the evolution system $U(\cdot, \cdot)$ generated by $A(\cdot)$ is hyperbolic, i.e, the condition (H2) clarified below holds. In Theorem 3.11.37 (Theorem 3.11.39), the inhomogenity $f(\cdot)$ is Stepanov p-q-aap. and the associated Green's function satisfies the condition (451) ((453)). In contrast to this, in our investigations of semilinear analogues of the abstract Cauchy problems (449) and (450) carried out in Subsection 3.11.7, we assume that the corresponding function $F(\cdot,\cdot)$ is q-aap.. This is essentially caused by the fact that the Stepanov q-q-aap. of function $F(\cdot, x(\cdot))$, established in composition principles Theorem 3.11.31 and Theorem 3.11.32, holds only if we additionally assume that the range of function $x(\cdot)$ is relatively compact, which need not be true for q-aap. functions and their Stepanov generalizations. Further study of semilinear nonautonomous differential equations with forcing term $F(\cdot,\cdot)$ belonging to Stepanov class of q-aap. functions is without scope of this book. Finally, in Example 3.11.44, we provide an instructive example of applications of our abstract theoretical results obtained, continuing thus the analyses raised by T. Diagana in [**145**, Section 4] and the author [**338**, Example 3.1].

Some notes and appendices to the third chapter are given in Section 3.13.

Some failures of the monograph are described as follows. Subordination principles are very actual and important theme in the theory of abstract Volterra integro-differential equations, degenerate or non-degenerate in time, and we will be imperatively forced to reconsider the statements like Theorem 2.2.6 and Theorem 2.2.13 several times throughout the book; some kind of duplicating, which we have tried to reduce to the minimum level, can be also easily recognized while reading the parts about differential and analytical properties of various types of degenerate operator families of solving operators, analytical properties of C-resolvents of multivalued linear operators, and while introducing the fundamental definitions of hypercyclic and topologically mixing properties of abstract degenerate equations under our consideration. In order to ensure better readability, we have decided to repeat some of the equations mentioned in the introductory part once more, but with different labels.

For the sake of brevity and better exposition, and because of some similarity with our previous researches of non-degenerate case, we have been forced to write the third chapter of monograph in a half-expository manner, including only the most relevant details of proofs of our structural results.

CHAPTER 1

PRELIMINARIES

1.1. Selected topics on vector-valued functions and closed linear operators

Vector-valued functions, closed operators. Unless specified otherwise, by E we denote a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. The abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E; if E is a Banach space and A is linear operator on E, then the norm of an element $x \in E$ is denoted by $\|x\|$. Assuming that X is another SCLCS, then by $L(E,X)$ we denote the space consisting of all continuous linear mappings from E into X; $L(E) \equiv L(E,E)$. Let $\mathcal{B}$ be the family of bounded subsets of E, let $\circledast_X$ denote the fundamental system of seminorms which defines the topology of X, and let $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast_X$, $B \in \mathcal{B}$, $T \in L(E,X)$. Then $p_B(\cdot)$ is a seminorm on $L(E,X)$ and the system $(p_B)_{(p,B)\in \circledast_X \times \mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E,X)$. Consequently, the Hausdorff locally convex topology on E^*, the dual space of E, defines the system $(|\cdot|_B)_{B\in\mathcal{B}}$ of seminorms on E^*, where $|x^*|_B := \sup_{x\in B} |\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. Here $\langle\, ,\rangle$ denotes the duality bracket between E and E^*, sometimes we shall also write $\langle x, x^*\rangle$ or $x^*(x)$ to denote the value of $\langle x^*, x\rangle$. It is well known that the spaces $L(E,X)$ and E^* are sequentially complete provided that E is barreled [**419**]. By E^{**} we denote the bidual of E. If X and Y are two SCLCSs such that Y is continuously embedded in X, then we write $Y \hookrightarrow X$. A linear operator $A\colon D(A) \to E$ is said to be closed if the graph of the operator A, defined by $G_A := \{(x, Ax) : x \in D(A)\}$, is a closed subset of $E \times E$. We identify A with its graph if there is no risk for confusion. The resolvent set, spectrum and range of A are denoted by $\rho(A)$, $\sigma(A)$ and $R(A)$, respectively. The null space of A is denoted by either $N(A)$ or $\mathrm{Kern}(A)$. As is well known, a necessary and sufficient condition for a linear operator $A\colon D(A) \to E$ to be closed is that, for every net $(x_\tau)_{\tau\in I}$ in $D(A)$ such that $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} Ax_\tau = y$, the following holds: $x \in D(A)$ and $Ax = y$ [**419**]. A linear operator A is said to be closable iff there exists a closed linear operator B such that $A \subseteq B$. Let us recall that the closability of the operator A is equivalent to say that, for every net $(x_\tau)_{\tau\in I}$ in $D(A)$ such that $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} Ax_\tau = 0$, we have $y = 0$ [**419**]. We assume the closedness of A in the sequel of this section, if not stated otherwise. We introduce the Hausdorff sequentially complete locally convex topology on $D(A)$ ($\overline{D(A)}$) by the following system of seminorms: $p_A(x) =: p(x) + p(Ax)$, $x \in D(A)$,

$p \in \circledast$ $((p_{|\overline{D(A)}})_{p\in\circledast})$. We denote the first of above spaces simply by $[D(A)]$ (if E is a Banach space, then $\|\cdot\|_{[D(A)]} := \|\cdot\| + \|A\cdot\|$ is known in the literature as the graph norm); a subspace $Y \subseteq D(A)$ is called a core for A iff Y is dense in $D(A)$ with respect to the graph norm. For any injective operator $C \in L(E)$, we define the C-resolvent set of A, $\rho_C(A)$ for short, by

$$\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}.$$

By the closed graph theorem [**419**], the following holds: If E is a webbed bornological space (this, in particular, holds if E is a Fréchet space), then any mapping $A\colon E \to E$ whose graph $G(A)$ is closed in $E \times E$ must be continuous on E, and the C-resolvent set of A consists of those complex numbers λ for which the operator $\lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)$. If F is a linear submanifold of E, then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Suppose $A\colon D(A) \to E$ is a linear operator. The power A^n of A is defined usually ($n \in \mathbb{N}_0$). Put $D_\infty(A) := \bigcap_{n\geqslant 1} D(A^n)$. For a closed linear operator A acting on E, we introduce the subset A^* of $E^* \times E^*$ by

$$A^* := \{(x^*, y^*) \in E^* \times E^* : x^*(Ax) = y^*(x) \text{ for all } x \in D(A)\}.$$

If A is densely defined, then A^* is also known as the adjoint operator of A and it is a closed linear operator on E^*. If $\alpha \in \mathbb{C} \smallsetminus \{0\}$, A and B are linear operators, then we define the operators αA, $A + B$ and AB in the usual way. A family Λ of continuous linear operators on E is said to be equicontinuous if for each $p \in \circledast$ there exist $c_p > 0$ and $q_p \in \circledast$ such that

$$p(Ax) \leqslant c_p q_p(x), \quad x \in E,\ A \in \Lambda.$$

The Gamma function will be denoted by $\Gamma(\cdot)$ and the principal branch will be always used to take the powers. Set, for every $\alpha > 0$, $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, $g_0(t) \equiv$ the Dirac delta distribution and, by common consent, $0^\zeta := 0$. The n-th convolution power of a locally integrable function $a(t) \in L^1_{loc}([0,\infty))$ is denoted by $a^{*,n}(t)$. Then $a^{*,n} \in L^1_{loc}([0,\infty))$. Given two numbers $s \in \mathbb{R}$ and $n \in \mathbb{N}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geqslant l\}$, $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leqslant l\}$, $\mathbb{N}_n := \{1, \ldots, n\}$ and $\mathbb{N}^0_n := \{0, 1, \ldots, n\}$.

Function spaces. The Schwartz spaces of test functions $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ are equipped with the usual inductive limit topologies; the topology of the space of rapidly decreasing functions $\mathcal{S}$ defines the following system of seminorms

$$p_{m,n}(\psi) := \sup_{x\in\mathbb{R}} |x^m \psi^{(n)}(x)|, \quad \psi \in \mathcal{S},\ m, n \in \mathbb{N}_0.$$

If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then we denote by $\mathcal{D}_\Omega$ the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\mathrm{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$. If φ, $\psi\colon \mathbb{R} \to \mathbb{C}$ are measurable functions, then we define the convolution products $\varphi * \psi$ and $\varphi *_0 \psi$ by

$$\varphi * \psi(t) := \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,ds \text{ and } \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\,ds, \quad t \in \mathbb{R}.$$

Sometimes we will use the symbol $*$ to denote the finite convolution product, if no confusion seems likely. Notice that $\varphi * \psi = \varphi *_0 \psi$, provided that $\operatorname{supp}(\varphi)$ and $\operatorname{supp}(\psi)$ are subsets of $[0,\infty)$. Given $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$, or $\varphi \in \mathcal{E}$ and $f \in \mathcal{E}'$, we define the convolution $f * \varphi$ by $(f * \varphi)(t) := f(\varphi(t - \cdot))$, $t \in \mathbb{R}$. For $f \in \mathcal{D}'$, or for $f \in \mathcal{E}'$, define $\check{f}$ by $\check{f}(\varphi) := f(\varphi(-\cdot))$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{E}$). Generally, the convolution of two distributions f, $g \in \mathcal{D}'$, denoted by $f * g$, is defined by $(f * g)(\varphi) := g(\check{f} * \varphi)$, $\varphi \in \mathcal{D}$. Then we know that $f * g \in \mathcal{D}'$ and $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.

The spaces $\mathcal{D}'(E) := L(\mathcal{D}, E)$, $\mathcal{E}'(E) := L(\mathcal{E}, E)$ and $\mathcal{S}'(E) := L(\mathcal{S}, E)$ are consisted of all continuous linear functions $\mathcal{D} \to E$, $\mathcal{E} \to E$ and $\mathcal{S} \to E$, respectively; $\mathcal{D}'_\Omega(E)$, $\mathcal{E}'_\Omega(E)$ and $\mathcal{S}'_\Omega(E)$ denote the subspaces of $\mathcal{D}'(E)$, $\mathcal{E}'(E)$ and $\mathcal{S}'(E)$, respectively, containing E-valued distributions whose supports are contained in Ω; $\mathcal{D}'_0(E) \equiv \mathcal{D}'_{[0,\infty)}(E)$, $\mathcal{E}'_0(E) \equiv \mathcal{E}'_{[0,\infty)}(E)$, $\mathcal{S}'_0(E) \equiv \mathcal{S}'_{[0,\infty)}(E)$. In the case that $E = \mathbb{C}$, then the above spaces are also denoted by $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$, $\mathcal{D}'_\Omega$, $\mathcal{E}'_\Omega$, $\mathcal{S}'_\Omega$, $\mathcal{D}'_0$, $\mathcal{E}'_0$ and $\mathcal{S}'_0$. Let G be an E-valued distribution, and let $f \colon \mathbb{R} \to E$ be a locally integrable function (cf. [**292**, Definition 1.1.4, Definition 1.1.5]). As in the scalar-valued case, we define the E-valued distributions $G^{(n)}$ ($n \in \mathbb{N}$) and hG ($h \in \mathcal{E}$); the regular E-valued distribution $\mathbf{f}$ is defined by $\mathbf{f}(\varphi) := \int_{-\infty}^{\infty} \varphi(t) f(t)\,dt$ ($\varphi \in \mathcal{D}$).

Suppose that $0 < \tau \leqslant \infty$, $n \in \mathbb{N}$. If $f \colon (0,\tau) \to E$ is a continuous function and

$$\int_0^\tau \varphi^{(n)}(t) f(t)\,dt = 0, \quad \varphi \in \mathcal{D}_{(0,\tau)},$$

then we know that there exist elements $x_0, \ldots, x_{n-1}$ in E such that $f(t) = \sum_{j=0}^{n-1} t^j x_j$, $t \in (0,\tau)$. Let $\tau > 0$, and let X be a general Hausdorff locally convex space (not necessarily sequentially complete). Following L. Schwartz [**479**, **480**], it will be said that a distribution $G \in \mathcal{D}'(X)$ is of finite order on the interval $(-\tau,\tau)$ iff there exist an integer $n \in \mathbb{N}_0$ and an X-valued continuous function $f \colon [-\tau,\tau] \to X$ such that

$$G(\varphi) = (-1)^n \int_{-\tau}^{\tau} \varphi^{(n)}(t) f(t)\,dt, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)}.$$

In the case that X is a quasi-complete (DF)-space, then we know from [**479**, Corollarie 2, p. 90] that each X-valued distribution is of finite order on any finite interval $(-\tau,\tau)$. Furthermore, if $G(\varphi) = 0$ for all $\varphi \in \mathcal{D}_{(-\infty,\tau_1)}$, where $0 < \tau_1 < \tau$, then $f(t)$ can be chosen so that $f(t) = 0$ for $t < \tau_1$. As it is well known, the above holds not only in quasi-complete (DF)-spaces but also in Banach spaces.

If $(M_p)_{p \in \mathbb{N}_0}$ is a sequence of positive real numbers with $M_0 = 1$, then we use the following conditions from the theory of ultradistributions (Komatsu's approach):

- $(M.1)$: $M_p^2 \leqslant M_{p+1} M_{p-1}$, $p \in \mathbb{N}$,
- $(M.2)$: $M_p \leqslant AH^p \min_{p_1,p_2 \in \mathbb{N}, p_1+p_2=p} M_{p_1} M_{p_2}$, $n \in \mathbb{N}$, for some $A > 1$ and $H > 1$,
- $(M.3)'$: $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$, and
- $(M.3)$: $\sup_{p \in \mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1} M_{p+1}}{p M_p M_q} < \infty$.

Recall that the condition $(M.3)$ is slightly stronger than $(M.3)'$ and that, for every $s > 1$, the Gevrey sequence $(p!^s)$ satisfies $(M.1)$–$(M.3)$.

We assume a priori that (M_p) satisfies only the first of these conditions, $(M.1)$; any employment of conditions $(M.2)$, $(M.3)'$ or $(M.3)$ will be explicitely emphasized. The associated function of sequence (M_p) is defined by $M(\rho) := \sup_{p\in\mathbb{N}} \ln \frac{\rho^p}{M_p}$, $\rho > 0$; $M(0) := 0$, $M(\lambda) := M(|\lambda|)$, $\lambda \in \mathbb{C} \smallsetminus [0,\infty)$. Set $m_p := M_p/M_{p-1}$, $p \in \mathbb{N}$. Then the condition $(M.1)$ implies that the sequence (m_p) is increasing. It is worth noting that the function $t \mapsto M(t)$, $t \geqslant 0$ is increasing as well as that $\lim_{\lambda\to\infty} M(\lambda) = \infty$ and the function $M(\cdot)$ vanishes in some open neighborhood of zero. Denote by $m(\lambda)$ the number of $m_p \leqslant \lambda$. Since (M_p) satisfies $(M.1)$, it follows that (cf. [**286**, p. 50]) $M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda$, $t \geqslant 0$. Hence, the mapping $t \mapsto M(t)$, $t \geqslant 0$ is absolutely continuous and the mapping $t \mapsto M(t)$, $t \in [0,\infty) \smallsetminus \{m_p : p \in \mathbb{N}\}$ is continuously differentiable with $M'(t) = \frac{m(t)}{t}$, $t \in [0,\infty) \smallsetminus \{m_p : p \in \mathbb{N}\}$.

Suppose now that (M_p) satisfies $(M.1)$, $(M.2)$ and $(M.3)'$. Let us recall that the spaces of Beurling, respectively, Roumieu ultradifferentiable functions are defined by $\mathcal{D}^{(M_p)} := \mathcal{D}^{(M_p)}(\mathbb{R}) := \operatorname{indlim}_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{(M_p)}$, respectively, $\mathcal{D}^{\{M_p\}} := \mathcal{D}^{\{M_p\}}(\mathbb{R}) := \operatorname{indlim}_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)} := \operatorname{projlim}_{h\to\infty} \mathcal{D}_K^{M_p,h}$, respectively, $\mathcal{D}_K^{\{M_p\}} := \operatorname{indlim}_{h\to 0} \mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \{\phi \in C^\infty(\mathbb{R}) : \operatorname{supp}(\phi) \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty\} \quad \text{and}$$
$$\|\phi\|_{M_p,h,K} := \sup\Big\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K,\ p \in \mathbb{N}_0\Big\}.$$

Henceforth the asterisk $*$ stands for the Beurling case (M_p) or for the Roumieu case $\{M_p\}$. Let $\emptyset \neq \Omega \subseteq \mathbb{R}$. The space of vector-valued ultradistributions of $*$-class $\mathcal{D}'^*(E) := L(\mathcal{D}^*, E)$ is consisted of all continuous linear mappings from $\mathcal{D}^*$ into E; $\mathcal{D}^*_\Omega$ denotes the subspace of $\mathcal{D}^*$ containing ultradifferentiable functions of $*$-class whose supports are compact subsets of Ω ($\mathcal{D}^*_0 \equiv \mathcal{D}^*_{[0,\infty)}$), while the symbol $\mathcal{E}'^*_\Omega$ denotes the space consisting of all scalar valued ultradistributions of $*$-class whose supports are compact subsets of Ω ($\mathcal{E}'^*_0 \equiv \mathcal{E}'^*_{[0,\infty)}$). Similarly we define the spaces $\mathcal{D}'^*_\Omega(E)$ and $\mathcal{D}'^*_0(E)$. An entire function of the form $P(\lambda) = \sum_{p=0}^\infty a_p\lambda^p$, $\lambda \in \mathbb{C}$ is of class (M_p), respectively, of class $\{M_p\}$, if there exist $l > 0$ and $C > 0$, respectively, for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leqslant Cl^p/M_p$, $p \in \mathbb{N}$; cf. [**286**] for further information. The corresponding ultradifferential operator $P(D) = \sum_{p=0}^\infty a_pD^p$ is of class (M_p), respectively, of class $\{M_p\}$. Since (M_p) satisfies $(M.2)$, the ultradifferential operator $P(D)$ of $*$-class

$$\langle P(D)G, \varphi\rangle := \langle G, P(-D)\varphi\rangle, \quad G \in \mathcal{D}'^*(E),\ \varphi \in \mathcal{D}^*,$$

is a continuous linear mapping from $\mathcal{D}'^*(E)$ into $\mathcal{D}'^*(E)$. The multiplication of a vector-valued ultradistribution G of $*$-class by a function $a \in \mathcal{E}^*(\Omega)$ is defined as in the scalar case. For every $t \in \mathbb{R}$, we define the scalar-valued (ultra-)distribution $\delta_t \in \mathcal{D}'$ ($\delta_t \in \mathcal{D}'^*$) by $\delta_t(\varphi) := \varphi(t)$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$). For more details about vector-valued (ultra-)distribution spaces used henceforth, see [**291**, Section 1.3], [**292**,**354**, Section 1.1] and the references cited therein.

A function $f\colon [0,T] \to X$, where $0 < T < \infty$, is said to be Hölder continuous with the exponent $r \in (0,1]$ iff for each $p \in \circledast_X$ there exists $M \geqslant 1$ such that $p(f(t) - f(s)) \leqslant M|t-s|^r$, provided $0 \leqslant t,\ s \leqslant T$, while a function $f\colon [0,\infty) \to X$ is said to be locally Hölder continuous with the exponent r iff its restriction on any finite interval $[0,T]$ is Hölder continuous with the same exponent. Define $C^r([0,T] : X)$ to be the vector space consisting of Hölder continuous functions $f\colon [0,T] \to X$ with the exponent r; if $r' \in (0,\infty) \smallsetminus \mathbb{N}$, then we define $C^{r'}([0,T] : X)$ as the vector space consisting of those functions $f\colon [0,T] \to X$ for which $f \in C^{\lfloor r' \rfloor}([0,T] : X)$ and $f^{(\lfloor r' \rfloor)} \in C^{r'-\lfloor r' \rfloor}([0,T] : X)$. Assume that X is a Banach space. Then the space of all X-valued functions that are absolutely continuous (of bounded variation) on any closed subinterval of $[0,\infty)$ will be denoted by $AC_{loc}([0,\infty) : X)$ ($BV_{loc}([0,\infty) : X)$). By $C^k(\Omega : E)$ we denote the space of k-times continuously differentiable functions ($k \in \mathbb{N}_0$)from a non-empty subset $\Omega \subseteq \mathbb{C}$ into a general sequentially complete locally convex space E, $C(\Omega : E) \equiv C^0(\Omega : E)$. If $X = \mathbb{C}$, then we also write $AC_{loc}([0,\infty))$ ($BV_{loc}([0,\infty))$) in place of $AC_{loc}([0,\infty) : X)$ ($BV_{loc}([0,\infty) : X)$); the spaces $BV[0,T]$, $BV_{loc}([0,\tau))$, $BV_{loc}([0,\tau) : X)$, as well as the space $L^p_{loc}(\Omega : X)$ for $1 \leqslant p \leqslant \infty$ are defined in a very similar way ($T, \tau > 0$); $L^p_{loc}(\Omega) \equiv L^p_{loc}(\Omega : \mathbb{C})$. Let $k \in \mathbb{N}$, let $p \in [1,\infty]$, and let Ω be an open non-empty subset of $\mathbb{R}^n$. Then the Sobolev space $W^{k,p}(\Omega : X)$, sometimes also denoted by $H^{k,p}(\Omega : X)$, consists of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every $i \in \mathbb{N}^0_k$ and for every multi-index $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leqslant k$, we have $D^\alpha u \in L^p(\Omega, X)$. It is needless to say that the derivative D^α is taken in the sense of distributions. The subspace of $\mathcal{D}'(\Omega : X)$ consisting of all X-valued distributions of the form

$$u = \sum_{|\alpha| \leqslant k} u^{(\alpha)}_\alpha, \tag{15}$$

where $u_\alpha \in L^p(\Omega : X)$, is denoted by $W^{-k,p}(\Omega : X)$ ($H^{-k,p}(\Omega : X)$). In this space, we introduce the norm

$$\|u\|_{-k,p,X} := \inf \Biggl\{ \Biggl(\sum_{|\alpha| \leqslant k} \|u_\alpha\|^p_{L^p(\Omega,X)} \Biggr)^{1/p} \Biggr\},$$

where the infimum is taken over all representations of distibution u of form (15); $W^{-k}(\Omega : X) \equiv W^{-k,2}(\Omega : X)$ ($H^{-k}(\Omega : X) \equiv H^{-k,2}(\Omega : X)$). It is worth noting that $W^{-k,p}(\Omega : X)$ is a Banach space and $W^{-k}(\Omega : X)$ is a Hilbert space. By $W^{k,p}_{loc}(\Omega : X)$ ($H^{k,p}_{loc}(\Omega : X)$) we denote the space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in W^{k,p}(\Omega' : X)$.

Integration of functions with values in locally convex spaces. By Ω we denote a locally compact and separable metric space, and by μ we denote a locally finite Borel measure defined on Ω.

Definition 1.1.1. (i) It is said that a function $f\colon \Omega \to E$ is simple iff there exist $k \in \mathbb{N}$, elements $z_i \in E$, $1 \leqslant i \leqslant k$ and pairwise disjoint Borel measurable subsets Ω_k, $1 \leqslant i \leqslant k$ of Ω, such that $\mu(\Omega_i) < \infty$, $1 \leqslant i \leqslant k$ and

$$f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t), \quad t \in \Omega. \tag{16}$$

(ii) It is said that a function $f\colon \Omega \to E$ is (strongly) μ-measurable, (strongly) measurable for short, iff there exists a sequence (f_n) in E^Ω such that, for every $n \in \mathbb{N}$, $f_n(\cdot)$ is a simple function and $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in \Omega$.

(iii) A function $f\colon \Omega \to E$ is said to be weakly μ-measurable, weakly measurable for short, iff for every $x^* \in E^*$, the function $t \mapsto x^*(f(t))$, $t \in \Omega$ is measurable.

(iv) A function $f\colon \Omega \to E$ is said to be μ-measurable by seminorms, measurable by seminorms for short, iff for every $p \in ⊛$ there exists a sequence (f_n^p) in E^Ω such that $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ a.e. $t \in \Omega$.

It is clear that every strongly measurable function is also weakly measurable and that the converse statement is not true in general. We define the Bochner integral of a simple function $f\colon \Omega \to E$, given by (16), as follows $\int_\Omega f\,d\mu := \sum_{i=1}^k z_i\mu(\Omega_i)$. Let us observe that this definition does not depend on the representation (16).

Let $1 \leqslant p < \infty$, let $(X, \|\cdot\|)$ be a complex Banach space, and let $(\Omega, \mathcal{R}, \mu)$ be a measure space. Then the space $L^p(\Omega, X, \mu)$ consists of all strongly μ-measurable functions $f\colon \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p d\mu)^{1/p}$ is finite, we also use the abbreviation $L^p(\Omega, \mu)$ in the case that $X = \mathbb{C}$. The space $L^\infty(\Omega, X, \mu)$ consists of all strongly μ-measurable, essentially bounded functions and is equipped with the norm $\|f\|_\infty :=\text{ess}\sup_{t\in\Omega} \|f(t)\|$, $f \in L^\infty(\Omega, X, \mu)$. The functions that are equal μ-almost everywhere on Ω will be identified; furthermore, if μ is the Lebesgue measure on the real line, then, for every $p \in [1, \infty]$, the space $L^p(\Omega, X, \mu)$ will be also denoted by $L^p(\Omega : X)$. The Riesz–Fischer theorem states that $(L^p(\Omega, X, \mu), \|\cdot\|_p)$ is a Banach space for all $p \in [1, \infty]$; as is well known, $(L^2(\Omega, X, \mu), \|\cdot\|_2)$ is a Hilbert space. Assuming that the Banach space X is reflexive, the space $L^p(\Omega, X, \mu)$ is reflexive for all $p \in (1, \infty)$ and its dual is isometrically isomorphic to $L^{\frac{p}{p-1}}(\Omega, X, \mu)$.

Definition 1.1.2. (C. Martinez, M. Sanz [**410**, pp. 99–102]; cf. [**292**, Section 1.2] for more details)

(i) Let $K \subseteq \Omega$ be a compact set, and let a function $f\colon K \to E$ be strongly measurable. Then it is said that $f(\cdot)$ is (μ-)integrable iff there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. $t \in K$ and for all $\varepsilon > 0$ and each $p \in ⊛$ there is a number $n_0 = n_0(\varepsilon, p)$ such that

$$\int_K p(f_n - f_m)d\mu \leqslant \varepsilon \quad (m, n \geqslant n_0). \tag{17}$$

In this case, we define

$$\int_K f\,d\mu := \lim_{n\to\infty} \int_K f_n d\mu.$$

(ii) A function $f\colon \Omega \to E$ is said to be locally μ-integrable iff, for every compact set $K \subseteq \Omega$, the restriction $f_{|K}\colon K \to E$ is μ-integrable.

(iii) A function $f\colon \Omega \to E$ is said to be μ-integrable iff it is locally integrable and

$$\int_\Omega p(f)d\mu < \infty, \quad p \in ⊛. \tag{18}$$

If this is the case, we define

$$\int_\Omega f\, d\mu := \lim_{n\to\infty} \int_{K_n} f\, d\mu,$$

with $(K_n)_{n\in\mathbb{N}}$ being an expansive sequence of compact subsets of Ω with the property that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$.

The definition in (ii) makes sense and does not depend on the choice of a sequence $(K_n)_{n\in\mathbb{N}}$. Moreover,

$$p\Big(\int_\Omega f\, d\mu\Big) \leqslant \int_\Omega p(f) d\mu, \quad p \in \circledast \tag{19}$$

and the μ-integrability of a function $f\colon K \to X$, resp. $f\colon \Omega \to X$, implies that for each $x^* \in X^*$, one has:

$$\Big\langle x^*, \int_K f\, d\mu \Big\rangle = \int_K \langle x^*, f\rangle d\mu, \text{ resp. } \Big\langle x^*, \int_\Omega f\, d\mu \Big\rangle = \int_\Omega \langle x^*, f\rangle d\mu. \tag{20}$$

Any continuous function $f\colon \Omega \to E$ satisfying (18) is μ-integrable and the following holds.

Theorem 1.1.3. (i) (*The Dominated Convergence Theorem*) *Suppose that (f_n) is a sequence of μ-integrable functions from E^Ω and (f_n) converges pointwisely to a function $f\colon \Omega \to E$. Assume that, for every $p \in \circledast$, there exists a μ-integrable function $F_p\colon \Omega \to [0,\infty)$ such that $p(f_n) \leqslant F_p$, $n \in \mathbb{N}$. Then $f(\cdot)$ is a μ-integrable function and $\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f\, d\mu$.*

(ii) *Let Y be an SCLCS, and let $T\colon X \to Y$ be a continuous linear mapping. If $f\colon \Omega \to X$ is μ-integrable, then $Tf\colon \Omega \to Y$ is likewise μ-integrable and*

$$T\int_\Omega f\, d\mu = \int_\Omega Tf\, d\mu. \tag{21}$$

(iii) *Let Y be an SCLCS, and let $T\colon D(T) \subseteq X \to Y$ be a closed linear mapping. If $f\colon \Omega \to D(T)$ is μ-integrable and $Tf\colon \Omega \to Y$ is likewise μ-integrable, then $\int_\Omega f\, d\mu \in D(T)$ and (21) holds.*

Assume that $\mu = dt$ is the Lebesgue measure on $\Omega = [0,\infty)$ and $f\colon [0,\infty) \to E$ is a locally Lebesgue integrable function. As in the Banach space case, we will denote the space consisting of such functions by $L^1_{loc}([0,\infty) : E)$; similarly we define the space $L^1([0,\tau] : E)$ for $0 < \tau < \infty$. It is clear that (20) implies $\langle x^*, f(\cdot)\rangle \in L^1_{loc}([0,\infty))$ for $x^* \in E^*$. The first normalized antiderivative $t \mapsto f^{[1]}(t) := F(t) := \int_0^t f(s)ds$, $t \geqslant 0$ of $f(\cdot)$ is continuous for $t \geqslant 0$, and we have that $\int_0^t p(f)d\mu < \infty$ for any $p \in \circledast$ and $t \geqslant 0$. Set $f^{[n]}(t) := \int_0^t g_n(t-s)f(s)ds$, $t \geqslant 0$. The formula for partial integration in the third part of subsequent theorem will be of crucial importance in our analysis of operational properties of Laplace transform of non-continuous functions with values in SCLCSs.

Theorem 1.1.4. (i) *Suppose that $g \in C([0,\infty))$ and $f \in L^1_{loc}([0,\infty) : E)$. Then $gf \in L^1_{loc}([0,\infty) : E)$.*

(ii) *If* $g \in L^1_{loc}([0,\infty))$ *and* $f \in C([0,\infty) : E)$, *then* $gf \in L^1_{loc}([0,\infty) : E)$.

(iii) (*The partial integration*) *Suppose that* $g \in AC_{loc}([0,\infty))$ *and* $f \in L^1_{loc}([0,\infty) : E)$. *Then, for every* $\tau \geqslant 0$, *we have*

$$\int_0^\tau g(t)f(t)dt = g(\tau)F(\tau) - \int_0^\tau g'(t)F(t)dt. \tag{22}$$

Proof. Fix a number $\tau \in (0,\infty)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions in $E^{[0,\tau]}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. $t \in K = [0,\tau]$ and for all $\varepsilon > 0$ and each $p \in \circledast$ there is a number $n_0 = n_0(\varepsilon, p)$ such that (17) holds. Then $\int_0^\tau f(t)dt = \lim_{n\to\infty} \int_0^\tau f_n(t)dt$ and the sequence $(p(f_n))_{n\in\mathbb{N}}$ is convergent in $L^1[0,\tau]$. By the proof of [**410**, Proposition 4.4.1], there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions in $\mathbb{C}^{[0,\tau]}$ such that $\lim_{n\to\infty} \|s_n - g\|_{L^\infty[0,\tau]} = 0$, $\sup_{n\in\mathbb{N}} \|s_n\|_{L^\infty[0,\tau]} \leqslant \|g\|_{L^\infty[0,\tau]}$ and that for all $\varepsilon > 0$ and $p = |\cdot|$ there is a number $n_0 = n_0(\varepsilon, p)$ such that (17) holds with the functions $f_n(\cdot)$ and $f_m(\cdot)$ replaced respectively with $s_n(\cdot)$ and $s_m(\cdot)$. Clearly, $(s_n f_n)_{n\in\mathbb{N}}$ is a sequence of simple functions in $E^{[0,\tau]}$ such that $\lim_{n\to\infty} s_n(t)f_n(t) = g(t)f(t)$ a.e. $t \in [0,\tau]$. Furthermore, it can be easily seen that

$$\begin{aligned}
&\int_0^t p(s_n(t)f_n(t) - s_m(t)f_m(t))dt \\
&\quad \leqslant \|s_n\|_{L^\infty[0,\tau]} \int_0^t p(f_n(t) - f_m(t))dt + \|s_n - s_m\|_{L^\infty[0,\tau]} \int_0^t p(f_m(t))dt \\
&\quad \leqslant \|g\|_{L^\infty[0,\tau]} \int_0^t p(f_n(t) - f_m(t))dt \\
&\qquad + \big(\|s_n - g\|_{L^\infty[0,\tau]} + \|s_m - g\|_{L^\infty[0,\tau]}\big) \int_0^t p(f_m(t))dt, \quad m,n \in \mathbb{N}.
\end{aligned}$$

This proves (i). To prove (ii), observe first that using Definition 1.1.2 we can directly prove that a function $g_1 f_1(\cdot)$ belongs to the space $L^1([0,\tau] : E)$, provided that $f_1 \colon [0,\tau] \to E$ is a simple function and $g_1 \in L^1[0,\tau]$. By the proof of [**410**, Proposition 4.4.1] once more, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions in $E^{[0,\tau]}$ such that, for every $p \in \circledast$, $\lim_{n\to\infty} p(f_n - f)_{L^\infty[0,\tau]} = 0$, $\sup_{n\in\mathbb{N}} p(f_n)_{L^\infty[0,\tau]} \leqslant p(f)_{L^\infty[0,\tau]}$ and that for all $\varepsilon > 0$ there is a number $n_0 = n_0(\varepsilon, p)$ such that (17) holds. Therefore, $(gf_n)_{n\in\mathbb{N}}$ is a sequence in $L^1([0,\tau] : E)$ and $\lim_{n\to\infty} g(t)f_n(t) = g(t)f(t)$ a.e. $t \in [0,\tau]$. Making use of the dominated convergence theorem (Theorem 1.1.3(i)), we get that $gf \in L^1_{loc}([0,\infty) : E)$, as claimed. By (i) and (ii), the both integrals in (22) are well-defined. Let $x^* \in E^*$. Using the partial integration in the Lebesgue integral and (20), we get that

$$\int_0^\tau g(t)\langle x^*, f(t)\rangle dt = g(\tau)\langle x^*, F(\tau)\rangle - \int_0^\tau g'(t)\langle x^*, F(t)\rangle dt.$$

Since x^* was arbitrary, it readily follows on account of (20) that (22) holds. The proof of the theorem is thereby complete. □

Analytical properties of functions with values in SCLCSs. A function $f \colon \Omega \to E$, where Ω is an open subset of $\mathbb{C}$, is said to be analytic iff it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in E. It is well known that the analyticity of $f(\cdot)$ is equivalent with the weak analyticity of $f(\cdot)$; in other words, the mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \Omega$ is analytic iff the mapping $\lambda \mapsto \langle x^*, f(\lambda)\rangle$, $\lambda \in \Omega$ is analytic for every $x^* \in E^*$. If the mapping $f\colon \Omega \to E$ is analytic, then the mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \Omega$ is infinitely differentiable and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\lambda)}{(\lambda - z)^{n+1}} d\lambda, \quad z \in \Omega \smallsetminus \Gamma,\ \mathrm{Ind}_\Gamma(z) = 1,\ n \in \mathbb{N}_0,$$

which simply implies that the equality

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

holds in a neighborhood of point $z_0 \in \Omega$. The identity theorem for analytic functions [**27**, Proposition A.2, p. 456] continues to hold in the case that X is a general locally convex space.

We need the following extension of Weierstrass theorem.

Lemma 1.1.5. (*E. Jordá [**261**, Theorem 3, p. 742]) Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and connected, and let $f_n\colon \Omega \to E$ be an analytic function ($n \in \mathbb{N}$). Assume further that, for every $z_0 \in \Omega$, there exists $r > 0$ such that the set $\bigcup_{n \in \mathbb{N}} f_n(B(z_0, r))$ is bounded and the set $\Omega_0 := \{z \in \Omega : \lim_{n\to\infty} f_n(z) \text{ exists}\}$ has a limit point in Ω. Then there exists an analytic function $f\colon \Omega \to E$ such that (f_n) converges locally uniformly to f.*

Complex powers of almost C-nonnegative operators. Let $m \in \mathbb{R}$, let A be a closed linear operator with domain and range contained in E, and let $C \in L(E)$ be an injective operator satisfying $CA \subseteq AC$. Then it is said that the operator A belongs to the class $\mathcal{M}_{C,m}$ iff $(-\infty, 0) \subseteq \rho_C(A)$ and the family

$$\{(\lambda^{-1} + \lambda^m)^{-1}(\lambda + A)^{-1}C : \lambda > 0\} \subseteq L(E)$$

is equicontinuous. Furthermore, it is said that A is almost C-nonnegative iff there exists $m \in \mathbb{R}$ such that A belongs to the class $\mathcal{M}_{C,m}$. It will be necessary to remind us of the following facts concerning the fractional powers of almost C-nonnegative operators; for further information, see [**292**, Section 2.9]. Set $p_n(x) := \sum_{i=0}^{n} p(A^i x)$ ($x \in D_\infty(A)$, $p \in \circledast$, $n \in \mathbb{N}_0$), $A_\infty := A_{|D_\infty(A)}$ and $C_\infty := C_{|D_\infty(A)}$. Then the system $(p_n)_{p\in\circledast, n\in\mathbb{N}_0}$ induces a Hausdorff sequentially complete locally convex topology on $D_\infty(A)$, $A_\infty \in L(D_\infty(A))$ and $C_\infty \in L(D_\infty(A))$ is injective.

In [**292**, Definition 2.9.11], we have generalized the notion of Balakrishnan's operators [**44**] as follows: Let $\alpha \in \mathbb{C}_+$ and $A \in \mathcal{M}_{C,-1}$. Then:

(i) If $0 < \operatorname{Re}\alpha < 1$, $D(J_C^\alpha) := D(A)$ and

$$J_C^\alpha x := \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + A)^{-1} CAx \, d\lambda, \quad x \in D(A).$$

(ii) If $\operatorname{Re}\alpha = 1$, $D(J_C^\alpha) := D(A^2)$ and

$$J_C^\alpha x := \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}\Big[(\lambda + A)^{-1}C - \frac{\lambda C}{\lambda^2 + 1}\Big] Ax \, d\lambda + \sin\frac{\alpha\pi}{2} CAx,$$

for any $x \in D(A^2)$.

(iii) If $n < \operatorname{Re}\alpha < n+1$, $n \in \mathbb{N}$, $D(J_C^\alpha) := D(A^{n+1})$ and
$$J_C^\alpha x := J_C^{\alpha-n} A^n x, \quad x \in D(A^{n+1}).$$

(iv) If $\operatorname{Re}\alpha = n+1$, $n \in \mathbb{N}$, $D(J_C^\alpha) := D(A^{n+2})$ and
$$J_C^\alpha x := J_C^{\alpha-n} A^n x, \quad x \in D(A^{n+2}).$$

Then we know, for every $\alpha \in \mathbb{C}_+$, the operator J_C^α is closable. Set $A_{C,\alpha} := C^{-1}\overline{J_C^\alpha}$ $(\alpha \in \mathbb{C}_+)$.

Consider now the case in which A belongs to the class $\mathcal{M}_{C,m}$ for some $m \geqslant -1$. Due to [**292**, Proposition 2.9.20], we have that the operator A_∞ is C_∞-nonnegative in the space $D_\infty(A)$, i.e., that $A_\infty \in \mathcal{M}_{C_\infty,-1}$. Therefore, we can construct the power $A_{\infty,\alpha} \equiv (A_\infty)_{C_\infty,\alpha}$ in the space $D_\infty(A)$ $(\alpha \in \mathbb{C}_+)$. In [**292**, Proposition 2.9.23], we have proved that $C^2(D(A^{2p+n})) \subseteq D(\overline{A_{\infty,\alpha}})$, provided that $\alpha \in \mathbb{C}_+$ satisfies $0 < \operatorname{Re}\alpha < n$ for some $n \in \mathbb{N}$ $(p \equiv \lfloor m+2 \rfloor)$. This is the most important auxiliary result which enables us to introduce the power A_α:

Definition 1.1.6. Suppose $m \geqslant -1$, $n \in \mathbb{N}$, $p = \lfloor m+2 \rfloor$, $A \in \mathcal{M}_{C,m}$, $\alpha \in \mathbb{C}_+$ and $0 < \operatorname{Re}\alpha < n$. Then we define the power A_α as follows
$$A_\alpha := C^{-2}((1+A)^{-1}C)^{-n(p+1)-p}\overline{A_{\infty,\alpha}}((1+A)^{-1}C)^{n(p+1)+p}C^2.$$

It is worth noting that Definition 1.1.6 does not depend on the particular choice of numbers m and n, as well as that A_α is a closed linear operator on E. Furthermore, the injectiveness of A implies the injectiveness of A_α. If this is the case, we define:
$$A_{-\alpha} := (A_\alpha)^{-1}, \quad A_0 := I \quad \text{and}$$
$$A_{i\tau} := C^{-3}((1+A)^{-1}C)^{-(3p+3)}A^{-1}A_{1+i\tau}((1+A)^{-1}C)^{3p+3}C^3.$$

The construction of fractional powers of the operator A which belongs to the class $A \in \mathcal{M}_{C,m}$ for some real number $m < -1$ strongly depends on the injectiveness of A: it is still an open problem to construct the power A_α $(\alpha \in \mathbb{C}_+)$ in the case that $A \in \mathcal{M}_{C,m}$ for some real number $m < -1$, and A is not injective. On the other hand, if the operator A is injective, then the operator A^{-1} belongs to the class $\mathcal{M}_{C,-m-2}$ and we can simply define the power A_α $(\alpha \in \mathbb{C})$ by setting $A_\alpha := (A^{-1})_{-\alpha}$ $(\alpha \in \mathbb{C})$. Observe only that $-m-2 > -1$ if $m < -1$.

1.2. Multivalued linear operators

In this section, we present a brief overview of the necessary definitions and properties of multivalued linear operators that will be necessary for our further work. In a separate subsection, we exhibit our original contributions on hypercyclic and disjoint hypercyclic multivalued linear operators (this is a part of a joint research study with C.-C. Chen, J. A. Conejero and M. Murillo-Arcila [**107**]; see also [**108**]). For more details about multivalued linear operators, we refer the reader to the monographs [**120**, **139**, **199**] as well as to the papers [**21**, **30**] and [**200**].

Let X and Y be two SCLCSs. A multivalued map (multimap) $\mathcal{A}\colon X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. An almost immediate consequence of definition is that $\mathcal{A}x + \mathcal{A}y = \mathcal{A}(x + y)$ for all $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x = \mathcal{A}(\lambda x)$ for all $x \in D(\mathcal{A})$, $\lambda \neq 0$. Furthermore, for any $x, y \in D(\mathcal{A})$ and $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted henceforth by $N(\mathcal{A})$ or $\mathrm{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. It is worth noting that $\mathcal{A}x = \mathcal{A}y$ for some two elements x and $y \in D(\mathcal{A})$, iff $\mathcal{A}x \cap \mathcal{A}y \neq \emptyset$; moreover, if $\mathcal{A}$ is injective, then the equality $\mathcal{A}x = \mathcal{A}y$ holds iff $x = y$. For any mapping $\mathcal{A}\colon X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is an MLO iff $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$.

If $\mathcal{A}, \mathcal{B}\colon X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It can be simply verified that $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Let $\mathcal{A}\colon X \to P(Y)$ and $\mathcal{B}\colon Y \to P(Z)$ be two MLOs, where Z is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A}\colon X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A}\colon X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A}\colon X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z, \omega \in \mathbb{C}$. Suppose that X' is a linear subspace of X, and $\mathcal{A}\colon X \to P(Y)$ is an MLO. Then we define the restriction of operator $\mathcal{A}$ to the subspace X', $\mathcal{A}_{|X'}$ for short, by $D(\mathcal{A}_{|X'}) := D(\mathcal{A}) \cap X'$ and $\mathcal{A}_{|X'}x := \mathcal{A}x$, $x \in D(\mathcal{A}_{|X'})$. Clearly, $\mathcal{A}_{|X'}\colon X \to P(Y)$ is an MLO. It is well known that an MLO $\mathcal{A}\colon X \to P(Y)$ is injective (resp., single-valued) iff $\mathcal{A}^{-1}\mathcal{A} = I_{|D(\mathcal{A})}$ (resp., $\mathcal{A}\mathcal{A}^{-1} = I^{Y}_{|R(\mathcal{A})}$).

The integer powers of an MLO $\mathcal{A}\colon X \to P(X)$ is defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\},$$

and

$$\mathcal{A}^n x := (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

We can prove inductively that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. Moreover, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$. If $\mathcal{A}\colon X \to P(Y)$ and $\mathcal{B}\colon X \to P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. Assume now that a linear single-valued operator $S\colon D(S) \subseteq X \to Y$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A}\colon X \to P(Y)$ is an MLO. Then S is called a section of $\mathcal{A}$; if this is the case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$. We say that an MLO operator $\mathcal{A}\colon X \to P(Y)$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in Y such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Following C. Knuckles and F. Neubrander [**285**], we introduce the notion of a relatively closed MLO as follows. We say that an MLO $\mathcal{A}\colon X \to P(Y)$ is relatively closed iff there exist auxiliary SCLCSs $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$ such that $D(\mathcal{A}) \subseteq X_{\mathcal{A}} \hookrightarrow X$, $R(\mathcal{A}) \subseteq Y_{\mathcal{A}} \hookrightarrow Y$ and $\mathcal{A}$ is closed in $X_{\mathcal{A}} \times Y_{\mathcal{A}}$; i.e., the assumptions $D(\mathcal{A}) \ni x_\tau \to x$ as $\tau \to \infty$ in $X_{\mathcal{A}}$ and $\mathcal{A}x_\tau \ni y_\tau \to y$ as $\tau \to \infty$ in $Y_{\mathcal{A}}$ implies that $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. A relatively closed operator will also be called $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. By way of illustration, let $A, B\colon D \subseteq X \to Y$ be closed linear operators with the same domain D. Then the operator $A + B$ is not necessarily closed but it is always $[D(A)] \times Y$-closed (cf. [**280**, p. 170]). Examples presented in [**285**] can be simply reformulated for operators acting on locally convex spaces, as well:

EXAMPLE 1.2.1. (i) If $\mathcal{A}\colon X \to P(Y)$ is an MLO, then $\bar{\mathcal{A}}\colon X \to P(Y)$ is likewise an MLO. This shows that any MLO has a closed linear extension, in contrast to the usually considered single-valued linear operators.

(ii) Let $A\colon D(A) \subseteq X \to Y$ be a single-valued linear operator that is $X_A \times Y_A$-closed, let $\mathcal{B}\colon X \to P(Y)$ be an MLO that is $X_{\mathcal{B}} \times Y_{\mathcal{B}}$-closed, and let $Y_A \hookrightarrow Y_{\mathcal{B}}$. Then the MLO $S = A + \mathcal{B}$ is $X_S \times Y_{\mathcal{B}}$-closed, where $X_S := D(A) \cap X_{\mathcal{B}}$ and the topology on X_S is induced by the system $(s_{p,q,r})$ of fundamental seminorms, defined as follows: $s_{p,q,r}(x) =: p(x) + p(Ax) + q(x) + r(Ax)$, $x \in X_S$ $(p \in \circledast_X,\ q \in \circledast_{X_{\mathcal{B}}},\ r \in \circledast_{Y_A})$.

(iii) Let $A\colon D(A) \subseteq X \to Y$ be a single-valued linear operator that is $X_A \times Y_A$-closed, let $\mathcal{B}\colon Y \to P(Z)$ be an MLO that is $Y_{\mathcal{B}} \times Z_{\mathcal{B}}$-closed, and let $Y_{\mathcal{B}} \hookrightarrow Y_A$. Then the MLO $C = \mathcal{B}A\colon X \to P(Z)$ is $X_C \times Z_{\mathcal{B}}$-closed, where $X_C := \{x \in D(A) : Ax \in Y_{\mathcal{B}}\}$ and the topology on X_C is induced by the system $(s_{p,q})$ of fundamental seminorms, defined as follows: $s_{p,q}(x) =: p(x) + p(Ax) + q(Ax)$, $x \in X_C$ $(p \in \circledast_X,\ q \in \circledast_{Y_{\mathcal{B}}})$.

(iv) Let $A\colon D(A) \subseteq X \to Y$ and $B\colon D(B) \subseteq X \to Y$ be two single-valued linear operators. Set

$$\mathcal{A} := B^{-1}A = \{(x, y) : x \in D(A),\ y \in D(B) \text{ and } Ax = By\}.$$

Then $\mathcal{A}$ is an MLO in X, and the following holds:

(a) If one of the operators A, B is bounded and the other closed, then $\mathcal{A}$ is closed.

(b) If A is closed and B is $X_B \times Y$-closed, then $\mathcal{A}$ is $[D(A)] \times X_B$-closed.

(c) If B is closed and A is $X_A \times Y$-closed, then $\mathcal{A}$ is $X_A \times [D(B)]$-closed.

(d) If A is $X_A \times Y_A$-closed and B is $X_B \times Y_B$-closed, where $Y_B \hookrightarrow Y_A$, then $\mathcal{A}$ is $X_C \times X_B$-closed, where X_C is defined as in (iii).

If $\mathcal{A}\colon X \to P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^*\colon Y^* \to P(X^*)$ of $\mathcal{A}$ by its graph

$$\mathcal{A}^* := \{(y^*, x^*) \in Y^* \times X^* : \langle y^*, y\rangle = \langle x^*, x\rangle \text{ for all pairs } (x, y) \in \mathcal{A}\}.$$

It is simpy verified that $\mathcal{A}^*$ is a closed MLO, and that $\langle y^*, y\rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$. Furthermore, the equations [**199**, (1.2)–(1.6)] continue to hold for adjoints of MLOs acting on locally convex spaces.

The following important lemma can be proved by means of the Hahn–Banach theorem and the argumentation from [**47**].

LEMMA 1.2.2. *Suppose that* $\mathcal{A}\colon X \to P(Y)$ *is an* MLO *and* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed. Assume, further, that* $x_0 \in X$, $y_0 \in Y$ *and* $\langle x^*, x_0\rangle = \langle y^*, y_0\rangle$ *for all pairs* $(x^*, y^*) \in X_{\mathcal{A}}^* \times Y_{\mathcal{A}}^*$ *satisfying that* $\langle x^*, x\rangle = \langle y^*, y\rangle$ *whenever* $y \in \mathcal{A}x$. *Then* $y_0 \in \mathcal{A}x_0$.

With Lemma 1.2.2 in view, we can simply prove the following extension of Theorem 1.1.3(iii) for relatively closed MLOs in locally convex spaces. Here, by Ω we denote a locally compact and separable metric space and by μ we denote a locally finite Borel measure defined on Ω.

THEOREM 1.2.3. *Suppose that* $\mathcal{A}\colon X \to P(Y)$ *is an* MLO *and* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed. Let* $f\colon \Omega \to X_{\mathcal{A}}$ *and* $g\colon \Omega \to Y_{\mathcal{A}}$ *be* μ*-integrable, and let* $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. *Then* $\int_\Omega f\,d\mu \in D(\mathcal{A})$ *and* $\int_\Omega g\,d\mu \in \mathcal{A}\int_\Omega f\,d\mu$.

Now we will analyze the C-resolvent sets of MLOs in locally convex spaces. Our standing assumptions will be that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$ (this is equivalent to say that, for any $(x, y) \in X \times X$, we have the implication $(x, y) \in \mathcal{A} \Rightarrow (Cx, Cy) \in \mathcal{A}$; by induction, we immediately get that $C\mathcal{A}^k \subseteq \mathcal{A}^k C$ for all $k \in \mathbb{N}$). Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). We can almost trivially construct examples of MLOs for which $\rho(\mathcal{A}) = \emptyset$ and $\rho_C(\mathcal{A}) \neq \emptyset$: Let Y be a proper closed linear subspace of X, let $\mathcal{A}$ be an MLO in Y, and let $\lambda \in \mathbb{C}$ so that $(\lambda - \mathcal{A})^{-1} \in L(Y)$. Taking any injective operator $C \in L(X)$ with $R(C) \subseteq Y$, and looking $\mathcal{A} = \mathcal{A}_X$ as an MLO in X, it is clear that $\lambda \in \rho_C(\mathcal{A}_X)$ and $\rho(\mathcal{A}_X) = \emptyset$. In general case, if $\rho_C(\mathcal{A}) \neq \emptyset$, then for any $\lambda \in \rho_C(\mathcal{A})$ we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1}C)$, as well as $\lambda \in \rho_C(\bar{\mathcal{A}})$, $\bar{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and $((\lambda - \mathcal{A})^{-1}C)^k(D(\mathcal{A}^l)) \subseteq D(\mathcal{A}^{k+l})$, $k, l \in \mathbb{N}_0$; here it is worth noting that the equality $\mathcal{A} = C^{-1}\mathcal{A}C$ holds provided, in addition, that $\rho(\mathcal{A}) \neq \emptyset$ (see e.g. the proofs of [**138**, Proposition 2.1, Lemma 2.3]). The basic properties of C-resolvent sets of single-valued linear operators [**291**, **292**] continue to hold in our framework (observe, however, that there exist certain differences that we will not discuss here). For example, if $\rho(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is closed; it is well known that this statement does not hold if $\rho_C(\mathcal{A}) \neq \emptyset$ for some $C \neq I$ (cf. [**138**, Example 2.2]). Arguing as in the proofs of [**199**, Theorem 1.7-Theorem 1.9], we can deduce the validity of the following theorem.

THEOREM 1.2.4. (i) *We have*

$$(\lambda - \mathcal{A})^{-1}C\mathcal{A} \subseteq \lambda(\lambda - \mathcal{A})^{-1}C - C \subseteq \mathcal{A}(\lambda - \mathcal{A})^{-1}C, \quad \lambda \in \rho_C(\mathcal{A}).$$

The operator $(\lambda-\mathcal{A})^{-1}C\mathcal{A}$ *is single-valued on* $D(\mathcal{A})$ *and* $(\lambda-\mathcal{A})^{-1}C\mathcal{A}x = (\lambda - \mathcal{A})^{-1}Cy$, *whenever* $y \in \mathcal{A}x$ *and* $\lambda \in \rho_C\mathcal{A}$.

(ii) *Suppose that* $\lambda, \mu \in \rho_C(\mathcal{A})$. *Then the resolvent equation*

$$(\lambda - \mathcal{A})^{-1}C^2x - (\mu - \mathcal{A})^{-1}C^2x = (\mu - \lambda)(\lambda - \mathcal{A})^{-1}C(\mu - \mathcal{A})^{-1}Cx, \quad x \in X$$

holds good. In particular, $[(\lambda-\mathcal{A})^{-1}C][(\mu-\mathcal{A})^{-1}C] = [(\mu-\mathcal{A})^{-1}C][(\lambda-\mathcal{A})^{-1}C]$.

(iii) *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are two given* MLOs, *as well as that* $C\mathcal{B} \subseteq \mathcal{B}C$ *and* $0 \in \rho_C(\mathcal{A}) \cap \rho_C(\mathcal{B})$. *Then we have*

$$(\lambda - \mathcal{A})^{-1}C^3x - (\mu - \mathcal{B})^{-1}C^3x = [\lambda(\lambda - \mathcal{A})^{-1}C - C]$$
$$\times [\mathcal{B}^{-1}C - \mathcal{A}^{-1}C][\lambda(\lambda - \mathcal{B})^{-1}C - C], \quad \lambda \in \rho_C(\mathcal{A}) \cap \rho_C(\mathcal{B}).$$

The following proposition can be proved by induction.

PROPOSITION 1.2.5. *Suppose that* $\lambda \in \rho_C(\mathcal{A})$, $n \in \mathbb{N}$, $x \in D(\mathcal{A}^n) = D((\lambda - \mathcal{A})^n)$ *and* $y \in (\lambda - \mathcal{A})^n x$. *Then we have*

$$((\lambda - \mathcal{A})^{-1}C)^n(\lambda - \mathcal{A})^n x = \{((\lambda - \mathcal{A})^{-1}C)^n y\} = \{C^n x\}.$$

By Theorem 1.2.4(i), it readily follows that the operator $\lambda(\lambda - \mathcal{A})^{-1}C - C \in L(X)$ is a bounded linear section of the MLO $\mathcal{A}(\lambda - \mathcal{A})^{-1}C$ ($\lambda \in \rho_C(\mathcal{A})$). Inductively, we can prove that, for every $x \in X$, $n \in \mathbb{N}_0$ and $\lambda \in \rho_C(\mathcal{A})$, we have $\operatorname{card}((\lambda - \bar{\mathcal{A}})^{-n}Cx) \leqslant 1$. Having in mind this fact, as well as the argumentation already seen many times in our previous research studies of C-resolvents of single-valued linear operators, we can prove the following extension of [**292**, Proposition 2.1.14] for MLOs in locally convex spaces.

PROPOSITION 1.2.6. *Let* $\emptyset \neq \Omega \subseteq \rho_C(\mathcal{A})$ *be open, and let* $x \in X$.

(i) *The local boundedness of the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$, *resp. the assumption that* X *is barreled and the local boundedness of the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \Omega$, *implies the analyticity of the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C^3x$, $\lambda \in \Omega$, *resp.* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C^3$, $\lambda \in \Omega$. *Furthermore, if* $R(C)$ *is dense in* X, *resp. if* $R(C)$ *is dense in* X *and* X *is barreled, then the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ *is analytic, resp. the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \Omega$ *is analytic.*

(ii) *Suppose that* $R(C)$ *is dense in* X. *Then the local boundedness of the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ *implies its analyticity as well as* $Cx \in \mathrm{R}((\lambda - \bar{\mathcal{A}})^n)$, $n \in \mathbb{N}$ *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda - \mathcal{A})^{-1}Cx = (-1)^{n-1}(n-1)!(\lambda - \bar{\mathcal{A}})^{-n}Cx \quad n \in \mathbb{N}. \tag{23}$$

Furthermore, if X *is barreled, then the local boundedness of the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \Omega$ *implies its analyticity as well as* $R(C) \subseteq R((\lambda -$

$\bar{\mathcal{A}})^n)$, $n \in \mathbb{N}$ and

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda - \mathcal{A})^{-1}C = (-1)^{n-1}(n-1)!(\lambda - \bar{\mathcal{A}})^{-n}C \in L(X), \quad n \in \mathbb{N}. \tag{24}$$

(iii) *The continuity of mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ implies its analyticity and* (23)*. Furthermore, if X is barreled, then the continuity of mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \Omega$ implies its analyticity and* (24)*.*

It is well known that $\rho_C(A)$ need not be an open subset of $\mathbb{C}$ if $C \neq I$ and A is a single-valued linear operator in X (cf. [**138**, Example 2.5]) and that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$, provided that X is a Banach space and $\mathcal{A}$ is an MLO in X (cf. [**199**, Theorem 1.6]). The regular C-resolvent set of $\mathcal{A}$, $\rho_C^r(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \rho_C(\mathcal{A})$ for which $(\lambda - \mathcal{A})^{-1}C \in R(X)$, where $R(X)$ denotes the set of all regular bounded linear operators $A \in L(X)$, i.e., the operators $A \in L(X)$ for which there exists a positive constant $c > 0$ such that for each seminorm $p \in \circledast$ there exists another seminorm $q \in \circledast$ such that $p(A^n x) \leqslant c^n q(x)$, $x \in X$, $n \in \mathbb{N}$; the regular resolvent set of $\mathcal{A}$, $\rho^r(\mathcal{A})$ for short, is then defined by $\rho^r(\mathcal{A}) := \rho_I^r(\mathcal{A})$. By the proof of [**199**, Theorem 1.6], it readily follows that $\rho^r(\mathcal{A})$ is always an open subset of $\mathbb{C}$. If $\mathcal{A}$ is an MLO, then for each complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$ we have

$$(\mathcal{A} - z_1)(\mathcal{A} - z_2)\ldots(\mathcal{A} - z_n) = \mathcal{A}^n + \sum_{k=1}^{n} a_{n-k}\mathcal{A}^{n-k},$$

where $a_{n-k} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} z_{i_1} z_{i_2} \ldots z_{i_k}$. The following polynomial spectral mapping theorem holds for multivalued linear operators in locally convex spaces (cf. [**537**, Theorem 7.5.3] for the case that $C = I$ and X is a Banach space):

THEOREM 1.2.7. *Suppose that $P_n(z) = \sum_{k=0}^{n} a_k z^k$ is a non-zero complex polynomial of degree $n \in \mathbb{N}_0$, and*

$$\lambda - P_n(z) = (-1)^{n+1}a_n(f_1(\lambda) - z)(f_2(\lambda) - z)\ldots(f_n(\lambda) - z), \quad \lambda, z \in \mathbb{C}.$$

Let $\emptyset \neq \Omega \subseteq \rho_C(\mathcal{A})$, and let $\lambda \in \mathbb{C}$ be such that $f_1(\lambda), f_2(\lambda), \ldots, f_n(\lambda) \in \Omega$. Then $\lambda \in \rho_{C^n}(P_n(\mathcal{A}))$ and

$$(\lambda - P_n(\mathcal{A}))^{-1}C^n = (-1)^{n+1}a_n(f_1() - \mathcal{A})^{-1}C(f_2(\lambda) - \mathcal{A})^{-1}C\ldots(f_n(\lambda) - \mathcal{A})^{-1}C.$$

We continue by observing that the generalized resolvent equations hold for C-resolvents of multivalued linear operators. More precisely, we have the following theorem which can be proved by induction.

THEOREM 1.2.8. (i) *Let $x \in X$, $k \in \mathbb{N}_0$ and $\lambda, z \in \rho_C(\mathcal{A})$ with $z \neq \lambda$. Then the following holds:*

$$(z - \mathcal{A})^{-1}C((\lambda - \mathcal{A})^{-1}C)^k x = \frac{(-1)^k}{(z-\lambda)^k}(z - \mathcal{A})^{-1}C^{k+1}x + \sum_{i=1}^{k} \frac{(-1)^{k-i}((\lambda - \mathcal{A})^{-1}C)^i C^{k+1-i}x}{(z-\lambda)^{k+1-i}}.$$

(ii) *Let* $k \in \mathbb{N}_0$, $x, y \in X$, $y \in (\lambda_0 - \mathcal{A})^k x$ *and* $\lambda_0, z \in \rho_C(\mathcal{A})$ *with* $z \neq \lambda_0$. *Then the following holds:*

$$(z-\mathcal{A})^{-1}C^{k+1}x = \frac{(-1)^k}{(z-\lambda_0)^k}(z-\mathcal{A})^{-1}C^{k+1}y + \sum_{i=1}^{k} \frac{(-1)^{k-i}((\lambda_0-\mathcal{A})^{-1}C)^i C^{k+1-i}y}{(z-\lambda_0)^{k+1-i}}.$$

1.2.1. Fractional powers. In this subsection, we assume that X is a Banach space, $(-\infty, 0] \subseteq \rho(\mathcal{A})$ as well as that there exist finite numbers $M \geqslant 1$ and $\beta \in (0,1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1+|\lambda|)^{-\beta}, \quad \lambda \leqslant 0.$$

Then there exist two positive real constants $c > 0$ and $M_1 > 0$ such that the resolvent set of $\mathcal{A}$ contains an open region $\Omega = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant (2M_1)^{-1}(c - \operatorname{Re}\lambda)^\beta,\ \operatorname{Re}\lambda \leqslant c\}$ of complex plane around the nonpositive half-line $(-\infty, 0]$, and we have the estimate $\|R(\lambda : \mathcal{A})\| = O((1+|\lambda|)^{-\beta})$, $\lambda \in \Omega$.

Let Γ' be the upwards oriented curve $\{\xi \pm i(2M_1)^{-1}(c-\xi)^\beta : -\infty < \xi \leqslant c\}$. We define the fractional power

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi i}\int_{\Gamma'} \lambda^{-\theta}(\lambda-\mathcal{A})^{-1}\,d\lambda \in L(E)$$

for $\theta > 1-\beta$. Set $\mathcal{A}^\theta := (\mathcal{A}^{-\theta})^{-1}$ $(\theta > 1-\beta)$. Then the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1+\theta_2)}$ and $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1+\theta_2}$ hold for $\theta_1, \theta_2 > 1-\beta$ (it is worth noting here that the fractional power $\mathcal{A}^\theta$ need not be injective and that the meaning of $\mathcal{A}^\theta$ is understood in the MLO sense for $\theta > 1-\beta$).

We endow the vector space $D(\mathcal{A})$ with the norm

$$\|\cdot\|_{[D(\mathcal{A})]} := \inf_{y \in \mathcal{A}\cdot} \|y\|.$$

Then $(D(\mathcal{A}), \|\cdot\|_{[D(\mathcal{A})]})$ is a Banach space and, since $0 \in \rho(\mathcal{A})$, the norm $\|\cdot\|_{[D(\mathcal{A})]}$ is equivalent with the following one $\|\cdot\| + \|\cdot\|_{[D(\mathcal{A})]}$. Since $0 \in \rho(\mathcal{A}^\theta)$, $(D(\mathcal{A}^\theta), \|\cdot\|_{[D(\mathcal{A}^\theta)]})$ is likewise a Banach space and we have the equivalence of norms $\|\cdot\|_{[D(\mathcal{A}^\theta)]}$ and $\|\cdot\| + \|\cdot\|_{[D(\mathcal{A}^\theta)]}$ for $\theta > 1-\beta$ (cf. the proof of [**199**, Proposition 1.1]).

For any $\theta \in (0,1)$, the vector space

$$E^\theta_{\mathcal{A}} := \Big\{x \in E : \sup_{\xi>0} \xi^\theta \|\xi(\xi+\mathcal{A})^{-1}x - x\| < \infty\Big\}$$

becomes one of Banach's when endowed with the norm

$$\|\cdot\|_{E^\theta_{\mathcal{A}}} := \|\cdot\| + \sup_{\xi>0} \xi^\theta \|\xi(\xi+\mathcal{A})^{-1}\cdot - \cdot\|.$$

It is clear that $E^\theta_{\mathcal{A}}$ is continuously embedded in E. We refer the reader to [**181**, **411**, **444**] and Section 2.12 for further information concerning interpolation spaces and fractional powers of multivalued linear operators. In Section 2.12, we will see that the main properties of C-resolvent sets of multivalued linear operators hold even if the injectivity of regularizing operator C is disregarded.

1.2.2. Hypercyclic and disjoint hypercyclic multivalued linear operators. The basic facts about topological dynamics of linear continuous operators in Banach and Fréchet spaces can be obtained by consulting the monographs [**60**] by F. Bayart, E. Matheron and [**238**] by K.-G. Grosse-Erdmann, A. Peris. Hypercyclicity and disjoint hypercyclicity of multivalued linear operators are relatively new topics in the field of linear topological dynamics. In this subsection, we also inquire into the topological transitivity, topologically mixing property and chaoticity of multivalued linear operators, as well as their disjoint analogues (cf. [**67**, **69**, **71**, **73**, **106**, **108**, **137**, **292**, **390**, **409**, **470**, **471**] and [**489**] for further information on single-valued linear case).

In this subsection, X and Y will be two separable SCLCSs. Set $SL_p(x,\varepsilon) := \{y \in X : p(x-y) < \varepsilon\}$, $\varepsilon > 0$, $x \in X$, $p \in \circledast$. Assume that $C \in L(X)$ is injective. Put $p_C(x) := p(C^{-1}x)$, $p \in \circledast$, $x \in R(C)$. Then $p_C(\cdot)$ is a seminorm on $R(C)$ and the calibration $(p_C)_{p\in\circledast}$ induces a locally convex topology on $R(C)$; we denote the above space by $[R(C)]_{\circledast}$. Notice that $[R(C)]_{\circledast}$ is a separable SCLCS, and $[R(C)]_{\circledast}$ is a (Fréchet, Banach space) provided that X is. Set $\mathfrak{S}_1 := \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $\mathcal{A}$ is an MLO in X. Then we say that a point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ iff there exists a vector $x \in X \smallsetminus \{0\}$ such that $\lambda x \in \mathcal{A}x$; we call x an eigenvector of operator $\mathcal{A}$ corresponding to the eigenvalue λ. Observe that, in purely multivalued case, a vector $x \in X \smallsetminus \{0\}$ can be an eigenvector of operator $\mathcal{A}$ corresponding to different values of scalars λ. The point spectrum of $\mathcal{A}$, $\sigma_p(\mathcal{A})$ for short, is defined as the union of all eigenvalues of $\mathcal{A}$. Let $k \in \mathbb{N}$. Then we define the MLO $\underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{k}$ on $\underbrace{X \oplus \cdots \oplus X}_{k}$ by $D(\underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{k}) := \underbrace{D(\mathcal{A}) \oplus \cdots \oplus D(\mathcal{A})}_{k}$ and

$$\underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{k}(x_1, x_2, \ldots, x_k) := \{(y_1, y_2, \ldots, y_k) : y_i \in \mathcal{A}x_i \text{ for all } i = 1, 2, \ldots, k\}.$$

We would like to propose the following definitions (for the sake of clearness and better exposition of material, we will not treat the subspace dynamical properties of MLOs here; cf. J. Banasiak, M. Moszyński [**48**] and [**292**, Chapter 3] for more details):

Definition 1.2.9. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of multivalued linear operators acting between the spaces X and Y, let $\mathcal{A}$ be an MLO in X, and let $x \in X$. Then we say that:

(i) x is a hypercyclic vector of the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ iff $x \in \bigcap_{n\in\mathbb{N}} D(\mathcal{A}_n)$ and for each $n \in \mathbb{N}_0$ there exists an element $y_n \in \mathcal{A}_n x$ such that the set $\{y_n : n \in \mathbb{N}\}$ is dense in Y; $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is said to be hypercyclic iff there exists a hypercyclic vector of $(\mathcal{A}_n)_{n\in\mathbb{N}}$;

(ii) $\mathcal{A}$ is hypercyclic iff the sequence $(\mathcal{A}^n)_{n\in\mathbb{N}}$ is hypercyclic; x is said to be a hypercyclic vector of $\mathcal{A}$ iff x is a hypercyclic vector of the sequence $(\mathcal{A}^n)_{n\in\mathbb{N}}$;

(iii) x is a periodic point of $\mathcal{A}$ iff $x \in D_\infty(\mathcal{A})$ and there exists an integer $n \in \mathbb{N}$ such that $x \in \mathcal{A}^n x$;

(iv) $\mathcal{A}$ is topologically transitive iff for every two open non-empty subsets U, V of X there exists $n \in \mathbb{N}$ such that

$$U \cap \mathcal{A}^{-n}(V) \neq \emptyset; \tag{25}$$

(v) $\mathcal{A}$ is topologically mixing iff for every two open non-empty subsets U, V of X there exists $n_0 \in \mathbb{N}$ such that (25) holds for $n \geqslant n_0$.

(vi) $\mathcal{A}$ is chaotic iff $\mathcal{A}$ is topologically transitive and the set constituted of all periodic points of $\mathcal{A}$ is dense in X.

Definition 1.2.10. Suppose that $N \in \mathbb{N}$, $(\mathcal{A}_{j,n})_{n\in\mathbb{N}}$ is a sequence of multivalued linear operators acting between the spaces X and Y $(1 \leqslant j \leqslant N)$, $\mathcal{A}_j$ is an MLO in X $(1 \leqslant j \leqslant N)$ and $x \in X$. Then we say that:

(i) x is a d-hypercyclic vector of the sequences $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$ iff for each $n \in \mathbb{N}$ there exist elements $y_{j,n} \in \mathcal{A}_{j,n}x$ $(1 \leqslant j \leqslant N)$ such that the set $\{(y_{1,n}, y_{2,n}, \dots, y_{N,n}) : n \in \mathbb{N}\}$ is dense in Y^N; the sequences $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$ are called d-hypercyclic iff there exists a d-hypercyclic vector of $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$;

(ii) x is a d-hypercyclic vector of the operators $\mathcal{A}_1, \dots, \mathcal{A}_N$ iff x is a d-hypercyclic vector of the sequences $(\mathcal{A}_1^n)_{n\in\mathbb{N}}, \dots, (\mathcal{A}_N^n)_{n\in\mathbb{N}}$; the operators $\mathcal{A}_1, \dots, \mathcal{A}_N$ are called d-hypercyclic iff there exists a d-hypercyclic vector of $\mathcal{A}_1, \dots, \mathcal{A}_N$.

Definition 1.2.11. Suppose that $N \in \mathbb{N}$, $(\mathcal{A}_{j,n})_{n\in\mathbb{N}}$ is a sequence of multivalued linear operators acting between the spaces X and Y $(1 \leqslant j \leqslant N)$, and $\mathcal{A}_j$ is an MLO in X $(1 \leqslant j \leqslant N)$. Then we say that:

(i) the sequences $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$ are d-topologically transitive iff for every open non-empty subset U of X and for every open non-empty subsets $V_1, \dots, V_N$ of Y, there exists $n \in \mathbb{N}$ such that

$$U \cap \mathcal{A}_{1,n}^{-1}(V_1) \cap \dots \cap \mathcal{A}_{N,n}^{-1}(V_N) \neq \emptyset; \tag{26}$$

(ii) the sequences $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$ are d-topologically mixing iff for every open non-empty subset $U \subseteq X$ and for every open non-empty subsets $V_1, \dots, V_N$ of Y, there exists $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, we have that (26) holds;

(iii) the operators $\mathcal{A}_1, \dots, \mathcal{A}_N$ are d-topologically transitive (d-topologically mixing) iff the sequences $(\mathcal{A}_1^n)_{n\in\mathbb{N}}, \dots, (\mathcal{A}_N^n)_{n\in\mathbb{N}}$ are d-topologically transitive (d-topologically mixing).

Definition 1.2.12. Given $N \geqslant 2$, the multivalued linear operators $\mathcal{A}_1, \dots, \mathcal{A}_N$ acting on X are said to be d-chaotic iff $\mathcal{A}_1, \dots, \mathcal{A}_N$ are d-topologically transitive and the set of periodic elements, denoted by $\mathcal{P}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N) := \{(x_1, x_2, \dots, x_N) \in X^N : \exists n \in \mathbb{N} \text{ with } x_j \in \mathcal{A}_j^n x_j,\ j \in \mathbb{N}_N\}$, is dense in X^N.

Remark 1.2.13. (i) Let the sequences $(\mathcal{A}_{1,n})_{n\in\mathbb{N}}, \dots, (\mathcal{A}_{N,n})_{n\in\mathbb{N}}$ be d-hypercyclic (d-topologically transitive, d-topologically mixing). Then for each $j \in \mathbb{N}_N$ the single operator sequence $(\mathcal{A}_{j,n})_{n\in\mathbb{N}}$ is hypercyclic (topologically transitive, topologically mixing).

(ii) Suppose that x is a periodic point of $\mathcal{A}$. Then $x \in D_\infty(\mathcal{A})$ and there exists an integer $n \in \mathbb{N}$ such that $x \in \mathcal{A}^n x$. This implies the existence of elements $y_j \in X$ $(1 \leqslant j \leqslant n-1)$ such that $(x, y_1) \in \mathcal{A}, (y_1, y_2) \in \mathcal{A}, \ldots, (y_{n-1}, x) \in \mathcal{A}$. Repeating this sequence, we easily get that $x \in \mathcal{A}^{kn} x$ for all $k \in \mathbb{N}$, so that the periodic points of $\mathcal{A}$ form a linear submanifold of X. Similarly, the set $\mathcal{P}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N)$ is a linear submanifold of X^N.

(iii) Suppose that Ω is an open connected subset of $\mathbb{K} = \mathbb{C}$ satisfying $\Omega \cap \mathfrak{S}_1 \neq \emptyset$, as well as that $N \geqslant 2$ and the multivalued linear operators $\mathcal{A}_1, \ldots, \mathcal{A}_N$ act on X. Let $f \colon \Omega \to X \setminus \{0\}$ be an analytic mapping such that $\lambda f(\lambda) \in \mathcal{A}_j f(\lambda)$ for all $\lambda \in \Omega$ and $j \in \mathbb{N}_N$. Set $\tilde{X} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. Then it is very simple to show that, for every $l \in \mathbb{N}$, the set $\mathcal{P}((\mathcal{A}_{1|\tilde{X}})^l, \ldots, (\mathcal{A}_{N|\tilde{X}})^l)$ is dense in $\tilde{X}^N$; cf. also the proof of Theorem 1.2.17 below.

Remark 1.2.14. In Definition 1.2.9–Definition 1.2.12, our assumptions on X and Y, as well as on $(\mathcal{A}_n)_{n\in\mathbb{N}}$, $\mathcal{A}$, $(\mathcal{A}_{j,n})_{n\in\mathbb{N}}$ and $\mathcal{A}_j$ can be substantially relaxed $(1 \leqslant j \leqslant N)$. It suffices to suppose that X and Y are topological spaces as well as that $\mathcal{A}_n$ and $\mathcal{A}_{j,n}$ ($\mathcal{A}$ and $\mathcal{A}_j$) are binary relations from X to Y (binary relations on X). To make this precise, we need to recall some basic facts about binary relations and their compositions. Suppose, for the time being, that X, Y, Z and T are given non-empty sets. The notion of reflexivity, symmetry, anti-symmetry and transitivity of a binary relation $\rho \subseteq X \times Y$ is defined in the usual way; the classes of RST relations and partial order relations on the set X are well-known. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and

$$\sigma \circ \rho := \{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \rho \text{ and } (y, t) \in \sigma\},$$

respectively. Domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x, \rho, y \Leftrightarrow (x, y) \in \rho$.

Assuming ρ is a binary relation on X and $n \in \mathbb{N}$, we define $\rho^n := \rho \circ \circ \circ \rho$ n-times inductively; $\rho^{-n} := (\rho^n)^{-1}$ and $\rho^0 := \Delta_X := \{(x, x) : x \in X\}$.

Set $D_\infty(\rho) := \bigcap_{n\in\mathbb{N}} D(\rho^n)$ and observe that the integer powers of an MLO $\mathcal{A}$ in an SCLCS are introduced, actually, by using the above definition for powers of the associated linear relation $\check{\mathcal{A}}$. Keeping in mind these notions, it is almost trivial to restate Definition 1.2.9–Definition 1.2.12 for (sequences of) binary relations between the topological spaces X and Y. In general, we do not assume that X or Y has a linear vector space structure; unquestionably, this assumption is not so pleasant because we are no longer in a position to clarify various types of (d-)hypercyclic and (d-)spectral criteria ensuring some kind of hypercyclic or topologically mixing behaviour of considered binary relations (cf. the continuation of this subsection for more details). In what follows, we will present a few illustrative examples:

(i) Let ρ be an RST relation on X. Then it is clear that ρ is hypercyclic iff there exists an element $x \in X$ such that $C_x := \{y \in X : x \ \rho \ y\}$ contains a sequence that is dense in X. If so, then any element of C_x is a hypercyclic element of ρ.

(ii) Let $X = G$ be finite and equipped with the discrete topology, let $|G| > 1$, and let ρ be a symmetric relation on G such that, for every $g \in G$, we have $(g, g) \notin \rho$. As is well-known, (G, ρ) is said to be a simple graph [**448**]. It can be simply proved that the graph G is connected iff G is hypercyclic iff G is chaotic; if this is the case, then any element of G is a hypercyclic element of ρ. Choosing some different topologies on G, as in part (iii) below, Definition 1.2.9 can be used to generalize the notion of connectivity of simple graphs. It is also worth noting that ρ need not be topologically mixing if it is topologically transitive; for example, if G is a square $x_1x_2x_3x_4$, then there does not exist an odd number $n \in \mathbb{N}$ such that $x_3 \in \rho^n(x_1)$; cf. Definition 1.2.9(v) with $U = \{x_3\}$ and $V = \{x_1\}$.

(iii) Let $X = G$ be finite, let (G, ρ) be a simple graph without isolated vertexes, and let τ be the topology on G. Denote by $G_1, \ldots, G_k$ the connected components of graph (G, ρ), where $k \in \mathbb{N}$. Then ρ is hypercyclic iff there exists a number $i \in \mathbb{N}_k$ such that G_i is dense in (G, τ), when any element of G_i is a hypercyclic element of ρ. For example, let $G = \{a, b, c, d\}$, and let $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}\}$ and $\tau_2 = \{\emptyset, \{a, b\}, \{a, b, c, d\}\}$ (according to Wikipedia, there exist 355 distinct topologies on G but only 33 inequivalent). Set

$$\rho := \{(a, d), (d, a), (b, c), (c, b)\}.$$

Then ρ is not hypercyclic in (G, τ_1) and ρ is hypercyclic in (G, τ_2). Observe, finally, that ρ is topologically transitive iff for every two open non-empty subsets U, V of X there exists $i \in \mathbb{N}_k$ such that $U \cap G_i \neq \emptyset$ and $V \cap G_i \neq \emptyset$.

Hypercyclic and disjoint hypercyclic extensions of binary relations, introduced here for multivalued linear operators, can be also analyzed; cf. our research study [**108**] for further information concerning this question. In that paper, we will scrutinize hypercyclic and disjoint hypercyclic properties of operators on Cayley graphs, as well.

Before proceeding further, it should be worthwhile to mention that, in single-valued linear case, J. Bés and A. Peris called the notion introduced in Definition 1.2.10 by d-universality (cf. also [**236**, Definition 2]).

EXAMPLE 1.2.15. It is clear that any multivalued linear extension of a hypercyclic (chaotic) single-valued linear operator is again hypercyclic (chaotic). It is also worth noting that Definition 1.2.9 prescribes some very strange situations in which even zero can be a hypercyclic vector: Let $\mathcal{A} := \{0\} \times W$, where W is a dense linear submanifold of X. Then $\mathcal{A}$ is hypercyclic, zero is the only hypercyclic vector of $\mathcal{A}$ and there is no single-valued linear restriction of $\mathcal{A}$ that is hypercyclic (in particular, a hypercyclic MLO need not be densely defined and the inverse of a hypercyclic MLO need not be hypercyclic, in contrast to the single-valued linear case, see also [**73**, Problem 1]); furthermore, Definition 1.2.10 is very strange in multivalued linear operators setting because the operators $\mathcal{A}, \ldots, \mathcal{A}$ are all the same and d-hypercyclic with zero being its only d-hypercyclic vector (as the analysis performed by J. Bés and A. Peris on p. 299 of [**73**] shows, a single-valued linear operator A and its constant multiple cA cannot be d-hypercyclic, $c \in \mathbb{K}$). Observe, finally, that the non-triviality of submanifold $\mathcal{A}0$ is not so directly and essentially connected with the hypercyclicity of $\mathcal{A}$: Let $\mathcal{A} := X \times W$, where W is a non-dense linear submanifold of X. Then $\mathcal{A}$ cannot be hypercyclic (topologically transitive).

Let $(O_n)_{n\in\mathbb{N}}$ be an open base of the topology of X, and let $O_n \neq \emptyset$ for every $n \in \mathbb{N}$. Then the set consisting of hypercyclic vectors of an arbitrary multivalued linear operator $\mathcal{A}$, denoted shortly by $\mathrm{HC}(\mathcal{A})$, can be computed by

$$\mathrm{HC}(\mathcal{A}) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \mathcal{A}^{-k}(O_n). \tag{27}$$

Similarly, the set consisting of all d-hypercyclic vectors of arbitrary multivalued linear operators $\mathcal{A}_1, \ldots, \mathcal{A}_N$, denoted shortly by $\mathrm{HC}(\mathcal{A}_1, \ldots, \mathcal{A}_N)$, can be computed by

$$\mathrm{HC}(\mathcal{A}_1, \ldots, \mathcal{A}_N) = \bigcap_{j_1,j_2,\ldots,j_N\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{l=1}^{N} \mathcal{A}_l^{-k}(O_{j_l}). \tag{28}$$

The formulae (27)–(28) become most important in the case that the powers of operator $\mathcal{A}$ (operators $\mathcal{A}_1, \ldots, \mathcal{A}_N$) are continuous; cf. [**120**, Section II.3] for the notion, and [**236**] for more details about this subject and universal families of continuous linear operators. The study of some important subcases of universality, like supercyclicity and positive supercyclicity, is also without the scope of our analysis (cf. [**291**,**292**,**390**,**462**] and references cited therein for more details on the subject).

Suppose now that $\mathcal{A}$ is an MLO in X, as well as that Y is another SCLCS and $\phi\colon X \to Y$ is a linear topological homeomorphism (cf. also [**236**, Proposition 4] and [**409**, Proposition 1.3.10]). Then we define the operator $\mathcal{A}_Y$ in Y by

$$D(\mathcal{A}_Y) := \phi(D(\mathcal{A})) \quad\text{and}\quad \mathcal{A}_Y(\phi(x)) := \phi(\mathcal{A}x), \quad x \in D(\mathcal{A}).$$

It is checked at once that $\mathcal{A}_Y$ is an MLO in Y. Keeping this fact in mind, we can formulate a great number of (d-)hypercyclic comparision principles for multivalued linear operators. We leave these questions to the interesed reader.

It is an elementary fact that the point spectrum of adjoint of a hypercyclic continuous single-valued linear operator has to be empty [**284**]. As the next proposition shows, the same thing holds for hypercyclic multivalued linear operators:

Proposition 1.2.16. *Let $\mathcal{A}$ be a hypercyclic* MLO *in X. Then $\sigma_p(\mathcal{A}^*) = \emptyset$.*

Proof. Let x be a hypercyclic vector of $\mathcal{A}$; hence, for every $n \in \mathbb{N}$, there exists an element $y_n \in \mathcal{A}^n x$ such that the set $\{y_n : n \in \mathbb{N}\}$ is dense in X. Suppose to the contrary that there exist $\lambda \in \mathbb{K}$ and $x^* \in X^* \smallsetminus \{0\}$ such that $\lambda x^* \in \mathcal{A}^* x^*$, i.e., that $\langle x^*, y\rangle = \lambda\langle x^*, x\rangle$, whenever $y \in \mathcal{A}x$. It is clear that $\langle x^*, y_n\rangle = \lambda^n\langle x^*, x\rangle$ for all $n \in \mathbb{N}$, so that the assumption $\langle x^*, x\rangle = 0$ implies $\langle x^*, y_n\rangle = 0$ for all $n \in \mathbb{N}$, and therefore, $x^* = 0$. So, $\langle x^*, x\rangle \neq 0$. If $|\lambda| \leqslant 1$, then $|\langle x^*, y_n\rangle| = |\lambda^n\langle x^*, x\rangle| \leqslant |\langle x^*, x\rangle|$ for all $n \in \mathbb{N}$; this would imply $|\langle x^*, u\rangle| \leqslant |\langle x^*, x\rangle|$ for all $u \in X$, which is a contradiction since $|\langle x^*, nu\rangle| \to +\infty$ for any $u \in X$ such that $\langle x^*, u\rangle \neq 0$. If

$|\lambda| > 1$, then $|\langle x^*, y_n\rangle| = |\lambda^n \langle x^*, x\rangle| \geqslant |\langle x^*, x\rangle|$ for all $n \in \mathbb{N}$; this would imply $|\langle x^*, u\rangle| \geqslant |\langle x^*, x\rangle|$ for all $u \in X$, which is a contradiction since $|\langle x^*, u/n\rangle| \to 0$ for any $u \in X$, and $\langle x^*, x\rangle \neq 0$. □

The following theorem extends the well-known result of R. M. Aron, J. B. Seoane-Sepúlveda and A. Weber [**31**, Theorem 2.1]. This is, actually, a kind of Godefroy–Shapiro and Dech–Schappacher–Webb Criterion (continuous version) for multivalued linear operators.

THEOREM 1.2.17. *Suppose that Ω is an open connected subset of $\mathbb{K} = \mathbb{C}$ satisfying $\Omega \cap \mathfrak{S}_1 \neq \emptyset$. Let $f \colon \Omega \to X \smallsetminus \{0\}$ be an analytic mapping such that $\lambda f(\lambda) \in \mathcal{A} f(\lambda)$ for all $\lambda \in \Omega$. Set $\tilde{X} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. Then the operator $\mathcal{A}_{|\tilde{X}}$ is topologically mixing in the space $\tilde{X}$ and the set of periodic points of $\mathcal{A}_{|\tilde{X}}$ is dense in $\tilde{X}$.*

PROOF. The proof of theorem is very similar to that of Theorem 2.10.3 and we will only outline the most relevant details. Without loss of generality, we may assume that $\tilde{X} = X$. If Ω_0 is an arbitrary subset of Ω which admits a cluster point in Ω, then the (weak) analyticity of mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \Omega$ shows that $\Psi(\Omega_0) := \text{span}\{f(\lambda) : \lambda \in \Omega_0\}$ is dense in X. Further on, it is clear that there exist numbers $\lambda_0 \in \Omega \cap \mathfrak{S}_1$ and $\delta > 0$ such that any of the sets $\Omega_{0,+} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta,\ |\lambda| > 1\}$ and $\Omega_{0,-} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta,\ |\lambda| < 1\}$ admits a cluster point in Ω. Suppose that U and V are two open non-empty subsets of X. Then there exist $y, z \in X$, $\varepsilon > 0$, $p, q \in \circledast$ such that $SL_p(y, \varepsilon) \subseteq U$ and $SL_q(z, \varepsilon) \subseteq V$. We may assume that $y \in \Psi(\Omega_{0,-})$, $z \in \Psi(\Omega_{0,+})$, $y = \sum_{i=1}^{n} \beta_i f(\lambda_i)$, $z = \sum_{j=1}^{m} \gamma_j f(\tilde{\lambda_j})$, where $\alpha_i, \beta_j \in \mathbb{C} \smallsetminus \{0\}$, $\lambda_i \in \Omega_{0,-}$ and $\tilde{\lambda_j} \in \Omega_{0,+}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Put $z_t := \sum_{j=1}^{m} \frac{\gamma_j}{\tilde{\lambda_j}^t} f(\tilde{\lambda_j})$ and $x_t := y + z_t$, $t \geqslant 0$. Then $\{x_t,\ y,\ z_t\} \subseteq D_\infty(\mathcal{A})$, $t \geqslant 0$ and it can be easily seen that $z \in \mathcal{A}^n z_n$, $n \in \mathbb{N}$ and $\omega_n := z + \sum_{i=1}^{n} \beta_i \lambda_i^n f(\lambda_i) \in \mathcal{A}^n x_n$, $n \in \mathbb{N}$, as well as that there exists $n_0(\varepsilon) \in \mathbb{N}$ such that, for every $n \geqslant n_0(\varepsilon)$, $x_n \in SL_p(y, \varepsilon)$ and $w_n \in SL_q(z, \varepsilon)$. Therefore, $\mathcal{A}$ is topologically mixing. Since the set $\Omega \cap \exp(2\pi i \mathbb{Q})$ has a cluster point in Ω, the proof that the set of periodic points of $\mathcal{A}$ is dense in X can be given as in that of [**31**, Theorem 2.1]. □

REMARK 1.2.18. It is clear that the validity of implication:
$$\langle x^*, f(\lambda)\rangle = 0,\ \lambda \in \Omega \text{ for some } x^* \in X^* \Rightarrow x^* = 0$$
yields that $\tilde{X} = X$.

Now we would like to illustrate Theorem 1.2.17 with a concrete example that is very similar to Example 2.10.6. It provides the existence of a substantially large class of topologically mixing multivalued linear operators with dense set of periodic points.

EXAMPLE 1.2.19. Suppose that A is a closed linear operator on X satisfying that there exist an open connected subset $\emptyset \neq \Lambda$ of $\mathbb{C}$ and an analytic mapping $g \colon \Lambda \to X \smallsetminus \{0\}$ such that $Ag(\nu) = \nu g(\nu)$, $\nu \in \Lambda$. Let $P(z)$ and $Q(z)$ be non-zero complex polynomials, let $R := \{z \in \mathbb{C} : P(z) = 0\}$, $\Lambda' := \Lambda \smallsetminus R$, and let

$\tilde{X} := \overline{span\{g(\lambda) : \lambda \in \Lambda\}}$. Suppose that

$$\frac{Q}{P}(\Lambda') \cap S_1 \neq \emptyset.$$

Then Theorem 1.2.17 implies that the parts of multivalued linear operators $Q(\mathcal{A})P(\mathcal{A})^{-1}$ and $P(\mathcal{A})^{-1}Q(\mathcal{A})$ in $\tilde{X}$ are topologically mixing in the space $\tilde{X}$, and that the sets of periodic points of these operators are dense in $\tilde{X}$.

It is worth noting that L. Bernal-González investigated in [**67**, Theorem 4.3] a disjoint analogue of Theorem 1.2.17 for continuous linear operators. Now we will state the following version of afore-mentioned theorem for multivalued linear operators without going into full details concerning the case in which there exists an integer $p_0 \in \mathbb{N}_N^0$ for which the set D_{p_0} is not total in X. The proof goes similarly and therefore omitted.

THEOREM 1.2.20. *Suppose that $N \in \mathbb{N}$ and $\mathcal{A}_1, \ldots, \mathcal{A}_N$ are given* MLOs *acting on X. Let for each $p \in \mathbb{N}_N^0$ there exists a total set D_p (that is, the linear span of D_p is dense in X) such that the following conditions hold:*

(i) *Any element of the set D_p is an eigenvector of any operator $\mathcal{A}_j$ ($p \in \mathbb{N}_N^0$, $j \in \mathbb{N}_N$); if $e \in D_p$, then there exists an eigenvalue $\lambda_{j,p}(e)$ of the operator $\mathcal{A}_j$ for which $\lambda_{j,p}(e)e \in \mathcal{A}_j e$ and* (ii)–(iii) *hold, where:*

(ii) $\lambda_{j,0}(e) \in int(\mathfrak{S}_1)$, $j \in \mathbb{N}_N$, $e \in D_0$ *and* $\lambda_{j,j}(e) \in ext(\mathfrak{S}_1)$, $j \in \mathbb{N}_N$, $e \in D_j$;

(iii) *Suppose $i, j \in \mathbb{N}_N$ and $i \neq j$. Then, for every $e \in D_i$, we have* $|\lambda_{j,i}(e)| < |\lambda_{i,i}(e)|$.

Then the operators $\mathcal{A}_1, \ldots, \mathcal{A}_N$ are d-topologically mixing.

Now we will illustrate Theorem 1.2.20 with a simple example pointing out that the continuity of operators can be neglected from the formulation of [**67**, Theorem 4.3] (the metrizability of state space X is also inrelevant in our approach), as well as that there exists a great number of Banach function spaces where this extended version of afore-mentioned theorem can be applied (cf. also [**67**, Final questions, 2.]). Numerous other examples involving multivalued linear operators can be given similarly, by using the analysis from Example 1.2.19; cf. also [**35**], Example 2.11.6 and [**457**] for certain unbounded differential operators that we can employ here.

EXAMPLE 1.2.21. [**259**] Suppose that $p > 2$. Let X be a symmetric space of non-compact type and rank one, let P_p be the parabolic domain defined in the proof of [**259**, Theorem 3.1], and let $c_p > 0$ be the apex of P_p. Then we know that $int(P_p) \subseteq \sigma_p(\Delta^{\natural}_{X,p})$, where $\Delta^{\natural}_{X,p}$ denotes the corresponding Laplace–Beltrami operator acting on $L^p_{\natural}(X)$; furthermore, there exists an analytic function $g \colon int(P_p) \to L^p_{\natural}(X)$ such that $\Delta^{\natural}_{X,p} g(\lambda) = \lambda g(\lambda)$, $\lambda \in int(P_p)$ and the set $\Psi(\Omega) := \{g(\lambda) : \lambda \in \Omega\}$ is total in X for any open, non-empty subset Ω of $int(P_p)$. Suppose that $N \geqslant 2$, $-1 - c_p < a_1 < a_2 < \cdots < a_N < 1 - c_p$ and, for every $i \in \mathbb{N}_N$, there exists a point $\lambda_i \in int(P_p)$ such that $|\lambda_i + a_i| > \max(1, |\lambda_i + a_j|)$ for all $j \in \mathbb{N}_N \setminus \{i\}$ (this case can really occur). An application of Theorem 1.2.20 (with $D_i = \Psi(\Omega_i)$, Ω_0 is a small ball around c_p+ and Ω_i is a small ball around λ_i, $i \in \mathbb{N}_N$) shows that the operators $\Delta^{\natural}_{X,p}+a_1, \Delta^{\natural}_{X,p}+a_2, \ldots, \Delta^{\natural}_{X,p}+a_N$ are d-topologically mixing; furthermore, by Remark 1.2.13, the set $\mathcal{P}(\Delta^{\natural}_{X,p}+a_1, \ldots, \Delta^{\natural}_{X,p}+a_N)$ is dense in $(L^p_{\natural}(X))^N$. Hence, the operators $\Delta^{\natural}_{X,p}+a_1, \Delta^{\natural}_{X,p}+a_2, \ldots, \Delta^{\natural}_{X,p}+a_N$ are d-chaotic.

It is almost straigtforward to state a continuous version of Theorem 1.2.20 for abstract degenerate fractional inclusions in separable SCLCSs (see also Theorem 3.6.3 below).

The Hypercyclicity Criterion for single-valued linear operators has been discovered independently by C. Kitai [**284**] and R. M. Gethner, J. H. Shapiro [**224**]. It is worth noting that this criterion can be formulated, in a certain way, for multivalued linear operators and that we do not need any type of continuity or closedness of operators under examination for its validity; the proof of next proposition can be obtained by the methods of Theorem 2.10.7 (continuous version), which is omitted here.

PROPOSITION 1.2.22. (Hypercyclicity Criterion for MLOs) *Let $\mathcal{A}$ be an* MLO *in X. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Let the set X_0, consisting of those elements $y \in X$ for which there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $y_n \in \mathcal{A}^{a_n}y$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = 0$, be dense in X. Furthermore, let the set X_∞, consisting of those elements $z \in X$ for which there exist a null sequence $(\omega_n)_{n\in\mathbb{N}}$ in X and a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that $u_n \in \mathcal{A}^{a_n}\omega_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = z$, be also dense in X. Then $\underbrace{\mathcal{A}\oplus\cdots\oplus\mathcal{A}}_{k}$ is topologically transitive ($k \in \mathbb{N}$).*

The question of whether any continuous hypercyclic single-valued linear operator on Banach space satisfies the Hypercyclicity Criterion was open for a long time; as is well-known, the negative answer was given by M. De La Rosa and C. Read in [**129**]. Now we will state a version of d-Hypercyclicity Criterion for MLOs; see [**73**, Proposition 2.6, Theorem 2.7, Remark 2.8] for single-valued linear case.

PROPOSITION 1.2.23. (d-Hypercyclicity Criterion for MLOs) *Let $N \in \mathbb{N}$, and let $\mathcal{A}_j$ be an* MLO *in X ($1 \leqslant j \leqslant N$). Suppose that $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Let the set X_0, consisting of those elements $y \in X$ satisfying that for each $j \in \mathbb{N}_N$ there exists a sequence $(y_{n,j})_{n\in\mathbb{N}}$ in X such that $y_{n,j} \in \mathcal{A}_j^{a_n}y$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_{n,j} = 0$, be dense in X. Furthermore, let for each $j \in \mathbb{N}_N$ there exists a dense set $X_{\infty,j}$, consisting of those elements $z \in X$ for which there exist elements $\omega_{n,i}(z)$ and $u_{n,i,j}(z)$ in X ($n \in \mathbb{N}$, $1 \leqslant i \leqslant N$) such that $(\omega_{n,j}(z))_{n\in\mathbb{N}}$ is a null sequence in X, $u_{n,i,j}(z) \in \mathcal{A}_j^{a_n}\omega_{n,i}(z)$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_{n,i,j} = \delta_{i,j}z$ ($1 \leqslant i \leqslant N$). Then the operators $\underbrace{\mathcal{A}_1\oplus\cdots\oplus\mathcal{A}_1}_{k}, \ldots, \underbrace{\mathcal{A}_N\oplus\cdots\oplus\mathcal{A}_N}_{k}$ are d-topologically transitive ($k \in \mathbb{N}$).*

PROOF. We will only outline the main details of proof. Since for any multivalued linear operator $\mathcal{A}$ we have

$$\Bigl(\underbrace{\mathcal{A}\oplus\cdots\oplus\mathcal{A}}_{k}\Bigr)^n = \underbrace{\mathcal{A}^n\oplus\cdots\oplus\mathcal{A}^n}_{k}, \quad k, n \in \mathbb{N},$$

we may assume without loss of generality that $k = 1$ (otherwise, we can pass to the subsets X_0^k, $X_{\infty,j}^k$ $(1 \leqslant j \leqslant N)$, as well as to the tuples $(z, \dots, z)$, $(y_{n,j}, \dots, y_{n,j})$, $(\omega_{n,i}, \dots, \omega_{n,i})$, $(u_{n,i,j}, \dots, u_{n,i,j})$, each of which have exactly k components $(1 \leqslant j \leqslant N)$). The remainder of the proof can be given by repeating verbatim the corresponding part of proof of [**73**, Proposition 2.6]. $\square$

Observe that the assertions of Proposition 1.2.22 and Proposition 1.2.23 can be formulated for the sequences of multivalued linear operators (cf. [**73**, Remark 2.8] and the proof of Theorem 1.2.24 below), as well as that Proposition 1.2.22 and Proposition 1.2.23 give sufficient conditions for (d-)topological transitivity of operators. A sufficient condition for hypercyclicity of linear operators in Banach spaces, the real Hypercyclicity Criterion for linear operators, if we could say something like that, was stated for the first time by J. Bès, K. C. Chen and S. M. Seubert [**69**, Theorem 2.1] (cf. also R. deLaubenfels, H. Emamirad and K.-G. Grosse-Erdmann [**137**, Theorem 2.3]). As mentioned on p. 49 of [**137**], neither of the assertions [**69**, Theorem 2.1] and [**137**, Theorem 2.3] is not contained in the other. We continue by stating the following d-hypercyclic analogue of [**137**, Theorem 2.3] for single-valued linear operators in Fréchet spaces:

THEOREM 1.2.24. *Suppose that X is a Fréchet space, $T_1, \dots, T_N$ are single-valued linear operators on X $(1 \leqslant j \leqslant N)$ and $C \in L(X)$ is injective. Suppose that there exists a subset X_0 of $D_\infty(T_1) \cap \cdots \cap D_\infty(T_N)$, dense in X, as well as dense subsets $X_1, \dots, X_N$ of X and mappings $S_{j,n} \colon X_j \to D_\infty(T_1) \cap \cdots \cap D_\infty(T_N)$ $(1 \leqslant j \leqslant N,\ n \in \mathbb{N})$ such that:*

(i) $\lim_{n\to\infty} T_j^n x_0 = 0$, $x_0 \in X_0$,
(ii) $\lim_{n\to\infty} S_{j,n} x_j = 0$, $x_j \in X_j$, $1 \leqslant j \leqslant N$,
(iii) $\lim_{n\to\infty} [T_j^n S_{i,n} x_j - \delta_{j,i} x_j] = 0$, $x_j \in X_j$, $1 \leqslant i, j \leqslant N$,
(iv) $R(C) \subseteq D_\infty(T_1) \cap \cdots \cap D_\infty(T_N)$ *and* $T_j^n C \in L(X)$, $1 \leqslant j \leqslant N$,
(v) $CT_j x = T_j C x$, $x \in D_\infty(T_j)$, $1 \leqslant j \leqslant N$,
(vi) $R(C)$ *is dense in* X.

Then the operators $T_1, \dots, T_N$ are d-hypercyclic.

PROOF. Define the operators $\mathfrak{T}_{j,n} \in L([R(C)]_{\circledast}, X)$ by $\mathfrak{T}_{j,n}(Cx) := T_j^n C x$, $x \in X$ (cf. (iv)). It suffices to show that the sequences $(\mathfrak{T}_{1,j})_{j\in\mathbb{N}}, \dots, (\mathfrak{T}_{N,j})_{j\in\mathbb{N}}$ are d-hypercyclic. Since the final conclusions of [**73**, Remark 2.8] also hold for sequences of continuous linear operators acting between different spaces, we need to prove the existence of a dense subset X_0' of $[R(C)]_{\circledast}$, dense subsets $X_1', \dots, X_N'$ of X and mappings $S_{j,n}' \colon X_j' \to [R(C)]$ $(1 \leqslant j \leqslant N,\ n \in \mathbb{N})$ such that the following holds:

(a) $\lim_{n\to\infty} \mathfrak{T}_{j,n} x_0' = 0$, $x_0' \in X_0'$,
(b) $\lim_{n\to\infty} S_{j,n}' x_j' = 0$, $x_j' \in X_j'$, $1 \leqslant j \leqslant N$, and
(c) $\lim_{n\to\infty} [\mathfrak{T}_{j,n} S_{i,n}' x_j' - \delta_{j,i} x_j'] = 0$, $x_j' \in X_j'$, $1 \leqslant i, j \leqslant N$.

Set $X_j' := C(X_j)$, $j \in \mathbb{N}_N^0$ and $S_{j,n}' \colon X_j' \to R(C)$ by $S_{j,n}'(Cx_j) := CS_{j,n} x$, $x_j \in X_j$ $(1 \leqslant j \leqslant N,\ n \in \mathbb{N})$. By (vi) and the density of X_j in X, we get that X_0' is dense in $[R(C)]_{\circledast}$ and X_j' is dense in X $(1 \leqslant j \leqslant N)$. The property (b) follows immediately from (ii) and definition of $S_{j,n}'$. The property (a) follows by making use the fact that X_0 belongs to $D_\infty(T_1) \cap \cdots \cap D_\infty(T_N)$, as well as from (v), (i) and definition of $\mathfrak{T}_{j,n}$. By (iii), (v) and inclusion $R(S_{j,n}) \subseteq D_\infty(T_1) \cap \cdots \cap D_\infty(T_N)$, we have

$$\begin{aligned}\lim_{n\to\infty}[\mathfrak{T}_j^n S'_{i,n}x'_j - \delta_{j,i}x'_j] &= \lim_{n\to\infty}[T_j^n CS_{i,n}x_j - C\delta_{j,i}x_j]\\ &= \lim_{n\to\infty} C[T_j^n S_{i,n}x_j - \delta_{j,i}x_j]\\ &= C\lim_{n\to\infty}[T_j^n S_{i,n}x_j - \delta_{j,i}x_j] = 0,\end{aligned}$$

provided that $x'_j = Cx_j \in X'_j$, $1 \leqslant j \leqslant N$. The proof of the theorem is thereby complete. $\square$

Now we would like to provide an illustrative application of Theorem 1.2.24. Let the requirements of Example 1.2.21 hold. Then we can simply prove with the help of Theorem 1.2.24 that the operators $\Delta^\natural_{X,p} + a_1, \Delta^\natural_{X,p} + a_2, \ldots, \Delta^\natural_{X,p} + a_N$ are d-hypercyclic, as well, by taking the injective operator

$$C := \frac{1}{2\pi i}\int_\Gamma e^{-(-\lambda)^b}\big(\lambda + \omega + \Delta^\natural_{X,p}\big)^{-1}d\lambda,\ \omega > 0 \text{ suff. large, } b \in (0, 1/2) \text{ fixed}$$

under advisement; here, Γ is the upwards oriented boundary of $\Sigma_\alpha \cup \{z \in \mathbb{C} : |z| \leqslant d\}$ with suitable chosen numbers $\alpha \in (0, \pi/2)$ and $d \in (0, 1]$. Then it is well known that (iv)–(vi) hold, see [**292**]; (i)–(iii) follows from the analyses already carried out in Example 1.2.21.

Now we want to turn our attention to the following theme:

1.3. Hypercyclic and disjoint hypercyclic MLO extensions

Let $\mathcal{A}$ be an MLO in X. In what follows, we will often identify $\mathcal{A}$ and its associated linear relation $\check{\mathcal{A}}$, which will be denoted by the same symbol.

It is clear that $\mathcal{A}$ is contained in $X \times X$, which is hypercyclic, chaotic and topologically mixing (transitive). Denote

$$S(\mathcal{A}) := \{Z : Z \text{ is a linear subspace of } X \times X \text{ and } \mathcal{A} \subseteq Z\}.$$

We say that an MLO $\mathcal{B}$ is a hypercyclic (chaotic, topologically mixing, topologically transitive) extension of $\mathcal{A}$ iff $\mathcal{B} \in S(\mathcal{A})$ and $\mathcal{B}$ is hypercyclic (chaotic, topologically mixing, topologically transitive).

Further on, let $N \in \mathbb{N}$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_N$ be given MLOs in X. Then the MLOs $X \times X, \ldots, X \times X$, totally counted N times, are d-hypercyclic, d-chaotic and d-topologically mixing (transitive). Denote

$$\begin{aligned}S(\mathcal{A}_1, \ldots, \mathcal{A}_N) := \{(\mathcal{B}_1, \ldots, \mathcal{B}_N) : \mathcal{B}_i &\text{ is a linear subspace of } X \times X\\ &\text{and } \mathcal{A}_i \subseteq \mathcal{B}_i \text{ for all } i \in \mathbb{N}_N\}.\end{aligned}$$

We say that the tuple $(\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of MLOs in X is a d-hypercyclic (d-chaotic, d-topologically mixing, d-topologically transitive) extension of $(\mathcal{A}_1, \ldots, \mathcal{A}_N)$ iff $(\mathcal{B}_1, \ldots, \mathcal{B}_N) \in S(\mathcal{A}_1, \ldots, \mathcal{A}_N)$ and the operators $\mathcal{B}_1, \ldots, \mathcal{B}_N$ are d-hypercyclic (d-chaotic, d-topologically mixing, d-topologically transitive).

We proceed with a simple example.

EXAMPLE 1.3.1. (i) Suppose that $X := \mathbb{C}^n$ and $\mathcal{A} := A \in L(X)$. Then $\mathcal{A}$ is not hypercyclic, and $\mathbb{C}^n \times \mathbb{C}^n$ is the only hypercyclic (chaotic, topologically mixing, topologically transitive) extension of $\mathcal{A}$; a similar statement holds for finite-dimensional spaces.

(ii) Suppose that X is infinite-dimensional, W is a non-dense linear subspace of X and $\mathcal{A} = X \times W$. Then any hypercyclic MLO extension of $\mathcal{A}$ has the form $X \times W'$, where W' is a dense linear subspace of X containing W.

Consider now the following MLO extension of $\mathcal{A}$:

$$\tilde{\mathcal{A}} := \bigcap \{Z \in S(\mathcal{A}) : Z \text{ is hypercyclic}\}.$$

We call $\tilde{\mathcal{A}}$ the quasi-hypercyclic extension of $\mathcal{A}$ and similarly define the quasi-chaotic (quasi-topologically transitive, quasi-topologically mixing) extension of $\mathcal{A}$.

EXAMPLE 1.3.2. Suppose that X is infinite-dimensional, $\{y_k : k \in \mathbb{N}\}$ is a dense subset of X, and $A \in L(X)$. Then any MLO extension $\mathcal{A}$ of A has the form $\mathcal{A}x = Ax + W$, where $W = \mathcal{A}0$ is a linear submanifold of X. Inductively,

$$\mathcal{A}^n x = A^n x + \sum_{j=0}^{n-1} A^j(W), \quad n \in \mathbb{N},\ x \in X. \tag{29}$$

Denote by $\mathfrak{T}$ the set consisting of all linear manifolds W' of X such that there exists an element $x \in X$ with the property that, for every $n \in \mathbb{N}$, there exist elements $z_{n,j}$ in W' $(0 \leqslant j \leqslant n-1)$ such that the set $\{A^n x + \sum_{j=0}^{n-1} A^j z_{n,j} : n \in \mathbb{N}\}$ is dense in X. Then $\mathfrak{T}$ is non-empty because $X \in \mathfrak{T}$ (with $x = 0$, $z_{n,j} = 0$, $1 \leqslant j \leqslant n-1$, $z_{n,0} = y_n$), and $\tilde{\mathcal{A}} = A + \bigcap \mathfrak{T}$. Now we will scrutinize some particular cases:

(i) Let X be an infinite-dimensional complex Hilbert space with the complete orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Define $A \sum_{n=1}^{\infty} x_n e_n := \sum_{n=1}^{\infty} x_n e_{n+1}$, for any $x = \sum_{n=1}^{\infty} x_n e_n \in X$. Then $\|A\| = 1$ and therefore $\mathcal{A}$ is not hypercyclic. It can be easily seen that any linear manifold W' belonging $\mathfrak{T}$ has to contain the linear span of $\{\omega\}$, for some element ω of X such that $\langle \omega, e_1 \rangle \neq 0$. Let W be the linear span of $\{e_1\}$. In what follows, we will prove that $A + W$ is a hypercyclic extension of A, with zero being the corresponding hypercyclic vector. It is clear that there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\Bigg\| y_k - \sum_{j=0}^{n_k - 1} \alpha_{n_k,j} e_{j+1} \Bigg\| < 2^{-k},$$

for some scalars $\alpha_{n_k,j}$ $(0 \leqslant j \leqslant n_k - 1)$. Clearly, the linear span of $\{e_1, \ldots, e_l\}$ is contained in $\sum_{j=0}^{l-1} A^j(W)$, for any $l \in \mathbb{N}$. Plugging $z_{n,j} := 0$, $0 \leqslant j \leqslant n-1$, if $n \neq n_k$ for all $k \in \mathbb{N}$, and $z_{n,j} := \alpha_{n_k,j} e_1$ $(0 \leqslant j \leqslant n_k - 1)$, if $n = n_k$ for some $k \in \mathbb{N}$, we can simply deduce that zero is a hypercyclic vector of $A + W$, as claimed.

(ii) Any hypercyclic MLO extension $\mathcal{A}$ of the identity operator I on X has the form $\mathcal{A}x = x + W$, $x \in X$, where $W = \mathcal{A}0$ is a dense linear submanifold of X (when any element $x \in X$ is a hypercyclic vector of $\mathcal{A}$).

(iii) Let $X := l_2(\mathbb{Z})$, let $(a_n)_{n\in\mathbb{Z}}$ be a bounded subset of $(0,\infty)$, and let

$$A \sum_{n=-\infty}^{\infty} x_n e_n := \sum_{n=-\infty}^{\infty} x_n a_n e_{n+1} \text{ for any } x = \sum_{n=-\infty}^{\infty} x_n e_n \in X; \tag{30}$$

here, $\{e_n : n \in \mathbb{Z}\}$ denotes the complete orthonormal basis of X. The hypercyclicity of bilateral weighted shift A has been characterized by H. Salas in [**470**, Theorem 2.1]: Given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists a sufficiently large integer $n \in \mathbb{N}$ such that $\prod_{s=0}^{n-1} a_{s+j} < \varepsilon$ and $\prod_{s=1}^{n} a_{j-s} > 1/\varepsilon$. Suppose that this condition does not hold. Then it can be simply verified with the help of (29) that $\operatorname{span}(\{e_{n_k} : k \in \mathbb{N}\}) \in \mathfrak{T}$ for any strictly decreasing sequence $(n_k)_{k\in\mathbb{N}}$ of negative integers. Hence, $\tilde{A} = A$.

(iv) Suppose that W is a dense linear submanifold of X, and $\mathcal{A}x = Ax + W$, $x \in X$. Let U and V be two arbitrary open non-empty subsets of X. Then (29) implies that, for every $n \in \mathbb{N}$ and $u \in U$, there exists an element $\omega \in W$ such that $A^n u + \omega \in V \cap \mathcal{A}^n u$; in particular, $\mathcal{A}$ is topologically mixing.

(v) Suppose that W is a linear submanifold of X, $\mathcal{A}x = Ax + W$, $x \in X$, and for each $\omega \in W$ there exist an integer $n \in \mathbb{N}$ and elements $\omega_j \in W$ $(1 \leqslant i \leqslant n-1)$ such that $\sum_{j=1}^{n-1} A^j \omega_j = -A^n \omega$ (this, in particular, holds provided that there exists an integer $n \in \mathbb{N}$ such that $A^n(W) \subseteq W$). Then (29) implies that $\omega \in \mathcal{A}^n \omega$, $w \in W$, so that W is consisted solely of periodic points of $\mathcal{A}$. Suppose now that $R(A)$ is dense in X, hence $R(A^j)$ is dense in X for all $j \in \mathbb{N}$. Using this fact and (iv), we get that $\mathcal{A}x = Ax + R(A^j)$, $x \in X$ is a chaotic extension of A for all $j \in \mathbb{N}$; this is no longer true if $R(A)$ is not dense in X, even if dimension of $X \smallsetminus \overline{R(A)}$ equals 1, cf. (i). In the case that the operator A is nilpotent and W is a dense submanifold of X, then it can be easily seen that $\mathcal{A}x = Ax + W$, $x \in X$ is a chaotic extension of A.

We can similarly introduce the notion of a d-quasi-hypercyclic extension $\widetilde{(\mathcal{A}_1, \dots, \mathcal{A}_N)}$ (d-quasi-topologically transitive extension, d-quasi-topologically mixing extension) of any MLO tuple $(\mathcal{A}_1, \dots, \mathcal{A}_N)$:

$$\widetilde{(\mathcal{A}_1, \dots, \mathcal{A}_N)} := \bigcap \{(\mathcal{B}_1, \dots, \mathcal{B}_N) \in S(\mathcal{A}_1, \dots, \mathcal{A}_N) : \mathcal{B}_1, \dots, \mathcal{B}_N\}.$$

are d-hypercyclic

The following is a continuation of Example 1.3.2:

Example 1.3.3. Suppose that X is infinite-dimensional and $A_1, \dots, A_N \in L(X)$. Then any MLO extension of tuple $(A_1, \dots, A_N)$ has the form $(A_1 + W_1, \dots, A_N + W_N)$, where W_i is a linear submanifold of X $(1 \leqslant i \leqslant N)$. Using (29), we have that $(A_1 + W_1, \dots, A_N + W_N)$ is d-hypercyclic iff there exists an element $x \in X$ with the property that, for every $n \in \mathbb{N}$, there exist elements $z_{n,j,l}$ in W_l $(0 \leqslant j \leqslant n-1,$

$1 \leqslant l \leqslant N$) such that the set $\{(A_1^n x+\sum_{j=0}^{n-1} A_1^j z_{n,j,1},\dots,A_N^n x+\sum_{j=0}^{n-1} A_N^j z_{n,j,N}) : n \in \mathbb{N}\}$ is dense in X^N; for example, $(A_1+X,\dots,A_N+X)$ is d-hypercyclic with $x=0$ being the corresponding d-hypercyclic vector. This immediately implies the profilation of $(\mathcal{A}_1,\dots,\mathcal{A}_N)$. Arguing as in Example 1.3.2, it can be easily seen that the following holds:

(i) Let A be the operator examined in Example 1.3.2(i). Then $(A+W_1, A^2+W_2,\dots,A^N+W^N)$ is a d-hypercyclic extension of tuple $(A,A^2,\dots,A^N)$, where W_i is the linear span of $\{e_1,e_2,\dots,e_i\}$ $(1 \leqslant i \leqslant N)$.

(ii) Any d-hypercyclic extension of the tuple $(I,\dots,I)$ has the form $(I+W_1,\dots,I+W_N)$, where W_i is a dense linear submanifold of X $(1 \leqslant i \leqslant N)$.

(iii) Let $X := l_2(\mathbb{Z})$, let $(a_n)_{n\in\mathbb{Z}}$ be a bounded subset of $(0,\infty)$, and let A be defined through (30). Suppose that $(n_{k,i})_{k\in\mathbb{N}}$ is any strictly decreasing sequence of negative integers possessing the property that, for every $s \in \mathbb{N}$ and $j \in \mathbb{N}_{i-1}^0$, there exists $l \in \mathbb{Z}$ such that $l < -s$ and $n_{l,i} \equiv j \pmod{i}$, $1 \leqslant i \leqslant N$. Set $W_i :=\operatorname{span}(\{e_{n_{k,i}} : k \in \mathbb{N}\})$, $1 \leqslant i \leqslant N$. Then $(A+W_1,A^2+W_2,\dots,A^N+W_N)$ is a d-hypercyclic extension of the tuple $(A,A^2,\dots,A^N)$.

(iv) Let $(\mathcal{A}_1,\dots,\mathcal{A}_N)=(A_1+W_1,\dots,A_N+W_N)$, where W_i is a dense linear submanifold of X $(1 \leqslant i \leqslant N)$. Then $(\mathcal{A}_1,\dots,\mathcal{A}_N)$ is a d-topologically mixing extension of $(A_1,\dots,A_N)$.

(v) Suppose that the range of A_i is dense in X $(1 \leqslant i \leqslant N)$. Then $(A_1+R(A_1^{j_1}),\dots,A_N+R(A_N^{j_N}))$ is a d-chaotic extension of $(A_1,\dots,A_N)$ for all $j_1,\dots,j_N \in \mathbb{N}$. In the case that the operator A_i is nilpotent and W_i is a dense submanifold of X $(1 \leqslant i \leqslant N)$, then it can be easily seen that $(A_1+W_1,\dots,A_N+W_N)$ is a d-chaotic extension of $(A_1,\dots,A_N)$.

Further analysis of hypercyclic and disjoint hypercyclic extensions of multivalued linear operators is without scope of this book. Fairly precise specification of chaotic and disjoint chaotic extensions of multivalued linear operators seems to be a more delicate problem.

We close this section with the observation that disjoint hypercyclic and disjoint topologically mixing properties of degenerate fractional differential equations have recently been considered in a joint paper with V. Fedorov [**215**].

1.4. Laplace transform of functions with values in SCLCSs

Without any doubt, two most important monographs on the Laplace transform of scalar valued functions are written by G. Doetsch [**158**] (1937) and D. V. Widder [**536**] (1941). The following conditions on a scalar valued function $k(t)$ will be used in the sequel:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(k) :=\inf\{\operatorname{Re}\lambda : \tilde{k}(\lambda) \text{ exists}\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

(P2): $k(t)$ satisfies (P1) and $\tilde{k}(\lambda) \neq 0$, $\operatorname{Re}\lambda > \beta$ for some $\beta \geqslant \operatorname{abs}(k)$.

For the basic theory of Laplace transform of Banach space valued functions, we refer the reader to the monograph [**27**] by W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. As mentioned in [**292**, Section 1.2], the Laplace transform of functions with values in sequentially complete locally convex spaces has attracted much less attention so far. The main purpose of this section is to introduce a relatively simple and new theoretical concept useful in the analysis of operational properties of Laplace transform of non-continuous functions with values in sequentially complete locally convex spaces. This concept extends the corresponding one introduced by T.-J. Xiao and J. Liang [**546**], 1997, and coincides with the classical concept of vector-valued Laplace transform in the case that the state space X is one of Banach's [**27**].

We are concerned with the existence of Laplace integral

$$(\mathcal{L}f)(\lambda) := \tilde{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)dt,$$

for $\lambda \in \mathbb{C}$. If $\tilde{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{C}$, then we define the abscissa of convergence of $\tilde{f}(\cdot)$ by

$$\mathrm{abs}_X(f) := \inf\{\operatorname{Re}\lambda : \tilde{f}(\lambda) \text{ exists}\};$$

otherwise, $\mathrm{abs}_X(f) := +\infty$. It is said that $f(\cdot)$ is Laplace transformable, or equivalently, that $f(\cdot)$ belongs to the class (P1)-X, iff $\mathrm{abs}_X(f) < \infty$. Assuming that there exists a number $\omega \in \mathbb{R}$ such that for each seminorm $p \in \circledast$ there exists a number $M_p > 0$ satisfying that $p(f(t)) \leqslant M_p e^{\omega t}$, $t \geqslant 0$, we define $\omega_X(f) \in [-\infty, \infty)$ as the infimum of all numbers $\omega \in \mathbb{R}$ with the above property; if there is no such a number $\omega \in \mathbb{R}$, then we define $\omega_X(f) := +\infty$. Further on, we abbreviate $\omega_X(f)$ $(\mathrm{abs}_X(f))$ to $\omega(f)$ $(\mathrm{abs}(f))$, if no confusion seems likely. Define

$$_w\,\mathrm{abs}(f) := \inf\left\{\lambda \in \mathbb{R} : \sup_{t>0}\left|\int_0^t e^{-\lambda s}\langle x^*, f(s)\rangle ds\right| < \infty \text{ for all } x^* \in X^*\right\},$$

$F_\infty := \lim_{\tau\to\infty} F(\tau)$, if the limit exists in X, and $F_\infty := 0$, otherwise.

Keeping in mind Theorem 1.1.4, we can repeat verbatim the argumentation from [**27**, Section 1.4, pp. 27-30] in order to see that the following theorem holds good (the only essential difference occurs on l. 6, p. 29, where we can use [**419**, Mackey's theorem 23.15] in place of the uniform boundedness principle):

THEOREM 1.4.1. *Let $f \in L^1_{loc}([0,\infty) : X)$. Then the following holds:*

- (i) *The Laplace integral $\tilde{f}(\lambda)$ converges if* $\operatorname{Re}\lambda > \mathrm{abs}(f)$ *and diverges if* $\operatorname{Re}\lambda < \mathrm{abs}(f)$. *If* $\operatorname{Re}\lambda = \mathrm{abs}(f)$, *then the Laplace integral may or may not be convergent.*
- (ii) $_w\,\mathrm{abs}(f) = \mathrm{abs}(f)$.
- (iii) *Suppose that $\lambda \in \mathbb{C}$ and the limit* $\lim_{t\to+\infty}\int_0^t e^{-\lambda s}p(f(s))ds$ *exists for any $p \in \circledast$. Then $\tilde{f}(\lambda)$ exists, as well.*
- (iv) *We have*

$$\mathrm{abs}(f) \leqslant \mathrm{abs}(p(f)) \leqslant \omega(f), \quad p \in \circledast.$$

In general, any of these two inequalities can be strict.

(v) *We have*

$$\operatorname{abs}(f)=\omega(F-F_\infty), \tag{31}$$

$$\tilde{f}(\lambda)=F_\infty+\lambda\int_0^\infty e^{-\lambda t}(F(t)-F_\infty)dt, \quad \operatorname{Re}\lambda>\omega(F-F_\infty), \tag{32}$$

$$\tilde{f}(\lambda)=\lambda\tilde{F}(\lambda), \quad \operatorname{Re}\lambda>\max(\operatorname{abs}(f),0) \tag{33}$$

and

$$\operatorname{abs}(f)\leqslant\omega\Leftrightarrow\omega(F)\leqslant\omega \quad (\textit{if } \omega\geqslant 0).$$

In particular, $f(\cdot)$ is Laplace transformable iff $\omega(F)<\infty$.

Recall [**541**], a function $h(\cdot)$ belongs to the class $LT-X$ iff there exist a function $g\in C([0,\infty):X)$ and a number $\omega\in\mathbb{R}$ such that $\omega(g)\leqslant\omega<\infty$ and $h(\lambda)=(\mathcal{L}g)(\lambda)$ for $\lambda>\omega$; as observed in [**292**, Section 1.2], the assumption $h\in LT-X$ immediately implies that the function $\lambda\mapsto h(\lambda)$, $\lambda>\omega$ can be analytically extended to the right half plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\omega\}$. In the sequel, the set of all originals $g(\cdot)$ whose Laplace transform belongs to the class $LT-X$ will be abbreviated to $LT_{or}-X$. Keeping the above observation and the equations (31)–(32)) in mind, we can simply prove that the mapping $\lambda\mapsto\tilde{f}(\lambda)$, $\operatorname{Re}\lambda>\operatorname{abs}(f)$ is analytic, provided that $f\in(\mathrm{P1})-X$. If this is the case, the following formula holds:

$$\frac{d^n}{d\lambda^n}\tilde{f}(\lambda)=(-1)^n\int_0^\infty e^{-\lambda t}t^n f(t)dt, \quad n\in\mathbb{N},\ \lambda\in\mathbb{C},\ \operatorname{Re}\lambda>\operatorname{abs}(f). \tag{34}$$

In the following theorem, we will collect various operational properties of vector-valued Laplace transform.

THEOREM 1.4.2. *Let $f\in$(P1)-X, $z\in\mathbb{C}$ and $s\geqslant 0$.*

(i) *Put $g(t):=e^{-zt}f(t)$, $t\geqslant 0$. Then $g(\cdot)$ is Laplace transformable,* $\operatorname{abs}(g)=\operatorname{abs}(f)-\operatorname{Re}z$ *and $\tilde{g}(\lambda)=\tilde{f}(\lambda+z)$, $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda>\operatorname{abs}(f)-\operatorname{Re}z$.*

(ii) *Put $f_s(t):=f(t+s)$, $t\geqslant 0$, $h_s(t):=f(t-s)$, $t\geqslant s$ and $h_s(t):=0$, $s\in[0,t]$. Then* $\operatorname{abs}(f_s)=\operatorname{abs}(h_s)=\operatorname{abs}(f)$, $\widetilde{f_s}(\lambda)=e^{\lambda s}(\tilde{f}(\lambda)-\int_0^s e^{-\lambda t}f(t)dt)$ *and $\widetilde{h_s}(\lambda)=e^{-\lambda s}\tilde{f}(\lambda)$ $(\lambda\in\mathbb{C},\ \operatorname{Re}\lambda>a)$.*

(iii) *Let $T\in L(X,Y)$. Then $T\circ f\in$ (P1)-Y and $T\tilde{f}(\lambda)=\widetilde{(T\circ f)}(\lambda)$ for $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda>\operatorname{abs}(f)$.*

(iv) *Suppose that $\mathcal{A}\colon X\to P(Y)$ is an* MLO *and $\mathcal{A}$ is $X_{\mathcal{A}}\times Y_{\mathcal{A}}$-closed, as well as $f\in(\mathrm{P1})-X_{\mathcal{A}}$, $l\in(\mathrm{P1})-Y_{\mathcal{A}}$ and $(f(t),l(t))\in\mathcal{A}$ for a.e. $t\geqslant 0$. Then $(\tilde{f}(\lambda),\tilde{l}(\lambda))\in\mathcal{A}$, $\lambda\in\mathbb{C}$ for* $\operatorname{Re}\lambda>\max(\operatorname{abs}(f),\operatorname{abs}(l))$.

(v) *Suppose, in addition, $\omega(f)<\infty$. Put*

$$j(t):=\int_0^\infty\frac{e^{-s^2/4t}}{\sqrt{\pi t}}f(s)ds:=\lim_{\tau\to\infty}\int_0^\tau\frac{e^{-s^2/4t}}{\sqrt{\pi t}}f(s)ds, \quad t>0$$

and

$$k(t):=\int_0^\infty\frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{\frac{3}{2}}}f(s)ds:=\lim_{\tau\to\infty}\int_0^\tau\frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{\frac{3}{2}}}f(s)ds, \quad t>0.$$

Then $j(\cdot)$ and $k(\cdot)$ are Laplace transformable,

$$\max(\operatorname{abs}(j),\operatorname{abs}(k)) \leqslant (\max(\omega(f),0))^2,\ \tilde{j}(\lambda)=\frac{\tilde{f}(\sqrt{\lambda})}{\sqrt{\lambda}} \text{ and } \tilde{k}(\lambda)=\tilde{f}(\sqrt{\lambda})$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > (\max(\omega(f),0))^2$.

(vi) *Let $f \in$ (P1)-X, $a \in L^1_{loc}([0,\infty))$ and $\operatorname{abs}(|a|) < \infty$. Suppose, in addition, that $f \in C([0,\infty) : X)$. Then the mapping $t \mapsto (a * f)(t) = \int_0^t a(t-s)f(s)ds$, $t \geqslant 0$ is continuous, $a * f \in$ (P1)-X, and*

$$\widetilde{a * f}(\lambda) = \tilde{a}(\lambda)\tilde{f}(\lambda),\ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \max(\operatorname{abs}(|a|),\operatorname{abs}(f)).$$

Proof. Factorizing in Theorem 1.1.4, Theorem 1.1.3(ii) and Theorem 1.2.3, the assertions (i)–(iv) can be proved as in the Banach space case (cf. [**27**, Proposition 1.6.1–Proposition 1.6.3] for more details). Consider now part (v). Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > (\max(\omega(f),0))^2$ be fixed. Then $\operatorname{Re}(\sqrt{\lambda}) > \max(\omega(f),0) \geqslant \max(\omega(F),0)$ so that [**292**, Theorem 1.2.1(v)] implies in combination with (33) that $\tilde{f}(\sqrt{\lambda})$ exists, as well as that

$$\tilde{f}(\sqrt{\lambda}) = \frac{\tilde{F}(\sqrt{\lambda})}{\sqrt{\lambda}} = \int_0^\infty e^{-\lambda t}\int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} f(s)ds\,dt.$$

On the other hand, we can simply prove with the help of dominated convergence theorem that the mapping $t \mapsto k(t)$, $t > 0$ is continuous as well as that for each seminorm $p \in ⊛$ there exists a finite number $m_p > 0$ such that $p(k(t)) \leqslant m_p t^{(-1)/2}$, $t \in (0,1]$. This simply implies $k \in L^1_{loc}([0,\infty) : X)$. Since

$$\int_0^\infty e^{-\lambda t}\langle x^*, k(t)\rangle dt = \int_0^\infty e^{-\sqrt{\lambda}t}\langle x^*, f(t)\rangle dt,\quad x^* \in X^*, \tag{35}$$

we obtain that

$$\lim_{\tau\to\infty}\left\langle x^*, \int_0^\tau e^{-\lambda t}k(t)dt\right\rangle = \langle x^*, \tilde{f}(\sqrt{\lambda})\rangle,\quad x^* \in X^*.$$

By Theorem 1.1.4(i), we get that the mapping $\tau \mapsto \int_0^\tau e^{-\lambda t}k(t)dt$, $\tau \geqslant 0$ is continuous so that the previous equality implies $\sup_{\tau\geqslant 0}|\langle x^*, \int_0^\tau e^{-\lambda t}k(t)dt\rangle| < \infty$ for all $x^* \in X^*$. Therefore, Theorem 1.4.1(ii) shows that $\lambda >_w \operatorname{abs}(k) = \operatorname{abs}(k)$ and $\tilde{k}(\lambda)$ exists. Using again (35), it readily follows that $\tilde{k}(\lambda) = f(\sqrt{\lambda})$, as claimed. Similarly we can prove that $\tilde{j}(\lambda) = f(\sqrt{\lambda})/\sqrt{\lambda}$. Suppose, finally, that the requirements of (vi) hold. Then it is very simple to prove that the mapping $t \mapsto (h * f)(t)$, $t \geqslant 0$ is continuous as well as that $\omega(1 * h * f) = \omega(h * (1 * f)) < \infty$. An application of Theorem 1.4.1(v) yields that $h * f \in$ (P1)-X. Fix now a number $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max(\operatorname{abs}(|h|),\operatorname{abs}(f))$. Since $\operatorname{abs}(\langle x^*, f(\cdot)\rangle) \leqslant \operatorname{abs}(f)$ for all $x^* \in X^*$, [**27**, Proposition 1.6.4] implies that $(\mathcal{L}(h * \langle x^*, f(\cdot)\rangle))(\lambda)$ exists. Using this fact, it readily follows that

$$\sup_{t>0}\left|\int_0^t e^{-s\operatorname{Re}\lambda}(h * \langle x^*, f(\cdot)\rangle)(s)ds\right| < \infty,\quad x^* \in X^*.$$

By Theorem 1.4.1(ii), we get that $\widetilde{h * f}(\lambda)$ exists. The equality $\widetilde{h * f}(\lambda) = \tilde{h}(\lambda)\tilde{f}(\lambda)$ can be proved in a routine manner. $\square$

The non-possibility of establishing Fubini–Tonelli theorem in this concept of integration does not able us to fully transfer some assertions from the Banach space case to the general locally convex space case; for example, in Theorem 1.4.2(vi) we consider the Laplace transform of finite convolution product and there it is almost inevitable to impose the condition that the function $f(t)$ is continuous (this is not the case in Fréchet space, when we can use the well-known extension of Bochner concept of integration; see [**410**, p. 100]). In a great number of our structrural results stated in the monograph [**292**], we have incorrectely applied Theorem 1.4.2(vi) by assuming only that the function $a(t)$ satisfies (P1). Strictly speaking, in all these assertions, we have to assume that $\mathrm{abs}(|a|) < \infty$. For the sequel, we need the notion of a Lebesgue point of a function $f \in L^1_{loc}([0,\infty) : X)$. A point $t \geqslant 0$ is said to be a Lebesgue point of $f(\cdot)$ iff for each seminorm $p \in \circledast$, we have

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} p(f(s) - f(t))\,ds = 0. \tag{36}$$

It is clear that any point of continuity of function $f(\cdot)$ is one of Lebesgue's points of $f(\cdot)$, as well as that the mapping $t \mapsto F(t)$, $t \geqslant 0$ is differentiable at any Lebesgue's point of $f(\cdot)$. Furthermore, a slight modification of the proof of [**27**, Proposition 1.2.2; a)/b)] shows that the following holds:

(Q1) For each seminorm $p \in \circledast$ there exists a set $N_p \subseteq [0,\infty)$ of Lebesgue's measure zero such that

$$\lim_{h \to 0} p\biggl(\frac{1}{h} \int_t^{t+h} f(s)\,ds - f(t)\biggr) = 0, \quad t \in [0,\infty) \smallsetminus N_p$$

and (36) holds for $t \in [0,\infty) \smallsetminus N_p$.

In the case that X is a Fréchet space, (Q1) immediately implies that almost every point $t > 0$ is a Lebesgue point of $f(\cdot)$. Using the proof of [**27**, Theorem 1.7.7], Theorem 1.1.4(iii), as well as the equations (19) and (34), we can simply prove that the Post–Widder inversion formula holds in our framework:

THEOREM 1.4.3. (Post–Widder) *Suppose $f \in (\mathrm{P1}) - X$ and $t > 0$ is a Lebesgue point of $f(\cdot)$. Then*

$$f(t) = \lim_{n \to \infty} (-1)^n \frac{1}{n!} \Bigl(\frac{n}{t}\Bigr)^{n+1} \tilde{f}^{(n)}\Bigl(\frac{n}{t}\Bigr).$$

The situation is much more complicated if we analyze the Phragmén–Doetsch inversion formula for the Laplace transform of functions with values in SCLCSs. The following result of this type will be sufficiently general for our purposes:

THEOREM 1.4.4. *Let $f \in$ (P1)-X and $t \geqslant 0$. Then the following holds:*

$$f^{[2]}(t) = \lim_{\lambda \to \infty} \sum_{n=1}^{\infty} (-1)^{n-1} n!^{-1} e^{n\lambda t} \frac{\tilde{f}(n\lambda)}{n\lambda}.$$

PROOF. In view of Theorem 1.4.1(v), we have $F \in C([0,\infty) : X)$ and $\omega(F) < \infty$. The result now follows easily from [**292**, Theorem 1.2.1(ix)]. $\square$

Now we will state and prove the following uniqueness type theorem for the Laplace transform.

THEOREM 1.4.5. (The uniqueness theorem for the Laplace transform) *Suppose $f \in (P1) - X$, $\lambda_0 > \mathrm{abs}(f)$ and $\tilde{f}(\lambda) = 0$ for all $\lambda > \lambda_0$. Then $F(t) = 0$, $t \geqslant 0$, $f(t) = 0$ if $t > 0$ is a Lebesgue point of $f(\cdot)$, and for each seminorm $p \in \circledast$ there exists a set $N_p \subseteq [0,\infty)$ of Lebesgue's measure zero such that $p(f(t)) = 0$, $t \in [0,\infty) \smallsetminus N_p$. In particular, if X is a Fréchet space, then $f(t) = 0$ for a.e. $t \geqslant 0$.*

PROOF. The function $t \mapsto F(t)$, $t \geqslant 0$ is continuous and by Theorem 1.4.1(v) we get that $\omega(F) < \infty$ and $\tilde{F}(\lambda) = 0$, $\lambda > \max(\lambda_0, 0)$. Now we can apply Theorem 1.4.3 in order to see that $F(t) = 0$, $t \geqslant 0$. The remainder of the proof is simple and therefore omitted. $\square$

REMARK 1.4.6. Suppose that $f \in L^1_{loc}([0,\infty) : X)$ and for each seminorm $p \in \circledast$ there exists a set $N_p \subseteq [0,\infty)$ of Lebesgue's measure zero such that $p(f(t)) = 0$, $t \in [0,\infty) \smallsetminus N_p$. Then $\mathrm{abs}(f) = \mathrm{abs}(p(f)) = -\infty$ ($p \in \circledast$) and $\tilde{f}(\lambda) = 0$ for all $\lambda \in \mathbb{C}$.

The following converse of Theorem 1.4.2(iv) simply follows from an application of Theorem 1.4.4.

PROPOSITION 1.4.7. *Suppose that $\mathcal{A} \colon X \to P(Y)$ is an* MLO *and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, as well as $f \in (\mathrm{P1}) - X_{\mathcal{A}}$, $l \in (\mathrm{P1}) - Y_{\mathcal{A}}$ and $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for* $\mathrm{Re}\,\lambda > \max(\mathrm{abs}(f), \mathrm{abs}(l))$. *Then $l^{[1]}(t) \in \mathcal{A} f^{[1]}(t)$, $t \geqslant 0$ and $l(t) \in \mathcal{A} f(t)$ for any $t > 0$ which is a Lebesgue point of both functions $f(t)$ and $l(t)$.*

Now we would like to briefly explain how we can extend the definition of Laplace transformable functions to the multivalued ones. Let $0 < \tau \leqslant \infty$ and $\mathcal{F} \colon [0,\tau) \to P(X)$. A single-valued function $f \colon [0,\tau) \to X$ is called a section of $\mathcal{F}$ iff $f(t) \in \mathcal{F}(t)$ for all $t \in [0,\tau)$. We denote the set of all sections, resp., all continuous sections, of $\mathcal{F}$ by $\mathrm{sec}(\mathcal{F})$, resp., $\mathrm{sec}_c(\mathcal{F})$. Suppose now that $\tau = \infty$ and any function $f \in \mathrm{sec}(\mathcal{F})$ belongs to the class (P1)-X. Then we define $\mathrm{abs}_X(\mathcal{F}) := \sup\{\mathrm{abs}_X(f) : f \in \mathrm{sec}(v)\}$; $\mathcal{F}(\cdot)$ is said to be Laplace transformable iff $\mathrm{abs}_X(\mathcal{F}) < \infty$.

The method proposed by T.-J. Xiao and J. Liang in [**546**] provides a sufficiently enough framework for the theoretical study of real and complex inversion methods for the Laplace transform of functions with values in SCLSCs, as well as for the studies of analytical properties and approximation of Laplace transform (see e.g. [**541**, Section 1.1.1] and [**292**, Section 1.2] for more details); this method can be successfully applied in the analysis of subordination principles for abstract time-fractional inclusions, as well (cf. Theorem 3.1.8 below). It is also worth noting that there exists a great number of theoretical results from the monograph [**27**], not mentioned so far, which can be reconsidered for the Laplace transformable functions with values in SCLSCs; for example, all structural results from [**27**, Section 4.1] continue to hold in our framework. Due primarily to the space limitations, we will not consider here numerous important questions concerning the vector-valued Laplace transform of functions with values in SCLCSs.

The complex inversion theorem for the vector-valued Laplace transform reads as follows.

THEOREM 1.4.8. *Assume $a > 0$, $r \in \mathbb{R}$, $q\colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a\} \to E$ is analytic, and for each $p \in \circledast$ there exists $M_p > 0$ such that*

$$p(q(\lambda)) \leqslant M_p|\lambda|^r, \quad \operatorname{Re}\lambda > a.$$

Then for each $\alpha > 1$ there exists a function $f_\alpha \in C([0,\infty) : E)$ with $f_\alpha(0) = 0$ and

$$p(h_\alpha(t)) \leqslant M_\alpha M_p e^{at}, \quad p \in \circledast,\ t \geqslant 0,$$

$$q(\lambda) = \lambda^{\alpha+r}\int_0^\infty e^{-\lambda t}h_\alpha(t)dt, \quad \operatorname{Re}\lambda > a,$$

where M_α is independent of p and $q(\cdot)$.

The following extension of Arendt–Widder theorem has been proved by T.-J. Xiao and J. Liang (see e.g. [**541**]).

THEOREM 1.4.9. *Let $a \geqslant 0$, $r \in (0,1]$, $\omega \in (-\infty, a]$, $M_p > 0$ for each $p \in \circledast$, and let $q\colon (a,\infty) \to E$ be an infinitely differentiable function. Then we have the equivalence of statements* (i) *and* (ii), *where:*

(i) *One has $p\big((\lambda-\omega)^{k+1}\frac{q^{(k)}(\lambda)}{k!}\big) \leqslant M$, $p \in \circledast$, $\lambda > a$, $k \in \mathbb{N}_0$.*
(ii) *There exists a function $F_r \in C([0,\infty) : E)$ satisfying $F_r(0) = 0$,*

$$q(\lambda) = \lambda^r\int_0^\infty e^{-\lambda t}F_r(t)dt, \quad \lambda > a,$$

$$p\bigg(\int_0^{t+h}\frac{(t+h-s)^{-r}}{\Gamma(1-r)}F_r(s)ds - \int_0^t\frac{(t-s)^{-r}}{\Gamma(1-r)}F_r(s)ds\bigg) \leqslant M_p h e^{\omega t}\max(e^{\omega h},1),$$

for any $t \geqslant 0$, $h \geqslant 0$ and $p \in \circledast$, if $r \in (0,1)$, and

$$p(F_r(t+h) - F_r(t)) \leqslant M_p h e^{\omega t}\max(e^{\omega h},1), \quad t \geqslant 0,\ h \geqslant 0,\ p \in \circledast,$$

if $r = 1$. Moreover, in this case,

$$p(F_r(t+h) - F_r(t)) \leqslant \frac{2M_p}{r\Gamma(r)}h^r\max(e^{\omega(t+h)},1), \quad t \geqslant 0,\ h \geqslant 0,\ p \in \circledast.$$

Recall that $\Sigma_\alpha = \{z \in \mathbb{C}\smallsetminus\{0\} : |\arg(z)| < \alpha\}$ $(\alpha \in (0,\pi])$. The most important analytical properties of Laplace transform of functions with values in SCLCSs are collected in the following theorem.

THEOREM 1.4.10. [**296**]

(i) *Let $\alpha \in \big(0,\frac{\pi}{2}\big]$, $\omega \in \mathbb{R}$ and $q\colon (\omega,\infty) \to E$. Then the following assertions are equivalent:*
   (a) *There exists an analytic function $f\colon \Sigma_\alpha \to E$ such that $q(\lambda) = \tilde{f}(\lambda)$, $\lambda \in (\omega,\infty)$ and the set $\{e^{-\omega z}f(z) : z \in \Sigma_\beta\}$ is bounded for all $\beta \in (0,\alpha)$.*

(b) *The function* $q(\cdot)$ *admits an analytic extension* $\tilde{q}\colon \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to E$ *which satisfies that the set* $\{(\lambda-\omega)\tilde{q}(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}\}$ *is bounded for all* $\gamma \in (0, \alpha)$.

If this is the case, then we have that, for every $k \in \mathbb{N}$ *and* $\beta \in (0, \alpha)$*, the set* $\{z^k e^{-\omega z} f^{(k)}(z) : z \in \Sigma_\beta\}$ *is bounded.*

(ii) *Let* $\alpha \in (0, \pi]$ *and let* $f\colon \Sigma_\alpha \to E$ *be an analytic function which satisfies that, for every* $\beta \in (0, \alpha)$*, the set* $\{f(z) : z \in \Sigma_\beta\}$ *is bounded. Let* $x \in E$*. Then the following holds:*

(a) *The assumption* $\lim_{t\to+\infty} f(t) = x$ *implies* $\lim_{z\to\infty, z\in\Sigma_\beta} f(z) = x$ *for all* $\beta \in (0, \alpha)$.

(b) *The assumption* $\lim_{t\to 0+} f(t) = x$ *implies* $\lim_{z\to 0, z\in\Sigma_\beta} f(z) = x$ *for all* $\beta \in (0, \alpha)$.

(iii) *Assume* $x \in E$, $\alpha \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$, $q\colon (\omega, \infty) \to E$ *and let* (i)(a) *of this theorem hold. Then:*

(a) $\lim_{t\to 0+} f(t) = x$ *iff* $\lim_{\lambda\to\infty} \lambda q(\lambda) = x$.

(b) *Let* $\omega = 0$*. Then* $\lim_{t\to+\infty} f(t) = x$ *iff* $\lim_{\lambda\to 0} \lambda q(\lambda) = x$.

We also need the following theorem on approximation of Laplace transform.

THEOREM 1.4.11. [**296**] *Let* $f_n \in C([0, \infty) : E)$, $n \in \mathbb{N}$*, let the set* $\{e^{-\omega t} f_n(t) : n \in \mathbb{N},\ t \geqslant 0\}$ *be bounded for some* $\omega \in \mathbb{R}$*, and let* $\lambda_0 \geqslant \omega$*. Then the following assertions are equivalent:*

(i) *The sequence* $(\tilde{f}_n)$ *converges pointwise on* (λ_0, ∞) *and the sequence* (f_n) *is equicontinuous at each point* $t \geqslant 0$.

(ii) *The sequence* (f_n) *converges uniformly on compact subsets of* $[0, \infty)$.

Assuming (ii) *holds and* $\lim_{n\to\infty} f_n(t) = f(t)$, $t \geqslant 0$*, one has* $\lim_{n\to\infty} \tilde{f}_n(\lambda) = \tilde{f}(\lambda)$, $\lambda > \lambda_0$.

Finally, it would be worthwhile to record the following slight extension of [**459**, Proposition 0.1, Theorem 0.4, p. 10–12].

THEOREM 1.4.12. [**296**]

(i) *Assume* $g\colon \mathbb{C}_+ \to E$ *is analytic and satisfies that the sets* $\{\lambda g(\lambda) : \lambda \in \mathbb{C}_+\}$ *and* $\{\lambda^2 g'(\lambda) : \lambda \in \mathbb{C}_+\}$ *are bounded. Then the set* $\{n!^{-1}\lambda^{n+1} g^{(n)}(\lambda) : \lambda \in \mathbb{C}_+,\ n \in \mathbb{N}_0\}$ *is bounded as well.*

(ii) *Assume* $k \in \mathbb{N}_0$, $g\colon \mathbb{C}_+ \to E$ *is analytic and satisfies that the set* $\{\lambda^{n+1} g^{(n)}(\lambda) : \lambda \in \mathbb{C}_+,\ 0 \leqslant n \leqslant k+1\}$ *is bounded. Then there exists a function* $u \in C^k((0, \infty) : E)$ *such that* $g(\lambda) = \tilde{u}(\lambda)$, $\lambda \in \mathbb{C}_+$ *and the sets* $\{t^n u^{(n)}(t) : t > 0,\ 0 \leqslant n \leqslant k\}$ *and* $\{(t-s)^{-1}(1 + \ln\frac{t}{t-s})^{-1}(t^{k+1}u^{(k)}(t) - s^{k+1}u^{(k)}(s)) : 0 \leqslant s < t < \infty\}$ *are bounded.*

1.5. Operators of fractional differentiation, Mittag-Leffler and Wright functions

The first congress on fractional calculus was held at the University of New Haven, in 1974 [**131**]. From then on, several applications of fractional calculus and fractional differential equations have emerged in engineering, physics, chemistry, biology and other sciences. For the basic information about fractional calculus and non-degenerate fractional differential equations, the reader may consult the monographs by M. H. Annaby, Z. S. Mansour [**23**], T. M. Atanacković, S. Pilipović, B. Stanković, D. Zorica [**36**,**37**], D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo [**45**], K. Diethelm [**153**], R. Hilfer [**255**], A. A. Kilbas, H. M. Srivastava, J. J. Trujillo [**279**], V. Kiryakova [**281**], F. Mainardi [**405**], K. B. Oldham, J. Spanier [**439**], I. Podlubny [**452**], S. G. Samko, A. A. Kilbas, O. I. Marichev [**472**], V. E. Tarasov [**516**] and the author [**291**, **292**]; we also hope that the doctoral dissertations of E. Bazhlekova [**61**] and Yu. V. Bogacheva [**82**], as well as the references [**8**,**33**,**34**,**102**, **105**, **157**, **165**, **166**, **169**, **227**, **231**, **270**, **271**, **274**, **277**, **345**, **382**, **477**, **529**, **550**], and those from author's ones [**295**,**355**], may be of some help to the reader. For the basic information on the history of fractional calculus, the reader may consult [**160**], [**292**, Section 1.3], [**439**] and [**472**].

Suppose that $\alpha > 0$, $m = \lceil \alpha \rceil$ and $I = (0, T)$ for some $T \in (0, \infty]$. Then the Riemann–Liouville fractional integral J_t^α of order α is defined by

$$J_t^\alpha f(t) := (g_\alpha * f)(t), \quad f \in L^1(I : E),\ t \in I,$$

while the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for those functions $u \in C^{m-1}([0,\infty) : E)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : E)$, by

$$\mathbf{D}_t^\alpha u(t) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

The existence of Caputo fractional derivative $\mathbf{D}_t^\zeta u$ for $t \geqslant 0$ implies $u \in C^{\lceil \zeta \rceil}((0,\infty) : E) \cap C^\zeta([0,T] : X)$, for each finite number $T > 0$. Suppose now, just for a few moments, that E is a Banach space. Then the Sobolev space $W^{m,1}(I : E)$ can be introduced in the following way (see e.g. [**61**, p. 7]):

$$W^{m,1}(I : E) := \Biggl\{ f \mid \exists \varphi \in L^1(I : E) \exists c_k \in \mathbb{C}\ (0 \leqslant k \leqslant m-1)$$
$$f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + (g_m * \varphi)(t) \text{ for a.e. } t \in (0, \tau) \Biggr\}.$$

If so, then we have $\varphi(t) = f^{(m)}(t)$ in the distributional sense, and $c_k = f^{(k)}(0)$ $(0 \leqslant k \leqslant m-1)$. The Riemann–Liouville fractional derivative D_t^α of order α is defined for those functions $f \in L^1(I : E)$ for which $g_{m-\alpha} * f \in W^{m,1}(I : E)$, by

$$D_t^\alpha f(t) := \frac{d^m}{dt^m} J_t^{m-\alpha} f(t), \quad t \in I.$$

Due to [**61**, Theorem 1.5], the Riemann–Liouville fractional integrals and derivatives satisfy the following equalities:

$$J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t), \quad D_t^\alpha J_t^\alpha f(t) = f(t),$$

for $f \in L^1(I : E)$ and

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t) \tag{37}$$

for any $f \in L^1(I : E)$ with $g_{m-\alpha} * f \in W^{m,1}(I : E)$. We are returning to the case in which E is a general SCLCS. Let $\beta > 0$ and $\beta \notin \mathbb{N}$. Then the Liouville right-sided fractional derivative of order β (see [**279**, (2.3.4)] for the scalar-valued case) is defined for those continuous functions $u\colon (0,\infty) \to E$ for which $\lim_{T\to\infty} \int_s^T g_{\lceil\beta\rceil-\beta}(t-s)u(t)dt = \int_s^\infty g_{\lceil\beta\rceil-\beta}(t-s)u(t)dt$ exists and defines a $\lceil\beta\rceil$-times continuously differentiable function on $(0,\infty)$, by

$$\mathbf{D}_-^\beta u(s) := (-1)^{\lceil\beta\rceil} \frac{d^{\lceil\beta\rceil}}{ds^{\lceil\beta\rceil}} \int_s^\infty g_{\lceil\beta\rceil-\beta}(t-s)u(t)dt, \quad s > 0.$$

We define the modified Liouville right-sided fractional derivative of order β, $D_-^\beta u(s)$ shortly, for those continuously differentiable functions $u\colon (0,\infty) \to E$ for which $\lim_{T\to\infty} \int_s^T g_{\lceil\beta\rceil-\beta}(t-s)u'(t)dt = \int_s^\infty g_{\lceil\beta\rceil-\beta}(t-s)u'(t)dt$ exists and defines a $\lceil\beta - 1\rceil$-times continuously differentiable function on $(0,\infty)$, by

$$D_-^\beta u(s) := (-1)^{\lceil\beta\rceil} \frac{d^{\lceil\beta-1\rceil}}{ds^{\lceil\beta-1\rceil}} \int_s^\infty g_{\lceil\beta\rceil-\beta}(t-s)u'(t)dt, \quad s > 0;$$

if $\beta = n \in \mathbb{N}$, then $\mathbf{D}_-^n u$ and $D_-^n u$ are defined for all n-times continuously differentiable functions $u(\cdot)$ on $(0,\infty)$, by $\mathbf{D}_-^n u := D_-^n u := (-1)^n d/d^n$, where d/d^n denotes the usual derivative operator of order n (cf. also [**279**, (2.3.5)]).

Let $\alpha \in (0,\infty) \setminus \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil\alpha\rceil$. Let us recall that the Weyl fractional derivative W_+^α of order α is defined by

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)ds, \quad t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}_0$, then we set $W_+^n := (-1)^n \frac{d^n}{dt^n}$. It is well known that the following equality holds: $W_+^{\alpha+\beta} f = W_+^\alpha W_+^\beta f$, $\alpha, \beta > 0$, $f \in \mathcal{S}$.

As already mentioned, in this book we will use the Caputo fractional derivatives, the Riemann–Liouville fractional derivatives, the modified Liouville right-sided fractional derivatives and Weyl fractional derivatives; for the brief summary of definitions of fractional order derivatives and integrals that usually appear in fractional calculus, we refer the reader to the recent paper [**131**] by E. C. de Oliveira and J. A. T. Machado.

The basic properties of Caputo fractional derivatives can be obtained by consulting [**61**] and [**292**, Preliminaries, Section 1.3]. For the later use, it would be very important to remind ourselves of the following facts. Suppose that $\beta > 0$, $\gamma > 0$ and $\mathbf{D}_t^{\beta+\gamma} u(t)$ is defined. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u(t)$ is defined for any number $\zeta \in (0, \beta+\gamma)$ but the equality $\mathbf{D}_t^{\beta+\gamma} u = \mathbf{D}_t^\beta \mathbf{D}_t^\gamma u$ does not hold in general (cf. [**292**, Preliminaries, Section 1.3, p. 14] and the equation (92) below). The validity of this equality can be proved in the following cases:

(1) $\gamma \in \mathbb{N}$,

(2) $\lceil \beta+\gamma \rceil = \lceil \gamma \rceil$,
(3) $u^{(j)}(0)=0$ for $\lceil \gamma \rceil \leqslant j \leqslant \lceil \beta+\gamma \rceil - 1$.

If $u \in C([0,\infty):E)$, resp. $u \in C^{m-1}([0,\infty):E)$ and $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty):E)$, then the following equality holds:

$$\mathbf{D}_t^\alpha J_t^\alpha u(t) = u(t), \quad t \geqslant 0, \text{ resp. } J_t^\alpha \mathbf{D}_t^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0) g_{k+1}(t), \quad t \geqslant 0. \tag{38}$$

The Laplace transform of function $\mathbf{D}_t^\alpha u(t)$ is computed by

$$\int_0^\infty e^{-\lambda t} \mathbf{D}_t^\alpha u(t)\,dt = \lambda^\alpha \tilde{u}(\lambda) - \sum_{k=0}^{m-1} u^{(k)}(0) \lambda^{\alpha-1-k}; \tag{39}$$

cf. the identity [**292**, (16)] for precise formulation.

The Mittag-Leffler and Wright functions are known to play a fundamental role in seeking of solutions of fractional differential equations. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}.$$

Here we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short,

$$E_\alpha(z) := E_{\alpha,1}(z), \quad z \in \mathbb{C}.$$

Like the function $E_1(z) = e^z$, for which the differential relation $(d/dt)e^{\omega t} = \omega e^{\omega t}$ holds, the function $E_\alpha(z)$ satisfies that $\mathbf{D}_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha)$. For $\alpha = 1/2$, $E_{1/2}(z)$ is the error function: $E_{1/2}(z) = \exp(z^2)\operatorname{erfc}(-z)$, and for $\alpha = 2$, $E_2(z)$ is the hyperbolic cosine: $E_2(z) = \cosh(\sqrt{z})$. The asymptotic expansion of the entire function $E_{\alpha,\beta}(z)$ is given in the following important theorem (see e.g. [**539**, Theorem 1.1]):

THEOREM 1.5.1. *Let $0 < \sigma < \frac{1}{2}\pi$. Then, for every $z \in \mathbb{C} \smallsetminus \{0\}$ and $m \in \mathbb{N} \smallsetminus \{1\}$,*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O(|z|^{-m}),$$

where Z_s is defined by $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers s satisfying $|\arg(z) + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$.

If $\alpha \in (0,2) \smallsetminus \{1\}$, $\beta > 0$ and $N \in \mathbb{N} \smallsetminus \{1\}$, then we have the following special cases of Theorem 1.5.1:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \varepsilon_{\alpha,\beta}(z), \quad |\arg(z)| < \alpha\pi/2, \tag{40}$$

and

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < \pi - \alpha\pi/2, \tag{41}$$

where

$$\varepsilon_{\alpha,\beta}(z) = \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad |z| \to \infty. \tag{42}$$

Perhaps the most interesting property of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is that

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \operatorname{Re} \lambda > \omega^{1/\alpha}, \ \omega > 0. \tag{43}$$

Recall that the function $t \mapsto E_{\alpha,\beta}(-t)$, $t \geqslant 0$ is completely monotonic (i.e., that $(-1)^n (d^n/dt^n) E_{\alpha,\beta}(-t) \geqslant 0$, $t \geqslant 0$, $n \in \mathbb{N}_0$) provided that $\alpha \in (0,1]$ or $\beta \geqslant \alpha$ **[405]**. Among a huge variety of identities for the Mittag-Leffler functions, we will single out the following ones:

(i) For every $j \in \mathbb{N}$ and $\alpha > 0$, there exist uniquely determined real numbers $c_{l,j,\alpha}$ $(1 \leqslant l \leqslant j)$ such that:

$$E_\alpha^{(j)}(z) = \sum_{l=1}^{j} c_{l,j,\alpha} E_{\alpha,\alpha j-(j-l)}(z), \quad z \in \mathbb{C}. \tag{44}$$

(ii) Let $\alpha > 0$, $\beta > 0$ and $r \in \mathbb{N}$. Then $z^r E_{\alpha,\beta+r\alpha}(z) = E_{\alpha,\beta}(z) + \sum_{n=0}^{r-1} \frac{z^n}{\Gamma(\beta+n\alpha)}$ for all $z \in \mathbb{C}$.

(iii) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $E_{\frac{m}{n}}(z) = \frac{1}{m} \sum_{r=1}^{m-1} E_{\frac{1}{n}}(z^{1/m} \exp(2\pi i r/m)$ for all $z \in \mathbb{C}$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ has the integral representation in the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_G \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} d\lambda, \quad z \in \mathbb{C},$$

where G is a contour (the Hankel path) which starts and ends at $-\infty$ and encircles the disc $|\lambda| \leqslant |z|^{1/\alpha}$ counter-clockwise.

Let $\gamma \in (0,1)$. Then the Wright function $\Phi_\gamma(\cdot)$ is defined by

$$\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t), \quad t \geqslant 0.$$

The Wright function $\Phi_\gamma(\cdot)$ can be analytically extended to the whole complex plane by the formula

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C},$$

and the following holds:

(i) $\Phi_\gamma(t) \geqslant 0$, $t \geqslant 0$,

(ii) $\int_0^\infty e^{-\lambda t} \gamma s t^{-1-\gamma} \Phi_\gamma(t^{-\gamma} s) dt = e^{-\lambda^\gamma s}$, $\lambda \in \mathbb{C}_+$, $s > 0$, and

(iii) $\int_0^\infty t^r \Phi_\gamma(t) dt = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$.

The asymptotic expansion of the Wright function $\Phi_\gamma(\cdot)$, as $|z| \to \infty$ in the sector $|\arg(z)| \leqslant \min((1-\gamma)3\pi/2, \pi) - \varepsilon$ is given by

$$\Phi_\gamma(z) = Y^{\gamma-1/2} e^{-Y} \left(\sum_{m=0}^{M-1} A_m Y^{-M} + O(|Y|^{-M}) \right),$$

where $Y = (1-\gamma)(\gamma^\gamma z)^{1/(1-\gamma)}$, $M \in \mathbb{N}$ and A_m are certain real numbers (see e.g. [**61**]). The Wright function $\Phi_\gamma(\cdot)$ can be integrally represented by the formula

$$\Phi_\gamma(z) = \frac{1}{2\pi i}\int_G \lambda^{\gamma-1}\exp(\lambda - z\lambda^\gamma)\,d\lambda, \quad z \in \mathbb{C},$$

where G is the Hankel path mentioned above.

We also need the following class of Wright type functions

$$\phi(\rho,\nu;z) := \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\rho n+\nu)}, \quad z \in \mathbb{C} \ \ (\rho > -1,\ \nu \in \mathbb{C}),$$

which is well known because of the following Laplace transform identity:

$$\int_0^\infty e^{-\lambda t}t^{v\rho}\phi(\rho,1+\rho v,-st^\rho)\,dt = \lambda^{-1-\rho v}e^{-s\lambda^{-\rho}}, \tag{45}$$

which is valid for $s>0$, $\operatorname{Re}\lambda > 0$ and $1+\rho v \geqslant 0$. If $0<\rho<1$, then we know that there exist two finite real constants $c>0$ and $L>0$ such that

$$|\phi(\rho,\nu;-r)| \leqslant Le^{-cr^{1/(1+\rho)}}, \quad r \geqslant 0; \tag{46}$$

if $\rho = 1/2$, then the function $\phi(\rho,\nu;-r)$ can be represented in terms of the well known special functions $\operatorname{erf}(r)$, $\operatorname{erfc}(r)$ and $\operatorname{daw}(r)$. For more details about the Wright functions, see [**234**] and [**401**].

In the continuation, we will also use the Bessel functions of first kind. Let us recall that the Bessel function of order $\nu>0$, denoted by J_ν, is defined by

$$J_\nu(z) := \Big(\frac{z}{2}\Big)^\nu \sum_{n=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{k!\Gamma(\nu+k+1)}, \quad z \in \mathbb{C}.$$

Then for each $\nu>0$ we have the existence of a finite real constant $M>0$ such that $\lim_{r\to+\infty}|r^{1/2}J_\nu(r)| = 0$. Then the following Laplace transform identity holds true for each $\beta \geqslant 0$:

$$\int_0^\infty e^{-\lambda t}J_{1+\beta}\big(2\sqrt{st}\big)s^{(1+\beta)/2}\,ds = t^{(1+\beta)/2}\lambda^{-2-\beta}e^{-t/\lambda}, \quad \operatorname{Re}\lambda>0,\ t>0. \tag{47}$$

CHAPTER 2

ABSTRACT DEGENERATE VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

In this chapter, we investigate abstract (non-scalar) degenerate Volterra integro-differential equations and abstract (multi-term) fractional differential equations in sequentially complete locally convex spaces. We use different methods to achieve what we are aiming for. Special attention is paid to the analysis of hypercyclic, topologically mixing and distributionally chaotic classes of abstract degenerate integro-differential equations.

Unless stated otherwise, throughout this chapter we assume that E is an infinite-dimensional sequentially complete locally convex space over the field $\mathbb{C}$, SCLCS for short, and that A and B are two closed linear operators acting on E. The abbreviation $\circledast$ stands for the fundamental system of seminorms which induces the topology on E; I denotes the identity operator on E, and $C \in L(E)$ denotes an injective operator satisfying $CA \subseteq AC$. In some sections, we need to have two different pivot spaces, so that we sometimes use the symbols $X, Y, Z \ldots$ in place of E. We start our work by investigating the basic properties of (C, B)-resolvents of closed linear operators.

2.1. (C, B)-resolvents

In this section, we would like to inscribe a few noteworthy facts about the main structural properties of (C, B)-resolvents. We assume that:

1. $A\colon D(A) \subseteq E \to E$ and $B\colon D(B) \subseteq E \to E$ are closed linear operators;
2. $C \in L(E)$ is an injective operator satisfying $CA \subseteq AC$ and $CB \subseteq BC$;
3. The closed graph theorem holds for mappings from E into E.

Then the set
$$\rho_C^B(A) := \{\lambda \in \mathbb{C} : (\lambda B - A)^{-1}C \in L(E)\}$$
is called the (C, B)-resolvent set of A; the (C, B)-spectrum of A is defined by $\sigma_C^B(A) := \mathbb{C} \setminus \rho_C^B(A)$. Sometimes we also write $\rho_C(A, B)$ ($\sigma_C(A, B)$) for $\rho_C^B(A)$ ($\sigma_C^B(A)$); $\rho^B(A) \equiv \rho_I^B(A)$ and $\sigma^B(A) \equiv \sigma_I^B(A)$. If $C \neq I$, then the (C, B)-resolvent set of the operator A need not be open (for a counterexample of this type, with $B = I$, E being the Hardy space $H^2(\{z \in \mathbb{C} : |z| \leqslant 1\})$, $A \in L(E)$ being injective and $C = A$, see [**138**, Example 2.5]). For any $\lambda \in \rho_C^B(A)$, we define the right (C, B)-resolvent of A, $R_\lambda^{C,B}(A)$ for short, and the left (C, B)-resolvent of A, $L_\lambda^{C,B}(A)$ for

short, by

$$R_\lambda^{C,B}(A) := (\lambda B - A)^{-1}CB \text{ and } L_\lambda^{C,B}(A) := B(\lambda B - A)^{-1}C \in L(E).$$

It is checked at once that the existence of operator $B^{-1} \in L(E)$ implies the closedness of the operator AB^{-1}, with domain and range contained in E, as well as that $\rho_C^B(A) \subseteq \rho_C(AB^{-1})$ and

$$(\lambda - AB^{-1})^{-1}C = B(\lambda B - A)^{-1}C, \quad \lambda \in \rho_C^B(A). \tag{48}$$

Now we would like to draw our attention to the case in which $\rho_C^B(A) \neq \emptyset$, the operator B is injective and $B^{-1} \notin L(E)$. Fix temporarily a number $\lambda \in \rho_C^B(A)$. Suppose that (x_τ) is a net in E as well as that $x_\tau \to x$ as $\tau \to \infty$ and $AB^{-1}x_\tau \to y$ as $\tau \to \infty$. This simply implies $B(\lambda B - A)^{-1}Cx_\tau \to B(\lambda B - A)^{-1}Cx$ as $\tau \to \infty$ and $B(\lambda B - A)^{-1}C(\lambda - AB^{-1})x_\tau = B(\lambda B - A)^{-1}C(\lambda B - A)B^{-1}x_\tau = Cx_\tau \to B(\lambda B - A)^{-1}C(\lambda x - y)$ as $\tau \to \infty$. Hence, $Cx = B(\lambda B - A)^{-1}C(\lambda x - y)$, $Cx \in D(AB^{-1})$ and $AB^{-1}Cx = AB^{-1}B(\lambda B - A)^{-1}C(\lambda x - y) = A(\lambda B - A)^{-1}C(\lambda x - y)$. Further on, $B(\lambda B - A)^{-1}Cy = \lambda B(\lambda B - A)^{-1}Cx - Cx = A(\lambda B - A)^{-1}Cx$, $(\lambda B - A)^{-1}Cy = B^{-1}A(\lambda B - A)^{-1}Cx = -B^{-1}Cx + \lambda(\lambda B - A)^{-1}Cx$, whence it easily follows that $Cy = -(\lambda B - A)B^{-1}Cx + Cx$ and $Cy = AB^{-1}Cx$. Hence, the operator AB^{-1} is closable and the supposition $C^{-1} \in L(E)$ implies that the operator AB^{-1} is closed; before proceeding further, we want to observe that the operator AB^{-1} need not be closed if the above requirements hold and $C^{-1} \notin L(E)$ (let $A = B = C$, and let $R(C)$ be a proper dense subspace of E; then $\overline{CC^{-1}} = I \neq CC^{-1}$, see [**138**, Example 2.2]). It is not problematic to verify that the operator $\overline{AB^{-1}}$ commutes with the operator $B(\lambda B - A)^{-1}C$ ($\lambda \in \rho_C^B(A)$), and that the operator $\lambda + \overline{AB^{-1}}$ is injective ($\lambda \in \rho_C^B(A)$). By the foregoing, we have $\rho_C^B(A) \subseteq \rho_C(\overline{AB^{-1}})$ and the following modification of (48):

$$(\lambda - \overline{AB^{-1}})^{-1}C = B(\lambda B - A)^{-1}C, \quad \lambda \in \rho_C^B(A). \tag{49}$$

If the operator B is not injective, then AB^{-1} is an MLO in E and, in this case, we can simply prove that $\rho_C^B(A) \subseteq \rho_C(AB^{-1})$ and (48) continues to hold. Therefore, we have arrived at the following propositions.

PROPOSITION 2.1.1. *Suppose that $\rho_C^B(A) \neq \emptyset$ and the operator B is injective.*

- (i) *If $B^{-1} \in L(E)$ or $C^{-1} \in L(E)$, then the operator AB^{-1} is closed, $\rho_C^B(A) \subseteq \rho_C(AB^{-1})$ and (48) holds.*
- (ii) *Suppose $B^{-1} \notin L(E)$ and $C^{-1} \notin L(E)$. Then the operator AB^{-1} is closable, $\rho_C^B(A) \subseteq \rho_C(\overline{AB^{-1}})$ and (49) holds.*

PROPOSITION 2.1.2. *Suppose that the operator B is not injective. Then AB^{-1} is an* MLO *in E, $\rho_C^B(A) \subseteq \rho_C(AB^{-1})$ and (48) holds in the sense of multivalued linear operators.*

The inclusion $\rho_C(AB^{-1}) \subseteq \rho_C^B(A)$ also holds in some cases (for example, if $B \in L(E)$), but we will not go into further details concerning this question here. Using the trivial identities

$$(\lambda B - A)(\mu B - A)^{-1}C = C + (\lambda - \mu)B(\mu B - A)^{-1}C, \quad \mu \in \rho_C^B(A),\ \lambda \in \mathbb{C},$$

$$(\mu B-A)^{-1}C(\lambda B-A)x=Cx+(\lambda-\mu)B(\mu B-A)^{-1}Cx,$$
$$\mu\in\rho_C^B(A),\ \lambda\in\mathbb{C},\ x\in D(A)\cap D(B),$$

and observing that for each $\lambda\in\rho_C^B(A)$ we have $B(\lambda B-A)^{-1}C^2=CB(\lambda B-A)^{-1}C$, the following version of Hilbert resolvent equation readily follows:

(50) $$(\lambda B-A)^{-1}C^2-(\mu B-A)^{-1}C^2=(\mu-\lambda)(\mu B-A)^{-1}CB(\lambda B-A)^{-1}C,$$

for any $\lambda,\mu\in\rho_C^B(A)$. From this, we may conclude the following:

(RE1) Suppose $\lambda,\mu\in\rho_C^B(A)$. Then

(51) $$L_\lambda^{C,B}(A)C-L_\mu^{C,B}(A)C=(\mu-\lambda)L_\mu^{C,B}(A)L_\lambda^{C,B}(A)$$

and
$$L_\mu^{C,B}(A)L_\lambda^{C,B}(A)=L_\lambda^{C,B}(A)L_\mu^{C,B}(A).$$

Hence, the nonemptiness of set $\rho_C^B(A)$ implies that the function $\lambda\mapsto L_\lambda^{C,B}(A)\in L(E)$, $\lambda\in\rho_C^B(A)$ is a C-pseudoresolvent in the sense of [**384**, Definition 3.1] and the following holds [**384**]:

(RE1)' The spaces $N(L_\lambda^{C,B}(A))$, $C^{-1}(R(L_\lambda^{C,B}(A)))$, $N(C-\lambda L_\lambda^{C,B}(A))$ and $C^{-1}(R(C-\lambda L_\lambda^{C,B}(A)))$ are independent of $\lambda\in\rho_C^B(A)$.

(RE1)" Suppose, additionally, that $N(L_\lambda^{C,B}(A))=\{0\}$ for some $\lambda\in\rho_C^B(A)$. Then we can define the closed linear operator W on E by

$$D(W):=C^{-1}(R(L_\lambda^{C,B}(A)))\text{ and }Wx:=(\lambda-(L_\lambda^{C,B}(A))^{-1}C)x\text{ for }x\in D(W);$$

observe that (RE1)' implies that the definition of W is independent of $\lambda\in\rho_C^B(A)$. Then $C^{-1}WC=W$, $\rho_C^B(A)\subseteq\rho_C(W)$ and $L_\lambda^{C,B}(A)=(\lambda-W)^{-1}C$, $\lambda\in\rho_C^B(A)$.

It is well known that the existence of operator W from (RE1)" cannot be proved in the case that there exists $\lambda\in\rho_C^B(A)$ such that the kernel space of operator $L_\lambda^{C,B}(A)$ is non-trivial (cf. also Example 2.1.6 below; then (RE1)" holds). It is not difficult to prove that

$$A(\lambda B-A)^{-1}CBx=B(\lambda B-A)^{-1}CAx,\quad x\in D(A)\cap D(B),\ \lambda\in\rho_C^B(A),$$

and (see the second equality in [**509**, Lemma 2.1.2] with $C=I$):

$$N(L_\lambda^{C,B}(A))=C^{-1}[\{Ax:x\in D(A)\cap N(B)\}],\quad\lambda\in\rho_C^B(A).$$

The proof of following resolvent equation follows from (reqefhil and the fact that, for every $x\in D(B)$, one has $B(\lambda B-A)^{-1}CBCx=CB(\lambda B-A)^{-1}CBx$:

(RE2) Suppose $\lambda,\mu\in\rho_C^B(A)$ and $x\in D(B)$. Then

$$R_\lambda^{C,B}(A)Cx-R_\mu^{C,B}(A)Cx=(\mu-\lambda)R_\mu^{C,B}(A)R_\lambda^{C,B}(A)x$$

and
$$R_\mu^{C,B}(A)R_\lambda^{C,B}(A)x=R_\lambda^{C,B}(A)R_\mu^{C,B}(A)x.$$

Taking into account (RE2) and proceeding as in the proofs of [**384**, Lemma 3.2, Lemma 3.3], we can deduce the following:

(RE2)' The spaces $N(R_\lambda^{C,B}(A))$, $C^{-1}(R(R_\lambda^{C,B}(A)))\cap D(B)$, $N(C-\lambda R_\lambda^{C,B}(A))$ and $C^{-1}(R(C-\lambda R_\lambda^{C,B}(A)))\cap D(B)$ are independent of $\lambda\in\rho_C^B(A)$.

Furthermore, if $B\in L(E)$ is injective, then it is not difficult to show that the operator $B^{-1}A$ is closed, as well as that $\rho_C(B^{-1}A)=\rho_C^B(A)$ and

$$(\lambda-B^{-1}A)^{-1}Cx=(\lambda B-A)^{-1}CBx,\quad x\in E;$$

cf. also (RE1)", [**384**, Theorem 3.4] and [**199**, Theorem 1.15]. Making use of [**296**, Lemma 3.3], (50) and the argumentation from [**138**, Section 2] (cf. [**138**, Proposition 2.6, Remark 2.7]), we can prove the following:

PROPOSITION 2.1.3. *Let $\emptyset\neq\Omega\subseteq\rho_C^B(A)$ be open, and let $x\in E$.*

(i) *The local boundedness of mapping $\lambda\mapsto B(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$, resp. the assumption that E is barreled and local boundedness of mapping $\lambda\mapsto B(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$, implies the analyticity of mappings $\lambda\mapsto(\lambda B-A)^{-1}C^3x$, $\lambda\in\Omega$ and $\lambda\mapsto B(\lambda B-A)^{-1}C^3x$, $\lambda\in\Omega$, resp. $\lambda\mapsto(\lambda B-A)^{-1}C^3\in L(E)$, $\lambda\in\Omega$ and $\lambda\mapsto B(\lambda B-A)^{-1}C^3\in L(E)$, $\lambda\in\Omega$. Furthermore, if $R(C)$ is dense in E, resp. if $R(C)$ is dense in E and E is barreled, then the mappings $\lambda\mapsto(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ and $\lambda\mapsto B(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ are analytic, resp. the mappings $\lambda\mapsto(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$ and $\lambda\mapsto B(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$ are analytic.*

(ii) *The continuity of mapping $\lambda\mapsto B(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ implies its analyticity. The continuity of mappings $\lambda\mapsto B(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ and $\lambda\mapsto(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ implies the analyticity of mapping $\lambda\mapsto(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$; the strong continuity of mapping $\lambda\mapsto(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$ ($\lambda\mapsto(\lambda B-A)^{-1}CB$, $\lambda\in\Omega$; with the meaning clear) implies the analyticity of mapping $\lambda\mapsto(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ ($\lambda\mapsto(\lambda B-A)^{-1}CBx$, $\lambda\in\Omega$, provided that $x\in D(B)$), as well. Furthermore, if E is barreled, then the continuity of mapping $\lambda\mapsto(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$ ($\lambda\mapsto B(\lambda B-A)^{-1}C\in L(E)$, $\lambda\in\Omega$) implies its analyticity; the same conclusion holds for the mapping $\lambda\mapsto(\lambda B-A)^{-1}CB\in L(E)$, $\lambda\in\Omega$, provided that E is barreled and $B\in L(E)$.*

For clarity's sake, we will prove parts (i) and (ii) of the following extension of [**138**, Corollary 2.8].

PROPOSITION 2.1.4. *Let $\emptyset\neq\Omega\subseteq\rho_C^B(A)$ be open, and let $x\in E$.*

(i) *Suppose that the mapping $\lambda\mapsto(\lambda B-A)^{-1}Cx$, $\lambda\in\Omega$ is analytic. Then*

$$(\lambda B-A)\frac{d^n}{d\lambda^n}(\lambda B-A)^{-1}Cx=(-n)B\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}Cx,\quad n\in\mathbb{N},\ \lambda\in\Omega,$$

$Cx\in D(((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1})$, $n\in\mathbb{N}$, $\lambda\in\Omega$, and for each $n\in\mathbb{N}$ and $\lambda\in\Omega$,

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}Cx=(-1)^{n-1}(n-1)!((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}Cx.$$

If, in addition, the mapping $\lambda \mapsto B(\lambda B-A)^{-1}Cx$, $\lambda \in \Omega$ *is analytic, then for each* $n \in \mathbb{N}$ *and* $\lambda \in \Omega$, $Cx \in D(B((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1})$, *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}B(\lambda B-A)^{-1}Cx = (-1)^{n-1}(n-1)!B((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}Cx.$$

(ii) *Suppose that the mapping* $\lambda \mapsto B(\lambda B-A)^{-1}Cx$, $\lambda \in \Omega$ *is analytic. Then for each* $n \in \mathbb{N}$ *and* $\lambda \in \Omega$, $((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}C^2x \in R(C)$, $n \in \mathbb{N}$, $\lambda \in \Omega$ *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}B(\lambda B-A)^{-1}Cx = C^{-1}(-1)^{n-1}(n-1)!B((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}C^2x.$$

(iii) *Suppose that* E *is barreled, and the mapping* $\lambda \mapsto (\lambda B-A)^{-1}C \in L(E)$, $\lambda \in \Omega$ *is analytic, resp.,* $B \in L(E)$ *and the mapping* $\lambda \mapsto (\lambda B-A)^{-1}CB \in L(E)$, $\lambda \in \Omega$ *is analytic. Then for each* $n \in \mathbb{N}$ *and* $\lambda \in \Omega$, $R(C) \subseteq D(((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1})$, *resp.,*
$R(CB) \subseteq D(((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1})$, *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}C = (-1)^{n-1}(n-1)!((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}C \in L(E),$$

resp.,

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}CB = (-1)^{n-1}(n-1)!((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}CB \in L(E).$$

(iv) *Suppose that* E *is barreled and the mapping* $\lambda \mapsto B(\lambda B-A)^{-1}C \in L(E)$, $\lambda \in \Omega$ *is analytic. Then* $R(((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}C^2) \subseteq R(C)$, $n \in \mathbb{N}$, $\lambda \in \Omega$ *and*

$$\begin{aligned}\frac{d^{n-1}}{d\lambda^{n-1}}B(\lambda B-A)^{-1}C &= C^{-1}(-1)^{n-1}(n-1)! \\ &\quad \times B((\lambda B-A)^{-1}B)^{n-1}(\lambda B-A)^{-1}C^2 \in L(E), \quad n \in \mathbb{N},\ \lambda \in \Omega.\end{aligned}$$

PROOF. Let $n \in \mathbb{N}$, let $\lambda \in \Omega$, and let Γ be a positively oriented circle around λ that is contained in Ω. Making use of the Cauchy integral formula, we get that

$$\frac{d^n}{d\lambda^n}(\lambda B-A)^{-1}Cx = \frac{n!}{2\pi i}\oint_\Gamma \frac{(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz.$$

Since the operators A and B are closed, we get from the above that $\frac{d^n}{d\lambda^n}(\lambda B-A)^{-1}Cx \in D(A) \cap D(B)$. Applying again the Cauchy integral formula, and taking into account that the operators $C^{-1}(\lambda B-A)C$ and B are closed, we get that

$$\begin{aligned}(\lambda B-A)\frac{d^n}{d\lambda^n}(\lambda B-A)^{-1}Cx &= C^{-1}(\lambda B-A)C\frac{n!}{2\pi i}\oint_\Gamma \frac{(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz \\ &= \frac{n!}{2\pi i}\oint_\Gamma \frac{C^{-1}(\lambda B-A)C(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz \\ &= \frac{n!}{2\pi i}\oint_\Gamma \frac{(\lambda B-A)(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz\end{aligned}$$

$$= \oint_\Gamma \frac{Cx}{(z-\lambda)^{n+1}}dz - \oint_\Gamma \frac{B(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz$$
$$= -\oint_\Gamma \frac{B(zB-A)^{-1}Cx}{(z-\lambda)^{n+1}}dz$$
$$= -B\oint_\Gamma \frac{(zB-A)^{-1}Cx}{(z-\lambda)^{n}}dz$$
$$= (-n)B\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}Cx,$$

which proves the first equality in (i). This implies

$$\frac{d^n}{d\lambda^n}(\lambda B-A)^{-1}Cx = (-n)[(\lambda B-A)^{-1}B]\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda B-A)^{-1}Cx,$$

and now the remainder of (i) simply follows by induction. To prove (ii), suppose that $\lambda_0 \in \Omega$. Since the mapping $\lambda \mapsto B(\lambda B-A)^{-1}Cx$, $\lambda \in \Omega$ is analytic, the Hilbert resolvent equation (50) shows that the mapping $\lambda \mapsto [(\lambda_0 B-A)^{-1}C]B(\lambda B-A)^{-1}Cx = (1/(\lambda_0-\lambda))[(\lambda B-A)^{-1}C^2x-(\lambda_0 B-A)^{-1}C^2x]$, $\lambda \in \Omega \smallsetminus \{\lambda_0\}$ is analytic, as well. From this, we may conclude that the mapping $\lambda \mapsto (\lambda B-A)^{-1}C^2x$, $\lambda \in \Omega$ is analytic. By our assumption, the mapping $\lambda \mapsto B(\lambda B-A)^{-1}C^2x$, $\lambda \in \Omega$ is likewise analytic so that part (ii) follows almost directly from (i). The proofs of (iii) and (iv) are simple and therefore omitted. $\square$

Summa summarum, Proposition 2.1.3 and Proposition 2.1.4 taken together provide a generalization of [**296**, Proposition 2.16] for degenerate (C, B)-resolvents.

Remark 2.1.5. In the case that $C = I$ and E is a Banach space, it is well known that the (I, B)-resolvent set of A is open, as well as that the (I, B)-resolvent, right (I, B)-resolvent and left (I, B)-resolvent of the operator A are analytic in $\rho_B(A)$ [**509**]. The corresponding statement in locally convex spaces has recently been analyzed in [**214**, Theorem 1].

Let us consider again the following abstract degenerate Volterra integral equation:

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in [0, \tau), \tag{52}$$

where $0 < \tau \leqslant \infty$, $t \mapsto f(t)$, $t \in [0, \tau)$ is a continuous mapping with values in a complex SCLCS E, $a \in L^1_{loc}([0, \tau))$ and A, B are closed linear operators with domain and range contained in E.

Example 2.1.6. Let the function $a(t)$ be a kernel on $[0, \tau)$ and let the operator W generate a (local) (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying $W\int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx$, $x \in X$, $t \in [0, \tau)$ (the use of operator W here seems to be much better than the use of operator $\overline{AB^{-1}}$, provided that B is injective and $C \neq I$; cf. [**292**] for the notion and (49)). Then a simple computation involving the definition of operator W shows that for each element $y \in E$ such that the element $x = C^{-1}L_\lambda^{C,B}(A)y$ is well-defined, we have

$$R(t)C^{-1}L_\lambda^{C,B}(A)y - k(t)L_\lambda^{C,B}(A)y = \int_0^t a(t-s)R(s)[C^{-1}A(\lambda B - A)^{-1}Cy]ds,$$

for any $t \in [0,\tau)$. In particular, if $y = (\lambda B - A)z$ for some $z \in D(A) \cap D(B)$, then the above requirements hold and we get

$$R(t)Bz - k(t)CBz = \int_0^t a(t-s)R(s)Az\,ds, \quad t \in [0,\tau).$$

Assuming additionally that $R(t)$ commutes with A and B for all $t \in [0,\tau)$, the above implies that the function $u(t) := R(t)z$, $t \in [0,\tau)$ is a unique strong solution of the abstract degenerate Volterra equation (52), with $f(t) = k(t)CBz$, $t \in [0,\tau)$.

Remark 2.1.7. Assume that the functions $k(t)$ and $|a|(t)$ satisfy the condition (P1), as well as that the operator B is injective. Using the definition of operator W, properties stated in (RE1)" and [**292**, Theorem 2.1.5], we have that W generates a global exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ (cf. [**292**] for the notion) provided that there exists a sufficiently large real number $\omega > 0$ such that the family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(E)$ is equicontinuous as well as that for each $\lambda \in \mathbb{C}$ with $\tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0$ and $\operatorname{Re}\lambda > \omega$, the operator $B - \tilde{a}(\lambda)A$ is injective and

$$\tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in E.$$

Combined with the conclusions clarified in the above example, we are in a position to prove Theorem 2.2.8(ii) in a much simpler way provided the injectiveness of B. It is also worth noting that we can use [**292**, Theorem 2.1.6, Proposition 2.1.7, Theorem 2.1.29, Proposition 2.1.32] here.

2.2. Degenerate (a,k)-regularized C-resolvent families in locally convex spaces

The main aim of this section is to present our recent results on degenerate (a,k)-regularized C-resolvent families in locally convex spaces and semilinear degenerate relaxation differential equations with abstract differential operators [**306, 319**].

Before starting our work, we need to remind ourselves of some well known facts and definitions. A nonnegative infinitely differentiable function $\varphi\colon (0,\infty) \to \mathbb{R}$ is called a Bernstein function iff the function $\varphi'(\cdot)$ is completely monotonic, i.e., $(-1)^n\varphi^{(n+1)}(t) \geqslant 0$, $n \in \mathbb{N}_0$, $t > 0$. Following [**459**, Definition 4.4], it will be said that a function $a\colon (0,\infty) \to \mathbb{R}$ is a creep function iff $a(t)$ is nonnegative, nondecreasing and concave. A creep function $a(t)$ has the standard form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(s)ds,$$

where $a_0 = a(0+) \geqslant 0$, $a_\infty = \lim_{t\to\infty} a(t)/t = \inf_{t>0} a(t)/t \geqslant 0$, and $a_1(t) = a'(t) - a_\infty$ is nonnegative, nonincreasing and $\lim_{t\to\infty} a_1(t) = 0$. Recall that, for every $j \in \mathbb{N}$ and $\alpha > 0$, there exist uniquely determined real numbers $c_{l,j,\alpha}$ $(1 \leqslant$

$l \leqslant j$) such that the equation (44) holds for the Mittag-Leffler function $E_\alpha(z)$. A function $a \in L^1_{loc}([0,\infty))$ is said to be completely positive iff for any $\eta \geqslant 0$, the solutions of the convolution equations

$$s(t) + \eta(a * s)(t) = 1 \text{ and } r(t) + \eta(a * r)(t) = a(t)$$

satisfy $s(t) \geqslant 0$ and $r(t) \geqslant 0$ on $[0,T]$; cf. also [**459**, Definition 4.5, p. 96] and [**395**, Remark 3.6, (3.3)].

Let $0 < \tau \leqslant \infty$ and $a \in L^1_{loc}([0,\tau))$. Then we say that the function $a(t)$ is a kernel on $[0,\tau)$ iff for each $f \in C([0,\tau))$ the assumption $\int_0^t a(t-s)f(s)ds = 0$, $t \in [0,\tau)$ implies $f(t) = 0$, $t \in [0,\tau)$. If $\tau = \infty$ and $a \neq 0$ in $L^1_{loc}([0,\infty))$, then the famous Titchmarsh theorem [**520**, Theorem VII] implies that the function $a(t)$ is automatically a kernel on $[0,\infty)$; the situation is quite different in the case that $\tau < \infty$, then we can apply the Titchmarsh–Foiaş theorem [**40**, Theorem 2.1] (cf. also [**291**, Theorem 3.4.40]) in order to see that the function $a(t)$ is a kernel on $[0,\tau)$ iff $0 \in \operatorname{supp}(a)$. In this place, it is worth noting that, for any function $a(t)$ satisfying (P1), the condition $0 \in \operatorname{supp}(a)$ is necessary and sufficient for the equality $\limsup_{\lambda\to\infty} \lambda^{-1} \ln|\tilde{a}(\lambda)| = 0$ to be true, or equivalently, for the convolution mapping $\mathcal{K}\colon f \mapsto a * f$ to be an injective operator on $C([0,\infty))$ with dense range in the Fréchet space $C_*([0,\infty))$ of all continuous functions $g\colon [0,\infty) \mapsto \mathbb{C}$ such that $g(0) = 0$, equipped with the seminorms $\|g\|_n := \sup_{t\in[0,n]} |g(t)|$ ($n \in \mathbb{N}$); see e.g. [**27**, p. 106].

In this section, we assume that $a \neq 0$ in $L^1_{loc}([0,\infty))$ and $k \neq 0$ in $C([0,\infty))$, so that the functions $a(t)$ and $k(t)$ will be kernels on $[0,\infty)$.

2.2.1. The main structural properties of degenerate (a,k)-regularized C-resolvent families. We start this subsection by introducing the following definition (cf. [**292**, Subsection 2.1.1] and [**459**] for the case $B = I$):

Definition 2.2.1. Let $0 < \tau \leqslant \infty$. A function $u \in C([0,\tau) : E)$ is said to be:

(i) a (mild) solution of (52) iff $(a * u)(t) \in D(A)$, $t \in [0,\tau)$, $A(a * u)(t) = Bu(t) - f(t)$, $t \in [0,\tau)$ and the mapping $t \mapsto Bu(t)$, $t \in [0,\tau)$ is continuous,
(ii) a strong solution of (52) iff the mapping $t \mapsto Au(t)$, $t \in [0,\tau)$ is continuous, $(a*Au)(t) = Bu(t) - f(t)$, $t \in [0,\tau)$ and the mapping $t \mapsto Bu(t)$, $t \in [0,\tau)$ is continuous,
(iii) a weak solution of (52) iff for every $(x^*, y^*) \in A^*$ and for every $t \in [0,\tau)$, one has $\langle x^*, Bu(t)\rangle = \langle x^*, f(t)\rangle + \langle y^*, (a * u)(t)\rangle$, $t \in [0,\tau)$.

It is clear that any strong solution of (52) is also a mild solution of the same problem; the converse statement is not true in general. Since [**295**, Lemma 2.4] continues to hold in SCLCSs, the concepts mild and weak solution of (52) coincide actually.

We introduce the notion of an exponentially equicontinuous (a,k)-regularized C-resolvent family for (52) as follows.

Definition 2.2.2. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t)\colon D(B) \to E$ is a linear mapping ($t \geqslant 0$). Let $C \in L(E)$ be injective, and let $CA \subseteq AC$. Then the operator family $(R(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous (a,k)-regularized C-resolvent family for (52) iff there exists $\omega \geqslant \max(0, \operatorname{abs}(a), \operatorname{abs}(k))$ such that the following holds:

(i) The mapping $t \mapsto R(t)x$, $t \geqslant 0$ is continuous for every fixed element $x \in D(B)$.

(ii) The family $\{e^{-\omega t}R(t) : t \geqslant 0\}$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that

$$p(e^{-\omega t}R(t)x) \leqslant cq(x), \quad x \in D(B),\ t \geqslant 0. \tag{53}$$

(iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ and

$$\tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in D(B). \tag{54}$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous r-times integrated (a,C)-regularized resolvent family for (52); an exponentially equicontinuous 0-times integrated (a,C)-regularized resolvent family for (52) is also said to be an exponentially equicontinuous (a,C)-regularized resolvent family for (52).

Remark 2.2.3. (i) If $B = I$, then the above simply means that A is a subgenerator of the exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ in the sense of [**296**, Definition 2.1]; cf. also [**296**, Theorem 2.7]. The case $B \neq I$ is more difficult to deal with; for example, the validity of some very simple equalities, like $R(t)Ax = AR(t)x$, $t \geqslant 0$ or $R(t)Bx = BR(t)x$, $t \geqslant 0$, cannot be proved without making some new assumptions. Furthermore, it is not clear how one can define, by using a method similar to that employed in Definition 2.2.2, the notion of an exponentially equicontinuous (a,k)-regularized (C_1, C_2)-existence and uniqueness family for (52) in a satisfactory way.

(ii) In contrast to [**542**,**543**], we do not assume in Definition 2.2.2 that $CB \subseteq BC$ or $R(C) \subseteq R(B - \tilde{a}(\lambda)A)$ ($\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$).

(iii) The uniqueness theorem for Laplace transform implies that there exists at most one exponentially equicontinuous (a,k)-regularized C-resolvent family for (52).

(iv) If E is complete and B is densely defined, then [**419**, Lemma 22.19] combined with (ii) of Definition 2.2.2 implies that, for every $t \geqslant 0$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, there exist two operators $\hat{R}(t), G(\lambda) \in L(E)$ such that $\hat{R}(t)x = R(t)x$, $x \in D(B)$ and $G(\lambda)x = (B - \tilde{a}(\lambda)A)^{-1}CBx$, $x \in D(B)$. The operator family $(\hat{R}(t))_{t\geqslant 0} \subseteq L(E)$ is strongly continuous and, for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that (53) holds for all $x \in E$ and $t \geqslant 0$, with $(R(t))_{t\geqslant 0}$ replaced by $(\hat{R}(t))_{t\geqslant 0}$. Furthermore, $\tilde{k}(\lambda)G(\lambda)x = \int_0^\infty e^{-\lambda t}R(t)x\,dt$, $x \in E$.

(v) The notion of a (local) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ for (52) can be defined in many different ways, but it seems that there is no satisfactory option that would provide us a general approach to the

Volterra problems of the kind (52). Observe also that we can simply construct a great number of examples of strongly continuous operator families $(R(t))_{t\in[0,\tau)} \subseteq L(E)$ for which neither $R(t)A \subseteq R(t)A$, $t \in [0,\tau)$ nor $R(t)B \subseteq R(t)B$, $t \in [0,\tau)$, or only $R(t)B \nsubseteq R(t)B$, $t \in [0,\tau)$, but the solution of (52) exists and has the form $u(t) = R(t)x$, $t \in [0,\tau)$ for some $x \in E$; see e.g. [**292**, Example 2.8.11] and [**295**, Example 2.31]. Because of that, in the sequel of this section we tend to pay attention primarily to the notion introduced in Definition 2.2.2.

The proof of following proposition is omitted for the sake of brevity (cf. [**296**, Proposition 2.4] for more details).

PROPOSITION 2.2.4. (i) *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k)-regularized C-resolvent family for* (52), *and let $b \neq 0$ in $L^1_{loc}([0,\infty))$. If there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\int_0^t |b(s)|ds \leqslant Me^{\omega t}$, $t \geqslant 0$, then $((b * R)(t))_{t\geqslant 0}$ is an exponentially equicontinuous $(a, k * b)$-regularized C-resolvent family for* (52).

(ii) *Let $(R_i(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k_i)-regularized C-resolvent family for* (52), *$i = 1,2$. Then $(k_2 * R_1)(t)x = (k_1 * R_2)(t)x$, $t \geqslant 0$, $x \in D(B)$.*

(iii) *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k)-regularized C-resolvent family for* (52). *Suppose that $k(0) \neq 0$, $b(t)$ satisfies* (P1), *$(b * k)(t) + k(t)k(0)^{-1} = 1$, $t \geqslant 0$ and the function $t \mapsto \int_0^t |b(s)|ds$, $t \geqslant 0$ is exponentially bounded. Then $(S(t)\cdot \equiv k(0)^{-1}R(t)\cdot + (b * R(\cdot)\cdot)(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,C)-regularized resolvent family for* (52).

It should be noted that our analysis covers many important subjects that have not been considered in [**542**, **543**]. For example, we are in a position to clarify Hille–Yosida's type theorems for degenerate exponentially equicontinuous (a,k)-regularized C-resolvent families.

THEOREM 2.2.5. (cf. [**292**, Theorem 2.1.6] for the case $B = I$)

(i) *Let $\omega_0 > \max(0, \operatorname{abs}(a), \operatorname{abs}(k))$, and let $a(t)$ and $k(t)$ satisfy* (P1). *Assume that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective and $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$. If for each $x \in D(B)$ there exists a function $\Upsilon_x\colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to E$ which satisfies:*
   (a) *$\Upsilon_x(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$,*
   (b) *the mapping $\lambda \mapsto \Upsilon_x(\lambda)$, $\operatorname{Re}\lambda > \omega_0$ is analytic, and*
   (c) *there exists $r \geqslant -1$ such that for each $p \in \circledast$ there exist $M_p > 0$ and $q_p \in \circledast$ satisfying*
$$p(\Upsilon_x(\lambda)) \leqslant M_p q_p(x)|\lambda|^r, \quad \operatorname{Re}\lambda > \omega_0,\ x \in D(B),$$

then, for every $\alpha > 1$, there exists an exponentially equicontinuous $(a, k * g_{\alpha+r})$-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$ for (52), *and there exists a constant $c_\alpha > 0$ such that*
$$p(R_\alpha(t)x) \leqslant c_\alpha M_p q_p(x) e^{\omega_0 t}, \quad p \in \circledast,\ x \in D(B),\ t \geqslant 0.$$

(ii) *Let* $\omega \in \mathbb{R}$, $\omega_0 > \max(0, \omega, \mathrm{abs}(a), \mathrm{abs}(k))$, *and let* $a(t)$ *and* $k(t)$ *satisfy* (P1). *Assume that for each* $x \in D(B)$ *there exists an infinitely differentiable function* $\Upsilon_x \colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to E$ *which satisfies the item* (i)(a) *for real values of parameter* λ, *as well as that for each* $p \in \circledast$ *there exist* $c_p > 0$ *and* $q_p \in \circledast$ *such that*

$$p\Big(n!^{-1}(\lambda-\omega)^{n+1}\frac{d^n}{d\lambda^n}\Upsilon_x(\lambda)\Big) \leqslant c_p q_p(x),$$

provided $\tilde{k}(\lambda) \neq 0$, $\lambda > \omega_0$, $x \in D(B)$, $n \in \mathbb{N}_0$. *Then, for every* $r \in (0,1]$, *there exists an exponentially equicontinuous* $(a, k * g_r)$*-regularized* C*-resolvent family* $(R_r(t))_{t\geqslant 0}$ *for* (52), *and*

$$p(R_r(t+h)x - R_r(t)x) \leqslant \frac{2c_p q_p(x)}{r\Gamma(r)}\max(e^{\omega(t+h)}, 1)h^r,$$

provided $p \in \circledast$, $t \geqslant 0$, $h > 0$, $x \in D(B)$. *Furthermore, if* B *is densely defined and the mapping* $t \mapsto R_1(t)x$, $t \geqslant 0$ *is continuously differentiable for all* $x \in D(B)$, *then there exists an exponentially equicontinuous* (a,k)*-regularized* C*-resolvent family for* (52).

In the subsequent theorem (cf. [**1**, **61**, **63**, **395**, **459**, **460**] and [**292**] for more details), we analyze subordination principles for degenerate (a,k)-regularized C-resolvent families.

THEOREM 2.2.6. (i) *Let* $a(t)$, $b(t)$ *and* $c(t)$ *satisfy* (P1) *and let* $\int_0^\infty e^{-\beta t}|b(t)|dt < \infty$ *for some* $\beta \geqslant 0$. *Let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\beta}, \quad \alpha = 0 \text{ otherwise},$$

and let $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda \geqslant \alpha$. *Assume that there exists an exponentially equicontinuous* (b,k)*-regularized* C*-resolvent family* $(R_b(t))_{t\geqslant 0}$ *for* (52), *with* $a(t)$ *replaced by* $b(t)$, *satisfying that the family* $\{e^{-\omega_b t}R_b(t) : t \geqslant 0\}$ *is equicontinuous for some* $\omega_b \geqslant 0$ *(with the meaning clear). Assume, further, that* $c(t)$ *is completely positive and there exists a function* $k_1(t)$ *satisfying* (P1) *and*

$$\widetilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big), \quad \lambda > \omega_0,\ \tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0, \text{ for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\omega_b}, \quad \omega_a = 0 \text{ otherwise}.$$

Then, for every $r \in (0,1]$, *there exists an exponentially equicontinuous* $(a, k_1 * g_r)$*-regularized* C*-resolvent family* $(R_r(t))_{t\geqslant 0}$ *for* (52), *satisfying that the family* $\{e^{-\omega_a t}R_r(t) : t \geqslant 0\}$ *is equicontinuous, if* $\omega_b = 0$ *or* $\omega_b\tilde{c}(0) \neq 1$, *resp., for every* $\varepsilon > 0$, *there exists* $M_\varepsilon \geqslant 1$ *such that the family* $\{e^{-\varepsilon t}R_r(t) : t \geqslant 0\}$ *is equicontinuous, if* $\omega_b > 0$ *and* $\omega_b\tilde{c}(0) = 1$.

(ii) *Suppose* $\alpha \geqslant 0$ *and there exists an exponentially equicontinuous* $(1, g_\alpha)$*-regularized* C*-resolvent family for 52. Assume, further, that* $a(t)$ *and* $k(t)$ *satisfies* (P1)*, as well as that* $\tilde{k}(\lambda) = \tilde{a}(\lambda)^\alpha$ *for* λ *sufficiently large, and* $a(t)$ *is completely positive. Then, for every* $r \in (0,1]$*, there exists an exponentially equicontinuous* $(a, k * g_r)$*-regularized* C*-resolvent family for* (52) *(*$(a, a^{*,n} * g_r)$*-regularized* C*-resolvent family for* (52) *if* $\alpha = n \in \mathbb{N}$*, resp.* (a, g_{r+1})*-regularized* C*-resolvent family if* $\alpha = 0$*).*

(iii) *Suppose* $\alpha \geqslant 0$ *and there exists an exponentially equicontinuous* (g_2, g_α)*-regularized* C*-resolvent function for* (52)*. Let* $L^1_{loc}([0,\infty)) \ni c$ *be completely positive and let* $a(t) = (c * c)(t)$*,* $t \geqslant 0$*. (Recall that for any function* $a \in L^1_{loc}([0,\infty))$ *given in advance, such a function* $c(t)$ *always exists provided* $a(t)$ *is completely positive or* $a(t) \neq 0$ *is a creep function and* $a_1(t)$ *is log-convex). Assume* $k(t)$ *satisfies* (P1) *and* $\tilde{k}(\lambda) = \tilde{c}(\lambda)^\alpha/\lambda$*,* λ *sufficiently large. Then, for every* $r \in (0,1]$*, there exists an exponentially equicontinuous* $(a, k * g_r)$*-regularized* C*-resolvent family for* (52) *(*$(a, c^{*,n} * g_r)$*-regularized* C*-resolvent family for* (52) *if* $\alpha = n \in \mathbb{N}$*, resp.* (a, g_{r+1})*-regularized* C*-resolvent family for* (52) *if* $\alpha = 0$*).*

REMARK 2.2.7. In the case that $B = I$ and A is densely defined, the mapping $t \mapsto R_1(t)x$, $t \geqslant 0$, which appears in the formulation of Theorem 2.2.5(ii), is continuously differentiable for all $x \in E$ and, in the situation of Theorem 2.2.6(i), there exists an exponentially equicontinuous (a, k_1)-regularized C-resolvent family $(R(t))_{t \geqslant 0}$ for (52) satisfying that the family $\{e^{-\omega_a t} R_r(t) : t \geqslant 0\}$ is equicontinuous, resp., for every $\varepsilon > 0$, the family $\{e^{-\varepsilon t} R_r(t) : t \geqslant 0\}$ is equicontinuous (similar statements hold in the case of Theorem 2.2.6(ii)–(iii), cf. [**292**, Theorem 2.1.8] for further information). It is not clear whether the above results can be reformulated for abstract degenerate Volterra equations.

The following theorem provides an extension of [**542**, Theorem 1.6] and [**543**, Theorem 3.1].

THEOREM 2.2.8. *Let* $\tau = \infty$*, let the functions* $|a|(t)$ *and* $k(t)$ *satisfy* (P1)*, and let* $(R(t))_{t \geqslant 0}$ *be an exponentially equicontinuous* (a,k)*-regularized* C*-resolvent family for* (52)*, satisfying* (ii) *of* Definition 2.2.2 *with* $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$*.*

(i) *Suppose that* $v_0 \in D(B)$ *and the following condition holds:*

(i.1) *for every* $x \in D(B)$*, there exist a number* $\omega_0 > \omega$ *and a function* $h(\lambda; x) \in LT - E$ *such that* $h(\lambda; x) = \tilde{k}(\lambda) B (B - \tilde{a}(\lambda) A)^{-1} C B x$ *provided* $\operatorname{Re} \lambda > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$*.*

Then the function $u(t) = R(t) v_0$*,* $t \geqslant 0$ *is a mild solution of* (52) *with* $f(t) = k(t) C B v_0$*,* $t \geqslant 0$*. The uniqueness of mild solutions holds if we suppose additionally that* $CB \subseteq BC$ *and the function* $k(t)$ *satisfies* (P2)*.*

(ii) *Suppose that* $v_0 \in D(A) \cap D(B)$*,* $CB \subseteq BC$*, and the following condition holds:*

(ii.1) *for every* $x \in E$*, there exist a number* $\omega_1 > \omega$ *and a function* $h(\lambda; x) \in LT - E$ *such that* $h(\lambda; x) = \tilde{k}(\lambda) B (B - \tilde{a}(\lambda) A)^{-1} C x$ *provided* $\operatorname{Re} \lambda > \omega_1$ *and* $\tilde{k}(\lambda) \neq 0$*.*

Then the function $u(t) = R(t)v_0$, $t \geqslant 0$ is a strong solution of (52) *with $f(t) = k(t)CBv_0$, $t \geqslant 0$. The uniqueness of strong solutions holds if we suppose additionally that the function $k(t)$ satisfies* (P2).

PROOF. Let $v_0 \in D(B)$. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have

$$\tilde{a}(\lambda)\tilde{k}(\lambda)A(B-\tilde{a}(\lambda)A)^{-1}CBv_0 = \tilde{k}(\lambda)[-CBv_0 + B(B-\tilde{a}(\lambda)A)^{-1}CBv_0]. \tag{55}$$

Taking into account (i.1), the equation (54), as well as the closedness of B and the uniqueness theorem for Laplace transform, it can be simply proved that $R(t)v_0 \in D(B)$ for all $t \geqslant 0$, as well as that the mapping $t \mapsto BR(t)v_0$, $t \geqslant 0$ is continuous and

$$\tilde{k}(\lambda)B(B-\tilde{a}(\lambda)A)^{-1}CBv_0 = \int_0^\infty e^{-\lambda t}BR(t)v_0\,dt, \quad \operatorname{Re}\lambda > \omega_0,\ \tilde{k}(\lambda) \neq 0.$$

The previous equality in combination with (55) and the uniqueness theorem for Laplace transform implies that

$$A(a*R)(t)v_0 = BR(t)v_0 - k(t)CBv_0, \quad t \geqslant 0, \tag{56}$$

so that the function $t \mapsto u(t) \equiv R(t)v_0$, $t \geqslant 0$ is a mild solution of (52) with $f(t) = k(t)CBv_0$, $t \geqslant 0$. In order to prove (ii), fix an element $v_0 \in D(A) \cap D(B)$. Since we have assumed that $CB \subseteq BC$, it readily follows that

$$\tilde{k}(\lambda)B(B-\tilde{a}(\lambda)A)^{-1}CBv_0 - \tilde{k}(\lambda)BCv_0 = \tilde{k}(\lambda)\tilde{a}(\lambda)B(B-\tilde{a}(\lambda)A)^{-1}CAv_0,$$

for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$. By (ii.1), we know that there exists a continuous function $t \mapsto G(t)$, $t \geqslant 0$ such that

$$\int_0^\infty e^{-\lambda t}G(t)\,dt = \tilde{k}(\lambda)B(B-\tilde{a}(\lambda)A)^{-1}CAv_0, \tag{57}$$

for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$ and $\tilde{k}(\lambda) \neq 0$. With the help of (56)–(57) we can show that:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}A\int_0^t a(t-s)R(s)v_0\,ds\,dt &= \int_0^\infty e^{-\lambda t}[BR(t)v_0 - k(t)CBv_0]\,dt \\ &= \int_0^\infty e^{-\lambda t}\int_0^t a(t-s)G(s)\,ds\,dt,\end{aligned}$$

for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$. By the uniqueness theorem for Laplace transform, we get that $A\int_0^t a(t-s)R(s)v_0\,ds = \int_0^t a(t-s)G(s)\,ds$, $t \geqslant 0$. Let $(x^*, y^*) \in A^*$. Then $\langle x^*, (a*G)(t)\rangle = \langle y^*, (a*R(\cdot)v_0)(t)\rangle$, $t \geqslant 0$, i.e., $\int_0^t a(t-s)\langle x^*, G(s)\rangle\,ds = \int_0^t a(t-s)\langle y^*, R(s)v_0\rangle\,ds$, $t \geqslant 0$. Since the function $a(t)$ is a kernel on $[0,\infty)$, we obtain that $\langle x^*, G(t)\rangle = \langle y^*, R(t)v_0\rangle$, $t \geqslant 0$. Because of that, the equality $AR(t)v_0 = G(t)$ holds for any $t \geqslant 0$, and the function $u(t) = R(t)v_0$, $t \geqslant 0$ is a strong solution of (52) with $f(t) = k(t)CBv_0$, $t \geqslant 0$. It remains to be proved the uniqueness of mild solutions under the additional assumptions $CB \subseteq BC$ and the function $k(t)$ satisfies (P2). Towards this end, suppose that the function $t \mapsto u(t)$, $t \geqslant 0$ is a mild solution of (52) with $f(t) \equiv 0$. Put $v(t) := Cu(t)$, $t \geqslant 0$. Since

$CB \subseteq BC$, the function $v(t)$ is also a mild solution of (52) with $f(t) \equiv 0$. Since $\mathrm{abs}(|a|) < \infty$, we have that there exist constants $M \geqslant 1$ and $\omega_2 \geqslant 0$ such that $\int_0^t |a(r)|dr \leqslant Me^{\omega_2 t}$, $t \geqslant 0$. Then, for every $\sigma > 0$, we have

$$\begin{aligned}(58)\quad \int_0^t \int_s^\infty e^{\lambda(s-r-\sigma)}|a(r)|dr\,ds &= e^{-\lambda\sigma}\int_0^t\int_0^\infty e^{-\lambda\eta}|a(s+\eta)|d\eta\,ds\\ &= e^{-\lambda\sigma}\int_0^\infty\int_0^t e^{-\lambda\eta}|a(s+\eta)|ds\,d\eta = e^{-\lambda\sigma}\int_0^\infty e^{-\lambda\eta}\int_\eta^{t+\eta}|a(r)|dr\,d\eta\\ &\leqslant Me^{\omega_2 t-\lambda\sigma}\int_0^\infty e^{-(\lambda-\omega_2)\eta}d\eta = Me^{\omega_2 t-\lambda\sigma}/(\lambda-\omega_2),\quad t\geqslant 0,\ \lambda \text{ suff. large.}\end{aligned}$$

Moreover,

$$\begin{aligned}(B-\tilde{a}(\lambda)A)(e^{\lambda\cdot} * a * v)(t) &= B(e^{\lambda\cdot} * a * v)(t) - \tilde{a}(\lambda)(e^{\lambda\cdot} * Bv)(t)\\ &= -\Bigl(Bv(\cdot) * \int_\cdot^\infty e^{\lambda(\cdot-s)}a(s)ds\Bigr)(t),\quad \operatorname{Re}\lambda > \omega,\ t \geqslant 0.\end{aligned}$$

Combined with (54), the above implies:

$$(59)\qquad (e^{\lambda\cdot} * a * v)(t) = -\Bigl(\int_0^\infty e^{-\lambda r}R(r)u(\cdot)dr * \int_\cdot^\infty e^{\lambda(\cdot-s)}a(s)ds\Bigr)(t),$$

for $\operatorname{Re}\lambda > \omega$ and $t \geqslant 0$. By (58)–(59), and (ii) of Definition 2.2.2, we get that for each $p \in \circledast$ there exist $c > 0$ and $q \in \circledast$ such that

$$(60)\quad e^{-\lambda\sigma}p((e^{\lambda\cdot} * a * v)(t)) \leqslant \frac{ce^{-\lambda\sigma}}{|\tilde{k}(\lambda)|(\lambda-\omega)}\int_0^t\int_s^\infty e^{\lambda(s-r)}|a(r)|q(u(t-s))dr\,ds,$$

for $\lambda > \omega$, $t \geqslant 0$ and $\sigma \geqslant 0$. On the other hand, we can always find constants $\sigma_0 > 0$ and $M \geqslant 1$ such that

$$(61)\qquad \frac{e^{-\lambda\sigma_0}}{|\tilde{k}(\lambda)|} \leqslant M,\quad \lambda > \omega + 1.$$

If not so, then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(\omega+1,\infty)$ such that $|\tilde{k}(\lambda_n)| \leqslant e^{-n\lambda_n}$, $n \in \mathbb{N}$. Making use of the condition (P2) and the Bolzano–Weierstrass theorem, it can be easily seen that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ must be unbounded; hence, $\limsup_{\lambda\to+\infty}(\ln|\tilde{k}(\lambda)|/\lambda) = -\infty$ and [**27**, Proposition 2.4.3] implies that $k(t) = 0$, $t \geqslant 0$, which is a contradiction. Applying now (60)–(61), we obtain that $\lim_{\lambda\to+\infty} e^{-\lambda\sigma}p((e^{\lambda\cdot} * a * v)(t)) = 0$, $t \geqslant 0$, $p \in \circledast$. Proceeding as in the proof of [**542**, Theorem 1.6], it readily follows that $(a*v)(t) = 0$, $t \geqslant 0$. Since the function $a(t)$ is a kernel on $[0,\infty)$ and C is injective, we get that $0 = v(t) = Cu(t) = u(t)$, $t \geqslant 0$. □

Remark 2.2.9. (i) Suppose $v_0 \in D(B)$ and, instead of (i.1), a slightly stronger condition

(i.1)' for every $x \in D(B)$, there exist a number $\omega_0 > \omega$ and two functions $h_1(\lambda; x)$, $h_2(\lambda) \in LT - E$ such that $h_1(\lambda; x) = \tilde{k}(\lambda)\tilde{a}(\lambda)^{-1}B(B-$

$\tilde{a}(\lambda)A)^{-1}CBx$ and $h_2(\lambda)=\tilde{k}(\lambda)\tilde{a}(\lambda)^{-1}$, provided $x\in D(B)$, $\operatorname{Re}\lambda>\omega_0$ and $\tilde{k}(\lambda)\neq 0$.

Then we can simply prove with the help of closedness of A and resolvent equation that there exists a function $h_3(\lambda;x)\in LT-E$ such that $h_3(\lambda;x)=\tilde{k}(\lambda)A(B-\tilde{a}(\lambda)A)^{-1}CBx$, provided $x\in D(B)$, $\operatorname{Re}\lambda>\omega_0$ and $\tilde{k}(\lambda)\neq 0$. Keeping in mind the uniqueness theorem for Laplace transform, it readily follows that the mapping $t\mapsto AR(t)v_0$, $t\geqslant 0$ is well defined and continuous. In conclusion, the function $u(t)=R(t)v_0$, $t\geqslant 0$ is a strong solution of (52) with $f(t)=k(t)CBv_0$, $t\geqslant 0$.

(ii) Observe that we must impose the condition $\operatorname{abs}(|a|)<\infty$ here because we need to apply Theorem 1.4.2(vi).

(iii) Regarding the question of whether the function $k(t)$ satisfies (P2) or not, the following comment should be made: Suppose that the assertion of [**445**, Lemma 4.1.1, p. 100] continues to hold with the sequence $\lambda_n=n$ replaced by any strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers tending to infinity (yet unproven or unsworn hypothesis, unknown in the existing literature to the best knowledge of the author). Then the uniqueness of solutions clarified in Theorem 2.2.8 holds even if the function $k(t)$ does not satisfy (P2), and this can be proved by using the estimate (60) and the fact that there exist a positive real number $\sigma'>0$ and a strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty}\lambda_n=\infty$ and $|\tilde{k}(\lambda_n)|\geqslant e^{-\sigma'\lambda_n}$, $n\in\mathbb{N}$ (cf. [**27**, Proposition 2.4.3]).

In the subsequent proposition, we state a rescaling result for degenerate K-convoluted C-semigroups in locally convex spaces; observe, however, that it is very difficult to say something more about perturbation properties of exponentially equicontinuous (a,k)-regularized C-resolvent families introduced in this section (cf. [**296**, Theorem 4.2(ii)] and [**292**, Section 2.6] for further information concerning non-degenerate case).

PROPOSITION 2.2.10. (i) *Suppose $z\in\mathbb{C}$, $K(t)$ satisfy* (P1), *$F(t)$ is exponentially bounded, $k(t)=\int_0^t K(s)ds$, $t\geqslant 0$, there exists $\omega_0>0$ such that*

$$\frac{\tilde{K}(\lambda)-\tilde{K}(\lambda+z)}{\tilde{K}(\lambda+z)}=\int_0^\infty e^{-\lambda t}F(t)dt,\quad \operatorname{Re}\lambda>\omega_0,\ \tilde{K}(\lambda+z)\neq 0,$$

and there exists an exponentially equicontinuous K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$ for (52), *i.e., (a,k)-regularized C-resolvent family for* (52) *with $a(t)=1$. Then there exists an exponentially equicontinuos K-convoluted C-semigroup $(S_{K,z}(t))_{t\in[0,\tau)}$ for* (52), *with A replaced by $A-zB$. Furthermore,*

$$S_{K,z}(t)x=e^{-tz}S_K(t)x+\int_0^t F(t-s)e^{-zs}S_K(s)x\,ds,\quad t\geqslant 0,\ x\in D(B).$$

(ii) *Suppose $z \in \mathbb{C}$, $\alpha > 0$ and there exists an exponentially equicontinuous α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$ for* (52), *i.e.* (a,k)-*regularized C-resolvent family for* (52) *with $a(t) = 1$ and $k(t) = g_{\alpha+1}(t)$. Then there exists an exponentially equicontinuous α-times integrated C-semigroup $(S_{\alpha,z}(t))_{t\in[0,\tau)}$ for* (52), *with A replaced by $A - zB$. Furthermore,*

$$S_{\alpha,z}(t)x = e^{-zt}S_\alpha(t)x + \int_0^t \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!} e^{-zs} S_\alpha(s)x\,ds, \quad t \geqslant 0,\ x \in D(B).$$

2.2.2. Differential and analytical properties of degenerate (a,k)-regularized C-resolvent families. In this subsection, we clarify the most important differential and analytical properties of degenerate (a,k)-regularized C-resolvent families. Let us recall that $\Sigma_\alpha = \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$ $(\alpha \in (0,\pi])$.

Definition 2.2.11. (cf. [**296**, Definition 3.1] for the case $B = I$) Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $C \in L(E)$ is an injective mapping satisfying $CA \subseteq AC$. Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k)-regularized C-resolvent family for (52), and let $0 < \alpha \leqslant \pi$. Then it is said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (equicontinuous), analytic (a,k)-regularized C-resolvent family for (52), of angle α, iff there exists $\omega \geqslant \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$ $(\omega = 0)$ such that the following holds:

(i) For every $x \in D(B)$, the mapping $t \mapsto R(t)x$, $t > 0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we shall denote the extension by the same symbol.

(ii) For every $x \in D(B)$ and $\beta \in (0,\alpha)$, one has $\lim_{z\to 0, z\in\Sigma_\beta} R(z)x = R(0)x$.

(iii) The family $\{e^{-\omega z}R(z) : z \in \Sigma_\beta\}$ is equicontinuous for all $\beta \in (0,\alpha)$, i.e., for every $p \in ⊛$, there exist $c > 0$ and $q \in ⊛$ such that

$$p(e^{-\omega z}R(z)x) \leqslant cq(x), \quad x \in D(B),\ z \in \Sigma_\beta.$$

Before going any further, we would like to observe that the assertion of [**296**, Theorem 3.6] cannot be transferred to degenerate (a,k)-regularized C-resolvent families without imposing some restrictive assumptions, including by all means the injectivity of the operator B. This is not the case with the assertion of [**296**, Theorem 3.7], as the following theorem shows.

Theorem 2.2.12. *Assume that $|a|(t)$ and $k(t)$ satisfy* (P1), *A and B are closed linear operators, $\omega \geqslant \max(0, \mathrm{abs}(|a|), \mathrm{abs}(k))$, $\alpha \in (0,\pi/2]$, $C \in L(E)$ is injective and satisfies $CA \subseteq AC$. Assume, further, that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have that the operator $B - \tilde{a}(\lambda)A$ is injective and $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$. Let for each $x \in D(B)$ there is an analytic function $q_x \colon \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to E$ such that*

$$q_x(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx, \quad \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0.$$

Suppose that, for every $\beta \in (0,\alpha)$ and $p \in ⊛$, there exist $c_{p,\beta} > 0$ and $r_{p,\beta} \in ⊛$ such that $p((\lambda-\omega)q_x(\lambda)) \leqslant c_{p,\beta}r_{p,\beta}(x)$, $x \in D(B)$, $\lambda \in \omega + \Sigma_{\beta+(\pi/2)}$ and that, for every $x \in D(B)$, there exists the limit $\lim_{\lambda\to+\infty} \lambda q_x(\lambda)$ in E. Then *there exists an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ for* (52)*, of angle α, and for each $\beta \in (0,\alpha)$ the family $\{e^{-\omega z}R(z) : z \in \Sigma_\beta\}$ is equicontinuous.*

Subordination principle [**292**, Theorem 2.4.2] can be reformulated for degenerate (a,k)-regularized C-resolvent families, as well:

THEOREM 2.2.13. *Assume that $k_\beta(t)$ satisfies* (P1)*, $0 < \alpha < \beta$, $\gamma = \alpha/\beta$ and there exists an exponentially equicontinuous (g_β, k_β)-regularized C-resolvent family $(R_\beta(t))_{t\geqslant 0}$ for* (52)*, with $a(t) = g_\beta(t)$ and $k(t) = k_\beta(t)$, satisfying that the family $\{e^{-\omega t}R_\beta(t) : t \geqslant 0\}$ is equicontinuous for some $\omega \geqslant \max(0, \mathrm{abs}(k_\beta))$. Assume that there exist a function $k_\alpha(t)$ satisfying* (P1) *and a number $\eta > 0$ such that $k_\alpha(0) = k_\beta(0)$ and $\tilde{k_\alpha}(\lambda) = \lambda^{\gamma-1}\tilde{k_\beta}(\lambda^\gamma)$, $\lambda > \eta$. Then there exists an exponentially equicontinuous (g_α, k_α)-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$ for* (52)*, with $a(t) = g_\alpha(t)$ and $k(t) = k_\alpha(t)$, satisfying that the family $\{e^{-\omega^{1/\gamma}t}S_\alpha(t) : t \geqslant 0\}$ is equicontinuous and*

$$R_\alpha(t)x = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})R_\beta(s)x\,ds, \quad x \in D(B),\ t > 0.$$

Furthermore, for every $\zeta \geqslant 0$, the equicontinuity of the family $\{e^{-\omega t}(1+t^\zeta)^{-1}R_\beta(t) : t \geqslant 0\}$, resp. $\{e^{-\omega t}t^{-\zeta}R_\beta(t) : t > 0\}$, implies the equicontinuity of the family $\{e^{-\omega^{1/\gamma}t}(1 + t^{\gamma\zeta})^{-1}(1 + \omega t^{\zeta(1-\gamma)})^{-1}R_\alpha(t) : t \geqslant 0\}$, resp. $\{e^{-\omega^{1/\gamma}t}t^{-\gamma\zeta}(1 + \omega t^{\zeta(1-\gamma)})^{-1}R_\alpha(t) : t > 0\}$, and the following holds:

- (i) *The mapping $t \mapsto R_\alpha(t)x$, $t > 0$ admits an analytic extension to the sector $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ for all $x \in D(B)$.*
- (ii) *If $\omega = 0$ and $\varepsilon \in (0, \min((\frac{1}{\gamma} - 1)\frac{\pi}{2}, \pi))$, then the family $\{R_\alpha(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}\}$ is equicontinuous and $\lim_{z\to 0, z\in\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}} R_\alpha(z)x = R_\alpha(0)x$ for all $x \in D(B)$.*
- (iii) *If $\omega > 0$ and $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2}))$, then there exists $\delta_{\gamma,\varepsilon} > 0$ such that the family $\{e^{-\delta_{\gamma,\varepsilon}\operatorname{Re} z}R_\alpha(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}\}$ is equicontinuous. Moreover, $\lim_{z\to 0, z\in\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}} R_\alpha(z)x = R_\alpha(0)x$ for all $x \in D(B)$.*

Concerning differential properties of degenerate (a,k)-regularized C-resolvent families, the following statements can be verified to be true based on the information provided in the final part of the proof of [**291**, Theorem 3.2.15] (cf. also [**296**, Theorem 3.18], [**296**, Theorem 3.20] with $m = 2$, the proof of [**291**, Theorem 2.4.8]), and Theorem 2.2.5(i); recall only that for each sequence (M_n) of positive real numbers satisfying $M_0 = 1$, $(M.1)$, $(M.2)$ and $(M.3)'$, we define the function $\omega_L(\cdot)$ by $\omega_L(t) := \sum_{n=0}^\infty \frac{t^n}{M_n}$, $t \geqslant 0$.

THEOREM 2.2.14. *Suppose A and B are closed linear operators, $|a|(t)$ and $k(t)$ satisfy* (P1)*, $r \geqslant -1$ and there exists $\omega \geqslant \max(0, \mathrm{abs}(|a|), \mathrm{abs}(k))$ such that, for every $z \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0\}$, the operator $B - \tilde{a}(z)A$ is injective and $C(R(B)) \subseteq R(B - \tilde{a}(z)A)$. Suppose, additionally, that for every $\sigma > 0$ and*

$x \in D(B)$, there exist a number $c_{\sigma,x} > 0$, an open neighborhood $\Omega_{\sigma,x,\omega}$ of the region

$$\Lambda_{\sigma,x,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + c_{\sigma,x}\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\},$$

and an analytic function $h_{\sigma,x} : \Omega_{\sigma,x,\omega} \to L(E)$ such that

$$h_{\sigma,x}(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx, \quad \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0$$

and the set

$$\{|\lambda|^{-r} h_{\sigma,x}(\lambda) : \lambda \in \Lambda_{\sigma,x,\omega},\ \operatorname{Re}\lambda \leqslant \omega\}$$

is bounded. If, for every $\sigma > 0$ and $p \in \circledast$, there exist $c_p > 0$ and $q_p \in \circledast$ such that $p(h_{\sigma,x}(\lambda)) \leqslant c_p|\lambda|^r q_p(x)$, $\operatorname{Re}\lambda > \omega$, $x \in D(B)$, then, for every $\zeta > 1$, there exists an exponentially equicontinuous $(a, k * g_{\zeta+r})$-regularized C-resolvent family $(R_\zeta(t))_{t\geqslant 0}$ for (52), *satisfying that the mapping $t \mapsto R_\zeta(t)x$, $t > 0$ is infinitely differentiable for all $x \in D(B)$.*

THEOREM 2.2.15. (i) *Suppose that $|a|(t)$ and $k(t)$ satisfy* (P1), *there exists an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ for* (52), *satisfying* (ii) *of* Definition 2.2.2 *with some number $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$. Let $\omega_0 > \omega$. Denote, for every $x \in D(B)$, $\varepsilon \in (0,1)$, and a corresponding $K_{\varepsilon,x} > 0$,*

$$F_{\varepsilon,\omega_0,x} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -\ln \omega_L(K_{\varepsilon,x}|\operatorname{Im}\lambda|) + \omega_0\}.$$

Assume that, for every $x \in D(B)$ and $\varepsilon \in (0,1)$, there exist $K_{\varepsilon,x} > 0$, an open neighborhood $O_{\varepsilon,\omega_0,x}$ of the region $G_{\varepsilon,\omega_0,x} := \{\lambda \in \mathbb{C} : Re\lambda \geqslant \omega_0,\ \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega_0} : \operatorname{Re}\lambda \leqslant \omega_0\}$ and the analytic mappings $h_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \to E$, $f_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \to \mathbb{C}$, $g_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \to \mathbb{C}$ such that:

(a) *$f_{\varepsilon,x}(\lambda) = \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega_0$; $g_{\varepsilon,x}(\lambda) = \tilde{a}(\lambda)$, $\operatorname{Re}\lambda > \omega_0$,*

(b) *for every $\lambda \in F_{\varepsilon,\omega_0,x}$, the operator $B - g_{\varepsilon,x}(\lambda)A$ is injective and $C(R(B)) \subseteq R(B - g_{\varepsilon,x}(\lambda)A)$,*

(c) *for every $x \in D(B)$, $h_{\varepsilon,x}(\lambda) = f_{\varepsilon,x}(\lambda)(B - g_{\varepsilon,x}(\lambda)A)^{-1}CBx$, $\lambda \in G_{\varepsilon,\omega_0,x}$,*

(d) *the set $\{(1 + |\lambda|)^{-m}e^{-\varepsilon|\operatorname{Re}\lambda|}h_{\varepsilon,x}(\lambda) : \lambda \in F_{\varepsilon,\omega},\ \operatorname{Re}\lambda \leqslant \omega_0\}$ is bounded.*

Then, for every $x \in D(B)$, the mapping $t \mapsto R(t)x$, $t > 0$ is infinitely differentiable and, for every compact set $K \subseteq (0,\infty)$, there exists $h_K > 0$ such that the set $\{\frac{h_K^n \frac{d^n}{dt^n}R(t)x}{M_n} : t \in K,\ n \in \mathbb{N}_0\}$ is bounded; furthermore, if $K_{\varepsilon,x}$ is independent of $x \in D(B)$ and if for each $p \in \circledast$ there exist $c_p > 0$ and $q_p \in \circledast$ such that $p((1+|\lambda|)^{-m}e^{-\varepsilon|\operatorname{Re}\lambda|}h_{\varepsilon,x}(\lambda)) \leqslant c_p q_p(x)$, $x \in D(B)$, $\lambda \in F_{\varepsilon,\omega_0}$, $\operatorname{Re}\lambda \leqslant \omega_0$, then the family $\{\frac{h_K^n \frac{d^n}{dt^n}R(t)}{M_n} : t \in K,\ n \in \mathbb{N}_0\}$ is equicontinuous.

(ii) *Suppose that $|a|(t)$ and $k(t)$ satisfy* (P1), *there exists an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ for* (52), *satisfying* (ii) *of* Definition 2.2.2 *with $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$. Let $\omega_0 > \omega$. Denote, for every $x \in D(B)$, $\varepsilon \in (0,1)$, $\rho \in [1,\infty)$ and a* *corresponding* $K_{\varepsilon,x}>0$,

$$F_{\varepsilon,\omega_0,\rho,x}:=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant -K_{\varepsilon,x}|\operatorname{Im}\lambda|^{1/\rho}+\omega_0\}.$$

Assume that, for every $x\in D(B)$ *and* $\varepsilon\in(0,1)$, *there exist* $K_{\varepsilon,x}>0$, *an open neighborhood* $O_{\varepsilon,\omega_0,x}$ *of the region* $G_{\varepsilon,\omega_0,\rho,x}:=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant\omega_0,\ \tilde{k}(\lambda)\neq 0\}\cup\{\lambda\in F_{\varepsilon,\omega_0,\rho,x}:\operatorname{Re}\lambda\leqslant\omega_0\}$, *and analytic mappings* $h_{\varepsilon,x}\colon O_{\varepsilon,\omega_0,x}\to E$, $f_{\varepsilon,x}\colon O_{\varepsilon,\omega_0,x}\to\mathbb{C}$ *and* $g_\varepsilon\colon O_{\varepsilon,\omega_0,x}\to\mathbb{C}$ *such that the conditions* (i)(a)–(d) *of this theorem hold with* $F_{\varepsilon,\omega_0,x}$, *resp.* $G_{\varepsilon,\omega_0,x}$, *replaced by* $F_{\varepsilon,\omega_0,\rho,x}$, *resp.* $G_{\varepsilon,\omega_0,\rho,x}$. *Then, for every* $x\in D(B)$, *the mapping* $t\mapsto R(t)$, $t>0$ *is infinitely differentiable and, for every compact set* $K\subseteq(0,\infty)$, *there exists* $h_K>0$ *such that the set* $\{\frac{h_K^n\frac{d^n}{dt^n}R(t)x}{n!^\rho}:t\in K,\ n\in\mathbb{N}_0\}$ *is equicontinuous; furthermore, if* $K_{\varepsilon,x}$ *is independent of* $x\in D(B)$ *and if for each* $p\in\circledast$ *there exist* $c_p>0$ *and* $q_p\in\circledast$ *such that* $p((1+|\lambda|)^{-m}e^{-\varepsilon|\operatorname{Re}\lambda|}h_{\varepsilon,x}(\lambda))\leqslant c_pq_p(x)$, $x\in D(B)$, $\lambda\in F_{\varepsilon,\omega_0,\rho,x}$, $\operatorname{Re}\lambda\leqslant\omega_0$, *then the family* $\{\frac{h_K^n\frac{d^n}{dt^n}R(t)}{n!^\rho}:t\in K,\ n\in\mathbb{N}_0\}$ *is equicontinuous.*

THEOREM 2.2.16 (The abstract Weierstrass formula). (i) *Assume that* $a(t)$ *and* $k(t)$ *satisfy* (P1), *and there exist* $M>0$ *and* $\omega>0$ *such that* $|k(t)|\leqslant Me^{\omega t}$, $t\geqslant 0$. *Assume, further, that there exist a number* $\omega'\geqslant\omega$ *and a function* $a_1(t)$ *such that* $\operatorname{abs}(a_1)<\infty$ *and* $\widetilde{a_1}(\lambda)=\tilde{a}(\sqrt{\lambda})$, $\operatorname{Re}\lambda>\omega'$ *(Let us recall that the above holds if* $a(t)$ *is exponentially bounded; in this case,* $a_1(t)=\int_0^\infty s\frac{e^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}a(s)ds$, $t>0$.*) Let there exist an exponentially equicontinuous* (a,k)*-regularized* C*-resolvent family* $(C(t))_{t\geqslant 0}$ *for* (52). *Then there exists an exponentially equicontinuous, analytic* (a_1,k_1)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *for* (52), *with* $a(t)$ *replaced by* $a_1(t)$, *of angle* $\frac{\pi}{2}$, *where:*

$$k_1(t):=\int_0^\infty\frac{e^{-s^2/4t}}{\sqrt{\pi t}}k(s)ds,\quad t>0,\ k_1(0):=k(0),\ \textit{and}\tag{62}$$

$$R(t)x:=\int_0^\infty\frac{e^{-s^2/4t}}{\sqrt{\pi t}}C(s)x\,ds,\quad t>0,\ x\in D(B),\ R(0)x:=C(0)x,\ x\in D(B).\tag{63}$$

(ii) *Assume that* $k(t)$ *satisfy* (P1), $\beta>0$ *aw well as there exist* $M>0$ *and* $\omega>0$ *such that* $|k(t)|\leqslant Me^{\omega t}$, $t\geqslant 0$. *Let there exist an exponentially equicontinuous* $(g_{2\beta},k)$*-regularized* C*-resolvent family* $(C(t))_{t\geqslant 0}$ *for* (52). *Then there exists an exponentially equicontinuous, analytic* (g_β,k_1)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *for* (52), *with* $a(t)$ *replaced by* $a_1(t)$, *of angle* $\frac{\pi}{2}$, *where* $k_1(t)$ *and* $R(t)$ *are defined through* (62)–(63).

REMARK 2.2.17. As Theorem 2.2.8 shows, the existence of an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ for (52) does not automatically imply the existence of mild (strong) solutions of this problem; we need to impose the conditions like (i.1), (i.1)' or (ii.1). Regarding this question, the following facts should be stated:

1. Proposition 2.2.4(i): If the function $b(t)$ satisfies the prescribed assumptions and $(R(t))_{t\geqslant 0}$ satisfies any of the conditions (i.1), (i.1)' or (ii.1), then $((b*R)(t))_{t\geqslant 0}$ satisfies the same condition as well, with the function $k(t)$ replaced by $(b*k)(t)$; Proposition 2.2.4(iii): If the functions $k(t)$ and $b(t)$ satisfy the prescribed assumptions and $(R(t))_{t\geqslant 0}$ satisfies any of the conditions (i.1), (i.1)' or (ii.1), then $(S(t))_{t\geqslant 0}$ satisfies the same condition as well, with the function $k(t)$ replaced by 1.
2. In the formulations of Theorem 2.2.5(i)–(ii), as well as Theorem 2.2.12 and Theorems 2.2.14-2.2.15, we must add some very natural conditions ensuring the validity of (i.1), (i.1)' or (ii.1) for the corresponding resolvent families.
3. The conditions (i.1) and (ii.1) are invariant under the action of subordination principles stated in Theorem 2.2.6, while some additional assumptions must be imposed for the invariance of the condition (i.1)'.
4. The rescaling of degenerate K-convoluted C-semigroups (cf. Proposition 2.2.10) preserves the conditions (i.1), (i.1)' and (ii.1).
5. Due to the proof of [**61**, Theorem 3.1], it is not difficult to verify that the conditions (i.1), (i.1)' and (ii.1) are invariant under the action of subordination principles stated in Theorem 2.2.13 and Theorem 2.2.16.

Now we want to illustrate our results with a concrete example.

EXAMPLE 2.2.18. (cf. [**291**, Example 2.8.3(iii)] and [**292**, Example 2.6.10]) Let $s>1$,

$$E:=\Big\{f\in C^{\infty}[0,1]\ ;\ \|f\|:=\sup_{p\geqslant 0}\frac{\|f^{(p)}\|_{\infty}}{p!^{s}}<\infty\Big\}$$

and

$$A:=-d/ds,\ D(A):=\{f\in E\ ;\ f'\in E,\ f(0)=0\}.$$

If $f\in E$, $t\in[0,1]$ and $\lambda\in\mathbb{C}$, set $f_{\lambda}^{1}(t):=\int_0^t e^{-\lambda(t-s)}f(s)ds$ and $f_{\lambda}^{2}(t):=\int_0^t e^{\lambda(t-s)}f(s)ds$. Then $f_{\lambda}^{1}(\cdot)$ and $f_{\lambda}^{2}(\cdot)\in E$, $\lambda\in\mathbb{C}$; moreover, there exist $b'>0$ and $M\geqslant 1$, independent of $f(\cdot)$, such that

$$\|f_{\lambda}^{1}(\cdot)\|\leqslant M\|f\|e^{b'|\lambda|^{1/s}},\quad \operatorname{Re}\lambda\geqslant 0,\ f\in E. \tag{64}$$

Furthermore, for each $\eta>1$ there exists $M_{\eta}\geqslant 1$, independent of $f(\cdot)$, such that

$$\|f_{\lambda}^{2}(\cdot)\|\leqslant M_{\eta}\|f\|e^{\eta|\lambda|},\quad \operatorname{Re}\lambda\geqslant 0,\ f\in E. \tag{65}$$

Let $P_1(z)=\sum_{j=0}^{N_1}a_{j,1}z^j$, $z\in\mathbb{C}$, $a_{N_1,1}\neq 0$ be a complex non-zero polynomial, and let $P_2(z)=\sum_{j=0}^{N_2}a_{j,2}z^j$, $z\in\mathbb{C}$, $a_{N_2,2}\neq 0$ be a complex non-zero polynomial so that $N_1=dg(P_1)>1+dg(P_2)=1+N_2$. For any complex non-zero polynomial $P(z)$, we define the operator $P(A)$ in the obvious way; then

$$\rho(P(A))=\mathbb{C}\text{ and }R(\lambda:A)f=f_{\lambda}^{1},\quad f\in E,\ \lambda\in\mathbb{C}. \tag{66}$$

Set $P_{\lambda}(z):=\lambda P_2(z)-P_1(z)$, $z\in\mathbb{C}$ ($\lambda\in\mathbb{C}$). Let $\{z_1,\ldots,z_s\}$ be the set which consists of joint multiple roots of polynomials $P_1(z)$ and $P_2(z)$. Then there exist uniquely determined integers $k_1, \ldots, k_s \in \mathbb{N}$ such that

$$P_\lambda(z) = (z - z_1)^{k_1} \ldots (z - z_s)^{k_s}(\lambda Q_2(z) - Q_1(z)), \quad z \in \mathbb{C},\ \lambda \in \mathbb{C},$$

with $Q_1(z)$ and $Q_2(z)$ being two non-zero complex polynomials without joint multiple roots, satisfying additionally that $dg(Q_1) > dg(Q_2) + 1$. This implies that there exists $d > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant d$, the polynomial $Q_\lambda(z) := \lambda Q_2(z) - Q_1(z)$ is square-free. Denote by $z_{1,\lambda}, \ldots, z_{N_1,\lambda}$ the roots of $P_\lambda(z)$ $(\lambda \in \mathbb{C})$. Using [**252**, Corollary 5.6] (this is an elementary result on root localization of complex polynomials), we get the existence of a positive real number $\vartheta \geqslant 1$ such that

$$|z_{i,\lambda}| \leqslant \vartheta(1 + |\lambda|)^{\frac{1}{N_1 - N_2}}, \quad 1 \leqslant i \leqslant N_1\ (\lambda \in \mathbb{C}). \tag{67}$$

It is quite easy to prove that the operator $\lambda P_2(A) - P_1(A)$ has the bounded inverse for all $\lambda \in \mathbb{C}$, as well as that

$$(\lambda P_2(A) - P_1(A))^{-1} = (-1)^{N_1+1} a_{N_1,1}^{-1} R(z_{1,\lambda} : A) \ldots R(z_{N_1,\lambda} : A), \quad \lambda \in \mathbb{C}. \tag{68}$$

Observe further that, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant d$, the discriminant of polynomial $Q_\lambda(z)$, for which it is well known that can be represented by a homogenous polynomial of degree $2(dg(Q_1) - 1)$ in the coefficients of $Q_\lambda(z)$, is a complex non-zero polynomial in λ. Hence, there exist numbers $d_1 \geqslant d$ and $\eta > 0$ such that $|D(Q_\lambda(z))| \geqslant \eta$, $|\lambda| \geqslant d_1$. Making use of this fact and [**467**, Theorem 1], we obtain the existence of a sufficiently small number $\zeta > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant d_1$, and for every two distinct roots $z_{i,\lambda}$, $z_{j,\lambda}$ of polynomial $Q_\lambda(z)$, we have $|z_{i,\lambda} - z_{j,\lambda}| \geqslant \zeta$. Then the calculation contained in the analysis made in [**292**, Example 2.6.10], combined with the equality (68) and the above fact, shows that the norm of operator $(\lambda P_2(A) - P_1(A))^{-1}$ does not exceed $M \sum \|R(z_{j,\lambda} : A)\|$, where the summation is taken over all roots $z_{j,\lambda}$ of polynomial $Q_\lambda(z)$. Taken together with (64)–(67) and the generalized resolvent equation, the above implies that there exist numbers $b > 0$, $c > 0$ and $\zeta > 0$ such that

$$\|(\lambda P_2(A) - P_1(A))^{-1}\| = O(e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}), \quad \lambda \in \mathbb{C},$$

and

$$\|(\lambda P_2(A) - P_1(A))^{-1} P_2(A) f\| \leqslant \zeta \|f\| e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}, \tag{69}$$

for all $\lambda \in \mathbb{C}$ and $f \in D(P_2(A))$. Before proceeding further, it should be noted that the above estimates can be used in proving the existence of convoluted solutions of fractional analogues of the linearized Benney–Luke equation (sometimes also called Dzektser equation)

$$(\lambda - \Delta)u_t = \alpha\Delta - \beta\Delta^2 \quad (\alpha, \beta > 0,\ \lambda \in \mathbb{R}),$$

in contrast with the assertions of Theorem 2.2.20–Theorem 2.2.21 below, which can be applied only in the case that $\lambda > 0$ (cf. [**307**] for more details); as is well known, this equation is important in evolution modeling of some problems appearing in the theory of liquid filtration, see e.g. [**199**, p. 6]. Suppose $N_1 - N_2 > \alpha \geqslant 1$, $\delta \in (0, \pi/2]$, $(\pi/2 + \delta)\alpha/(N_1 - N_2) < \pi/2$, $\varrho > c/\cos((\pi/2 + \delta)\alpha/(N_1 - N_2))$ and $k(t) = \mathcal{L}^{-1}(e^{-\varrho\lambda^{\alpha/n}})(t)$, $t \geqslant 0$. By Theorem 2.2.12 and (69), there exists an exponentially bounded, analytic (g_α, k)-regularized resolvent family $(R_\alpha(t))_{t\geqslant 0}$ for the corresponding problem (52), of angle δ; it is clear that the conditions (i.1)' and (ii.1) stated in Theorem 2.2.8 and Remark 2.2.9(i) holds for $(R_\alpha(t))_{t\geqslant 0}$. Observe finally that the case $N_1 = N_2 + 1$ is critical and that we always have the existence of integrated solution families in the case $N_2 \geqslant N_1$ (this follows from the above analysis and the fact that, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant d$, the roots $z_{1,\lambda}, \ldots, z_{N_2,\lambda}$ of polynomial $P_\lambda(z)$ belong to a compact set $K \subseteq \mathbb{C}$ which does not depend on λ; see e.g. [**252**, Theorem 5.4]). Computing the optimal rate of integration is a non-trivial problem which will not be discussed here.

2.2.3. Degenerate time-fractional equations associated with abstract differential operators. With the exceptions of Remark 2.2.23 and Remark 2.2.26(ii), we assume in this subsection, and the next one, that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E. Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Put $k := 1 + \lfloor n/2 \rfloor$, $A := (A_1, \ldots A_n)$ and $A^\eta := A_1^{\eta_1} \ldots A_n^{\eta_n}$ for any $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$. Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform on $\mathbb{R}^n$ and its inverse transform, respectively. For every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $u \in \mathcal{F}L^1(\mathbb{R}^n) = \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$, we set $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$, $(\xi, A) := \sum_{j=1}^n \xi_j A_j$ and

$$u(A)x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi) e^{-i(\xi,A)} x \, d\xi, \quad x \in E.$$

Then $u(A) \in L(E)$, $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and there exists a finite constant $M \geqslant 1$ such that

$$\|u(A)\| \leqslant M \|\mathcal{F}^{-1}u\|_{L^1(\mathbb{R}^n)}, \quad u \in \mathcal{F}L^1(\mathbb{R}^n).$$

Let $N \in \mathbb{N}$, and let $P(x) = \sum_{|\eta| \leqslant N} a_\eta x^\eta$, $x \in \mathbb{R}^n$ be a complex polynomial. Then we define $P(A) := \sum_{|\eta| \leqslant N} a_\eta A^\eta$ and $E_0 := \{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n),\ x \in E\}$. We know that the operator $P(A)$ is closable and the following holds (cf. [**292**,**375**,**561**] and [**304**] for further information):

($\triangleright$) $\overline{E_0} = E$, $E_0 \subseteq \bigcap_{\eta \in \mathbb{N}_0^n} D(A^\eta)$, $\overline{P(A)_{|E_0}} = \overline{P(A)}$ and $\phi(A)P(A) \subseteq P(A)\phi(A) = (\phi P)(A)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Assuming that E is a function space on which translations are uniformly bounded and strongly continuous, the obvious choice for A_j is $-i\partial/\partial x_j$ (notice also that E can be consisted of functions defined on some bounded domain [**132**,**375**,**561**], [**560**, pp. 101-103]). If $P(x) = \sum_{|\eta| \leqslant N} a_\eta x^\eta$, $x \in \mathbb{R}^n$ and E is such a space (for example, $L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$), then $\overline{P(A)}$ is nothing else but the operator $\sum_{|\eta| \leqslant N} a_\eta (-i)^{|\eta|} \partial^{|\eta|}/\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n} \equiv \sum_{|\eta| \leqslant N} a_\eta D^\eta$, acting with its maximal distributional domain. Recall that $P(x)$ is called r-coercive ($0 < r \leqslant N$) if there exist $M, L > 0$ such that $|P(x)| \geqslant M|x|^r$, $|x| \geqslant L$; by a corollary of the Seidenberg–Tarski theorem, the equality $\lim_{|x| \to \infty} |P(x)| = \infty$ implies in particular that $P(x)$ is r-coercive for some $r \in (0, N]$ (cf. [**27**, Remark 8.2.7]). In the sequel of this subsection, $M > 0$ denotes a generic constant whose value may change from line to line.

Let $p \in [1,\infty]$. Following L. Hörmander [**257**], it will be said that a function $u \in L^{\infty}(\mathbb{R}^n)$ is a Fourier multiplier on $L^p(\mathbb{R}^n)$ iff $\mathcal{F}^{-1}(u\mathcal{F}\phi) \in L^p(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\|u\|_{\mathcal{M}_p} := \sup\{\|\mathcal{F}^{-1}(u\mathcal{F}\phi)\|_{L^p(\mathbb{R}^n)} : \phi \in \mathcal{S}(\mathbb{R}^n),\ \|\phi\|_{L^p(\mathbb{R}^n)} \leqslant 1\} < \infty.$$

We use the abbreviation $\mathcal{M}_p$ for the space of all Fourier multipliers on $L^p(\mathbb{R}^n)$; cf. [**254**] for more details. Then $\mathcal{M}_p$ is a Banach algebra under pointwise multiplication and $\mathcal{F}L^1(\mathbb{R}^n)$ is continuously embedded in $\mathcal{M}_p$. We need the following lemma (see e.g. [**541**, Lemma 5.2, Lemma 5.4, pp. 20-22]).

LEMMA 2.2.19. (i) *Let $1 \leqslant p \leqslant \infty$, $j,n \in \mathbb{N}$, $j > n/2$ and $\{f_t\}_{t\geqslant 0}$ be a family of $C^j(\mathbb{R}^n)$-functions. Assume that for each $x \in \mathbb{R}^n$, $\eta \in \mathbb{N}_0^n$ with $|\eta| \leqslant j$, $t \mapsto D^{\alpha} f_t(x)$, $t \geqslant 0$ is continuous and that there exist $a > 0$, $r > n|\frac{1}{p} - \frac{1}{2}|$ and $M_t > 0$ (M_t is bounded on compacts of $t \geqslant 0$) such that*

$$|D^{\eta} f_t(x)| \leqslant M_t^{|\eta|}(1+|x|)^{(a-1)|\eta|-ar}, \quad |\eta| \leqslant j,\ x \in \mathbb{R}^n,\ t \geqslant 0.$$

Then, for any $t \geqslant 0$, $p = 1, \infty$ (resp. $1 < p < \infty$), we have $f_t \in \mathcal{F}L^1(\mathbb{R}^n)$ (resp. $f_t \in \mathcal{M}_p$), $t \mapsto f_t$, $t \geqslant 0$ is continuous with respect to $||\cdot||_{\mathcal{F}L^1(\mathbb{R}^n)}$ (resp. $||\cdot||_{\mathcal{M}_p}$) and there exists a constant $M > 0$ independent of $t \geqslant 0$ such that

$$\|f_t\|_{\mathcal{F}L^1(\mathbb{R}^n)}\ (\textit{resp. } \|f_t\|_{\mathcal{M}_p}) \leqslant M M_t^{n|\frac{1}{p}-\frac{1}{2}|}, \quad t \geqslant 0.$$

(ii) *Let $1 < p < \infty$, $j,n \in \mathbb{N}$, $j > n/2$ and $f \in C^j(\mathbb{R}^n)$. Assume that there exist $a \geqslant 0$, $r \geqslant n|\frac{1}{p} - \frac{1}{2}|$, $M_f \geqslant 1$ and $L_f > 0$ such that*

$$|D^{\eta} f(x)| \leqslant L_f M_f^{|\eta|}(1+|x|)^{(a-1)|\eta|-ar}, \quad |\eta| \leqslant j,\ x \in \mathbb{R}^n,\ t \geqslant 0.$$

Then $f \in \mathcal{M}_p$ and there exists a constant $M > 0$ independent of $f(\cdot)$ such that

$$\|f\|_{\mathcal{M}_p} \leqslant M L_f M_f^{n|\frac{1}{p}-\frac{1}{2}|}.$$

Suppose now that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials in n variables and $0 < \alpha \leqslant 2$; put $N_1 := dg(P_1(x))$, $N_2 := dg(P_2(x))$ and $m := \lceil \alpha \rceil$. We investigate the generation of some very specific classes of (g_α, C)-regularized resolvent families associated with the following fractional degenerate abstract Cauchy problem

$$(\mathrm{DFP}) : \begin{cases} \mathbf{D}_t^{\alpha} \overline{P_2(A)} u(t) = \overline{P_1(A)} u(t), & t \geqslant 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0,\ 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

Convoluting both sides of (DFP) with $g_\alpha(t)$, and using the equality (38) it follows that every solution of (DFP) is, in fact, a strong solution of problem (52) with $B = \overline{P_2(A)}$, $\tau = \infty$, $a(t) = g_\alpha(t)$ and $f(t) \equiv \overline{P_2(A)}Cx$. It can be easily checked that any such a solution of problem (52) is also a strong solution of problem (DFP); cf. Definition 2.3.1 below. A continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ is said to be a mild solution of (DFP) iff $\overline{P_1(A)}(g_\alpha * u)(t) = \overline{P_2(A)}u(t) - \overline{P_2(A)}Cx$, $t \geqslant 0$. We start by stating the following extension of [**304**, Theorem 2.1]; observe only that we do not assume here the coercivity of $P_1(x)$ or $P_2(x)$, and that in the formulation of the afore-mentioned theorem we have that $P_1(x) = Q(x) = P(x)$ and $P_2(x) = 1$ $(x \in \mathbb{R}^n)$.

THEOREM 2.2.20. *Suppose* $0 < \alpha < 2$, $\omega \geqslant 0$, $P_1(x)$ *and* $P_2(x)$ *are non-zero complex polynomials* $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $N \in \mathbb{N}$ *and* $r \in (0, N]$. *Let* $Q(x)$ *be an* r*-coercive complex polynomial of degree* N, $a \in \mathbb{C} \smallsetminus Q(\mathbb{R}^n)$, $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ (*resp.* $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}| \max(N, \frac{N_1+N_2}{\min(1,\alpha)})$, *if* $E = L^p(\mathbb{R}^n)$ *for some* $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ *and*

$$\sup_{x \in \mathbb{R}^n} \operatorname{Re} \Big(\Big(\frac{P_1(x)}{P_2(x)} \Big)^{1/\alpha} \Big) \leqslant \omega. \tag{70}$$

Set

$$R_\alpha(t) := \Big(E_\alpha \Big(t^\alpha \frac{P_1(x)}{P_2(x)} \Big) (a - Q(x))^{-\gamma} \Big)(A), \quad t \geqslant 0.$$

Then $(R_\alpha(t))_{t \geqslant 0} \subseteq L(E)$ *is a global exponentially bounded* $(g_\alpha, R_\alpha(0))$*-regularized resolvent family for* (DFP), $(R_\alpha(t))_{t \geqslant 0}$ *is norm continuous provided* $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$, *and the following holds:*

$$\|R_\alpha(t)\| \leqslant M(1 + t^{\max(1,\alpha)n/2}) e^{\omega t}, \quad t \geqslant 0, \ \textit{resp.}, \tag{71}$$

$$\|R_\alpha(t)\| \leqslant M(1 + t^{\max(1,\alpha)n|\frac{1}{p} - \frac{1}{2}|}) e^{\omega t}, \quad t \geqslant 0.$$

PROOF. Put $C := R_\alpha(0)$. Then C is injective and it can be easily proved with the help of ($\triangleright$) that $C\overline{P(A)} \subseteq \overline{P(A)}C$ for any complex polynomial $P(x)$; see e.g. **[304]**. Furthermore, $\sup_{x \in \mathbb{R}^n} |P_2(x)|^{-1} < \infty$ and, for every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, there exists $c_\eta > 0$ such that

$$\Big| D^\eta \Big(\frac{P_1(x)}{P_2(x)} \Big) \Big| \leqslant c_\eta (1 + |x|)^{|\eta|(N_1+N_2-1)}, \quad x \in \mathbb{R}^n. \tag{72}$$

By induction, one can prove that, for every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, the following holds:

$$D^\eta E_\alpha \Big(t^\alpha \frac{P_1(x)}{P_2(x)} \Big) = \sum_{j=1}^{|\eta|} t^{\alpha j} E_\alpha^{(j)} \Big(t^\alpha \frac{P_1(x)}{P_2(x)} \Big) R_{\eta,j}(x), \quad t \geqslant 0, \ x \in \mathbb{R}^n, \tag{73}$$

where $R_{\eta,j}(x)$ can be represented as a finite sum of terms like $\prod_{w=1}^{s_j} D^{\eta_{j,w}} (\frac{P_1(x)}{P_2(x)})$ with $|\eta_{j,w}| > 0$ $(1 \leqslant w \leqslant s_j)$ and $|\eta_{j,1}| + \cdots + |\eta_{j,s_j}| \leqslant |\eta|$. Consider now the assertion of Theorem 1.5.1. Taking the number $\sigma > 0$ sufficiently small, and keeping in mind that $0 < \alpha < 2$, we obtain that, for every $m \in \mathbb{N} \smallsetminus \{1\}$ and for every $t \geqslant 0$, $x \in \mathbb{R}^n$ with $|t^\alpha P_1(x)/P_2(x)| \geqslant 1$, the term

$$\Big| E_{\alpha,\alpha j-(j-l)} \Big(t^\alpha \frac{P_1(x)}{P_2(x)} \Big) - \frac{1}{\alpha} \Big(t^\alpha \frac{P_1(x)}{P_2(x)} \Big)^{(1-(\alpha j-(j-l)))/\alpha} e^{(t^\alpha P_1(x)/P_2(x))^{1/\alpha}}$$
$$- \sum_{j=1}^{m-1} \frac{(t^\alpha P_1(x)/P_2(x))^{-j}}{\Gamma(\alpha j - (j-l) - \alpha j)} \Big|$$

can be majorized by $M|t^\alpha P_1(x)/P_2(x)|^{-m}$. Clearly, the function $E_{\alpha,\alpha j-(j-l)}(\cdot)$ is bounded on compacts of $\mathbb{C}$, and (70) implies that $\operatorname{Re}((t^\alpha(P_1(x)/P_2(x))^{1/\alpha}) \leqslant \omega t$, $t \geqslant 0$, $x \in \mathbb{R}^n$. As in the proof of [**296**, Theorem 2.1], the above implies that, for every $t \geqslant 0$, $x \in \mathbb{R}^n$ and $1 \leqslant l \leqslant j \leqslant k$,

$$(74)\quad |E_{\alpha,\alpha j-(j-l)}(t^\alpha P_1(x)/P_2(x))| \leqslant M\big[1 + t^{1-(\alpha j-(j-l))}|P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l))}{\alpha}} e^{\omega t}\big].$$

By (44), (72)–(73) and the boundedness of derivatives of the Mittag-Leffler function $E_\alpha(\cdot)$ on compacts of $\mathbb{C}$, we obtain that, for every $t \geqslant 0$ and $x \in \mathbb{R}^n$ with $|t^\alpha P_1(x)/P_2(x)| \leqslant 1$:

$$(75)\qquad \Big|D^\eta E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)\Big| \leqslant M(t^\alpha + t^{\alpha|\eta|})(1+|x|)^{|\eta|(N_1+N_2-1)}, \quad 0 < |\eta| \leqslant k.$$

If $0 < |\eta| \leqslant k$, $t \geqslant 0$, $x \in \mathbb{R}^n$ and $|t^\alpha P_1(x)/P_2(x)| \geqslant 1$, then the following holds (cf. (74) and [**304**, (2.6)–(2.7)]):

$$\begin{aligned}
&\Big|D^\eta E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)\Big| \\
&\quad \leqslant M \sum_{j=1}^{|\eta|} t^{\alpha j} \sum_{l=1}^{j} \big[1 + |t^\alpha P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l))}{\alpha}} e^{\omega t}\big](1+|x|)^{|\eta|(N_1+N_2-1)} \\
&\quad \leqslant M \sum_{j=1}^{|\eta|} t^{\alpha j} \sum_{l=1}^{j} [1 + e^{\omega t}(1 + t^{1-(\alpha j-(j-l))})](1+|x|)^{|\eta|(\frac{N_1+N_2}{\min(1,\alpha)}-1)} \\
&\quad \leqslant M(1 + t^{\max(1,\alpha)|\eta|})e^{\omega t}(1+|x|)^{|\eta|(\frac{N_1+N_2}{\min(1,\alpha)}-1)}.
\end{aligned}$$

Taking into account (75), we obtain from the previous estimate that, for every $t \geqslant 0$ and $x \in \mathbb{R}^n$,

$$(76)\quad \Big|D^\eta E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)\Big| \leqslant M(1+t^{\max(1,\alpha)|\eta|})e^{\omega t}(1+|x|)^{|\eta|(\frac{N_1+N_2}{\min(1,\alpha)}-1)},\ 0 < |\eta| \leqslant k;$$

observe that the inequality $\operatorname{Re}((t^\alpha P_1(x)/P_2(x))^{1/\alpha}) \leqslant \omega t$, $t \geqslant 0$, $x \in \mathbb{R}^n$ and Theorem 1.5.1 together imply that the previous estimate also holds in the case that $|\eta| = 0$. Set $f_t(x) := E_\alpha(t^\alpha P_1(x)/P_2(x))(a - Q(x))^{-\gamma}$, $t \geqslant 0$, $x \in \mathbb{R}^n$. It is clear that there exists $L > 0$ such that $|Q(x)| \geqslant M|x|^r$, $|x| \geqslant L$ and $|a - Q(x)| \geqslant M|x|^r$, $|x| \geqslant L$. Using [**380**, (3.19)], (76) and the product rule, it readily follows that, for every $t \geqslant 0$, $x \in \mathbb{R}^n$, and for every $\eta \in \mathbb{N}_0^n$ with $|\eta| \leqslant k$,

$$\begin{aligned}
(77)\quad &\Big|D^\eta\Big(E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a-Q(x))^{-\gamma}\Big)\Big| \\
&\qquad \leqslant M(1 + t^{\max(1,\alpha)|\eta|})e^{\omega t}(1+|x|)^{|\eta|(\max(N,\frac{N_1+N_2}{\min(1,\alpha)})-1)-r\gamma}.
\end{aligned}$$

We obtain similarly that, for every $t \geqslant 0$, $x \in \mathbb{R}^n$, and for every $\eta \in \mathbb{N}_0^n$ with $|\eta| \leqslant k$,

$$\begin{aligned}
(78)\quad &\Big|D^\eta\Big(P_2(x)^{-1}E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a-Q(x))^{-\gamma}\Big)\Big| \\
&\qquad \leqslant M(1 + t^{\max(1,\alpha)|\eta|})e^{\omega t}(1+|x|)^{|\eta|(\max(N,\frac{N_1+N_2}{\min(1,\alpha)})-1)-r\gamma}.
\end{aligned}$$

Suppose first that $\gamma > \frac{n\max(N,\frac{N_1+N_2}{\min(1,\alpha)})}{2r}$. By performing the Laplace transform and using (43) we get that for each $t \geqslant 0$ and $x \in \mathbb{R}^n$,

$$(a - Q(x))^{-\gamma} P_2(x) = P_2(x) E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a - Q(x))^{-\gamma} \\ - \int_0^t g_\alpha(t-s) P_1(x) E_\alpha\Big(s^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a - Q(x))^{-\gamma} ds.$$

Keeping in mind this equality, Lemma 2.2.19 and the fact that $R_\alpha(t)\overline{P(A)} \subseteq \overline{P(A)}R_\alpha(t)$, $t \geqslant 0$ for any complex polynomial $P(x)$, we can repeat verbatim the arguments used in the proof of [**304**, Theorem 2.1] so as to conclude that

$$\overline{P_1(A)} \int_0^t g_\alpha(t-s) R_\alpha(s)x\, ds = R_\alpha(t)\overline{P_2(A)}x - C\overline{P_2(A)}x, \quad t \geqslant 0,\ x \in D(\overline{P_2(A)}),$$

which clearly implies by Theorem 2.2.8 that for each $x \in D(\overline{P_2(A)})$ ($x \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$) the function $t \mapsto u(t) \equiv R_\alpha(t)x$, $t \geqslant 0$ is a mild (strong) solution of (DFP). In general case, the proof of theorem can be completed by performing the Laplace transform once more. The proof is quite similar if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, and the only non-trivial thing in this case is to show the strong continuity of the operator family $(R_\alpha(t))_{t\geqslant 0}$. The arguments used in the proof of [**304**, Theorem 2.1] show that the mapping $t \mapsto R_\alpha(t)f$, $t \geqslant 0$ is continuous for every $f \in R(\overline{P_2(A)})$. But, $\overline{P_2(A)}_{|\mathcal{S}(\mathbb{R}^n)}$ is a linear topological homeomorphism of the space $\mathcal{S}(\mathbb{R}^n)$, which along with the exponential boundedness of $(R_\alpha(t))_{t\geqslant 0}$ implies the continuity of mapping $t \mapsto R_\alpha(t)f$, $t \geqslant 0$ for any $f \in L^p(\mathbb{R}^n)$. $\square$

We can prove in a similar way the following extension of [**304**, Theorem 2.2]; let us only note that the choice of regularizing operator C is slightly different now.

THEOREM 2.2.21. *Suppose* $0 < \alpha < 2$, $\omega \geqslant 0$, $P_1(x)$ *and* $P_2(x)$ *are non-zero complex polynomials,* $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ *(resp.* $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\alpha)}$, *if* $E = L^p(\mathbb{R}^n)$ *for some* $1 < p < \infty$*),* $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ *and* (70) *holds. Set*

$$R_\alpha(t) := \Big(E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(1 + |x|^2)^{-\beta/2}\Big)(A), \quad t \geqslant 0.$$

Then $(R_\alpha(t))_{t\geqslant 0} \subseteq L(E)$ *is a global exponentially bounded* $(g_\alpha, R_\alpha(0))$*-regularized resolvent family for* (DFP), $(R_\alpha(t))_{t\geqslant 0}$ *is norm continuous provided* $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$, *and* (71) *holds.*

REMARK 2.2.22. (i) The assumption $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ implies that the operator $\overline{P_2(A)}$ is injective so that the character of degeneracy of problems considered in Subsection 2.2.3 is very mild. In actual fact, the assumption $\overline{P_2(A)}f = 0$, in combination with ($\triangleright$) and the fact that $P_2(\cdot)^{-1}\phi(\cdot) \in \mathcal{S}(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ implies that $\phi(A)f = 0$, $\phi \in \mathcal{S}(\mathbb{R}^n)$; hence, $f = 0$. Consider now the situation of Theorem 2.2.20, with E being a general space and $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$. Set $G_\alpha(t) := (P_2(\cdot)^{-1} f_t(\cdot))(A)$, $t \geqslant 0$. Applying again $(\triangleright)$, we get that $\overline{P_2(A)} G_\alpha(t)x = R_\alpha(t)x$, $t \geqslant 0$, $x \in E_0$. By the closedness of $\overline{P_2(A)}$, the above equality holds for any $x \in E$ so that $G_\alpha(t) = \overline{P_2(A)}^{-1} R_\alpha(t)$, $t \geqslant 0$; furthermore, $(G_\alpha(t))_{t\geqslant 0} \subseteq L(E)$ is a strongly continuous operator family. Then the Laplace transform and the identity $\overline{P_1(A)}(g_\alpha * G_\alpha)(t)x = R_\alpha(t)x - Cx$, $t \geqslant 0$, $x \in E$ can be used to prove that $\lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx = \int_0^\infty e^{-\lambda t} G_\alpha(t)x\,dt$ for any $x \in E$ and $\lambda > 0$ sufficiently large; therefore, the condition (ii.1) stated in Theorem 2.2.8 holds, with $a(t) = g_\alpha(t)$ and $k(t) = 1$, which continues to hold in any case set out in Theorem 2.2.20–Theorem 2.2.21 and Remark 2.2.23 below. It should be also observed that $(G_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by $\overline{P_1(A)}$, $\overline{P_2(A)}$ (cf. the next section for the notion and more details), and that for each $f \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$, the function $u(t) := R_\alpha(t)x$, $t \geqslant 0$ is a unique solution of the following Cauchy problem:

$$(P)_L : \begin{cases} u \in C([0,\infty) : [D(\overline{P_1(A)})]) \cap C([0,\infty) : [D(\overline{P_2(A)})]), \\ \overline{P_2(A)} \mathbf{D}_t^\alpha u(t) = \overline{P_1(A)} u(t), \quad t \geqslant 0, \\ u(0) = Cx; \; u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

A similar result holds for the second order degenerate equations in the next subsection, and for the first order degenerate equations in the case that the requirements of Theorem 2.2.21 hold; cf. [**307**].

(ii) It is worth noting that Theorem 2.2.20 and Theorem 2.2.21 can be strengthened in the following way. Suppose that the estimate (72) holds with the number $N_1 + N_2$ replaced by some other number $\sigma \geqslant 0$, and $|P_1(x)/P_2(x)| \leqslant M(1 + |x|)^{\sigma_1}$, $x \in \mathbb{R}^n$ for some $\sigma_1 \in [0, N_1]$. Put $W := \sigma + \chi_{(0,1)}(\alpha)\sigma_1(\alpha^{-1} - 1)$. Based on the evidence used in proving the estimate (76), along with the inequality [**380**, (3.19)] and the first estimate appearing in the proof of [**380**, Theorem 4.3], implies that the following holds:

Theorem 2.2.20: The assertion of this theorem continues to hold for any number $\gamma > n \max(N, W)/2r$ (resp. $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}| \max(N, W)$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$).

Theorem 2.2.21: The assertion of this theorem continues to hold for any number $\beta > \frac{nW}{2}$ (resp. $\beta = n|\frac{1}{p} - \frac{1}{2}|W$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$).

If $V_2 \geqslant 0$ and if for each $\eta \in \mathbb{N}_0^n$ there exists $M_\eta > 0$ such that

$$|D^\eta(P_2(x)^{-1})| \leqslant M_\eta(1 + |x|)^{|\eta|(V_2-1)}, \quad x \in \mathbb{R}^n$$

(this holds with $V_2 = N_2$), then similarly as in the first part of this remark we can prove that the condition (ii.1) stated in Theorem 2.2.8 holds, with $a(t) = g_\alpha(t)$ and $k(t) = 1$, if: $\gamma > \frac{n \max(N, V_2, W)}{2r}$, resp. $\gamma \geqslant \frac{n}{r}|\frac{1}{p} - \frac{1}{2}| \max(N, V_2, W)$ (Theorem 2.2.20); $\beta > \frac{n \max(V_2, W)}{2}$, resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}| \max(V_2, W)$ (Theorem 2.2.21). Observe, finally, that there is a great number of concrete examples where we can further refine the obtained theoretical results by using direct calculations [**291**].

(iii) The estimate (70) is very restrictive in the case that $\alpha \in (0, 1)$. If $1 < \alpha < 2$ and $\omega \geqslant 0$, then by the proof of [**380**, Theorem 4.2], cf. also [**304**, Remark 2.1(i)], the condition $\frac{P_1}{P_2}(\mathbb{R}^n) \subseteq \mathbb{C} \smallsetminus (\omega + \Sigma_{\alpha\pi/2})$ implies the validity of (70).

(iv) Let $t \mapsto u(t)$, $t \geqslant 0$ be a mild solution of the problem (52) with the operators A and B replaced respectively by $\overline{P_1(A)}$ and $\overline{P_2(A)}$. Then it can be simply proved that the variation of parameters formula $(R_\alpha * f)(t) = (C\overline{P_2(A)} * u)(t)$ holds for any $t \geqslant 0$. This implies that we can look into the C-wellposedness of the inhomogeneous degenerate Cauchy problem $(\mathrm{DFP})_f$, obtained by adding the term $f(t)$ on the right hand side of (DFP). All this has been seen many times and we shall skip details for the sake of brevity; the interested reader may consult [**292**, Subsection 2.1.1] for further information concerning the C-wellposedness of abstract fractional Cauchy problems.

(v) Let $0 < \alpha_0 < \alpha < 2$, and let the assumptions of Theorem 2.2.20 (Theorem 2.2.21) hold. Using Theorem 2.2.13(ii), it readily follows that there exists an exponentially bounded, analytic $(g_{\alpha_0}, R_{\alpha_0}(0))$-regularized resolvent family $(R_{\alpha_0}(t))_{t\geqslant 0} \subseteq L(E)$ for (DFP), of angle $\min(((\alpha/\alpha_0)-1)\pi/2, \pi/2)$; furthermore, if $E = L^2(\mathbb{R}^n)$ and $\omega = 0$, then the angle of analyticity equals $\min(((\alpha/\alpha_0) - 1)\pi/2, \pi)$ and can be strictly greater than $\pi/2$.

Remark 2.2.23. In this remark, we would like to explain how one can reformulate the assertions of Theorem 2.2.20 and Theorem 2.2.21 in E_l-type spaces. Let E be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and let $0 \leqslant l \leqslant n$. Let us recall that the space E_l is defined by $E_l := \{f \in E : f^{(\alpha)} \in E \text{ for all } \alpha \in \mathbb{N}_0^l\}$. The totality of seminorms $(q_\alpha(f) := ||f^{(\alpha)}||_E,\ f \in E_l;\ \alpha \in \mathbb{N}_0^l)$ induces a Fréchet topology on E_l. Let $\mathbf{T}_l\langle\cdot\rangle$ possess the same meaning as in [**542**], let $a_\eta \in \mathbb{C}$, $0 \leqslant |\eta| \leqslant N_1$, and let $b_\eta \in \mathbb{C}$, $0 \leqslant |\eta| \leqslant N_2$. Assume that the operators $P_1(D)f \equiv \sum_{|\eta|\leqslant N_1} a_\eta D^\eta f$ and $P_2(D)f \equiv \sum_{|\eta|\leqslant N_2} b_\eta D^\eta f$ act with their maximal distributional domains. Then $P_1(D)$ and $P_2(D)$ are closed linear operators on E_l; in the sequel, we assume that $P_1(D) \neq 0$ and $P_2(D) \neq 0$. Let $\omega \geqslant 0$ be such that (70) holds. Then it can be easily seen that $P(D)$ generates an exponentially equicontinuous (g_α, I)-regularized resolvent family $(S_\alpha(t))_{t\geqslant 0}$ in the space E_n and the condition (ii.1) stated in Theorem 2.2.8 holds, with $a(t) = g_\alpha(t)$ and $k(t) = 1$. Let γ (β) have the same value as in the formulation of Theorem 2.2.20 (Theorem 2.2.21). Then the estimates (77)–(78) continue to hold, and slight modifications of the proofs of [**542**, Theorem 2.2, Theorem 2.4] show that the following holds (in our opinion, the proofs of Theorem 2.2.20 and Theorem 2.2.21 are much simpler than those of [**542**, Theorem 2.2, Theorem 2.4] in the case that $l = 0$ and $E \neq L^\infty(\mathbb{R}^n)$, $E \neq C_b(\mathbb{R}^n)$):

(i) Theorem 2.2.20: Set $R_\alpha(t) =: \mathbf{T}_1\langle E_\alpha(t^\alpha P_1(x)/P_2(x))(a-Q(x))^{-\gamma}\rangle$, $t \geqslant 0$. Then $(R_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, R_\alpha(0))$-regularized resolvent family for (DFP), $(R_\alpha(t))_{t\geqslant 0}$ is 'norm continuous' provided

$\gamma > \frac{n\max(N,\frac{N_1+N_2}{\min(1,\alpha)})}{2r}$, in the sense that, for every bounded subset B of E_l and for every $\eta \in \mathbb{N}_0^l$, the mapping $t \mapsto \sup_{f\in B} q_\eta(R_\alpha(t)f)$, $t \geqslant 0$ is continuous. The estimate (71) reads as follows:

$$(79)\quad q_\eta(R_\alpha(t)f) \leqslant M(1+t^{\max(1,\alpha)n/2})e^{\omega t}q_\eta(f), \quad t\geqslant 0,\ f\in E_l,\ \eta\in\mathbb{N}_0^l, \text{ resp.,}$$
$$q_\eta(R_\alpha(t)f) \leqslant M(1+t^{\max(1,\alpha)n|\frac{1}{p}-\frac{1}{2}|})e^{\omega t}q_\eta(f), \quad t\geqslant 0,\ f\in E_l,\ \eta\in\mathbb{N}_0^l,$$

with M being independent of $f\in E_l$ and $\eta\in\mathbb{N}_0^l$.

(ii) Theorem 2.2.21: Set $R_\alpha(t) =: \mathbf{T_1}\langle E_\alpha(t^\alpha P_1(x)/P_2(x))(1+|x|^2)^{-\beta/2}\rangle$, $t \geqslant 0$. Then $(R_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, R_\alpha(0))$-regularized resolvent family for (DFP), $(R_\alpha(t))_{t\geqslant 0}$ is 'norm continuous' provided $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ and (79) holds.

Notice also that $(g_\alpha, R_\alpha(0))$-regularized resolvent families for (DFP), constructed in this remark (Theorem 2.2.20–Theorem 2.2.21), satisfy $R_\alpha(t)R_\alpha(s) = R_\alpha(s)R_\alpha(t)$, $t,s\geqslant 0$, as well as that $R_\alpha(t)P(D)\subseteq P(D)R_\alpha(t)$, $t\geqslant 0$ ($R_\alpha(t)\overline{P(A)}\subseteq \overline{P(A)}R_\alpha(t)$, $t\geqslant 0$) for any complex polynomial $P(x)$. The final conclusions of Remark 2.2.22(ii) remain true for $(g_\alpha, R_\alpha(0))$-regularized resolvent families for (DFP) in the setting of E_l-type spaces.

Before proceeding to the next subsection, it would be worthwhile to note that Theorem 2.2.20 and Theorem 2.2.21 can be successfully applied in the analysis of the fractional analogues of the Barenblatt–Zheltov–Kochina equation in infinite domains (cf. [**140**, Example 1.6, p. 50] and [**52**]):

$$(\eta\Delta - 1)\mathbf{D}_t^\alpha u(t) + \Delta u = 0 \quad (\eta > 0),$$

where $0<\alpha<2$ and $\cos(\pi/\alpha)\leqslant 0$. Details can be left to the reader.

2.2.4. Degenerate second order equations associated with abstract differential operators. The main objective in this subsection is to prove some results on the C-wellposedness of the following abstract degenerate Cauchy problem of second order

$$(DFP)_2 : \begin{cases} \frac{d^2}{dt^2}\overline{P_2(A)}u(t) = \overline{P_1(A)}u(t), \quad t\geqslant 0,\\ u(0)=Cx,\ u'(0)=0.\end{cases}$$

Keeping in mind the results clarified by now, as well as the analyses contained in the papers [**559**] and [**542**,**543**], the consideration of degenerate second order equations is similar to that of degenerate fractional equations of order $\alpha\in(0,2)$; because of that, we shall only outline the main details and omit the proofs. As before, we assume that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials in n variables, as well as that $N_1 = dg(P_1(x))$ and $N_2 = dg(P_2(x))$. Set $F_t(z) := E_2(t^2 z)$, $t\geqslant 0$, $z\in\mathbb{C}$, and $\Omega(\omega) := \{\lambda^2 : \operatorname{Re}\lambda > \omega\}$, if $\omega>0$ and $\Omega(\omega) := \mathbb{C}\smallsetminus(-\infty,-\omega^2]$, if $\omega\leqslant 0$. Given $l\geqslant 0$ and $t\geqslant 0$ in advance, set

$$Q_l(t) := \begin{cases} (1+t^l)e^{\omega t}, & \text{if } \omega>0,\\ 1+t^{2l}, & \text{if } \omega = 0,\\ 1+t^l, & \text{if } \omega<0.\end{cases}$$

Suppose now that $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and $P_1(x)/P_2(x) \notin \Omega(\omega)$, $x \in \mathbb{R}^n$. Then, for every $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, the equation (73) reads as follows:

$$D^\eta E_2\Big(t^2 \frac{P_1(x)}{P_2(x)}\Big) = \sum_{j=1}^{|\eta|} F_t^{(j)}\Big(\frac{P_1(x)}{P_2(x)}\Big) R_{\eta,j}(x), \quad t \geqslant 0,\ x \in \mathbb{R}^n, \tag{80}$$

where $R_{\eta,j}(x)$ is a finite sum of terms like $\prod_{w=1}^{s_j} D^{\eta_{j,w}}(\frac{P_1(x)}{P_2(x)})$ with $|\eta_{j,w}| > 0$ $(1 \leqslant w \leqslant s_j)$ and $|\eta_{j,1}| + \cdots + |\eta_{j,s_j}| \leqslant |\eta|$. Due to the computation established in [**559**, Lemma 2.1], we have that, for every $l \in \mathbb{N}_0$, $|F_t^{(l)}(P_1(x)/P_2(x))| \leqslant MQ_l(t)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Combining this estimate with (80), and repeating verbatim the arguments given in the proof of Theorem 2.2.20 (cf. also Remark 2.2.22), it can be easily seen that the following two theorems hold good.

THEOREM 2.2.24. *Suppose that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}$, $P_1(x)/P_2(x) \notin \Omega(\omega)$, $x \in \mathbb{R}^n$, $N \in \mathbb{N}$ and $r \in (0, N]$. Let $Q(x)$ be an r-coercive complex polynomial of degree N, $a \in \mathbb{C} \smallsetminus Q(\mathbb{R}^n)$, let for each $\eta \in \mathbb{R}^n$ with $|\eta| > 0$ the estimate* (72) *hold with the number $N_1 + N_2$ replaced by $\sigma \geqslant 0$ (the choice $\sigma = N_1 + N_2$ is always possible), and let $\gamma > \frac{n \max(N,\sigma)}{2r}$ (resp. $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}|\max(N, \sigma)$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$). Set*

$$R_2(t) := \Big(E_2\Big(t^2 \frac{P_1(x)}{P_2(x)}\Big)(a - Q(x))^{-\gamma}\Big)(A), \quad t \geqslant 0.$$

Then $(R_2(t))_{t \geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_2, R_2(0))$-regularized resolvent family for $(\mathrm{DFP})_2$, *$(R_2(t))_{t\geqslant 0}$ is norm continuous provided $\gamma > \frac{n \max(N,\sigma)}{2r}$, and the following holds:*

$$\|R_2(t)\| \leqslant MQ_{n/2}(t), \quad t \geqslant 0,\ \textit{resp.},\ \|R_2(t)\| \leqslant MQ_{n|\frac{1}{p}-\frac{1}{2}|}(t), \quad t \geqslant 0. \tag{81}$$

THEOREM 2.2.25. *Suppose that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}$ and $P_1(x)/P_2(x) \notin \Omega(\omega)$, $x \in \mathbb{R}^n$. Let for each $\eta \in \mathbb{R}^n$ with $|\eta| > 0$ the estimate* (72) *hold with the number $N_1 + N_2$ replaced by $\sigma \geqslant 0$ (the choice $\sigma = N_1 + N_2$ is always possible), and let $\beta > \frac{n\sigma}{2}$ (resp. $\beta = n|\frac{1}{p} - \frac{1}{2}|\sigma$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$). Set*

$$R_2(t) := \Big(E_2\Big(t^2 \frac{P_1(x)}{P_2(x)}\Big)(1 + |x|^2)^{-\beta/2}\Big)(A), \quad t \geqslant 0.$$

Then $(R_2(t))_{t \geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_2, R_2(0))$-regularized resolvent family for $(\mathrm{DFP})_2$, *$(R_2(t))_{t\geqslant 0}$ is norm continuous provided $\beta > \frac{n\sigma}{2}$, and* (81) *holds.*

REMARK 2.2.26. (i) Suppose that

$$\mathrm{Re}(P_1(x)/P_2(x)) \leqslant -\zeta|x|^r + \zeta_1, \quad x \in \mathbb{R}^n,$$

for some positive real numbers $r, \zeta, \zeta_1 > 0$. Then the Lagrange mean value theorem for vector-valued functions implies that $\sigma \geqslant r$, and by the proof of [**559**, Lemma 2.1], we have that there exist numbers $L \geqslant 1$ and $M \geqslant 1$ such that, for every $j \in \mathbb{N}_0$ with $j < n/2$, the following holds $|F_t^{(j)}(P_1(x)/P_2(x))| \leqslant MQ_j(t)(1+|x|)^{-jr/2}$, $t \geqslant 0$, $|x| \geqslant L$. Unfortunately, the above does not guarantee that we can refine the results clarified in Theorem 2.2.24 and Theorem 2.2.25 by replacing the number σ with $\sigma - (r/2)$, unless $P_2(x) \equiv 1$. Observe, however, that the refinement of this type is possible if $\omega \leqslant 0$ (in this case we can estimate the derivatives of function $E_2(t^2 P_1(x)/P_2(x))$ by using the formula appearing in the second line of the proof of [**543**, Theorem 4.1]).

(ii) Let $V_2 \geqslant 0$. If for each $\eta \in \mathbb{N}_0^n$ there exists $M_\eta > 0$ such that $|D^\eta(P_2(x)^{-1})| \leqslant M_\eta(1+|x|)^{|\eta|(V_2-1)}$, $x \in \mathbb{R}^n$ (recall that the choice $V_2 = N_2$ is always possible), and if we replace, in the formulations of Theorem 2.2.24 and Theorem 2.2.25, the number σ with $\max(\sigma, V_2)$, then condition (ii.1) stated in Theorem 2.2.8 holds, with $a(t) = t$ and $k(t) = 1$.

(iii) The assertions of Theorem 2.2.24 and Theorem 2.2.25, as well as the conclusions stated in the first and second part of this remark, continue to hold with suitable modifications in the setting of E_l-type spaces; cf. also [**543**, Theorem 4.1-Theorem 4.2].

Before closing this subsection with an illustrative example, it is our duty to say that the results from Subsection 2.1.3–Subsection 2.1.4 are inapplicable in the analysis of a great number of equations of mathematical physics that are not solvable relative to the highest-order time-derivative (cf. Example 2.3.22 for further information in this direction).

EXAMPLE 2.2.27. (i) Let $1 < p < \infty$, $0 < \alpha < 2$, $l \in \mathbb{N}$, $E = L^p(\mathbb{R}^2)$, and let the fractional Sobolev space $\mathrm{S}^{\alpha,p}(\mathbb{R}^2)$ be defined in the sense of [**410**, Definition 12.3.1, p. 297]; that is,

$$\mathrm{S}^{\alpha,p}(\mathbb{R}^2) := D((1-\Delta_p)^{\alpha/2}),$$

where Δ_p acts on $L^p(\mathbb{R}^2)$ with its maximal distributional domain. Consider the following degenerate fractional Cauchy problem:

$$(P): \begin{cases} \mathbf{D}_t^\alpha[u_{xx}+u_{xy}+u_{yy}-u] = e^{-i\alpha\frac{\pi}{2}}\big[(-1)^{l+1}\frac{\partial^{2l}}{\partial x^{2\alpha}}u + u_{yy}\big], \quad t \geqslant 0, \\ u(0,x) = \phi(x);\ u_t(0,x) = 0 \ \text{ if } \alpha \geqslant 1, \end{cases}$$

cf. Theorem 2.2.21 with $P_1(x,y) = e^{-i\alpha\frac{\pi}{2}}(x^{2l}+y^2)$, $P_2(x,y) = x^2+xy+y^2+1$ and $\omega = 0$. Then it can be easily seen that the conditions stated in Remark 2.1.23(ii) hold with $\sigma = \sigma_1 = 2l-2$, so that for each $\beta \geqslant n|\frac{1}{p}-\frac{1}{2}|(2l-2)(1+\chi_{(0,1)}(\alpha)(\alpha^{-1}-1))$ there exists a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family for the corresponding problem (DFP), obeying the property (ii.1) of Theorem 2.2.8 with $a(t) = g_\alpha(t)$ and $k(t) = 1$. Hence, there exists a unique strong solution of problem (P) provided that $\phi \in \mathrm{S}^{2l+\beta,p}(\mathbb{R}^2)$.

(ii) Let $1 < p < \infty$, $E = L^p(\mathbb{R}^n)$ and $Q \in \mathbb{N} \setminus \{1\}$. Consider the following degenerate second order Cauchy problem:

$$(P_2) : \begin{cases} \frac{\partial^2}{\partial t^2}(\Delta u(t,x) - u(t,x)) = \sum_{|\eta| \leqslant Q} a_\eta D^\eta u(t,x), & t \geqslant 0,\ x \in \mathbb{R}^n, \\ u(0,x) = \phi(x),\ u_t(0,x) = 0; \end{cases}$$

then $P_1(x) = \sum_{|\eta| \leqslant Q} a_\eta x^\eta$ and $P_2(x) = -1 - |x|^2$ $(x \in \mathbb{R}^n)$. Assuming that the polynomial $P_1(x)$ is positive, as well as that the estimate (72) holds with some number $\sigma \geqslant 0$ and the condition stated in Remark 2.2.26(ii) holds with some number $V_2 \geqslant 0$, then there exists a unique strong solution of problem (P_2) provided that

$$\phi \in \mathrm{S}^{Q+n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n).$$

2.2.5. Semilinear degenerate relaxation equations associated with abstract differential operators. In this subsection, we shall present our recent results from [**310**] and [**319**] concerning the existence and uniqueness of mild solutions of the following semilinear degenerate relaxation equation

$$(\mathrm{DFP})_{sl} : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} u(t) = \overline{P_1(A)} u(t) + f(t,u(t)), & t \geqslant 0, \\ u(0) = x, \end{cases}$$

where $0 < \alpha < 1$, the function $f(\cdot,\cdot)$ satisfies certain properties and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E.

We need the following definition.

Definition 2.2.28. Let $0 < \alpha < 1$, let $C \in L(E)$ be injective and let $C^{-1}\overline{P_1(A)}C = \overline{P_1(A)}$, $C^{-1}\overline{P_2(A)}C = \overline{P_2(A)}$. A strongly continuous operator family $(P_\alpha(t))_{t>0} \subseteq L(E)$ is said to be an $(\alpha,\alpha,\overline{P_1(A)},\overline{P_2(A)},C)$-resolvent family iff there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that the mapping $t \mapsto \|t^{1-\alpha}P_\alpha(t)\|$, $t \in (0,1]$ is bounded, $\|P_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 1$ and

$$(\lambda^\alpha \overline{P_2(A)} - \overline{P_1(A)})^{-1}Cx = \int_0^\infty e^{-\lambda t} P_\alpha(t)x\,dt, \quad \operatorname{Re}\lambda > \omega,\ x \in E.$$

The following theorem is backbone of this subsection.

Theorem 2.2.29. *Suppose* $0 < \alpha < 1$, $\omega \geqslant 0$, $P_1(x)$ *and* $P_2(x)$ *are non-zero complex polynomials,* $N_1 = dg(P_1(x))$ *and* $N_2 = dg(P_2(x))$.

(i) *Let* $N \in \mathbb{N}$, $r \in (0,N]$, *let* $Q(x)$ *be an* r*-coercive complex polynomial of degree* N, $a \in \mathbb{C} \setminus Q(\mathbb{R}^n)$, $\gamma > \frac{n\max(N,\frac{N_1+N_2}{\alpha})}{2r}$ *(resp.* $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}|\max(N,\frac{N_1+N_2}{\alpha})$, *if* $E = L^p(\mathbb{R}^n)$ *for some* $1 < p < \infty$*),* $\gamma' > \frac{\max(N,\frac{N_1+N_2}{\alpha})}{r}\frac{n}{2} + \frac{N_1+N_2}{r\alpha}(1-\alpha)$ $(\gamma' = n|\frac{1}{p} - \frac{1}{2}|\frac{\max(N,\frac{N_1+N_2}{\alpha})}{r} + \frac{N_1+N_2}{r\alpha}(1-\alpha)$, *if* $E = L^p(\mathbb{R}^n)$ *for some* $1 < p < \infty)$, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, *and let* (70) *hold. Set*

$$R_{\alpha,\gamma}(t) := \Bigl(E_\alpha\Bigl(t^\alpha \frac{P_1(x)}{P_2(x)}\Bigr)(a - Q(x))^{-\gamma}\Bigr)(A), \quad t \geqslant 0,$$

$C := R_{\alpha,\gamma'}(0)$ and

$$P_{\alpha,\gamma'}(t) := t^{\alpha-1}\Big(P_2(x)^{-1}E_{\alpha,\alpha}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big)(a-Q(x))^{-\gamma'}\Big)(A), \quad t>0.$$

Then $(R_{\alpha,\gamma}(t))_{t\geqslant 0}\subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_{\alpha,\gamma}(0))$-regularized resolvent family for (DFP)*, $(P_{\alpha,\gamma'}(t))_{t>0}\subseteq L(E)$ is a global $(\alpha,\alpha,\overline{P_1(A)},\overline{P_2(A)},C)$-resolvent family, $(R_{\alpha,\gamma}(t))_{t\geqslant 0}$ is norm continuous provided $\gamma > \frac{n\max(N,\frac{N_1+N_2}{\alpha})}{2r}$, $(P_{\alpha,\gamma'}(t))_{t>0}$ is norm continuous provided $\gamma' > \frac{\max(N,\frac{N_1+N_2}{\alpha})}{r}\frac{n}{2}+\frac{N_1+N_2}{r\alpha}(1-\alpha)$,* (71) *holds with $(R_\alpha(t))_{t\geqslant 0}$ replaced by $(R_{\alpha,\gamma}(t))_{t\geqslant 0}$ therein, and*

(82)
$$\|P_{\alpha,\gamma'}(t)\| \leqslant Mt^{\alpha-1}(1+t^{1-\alpha+\frac{n}{2}})e^{\omega t}, \quad t>0, \text{ resp.,}$$
$$\|P_{\alpha,\gamma'}(t)\| \leqslant Mt^{\alpha-1}(1+t^{1-\alpha+n|\frac{1}{p}-\frac{1}{2}|})e^{\omega t}, \quad t>0.$$

(ii) *Suppose $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\alpha}$ (resp. $\beta \geqslant n|\frac{1}{p}-\frac{1}{2}|\frac{(N_1+N_2)}{\alpha}$, if $E=L^p(\mathbb{R}^n)$ for some $1<p<\infty$), $\beta' > (1-\alpha+\frac{n}{2})\frac{(N_1+N_2)}{\alpha}$ (resp. $\beta' \geqslant (1-\alpha+n|\frac{1}{p}-\frac{1}{2}|)\frac{(N_1+N_2)}{\alpha}$, if $E=L^p(\mathbb{R}^n)$ for some $1<p<\infty$), $P_2(x)\neq 0$, $x\in\mathbb{R}^n$ and* (70) *holds. Set*

$$R_{\alpha,\beta}(t) := \Big(E_\alpha\Big(t^\alpha\frac{P_1(x)}{P_2(x)}\Big)(1+|x|^2)^{-\beta/2}\Big)(A), \quad t\geqslant 0$$

$C := R_{\alpha,\beta'}(0)$, and

$$P_{\alpha,\beta'}(t) := t^{\alpha-1}\Big(P_2(x)^{-1}E_{\alpha,\alpha}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big)(1+|x|^2)^{-\beta'/2}\Big)(A), \quad t>0.$$

Then $(R_{\alpha,\beta}(t))_{t\geqslant 0}\subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_{\alpha,\beta}(0))$-regularized resolvent family for (DFP)*, $(P_{\alpha,\beta'}(t))_{t>0}\subseteq L(E)$ is a global $(\alpha,\alpha,\overline{P_1(A)},\overline{P_2(A)},C)$-resolvent family, $(R_{\alpha,\beta}(t))_{t\geqslant 0}$ is norm continuous provided $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\alpha}$, $(P_{\alpha,\beta'}(t))_{t>0}$ is norm continuous provided $\beta' > (1-\alpha+\frac{n}{2})\frac{(N_1+N_2)}{\alpha}$,* (71) *holds with $(R_\alpha(t))_{t\geqslant 0}$ replaced by $(R_{\alpha,\beta}(t))_{t\geqslant 0}$ therein, and* (82) *holds with $(P_{\alpha,\gamma'}(t))_{t\geqslant 0}$ replaced by $(P_{\alpha,\beta'}(t))_{t\geqslant 0}$ therein.*

Proof. Recall that $k=1+\lfloor n/2\rfloor$. The results for $(R_{\alpha,\gamma}(t))_{t\geqslant 0}$ and $(R_{\alpha,\beta}(t))_{t\geqslant 0}$ have been proved in Theorem 2.2.20 and Theorem 2.2.21, respectively. In either choice of the regularizing operator C, we have $C^{-1}\overline{P_1(A)}C=\overline{P_1(A)}$ and $C^{-1}\overline{P_2(A)}C=\overline{P_2(A)}$ [**304**]. Furthermore, for every $j\in\mathbb{N}$, there exist uniquely determined real numbers $c_{l,j,\alpha}$ $(1\leqslant l\leqslant j)$ such that $E'_\alpha(z)=\alpha^{-1}E_{\alpha,\alpha}(z)$, $z\in\mathbb{C}$, as well as that $E^{(j)}_\alpha(z)=\sum_{l=1}^{j}c_{l,j,\alpha}E_{\alpha,\alpha j-(j-l)}(z)$, $z\in\mathbb{C}$ (cf. (44)). Using these facts and the proof of Theorem 2.2.20, we may conclude that:

1. For every multi-index $\eta\in\mathbb{N}_0^n$ with $|\eta|>0$, there exists $c_\eta>0$ such that

$$\Big|D^\eta\Big(\frac{P_1(x)}{P_2(x)}\Big)\Big| \leqslant c_\eta(1+|x|)^{|\eta|(N_1+N_2-1)}, \quad x\in\mathbb{R}^n.$$

2. For every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, for every $t \geqslant 0$ and for every $x \in \mathbb{R}^n$, we have:

$$D^{\eta}E_{\alpha,\alpha}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big) = \sum_{j=1}^{|\eta|} t^{\alpha j}E^{(j)}_{\alpha,\alpha}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big)R_{\eta,j}(x) \tag{83}$$

$$= \sum_{j=1}^{|\eta|} t^{\alpha j}\sum_{l=1}^{j+1}\alpha c_{l,j+1,\alpha}E_{\alpha,\alpha j-j+l+\alpha-1}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big)R_{\eta,j}(x), \tag{84}$$

where $R_{\eta,j}(x)$ can be represented as a finite sum of terms like $\prod_{q=1}^{s_j} D^{\eta_{j,q}}(\frac{P_1(x)}{P_2(x)})$ with $|\eta_{j,q}| > 0$ $(1 \leqslant q \leqslant s_j)$ and $|\eta_{j,1}|+\cdots+|\eta_{j,s_j}| \leqslant |\eta|$.

In the remainder of the proof, by M we denote a generic constant whose value may vary at each occurrence. Owing to [1./2.], we get that

$$|R_{\eta,j}(x)| \leqslant M(1+|x|)^{|\eta|(N_1+N_2-1)}, \quad \text{provided } \ 1 \leqslant j \leqslant |\eta| \leqslant k \ \text{ and } \ x \in \mathbb{R}^n. \tag{85}$$

Arguing as in the proof of [**296**, Theorem 2.1], we can prove that, for every $t \geqslant 0$, $x \in \mathbb{R}^n$, and for every $j, l \in \mathbb{N}$ such that $1 \leqslant j \leqslant k$ and $1 \leqslant l \leqslant j+1$, the following holds:

$$\begin{aligned}&|E_{\alpha,\alpha j-(j-l)+\alpha-1}(t^{\alpha}P_1(x)/P_2(x))|\\ &\qquad \leqslant M\big[1+t^{1-(\alpha j-(j-l)+\alpha-1)}|P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}e^{\omega t}\big].\end{aligned} \tag{86}$$

If $t \geqslant 0$ and $x \in \mathbb{R}^n$ satisfies $|t^{\alpha}P_1(x)/P_2(x)| \leqslant 1$, then the equation (83) yields:

$$|D^{\eta}(E_{\alpha,\alpha}(t^{\alpha}P_1(x)/P_2(x)))| \leqslant M(t^{\alpha}+t^{\alpha|\eta|})(1+|x|)^{|\eta|(N_1+N_2-1)}, \quad |\eta| \leqslant k.$$

Suppose now that $1 \leqslant l \leqslant j+1$, $1 \leqslant j \leqslant |\eta| \leqslant k$, $t \geqslant 0$, $x \in \mathbb{R}^n$ and $|t^{\alpha}P_1(x)/P_2(x)| \geqslant 1$. Then it can be easily seen that the supposition $1-(\alpha j-(j-l)+\alpha-1) \geqslant 0$ implies

$$\begin{aligned}&(N_1+N_2)\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}+|\eta|(N_1+N_2-1)\\ &\qquad \leqslant |\eta|\Big(\frac{N_1+N_2}{\alpha}-1\Big)+(N_1+N_2)\frac{1-\alpha}{\alpha}.\end{aligned}$$

Using this estimate and (85), it readily follows that

$$\begin{aligned}&|t^{\alpha}P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}(1+|x|)^{|\eta|(N_1+N_2-1)}\\ &\qquad \leqslant Mt^{1-(\alpha j-(j-l)+\alpha-1)}(1+|x|)^{|\eta|(\frac{N_1+N_2}{\alpha}-1)+(N_1+N_2)\frac{1-\alpha}{\alpha}},\end{aligned}$$

provided $1-(\alpha j-(j-l)+\alpha-1) \geqslant 0$. On the other hand, it is clear that

$$|t^{\alpha}P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}(1+|x|)^{|\eta|(N_1+N_2-1)} \leqslant M(1+|x|)^{|\eta|(N_1+N_2-1)}, \tag{87}$$

provided $1-(\alpha j-(j-l)+\alpha-1) \leqslant 0$. Then, for $t \geqslant 0$, $x \in \mathbb{R}^n$ and $0 < |\eta| \leqslant k$, the following holds (cf. (84), (86)–(87) and [**304**, (2.6)–(2.7)]):

$$\Big|D^{\eta}E_{\alpha,\alpha}\Big(t^{\alpha}\frac{P_1(x)}{P_2(x)}\Big)\Big| \leqslant M(1+t^{1-\alpha})(1+t^{|\eta|})e^{\omega t}(1+|x|)^{|\eta|(\frac{N_1+N_2}{\alpha}-1)+(N_1+N_2)\frac{1-\alpha}{\alpha}}; \tag{88}$$

observe that the inequality $\mathrm{Re}((t^\alpha P_1(x)/P_2(x))^{1/\alpha}) \leqslant \omega t$, $t \geqslant 0$, $x \in \mathbb{R}^n$ and Theorem 1.5.1 together imply that the previous estimate also holds in the case that $|\eta| = 0$. Set

$$G_{\alpha,\gamma'}(t) := \Bigl(P_2(x)^{-1}E_\alpha\Bigl(t^\alpha \frac{P_1(x)}{P_2(x)}\Bigr)(a-Q(x))^{-\gamma'}\Bigr)(A), \quad t \geqslant 0,$$

in the case of examination of (i), resp.

$$G_{\alpha,\beta'}(t) := \Bigl(P_2(x)^{-1}E_\alpha\Bigl(t^\alpha \frac{P_1(x)}{P_2(x)}\Bigr)(1+|x|^2)^{-\beta'/2}\Bigr)(A), \quad t \geqslant 0,$$

in the case of examination of (ii). Then the identity

$$E_\alpha(t^\alpha P_1(x)/P_2(x)) = \int_0^t g_{1-\alpha}(t-s)s^{\alpha-1}E_{\alpha,\alpha}(s^\alpha P_1(x)/P_2(x))\,ds, \quad t > 0,\ x \in \mathbb{R}^n$$

(cf. [**529**, p. 212, l.4] and the proof of [**310**, Theorem 2.1]), shows that

$$G_{\alpha,\gamma'}(t) = (g_{1-\alpha} * P_{\alpha,\gamma'})(t), \quad t > 0 \text{ and } G_{\alpha,\beta'}(t) = (g_{1-\alpha} * P_{\alpha,\beta'})(t), \quad t > 0.$$

The proof can be completed routinely by using the estimate (88), the above equalities and the argumentation used in the proofs of Theorem 2.2.20 and [**304**, Theorem 2.1]; see also Remark 2.1.23(i). □

It is worth noting that the assertion of Theorem 2.2.29 continues to hold, with appropriate technical modifications, in the case that $E = C_b(\mathbb{R}^n)$ or $E = L^\infty(\mathbb{R}^n)$, and that the additional refinement of lower bounds for the numbers γ, γ', β, β' can be proved following the approach from Remark 2.1.23(ii). Suppose now that $T > 0$, the requirements of Theorem 2.2.29(i) or Theorem 2.2.29(ii) hold, $C := R_{\alpha,\gamma'}(0)$ ($C := R_{\alpha,\beta'}(0)$) in the case of consideration Theorem 2.2.29(i) (Theorem 2.2.29(ii)) and $x \in R(C)$. Following the analyses of non-degenerate case from [**529**] and [**277**] (another way to see that the subsequent definition of a mild solution of the problem $(\mathrm{DFP})_{sl}$ is correct in the case that $f(t, u(t)) = f(t)$ satisfies (P1) is to take the Laplace transform of both sides of the equality $\mathbf{D}_t^\alpha \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t) + f(t)$ by making use of the formula (39), on the one hand, and to compute the Laplace transform of the right hand side of equality (89) below, on the other hand), it will be said that a continuous function $t \mapsto u(t)$, $t \in [0, T]$ is a mild solution of the semilinear abstract degenerate Cauchy problem $(\mathrm{DFP})_{sl}$ on $[0, T]$ iff the mapping $t \mapsto C^{-1}f(t, u(t))$, $t \in [0, T]$ is well-defined and continuous, as well as

$$u(t) = R_\alpha(t)C^{-1}x + \int_0^t P_\alpha(t-s)C^{-1}f(s, u(s))\,ds, \quad t \in [0, T]. \tag{89}$$

Define the operator $Q_\alpha \colon C([0, T] : E) \to C([0, T] : E)$ by

$$(Q_\alpha u)(t) := R_\alpha(t)C^{-1}x + \int_0^t P_\alpha(t-s)C^{-1}f(s, u(s))\,ds, \quad t \in [0, T].$$

The most common technique to proving existence and uniqueness of mild solutions of semilinear fractional evolution equations is to apply some of the fixed point theorems; in our concrete situation, we must prove that the mapping $Q_\alpha(\cdot)$ has a unique fixed point. Not aspiring completeness of analysis here, we shall only state and prove the following adaptation of [**445**, Theorem 1.2, p. 184] to close the whole section.

THEOREM 2.2.30. *Let $T > 0$, let $x \in R(C)$, and let the requirements of* Theorem 2.2.29(i) *or* Theorem 2.2.29(ii) *hold. Put $C := R_{\alpha,\gamma'}(0)$, in the case of* Theorem 2.2.29(i), *and $C := R_{\alpha,\beta'}(0)$, in the case of* Theorem 2.2.29(ii). *Suppose that the mapping $C^{-1}f\colon [0,T] \times E \to E$ is continuous in t on $[0,T]$ and uniformly Lipschitz continuous (with constant L) on E. Then the semilinear fractional Cauchy problem* $(\mathrm{DFP})_{sl}$ *has a unique mild solution $u \in C([0,T] : E)$. Moreover, the mapping $x \to u(\cdot)$ is Lipschitz continuous from $R(C)$ (endowed with the norm $\|\cdot\|_{R(C)} := \|C^{-1}\cdot\|$, $R(C)$ becomes a Banach space) into $C([0,T] : E)$.*

PROOF. Set $M := \max_{t\in(0,T]}(t^{1-\alpha}\Gamma(\alpha)\|P_\alpha(t)\|)$. Arguing as in the proof of [**445**, Theorem 1.2, p. 184], we get that, for every $u, v \in C([0,T] : E)$,

$$\|(Q_\alpha^n u)(t) - (Q_\alpha^n v)(t)\|_{C([0,T]:E)} \leqslant \frac{(MLT^\alpha)^n}{\Gamma(n\alpha+1)}\|u-v\|_{C([0,T]:E)}, \quad n \in \mathbb{N},\ t \in [0,T].$$

For a sufficiently large number $n \in \mathbb{N}$, one has

$$\frac{(MLT^\alpha)^n}{\Gamma(n\alpha+1)} < 1,$$

so that a well known extension of the Banach contraction principle implies that the mapping $Q_\alpha(\cdot)$ has a unique fixed point, finishing the proof of existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{sl}$ on $[0,T]$. Keeping in mind a Gronwall-type inequality [**153**, Lemma 6.19, p. 111], the remainder of the proof follows similarly as in that of [**445**, Theorem 1.2, p. 184]. $\square$

2.3. Degenerate multi-term fractional differential equations in locally convex spaces

Suppose that $n \in \mathbb{N} \smallsetminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_n$, let $A_1, \ldots, A_{n-1}$ are closed linear operators on a Hausdorff sequentially complete locally convex space E, and $f\colon [0,\infty) \to E$ is a continuous function. Let us recall (cf. the introductory part) that $m_i := \lceil \alpha_i \rceil$, $i \in \mathbb{N}_{n-1}$, $T_{i,L}u(t) := A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geqslant 0$, $i \in \mathbb{N}_{n-1}$ and $\alpha_i > 0$, and $T_{i,R}u(t) := \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$. Henceforth it will be assumed that, for every $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$, $T_i u(t)$ denotes either $T_{i,L}u(t)$ or $T_{i,R}u(t)$. As already announced, we will deal with the following degenerate multi-term problem:

$$\sum_{i=1}^{n-1} T_i u(t) = f(t), \quad t \geqslant 0. \tag{90}$$

Set

$$P_\lambda := \sum_{i=1}^{n-1} \lambda^{\alpha_i - \alpha_{n-1}} A_i, \quad \lambda \in \mathbb{C} \smallsetminus \{0\},$$

$\mathcal{I} := \{i \in \mathbb{N}_{n-1} : \alpha_i > 0$ and $T_{i,L}u(t)$ appears on the left hand side of (90)$\}$, $Q := \max \mathcal{I}$, if $\mathcal{I} \neq \emptyset$ and $Q := m_Q := 0$, if $\mathcal{I} = \emptyset$. Assume that $\alpha > 0$ and $m = \lceil \alpha \rceil$. The following facts can be proved by using the equality (38), induction and closedness of A:

(a) Suppose that $l \in \mathbb{N}$ and $u, Au \in C^l([0,\infty) : E)$. Then $u^{(j)}(t) \in D(A)$, $t \geqslant 0$ and $Au^{(j)}(t) = (Au)^{(j)}(t)$, $t \geqslant 0$ $(0 \leqslant j \leqslant l)$.

(b) Suppose that the Caputo fractional derivatives $\mathbf{D}_t^{\alpha} u$ and $\mathbf{D}_t^{\alpha} Au$ are defined. Then $u^{(j)}(t) \in D(A)$, $t \geqslant 0$, $Au^{(j)}(t) = (Au)^{(j)}(t)$, $t \geqslant 0$ $(0 \leqslant j \leqslant m-1)$, $\mathbf{D}_t^{\alpha} u(t) \in D(A)$, $t \geqslant 0$,

$$A\mathbf{D}_t^{\alpha} u(t) = \mathbf{D}_t^{\alpha} Au(t), \quad t \geqslant 0,$$

and

$$J_t^{\alpha} A\mathbf{D}_t^{\alpha} u(t) = J_t^{\alpha} \mathbf{D}_t^{\alpha} Au(t) = Au(t) - \sum_{j=0}^{m-1} Au^{(j)}(0) g_{j+1}(t), \quad t \geqslant 0.$$

Taking into account (a)–(b), it seems reasonable to consider the equation (90) with the initial conditions

$$(91) \quad u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant m_Q - 1 \text{ and } (A_i u)^{(j)}(0) = u_{i,j} \text{ if } m_i - 1 \geqslant j \geqslant m_Q.$$

If $T_{n-1}u(t) = T_{n-1,L}u(t)$, then (91) takes the following simple form: $u^{(j)}(0) = u_j$, $0 \leqslant j \leqslant m_{n-1} - 1$. If this is not the case, then the choice (91) may be non-optimal, the index $i \in \mathbb{N}_{n-1}$ has to satisfy the inequality $m_i - 1 \geqslant m_Q$ in the second equality appearing in (91), and we cannot expect the existence of solutions of problem [(90)–(91)], in general (consider, for example, the case $n = 3$, $A_2 = A_1$ and $u_{2,0} \neq u_{1,0}$); furthermore, for any index $i \in \mathbb{N}_{n-1}$ satisfying the inequality $m_i - 1 \geqslant m_Q$ and for every non-negative integer $k \in [m_Q, m_i - 1]$, we need to introduce exactly one initial value $u_{i,k}$. Throughout the section, we assume that the function $k(t)$ is a scalar-valued continuous kernel on $[0,\infty)$. We will use the following fact about Caputo fractional derivatives: Assume that $\alpha > 0$, $m = \lceil \alpha \rceil$, $\beta \in (0,\alpha)$ and the Caputo fractional derivative $\mathbf{D}_t^{\alpha} u(\cdot)$ is defined. Then the Caputo fractional derivative $\mathbf{D}_t^{\beta} u(\cdot)$ is also defined and the following equality holds:

$$(92) \qquad \mathbf{D}_t^{\beta} u(t) = (g_{\alpha-\beta} * \mathbf{D}_t^{\alpha} u(\cdot))(t) + \sum_{j=\lceil \beta \rceil}^{m-1} u^{(j)}(0) g_{j+1-\beta}(t), \quad t \geqslant 0.$$

2.3.1. Abstract Cauchy problem [(90)–(91)]. Let us recall that $n \in \mathbb{N} \setminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_{n-1}$, as well as that $A_1, \ldots, A_{n-1}$ are closed linear operators on E and $f \colon [0,\infty) \to E$ is a continuous function. Let the set $\mathcal{I}$ and number Q be defined as above.

Suppose, for the time being, that the initial values $u_j \in E$ $(0 \leqslant j \leqslant m_Q - 1)$ satisfy $u_j \in D(A_i)$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R} u(t)$ and $0 \leqslant j \leqslant m_i - 1$ (put $u_{i,j} := A_i u_j$ in this case), and $u_{i,j} \in E$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R} u(t)$ and $m_i - 1 \geqslant j \geqslant m_Q$. We start this section by introducing the notion of a strong solution of problem [(90)–(91)].

Definition 2.3.1. A function $u \in C([0,\infty) : E)$ is said to be a strong solution of problem [(90)–(91)] iff the term $T_i u(t)$ is well defined and continuous for any $t \geqslant 0$, $i \in \mathbb{N}_{n-1}$, and [(90)–(91)] holds identically on $[0,\infty)$.

Now we would like to observe the following fact. If $Q > 0$, then we can consider the problem obtained from the problem (90) by replacing some of terms $T_{i,R}(t)$, for $1 \leqslant i \leqslant Q$, with the corresponding terms of form $T_{i,L}(t)$. By (92) and (b), it readily follows that a strong solution of problem [(90)–(91)] is also a strong solution of the problem described above, when endowed with the initial conditions (91).

Define, for every $i \in \mathbb{N}_{n-1}$ and $t \geqslant 0$,

$$\mathcal{T}_{i,L}u(t) := g_{\alpha_{n-1}-\alpha_i} * A_i\Bigg[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\Bigg](t),$$

if $T_i u(t) = T_{i,L}u(t)$, and

$$\mathcal{T}_{i,R}u(t) := g_{\alpha_{n-1}-\alpha_i} * \Bigg[A_i u(\cdot) - \sum_{j=0}^{m_i-1} u_{i,j} g_{j+1}(\cdot)\Bigg](t),$$

if $T_i u(t) = T_{i,R}u(t)$. Let $\mathcal{T}_i u(t)$ denote exactly one of terms $\mathcal{T}_{i,L}u(t)$ or $\mathcal{T}_{i,R}u(t)$. Integrating the equation (90) α_{n-1} times, the foregoing arguments imply that any strong solution $t \mapsto u(t)$, $t \geqslant 0$ of problem [(90)–(91)] satisfies the following integral equation

$$\sum_{i=1}^{n-1} \mathcal{T}_i u(t) = (g_{\alpha_{n-1}} * f)(t), \quad t \geqslant 0. \tag{93}$$

This motivates us to introduce the following definition.

Definition 2.3.2. Let $u_j \in E$ $(0 \leqslant j \leqslant m_Q - 1)$, let $u_{i,j} \in E$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R}u(t)$ and $0 \leqslant j \leqslant m_i - 1$, and let $\mathcal{V} \subseteq \mathbb{N}_{n-1}$. Then a continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ is said to be a $\mathcal{V}$-mild solution of (93) iff (i)–(v) hold, where:

(i) $g_{\alpha_{n-1}-\alpha_i} * [u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)](t) \in D(A_i)$ for all $t \geqslant 0$ and $i \in \mathcal{I} \cap \mathcal{V}$, the mapping $t \mapsto A_i(g_{\alpha_{n-1}-\alpha_i} * [u - \sum_{j=0}^{m_i-1} u_j g_{j+1}])(t)$, $t \geqslant 0$ is well-defined and continuous for all $i \in \mathcal{I} \cap \mathcal{V}$,

(ii) the mapping $t \mapsto (g_{\alpha_{n-1}-\alpha_i} * A_i[u - \sum_{j=0}^{m_i-1} u_j g_{j+1}])(t)$, $t \geqslant 0$ is continuous for all $i \in \mathcal{I} \smallsetminus \mathcal{V}$,

(iii) $(g_{\alpha_{n-1}-\alpha_i} * u)(t) \in D(A_i)$ for all $t \geqslant 0$ and $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \cap \mathcal{V}$, the mapping $t \mapsto A_i(g_{\alpha_{n-1}-\alpha_i} * u)(t)$, $t \geqslant 0$ is continuous for all $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \cap \mathcal{V}$,

(iv) the mapping $t \mapsto (g_{\alpha_{n-1}-\alpha_i} * A_i u)(t)$, $t \geqslant 0$ is well-defined and continuous for all $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \smallsetminus \mathcal{V}$,

(v) for every $t \geqslant 0$, the following holds:

$$\sum_{i \in \mathcal{I} \cap \mathcal{V}} A_i\Bigg(g_{\alpha_{n-1}-\alpha_i} * \Bigg[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\Bigg]\Bigg)(t) \tag{94}$$
$$+ \sum_{i \in \mathcal{I} \smallsetminus \mathcal{V}} \Bigg(g_{\alpha_{n-1}-\alpha_i} * A_i\Bigg[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\Bigg]\Bigg)(t)$$

$$+ \sum_{i\in(\mathbb{N}_{n-1}\smallsetminus\mathcal{I})\smallsetminus\mathcal{V}} (g_{\alpha_{n-1}-\alpha_i} * A_i u)(t) + \sum_{i\in(\mathbb{N}_{n-1}\smallsetminus\mathcal{I})\cap\mathcal{V}} A_i(g_{\alpha_{n-1}-\alpha_i} * u)(t)$$
$$= \sum_{i\in\mathbb{N}_{n-1}\smallsetminus\mathcal{I}} \sum_{j\in\mathbb{N}^0_{m_i-1}} g_{\alpha_{n-1}-\alpha_i+1+j}(t)u_{i,j} + (g_{\alpha_{n-1}} * f)(t), \quad t \geqslant 0.$$

If $\mathcal{V} = \emptyset$ ($\mathcal{V} = \mathbb{N}_{n-1}$), then we also say that $u(t)$ is a strong (mild) solution of (93).

Any strong solution of problem [(90)–(91)] is also a strong solution of problem (93), and any $\mathcal{V}$-mild solution of problem (93) is also a $\mathcal{V}'$-mild solution of (93) provided that $\mathcal{V}, \mathcal{V}' \subseteq \mathbb{N}_{n-1}$ and $\mathcal{V} \subseteq \mathcal{V}'$. As already observed in [**308**] for the problem $(\mathrm{DFP})_L$, a sufficiently smooth strong solution of the problem (93) need not be a strong solution of problem [(90)–(91)] in the case that $\mathcal{I} \neq \emptyset$. The situation is quite intricate even if $\mathcal{I} = \emptyset$ because then we can only prove that a strong solution of problem (93) satisfies the equation

$$\sum_{i\in\mathbb{N}_{n-1}} \left(g_{m_{n-1}-m_i} * g_{m_i-\alpha_i} * \left[A_i u(\cdot) - \sum_{j=0}^{m_i-1} u_{i,j} g_{1+j}(\cdot) \right] \right)(t) = (g_{\alpha_{n-1}} * f)(t), \quad t \geqslant 0,$$

which does not imply, in general, that the function

$$t \mapsto g_{m_i-\alpha_i} * \left[A_i u - \sum_{j=0}^{m_i-1} u_{i,j} g_{1+j} \right](t), \quad t \geqslant 0$$

is m_i-times continuously differentiable for $i \in \mathbb{N}_{n-1}$ (the problem $(\mathrm{DFP})_R$ is an exception, cf. [**308**]). Because of that, we shall primarily consider degenerate integral equation (93) in the sequel.

Remark 2.3.3. It should be observed that we can further generalize the abstract form of problem (90) by assuming that some of terms $T_i u(t)$ can be expressed as sums of terms like $A_i'\mathbf{D}_t^{\alpha_i}(B_i'\mathbf{D}_t^{\beta_i}u(t))$ and $\mathbf{D}_t^{\alpha_i}A_i''(\mathbf{D}_t^{\beta_i}B_i''u(t))$, with A_i', B_i', A_i'', B_i'' being closed linear operators on E and $\beta_i \geqslant 0$ (cf. [**490**, Chapter VI] for a great number of such examples of degenerate equations of second order possessing a certain physical meaning). It would take too long to go into further details concerning this topic here.

2.3.2. Exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation families. Following the method employed in the papers [**542, 543**] and [**306**], we introduce the notion of an exponentially equicontinuous k-regularized C-resolvent propagation family for problem [(90)–(91)] as follows (cf. the problem (93) with $\mathcal{I} = \emptyset$, $x = u_{i,j}$, the other initial values being zeroes, and then apply the formula (39) for the Laplace transform of Caputo derivatives of the α^{th} order).

Definition 2.3.4. Suppose that the function $k(t)$ satisfies (P1), as well as that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ and $R_{i,j}(t)\colon D(A_i) \to E$ is a linear mapping ($t \geqslant 0$). Let the operator $C \in L(E)$ be injective. Then the operator family $(R_{i,j}(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem [(90)–(91)] iff there exists $\omega \geqslant \max(0, \mathrm{abs}(k))$ such that the following holds:

(i) The mapping $t \mapsto R_{i,j}(t)x$, $t \geqslant 0$ is continuous for every fixed element $x \in D(A_i)$.
(ii) The family $\{e^{-\omega t}R_{i,j}(t) : t \geqslant 0\}$ is equicontinuous, i.e., for every $p \in ⊛$, there exist $c > 0$ and $q \in ⊛$ such that

$$p(e^{-\omega t}R_{i,j}(t)x) \leqslant cq(x), \quad x \in D(A_i),\ t \geqslant 0.$$

(iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator P_λ is injective, $C(R(A_i)) \subseteq R(P_\lambda)$ and

$$\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)P_\lambda^{-1}CA_ix = \int_0^\infty e^{-\lambda t}R_{i,j}(t)x\,dt, \quad x \in D(A_i). \tag{95}$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R_{i,j}(t))_{t\geqslant 0}$ is an exponentially equicontinuous r-times integrated C-regularized resolvent (i,j)-propagation family for [(90)–(91)]; an exponentially equicontinuous 0-times integrated C-regularized resolvent (i,j)-propagation family for [(90)–(91)] is also said to be an exponentially equicontinuous C-regularized resolvent (i,j)-propagation family for [(90)–(91)].

Before we state the following important extension of [**543**, Theorem 3.1], it is worth noting that we do not use here the condition $CA_i \subseteq A_iC$, in contrast to the corresponding definitions from [**542**, **543**] and [**306**], and that the existence of an exponentially equicontinuous k-regularized C-resolvent $(i,0)$-propagation family for problem [(90)–(91)] implies the existence of an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem [(90)–(91)] ($j \in \mathbb{N}^0_{m_i-1}$); if this is the case, we have $R_{i,j}(t)x = (g_j * R_{i,0}(\cdot)x)(t)$, $t \geqslant 0$, $j \in \mathbb{N}^0_{m_i-1}$, $x \in D(A_i)$. Observe also that the uniqueness theorem for Laplace transform implies that there exists at most one exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem [(90)–(91)] and that the assertions of [**306**, Remark 2.3(iv), Proposition 2.4, Theorem 2.5] can be reformulated in our context.

THEOREM 2.3.5. *Suppose that* $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ *and there exists an exponentially equicontinuous* k*-regularized* C*-resolvent* (i,j)*-propagation family* $(R_{i,j}(t))_{t\geqslant 0}$ *for problem* [(90)–(91)].

(i) *Assume that there exists* $l \in \mathbb{N}_{n-1}$ *such that the following condition:*
(C.1) *For every* $v \in \mathbb{N}_{n-1} \setminus \{l\}$ *and* $x \in D(A_i)$, *there exist a number* $\omega_0 > \omega$ *and a continuous* E*-valued function* $t \mapsto f_{i,j,v}(t;x)$, $t \geqslant 0$ *such that, for every* $p \in ⊛$, *there exists* $M_p > 0$ *with* $p(f_{i,j,v}(t;x)) \leqslant M_pe^{\omega t}$, $t \geqslant 0$ $(v \in \mathbb{N}_{n-1} \setminus \{l\})$ *and that, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$,

$$\int_0^\infty e^{-\lambda t}f_{i,j,v}(t;x)dt = \lambda^{\alpha_i-\alpha_{n-1}-j+\alpha_v-\alpha_{n-1}}\tilde{k}(\lambda)A_vP_\lambda^{-1}CA_ix$$

holds. Then for each $v_0 \in D(A_i)$ *the function* $u(t) := R_{i,j}(t)v_0$, $t \geqslant 0$ *is a mild solution of the integral equation*

$$\sum_{v=1}^{n-1} A_v(g_{\alpha_{n-1}-\alpha_v} * u)(t) = (g_{\alpha_{n-1}-\alpha_i+j} * k)(t)CA_iv_0, \quad t \geqslant 0, \tag{96}$$

defined in the same way as in Definition 2.3.2(ii).

(ii) *Let* $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$. *If the conditions:*

(C.2) *For every* $l \in \mathcal{K}$ *and* $x \in D(A_i)$, *and for every* $v \in \mathbb{N}_{n-1} \setminus \{l\}$, *there exist a number* $\omega_{l,v} > \omega$ *and a continuous* E*-valued function* $t \mapsto g_{i,j,l,v}(t;x)$, $t \geqslant 0$ *such that, for every* $p \in \circledast$, *there exists* $M_{p,l,v} > 0$ *with* $p(g_{i,j,l,v}(t;x)) \leqslant M_{p,l,v}e^{\omega_{l,v}t}$, $t \geqslant 0$ $(l \in \mathcal{K},\ v \in \mathbb{N}_{n-1} \setminus \{l\})$ *and that, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_{l,v}$ *and* $\tilde{k}(\lambda) \neq 0$,

$$\int_0^\infty e^{-\lambda t} g_{i,j,l,v}(t;x)dt = \lambda^{\alpha_i-\alpha_{n-1}-j+\alpha_v-\alpha_l}\tilde{k}(\lambda)A_vP_\lambda^{-1}CA_ix$$

and

(C.3) *For every* $l \in \mathcal{K}$, *there exist a number* $\omega_l > \omega$ *and a continuous function* $h_l \colon [0,\infty) \to \mathbb{C}$ *satisfying* (P1) *and*

$$\widetilde{h_l}(\lambda) = \tilde{k}(\lambda)\lambda^{\alpha_i-\alpha_l-j}, \quad \operatorname{Re}\lambda > \omega_l,$$

holds, then for each $v_0 \in D(A_i)$ *the function* $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ *satisfies that the mappings* $t \mapsto A_lu(t)$, $t \geqslant 0$ *are well-defined, continuous and that for each* $p \in \circledast$ *there exist* $M_p > 0$ *and* $\omega_0 > \omega$ *with* $p(A_lu(t) - h_l(t)CA_iv_0) \leqslant M_pe^{\omega_0 t}$, $t \geqslant 0$ $(l \in \mathcal{K})$. *Furthermore, for every* $t \geqslant 0$,

$$\sum_{l\in\mathcal{K}}(g_{\alpha_{n-1}-\alpha_l} * A_lu)(t) + \sum_{l\in\mathbb{N}_{n-1}\setminus\mathcal{K}} A_l(g_{\alpha_{n-1}-\alpha_l} * u)(t) = (g_{\alpha_{n-1}-\alpha_i+j} * k)(t)CA_iv_0. \tag{97}$$

(iii) *Suppose that* (C.1) *holds. Let* $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$, *let* $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$, *and let* $CA_p \subseteq A_pC$ *for all* $p \in \mathbb{N}_{n-1}$. *If the condition:*

(C.4) *For every* $l \in \mathcal{K}$ *and for every* $v \in \mathbb{N}_{n-1} \setminus \{i\}$, *there exist a number* $\omega_{l,v} > \omega$ *and a continuous function* $h_{l,v} \colon [0,\infty) \to E$ *satisfying that, for every* $p \in \circledast$, *there exists* $M_{p,l,v} > 0$ *with* $p(h_{l,v}(t)) \leqslant M_{p,l,v}e^{\omega_{l,v}t}$, $t \geqslant 0$ *and*

$$\widetilde{h_{l,v}}(\lambda) = \tilde{k}(\lambda)\lambda^{\alpha_v-\alpha_{n-1}-j}A_lP_\lambda^{-1}CA_vv_0, \quad \operatorname{Re}\lambda > \omega_{l,v},\ \tilde{k}(\lambda) \neq 0$$

holds, then the function $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ *satisfies that the mappings* $t \mapsto A_lu(t)$, $t \geqslant 0$ *are well-defined, continuous and that for each* $p \in \circledast$ *there exist* $M_p > 0$ *and* $\omega_0 > \omega$ *with* $p(A_lu(t) - (g_j * k)(t)CA_lv_0) \leqslant M_pe^{\omega_0 t}$, $t \geqslant 0$ $(l \in \mathcal{K})$. *Furthermore, for every* $t \geqslant 0$, (97) *holds.*

(iv) *Suppose that* $CA_p \subseteq A_pC$, $p \in \mathbb{N}_{n-1}$ *and* $k(t)$ *satisfies* (P2), *as well as that* $n = 3$ *or that* $n \geqslant 4$ *and the following condition holds:*

(C.5) *For every* $p \in \circledast$ *and* $l \in \mathbb{N}_{n-1} \smallsetminus \{i\}$, *there exist numbers* $\lambda_{p,l}, \sigma_{p,l} > 0$, *a seminorm* $q_{p,l} \in \circledast$ *and a function* $h_{p,l} \colon (\lambda_{p,l}, \infty) \to (0, \infty)$ *such that:*

$$p(P_\lambda^{-1} C A_l x) \leqslant [q_{p,l}(x) + q_{p,l}(A_l x)] h_{p,l}(\lambda), \quad \lambda > \lambda_{p,l}, \ x \in D(A_l),$$

and

$$\lim_{\lambda \to +\infty} e^{-\lambda \sigma_{p,l}} h_{p,l}(\lambda) = 0.$$

Then the function $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ *is a unique mild solution of the integral equation* (96), *provided that* $v_0 \in D(A_i)$ *and the assumptions of* (i) *hold. Furthermore, the function* $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ *is a unique function satisfying that the mapping* $t \mapsto A_l u(t)$, $t \geqslant 0$ *is well-defined, continuous* ($l \in \mathcal{K}$) *and that* (97) *holds, provided that* $v_0 \in D(A_i)$ *and the assumptions of* (ii) *hold, resp.* $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$ *and the assumptions of* (iii) *hold.*

Proof. Let $v_0 \in D(A_i)$. Due to the condition (C.1) and the uniqueness theorem for Laplace transform, we have that the function $t \mapsto A_v(g_{\alpha_{n-1}-\alpha_v} * R_{i,j}(\cdot)v_0)(t)$, $t \geqslant 0$ is well-defined, continuous and for each $p \in \circledast$ there exist $M_p' > 0$ and $\omega' > \omega$ with $p(A_v(g_{\alpha_{n-1}-\alpha_v} * R_{i,j}(\cdot)v_0)(t)) \leqslant M_p' e^{\omega' t}$, $t \geqslant 0$ ($v \in \mathbb{N}_{n-1} \smallsetminus \{l\}$); furthermore,

$$\int_0^\infty e^{-\lambda t} A_v(g_{\alpha_{n-1}-\alpha_v} * R_{i,j}(\cdot)v_0)(t)\,dt = \tilde{k}(\lambda)\lambda^{\alpha_i - \alpha_{n-1} - j + \alpha_v - \alpha_{n-1}} A_v P_\lambda^{-1} C A_i v_0,$$

for any $v \in \mathbb{N}_{n-1} \smallsetminus \{l\}$ and for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega'$ and $\tilde{k}(\lambda) \neq 0$. Using the identity

$$\begin{aligned} (98) \quad & \tilde{k}(\lambda)\lambda^{\alpha_i - \alpha_{n-1} - j + \alpha_l - \alpha_{n-1}} A_l P_\lambda^{-1} C A_i v_0 \\ &= \tilde{k}(\lambda)\lambda^{\alpha_i - \alpha_{n-1} - j} \Big[C A_i v_0 - \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{l\}} \lambda^{\alpha_v - \alpha_{n-1}} A_v P_\lambda^{-1} C A_i v_0 \Big], \end{aligned}$$

for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega'$ and $\tilde{k}(\lambda) \neq 0$, and the uniqueness theorem for Laplace transform, it readily follows that the function $t \mapsto A_l(g_{\alpha_{n-1}-\alpha_l} * R_{i,j}(\cdot)v_0)(t)$, $t \geqslant 0$ is well-defined, continuous and

$$\begin{aligned} A_l(g_{\alpha_{n-1}-\alpha_l} * R_{i,j}(\cdot)v_0)(t) &= (g_{\alpha_{n-1}-\alpha_i+j} * k)(t) C A_i v_0 \\ &\quad - \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{l\}} A_l(g_{\alpha_{n-1}-\alpha_v} * R_{i,j}(\cdot)v_0)(t), \quad t \geqslant 0, \end{aligned}$$

proving that the function $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ is a mild solution of the integral equation (96). Suppose now that the conditions (C.2)–(C.3) hold, as well as $v_0 \in D(A_i)$ and $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$. Clearly, (C.2) implies (C.1) with any $l \in \mathcal{K}$. Similarly as in the proof of (i), the conditions (C.2)–(C.3) in combination with the equation (98), multiplied by $\lambda^{\alpha_{n-1}-\alpha_l}$, imply that there exists a sufficiently large number

$\omega_l' > \omega$ such that

$$A_l \int_0^\infty e^{-\lambda t} R_{i,j}(t) v_0 \, dt = \widetilde{h}_l(\lambda) C A_i v_0 - \int_0^\infty e^{-\lambda t} \sum_{v \in \mathbb{N}_{n-1} \setminus \mathcal{K}} g_{i,j,l,v}(t; v_0) \, dt,$$

for any $l \in \mathcal{K}$ and for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega_l'$. Then we can use the assertion (i) and the uniqueness theorem for Laplace transform to complete the proof of (ii). In order to prove (iii), observe first that the assumptions $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$ and $CA_p \subseteq A_p C$, $p \in \mathbb{N}_{n-1}$ imply

$$(99) \quad P_\lambda^{-1} C(\lambda^{\alpha_i - \alpha_{n-1}} A_i v_0) + \sum_{v \in \mathbb{N}_{n-1} \setminus \{i\}} P_\lambda^{-1} C(\lambda^{\alpha_v - \alpha_{n-1}} A_v v_0) = P_\lambda^{-1} C P_\lambda v_0 = C v_0,$$

provided $\operatorname{Re} \lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$. Making use of (95) and (99), we obtain that, for any such a value of complex parameter λ, the following holds:

$$\begin{aligned} A_l \int_0^\infty e^{-\lambda t} R_{i,j}(t) v_0 \, dt &= \lambda^{-j} \tilde{k}(\lambda) A_l P_\lambda^{-1} C(\lambda^{\alpha_i - \alpha_{n-1}} A_i v_0) \\ &= \lambda^{-j} \tilde{k}(\lambda) A_l \Bigg[C v_0 - \sum_{v \in \mathbb{N}_{n-1} \setminus \{i\}} \lambda^{\alpha_v - \alpha_{n-1}} P_\lambda^{-1} C A_v v_0 \Bigg]. \end{aligned}$$

Keeping in mind the last equation, as well as the condition (C.4) and the uniqueness theorem for Laplace transform, the proof of (iii) follows instantly. We will prove the uniqueness of solutions in (iv) only in the case that $v_0 \in D(A_i)$ and the assumptions of (i) hold. Let $t \mapsto u(t)$, $t \geqslant 0$ be a mild solution of the integral equation (96) with $v_0 = 0$. Convoluting the function $u(\cdot)$ with $g_\xi(\cdot)$, for a sufficiently large number $\xi > 0$, we may assume without of generality that, for every $v \in \mathbb{N}_{n-1}$, the mapping $t \mapsto A_v u(t)$, $t \geqslant 0$ is well-defined and continuous. Set, for every $t \geqslant 0$ and $\zeta > 0$, $v_{t,\zeta}(\lambda) := (g_\zeta * e^{\lambda \cdot})(t) - \lambda^{-\zeta} e^{t\lambda}$, $\lambda > 0$; $v_{t,0}(\lambda) := 0$ $(t \geqslant 0,\ \lambda > 0)$. Then the mapping $t \mapsto v_{t,\zeta}(\lambda)$ is continuous in $t \geqslant 0$, for any fixed numbers $\zeta \geqslant 0$ and $\lambda > 0$, and by [**541**, Lemma 1.5.5, p. 23], there exists $M \geqslant 1$ such that the mapping $\lambda \mapsto v_{t,\zeta}(\lambda)$, $\lambda > 0$ satisfies

$$(100) \quad |v_{t,\zeta}(\lambda)| \leqslant M[(1+t)^{\zeta-1} \lambda^{-1}(1 + \lambda^{1-\zeta}) + t^{\zeta-1} \lambda^{-1}], \quad \lambda > 0,\ t > 0,\ \zeta > 0.$$

Keeping in mind that $CA_p \subseteq A_p C$, $p \in \mathbb{N}_{n-1}$, we have that, for every $t \geqslant 0$ and $\lambda > 0$,

$$\begin{aligned} &\lambda^{\alpha_i - \alpha_{n-1}} \int_0^t e^{\lambda(t-s)} A_i C u(s) \, ds + \int_0^t v_{t-s, \alpha_{n-1} - \alpha_i}(\lambda) A_i C u(s) \, ds \\ &= C \int_0^t e^{\lambda(t-s)} (g_{\alpha_{n-1} - \alpha_i} * A_i u)(s) \, ds \\ &= (-C) \sum_{v \in \mathbb{N}_{n-1} \setminus \{i\}} \int_0^t e^{\lambda(t-s)} (g_{\alpha_{n-1} - \alpha_v} * A_v u)(s) \, ds \\ &= - \sum_{v \in \mathbb{N}_{n-1} \setminus \{i\}} \Bigg[\lambda^{\alpha_v - \alpha_{n-1}} \int_0^t e^{\lambda(t-s)} A_v C u(s) \, ds + \int_0^t v_{t-s, \alpha_{n-1} - \alpha_v}(\lambda) A_v C u(s) \, ds \Bigg], \end{aligned}$$

which clearly implies that, for every $\lambda > \omega$, $\sigma > 0$ and $t \geqslant 0$, the following holds:

$$\begin{aligned}(101)\quad &\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}\int_0^t e^{\lambda(t-s)}Cu(s)ds\\ &= -\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}P_\lambda^{-1}C\int_0^t v_{t-s,\alpha_{n-1}-\alpha_i}(\lambda)A_iu(s)ds\\ &- \lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}C\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)A_vu(s)ds.\end{aligned}$$

By (95) and (101), we obtain that, for every $\lambda > \omega$, $\sigma > 0$ and $t \geqslant 0$, the following holds:

$$\begin{aligned}&e^{-\lambda\sigma}\int_0^t e^{\lambda(t-s)}Cu(s)ds\\ &= -\frac{\lambda^{\alpha_{n-1}+j-\alpha_i}e^{-\lambda\sigma}}{\tilde{k}(\lambda)}\int_0^\infty e^{-\lambda s}R_{i,j}(s)\Big(\int_0^t v_{t-r,\alpha_{n-1}-\alpha_i}(\lambda)u(r)dr\Big)ds\\ &\qquad - e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}CA_v\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)u(s)ds.\end{aligned}$$

For the estimation of the first addend on the right hand side of the above equality, we can use the already employed fact that there exist numbers $\sigma_0 > 0$ and $M' \geqslant 1$ such that

$$(102)\qquad \frac{e^{-\lambda\sigma_0}}{|\tilde{k}(\lambda)|} \leqslant M', \quad \lambda > \omega + 1.$$

Keeping in mind (100) and (102), it can be simply proved that, for every $\sigma > \sigma_0$ and for every $p \in \circledast$, we have

$$(103)\quad \lim_{\lambda\to+\infty} p\Big(\frac{\lambda^{\alpha_{n-1}+j-\alpha_i}e^{-\lambda\sigma}}{\tilde{k}(\lambda)}\int_0^\infty e^{-\lambda s}R_{i,j}(s)\Big(\int_0^t v_{t-r,\alpha_{n-1}-\alpha_i}(\lambda)u(r)dr\Big)ds\Big) = 0.$$

If $n \geqslant 4$, then the condition (C.5) in combination with the previous equality and (100) shows that, for every $p \in \circledast$, there exists a sufficiently large number $\sigma_p > 0$ such that $\lim_{\lambda\to+\infty} e^{-\lambda\sigma_p}p((e^{\lambda\cdot} * Cu)(t)) = 0$, $t \geqslant 0$; the same holds in the case that $n = 3$ because then we can use, instead of condition (C.5), the equation (99) and the arguments already seen in proving the equation (103), to conclude that

$$\lim_{\lambda\to+\infty} p\Big(e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}CA_v\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)v(s)ds\Big) = 0,$$

for any $\sigma > \sigma_0$ and $t \geqslant 0$. In such a way, we obtain that for each $p \in \circledast$ the following holds: $\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-s-\sigma)}Cu(s)ds = 0$, $t \geqslant 0$, $\sigma > \sigma_p$. By the dominated convergence theorem, it readily follows that for each $p \in \circledast$ we have: $\lim_{\lambda\to+\infty} p(\int_0^{t-\sigma} e^{\lambda(t-s-\sigma)}Cu(s)ds) = 0$, $t \geqslant \sigma > \sigma_p$. Therefore,

$$\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-s)}Cu(s)ds = 0, \quad t \geqslant 0.$$

Since C is injective, we can apply [**292**, Lemma 2.1.33(iii)] (cf. [**445**, Lemma 1.4.4, p. 100] for the Banach space case) to complete the proof. $\square$

The uniqueness of solutions of integral equation (96), resp. (97), can be proved even in the case of non-existence of an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem [(90)–(91)]. Strictly speaking, the proof of Theorem 2.3.5 implies the following uniqueness type theorem for degenerate multi-term problems (cf. [**197**, Theorem 3.1] for a pioneering result on the uniqueness of solutions to abstract degenerate first-order equations):

THEOREM 2.3.6. *Suppose that $CA_p \subseteq A_pC$ for all $p \in \mathbb{N}_{n-1}$, $\mathcal{V} \subseteq \mathbb{N}_{n-1}$ and the requirements in* (C.5) *hold for every seminorm $p \in \circledast$ and for every number $l \in \mathbb{N}_{n-1}$. Then there exists at most one mild solution $t \mapsto u(t)$, $t \geqslant 0$ of the integral equation* (96) *with $v_0 = 0$, resp. there exists at most one continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ satisfying that the mapping $t \mapsto A_l u(t)$, $t \geqslant 0$ is well-defined, continuous ($l \in \mathcal{K}$) and* (97) *holds with $v_0 = 0$. In particular, there exists at most one $\mathcal{V}$-mild solution of problem* (93) *and there exists at most one strong solution of problem* [(90)–(91)].

REMARK 2.3.7. Suppose again that the general assumptions of Theorem 2.3.5 hold, i.e., that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ and there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t \geqslant 0}$ for problem [(90)–(91)].

- (i) Suppose that $k(t)$ satisfies (P2) and, for every $l \in \mathbb{N}_{n-1} \setminus \{i\}$, there exists $j_l \in \mathbb{N}^0_{m_l-1}$ such that there exists an exponentially equicontinuous k-regularized C-resovent (l, j_l)-propagation family for problem [(90)–(91)]. By the proof of Theorem 2.3.5(iv), we have that the condition (C.5) automatically holds.
- (ii) The uniqueness of solutions of non-degenerate integral equations has recently been considered in [**347**]. It ought to be observed that we must impose the additional condition $CA_p \subseteq A_pC$, $p \in \mathbb{N}^0_{n-1}$ in the formulation of Theorem 3.2 in [**347**] in order for its proof to work.
- (iii) Let $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$. Suppose that $v_0 \in D(A_i)$ and the conditions (C.2)–(C.3) hold, or $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$, $CA_p \subseteq A_pC$ for all $p \in \mathbb{N}_{n-1}$, and the conditions (C.1) and (C.4) hold. Let $u(t)$ be the solution of (97), satisfying the properties stated above. Consider now the equation (93) and the notion introduced in Definition 2.3.2 with indexes i, j replaced by i', j'. Then the following holds:
  - (a) If $i \in \mathbb{N}_{n-1} \setminus \mathcal{I}$, $k(t) = 1$, $u_{j'} = 0$ $(0 \leqslant j' \leqslant m_{i'} - 1)$, $u_{i',j'} = CA_iv_0$, provided $i' = i$ and $j' = j$, and $u_{i',j'} = 0$, otherwise, then $u(t)$ is a $(\mathbb{N}_{n-1} \setminus \mathcal{K})$-mild solution of (93) with $f(t) = 0$.
  - (b) If $i \in \mathcal{I}$, $k(t) = 1$, $u_{i',j'} = 0$ $(i' \in \mathbb{N}_{n-1}$, $j' \in \mathbb{N}^0_{m_{i'}-1})$, $u_{j'} = Cv_0$, provided $j' = j$, $u_{j'} = 0$, otherwise, and $CA_i \subseteq A_iC$, then $u(t)$ is a $(\mathbb{N}_{n-1} \setminus \mathcal{K})$-mild solution of (93) with $f(t) = 0$, provided that for each $i' \in \{s \in \mathcal{I} \setminus \{i\} : m_s - 1 \geqslant j\}$ one has $A_{i'}Cv_0 = 0$.

(iv) Making use of [**541**, Theorem 1.1.9], the uniqueness theorem for Laplace transform and the formula (39), we can clarify some sufficient conditions for the existence of terms $A_p\mathbf{D}_t^{\alpha_p}u(t)$ and $\mathbf{D}_t^{\alpha_p}A_pu(t)$ ($p \in \mathbb{N}_{n-1}$). Unfortunately, it is very hard to verify these conditions in practical situations because we do not know the precise values of elements $R_{i,j}(0)x$, $R'_{i,j}(0)x, \dots$ ($x \in D(A_i)$).

The notion of an exponentially equicontinuous (equicontinuous), analytic k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem [(90)–(91)] is introduced in the following definition.

Definition 2.3.8. Suppose that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $0 < \alpha \leqslant \pi$ and there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem [(90)–(91)]. Suppose, further, that the function $k(t)$ satisfies (P1), as well as $C \in L(E)$ is an injective mapping. Then it is said that $(R_{i,j}(t))_{t\geqslant 0}$ is an exponentially equicontinuous (equicontinuous), analytic k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem [(90)–(91)], of angle α, iff the following holds:

(i) For every $x \in D(A_i)$, the mapping $t \mapsto R_{i,j}(t)x$, $t > 0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we denote the extension by the same symbol.

(ii) For every $x \in D(A_i)$ and $\beta \in (0,\alpha)$, one has $\lim_{z\to 0, z\in\Sigma_\beta} R_{i,j}(z)x = R_{i,j}(0)x$.

(iii) For every $\beta \in (0,\alpha)$, there exists $\omega_\beta \geqslant \max(0, \operatorname{abs}(k))$ ($\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z}R_{i,j}(z) : z \in \Sigma_\beta\}$ is equicontinuous, i.e., for every $p \in ⊛$, there exist $c > 0$ and $q \in ⊛$ such that

$$p(e^{-\omega_\beta z}R_{i,j}(z)x) \leqslant cq(x), \quad x \in D(A_i),\ z \in \Sigma_\beta.$$

The proof of following theorem can be deduced similarly as that of [**296**, Theorem 3.7].

Theorem 2.3.9. *Assume that the function $k(t)$ satisfies* (P1), $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $\omega \geqslant \max(0, \operatorname{abs}(k))$, $\alpha \in (0, \pi/2]$ *and the operator* $C \in L(E)$ *is injective. Assume, further, that for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$, *we have that the operator* P_λ *is injective and* $C(R(A_i)) \subseteq R(P_\lambda)$. *Let for each* $x \in D(A_i)$ *there is an analytic function* $q_x : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to E$ *such that*

$$q_x(\lambda) = \lambda^{\alpha_i - \alpha_{n-1} - j}\tilde{k}(\lambda)P_\lambda^{-1}CA_ix, \quad \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0.$$

Suppose that, for every $\beta \in (0,\alpha)$ *and* $p \in ⊛$, *there exist* $c_{p,\beta} > 0$ *and* $r_{p,\beta} \in ⊛$ *such that* $p((\lambda - \omega)q_x(\lambda)) \leqslant c_{p,\beta}r_{p,\beta}(x)$, $x \in D(A_i)$, $\lambda \in \omega + \Sigma_{\beta+(\pi/2)}$ *and, for every* $x \in D(A_i)$, *there exists the limit* $\lim_{\lambda\to+\infty}\lambda q_x(\lambda)$ *in* E. *Then there exists an exponentially equicontinuous* k*-regularized* C*-resolvent* (i,j)*-propagation family* $(R_{i,j}(t))_{t\geqslant 0}$ *for problem* [(90)–(91)], *of angle* α, *and for each* $\beta \in (0,\alpha)$ *the family* $\{e^{-\omega z}R_{i,j}(z) : z \in \Sigma_\beta\}$ *is equicontinuous.*

Differential properties of exponentially equicontinuous (analytic) k-regularized C-resolvent (i,j)-propagation families in locally convex spaces can be clarified following the method employed in Section 2.2. In the following theorem, we state the subordination principle for exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation families.

THEOREM 2.3.10. *Suppose that* $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $0 < \gamma < 1$, $0 \leqslant j' \leqslant m_i - 1$, *and there exists an exponentially equicontinuous* k*-regularized* C*-resolvent* (i,j)*-propagation family* $(R_{i,j}(t))_{t \geqslant 0}$ *for problem* [(90)–(91)] *satisfying that the family* $\{e^{-\omega t} R_{i,j}(t) : t \geqslant 0\}$ *is equicontinuous for some* $\omega \geqslant \max(0, \operatorname{abs}(k))$. *Assume that* $k(t)$ *satisfies* (P1) *and there exist a scalar-valued continuous kernel* $k_\gamma(t)$ *on* $[0,\infty)$, *satisfying* (P1), *and a positive real number* $\eta > 0$ *such that*

$$\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma - 1 + j' - \gamma j} \tilde{k}(\lambda^\gamma), \quad \lambda > \eta.$$

Then there exists an exponentially equicontinuous k*-regularized* C*-resolvent* (i,j')*-propagation family* $(R_{i,j',\gamma}(t))_{t \geqslant 0}$ *for problem* [(90)–(91)], *with* α_i *replaced by* $\gamma\alpha_i$ $(i \in \mathbb{N}_{n-1})$, *and* $(R_{i,j',\gamma}(t))_{t \geqslant 0}$ *is given by* $R_{i,j',\gamma}(0) := R_{i,j}(0)$,

$$R_{i,j',\gamma}(t)x := \int_0^\infty t^{-\gamma} \Phi_\gamma(st^{-\gamma}) R_{i,j}(s)x\,ds, \quad x \in D(A_i),\ t > 0.$$

Furthermore, the family $\{e^{-\omega^{1/\gamma} t} R_{i,j',\gamma}(t) : t \geqslant 0\}$ *is equicontinuous and, for every* $\zeta \geqslant 0$, *the equicontinuity of the family* $\{e^{-\omega t}(1 + t^\zeta)^{-1} R_{i,j}(t) : t \geqslant 0\}$, *resp.* $\{e^{-\omega t} t^{-\zeta} R_{i,j}(t) : t > 0\}$, *implies the equicontinuity of the family* $\{e^{-\omega^{1/\gamma} t}(1 + t^{\gamma\zeta})^{-1}(1 + \omega t^{\zeta(1-\gamma)})^{-1} R_{i,j',\gamma}(t) : t \geqslant 0\}$, *resp.* $\{e^{-\omega^{1/\gamma} t} t^{-\gamma\zeta}(1 + \omega t^{\zeta(1-\gamma)})^{-1} R_{i,j',\gamma}(t) : t > 0\}$, *and the following holds:*

- (i) *The mapping* $t \mapsto R_{i,j',\gamma}(t)x$, $t > 0$ *admits an analytic extension to the sector* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *for all* $x \in D(A_i)$.
- (ii) *If* $\omega = 0$ *and* $\varepsilon \in (0, \min((\frac{1}{\gamma} - 1)\frac{\pi}{2}, \pi))$, *then the family* $\{R_{i,j',\gamma}(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}\}$ *is equicontinuous and* $\lim_{z \to 0, z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}} R_{i,j',\gamma}(z)x = R_{i,j',\gamma}(0)x$ *for all* $x \in D(A_i)$.
- (iii) *If* $\omega > 0$ *and* $\varepsilon \in (0, \min((\frac{1}{\gamma} - 1)\frac{\pi}{2}, \frac{\pi}{2}))$, *then there exists* $\delta_{\gamma,\varepsilon} > 0$ *such that the family* $\{e^{-\delta_{\gamma,\varepsilon} \operatorname{Re} z} R_{i,j',\gamma}(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}\}$ *is equicontinuous. Moreover,* $\lim_{z \to 0, z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}} R_{i,j',\gamma}(z)x = R_{i,j',\gamma}x$ *for all* $x \in D(A_i)$.

REMARK 2.3.11. Using the proof of [**61**, Theorem 3.1], it can be simply verified that any of conditions (C.1)–(C.5) is invariant under the action of subordination principle described in Theorem 2.3.10.

EXAMPLE 2.3.12. (cf. also [**346**, Example 5.1(i)]) Suppose that $c_l \in \mathbb{C} \smallsetminus 0$ $(1 \leqslant l \leqslant n-1)$, as well as A and B are closed linear operators on E, and $A_l = c_l B$ for $1 \leqslant l \leqslant n-1$. We consider the following degenerate multi-term problem:

$$\mathbf{D}_t^{\alpha_n} Bu(t) + \sum_{l=1}^{n-1} c_l \mathbf{D}_t^{\alpha_l} Bu(t) = \mathbf{D}_t^{\alpha} Au(t), \quad t \geqslant 0, \tag{104}$$

equipped with the initial conditions of the form (91). Here $0 \leqslant \alpha_1 < \cdots < \alpha_n$, $0 \leqslant \alpha < \alpha_n$, and

$$P_\lambda = \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha_n} B - \lambda^{\alpha - \alpha_n} A + B, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

(i) (a) Suppose $0 < \delta \leqslant 2$, $\sigma \geqslant 1$, $\frac{\pi\delta}{2(\alpha_n - \alpha)} - \frac{\pi}{2} > 0$, $0 \leqslant j \leqslant \lceil \alpha_n \rceil - 1$, and there exists an exponentially equicontinuous $(\sigma - 1)$-times integrated C-resolvent propagation family $(R(t))_{t \geqslant 0}$ for problem (52), with $a(t) = g_\delta(t)$. Put $\sigma' := \max(1, 1 - j + (\alpha_n - \alpha)(\sigma - 1)\delta^{-1})$ and $\theta := \min(\frac{\pi}{2}, \frac{\pi\delta}{2(\alpha_n - \alpha)} - \frac{\pi}{2})$. Then, for every sufficiently small number $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that $C(R(B)) \subseteq R(P_\lambda)$ for all $\lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon}$ and the family $\{|\lambda|^{\frac{\delta - \sigma}{\delta}}(1 + |\lambda|^{\frac{1}{\delta}})(\lambda B - A)^{-1} CBx : \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon},\ x \in D(B)\}$ is equicontinuous. Noticing also that

$$\arg\left(\lambda^{\alpha_n - \alpha} + \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha}\right) \approx (\alpha_n - \alpha)\arg(\lambda), \quad \lambda \to \infty,\ \arg(\lambda) < \frac{\pi}{\alpha_n - \alpha},$$

our choice of θ implies that, for every sufficiently small number $\varepsilon > 0$, there exists $\omega'_\varepsilon > 0$ such that, for every $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$, one has:

$$\lambda^{\alpha_n - \alpha} + \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha} \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon}.$$

Put now, for every $x \in D(B)$ and $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$,

$$q_x(\lambda) := \lambda^{-j - \sigma'} P_\lambda^{-1} CBx.$$

Then $q_x \colon \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon} \to E$ is an analytic function and, for every $\beta \in (0, \theta)$ and $p \in \circledast$, there exist $c_{p,\beta} > 0$ and $r_{p,\beta} \in \circledast$ such that $p((\lambda - \omega'_\varepsilon) q_x(\lambda)) \leqslant c_{p,\beta} r_{p,\beta}(x)$, $x \in D(B)$, $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$. By the proof of [**27**, Proposition 4.1.3, p. 248], we have that $\lim_{\operatorname{Re}\lambda \to +\infty} \lambda^{\delta - \sigma + 1}(\lambda^\delta B - A)^{-1} CBx = R_{i,j}(0)x$, $x \in D(B)$, which simply implies that, for every $x \in D(B)$, there exists the limit $\lim_{\lambda \to +\infty} \lambda q_x(\lambda)$ in E. Therefore, Theorem 2.3.9 implies that there exists an exponentially equicontinuous, analytic $(\sigma' - 1)$-times integrated C-resolvent (n, j)-propagation family $(R_{n,j}(t))_{t \geqslant 0}$ for problem (104), of angle θ (with the clear meaning).

(b) Suppose $0 < \delta \leqslant 2$, $\sigma \geqslant 1$, $0 \leqslant j \leqslant \lceil \alpha_n \rceil - 1$, $\gamma \in (0, \frac{\pi}{2}]$ and $\frac{\delta(\frac{\pi}{2} + \gamma)}{(\alpha_n - \alpha)} - \frac{\pi}{2} > 0$. Put $\sigma_1 := \sigma'$ and $\theta_1 := \min(\frac{\pi}{2}, \frac{\delta(\frac{\pi}{2} + \gamma)}{(\alpha_n - \alpha)} - \frac{\pi}{2})$. Arguing similarly as in (a), one can prove the following: Suppose that for each $\varepsilon \in (0, \frac{\pi}{2} + \gamma)$ there exists $\omega_\varepsilon > 0$ such that for each $x \in D(B)$ there exists an analytic function $q_x \colon \omega_\varepsilon + \Sigma_{\frac{\pi}{2} + \gamma - \varepsilon} \to E$ satisfying that

$$q_x(\lambda) = \lambda^{\delta - \sigma}(\lambda^\delta B - A)^{-1} CBx, \quad \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2} + \gamma - \varepsilon},\ x \in D(B),$$

and for each $p \in ⊛$ there exist $c_p > 0$ and $q_p \in ⊛$ so that

$$p(q_x(\lambda)) \leqslant c_p \frac{q_p(x)}{1+|\lambda|}, \quad \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}+\gamma-\varepsilon}, \ x \in D(B).$$

Then the existence of limit $\lim_{\mathrm{Re}\,\lambda\to+\infty} \lambda^{\delta-\sigma+1}(\lambda^\delta B - A)^{-1}CBx$ in E, for all $x \in D(B)$, implies that there exists an exponentially equicontinuous, analytic $(\sigma_1 - 1)$-times integrated C-resolvent (n,j)-propagation family $(R_{n,j}(t))_{t\geqslant 0}$ for problem (104), of angle θ_1; if there is an element $x \in D(B)$ such that the limit $\lim_{\mathrm{Re}\,\lambda\to+\infty} \lambda^{\delta-\sigma+1}$ $(\lambda^\delta B - A)^{-1}CBx$ does not exist in E, then the above holds with any number $\sigma_2 > \sigma_1$. For the purpose of illustration of obtained results, assume now that $n \in \mathbb{N}$ and $iA'_l, 1 \leqslant l \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E. Put $A' := (A'_1, \ldots, A'_n)$; cf. the previous section for the definition of a closable operator $P(A')$, where $P(x)$ is a complex polynomial in n variables. Suppose $0 < \delta < 2$, $\omega \geqslant 0$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\delta)}$ (resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\delta)}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and (70) holds with the number α replaced by δ therein. Set

$$R_\delta(t) := \Bigl(E_\delta\Bigl(t^\delta \frac{P_1(x)}{P_2(x)}\Bigr)(1+|x|^2)^{-\beta/2}\Bigr)(A'), \quad t \geqslant 0. \tag{105}$$

By Theorem 2.2.21, we know that $(R_\delta(t))_{t\geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_\delta, R_\delta(0))$-regularized resolvent family for the problem (52) with $B = \overline{P_2(A')}$, $A = \overline{P_1(A')}$ and $a(t) = g_\delta(t)$. By the conclusion stated in (a), it readily follows that there exists an exponentially equicontinuous, analytic C-resolvent (n,j)-propagation family $(R_{n,j}(t))_{t\geqslant 0}$ for problem (104), of angle $\theta = \min(\frac{\pi}{2}, \frac{\pi\delta}{2(\alpha_n-\alpha)} - \frac{\pi}{2})$. Since the condition (ii.1) given in the formulation of Theorem 2.2.8 holds, with $a(t) = g_\delta(t)$ and $k(t) = 1$, it is not problematic to show, with the help of our previous consideration and the results concerning the Laplace transform of vector-valued analytic functions (see e.g. Theorem 1.4.10 and [**296**, Section 3]) that the conditions (C.1) and (C.5) hold for (104), as well as that the condition (C.4) holds for (104) provided that $\alpha_{n-1} \leqslant \alpha$; we need the last condition because the inclusion $\lambda^{\alpha_{n-1}-\alpha_n-1}AP_\lambda^{-1}Cx \in LT-E$ has to be satisfied ($x \in E$). It is also worth noting that we do not need the condition $\alpha_{n-1} \leqslant \alpha$ for the existence of solutions of the integral equation

$$Bu(t) + \sum_{l=1}^{n-1} c_l(g_{\alpha_n-\alpha_l} * Bu)(t) = A(g_{\alpha_n-\alpha} * u)(t) + CBv_0, \quad t \geqslant 0;$$

cf. (97). It is also worth noting that we can refine the results on C-wellposedness of equation (104) by using the estimates quoted in

Remark 2.1.23(ii) and that we can similarly consider the equation (104) in E_l-type spaces.

(ii) (cf. [**346**, Example 5.1(i)–(b)] and Example 2.2.18 for more details) Let $s > 1$, $0 \leqslant j \leqslant \lceil \alpha_n \rceil - 1$, $k_{a,b}(t) := \mathcal{L}^{-1}(\exp(-a\lambda^b))(t)$, $t \geqslant 0$ ($a > 0$, $b \in (0,1)$),

$$E := \Bigl\{ f \in C^\infty[0,1] \ ; \ \|f\| := \sup_{p \geqslant 0} \frac{\|f^{(p)}\|_\infty}{p!^s} < \infty \Bigr\}$$

and

$$A' := -d/ds, \ D(A') := \{ f \in E \ ; \ f' \in E, \ f(0) = 0 \}.$$

Let $P_1(z) = \sum_{l=0}^{N_1} a_{l,1} z^l$, $z \in \mathbb{C}$, $a_{N_1,1} \neq 0$ be a complex non-zero polynomial, and let $P_2(z) = \sum_{l=0}^{N_2} a_{l,2} z^l$, $z \in \mathbb{C}$, $a_{N_2,2} \neq 0$ be a complex non-zero polynomial so that $N_1 = dg(P_1) \geqslant 1 + dg(P_2) = 1 + N_2$. Set $A := P_1(A')$ and $B := P_2(A')$. We have proved so far that there exist numbers $b > 0$ and $c > 0$ such that

$$\|(\lambda B - A)^{-1}\| = O(e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}), \quad \lambda \in \mathbb{C}, \tag{106}$$

and, for every complex non-zero polynomial $P(z)$ with $dg(P) \leqslant N_1$, there exists $\zeta > 0$ such that

$$\|(\lambda B - A)^{-1} P(A') f\| \leqslant \zeta \|f\| e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}, \tag{107}$$

for all $\lambda \in \mathbb{C}$ and $f \in D(P(A'))$. Let $\theta \in (0, \pi/2]$, $b' = (\alpha_n - \alpha)/(N_1 - N_2)$ and let $b' \leqslant \pi/(\pi + 2\theta)$. Owing to (106)–(107) and Theorem 2.3.9, we obtain that there is a sufficiently large number $a' > 0$ such that there exists an exponentially equicontinuous, analytic $k_{a',b'}$-regularized I-resolvent (n,j)-propagation family $(R_{n,j}(t))_{t \geqslant 0}$ for problem (104), of angle θ, satisfying the conditions (C.1)–(C.5). Denote, as before, $T_{l,L}u(t) = B\mathbf{D}_t^{\alpha_l}u(t)$, $t \geqslant 0$ if $l \in \mathbb{N}_n$ and $\alpha_l > 0$, $T_{l,R}u(t) = \mathbf{D}_t^{\alpha_l}Bu(t)$, $t \geqslant 0$ if $l \in \mathbb{N}_n$, $T_{0,L}u(t) = A\mathbf{D}_t^{\alpha}u(t)$, $t \geqslant 0$ if $\alpha > 0$, and $T_{0,R}u(t) = \mathbf{D}_t^{\alpha}Au(t)$, $t \geqslant 0$. Let $T_l u(t)$ be either $T_{l,L}u(t)$ or $T_{l,R}u(t)$ ($l \in \mathbb{N}_n^0$). Then it can be easily seen that for each $x \in D(B)$ the function $u(t) = R_{n,j}(t)x$, $t \geqslant 0$ is a unique strong solution of problem

$$T_n u(t) + \sum_{l=1}^{n-1} c_l T_l u(t) = T_0 u(t) + \bigl(k_{a',b'}^{(m_n)} * g_{j+m_n-\alpha_n}\bigr)(t), \quad t \geqslant 0,$$

with all initial values chosen to be zeroes. Observe, finally, that the analysis contained in [**346**, Example 5.4] can be used for the construction of hypoanalytic exponentially equicontinuous k-regularized I-resolvent (n,j)-propagation families for the problem

$$\sum_{l \in \mathcal{K}} P_l(A')\mathbf{D}_t^{\alpha_l}u(t) + \sum_{l \in \mathbb{N}_{n-1} \smallsetminus \mathcal{K}} \mathbf{D}_t^{\alpha_l} P_l(A')u(t) = 0, \quad t \geqslant 0,$$

where $P_1(z), \ldots, P_{n-1}(z)$ are complex non-zero polynomials satisfying certain properties, and $\mathcal{K} \subseteq \mathbb{N}_{n-1}$.

2.3.3. Exponentially equicontinuous (a,k)-regularized C-resolvent families generated by A, B; exponentially equicontinuous $(k;C)$-regularized resolvent (i,j)-propagation families. In this subsection, our main task will be to investigate the C-wellposedness of problem $(\mathrm{DFP})_L$ with A and B being closed linear operators on E. Following the examination from [**6**, Section 2], we introduce the following definition.

Definition 2.3.13. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as $R(t) \in L(E,[D(B)])$ for all $t \geqslant 0$. Let $C \in L(E)$ be injective, and let $CA \subseteq AC$ and $CB \subseteq BC$. Then the operator family $(R(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B iff there exists $\omega \geqslant \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$ such that the following holds:

(i) The mappings $t \mapsto R(t)x$, $t \geqslant 0$ and $t \mapsto BR(t)x$, $t \geqslant 0$ are continuous for every fixed element $x \in E$.

(ii) The family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(E,[D(B)])$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that
$$p(e^{-\omega t}R(t)x) + p(e^{-\omega t}BR(t)x) \leqslant cq(x), \quad x \in E,\ t \geqslant 0.$$

(iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $R(C) \subseteq R(B - \tilde{a}(\lambda)A)$ and
$$\tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in E.$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous r-times integrated (a,C)-regularized resolvent family generated by A, B; an exponentially equicontinuous 0-times integrated (a,C)-regularized resolvent family generated by A, B is also said to be an exponentially equicontinuous (a,C)-regularized resolvent family generated by A, B.

Before going any further, it should be noticed that we have already constructed some examples of (g_α,k)-regularized C-resolvent families generated by A, B in Example 2.3.12(ii).

Remark 2.3.14. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $CA \subseteq AC$ and $CB \subseteq BC$.

(i) It is clear that an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B, if exists, must be unique.

(ii) If for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$ the operator B commutes with $(B - \tilde{a}(\lambda)A)^{-1}C$, then the operator family $(BR(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family for (52), and the condition (ii.1) stated in the formulation of Theorem 2.2.8 holds. Furthermore, for each $t \geqslant 0$ the operator $BR(t)$ can be continuously extended from $D(B)$ to the whole space E.

(iii) Assume $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family for (52) and there exists a strongly continuous operator family $(\hat{R}(t))_{t\geqslant 0} \subseteq L(E)$ such that $\hat{R}(t)x = R(t)x$, $t \geqslant 0$, $x \in D(B)$ (the last condition automatically holds provided that E is complete and B is densely defined). If $B^{-1} \in L(E)$ and $BR(t) \subseteq R(t)B$, $t \geqslant 0$, then $(R(t)B^{-1})_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B.

The proof of following theorem can be deduced by using slight modifications of the proofs of [**6**, Proposition 2.1, Lemma 2.2] and the fact that the assertion of [**295**, Lemma 2.4] continues to hold in SCLCSs.

THEOREM 2.3.15. *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B, and let* $\operatorname{abs}(|a|) < \infty$*. Then the following holds:*

(i) *For every $x \in E$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have:*
$$\tilde{k}(\lambda)B(B-\tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}BR(t)x\,dt.$$
(ii) $R(t)Bx = k(t)Cx + \int_0^t a(t-s)R(s)Ax\,ds$*, $t \geqslant 0$, $x \in D(A)\cap D(B)$.*
(iii) $\int_0^t a(t-s)R(s)x\,ds \in D(A)\cap D(B)$*, $t \geqslant 0$, $x \in E$.*
(iv) $BR(t)x = k(t)Cx + A\int_0^t a(t-s)R(s)x\,ds$*, $t \geqslant 0$, $x \in E$.*
(v) $R(t)B(D(A)\cap D(B)) \subseteq D(A)\cap D(B)$*, $t \geqslant 0$.*
(vi) $B(B-\tilde{a}(\lambda)A)^{-1}CAx = A(B-\tilde{a}(\lambda)A)^{-1}CBx$ *for every $x \in D(A)\cap D(B)$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda)\neq 0$; $AR(t)Bx = BR(t)Ax$, $t \geqslant 0$, $x \in D(A)\cap D(B)$.*
(vii) *Suppose that the function $k(t)$ is differentiable in a point $t_0 \geqslant 0$ and $a \in AC_{loc}([0,\infty))$. If $\lambda \in \mathbb{C}$ satisfies $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, then for every $j \in \mathbb{N}_0$, $z \in \mathbb{C}$ and for every complex polynomial $P(\cdot)$, we have:*

$$\begin{aligned}(108)\quad &\Big(\frac{d}{dt}[(z(B-\tilde{a}(\lambda)A)^{-1}C - P(C))^j R(t)Bx]\Big)_{t=t_0}\\ &\qquad = (z(B-\tilde{a}(\lambda)A)^{-1}C - P(C))^j\Big(\frac{d}{dt}R(t)Bx\Big)_{t=t_0}.\end{aligned}$$

(viii) *Let $x \in D(A)\cap D(B)$. Then the function $t \mapsto u(t)$, $t \geqslant 0$, defined by $u(t) := R(t)Bx$, $t \geqslant 0$ satisfies $u \in C([0,\infty):[D(A)])\cap C([0,\infty):[D(B)])$ and*
$$Bu(t) = k(t)CBx + \int_0^t a(t-s)Au(s)ds, \quad t \geqslant 0.$$

REMARK 2.3.16. (i) Suppose that $x \in D(A)\cap D(B)$, $\alpha > 0$ and there exists an exponentially equicontinuous (g_α, C)-regularized resolvent family $(R(t))_{t\geqslant 0}$ generated by A, B. Using the identity $R(t)Bx = Cx + \int_0^t g_\alpha(t-s)R(s)Ax\,ds$, $t \geqslant 0$, it readily follows that the mapping $t \mapsto R(t)Bx$, $t \geqslant 0$

is $(m-1)$-times continuously differentiable on $[0,\infty)$, where $m = \lceil \alpha \rceil$. Furthermore, it can be easily verified that the Caputo derivative $\mathbf{D}_t^\alpha R(t)Bx$ is well defined as well as that $\mathbf{D}_t^\alpha R(t)Bx = R(t)Ax$, $t \geqslant 0$. Keeping in mind Remark 2.3.7(ii) and Proposition 2.3.15(vi), we get that the function $u(t) := R(t)Bx$, $t \geqslant 0$ is a unique solution of the following Cauchy problem:

$$\begin{cases} u \in C([0,\infty) : [D(A)]) \cap C([0,\infty) : [D(B)]) \cap C^{m-1}([0,\infty) : E), \\ B\mathbf{D}_t^\alpha u(t) = Au(t), \quad t \geqslant 0, \\ u(0) = Cx; \ u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant m-1. \end{cases}$$

In Theorem 2.3.18, we shall extend this result to the class of exponentially equicontinuous $(g_\alpha, g_{\alpha l+1})$-regularized C-resolvent families generated by A, B ($l \in \mathbb{N}$).

(ii) Now we would like to illustrate the conclusion deduced in the first part of this remark to degenerate fractional equations associated with the abstract differential operators. For the sake of simplicity, we shall only consider the equations of order $\alpha \in (0,2)$; the case $\alpha = 2$ has been considered in [**306**, Subsection 4.1] and here we only want to point out that the results from Subsection 2.1.4 and this remark can be also applied in the analysis of equation

$$\frac{\partial^2}{\partial t^2}(u_{zz} - \beta^2 u) + \omega_0^2 u_{yy} = 0 \quad (\beta > 0, \ \omega_0 > 0 \text{ is the Väisälä–Brunt frequency}),$$

which is important in the linear theory of internal waves in stratified fluid ([**483**]), and the Boussinesq equation (see e.g. [**140**, Example 1.5, p. 50])

$$(\sigma^2\Delta - 1)u_{tt} + \gamma^2 \Delta u = 0 \quad (\sigma > 0, \ \gamma > 0).$$

Assume that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E. Suppose again that $0 < \alpha < 2$, $\omega \geqslant 0$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ (resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\alpha)}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and (70) holds with δ replaced by α. Define $(R_\alpha(t))_{t\geqslant 0}$ as in (105), with δ replaced by α; $C \equiv R_\alpha(0)$. Then we know that $(R_\alpha(t))_{t\geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family for the problem

$$(P)_R : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t), \quad t \geqslant 0, \\ u(0) = Cx; \ u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

Furthermore, the analysis contained in Remark 2.1.23(i) implies that there exists an exponentially bounded, strongly continuous operator family $(G_\alpha(t))_{t\geqslant 0}$ such that $G_\alpha(t)x = \overline{P_2(A)}^{-1} R_\alpha(t)x$, $t \geqslant 0$, $x \in E$ and $\lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx = \int_0^\infty e^{-\lambda t} G_\alpha(t)x\,dt$ for any $x \in E$ and $\lambda > 0$ sufficiently large. Hence, $(G_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous

(g_α, C)-regularized resolvent family generated by $\overline{P_1(A)}$, $\overline{P_2(A)}$, as announced before. The consideration is quite similar in the case that the requirements of Theorem 2.2.21 hold.

We employ the following auxiliary lemma in the proof of Theorem 2.3.18 mentioned above.

LEMMA 2.3.17. (cf. [**292**, Corollary 2.1.20]) *Suppose $\alpha > 0$, $l \in \mathbb{N}$, $z \in \mathbb{C}$, A is a subgenerator of an exponentially equicontinuous $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family $(S_{l,\alpha}(t))_{t\geqslant 0}$ on E, $z - A$ is injective, $R(C) \subseteq R((z - A)^l)$ and $(z - A)^{-1}C \in L(E)$, ..., $(z - A)^{-l}C \in L(E)$. Set, for every $x \in E$ and $t \geqslant 0$,*

$$S_\alpha(t)x := (-1)^l S_{l,\alpha}(t)x + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j} z^{l-j}\Big[\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha - z)^l}\Big) * S_{l,\alpha}(\cdot)x\Big](t)$$
$$+ \sum_{j=1}^{l}(-1)^{l-j}\mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}}\Big)(t)(z - A)^{-j}Cx.$$

Then $(S_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, (z - A)^{-l}C)$-regularized resolvent family with a subgenerator A.

Now we state the following important extension of [**6**, Theorem 2.2].

THEOREM 2.3.18. *Suppose that $\alpha > 0$, $l \in \mathbb{N}$, $z \in \mathbb{C}$, there exists an exponentially equicontinuous $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family $(S_{l,\alpha}(t))_{t\geqslant 0}$ generated by A, B, the operator $zB - A$ is injective and $x \in D(A) \cap D(B) \cap D(((zB - A)^{-1}B)^l C)$. Define*

$$u(t) := (-1)^l S_{l,\alpha}(t)Bx + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j} z^{l-j}\Big[\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha - z)^l}\Big) * S_{l,\alpha}(\cdot)Bx\Big](t)$$
$$+ \sum_{j=1}^{l}(-1)^{l-j}\mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}}\Big)(t)((zB - A)^{-1}B)^j Cx, \quad t \geqslant 0.$$

Then the function $u(t)$ is a unique solution of the problem (DFP)$_L$ *with $f(t) \equiv 0$ and the initial value x replaced by $((zB-A)^{-1}B)^l Cx$ (we will designate this problem by* (DFP)$_{L,l}$ *in the sequel).*

PROOF. The uniqueness of solutions follows similarly as in Remark 2.3.16(i) and we will only prove that the function $u(t)$ is a solution of the problem (DFP)$_{L,l}$. Denote $x_j := ((zB - A)^{-1}B)^j Cx$ ($j \in \mathbb{N}_l^0$), $F_{j,l}(t) := \mathcal{L}^{-1}(\frac{r^{\alpha j}}{(r^\alpha - z)^l})(t)$, $t > 0$ ($0 \leqslant j \leqslant l - 1$) and $G_{j,l}(t) := \mathcal{L}^{-1}(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}})(t)$, $t \geqslant 0$ ($1 \leqslant j \leqslant l$). Then the function $F_{j,l}(t)$ is continuous on $(0, \infty)$, locally integrable on $[0, \infty)$ and exponentially bounded on $[1, \infty)$ ($0 \leqslant j \leqslant l - 1$), while the function $G_{j,l}(t)$ is continuous and exponentially bounded on $[0, \infty)$ ($1 \leqslant j \leqslant l$); cf. [**292**]. Set $m := \lceil \alpha \rceil$. By Theorem 2.3.15(ii), we have that the mapping $t \mapsto S_{l,\alpha}(t)Bx$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable and

$$\frac{d^{m-1}}{dt^{m-1}}S_{l,\alpha}(t)Bx = g_{\alpha l+2-m}(t)Cx + \int_0^t g_{\alpha+1-m}(t-s)S_{l,\alpha}(s)Ax\,ds, \quad t \geqslant 0;$$

hence $(\frac{d^j}{dt^j}S_{l,\alpha}(t)Bx)_{t=0} = 0$, $0 \leqslant j \leqslant m-1$. This simply implies that the mapping $t \mapsto [F_{j,l} * S_{l,\alpha}(\cdot)Bx](t)$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable as well as

$$\frac{d^{m-1}}{dt^{m-1}}[F_{j,l} * S_{l,\alpha}(\cdot)Bx](t) = \Big[F_{j,l} * \frac{d^{m-1}}{dt^{m-1}}S_{l,\alpha}(\cdot)Bx\Big](t), \quad t \geqslant 0,$$

provided $0 \leqslant j \leqslant l-1$; hence, $(\frac{d^p}{dt^p}[F_{j,l} * S_{l,\alpha}(\cdot)Bx](t))_{t=0} = 0$, $0 \leqslant p \leqslant m-1$ $(0 \leqslant j \leqslant l-1)$. Now it is very simple to show that

$$\mathbf{D}_t^\alpha S_{l,\alpha}(t)Bx = g_{l\alpha+1-\alpha}(t)Cx + S_{l,\alpha}(t)Ax, \quad t \geqslant 0,$$

and

$$\mathbf{D}_t^\alpha [F_{j,l} * S_{l,\alpha}(\cdot)Bx](t) = g_{l\alpha+1-\alpha}(t)Cx + S_{l,\alpha}(t)Ax, \quad t \geqslant 0 \ (0 \leqslant j \leqslant l-1).$$

Suppose, for the time being, that the assumptions of Lemma 2.3.17 hold. Since for each $x \in D(A)$ the function $v(t) := S_\alpha(t)x$, $t \geqslant 0$ is a unique solution of the problem

$$\begin{cases} \mathbf{D}_t^\alpha v(t) = Av(t), \quad t \geqslant 0, \\ v(0) = Cx; \ v^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant m-1, \end{cases}$$

we may conclude from the above (by plugging $l = 1, 2, \ldots$ successively in Lemma 2.3.17) that for each $j \in \mathbb{N}_l$ the function $G_{j,l}(t)$ is $(m-1)$-times continuously differentiable on $[0,\infty)$ as well as that $(\frac{d^p}{dt^p}G_{j,l}(t))_{t=0} = 0$, $1 \leqslant p \leqslant m-1$ $(1 \leqslant j \leqslant l)$ and the Caputo derivative $\mathbf{D}_t^\alpha G_{j,l}(t)$ is well defined $(1 \leqslant j \leqslant l)$. Since $G_{l,l}(t) = E_\alpha(zt^\alpha)$, $t \geqslant 0$, it readily follows that the function $u(t)$ satisfies $u(0) = x_l$ and $u^{(j)}(0) = 0$, $1 \leqslant j \leqslant m-1$. It remains to be proved that $B\mathbf{D}_t^\alpha = Au(t)$, $t \geqslant 0$. Carrying out a straightforward computation, it can be easily seen that this equality holds iff

$$(-1)^l g_{\alpha l+1-\alpha}(t)BCx + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j}z^{l-j}[F_{j,l} * g_{\alpha l+1-\alpha}](t)BCx$$
$$+ \sum_{j=1}^{l}(-1)^{l-j}\mathbf{D}_t^\alpha G_{j,l}(t)Bx_j = \sum_{j=1}^{l}(-1)^{l-j}G_{j,l}(t)Ax_j, \quad t \geqslant 0$$

iff

$$(-1)^l g_{\alpha l+1-\alpha}(t)BCx + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j}z^{l-j}[F_{j,l} * g_{\alpha l+1-\alpha}](t)BCx$$
$$+ \sum_{j=1}^{l}(-1)^{l-j}\mathbf{D}_t^\alpha G_{j,l}(t)Bx_j = \sum_{j=1}^{l}(-1)^{l-j}G_{j,l}(t)[zBx_j - Bx_{j-1}], \quad t \geqslant 0.$$

This is true because the coefficients of Bx_j, for every fixed number $j \in \mathbb{N}_l^0$, on both sides of previous equality are equal (cf. also the proof of [**292**, Theorem 2.1.19]). $\square$

Suppose that the operator B is injective, $x \in D(AB^{-1})$, $\alpha > 0$ and there exists an exponentially equicontinuous (g_α, C)-regularized resolvent family $(R(t))_{t\geqslant 0}$ generated by A, B. Then it is readily seen that the function $u(t) := R(t)x$, $t \geqslant 0$ is a unique solution of problem $(\mathrm{DFP})_L$ with $f(t) \equiv 0$ and the initial value x replaced by $CB^{-1}x$. We leave to the interested reader the problem of transferring this conclusion, as well as the others from [**6**, Remark 2.4], to degenerate fractional equations whose solutions are governed by $(g_\alpha, g_{\alpha l+1})$-regularized C-resolvent families generated by A, B ($l \in \mathbb{N}$).

Assume now that $n \in \mathbb{N} \smallsetminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_{n-1}$, and $A_1, \ldots, A_{n-1}$ are closed linear operators on E. In the analysis of existence and uniqueness of integral equations associated with the problem [(90)–(91)], we can also use the notion of an exponentially equicontinuous (analytic) $(k; C)$-regularized resolvent (i, j)-propagation family:

Definition 2.3.19. (cf. Definition 2.3.4 and Definition 2.3.8) Suppose that the function $k(t)$ satisfies (P1), as well as $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ and $R_{i,j}(t) \in L(E, [D(A_i)])$ for all $t \geqslant 0$. Let the operator $C \in L(E)$ be injective.

(i) Then the operator family $(R_{i,j}(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation family for problem [(90)–(91)] iff there exists $\omega \geqslant \max(0, \mathrm{abs}(k))$ such that the following holds:
  (a) The mappings $t \mapsto R_{i,j}(t)x$, $t \geqslant 0$ and $t \mapsto A_i R_{i,j}(t)x$, $t \geqslant 0$ are continuous for every fixed element $x \in E$.
  (b) The family $\{e^{-\omega t} R_{i,j}(t) : t \geqslant 0\} \subseteq L(E, [D(A_i)])$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that
$$p(e^{-\omega t} R_{i,j}(t)x) + p(e^{-\omega t} A_i R_{i,j}(t)x) \leqslant cq(x), \quad x \in E,\ t \geqslant 0.$$
  (c) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator P_λ is injective, $R(C) \subseteq R(P_\lambda)$ and
$$\lambda^{\alpha_i - \alpha_{n-1} - j} \tilde{k}(\lambda) P_\lambda^{-1} C x = \int_0^\infty e^{-\lambda t} R_{i,j}(t)x\, dt, \quad x \in E.$$

(ii) Let $(R_{i,j}(t))_{t\geqslant 0}$ be an exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation family for problem [(90)–(91)]. Then it is said that $(R_{i,j}(t))_{t\geqslant 0}$ is an exponentially equicontinuous (equicontinuous), analytic $(k; C)$-regularized resolvent (i, j)-propagation family for problem [(90)–(91)], of angle α, iff the following holds:
  (a) For every $x \in E$, the mappings $t \mapsto R_{i,j}(t)x$, $t > 0$ and $t \mapsto A_i R_{i,j}(t)x$, $t > 0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we denote these extensions by the same symbols.
  (b) For every $x \in E$ and $\beta \in (0, \alpha)$, one has $\lim_{z\to 0, z\in\Sigma_\beta} R_{i,j}(z)x = R_{i,j}(0)x$ and $\lim_{z\to 0, z\in\Sigma_\beta} A_i R_{i,j}(z)x = A_i R_{i,j}(0)x$.
  (c) For every $\beta \in (0, \alpha)$, there exists $\omega_\beta \geqslant \max(0, \mathrm{abs}(k))$ ($\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z} R_{i,j}(z) : z \in \Sigma_\beta\} \subseteq L(E, [D(A_i)])$ is

equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that

$$p(e^{-\omega_\beta z}R_{i,j}(z)x) + p(e^{-\omega_\beta z}A_i R_{i,j}(z)x) \leqslant cq(x), \quad x \in E, \ z \in \Sigma_\beta.$$

Exponentially equicontinuous (analytic) $(k; C)$-regularized resolvent (i, j)-propagation families yield results very similar to those obtained by$k$-regularized C-resolvent (i, j)-propagation families. Without going into a deeper analysis, we shall only observe that the assertions of Theorem 2.3.5(i)–(iii), Remark 2.3.7(i),(iii), Theorem 2.3.9 and Theorem 2.3.10 can be restated for exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation families. Details can be left to the interested reader.

Before we move to the next subsection, it would be worthwhile to reconsider the assertion of [**292**, Theorem 2.3.3] for systems of abstract degenerate differential equations here. In order to do that, assume that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E, as well that $A = (A_1, \ldots, A_n)$; cf. Subsection 2.2.3 for more details. Denote by $\mathbb{C}^{m,m}$ the ring of $m \times m$ matrices over $\mathbb{C}$, and by I_m the identity matrix of format $m \times m$ ($m \in \mathbb{N}$). If $P(x) = [p_{ij}(x)]$ is an $m \times m$ matrix of polynomials of $x \in \mathbb{R}^n$, then there exist $d \in \mathbb{N}$ and matrices $P_\eta \in \mathbb{C}^{m,m}$ such that $P(x) = \sum_{|\eta| \leqslant d} P_\eta x^\eta$, $x \in \mathbb{R}^n$. Then we know that the operator $P(A) := \sum_{|\eta| \leqslant d} P_\eta A^\eta$ is closable on E^m (cf. [**132**, Theorem 14.1]).

Now we are ready to formulate the following theorem.

THEOREM 2.3.20. *Let $(E, \|\cdot\|)$ be a complex Banach space and let $iA_j, 1 \leqslant j \leqslant n$ be commuting generators of bounded C_0-groups on E. Suppose $\alpha > 0$, $d \in \mathbb{N}$ and $P_i(x) = \sum_{|\eta| \leqslant d} P_{\eta,i} x^\eta$ ($P_{\eta,i} \in \mathbb{C}^{m,m}$, $x \in \mathbb{R}^n$, $i = 1, 2$) are two given polynomial matrices. Suppose that for each $x \in \mathbb{R}^n$ the matrix $P_2(x)$ is regular. Then there exists a dense subset $E_{\alpha,m}$ of E^m such that, for every $\vec{x} \in E_{\alpha,m}$, there exists a unique solution of the following abstract Cauchy problem:*

$$(\mathrm{DFP})' : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} \vec{u}(t) = \overline{P_2(A)} \mathbf{D}_t^\alpha \vec{u}(t) = \overline{P_1(A)} \vec{u}(t), & t \geqslant 0, \\ \vec{u}(0) = \vec{x}; \ \vec{u}^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

Furthermore, for every $\vec{x} \in E_{\alpha,m}$, the mapping $t \mapsto \vec{u}(t)$, $t \geqslant 0$ can be extended to the whole complex plane (the extension of this mapping will be denoted by the same symbol in the sequel), and the following holds:

(i) *The mapping $z \mapsto \vec{u}(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic.*
(ii) *The mapping $z \mapsto \vec{u}(z)$, $z \in \mathbb{C}$ is entire provided that $\alpha \in \mathbb{N}$.*

PROOF. Let us recall that $k = 1 + \lfloor n/2 \rfloor$. Suppose that $P_1(x) = [p_{ij;1}(x)]_{1 \leqslant i,j \leqslant m}$ and $P_2(x) = [p_{ij;2}(x)]_{1 \leqslant i,j \leqslant m}$ ($x \in \mathbb{R}^n$), and d is the maximal degree of all non-zero polynomials $p_{ij;1}(x)$ and $p_{ij;2}(x)$ ($1 \leqslant i, j \leqslant m$). Then $\sup_{x \in \mathbb{R}^n} (\det(P_2(x)))^{-1} < \infty$ and we can inductively prove that there exist numbers $M_1 \geqslant 1$ and $M_2 \geqslant 1$ such that for each $l \in \mathbb{N}_0$ there exist polynomials $R_{ij;l}(x)$ ($1 \leqslant i, j \leqslant m$) of degree $\leqslant lmd$

satisfying that

$$(P_2(x)^{-1}P_1(x))^l = \frac{[R_{ij;l}(x)]_{1\leqslant i,j\leqslant m}}{(\det(P_2(x)))^l}, \quad x \in \mathbb{R}^n$$

and the following holds:

$$\Bigl|D^\eta\Bigl(\frac{R_{ij;l}(x)}{(\det(P_2(x)))^l}\Bigr)\Bigr| + \Bigl|D^\eta\Bigl(p_{ij;1}(x)\frac{R_{ij;l}(x)}{(\det(P_2(x)))^l}\Bigr)\Bigr| + \Bigl|D^\eta\Bigl(p_{ij;2}(x)\frac{R_{ij;l}(x)}{(\det(P_2(x)))^l}\Bigr)\Bigr| \leqslant M_1^l(1+|x|)^{lmdM_2}, \tag{109}$$

provided $l \in \mathbb{N}_0$, $x \in \mathbb{R}^n$, $0 \leqslant |\eta| \leqslant k$ and $1 \leqslant i,j \leqslant m$. It is very simple to prove that there exists a sufficiently large natural number k' satisfying $2|k'$ and

$$\lim_{l\to+\infty} \frac{\bigl(\Gamma(\frac{2M_2(l+1)md+n}{k'd})\bigr)^{1/2l}}{(\Gamma(\alpha l+1))^{1/l}} = 0. \tag{110}$$

Let $a > 0$ be fixed. Set $C := (e^{-a|x|^{k'd}})(A)$ and $E_{\alpha,m} := (R(C))^m$. Then $C \in L(E)$, C is injective and $D_\infty(A_1^2 + \cdots + A_n^2) \supseteq R(C)$ is dense in E [**132**]. Define

$$W_\alpha(z) := \Biggl[\sum_{l=0}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)}\Bigl(\frac{R_{ij;l}(x)}{(\det(P_2(x)))^l}e^{-a|x|^{k'd}}\Bigr)(A)\Biggr]_{1\leqslant i,j\leqslant m}, \ z \in \mathbb{C}. \tag{111}$$

Using (109)–(110) and the proof of [**292**, Theorem 2.3.3], it readily follows that $W_\alpha(z) \in L(E^m)$ for all $z \in \mathbb{C}$, as well as that the expressions

$$\Biggl[\sum_{l=0}^{\infty}\sum_{v=1}^{m} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)}\Bigl(p_{iv;2}(x)\frac{R_{vj;l+1}(x)}{(\det(P_2(x)))^{l+1}}e^{-a|x|^{k'd}}\Bigr)(A)\Biggr]_{1\leqslant i,j\leqslant m}$$

and

$$\Biggl[\sum_{l=0}^{\infty}\sum_{v=1}^{m} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)}\Bigl(p_{iv;1}(x)\frac{R_{vj;l}(x)}{(\det(P_2(x)))^{l}}e^{-a|x|^{k'd}}\Bigr)(A)\Biggr]_{1\leqslant i,j\leqslant m}$$

define the bounded linear operators on E^m ($z \in \mathbb{C}$). Furthermore, the mapping $z \mapsto W_\alpha(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic, and the mapping $z \mapsto W_\alpha(z)$, $z \in \mathbb{C}$ is entire provided that $\alpha \in \mathbb{N}$. Suppose now $\vec{x} \in E_{\alpha,m}$. Then there exists $\vec{x'} \in E^m$ such that $\vec{x} = C_m\vec{x'}$, where $C_m = CI_m$. Setting $\vec{u}(z) := W_\alpha(z)\vec{x'}$, $z \in \mathbb{C}$, we immediately obtain that (i) and (ii) hold. It is not difficult to prove that $\mathbf{D}_t^\alpha(t)\vec{u}(t)$ is well-defined, as well as that

$$\mathbf{D}_t^\alpha(t)\vec{u}(t) = \Biggl[\sum_{l=1}^{\infty} \frac{t^{\alpha(l-1)}}{\Gamma(\alpha(l-1)+1)}\Bigl(\frac{R_{ij;l}(x)}{(\det(P_2(x)))^l}e^{-a|x|^{k'd}}\Bigr)(A)\Biggr]_{1\leqslant i,j\leqslant m}\vec{x'}, \quad t \geqslant 0,$$

and $\vec{u}(0) = \vec{x}$, $\vec{u}^{(j)}(0) = 0$, $1 \leqslant j \leqslant \lceil\alpha\rceil - 1$. Since $\overline{P_1(A)}$ and $\overline{P_2(A)}$ are closed, we can prove with the help of ($\triangleright$) that $\vec{u}(t) \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$, $\mathbf{D}_t^\alpha\vec{u}(t) \in D(\overline{P_2(A)})$, the term $\mathbf{D}_t^\alpha\overline{P_2(A)}\vec{u}(t)$ is well defined, with

$$\overline{P_2(A)}\mathbf{D}_t^\alpha\vec{u}(t) = \mathbf{D}_t^\alpha\overline{P_2(A)}\vec{u}(t)$$

$$= \Big[\sum_{l=0}^{\infty} \sum_{v=1}^{m} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)} \Big(p_{iv;2}(x) \frac{R_{iv;l+1}(x)}{(\det(P_2(x)))^{l+1}} e^{-a|x|^{k'd}} \Big)(A) \Big]_{1 \leqslant i,j \leqslant m} \vec{x'}$$

and

$$\overline{P_1(A)}\vec{u}(t) = \Big[\sum_{l=0}^{\infty} \sum_{v=1}^{m} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)} \Big(p_{iv;1}(x) \frac{R_{iv;l}(x)}{(\det(P_2(x)))^{l}} e^{-a|x|^{k'd}} \Big)(A) \Big]_{1 \leqslant i,j \leqslant m} \vec{x'},$$

for any $t \geqslant 0$. Since

$$P_2(x) \frac{[R_{ij;l+1}(x)]_{1 \leqslant i,j \leqslant m}}{(\det(P_2(x)))^{l+1}} = P_1(x) \frac{[R_{ij;l}(x)]_{1 \leqslant i,j \leqslant m}}{(\det(P_2(x)))^{l}}, \quad l \in \mathbb{N}_0, \ x \in \mathbb{R}^n,$$

a simple matricial calculation shows that the function $t \mapsto \vec{u}(t)$, $t \geqslant 0$ is a solution of problem (DFP)'. Now we will prove the uniqueness of solutions of problem (DFP)'. Let $t \mapsto \vec{u}(t)$, $t \geqslant 0$ be a solution of (DFP)' with $\vec{x} = \vec{0}$. Integrating α-times (DFP)', we get that $\overline{P_2(A)}\vec{u}(t) = \int_0^t g_\alpha(t-s)\overline{P_1(A)}\vec{u}(s)\,ds$, $t \geqslant 0$. Using this equality, as well as the fact that $\overline{P_2(A)}W_\alpha(t) - \overline{P_2(A)}C_m = \overline{P_1(A)}(g_\alpha * W_\alpha(\cdot))(t) \in L(E^m)$, $t \geqslant 0$, and the proof of [**459**, Proposition 1.1], we obtain that $0 = (W_\alpha * 0)(t) = (\overline{P_2(A)}C_m * \vec{u})(t)$, $t \geqslant 0$, so that it suffices to prove that the operator $\overline{P_2(A)}$ is injective. Suppose that $\overline{P_2(A)}\vec{x} = \vec{0}$ for some $\vec{x} \in E^m$. By [**561**, Lemma 1.1(a)], we may assume without loss of generality that $\vec{x} \in E_0^m$ (cf. ($\triangleright$)). It is clear that there exist polynomials $q_{ij}(x)$ $(1 \leqslant i,j \leqslant m)$ such that $P_2(x)^{-1} = (\det(P_2(x)))^{-1}[q_{ij}(x)]_{1 \leqslant i,j \leqslant m}$. Using ($\triangleright$), one can prove that $[(\det(P_2(x)))(A)I_m][\phi(A)I_m]\vec{x} = [(\phi(x)q_{ij}(x))(A)]_{1 \leqslant i,j \leqslant m}P_2(A)\vec{x} = \vec{0}$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. By [**306**, Remark 4.4(i)], the operator $(\det(P_2(x)))(A)$ is injective, whence we may conclude that $[\phi(A)I_m]\vec{x} = \vec{0}$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. This, in turn, implies $\vec{x} = \vec{0}$ and completes the proof of theorem. $\square$

REMARK 2.3.21. (i) Suppose that $m = 1$, $P_1(x) = \sum_{|\alpha| \leqslant d} a_\alpha x^\alpha$, $P_2(x) = \sum_{|\alpha| \leqslant d} b_\alpha x^\alpha$, $x \in \mathbb{R}^n$ $(a_\alpha,\ b_\alpha \in \mathbb{C})$, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and E is a function space on which translations are uniformly bounded and strongly continuous (for example, $L^p(\mathbb{R}^n)$ with $p \in [1,\infty)$). Then the obvious choice for A_j is $i\partial/\partial x_j$ $(1 \leqslant j \leqslant n)$ and then we have that $\overline{P_1(A)}$ and $\overline{P_2(A)}$ are just the operators $\sum_{|\alpha| \leqslant d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha$ and $\sum_{|\alpha| \leqslant d} b_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha$, respectively, acting with their maximal distributional domains. Making use of Theorem 2.3.20 and a slight modification of the formula appearing on l. 1, p. 49 of [**292**], we can prove that for each $\alpha > 0$ there exists a dense subset $E_{\alpha,1}$ of $L^p(\mathbb{R}^n)$ such that the abstract Cauchy problem:

$$\begin{aligned} \sum_{|\alpha| \leqslant d} b_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha \mathbf{D}_t^\alpha u(t,x) &= \mathbf{D}_t^\alpha \sum_{|\alpha| \leqslant d} b_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x) \\ &= \sum_{|\alpha| \leqslant d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x), \quad t > 0, \ x \in \mathbb{R}^n; \end{aligned}$$

$$\frac{\partial^l}{\partial t^l} u(t,x)_{|t=0} = f_l(x), \quad x \in \mathbb{R}^n, \ l = 0,1,\ldots,\lceil \alpha \rceil - 1,$$

has a unique solution provided $f_l(\cdot) \in E_{\alpha,1}$, $l = 0, 1, \ldots, \lceil \alpha \rceil - 1$. A similar assertion can be formulated in E_l-type spaces [**541**]; we can also move to the spaces $L^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$ or $C^\sigma(\mathbb{R}^n)$ $(0 < \sigma < 1)$ by using distributional techniques.

(ii) Denote $\Omega(\omega) = \{\lambda^2 : \operatorname{Re} \lambda > \omega\}$, if $\omega > 0$, and $\Omega(\omega) = \mathbb{C} \smallsetminus (-\infty, -\omega^2]$, if $\omega \leqslant 0$. In the previous part of Section 2.1 and Section 2.2, we have considered the C-wellposedness of the abstract degenerate Cauchy problem

$$(\mathrm{DFP})_2 : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} u(t) = \overline{P_2(A)} \mathbf{D}_t^\alpha u(t) = \overline{P_1(A)} u(t), \quad t \geqslant 0, \\ u(0) = Cx; \ u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

where $0 < \alpha \leqslant 2$, $P_1(x)$ and $P_2(x)$ are complex polynomials, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E, and $A = (A_1, \ldots, A_n)$. The basic assumption was that (70) holds for $0 < \alpha < 2$, and that $P_1(x)/P_2(x) \notin \Omega(\omega)$, $x \in \mathbb{R}^n$, provided $\alpha = 2$. Observe that our results from the part (i) of this remark can be applied in the analysis of problem (DFP) in the general case $\alpha > 0$, and also in the case that $0 < \alpha \leqslant 2$ and the above-mentioned requirements are not satisfied.

(iii) The assertion of [**292**, Theorem 2.3.5] can be reformulated in the degenerate case, as well. We leave the precise details as an exercise for the interested reader.

Example 2.3.22. Unfortunately, Theorem 2.3.20 and the conclusions from the parts of (i) and (ii) of former remark cannot be applied in the analysis of L^p-wellposedness of a great number of very important degenerate equations of mathematical physics, like (cf. the monograph by G. V. Demidenko–S. V. Uspenskii [**140**] for more details):

(a) (The Rossby wave equation, 1939)

$$\Delta u_t + \beta u_y = 0 \quad (n = 2), \qquad u(0, x, y) = u_0(x, y);$$

(b) (The Sobolev equation, 1940)

$$\Delta u_{tt} + \omega^2 u_{zz} = 0 \quad (n = 3),$$
$$u(0, x, y, z) = u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z),$$

here $\omega/2$ is the angular velocity;

(c) (The internal wave equation in the Boussinesq approximation, 1903)

$$\Delta u_{tt} + N^2(u_{xx} + u_{yy}) = 0 \quad (n = 3),$$
$$u(0, x, y, z) = u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z);$$

(d) (The gravity-gyroscopic wave equation, cf. [**222**] and [**535**])

$$\Delta u_{tt} + N^2(u_{xx} + u_{yy}) + \omega^2 u_{zz} = 0 \quad (n = 3),$$
$$u(0, x, y, z) = u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z);$$

(e) (Small amplitude oscillations of a rotating viscous fluid)

$$\Delta u_{tt} - 2\nu\Delta^2 u_t + v^2\Delta^3 u + \omega^2 u_{zz} = 0 \quad (n = 3),$$
$$u(0, x, y, z) = u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z).$$

Here $\omega/2$ is the angular velocity and $\nu > 0$ is the viscosity coefficient.

Before including some details on the wellposedness of equations (a)–(e) in L^p spaces, we need to explain how one can reformulate the assertion of Theorem 2.3.20 in the case that there exist a vector $x_0 \in \mathbb{R}^n$ and a non-empty compact subset K of $\mathbb{R}^n$ such that the matrix $P_2(x_0)$ is singular and $\{x \in \mathbb{R}^n : \det(P_2(x)) = 0\} \subseteq K$; some fractional analogues of (a)–(e) can be analyzed similarly. Denote by $\mathcal{G}$ the class consisting of those $C^\infty(\mathbb{R}^n)$-functions $\phi(\cdot)$ satisfying that there exist two open relatively compact neighborhoods Ω and Ω' of K in $\mathbb{R}^n$ such that $\phi(x) = 0$ for all $x \in \Omega$ and $\phi(x) = 1$ for all $x \in \mathbb{R}^n \smallsetminus \Omega'$. Since the estimate (109) holds for all $x \in \mathbb{R}^n \smallsetminus \Omega$, for each $z \in \mathbb{C}$ we can define the matricial operator $W_\alpha(z)$ (cf. the proof of Theorem 2.3.20) by replacing the function $e^{-a|x|^{k'd}}$ in (111) with the function $\phi(x)e^{-a|x|^{k'd}}$. Setting $C_\phi := (\phi(x)e^{-a|x|^{k'd}})(A)$ for $\phi \in \mathcal{G}$ (then we do not know any longer whether the set $\bigcup_{\phi\in\mathcal{A}} R(C_\phi)$ is dense in E, and we cannot clarify whether the operator C_ϕ is injective or not) and $E'_{\alpha,m} := (\bigcup_{\phi\in\mathcal{A}} R(C_\phi))^m$, and assuming additionally the injectivity of matricial operator $\overline{P_2(A)}$ on E^m, then for each $\vec{x} \in E'_{\alpha,m}$ there exists a unique solution $t \mapsto \vec{u}(t)$, $t \geqslant 0$ of the abstract Cauchy problem (DFP)', which can be extended to the whole complex plane, and (i)-(ii) from the formulation of Theorem 2.3.20 continues to hold. Rewriting any of the equations (a)–(e) in the matricial form, and using the following

LEMMA 2.3.23. *Suppose that $1 \leqslant p < \infty$, $n \in \mathbb{N}$ and $E := L^p(\mathbb{R}^n)$. Denote by $\Delta_{p,n}$ the operator Δ acting on E with its maximal distributional domain. Then $\Delta_{p,n}$ is injective.*

PROOF. If $1 < p < \infty$, then the statement immediately follows from the fact that the operator $-\Delta_{p,n}$ is non-negative, with dense domain and range (cf. [**412**, pp. 256, 266]). Suppose now that $p = 1$ and $\Delta_{p,n} f = 0$ for some $f \in E$. Then [**412**, Lemma 3.2] implies that, for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and for every multi-index $\eta \in \mathbb{N}_0^n$, the function $\varphi * f$ belongs to the space $\mathcal{T}$consisting of those $C^\infty(\mathbb{R}^n)$-functions whose any partial derivative belongs to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Since $\Delta_{p,n}(\varphi * f) = \varphi * \Delta_{p,n} f = 0$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and the operator $\Delta_{\mathcal{T}}$ is injective by [**412**, Remark 3.3], we have that $\varphi * f = 0$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Hence, $f = 0$. □

We obtain that there exists a non-trivial subspace $E'_{1,1}$ of $L^p(\mathbb{R}^2)$, resp. $E'_{1,2}$ of $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, such that the equation (a), resp. any of the equations (b)–(e), has a unique entire solution provided $u_0(x, y) \in E'_{1,1}$, resp. $(u_0(x, y, z), u_1(x, y, z)) \in E'_{1,2}$. In the present situation, we do not know whether the subspace $E'_{1,1}$, resp. $E'_{1,2}$, can be chosen to be dense in $L^p(\mathbb{R}^2)$, resp. $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$.

2.3.4. Degenerate k-regularized (C_1, C_2)-existence and uniqueness propagation families for (112)**.** The main purpose of this subsection is to report how the techniques established in a joint paper of C.-G. Li, M. Li and the author [**346**] can be successfully applied in the analysis of a wide class of abstract degenerate multi-term fractional differential equations with Caputo derivatives. We propose an important theoretical novelty method capable of seeking of solutions of some very atypical degenerate differential equations in L^p-spaces. Furthermore, we provide generalizations of [**347**, Theorem 2.3, Theorem 3.1] for degenerate multi-term problems.

We assume that $n \in \mathbb{N} \setminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_n$, $m = \lceil \alpha \rceil$, $\alpha_0 = \alpha$ and $m_i = \lceil \alpha_i \rceil$, $i \in \mathbb{N}_n^0$. Set $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geqslant i\}$ $(i \in \mathbb{N}^0_{m_n-1})$. The state space X is an SCLCS and $A = A_0, A_1, \ldots, A_{n-1}, A_n = B$ denote closed linear operators acting on X. Let $T > 0$ and $f \in C([0,T] : X)$. Consider the following degenerate multi-term problem:

$$
\begin{gathered}
B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = A\mathbf{D}_t^{\alpha} u(t) + f(t), \quad t \geqslant 0; \\
u^{(j)}(0) = u_j, \quad j = 0, \ldots, \lceil \alpha_n \rceil - 1,
\end{gathered} \tag{112}
$$

where $0 \leqslant \alpha < \alpha_n$. By a strong solution of problem (112) on the interval $[0,T]$ we mean any continuous function $t \mapsto u(t)$, $t \in [0,T]$ satisfying that the term $A_i \mathbf{D}_t^{\alpha_i} u(t)$ is well-defined and continuous on $[0,T]$ $(i \in \mathbb{N}_n^0)$, as well as that (112) holds identically on $[0,T]$. Convoluting both sides of (112) with $g_{\alpha_n}(t)$, we get that:

$$
\begin{aligned}
B\Bigg[u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot)\Bigg] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \Bigg[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\Bigg] \\
= g_{\alpha_n-\alpha} * A\Bigg[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\Bigg] + (g_{\alpha_n} * f)(\cdot), \quad t \in [0,T].
\end{aligned} \tag{113}
$$

By a mild solution of (112) on $[0,T]$ we mean any continuous X-valued function $t \mapsto u(t)$, $t \in [0,T]$ satisfying

$$
\begin{aligned}
B\Bigg[u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot)\Bigg] + \sum_{j=1}^{n-1} A_j \Bigg(g_{\alpha_n-\alpha_j} * \Bigg[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\Bigg]\Bigg) \\
= A\Bigg(g_{\alpha_n-\alpha} * \Bigg[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\Bigg]\Bigg) + (g_{\alpha_n} * f)(\cdot), \quad t \in [0,T].
\end{aligned}
$$

Consider the following inhomogeneous equation:

$$
Bu(t) + \sum_{j=1}^{n-1} (g_{\alpha_n-\alpha_j} * A_j u)(t) = f(t) + (g_{\alpha_n-\alpha} * Au)(t), \quad t \in [0,T]. \tag{114}
$$

It is said that a function $u \in C([0,T] : X)$ is:

(i) a strong solution of (114) iff $A_j u \in C([0,T] : X)$, $j \in \mathbb{N}^0_{n-1}$ and (114) holds for every $t \in [0,T]$.

(ii) a mild solution of (114) iff $(g_{\alpha_n-\alpha_j} * u)(t) \in D(A_j)$, $t \in [0,T]$, $j \in \mathbb{N}_{n-1}^0$ and
$$Bu(t) + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * u)(t) = f(t) + A(g_{\alpha_n-\alpha} * u)(t), \quad t \in [0,T].$$

A mild (strong) solution of problem (112), resp. (114), on $[0,\infty)$ is defined analogously.

We will be interested in the following notions.

Definition 2.3.24. (cf. [**346**, Definition 2.2] for the case $B = I$) Suppose $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $C, C_1, C_2 \in L(X)$, C and C_2 are injective.

(i) A sequence $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X, [D(B)])$ is called a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (112) iff the following holds:
$$\begin{aligned} B[R_i(\cdot)x - (k * g_i)(\cdot)C_1x] &+ \sum_{j\in D_i} A_j[g_{\alpha_n-\alpha_j} * (R_i(\cdot)x - (k * g_i)(\cdot)C_1x)] \\ &+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j(g_{\alpha_n-\alpha_j} * R_i)(\cdot)x \\ &= \begin{cases} A(g_{\alpha_n-\alpha} * R_i)(\cdot)x, & m-1 < i,\ x \in X, \\ A[g_{\alpha_n-\alpha} * (R_i(\cdot)x - (k * g_i)(\cdot)C_1x)](\cdot), & m-1 \geqslant i,\ x \in X, \end{cases} \end{aligned}$$
for any $i = 0, \dots, m_n - 1$.

(ii) A sequence $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$) k-regularized C_2-uniqueness propagation family for (112) iff
$$\begin{aligned} [R_i(\cdot)Bx - (k * g_i)(\cdot)C_2Bx] &+ \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * [R_i(\cdot)A_jx - (k * g_i)(\cdot)C_2A_jx] \\ &+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} (g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx)(\cdot) \\ &= \begin{cases} (g_{\alpha_n-\alpha} * R_i(\cdot)Ax)(\cdot), & m-1 < i, \\ g_{\alpha_n-\alpha} * [R_i(\cdot)Ax - (k * g_i)(\cdot)C_2Ax](\cdot), & m-1 \geqslant i, \end{cases} \end{aligned}$$
for any $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$ and $i \in \mathbb{N}_{m_n-1}^0$.

(iii) A sequence $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$) k-regularized C-resolvent propagation family for (112), in short k-regularized C-propagation family for (112), iff $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (112), and for every $t \in [0,\tau)$, $i \in \mathbb{N}_{m_n-1}^0$ and $j \in \mathbb{N}_n^0$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA_j \subseteq A_jC$.

In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \geqslant 0$, it is also said that $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a ζ-times integrated C_1-existence propagation family for (112);

0-times integrated C_1-existence propagation family for (112) is simply called C_1-existence propagation family for (112). For a k-regularized C_1-existence propagation family $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$, it is said that is locally equicontinuous (exponentially equicontinuous) iff each single operator family $(R_0(t))_{t\in[0,\tau)}\subseteq L(X,[D(B)])$, $\ldots$, $(R_{m_n-1}(t))_{t\in[0,\tau)}\subseteq L(X,[D(B)])$ is; $((R_0(t))_{t\geqslant 0},\ldots,(R_{m_n-1}(t))_{t\geqslant 0})$ is said to be an exponentially equicontinuous, analytic k-regularized C_1-existence propagation family for problem (112), of angle $\alpha\in(0,\pi/2]$, iff the following holds:

(a) For every $x\in X$ and $i\in\mathbb{N}^0_{m_n-1}$, the mappings $t\mapsto R_i(t)x$, $t>0$ and $t\mapsto BR_i(t)x$, $t>0$ can be analytically extended to the sector Σ_α; the extensions will be denoted by the same symbols.
(b) For every $x\in X$, $\beta\in(0,\alpha)$ and $i\in\mathbb{N}^0_{m_n-1}$, we have $\lim_{z\to 0,z\in\Sigma_\beta}R_i(z)x=R_i(0)x$ and $\lim_{z\to 0,z\in\Sigma_\beta}BR_i(z)x=BR_i(0)x$.
(c) For every $\beta\in(0,\alpha)$ and $i\in\mathbb{N}^0_{m_n-1}$, there exists $\omega_\beta\geqslant\max(0,\operatorname{abs}(k))$ $(\omega_\beta=0)$ such that the family $\{e^{-\omega_\beta z}R_i(z): z\in\Sigma_\beta\}\subseteq L(X,[D(B)])$ is equicontinuous.

The above terminological agreements and abbreviations are also introduced for the classes of k-regularized C_2-uniqueness propagation families for (112) and k-regularized C-propagation families for (112).

Immediately from definition of a k-regularized C_1-existence propagation family for (112) (k-regularized C_2-uniqueness propagation family for (112)), we can state some obvious facts about the existence and uniqueness of mild (strong) solutions of problem (114); details can be skipped.

The proof of following extension of [**346**, Proposition 2.3] is omitted, too.

PROPOSITION 2.3.25. *Let $i\in\mathbb{N}^0_{m_n-1}$ and $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ be a locally equicontinuous k-regularized C_1-existence propagation family for* (112). *If $R_i(t)A_j\subseteq A_jR_i(t)$ $(j\in\mathbb{N}^0_n,\ t\in[0,\tau))$, $R_i(t)C_1=C_1R_i(t)$ $(t\in[0,\tau))$, C_1 is injective, $k(t)$ is a kernel on $[0,\tau)$ and $C_1A_j\subseteq A_jC_1$ $(j\in\mathbb{N}^0_n)$, then the following holds:*

(i) *The equality*

$$R_i(t)R_i(s)=R_i(s)R_i(t),\quad 0\leqslant t,\ s<\tau \tag{115}$$

holds, provided that $m-1<i$ and the condition
($\diamond$) *The assumption $Bf(t)+\sum_{j\in D_i}A_j(g_{\alpha_n-\alpha_j}*f)(t)=0$, $t\in[0,\tau)$ for some $f\in C([0,\tau):X)$, implies $f(t)=0$, $t\in[0,\tau)$,*
holds.

(ii) *The equality* (115) *holds provided that $m-1\geqslant i$, $\mathbb{N}_{n-1}\smallsetminus D_i\neq\emptyset$, and the condition*
($\diamond\diamond$) *If $\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}A_j(g_{\alpha_n-\alpha_j}*f)(t)=0$, $t\in[0,\tau)$, for some $f\in C([0,\tau):X)$, then $f(t)=0$, $t\in[0,\tau)$,*
holds.

The assertions of [**346**, Proposition 2.5, Proposition 2.6] can be reformulated for degenerate multi-term problems; the same holds for the generalized variation of parameters formula [**346**, Proposition 2.8]:

THEOREM 2.3.26. *Let $C_2 \in L(X)$ be injective. Suppose $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous k-regularized C_2-uniqueness propagation family for* (112), *$T \in (0,\tau)$ and $f \in C([0,T] : X)$. Then the following holds:*

(i) *If $m-1 < i$, then any strong solution $u(t)$ of* (114) *satisfies the equality:*
$$(R_i * f)(t) = (k * g_i * C_2Bu)(t) + \sum_{j\in D_i} (g_{\alpha_n-\alpha_j+i} * k * C_2A_ju)(t),$$
for any $t \in [0,T]$. Therefore, there is at most one strong (mild) solution for (114), *provided that $k(t)$ is a kernel on $[0,\tau)$ and $(\diamond)$ holds.*

(ii) *If $m-1 \geqslant i$, then any strong solution $u(t)$ of* (114) *satisfies the equality:*
$$(R_i * f)(t) = -\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i} (g_{\alpha_n-\alpha_j+i} * k * C_2A_ju)(t), \quad t \in [0,T].$$
Therefore, there is at most one strong (mild) solution for (114), *provided that $k(t)$ is a kernel on $[0,\tau)$, $\mathbb{N}_{n-1} \smallsetminus D_i \neq \emptyset$ and $(\diamond\diamond)$ holds.*

As explained in [**292**, Section 2.10], the notion of a k-regularized C_1-existence propagation family is probably the best theoretical concept for the investigation of integral solutions of non-degenerate abstract time-fractional equation (112) with $A_j \in L(E)$, $1 \leqslant j \leqslant n-1$. If $A_j \notin L(E)$ for some $j \in \mathbb{N}_{n-1}$, then the vector-valued Laplace transform cannot be so easily applied, which certainly implies that there exist some limitations to this class of propagation families. A similar problem appears in the analysis od degenerate multi-term fractional differential equation (112); because of that, we will only leave as an interesting problem to the reader to try to reconsider the assertions of [**346**, Theorem 2.9(i), Theorem 2.10–Theorem 2.12] in our new framework. In contrast to the above, it is very simple to reformulate the assertion of [**346**, Theorem 2.9(ii)] to degenerate equations, without imposing any additional barriers at:

THEOREM 2.3.27. *Suppose $k(t)$ satisfies* (P1), *$\omega \geqslant \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geqslant 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous, provided $0 \leqslant i \leqslant m_n - 1$. Let $C_2 \in L(X)$ be injective. Then $((R_0(t))_{t\geqslant 0}, \dots, (R_{m_n-1}(t))_{t\geqslant 0})$ is a global k-regularized C_2-uniqueness propagation family for* (112) *iff, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, and for every $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$, the following equality holds:*

$$\int_0^\infty e^{-\lambda t}[R_i(t)Bx - (k * g_i)(t)C_2Bx]dt$$
$$+ \sum_{j\in D_i} \lambda^{\alpha_j-\alpha_n} \int_0^\infty e^{-\lambda t}[R_i(t)x - (k * g_i)(t)C_2A_jx]dt$$
$$+ \sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i} \lambda^{\alpha_j-\alpha_n} \int_0^\infty e^{-\lambda t}R_i(t)A_jx\, dt$$

$$= \begin{cases} \lambda^{\alpha-\alpha_n} \int_0^\infty e^{-\lambda t} R_i(t)Ax\, dt, & m-1 < i, \\ \lambda^{\alpha-\alpha_n} \int_0^\infty e^{-\lambda t} [R_i(t)Ax - (k * g_i)(t)C_2 Ax] dt, & m-1 \geqslant i. \end{cases}$$

Now we would like to present an instructive example of a local k-regularized I-resolvent propagation family for (112):

EXAMPLE 2.3.28. (cf. [**346**, Example 5.2] for non-degenerate case) Suppose $1 \leqslant p \leqslant \infty$, $E := L^p(\mathbb{R})$, $m\colon \mathbb{R} \to \mathbb{C}$ is measurable, $a_j \in L^\infty(\mathbb{R})$, $(A_j f)(x) := a_j(x)f(x)$, $x \in \mathbb{R}$, $f \in E$ $(1 \leqslant j \leqslant n)$, $(Af)(x) := m(x)f(x)$, $x \in \mathbb{R}$, with maximal domain, and $\alpha = 0$. Assume $s \in (1, 2)$, $\delta = 1/s$, $M_p = p!^s$ and $k_\delta(t) = \mathcal{L}^{-1}(\exp(-\lambda^\delta))(t)$, $t \geqslant 0$. Denote by $M(t)$ the associated function of sequence (M_p) and put $\Lambda'_{\alpha',\beta',\gamma'} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geqslant \gamma'^{-1} M(\alpha'\lambda) + \beta'\}$, $\alpha', \beta', \gamma' > 0$. Clearly, there exists a constant $C_s > 0$ such that $M(\lambda) \leqslant C_s|\lambda|^{1/s}$, $\lambda \in \mathbb{C}$. Assume that the following condition holds:

(CH): For every $\tau > 0$, there exist $\alpha' > 0$, $\beta' > 0$ and $d > 0$ such that $\tau \leqslant \frac{\cos(\frac{\delta\pi}{2})}{C_s(\alpha')^{1/s}}$ and

$$\Bigl| \sum_{j=1}^n \lambda^{\alpha_j - \alpha} a_j(x) - m(x) \Bigr| \geqslant d, \quad x \in \mathbb{R},\ \lambda \in \Lambda_{\alpha',\beta',1}.$$

Notice that the above condition holds provided $n = 2$, $\alpha_2 = 2$, $\alpha_1 = 1$, $c_1 \in L^\infty(\mathbb{R})$, $|c_1(x)| \geqslant d_1 > 0$ for a.e. $x \in \mathbb{R}$, $a_2(x) \in L^\infty(\mathbb{R})$, $a_2(x) = 0$, $x \in (-1, 1)$, $a_1(x) = a_2(x)c_1(x)$ and $m(x) = \frac{1}{4}c_1^2(x)a_2(x) - \frac{1}{16}c_1^4(x)a_2(x) - a_2(x)$, $x \in \mathbb{R}$ (cf. [**346**, (5.7)]), and that the validity of condition (CH) does not imply, in general, the essential boundedness of function $m(\cdot)$ or the injectivity of the operator B. We will prove that there exists a global (not exponentially bounded, in general) k_δ-regularized I-resolvent propagation family $((R_0(t))_{t\geqslant 0}, \ldots, (R_{m_n-1}(t))_{t\geqslant 0})$ for (112). Clearly, it suffices to show that, for every $\tau > 0$, there exists a local k_δ-regularized I-resolvent propagation family for (112) on $[0, \tau)$. Suppose $\tau > 0$ is given in advance, and $\alpha' > 0$, $\beta' > 0$ and $d > 0$ satisfy (CH), with this τ. Let Γ denote the upwards oriented boundary of ultra-logarithmic region $\Lambda_{\alpha',\beta',1}$. Put, for every $t \in [0, \tau)$, $f \in E$ and $x \in \mathbb{R}$,

$$(R_i(t)f)(x) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t - \lambda^\delta} \frac{\bigl[\lambda^{\alpha_n - \alpha - i} a_n(x) + \sum_{j \in D_i} \lambda^{\alpha_j - \alpha - i} a_j(x)\bigr] f(x)}{\lambda^{\alpha_n - \alpha} a_n(x) + \sum_{j=1}^{n-1} \lambda^{\alpha_j - \alpha} a_j(x) - m(x)} d\lambda.$$

Then the analysis contained in [**346**, Example 5.2] shows that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a local k_δ-regularized I-resolvent propagation family for (112), as well as that, for every compact set $K \subseteq [0, \infty)$, there exists $h_K > 0$ such that

$$\sup_{t \in K, p \in \mathbb{N}_0, i \in \mathbb{N}^0_{m_n-1}} \frac{\bigl\| h_k^p \frac{d^p}{dt^p} R_i(t) \bigr\|}{p!^s} < \infty.$$

We can similarly consider the existence of local $k_{1/2}$-regularized I-resolvent propagation families for (112) which obey slight modifications of the properties stated above with $s = 2$, and with the operators A_j not belonging to the space $L(E)$ for some indexes $j \in \mathbb{N}_n$. Furthermore, we can similarly construct some examples of (local) k-regularized I-resolvent propagation family for (112) in certain classes of Fréchet function spaces.

2.3.5. Degenerate k-regularized (C_1, C_2)-existence and uniqueness families. In this subsection, we assume that X and Y are two Hausdorff sequentially complete locally convex spaces over the field of complex numbers. By $\circledast_X$ ($\circledast$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of X. The fundamental system of seminorms which defines the topology on Y is denoted by $\circledast_Y$. The symbol I denotes the identity operator on X. Let $0 < \tau \leqslant \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous iff, for every $T \in (0, \tau)$ and for every $p \in \circledast_X$, there exist $q_p \in \circledast_Y$ and $c_p > 0$ such that $p(W(t)y) \leqslant c_p q_p(y)$, $y \in Y$, $t \in [0, T]$; the notion of equicontinuity of $(W(t))_{t\in[0,\tau)}$ is defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is always locally equicontinuous in the case that the space Y is barreled.

In the following definition, we will generalize the notion introduced in [**346**, Definition 3.1] (cf. also R. deLaubenfels [**136**] and T.-J. Xiao–J. Liang [**545**] for some other known concepts in the case $B = I$).

Definition 2.3.29. Suppose $0 < \tau \leqslant \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (112) iff, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n-1}([0, \tau) : [D(B)])$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n-1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0, \tau) : X)$ for $0 \leqslant j \leqslant n$, and

$$
\begin{aligned}
BE^{(m_n-1)}(t)y + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(t)y& \\
- A(g_{\alpha_n-\alpha} * E^{(m_n-1)})(t)y &= k(t)C_1 y,
\end{aligned}
\tag{116}
$$

for any $t \in [0, \tau)$.

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (112) iff, for every $\tau \in [0, \tau)$ and $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$, the following holds:

$$
\begin{aligned}
U(t)Bx + \sum_{j=1}^{n-1} (g_{\alpha_n-\alpha_j} * U(\cdot)A_j x)(t)& \\
- (g_{\alpha_n-\alpha} * U(\cdot)Ax)(t)y &= (k * g_{m_n-1})(t)C_2 x.
\end{aligned}
\tag{117}
$$

(iii) A strongly continuous family $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)}) \subseteq L(Y, X) \times L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized (C_1, C_2)-existence and uniqueness family for (112) iff $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (112), and $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family for (112).

(iv) Suppose $Y = X$ and $C = C_1 = C_2$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C-resolvent family for (112) iff $(R(t))_{t\in[0,\tau)}$ is a k-regularized C-uniqueness family for (112), $R(t)A_j \subseteq A_jR(t)$, for $0 \leqslant j \leqslant n$ and $t \in [0,\tau)$, as well as $R(t)C = CR(t)$, $t \in [0,\tau)$, and $CA_j \subseteq A_jC$, for $0 \leqslant j \leqslant n$.

If $k(t) = g_{\zeta+1}(t)$, where $\zeta \geqslant 0$, then it is also said that $(E(t))_{t\in[0,\tau)}$ is a ζ-times integrated C_1-existence family for (112); 0-times integrated C_1-existence family for (112) is also said to be a C_1-existence family for (112). The above terminological agreement can be simply understood for all other classes of uniqueness and resolvent families introduced in Definition 2.3.29.

Albeit the choice of an SCLCS space Y different from X can produce a larger set of initial data for which the abstract Cauchy problem (112) has a strong solution (see e.g. [**545**, Example 2.5]), in our further work the most important case will be that in which $Y = X$. Keeping in mind that the operators A, B, $A_1, \ldots, A_{n-1}$ are closed, we can integrate the both sides of (116) sufficiently many times in order to see that:

(118)

$$BE^{(l)}(t)y + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * E^{(l)})(t)y - A(g_{\alpha_n-\alpha} * E^{(l)})(t)y = (k * g_{m_n-1-l})(t)C_1y,$$

for any $t \in [0,\tau)$, $y \in Y$ and $l \in \mathbb{N}^0_{m_n-1}$.

PROPOSITION 2.3.30. *Suppose that $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)})$ is a k-regularized (C_1, C_2)-existence and uniqueness family for* (112), *and $(U(t))_{t\in[0,\tau)}$ is locally equicontinuous. Then $C_2E(t)y = U(t)C_1y$, $t \in [0,\tau)$, $y \in Y$.*

PROOF. The proof of proposition is almost the same as the corresponding proof of [**346**, Proposition 3.2]. Observe only that we can always assume, without loss of generality, that the number α is less than or equal to α_1. $\square$

DEFINITION 2.3.31. (cf. [**346**, Definition 3.3]) Suppose $0 \leqslant i \leqslant m_n - 1$. Then we define $D_i' := \{j \in \mathbb{N}^0_{n-1} : m_j - 1 \geqslant i\}$, $D_i'' := \mathbb{N}^0_{n-1} \smallsetminus D_i'$ and

$$\mathbf{D}_i := \Big\{x \in \bigcap_{j\in D_i''} D(A_j) : A_ju_i \in R(C_1),\ j \in D_i''\Big\}.$$

It is not so predictable that [**346**, Theorem 3.4] continues to hold in the degenerate case without any terminological changes, and that the operator B does not appear in the definition of set $\mathbf{D}_i$, for which it is well known that represents, in non-degenerate case, the set which consists of all initial values for which the homogeneous counterpart of abstract Cauchy problem (112), with $B = I$ and $u_j = 0$, $j \in \mathbb{N}^0_{m_n-1} \smallsetminus \{i\}$, has a strong solution (provided that there exists a C_1-existence family for (112)). It is also worth nothing that we do not use the injectiveness of the operator B in (ii):

THEOREM 2.3.32. (i) *Suppose* $(E(t))_{t\in[0,\tau)}$ *is a* C_1*-existence family for* (112), $T\in(0,\tau)$, *and* $u_i\in\mathbf{D}_i$ *for* $0\leqslant i\leqslant m_n-1$. *Then the function*

$$u(t)=\sum_{i=0}^{m_n-1}u_ig_{i+1}(t)-\sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{\alpha_n-\alpha_j}*E^{(m_n-1-i)})(t)v_{i,j} \\ +\sum_{i=m}^{m_n-1}(g_{\alpha_n-\alpha}*E^{(m_n-1-i)})(t)v_{i,0},\quad 0\leqslant t\leqslant T, \tag{119}$$

is a strong solution of the problem (112) *on* $[0,T]$, *with* $f(t)\equiv 0$, *where* $v_{i,j}\in Y$ *satisfy* $A_ju_i=C_1v_{i,j}$ *for* $0\leqslant j\leqslant n-1$.

(ii) *Suppose* $(U(t))_{t\in[0,\tau)}$ *is a locally equicontinuous k-regularized* C_2*-uniqueness family for* (112), *and* $T\in(0,\tau)$. *Then there exists at most one strong (mild) solution of* (112) *on* $[0,T]$, *with* $u_i=0$, $i\in\mathbb{N}^0_{m_n-1}$.

PROOF. We will provide all the relevant details for the sake of completeness. Making use of (118), it can be easily verified that:

$$\begin{aligned}
&B\Big[u(\cdot)-\sum_{i=0}^{m_n-1}u_ig_{i+1}(\cdot)\Big]+\sum_{j=1}^{n-1}A_j\Big(g_{\alpha_n-\alpha_j}*\Big[u(\cdot)-\sum_{i=0}^{m_j-1}u_ig_{i+1}(\cdot)\Big]\Big)\\
&\quad=-\sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{\alpha_n-\alpha_j}*BE^{(m_n-1-i)})(\cdot)v_{i,j}\\
&\qquad+\sum_{i=m}^{m_n-1}(g_{\alpha_n-\alpha}*BE^{(m_n-1-i)})(\cdot)v_{i,0}\\
&+\sum_{j=1}^{n-1}A_j\Big(g_{\alpha_n-\alpha_j}*\Big\{\sum_{i=m_j}^{m_n-1}g_{i+1}(\cdot)u_i-\sum_{i=0}^{m_n-1}\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{\alpha_n-\alpha_l}*E^{(m_n-1-i)})(\cdot)v_{i,l}\\
&\qquad+\sum_{i=m}^{m_n-1}(g_{\alpha_n-\alpha}*E^{(m_n-1-i)})(\cdot)v_{i,0}\Big\}\Big)\\
&\quad=-\sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{\alpha_n-\alpha_j}*BE^{(m_n-1-i)})(\cdot)v_{i,j}\\
&\qquad+\sum_{i=m}^{m_n-1}(g_{\alpha_n-\alpha}*BE^{(m_n-1-i)})(\cdot)v_{i,0}\\
&+\sum_{j=1}^{n-1}\sum_{i=m_j}^{m_n-1}C_1v_{i,j}g_{\alpha_n-\alpha_j+i+1}(\cdot)-\sum_{i=0}^{m_n-1}\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_i}g_{\alpha_n-\alpha_l}*[-BE^{(m_n-1-i)}(\cdot)v_{i,l}\\
&\qquad+A(g_{\alpha_n-\alpha}*E^{(m_n-1-i)})(\cdot)v_{i,l}+g_{i+1}(\cdot)C_1v_{i,l}]\\
&\qquad+\sum_{i=m}^{m_n-1}g_{\alpha_n-\alpha}*[-BE^{(m_n-1-i)}(\cdot)v_{i,0}
\end{aligned}$$

$$+ A(g_{\alpha_n-\alpha} * R^{(m_n-1-i)})(\cdot)v_{i,0} + g_{i+1}(\cdot)C_1 v_{i,0}]$$
$$= g_{\alpha_n-\alpha} * A\Big[u(\cdot) - \sum_{i=0}^{m-1} u_i g_{i+1}(\cdot)\Big],$$

since

$$\sum_{j=1}^{n-1}\sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot) = \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot).$$

This implies that $u(t)$ is a mild solution of (112) on $[0,T]$. In order to complete the proof of (i), it suffices to show that $\mathbf{D}_t^{\alpha_n}u(t) \in C([0,T] : X)$ and $A_i\mathbf{D}_t^{\alpha_i}u \in C([0,T] : X)$ for all $i \in \mathbb{N}_n^0$. Towards this end, notice that the partial integration implies that, for every $t \in [0,T]$,

$$g_{m_n-\alpha_n} * \Big[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\Big](t) = \sum_{i=m}^{m_n-1}(g_{m_n-\alpha+i} * E^{(m_n-1)})(t)v_{i,0}$$
$$- \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{m_n-\alpha_j+i} * E^{(m_n-1)})(t)v_{i,j}.$$

Therefore, $\mathbf{D}_t^{\alpha_n}u \in C([0,T] : X)$ and, for every $t \in [0,T]$,

$$\mathbf{D}_t^{\alpha_n}u(t) = \frac{d^{m_n}}{dt^{m_n}}\Big\{g_{m_n-\alpha_n} * \Big[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\Big](t)\Big\}$$
$$= \sum_{i=m}^{m_n-1}(g_{i-\alpha} * E^{(m_n-1)})(t)v_{i,0} - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{i-\alpha_j} * E^{(m_n-1)})(t)v_{i,j}, \tag{120}$$

whence we may directly conclude that $B\mathbf{D}_t^{\alpha_n}u \in C([0,T] : X)$. Suppose, for the time being, $i \in \mathbb{N}_{n-1}^0$. Then $A_iu_j \in R(C_1)$ for $j \geqslant m_i$. Moreover, the inequality $l \geqslant \alpha_j$ holds provided $0 \leqslant l \leqslant m_n - 1$ and $j \in \mathbb{N}_{n-1} \smallsetminus D_l$, and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,T] : X)$ for $0 \leqslant j \leqslant n-1$ and $y \in Y$. Using (120), we have:

$$A_i\mathbf{D}_t^{\alpha_i}u(\cdot) = \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot)A_iu_j$$
$$- \sum_{l=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_l}[g_{l-\alpha_j} * A_i(g_{\alpha_n-\alpha_i} * E^{(m_n-1)})](\cdot)v_{l,j}$$
$$+ \sum_{l=m}^{m_n-1}[g_{l-\alpha} * A_i(g_{\alpha_n-\alpha_i} * E^{(m_n-1)})](\cdot)v_{l,0} \in C([0,T] : X),$$

finishing the proof of (i). The second part of theorem can be proved as follows. Suppose $u(t)$ is a strong solution of (112) on $[0,T]$, with $u_i = 0$, $i \in \mathbb{N}_{m_n-1}^0$. Making use of (117) and the equality

$$\int_0^t\int_0^{t-s} g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_ju(s)\,dr\,ds = \int_0^t\int_0^s g_{\alpha_n-\alpha_j}(r)U(t-s)A_ju(s-r)\,dr\,ds,$$

holding for any $t \in [0,T]$ and $j \in \mathbb{N}^0_{n-1}$, imply that

$$\begin{aligned}(UB*u)(t) &= (k*g_{m_n-1}C_2*u)(t)\\ &+ \int_0^t\int_0^{t-s}[g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_ju(s) - g_{\alpha_n-\alpha}(r)U(t-s-r)Au(s)]dr\,ds\\ &= (k*g_{m_n-1}C_2*u)(t) + (U*Bu)(t), \quad t \in [0,T].\end{aligned}$$

Therefore, $(k*g_{m_n-1}C_2*u)(t) = 0$, $t \in [0,T]$ and $u(t) = 0$, $t \in [0,T]$. $\square$

The standard proof of following theorem is omitted.

THEOREM 2.3.33. *Suppose $k(t)$ satisfies* (P1), *$(E(t))_{t\geqslant 0} \subseteq L(Y,X)$, $(U(t))_{t\geqslant 0} \subseteq L(X)$, $\omega \geqslant \max(0, \operatorname{abs}(k))$, $C_1 \in L(Y,X)$ and $C_2 \in L(X)$ is injective. Set $\mathbf{P}_\lambda := B + \sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}A_j - \lambda^{\alpha-\alpha_n}A$, $\lambda \in \mathbb{C}\smallsetminus\{0\}$.*

(i) (a) *Let $(E(t))_{t\geqslant 0}$ be a k-regularized C_1-existence family for* (112), *let the family $\{e^{-\omega t}E(t) : t \geqslant 0\}$ be equicontinuous, and let the family $\{e^{-\omega t}A_j(g_{\alpha_n-\alpha_j}*E)(t) : t \geqslant 0\}$ be equicontinuous* $(0 \leqslant j \leqslant n)$. *Then the following holds:*

$$\mathbf{P}_\lambda\int_0^\infty e^{-\lambda t}E(t)y\,dt = \tilde{k}(\lambda)\lambda^{1-m_n}C_1y, \quad y \in Y,\ \operatorname{Re}\lambda > \omega.$$

(b) *Let the operator $\mathbf{P}_\lambda$ be injective for every $\lambda > \omega$ with $\tilde{k}(\lambda) \neq 0$. Suppose, additionally, that there exist strongly continuous operator families $(W(t))_{t\geqslant 0} \subseteq L(Y,X)$ and $(W_j(t))_{t\geqslant 0} \subseteq L(Y,X)$ such that $\{e^{-\omega t}W(t) : t \geqslant 0\}$ and $\{e^{-\omega t}W_j(t) : t \geqslant 0\}$ are equicontinuous $(0 \leqslant j \leqslant n)$ as well as that:*

$$\int_0^\infty e^{-\lambda t}W(t)y\,dt = \tilde{k}(\lambda)\mathbf{P}_\lambda^{-1}C_1y$$

and

$$\int_0^\infty e^{-\lambda t}W_j(t)y\,dt = \tilde{k}(\lambda)\lambda^{\alpha_j-\alpha_n}A_j\mathbf{P}_\lambda^{-1}C_1y,$$

for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, $y \in Y$ and $j \in \mathbb{N}^0_n$. Then there exists a k-regularized C_1-existence family for (112), *denoted by $(E(t))_{t\geqslant 0}$. Furthermore, $E^{(m_n-1)}(t)y = W(t)y$, $t \geqslant 0$, $y \in Y$ and $A_j(g_{\alpha_n-\alpha_j}*E^{(m_n-1)})(t)y = W_j(t)y$, $t \geqslant 0$, $y \in Y$, $j \in \mathbb{N}^0_{n-1}$.*

(ii) *Suppose $(U(t))_{t\geqslant 0}$ is strongly continuous and the operator family $\{e^{-\omega t}U(t) : t \geqslant 0\}$ is equicontinuous. Then $(U(t))_{t\geqslant 0}$ is a k-regularized C_2-uniqueness family for* (112) *iff, for every $x \in \bigcap_{j=0}^n D(A_j)$, the following holds:*

$$\int_0^\infty e^{-\lambda t}U(t)\mathbf{P}_\lambda x\,dt = \tilde{k}(\lambda)\lambda^{1-m_n}C_2x, \quad \operatorname{Re}\lambda > \omega.$$

The assertion of [**346**, Theorem 3.7], concerning the inhomogeneous Cauchy problem (112), can be stated for degenerate multi-term problems without any terminological changes, as well:

THEOREM 2.3.34. *Suppose $(E(t))_{t\in[0,\tau)}$ is a locally equicontinuous C_1-existence family for* (112), *$T \in (0,\tau)$, and $u_i \in \mathbf{D}_i$ for $0 \leqslant i \leqslant m_n-1$. Let $f \in C([0,T] : X)$, let $g \in C([0,T] : Y)$ satisfy $C_1 g(t) = f(t)$, $t \in [0,T]$, and let $G \in C([0,T] : Y)$ satisfy $(g_{\alpha_n - m_n+1} * g)(t) = (g_1 * G)(t)$, $t \in [0,T]$. Then the function*

$$(121)\quad u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} (g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)})(t)v_{i,j} + \sum_{i=m}^{m_n-1} (g_{\alpha_n-\alpha} * E^{(m_n-1-i)})(t)v_{i,0} + \int_0^t E(t-s)G(s)ds, \quad 0 \leqslant t \leqslant T,$$

is a mild solution of the problem (113) *on $[0,T]$, where $v_{i,j} \in Y$ satisfy $A_j u_i = C_1 v_{i,j}$ for $0 \leqslant j \leqslant n-1$. If, additionally, $g \in C^1([0,T] : Y)$ and $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is locally equicontinuous, then the solution $u(t)$, given by* (121), *is a strong solution of* (112) *on $[0,T]$.*

Contrary to the assertion of [**346**, Theorem 3.7], the final conclusions of [**346**, Remark 3.8] cannot be proved for degenerate equations without imposing some additional conditions.

Concerning the action of subordination principles, we can state the following analogue of [**346**, Theorem 4.1] for degenerate multi-term problems (the final conclusions of [**346**, Remark 4.2] can be restated in our new setting, too).

THEOREM 2.3.35. *Suppose $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective and $\gamma \in (0,1)$.*

(i) *Let $\omega \geqslant \max(0, \mathrm{abs}(k))$, and let the assumptions of* Theorem 2.3.33(i)–(b) *hold. Put*

$$(122)\quad W_\gamma(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(t^{-\gamma}s)W(s)y\,ds, \quad t>0,\ y \in Y \text{ and } W_\gamma(0) := W(0).$$

Define, for every $j \in \mathbb{N}_n^0$ and $t \geqslant 0$, $W_{j,\gamma}(t)$ by replacing $W(t)$ in (122) *with $W_j(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ on $[0,\infty)$ satisfying* (P1) *and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma-1}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists an exponentially equicontinuous k_γ-regularized C_1-existence family $(E_\gamma(t))_{t\geqslant 0}$ for* (112), *with α_j replaced by $\alpha_j\gamma$ therein $(0 \leqslant j \leqslant n)$. Furthermore, the family $\{(1+t^{\lceil \alpha_n\gamma\rceil-2})^{-1}e^{-\omega^{1/\gamma}t}E_\gamma(t) : t \geqslant 0\}$ is equicontinuous.*

(ii) *Suppose $(U(t))_{t\geqslant 0}$ is a k-regularized C_2-uniqueness family for* (112), *and the family $\{e^{-\omega t}U(t) : t \geqslant 0\}$ is equicontinuous. Define, for every $t \geqslant 0$, $U_\gamma(t)$ by replacing $W(t)$ in* (122) *with $U(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ on $[0,\infty)$ satisfying* (P1) *and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma(2-m_n)-2+\lceil\alpha_n\gamma\rceil}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists a*

k_γ-regularized C_2-uniqueness family for (112)*, with α_j replaced by $\alpha_j\gamma$ therein ($0 \leqslant j \leqslant n$). Furthermore, the family $\{e^{-\omega^{1/\gamma}t}U_\gamma(t) : t \geqslant 0\}$ is equicontinuous.*

Of importance is the following abstract degenerate Volterra equation:

$$Bu(t) = f(t) + \sum_{j=0}^{n-1}(a_j * A_ju)(t), \quad t \in [0,\tau), \tag{123}$$

where $0 < \tau \leqslant \infty$, $f \in C([0,\tau) : X)$, $a_0,\dots,a_{n-1} \in L^1_{loc}([0,\tau))$, and $A = A_0,\dots,A_{n-1}, B$ are closed linear operators on X. By a mild solution, resp. strong solution, of (123), we mean any function $u \in C([0,\tau) : [D(B)])$ such that $A_j(a_j * u)(t) \in C([0,\tau) : X)$, $j \in \mathbb{N}^0_{n-1}$ and

$$Bu(t) = f(t) + \sum_{j=0}^{n-1} A_j(a_j * u)(t), \quad t \in [0,\tau),$$

resp. any function $u \in C([0,\tau) : [D(B)])$ such that $u(t) \in \bigcap_{j=0}^{n-1} D(A_j)$, $t \in [0,\tau)$, the mapping $t \mapsto A_ju(t)$, $t \in [0,\tau)$ is continuous for $j \in \mathbb{N}^0_{n-1}$, and (123) holds.

The following definition plays a crucial role in our investigation of problem (123).

Definition 2.3.36. (cf. [**346**, Definition 4.3] for the case $B = I$) Suppose $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y,[D(B)])$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (123) iff

$$BE(t)y = k(t)C_1y + \sum_{j=0}^{n-1} A_j(a_j * E)(t)y, \quad t \in [0,\tau),\ y \in Y.$$

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (123) iff

$$U(t)Bx = k(t)C_2x + \sum_{j=0}^{n-1}(a_j * A_jU)(t)x, \quad t \in [0,\tau),\ x \in \bigcap_{j=0}^{n} D(A_j).$$

As in non-degenerate case, we have the following:

(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (123). Then, for every $y \in Y$, the function $u(t) = E(t)y$, $t \in [0,\tau)$ is a mild solution of (123) with $f(t) = k(t)C_1y$, $t \in [0,\tau)$.

(ii) Let $(U(t))_{t\in[0,\tau)}$ be a locally equicontinuous k-regularized C_2-uniqueness family for (123). Then there exists at most one mild (strong) solution of (123).

The most important structural properties of k-regularized C_1-existence families for (123) and k-regularized C_2-uniqueness families for (123) are stated in the following analogue of Theorem 2.3.33.

THEOREM 2.3.37. *Suppose* $k(t)$ *and* $|a_0|(t),\ldots,|a_{n-1}|(t)$ *satisfy* (P1), $(E(t))_{t\geqslant 0} \subseteq L(Y,X)$, $(U(t))_{t\geqslant 0} \subseteq L(X)$, $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a_0|),\ldots, \operatorname{abs}(|a_{n-1}|))$, $C_1 \in L(Y,X)$ *and* $C_2 \in L(X)$ *is injective. Set*

$$\mathcal{P}_\lambda := B - \sum_{j=0}^{n-1} \widetilde{a_j}(\lambda) A_j, \quad \operatorname{Re}\lambda > \omega.$$

(i) (a) *Let* $(E(t))_{t\geqslant 0}$ *be a* k*-regularized* C_1*-existence family for* (123), *let the family* $\{e^{-\omega t}E(t) : t \geqslant 0\} \subseteq L(Y,[D(B)])$ *be equicontinuous, and let the family* $\{e^{-\omega t}A_j(a_j * E)(t) : t \geqslant 0\} \subseteq L(Y,X)$ *be equicontinuous* $(0 \leqslant j \leqslant n-1)$. *Then the following holds:*

$$\mathcal{P}_\lambda \int_0^\infty e^{-\lambda t} E(t) y\, dt = \tilde{k}(\lambda) C_1 y, \quad y \in Y, \ \operatorname{Re}\lambda > \omega.$$

(b) *Let the operator* $\mathcal{P}_\lambda$ *be injective for every* $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$. *Suppose, additionally, that there exist strongly continuous operator families* $(E(t))_{t\geqslant 0} \subseteq L(Y,X)$, $(E_B(t))_{t\geqslant 0} \subseteq L(Y,X)$, *and* $(E_j(t))_{t\geqslant 0} \subseteq L(Y,X)$ *such that the operator families* $\{e^{-\omega t}E(t) : t \geqslant 0\}$, $\{e^{-\omega t}E_B(t) : t \geqslant 0\}$, *and* $\{e^{-\omega t}E_j(t) : t \geqslant 0\}$ *are equicontinuous* $(0 \leqslant j \leqslant n-1)$ *as well as that:*

$$\int_0^\infty e^{-\lambda t} E(t) y\, dt = \tilde{k}(\lambda) \mathcal{P}_\lambda^{-1} C_1 y, \quad \int_0^\infty e^{-\lambda t} E_B(t) y\, dt = \tilde{k}(\lambda) B \mathcal{P}_\lambda^{-1} C_1 y$$

and

$$\int_0^\infty e^{-\lambda t} E_j(t) y\, dt = \tilde{k}(\lambda) \widetilde{a_j}(\lambda) A_j \mathcal{P}_\lambda^{-1} C_1 y,$$

for every $\lambda > \omega$ *with* $\tilde{k}(\lambda) \neq 0$, $y \in Y$ *and* $j \in \mathbb{N}_{n-1}^0$. *Then* $(E(t))_{t\geqslant 0}$ *is a* k*-regularized* C_1*-existence family for* (123). *Furthermore,* $BE(t)y = E_B(t)y$, $t \geqslant 0$, $y \in Y$ *and* $A_j(a_j * E)(t)y = E_j(t)y$, $t \geqslant 0$, $y \in Y$, $j \in \mathbb{N}_{n-1}^0$.

(ii) *Suppose* $(U(t))_{t\geqslant 0}$ *is strongly continuous and the operator family* $\{e^{-\omega t}U(t) : t \geqslant 0\} \subseteq L(X)$ *is equicontinuous. Then* $(U(t))_{t\geqslant 0}$ *is a* k*-regularized* C_2*-uniqueness family for* (123) *iff, for every* $x \in \bigcap_{j=0}^n D(A_j)$, *the following holds:*

$$\int_0^\infty e^{-\lambda t} U(t) \mathcal{P}_\lambda x\, dt = \tilde{k}(\lambda) C_2 x, \quad \operatorname{Re}\lambda > \omega.$$

The proof of following subordination principle is standard and therefore omitted; observe that we correct here some inconsistencies made in the formulation of [**346**, Theorem 4.4].

THEOREM 2.3.38. (i) *Suppose that the requirements of* Theorem 2.3.37 (i)–(b) *hold. Let* $c(t)$ *be completely positive, let* $c(t)$, $k(t)$, $k_1(t)$, $|a_0|(t),\ldots, |a_{n-1}|(t)$ *and* $|b_0|(t),\ldots,|b_{n-1}|(t)$ *satisfy* (P1), *and let* $\omega_0 > 0$ *be such that,* *for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, the following holds:*

$$\widetilde{a_j}(1/\tilde{c}(\lambda)) = \widetilde{b_j}(\lambda), \quad j \in \mathbb{N}^0_{n-1} \text{ and } \widetilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}(1/\tilde{c}(\lambda)). \tag{124}$$

Then for each $r \in (0,1]$ there exists a locally Hölder continuous (with exponent r), exponentially equicontinuous $(k_1 * g_r)$-regularized C_1-existence family for

$$Bu(t) = f(t) + \sum_{j=0}^{n-1}(b_j * A_j u)(t), \quad t \in [0,\tau). \tag{125}$$

(ii) *Suppose that the requirements of* Theorem 2.3.37(ii) *hold. Let $c(t)$ be completely positive, let $c(t)$, $k(t)$, $k_1(t)$, $|a_0|(t), \ldots, |a_{n-1}|(t)$ and $|b_0|(t), \ldots, |b_{n-1}|(t)$ satisfy* (P1)*, and let $\omega_0 > 0$ be such that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$,* (124) *holds. Then for each $r \in (0,1]$ there exists a locally Hölder continuous (with exponent r), exponentially equicontinuous $(k_1 * g_r)$-regularized C_2-uniqueness family for* (125).

The interested reader may try to transfer the final conclusions of [**347**, Theorem 2.1, Theorem 2.2, Remark 2.1, Proposition 2.1] to degenerate multi-term fractional differential equations. Concerning [**347**, Theorem 2.3], we first need to introduce the following notion.

Definition 2.3.39. A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) (k, C_2)-uniqueness family for (112) iff, for every $t \in [0,\tau)$ and $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$, the following holds:

$$U(t)Bx + \sum_{j=1}^{n-1}(g_{\alpha_n-\alpha_j} * U(\cdot)A_j x)(t) - (g_{\alpha_n-\alpha} * U(\cdot)Ax)(t)x = k(t)C_2 x.$$

Then it is clear that for any strongly continuous operator family $(U(t))_{t\in[0,\tau)}$ the following equivalence relation holds: $(U(t))_{t\in[0,\tau)}$ is a (local) $(k * g_{m_n-1}, C_2)$-uniqueness family for (112) iff $(U(t))_{t\in[0,\tau)}$ is a (local) k-regularized C_2-uniqueness family for (112).

Consider now the perturbed equation:

$$\begin{aligned} B\mathbf{D}_t^{\alpha_n}u(t) + \sum_{i=1}^{n-1}(A_i + F_i)\mathbf{D}_t^{\alpha_i}u(t) &= (A+F)\mathbf{D}_t^{\alpha}u(t) + f(t), \quad t \geqslant 0; \\ u^{(j)}(0) &= u_j, \quad j = 0, \ldots, \lceil \alpha_n \rceil - 1, \end{aligned} \tag{126}$$

where $F_i \in L(X)$ for $0 \leqslant i \leqslant n-1$ and $F_0 \equiv F$. A similar line of reasoning as in the proof of [**347**, Theorem 2.3] shows that the following result about the C-wellposedness of problem (126) holds good (observe that the employed method is based on the arguments contained in the proof of [**459**, Theorem 6.1], which does not work any longer if we replace the term $B\mathbf{D}_t^{\alpha_n}u(t)$, in (126), with $(B + F_n)\mathbf{D}_t^{\alpha_n}u(t)$):

THEOREM 2.3.40. (i) *Suppose* $Y = X$, $(E(t))_{t\in[0,\tau)} \subseteq L(X)$ *is a (local)* C_1*-existence family for* (112), $E_j \in L(X)$ *and* $F_j = C_1E_j$ $(j \in \mathbb{N}_{n-1}^0)$. *Suppose that the following conditions hold:*

(a) *For every* $p \in \circledast_X$ *and for every* $T \in (0,\tau)$, *there exists* $c_{p,T} > 0$ *such that*

$$p(E^{(m_n-1)}(t)x) \leqslant c_{p,T}p(x), \quad x \in X, \ t \in [0,T].$$

(b) *For every* $p \in \circledast_X$, *there exists* $c_p > 0$ *such that*

$$p(E_jx) \leqslant c_pp(x), \quad j \in \mathbb{N}_{n-1}^0, \ x \in X.$$

(c) $\alpha_n - \alpha_{n-1} \geqslant 1$ *and* $\alpha_n - \alpha \geqslant 1$.

Then there exists a (local) C_1*-existence propagation family* $(R(t))_{t\in[0,\tau)}$ *for* (126). *If* $\tau = \infty$ *and if, for every* $p \in \circledast_X$, *there exist* $M \geqslant 1$ *and* $\omega \geqslant 0$ *such that*

$$p(E^{(m_n-1)}(t)x) \leqslant Me^{\omega t}p(x), \quad t \geqslant 0, \ x \in X, \tag{127}$$

respectively (127) *and*

$$p(BE^{(m_n-1)}(t)x) \leqslant Me^{\omega t}p(x), \quad t \geqslant 0, \ x \in X, \tag{128}$$

then $(R(t))_{t\geqslant 0}$ *is exponentially equicontinuous, and moreover,* $(R(t))_{t\geqslant 0}$ *also satisfies the condition* (127), *repectively* (127) *and* (128), *with possibly different numbers* $M \geqslant 1$ *and* $\omega > 0$.

(ii) *Suppose* $Y = X$, $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ *is a (local)* $(1, C_2)$*-uniqueness family for* (112), $E_j \in L(E)$ *and* $F_j = E_jC_2$ $(j \in \mathbb{N}_{n-1}^0)$. *Suppose that* (b)–(c) *hold, and that* (a) *holds with* $(E^{(m_n-1)}(t))_{t\in[0,\tau)}$ *replaced by* $(U(t))_{t\in[0,\tau)}$ *therein. Then there exists a (local)* $(1, C_2)$*-uniqueness family* $(W(t))_{t\in[0,\tau)}$ *for* (126). *If* $\tau = \infty$ *and if, for every* $p \in \circledast_X$, *there exist* $M \geqslant 1$ *and* $\omega \geqslant 0$ *such that* (127) *holds, then* $(W(t))_{t\geqslant 0}$ *is exponentially equicontinuous, and moreover,* $(W(t))_{t\geqslant 0}$ *also satisfies the condition* (127), *with possibly different numbers* $M \geqslant 1$ *and* $\omega > 0$.

Concerning the existence of strong solutions of (112), we can prove the following slight extension of [**347**, Theorem 3.1]; this result can be viewed of some independent interest and details of proof will be given later, when we will be analyzing abstract degenerate multi-term fractional differential equations with Riemann–Liouville derivatives.

THEOREM 2.3.41. (cf. alsoTheorem 2.4.6) *Suppose* A, B, $A_1, \ldots, A_{n-1}$ *are closed linear operators on* X, $\omega > 0$, $L(X) \ni C$ *is injective and* $u_0, \ldots, u_{m_n-1} \in X$. *Set* $P_\lambda := \lambda^{\alpha_n-\alpha}B + \sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - A$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$. *Let the following conditions hold:*

(i) *The operator* P_λ *is injective for* $\lambda > \omega$ *and* $D(P_\lambda^{-1}C) = X$, $\lambda > \omega$.

(ii) *If* $0 \leqslant j \leqslant n$, $0 \leqslant k \leqslant m_n - 1$, $m - 1 < k$, $1 \leqslant l \leqslant n$, $m_l - 1 \geqslant k$ *and* $\lambda > \omega$, *then* $Cu_k \in D(P_\lambda^{-1}A_l)$,

$$A_j\Biggl\{\lambda^{\alpha_j}\Biggl[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k+\sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\Biggr] \\ -\sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\Biggr\}\in LT-X \tag{129}$$

and

$$\lambda^{\alpha_n}\Biggl[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k+\sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\Biggr] \\ -\lambda^{\alpha_n-1-k}Cu_k\in LT-X. \tag{130}$$

(iii) *If* $0\leqslant j\leqslant n$, $0\leqslant k\leqslant m_n-1$, $m-1\geqslant k$, $\mathbb{N}_{n-1}\smallsetminus D_k\neq\emptyset$, $s\in\mathbb{N}_{n-1}\smallsetminus D_k$ *and* $\lambda>\omega$, *then* $Cu_k\in D(A_s)$, $\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\in D(P_\lambda^{-1})$,

$$A_j\Biggl\{\lambda^{\alpha_j}\Biggl[\lambda^{-k-1}Cu_k-P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\Biggr] \\ -\sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\Biggr\}\in LT-X \tag{131}$$

and

$$\lambda^{\alpha_n}\Biggl[\lambda^{-k-1}Cu_k-P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\Biggr] \\ -\lambda^{\alpha_n-1-k}Cu_k\in LT-X. \tag{132}$$

Then the abstract Cauchy problem (112) *has a strong solution, with* u_k *replaced by* Cu_k $(0\leqslant k\leqslant m_n-1)$.

REMARK 2.3.42. Let $0\leqslant k\leqslant m_n-1$ and $m-1<k$. Then Theorem 2.3.41 continues to hold if we replace the term

$$\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k+\sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k$$

i.e., the Laplace transform of $u_k(t)$, in (129)–(130) by

$$\lambda^{-k-1}Cu_k-\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k+\lambda^{-k-1}P_\lambda^{-1}ACu_k;$$

in this case, it is indispensable to assume that $Cu_k\in D(P_\lambda^{-1}A_l)$, provided $0\leqslant l\leqslant n-1$, $k>m_l-1$ and $\lambda>\omega$. Let us also observe that a similar modification can be made in the case $0\leqslant k\leqslant m_n-1$ and $m-1\geqslant k$. Strictly speaking, one can replace the term

$$\lambda^{-k-1}Cu_k-P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\smallsetminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k$$

i.e., the Laplace transform of $u_k(t)$, in (131)–(132) by

$$\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}BCu_k+\sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k-\lambda^{-k-1}P_\lambda^{-1}ACu_k;$$

in this case, one has to assume that $Cu_k \in D(P_\lambda^{-1}A_l)$, provided $0 \leqslant l \leqslant n$, $m_l - 1 \geqslant k$ and $\lambda > \omega$.

This is the right time to illustrate our results with some examples.

EXAMPLE 2.3.43. Suppose $1 \leqslant p < \infty$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded domain with smooth boundary, and $X := L^p(\Omega)$. Consider the equation

$$\begin{gathered}(\alpha-\Delta)u_{tt}=\beta\Delta u_t+\Delta u+\int_0^t g(t-s)\Delta u(s,x)ds, \quad t>0,\ x\in\Omega;\\ u(0,x)=\phi(x), \quad u_t(0,x)=\psi(x),\end{gathered} \tag{133}$$

where $g \in L^1_{loc}([0,\infty))$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \smallsetminus \{0\}$. As explained by M. V. Falaleev and S. S. Orlov in [**174**], the equation (133) appears in the models of nonlinear viscoelasticity provided $n = 3$. Integrating (133) twice with the respect to the time-variable t, we obtain the associated integral equation

$$\begin{aligned}(\alpha-\Delta)u(t)=(\alpha+(\beta-1)\Delta)\phi(x)+t(\alpha-\Delta)\psi+\beta\Delta(g_1*u)(t)\\ +\Delta(g_2*u)+\Delta(g_2*g*u)(t),\end{aligned} \tag{134}$$

which is of the form (123) with $B := \alpha - \Delta$, $A_2 := \beta\Delta$, $A_1 = A_0 := \Delta$ (acting with the Dirichlet boundary conditions) and $a_2(t) := g_1(t)$, $a_1(t) := g_2(t)$, $a_0(t) := (g_2 * g)(t)$. Then

$$\mathcal{P}_\lambda=\frac{\lambda^2+\beta\lambda+\tilde{g}(\lambda)+1}{\lambda^2}\Bigg[\frac{\alpha\lambda^2}{\lambda^2+\beta\lambda+\tilde{g}(\lambda)+1}-\Delta\Bigg].$$

(i) In the first part of our analysis, we assume that $\alpha > 0$ and $g(t)$ is of the following form:

$$g(t)=\sum_{j=0}^{l}c_jg_{\beta_j}(t)+f(t), \quad t>0,$$

where $l \in \mathbb{N}$, $c_j \in \mathbb{C}$ $(0 \leqslant j \leqslant l)$, $0 < \beta_1 < \cdots < \beta_l < 1$ and the function $f(t)$ satisfies the requirements of [**296**, Theorem 3.4(i)-(a)] with $\alpha = \pi/2$ and $\omega > 0$ sufficiently large. Using the fact that the operator Δ generates a bounded analytic C_0-semigroup of angle $\pi/2$, and the resolvent equation, it can be simply verified that

$$\frac{1}{\lambda}\mathcal{P}_\lambda^{-1}\in LT{-}L(X),\ \frac{1}{\lambda}B\mathcal{P}_\lambda^{-1}\in LT{-}L(X)\text{ and }\frac{\widetilde{a_j}(\lambda)}{\lambda}\mathcal{P}_\lambda^{-1}\in LT{-}L(X),\ j=0,1,2.$$

This implies by Theorem 2.3.37 that there exists an exponentially bounded I-existence family $(E(t))_{t\geqslant 0}$ for (134), satisfying additionally that for each $f \in X$ the mappings $t \mapsto E(t)f$, $t > 0$, $t \mapsto BE(t)f$, $t > 0$ and $t \mapsto A_j(a_j * E)(t)f$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$; furthermore, $(E(t))_{t\geqslant 0}$ is an exponentially bounded I-uniqueness family for (134). Therefore, for every $\phi,\ \psi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, there exists a unique strong solution

$$u(t) = E(t)(\alpha + (\beta - 1)\Delta)\phi + \int_0^t E(s)(\alpha - \Delta)\psi\, ds, \quad t \geqslant 0,$$

of the integral equation (134), and $u(t)$ can be analytically extended to the sector $\Sigma_{\pi/2}$.

(ii) Denote by $\{\lambda_k\}$ $[= \sigma(\Delta)]$ the eigenvalues of the Dirichlet Laplacian Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leqslant -\lambda_2 \ldots \leqslant -\lambda_k \leqslant \cdots \to +\infty$ as $k \to \infty$; cf. Example 2.3.48-Example 2.3.53 for more details); by $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal [in the sense of $L^2(\Omega)$] eigenfunctions. In this part, we will delve into the case in which $\alpha = \lambda_{k_0} \in \sigma(\Delta)$ for some $k_0 \in \mathbb{N}$ and the function $g(t)$ satisfies (P1). Then there exist finite constants $M \geqslant 1$ and $\omega \geqslant 0$ such that $|\int_0^t g(s)ds| \leqslant Me^{\omega t}$, $t \geqslant 0$ and $\lambda \int_0^\infty e^{-\lambda t} \int_0^t g(s)ds\, dt = \int_0^\infty e^{-\lambda t} g(t)dt$, $\lambda > \omega$ [**27**], which simply implies that the set $\{\tilde{g}(\lambda) : \lambda > \omega + 1\}$ is bounded. Define $D\colon L^2(\Omega) \to L^2(\Omega)$ by $Df := (-1)\beta^{-1} \sum_{k=1}^\infty \langle \phi_k, f\rangle \phi_k$, $f \in L^2(\Omega)$. Using Parseval's equality, it can be simply verified that $D,\ BD \in L(L^2(\Omega))$; furthermore, $\|R(\lambda : \Delta)\| = O(|\alpha - \lambda|^{-1})$ as $\lambda \to \alpha$ (see e.g. Example 2.3.48). Using the resolvent equation and these facts, we obtain the existence of a sufficiently large real number $R > 0$ such that $\mathbf{P}_\lambda^{-1} \in L(L^2(\Omega))$ for $|\lambda| \geqslant R$, as well as that

$$|\lambda|^{-2} \Bigg[\|\mathbf{P}_\lambda^{-1}\| + \|B\mathbf{P}_\lambda^{-1}\| + \sum_{j=0}^2 \|\tilde{a_j}(\lambda) A_j \mathbf{P}_\lambda^{-1}\| \Bigg] \leqslant M, \quad |\lambda| \geqslant R, \tag{135}$$

$\lim_{|\lambda|\to\infty} \lambda^{-1}\mathbf{P}_\lambda^{-1} f = Df$, $\lim_{|\lambda|\to\infty} \lambda^{-1} B\mathbf{P}_\lambda^{-1} f = BDf$ and $\lim_{|\lambda|\to\infty} \lambda^{-1}\tilde{a_j}(\lambda)\mathbf{P}_\lambda^{-1} f = 0$, $0 \leqslant j \leqslant 2$ $(f \in L^2(\Omega))$. Making use of Theorem 2.3.37, we obtain that there exists an exponentially bounded once integrated I-existence family $(E_1(t))_{t\geqslant 0}$ for (134), satisfying additionally that for each $f \in L^2(\Omega)$ the mappings $t \mapsto E_1(t)f$, $t > 0$, $t \mapsto BE_1(t)f$, $t > 0$ and $t \mapsto A_j(a_j * E_1)(t)f$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$; furthermore, $(E_1(t))_{t\geqslant 0}$ is an exponentially bounded once integrated I-uniqueness family for (134). Therefore, for every $\phi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique strong solution of the associated once integrated problem

$$\begin{aligned}(\alpha - \Delta)u(t) = {} & t(\alpha + (\beta - 1)\Delta)\phi(x) + \frac{t^2}{2}(\alpha - \Delta)\psi \\ & + \beta\Delta(g_1 * u)(t) + \Delta(g_2 * u) + \Delta(g_2 * g * u)(t), \quad t \geqslant 0,\end{aligned} \tag{136}$$

given by $u(t) = E_1(t)(\alpha + (\beta - 1)\Delta)\phi + \int_0^t E_1(s)(\alpha - \Delta)\psi\, ds$, $t \geqslant 0$. It should be noticed that the mappings $t \mapsto u(t)$, $t > 0$ and $t \mapsto Bu(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$ and that it is not clear whether there exist analytical extensions of these mappings to the sector Σ_π in our general choice of function $g(t)$ (in [**174**, Theorem 5], the authors have investigated the existence and uniqueness of generalized ($C^2([0,\infty) : L^2(\Omega))$) solutions of problem (134)). On the other hand, (135), the equality $\lim_{|\lambda|\to\infty} \lambda^{-1} B\mathcal{P}_\lambda^{-1} f = BDf$, Theorem 3.2.19 and Remark 3.1.2(v) taken together imply that for each $\theta \in (-\pi, \pi]$ the MLO $e^{i\theta} AB^{-1}$ generates an exponentially bounded, analytic once integrated $(b+g_2(t)+(g_2*g)(t), I)$-regularized resolvent family $(E_{1,B}(t) \equiv BE_1(t))_{t\geqslant 0}$ of angle $\pi/2$; cf. Section 3.2 for the notion. Since [**292**, Theorem 2.1.29(ii)] holds in the MLO framework, this immediately yields some results on the existence and uniqueness of analytical (entire) solutions of the problem (136) with the term $t(\alpha + (\beta - 1)\Delta)\phi + \frac{t^2}{2}(\alpha - \Delta)\psi$ replaced by a general inhomogeneity $f(t)$.

(iii) Consider the equation

$$(137)\qquad \begin{aligned} (\nu - \Delta)u_t = \Delta u + \int_0^t g(t-s)\Delta u(s,x)ds, \quad t > 0,\ x \in \Omega;\\ u(0,x) = \phi(x), \quad x \in \Omega, \end{aligned}$$

in $L^2(\Omega)$, where $g \in L^1_{loc}([0,\infty))$ and $\nu = \lambda_{k_0} \in \sigma(\Delta)$. This equation appears in the study of non-linear dynamics of hereditarily elastic bodies (cf. [**174**, Theorem 4]). Clearly, in our new setting, we have that $B = v - \Delta$, $n = 1$, $a_0(t) = g_1(t)$, $a_1(t) = (g_1 * g)(t)$ and $A_0 = A_1 = \Delta$. Concerning the function $g(t)$, we assume that $\mathrm{abs}(|g|) < \infty$. Arguing as in (ii), we get that there exists an exponentially bounded once integrated I-existence family $(E_1(t))_{t\geqslant 0}$ for (137), satisfying additionally that for each $f \in L^2(\Omega)$ the mappings $t \mapsto E_1(t)f$, $t > 0$, $t \mapsto BE_1(t)f$, $t > 0$ and $t \mapsto A_j(a_j * E_1)(t)f$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$, as well as that $(E_1(t))_{t\geqslant 0}$ is an exponentially bounded once integrated I-uniqueness family for (137). This, in turn, implies that there exists a unique strong solution $t \mapsto u(t)$, $t \geqslant 0$ of the associated once integrated equation:

$$(v - \Delta)u(t) = t(v - \Delta)\phi + \Delta(g_1 * u)(t) + \Delta(g_1 * g * u), \quad t \geqslant 0.$$

Furthermore, the mappings $t \mapsto u(t)$, $t > 0$ and $t \mapsto Bu(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$.

EXAMPLE 2.3.44. Suppose $1 < p < \infty$, $X := L^p(\mathbb{R}^n)$, $1/2 < \gamma \leqslant 1$, $Q \in \mathbb{N} \setminus \{1\}$, $P_1(x) = \sum_{|\eta|\leqslant Q} a_\eta x^\eta$, $P_2(x) = -1 - |x|^2$ ($x \in \mathbb{R}^n$), $P_1(x)$ is positive, $\sigma \geqslant 0$, the estimate

$$\Bigl| D^\eta \Bigl(\frac{P_1(x)}{P_2(x)} \Bigr) \Bigr| \leqslant c_\eta (1 + |x|)^{|\eta|(\sigma - 1)}, \quad x \in \mathbb{R}^n$$

holds for each multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, $V_2 \geqslant 0$ and for each $\eta \in \mathbb{N}_0^n$ there exists $M_\eta > 0$ such that $|D^\eta (P_2(x)^{-1})| \leqslant M_\eta (1 + |x|)^{|\eta|(V_2 - 1)}$, $x \in \mathbb{R}^n$. Set $A_2 := \Delta - I$, $A_0 f := \sum_{|\eta|\leqslant Q} a_\eta D^\eta f$ with maximal distributional domain, where $D^\eta \equiv (-i)^{|\eta|} f^\eta$, and $C_1 := (I - \Delta)^{-\frac{n}{2}|\frac{1}{p} - \frac{1}{2}|\max(\sigma, V_2)}$. Let $E_i \in L(X)$ and $F_i = C_1 E_i$ ($i = 0, 1$). Then we know (cf. [**306, 307**]) that $\lambda(\lambda^2 A_2 - A_0)^{-1} C_1 \in LT - L(X)$

and $\lambda A_2(\lambda^2 A_2 - A_0)^{-1}C_1 \in LT - L(X)$, which implies by Theorem 2.3.37(i)–(b) that there exists an exponentially bounded C_1-existence family $(E(t))_{t\geqslant 0}$ for the following degenerate second order Cauchy problem:

$$\begin{cases}(\Delta - I)u_{tt}(t,x) = \sum_{|\eta|\leqslant Q} a_\eta D^\eta u(t,x), \quad t \geqslant 0,\ x \in \mathbb{R}^n,\\ u(0,x) = u_0(x) = \phi(x), \quad u_t(0,x) = u_1(x) = \psi(x),\end{cases}$$

obeying the properties (127)–(128) stated in the formulation of Theorem 2.3.40. Applying Theorem 2.3.40, we get there exists an exponentially bounded C_1-existence family $(R(t))_{t\geqslant 0}$ for the following degenerate second order Cauchy problem:

$$(P)' : \begin{cases}(\Delta - I)u_{tt}(t,x) + F_1 u_t(t,x) = (\sum_{|\eta|\leqslant Q} a_\eta D^\eta + F_0)u(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^n,\\ u(0,x) = u_0(x) = \phi(x), \quad u_t(0,x) = u_1(x) = \psi(x).\end{cases}$$

Then Theorem 2.3.32(i) shows that there exists a strong solution of problem $(P)'$ provided that

$$\phi, \psi \in \mathrm{S}^{Q+n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n), \quad (A_0 + F)\phi \in \mathrm{S}^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n),$$
$$F_1\psi \in \mathrm{S}^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n) \text{ and } (A_0 + F)\psi \in \mathrm{S}^{n|\frac{1}{p}-\frac{1}{2}|\max(\sigma,V_2),p}(\mathbb{R}^n).$$

If we denote by $U(t,x)$, resp. $V(t,x)$, the corresponding strong solution of problem $(P)'$ with the initial values $\phi(x)$ and $\psi(x) \equiv 0$, resp. $\phi(x) \equiv 0$ and $\psi(x)$, then one can simply verify that the function

$$u(t,x) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})U(s,x)ds + \int_0^t g_{1-\gamma}(t-s)\int_0^\infty s^{-\gamma}\Phi_\gamma(rs^{-\gamma})V(r,x)dr\,ds,$$

is a strong solution of the following integral equation

$$(138)\quad A_2[u(t,x) - \phi(x) - t\psi(x)] + F_1\int_0^t g_\gamma(t-s)[u(s,x) - \phi(x)]ds$$
$$= \int_0^t g_{2\gamma}(t-s)(A_0 + F)u(s,x)ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n;$$

furthermore, the function $t \mapsto u(t,\cdot) \in X$ can be analytically extended to the sector $\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}$. In the present situation, we can only prove that there is at most one strong solution of the integral equation (138) provided that $p = 2$. Strictly speaking, suppose that $u(t,x)$ is a strong solution of (138) with $\phi(x) \equiv \psi(x) \equiv 0$. Then $A_2^{-1} \in L(X)$, $C_1 = I$ and the function $v(t,x) := A_2 u(t,x)$ is a strong solution of the following non-degenerate integral equation

$$(139)\quad u(t,x) + \int_0^t g_\gamma(t-s)F_1 A_2^{-1}u(s,x)ds$$
$$= \int_0^t g_{2\gamma}(t-s)(A_0A_2^{-1} + FA_2^{-1})u(s,x)ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Since $\lambda(\lambda^2 - A_0A_2^{-1})^{-1} = \lambda A_2(\lambda^2 A_2 - A_0)^{-1} \in LT - L(X)$, the operator $A_0A_2^{-1}$ generates a cosine operator function and we can apply Theorem 2.3.40(ii) in order to see that there exists an exponentially bounded $(1, I)$-uniqueness family for (139), with the meaning clear. This proves the claimed assertion on the uniqueness of strong solutions of problem (138). In general case $p \neq 2$, it is not clear how we can prove that there is at most one strong solution of the integral equation (138) without assuming that F_1 and F take some specific forms.

In the remainder of this section, we analyze entire and analytical properties of degenerate resolvent operator families introduced above and apply obtained results in the study of abstract Boussinesq-Love equation and the abstract Barenblatt–Zheltov–Kochina equation (cf. also Example 2.11.19 and the paragraph directly before Subsection 2.2.4; for some other references, one may refer e.g. to [**96**] and [**98**]). We reconsider some results obtained by G. A. Sviridyuk and A. A. Zamyshlyaeva in [**515**, Section 5], and slightly improve the assertion of [**509**, Theorem 5.1.3(ii)] in L^2 type spaces. We divide our investigation into two parts. In the first part, our standing hypothesis is that the orders α, $\alpha_1, \ldots, \alpha_n$ of Caputo derivatives $\mathbf{D}_t^{\alpha}u(t)$, $\mathbf{D}_t^{\alpha_1}u(t), \ldots, \mathbf{D}_t^{\alpha_n}u(t)$, appearing in (112), are non-negative integers; in the second part, where we analyze some fractional analogues of the abstract Boussinesq–Love equation and the abstract Barenblatt–Zheltov–Kochina equation, some of orders α, $\alpha_1, \ldots, \alpha_n$ can be purely fractional numbers.

DEFINITION 2.3.45. (cf. also Definition 2.3.50 and Definition 2.6.1 below) Let $\alpha_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}_n^0$, and let the function $u \in C^{\alpha_n - 1}([0,\infty) : X)$ be a strong solution of problem (112). Then we say that $u(\cdot)$ is an entire solution of problem (112) iff the functions $u(\cdot)$ and $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\ldots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be analytically extended from the interval $[0,\infty)$ to the whole complex plane.

Now we are ready to formulate the following theorem.

THEOREM 2.3.46. *Suppose $k(t)$ satisfies* (P1), *$C_1 \in L(Y,X)$ and $C_2 \in L(X)$ is injective. Let $\alpha_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}_n^0$, and let there exist a locally equicontinuous k-regularized C_2-uniqueness family for* (112). *Let the following hold:*

- *Suppose that there exists a sufficiently large number $R > 0$ such that the operator $\mathbf{P}_z$ is injective for $|z| \geqslant R$, as well as that the operator families $\{\mathbf{P}_z^{-1}C_1 : |z| \geqslant R\} \subseteq L(Y,X)$ and $\{z^{\alpha_j - \alpha_n}A_j\mathbf{P}_z^{-1}C_1 : |z| \geqslant R$, $j \in \mathbb{N}_n^0\} \subseteq L(Y,X)$ are equicontinuous.*
- *Let the mappings $z \mapsto \mathbf{P}_z^{-1}C_1y \in X$, $|z| > R$ and $z \mapsto z^{\alpha_j - \alpha_n}A_j\mathbf{P}_z^{-1}C_1y \in X$, $|z| > R$ be analytic for any $y \in Y$, $j \in \mathbb{N}_n^0$, and let there exist operators $D, D_0, D_1, \ldots, D_n \in L(Y,X)$ such that $\lim_{z\to\infty} \mathbf{P}_z^{-1}C_1y = Dy$ and $\lim_{z\to\infty} z^{\alpha_j - \alpha_n}A_j\mathbf{P}_z^{-1}C_1y = D_jy$ for any $y \in Y$, $j \in \mathbb{N}_n^0$.*
- *Suppose that $u_i \in \mathbf{D}_i$ for $0 \leqslant i \leqslant m_n - 1$, $v_{i,j} \in Y$ satisfy $A_ju_i = C_1v_{i,j}$ for $0 \leqslant j \leqslant n-1$, as well as that $g \in C^1([0,\infty) : Y)$ and $C_1g(t) = f(t)$, $t \geqslant 0$.*

Then there exists a unique strong solution of (112).

- *Assume, additionally, that the function $t \mapsto g(t)$, $t \geqslant 0$ can be analytically extended to the whole complex plane, resp., to a continuously differentiable function $\mathbb{R} \mapsto Y$.*

Then there exists a unique entire solution $u(\cdot)$ of (112), *resp., the function $t \mapsto u(t)$, $t \geqslant 0$ can be extended to an α_n-times continuously differentiable function*

$\mathbb{R} \mapsto X$ and the functions $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\dots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to continuously differentiable functions $\mathbb{R} \mapsto X$. Furthermore, in any case set out above, the existence of a positive real number $\omega' > 0$ such that the set $\{e^{-\omega' s}g(s) : s \geqslant 0\}$, resp., $\{e^{-\omega' z}g(z) : z \in \mathbb{C}\}$ ($\{e^{-\omega'|s|}g(s) : s \in \mathbb{R}\}$) is bounded in Y, implies the existence of a positive real number $\omega'' > 0$ such that the set $\{e^{-\omega'' s}u(s) : s \geqslant 0\}$, resp., $\{e^{-\omega'' z}u(z) : z \in \mathbb{C}\}$ ($\{e^{-\omega''|s|}u(s) : s \in \mathbb{R}\}$) is bounded in X.

PROOF. Let $\beta \in (-\pi, \pi]$. Then, for every $\theta \in (0, \pi/2)$, there exists a sufficiently large number $\omega_{\beta,\theta} > 0$ satisfying that the function $q_{\beta,\theta}(z) := z^{-1}\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 \in L(Y, X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0, \theta)$ the operator family $\{z^{-1}(z - \omega_{\beta,\theta})\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 : z \in \omega_{\beta,\theta} + \Sigma_{\theta'+(\pi/2)}\} \subseteq L(Y, X)$ is equicontinuous. By Theorem 1.4.10(i), we obtain that for each $y \in Y$ there exists an X-valued analytic mapping $z \mapsto W_{\beta,y}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0, \pi/2)$, we have that $\int_0^\infty e^{-zt}W_{\beta,y}(t)\,dt = z^{-1}\mathbf{P}^{-1}_{ze^{-i\beta}}C_1y$, $\operatorname{Re} z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta}z}W_{\beta,y}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($y \in Y$, $\theta' \in (0, \theta)$). Define $W_\beta(z)y := W_{\beta,y}(z)$, $z \in \Sigma_{\pi/2}$, $y \in Y$. By the uniqueness theorem for Laplace transform, it readily follows that $W_\beta(z)\colon Y \to X$ is a linear mapping ($z \in \Sigma_{\pi/2}$); furthermore, we can argue as in the proofs of [**292**, Theorem 2.2.5] and [**27**, Theorem 2.6.1] so as to conclude that, for every $\theta \in (0, \pi/2)$, $\{e^{-\omega_{\beta,\theta}z}W_\beta(z) : z \in \Sigma_{\theta'}\} \subseteq L(Y, X)$ is an equicontinuous operator family ($\theta' \in (0, \theta)$). Since $\lim_{z\to\infty} \mathbf{P}_z^{-1}C_1y = Dy$ ($y \in Y$), we can apply Theorem 1.4.10(ii)/(iii) in order to see that, for every $y \in Y$ and $\theta \in (0, \pi/2)$, we have $\lim_{z\to 0, z\in\Sigma_\theta} W_\beta(z)y = Dy$. Now we will prove that, for every $z \in \Sigma_{\pi/2} \cap e^{i\pi/2}\Sigma_{\pi/2}$, we have $W_0(z) = W_{\pi/2}(ze^{-i\pi/2})$. Let $y \in Y$ be fixed, and let $\arg(z) = \theta$. Set $\Gamma_\theta := \{e^{i\theta}t : t \geqslant 0\}$. Using Cauchy's formula, it is not difficult to see that, for all sufficiently large values of positive real parameter $s > 0$, we have

$$\begin{aligned}\int_0^\infty e^{-st}W_0(e^{i\theta}t)y\,dt &= e^{-i\theta}\int_{\Gamma_\theta} e^{-se^{-i\theta}v}W_0(v)y\,dv\\ &= e^{-i\theta}\int_0^\infty e^{-se^{-i\theta}v}W_0(v)y\,dv = s^{-1}\mathbf{P}^{-1}_{se^{-i\theta}}C_1y, \quad y \in Y.\end{aligned}$$

Similarly, $\int_0^\infty e^{-st}W_{\pi/2}(e^{i(\theta-\pi/2)}t)y\,dt = s^{-1}\mathbf{P}^{-1}_{se^{i\theta}}C_1y$, $y \in Y$ so that the uniqueness theorem for Laplace transform implies that $W_0(e^{i\theta}t) = W_{\pi/2}(e^{i(\theta-\pi/2)}t)$ for all $t \geqslant 0$. Plugging $t = |z|$, we get that $W_0(z) = W_{\pi/2}(ze^{-i\pi/2})$, as claimed. A similar line of reasoning shows that the operator family $(W(z))_{z\in\mathbb{C}}$, where

$$W(z) := \begin{cases} W_0(z), & z \in \Sigma_{\pi/2},\\ W_{\pi/2}(ze^{-i\pi/2}), & \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2},\\ W_{-\pi/2}(ze^{i\pi/2}), & \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2},\\ W_\pi(ze^{-i\pi}), & \text{if } z \in e^{i\pi}\Sigma_{\pi/2},\\ D, & \text{if } z = 0,\end{cases}$$

is well-defined. By the foregoing, we obtain that there exists $\omega > 0$ such that the operator family $\{e^{-\omega z}W(z) : z \in \mathbb{C}\} \subseteq L(Y, X)$ is equicontinuous as well as that, for every $y \in Y$, the mapping $z \mapsto W(z)y$, $z \in \mathbb{C}$ is entire (because it is weakly entire; this follows from the fact that for each $x^* \in X^*$ the mapping $z \mapsto \langle x^*, W(z)y\rangle$, $z \in \mathbb{C} \setminus \{0\}$ is analytic and has the limit $\langle x^*, Dy\rangle$ as $z \to 0$). Replacing the function $z \mapsto q_{\beta,\theta}(z) = z^{-1}\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 \in L(Y,X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ with the function $z \mapsto q_{\beta,\theta,j}(z) := z^{-1}(ze^{-i\beta})^{\alpha_j-\alpha_n}A_j\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 \in L(Y,X)$, $z \in \omega_{\beta,\theta}+\Sigma_{\theta+(\pi/2)}$ in the first part of proof ($\theta \in (0,\pi/2)$, $j \in \mathbb{N}_n^0$), for each $y \in Y$ we can define an X-valued analytic mapping $z \mapsto W_{\beta,j,y}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0,\pi/2)$, we have that $\int_0^\infty e^{-zt}W_{\beta,j,y}(t)\,dt = z^{-1}(ze^{-i\beta})^{\alpha_j-\alpha_n}A_j\mathbf{P}^{-1}_{ze^{-i\beta}}C_1y$, $\operatorname{Re} z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta}z}W_{\beta,j,y}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($y \in Y$, $\theta' \in (0,\theta)$). Define now $W_{\beta,j}(z)y := W_{\beta,j,y}(z)$, $z \in \Sigma_{\pi/2}$, $y \in Y$, and $W^j(\cdot)$ by replacing $W_0(\cdot)$, $W_{\pi/2}(\cdot)$, $W_{-\pi/2}(\cdot)$, $W_\pi(\cdot)$ and D in the definition of $W(\cdot)$ with $W_{0,j}(\cdot)$, $W_{\pi/2,j}(\cdot)$, $W_{-\pi/2,j}(\cdot)$, $W_{\pi,j}(\cdot)$ and D_j, respectively ($j \in \mathbb{N}_n^0$). Then there exists $\omega_j > 0$ such that the operator family $\{e^{-\omega_j z}W^j(z) : z \in \mathbb{C}\} \subseteq L(Y,X)$ is equicontinuous and, for every $y \in Y$, the mapping $z \mapsto W^j(z)y$, $z \in \mathbb{C}$ is entire ($j \in \mathbb{N}_n^0$). By Theorem 2.3.33(i)/(b), we get that there exists an exponentially equicontinuous C_1-existence family for (112), denoted by $(E(t))_{t\geqslant 0}$. Furthermore, for every $y \in Y$, the mapping $t \mapsto E(t)y$, $t \geqslant 0$ can be analytically extended to the whole complex plane so that $E^{(\alpha_n-1)}(z)y = W(z)y$, $z \in \mathbb{C}$, $y \in Y$ and $A_j(g_{\alpha_n-\alpha_j} * E^{(\alpha_n-1)})(z)y = W^j(z)y$, $z \in \mathbb{C}$, $y \in Y$, $j \in \mathbb{N}_n^0$. Making use of the closedness of operators A_j for $j \in \mathbb{N}_n^0$, the above implies that the functions $z \mapsto A_jE^{(\alpha_n-1)}(z)y$, $z \in \mathbb{C}$ are well-defined and entire ($y \in Y$, $j \in \mathbb{N}_n^0$). By Theorem 2.3.34 and Theorem 2.3.33(ii), we get that the function $t \mapsto u(t)$, $t \geqslant 0$, given by (121), with $g(t) = G(t)$, $t \geqslant 0$, is a unique strong solution of problem (112). Define $v(t) := u(t) - \int_0^t E(t-s)g(s)\,ds$, $t \geqslant 0$. By the proof of Theorem 2.3.32(i), we have:

$$
\begin{aligned}
v^{(\alpha_n)}(\cdot) = {} & \sum_{i=m}^{m_n-1} (g_{i-\alpha} * E^{(m_n-1)})(\cdot)v_{i,0} \\
& - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} (g_{i-\alpha_j} * E^{(m_n-1)})(\cdot)v_{i,j} \in C([0,\infty) : X),
\end{aligned}
$$

$Bv^{(\alpha_n)}(\cdot) \in C([0,\infty) : X)$ and

$$
\begin{aligned}
A_iv^{(\alpha_i)}(\cdot) = {} & \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot)A_iu_j \\
& - \sum_{l=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_l} \Big[g_{l-\alpha_j} * A_i(g_{\alpha_n-\alpha_i} * E^{(m_n-1)})\Big](\cdot)v_{l,j} \\
& + \sum_{l=m}^{m_n-1} \Big[g_{l-\alpha} * A_i(g_{\alpha_n-\alpha_i} * E^{(m_n-1)})\Big](\cdot)v_{l,0} \in C([0,\infty) : X),
\end{aligned}
$$

for all $i \in \mathbb{N}_{n-1}^0$. These representation formulae imply that the functions $v(\cdot)$ and $Bv^{(\alpha_n)}(\cdot)$, $A_1v^{(\alpha_1)}(\cdot)$, $\ldots$, $A_{n-1}v^{(\alpha_{n-1})}(\cdot)$, $Av^{(\alpha)}(\cdot)$ can be analytically extended from the interval $[0,\infty)$ to the whole complex plane. Furthermore, $(u -$

$v)^{(\alpha_n-1)}(t) = \int_0^t E^{(\alpha_n-1)}(t-s)g(s)ds$, $t \geqslant 0$ and $(u-v)^{(\alpha_n)}(t) = \int_0^t E^{(\alpha_n-1)}(t-s)g'(s)ds + E^{(\alpha_n-1)}(t)g(0)$, $t \geqslant 0$. Now it is quite simple to prove that if the function $t \mapsto g(t)$, $t \geqslant 0$ can be analytically extended to the whole complex plane, resp., to a continuously differentiable function $\mathbb{R} \mapsto Y$, then $u(\cdot)$ is an entire solution of problem (112), resp., the function $t \mapsto u(t)$, $t \geqslant 0$ can be extended to an α_n-times continuously differentiable function $\mathbb{R} \mapsto X$ and the functions $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\dots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to continuously differentiable functions $\mathbb{R} \mapsto X$. The rest of the proof can be left to the reader. $\square$

REMARK 2.3.47. (i) Suppose that $Y = X$, $C_1 \in L(X)$ is injective, $C_1A_j \subseteq A_jC_1$, $j \in \mathbb{N}_n^0$, as well as the operator families $\{\mathbf{P}_z^{-1}C_1 : |z| \geqslant R\} \subseteq L(X)$ and $\{z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1 : |z| \geqslant R,\ j \in \mathbb{N}_n^0\} \subseteq L(X)$ are equicontinuous and strongly continuous. Then the analyticity of mappings $z \mapsto \mathbf{P}_z^{-1}C_1x \in X$, $|z| > R$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1x \in X$, $|z| > R$ automatically follows for any $x \in X$, $j \in \mathbb{N}_n^0$ (cf. the proof of Lemma 2.6.3 below).

(ii) Suppose that $g \in C^\infty([0,\infty) : Y)$, resp., $g(\cdot)$ can be extended to an infinitely differentiable function $\mathbb{R} \mapsto Y$. Then $u \in C^\infty([0,\infty) : X)$ and $Bu^{(\alpha_n)}$, $A_1u^{(\alpha_1)}$, $\dots$, $A_{n-1}u^{(\alpha_{n-1})}$, $Au^{(\alpha)} \in C^\infty([0,\infty) : X)$, resp., the functions $u(\cdot)$ and $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\dots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to infinitely differentiable functions $\mathbb{R} \mapsto X$.

(iii) Let $0 \leqslant i \leqslant m_n - 1$, $0 \leqslant j \leqslant n-1$ and $i \geqslant \alpha_j$. If a strong solution $u(\cdot)$ of problem (112) has the property that $u \in C^\infty([0,\infty) : X)$ and $Bu^{(\alpha_n)}$, $A_1u^{(\alpha_1)}$, $\dots$, $A_{n-1}u^{(\alpha_{n-1})}$, $Au^{(\alpha)} \in C^\infty([0,\infty) : X)$, then it can be easily seen that the mapping $t \mapsto A_ju^{(\alpha_j')}(t)$, $t \geqslant 0$ is well-defined and infinitely differentiable for $\alpha_j' \geqslant \alpha_j$; hence, $u_i \in D(A_j)$ for $0 \leqslant i \leqslant m_n - 1$, $j \in D_i''$ and our result on the well-posedness of (112) is optimal provided that $R(C_1) = X$.

Now we would like to present how Theorem 2.3.46 can be applied in the analysis of abstract Boussinesq–Love equation in finite domains.

EXAMPLE 2.3.48. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. In the cylinder $\mathbb{R} \times \Omega$, we consider the following Cauchy–Dirichlet problem for linearized Boussinesq–Love equation:

$$(140)\quad (\lambda-\Delta)u_{tt}(t,x)-\alpha(\Delta-\lambda')u_t(t,x) = \beta(\Delta-\lambda'')u(t,x)+f(t,x),\ t \in \mathbb{R},\ x \in \Omega,$$

$$(141)\quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ (t,x) \in \mathbb{R}\times\Omega;\ u(t,x) = 0,\ (t,x) \in \mathbb{R}\times\partial\Omega,$$

where $\lambda, \lambda', \lambda'' \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$ (in [**515**], the standing hypothesis was that $\alpha \neq 0$; as explained later in [**556**], the case $\alpha = 0$ is meaningful and has a certain physical meaning). By $\{\lambda_k\}$ [$= \sigma(\Delta)$] we denote the eigenvalues of the Dirichlet Laplacian Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leqslant -\lambda_2 \ldots \leqslant -\lambda_k \leqslant \cdots \to +\infty$ as $k \to \infty$; cf. [**521**, Section 5.6], [**27**, Section 6] and [**509**, Section 1.3] for more details) numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal [in the sense of $L^2(\Omega)$] eigenfunctions. In [**515**], G. A. Sviridyuk and A. A. Zamyshlyaeva have considered the well-posedness of problem (140)–(141) in the Sobolev space $W^{p,l}(\Omega)$, where $1 < p < \infty$ and $l \in \mathbb{N}_0$, and the Hölder space $C^{l+\gamma}(\Omega)$, where $0 < \gamma < 1$ and $l \in \mathbb{N}_0$. In order to apply [**515**, Theorem 4.1], the authors imposed the following condition:

(i) $\lambda \in \rho(\Delta)$, or
(iii) $\lambda \in \sigma(\Delta) \wedge \lambda = \lambda' \wedge \lambda \neq \lambda''$.

Although our results on the well-posedness of problem (140)–(141) in cases (i) or (iii) give some new information about qualitative properties of strong solutions of (140)–(141), in the remainder of this example we will completely focus our attention on the following case:

(ii) $\lambda \in \sigma(\Delta) \wedge \lambda \neq \lambda' \wedge (\alpha = 0 \Rightarrow \lambda \neq \lambda'')$.

If (ii) holds with $\alpha \neq 0$, then we cannot apply [**515**, Theorem 4.1] (despite of the validity of requirement stated in the formulation of [**515**, Lemma 5.1]) because of the violation of condition [**515**, (A), p. 271]. Here it is also worth noting that the existence and uniqueness of two times continuously differentiable solutions of problem (140)–(141) on the non-negative real axis (understood in a broader sense of [**509**, Definition 5.6.2]) have been studied in [**509**, Example 5.7.1, Lemma 5.7.1(ii), Theorem 5.7.3] provided that $Y = H^2(\Omega) \cap H_0^1(\Omega)$, $X = L^2(\Omega)$, as well as that (ii) holds and that, additionally, $\lambda'' \neq 0$; even in this case, we obtain from Theorem 2.3.46 and a simple analysis, with $X = Y = L^2(\Omega)$ and C_1 being the identity operator on X, that a strong solution $t \mapsto u(t)$, $t \geqslant 0$ of this problem has the property that the mapping $t \mapsto \Delta u(t)$, $t \geqslant 0$ belongs to the space $C^2([0,\infty) : L^2(\Omega))$ provided that $f \in C^1([0,\infty) : L^2(\Omega))$ needless to say that we obtain the existence and uniqueness of entire solutions of problem (140)–141 provided that the function $f(\cdot)$ can be extended analytically to the whole complex plane.

With a little abuse of notation, we have that $n = 2$, $B = \lambda - \Delta$, $A_1 = -\alpha(\Delta - \lambda')$, $A = \beta(\Delta - \lambda'')$, $\alpha_2 = 2$, $\alpha_1 = 1$ and $\alpha = 0$ (the use of symbols α and β will be clear from the context). Hence,

$$\mathbf{P}_z = z^{-2}[(z^2\lambda + \alpha z\lambda' + \beta\lambda'') + (-z^2 - \alpha z - \beta)\Delta], \quad z \in \mathbb{C} \setminus \{0\}.$$

It is clear that (ii) implies that

$$\lambda \neq \frac{z^2\lambda + \alpha z\lambda' + \beta\lambda''}{z^2 + \alpha z + \beta} \to \lambda \text{ as } |z| \to \infty.$$

We assume that $X = Y = L^p(\Omega)$ for some $p \in (1,\infty)$, C_1 is the identity operator on X, Δ is the Dirichlet Laplacian on $L^p(\Omega)$ acting with domain $D(\Delta) := W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$, as well as that the following condition holds:

HP. There exist a sufficiently large real number $R > 0$ and a positive real number number $l < 4$, resp., $l < 2$, provided that (ii) holds with $\alpha \neq 0$, resp., $\alpha = 0$, such that

$$\|R(z : \Delta)\| = O(|\lambda - z|^{-l}) \text{ as } z \to \lambda. \tag{142}$$

Before proceeding further, it should be observed that the condition HP. holds in the case that $p = 2$, with $l = 1$: Suppose that $\lambda = \lambda_{k_0}$ for some $k_0 \in \mathbb{N}$. Then $g = R(z : \Delta)f = \sum_{k=1}^{\infty} \frac{\langle \phi_k, f\rangle}{z - \lambda_k}\phi_k$ as $z \to \lambda_{k_0}$, so that Parseval's equality implies

$|z-\lambda_{k_0}|^2\|g\|^2 = \sum_{k=1}^{\infty} \frac{|z-\lambda_{k_0}|^2|\langle \phi_k,f\rangle|^2}{|z-\lambda_k|^2} \leqslant \text{Const.} \sum_{k=1}^{\infty} |\langle \phi_k, f\rangle|^2 = \|f\|^2$ as $z \to \lambda_{k_0}$ (let us recall that $\lambda_k \to -\infty$ as $k \to \infty$). Using now the condition HP., the expression

$$\mathbf{P}_z^{-1} = z^{-2}(z^2+\alpha z+\beta)^{-1}\Big[\frac{z^2\lambda+\alpha z\lambda'+\beta\lambda''}{z^2+\alpha z+\beta} - \Delta\Big]^{-1}, \quad |z| \geqslant R,$$

and the resolvent equation, it readily follows that there exists a positive real number $\zeta > 0$ such that the operator families $\{(1+|z|)^{\zeta}\mathbf{P}_z^{-1} : |z| \geqslant R\} \subseteq L(X)$ and $\{(1+|z|)^{\zeta} z^{\alpha_j-\alpha_n} A_j \mathbf{P}_z^{-1} : |z| \geqslant R,\ j \in \mathbb{N}_2^0\} \subseteq L(X)$ are equicontinuous, as well as that $\lim_{z\to\infty} \mathbf{P}_z^{-1}x = 0$ and $\lim_{z\to\infty} z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}x = 0$ for any $x \in X$, $j \in \mathbb{N}_n^0$. The strong analyticity of mappings $z \mapsto \mathbf{P}_z^{-1}$, $|z| > R$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}$, $|z| > R$ follows from Remark 2.3.47(i), while the existence of an exponentially bounded I-uniqueness family for the corresponding problem (112) simply follows from Theorem 2.3.33(ii) and the above argumentation; here I stands for the identity operator on X. Hence, there exists a unique entire solution $z \mapsto u(z)$, $z \in \mathbb{C}$ of problem (140)–(141), provided that $u_0(x) \in W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$, $u_1(x) \in W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$ and the function $f(\cdot)$ can be analytically extended to the whole complex plane; moreover, we have the existence of a positive real number $\omega' > 0$ such that the set $\{e^{-\omega' z}g(z) : z \in \mathbb{C}\}$ is bounded in $L^p(\Omega)$. Since C_1 is the identity operator on X, this is an optimal result as long as the condition HP. holds (cf. Remark 2.3.47(iii)).

We continue our analysis by enquiring into the existence and uniqueness of entire solutions to the abstract Barenblatt–Zheltov–Kochina equation in finite domains. We use the argumentation contained in the proof of Theorem 2.3.46 and the approach of N. H. Abdelaziz, F. Neubrander (cf. [**6**] and Subsection 2.3.3); for the sake of simplicity, we will deal only with the homogenous case.

EXAMPLE 2.3.49. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$, $\{\lambda_k\}$, $\{\phi_k\}$ and Δ possess the same meanings as in the previous example, let $X = Y = L^2(\Omega)$, and let C_1 be the identity operator on X. As mentioned above, we analyze entire solutions of the Barenblatt–Zheltov–Kochina equation

$$(\lambda - \Delta)u_t(t,x) = \zeta\Delta u(t,x), \quad t \in \mathbb{R},\ x \in \Omega; \tag{143}$$

$$u(0,x) = u_0(x), \quad x \in \Omega,\ u(t,x) = 0,\ (t,x) \in \mathbb{R} \times \partial\Omega, \tag{144}$$

where $\zeta \in \mathbb{R} \smallsetminus \{0\}$ and $\lambda = \lambda_{k_0} \in \sigma(\Delta)$ (cf. the equation (112) with $n = 2$, $B = \lambda - \Delta$, $A_1 = 0$, $A = \zeta\Delta$, $\alpha_2 = 1$ and $\alpha_1 = \alpha = 0$; then we have $\mathbf{P}_z = \lambda - (1+\zeta z^{-1})\Delta$). Using Parseval's equality, it can be easily seen that the operator $D : f \mapsto (-1)(\zeta\lambda)^{-1}\sum_{\lambda=\lambda_k}\langle \phi_k, f\rangle\phi_k$ ($f \in L^2(\Omega)$) belongs to the space $L(L^2(\Omega))$. Let $\beta \in (-\pi, \pi]$. Then the equation (142) holds with $l = 1$, which enables us to verify that, for every $\theta \in (0, \pi/2)$, there exists a sufficiently large number $\omega_{\beta,\theta} > 0$ satisfying that the function $q_{\beta,\theta}(z) := z^{-2}\mathbf{P}_{ze^{-i\beta}}^{-1} \in L(X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0,\theta)$ the operator family $\{z^{-2}(z-\omega_{\beta,\theta})\mathbf{P}_{ze^{-i\beta}}^{-1} : z \in \omega_{\beta,\theta} + \Sigma_{\theta'+(\pi/2)}\} \subseteq L(X)$ is equicontinuous. As in the proof of Theorem 2.3.46, we obtain that for each $f \in X$ there exists an X-valued analytic mapping $z \mapsto W_{\beta,f}^1(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0, \pi/2)$, one has $\int_0^\infty e^{-zt}W^1_{\beta,f}(t)dt = z^{-2}\mathbf{P}^{-1}_{ze^{-i\beta}}f$, $\operatorname{Re} z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta}z}W^1_{\beta,f}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($f \in X$, $\theta' \in (0,\theta)$). Define $W^1_\beta(z)f := W^1_{\beta,f}(z)$, $z \in \Sigma_{\pi/2}$, $f \in X$. Then, for every $\theta \in (0,\pi/2)$, $\{e^{-\omega_{\beta,\theta}z}W^1_\beta(z) : z \in \Sigma_\theta\} \subseteq L(X)$ is an equicontinuous operator family. On the other hand, there exist finite constants $R > 0$ and $c > 0$ such that the set $\{|z|^{-1}|(\lambda-\lambda_k)z - \zeta\lambda_k| : |z| \geqslant R,\ k \in \mathbb{N} \smallsetminus \{k_0\}\}$ is bounded from below by c, so that we can apply Parseval's equality once more in order to see that:

$$z^{-1}\mathbf{P}_z^{-1}f = \frac{1}{\zeta+z}\Big[\frac{\lambda z}{\zeta+z} - \Delta\Big]^{-1}f = \sum_{k=1,k\neq k_0}^{\infty} \frac{\langle \phi_k, f\rangle}{(\lambda-\lambda_k)z - \zeta\lambda_k}\phi_k + Df \to Df,$$

as $|z| \to \infty$ ($f \in X$); similarly, we have that the operator family $\{z^{-2}B\mathbf{P}^{-1}_{ze^{-i\beta}} : |z| \geqslant R\} \in L(X)$ is equicontinuous and $z^{-1}B\mathbf{P}_z^{-1}f \to 0$, $|z| \to \infty$ ($f \in X$), so that we can define a strongly analytic operator family $(W^1_{\beta,B}(z))_{z\in\Sigma_{\pi/2}} \subseteq L(X)$ satisfying that, for every $\theta \in (0,\pi/2)$, the operator family $\{e^{-\omega'_{\beta,\theta}z}W^1_{\beta,B}(z) : z \in \Sigma_\theta\} \subseteq L(X)$ is equicontinuous for some number $\omega'_{\beta,\theta} > 0$. Since $\lim_{|z|\to\infty} z^{-1}\mathbf{P}_z^{-1}f = Df$ ($f \in X$), an application of Theorem 1.4.10(ii)/(iii) yields that, for every $f \in X$ and $\theta \in (0,\pi/2)$, we have $\lim_{z\to 0, z\in\Sigma_\theta} W^1_\beta(z)f = Df$. Define

$$W^1(z) := \begin{cases} W^1_0(z), & z \in \Sigma_{\pi/2}, \\ e^{i\pi/2}W^1_{\pi/2}(ze^{-i\pi/2}), & \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ e^{-i\pi/2}W^1_{-\pi/2}(ze^{i\pi/2}), & \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ e^{i\pi}W^1_\pi(ze^{-i\pi}), & \text{if } z \in e^{i\pi}\Sigma_{\pi/2}, \\ D, & \text{if } z = 0, \end{cases}$$

and $W^1_B(z)$ by replacing the operators $W^1_0(z)$, $W^1_{\pi/2}(ze^{-i\pi/2})$, $W^1_{-\pi/2}(ze^{i\pi/2})$, $W^1_\pi(ze^{-i\pi})$ and D in the above definition by the operators $W^1_{0,B}(z)$, $W^1_{\pi/2,B}(ze^{-i\pi/2})$, $W^1_{-\pi/2,B}(ze^{i\pi/2})$, $W^1_{\pi,B}(ze^{-i\pi})$ and 0, respectively ($z \in \mathbb{C}$). Then there exists a finite constant $\omega > 0$ such that the operator families $\{e^{-\omega z}W(z) : z \in \mathbb{C}\} \subseteq L(X)$ and $\{e^{-\omega z}BW(z) : z \in \mathbb{C}\} \subseteq L(X)$ are equicontinuous as well as that, for every $f \in X$, the mappings $z \mapsto W(z)f$, $z \in \mathbb{C}$ and $z \mapsto BW(z)f$, $z \in \mathbb{C}$ are entire; cf. also [**291**, Proposition 2.4.2, Corollary 2.4.3]. Furthermore, it is not difficult to see that $(W^1(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ is a once integrated evolution family generated by A, B in the sense of considerations from [**6**, Section 2]. By [**6**, Theorem 2.3] and an elementary analysis, we may conclude that for each function $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, orthogonal to the eigenfunction(s) ϕ_k for $k = k_0$, there exists a unique strong solution $t \mapsto u(t)$, $t \geqslant 0$ of problem (143)–(144) satisfying that there exists a finite constant $\omega' > 0$ such that the mappings $t \mapsto u(t)$, $t \geqslant 0$ and $t \mapsto Bu(t)$, $t \geqslant 0$ can be analytically extended to the whole complex plane, as well as that the sets $\{e^{-\omega' z}u(z) : z \in \mathbb{C}\}$ and $\{e^{-\omega' z}Bu(z) : z \in \mathbb{C}\}$ are bounded. This result slightly improves the assertion of [**509**, Theorem 5.1.3(ii)] in L^2 spaces.

Now we will investigate the existence and uniqueness of analytical solutions of abstract degenerate Cauchy problem (112), provided that there exists an index

$i \in \mathbb{N}_n^0$ such that $\alpha_i \notin \mathbb{N}_0$; for the sake of brevity, we will consider only homogeneous equations.

DEFINITION 2.3.50. A function $u \in C([0,\infty) : X)$ is said to be an analytical solution of problem (112) on the region $\mathbb{C} \setminus (-\infty, 0]$ iff $u(\cdot)$ is a strong solution of (112) and it can be extended to the whole complex plane, analytically on the region $\mathbb{C} \setminus (-\infty, 0]$ and continuously on the region $\mathbb{C} \setminus (-\infty, 0)$, as well as any of the terms $A_i \mathbf{D}_t^{\alpha_i} u(t)$ ($0 \leqslant i \leqslant n$, $t \geqslant 0$) is well defined and can be extended to the whole complex plane, analytically on the region $\mathbb{C} \setminus (-\infty, 0]$ and continuously on the region $\mathbb{C} \setminus (-\infty, 0)$.

Set, for every $\gamma \in (-\pi, \pi]$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\mathbf{P}_{z,\gamma} := B + \sum_{j=1}^{n-1} z^{\alpha_j - \alpha_n} e^{i\gamma(\alpha_n - \alpha_j)} A_j - z^{\alpha - \alpha_n} e^{i\gamma(\alpha_n - \alpha)} A.$$

Before stating the following theorem, it should be observed that $\mathbf{P}_{z,\gamma}$ need not be equal to $\mathbf{P}_{ze^{-i\gamma}}$ for some particular values of γ and z.

THEOREM 2.3.51. *Suppose $k(t)$ satisfies* (P1), *$C_1 \in L(Y, X)$, $C_2 \in L(X)$ is injective, and there exists a locally equicontinuous k-regularized C_2-uniqueness family for problem* (112), *with $f(t) \equiv 0$. Let the following hold:*

- *Suppose that for each number $\gamma \in (-\pi, \pi]$ there exists a sufficiently large number $R > 0$ such that the operator $\mathbf{P}_{z,\gamma}$ is injective for $|z| \geqslant R$, as well as that the operator families $\{\mathbf{P}_{z,\gamma}^{-1} C_1 : |z| \geqslant R,\ z \notin (-\infty, 0]\} \subseteq L(Y, X)$ and $\{z^{\alpha_j - \alpha_n} A_j \mathbf{P}_{z,\gamma}^{-1} C_1 : |z| \geqslant R,\ j \in \mathbb{N}_n^0,\ z \notin (-\infty, 0]\} \subseteq L(Y, X)$ are equicontinuous.*
- *Let the mappings $z \mapsto \mathbf{P}_{z,\gamma}^{-1} C_1 y \in X$, $|z| > R$, $z \notin (-\infty, 0]$ and $z \mapsto z^{\alpha_j - \alpha_n} A_j \mathbf{P}_{z,\gamma}^{-1} C_1 y \in X$, $|z| > R$, $z \notin (-\infty, 0]$ be analytic for any $y \in Y$, $j \in \mathbb{N}_n^0$, $\gamma \in (-\pi, \pi]$.*
- *Let there exist operators $D, D_0, D_1, \ldots, D_n \in L(Y, X)$ such that $\lim_{z \to +\infty} \mathbf{P}_{z,\gamma}^{-1} C_1 y = Dy$ and $\lim_{z \to +\infty} z^{\alpha_j - \alpha_n} e^{i\gamma(\alpha_n - \alpha_j)} A_j \mathbf{P}_{z,\gamma}^{-1} C_1 y = D_j y$ for any $y \in Y$, $j \in \mathbb{N}_n^0$, $\gamma \in (-\pi, \pi]$.*
- *Suppose that $u_i \in \mathbf{D}_i$ for $0 \leqslant i \leqslant m_n - 1$.*

Then there exists a unique analytical solution of problem (112) *on the region $\mathbb{C} \setminus (-\infty, 0]$, with $f(t) \equiv 0$. Denote by $A_i \mathbf{D}_z^{\alpha_i} u(z)$ the analytical extensions of terms $A_i \mathbf{D}_t^{\alpha_i} u(t)$ ($0 \leqslant i \leqslant n$, $t \geqslant 0$) on region $\mathbb{C} \setminus (-\infty, 0]$. Then, for every $\zeta \in (0, \pi)$, we have the existence of a positive real number $\omega' > 0$ such that the sets $\{e^{-\omega' z} u(z) : z \in \overline{\Sigma_\zeta}\}$ and $\{e^{-\omega' z} A_i \mathbf{D}_z^{\alpha_i} u(z) : z \in \overline{\Sigma_\zeta}\}$ ($0 \leqslant i \leqslant n$) are bounded in X.*

PROOF. The proof of theorem is very similar to that of Theorem 2.3.46 and, because of that, we will only outline the most relevant details. Fix numbers $\gamma \in (-\pi, \pi]$ and $\zeta \in (0, \pi)$. Then, for every $\theta \in (0, \pi/2)$, there exists a sufficiently large number $\omega_{\gamma,\theta} > 0$ satisfying that the function $r_{\gamma,\theta}(z) := z^{-1} \mathbf{P}_{z,\gamma}^{-1} C_1 \in L(Y, X)$, $z \in \omega_{\gamma,\theta} + \Sigma_{\theta + (\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0, \theta)$ the operator family $\{z^{-1}(z - \omega_{\gamma,\theta}) \mathbf{P}_{z,\gamma}^{-1} C_1 : z \in \omega_{\gamma,\theta} + \Sigma_{\theta' + (\pi/2)}\} \subseteq L(Y, X)$ is equicontinuous. Arguing as in the proof of Theorem 2.3.46, we obtain the existence of a strongly analytic operator family $(W_\gamma(z))_{z\in\Sigma_{\pi/2}} \subseteq L(Y,X)$ satisfying that $\lim_{z\to 0, z\in\overline{\Sigma_\theta}} W_\gamma(z)y = Dy$ ($\theta \in (0,\pi/2)$, $y \in Y$), $\int_0^\infty e^{-zt}W_\gamma(t)y\,dt = z^{-1}\mathbf{P}_{z,\gamma}^{-1}C_1y$, $\operatorname{Re} z > \omega_{\gamma,\theta}$, $y \in Y$ and that, for every $\theta \in (0,\pi/2)$, $\{e^{-\omega_{\gamma,\theta}z}W_\gamma(z) : z \in \Sigma_{\theta'}\} \subseteq L(Y,X)$ is an equicontinuous operator family ($\theta' \in (0,\theta)$). Define

$$W(z) := \begin{cases} W_0(z), & z \in \Sigma_{\pi/2}, \\ W_{\pi/2}(ze^{-i\pi/2}), & \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ W_{-\pi/2}(ze^{i\pi/2}), & \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ 0, & \text{if } z < 0, \\ D, & \text{if } z = 0. \end{cases}$$

Then it can be simply verified that $W(\cdot)$ is well-defined and there exists $\omega > 0$ such that the operator family $\{e^{-\omega z}W(z) : z \in \overline{\Sigma_\zeta}\} \subseteq L(Y,X)$ is equicontinuous as well as that, for every $y \in Y$, the mapping $z \mapsto W(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0]$ is analytic. Replacing the function $z \mapsto q_{\gamma,\theta}(z)$, $z \in \omega_{\gamma,\theta} + \Sigma_{\theta+(\pi/2)}$ with the function $z \mapsto q_{\gamma,\theta,j}(z) := z^{-1}z^{\alpha_j-\alpha_n}e^{i\gamma(\alpha_n-\alpha_j)}A_j\mathbf{P}_{z,\gamma}^{-1}C_1 \in L(Y,X)$, $z \in \omega_{\gamma,\theta} + \Sigma_{\theta+(\pi/2)}$ in the first part of proof ($\theta \in (0,\pi/2)$, $j \in \mathbb{N}_n^0$), we can define a strongly analytic operator family $(W_{\gamma,j}(z))_{z\in\Sigma_{\pi/2}} \subseteq L(Y,X)$ satisfying that, for every $\theta \in (0,\pi/2)$ and $y \in Y$, we have that $\int_0^\infty e^{-zt}W_{\gamma,j}(t)y\,dt = z^{-1}z^{\alpha_j-\alpha_n}e^{i\gamma(\alpha_n-\alpha_j)}A_j\mathbf{P}_{z,\gamma}^{-1}C_1y$, $\operatorname{Re} z > \omega_{\beta,\theta}$ and the operator family $\{e^{-\omega_{\beta,\theta}z}W_{\gamma,j}(z) : z \in \Sigma_{\theta'}\} \subseteq L(Y,X)$ is equicontinuous ($\theta' \in (0,\theta)$, $j \in \mathbb{N}_n^0$). Define now $W^j(\cdot)$ by replacing $W_0(\cdot)$, $W_{\pi/2}(\cdot)$, $W_{-\pi/2}(\cdot)$ and D in the definition of $W(\cdot)$ with $W_{0,j}(\cdot)$, $W_{\pi/2,j}(\cdot)$, $W_{-\pi/2,j}(\cdot)$ and D_j, respectively ($j \in \mathbb{N}_n^0$). Then there exists $\omega_j > 0$ such that the operator family $\{e^{-\omega_j z}W^j(z) : z \in \overline{\Sigma_\zeta}\} \subseteq L(Y,X)$ is equicontinuous as well as that, for every $y \in Y$, the mapping $z \mapsto W^j(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0]$ is analytic and the mapping $z \mapsto W^j(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0)$ is continuous ($j \in \mathbb{N}_n^0$). By Theorem 2.3.33(i)/(b), we get that there exists an exponentially equicontinuous C_1-existence family for (112), denoted by $(E(t))_{t\geqslant 0}$. Furthermore, for every $y \in Y$, the mapping $t \mapsto E(t)y$, $t \geqslant 0$ can be analytically extended to the region $z \in \mathbb{C} \smallsetminus (-\infty,0]$ so that the mapping $z \mapsto E(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0)$ is continuous as well as that $E^{(m_n-1)}(z)y = W(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0]$, $y \in Y$ and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(z)y = W^j(z)y$, $z \in \mathbb{C} \smallsetminus (-\infty,0]$, $y \in Y$, $j \in \mathbb{N}_n^0$ (this equality can be proved by using the closedness of operators A_j for $j \in \mathbb{N}_n^0$, Proposition 1.4.7 and the arguments already used in the proofs of Kato's analyticity criteria from [**291**, Section 2.4]). By Theorem 2.3.34 and Theorem 2.3.33(ii), we get that the function $t \mapsto u(t)$, $t \geqslant 0$, given by (121), with $g(t) = G(t) = 0$, $t \geqslant 0$, is a unique strong solution of problem (112). The rest of the proof can be obtained by using a slight modification of the corresponding parts of the proof of Theorem 2.3.46. $\square$

REMARK 2.3.52. Suppose that $Y = X$, $C_1 \in L(X)$ is injective, $C_1A_j \subseteq A_jC_1$, $j \in \mathbb{N}_n^0$, as well as that the operator families $\{\mathbf{P}_{z,\gamma}^{-1}C_1 : |z| \geqslant R,\ z \notin (-\infty,0]\} \subseteq L(X)$ and $\{z^{\alpha_j-\alpha_n}A_j\mathbf{P}_{z,\gamma}^{-1}C_1 : |z| \geqslant R,\ z \notin (-\infty,0],\ j \in \mathbb{N}_n^0\} \subseteq L(X)$ are equicontinuous and strongly continuous for any $\gamma \in (-\pi,\pi]$. Then the mappings

$z \mapsto \mathbf{P}_{z,\gamma}^{-1}C_1x \in X$, $|z| > R$, $z \notin (-\infty, 0]$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_{z,\gamma}^{-1}C_1x \in X$, $|z| > R$, $z \notin (-\infty, 0]$ are analytic for any $x \in X$, $j \in \mathbb{N}_n^0$, $\gamma \in (-\pi, \pi]$.

Now we will continue our analysis of abstract Boussinesq–Love equation and abstract Barenblatt–Zheltov–Kochina equation in finite domains (fractional-order case).

EXAMPLE 2.3.53. (i) Suppose, as in Exampe 2.3.48, that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $0 < \alpha_1 < \alpha_2 \leqslant 2$. We analyze the following fractional degenerate Cauchy-Dirichlet problem:

$$(145)\quad (\lambda - \Delta)\mathbf{D}_t^{\alpha_2}(t,x) - \alpha(\Delta - \lambda')\mathbf{D}_t^{\alpha_1}(t,x) = \beta(\Delta - \lambda'')u(t,x), \quad t \geqslant 0,\ x \in \Omega,$$

$$(146)\quad u(0,x) = u_0(x); \quad u_t(0,x) = u_1(x),\ (t,x) \in [0,\infty) \times \Omega,\ \text{if } \alpha_2 > 1;$$

$$(147)\quad u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega,$$

where Δ denotes the Dirichlet Laplacian on $L^2(\Omega)$, $\lambda, \lambda', \lambda'' \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Assume that the following condition holds (cf. Example 2.3.48(ii)):

$$\lambda \in \sigma(\Delta) \wedge \lambda \neq \lambda' \wedge (\alpha = 0 \Rightarrow \lambda \neq \lambda'').$$

Arguing similarly as in the afore-mentioned example, we can prove that the requirements of Theorem 2.3.51 holds with $D = D_0 = D_1 = D_2 = 0$, so that the abstract Cauchy problem (145)–(147) has a unique analytical solution on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ for any $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, and $u_1(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, if $\alpha_2 > 1$.

(ii) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, let $\alpha \in (0, 2) \smallsetminus \{1\}$, and let $X = Y = L^2(\Omega)$. Suppose that Δ denotes the Dirichlet Laplacian on X and the operator D has the same meaning as in Example 2.3.49 (we will use the same notation henceforth). Consider the following fractional Barenblatt–Zheltov–Kochina equation:

$$(148)\quad (\lambda - \Delta)\mathbf{D}_t^{\alpha}u(t,x) = \zeta\Delta u(t,x), \quad t \in \mathbb{R},\ x \in \Omega;$$

$$(149)\quad u(0,x) = u_0(x),\ \ x \in \Omega; \quad u_t(0,x) = u_1(x),\ (t,x) \in [0,\infty) \times \Omega,\ \text{if } \alpha > 1;$$

$$(150)\quad u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \partial\Omega,$$

where $\zeta \in \mathbb{R} \smallsetminus \{0\}$ and $\lambda = \lambda_{k_0} \in \sigma(\Delta)$ (let us recall that the problem (148)–(150) is a special case of problem (112) with $n = 2$, $B = \lambda - \Delta$, $A_1 = 0$, $A = \zeta\Delta$, $\alpha_2 = \alpha$ and $\alpha_1 = \alpha = 0$; now we have $\mathbf{P}_z = \lambda - (1 + \zeta z^{-\alpha})\Delta)$. Let $\gamma \in (-\pi, \pi]$. Then it is not difficult to verify that, for every $\theta \in (0, \pi/2)$, there exists a sufficiently large number $\omega_{\gamma,\theta} > 0$ satisfying that the function $q_{\gamma,\theta}(z) := z^{-\alpha-1}\mathbf{P}_{z,\gamma}^{-1} \in L(X)$, $z \in \omega_{\gamma,\theta} + \Sigma_{\theta+(\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0, \theta)$ the operator family $\{z^{-\alpha-1}(z - \omega_{\gamma,\theta})\mathbf{P}_{z,\gamma}^{-1} : z \in \omega_{\gamma,\theta} + \Sigma_{\theta'+(\pi/2)}\} \subseteq L(X)$ is equicontinuous. Furthermore, for every $f \in X$, there exists an X-valued analytic mapping $z \mapsto W_{\gamma,f}^{\alpha}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0, \pi/2)$, one has $\int_0^\infty e^{-zt}W_{\gamma,f}^{\alpha}(t)\,dt = z^{-\alpha-1}\mathbf{P}_{z,\gamma}^{-1}f$, $\operatorname{Re} z > \omega_{\gamma,\theta}$ and the set $\{e^{-\omega_{\gamma,\theta}z}W_{\gamma,f}^{\alpha}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($f \in X$, $\theta' \in (0, \theta)$). Define $W_\gamma^{\alpha}(z)f := W_{\gamma,f}^{\alpha}(z)$, $z \in \Sigma_{\pi/2}$, $f \in X$. Then, for every $\theta \in$

$(0,\pi/2)$, $\{e^{-\omega_{\gamma,\theta}z}W_{\gamma}^{\alpha}(z) : z \in \Sigma_{\theta}\} \subseteq L(X)$ is an equicontinuous operator family. By the foregoing, we have $z^{-\alpha}\mathbf{P}_{z}^{-1}f \to Df$, as $|z| \to \infty$, $z \notin (-\infty,0]$ $(f \in X)$; similarly, the operator family $\{z^{-\alpha-1}B\mathbf{P}_{z,\gamma}^{-1} : |z| \geqslant R,\ z \notin (-\infty,0]\} \in L(X)$ is equicontinuous and $z^{-\alpha}B\mathbf{P}_{z}^{-1}f \to 0$, $|z| \to \infty$, $z \notin (-\infty,0]$ $(f \in X)$, so that we can define a strongly analytic operator family $(W_{\gamma,B}^{\alpha}(z))_{z\in\Sigma_{\pi/2}} \subseteq L(X)$ satisfying that, for every $\theta \in (0,\pi/2)$, the operator family $\{e^{-\omega_{\gamma,\theta}' z}W_{\beta,B}^{\alpha}(z) : z \in \Sigma_{\theta}\} \subseteq L(X)$ is equicontinuous for some number $\omega_{\gamma,\theta}' > 0$. We have that, for every $f \in X$ and $\theta \in (0,\pi/2)$, $\lim_{z\to 0, z\in\Sigma_{\theta}} W_{\gamma}^{\alpha}(z)f = Df$. Define

$$W^{\alpha}(z) := \begin{cases} W_{0}^{\alpha}(z), & z \in \Sigma_{\pi/2}, \\ e^{i\alpha\pi/2}W_{\pi/2}^{\alpha}(ze^{-i\pi/2}), & \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ e^{-i\alpha\pi/2}W_{-\pi/2}^{\alpha}(ze^{i\pi/2}), & \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ 0, & \text{if } z < 0, \\ D, & \text{if } z = 0, \end{cases}$$

and $W_{B}^{\alpha}(z)$ by replacing the operators $W_{0}^{\alpha}(z)$, $W_{\pi/2}^{\alpha}(ze^{-i\pi/2})$, $W_{-\pi/2}^{\alpha}(ze^{i\pi/2})$ and D in the above definition by the operators $W_{0,B}^{\alpha}(z)$, $W_{\pi/2,B}^{\alpha}(ze^{-i\pi/2})$, $W_{-\pi/2,B}^{\alpha}(ze^{i\pi/2})$ and 0, respectively $(z \in \mathbb{C})$. Then, for every $\nu \in (0,\pi)$, there exists a finite constant $\omega_{\nu} > 0$ such that the operator families $\{e^{-\omega_{\nu}z}W^{\alpha}(z) : z \in \overline{\Sigma_{\nu}}\} \subseteq L(X)$ and $\{e^{-\omega_{\nu}z}BW^{\alpha}(z) : z \in \overline{\Sigma_{\nu}}\}$ $= \{e^{-\omega_{\nu}z}W^{\alpha,B}(z) : z \in \overline{\Sigma_{\nu}}\} \subseteq L(X)$ are equicontinuous as well as that, for every $f \in X$, the mappings $z \mapsto W^{\alpha}(z)f$ and $z \mapsto BW^{\alpha}(z)f$ are analytical on $\mathbb{C} \smallsetminus (-\infty,0]$ and continuous on $\mathbb{C} \smallsetminus (-\infty,0)$. Now it can be easily seen that $(W^{\alpha}(t))_{t\geqslant 0} \subseteq L(X,[D(B)])$ is an exponentially bounded $(g_{\alpha},g_{\alpha+1})$-regularized resolvent family generated by A, B. Furthermore, there exists a sufficiently large number $R' > 0$ such that $(zB-A)^{-1} \in L(X)$ for all $z \in S_{R'} := \{z \in \mathbb{C} \smallsetminus (-\infty,0] : |z| = R'\}$. Denote, for every $f \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ and $z \in S_{R'}$,

$$\begin{aligned} u(t;f,z) := &- W^{\alpha}(t)Bf - \int_{0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^{\alpha-1}z)W^{\alpha}(s)x\,ds \\ &+ E_{\alpha}(t^{\alpha}z)(zB-A)^{-1}Bf, \quad t \geqslant 0. \end{aligned}$$

Fix a number $z_{0} \in S_{R'}$ such that $z_{0}e^{\pm i\alpha/2} \in S_{R'}$. Using Theorem 2.3.18, it readily follows that the function $t \mapsto u(t;u_{0},z_{0}) + \int_{0}^{t}u(s;u_{1},z_{0})ds$, $t \geqslant 0$ is a unique solution of problem (148)–(150) with the initial values u_{0} and u_{1} replaced by $(z_{0}B-A)^{-1}Bu_{0}$ and $(z_{0}B-A)^{-1}Bu_{1}$, respectively. On the other hand, using analytical properties of vector-valued Laplace transform we can prove that the function $t \mapsto \int_{0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^{\alpha-1}z)W^{\alpha}(s)x\,ds$, $t \geqslant 0$ can be continuously extended on the region

$\mathbb{C} \smallsetminus (-\infty, 0)$, analytically on the region $\mathbb{C} \smallsetminus (-\infty, 0]$, by the formula

$$\begin{cases} \mathcal{L}^{-1}\big(\frac{1}{z^\alpha - z_0}\frac{1}{z^{\alpha+1}}\mathbf{P}_z^{-1}\big), & \text{if } z \in \Sigma_{\pi/2}, \\ \mathcal{L}^{-1}\big(\frac{1}{z^\alpha - e^{i\pi\alpha/2}z_0}\frac{1}{z^{\alpha+1}}\mathbf{P}_{z,\pi/2}^{-1}\big), & \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ \mathcal{L}^{-1}\big(\frac{1}{z^\alpha - e^{-i\pi\alpha/2}z_0}\frac{1}{z^{\alpha+1}}\mathbf{P}_{z,-\pi/2}^{-1}\big), & \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ 0, & \text{if } z \leqslant 0. \end{cases}$$

This simply implies that, for any two functions $u_{0,1} \in H^2(\Omega) \cap H_0^1(\Omega)$, orthogonal to the eigenfunction(s) ϕ_k for $k = k_0$, there exists a unique analytical solution $t \mapsto u(t)$, $t \geqslant 0$ of problem (148)–(150) on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ with the property that for each $\nu \in (0, \pi)$ there exists a finite constant $\omega_\nu > 0$ such that the sets $\{e^{-\omega_\nu z}u(z) : z \in \overline{\Sigma_\nu}\}$ and $\{e^{-\omega_\nu z}Bu(z) : z \in \overline{\Sigma_\nu}\}$ are bounded.

2.4. Abstract degenerate multi-term fractional differential equations with Riemann–Liouville derivatives

Let $n \in \mathbb{N} \smallsetminus \{1\}$, let A, B and $A_1, \dots, A_{n-1}$ be closed linear operators on a complex Banach space E. Further on, assume that $0 \leqslant \alpha_1 < \cdots < \alpha_n$, $0 \leqslant \alpha < \alpha_n$, $0 < \tau \leqslant \infty$, $f(t)$ is an E-valued function, and D_t^α denotes the Riemann–Liouville fractional derivative of order α. In this section, we investigate the abstract multi-term fractional differential equation

$$BD_t^{\alpha_n}u(t) + \sum_{j=1}^{n-1} A_j D_t^{\alpha_j}u(t) = AD_t^\alpha u(t) + f(t), \quad t \in (0, \tau). \tag{151}$$

We introduce and systematically analyze some new types of degenerate k-regularized (C_1, C_2)-existence and uniqueness (propagation) families for (151). Recall that $\alpha_0 = \alpha$, $m = \lceil \alpha \rceil$, $A_0 = A$, $A_n = B$ and $m_i = \lceil \alpha_i \rceil$ for $1 \leqslant i \leqslant n$. The following vector-valued modification of the condition (P1) will be used henceforth:

(P1)': $h(t)\colon [0, \infty) \to E$ is Laplace transformable, i.e., $h \in L^1_{loc}([0, \infty) : E)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{h}(\lambda) := \mathcal{L}(h)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t}h(t)\,dt := \int_0^\infty e^{-\lambda t}h(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(h) := \inf\{\operatorname{Re}\lambda : \tilde{h}(\lambda) \text{ exists}\}$.

The inclusion $H(\lambda) \in LT_E$ means that there exist a function $h(t)\colon [0, \infty) \to E$ satisfying (P1) and a number $a > \operatorname{abs}(h)$ so that $\tilde{h}(\lambda) = H(\lambda)$, $\lambda > a$.

Definition 2.4.1. Suppose $0 < \tau \leqslant \infty$ and $f \in L^1((0, \tau) : E)$. By a strong solution of (151) we mean any function $u \in L^1((0, \tau) : E)$ for which $g_{m_j - \alpha_j} * u \in W^{m_j,1}((0, \tau) : E)$ $(0 \leqslant j \leqslant n)$, $A_j D_t^{\alpha_j}u(t) \in L^1((0, \tau) : E)$ $(0 \leqslant j \leqslant n)$, and (151) holds for a.e. $t \in (0, \tau)$.

Now we would like to introduce the concept mild solution of (151), and to endow the equation (151) with corresponding inital conditions. In order to do that, let us assume that $u(t)$ is a strong solution of (151). Then we can integrate the equation

(151) α_n-times by using the formula (37) and the closedness of the operators A_j for $j \in \mathbb{N}_{n-1}^0$. In such a way, we get

$$(152)\quad B\Biggl[u(t) - \sum_{i=0}^{m_n-1} (g_{m_n-\alpha_n} * u)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] \\ + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j\Biggl[u(t) - \sum_{i=0}^{m_j-1} (g_{m_j-\alpha_j} * u)^{(i)}(0) g_{\alpha_j+i+1-m_j}(t)\Biggr] \\ = g_{\alpha_n-\alpha} * A\Biggl[u(t) - \sum_{i=0}^{m-1} (g_{m-\alpha} * u)^{(i)}(0) g_{\alpha+i+1-m}(t)\Biggr] + (g_{\alpha_n} * f)(t),$$

for all $t \in [0, \tau)$. This immediately implies that any strong solution of (151) satisfies

$$(153)\qquad B\Biggl[u(t) - \sum_{i=0}^{m_n-1} (g_{m_n-\alpha_n} * u)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] \in C([0,\tau) : E),$$

whence we may conclude that $u \in C([0, \tau) : E)$ provided that $B^{-1} \in L(E)$.

Definition 2.4.2. Suppose $0 < \tau \leqslant \infty$ and $f \in L^1((0, \tau) :)$. By a mild solution of (151) we mean any function $u \in L^1((0, \tau) : E)$ for which $g_{m_j-\alpha_j} * u \in W^{m_j,1}((0, \tau) : E)$ $(0 \leqslant j \leqslant n)$, $A_j(g_{\alpha_n} * D_t^{\alpha_j} u(\cdot))(\cdot) \in C([0, \tau) : E)$ $(0 \leqslant j \leqslant n-1)$, (153) holds, and

$$(154)\quad B\Biggl[u(t) - \sum_{i=0}^{m_n-1} (g_{m_n-\alpha_n} * u)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] + \sum_{j=1}^{n-1} A_j(g_{\alpha_n} * D_t^{\alpha_j} u(\cdot))(t) \\ = A(g_{\alpha_n} * D_t^{\alpha} u(\cdot))(t) + (g_{\alpha_n} * f)(t), \quad t \in [0, \tau).$$

By the foregoing, any strong solution of (151) is also a mild solution of the same problem; the converse statement is not true, in general. Observe that the equation (154) can be written in the following way:

$$(155)\quad B\Biggl[u(t) - \sum_{i=0}^{m_n-1} (g_{m_n-\alpha_n} * u)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] \\ + \sum_{j=1}^{n-1} A_j\Biggl(g_{\alpha_n-\alpha_j} * \Biggl[u(\cdot) - \sum_{i=0}^{m_j-1} (g_{m_j-\alpha_j} * u)^{(i)}(0) g_{\alpha_j+i+1-m_j}(\cdot)\Biggr]\Biggr)(t) \\ = A\Biggl(g_{\alpha_n-\alpha} * \Biggl[u(\cdot) - \sum_{i=0}^{m-1} (g_{m-\alpha} * u)^{(i)}(0) g_{\alpha+i+1-m}(\cdot)\Biggr]\Biggr)(t) + (g_{\alpha_n} * f)(t),$$

for all $t \in [0, \tau)$. Put

$$\mathcal{T}_{(151)} := \begin{cases} 1, & \text{if there exists } j \in \mathbb{N}_n^0 \text{ such that } \alpha_j \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and $S := \{j \in \mathbb{N}_n^0 : \alpha_j \in \mathbb{N}\}$. For the sequel, it will be of crucial importance to recognise the following three subcases of (151):

(SC1) $\alpha_n > 1$: Then for each number $i \in \mathbb{N}_{m_n-1}$ we define the set $\mathcal{D}_i$ by $\mathcal{D}_i := \{j \in \mathbb{N}_n^0 : m_j - 1 \geqslant i\}$. Observe that $n \in \mathcal{D}_i$ ($i \in \mathbb{N}_{m_n-1}$) and $\mathcal{D}_{m_n-1} \subseteq \cdots \subseteq \mathcal{D}_1$. Set $S_i := \{m_j - \alpha_j : j \in \mathcal{D}_i\}$ and, after that, $s_i := \operatorname{card}(S_i)$. Then $S_i \subseteq [0,1)$ and S_i can be written in the following way

$$S_i = \{a_{i,1}, \ldots, a_{i,s_i}\},$$

where $0 \leqslant a_{i,1} < \cdots < a_{i,s_i} \leqslant 1$ ($i \in \mathbb{N}_{m_n-1}$). Define $\mathcal{D}_i^l := \{j \in \mathcal{D}_i : m_j - \alpha_j = a_{i,l}\}$ ($i \in \mathbb{N}_{m_n-1}$, $1 \leqslant l \leqslant s_i$). Then for each number $i \in \mathbb{N}_{m_n-1}$ we introduce s_i initial values $x_{i,1}, \ldots, x_{i,s_i}$ for terms $(g_{m_j-\alpha_j} * u)^{(i)}(0)$, where $j \in \mathcal{D}_i$. In addition, if there exists $j \in \mathbb{N}_n^0$ such that $\alpha_j \in \mathbb{N}$, i.e., if $S \neq \emptyset$, then one has to introduce a new initial value x_0 for term $(g_0 * u)(0) \equiv u(0)$.

(SC2) $\alpha_n = 1$: Then we introduce only one initial value for term $(g_0 * u)(0) \equiv u(0)$.

(SC3) $\alpha_n < 1$: Then we consider the equation (151) without initial conditions.

Define

$$\mathcal{B}_{(151)} := \begin{cases} s_1 + \cdots + s_{m_n-1} + \mathcal{T}_{(151)}, & \text{if } \alpha_n > 1, \\ 1, & \text{if } \alpha_n = 1, \\ 0, & \text{if } \alpha_n < 1. \end{cases}$$

Summa summarum, there will be exactly $\mathcal{B}_{(151)}$ initial conditions for (151).

The subcase (SC3) is very specific and therefore, not discussed henceforth. Consider now, for the sake of illustration and better understanding, the subcase (SC1). Plugging $x_{i,l} = (g_{m_j-\alpha_j} * u)^{(i)}(0)$ in (152), where $j \in \mathcal{D}_i^l$, and choosing other initial values to be zeroes, we obtain

$$\begin{aligned} (156) \quad & B[u(t) - \chi_{\mathcal{D}_i^l}(n) g_{\alpha_n+i+1-m_n}(t) x_{i,l}] \\ & \quad + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j[u(t) - \chi_{\mathcal{D}_i^l}(j) g_{\alpha_j+i+1-m_j}(t) x_{i,l}] \\ & \quad = g_{\alpha_n-\alpha} * A[u(t) - \chi_{\mathcal{D}_i^l}(0) g_{\alpha+i+1-m}(t) x_{i,l}] \quad \text{for all } t \in [0,\tau). \end{aligned}$$

If $S \neq \emptyset$, then inserting the initial value x_0 for $u(0)$ in (152), and choosing $x_{i,l}$ to be zero for $i \in \mathbb{N}_{m_n-1}$ and $1 \leqslant l \leqslant s_i$, we obtain similarly that

$$\begin{aligned} (157) \quad & B[u(t) - \chi_S(n) x_0] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j[u(t) - \chi_S(j) x_0] \\ & \quad = g_{\alpha_n-\alpha} * A[u(t) - \chi_S(0) x_0] \quad \text{for all } t \in [0,\tau). \end{aligned}$$

Suppose now, only for the purpose of further analysis, that $0 < \tau \leqslant \infty$, $K(t) \neq 0$ in $L^1_{loc}([0,\tau))$ and $k(t) = \int_0^t K(s)\,ds$, $t \in [0,\tau)$. Convoluting the above equations with $K(t)$ and using the procedure similar to that already employed for abstract multi-term problems with Caputo fractional derivatives, we come to the following definition.

DEFINITION 2.4.3. Let $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $C, C_1, C_2 \in L(E)$, and let C and C_2 be injective.

(i) (SC1) Suppose that, for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(R_{i,l}(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ is strongly continuous, as well as that, for every $t \in [0,\tau)$, $x \in E$, $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, the following functional equation

$$B[R_{i,l}(t)x - \chi_{\mathcal{D}_i^l}(n)(k * g_{\alpha_n+i-m_n})(t)C_1x] \\ + \sum_{j=1}^{n-1} A_j[g_{\alpha_n-\alpha_j} * (R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(j)(k * g_{\alpha_j+i-m_j})(\cdot)C_1x)](t) \\ = A[g_{\alpha_n-\alpha} * (R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(0)(k * g_{\alpha+i-m})(\cdot)C_1x)](t)$$

holds. If $S \neq \emptyset$, then we also introduce a strongly continuous family $(R_{0,1}(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ satisfying that, for every $t \in [0,\tau)$ and $x \in E$,

$$B[R_{0,1}(t)x - \chi_S(n)k(t)C_1x] + \sum_{j=1}^{n-1} A_j[g_{\alpha_n-\alpha_j} * (R_{0,1}(\cdot)x - \chi_S(j)k(\cdot)C_1x)](t) \\ = A[g_{\alpha_n-\alpha} * (R_{0,1}(\cdot)x - \chi_S(0)k(\cdot)C_1x)](t).$$

Then the sequence $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1, 1\leqslant l\leqslant s_i}$ if $S = \emptyset$, resp., $((R_{i,l}(t))_{t\in[0,\tau)}, (R_{0,1}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1, 1\leqslant l\leqslant s_i}$ if $S \neq \emptyset$, is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (151).
(SC2) A strongly continuous family $(R(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ satisfying that, for every $t \in [0,\tau)$ and $x \in E$,

$$B[R(t)x - k(t)C_1x] + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * R(\cdot)x)(t) = A(g_{\alpha_n-\alpha} * R(\cdot)x)(t),$$

is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (151).

(ii) (SC1) Suppose that, for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(W_{i,l}(t))_{t\in[0,\tau)} \subseteq L(E)$ is strongly continuous, as well as

$$[W_{i,l}(t)Bx - \chi_{\mathcal{D}_i^l}(n)(k * g_{\alpha_n+i-m_n})(t)C_2Bx] \\ + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * [W_{i,l}(\cdot)A_jx - \chi_{\mathcal{D}_i^l}(j)(k * g_{\alpha_j+i-m_j})(\cdot)C_2A_jx] \\ = g_{\alpha_n-\alpha} * [W_{i,l}(\cdot)Ax - \chi_{\mathcal{D}_i^l}(0)(k * g_{\alpha+i-m})(\cdot)C_2Ax],$$

for every $i \in \mathbb{N}_{m_n-1}$, $l \in \mathbb{N}_{s_i}$, $t \in [0,\tau)$ and $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$. If $S \neq \emptyset$, then we also introduce a strongly continuous family $(W_{0,1}(t))_{t\in[0,\tau)} \subseteq L(E)$ satisfying that, for every $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$,

$$[W_{0,1}(\cdot)Bx - \chi_S(n)k(\cdot)C_2Bx]$$

$$+\sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * [W_{0,1}(\cdot)A_jx - \chi_S(j)k(\cdot)C_2A_jx]$$
$$= g_{\alpha_n-\alpha} * [W_{0,1}(\cdot)Ax - \chi_S(0)k(\cdot)C_2Ax].$$

Then the sequence $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ if $S=\emptyset$, resp., $((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ if $S\neq\emptyset$, is said to be a (local, if $\tau<\infty$) k-regularized C_2-uniqueness propagation family for (151).

(SC2) A strongly continuous family $(W(t))_{t\in[0,\tau)}\subseteq L(E)$ satisfying that, for every $x\in\bigcap_{0\leqslant j\leqslant n}D(A_j)$ and $t\in[0,\tau)$,

$$[W(t)Bx-k(t)C_2Bx]+\sum_{j=1}^{n-1}(g_{\alpha_n-\alpha_j}*W(\cdot)C_2A_jx)(t)=(g_{\alpha_n-\alpha}*W(\cdot)C_2Ax)(t),$$

is said to be a (local, if $\tau<\infty$) k-regularized C_2-uniqueness propagation family for (151).

If $k(t)=g_{\zeta+1}(t)$, where $\zeta\geqslant 0$, then a k-regularized C_1-existence propagation family for (151) is also said to be ζ-times integrated C_1-existence propagation family for (151); 0-times integrated C_1-existence propagation family for (151) is simply called C_1-existence propagation family for (151); similar noions will be used for the classes of C_2-uniqueness propagation families for (151) and C-resolvent propagation families for (151).

A k-regularized C_1-existence propagation family for (151) is said to be locally equicontinuous (exponentially equicontinuous) iff each single operator family of it, considered as an element of the space $L(E,[D(B)])$, is locally equicontinuous (exponentially equicontinuous). The notion of an exponential equicontinuity of a k-regularized C_1-existence propagation family for (151), of angle $\alpha\in(0,\pi/2]$, will be understood in the sense of Definition 2.3.24.

We define the notion of a mild (strong) solution of problem (156), resp. (157), as before: Let $\tau\in(0,\infty]$. By a mild solution of problem (156) on $[0,\tau)$ we mean any continuous function $t\mapsto u(t)$, $t\in[0,\tau)$ satisfying that

$$B[u(t)-\chi_{\mathcal{D}_i^l}(n)g_{\alpha_n+i+1-m_n}(t)x_{i,l}]$$
$$+\sum_{j=1}^{n-1}A_j(g_{\alpha_n-\alpha_j}*[u(\cdot)-\chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(\cdot)x_{i,l}])(t)$$
$$=A(g_{\alpha_n-\alpha}*[u(\cdot)-\chi_{\mathcal{D}_i^l}(0)g_{\alpha+i+1-m}(\cdot)x_{i,l}])(t),\quad t\in[0,\tau).$$

By a strong solution of problem (156) on $[0,\tau)$ we mean any continuous function $t\mapsto u(t)$, $t\in[0,\tau)$ satisfying that $t\mapsto A_j[u(t)-\chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(t)x_{i,l}]$, $t\in[0,\tau)$ is continuous, as well as

$$B[u(t)-\chi_{\mathcal{D}_i^l}(n)g_{\alpha_n+i+1-m_n}(t)x_{i,l}]$$
$$+\sum_{j=1}^{n-1}(g_{\alpha_n-\alpha_j}*A_j[u(\cdot)-\chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(\cdot)x_{i,l}])(t)$$

$$= (g_{\alpha_n-\alpha} * A[u(\cdot) - \chi_{\mathcal{D}_i^l}(0)g_{\alpha+i+1-m}(\cdot)x_{i,l}])(t), \quad t \in [0,\tau).$$

The notion of a mild (strong) solution of problem (157) on $[0,\tau)$ is defined similarly.

Then the following holds (for the sake of brevity, we shall consider only the subcase (SC1)):

(A) If $S = \emptyset$ and $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a C_1-existence propagation family for (151), then the function $u_{i,l}(t) := R_{i,l}(t)C_1x$, $t \in [0,\tau)$ is a mild solution of (156) with $x_{i,l} = C_1x$ for $1 \leqslant i \leqslant m_n - 1, 1 \leqslant l \leqslant s_i$. If $S \neq \emptyset$ and $((R_{i,l}(t))_{t\in[0,\tau)}, (R_{0,1}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a C_1-existence propagation family for (151), then the function $u_{0,1}(t) := R_{0,1}(t)C_1x$, $t \in [0,\tau)$ is a mild solution of (157) with $x_0 = C_1x$.

(B) If $S = \emptyset$ and $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a C_2-uniqueness propagation family for (151), as well as $A_jW_{i,l}(t)x = W_{i,l}(t)A_jx$, $t \in [0,\tau)$, $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$ and $C_2A_j \subseteq A_jC_2$, $j \in \mathbb{N}_n^0$, then the function $u_{i,l}(t) := W_{i,l}(t)C_2^{-1}x_{i,l}$, $t \in [0,\tau)$ is a strong solution of (156) for $1 \leqslant i \leqslant m_n - 1, 1 \leqslant l \leqslant s_i$ and $x_{i,l} \in C_2(\bigcap_{0\leqslant j\leqslant n} D(A_j))$. If $S \neq \emptyset$ and $((W_{i,l}(t))_{t\in[0,\tau)}, (W_{0,1}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a C_2-uniqueness propagation family for (151), as well as (in addition to the assumptions employed in the case that $S = \emptyset$) $A_jW_{0,1}(t)x = W_{0,1}(t)A_jx$, $t \in [0,\tau)$, $x \in \bigcap_{0\leqslant j\leqslant n} D(A_j)$, then the function $u_0(t) := W_{0,1}(t)C_2^{-1}x_0$, $t \in [0,\tau)$ is a strong solution of (156) for $1 \leqslant i \leqslant m_n - 1, 1 \leqslant l \leqslant s_i$ and $x_0 \in C_2(\bigcap_{0\leqslant j\leqslant n} D(A_j))$.

The assertions of [**346**, Proposition 2.3, Proposition 2.5, Proposition 2.6] admit reformulations in our new framework (cf. also Theorem 2.3.26). This is also the case with [**346**, Theorem 2.8], as the following theorem shows:

THEOREM 2.4.4. (SC1) *Suppose that $S = \emptyset$, $C_2 \in L(E)$ is injective, $f \in C([0,\tau) : E)$, and $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a k-regularized C_2-uniqueness propagation family for* (151). *If a function $u(t)$ is a strong solution of problem* (114), *then*

$$\begin{aligned}
&[W_{i,l}(\cdot) - \chi_{\mathcal{D}_i^l}(n)(k * g_{\alpha_n+i-m_n})(\cdot)C_2] * f \\
&= \sum_{j=1}^{n-1}[\chi_{\mathcal{D}_i^l}(j)(g_{\alpha_n+i-m_j} * kC_2 * A_ju)(\cdot) - \chi_{\mathcal{D}_i^l}(n)(g_{2\alpha_n-\alpha_j+i-m_n} * kC_2 * A_ju)(\cdot)] \\
&\quad - [\chi_{\mathcal{D}_i^l}(0)(g_{\alpha_n+i-m} * kC_2 * Au)(\cdot) - \chi_{\mathcal{D}_i^l}(n)(g_{2\alpha_n-\alpha+i-m_n} * kC_2 * Au)(\cdot)].
\end{aligned}$$

If $S \neq \emptyset$ and $((W_{i,l}(t))_{t\in[0,\tau)}, (W_{0,1}(t))_{t\in[0,\tau)})_{1\leqslant i\leqslant m_n-1,1\leqslant l\leqslant s_i}$ is a k-regularized C_2-uniqueness propagation family for (151), *then we also have the following equality on $[0,\tau)$:*

$$\begin{aligned}
[W_{0,1}(\cdot)x - \chi_S(n)k(\cdot)C_2] * f &= \sum_{j=1}^{n-1}(\chi_S(j) - \chi_S(n))(g_{\alpha_n-\alpha_j} * kC_2 * A_ju)(\cdot) \\
&\quad - (\chi_S(0) - \chi_S(n))(g_{\alpha_n-\alpha} * kC_2 * Au)(\cdot).
\end{aligned}$$

(SC2) *Suppose that $C_2 \in L(E)$ is injective, $(W(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness propagation family for* (151), *and $f \in C([0,\tau) : E)$. If a function $u(t)$ is a strong solution of problem* (114), *then the following equality holds on $[0,\tau)$:*

$$[W(\cdot)x - k(\cdot)C_2] * f = (g_{\alpha_n-\alpha} * kC_2 * Au)(\cdot) - \sum_{j=1}^{n-1}(g_{\alpha_n-\alpha_j} * kC_2 * A_j u)(\cdot).$$

As explained before for the equations with Caputo fractional derivatives, the notion of a k-regularized C_1-existence propagation family is probably the best theoretical concept for the investigation of existence of integral solutions of non-degenerate abstract time-fractional equation (151) with $A_j \in L(E)$, $1 \leqslant j \leqslant n-1$; the Laplace transform cannot be so simply applied in the case that there is an index $j \in \mathbb{N}_{n-1}$ such that $A_j \notin L(E)$. In contrast to the above, it is very simple to reword the assertions of [**346**, Theorem 2.9(ii)] and [**314**, Theorem 2.4], concerning the Laplace transform of k-regularized C_2-uniqueness propagation families, to degenerate differential equations with Riemann–Liouville fractional derivatives. The assertions of [**346**, Theorem 2.10-Theorem 2.12] can be rephrased for abstract degenerate multi-term problems with Riemann–Liouville fractional derivatives, as well. Having these done, it is not difficult to reconsider [**346**, Example 5.1(i)] in our new setting. Furthermore, it is not so difficult to construct some examples of local k-regularized I-resolvent propagation families for (151); see e.g. Example 2.3.28.

Before proceeding further, we would like to present three interesting examples of non-degenerate fractional differential equations that can be analyzed with the help of k-regularized C-resolvent propagation families:

EXAMPLE 2.4.5. [**345**]

(i) Based on the so-called CTRW method, T. A. M. Langlands, B. I. Henry and S. L. Wearne have suggested in [**372**] using Riemann–Liouville fractional derivatives for the approximation of passive electrotonic properties of spiny dendrites. In the abstract form (see e.g. [**372**, (1.18)]), the equation suggested by them takes the following form

$$D_t^1 u(t,x) + \mu^2 D_t^{1-\kappa} u(t,x) = D_t^{1-\gamma}\Delta u(t,x), \tag{158}$$

where $0 < \gamma < \kappa < 1$ and $\mu \in \mathbb{R} \setminus \{0\}$. The equation (158) is of type (SC2) and repeating verbatim the corresponding parts the analysis of equation [**346**, (5.4)] we can simply prove that there exists an exponentially equicontinuous, analytic resolvent propagation family for (158), of angle $\theta = \pi/2$ (recall that it is not clear whether the angle of analyticity can be improved by allowing that θ takes the value $\min(\pi, \pi/\gamma - \pi/2)$), on a large class of Banach function spaces (consisted of functions acting on finite or infinite domain).

(ii) [**63**] The fractional order differential equation

$$u'(t) = Au(t) + \gamma D_t^\alpha Au(t) + f(t), \quad t > 0; \quad u(0) = x_0 \in E$$

has recently been studied in [**63**], provided that the operator A generates a strongly continuous semigroup on E, $\gamma > 0$, $\alpha \in (0,1)$ and $f(t)$ is an E-valued function. Equations of this type appear in the modelling of unidirectional viscoelastic flows; in particular, they can be of some importance in the analysis of the Rayleigh–Stokes problem for generalized second-grade fluids.

(iii) Suppose that $n \in \mathbb{N} \setminus \{1\}$, $c_1, \ldots, c_{n-1} \in \mathbb{C}$, $0 \leqslant \alpha_1 < \cdots < \alpha_n \leqslant 2$, $0 < \beta < 1$, $1 < \gamma \leqslant 2$, $k_\beta, k_\gamma > 0$ and $L > 0$. The following scalar multi-term time-space Caputo–Riesz fractional advection diffusion equation, (MT-TSCR-FADE) for short,

$$\begin{gathered} \mathbf{D}_t^{\alpha_n} u(t,x) + c_{n-1}\mathbf{D}_t^{\alpha_{n-1}} u(t,x) \\ + \cdots + c_1 \mathbf{D}_t^{\alpha_1} u(t,x) = k_\beta \frac{\partial^\beta u(t,x)}{\partial |x|^\beta} + k_\gamma \frac{\partial^\gamma u(t,x)}{\partial |x|^\gamma} \\ \Bigl(\frac{\partial^k}{\partial t^k} u(t,x)\Bigr)_{t=0} = u_k(x), \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1,\ 0 \leqslant x \leqslant L, \end{gathered} \tag{159}$$

where $\frac{\partial^\beta u(t,x)}{\partial |x|^\beta}$ denotes the Riesz fractional operator of order β, has recently been analyzed by H. Jiang, F. Liu, I. Turner and K. Burrage in [**260**]. In [**292**, Example 2.9.53], we have considered two different evolution modelings of problem (159). In the first of these modelings, the equation (159) has been rewritten in the form of the following multi-term fractional differential equation:

$$\begin{gathered} \mathbf{D}_t^{\alpha_n} u(t) + c_{n-1}\mathbf{D}_t^{\alpha_{n-1}} u(t) + \cdots + c_1 \mathbf{D}_t^{\alpha_1} u(t) = -k_\beta A_{\beta/2} - k_\gamma A_{\gamma/2} \\ u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1, \end{gathered} \tag{160}$$

where the operator A belongs to the class $\mathcal{M}_{C,m}$ for some $m \in \mathbb{R}$, and acts on an appropriately chosen space of functions defined on $[0, L]$ (cf. Section 1.1 and [**292**, Section 2.9] for more details about almost C-sectorial operators). As announced in [**292**], the analysis of equation (160) is quite complicated in the case that $C \neq I$ or $m \neq -1$; suppose, because of that and for the sake of simplicity, that A is a sectorial operator of angle $\omega \in [0, \pi)$, with $D(A)$ and $R(A)$ being dense in E. Let the so-called parabolicity condition $2\pi > (\beta+\gamma)\omega$ hold, let $\alpha_n^{-1}(\pi-(\gamma\omega/2))-(\pi/2) > 0$, and let $1 < \alpha_n < 2$. Then we can prove with the help of Da Prato–Grisvard theorem (see e.g. [**245**, Theorem 9.3.1, Corollary 9.3.2]) that the operator $k_\beta A_{\beta/2} + k_\gamma A_{\gamma/2}$ is sectorial of angle $\gamma\omega/2$. Using this fact, it can be easily verified that for each $x_{1,1} \in E$ there exists a unique mild solution $t \mapsto u(t)$, $t > 0$ of the corresponding integral equation (156) and that this solution can be analytically extended to the sector Σ_δ, with $\delta = \min(\pi/2, \alpha_n^{-1}(\pi - (\gamma\omega/2)) - (\pi/2))$. In the second modeling, we have analyzed the C-wellposedness of the backwards equation (MT-TSCR-FADE):

$$\begin{aligned} \mathbf{D}_t^{\alpha_n}u(t)+c_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u(t)+\cdots+c_1\mathbf{D}_t^{\alpha_1}u(t)&=-k_\beta A_\beta-k_\gamma A_\gamma\\ u(0)&=u_0, \end{aligned}\tag{161}$$

where $0<\beta<1$, $1<\gamma<2$, $k_\beta, k_\gamma>0$, $0\leqslant\alpha_1<\cdots<\alpha_n<1$, $\alpha_n^{-1}(\pi-(\gamma\pi/2))>\pi/2$, $E:=\{f\in C^\infty[0,1]\ ;\ \|f\|:=\sup_{p\in\mathbb{N}_0}\frac{\|f^{(p)}\|_\infty}{p!^\zeta}<\infty\}$ $(\zeta>1)$ and $A:=-d/ds$ with $D(A):=\{f\in E: f'\in E,\ f(0)=0\}$. Regrettably, in this approach, (161) is a modified abstract time-relaxation equation and almost nothing interesting can be said about the corresponding equation with Riemann–Liouville fractional derivatives (the subcase (SC3)).

Now we would like to state the following result on the existence of strong solutions of equation (151) (cf. [**347**, Theorem 3.1] and [**314**, Theorem 3.13] for the corresponding statement in the case of equations with Caputo fractional derivatives).

THEOREM 2.4.6. *Suppose $A, B, A_1, \ldots, A_{n-1}$ are closed linear operators on E, $\omega>0$, $0<\tau<\infty$, $C\in L(E)$ is injective, $f(t)\equiv 0$, the operator P_λ is injective for $\lambda>\omega$ and $D(P_\lambda^{-1}C)=E$, $\lambda>\omega$.*

(SC1) *Suppose $1\leqslant i\leqslant m_n-1$, $1\leqslant l\leqslant s_i$, $n\in\mathcal{D}_i^l$, $Cx_{i,l}\in D(P_\lambda^{-1}A_j)$, provided $\lambda>\omega$ and $j\in\mathbb{N}_n^0\cap\mathcal{D}_i^l$, as well as $\alpha_j-\alpha_n+m_n-1-i<0$, provided $j\in\mathbb{N}_{n-1}^0\smallsetminus\mathcal{D}_i^l$, and the following holds:*

$$\begin{aligned}\lambda^{\alpha_n}P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha}BCx_{i,l}+\sum_{j=1}^{n-1}A_j(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l})\\ -A(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l})\Biggr]-\lambda^{m_n-1-i}Cx_{i,l}\in LT_E\end{aligned}\tag{162}$$

and

$$\begin{aligned}A_j\Biggl\{\lambda^{\alpha_j}P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha}BCx_{i,l}+\sum_{j=1}^{n-1}A_j(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l})\\ -A(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l})\Biggr]-\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l}\Biggr\}\in LT_E.\end{aligned}\tag{163}$$

Then there exists a strong solution of (151) on $(0,\tau)$, with initial value $x_{i,l}$ replaced by $Cx_{i,l}$ and other initial values chosen to be zeroes. If $S\neq\emptyset$ and the above conditions hold for the initial value $x_{0,1}=u(0)$ with set $\mathcal{D}_i^l$ replaced by S ($i=0$, $l=1$), then there exists a strong solution of (151) on $(0,\tau)$, with initial value $x_{0,1}$ replaced by $Cx_{0,1}$ and other initial values chosen to be zeroes.

(SC2) *Suppose $Cx_{0,1}\in D(B)$, $\lambda^{1-\alpha}P_\lambda^{-1}BCx_{0,1}-Cx_{0,1}\in LT_E$ and $\lambda^{\alpha_j-\alpha}A_jP_\lambda^{-1}BCx_{0,1}\in LT_E$.*

Then there exists a strong solution of (151) *on* $(0, \tau)$, *with initial value* $x_{0,1}$ *replaced by* $Cx_{0,1}$ *and other initial values chosen to be zeroes.*

PROOF. We will prove the assertion of theorem only in the case (SC1) with $1 \leqslant i \leqslant m_n - 1$ and $1 \leqslant l \leqslant s_i$. Let $u_{i,l} \in L^1_{loc}([0,\infty) : E)$ and $F_{i,l,n} \in L^1_{loc}([0,\infty) : E)$ satisfy:

$$\int_0^\infty e^{-\lambda t} u_{i,l}(t)dt = P_\lambda^{-1}\Bigg[\lambda^{m_n-i-1-\alpha}BCx_{i,l} \\ + \sum_{j=1}^{n-1} A_j(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}) - A(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l})\Bigg]$$

and

$$\int_0^\infty e^{-\lambda t} F_{i,l,n}(t)dt = \lambda^{\alpha_n} P_\lambda^{-1}\Bigg[\lambda^{m_n-i-1-\alpha}BCx_{i,l} - A(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l}) \\ + \sum_{j=1}^{n-1} A_j(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l})\Bigg] - \lambda^{m_n-1-i}Cx_{i,l},$$

for $\lambda > \omega$ suff. large; cf. (162). By performing the Laplace transform, it can be easily checked that:

$$(g_{m_n} * F_{i,l,n})(t) = (g_{m_n-\alpha_n} * u_{i,l})(t) - g_{i+1}(t)Cu_{i,l}, \quad t > 0.$$

This implies that $D_t^{\alpha_n} u_{i,l}(t)$ is well defined for $t > 0$ (more precisely, on any finite subinterval of $(0,\infty)$) and $F_{i,l,n}(t) = D_t^{\alpha_n} u_{i,l}(t)$, $t > 0$. Keeping in mind that $n \in \mathcal{D}_i^l$ and $\alpha_j - \alpha_n + m_n - 1 - i < 0$ for $j \in \mathbb{N}_{n-1}^0 \smallsetminus \mathcal{D}_i^l$, we can conclude from (162) that $\lambda^{\alpha_j}\widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l} \in LT_E$ for all $j \in \mathbb{N}_{n-1}^0$, as well as that $D_t^{\alpha_j} u_{i,l}(t)$ is well defined for $t > 0$, with

$$\int_0^\infty e^{-\lambda t} D_t^{\alpha_j} u_{i,l}(t)dt = \lambda^{\alpha_j}\widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l} \in LT_E,$$

for all $j \in \mathbb{N}_{n-1}^0$. Using (163) and [**27**, Proposition 1.7.6], it readily follows that $A_j D_t^{\alpha_j} u_{i,l}(t)$ is well defined for $t > 0$, and

$$\int_0^\infty e^{-\lambda t} A_j D_t^{\alpha_j} u_{i,l}(t)dt = A_j[\lambda^{\alpha_j}\widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l}] \in LT_E,$$

for all $j \in \mathbb{N}_{n-1}^0$. Finally, a simple calculation yields that:

$$\int_0^\infty e^{-\lambda t}[BD_t^{\alpha_n} u_{i,l}(t) + A_{n-1}D_t^{\alpha_{n-1}} u_{i,l}(t) + \cdots + A_1 D_t^{\alpha_1} u_{i,l}(t) - AD_t^{\alpha} u_{i,l}(t)]dt = 0,$$

which implies by the uniqueness theorem for the Laplace tranform that $u_{i,l}(\cdot)$ is a strong solution of the problem (151) with initial value $x_{i,l}$ replaced by $Cx_{i,l}$ and other initial values chosen to be zeroes. $\square$

REMARK 2.4.7. Consider the subcase (SC1) and suppose first that $1 \leqslant i \leqslant m_n - 1$, $1 \leqslant l \leqslant s_i$, $n \in \mathcal{D}_i^l$, as well as $\alpha_j - \alpha_n + m_n - 1 - i < 0$, provided

$j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$. Then a straightforward calculation shows that the assertion of Theorem 2.4.6 continues to hold if we replace the term

$$P_\lambda^{-1}\Bigg[\lambda^{m_n-i-1-\alpha}BCx_{i,l} + \sum_{j=1}^{n-1} A_j(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}) \\ - A(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l})\Bigg],$$

i.e., the Laplace transform of solution $t \mapsto u_{i,l}(t)$, $t > 0$ with the term

$$\lambda^{m_n-i-1-\alpha_n}Cx_{i,l} - P_\lambda^{-1}\Bigg[\sum_{j\in\mathbb{N}_{n-1}\setminus\mathcal{D}_i^l} A_j(\lambda^{m_n-i-1-\alpha_n+\alpha_j-\alpha}Cx_{i,l}) \\ + A(\chi_{\mathbb{N}_{n-1}\setminus\mathcal{D}_i^l}(0)\lambda^{m_n-i-1-\alpha_n}Cx_{i,l})\Bigg],$$

and suppose that $Cx_{i,l} \in D(P_\lambda^{-1}A_j)$ for $\lambda > \omega$ and $j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$, instead of $Cx_{i,l} \in D(P_\lambda^{-1}A_j)$ for $\lambda > \omega$ and $j \in \mathbb{N}_{n-1}^0 \cap \mathcal{D}_i^l$. If $S \neq \emptyset$, $i = 0$ and $l = 1$, then one has to replace the set $\mathcal{D}_i^l$ with S. The corresponding analysis of the subcase (SC2) is left to the interested reader.

Before proceeding further, we would like to present an illustrative example in which the existence of strong solutions of problem (151) can be proved trivially and which also shows that there exists a large class of (degenerate) multi-term problems which do have strong solutions that are completely independent of the choice of operators A_j.

EXAMPLE 2.4.8. Consider the subcase (SC1) with $1 \leqslant i \leqslant m_n - 1$, $l = 1$, $m_j \geqslant i+1$, $j \in \mathbb{N}_0$ and $m_j - \alpha_j = a_{i,1}$, $j \in \mathbb{N}_0$. Let $0 < \tau < \infty$ and $f(t) \equiv 0$. Then $D_t^{\alpha_j} g_{i+1-a_{i,1}}(t) = 0$, $j \in \mathbb{N}_0$ so that the function $u(t) = g_{i+1-a_{i,1}}(t)x_{i,1}$, $t \in [0, \tau)$ is a strong solution of problem (151) with $B = I$, $x_{i,1} = (g_{m_j-\alpha_j} * u)^{(i)}(0) \in X$ and other initial values chosen to be zeroes.

In the remainder of this section, we assume that X (the state space) and Y are complex Banach spaces; the norm of an element $y \in Y$ will be denoted by $\|y\|_Y$. Now the closed linear operators A, B, $A_1, \ldots, A_{n-1}$ are acting on X. By a (local) C_1-existence family, resp. k-regularized C_2-uniqueness family, we mean (local) C_1-existence family for (112), resp. k-regularized C_2-uniqueness family for (112).

The first part of subsequent theorem can be proved following the analysis carried out in [**345**, Section 3]; the second part of this theorem can be proved by using Theorem 2.3.32(ii) and the fact that $D_t^\zeta u(t) = \mathbf{D}_t^\zeta u(t)$, $t > 0$, provided that $\zeta > 0$, $\mathbf{D}_t^\zeta u(t)$ is defined and $u^{(i)}(0) = 0$ for all $i \in \mathbb{N}_{\lceil\zeta\rceil-1}^0$. Observe also that we can reformulate the final conclusions from Theorem 2.3.6 for degenerate multi-term problems with Riemann–Liouville fractional derivatives and that it is very difficult to state, in contrast to the equations with Caputo fractional derivatives, some satisfactory results on the existence of strong solutions of (151), provided that there exists a (local) C_1-existence family.

THEOREM 2.4.9. *Let $0 < \tau \leqslant \infty$, let $C_1 \in L(Y,X)$, and let $C_2 \in L(X)$ be injective.*

(i) *Suppose that $(E(t))_{t\in[0,\tau)}$ is a (local) C_1-existence family, $1 \leqslant i \leqslant m_n - 1$ and $1 \leqslant l \leqslant s_i$. Assume further that, for every $j \in \mathbb{N}^0_{n-1} \cap \mathcal{D}^l_i$, there exists an element $y_{i,l,j} \in Y$ such that $A_j x_{i,l} = C_1 y_{i,l,j}$, as well as that the condition $n \in \mathcal{D}^l_i$ implies the existence of an element $y_{i,l} \in Y$ such that $Bx_{i,l} = C_1 y_{i,l}$. Define*

$$u(t) := \int_0^t E^{(m_n-1)}(t-s)\Bigg[\chi_{\mathcal{D}^l_i}(n) g_{\alpha_n+i-m_n}(s) y_{i,l} + \sum_{j\in\mathbb{N}_{n-1}\cap\mathcal{D}^l_i} g_{\alpha_n+i-m_j}(s) y_{i,l,j} - \chi_{\mathcal{D}^l_i}(0) g_{\alpha_n+i-m}(s) y_{i,l,0}\Bigg]\,ds, \quad t \in [0,\tau).$$

Then the function $t \mapsto u(t)$, $t \in [0,\tau)$ is a mild solution of the problem (156). *If $S \neq \emptyset$, $i = 0$ and $l = 1$, then the above holds with the set $\mathcal{D}^l_i$ replaced by S.*

(ii) *Suppose that $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family and $k(t)$ is a kernel on $[0,\tau)$. Then every two strong (mild) solutions of the equation* (151) *possessing the same initial conditions* (*cf.* (152) *and* (155)) *are identically equal on $[0,\tau)$.*

In Theorem 2.3.34, we have analyzed inhomogeneous multi-term problems with Caputo fractional derivatives. Similarly we can analyze the inhomogeneous multi-term problems with Riemann–Liouville fractional derivatives.

2.5. The existence and uniqueness of solutions of abstract degenerate fractional differential equations: ultradistribution theory

Our first task in this section will be to extend the assertions of [**303**, Theorem 2.1, Corollary 2.1] to abstract degenerate fractional differential equations (the Gevrey case). In order to do that, fix the numbers $\zeta \in (0,1]$, $\alpha > 0$, $\beta > 0$, $l \geqslant 1$, $\xi \geqslant 0$, $b \in (0,1)$, and denote by $M_v(\cdot)$ the associated function of the sequence $(p^{\frac{p}{v}})$ $(v \in (0,1))$. Then we know that $M_v(t) \sim (ve)^{-1}t^v$ as $t \to +\infty$. If (N_p) and (R_p) are two sequences of positive real numbers, then we write $N_p \prec R_p$ iff for each number $\sigma > 0$ we have

$$\sup_{p\in\mathbb{N}_0} \frac{N_p \sigma^p}{R_p} < \infty.$$

Henceforth we shall always assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and the condition $(M.1)$ holds. Suppose that

$$p^{\frac{p}{b}} \prec M_p. \tag{164}$$

Then, for every $\mu > 0$, there exist positive real constants $c_\mu > 0$ and $C_\mu > 0$ such that $\lim_{\mu\to 0} c_\mu = 0$ and

$$M(l\lambda) \leqslant M_b(\mu l\lambda) + C_\mu \leqslant c_\mu|\lambda|^b + C_\mu, \quad \lambda \geqslant 0. \tag{165}$$

As mentioned earlier, the (M_p)-ultralogarithmic region of type l

$$\Lambda_{\alpha,\beta,l} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha M(l|\operatorname{Im}\lambda|) + \beta\},$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $l \geqslant 1$, was defined for the first time by J. Chazarain in 1971 [**99**]. We assume that the boundary of the ultra-logarithmic region $\Lambda_{\alpha,\beta,l}$, denoted by Γ_l, is upwards oriented. Set

$$\Lambda^{\zeta}_{\alpha,\beta,l} := \{\lambda^{\zeta} : \lambda \in \Lambda_{\alpha,\beta,l}\} \text{ and } \Omega := \mathbb{C} \smallsetminus \Lambda^{\zeta}_{\alpha,\beta,l}.$$

By $\mathcal{A}$ we denote the class consisting of all continuous functions $f : \bar{\Omega} \to \mathbb{C}$ that are analytic in Ω and satisfy that there exist constants $a_1 > 0$ and $a_2 > \xi$ such that:

$$|f(\lambda)| \leqslant a_1 e^{-a_2|\lambda|^{b/\zeta}}, \quad \lambda \in \bar{\Omega}. \tag{166}$$

Suppose that $f \in \mathcal{A}$, $f \neq 0$. Then we define $F(\cdot)$ by

$$F(\lambda) := f\bigl(-\lambda^{\frac{\pi-(\zeta\pi/2)}{\pi/2}}\bigr), \quad \lambda \in \overline{\Sigma_{\pi/2}}.$$

The function $F(\cdot)$ can be analytically extended to an open neighborhood of the region $\overline{\Sigma_{\pi/2}}$ and satisfies that:

$$|F(\lambda)| \leqslant a_1 e^{-a_2|\lambda|^{\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta}}}, \quad \lambda \in \overline{\Sigma_{\pi/2}}.$$

Now we can apply the Phragmén-Lindelöf type theorems (see e.g. [**376**, p. 40]) in order to see that the inequality $\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta} \geqslant 1$ implies $f = 0$ identically. Hence, one has to assume that $\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta} < 1$, i.e., that

$$\frac{1}{2-\zeta} > \frac{b}{\zeta} \tag{167}$$

in order to ensure the non-triviality of class $\mathcal{A}$ (observe that $1/(2-\zeta) \in (1/2, 1]$ for $\zeta \in (0, 1]$, so that (167) automatically implies $b < \zeta$). Suppose now that (167) holds. Then the class $\mathcal{A}$ is non-trivial, indeed, and we can simply prove this fact as follows. Put $\theta := \arctan(\cos(\frac{b}{\zeta}(\pi - \frac{\pi\zeta}{2})))$. Then the function

$$f(\lambda) = f_t(\lambda) := e^{-t(-\lambda+\omega)^{b/\zeta}}, \quad \lambda \in \bar{\Omega}$$

belongs to $\mathcal{A}$ provided $t = t_1 + it_2 \in \Sigma_\theta$, $\omega > \beta^{\zeta}$ and $t_1 \tan\theta - |t_2| > \xi$, because $\arg(-\lambda^{\zeta} + \omega) \to \pi - \frac{\pi}{2}\zeta$ as $|\lambda| \to \infty$, $\lambda \in \Gamma_l$ and there exists $R > 0$ such that, for every $t = t_1 + it_2 \in \Sigma_\theta$,

$$\bigl|e^{-t(-\lambda+\omega)^{b/\zeta}}\bigr| = e^{-t_1|-\lambda+\omega|^{b/\zeta}\cos(\frac{b}{\zeta}\arg(-\lambda+\omega))+t_2|-\lambda+\omega|^{b/\zeta}\sin(\frac{b}{\zeta}\arg(-\lambda+\omega))}$$

$$\leqslant e^{-(t_1\cos(\frac{b}{\zeta}\arg(-\lambda+\omega))-|t_2|)|-\lambda+\omega|^{b/\zeta}} \leqslant e^{-(t_1\tan\theta-|t_2|)|-\lambda+\omega|^{b/\zeta}}, \quad \lambda \in \bar{\Omega},\ |\lambda| \geqslant R.$$

It is clear that $f \cdot g, f + g, zf \in \mathcal{A}$, provided $f, g \in \mathcal{A}$ and $z \in \mathbb{C}$. In order to stay consistent with the notation used in our joint paper with V. Fedorov [**213**], the time-variable in this section will be denoted by s. Further on, let $n \in \mathbb{N}$, let $p_0, p_1, \ldots, p_n$ and $q_0, q_1, \ldots, q_n$ be given non-negative integers satisfying $p_0 = q_0 = 0$ and $0 < p_1 + q_1 \leqslant p_2 + q_2 \leqslant \ldots \leqslant p_n + q_n$. Let $A_0 = A, A_1, \ldots, A_{n-1}, A_n = B$ be closed linear operators acting on an SCLCS E. Set, with a little abuse of notation, $T_iu(s) := (\mathbf{D}_s^\zeta)^{p_i}A_i(\mathbf{D}_s^\zeta)^{q_i}u(s)$, $s \geqslant 0$, $i \in \mathbb{N}_n^0$, $S_l := \{i \in \mathbb{N}_n : q_i \geqslant 1\}$, $S_r := \{i \in \mathbb{N}_n : p_i \geqslant 1\}$,

$$P_\lambda := \lambda^{(p_n+q_n)\zeta}B + \sum_{i=0}^{n-1}\lambda^{(p_i+q_i)\zeta}A_i, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

and conventionally, $\max(\emptyset) := \emptyset$, $\mathbb{N}_\emptyset^0 := \emptyset$. We analyze the abstract degenerate Cauchy problem [(168)–(169)], where

$$\sum_{i=0}^{n} T_iu(s) = 0, \quad s \geqslant 0, \tag{168}$$

and

$$\begin{aligned} ((\mathbf{D}_s^\zeta)^j u(s))_{s=0} &= u_j, \quad j \in \mathbb{N}^0_{\max\{q_i-1 : i\in S_l\}}, \text{ and} \\ ((\mathbf{D}_s^\zeta)^j A_i(\mathbf{D}_s^\zeta)^{q_i}u(s))_{s=0} &= u_{i,j} \quad (i \in S_r,\ j \in \mathbb{N}^0_{p_i-1}). \end{aligned} \tag{169}$$

Denote by $(\mathrm{DFP})'_R$ ($(\mathrm{DFP})'_L$) the homogeneous abstract fractional Cauchy problem obtained by replacing the number α and the operator A in problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) with the number $\zeta \in (0,1]$ and the operator $-A$, respectively:

$$(\mathrm{DFP})'_R : \begin{cases} \mathbf{D}_s^\zeta Bu(s) + Au(s) = 0, & s \geqslant 0, \\ Bu(0) = Bx, \end{cases}$$

$$(\mathrm{DFP})'_L : \begin{cases} B\mathbf{D}_s^\zeta u(s) + Au(s) = 0, & s \geqslant 0, \\ u(0) = x. \end{cases}$$

The problems $(\mathrm{DFP})'_R$ and $(\mathrm{DFP})'_L$ are special cases of the problem [(168)–(169)], with $n = 1$, $q_1 = 0$, $p_1 = 1$ and $u_{1,0} = Bx$, in the case of problem $(\mathrm{DFP})'_R$, and $n = 1$, $q_1 = 1$, $p_1 = 0$, $u_1 = x$, in the case of problem $(\mathrm{DFP})'_L$. A function $u \in C([0,\infty) : E)$ is said to be a strong solution of problem [(168)–(169)] iff the term $T_iu(s)$ is well defined and continuous for any $s \geqslant 0$, $i \in \mathbb{N}_n^0$, and [(168)–(169)] holds identically on $[0,\infty)$. If the function $u \in C([0,\infty) : E)$ is a strong solution of problem [(168)-(169)] with all initial values chosen to be zeroes, then we can integrate the equation (168) $((p_n + q_n)\zeta)$-times; taking into account the equality (38) and our choice of initial values in (169), we get that

$$Bu(s) + \sum_{i=0}^{n-1} A_i(g_{((p_n+q_n)-(p_i+q_i))\zeta} * u)(s) = 0, \quad s \geqslant 0. \tag{170}$$

Now we are ready to formulate the following extension of [**303**, Theorem 2.1]:

Theorem 2.5.1. *Suppose that* (M_p) *satisfies* $(M.1)$, $b \in (0,1)$, $\zeta \in (0,1]$ *and* (164) *holds. Let* $\nu > -1$, $\xi \geqslant 0$, $\alpha > 0$, $\beta > 0$, $l \geqslant 1$, *and let* (167) *hold. Suppose, further, that the operator* P_λ *is injective for all* $\lambda \in \Lambda_{\alpha,\beta,l}$, *as well as that* $P_\lambda^{-1}C \in L(E)$, $\lambda \in \Lambda_{\alpha,\beta,l}$, *the mapping* $\lambda \mapsto P_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ *is continuous for every fixed element* $x \in E$, *and the operator family*

$$\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\} \subseteq L(E)$$

is equicontinuous. Set, for every function $f \in \mathcal{A}$,

$$S_f(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}E_\zeta(s^\zeta\lambda^\zeta)P_\lambda^{-1}Cx\,d\lambda, \quad s \geqslant 0,\ x \in E. \tag{171}$$

Then $(S_f(s))_{s\geqslant 0} \subseteq L(E)$ is strongly continuous, the mapping $s \mapsto S_f(s) \in L(E)$, $s \geqslant 0$ ($s \mapsto S_f(s) \in L(E)$, $s > 0$) is infinitely differentiable provided $\zeta = 1$, $f \in \mathcal{A}$ ($\zeta \in (0,1)$, $f \in \mathcal{A}$) and, for every $p \in \mathbb{N}_0$ and $f \in \mathcal{A}$, the mapping $s \mapsto (\mathbf{D}_s^\zeta)^p S_f(s) \in L(E)$, $s \geqslant 0$ is well-defined, with

$$(\mathbf{D}_s^\zeta)^p S_f(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}\lambda^{p\zeta}E_\zeta(s^\zeta\lambda^\zeta)P_\lambda^{-1}Cx\,d\lambda, \quad s \geqslant 0,\ x \in E. \tag{172}$$

Furthermore, the following holds:

(i) *Suppose that there exists $i \in \mathbb{N}_n^0$ such that the mappings $\lambda \mapsto A_jP_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ are continuous for some $x \in E$ ($j \in \mathbb{N}_n^0 \smallsetminus \{i\}$) and for each seminorm $\mathbf{p} \in \circledast$ the set*

$$\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}\mathbf{p}(A_jP_\lambda^{-1}Cx) : \lambda \in \Lambda_{\alpha,\beta,l},\ j \in \mathbb{N}_n^0 \smallsetminus \{i\}\}$$

is bounded. Then we have

$$(\mathbf{D}_s^\zeta)^p A_i (\mathbf{D}_s^\zeta)^q S_f(s)x = \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}\lambda^{(p+q)\zeta}E_\zeta(s^\zeta\lambda^\zeta)A_iP_\lambda^{-1}Cx\,d\lambda, \tag{173}$$

for any $x \in E$, $s \geqslant 0$, $i \in \mathbb{N}_n^0$ and p, $q \in \mathbb{N}_0$. Moreover, the mapping $s \mapsto u(s) := S_f(s)x$, $s \geqslant 0$ is a strong solution of the problem [(168)–(169)], with the initial value u_j obtained by plugging $p = j$ and $s = 0$ into the right-hand side of (172), for $j \in \mathbb{N}^0_{\max\{q_i-1:i\in S_l\}}$, and the initial value $u_{i,j}$ obtained by plugging $p = j$, $q = q_i$ and $s = 0$ into the right-hand side of (173), for $i \in S_r$ and $j \in \mathbb{N}^0_{p_i-1}$ ($f \in \mathcal{A}$). If $CA_i \subseteq A_iC$ for all $i \in \mathbb{N}_n^0$, then there exists at most one strong solution of problem [(168)–(169)].

(ii) *Suppose that $f \in \mathcal{A}$, $q \in \circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of $[0,\infty)$. Then there exists $h_0 > 0$ such that*

$$\sup_{p\in\mathbb{N}_0, s\in K, x\in\mathbb{B}} \frac{(h_0)^p q((\mathbf{D}_s^\zeta)^p S_f(s)x)}{p^{p\zeta/b}} < \infty. \tag{174}$$

Proof. We will basically follow the proof of [**303**, Theorem 2.1]. Let $f \in \mathcal{A}$ be such that (166) holds with some numbers $a_1 > 0$ and $a_2 > \xi$. In order to prove that $S_f(s) \in L(E)$ for all $s \geqslant 0$, observe that Theorem 1.5.1 in combination with (165) and the equicontinuity of operator family $\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\}$ implies that for each $\mathbf{p} \in \circledast$ there exist $c_\mathbf{p} > 0$ and $\mathbf{q} \in \circledast$ such that, for any sufficiently small number $\mu > 0$, the following holds with an appropriate constant $M_\mu > 0$:

$$\begin{aligned} &|f(\lambda^\zeta)\lambda^{\zeta-1}E_\zeta(s^\zeta\lambda^\zeta)\mathbf{p}(P_\lambda^{-1}Cx)| \\ &\leqslant a_1M_\mu c_\mathbf{p}e^{-(a_2-\xi)|\lambda|^b}e^{s(\beta+c_\mu|\lambda|^b)}(1+|\lambda|)^{\nu+\zeta}\mathbf{q}(x), \quad \lambda \in \Gamma_l,\ |\lambda| \geqslant R,\ x \in E. \end{aligned} \tag{175}$$

Keeping in mind that $\lim_{\mu\to 0} c_\mu = 0$, we obtain from (175) that $S_f(s) \in L(E)$ for all $s \geqslant 0$, and that the operator family $(S_f(s))_{s\geqslant 0} \subseteq L(E)$ is strongly continuous. The infinite differentiability of mapping $s \mapsto S_f(s) \in L(E)$, $s \geqslant 0$ for $\zeta = 1$ and $f \in \mathcal{A}$ can be simply proved. In order to prove that the mapping $s \mapsto S_f(s) \in L(E)$, $s > 0$ is infinitely differentiable for $\zeta < 1$ and $f \in \mathcal{A}$, we need to recall the well known fact that, for every $l \in \mathbb{N}$, there exist real numbers $(c_{j,\zeta})_{1 \leqslant j \leqslant l}$ and $(c_{j,l,\zeta})_{1 \leqslant j \leqslant l}$ such that

$$\frac{d^l}{ds^l} E_\zeta(zs^\zeta) = \sum_{j=1}^{l} c_{j,\zeta} s^{j\zeta - l} E_\zeta^{(j)}(zs^\zeta), \quad s > 0, \ z \in \mathbb{C}$$

and

$$\frac{d^l}{dz^l} E_\zeta(z) = \sum_{j=1}^{l} c_{j,l,\zeta} E_{\zeta,\zeta l-(l-j)}(z), \quad z \in \mathbb{C}$$

(cf. Section 1.2 and [**292**, Section 1.3]). This implies that, for every $l \in \mathbb{N}$, and for every $h > 0$ suff. small, we have:

$$(176) \quad \frac{E_\zeta^{(l)}((s+h)^\zeta \lambda^\zeta) - E_\zeta^{(l)}(s^\zeta \lambda^\zeta)}{h} - \frac{d^{l+1}}{ds^{l+1}} E_\zeta(s^\zeta \lambda^\zeta)$$
$$= \frac{1}{h} \sum_{j=1}^{l+2} \sum_{i=1}^{j} \int_s^{s+h} \int_s^r c_{j,\zeta} c_{i,j,\zeta} \tau^{j\zeta-(l+2)} E_{\zeta,\zeta j-(i-j)}(\tau^\zeta \lambda^\zeta) d\tau \, dr, \quad s > 0, \ \lambda \in \Gamma_l.$$

An application of Theorem 1.5.1 yields that, for every $l \in \mathbb{N}$, there exists a constant $\delta > 0$ satisfying that, for every $j \in \mathbb{N}$ with $j \leqslant l+2$, and for every $i \in \mathbb{N}$ with $i \leqslant j$, we have

$$|E_{\zeta,\zeta j-(i-j)}(\tau^\zeta \lambda^\zeta)| \leqslant \delta[1 + (\tau\lambda)^{(1+(i-j)-\zeta j)/\zeta} e^{\tau \operatorname{Re} \lambda}], \quad \tau > 0, \ \lambda \in \Gamma_l.$$

Combining this estimate with (176), it readily follows that the mapping $s \mapsto S_f(s) \in L(E)$, $s > 0$ is l-times continuously differentiable, with

$$(177) \quad \frac{d^l}{ds^l} S_f(s)x = \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta) \lambda^{\zeta-1} \frac{d^l}{ds^l} [E_\zeta(s^\zeta \lambda^\zeta)] P_\lambda^{-1} Cx \, d\lambda, \quad s > 0, \ x \in E.$$

Using the identity $\lambda^\zeta (g_{\lceil \zeta \rceil} * E_\zeta(\cdot^\zeta \lambda^\zeta))(s) = (g_{\lceil \zeta \rceil - \zeta} * [E_\zeta(\cdot^\zeta \lambda^\zeta) - 1])(s)$, $s \geqslant 0$, $\lambda \in \Gamma_l$ (see e.g. [**61**, (1.25)] and the proof of [**292**, Lemma 3.3.1]) and a straightforward integral computation, it is checked at once that for each $x \in E$ and $s \geqslant 0$ we have:

$$[g_{\lceil \zeta \rceil - \zeta} * (S_f(\cdot)x - S_f(0)x)](s) = \left[g_{\lceil \zeta \rceil} * \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta) \lambda^{\zeta-1} \lambda^\zeta E_\zeta(\cdot^\zeta \lambda^\zeta) P_\lambda^{-1} Cx \, d\lambda \right](s).$$

This implies the validity of (172) with $p = 1$. Inductively, we obtain that (172) holds for any integer $p \in \mathbb{N}$ by repeating verbatim the above arguments.

Suppose now that the requirements of (i) hold for some element $x \in E$. Using the resolvent equation, we obtain that the mappings $\lambda \mapsto A_i P_\lambda^{-1} Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ are continuous for all $i \in \mathbb{N}_n^0$ and there exists a number $\nu' > 0$ such that for each seminorm $\mathbf{p} \in \circledast$ the set $\{(1+|\lambda|)^{-\nu'} e^{-\xi|\lambda|^b} \mathbf{p}(A_i P_\lambda^{-1} Cx) : \lambda \in \Lambda_{\alpha,\beta,l}, \ i \in \mathbb{N}_n^0\}$ is bounded, which clearly implies that the mapping $s \mapsto A_i S_f(s)x$, $s \geqslant 0$ is well defined for any $x \in E$ and $i \in \mathbb{N}_n^0$. By the foregoing, we have that (173) holds for any $x \in E$, $s \geqslant 0$, $i \in \mathbb{N}_n^0$ and $p, q \in \mathbb{N}_0$. Using the substitution $z = \lambda^\zeta$, Theorem 1.5.1 and the Cauchy formula, we get that

$$\int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}E_\zeta(s^\zeta\lambda^\zeta)\,d\lambda = 0, \quad s \geqslant 0. \tag{178}$$

By (173)–(178), it readily follows that the mapping $s \mapsto u(s) = S_f(s)x$, $s \geqslant 0$ is a strong solution of the problem [(168)–(169)] with the prescribed set of initial values. If $CA_i \subseteq A_iC$ for all $i \in \mathbb{N}_n^0$, then the uniqueness of strong solutions of associated integral equation (170) is an immediate consequence of [**307**, Theorem 2.2], finishing in a routine manner the proof of (i). The existence of number $h_0 > 0$ in (ii) and the proof of inequality (174) follows from (165), (172) and a simple calculus. $\square$

For the sequel, we need some preliminaries concerning abstract Beurling spaces. We define the abstract Beurling space of (M_p) class associated to A, $E^{(M_p)}(A)$ for short, as in the Banach space case (cf. [**109**, **291**] for more details): Put $D_\infty(A) =: \bigcap_{n\in\mathbb{N}} D(A^n)$,

$$E^{(M_p)}(A) := \operatorname{projlim}_{h\to+\infty} E_h^{(M_p)}(A),$$

where for each $h > 0$,

$$E_h^{(M_p)}(A) := \Big\{x \in D_\infty(A) : \|x\|_{h,q}^{(M_p)} = \sup_{p\in\mathbb{N}_0} \frac{h^p q(A^p x)}{M_p} < \infty \text{ for all } q \in \circledast\Big\}.$$

In this place, it is worth noting that for each $h > 0$ the calibration $(\|\cdot\|_{h,q}^{(M_p)})_{q\in\circledast}$ induces a Hausdorff locally convex space topology on $E_h^{(M_p)}(A)$, as well as that $E_{h'}^{(M_p)}(A) \subseteq E_h^{(M_p)}(A)$ provided $0 < h < h' < \infty$, and that the spaces $E_h^{(M_p)}(A)$ and $E^{(M_p)}(A)$ are continuously embedded in E; cf. [**291**]. Following the ideas of R. Beals [**65**], we define the space $E^{\langle M_p\rangle}(A)$ as the inductive limit of spaces $E_h^{(M_p)}(A)$ as $h \to 0+$; that is

$$E^{\langle M_p\rangle}(A) := \operatorname{indlim}_{h\to 0+} E_h^{(M_p)}(A).$$

In the following rather long remark, we will collect a great number of various thoughts and insights about Theorem 2.5.1.

Remark 2.5.2. (i) In the case that $\zeta < 1$, Theorem 2.5.1 seems to be new and not considered elsewhere (even supposing that $B = I$). If $\zeta = 1$, then there exist two possibilities: $n = 1$ and $n > 1$. If $n = 1$ and $\zeta = 1$, then the assertion of Theorem 2.5.1 seems to be new in the case that E is not a Banach space and $B \neq I$, or that $B \neq I$ and $C \neq I$ (cf. [**296**, Theorem 3.16, Example 4.5] for some results in locally convex spaces, with $B = I$). If $n = 1$, $\zeta = 1$ and E is a Banach space, then it is worth noting that A. Favini [**185**] was the first who considered R. Beals's type regularization process [**64**, **65**] for seeking of solutions of degenerate equations of first order, provided in addition that $C = I$ (cf. also [**199**, Section 5.4] for the case $B \neq I$, as well as [**132**, **243**, **244**, Section XXIII], [**291**, Section 1.4], [**292**, **303**, **364**, **502**, Section 2.9] and [**541**, Section 4.4] for more details concerning the case $B = I$). If $n > 1$ and $\zeta = 1$, then the assertion of Theorem 2.5.1 seems to be considered only in the case that $C = I$, $\xi = 0$, $p_i = 0$ for all $i \in \mathbb{N}_n^0$, and E is a Banach space (cf. [**185**, Application 2, Assumption H.10] and compare with our assumptions made in (i) of Theorem 2.5.1). There is no need to say that the usual converting of higher-order (degenerate) differential equations into first order matrix (degenerate) differential equations, used in numerous papers on higher-order abstract differential equations and, in particular, in the above-mentioned Application 2 of [**185**], cannot offer significant help in the analysis of problem [(168)–(169)], in general.

(ii) Let $v \in \mathbb{Z}$, let $f \in \mathcal{A}$, and let an element $x \in E$ satisfy the requirements of (i). Define

$$S_{f,v}(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}\lambda^{v\zeta} E_\zeta(s^\zeta\lambda^\zeta)P_\lambda^{-1}Cx\,d\lambda, \quad s \geqslant 0,\ x \in E.$$

Then the mapping $s \mapsto S_{f,v}(s)x$, $s \geqslant 0$ is likewise a strong solution of problem (168), with the initial values (169) endowed similarly as in the formulation of (i).

(iii) In the formulation of [**303**, Theorem 2.1], it has been additionally assumed that the sequence (M_p) satisfies the condition $(M.2)$. The proof of Theorem 2.5.1 shows that we can completely neglect this condition from our analysis.

(iv) (cf. also Remark 2.10.25(iii)) The worth noticing is that the term $\mathbf{D}_s^{\nu_1+\nu_2}u(s)$ need not be defined for some functions $s \mapsto u(s)$, $s \geqslant 0$ for which the term $\mathbf{D}_s^{\nu_1}\mathbf{D}_s^{\nu_2}u(s)$ is defined. Consider, by way of illustration, the case $\nu_1 = \nu_2 = 1/2$, $\lambda > 0$ and $u(s) = E_{1/2}(\lambda^{1/2}s^{1/2})$, $s \geqslant 0$; then [**61**, (1.25)] implies that $\mathbf{D}_s^{\nu_1}\mathbf{D}_s^{\nu_2}u(s) = \lambda E_{1/2}(\lambda^{1/2}s^{1/2})$, $s \geqslant 0$. On the other hand, $\mathbf{D}_s^1 u(s)$ is not defined for $s \geqslant 0$ because the function $s \mapsto u(s)$, $s \geqslant 0$ is not continuously differentiable for $s \geqslant 0$; even if we accept a slightly weaker definition of Caputo fractional derivatives from [**61**], when $\mathbf{D}_s^1 E_{1/2}(\lambda^{1/2}s^{1/2})$ exists and equals to $\sum_{k=1}^{\infty} \frac{\lambda^{k/2}s^{(k/2)-1}}{\Gamma(k/2)}$ for $s > 0$, the equality

$$\mathbf{D}_s^{1/2}\mathbf{D}_s^{1/2}E_{1/2}(\lambda^{1/2}s^{1/2}) = \mathbf{D}_s^1 E_{1/2}(\lambda^{1/2}s^{1/2}), \quad s > 0$$

does not hold for any $\lambda > 0$ because $\lambda E_{1/2}(\lambda^{1/2}s^{1/2}) \sim \lambda$ as $s \to 0+$ while $\sum_{k=1}^{\infty} \frac{\lambda^{k/2}s^{(k/2)-1}}{\Gamma(k/2)} \sim (\frac{\lambda}{\pi s})^{1/2}$ as $s \to 0+$ (cf. [**302**], Remark 2.10.25(iv) and the equation (241) below). Hence, we will have to make a strict distinction between the operator $(\mathbf{D}_s^\zeta)^p$ and the operator $\mathbf{D}_s^{\zeta p}$. As explained in Remark 2.10.25(iii), the method proposed in Theorem 2.5.1 cannot be used for proving the existence of strong solutions of (non-degenerate) problem

$$B\mathbf{D}_s^{\alpha_n}u(s) + \sum_{i=0}^{n-1} A_i\mathbf{D}_s^{\alpha_i}u(s) = 0, \quad s \geqslant 0,$$

provided that $n > 1$ and there exists an index $i \in \mathbb{N}_n^0$ such that the order α_i of the Caputo fractional derivative $\mathbf{D}_s^{\alpha_i} u(s)$ does not belong to $\mathbb{N}_0$. Here, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$.

(v) It can be simply verified that $\int_0^s g_\zeta(s-r)(-A)S_f(r)x\,dr = BS_f(s)x - BS_f(0)x$, provided that $n = 1$, $s \geqslant 0$, $f \in \mathcal{A}$ and $x \in E$ satisfies the requirements of Theorem 2.5.1(i).

(vi) It is well known that the notion of an abstract Beurling space plays an important role in the theory of ultradistribution semigroups in Banach spaces (cf. Theorem 2.5.1(i) with $n = 1$ and $B = I$). Unfortunately, it is very problematic to introduce a similar concept for degenerate differential equations of first order, especially in the case that the operator B is not injective. For the purpose of illustration of Theorem 2.5.1(i), we will present two examples in which we use the abstract Beurling spaces:

(vi.1) Suppose that $n = 1$, $x \in D_\infty(B)$, the element $B^p x$ satisfies the requirements of Theorem 2.5.1(i) for all $p \in \mathbb{N}_0$, and $B(zB+A)^{-1}CB^p x = (zB+A)^{-1}CB^{p+1}x$, $p \in \mathbb{N}_0$, $z \in \Lambda^\zeta_{\alpha,\beta,l}$. Then $A^p S_f(s)x = \frac{(-1)^p \zeta}{2\pi i} \int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}\lambda^{p\zeta} E_\zeta(s^\zeta \lambda^\zeta) P_\lambda^{-1} CB^p x\,d\lambda$, $s \geqslant 0$, $p \in \mathbb{N}$, $f \in \mathcal{A}$. This, in turn, implies $\bigcup_{s\geqslant 0, f\in\mathcal{A}}\{S_f(s)x\} \subseteq E^{\langle p^{p\zeta/b}\rangle}(A)$, provided that the orbit $\{B^p x : p \in \mathbb{N}_0\}$ is bounded, and $\bigcup_{s\geqslant 0, f\in\mathcal{A}}\{S_f(s)x\} \subseteq E^{\langle p^{2p\zeta/b}\rangle}(A)$, provided that $Bx \in E^{\langle p^{p\zeta/b}\rangle}(A)$.

(vi.2) (cf. also Remark 2.5.5) Suppose that $n = 1$, B is injective and an element $x \in E$ satisfies the requirements of Theorem 2.5.1(i). Then B^{-1} is closed and we can inductively prove that $(B^{-1}A)^p S_f(s)x = (-1)^p(\mathbf{D}_s^\zeta)^p S_f(s)x$, $s \geqslant 0$, $p \in \mathbb{N}$, $f \in \mathcal{A}$. Taking into account (174), the above implies $\bigcup_{s\geqslant 0, f\in\mathcal{A}}\{S_f(s)x\} \subseteq E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A)$.

(vii) Let $f \in \mathcal{A}$, let $\varepsilon > 0$, and let $g \colon \mathbb{C} \smallsetminus \Lambda^\zeta_{\alpha,\beta,l+\varepsilon} \to \mathbb{C}$ be continuous in $D(g)$ and analytic in $\operatorname{int}(D(g))$. Suppose, further, that there exist constants $a_1' > 0$ and $a_2' > \xi$ such that (166) holds with $f = g$, $a_1 = a_1'$, $a_2 = a_2'$, $\lambda \in D(g)$, as well as that $n = 1$ and the family $\{(1+|\lambda|)^{-\nu} e^{-\xi|\lambda|^b} BP_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\} \subseteq L(E)$ is both equicontinuous and strongly continuous. Let

$$CB(zB+A)^{-1}C = B(zB+A)^{-1}C^2, \quad z \in \Lambda^\zeta_{\alpha,\beta,l}, \tag{179}$$

and let Γ_l^ζ ($\Gamma^\zeta_{l,\varepsilon}$) denote the upwards oriented boundary of $\Lambda^\zeta_{\alpha,\beta,l}$ ($\Lambda^\zeta_{\alpha,\beta,l+\varepsilon}$). Then, for every $z, z' \in \Lambda^\zeta_{\alpha,\beta,l}$ and $x \in E$, the resolvent equation

$$(zB+A)^{-1}C^2x - (z'B+A)^{-1}C^2x = (z'-z)(zB+A)^{-1}CB(z'B+A)^{-1}Cx,$$

holds, which clearly implies that the mapping $z \mapsto B(zB+A)^{-1}C^2x$, $z \in \operatorname{int}(\Lambda^\zeta_{\alpha,\beta,l})$ is analytic ($x \in E$). Using (179) and the consideration from [**138**, Remark 2.7], the above implies that the mapping $z \mapsto B(zB+A)^{-1}Cx$, $z \in \operatorname{int}(\Lambda^\zeta_{\alpha,\beta,l})$ is analytic, as well ($x \in E$). Applying the substitution $z = \lambda^\zeta$ and the Cauchy formula, we then get that $BS_g(0)x = (2\pi i)^{-1}\int_{\Gamma^\zeta_{l,\varepsilon}} g(z)B(zB+A)^{-1}Cx\,dz$, $x \in E$. Proceeding as in the proof of [**65**, Lemma 4.2], it readily follows that

$$S_f(0)BS_g(0)x = S_{fg}(0)Cx, \quad x \in E.$$

If $\xi = 0$, $f(\lambda) = f_t(\lambda)$ and $g(\lambda) = f_s(\lambda)$, with $t, s \in \Sigma_\theta$, the above means that $T(t)BT(s) = T(t+s)C$. In the notion of Section 2.7 (cf. Definition 2.7.1), we have that $(T(t))_{t\in\Sigma_\theta}$ is an analytic (B,C)-regularized semigroup of growth order $(\nu+1)\zeta/b$.

(viii) If $B = I$, $n = 1$, $P_\lambda^{-1}C$ exists and is polynomially bounded on the region $\Lambda_{\alpha,\beta,l}$ (with the meaning clear), then it might be surprising that we must impose the condition (167) in order to ensure the existence of strong solutions of problem $(\mathrm{DFP})'_R$ with the initial value $x \neq 0$. If we replace the condition (167) with the condition that $\frac{1}{2-\zeta} \leqslant \frac{b}{\zeta}$ (which clearly implies $\zeta < 1$ and the triviality of class $\mathcal{A}$), and accept all the remaining assumptions from the formulation of this theorem, with $B = I$, $n = 1$ and $\xi = 0$, then it is not clear whether there exist a Hilbert space (Banach space, sequentially complete locally convex space) E and a closed linear operator A acting on E such that the problem $(\mathrm{DFP})'_R$ has no local strong solutions unless $x = 0$ (cf. [**64**, Theorem 2, Theorem 2'] for more details concerning the case $\zeta = 1$). This is an open problem we would like to address to our readers.

(ix) If the assumptions of Theorem 2.5.1 hold with the region $\Lambda_{\alpha,\beta,l}$ replaced by the right half-plane $RHP_{\bar{\omega}} \equiv \{z \in \mathbb{C} : \operatorname{Re} z > \bar{\omega}\}$ (and with the set Ω replaced by $\mathbb{C} \smallsetminus (RHP_{\bar{\omega}})^\zeta$), then for each $p \in \mathbb{N}_0$ and $f \in \mathcal{A}$ the operator family $\{e^{-\bar{\omega}s}(\mathbf{D}_s^\zeta)^p S_f(s) : s \geqslant 0\}$ is equicontinuous ($\bar{\omega} > 0$); cf. [**29**] for some interesting examples given in the case that $\zeta = 1$, $\xi = 0$ and $C = I$. It is also worth noting that we can consider, instead of the region $\Lambda_{\alpha,\beta,l}$ considered above, a region of the form $\Omega(\omega) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \max(x_0, \omega(|\operatorname{Im}\lambda|))\}$, where $x_0 > 0$, $\omega\colon [0,\infty) \to [0,\infty)$ is a continuous, concave, increasing function satisfying $\lim_{t\to\infty}\omega(t) = \infty$, $\lim_{t\to\infty}(\omega(t)/t) = 0$ and $\int_1^\infty(\omega(t)/t^2)dt < \infty$ (cf. [**64, 65, 111, 291, 303**] and [**296**, Example 4.5]), or the exponential region $E(a,b) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\}$ ($a, b > 0$), introduced for the first time by W. Arendt, O. El-Mennaoui and V. Keyantuo in [**28**] (cf. also C. Foiaş [**220**] for a very similar notion of the logarithmic region $\Lambda(\alpha,\beta,\omega)$ which can also be used here). We will not go into further details concerning these questions.

(x) Suppose $x \in \bigcap_{v=0}^n D(A_v)$, $i \in \mathbb{N}_n^0$, $j \in \mathbb{N}_0$, $(f_\varepsilon(\lambda))_{\varepsilon>0}$ is a net of functions from $\mathcal{A}$ and $CA_v \subseteq A_vC$, $v \in \mathbb{N}_n^0$. Denote $u_{i,\varepsilon}^j := ((\mathbf{D}_s^\zeta)^j S_{f_\varepsilon}(s)A_ix)_{s=0}$ ($\varepsilon > 0$). Then the following equality holds: $P_\lambda^{-1}CA_ix = \lambda^{-(p_i+q_i)\zeta}[Cx - \sum_{v\in\mathbb{N}_n^0\smallsetminus\{i\}}\lambda^{(p_v+q_v)\zeta}P_\lambda^{-1}CA_vx]$, $\lambda \in \Lambda_{\alpha,\beta,l}$, which implies that

$$u_{i,\varepsilon}^j = \frac{1}{2\pi i}\int_{\Gamma_l^\zeta} f_\varepsilon(\lambda)\lambda^{j-(p_i+q_i)}\Bigg[Cx - \sum_{v\in\mathbb{N}_n^0\smallsetminus\{i\}}\lambda^{(p_v+q_v)}P_{\lambda^{1/\zeta}}^{-1}CA_vx\Bigg]d\lambda.$$

If we impose some additional conditions on the net $(f_\varepsilon(\lambda))_{\varepsilon>0}$ (for example, the condition that $f_\varepsilon(0)\neq 0$, $\varepsilon>0$, as well as $f_\varepsilon^{(p_i+q_i-j-1)}(0)\to z_0^{i,j}$ as $\varepsilon\to 0$, provided $p_i+q_i-j-1\geqslant 0$, and the limit equality $f_\varepsilon(\lambda)\to 1$ as $\varepsilon\to 0$ ($\lambda\in\Gamma_l^\zeta$), uniformly on compacts of Γ_l^ζ, at least) and if suppose that the operator family $\{(1+|\lambda|)^{-\nu'}P_\lambda^{-1}C:\lambda\in\Lambda_{\alpha,\beta,l}\}\subseteq L(E)$ is both equicontinuous and strongly continuous for a sufficiently large negative number $\nu'<0$ (cf. also [**541**, Theorem 4.2, p. 168]), then we may apply the dominated convergence theorem and the residue theorem in order to see that $\lim_{\varepsilon\to 0}u_{i,\varepsilon}^j$ equals0, if $j\geqslant p_i+q_i$, and $[(p_i+q_i-j-1)!]^{-1}z_0^{i,j}Cx$, otherwise. If we use the net of functions of form $f_\varepsilon(\lambda)=e^{-\varepsilon(-\lambda+\omega)^{b/\zeta}}$ ($\varepsilon>0$), then we have that $z_0^{i,j}=1$ if $p_i+q_i-j-1=0$, and $z_0^{i,j}=0$ if $p_i+q_i-j-1>0$ [**502, 541**]. Using this idea, we can prove the following (cf. also Theorem 2.6.2 below): Suppose that $x_v\in\bigcap_{p=0}^n D(A_p)$ for all $v\in\mathbb{N}_{q_n-1}^0$. Set $S_v:=\{j\in\mathbb{N}_n^0:q_j-1\geqslant v\}$, $v\in\mathbb{N}_{q_n-1}^0$; then $n\in S_v$ for all $v\in\mathbb{N}_{q_n-1}^0$. Suppose that

$$-\infty<\nu'<\min_{v\in\mathbb{N}_{q_n-1}^0}[-(q_n-1-v+\max\{q_j:j\in\mathbb{N}_n^0\smallsetminus S_v\})] \tag{180}$$

and the operator family $\{(1+|\lambda|)^{-\nu'}P_\lambda^{-1}C:\lambda\in\Lambda_{\alpha,\beta,l}\}\subseteq L(E)$ is both equicontinuous and strongly continuous. Assume, further, that for each $x\in E$ and $i\in\mathbb{N}_{n-1}^0$ the mapping $\lambda\mapsto A_iP_{\lambda^{1/\zeta}}^{-1}Cx$, $\lambda\in\Lambda_{\alpha,\beta,l}^\zeta$ is continuous and there exists $v_i\in\mathbb{N}$ such that the operator family $\{(1+|\lambda|)^{-v_i}A_iP_{\lambda^{1/\zeta}}^{-1}C:\lambda\in\Lambda_{\alpha,\beta,l}^\zeta\}\subseteq L(E)$ is equicontinuous. Let the function $f_\varepsilon(\lambda)$ be chosen as above, and let $p_v=0$, $v\in\mathbb{N}_n^0$ (with the exception of problem $(\mathrm{DFP})_R$, the analysis becomes very difficult in the case that there exists $v_0\in\mathbb{N}_n^0$ such that $p_{v_0}>0$). Then the function

$$\begin{aligned}s\mapsto u_\varepsilon(s):=\sum_{v=0}^{q_n-1}\sum_{j\in S_v}\frac{1}{2\pi i}\int_{\Gamma_l^\zeta}&e^{-\varepsilon(-\lambda+\omega)^{b/\zeta}}\\ &\times E_\zeta(s^\zeta\lambda)\lambda^{q_j-1-v}P_{\lambda^{1/\zeta}}^{-1}CA_jx_v\,d\lambda,\quad s\geqslant 0\end{aligned}$$

is a strong solution of problem (168) with the initial values $(u_0^\varepsilon,\dots,u_{q_n-1}^\varepsilon)$, converging to $(Cx_0,\dots,Cx_{q_n-1})$ as $\varepsilon\to 0+$. Hence, the set $\mathfrak{W}$ consisting of all initial values $(y_0,\dots,y_{q_n-1})\in E^{q_n}$ subjected to some strong solution $s\mapsto u(s)$, $s\geqslant 0$ of problem (168) is dense in $(C(\bigcap_{v=0}^n D(A_v)))^{q_n}$ (cf. Example 2.5.8 below for an interesting application of this result, with C not being the identity operator). Generally, it is very difficult to say anything else about the set $\mathfrak{W}$ in the case that $n>1$.

(xi) Following the method employed in the proof of Theorem 2.5.1, one can extend the assertions of [**298**, Theorem 2.1, Theorem 2.2] to abstract degenerate (multi-term) fractional differential equations, thus proving some results on the C-wellposedness of problem [(168)–(169)] in the case that $\zeta>2$ [**298**, Theorem 2.1] and $2\geqslant\zeta>1$ [**298**, Theorem 2.2]. Consider, for example, the case $2\geqslant\zeta>1$. Let $\vartheta\in(\pi(2-\zeta)/2,\pi/2)$, let

$b \in (1/\zeta, \pi/(2(\pi-\vartheta)))$ and let $z \in \Sigma_{\vartheta'}$, where $\vartheta' := \arctan(\cos(b(\pi-\vartheta)))$. If there exist $d \in (0,1]$ and $\nu > -1$ such that the operator family $\{(1+|\lambda|)^{-\nu} P_\lambda^{-1} C : \lambda \in \Sigma_{\vartheta/\zeta} \cup B_d\} \subseteq L(E)$ is both equicontinuous and strongly continuous (for the sake of simplicity, we shall only deal with the case in which $\xi = 0$), then for each number $s \geqslant 0$ we can define the bounded linear operator $S(s)$ by

$$S(s)x := \frac{1}{2\pi i} \int_{\Gamma_{\zeta,d}} e^{-z(-\lambda)^b} E_\zeta(s^\zeta \lambda) P_{\lambda^{1/\zeta}}^{-1} Cx \, d\lambda, \quad x \in E, \; s \geqslant 0,$$

where $c \in (0, \vartheta)$ is chosen so that $b \in (1/\zeta, \pi/(2(\pi-c)))$ and $\vartheta < \arctan(\cos(b(\pi-c)))$ (cf. (171) and apply the substitution $\lambda \mapsto \lambda^\zeta$). Suppose, further, that there exists $i \in \mathbb{N}_n^0$ such that the mappings $\lambda \mapsto A_j P_\lambda^{-1} Cx$, $\lambda \in \Sigma_{\vartheta/\zeta} \cup B_d$ are continuous for some $x \in E$ ($j \in \mathbb{N}_n^0 \smallsetminus \{i\}$) and for each seminorm $p \in \circledast$ the set $\{(1+|\lambda|)^{-\nu} p(A_j P_\lambda^{-1} Cx) : \lambda \in \Sigma_{\vartheta/\zeta} \cup B_d, \; j \in \mathbb{N}_n^0 \smallsetminus \{i\}\}$ is bounded. Then the final conclusions stated in Theorem 2.5.1 continue to hold after some obvious modifications. In the situation of [**298**, Theorem 2.1] (the case $\zeta > 2$), which is very specific, we can assume that the operators $P_\lambda^{-1} C$ exist on a certain region of the complex plane which does not contain any acute angle. The interested reader should carry out details concerning the transmitting our previous results and comments from the items (i)–(x) of this remark to the case in which $\zeta > 1$. The method proposed in [**298**, **541**, Section 4.4, pp. 167–175] as well as in parts (x)–(xi) of this remark can serve one to prove some results on the existence of entire and analytical solutions of degenerate (multi-term) fractional differential equations and their systems. The reader may consult [**205**, **311**], Theorem 2.3.20 and Theorem 2.6.2 below for similar results in this direction.

The proof of following extension of [**303**, Corollary 2.1] is omitted.

THEOREM 2.5.3. *Suppose that* $0 < c < b < \zeta \leqslant 1$, $\sigma > 0$, $\nu > -1$, $\xi \geqslant 0$, $\varsigma > 0$ *and* (167) *holds. Denote*

$$\Pi_{c,\sigma,\varsigma} := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant \sigma |\operatorname{Im} \lambda|^c + \varsigma\},$$

$$\Pi_{c,\sigma,\varsigma}^\zeta := \{\lambda^\zeta : \lambda \in \Pi_{c,\sigma,\varsigma}\} \text{ and } \Omega' := \mathbb{C} \smallsetminus \Pi_{c,\sigma,\varsigma}^\zeta.$$

Let $f : \overline{\Omega'} \to \mathbb{C}$ *be a continuous function that is analytic in* Ω' *and satisfies that there exist constants* $a_1 > 0$ *and* $a_2 > \xi$ *such that* $|f(\lambda)| \leqslant a_1 e^{-a_2 |\lambda|^{b/\zeta}}$, $\lambda \in \overline{\Omega'}$. *Suppose, further, that the operator* P_λ *is injective for all* $\lambda \in \Pi_{c,\sigma,\varsigma}$, *as well as that* $P_\lambda^{-1} C \in L(E)$, $\lambda \in \Pi_{c,\sigma,\varsigma}$, *the mapping* $\lambda \mapsto P_\lambda^{-1} Cx$, $\lambda \in \Pi_{c,\sigma,\varsigma}$ *is continuous for every fixed element* $x \in E$, *and the operator family*

$$\{(1+|\lambda|)^{-\nu} e^{-\xi|\lambda|^b} P_\lambda^{-1} C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

is equicontinuous. Set

$$T_f(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_c} f(\lambda^\zeta) \lambda^{\zeta-1} E_\zeta(s^\zeta \lambda^\zeta) P_\lambda^{-1} Cx \, d\lambda, \quad s \geqslant 0, \; x \in E,$$

where Γ_c *denotes the upwards oriented boundary of* $\Pi_{c,\sigma,\varsigma}$*. Then* $(T_f(s))_{s\geqslant 0}\subseteq L(E)$ *is strongly continuous, the mapping* $s\mapsto T_f(s)\in L(E)$*,* $s\geqslant 0$ $(s\mapsto T_f(s)\in L(E)$*,* $s>0)$ *is infinitely differentiable provided* $\zeta=1$ $(\zeta\in(0,1))$ *and, for every* $p\in\mathbb{N}_0$*, the mapping* $s\mapsto(\mathbf{D}_s^{\zeta})^pT_f(s)\in L(E)$*,* $s\geqslant 0$ *is well-defined. Furthermore,* (172) *holds with* $(S_f(s))_{s\geqslant 0}$ *and* Γ_l *replaced respectively by* $(T_f(s))_{s\geqslant 0}$ *and* Γ_c*, and we have the following:*

(i) *Suppose that there exists* $i\in\mathbb{N}_n^0$ *such that the mappings* $\lambda\mapsto A_jP_\lambda^{-1}Cx$*,* $\lambda\in\Pi_{c,\sigma,\varsigma}$ *are continuous for some* $x\in E$ $(j\in\mathbb{N}_n^0\smallsetminus\{i\})$ *and for each seminorm* $\mathbf{p}\in\circledast$ *the set* $\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}\mathbf{p}(A_jP_\lambda^{-1}Cx):\lambda\in\Pi_{c,\sigma,\varsigma},\ j\in\mathbb{N}_n^0\smallsetminus\{i\}\}$ *is bounded. Then* (173) *holds with* $(S_f(s))_{s\geqslant 0}$ *and* Γ_l *replaced respectively by* $(T_f(s))_{s\geqslant 0}$ *and* Γ_c*, the mapping* $s\mapsto u(s):=T_f(s)x$*,* $s\geqslant 0$ *is a strong solution of the problem* [(168)*–169*]*, with the initial value* u_j *obtained by plugging* $p=j$ *and* $s=0$ *into the right-hand side of* (172)*, for* $j\in\mathbb{N}^0_{\max\{q_i-1:i\in S_l\}}$*, and the initial value* $u_{i,j}$ *obtained by plugging* $p=j$*,* $q=q_i$ *and* $s=0$ *into the right-hand side of* (173)*, for* $i\in S_r$ *and* $j\in\mathbb{N}^0_{p_i-1}$ *(with the obvious replacements described above). If* $CA_i\subseteq A_iC$ *for all* $i\in\mathbb{N}_n^0$*, then there exists at most one strong solution of problem* [(168)–(169)].

(ii) *Suppose that* $q\in\circledast$*,* $\mathbb{B}$ *is a bounded subset of* E *and* K *is a compact subset of* $[0,\infty)$*. Then there exists* $h_0>0$ *such that* (174) *holds.*

Assume now that $\alpha>0$, $\beta>0$, $l\geqslant 1$, $0<\zeta\leqslant 1$ and, as before, that (M_p) is a sequence of positive real numbers such that $M_0=1$ and the condition $(M.1)$ is fulfilled for (M_p). Recall that $\Omega=\mathbb{C}\smallsetminus\Lambda^{\zeta}_{\alpha,\beta,l}$. If $g\colon[0,\infty)\to[0,\infty)$ is a monotonically increasing, continuous function satisfying

$$\lim_{t\to+\infty}(1+t)^ve^{\sigma M(st)-g(t)}=0,\quad v\in\mathbb{N},\ s\geqslant 0,\ \sigma>0,\tag{181}$$

then we denote by $\mathcal{A}_g$ the class consisting of all continuous functions $f\colon\bar{\Omega}\to\mathbb{C}$ that are analytic in Ω and satisfy that $|f(z)|\leqslant\mathrm{Const}\cdot e^{-g(|z|^{1/\zeta})}$, $z\in\bar{\Omega}$. The main purpose of following theorem is to consider a non-Gevrey analogue of Theorem 2.5.1 and Theorem 2.5.3 (the proof is very similar to that of Theorem 2.5.1 and we shall skip details for the sake of brevity):

THEOREM 2.5.4. *Suppose that* (M_p) *satisfies* $(M.1)$*, as well as that there exists a monotonically increasing, continuous function* $g\colon[0,\infty)\to[0,\infty)$ *satisfying that* (181) *holds and the class* $\mathcal{A}_g$ *is non-trivial. Let* $0<\zeta\leqslant 1$*,* $\nu>-1$*,* $\xi\geqslant 0$*,* $\alpha>0$*,* $\beta>0$ *and* $l\geqslant 1$*. Suppose, further, that the operator* P_λ *is injective for all* $\lambda\in\Lambda_{\alpha,\beta,l}$*, as well as that* $P_\lambda^{-1}C\in L(E)$*,* $\lambda\in\Lambda_{\alpha,\beta,l}$*, the mapping* $\lambda\mapsto P_\lambda^{-1}Cx$*,* $\lambda\in\Lambda_{\alpha,\beta,l}$ *is continuous for every fixed element* $x\in E$*, and the operator family*

$$\{(1+|\lambda|)^{-\nu}e^{-M(\xi\lambda)}P_\lambda^{-1}C:\lambda\in\Lambda_{\alpha,\beta,l}\}\subseteq L(E)$$

is equicontinuous. Set, for every function $f\in\mathcal{A}_g$*, the bounded linear operator* $S_f(s)$ $(s\geqslant 0)$ *through* (171)*. Then* $(S_f(s))_{s\geqslant 0}\subseteq L(E)$ *is strongly continuous, the mapping* $s\mapsto S_f(s)\in L(E)$*,* $s\geqslant 0$ $(s\mapsto S_f(s)\in L(E)$*,* $s>0)$ *is infinitely differentiable provided* $\zeta=1$*,* $f\in\mathcal{A}_g$ $(\zeta\in(0,1)$*,* $f\in\mathcal{A}_g)$ *and, for every* $p\in\mathbb{N}_0$ *and*

$f \in \mathcal{A}_g$, the mapping $s \mapsto (\mathbf{D}_s^{\zeta})^p S_f(s) \in L(E)$, $s \geqslant 0$ is well-defined. Furthermore, (172) *and* (i)–(ii) *hold, where:*

(i) *Suppose that there exists $i \in \mathbb{N}_n^0$ such that the mappings $\lambda \mapsto A_j P_\lambda^{-1} Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ are continuous for some $x \in E$ ($j \in \mathbb{N}_n^0 \setminus \{i\}$) and for each seminorm $\mathbf{p} \in \circledast$ the set $\{(1+|\lambda|)^{-\nu} e^{-M(\xi\lambda)} \mathbf{p}(A_j P_\lambda^{-1} Cx) : \lambda \in \Lambda_{\alpha,\beta,l},\ j \in \mathbb{N}_n^0 \setminus \{i\}\}$ is bounded. Then* (173) *holds for any $x \in E$, $s \geqslant 0$, $i \in \mathbb{N}_n^0$ and $p, q \in \mathbb{N}_0$. Moreover, the mapping $s \mapsto u(s) := S_f(s)x$, $s \geqslant 0$ is a strong solution of the problem* [(168)–(169)], *with the initial value u_j obtained by plugging $p = j$ and $s = 0$ into the right-hand side of* (172), *for $j \in \mathbb{N}^0_{\max\{q_i - 1 : i \in S_l\}}$, and the initial value $u_{i,j}$ obtained by plugging $p = j$, $q = q_i$ and $s = 0$ into the right-hand side of* (173), *for $i \in S_r$ and $j \in \mathbb{N}^0_{p_i - 1}$ ($f \in \mathcal{A}$). If $CA_i \subseteq A_i C$ for all $i \in \mathbb{N}_n^0$, then there exists at most one strong solution of problem* [(168)–(169)].

(ii) *Let $(N_p)_{p \in \mathbb{N}_0}$ be a sequence of positive real numbers satisfying $N_0 = 1$, $(M.1)$ and the property that for each $v \in \mathbb{N}$, $s \geqslant 0$ and $\sigma > 0$ there exists $h > 0$ such that $\lim_{t \to +\infty} (1+t)^v e^{\sigma M(st) + N(ht^{\zeta}) - g(t)} = 0$. Suppose that $f \in \mathcal{A}$, $q \in \circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of $[0, \infty)$. Then there exists $h_0 > 0$ such that* (174) *holds with the sequence $(p^{p\zeta/b})$ replaced by (N_p) therein.*

It is worth noting that we have analyzed a slightly different growth rate of $P_\lambda^{-1} C$ in Theorem 2.5.1 (Theorem 2.5.3), and that one has to assume that for each $v \in \mathbb{N}$, $s \geqslant 0$ and $\sigma > 0$ there exists $h > 0$ such that $\lim_{t \to +\infty} (1+t)^v e^{\sigma M(st) + \xi|\lambda|^b + N(ht^{\zeta}) - g(t)} = 0$ ($\lim_{t \to +\infty} (1+t)^v e^{st^c + \xi|\lambda|^b + N(ht^{\zeta}) - g(t)} = 0$) in order to deduce Theorem 2.5.1 (Theorem 2.5.3) from Theorem 2.5.4. Also, it is worth noting that Theorem 2.5.4 is closely linked with the assertions of [**303**, Theorem 3.3] and [**354**, Theorem 7.6], where we have studied the regularization of ultradistribution semigroups in Banach spaces ($B = I$, $n = 1$, $\zeta = 1$, $\xi \geqslant 0$, $-A$ generates an ultradistribution semigroup of (M_p)-class; cf. [**291**] and [**354**] for the notion, as well as [**109**, **111**, **289**, **291**, **330**, **349**, **355**, **363**] and [**424**] for more details concerning ultradistribution semigroups). If the corresponding sequence (M_p) satisfies the conditions $(M.1)$, $(M.2)$ and $(M.3)$, then we have proved in the afore-mentioned theorems that there exist two functions, $g(\cdot)$ and $f \in \mathcal{A}_g$, such that the operator $-A$ generates a global locally equicontinuous C-regularized semigroup $(S_f(s))_{s \geqslant 0}$, with $C = S_f(0)$ being injective, satisfying additionally that the mapping $s \mapsto S_f(s) \in L(E)$, $s \geqslant 0$ is infinitely differentiable and $E^{(M_p)}(A) \subseteq C(D_\infty(A))$. The proof of this fact is based upon the existence of a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, $(M.1)$, $(M.2)$, $(M.3)$ and $N_p \prec M_p$ (cf. [**303**, Lemma 3.2]), and by putting $f(\cdot) = 1/\omega_{l',(N_p)}(-\cdot)$ ($l' \in \mathbb{N}$ sufficiently large) after that, where $\omega_{l',(N_p)}(\lambda) := \prod_{p=1}^{\infty} (1 + \frac{l' \lambda N_{p-1}}{N_p})$, $\lambda \in \mathbb{C}$ ($l' > 0$).

Remark 2.5.5. The comments from Remark 2.5.2 can be reformulated in the case that the assumptions of Theorem 2.5.3 or Theorem 2.5.4 below hold. Here we would like to point out some facts closely linked with Remark 2.5.2[(ii),(vi.1)] and [**65**, Lemma 1, Lemma 4]. Consider the situation of Theorem 2.5.3 with $\xi = 0$ and

$n = 1$ (the final conclusions continue to hold in the case of examination of Theorem 2.5.1; after the replacement of region $\Pi^{\zeta}_{c,\sigma,\varsigma}$ with $\Lambda^{\zeta}_{\alpha,\beta,l}$, one just has to make some obvious terminological changes). Suppose, additionally, that $b/\zeta < 1/(2-\zeta)$, $\omega > \varsigma^{\zeta}$, $CA \subseteq AC$, $CB \subseteq BC$, B is injective,

$$B^{-1}A(zB+A)^{-1}Cx = (zB+A)^{-1}CB^{-1}Ax, \quad x \in D_{\infty}(B^{-1}A),\ z \in \Pi^{\zeta}_{c,\sigma,\varsigma}, \tag{182}$$

the family $\{(1+|z|)^{-\nu}B(zB+A)^{-1}C : z \in \Pi^{\zeta}_{c,\sigma,\varsigma}\} \subseteq L(E)$ is both equicontinuous and strongly continuous, $y \in D(B)$ satisfies that $Cy \in D(A)$, $BACy = ACBy$ and there exists $h_0 > 0$ such that the set $\{h_0^p p^{(-p\zeta)/b}(B^{-1}A)^p By : p \in \mathbb{N}_0\}$ is bounded. Then it is readily seen that $C(zB+A)^{-1}C = (zB+A)^{-1}C^2$, $z \in \Pi^{\zeta}_{c,\sigma,\varsigma}$ and the mapping $z \mapsto (zB+A)^{-1}Cx$, $z \in \operatorname{int}(\Pi^{\zeta}_{c,\sigma,\varsigma})$ is analytic ($x \in E$); cf. Remark 2.5.2(vii). Using the Cauchy formula and the foregoing arguments, we have that $S_{f_t}(0)x = (2\pi i)^{-1}\int_{\Gamma^{\zeta}_c - \omega} e^{-t(-\lambda)^{b/\zeta}}((\lambda+\omega)B+A)^{-1}Cx\,d\lambda$ ($x \in E$, $t > 0$), where Γ^{ζ}_c denotes the upwards oriented boundary of $\Pi^{\zeta}_{c,\sigma,\varsigma}$. Let the curve Γ' be sufficiently close to Γ^{ζ}_c, on the right of Γ^{ζ}_c, and let the curve $\Gamma'_{\omega} := \Gamma' - \omega$ be upwards oriented. Modifying slightly the second part of proof of [**65**, Lemma 4] (the proof of this lemma contains some typographical mistakes but the essence and final conclusions are true; we can apply Stirling's formula), and keeping in mind the boundedness of set $\{h_0^p p^{(-p\zeta)/b}(B^{-1}A)^p By : p \in \mathbb{N}_0\}$, we get that there exists a number $\delta > 0$ such that for each integer $p \in \mathbb{N}$ there exists an integer $n(p) \in \mathbb{N} \cap (\frac{bp}{\zeta}+\nu+2, \frac{bp}{\zeta}+\nu+3]$ such that the series $\sum_{p=0}^{\infty} x_p$ and $\sum_{p=0}^{\infty} Bx_p$ are convergent, where

$$x_p := \frac{\delta^p}{2\pi i p!}\int_{\Gamma'_{\omega}} (-\lambda)^{bp/\zeta}(\lambda+\omega)^{-n(p)}((\lambda+\omega)B+A)^{-1}C(B^{-1}A)^{n(p)}By\,d\lambda.$$

Let $x = \sum_{p=0}^{\infty} x_p$ and $Bx = \sum_{p=0}^{\infty} Bx_p$; arguing as in Remark 2.5.2(vii), we obtain with the help of equation (182), the Cauchy formula and the resolvent equation that

$$S_{f_t}(0)Bx = \frac{1}{2\pi i}\int_{\Gamma'_{\omega}} e^{-(t-\delta)(-\lambda)^{b/\zeta}}((\lambda+\omega)B+A)^{-1}C^2By\,d\lambda, \quad t > \delta. \tag{183}$$

Let $\lambda_0 \in \mathbb{C}$ be on the right of Γ^{ζ}_c, and simultaneously, on the left of Γ'. Making use of the identity [**296**, (3.16)], with the operator A replaced by $-B^{-1}A$ therein (we only need the linearity of operator $B^{-1}A$, not its closedness), we get that

$$\begin{aligned} CBy = \sum_{j=0}^{\lceil\nu\rceil+2} \frac{(-1)^j}{((\lambda+\omega)-\lambda_0)^{j+1}}((\lambda+\omega)+B^{-1}A)CBy \\ + (-1)^{\lceil\nu\rceil+1}\frac{C(\lambda_0 I + B^{-1}A)^{\lceil\nu\rceil+3}By}{((\lambda+\omega)-\lambda_0)^{\lceil\nu\rceil+3}}, \quad \lambda \in \Gamma'_{\omega}. \end{aligned} \tag{184}$$

Since $Cy \in D(A)$, $BACy = ACBy$ and $CB \subseteq BC$, we have that $((\lambda+\omega)+B^{-1}A)CBy = ((\lambda+\omega)B+A)Cy$, $\lambda \in \Gamma'_{\omega}$. Applying the operator $((\lambda+\omega)B+A)^{-1}C$ on the both sides of (184), the above implies

$$((\lambda+\omega)B+A)^{-1}C^2By=\sum_{j=0}^{\lceil\nu\rceil+2}\frac{(-1)^j}{((\lambda+\omega)-\lambda_0)^{j+1}}C^2y$$
$$+(-1)^{\lceil\nu\rceil+1}\frac{((\lambda+\omega)B+A)^{-1}C}{((\lambda+\omega)-\lambda_0)^{\lceil\nu\rceil+3}}(\lambda_0 I+B^{-1}A)^{\lceil\nu\rceil+3}CBy,\quad \lambda\in\Gamma'_\omega.$$

Inserting this expression in (183), and using after that the limit equality [**502**, Lemma 2.7; p. 543, l. -7], as well as the residue theorem and the dominated convergence theorem, we obtain that $S_{f_\delta}(0)Bx=\lim_{t\to\delta+}S_{f_t}(0)Bx=C^2y$. Keeping in mind Remark 2.5.2(vi.2), the above implies

$$C^2(E(A;B))\subseteq\bigcup_{t>0}R(S_{f_t}(0)B)\subseteq\bigcup_{t>0}R(S_{f_t}(0))\subseteq E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A),\tag{185}$$

where

$$E(A;B):=\{y\in B^{-1}(E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A)):Cy\in D(A),\ BACy=ACBy\}.$$

In the present situation, we do not know whether the equation (185) continues to hold if we replace the term $C^2(E(A;B))$ with $C(E(A;B))$.

Remark 2.5.6. Consider the case $B=I$, $n=1$ and $\zeta\in(0,1)$ (cf. [**354**] for the case $\zeta=1$). As before, we assume that (M_p) is a sequence of positive real numbers satisfying $M_0=1$ and $(M.1)$, as well as that there exist constants $l\geqslant 1$, $\alpha>0$, $\beta>0$, $\nu>-1$ and $\xi\geqslant 0$ such that $\Lambda^\zeta_{\alpha,\beta,l}\subseteq\rho_C(-A)$. Suppose that the operator family $\{(1+|\lambda|)^{-\nu}e^{-M(\xi|\lambda|)}(\lambda^\zeta+A)^{-1}C:\lambda\in\Lambda^\zeta_{\alpha,\beta,l}\}$ is both equicontinuous and strongly continuous. Then we have found that the problem

(PR) In which cases does there exist an injective operator $C'\in L(E)$ such that the operator $-A$ generates a global locally equicontinuous (g_ζ,C')-regularized resolvent family $(S(t))_{t\geqslant 0}$ on E?

is very difficult to answer in general. Here we will prove that the problem (PR) can be answered in the affirmative provided that (164) holds with some $b\in(0,1)$, as well as that $\xi=0$ and $1/(2-\zeta)>b/\zeta$ (cf. Theorem 2.5.1, Theorem 2.5.3 and Remark 2.5.2(v)). Then $(S_{f_t}(0))_{t\in\Sigma_\theta}$ is an analytic C-regularized semigroup of growth order $(\nu+1)\zeta/b$, consisting of injective operators, with the closed linear operator $-(-A-\omega)_{b/\zeta}$ being its integral generator ($\omega>0$ is a sufficiently large real number; cf. [**103**, Theorem 3.5, Theorem 3.7]), and the following holds:

(a) $(S_{f_t}(s))_{s\geqslant 0}$ is a locally equicontinuous $(g_\zeta,S_{f_t}(0))$-regularized resolvent family generated by $-A$ ($t\in\Sigma_\theta$). If $q\in\circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of $[0,\infty)$, then there exists $h_0>0$ such that (174) holds with $f=f_t$ ($t\in\Sigma_\theta$).

(b) Suppose that $0<c<b$, $\sigma>0$, $\nu>-1$, $\varsigma>0$, $\Pi^\zeta_{c,\sigma,\varsigma}\subseteq\rho_C(-A)$, and the operator family $\{(1+|\lambda|)^{-\nu}(\lambda^\zeta+A)^{-1}C:\lambda\in\Pi_{c,\sigma,\varsigma}\}\subseteq L(E)$ is both equicontinuous and strongly continuous. Then the conclusions stated in (a) continue to hold.

Therefore, a great number of multiplication and (pseudo-)differential operators in L^p-spaces can serve as examples of the integral generators of fractional C-regularized resolvent families. Although the applications of theoretical results from statements (a)–(b) and Remark 2.5.5 can be also made to (pseudo-)differential operators with empty resolvent set, and to the operators considered in certain classes of Fréchet function spaces (cf. also Example 2.7.6 below), we shall present only one illustrative example of application of the results from (b) and Remark 2.5.5 to abstract non-degenerate fractional differential equations. Assume $0 < c < b < 1$, $1/(2-\zeta) > b/\zeta$, $\sigma > 0$, $\varsigma > 0$, $p \in [1, \infty)$, $m > 0$, $\rho \in [0, 1]$, $r > 0$, $a \in S^m_{\rho,0}$ satisfies (H_r), the inequality

$$n\Big|\frac{1}{2} - \frac{1}{p}\Big|\Big(\frac{m-r-\rho+1}{r}\Big) < 1 \tag{186}$$

holds, $E = L^p(\mathbb{R}^n)$ or $E = C_0(\mathbb{R}^n)$ (in the last case, we assume that (186) holds with $p = \infty$), and $A := -\mathrm{Op}_E(a)$ (cf. [**27**, Chapter 8] for the notion and terminology). If $\operatorname{dist}(a(\mathbb{R}^n), \Pi^{\zeta}_{c,\sigma,\varsigma}) > 0$, then there exists a number $\nu > -1$ such that the operator family $\{(1 + |\lambda|)^{-\nu}(\lambda^\zeta + A)^{-1} : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$ is both equicontinuous and strongly continuous ($C = I$), so that $(S_{f_t}(s))_{s\geqslant 0}$ is a global $(g_\zeta, S_{f_t}(0))$-regularized resolvent family generated by $-A$ ($t \in \Sigma_\theta$); furthermore, if K is a compact subset of $[0, \infty)$ and $t \in \Sigma_\theta$, then there exists $h_0 > 0$ such that

$$\sup_{p\in\mathbb{N}_0, s\in K} \frac{(h_0)^p \|(\mathbf{D}^\zeta_s)^p S_{f_t}(s)\|}{p^{p\zeta/b}} < \infty.$$

The proof of (185) implies that $\bigcup_{t>0} S_{f_t}(0)(D(A)) = E^{\langle p^{p\zeta/b}\rangle}(A)$, so that the problem $(\mathrm{DFP})'_R$, with $B = I$, has a unique strong solution for all $x \in E^{\langle p^{p\zeta/b}\rangle}(A)$, given by $u(s) := S_{f_t}(s)S_{f_t}(0)^{-1}x$, $s \geqslant 0$, where $t > 0$ satisfies $x \in S_{f_t}(0)(D(A))$. A concrete example can be simply constructed.

Example 2.5.7. Suppose that $\zeta = 1 - c > c(1 + c)$. This, in turn, implies $1/(2-\zeta) > b/\zeta > c/\zeta$ as $1 > b \to c+$. Since $(x + ix^{1/c})^\zeta = (x^2 + x^{2/c})^{\zeta/2}[\cos(\zeta \arctan x^{(1/c)-1}) + i \sin(\zeta \arctan x^{(1/c)-1})]$, $x > 0$, an elementary calculus shows that

$$\operatorname{Re}((x + ix^{1/c})^\zeta)/\operatorname{Im}((x + ix^{1/c})^\zeta) \sim 1/\tan(\zeta\pi/2) \text{ as } x \to +\infty,$$

and

$$\begin{aligned}\operatorname{Re}((x + ix^{1/c})^\zeta) &- (\tan(\zeta\pi/2))^{-1} \operatorname{Im}((x + ix^{1/c})^\zeta)\\ &\sim \zeta(\sin(\zeta\pi/2))^{-1}x^{((\zeta-1)/c)+1} = \zeta(\sin(\zeta\pi/2))^{-1} \text{ as } x \to +\infty\end{aligned}$$

(similar formulae hold if we look into the term $(x - ix^{1/c})^\zeta$ in place of $(x + ix^{1/c})^\zeta$). Using these asymptotic expansions, it readily follows that for each number $d \in (0, \zeta/\sin(\zeta\pi/2))$ there exists a sufficiently large number $r_d > 0$ such that $\operatorname{dist}(a(\mathbb{R}^n), \Pi^\zeta_{c,\sigma_d,\varsigma_d}) > 0$ for suitable chosen numbers $\sigma_d > 0$ and $\varsigma_d > 0$, provided that $a(x) = d + (r_d + P(x))e^{\pm i\pi\zeta/2}$, where $P(x)$ is a positive real elliptic polynomial in n variables, of order m (then (186) holds with $m = r$ and $\rho = 1$).

The polynomials of the operator $A := -d/ds$, $D(A) := \{f \in E\ ;\ f' \in E,\ f(0) = 0\}$, acting on the Banach space

$$E := \Big\{ f \in C^\infty[0,1]\ ;\ \|f\| := \sup_{p \geqslant 0} \frac{\|f^{(p)}\|_\infty}{p!^s} < \infty \Big\} \quad (s > 1),$$

have been considered multiple times by now. The interested reader may throws oneself into the problem of proving some upper bounds on the growth rate of the term

$$\Big\| \Big(\lambda^{(p_n+q_n)\zeta} P_n(A) + \sum_{i=0}^{n-1} \lambda^{(p_i+q_i)\zeta} P_i(A) \Big)^{-1} \Big\|,$$

where $P_i(z)$ is a complex non-zero polynomial $(1 \leqslant i \leqslant n)$, thus providing certain applications of Theorem 2.5.1 and Theorem 2.5.3.

Example 2.5.8. Suppose that E is a general SCLCS, $b \in (0,1)$, (M_p) satisfies $(M.1)$ and (164), $\zeta = 1$, $p_i = 0$ for all $i \in \mathbb{N}_n^0$, $q_n > q_{n-1}$, $\alpha > 0$, $\beta > 0$, $l \geqslant 1$, $\emptyset \neq \Omega' \subseteq \mathbb{C}$, $N \in \mathbb{N}$, A is a densely defined closed linear operator in E satisfying that $\Omega' \subseteq \rho(A)$ and the operator family $\{(1+|\lambda|)^{-N}(\lambda - A)^{-1} : \lambda \in \Omega'\} \subseteq L(E)$ is equicontinuous. Suppose, further, that $P_i(z)$ is a complex polynomial $(i \in \mathbb{N}_n^0)$, $P_n(z) \not\equiv 0$, $\lambda_0 \in \rho(A) \smallsetminus \{z \in \mathbb{C} : P_n(z) = 0\}$, $\operatorname{dist}(\lambda_0, \Omega') > 0$, as well as that for each $\lambda \in \Lambda_{\alpha,\beta,l}$ all roots of the polynomial

$$z \mapsto \lambda^{q_n} P_n(z) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(z), \quad z \in \mathbb{C}$$

belong to Ω'. Set $B := P_n(A)$ and $A_i := P_i(A)$ $(i \in \mathbb{N}_{n-1}^0)$. Then there exist $M \in \mathbb{N}$, λ-polynomials $F_0(\lambda), \ldots, F_M(\lambda)$ and not necessarily distinct numbers $f_1(\lambda) \in \Omega', \ldots, f_M(\lambda) \in \Omega'$, continuously depending on λ, such that $\lambda^{q_n} P_n(z) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(z) = F_M(\lambda) z^M + \cdots + F_1(\lambda) z + F_0(\lambda) = (-1)^M F_M(\lambda)(f_M(\lambda) - z) \ldots (f_1(\lambda) - z)$ for all $\lambda \in \Lambda_{\alpha,\beta,l} \smallsetminus \mathcal{P}$ and $z \in \mathbb{C}$, where $\mathcal{P} \equiv \{\lambda \in \mathbb{C} : F_M(\lambda) = 0\}$; furthermore, for each $\lambda \in \Lambda_{\alpha,\beta,l} \smallsetminus \mathcal{P}$ the following holds:

$$\Big(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A) \Big)^{-1} = (-1)^M (F_M(\lambda))^{-1} (f_M(\lambda) - A)^{-1} \ldots (f_1(\lambda) - A)^{-1}.$$

Using the generalized resolvent equation, it readily follows that for any integer $Q \geqslant N+2$ the operator family $\{(f_i(\lambda) - \lambda_0)(f_i(\lambda) - A)^{-1}(\lambda_0 - A)^{-Q} : \lambda \in \Lambda_{\alpha,\beta,l} \smallsetminus \mathcal{P}\} \subseteq L(E)$ is equicontinuous $(1 \leqslant i \leqslant M)$. This implies that there exists a sufficiently large integer $Q' \geqslant N+2$ such that for each seminorm $p \in \circledast$ there exist $c_p > 0$ and $q \in \circledast$ such that, for every $j \in \mathbb{N}_{n-1}^0$, $\lambda \in \Lambda_{\alpha,\beta,l} \smallsetminus \mathcal{P}$ and $x \in E$,

$$\begin{aligned} & p\Big(\Big(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A) \Big)^{-1} (\lambda_0 - A)^{-Q'} x \Big) \\ & \quad + p\Big(P_j(A) \Big(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A) \Big)^{-1} (\lambda_0 - A)^{-Q'} x \Big) \\ & = p\big((F_M(\lambda))^{-1} (f_M(\lambda) - A)^{-1} \ldots (f_1(\lambda) - A)^{-1} (\lambda_0 - A)^{-Q'} x \big) \end{aligned}$$

$$
\begin{aligned}
&+ p\big(P_j(A)(F_M(\lambda))^{-1}(f_M(\lambda)-A)^{-1}\dots(f_1(\lambda)-A)^{-1}(\lambda_0-A)^{-Q'}x\big)\\
&\leqslant c_p|F_M(\lambda)|^{-1}|(f_M(\lambda)-\lambda_0)\dots(f_1(\lambda)-\lambda_0)|^{-1}q(x)\\
&= c_p|F_M(\lambda)|^{-1}|F_M(\lambda)||F_M(\lambda)\lambda_0^m+\dots+F_1(\lambda)\lambda_0+F_0(\lambda)|^{-1}q(x)\\
&= c_p\bigg|\lambda^{q_n}P_n(\lambda_0)+\sum_{i=0}^{n-1}\lambda^{q_i}P_i(\lambda_0)\bigg|^{-1}q(x)\sim c_p|P_n(\lambda_0)|^{-1}|\lambda|^{-q_n}q(x)\text{ as }|\lambda|\to\infty.
\end{aligned}
$$

Therefore, there exists a sufficiently large number $\beta'>\beta$ such that the operator families $\{(1+|\lambda|)^{q_n}(\lambda^{q_n}P_n(A)+\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A))^{-1}(\lambda_0-A)^{-Q'}:\lambda\in\Lambda_{\alpha,\beta',l}\}\subseteq L(E)$ and $\{(1+|\lambda|)^{q_n}P_j(A)(\lambda^{q_n}P_n(A)+\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A))^{-1}(\lambda_0-A)^{-Q'}:\lambda\in\Lambda_{\alpha,\beta',l}\}\subseteq L(E)$ are equicontinuous ($j\in\mathbb{N}_{n-1}^0$). Since $q_n>q_{n-1}$ and $P(A)$ is dense in E for any complex polynomial $P(z)\in\mathbb{C}[z]$, the analysis contained in Remark 2.5.2(x), with $C\equiv(\lambda_0-A)^{-Q'}$, shows that for each $(x_0,\dots,x_{q_n-1})\in E^{q_n}$ there exists a net $(u_\varepsilon(t))_{\varepsilon>0}$ of strong solutions of problem (168) with the subjected initial values $(u_0^\varepsilon,\dots,u_{q_n-1}^\varepsilon)$, converging to $(x_0,\dots,x_{q_n-1})$ as $\varepsilon\to0+$ (for the topology of E^{q_n}). This example can be also used to provide the possible applications of Theorem 2.6.2 below, with C not being the identity operator on E.

2.6. Entire and analytical solutions of abstract degenerate Cauchy problem (PN)

Before starting our work in this section, we need to repeat some notations and preliminaries from the previous one; cf. also Introduction. We assume that $n\in\mathbb{N}$, $0<\zeta\leqslant2$, $q_0,q_1,\dots,q_n$ are given non-negative integers satisfying $q_0=0$ and $0<q_1\leqslant q_2\leqslant\dots\leqslant q_n$, as well as that $p_i=0$ for all $i\in\mathbb{N}_n^0$, and $A_0,A_1,\dots,A_{n-1},A_n$ are closed linear operators acting on E; we also write B for A_n. Hence, $T_iu(s)=A_i(\mathbf{D}_s^\zeta)^{q_i}u(s)$, $s\geqslant0$, $i\in\mathbb{N}_n^0$ and $P_\lambda=\lambda^{q_n\zeta}B+\sum_{i=0}^{n-1}\lambda^{q_i\zeta}A_i$, $\lambda\in\mathbb{C}\setminus\{0\}$. Set $S_\omega:=\{j\in\mathbb{N}_n^0:q_j-1\geqslant\omega\}$, $\omega\in\mathbb{N}_{q_n-1}^0$; then $n\in S_\omega$ for all $\omega\in\mathbb{N}_{q_n-1}^0$. Suppose that the equation (180) holds with the number v replaced with ω therein.

In this section, we will take up the study of entire and analytical solutions of the abstract Cauchy problem (168) subjected with the initial conditions of the form

$$
\begin{aligned}
&((\mathbf{D}_s^\zeta)^ju(s))_{s=0}=u_j,\ j\in\mathbb{N}_{q_n-1}^0,\ \text{if }\zeta\in(0,1],\ \text{resp.,}\\
&((\mathbf{D}_s^\zeta)^ju(s))_{s=0}=u_j,\ j\in\mathbb{N}_{q_n-1}^0;\\
&\Big(\frac{d}{ds}(\mathbf{D}_s^\zeta)^ju(s)\Big)_{s=0}=v_j,\ j\in\mathbb{N}_{q_n-1}^0,\ \text{if }\zeta\in(1,2].
\end{aligned}\tag{187}
$$

To simplify the notation, this initial value problem will be also called (PN) problem.

Definition 2.6.1. (ii) A function $u\in C([0,\infty):E)$ is said to be an entire solution of problem (PN) iff $u(\cdot)$ is a strong solution of (PN) and it can be analytically extended to the whole complex plane, as well as any of the terms $A_iu^{(p)}(\cdot)$ $(0\leqslant i\leqslant n,\ p\in\mathbb{N}_0)$ can be analytically extended to the whole complex plane.

(iii) A function $u \in C([0,\infty) : E)$ is said to be an analytical solution of problem (PN) on the region $\mathbb{C} \setminus (-\infty, 0]$ iff $u(\cdot)$ is a strong solution of (PN) and it can be extended to the whole complex plane, analytically on the region $\mathbb{C} \setminus (-\infty, 0]$ and continuously on the region $\mathbb{C} \setminus (-\infty, 0)$, as well as any of the terms $A_i(\mathbf{D}_s^{\zeta})^p u(s)$ $(0 \leqslant i \leqslant n,\ p \in \mathbb{N}_0,\ s \geqslant 0)$ is well defined and can be extended to the whole complex plane, analytically on the region $\mathbb{C} \setminus (-\infty, 0]$ and continuously on the region $\mathbb{C} \setminus (-\infty, 0)$.

Now we are ready to formulate the following theorem.

THEOREM 2.6.2. *Suppose that the operator* $C \in L(E)$ *is injective,* $CA_i \subseteq A_iC$, $i \in \mathbb{N}_n^0$, $0 < \zeta \leqslant 2$, $\phi \in (-\pi, \pi]$, $\theta \in (\pi - \pi\zeta, \pi - (\pi\zeta)/2)$, $a > r > 0$ *and* ν' *satisfies* (180). *Assume, further, that the following holds:*

(i) *The operator family* $\{(1+|\lambda|)^{-\nu'} P_{\lambda^{1/\zeta}}^{-1} C : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta},\ |\lambda| \geqslant r\} \subseteq L(E)$ *is both strongly continuous and equicontinuous.*

(ii) *For every* $x \in E$ *and* $i \in \mathbb{N}_{n-1}^0$, *the mapping* $\lambda \mapsto A_i P_{\lambda^{1/\zeta}}^{-1} Cx$, $\lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta}$, $|\lambda| \geqslant r$ *is continuous and there exists* $v_i \in \mathbb{N}$ *such that the operator family* $\{(1+|\lambda|)^{-v_i} A_i P_{\lambda^{1/\zeta}}^{-1} C : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta},\ |\lambda| \geqslant r\} \subseteq L(E)$ *is equicontinuous.*

Denote by $\mathfrak{W}$ ($\mathfrak{W}_e$) *the subspace of* E^{q_n}, *resp.* E^{2q_n}, *consisting of all initial values* $(u_0, \ldots, u_{q_n-1}) \in E^{q_n}$, *resp.* $(u_0, \ldots, u_{q_n-1}; v_0, \ldots, v_{q_n-1}) \in E^{2q_n}$, *subjected to some analytical solution* $u(\cdot)$ *of problem* (168) *on the region* $\mathbb{C} \setminus (-\infty, 0]$ (*entire solution* $u(\cdot)$ *of problem* (168)). *Then* $\mathfrak{W}$ *is dense in* $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$ *for the topology of* E^{q_n}, *provided that* $0 < \zeta < 1$, *resp.* $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$ *for the topology of* E^{2q_n}, *provided that* $1 < \zeta < 2$. *Furthermore, if* $\zeta = 1$, *resp.* $\zeta = 2$, *then the set* $\mathfrak{W}_e$ *is dense in* $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$ *for the topology of* E^{q_n}, *resp.* $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$ *for the topology of* E^{2q_n}.

To prove Theorem 2.6.2, we need the following lemma (cf. also [**515**, Lemma 1.1, Theorem 1.1]).

LEMMA 2.6.3. *Let* $x \in E$. *Then the mapping*

$$\lambda \mapsto P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} Cx, \quad \lambda \in \Sigma_{(\zeta\pi/2)+\theta},\ |\lambda| > r$$

is analytic.

PROOF. Without loss of generality, we may assume that $q_i = i$ $(i \in \mathbb{N}_n^0)$, $\zeta = 1$ and $\phi = 0$. Clearly, (ii) holds for every $x \in E$ and $i \in \mathbb{N}_n^0$. Furthermore, the following analogue of the Hilbert resolvent equation holds:

$$P_\lambda^{-1} C^2 x - P_z^{-1} C^2 x = (z-\lambda) P_\lambda^{-1} C$$
$$\times \Bigg[\sum_{k=1}^{n-1} \binom{n}{k} (z-\lambda)^{k-1} \lambda^{n-k} B + \sum_{k=1}^{n-2} \binom{n-1}{k} (z-\lambda)^{k-1} \lambda^{n-1-k} A_{n-1} + \cdots + A_1 \Bigg]$$
$$\times P_z^{-1} Cx, \text{ provided } \lambda,\ z \in \Sigma_{(\zeta\pi/2)+\theta} \text{ and } |\lambda|,\ |z| > r.$$

This implies that the mapping $\lambda \mapsto P_\lambda^{-1}C^2x$, $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$, $|\lambda| > r$ is weakly analytic and therefore analytic, as well as that

$$\frac{d}{d\lambda}\langle x^*, P_\lambda^{-1}C^2x\rangle = -\langle x^*, P_\lambda^{-1}[n\lambda^{n-1}B + (n-1)\lambda^{n-2}A_{n-1} + \cdots + A_1]P_\lambda^{-1}Cx\rangle,$$

provided $x^* \in E^*$, $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$ and $|\lambda| > r$. Using the Morera theorem and the observation from [**138**, Remark 2.7], the above implies that the mapping $\lambda \mapsto P_\lambda^{-1}Cx$, $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$, $|\lambda| > r$ is analytic, as claimed. □

Now we can proceed to the proof of Theorem 2.6.2.

PROOF. Clearly, $(\zeta\pi/2) + \theta < \pi$, $\pi\zeta/2 > \pi - (\zeta\pi/2) - \theta$ and we can find a number $b \in \mathbb{R}$ satisfying

$$1 < b < \frac{\pi\zeta/2}{\pi - (\zeta\pi/2) - \theta}.$$

Denote by Γ the upwards oriented boundary of the region $\{\lambda \in \Sigma_{(\zeta\pi/2)+\theta} : |\lambda| \geqslant r\}$. Let Ω' be the open region on the left of Γ. Then there exists a sufficiently large number $R > 0$ such that $a - \lambda \in \Sigma_{\pi-(\zeta\pi/2)-\theta}$ for all $\lambda \in \Omega' \cup \Gamma$ with $|\lambda| \geqslant R$. This implies $|e^{-\varepsilon(a-\lambda)^{b/\zeta}}| = e^{-\varepsilon\,\mathrm{Re}((a-\lambda)^{b/\zeta})} \leqslant e^{-\varepsilon|a-\lambda|^{b/\zeta}\cos(b\zeta^{-1}(\pi-(\pi\zeta/2)-\theta))}$, provided $\varepsilon > 0$, $\lambda \in \Omega' \cup \Gamma$ and $|\lambda| \geqslant R$. Keeping in mind Theorem 1.5.1, we obtain the existence of a constant $c_\zeta' > 0$ such that $|E_\zeta(z^\zeta\lambda e^{i\phi})| \leqslant E_\zeta(|z|^\zeta|\lambda|) \leqslant c_\zeta' e^{|z||\lambda|^{1/\zeta}}$ for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Hence, there exists a constant $c_\zeta > 0$ such that

$$|e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(z^\zeta\lambda e^{i\phi})| \leqslant c_\zeta e^{-\varepsilon|a-\lambda|^{b/\zeta}\cos(b\zeta^{-1}(\pi-(\pi\zeta/2)-\theta))+|z||\lambda|^{1/\zeta}}, \tag{188}$$

for any $z \in \mathbb{C}$, $\varepsilon > 0$ and $\lambda \in \Omega' \cup \Gamma$. Suppose now that $x_w \in \bigcap_{j=0}^n D(A_j)$ for all $w \in \mathbb{N}_{q_n-1}^0$. Then (i) and the estimate (188) enable us to define the function $z \mapsto u_\varepsilon(z)$, $z \in \mathbb{C}$, for any $\varepsilon > 0$, by

$$u_\varepsilon(z) := \sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(z^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{q_j-1-w}P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1}CA_jx_w\,d\lambda.$$

It can be simply verified that the mapping $z \mapsto u_\varepsilon(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0)$ is continuous ($\varepsilon > 0$). Using Theorem 1.5.1 and the proof of Theorem 2.5.1, it readily follows that the mapping $z \mapsto u_\varepsilon(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic ($\varepsilon > 0$), with

$$u_\varepsilon'(z) = \sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}z^{\zeta-1}E_{\zeta,\zeta}(z^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{q_j-w}P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1}CA_jx_w\,d\lambda, \tag{189}$$

for any $\varepsilon > 0$ and $z \in \mathbb{C} \smallsetminus (-\infty, 0]$; furthermore, the mapping $z \mapsto u_\varepsilon(z)$, $z \in \mathbb{C}$ is entire provided $\varepsilon > 0$, $\zeta = 1$ and, in this case, (189) holds for any $\varepsilon > 0$ and $z \in \mathbb{C}$. The proof of Theorem 2.5.1 also shows that the term $(\mathbf{D}_s^\zeta)^p u_\varepsilon(s)$, $s \geqslant 0$ is well defined for any $p \in \mathbb{N}_0$ and $\varepsilon > 0$, with

$$\begin{aligned}(\mathbf{D}_s^\zeta)^p u_\varepsilon(s) &\\ = \sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}&E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{p+q_j-1-w}P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1}CA_jx_w\,d\lambda;\end{aligned} \tag{190}$$

cf. also the formula [**61**, (1.25)]. Combined with the Cauchy theorem, (ii) and Theorem 1.5.1, the above implies that the term $A_i(\mathbf{D}_s^\zeta)^p u_\varepsilon(s)$ is well defined for $s \geqslant 0$, $i \in \mathbb{N}_n^0$, $p \in \mathbb{N}_0$ and $\varepsilon > 0$, with

$$\begin{aligned}&A_i(\mathbf{D}_s^\zeta)^p u_\varepsilon(s)\\&\quad=\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{p+q_j-1-w}A_iP^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_wd\lambda.\end{aligned}$$

This implies that, for every $\varepsilon > 0$, any of the terms $A_i(\mathbf{D}_s^\zeta)^p u_\varepsilon(\cdot)$ $(0 \leqslant i \leqslant n$, $p \in \mathbb{N}_0)$ can be extended to the whole complex plane, analytically on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ and continuously on the region $\mathbb{C} \smallsetminus (-\infty, 0)$; we only need to replace the variable $s \geqslant 0$, appearing in the above formula, with the variable $z \in \mathbb{C}$. Furthermore,

$$\begin{aligned}&\sum_{i=0}^n A_i(\mathbf{D}_s^\zeta)^{q_i}u_\varepsilon(s)\\&\quad=\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\sum_{i=0}^n\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{q_i+q_j-1-w}A_iP^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_wd\lambda\\&\quad=\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{q_j-1-w}CA_jx_\omega d\lambda=0,\ s\geqslant 0,\ \varepsilon>0,\end{aligned}$$

so that for each $\varepsilon > 0$ the mapping $s \mapsto u_\varepsilon(s)$, $s \geqslant 0$ is an analytical solution of problem (168) on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ (entire solution of problem (168), provided that $\zeta = 1$), with the initial values $(u_0^\varepsilon, \dots, u_{q_n-1}^\varepsilon)$ subjected. Let $u_l^\varepsilon = ((\mathbf{D}_s^\zeta)^l u_\varepsilon(s))_{s=0}$, $l \in \mathbb{N}_{q_n-1}^0$ $(\varepsilon > 0)$. Now we will prove that $(u_0^\varepsilon, \dots, u_{q_n-1}^\varepsilon)$ converges to $e^{-i\phi}(Cx_0, \dots, Cx_{q_n-1})$ as $\varepsilon \to 0+$, for the topology of E^{q_n}. Let $\omega \in \mathbb{N}_{q_n-1}^0$ and $l \in \mathbb{N}_{q_n-1}^0$ be fixed. Keeping in mind (190), it suffices to prove that the following holds:

$$\lim_{\varepsilon\to 0+}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}(\lambda e^{i\phi})^{l+q_j-1-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_wd\lambda=e^{-i\phi}\delta_{\omega,l}Cx_\omega,$$

i.e., that

$$\begin{aligned}(191)\quad&\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_\Gamma e^{-\varepsilon(a-\lambda)^{b/\zeta}}(\lambda e^{i\phi})^{l-1-w}\\&\qquad\times\Bigg[Cx_\omega-\sum_{j\in\mathbb{N}_n^0\smallsetminus S_\omega}(\lambda e^{i\phi})^{q_j}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\Bigg]d\lambda=e^{-i\phi}\delta_{\omega,l}Cx_\omega,\end{aligned}$$

where $\delta_{\omega,l}$ denotes the Kronecker delta. Since $|e^{-\varepsilon(a-\lambda)^{b/\zeta}}| \leqslant 1$, $\lambda \in \Gamma$, $\varepsilon > 0$, (180) and (i) holds, we have that there exists $\sigma > 0$ such that

$$|e^{-\varepsilon(a-\lambda)^{b/\zeta}}(\lambda e^{i\phi})^{l-1-w}(\lambda e^{i\phi})^{q_j}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w|\leqslant \text{Const. }|\lambda|^{-1-\sigma},$$

for any $\lambda \in \Gamma$, $\varepsilon > 0$ and $j \in \mathbb{N}_n^0 \smallsetminus S_\omega$. Applying the dominated convergence theorem, Lemma 2.6.3 and the Cauchy theorem, we get that

$$\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}(\lambda e^{i\phi})^{l-1-w}\sum_{j\in\mathbb{N}_n^0\smallsetminus S_\omega}(\lambda e^{i\phi})^{q_j}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\,d\lambda$$
$$=\frac{1}{2\pi i}\int_{\Gamma}(\lambda e^{i\phi})^{l-1-w}\sum_{j\in\mathbb{N}_n^0\smallsetminus S_\omega}(\lambda e^{i\phi})^{q_j}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\,d\lambda=0.$$

Taking into account the last formula on p. 170 of [**541**], it readily follows that (191) golds good. The proof of theorem is thereby complete in the case that $0<\zeta\leqslant 1$. Suppose now $1<\zeta\leqslant 2$. Then it is not difficult to show that there exists a finite constant $d_\zeta>0$ such that the function $F_\lambda(z):=zE_{\zeta,2}(z^\zeta\lambda e^{i\phi})$, $z\in\mathbb{C}$ $(\lambda\in\mathbb{C})$ satisfies $F_\lambda'(z)=E_\zeta(z^\zeta\lambda e^{i\phi})$, $z\in\mathbb{C}\smallsetminus(-\infty,0]$ and $|F_\lambda(z)|\leqslant d_\zeta(1+|z|)e^{|z||\lambda|^{1/\zeta}}$, $z\in\mathbb{C}$ $(\lambda\in\mathbb{C})$. Since for any function $u\in C^1([0,\infty):X)$ with $u'(0)=0$ we have $\mathbf{D}_s^\zeta(g_1*u)(s)=(g_1*\mathbf{D}_\cdot^\zeta u)(s)$, $s\geqslant 0$, provided in addition that the term $\mathbf{D}_s^\zeta u(s)$ is defined for $s\geqslant 0$, it readily follows that $\mathbf{D}_s^\zeta F_\lambda(s)=(g_1*\mathbf{D}_s^\zeta E_\zeta(\cdot^\zeta\lambda e^{i\phi}))(s)=\lambda e^{i\phi}(g_1*E_\zeta(\cdot^\zeta\lambda e^{i\phi}))(s)=\lambda e^{i\phi}F_\lambda(s)$, $s\geqslant 0$ $(\lambda\in\mathbb{C})$. Let $x_w,y_w\in\bigcap_{j=0}^n D(A_j)$ for all $w\in\mathbb{N}^0_{q_n-1}$. Define now the solution $u_\varepsilon(\cdot)$ by

$$u_\varepsilon(z):=\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(z^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{q_j-1-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\,d\lambda$$
$$+\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}F_\lambda(z)(\lambda e^{i\phi})^{q_j-1-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jy_w\,d\lambda,$$

for any $z\in\mathbb{C}$ and $\varepsilon>0$. It is not difficult to prove that, for every $p\in\mathbb{N}_0$, $s\geqslant 0$ and $\varepsilon>0$, the following holds:

$$(\mathbf{D}_s^\zeta)^p u_\varepsilon(s)$$
$$=\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{p+q_j-1-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\,d\lambda$$
$$+\sum_{w=0}^{q_n-1}\sum_{j\in S_\omega}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}F_\lambda(s)(\lambda e^{i\phi})^{p+q_j-1-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jy_w\,d\lambda$$

and

$$\frac{d}{ds}(\mathbf{D}_s^\zeta)^p u_\varepsilon(s)$$
$$=\sum_{w=0}^{q_n-1}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}s^{\zeta-1}E_{\zeta,\zeta}(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{p+q_j-w}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jx_w\,d\lambda$$
$$+\sum_{w=0}^{q_n-1}\frac{1}{2\pi i}\int_{\Gamma} e^{-\varepsilon(a-\lambda)^{b/\zeta}}E_\zeta(s^\zeta\lambda e^{i\phi})(\lambda e^{i\phi})^{p+q_j-w-1}P^{-1}_{(\lambda e^{i\phi})^{1/\zeta}}CA_jy_w\,d\lambda.$$

The remainder of the proof can be deduced by repeating almost verbatim the corresponding parts of the first part of proof (the case $0<\zeta\leqslant 1$). $\square$

REMARK 2.6.4. (i) Observe that Theorem 2.6.2 seems to be new and not considered elsewhere provided that $B \neq I$ or $\zeta \neq 1$. It is also worth noting that the assertion of Theorem 2.6.2 can be reformulated in the case that $\zeta > 2$ (cf. also [**298**, Theorem 2.1, Theorem 2.2]) and the condition (180) holds provided that $-\infty < \nu' < 1 - q_n$. We can prove a similar result on the existence and uniqueness of entire and analytical solutions of problem (DFP)$_R$; we leave the reader to make this precise.

(ii) The notion of an entire solution of the abstract Cauchy problem (ACP_n), introduced in [**544**, Definition 1.1], is slightly different from the corresponding notion introduced in Definition 2.6.1(ii). Strictly speaking, if $u(\cdot)$ is an entire solution of the abstract Cauchy problem (ACP_n) in the sense of Definition 2.6.1(ii), then $u(\cdot)$ is an entire solution of problem (ACP_n) in the sense of [**544**, Definition 1.1]. The converse statement holds provided that for each index $i \in \mathbb{N}_{n-1}$ the initial values $u_0, \ldots, u_{i-1}$ belong to $D(A_i)$.

(ii) The uniqueness of analytical solutions of problem (PN) on the region $\mathbb{C} \smallsetminus (-\infty, 0]$ can be proved as follows. Let $u(\cdot)$ be an analytical solution of problem (PN) on the region $\mathbb{C} \smallsetminus (-\infty, 0]$, with the initial values u_j, resp. u_j, v_j, being zeroes $(0 \leqslant j \leqslant q_n - 1)$. Then the choice of initial values in (187) enables us to integrate the equation (168) $(q_n\zeta)$-times by using the formula (38). Keeping in mind the analyticity of $u(\cdot)$, we easily infer that for each $i \in \mathbb{N}_{n-1}^0$ the mappings $z \mapsto Bu(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0)$ $(z \mapsto Bu(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0])$ and $z \mapsto (g_{(q_n-q_i)\zeta} * A_i u)(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0)$ $(z \mapsto (g_{(q_n-q_i)\zeta} * A_i u)(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0])$ are well defined and continuous (analytical), as well as that

$$Bu(se^{i\gamma}) + \sum_{i=0}^{n-1} \int_0^{se^{i\gamma}} g_{(q_n-q_i)\zeta}(v) A_i u(se^{i\gamma} - v)\,dv = 0, \quad s \geqslant 0,\ \gamma \in (-\pi, \pi),$$

i.e., that

$$Bu(se^{i\gamma}) + \sum_{i=0}^{n-1} (e^{i\gamma})^{(q_n-q_i)\zeta} \int_0^s g_{(q_n-q_i)\zeta}(s-v) A_i u(ve^{i\gamma})\,dv = 0, \quad t \geqslant 0,\ \gamma \in (-\pi, \pi).$$

It is clear that there exists $\gamma \in (-\pi, \pi)$ such that $2re^{-i\gamma\zeta} \in \{\lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta} : |\lambda| \geqslant r\}$. Setting $\phi' := \gamma\zeta$, $u_\gamma(s) := u(se^{i\gamma})$, $s \geqslant 0$, we obtain that $u_\gamma \in C([0, \infty) : E)$ and

$$e^{-iq_n\phi'} Bu_\gamma(s) + \sum_{i=0}^{n-1} e^{-iq_i\phi'} A_i (g_{(q_n-q_i)\zeta} * u_\gamma)(s) = 0, \quad s \geqslant 0.$$

On the other hand,

$$P_{(\lambda e^{-i\phi'})^{1/\zeta}} = \lambda^{q_n} e^{-iq_n\phi'} B + \sum_{i=0}^{n-1} \lambda^{q_i} e^{-iq_i\phi'} A_i, \quad \lambda \in \mathbb{C} \smallsetminus \{0\}.$$

Using the previous two equalities and Theorem 2.3.6 (applied to the operators $e^{-iq_i\phi'} A_i$ in place of the operators A_i appearing in the formulation of this theorem), we get that $u_\gamma(s) = 0$, $s \geqslant 0$, which clearly implies that $u(z) = 0$, $z \in \mathbb{C} \smallsetminus (-\infty, 0)$.

It is also worth noting that Theorem 2.6.2 is an extension of [**544**, Theorem 2.1] (cf. also [**541**, Theorem 4.2, p. 168]), where it has been assumed that $B = C = I$, $\zeta = 1$, E is a Banach space and $\bigcap_{j=0}^{n} D(A_j)$ is dense in E (in our opinion, the strong continuity in (ii) is very important for the validity of Theorem 2.6.2 and cannot be so simply neglected here (cf. [**541**, (4.8), p. 169]); also, it ought to be observed that Lemma 2.6.3 is very important for filling some absences in the proof of [**541**, Theorem 4.2], appearing on the lines 1–6, p. 171 in [**541**], where the Cauchy formula has been used by assuming the analyticity of mapping $\lambda \mapsto R_{e^{i\phi}\lambda}$, $\lambda \in \Sigma_{(\pi/2)+\theta}$, $|\lambda| > r$ a priori); observe also that, in the concrete situation of abstract Cauchy problem (ACP_n), our estimate on the growth rate of $P^{-1}_{e^{i\phi}\cdot}$ (cf. the equation (180) with $\nu' < -(n-1)$) is slightly better than the corresponding estimate [**541**, (4.2)], where it has been required that $\nu' \leqslant -n$. If $B = C = I$ and $\zeta = 2$, then Theorem 2.6.2 strengthens [**544**, Theorem 2.1] in a drastic manner. In actual fact, our basic requirement in (i) is that the operator $P_\lambda^{-1} = (\lambda^{2n} + \lambda^{2n-2}A_{n-1} + \cdots + A_0)^{-1}$ exists on the region $\{\lambda^{1/2} : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta} : |\lambda| \geqslant r\}$, which can be contained in an arbitrary acute angle at vertex $(0, 0)$; on the other hand, in the formulation of [**544**, Theorem 2.1], T.-J. Xiao and J. Liang require the existence of operator P_λ^{-1} for any complex number λ having the modulus greater than or equal to r and belonging to the obtuse angle $e^{i\phi}\Sigma_{(\pi/2)+\theta}$.

Before proceeding further, we would like to note that an illustrative example of application of Theorem 2.5.3 and the conclusions from Remark 2.5.5 to (multi-term) degenerate Cauchy problems will be presented within Example 2.7.6.

2.7. Abstract incomplete degenerate differential equations

Let E be an SCLCS. We start this section by introducing the following definition.

Definition 2.7.1. Suppose that B is a closed linear operator on E and $C \in L(E)$ is an injective operator.

(i) An operator family $(T(t))_{t>0} \subseteq L(E)$ is said to be a pre-(B, C)-regularized semigroup of growth order $r > 0$ iff $R(T(t)) \subseteq D(B)$, $t > 0$ and the following holds:
  (a) $T(t + s)C = T(t)BT(s)$ for all t, $s > 0$,
  (b) for every $x \in E$, the mapping $t \mapsto T(t)x$, $t > 0$ is continuous, and
  (c) the family $\{t^r T(t) : t \in (0, 1]\} \subseteq L(E)$ is equicontinuous.

  $(T(t))_{t>0} \subseteq L(E)$ is said to be a (B, C)-regularized semigroup of growth order $r > 0$ iff, in addition to (a)–(c), we have that:
  (d) for every $x \in E$, the mapping $t \mapsto BT(t)x$, $t > 0$ is continuous, and
  (e) the family $\{t^r BT(t) : t \in (0, 1]\} \subseteq L(E)$ is equicontinuous.

(ii) Suppose $0 < \gamma \leqslant \pi/2$, $(T(t))_{t>0}$ is a pre-(B, C)-regularized semigroup of growth order $r > 0$, and the mapping $t \mapsto T(t)x$, $t > 0$ has an analytic extension to the sector Σ_γ, denoted by the same symbol ($x \in E$). If there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that, for every $\delta \in (0, \gamma)$, the family $\{|z|^r e^{-\omega \operatorname{Re} z} T(z) : z \in \Sigma_\delta\} \subseteq L(E)$ is equicontinuous, then $(T(z))_{z \in \Sigma_\gamma} \subseteq L(E)$ is said to be an (equicontinuous) analytic pre-(B, C)-regularized semigroup of growth order r.

(iii) If $(T(t))_{t>0}$ is a (B, C)-regularized semigroup of growth order $r > 0$, the mappings $t \mapsto T(t)x$, $t > 0$ and $t \mapsto BT(t)x$, $t > 0$ admit analytic extensions to the sector Σ_γ, denoted by the same symbols $(x \in E)$, and if there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that, for every $\delta \in (0, \gamma)$, the families $\{|z|^r e^{-\omega \operatorname{Re} z} T(z) : z \in \Sigma_\delta\} \subseteq L(E)$ and $\{|z|^r e^{-\omega \operatorname{Re} z} BT(z) : z \in \Sigma_\delta\} \subseteq L(E)$ are equicontinuous, then $(T(z))_{z \in \Sigma_\gamma} \subseteq L(E)$ is said to be an (equicontinuous) analytic (B, C)-regularized semigroup of growth order r.

REMARK 2.7.2. (i) Our assumption $R(T(t)) \subseteq D(B)$, $t > 0$ immediately implies that $BT(t)$ is a closed linear operator for all $t > 0$. In the case that E is a webbed bornological space, then the above implies by the closed graph theorem that $BT(t) \in L(E)$ for all $t > 0$.

(ii) If $(T(t))_{t>0}$ is a (B, C)-regularized semigroup of growth order $r > 0$ satisfying additionally that the preassumption $BT(t)x = 0$, $t > 0$ implies $x = 0$, then $(BT(t))_{t>0} \subseteq L(E)$ is a C-regularized semigroup of growth order $r > 0$ (in the sense of [**103**, Definition 3.4]). A similar statement can be formulated for the class of analytic (B, C)-regularized semigroups.

If $(T(t))_{t>0} \subseteq L(E)$ is a pre-(B, C)-regularized semigroup of growth order $r > 0$, then we define the integral generator of $(T(t))_{t>0}$ by

$$\mathcal{A}_B := \Big\{(x, y) \in E \times E : \; BT(t)x - BT(s)x = B \int_s^t T(r)y \, dr \text{ for all } t, s > 0 \text{ with } t \geqslant s\Big\}.$$

Then it is clear that $\mathcal{A}_B$ is a multivalued linear operator in E and that the local equicontinuity of operator family $(T(t))_{t>0}$ combined with the assumption $B \in L(E)$ implies that $\mathcal{A}_B$ is closed; in the case that there exists a linear manifold Y of E such that $E = \mathcal{A}_B 0 \oplus Y$ (cf. [**424**, Proposition 1.6.4]), we can single out a single-valued branch of $\mathcal{A}_B$. It is very difficult to say what will be the complete infinitesimal generator of $(T(t))_{t>0}$ (cf. [**103**] for the case $B = I$) in the degenerate case.

The reader may consult [**291**, Section 1.2] for further information about non-degenerate semigroups of growth order $r > 0$, i.e., (I, I)-regularized semigroups of growth order $r > 0$. It is an open problem to find a necessary and sufficient condition for generation of degenerate semigroups of growth order $r > 0$ in terms of spectral properties of their integral generators.

In the sequel of this section, we shall always assume that A and B are two closed linear operators acting on E, as well as that $CA \subseteq AC$ and $CB \subseteq BC$. Sometimes we use the following condition.

(H): A and B are closed linear operators on E, $C \in L(E)$ is injective, $0 \leqslant \omega < \pi$, $q \in \mathbb{R}$, $\mathbb{C} \setminus \overline{\Sigma_\omega} \subseteq \rho_C(B, A)$, the families

$$\{(|\lambda|^{-1} + |\lambda|^q)^{-1}(\lambda B - A)^{-1}C : \lambda \notin \Sigma_{\omega'} \cup \{0\}\} \subseteq L(E) \text{ and}$$
$$\{(|\lambda|^{-1} + |\lambda|^q)^{-1}B(\lambda B - A)^{-1}C : \lambda \notin \Sigma_{\omega'} \cup \{0\}\} \subseteq L(E)$$

are equicontinuous for every $\omega < \omega' < \pi$, as well as the mappings $\lambda \mapsto (\lambda B - A)^{-1}Cx$, $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\omega}$ and $\lambda \mapsto B(\lambda B - A)^{-1}Cx$, $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\omega}$ are continuous for every fixed element $x \in E$.

Since we have assumed that $CA \subseteq AC$ and $CB \subseteq BC$, the analysis contained in Remark 2.5.2(vii) shows that the validity of condition (H) implies that the mappings $\lambda \mapsto (\lambda B - A)^{-1}Cx$, $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\omega}$ and $\lambda \mapsto B(\lambda B - A)^{-1}Cx$, $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\omega}$ are analytic, indeed, for every fixed element $x \in E$. By $\mathcal{M}_{B,C,q}$ we denote the class consisting of all closed linear operators A' on E, satisfying that the families $\{(|\lambda|^{-1}+|\lambda|^q)^{-1}(\lambda B-A')^{-1}C : \lambda \in (-\infty, 0)\} \subseteq L(E)$ and $\{(|\lambda|^{-1}+|\lambda|^q)^{-1}B(\lambda B-A')^{-1}C : \lambda \in (-\infty, 0)\} \subseteq L(E)$ are equicontinuous, as well as that the mappings $\lambda \mapsto (\lambda B-A')^{-1}Cx$, $\lambda \in (-\infty, 0)$ and $\lambda \mapsto B(\lambda B-A')^{-1}Cx$, $\lambda \in (-\infty, 0)$ are continuous for every fixed element $x \in E$. Following A. V. Balakrishnan [**44**], we introduce the function $f_{t,\gamma}(\lambda)$ by

$$\begin{aligned} f_{t,\gamma}(\lambda) &:= \frac{1}{\pi} e^{-t\lambda^\gamma \cos \pi\gamma} \sin(t\lambda^\gamma \sin \pi\gamma) \\ &= \frac{1}{2\pi i}\left(e^{-t\lambda^\gamma e^{-i\pi\gamma}} - e^{-t\lambda^\gamma e^{i\pi\gamma}}\right), \quad \lambda > 0 \ (t > 0,\ \gamma \in (0, 1/2)). \end{aligned}$$

This function enjoys the following properties [**413**]:

1. $|f_{t,\gamma}(\lambda)| \leqslant \pi^{-1} e^{-\lambda^\gamma \varepsilon_{t,\gamma}}$, $\lambda > 0$, where $\varepsilon_{t,\gamma} := t \cos \pi\gamma > 0$.
2. $|f_{t,\gamma}(\lambda)| \leqslant \gamma t \lambda^\gamma e^{-t\lambda^\gamma \sin \varepsilon_{t,\gamma}}$, $\lambda > 0$.
3. $\int_0^\infty \lambda^n f_{t,\gamma}(\lambda)\, d\lambda = 0$, $n \in \mathbb{N}_0$.

Set

$$H_n(\omega, z) := \frac{d^n}{dz^n} \exp(-\omega z^\gamma), \quad \omega \in \mathbb{C},\ z \in \mathbb{C} \setminus (-\infty, 0],$$

and, if that makes any sense,

$$W_\gamma(t)x := \int_0^\infty f_{t,\gamma}(\lambda)(\lambda B + A)^{-1}Cx\, d\lambda, \quad t > 0,\ x \in E \ (\gamma \in (0, 1/2)).$$

Then the function $H_n(\omega, z)$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ for every fixed number ω, and entire in $\mathbb{C}$ for every fixed number z (cf. also the proof of [**413**, Proposition 3.5]).

The main purpose of following theorem is to transmit the assertion of [**292**, Theorem 2.9.48] to abstract degenerate differential equations (some parts of this theorem cannot be so simply formulated for degenerate differential equations because we do not know what would be the fractional power A_z ($z \in \mathbb{C}_+$) in the newly arisen situation; cf. also [**181**]).

THEOREM 2.7.3. *Let $0 < \gamma < 1/2$, and let $A \in \mathcal{M}_{B,C,q}$ where $q+\gamma > -1$. Suppose that $CA \subseteq AC$ and $CB \subseteq BC$. Then the families $\{(1 + t^{-(q+1)/\gamma})^{-1}W_\gamma(t) : t > 0\} \subseteq L(E)$ and $\{(1 + t^{-(q+1)/\gamma})^{-1}BW_\gamma(t) : t > 0\} \subseteq L(E)$ are equicontinuous, and there exists an operator family $(\mathbf{W}_\gamma(z))_{z \in \Sigma_{(\pi/2)-\gamma\pi}} \subseteq L(E)$, resp.,*

$(\mathbf{W}_{\gamma,B}(z))_{z\in\Sigma_{(\pi/2)-\gamma\pi}} \subseteq L(E)$, *such that, for every* $x \in E$, *the mapping* $z \mapsto \mathbf{W}_{\gamma}(z)x$, $z \in \Sigma_{(\pi/2)-\gamma\pi}$, *resp.,* $z \mapsto \mathbf{W}_{\gamma,B}(z)x$, $z \in \Sigma_{(\pi/2)-\gamma\pi}$, *is analytic as well as that* $\mathbf{W}_{\gamma}(t) = W_{\gamma}(t)$, $t > 0$, *resp.,* $\mathbf{W}_{\gamma,B}(t) = BW_{\gamma}(t)$, $t > 0$ *(in the sequel, we will not make any difference between* $\mathbf{W}_{\gamma}(\cdot)$ *and* $W_{\gamma}(\cdot)$, $\mathbf{W}_{\gamma,B}(\cdot)$ *and* $W_{\gamma,B}(\cdot)$*). Furthermore, the following holds:*

(i) $W_{\gamma}(z_1)BW_{\gamma}(z_2) = W_{\gamma}(z_1 + z_2)C$ *for all* $z_1, z_2 \in \Sigma_{(\pi/2)-\gamma\pi}$.

(ii) *Let* $-1-\gamma < q \leqslant -1$. *If* $(\lambda B - A)^{-1}CA \subseteq A(\lambda B - A)^{-1}C$, $\lambda \in \rho_C(A, B)$ *and* $x \in \overline{D(A)}$, *then*

$$\lim_{z\to 0,z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} BW_{\gamma}(z)x = Cx, \quad \varepsilon \in (0, (\pi/2) - \gamma\pi); \tag{192}$$

if, in addition to this, the condition (H) *holds, then we can extend the operator family* $W_{\gamma}(\cdot)$ *to the sector* $\Sigma_{\frac{\pi}{2}-\omega\gamma}$ *and the limit equality* (192) *remains true for each* $x \in \overline{D(A)}$, *with the number* $(\pi/2) - \gamma\pi$ *replaced by* $(\pi/2) - \omega\gamma$.

(iii) *Let* $q > -1$. *Then* $(W_{\gamma}(z))_{z\in\Sigma_{(\pi/2)-\gamma\pi}}$ *is an equicontinuous analytic* (B, C)*-regularized semigroup of growth order* $(q + 1)/\gamma$. *If, additionally, the condition* (H) *holds, then* $(W_{\gamma}(t))_{t>0}$ *can be extended to an equicontinuous analytic* (B, C)*-regularized semigroup* $(\mathbf{W}_{\gamma}(z))_{z\in\Sigma_{(\pi/2)-\gamma\omega}}$ *of growth order* $(q+1)/\gamma$. *Suppose that the operator* B *is injective,* $x \in D((B^{-1}A)^{\lfloor q+2\rfloor}) \cap D(B)$ *and* $(\lambda B - A)^{-1}CB \subseteq B(\lambda B - A)^{-1}C$ $(\lambda \in \rho_C(A, B))$. *Then* $\lim_{z\to 0,z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} BW_{\gamma}(z)x = Cx$, $\varepsilon \in (0, (\pi/2) - \gamma\pi)$; *if, in addition to this, the condition* (H) *holds, then the above limit equality remains true with the number* $(\pi/2) - \gamma\pi$ *replaced by* $(\pi/2) - \omega\gamma$.

(iv) *Suppose that* $q < 0$ *and* $x \in D(A) \cap D(B)$. *Then* $\lim_{z\to 0,z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} W_{\gamma}(z)Bx = Cx$, $\varepsilon \in (0, (\pi/2) - \gamma\pi)$; *if, in addition to this, the condition* (H) *holds, then the above limit equality remains true with the number* $(\pi/2) - \gamma\pi$ *replaced by* $(\pi/2) - \omega\gamma$.

(v)(v.1) *Let* $q + \gamma \geqslant 0$, $z_0 \in \mathbb{C}_+$, *let* B *be injective,* $x \in D((B^{-1}A)^{\lfloor q+\gamma\rfloor+2}) \cap D(B)$, *and let* $(\lambda B - A)^{-1}CB \subseteq B(\lambda B - A)^{-1}C$ $(\lambda \in \rho_C(A, B))$. *Then*

$$\begin{aligned} \lim_{z\to 0,z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} & \frac{BW_{\gamma}(z)x - Cx}{z} \\ &= -z_0^{\gamma}Cx + \sum_{k=2}^{\lfloor q+\gamma\rfloor+2} \frac{(-1)^{k-1}}{(k-1)!} H_{k-1}(0, z_0)(z_0 - B^{-1}A)^{k-1}Cx \\ - \sin(\pi\gamma) \int_0^{\infty} & \lambda^{\gamma} \frac{B(\lambda + A)^{-1}C(z_0 - B^{-1}A)^{\lfloor q+\gamma\rfloor+2}x}{(\lambda + z_0)^{\lfloor q+\gamma\rfloor+2}} d\lambda, \quad \varepsilon \in (0, (\pi/2) - \gamma\pi); \end{aligned} \tag{193}$$

if the condition (H) *holds, then the formula* (193) *remains true with the number* $(\pi/2) - \gamma\pi$ *replaced by* $(\pi/2) - \omega\gamma$.

(v.2) *Let* $q + \gamma < 0$, $z_0 \in \mathbb{C}_+$ *and* $x \in D(A) \cap D(B)$. *Then*

$$\lim_{z\to 0,z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} \frac{W_{\gamma}(z)Bx - Cx}{z} = -z_0^{\gamma}Cx \tag{194}$$

$$- \sin(\pi\gamma) \int_0^{\infty} \lambda^{\gamma} \frac{(\lambda + A)^{-1}C(z_0 B - A)x}{(\lambda + z_0)} d\lambda, \quad \varepsilon \in (0, (\pi/2) - \gamma\pi);$$

if the condition (H) *holds, then the formula* (194) *remains true with the number* $(\pi/2) - \gamma\pi$ *replaced by* $(\pi/2) - \omega\gamma$.

PROOF. The existence of operator families $(\mathbf{W}_{\gamma}(z))_{z \in \Sigma_{(\pi/2)-\gamma\pi}}$ and $(\mathbf{W}_{\gamma,B}(z))_{z \in \Sigma_{(\pi/2)-\gamma\pi}}$, satisfying the properties stated in the first part of formulation of theorem, before the assertion (i), follows similarly as in the case that $B = I$. Suppose $\delta \in (0, (\pi/2) - \pi\gamma)$ and $r \in \circledast$. Arguing as in the proof of [**292**, Theorem 2.9.48], we obtain that there exist $r' \in \circledast$, $m > 0$ and $c > 0$ such that, for every $x \in E$ and $z \in \Sigma_{\delta}$, we have

$$r(W_{\gamma}(z)x) \leqslant \Bigl(1 + |z|^{-\frac{q+1}{\gamma}}\Bigr) r'(x) \text{ and } r(BW_{\gamma}(z)x) \leqslant \Bigl(1 + |z|^{-\frac{q+1}{\gamma}}\Bigr) r'(x). \tag{195}$$

The semigroup property stated in (i) can be proved by means of the resolvent equation from Remark 2.5.2(vii) and direct computations, similar to those appearing in [**44**, Section 3]. Therefore, $(W_{\gamma}(z))_{z \in \Sigma_{(\pi/2)-\gamma\pi}}$ is an equicontinuous analytic (B, C)-regularized semigroup of growth order $(q + 1)/\gamma$, provided that $q > -1$. Suppose temporarily that the operator B is injective and $B(\lambda B - A)^{-1}C \subseteq (\lambda B - A)^{-1}CB$ $(\lambda \in \rho_C(A, B))$. Then the following equality holds

$$(\lambda + B^{-1}A)^{-1}Cx = B(\lambda B + A)^{-1}Cx, \quad \lambda > 0, \ x \in D(B). \tag{196}$$

Combined with the identity [**413**, p. 212, l. 8], (196) implies that, for every $\lambda > 0$ and $z_0 \in \mathbb{C}_+$, we have

$$\begin{aligned} B(\lambda B + A)^{-1}Cx = &\sum_{j=1}^{\lfloor q+2 \rfloor} \frac{(z_0 - B^{-1}A)^{j-1}Cx}{(\lambda + z_0)^j} \\ &+ \frac{B(\lambda B + A)^{-1}C(z_0 - B^{-1}A)^{\lfloor q+2 \rfloor}Cx}{(\lambda + z_0)^{\lfloor q+2 \rfloor}}, \end{aligned}$$

so that the proof of limit equality in (iii), on proper subsectors of $\Sigma_{(\pi/2)-\gamma\pi}$, follows in almost the same way as in the proof of [**413**, Proposition 3.5]. Using the proof of [**292**, Theorem 2.9.48(v)], we obtain similarly that the limit equality in (v.1) holds on proper subsectors of $\Sigma_{(\pi/2)-\gamma\pi}$. We continue the proof of (iii). Suppose that $\varepsilon > 0$ is sufficiently small. Let $q > -1$, and let the condition (H) hold. Then one can take numbers $\theta_1 \in (0, \pi - \omega)$ and $\theta_2 \in (\omega - \pi, 0)$ such that $(\pi/2) - \gamma\omega > (\pi/2) - \gamma\pi + \gamma\theta_1 > (\pi/2) - \gamma\omega - \varepsilon$ and $\omega\gamma - (\pi/2) + \varepsilon > \gamma\pi - (\pi/2) + \gamma\theta_2 > \omega\gamma - (\pi/2)$. Set, for every $\theta \in (\omega - \pi, \pi - \omega)$,

$$W_{\theta,\gamma}(z)x := \int_0^{\infty} f_{z,\gamma}(\lambda)(\lambda B + e^{i\theta}A)^{-1}Cx \, d\lambda, \quad x \in E, \ z \in \Sigma_{(\pi/2)-\gamma\pi}.$$

Then it can be simply verified that $(W_{\theta,\gamma}(z))_{z \in \Sigma_{(\pi/2)-\gamma\pi}}$ is an equicontinuous analytic (B, C)-regularized semigroup of growth order $(q + 1)/\gamma$. Define, for every

$x \in E$,

$$W_\gamma(z)x := \begin{cases} W_\gamma(z)x, & \text{if } z \in \Sigma_{(\pi/2)-\gamma\pi}, \\ W_{\theta_1,\gamma}(ze^{-i\theta_1\gamma})x, & \text{if } z \in e^{i\theta_1\gamma}\Sigma_{(\pi/2)-\gamma\pi}, \\ W_{\theta_2,\gamma}(ze^{-i\theta_2\gamma})x, & \text{if } z \in e^{i\theta_2\gamma}\Sigma_{(\pi/2)-\gamma\pi}. \end{cases}$$

By Cauchy formula, we have that $W_\gamma(z)x = W_{\theta_1,\gamma}(ze^{-i\theta_1\gamma})$, if $z \in \Sigma_{(\pi/2)-\gamma\pi} \cap e^{i\theta_1\gamma}\Sigma_{(\pi/2)-\gamma\pi}$, and $W_\gamma(z)x = W_{\theta_2,\gamma}(ze^{-i\theta_2\gamma})$, if $z \in \Sigma_{(\pi/2)-\gamma\pi} \cap e^{i\theta_2\gamma}\Sigma_{(\pi/2)-\gamma\pi}$, whence we may conclude that the operator family $(W_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}} \subseteq L(E)$ is well defined. We define the operator family $(BW_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}} \subseteq L(E)$ similarly. Then it is checked at once that $(W_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}}$ is an equicontinuous analytic (B, C)-regularized semigroup of growth order $(q + 1)/\gamma$. The way of construction of $(W_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}} \subseteq L(E)$ shows that the limit equality stated in (iii) continues to hold for each $x \in D((B^{-1}A)^{\lfloor q+2\rfloor}) \cap D(B)$, with the number $(\pi/2) - \gamma\pi$ replaced by $(\pi/2) - \gamma\omega$ (if $q \leqslant -1$, then we define $(W_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}} \subseteq L(E)$ and $(BW_\gamma(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega\gamma}} \subseteq L(E)$ in the same way as above, showing also that any considered limit equality from (ii)–(v) continues to hold with the number $(\pi/2) - \gamma\pi$ replaced by $(\pi/2) - \gamma\omega$). The proof of (iii) is therefore completed. Suppose now that $-1-\gamma < q \leqslant -1$. Then an insignificant modification of the proof of [**410**, Theorem 5.5.1(iv)] shows that $\lim_{t\to 0+} BW_\gamma(t)x = Cx$, provided that $x \in E$ satisfies $\lim_{\lambda\to+\infty} \lambda B(\lambda B + A)^{-1}Cx = Cx$. Since we have assumed that A commutes with $(\lambda B+A)^{-1}C$ ($\lambda \in \rho_C(A, B)$), and that $\lambda B(\lambda B+A)^{-1}Cx-Cx = A(\lambda B+A)^{-1}Cx = (\lambda B + A)^{-1}CAx$, $x \in D(A)$, $\lambda > 0$, one has $\lim_{t\to 0+} BW_\gamma(t)x = Cx$, $x \in D(A)$. Using (195), it readily follows that $\lim_{t\to 0+} BW_\gamma(t)x = Cx$, $x \in \overline{D(A)}$, so that the limit equality (192) follows from an application of [**296**, Theorem 3.4(ii)]. The limit equalities stated in (iv) and (v.2), on proper subsectors of $\Sigma_{(\pi/2)-\gamma\pi}$, can be proved by using the equality

$$(\lambda B + A)^{-1}CBx = \frac{Cx}{\lambda + z_0} + \frac{(\lambda B + A)^{-1}C}{\lambda + z_0}(z_0Bx - Ax),$$

which holds for $z_0 \in \mathbb{C}_+, \lambda > 0$, $x \in D(A) \cap D(B)$, and a slight modification of the proofs of [**413**, Lemma 3.4, Proposition 3.5]. The proof of the theorem is thereby complete. $\square$

In Theorem 2.7.4, we use the same terminology as in the formulation and proof of Theorem 2.7.3. We continue our previous analysis by investigating the existence of solutions of abstract incomplete degenerate Cauchy problems

$$(FP_{\alpha_1,\beta_1,\theta}) : \begin{cases} u \in C^\infty((0, \infty) : E), \\ D_-^{\alpha_1} BD_-^{\beta_1} u(s) = e^{i\theta/\gamma}Au(s),\ s > 0, \\ \lim_{s\to 0+} Bu(s) = Cx, \\ \text{the set } \{Bu(s) : s > 0\} \text{ is bounded in } E, \end{cases}$$

and

$$(FP_{\alpha_1,\beta_1,\theta})' : \begin{cases} u \in C^\infty((0,\infty) : E), \\ D_-^{\alpha_1} B D_-^{\beta_1} u(s) = e^{i\theta/\gamma} A u(s), \ s > 0, \\ \lim_{s\to 0+} Bu(s) = Cx, \\ \text{the sets } \{(1+s^{-(q+1)/\gamma})^{-1}u(s) : s > 0\} \\ \ \text{and } \{(1+s^{-(q+1)/\gamma})^{-1}Bu(s) : s > 0\} \text{ are bounded in } E. \end{cases}$$

THEOREM 2.7.4. *Let $0 < \gamma < 1/2$, and let $A \in \mathcal{M}_{B,C,q}$ where $q + \gamma > -1$. Suppose that $CA \subseteq AC$, $CB \subseteq BC$, $\alpha_1 \geqslant 0$, $\beta_1 \geqslant 0$, $\alpha_1 + \beta_1 = 1/\gamma$ and $\theta \in (\gamma\pi - (\pi/2), (\pi/2) - \gamma\pi)$. Then the following holds:*

(i) *Let $q > -1$. Denote by $\Omega_{\theta,\gamma}$, resp. Ψ_γ, the set consisting of those elements $x \in E$ for which $\lim_{s\to 0+} BW_\gamma(se^{i\theta})x = Cx$, resp., $\lim_{z\to 0, z\in\Sigma_{(\pi/2)-\gamma\pi-\varepsilon}} BW_\gamma(z)x = Cx$ for all $\varepsilon \in (0,(\pi/2)-\gamma\pi)$. Then, for every $x \in \Omega_{\theta,\gamma}$, the incomplete abstract degenerate Cauchy problem $(FP_{\alpha_1,\beta_1,\theta})$ has a solution $u(s) = W_\gamma(se^{i\theta})x$, $s > 0$, which can be analytically extended to the sector $\Sigma_{(\pi/2)-\gamma\pi-|\theta|}$. If, additionally, $x \in \Psi_\gamma$, then for every $\delta \in (0,(\pi/2)-\gamma\pi-|\theta|)$ and $j \in \mathbb{N}_0$, we have that the set $\{z^j Bu^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded in E. Assume that the condition* (H) *holds. Then the solution $s \mapsto u(s)$, $s > 0$ can be analytically continued to the sector $\Sigma_{(\pi/2)-\gamma\omega}$; if, in addition to this,*

$$\lim_{z\to 0, z\in\Sigma_{(\pi/2)-\gamma\omega-\varepsilon}} BW_\gamma(z)x = Cx \text{ for all } \varepsilon \in (0,(\pi/2)-\gamma\omega),$$

then for every $\delta \in (0,(\pi/2)-\gamma\omega-|\theta|)$ and $j \in \mathbb{N}_0$, we have that the set $\{z^j Bu^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded in E.

(ii) *Let $-1-\gamma < q \leqslant -1$, $(\lambda B - A)^{-1}CA \subseteq A(\lambda B - A)^{-1}C$, $\lambda \in \rho_C(A,B)$, and let $x \in \overline{D(A)}$. Then the incomplete abstract degenerate Cauchy problem $(FP_{\alpha_1,\beta_1,\theta})'$ has a solution $u(s) = W_\gamma(se^{i\theta})x$, $s > 0$, which can be analytically extended to the sector $\Sigma_{(\pi/2)-\gamma\pi-|\theta|}$. Moreover, for every $\delta \in (0,(\pi/2)-\gamma\pi-|\theta|)$ and $j \in \mathbb{N}_0$, the sets $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}u^{(j)}(z) : z \in \Sigma_\delta\}$ and $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}Bu^{(j)}(z) : z \in \Sigma_\delta\}$ are bounded in E. If, additionally, the condition* (H) *holds, then the above statements remain true with the number $(\pi/2) - \gamma\pi$ replaced by $(\pi/2) - \omega\gamma$.*

PROOF. The proof of theorem almost completely follows from the arguments used in those of [**292**, Theorem 2.9.48] and [**103**, Theorem 3.5(i)/b'], and the only non-trivial thing that should be explained here is the way how we can prove that, for every $\delta \in (0,(\pi/2)-\gamma\pi-|\theta|)$ and $j \in \mathbb{N}_0$, the solution $s \mapsto u(s)$, $s > 0$ has the property that the sets $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}u^{(j)}(z) : z \in \Sigma_\delta\}$ and $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}Bu^{(j)}(z) : z \in \Sigma_\delta\}$ are bounded in E (cf. (ii)). In order to show this, observe first that for each $x \in E$, $z \in \Sigma_{\frac{\pi}{2}-\gamma\pi}$ and $j \in \mathbb{N}$ we have

$$\frac{d^j}{dz^j} W_\gamma(z)x = \frac{1}{2\pi i}\int_0^\infty (-\lambda^\gamma e^{-i\pi\gamma})^j e^{-z\lambda^\gamma e^{-i\pi\gamma}} (\lambda B + A)^{-1} Cx \, d\lambda \tag{197}$$

$$-\frac{1}{2\pi i}\int_0^{\infty}(-\lambda^{\gamma}e^{i\pi\gamma})^j e^{-z\lambda^{\gamma}e^{i\pi\gamma}}(\lambda B+A)^{-1}Cx\,d\lambda.$$

Hence, $B\frac{d^j}{dz^j}W_{\gamma}(z)x=\frac{d^j}{dz^j}BW_{\gamma}(z)x$ ($x\in E$, $z\in\Sigma_{\frac{\pi}{2}-\gamma\pi}$, $j\in\mathbb{N}$), whence it easily follows that $Bu^{(j)}(z)=(Bu)^{(j)}(z)$ ($z\in\Sigma_{\frac{\pi}{2}-\gamma\pi}$, $j\in\mathbb{N}$). Having in mind this equality, the boundedness of sets $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}u^{(j)}(z):z\in\Sigma_{\delta}\}$ and $\{|z|^j(1+|z|^{-(q+1)/\gamma})^{-1}Bu^{(j)}(z):z\in\Sigma_{\delta}\}$ can be proved by using (197) and a direct computation involving the estimates used in the proof of [**291**, Theorem 1.4.15]. $\square$

In this monograph, we will not consider again the assertions of [**292**, Theorem 2.9.39, Theorem 2.9.40, Theorem 2.9.58, Theorem 2.9.60] for degenerate differential equations. In the following theorem, we will focus entirely on the assertion of [**292**, Theorem 2.9.51(iii)].

Theorem 2.7.5. *(cf. [**292**, Theorem 2.9.51(iii)] for non-degenerate case) Suppose that the condition* (H) *holds, as well as that $CA\subseteq AC$ and $CB\subseteq BC$.*

(i) *Let $-1<q<(-1)/2$, and let $(\lambda B-A)^{-1}CA\subseteq A(\lambda B-A)^{-1}C$ ($\lambda\in\rho_C(A,B)$). Then, for every $x\in D(A)$, the incomplete abstract degenerate Cauchy problem*

$$(P_{2,q,B}):\begin{cases}u\in\mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)}:E),\quad Bu\in\mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)}:E),\\ Bu''(z)=\frac{d^2}{dz^2}Bu(z)=Au(z),\quad z\in\Sigma_{(\pi/2)-(\omega/2)},\\ \lim_{z\to0,z\in\Sigma_{\delta}}Bu(z)=Cx,\quad \textit{for every } \delta\in(0,(\pi/2)-(\omega/2)),\\ \textit{the sets } \{(1+|z|^{-(2q+2)})^{-1}u(z):z\in\Sigma_{\delta}\}\quad \textit{and}\\ \{(1+|z|^{-(2q+2)})^{-1}Bu(z):z\in\Sigma_{\delta}\}\quad \textit{are bounded in } E\\ \textit{for every } \delta\in(0,(\pi/2)-(\omega/2)),\end{cases}$$

has a solution $u(z)$ ($z\in\Sigma_{(\pi/2)-(\omega/2)}$). Moreover, for every $\delta\in(0,(\pi/2)-(\omega/2))$ and $j\in\mathbb{N}_0$, we have that the sets $\{|z|^j(1+|z|^{-(2q+2)})^{-1}u^{(j)}(z):z\in\Sigma_{\delta}\}$ and $\{|z|^j(1+|z|^{-(2q+2)})^{-1}(Bu)^{(j)}(z):z\in\Sigma_{\delta}\}$ are bounded in E.

(ii) *Let $-1\geqslant q>(-3)/2$, and let $(\lambda B-A)^{-1}CA\subseteq A(\lambda B-A)^{-1}C$ ($\lambda\in\rho_C(A,B)$). Then, for every $x\in\overline{D(A)}$, the problem $(P_{2,q,B})$ has a solution $u(z)$ ($z\in\Sigma_{(\pi/2)-(\omega/2)}$). Moreover, for every $\delta\in(0,(\pi/2)-(\omega/2))$ and $j\in\mathbb{N}_0$, we have that the sets $\{|z|^j(1+|z|^{-(2q+2)})^{-1}u^{(j)}(z):z\in\Sigma_{\delta}\}$ and $\{|z|^j(1+|z|^{-(2q+2)})^{-1}(Bu)^{(j)}(z):z\in\Sigma_{\delta}\}$ are bounded in E.*

Proof. Suppose first that $q=-1$ and $(\lambda B-A)^{-1}CA\subseteq A(\lambda B-A)^{-1}C$ ($\lambda\in\rho_C(A,B)$); cf. (ii). Let $0<\delta'<\delta<(\pi-\omega)/2$, $1/2>\gamma_0>\delta/(\pi-\omega)$ and $\theta\in(\omega-\pi,(-\delta)/\gamma_0)$. Then, for every $\gamma\in(\gamma_0,1/2)$, we have $\theta\in(\omega-\pi,(-\delta)/\gamma)$ and $\gamma>\delta/(\pi-\omega)$. Suppose that $\varepsilon\in(0,\omega/2)$ is sufficiently small. Define, for every $\gamma\in(\gamma_0,1/2)$ and $x\in E$,

$$F_{\gamma}(\lambda)x:=\begin{cases}\int_0^{\infty}\frac{e^{i\theta\gamma}\sin(\gamma\pi)v^{\gamma}(vB+e^{i\theta}A)^{-1}Cx\,dv}{\pi(\lambda e^{i\theta\gamma}+v^{\gamma}\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}, & \text{if } \arg(\lambda)\in(-\varepsilon,(\pi/2)+\delta),\\ \int_0^{\infty}\frac{e^{-i\theta\gamma}\sin(\gamma\pi)v^{\gamma}(vB+e^{-i\theta}A)^{-1}Cx\,dv}{\pi(\lambda e^{-i\theta\gamma}+v^{\gamma}\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}, & \text{if } \arg(\lambda)\in(-(\pi/2)-\delta,\varepsilon).\end{cases}\tag{198}$$

If $x \in E$ and $\arg(\lambda) \in (-\varepsilon, (\pi/2)+\delta)$, resp., $\arg(\lambda) \in (-(\pi/2)-\delta, \varepsilon)$, then it is very simple to prove that

$$\int_0^\infty e^{-\lambda e^{i\theta\gamma}t} W_{\theta,\gamma}(t)x\,dt = \frac{\sin\gamma\pi}{\pi}\int_0^\infty \frac{v^\gamma (vB+e^{i\theta}A)^{-1}Cx}{(\lambda e^{i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}\,dv,$$

resp.,

$$\int_0^\infty e^{-\lambda e^{-i\theta\gamma}t} W_{-\theta,\gamma}(t)x\,dt = \frac{\sin\gamma\pi}{\pi}\int_0^\infty \frac{v^\gamma (vB+e^{-i\theta}A)^{-1}Cx}{(\lambda e^{-i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}\,dv,$$

where we use the notation from the proof of Theorem 2.7.3. Denote, with a little abuse of notation, $\Gamma_\vartheta := \{te^{i\vartheta} : t \geqslant 0\}$ $(\vartheta \in (-\pi,\pi])$. Since, for every $x \in E$,

$$e^{i\theta\gamma}\int_0^\infty e^{-\lambda e^{i\theta\gamma}t} W_{\theta,\gamma}(t)x\,dt = \int_{\Gamma_{\theta\gamma}} e^{-\lambda v} W_{\theta,\gamma}(ve^{-i\theta\gamma})x\,dv = \int_{\Gamma_{\theta\gamma}} e^{-\lambda v} W_\gamma(v)x\,dv$$

and

$$\begin{aligned} e^{-i\theta\gamma}\int_0^\infty e^{-\lambda e^{-i\theta\gamma}t} W_{-\theta,\gamma}(t)x\,dt &= \int_{\Gamma_{-\theta\gamma}} e^{-\lambda v} W_{-\theta,\gamma}(ve^{i\theta\gamma})x\,dv \\ &= \int_{\Gamma_{-\theta\gamma}} e^{-\lambda v} W_\gamma(v)x\,dv \end{aligned}$$

(cf. the construction of $(W_\gamma(z))_{z\in\Sigma_{(\pi/2)-\omega\gamma}}$, and observe that $|\theta\gamma| \in (\delta, \gamma(\pi-\omega)))$, the Cauchy formula yields that

$$(199)\quad e^{i\theta\gamma}\int_0^\infty e^{-\lambda e^{i\theta\gamma}t} W_{\theta,\gamma}(t)x\,dt = e^{-i\theta\gamma}\int_0^\infty e^{-\lambda e^{-i\theta\gamma}t} W_{-\theta,\gamma}(t)x\,dt, \quad \lambda \in \Sigma_\varepsilon.$$

This, in turn, implies that the function $\lambda \mapsto F_\gamma(\lambda)x$, $\lambda \in \Sigma_{(\pi/2)+\delta}$ is well defined, analytic and bounded by $\mathrm{Const}_{\delta'}|\lambda|^{-1}$ on $\Sigma_{(\pi/2)+\delta'}$ $(x \in E)$. Further on, with the help of [**296**, Theorem 3.4] (cf. also [**27**, Theorem 2.6.1] for the Banach space case) and the uniqueness theorem for the Laplace transform we can prove that

$$W_\gamma(z)x = \frac{1}{2\pi i}\int_{\Gamma_{\delta',z}} e^{\lambda z}F_\gamma(\lambda)x\,d\lambda, \quad x \in E,\ z \in \Sigma_{\delta'},\ \gamma \in (\gamma_0, 1/2),$$

where $\Gamma_{\delta',z} := \Gamma_{\delta',z,1} \cup \Gamma_{\delta',z,2}$, $\Gamma_{\delta',z,1} := \{re^{i((\pi/2)+\delta')} : r \geqslant |z|^{-1}\} \cup \{|z|^{-1}e^{i\vartheta} : \vartheta \in [0, (\pi/2)+\delta']\}$ and $\Gamma_{\delta',z,2} := \{re^{-i((\pi/2)+\delta')} : r \geqslant |z|^{-1}\} \cup \{|z|^{-1}e^{i\vartheta} : \vartheta \in [-(\pi/2)-\delta', 0]\}$ are oriented counterclockwise. Applying the dominated convergence theorem, we get that

$$(200)\quad \begin{aligned} \lim_{\gamma\to\frac{1}{2}-} W_\gamma(z)x &= \frac{e^{i\theta/2}}{2\pi^2 i}\int_{\Gamma_{\delta',z,1}} e^{\lambda z}\int_0^\infty \frac{v^{1/2}(vB+e^{i\theta}A)^{-1}Cx}{\lambda^2 e^{i\theta}+v}\,dv\,d\lambda \\ &\quad + \frac{e^{-i\theta/2}}{2\pi^2 i}\int_{\Gamma_{\delta',z,2}} e^{\lambda z}\int_0^\infty \frac{v^{1/2}(vB+e^{-i\theta}A)^{-1}Cx}{\lambda^2 e^{-i\theta}+v}\,dv\,d\lambda \\ &:= W_{1/2}(z)x, \quad x \in E,\ z \in \Sigma_{\delta'}. \end{aligned}$$

Define $F_{1/2}(\lambda)$ by replacing the number γ with the number $1/2$ in definition of $W_\gamma(\lambda)$. Then, for every $x \in E$, the function $\lambda \mapsto F_{1/2}(\lambda)x$, $\lambda \in \Sigma_{(\pi/2)+\delta}$ is well defined and analytic on $\Sigma_{(\pi/2)+\delta}$ because $F_{1/2}(\lambda)x = \lim_{\gamma\to 1/2-} F_\gamma(\lambda)x$, $\lambda \in \Sigma_{(\pi/2)+\delta}$

and the convergence is uniform on compacts of $\Sigma_{(\pi/2)+\delta}$ (cf. [**296**, Lemma 3.3]). Furthermore, we can argue as in the proof of estimate [**292**, (334)] so as to conclude that for each $q \in \circledast$ there exists $r_q \in \circledast$ such that $q(F_{1/2}(\lambda)x) \leqslant r_q(x) \operatorname{Const}_{\delta'} |\lambda|^{-1}$, $\lambda \in \Sigma_{(\pi/2)+\delta'}$, $x \in E$. Now it becomes apparent that we can define the operator family $(W_{1/2}(z))_{z\in\Sigma_{(\pi-\omega)/2}} \subseteq L(E)$, equicontinuous on any proper subsector of $\Sigma_{(\pi-\omega)/2}$, satisfying additionally that $\lim_{\gamma\to\frac{1}{2}-} W_\gamma(z)x = W_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$, $x \in E$, and that the mapping $z \mapsto W_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$ is analytic for all $x \in E$. Let us prove that for each $x \in \overline{D(A)}$ the function $u(z) := W_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$ is a solution of problem $(P_{2,q,B})$ with $q = -1$. Suppose first that $x \in D(A)$. Then the condition $(\lambda B - A)^{-1}CA \subseteq A(\lambda B - A)^{-1}C$ ($\lambda \in \rho_C(A, B)$) in combination with the closedness of A shows that $W_{1/2}(z)Ax = AW_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$. By the foregoing, we also have that the operator family $(BW_{1/2}(z))_{z\in\Sigma_{(\pi-\omega)/2}} \subseteq L(E)$ is equicontinuous and the mapping $z \mapsto BW_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$ is analytic ($x \in E$), as well as that $(d^2/dz^2)BW_{1/2}(z)x = B(d^2/dz^2)W_{1/2}(z)x$, $z \in \Sigma_{(\pi-\omega)/2}$, $x \in E$ and $BW_{1/2}(z) \in L(E)$, $(d^2/dz^2)BW_{1/2}(z) \in L(E)$, $B(d^2/dz^2)W_{1/2}(z) \in L(E)$ for $z \in \Sigma_{(\pi-\omega)/2}$. Furthermore, the dominated convergence theorem yields that

$$\lim_{\gamma\to 1/2-} BW_\gamma^{(n)}(z) = BW_{1/2}^{(n)}(z)x, \quad z \in \Sigma_{(\pi-\omega)/2},\ x \in E,\ n \in \mathbb{N}_0.$$

By Theorem 2.7.3(vi.2) and definition of modified Liouville right-sided fractional derivatives, we get that

$$\frac{d^2}{ds^2}\int_0^\infty g_{3-\frac{1}{\gamma}}(t)BW_\gamma'(t+s)x\,dt = -W_\gamma(s)Ax, \quad s > 0,\ \gamma \in (\gamma_0, 1/2), \tag{201}$$

i.e.,

$$\int_0^\infty g_{3-\frac{1}{\gamma}}(t)BW_\gamma'''(t+s)x\,dt = -W_\gamma(s)Ax, \quad s > 0,\ \gamma \in (\gamma_0, 1/2).$$

The integration by parts leads us to the following

$$-\int_0^\infty g_{4-\frac{1}{\gamma}}(t)BW_\gamma^{(iv)}(t+s)x\,dt = -W_\gamma(s)Ax, \quad s > 0,\ \gamma \in (\gamma_0, 1/2).$$

Using again the dominated convergence theorem, we obtain by letting $\gamma \to 1/2-$ that

$$\int_0^\infty tBW_{1/2}^{(iv)}(t+s)x\,ds = W_{1/2}(s)Ax, \quad s > 0,$$

which clearly implies after an application of the partial integration that $BW_{1/2}''(s) = AW_{1/2}(s)x$, $s > 0$. By the uniqueness theorem for analytic functions, this equality continues to hold for all $z \in \Sigma_{(\pi-\omega)/2}$. Further on, we can compute $F_\gamma(\lambda)x$ for $\lambda > 0$ ($\gamma \in (\gamma_0, 1/2)$) by plugging $\theta = 0$ in either of two terms appearing in (198). Taking the limit as $\gamma \to 1/2-$, we get that $F_{1/2}(\lambda)x = \pi^{-1}\int_0^\infty v^{1/2}(\lambda^2 + \nu)^{-1}(vB + A)^{-1}Cx\,dv$, $\lambda > 0$. Now we will prove that $BW_{1/2}(z)x - Cx \to 0$ as $z \to 0$, $z \in \Sigma_{(\pi/2)+\delta'}$. Due to Theorem 1.4.10(iii), it suffices to show that $\lim_{\lambda\to+\infty} \lambda BF_{1/2}(\lambda)x = Cx$. This follows by applying the dominated convergence theorem on the integral appearing on the right-hand side of the equality

$$\lambda BF_{1/2}(\lambda)x - Cx = \frac{1}{\pi}\int_0^{\infty}\frac{v^{1/2}\lambda}{\lambda^2+v}\Big[B(vB+A)^{-1}Cx-\frac{Cx}{v}\Big]dv,$$

and by observing that

$$(202)\quad q\Big(B(vB+A)^{-1}Cx-\frac{Cx}{v}\Big)=\frac{1}{v}q((vB+A)^{-1}CAx)=O(v^{-2}),\quad q\in\circledast,\ v>1$$

and

$$(203)\qquad q\Big(B(vB+A)^{-1}Cx-\frac{Cx}{v}\Big)=O(v^{-1}),\quad q\in\circledast,\ v\in(0,1).$$

Keeping in mind the Cauchy integral formula, the proof of (ii) follows immediately in the case that $q=-1$ and $x\in D(A)$. The proof of (ii) in the case that $q=-1$ and $x\in\overline{D(A)}$ follows from the standard limit procedure. If $(-3)/2<q<-1$, then for each $\varepsilon>0$ the condition (H), with the operators A and B replaced respectively by $A+\varepsilon B$ and B, holds with $q=-1$ and the same spectral angle ω; in this case, the proof of (ii) can be deduced by slightly modifying the corresponding part of the proof of [**292**, Theorem 2.9.51(iii.2)]. The proof of (i) is very similar to that of (ii); for the sake of completeness, we will include almost all relevant details. As in the proof of (ii), it will be assumed that $0<\delta'<\delta<(\pi-\omega)/2$, $1/2>\gamma_0>\delta/(\pi-\omega)$, $\theta\in(\omega-\pi,(-\delta)/\gamma_0)$ and that $\varepsilon\in(0,\omega/2)$ is sufficiently small. Due to the proof of [**292**, Theorem 2.9.48] (cf. also the estimate (195)), we have that, for every $\theta'\in(\omega-\pi,\pi-\omega)$ and $\gamma\in(q+1,1/2)$, the mapping $t\mapsto W_{\theta',\gamma}(t)x$, $t\geqslant 0$ $(x\in E)$ satisfies the condition (P1), as well as that

$$\mathcal{L}(W_{\theta',\gamma}(t)x)(\mu)=\frac{\sin\gamma\pi}{\pi}\int_0^{\infty}\frac{v^{\gamma}(vB+e^{i\theta'}A)^{-1}Cx}{(\mu+v^{\gamma}\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}dv,\quad \mu\in\mathbb{C}_+,\ x\in E,$$

and that for each $q\in\circledast$ there exists $r_q\in\circledast$ such that

$$q(\mathcal{L}(W_{\theta',\gamma}(t)x)(\mu))=r_q(x)O\big(|\mu|^{-1}+|\mu|^{\frac{q+1}{\gamma}-1}\big),\ \ \mu\in\Sigma_{\frac{\pi}{2}-\varepsilon},\ x\in E\ \ (\varepsilon\in(0,\pi/2)).$$

This implies that, for every $\gamma\in(\max(\gamma_0,q+1),1/2)$ and $x\in E$, we can define $F_\gamma(\lambda)x$ through (198). If $x\in E$ and $\arg(\lambda)\in(-\varepsilon,(\pi/2)+\delta)$, resp., $\arg(\lambda)\in(-(\pi/2)-\delta,\varepsilon)$, then the equality (410), resp., (411), continues to hold for those values of parameter γ. Furthermore, (199) holds for any $\lambda\in\Sigma_\varepsilon$; hence, the function $\lambda\mapsto F_\gamma(\lambda)x$, $\lambda\in\Sigma_{(\pi/2)+\delta}$ is well defined, analytic and bounded by $\mathrm{Const}_{\delta'}(|\lambda|^{-1}+|\lambda|^{((q+1)/\gamma)-1})$ on $\Sigma_{(\pi/2)+\delta'}$ $(x\in E,\ \gamma\in(\max(\gamma_0,q+1),1/2))$. If $x\in E$, $z\in\Sigma_{\delta'}$, $\gamma\in(\max(\gamma_0,q+1),1/2)$ and $\zeta\geqslant 0$, then we define

$$W_\gamma^{(-\zeta)}(z)x:=\frac{1}{2\pi i}\int_{\Gamma^{\omega'}_{\delta',z}}e^{\lambda z}\lambda^{-\zeta}F_\gamma(\lambda)x\,d\lambda,$$

where $\omega'>0$ is taken arbitrarily, $\Gamma^{\omega'}_{\delta',z}:=\Gamma^{\omega'}_{\delta',z,1}\cup\Gamma^{\omega'}_{\delta',z,2}$, $\Gamma^{\omega'}_{\delta',z,1}:=\{\omega'+re^{i((\pi/2)+\delta')}:r\geqslant|z|^{-1}\}\cup\{\omega'+|z|^{-1}e^{i\vartheta}:\vartheta\in[0,(\pi/2)+\delta']\}$ and $\Gamma^{\omega'}_{\delta',z,2}:=\{\omega'+re^{-i((\pi/2)+\delta')}:r\geqslant|z|^{-1}\}\cup\{\omega'+|z|^{-1}e^{i\vartheta}:\vartheta\in[-(\pi/2)-\delta',0]\}$ are oriented counterclockwise. By Theorem 1.4.10, we get that $\mathcal{L}(W_\gamma^{(-1)}(t)x)(\lambda)=\lambda^{-1}F_\gamma(\lambda)x$, $\lambda>0$, $x\in E$; using this fact, as well as the equality $\mathcal{L}(W_\gamma(t)x)(\lambda)=F_\gamma(\lambda)x$, $\lambda>0$, $x\in E$, the uniqueness theorem for Laplace transform and the uniqueness theorem for analytic functions, we obtain that $(d/dz)W_{\gamma}^{(-1)}(z)x = W_{\gamma}(z)x$, $x \in E$, $z \in \Sigma_{\delta'}$ ($\gamma \in (\max(\gamma_0, q+1), 1/2)$). On the other hand, the dominated convergence theorem yields that $(d/dz)W_{\gamma}^{(-1)}(z)x = W_{\gamma}^{(0)}(z)x$, so that $W_{\gamma}^{(0)}(z)x = W_{\gamma}(z)x$, $x \in E$, $z \in \Sigma_{\delta'}$ ($\gamma \in (\max(\gamma_0, q+1), 1/2)$). Define $F_{1/2}(\lambda)$ ($\lambda \in \Sigma_{(\pi/2)+\delta}$) and $(W_{1/2}(z))_{z\in\Sigma_{\delta'}} \subseteq L(E)$ in exactly the same vein as in the proof of (ii). As before, we have that, for every $x \in E$, the function $\lambda \mapsto F_{1/2}(\lambda)x$, $\lambda \in \Sigma_{(\pi/2)+\delta}$ is well defined and analytic on $\Sigma_{(\pi/2)+\delta}$ because $F_{1/2}(\lambda)x = \lim_{\gamma\to 1/2-} F_{\gamma}(\lambda)x$, $\lambda \in \Sigma_{(\pi/2)+\delta}$ and the convergence is uniform on compacts of $\Sigma_{(\pi/2)+\delta}$; furthermore, for each $q \in \circledast$ there exists $r_q \in \circledast$ such that $q(F_{1/2}(\lambda)x) \leqslant r_q(x)\,\mathrm{Const}_{\delta'}(|\lambda|^{-1} + |\lambda|^{2q+1})$, $\lambda \in \Sigma_{(\pi/2)+\delta'}$, $x \in E$. Then the limit equality (200) continues to hold (with the replacements of contours $\Gamma_{\delta',z,1}$, $\Gamma_{\delta',z,2}$ with $\Gamma^{\omega'}_{\delta',z,1}$, $\Gamma^{\omega'}_{\delta',z,2}$, respectively), the operator families $(W_{1/2}(z))_{z\in\Sigma_{(\pi-\omega)/2}} \subseteq L(E)$ and $(BW_{1/2}(z))_{z\in\Sigma_{(\pi-\omega)/2}} \subseteq L(E)$ can be defined in the very obvious way, and we have that $\lim_{\gamma\to 1/2-} BW_{\gamma}^{(n)}(z)x = BW_{1/2}^{(n)}(z)x$ for all $x \in E$, $z \in \Sigma_{(\pi-\omega)/2}$ and $n \in \mathbb{N}_0$. Arguing as in the proof of [**292**, Theorem 2.9.51(iii.1)], we can show that, for every $j \in \mathbb{N}_0$, the families $\{|z|^j(1+|z|^{-(2q+2)})^{-1}W_{1/2}^{(j)}(z) : z \in \Sigma_{\delta'}\} \subseteq L(E)$ and $\{|z|^j(1+|z|^{-(2q+2)})^{-1}(BW_{1/2})^{(j)}(z) : z \in \Sigma_{\delta'}\} \subseteq L(E)$ are equicontinuous. Suppose now that $x \in D(A)$. Then $AW_{1/2}(z)x = W_{1/2}(z)Ax$, $z \in \Sigma_{(\pi-\omega)/2}$ and for each $q \in \circledast$ there exist $c_{q,\delta'} > 0$ and $r_q \in \circledast$ such that (cf. (202)–(203) with $\theta = 0$):

$$
\begin{aligned}
(204)\quad & q\Big(\frac{e^{i\theta/2}}{\pi}\int_0^{\infty}\frac{v^{1/2}B(vB+e^{i\theta}A)^{-1}Cx}{\lambda^2 e^{i\theta}+v}\,dv - \frac{Cx}{\lambda}\Big) \\
&\leqslant c_{q,\delta'}[r_q(x)+r_q(Ax)]\Big[\int_0^1 \frac{v^{1/2}}{|\lambda|^2+v}\Big(\frac{1}{v}+v^q\Big)dv + \int_1^{\infty}\frac{v^{1/2}}{|\lambda|^2+v}\frac{1}{v}\Big(\frac{1}{v}+v^q\Big)dv\Big] \\
&\leqslant c_{q,\delta'}[r_q(x)+r_q(Ax)]|\lambda|^{-1}, \quad \arg(\lambda) \in (-\varepsilon, (\pi/2)+\delta').
\end{aligned}
$$

If $\arg(\lambda) \in (-((\pi/2)+\delta'), -\varepsilon)$, then we can estimate the term

$$
q\Big(\frac{e^{-i\theta/2}}{\pi}\int_0^{\infty}\frac{v^{1/2}B(vB+e^{-i\theta}A)^{-1}Cx}{\lambda^2 e^{-i\theta}+v}\,dv - \frac{Cx}{\lambda}\Big)
$$

in the same way as above, from which we may conclude that $q(BF_{1/2}(\lambda)Ax) = (r_q(x)+r_q(Ax))O(|\lambda|^{-1})$, $\lambda \in \Sigma_{(\pi/2)+\delta'}$. A similar line of reasoning as performed in the case that $q = -1$ and $x \in D(A)$ enables us to deduce that

$$
(205)\qquad \lambda\Big[\frac{e^{i\theta/2}}{\pi}\int_0^{\infty}\frac{v^{1/2}B(vB+e^{i\theta}A)^{-1}Cx}{\lambda^2 e^{i\theta}+v}\,dv - \frac{Cx}{\lambda}\Big] \to 0, \quad \lambda \to +\infty.
$$

Applying [**296**, Theorem 3.4] and (205), we get that $\mathcal{L}(BW_{1/2}(t)x)(\lambda) = BF_{1/2}(\lambda)x$, $\lambda > 0$ and $\lim_{z\to 0, z\in\Sigma_{\delta'}} BW_{1/2}(z)x = Cx$. In the final part of proof of Theorem 2.7.3, we have proved that $B\frac{d^j}{dz^j}W_{\gamma}(z)x = \frac{d^j}{dz^j}BW_{\gamma}(z)x$ ($z \in \Sigma_{\frac{\pi}{2}-\gamma\pi}$, $j \in \mathbb{N}$). As a consequence of this equality and the Cauchy integral formula, we have that $q(B\frac{d^j}{dt^j}W_{\gamma}(t)x) = O(t^{-j})$, $t > 0$. On the other hand, the proof of [**292**, Theorem 2.9.48] shows that the equation (201) continues to hold, which simply implies that $BW''_{1/2}(z)x = AW_{1/2}(z)x,\ z \in \Sigma_{(\pi-\omega)/2}$. The proof of theorem is thereby completed. ☐

The main problem in application of Theorem 2.7.3–Theorem 2.7.5 to incomplete abstract degenerate differential equations lies in the fact that, in many concrete situations, any of conditions $(\lambda B - A)^{-1}CA \subseteq A(\lambda B - A)^{-1}C$ ($\lambda \in \rho_C(A, B)$) and $(\lambda B - A)^{-1}CB \subseteq B(\lambda B - A)^{-1}C$ ($\lambda \in \rho_C(A, B)$) is not satisfied. Suppose, for example, that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded set with C^∞-boundary, $E := L^2(\Omega)$, $A := \Delta$ with the Dirichlet boundary conditions, $a(x) \in L^\infty(\Omega)$, $a(x) \leqslant 0$ on $\bar{\Omega}$, $a(x) < 0$ almost everywhere in Ω, and $Bf(x) := a(x)^{-1}f(x)$ with maximal domain (cf. [**199**, Example 3.8, pp. 81-83]). Then $B^{-1} \in L(E)$ and the operator AB^{-1} is closed. Suppose, in addition, that $a^{-1} \in L^r(\Omega)$ for some $r \geqslant 2$ (resp., $r > 2$, $r > n$), if $n = 1$ (resp., $n = 2$, $n \geqslant 3$). Then it has been proved in the above-mentioned example that the condition [**292**, (HQ), p. 207] holds with the number $\omega = 0$, $C = I$, $m = -1+(n/2r) \in (-1, (-1/2))$, and with the operator A replaced by AB^{-1} therein. This implies by [**292**, Theorem 2.9.51(i.3)] that the operator $-(AB^{-1})_{1/2}$ is the integral generator of an exponentially bounded, analytic (n/r)-times integrated semigroup of angle $\pi/2$ on E, and that the abstract incomplete Cauchy problem [**292**, $(P_{2,m})$], which corresponds to the equation $u_{tt}(t, x) = \Delta\{a(x)u(t, x)\}$, $t > 0$, has a unique solution that is analytically extensible to the right half plane. It is clear that Theorem 2.7.5 cannot be applied here directly, by regarding the problem $u_{tt}(t, x) = \Delta\{a(x)u(t, x)\}$, $t > 0$ as a problem of the form $(P_{2,q,B})$ dealt with above.

EXAMPLE 2.7.6. Assume that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E; set $\mathbb{A} := (A_1, \dots, A_n)$. Assume, further, that $0 < \delta < 2$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\delta)}$ (resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\delta)}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and

$$\sup_{x\in\mathbb{R}^n} \operatorname{Re}\Bigl(\Bigl(\frac{P_1(x)}{P_2(x)}\Bigr)^{1/\delta}\Bigr) \leqslant 0. \tag{206}$$

Define

$$R_\delta(t) := \Bigl(E_\delta\Bigl(t^\delta \frac{P_1(x)}{P_2(x)}\Bigr)(1 + |x|^2)^{-\beta/2}\Bigr)(\mathbb{A}),\ t \geqslant 0,\ G_\delta(t) := \overline{P_2(\mathbb{A})}^{-1} R_\delta(t),\ t \geqslant 0,$$

$A' := \overline{P_1(\mathbb{A})}$, $B' := \overline{P_2(\mathbb{A})}$ and $C := R_\delta(0)$. Then we know that $(G_\delta(t))_{t\geqslant0}$ is an exponentially equicontinuous (g_δ, C)-regularized resolvent family generated by A', B', as well as that $\|R_\delta(t)\| + \|G_\delta(t)\| = O(1 + t^{\max(1,\delta)n/2})$, $t \geqslant 0$ ($\|R_\delta(t)\| + \|G_\delta(t)\| = O(1 + t^{\max(1,\delta)n|\frac{1}{p}-\frac{1}{2}|})$, $t \geqslant 0$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$),

$$\lambda^{\delta-1}(\lambda^\delta B' - A')^{-1}Cx = \int_0^\infty e^{-\lambda t}G_\delta(t)x\,dt, \quad \operatorname{Re}\lambda > 0,\ x \in E,$$

and

$$\lambda^{\delta-1}B'(\lambda^\delta B' - A')^{-1}Cx = \int_0^\infty e^{-\lambda t}B'G_\delta(t)x\,dt = \int_0^\infty e^{-\lambda t}R_\delta(t)x\,dt, \quad \operatorname{Re}\lambda > 0,\ x \in E.$$

This implies

$$\|(\lambda B' - A')^{-1}C\| + \|B'(\lambda B' - A')^{-1}C\| = O\bigl(|\lambda|^{-1} + |\lambda|^{-1-\frac{\max(1,\delta)n}{2\delta}}\bigr), \ \lambda \in \Sigma_{\delta\pi/2},$$

in the case of a general space E, and

$$\|(\lambda B'-A')^{-1}C\|+\|B'(\lambda B'-A')^{-1}C\| = O\bigl(|\lambda|^{-1}+|\lambda|^{-1-\frac{\max(1,\delta)n|\frac{1}{p}-\frac{1}{2}|}{\delta}}\bigr), \ \lambda \in \Sigma_{\delta\pi/2},$$

in the case that $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Setting $A := -A'$ and $B := B'$, we have that the condition (H) holds wih $\omega = \pi - (\delta\pi/2)$ as well as that $C^{-1}AC = A$, $C^{-1}BC = B$, $(\lambda B-A)^{-1}CA \subseteq A(\lambda B-A)^{-1}C$ and $(\lambda B-A)^{-1}CB \subseteq B(\lambda B - A)^{-1}C$ $(\lambda \in \rho_C(A, B))$, which implies that Theorem 2.7.3(vi.2), resp., Theorem 2.5.4(ii) is susceptible to applications in the case that $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$ satisfying

$$\frac{\max(1,\delta)n\bigl|\frac{1}{p}-\frac{1}{2}\bigr|}{\delta} < \gamma < 1/2, \text{ resp., } \frac{\max(1,\delta)n\bigl|\frac{1}{p}-\frac{1}{2}\bigr|}{\delta} < 1/2.$$

Before proceeding further, we would like to mention in passing that the operator B is injective and that the operator $B^{-1}A$ is closable because $\rho_{C'}(B^{-1}A) \neq \emptyset$ $(C' := C(B + A)^{-1}C)$, which follows from the equality $(\lambda + B^{-1}A)^{-1}C(B + A)^{-1}Cx = B(\lambda B + A)^{-1}C(B + A)^{-1}Cx$, $\lambda > 0$, $x \in E$ (cf. (196)). It is also worth noting that the operator AB^{-1} is closable because $\rho_C(AB^{-1}) \neq \emptyset$; this is a consequence of the equality $(\lambda + AB^{-1})^{-1}Cx = B(\lambda B + A)^{-1}Cx$, $\lambda > 0$, $x \in E$. Now we will prove the uniqueness of solution of problem $(P_{2,q,B})$ in our particular case; recall that $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$ satisfying $\zeta := \frac{\max(1,\delta)n|\frac{1}{p}-\frac{1}{2}|}{\delta} < 1/2$. Denote by $\mathcal{M}_{C,q,\omega}$ the class which consists of all closed linear operators D acting on E such that $\mathbb{C} \smallsetminus \overline{\Sigma_\omega} \subseteq \rho_C(D)$, $DC \subseteq CD$ and the family

$$\bigl\{(|\lambda|^{-1} + |\lambda|^q)^{-1}(\lambda - D)^{-1}C : \lambda \notin \Sigma_{\omega'}\bigr\}$$

is equicontinuous for every $\omega < \omega' < \pi$; here $q \in \mathbb{R}$ and $\omega \in [0, \pi)$. Then we have $\zeta \in [0, 1/2)$ and our previous examinations show that $\overline{AB^{-1}} \in \mathcal{M}_{C,-1-\zeta,\pi-(\delta\pi/2)}$. If $z \mapsto u(z)$, $z \in \Sigma_{(\pi-\omega)/2}$ is a solution of problem $(P_{2,q,B})$ with $x = 0$, then $v(z) := Bu(z)$, $z \in \Sigma_{(\pi-\omega)/2}$ is a solution of problem

$$(P_{2,q}) : \begin{cases} v \in \mathcal{A}(\Sigma_{(\pi/2)-(\omega/2)} : E), \\ \frac{d^2}{dz^2}v(z) = \overline{AB^{-1}}v(z), \ z \in \Sigma_{(\pi/2)-(\omega/2)}, \\ \lim_{z\to 0, z\in\Sigma_\delta} v(z) = 0, \ \text{for every } \delta \in (0, (\pi/2) - (\omega/2)), \\ \text{the set } \{(1 + |z|^{-(2q+2)})^{-1}v(z) : z \in \Sigma_\delta\} \text{ is bounded in } E \\ \text{for every } \delta \in (0, (\pi/2) - (\omega/2)). \end{cases}$$

An application of [**292**, Theorem 2.9.51(iii.2)] yields $v(z) = 0$, $z \in \Sigma_{(\pi-\omega)/2}$, so that $u(z) = 0$, $z \in \Sigma_{(\pi-\omega)/2}$ by the injectiveness of B. In the remainder of this example, we will provide certain applications of Theorem 2.5.3. It will be assumed that $0 < c < b < 1$, $0 < \zeta \leqslant 1$, $1/(2 - \zeta) > c/\zeta$, $\sigma > 0$ and $\varsigma > 0$; we define the operators A and B in the same way as above but, instead of estimate (206), we assume that

$$\text{dist}\bigl(\{-P_1(x)P_2(x)^{-1} : x \in \mathbb{R}^n\}, \Pi^{\zeta}_{c,\sigma,\varsigma}\bigr) > 0. \tag{207}$$

Then there exist sufficiently large numbers $\beta' \geqslant 0$ and $\nu \geqslant 0$ (the proofs of [**292**, Theorem 2.5.2] and Theorem 2.2.20 can give more detailed and accurate information about β' and ν; we leave the reader to make this precise) such that

$$\Big(\frac{1}{\lambda^{\zeta}P_2(x)+P_1(x)}(1+|x|^2)^{-\beta'/2}\Big)(\mathbb{A}) = (\lambda^{\zeta}B+A)^{-1}C, \quad \lambda \in \Pi_{c,\sigma,\varsigma},$$
$$\Big(\frac{P_2(x)}{\lambda^{\zeta}P_2(x)+P_1(x)}(1+|x|^2)^{-\beta'/2}\Big)(\mathbb{A}) = B(\lambda^{\zeta}B+A)^{-1}C, \quad \lambda \in \Pi_{c,\sigma,\varsigma},$$

and the operator families $\{(1+|\lambda|)^{-\nu}(\lambda^{\zeta}B+A)^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$ and $\{(1+|\lambda|)^{-\nu}B(\lambda^{\zeta}B+A)^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$ are both equicontinuous and strongly continuous ($C := ((1+|x|^2)^{-\beta'/2})(\mathbb{A})$), so that Theorem 2.5.3 can be applied. Although the equation (185) of Remark 2.5.5 holds in our concrete situation, it is our duty to say that Theorem 2.5.3–Remark 2.5.5 certainly have some disadvantages in the degenerate case because it is very difficult to say whether an element $x \in E$ belongs to the space $E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A)$ or not, with the exception of some very special cases. Suppose now that the operators A_k and B_k are defined by $A_k := -\overline{P_{1,k}(\mathbb{A})}$, $B_k := \overline{P_{2,k}(\mathbb{A})}$, and that the estimate (207) holds with the polynomials $P_1(x)$ and $P_2(x)$ replaced respectively with the polynomials $P_{1,k}(x)$ and $P_{2,k}(x)$. Then Theorem 2.5.3 can be applied to a large class of multi-term (non-)degenerate differential equations of the form (168), whose P_λ looks like

$$P_\lambda = (\lambda^{\zeta}B_1 + A_1)(\lambda^{\zeta}B_2 + A_2)\dots(\lambda^{\zeta}B_k + A_k).$$

The choice of regularizing operator C is essentially the same as above but we must eventually increase the value of β'.

The analysis contained in Example 2.7.6 shows that there exists at most one solution of problem $(P_{2,q,B})$ stated in Theorem 2.5.4(i), resp., Theorem 2.5.4(ii), provided that the operator B is injective, the operator AB^{-1} is closable and $\overline{AB^{-1}} \in \mathcal{M}_{C,q,\omega}$ for some $q \in (-1,(-1)/2)$, resp., $q \in ((-3/2),-1]$, and $\omega \in [0,\pi)$. It is not clear, however, in which other cases the uniqueness of solutions of problem $(P_{2,q,B})$ can be proved ($B \neq I$).

2.8. Abstract degenerate fractional differential equations in locally convex spaces with a σ-regular pair of operators

The results presented in this section are obtained jointly with V. E. Fedorov [**214**]. Suppose that X is an SCLCS and A is a closed linear operator acting on X. Let us recall that the regular resolvent set of A, $\rho^r(A)$ shortly, is defined as the union of those complex numbers $\lambda \in \rho(A)$ for which $(\lambda - A)^{-1} \in R(X)$, where $R(X)$ denotes the set of all regular bounded linear operators $D \in L(X)$.

If $D \in R(X)$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([0,\tau) : X)$ and $x_k \in X$ $(0 \leqslant k \leqslant m-1)$, then we have proved in [**214**, Theorem 1] that the unique solution of the non-degenerate differential equation

$$\mathbf{D}_t^{\alpha}u(t) = Du(t) + f(t), \quad t \in [0,\tau); \ u^{(k)}(0) = x_k \ \ (0 \leqslant k \leqslant m-1)$$

is given by

$$(208)\quad u(t) = \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(t^\alpha D)x_k + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(D(t-s)^\alpha)f(s)ds,\ t \in [0,\tau),$$

where the operator $E_{\alpha,\nu}(t^\alpha D)$ is defined in the very obvious way ($0 < \tau \leqslant \infty$, $\nu \in \mathbb{R}$).

In the remainder of this section, we assume that Y is an SCLCS as well as that A and B are two closed linear operators acting between the spaces X and Y. Set, as before, $R_\lambda^B(A) := (\lambda B - A)^{-1}B$ and $L_\lambda^B(A) := B(\lambda B - A)^{-1}$. Then the regular B-resolvent set of the operator A, $\rho_r^B(A)$ for short, is defined to be

$$\rho_r^B(A) := \bigl\{\lambda \in \mathbb{C} : (\lambda B - A)^{-1} \in L(Y,X),\ R_\lambda^B(A) \in L(X),\ L_\lambda^B(A) \in L(Y)\bigr\}.$$

The following proposition holds true:

PROPOSITION 2.8.1. *Suppose that $D(A) \cap D(B)$ is sequentially dense in X. Then we have:*

(i) *$\rho_r^B(A)$ is an open subset of $\mathbb{C}$.*
(ii) *If $\rho_r^B(A) \neq \emptyset$, then the mappings $\lambda \mapsto (\lambda B - A)^{-1}$, $\lambda \in \rho_r^B(A)$, $\lambda \mapsto L_\lambda^B(A)$, $\lambda \in \rho_r^B(A)$ and $\lambda \mapsto L_\lambda^B(A)$, $\lambda \in \rho_r^B(A)$ are strongly analytic.*
(iii) *If $\rho_r^B(A) \neq \emptyset$, then the constants of regularity of operators $L_\lambda^B(A)$ and $R_\lambda^B(A)$ are continuously dependent of parameter $\lambda \in \rho_r^B(A)$.*

It is said that the operator A is (B,σ)-regular iff there exists a finite number $a > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| > a$, we have $\lambda \in \rho_r^B(A)$. If so, then we also say that a pair of operators (A,B) is σ-regular.

Suppose that the pair of operators (A,B) is σ-regular and $R > a$. Denote by γ the positively oriented contour $\gamma = \{\lambda \in \mathbb{C} : |\lambda| = R\}$. Set

$$Px := \frac{1}{2\pi i}\int_\gamma R_\lambda^B(A)x\, d\lambda \text{ and } Qy := \frac{1}{2\pi i}\int_\gamma L_\lambda^B(A)y\, d\lambda \quad (x \in X,\ y \in Y).$$

Then it is very simple to prove that $P \in L(X)$ and $Q \in L(Y)$ are projections. Define $X_0 := N(P)$, $Y_0 := N(Q)$, $X_1 := R(P)$ and $Y_1 := R(Q)$. Clearly, $X = X_0 \oplus X_1$ and $Y = Y_0 \oplus Y_1$. By A_k (B_k) we denote the restriction of operator A (B) to the space X_k, $k = 0,1$. By $\rho_{r,k}^B(A)$ we denote the regular B_k-resolvent set of the operator A_k, $k = 0,1$.

We have the following:

PROPOSITION 2.8.2. **[214]** *Suppose that the pair of operators (A,B) is σ-regular. Then the following holds:*

(i) *For every $x \in X$, we have $Px \in D(A)$.*
(ii) *$B_k \in L(X_k, Y_k)$, $k = 0,1$.*
(iii) *A_0 is a closed linear operator acting between the spaces X_0 and Y_0; $A_1 \in L(X_1, Y_1)$.*
(iv) *There exists operator $B_1^{-1} \in L(Y_1, X_1)$.*
(v) *$\rho_{r,0}^B(A) = \mathbb{C}$ and, in particular, there exists operator $A_0^{-1} \in L(Y_0, X_0)$.*
(vi) *The operators $S_1 := B_1^{-1}A_1$ and $T_1 := A_1B_1^{-1}$ are regular.*

Set $H := A_0^{-1}B_0$. If there exists $p \in \mathbb{N}_0$ such that $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$, then we say that the operator A is (B,p)-regular or, equivalently, that the pair of operators (A,B) is p-regular (the symbol $\mathbb{O}$ denotes the null operator in X_0). It is well known (see e.g. [**509**, p. 92]) that the existence or non-existence of number $p \in \mathbb{N}_0$ with the above property characterizes the behaviour of relative resolvent $(\lambda B - A)^{-1}$ at the point ∞. Assume that $Bx_0 = 0$ for some $x_0 \in X \setminus \{0\}$ (then we simply say that x is an eigenvector of B). An ordered set of vectors $\{x_0, x_1, \dots\}$ is called a chain of A-adjoint vectors of x_0 iff $x_k \notin N(B)$ for $k \in \mathbb{N}$, and

$$Bx_{k+1} = Ax_k, \quad k = 0, 1, \dots.$$

Call the index of a vector in the chain (starting from 0) its height, and the eigenvectors the A-adjoint vectors of height 0. It is well known that a vector $x \neq 0$ is an A-adjoint vector of height at most l of B iff $(R_\lambda^B(A))^{l+1}x = 0$ for some (all) $\lambda \in \rho_r^B(A)$. Furthermore, the operator A is $(B,0)$-regular iff $N(B) = X_0$; if this is the case, then $R(B) = Y_1$ and, for every $x_0 \in N(B) \setminus \{0\}$, we have that $x_0 \notin D(A)$ or $Ax_0 \notin R(B)$. The (B,p)-regularity of operator A has been profiled in [**214**, Theorem 3(i)].

Suppose that the pair of operators (A,B) is σ-regular, $\alpha > 0$ and $\beta > 0$. Set

$$U_{\alpha,\beta}(t)x := \frac{1}{2\pi i}\int_\gamma E_{\alpha,\beta}(\lambda t^\alpha) R_\lambda^B(A)x \, d\lambda, \quad x \in X,$$

and

$$V_{\alpha,\beta}(t)y := \frac{1}{2\pi i}\int_\gamma E_{\alpha,\beta}(\lambda t^\alpha) R_\lambda^B(A)y \, d\lambda, \quad y \in Y.$$

Then the following theorem holds:

THEOREM 2.8.3. (i) *$(U_{\alpha,\beta}(t))_{t \geqslant 0} \subseteq L(X)$ and $(V_{\alpha,\beta}(t))_{t\geqslant 0} \subseteq L(Y)$ are strongly continuous.*

(ii) *$U_{\alpha,\beta}(t)P = PU_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)$ and $V_{\alpha,\beta}(t)Q = QV_{\alpha,\beta}(t) = V_{\alpha,\beta}(t)$ for all $t \geqslant 0$.*

(iii) *$X_0 \subseteq N(U_{\alpha,\beta}(t))$, $R(U_{\alpha,\beta}(t)) \subseteq X_1$, $Y_0 \subseteq N(V_{\alpha,\beta}(t))$ and $R(V_{\alpha,\beta}(t)) \subseteq Y_1$ for all $t \geqslant 0$.*

(iv) *$U_{\alpha,\beta}(t) = E_{\alpha,\beta}(B_1^{-1}A_1 t^\alpha)$ and $V_{\alpha,\beta}(t) = E_{\alpha,\beta}(A_1B_1^{-1}t^\alpha)$ for all $t \geqslant 0$.*

PROOF. The proof of (i) is trivial and therefore omitted. Let $t \geqslant 0$ and $x \in X$. By the Fubini theorem, we have that $U_{\alpha,\beta}(t)Px = PU_{\alpha,\beta}(t)x$. In order to see that $PU_{\alpha,\beta}(t)x = U_{\alpha,\beta}(t)x$, let $\gamma_1 = \{z \in \mathbb{C} : |z| = R+1\}$ be positively oriented, where $R > a$. Then the residue theorem and the resolvent equation together imply

$$\begin{aligned}
PU_{\alpha,\beta}(t)x &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma_1} R_\lambda^B(A)R_\mu^B(A)E_{\alpha,\beta}(t^\alpha\mu)x \, d\mu \, d\lambda \\
&= \frac{1}{(2\pi i)^2}\int_\gamma R_\mu^B(A)E_{\alpha,\beta}(t^\alpha\mu)\left[\int_{\gamma_1}\frac{x\, d\lambda}{\lambda-\mu}\right]d\mu \\
&\quad - \frac{1}{(2\pi i)^2}\int_{\gamma_1} R_\lambda^B(A)\left[\int_\gamma \frac{E_{\alpha,\beta}(t^\alpha\mu)x}{\lambda-\mu}d\mu\right]d\lambda \\
&= U_{\alpha,\beta}(t)x - 0 = U_{\alpha,\beta}(t)x.
\end{aligned}$$

The proof of second equality in (ii) can be deduced analogically; (iii) is an immediate consequence of (ii). The proof of (iv) essentially follows from the following calculus ($S \equiv A_1^{-1}B_1$):

$$\begin{aligned}
U_{\alpha,\beta}(t)x &= PU_{\alpha,\beta}(t)x \\
&= \frac{1}{2\pi i}\int_\gamma R_\mu^{B_1}(A_1)PE_{\alpha,\beta}(t^\alpha\mu)x\,d\mu \\
&= \frac{1}{2\pi i}\int_\gamma (\mu I - S)^{-1}PE_{\alpha,\beta}(t^\alpha\mu)x\,d\mu \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^\infty \mu^{-k-1}S^kP\sum_{n=0}^\infty \frac{t^{\alpha n}\mu^n}{\Gamma(\alpha n+\beta)}x\,d\mu \\
&= \sum_{k=0}^\infty \frac{t^{\alpha k}S^k}{\Gamma(\alpha k+\beta)}Px = E_{\alpha,\beta}(t^\alpha S)Px. \qquad \square
\end{aligned}$$

Along with the problem

$$(\mathrm{DF}) : \begin{cases} \mathbf{D}_t^\alpha Bu(t) = Au(t), \quad t \geqslant 0, \\ u(0) = x_0,\ u^{(k)}(0) = 0, \quad 1 \leqslant k \leqslant m-1, \end{cases}$$

we examine the following one:

$$(\mathrm{DF})_1 : \begin{cases} \mathbf{D}_t^\alpha L_r^B(A)v(t) = L_r^B(A)v(t), \quad t \geqslant 0, \\ v(0) = (\zeta B - A)^{-1}x_0,\ v^{(k)}(0) = 0, \quad 0 \leqslant k \leqslant m-1, \end{cases}$$

where $\zeta \in \rho_r^B(A)$. The solution $u(t)$ of (DF) and solution $v(t)$ of $(\mathrm{DF})_1$ are linked by the relation $u(t) = (\zeta B - A)^{-1}v(t)$, $t \geqslant 0$. Suppose that $x \in X$ and $y \in Y$. Set $u(t) := U_{\alpha,1}(t)x$, $t \geqslant 0$ and $v(t) := V_{\alpha,1}(t)y$, $t \geqslant 0$. Then it can be easily seen that

$$Bu(t) = \frac{1}{2\pi i}\int_\gamma \mu AR_\mu^B(A)E_\alpha(t^\alpha\mu)x\,d\mu,$$

which is a continuous function for $t \geqslant 0$. Evidently, the function $t \mapsto Au(t)$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable for $t \geqslant 0$. Making use of the Cauchy theorem and the closedness of operator B, we have ($m = \lceil\alpha\rceil$):

$$\begin{aligned}
\mathbf{D}_t^\alpha Au(t) &= \frac{1}{2\pi i}\int_\gamma AR_\mu^B(A)J_t^{m-\alpha}\sum_{n=1}^\infty \frac{\mu^n\alpha^{\alpha n-m}}{\Gamma(\alpha n-m+1)}x\,d\mu \\
&= \frac{1}{2\pi i}\int_\gamma AR_\mu^B(A)\sum_{n=1}^\infty \frac{\mu^n\alpha^{\alpha(n-1)}}{\Gamma(\alpha(n-1)+1)}x\,d\mu \\
&= Bu(t), \quad t \geqslant 0.
\end{aligned}$$

This shows that the mapping $t \mapsto u(t)$, $t \geqslant 0$ is a solution of problem (DF) considered without initial conditions; we can similarly prove that the mapping $t \mapsto v(t)$, $t \geqslant 0$, is a solution of problem $(\mathrm{DF})_1$ considered without initial conditions; cf. [**214**, Lemma 4]. Since no confusion seems likely, the above problems will be denoted by the same symbols (DF) and $(\mathrm{DF})_1$.

DEFINITION 2.8.4. A set $\mathcal{P} \subseteq X$ is said to be the phase space of problem (DF) iff the following holds:

(i) for any strong solution $u(t)$ of (DF), we have that $u(t) \in \mathcal{P}$ for all $t \geqslant 0$,
(ii) for any $x_0 \in \mathcal{P}$, there exists a unique strong solution of (DF).

In order to profile the phase space of problem (DF), we need the following technical result.

LEMMA 2.8.5. *Suppose that the operator G is nilpotent of order $p \in \mathbb{N}_0$. Let the function $t \mapsto g(t)$, $t \geqslant 0$ satisfy that $(\mathbf{D}_t^\alpha G)^k g \in C([0,\infty) : X)$ for all $k \in \mathbb{N}_p^0$. Then there exists a unique solution of the following time-fractional initial value problem*

$$\mathbf{D}_t^\alpha G z(t) = z(t) + g(t), \quad t \geqslant 0, \tag{209}$$

and the solution $z(t)$ is given by the formula

$$z(t) = -\sum_{k=0}^{p} (\mathbf{D}_t^\alpha G)^k g(t), \quad t \geqslant 0. \tag{210}$$

PROOF. Suppose that $z(t)$ is a solution of (209). Applying the operator G on the both sides of this equation, we get that $G\mathbf{D}_t^\alpha G z(t) = Gz(t) + Gg(t)$, $t \geqslant 0$. Since the Caputo fractional derivative of the right hand side of the last equality exists, it is clear that the same holds for the left hand side. Performing now the operator $\mathbf{D}_t^\alpha$, we obtain $(\mathbf{D}_t^\alpha G)^2 z(t) = \mathbf{D}_t^\alpha G z(t) + \mathbf{D}_t^\alpha G g(t) = z(t) + g(t) + \mathbf{D}_t^\alpha G g(t)$, $t \geqslant 0$. Repeating this procedure, it readily follows that $(\mathbf{D}_t^\alpha G)^{p+1} z(t) = z(t) + \sum_{k=0}^{p} (\mathbf{D}_t^\alpha G)^k g(t)$, $t \geqslant 0$. Taking into account the nilpotency of the operator G, it is checked at once that $(\mathbf{D}_t^\alpha G)^{p+1} z(t) = (\mathbf{D}_t^\alpha)^{p+1} G^{p+1} z(t) = 0$, $t \geqslant 0$. This implies that any solution of (209) has the form (210). Direct computation shows that the function $z(t)$, given by (210), is a solution of (209), finishing the proof of lemma. $\square$

In [**214**, Theorem 5], we have proved that the σ-regularity of pair (A, B) for some non-negative integer p implies that the phase space of problem (DF), resp., $(\mathrm{DF})_1$, coincides with X_1 (Y_1). We will include all relevant details of proof for problem (DF): Suppose that $u(t)$ is a solution of (DF). Set $u_0(t) := (I - P)u(t)$, $t \geqslant 0$ and $u_1(t) := Pu(t)$, $t \geqslant 0$. Then

$$\begin{aligned} \mathbf{D}_t^\alpha H u_0(t) &= u_0(t), \quad t \geqslant 0 \ ; \ H_0 \equiv B_0^{-1} A_0, \\ \mathbf{D}_t^\alpha u_1(t) &= S u_1(t), \quad t \geqslant 0 \ ; \ S \equiv B_1^{-1} A_1. \end{aligned}$$

By Lemma 2.8.5, we get that $u_0(t) = 0$, $t \geqslant 0$ and $u_1(t) \in X_1$ for all $t \geqslant 0$. Since the operator S is regular in X_1, by the foregoing we have that for each $Pu_0 = u_0^1 \in X_1$ a unique solution of Cauchy problem (DF) is given by $u(t) = E_\alpha(t^\alpha S) u_0^1 = U_{\alpha,1}(t) u_0$, $t \geqslant 0$. The main results of [**214**] are given in the following theorem:

THEOREM 2.8.6. (i) *Suppose that the pair (A, B) is p-regular, $0 < \tau \leqslant \infty$, $Qf \in C^m([0,\tau) : Y)$, there exist fractional operators $(\mathbf{D}_t^\alpha H)^n A_0^{-1}(I - Q)f \in C([0,\tau) : X)$ for $0 \leqslant n \leqslant p$, $x_k \in X$ for $0 \leqslant k \leqslant m-1$, and the* *compatibility condition*

$$-\mathbf{D}_t^k \sum_{n=0}^{p} (\mathbf{D}_t^\alpha H)^n A_0^{-1}(I-Q)f(t)_{|t=0} = (I-P)x_k, \quad 0 \leqslant k \leqslant m-1$$

holds. Then there exists a unique strong solution $u(t)$ of problem

$$(\mathrm{DF})_f : \begin{cases} \mathbf{D}_t^\alpha Bu(t) = Au(t) + f(t), & t \geqslant 0, \\ u^{(k)}(0) = x_k, & 0 \leqslant k \leqslant m-1. \end{cases}$$

Furthermore, $u(t)$ is given by the following formula:

$$\begin{aligned} u(t) = & \sum_{k=0}^{m-1} t^k U_{\alpha,k+1}(t)x_k + \int_0^t (t-s)^{\alpha-1} U_{\alpha,\alpha}(t-s) B_1^{-1} Qf(s)\,ds \\ & - \sum_{n=0}^{p} (\mathbf{D}_t^\alpha H)^n A_0^{-1}(I-Q)f(t), \quad t \in [0,\tau). \end{aligned}$$

(ii) *Suppose that the pair (A,B) is p-regular, $0 < \tau \leqslant \infty$, $Qf \in C^m([0,\tau) : Y)$, there exist fractional operators $(\mathbf{D}_t^\alpha H)^n A_0^{-1}(I-Q)f \in C([0,\tau) : X)$ for $0 \leqslant n \leqslant p$, and $x_k \in X$ for $0 \leqslant k \leqslant m-1$. Then there exists a unique strong solution of problem*

$$(\mathrm{DF})_{f,P} : \begin{cases} \mathbf{D}_t^\alpha Bu(t) = Au(t) + f(t), & t \geqslant 0, \\ (Pu)^{(k)}(0) = x_k, & 0 \leqslant k \leqslant m-1. \end{cases}$$

Furthermore, the form of solution $u(t)$ is the same as in (i).

PROOF. We will prove only (i). It is not difficult to show that

$$\mathbf{D}_t^\alpha Hu_0(t) = u_0(t) + B_0^{-1}(I-Q)f(t), \quad t \geqslant 0, \tag{211}$$

$$\mathbf{D}_t^\alpha u_1(t) = Su_1(t) + h(t), \quad t \geqslant 0 \ ; \ h(t) \equiv A_1^{-1}Qf(t). \tag{212}$$

In view of Lemma 2.8.5, the unique solution of problem (211) is given by $u_0(t) = -\sum_{n=0}^{p} (\mathbf{D}_t^\alpha H)^n B_0^{-1}(I-Q)f(t)$, $t \geqslant 0$. Since the operator S is regular in X_1, the result now simply follows from the equality (208), which gives the form of solution of equation (212). $\square$

Now we would like to illustrate Theorem 2.8.6 with the following example.

EXAMPLE 2.8.7. Let $(E, \|\cdot\|)$ be a complex Banach space, and let A be a closed linear operator acting on E. For any $\tau > 0$, we put

$$D_{\infty,\tau,0}(A) := \Big\{ x \in D_\infty(A) : \limsup_{k\to\infty} \|A^k x\|^{1/k} \leqslant \tau \Big\}.$$

Let $D_{\infty,\tau}(A)$ be the biggest closed subspace of $D_{\infty,\tau,0}(A)$ with respect to the topology inherited from $D_\infty(A)$. Then it is well known that $D_{\infty,\tau}(A)$ is a Fréchet space and that $A_{|D_{\infty,\tau}(A)}$ is a regular operator on $D_{\infty,\tau}(A)$. Set $\mathbb{D}_\infty(A)$ be the inductive limit of spaces $D_{\infty,n}(A)$ as $n \to \infty$. Then, for every two entire functions $G(z)$ and $J_1(z)$, we have that $G(A)$ and $J_1(A)$ are continuous linear operators on $\mathbb{D}_\infty(A)$, which is known to be a separable and sequentially complete locally convex space [**226**]. In [**214**, Theorem 8], we have proved that the assumptions E is a Hilbert space and A is a self-adjoint operator on E imply that the operator $J_1(A)$ is $(G(A), 0)$-radial, provided in addition to the above that the functions $G(z)$ and $J_1(z)$ do not have common roots on $\sigma(A)$ as well as that there exists a finite constant $a > 0$ such that $|J_1(\lambda)/G(\lambda)| \leqslant a$ for all $\lambda \in \sigma(A)$ with $G(\lambda) \neq 0$. The main feature of [**214**, Theorem 9] is to analyze the following special case: $E =: L^2[0, 1]$ and

$$A := id/dx, \quad D(A) := \{u \in L^2[0, 1] : u' \in L^2[0, 1], \ u(0) = u(1)\}.$$

Applying [**214**, Theorem 8], we have proved that the following initial boundary value problem

$$\begin{gathered}
\mathbf{D}_t^\alpha G(A)u(x, t) = J_1(A)u(x + h, t) + f(x, t), \quad x \in \mathbb{R}, \ t \geqslant 0, \\
u(x, t) = u(x + 1, t), \quad x \in \mathbb{R}, \ t \geqslant 0, \\
\frac{\partial^n u}{\partial x^n} u(x, 0) = u_n(x), \quad n = 0, 1, \ldots, m - 1, \ x \in [0, 1]
\end{gathered}$$

has a unique solution provided that $|G(2\pi k)| + |J_1(2\pi k)| \neq 0$, $k \in \mathbb{Z}$, the set $\{|J_1(2\pi k)|/|G(2\pi k)| : k \in \mathbb{Z}, \ G(2\pi k) \neq 0\}$ is bounded, $f(x, t) = f(x + 1, t)$, $x \in \mathbb{R}$, $t \geqslant 0$, $f \in C^m([0, \infty) : \mathbb{D}_\infty(A))$, $u_n \in \mathbb{D}_\infty(A)$ $(n = 0, 1, \ldots, m - 1)$, and

$$\int_{G_0} d\mathcal{E}_\lambda \Big(J_1(\lambda)e^{-ih\lambda}u_n + f^{(n)}(\cdot, 0) \Big) = 0, \quad n = 0, 1, \ldots, m - 1,$$

where $\mathcal{E}_\lambda$ $(\lambda \in \mathbb{R})$ is the resolution of the identity for A and $G_0 := \{\lambda \in \mathbb{R} : G(\lambda) \neq 0\}$. Furthermore, the solution $u(\cdot, t)$ is given by the following formula:

$$\begin{aligned}
u(\cdot, t) = {} & \sum_{n=0}^{m-1} t^n \int_{\sigma(A) \smallsetminus G_0} E_{\alpha, n+1}\Big(t^\alpha \frac{J_1(\lambda)e^{-ih\lambda}}{G(\lambda)} \Big) d\mathcal{E}_\lambda u_n \\
& + \int_0^t (t - s)^{\alpha - 1} \int_{\sigma(A) \smallsetminus G_0} E_{\alpha, \alpha}\Big(t^\alpha \frac{J_1(\lambda)e^{-ih\lambda}}{G(\lambda)} \Big) \frac{d\mathcal{E}_\lambda f(\cdot, s)}{G(\lambda)} ds \\
& - \int_{G_0} \frac{e^{ih\lambda} d\mathcal{E}_\lambda f(\cdot, t)}{J_1(\lambda)}, \quad t \geqslant 0.
\end{aligned}$$

The solution of this equation is analytic with repect to the variable t, in $\mathbb{C} \smallsetminus (-\infty, 0]$, if $f(\cdot, \cdot)$ has some expected properties, as well.

We end this section with the observation that Fedorov and Gordievskikh [**211**] have analyzed the following abstract degenerate fractional Cauchy problem:

$$\begin{gathered}
\mathbf{D}_t^\alpha P_n(\Delta) = Q_r(\Delta)u(x, t), \quad (x, t) \in \Omega \times [0, \infty), \\
(1 - \theta)\Delta^k + \theta \frac{\partial \Delta^k u}{\partial n} u(x, t) = 0, \quad k = 0, 1, \ldots, n - 1, \ (x, t) \in \partial\Omega \times [0, \infty), \\
\frac{\partial^k u}{\partial t^k}(x, 0) = 0, \quad k = 0, 1, \ldots, m - 1;
\end{gathered}$$

here, Ω is a smooth domain in $\mathbb{R}^n$, $\alpha > 0$, $\theta \in \mathbb{R}$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$ and P_n, Q_r are non-zero complex polynomials of degrees n, r. In [**211**, Theorem 8] and [**211**, Theorem 9], the authors explore separately the cases $r \leqslant n$ and $r > n$, respectively.

2.9. Abstract degenerate non-scalar Volterra equations

In this section, we delve into the details regarding abstract degenerate Volterra equations of non-scalar type. Let X and Y be two complex Banach spaces satisfying that Y is continuously embedded in X, let the operator $C \in L(X)$ be injective, and let $\tau \in (0, \infty]$. The norm in X, resp. Y, will be denoted by $\| \cdot \|_X$, resp. $\| \cdot \|_Y$; as before, $[R(C)]$ denotes the Banach space $R(C)$ equipped with the norm $\|x\|_{R(C)} = \|C^{-1}x\|_X$, $x \in R(C)$. By B we denote a closed linear operator with domain and range contained in X. If Z is a general topological space and $Z_0 \subseteq Z$, then by $\overline{Z_0}^Z$ we denote the adherence of Z_0 in Z (we will use the abbreviation $\overline{Z_0}$, if there is no risk for confusion).

Let $A(t)$ be a locally integrable function from $[0, \tau)$ into $L(Y, X)$. Unless stated otherwise, we assume that $A(t)$ is not of scalar type, i.e., that there does not exist $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, and a closed linear operator A in X such that $Y = [D(A)]$ and $A(t) = a(t)A$ for a.e. $t \in [0, \tau)$. In the sequel, we will basically follow the notation employed in the monograph of J. Prüss [**459**] and our previous paper [**299**] on abstract non-degenerate equations of non-scalar type (cf. also [**262**] and [**269**] for some other references in this direction); the meaning of symbol A will be clear from the context.

Definition 2.9.1. Let $k \in C([0, \tau))$ and $k \neq 0$, let $\tau \in (0, \infty]$, $f \in C([0, \tau) : X)$, and let $A \in L^1_{loc}([0, \tau) : L(Y, X))$. Consider the linear degenerate Volterra equation

$$Bu(t) = f(t) + \int_0^t A(t-s)u(s)\,ds, \quad t \in [0, \tau). \tag{213}$$

Then a function $u \in C([0, \tau) : [D(B)])$ is said to be:

(i) a strong solution of (213) iff $u \in L^\infty_{loc}([0, \tau) : Y)$ and (213) holds on $[0, \tau)$,
(ii) a mild solution of (213) iff there exist a sequence (f_n) in $C([0, \tau) : X)$ and a sequence (u_n) in $C([0, \tau) : [D(B)])$ such that $u_n(t)$ is a strong solution of (213) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0, \tau)$.

The abstract Cauchy problem (213) is said to be (kC)-well posed (C-well posed, if $k(t) \equiv 1$) iff for every $y \in Y$, there exists a unique strong solution of

$$Bu(t; y) = k(t)Cy + \int_0^t A(t-s)u(s; y)\,ds, \quad t \in [0, \tau) \tag{214}$$

and if $u(t; y_n) \to 0$ in $[D(B)]$, uniformly on compact subsets of $[0, \tau)$, whenever (y_n) is a zero sequence in Y; (213) is said to be a-regularly (kC)-well posed (a-regularly C-well posed, if $k(t) \equiv 1$), where $a \in L^1_{loc}([0, \tau))$, iff (213) is (kC)-well posed and if the equation

$$Bu(t) = (a * k)(t)Cx + \int_0^t A(t-s)u(s)\,ds, \quad t \in [0, \tau)$$

admits a unique strong solution for every $x \in X$.

We would like to point out that every strong solution of (213) is also a mild solution of (213) as well as that the notion introduced in Definition 2.9.1 generalizes the corresponding one from [**299**, Definition 1], given in the case that $B = I$. It is also clear that the concept of a strong (mild) solution of (213) and the concept of a (kC)-well posedness of (213) can be defined in some other ways; we will skip all related details.

The following definition will be crucial for our further work.

DEFINITION 2.9.2. Let $\tau \in (0, \infty]$, $k \in C([0, \tau))$, $k \neq 0$ and $A \in L^1_{loc}([0, \tau) : L(Y, X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X, [D(B)])$ is called an (A, k, B)-regularized C-pseudoresolvent family iff the following holds:

(S1) The mappings $t \mapsto S(t)x$, $t \in [0, \tau)$ and $t \mapsto BS(t)x$, $t \in [0, \tau)$ are continuous in X for every fixed $x \in X$, $BS(0) = k(0)C$ and $S(t)C = CS(t)$, $t \in [0, \tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0, \tau)$. Then (S2) means $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0, \tau)$ and $(U(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$BS(t)y = k(t)Cy + \int_0^t A(t - s)dU(s)y, \quad t \in [0, \tau),\ y \in Y, \tag{215}$$

$$BS(t)y = k(t)Cy + \int_0^t S(t - s)A(s)y\,ds, \quad t \in [0, \tau),\ y \in Y, \tag{216}$$

hold; (215), resp. (216), is called the first resolvent equation, resp. the second resolvent equation.

An (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A, k, B)-regularized C-resolvent family if additionally:

(S4) For every $y \in Y$, $S(\cdot)y \in L^\infty_{loc}([0, \tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ in $L(X, [D(B)])$ is called a weak (A, k, B)-regularized C-pseudoresolvent family iff (S1) and (216) hold. Finally, a weak (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be a-regular $(a \in L^1_{loc}([0, \tau)))$ iff $a * S(\cdot)x \in C([0, \tau) : Y)$, $x \in \bar{Y}^X$.

Let us agree on the following: A (weak) (A, k, B)-regularized C-(pseudo)resolvent family with $k(t) \equiv g_{\alpha+1}(t)$, where $\alpha \geqslant 0$, is also called a (weak) α-times integrated (A, B)-regularized C-(pseudo)resolvent family, whereas a (weak) 0-times integrated (A, B)-regularized C-(pseudo)resolvent family is also said to be a (weak) (A, B)-regularized C-(pseudo)resolvent family. A (weak) (A, k, B)-regularized C-(pseudo)resolvent family is also called a (weak) (A, k, B)-regularized (pseudo)resolvent family ((weak) (A, B)-regularized (pseudo)resolvent family) if $C = I$ (if $C = I$ and $k(t) \equiv 1$).

As in non-degenerate case, the integral appearing in the first resolvent equation (215) is understood in the sense of discussion following [**459**, Definition 6.2, p. 153].

Observe also that the condition (S3) can be rewritten in the following equivalent form:

(S3)' $$BU(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds, \quad t \in [0,\tau),\ y \in Y,$$
$$BU(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds, \quad t \in [0,\tau),\ y \in Y.$$

By the norm continuity we mean the continuity in $L(X)$ and, in many places, we do not distinguish $S(\cdot)$ ($U(\cdot)$) and its restriction to Y.

The notion of an (A,k,B)-regularized C-uniqueness family plays a crucial role in proving the uniqueness of solutions of abstract degenerate Cauchy problem (213).

Definition 2.9.3. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A strongly continuous operator family $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be an (A,k,B)-regularized C-uniqueness family iff

$$V(t)By = k(t)Cy + \int_0^t V(t-s)A(s)y\,ds, \quad t \in [0,\tau),\ y \in Y \cap D(B).$$

Before stating the following propositions, whose proofs can be deduced as in non-degenerate case (cf. [**459**] and [**299**]), we want to observe that the notion of an (A,k,I)-regularized C-uniqueness family is a special case of the notion of a weak (A,k,I)-regularized C-pseudoresolvent family and the assertion of [**299**, Proposition 2(i)] holds even if the condition $S(t)C = CS(t)$, $t \in [0,\tau)$ is disregarded (cf. also Proposition 2.9.5(i) below).

Proposition 2.9.4. (i) *Suppose that $(S_1(t))_{t\in[0,\tau)}$ is an (A,k_1,B)-regularized C_1-pseudoresolvent family and $(S_2(t))_{t\in[0,\tau)}$ is an (A,k_2,B)-regularized C_2-uniqueness family. Then $C_2(k_2 * S_1)(t)x = (k_1 * S_2)(t)C_1x$, $t \in [0,\tau)$, $x \in \bar{Y}^X$.*

(ii) *Let $(S(t))_{t\in[0,\tau)}$ be an (A,k,B)-regularized C-pseudoresolvent family. Assume that Y has the Radon-Nikodym property. Then $(S(t))_{t\in[0,\tau)}$ is an (A,k,B)-regularized C-resolvent family. Furthermore, if Y is reflexive, then $S(t)(Y) \subseteq Y$, $t \in [0,\tau)$ and the mapping $t \mapsto S(t)y$, $t \in [0,\tau)$ is weakly continuous in Y for all $y \in Y$.*

Proposition 2.9.5. (i) *Assume that $(V(t))_{t\in[0,\tau)}$ is an (A,k,B)-regularized C-uniqueness family, $f \in C([0,\tau) : X)$ and $u(t)$ is a mild solution of* (213). *Then $(kC * u)(t) = (V * f)(t)$, $t \in [0,\tau)$ and mild solutions of* (213) *are unique provided in addition that $k(t)$ is a kernel on $[0,\tau)$.*

(ii) *Assume $n \in \mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is an $(n-1)$-times integrated (A,B)-regularized C-pseudoresolvent family, $C^{-1}f \in C^{n-1}([0,\tau) : X)$ and $f^{(i)}(0) = 0$, $0 \leqslant i \leqslant n-1$. Then the following assertions hold:*

(a) *Let $(C^{-1}f)^{(n-1)} \in AC_{loc}([0,\tau) : Y)$ and $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : Y)$. Then the function $t \mapsto u(t)$, $t \in [0,\tau)$ given by*

$$u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)ds = \int_0^t dU(s)(C^{-1}f)^{(n)}(t-s)$$

is a strong solution of (213). *Moreover,* $u \in C([0,\tau) : Y)$.

(b) *Let* $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : X)$ *and* $\bar{Y}^X = X$. *Then the function* $u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)ds$, $t \in [0,\tau)$ *is a mild solution of* (213).

(c) *Let* $C^{-1}g \in W^{n,1}_{loc}([0,\tau) : \bar{Y}^X)$, $a \in L^1_{loc}([0,\tau))$, $f(t) = (g_n * a * g^{(n)})(t)$, $t \in [0,\tau)$ *and let* $(S(t))_{t\in[0,\tau)}$ *be a-regular. Then the function* $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)^{(n)})(s)ds$, $t \in [0,\tau)$ *is a strong solution of* (213).

The uniqueness of solutions in (a), (b) *or* (c) *holds provided that for each* $y \in Y \cap D(B)$ *we have* $S(t)By = BS(t)y$, $t \in [0,\tau)$.

(iii) *Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k,B)*-regularized* C*-resolvent family. Put* $u(t;y) := S(t)y$, $t \in [0,\tau)$, $y \in Y$. *Then* $u(t;y)$ *is a strong solution of* (214), *and* (214) *is* (kC)*-well posed if* $k(t)$ *is a kernel on* $[0,\tau)$ *and* $S(t)By = BS(t)y$, $t \in [0,\tau)$, $y \in Y \cap D(B)$.

Before we clarify a Hille–Yosida type theorem for (A,k,B)-regularized C-pseudoresolvent families, it should be observed that there exists a great number of statements from [**299**] which can be reconsidered in the degenerate case; without going into details, we only want to capture our readers' attention to the assertions of [**299**, Proposition 1(iii)(b)/(c), Proposition 3(ii)–(iii), Proposition 4, Remark 1, Theorem 2, Remark 2, Proposition 5]. It is also worth noting that the class of (A,k,B)-regularized C-uniqueness families can be characterized through the vector-valued Laplace transform and we need the condition $S(t)By = BS(t)y$, $t \geqslant 0$, $y \in Y \cap D(B)$ (see the formulation of Theorem 2.9.6) in order to show the injectiveness of operator $B - \tilde{A}(\lambda)$ for $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$ (in Theorem 2.9.10 below, this condition will not be used).

THEOREM 2.9.6. *Assume* $A \in L^1_{loc}([0,\tau) : L(Y,X))$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $|a|(t)$ *and* $k(t)$ *satisfy* (P1), $\varepsilon_0 \geqslant 0$ *and*

$$\int_0^\infty e^{-\varepsilon t}\|A(t)\|_{L(Y,X)}dt < \infty, \quad \varepsilon > \varepsilon_0. \tag{217}$$

(i) *Let* $(S(t))_{t\geqslant 0}$ *be an* (A,k,B)*-regularized* C*-pseudoresolvent family such that* $S(t)By = BS(t)y$, $t \geqslant 0$, $y \in Y \cap D(B)$ *and there exists* $\omega \geqslant 0$ *with*

$$\sup_{t>0} e^{-\omega t}\Big(\|S(t)\|_{L(X)} + \|BS(t)\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\|U(t)-U(s)\|_{L(Y)}\Big) < \infty. \tag{218}$$

Put $\omega_0 := \max(\omega, \operatorname{abs}(k), \varepsilon_0)$ *and* $H(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x \in X$, $\operatorname{Re}\lambda > \omega_0$. *Then the following holds:*

(N1) $(\tilde{A}(\lambda))_{\operatorname{Re}\lambda>\varepsilon_0}$ *is analytic in* $L(Y,X)$, $R(C_{|Y}) \subseteq R(B - \tilde{A}(\lambda))$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$, *and* $B - \tilde{A}(\lambda)$ *is injective,* $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$.

(N2) $H(\lambda)y = \lambda\tilde{U}(\lambda)y$, $y \in Y$, $\operatorname{Re}\lambda > \omega_0$, $(B - \tilde{A}(\lambda))^{-1}C_{|Y} \in L(Y)$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$, $(H(\lambda))_{\operatorname{Re}\lambda>\omega_0}$ *is analytic in both spaces,* $L(X)$

and $L(Y)$, $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$, $H(\lambda)By = BH(\lambda)y$, $\operatorname{Re}\lambda > \omega_0$, $y \in Y \cap D(B)$, *and for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$, *the following holds:*

$$(B-\tilde{A}(\lambda))H(\lambda)y = \tilde{k}(\lambda)Cy,\ y \in Y \text{ and } H(\lambda)(B-\tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ y \in Y\cap D(B).$$

$$\text{(N3)}\quad \sup_{n\in\mathbb{N}_0}\ \sup_{\lambda>\omega_0,\tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!}\Big(\Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}BH(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(Y)}\Big) < \infty.$$

(ii) *Assume that* (N1)–(N3) *hold. Then there exists an exponentially bounded* (A,Θ,B)*-regularized* C*-resolvent family* $(S_1(t))_{t\geqslant 0}$ *satisfying* $S_1(t)By = BS_1(t)y$, $t \geqslant 0$, $y \in Y \cap D(B)$.

(iii) *Assume that* (N1)–(N3) *hold,* $B^{-1} \in L(X)$ *and* $\bar{Y}^X = X$. *Then there exists an exponentially bounded* (A,k,B)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geqslant 0}$ *such that* (218) *holds and* $S(t)By = BS(t)y$, $t \geqslant 0$, $y \in Y \cap D(B)$.

(iv) *Assume* $(S(t))_{t\geqslant 0}$ *is an* (A,k,B)*-regularized* C*-pseudoresolvent family satisfying* (218) *with some* $\omega \geqslant 0$. *Let* $\omega' \geqslant \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k), \varepsilon_0)$. *Then* $(S(t))_{t\geqslant 0}$ *is* a*-regular and* $\sup_{t\geqslant 0} e^{-\omega' t}\|a * S(t)\|_{L(\bar{Y}^X,Y)} < \infty$ *iff there exists a number* $\omega_1 \geqslant \omega'$ *such that*

$$\sup_{n\in\mathbb{N}_0}\ \sup_{\lambda>\omega_1,\tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega')^{n+1}}{n!}\Big\|\frac{d^n}{d\lambda^n}(\tilde{a}(\lambda)H(\lambda))\Big\|_{L(\bar{Y}^X,Y)} < \infty.$$

The hyperbolic perturbation results for non-scalar Volterra equations have been studied in [**459**, Theorem 6.1, p. 159] and [**299**, Theorem 3]. It is worth noting that the above-mentioned results can be generalized to degenerate non-scalar Volterra equations. More precisely, the following theorem holds true (the proof can be deduced by slightly modifying the corresponding proof of [**459**, Theorem 6.1], with $K_0 = S * C^{-1}BD_0$ and $K_1 = S * C^{-1}BD_1$):

THEOREM 2.9.7. *Assume* $L^1_{loc}([0,\tau)) \ni a$ *is a kernel on* $[0,\tau)$, $C(Y) \subseteq Y$, $\bar{Y}^X = X$, $CB \subseteq BC$,

$$D(t)y = D_0(t)y + (a * D_1)(t)y, \quad t \in [0,\tau),\ y \in Y,$$

where $(D_0(t))_{t\in[0,\tau)} \subseteq L(Y)$, $(BD_0(t))_{t\in[0,\tau)} \subseteq L(X,[R(C)])$, $(BD_1(t))_{t\in[0,\tau)} \subseteq L(Y,[R(C)])$,

(i) $C^{-1}BD_0(\cdot)y \in BV_{loc}([0,\tau) : Y)$ *for all* $y \in Y$, $C^{-1}BD_0(\cdot)x \in BV_{loc}([0,\tau) : X)$ *for all* $x \in X$,

(ii) $C^{-1}BD_1(\cdot)y \in BV_{loc}([0,\tau) : X)$ *for all* $y \in Y$, *and*

(iii) $CBD(t)y = BD(t)Cy$, $y \in Y$, $t \in [0,\tau)$.

Then the existence of an a*-regular* (A,B)*-regularized* C*-(pseudo)resolvent family* $(S(t))_{t\in[0,\tau)}$ *is equivalent with the existence of an* a*-regular* $(A+BD,B)$*-regularized* C*-(pseudo)resolvent family* $(R(t))_{t\in[0,\tau)}$.

Theorem 2.9.7 can be applied to abstract degenerate non-scalar Volterra equations involving abstract differential operators. For example, let $1 < p < \infty$, let $X := L^p(\mathbb{R}^n)$, and let $0 < \alpha < 2$. Then it is clear that the operators $\partial/\partial x_j$, acting with their maximal distributional domains, are commuting generators of bounded C_0-groups on X; set $A := (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. Suppose that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials satisfying $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and the estimate (70). Define the strongly continuous operator family $(G_\alpha(t))_{t\geqslant 0} \subseteq L(X)$ in the same way as it has been done in Remark 2.3.16(ii), and the operators $\overline{P_i(A)}$, $i = 1, 2$ in the usual way. Set $Y := D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$, $\|f\| := \|f\|_X + \|\overline{P_1(A)}f\|_X + \|\overline{P_2(A)}f\|_X$ $(f \in Y)$, $A(t) := g_\alpha(t)\overline{P_1(A)}_{|Y}$ $(t > 0)$ and $C := \overline{P_2(A)}G_\alpha(0)$. Then $(Y, \|\cdot\|_Y)$ is a Banach space continuously embedded in X, $\bar{Y}^X = X$, $C\overline{P_2(A)} \subseteq \overline{P_2(A)}C$, $C(Y) \subseteq Y$, and a simple analysis shows that $(G_\alpha(t))_{t\geqslant 0}$ is an $(A, \overline{P_2(A)})$-resolvent C-regularized resolvent family obeying the property that for each $f \in Y$ the mapping $t \mapsto \int_0^t G_\alpha(s)f\,ds$, $t \geqslant 0$ is continuously differentiable in Y. Therefore, Theorem 2.9.7 can be applied with the regularizing operator C being in general the non-identity operator on X. We continue by introducing the following definition (cf. [**299**, Definition 3(i)] for non-degenerate case).

Definition 2.9.8. Let $k \in C([0,\infty))$, $k \neq 0$, $A \in L^1_{loc}([0,\infty) : L(Y,X))$, $\alpha \in (0,\pi]$, and let $(S(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ be a (weak) (A,k,B)-regularized C-(pseudo)resolvent family. Then it is said that $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α, if there exists an analytic function $\mathbf{S}\colon \Sigma_\alpha \to L(X, [D(B)])$ satisfying $\mathbf{S}(t) = S(t)$, $t > 0$, $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}(z)x = S(0)x$ and $\lim_{z\to 0, z\in\Sigma_\gamma} B\mathbf{S}(z)x = BS(0)x$ for all $\gamma \in (0,\alpha)$ and $x \in X$. We say that $(S(t))_{t\geqslant 0}$ is an exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, resp. bounded analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, of angle α, if $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α and for each $\gamma \in (0,\alpha)$ there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $||\mathbf{S}(z)||_{L(X)} + ||B\mathbf{S}(z)||_{L(X)} \leqslant M_\gamma e^{\omega_\gamma |z|}$, $z \in \Sigma_\gamma$. Since no confusion seems likely, we shall identify $S(\cdot)$ and $\mathbf{S}(\cdot)$ in the sequel.

The most important properties of exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent families are collected in the subsequent theorems.

Theorem 2.9.9. (i) *Assume* $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1), $\omega \geqslant \max(\mathrm{abs}(k), \varepsilon_0)$, (217) *holds,* $(S(t))_{t\geqslant 0}$ *is a weak analytic* (A,k,B)-*regularized* C-*pseudoresolvent family of angle* $\alpha \in (0,\pi/2]$ *and*

$$\sup_{z\in\Sigma_\gamma} \left[\|e^{-\omega z}S(z)\|_{L(X)} + \|e^{-\omega z}BS(z)\|_{L(X)}\right] < \infty \quad \textit{for all } \gamma \in (0,\alpha). \tag{219}$$

Then there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *such that:*

(a) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega$,

(b) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\left[\|(\lambda-\omega)H(\lambda)\|_{L(X)}+\|(\lambda-\omega)BH(\lambda)\|_{L(X)}\right]<\infty$ *for all* $\gamma\in(0,\alpha)$,

(c) *there exists an operator* $F\in L(X,[D(B)])$ *such that* $BFx=k(0)Cx$, $x\in X$ *and* $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda H(\lambda)x=Fx$, $x\in X$, *and*

(d) $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda BH(\lambda)x=k(0)Cx$, $x\in X$.

(ii) *Assume* $\varepsilon_0\geqslant 0$, $k(t)$ *satisfies* (P1), (217) *holds,* $\omega\geqslant\max(\mathrm{abs}(k),\varepsilon_0)$, $\alpha\in(0,\pi/2]$, *there exists an analytic mapping* $H:\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(X,[D(B)])$ *such that* (a), (b) *and* (c) *of the item* (i) *hold and that, in the case* $\bar{Y}^X\neq X$, (d) *also holds. Then there exists a weak analytic* (A,k,B)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geqslant 0}$ *of angle* α *such that* (219) *holds.*

THEOREM 2.9.10. (i) *Assume* $\varepsilon_0\geqslant 0$, $k(t)$ *satisfies* (P1), $\omega_0\geqslant\max(\mathrm{abs}(k),\varepsilon_0)$, (217) *holds,* $\alpha\in(0,\pi/2]$, $(S(t))_{t\geqslant 0}$ *is an analytic* (A,k,B)*-regularized* C*-resolvent family of angle* α, *the mapping* $t\mapsto U(t)\in L(Y)$, $t>0$ *can be analytically extended to the sector* Σ_α *(we shall denote the analytical extensions of* $U(\cdot)$ *and* $S(\cdot)$ *in the space* $L(Y)$ *by the same symbols), and for each* $\gamma\in(0,\alpha)$ *one has:*

$$\sup_{z\in\Sigma_\gamma}\left[\|e^{-\omega_0 z}S(z)\|_{L(X)}+\|e^{-\omega_0 z}BS(z)\|_{L(X)}+\sup_{z\in\Sigma_\gamma}\|e^{-\omega_0 z}S(z)\|_{L(Y)}\right]<\infty. \tag{220}$$

Denote $H(\lambda)x=\int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x\in X$, $\operatorname{Re}\lambda>\omega_0$. *Then* $H\colon\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(X,[D(B)])$ *is an analytic mapping,* $H_{|Y}\colon\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(Y)$ *is likewise an analytic mapping, and the following holds:*

(a) $\sup_{\lambda\in\omega_0+\Sigma_{\frac{\pi}{2}+\gamma}}\left[\|(\lambda-\omega_0)H(\lambda)\|_{L(X)}+\|(\lambda-\omega_0)BH(\lambda)\|_{L(X)}+\|(\lambda-\omega_0)H(\lambda)\|_{L(Y)}\right]<\infty$ *for all* $\gamma\in(0,\alpha)$,

(b) $BH(\lambda)y-H(\lambda)\tilde{A}(\lambda)y=\tilde{k}(\lambda)Cy$, $y\in Y$, $\operatorname{Re}\lambda>\omega$, $\tilde{k}(\lambda)\neq 0$; $BH(\lambda)y-\tilde{A}(\lambda)H(\lambda)y=\tilde{k}(\lambda)Cy$, $y\in Y$, $\operatorname{Re}\lambda>\omega$, $\tilde{k}(\lambda)\neq 0$; $H(\lambda)C=CH(\lambda)$, $\operatorname{Re}\lambda>\omega_0$,

(c) *there exists an operator* $F\in L(X,[D(B)])$ *such that* $BFx=k(0)Cx$, $x\in X$, $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda H(\lambda)x=Fx$, $x\in X$, *and*

(d) $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda BH(\lambda)x=k(0)Cx$, $x\in X$.

(ii) *Assume* $\alpha\in(0,\pi/2]$, $\varepsilon_0\geqslant 0$, $k(t)$ *satisfies* (P1) *and* (217) *holds. Let* $\omega_0\geqslant\max(\mathrm{abs}(k),\varepsilon_0)$, *and let there exist an analytic mapping* $H\colon\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(X,[D(B)])$ *such that* $H_{|Y}\colon\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(Y)$ *is an analytic mapping, as well as that* (a)–(c) *of the item* (i) *of this theorem hold and, in the case* $\bar{Y}^X\neq X$, (d) *also holds. Then there exists an analytic* (A,k,B)*-regularized* C*-resolvent family* $(S(t))_{t\geqslant 0}$ *of angle* α *such that* (220) *holds and the mapping* $t\mapsto U(t)\in L(Y)$, $t>0$ *can be analytically extended to the sector* Σ_α.

REMARK 2.9.11. If $B^{-1}\in L(X)$, then the condition (i)/(c) in Theorem 2.9.9, i.e., the condition (i)/(c) in Theorem 2.9.10, automatically holds. Therefore, Theorem 2.9.9 and Theorem 2.9.10 taken together provide extensions of [**299**, Theorem 4-Theorem 5].

Based on the evidence contained in the proofs of [**299**, Theorem 6-Theorem 7], we can simply clarify the basic results about differentiability of (A, k, B)-regularized C-pseudoresolvent families in Banach spaces. Although interesting, this theme will not be considered here; let us only mention that the considerations from [**299**, Example 3] and [**314**, Example 2.5] enable one to construct some important examples of (local) (A, k, B)-regularized resolvent families with a certain hypoanalytic behaviour.

2.10. Hypercyclic and topologically mixing properties of abstract degenerate fractional differential equations

In this section, which is broken down into four subsections, we analyze hypercyclic and topologically mixing properties of various classes of abstract (multi-term) degenerate fractional differential equations with Caputo derivatives. In Subsection 2.10.1–Subsection 2.10.3, by $(E, \|\cdot\|)$ we denote a non-trivial separable Banach space over the field $\mathbb{C}$ (some of our results presented in these subsections continue to hold in the setting of separable, infinite-dimensional, Hausdorff, sequentially complete locally convex spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; cf. [**112**, **292**, Section 3.1] and [**309**] for more details).

2.10.1. Hypercyclic and topologically mixing properties of solutions of the equations $\mathbf{D}_t^\alpha Bu(t) = Au(t)$ and $B\mathbf{D}_t^\alpha u(t) = Au(t)$ $(\alpha > 0)$. Let A and B be closed linear operators acting on E, and let $C \in L(E)$ be an injective operator. If $\emptyset \neq \Omega \subseteq \mathbb{C} \smallsetminus (-\infty, 0]$ and $\alpha > 0$, set $\Omega^\alpha := \{\lambda^\alpha : \lambda \in \Omega\}$. By $\mathcal{D}$ we denote the set which consists of all infinitely differentiable, compactly supported functions from $\mathbb{R}$ into $\mathbb{C}$; if $\emptyset \neq K \subseteq \mathbb{R}$, then we define $\mathcal{D}_K := \{\varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subseteq K\}$. We enquire into the hypercyclic and topologically mixing solutions of homogeneous counterparts of fractional Sobolev equations

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^\alpha Bu(t) = Au(t) + f(t), \quad t \geqslant 0, \\ Bu(0) = Bx; \quad (Bu)^{(j)}(0) = 0,\ 1 \leqslant j \leqslant \lceil \alpha \rceil - 1 \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B\mathbf{D}_t^\alpha u(t) = Au(t) + f(t), \quad t \geqslant 0, \\ u(0) = x; \quad u^{(j)}(0) = 0,\ 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

Throughout the whole section, we will use the same symbols to denote such problems; this little abuse of notation will not create any confusion henceforward. Suppose that the function $u(t)$ is a strong solution of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$); cf. Definition 2.3.1 (in the existing literature on non-degenerate first order equations, this is equivalent to say that $u(t)$ is a classical solution of problem $(\mathrm{DFP})_R$; cf. [**445**]). Convoluting both sides of $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) with $g_\alpha(t)$, and using the equality (38), it readily follows that every solution of $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) satisfies the equality

$$Bu(t) - Bx = (g_\alpha * Au(\cdot))(t), \quad t \geqslant 0 \qquad (B(u(t) - x) = (g_\alpha * Au(\cdot))(t), \quad t \geqslant 0).$$

If a continuous function $t \mapsto u(t)$, $t \geqslant 0$ satisfies the first of these equations with $Au, Bu \in C([0,\infty) : E)$, then it can be easily checked that $u(t)$ is a strong solution of problem $(\mathrm{DFP})_R$; observe, however, that we cannot prove the corresponding statement for the problem $(\mathrm{DFP})_L$. Let us recall that a mild solution of the problem $(\mathrm{DFP})_R$ is any continuous function $t \mapsto u(t)$, $t \geqslant 0$ such that the mapping $t \mapsto Bu(t)$, $t \geqslant 0$ is continuous and $A(g_\alpha * u)(t) = Bu(t) - Bx$, $t \geqslant 0$. Denote by $Z_{\alpha,R}(A,B)$ $(Z_{\alpha,L}(A,B))$ the set which consists of those elements $x \in D(B)$ $(x \in D(A))$ for which there exists a strong solution of $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$. By $Z^{uniq}_{\alpha,R}(A,B)$ $(Z^{uniq}_{\alpha,L}(A,B))$ we denote the set which consists of those elements $x \in D(B)$ $(x \in D(A))$ for which there exists a unique strong solution of $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$. Then $Z_{\alpha,R}(A,B)$ $(Z_{\alpha,L}(A,B))$ is a linear subspace of E, and $Z^{uniq}_{\alpha,R}(A,B)$ $(Z^{uniq}_{\alpha,L}(A,B))$ is a linear subspace of E provided that the zero function is a unique strong solution of the problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ with $x = 0$. In this case, we define for each $t \geqslant 0$ the linear operator $T_R(t;\cdot) : Z^{uniq}_{\alpha,R}(A,B) \rightarrow E$ $(T_L(t;\cdot) : Z^{uniq}_{\alpha,L}(A,B) \rightarrow E)$ by $T_R(t;x) := u(t;x)$, $x \in Z^{uniq}_{\alpha,R}(A,B)$ $(x \in Z^{uniq}_{\alpha,L}(A,B))$, where $u(\cdot;x)$ denotes the strong solution of $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$.

In this subsection, we will restrict ourselves to the analysis of (subspace) hypercyclicity, topological transitivity and topological mixing property of problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$; in Subsection 2.10.3, we will also look into the (subspace) chaoticity and weakly mixing property of first and second order degenerate equations (cf. [**292**, Chapter 3] and references cited there, as well as Subsection 2.10.4, for some other dynamical properties of abstract degenerate differential equations that are not considered in Subsection 2.10.1–Subsection 2.10.3).

Definition 2.10.1. Let $\alpha > 0$, and let $\tilde{E}$ be a closed linear subspace of E. Then it is said that:

(i) an element $x \in Z_{\alpha,R}(A,B) \cap \tilde{E}$ $(x \in Z_{\alpha,L}(A,B) \cap \tilde{E})$ is a $\tilde{E}$-hypercyclic vector for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ iff there exists a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ with the property that the set $\{u(t;x) : t \geqslant 0\}$ is a dense subset of $\tilde{E}$.

Furthermore, we say that the abstract Cauchy problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ is:

(ii) $\tilde{E}$-topologically transitive iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exist $x \in Z_{\alpha,R}(A,B) \cap \tilde{E}$ $(x \in Z_{\alpha,L}(A,B) \cap \tilde{E})$, a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ and $t \geqslant 0$ such that $\|y - x\| < \varepsilon$ and $\|z - u(t;x)\| < \varepsilon$;

(iii) $\tilde{E}$-topologically mixing iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exist $x_t \in Z_{\alpha,R}(A,B) \cap \tilde{E}$ $(x_t \in Z_{\alpha,L}(A,B) \cap \tilde{E})$ and a strong solution $t \mapsto u(t;x_t)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ such that $\|y - x_t\| < \varepsilon$ and $\|z - u(t;x_t)\| < \varepsilon$.

In the case $\tilde{E} = E$, it is also said that a $\tilde{E}$-hypercyclic vector for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ is a hypercyclic vector for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$, and that $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ is topologically transitive, resp. topologically mixing.

Before proceeding further, it should be noted that the study we have taken up shows that there is no substantial difference in the analysis of hypercyclic properties of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, and there is no ideal option that would provide us a general method for studying hypercyclic properties of abstract degenerate equations. In Definition 2.10.1, which is, in our opinion, the best theorethical concept for the analysis of hypercyclicity of abstract degenerate equations, we work only with strong solutions. This is primarily caused by the fact that it is very difficult to define the notion of a mild solution of the problem $(\mathrm{DFP})_L$ in a satisfactory way, as well as that the solutions appearing in Theorem 2.10.3 below, which is the main result of this subsection, are strong in fact (in the case of problem $(\mathrm{DFP})_R$, the notions from Definition 2.10.1 can be introduced with mild solutions instead of strong ones, in the very obvious way; this has been the usual way for introducing $\tilde{E}$-hypercyclic vectors, $\tilde{E}$-topological transitivity and $\tilde{E}$-topologically mixing property of non-degenerate equations so far). As a large class of important examples shows (see e.g. Example 2.10.5–Example 2.10.6 below), it is much better to introduce the notions from Definition 2.10.1 with the set $Z_{\alpha,R}(A,B)$ $(Z_{\alpha,L}(A,B))$ than with $Z^{uniq}_{\alpha,R}(A,B)$ $(Z^{uniq}_{\alpha,L}(A,B))$; cf. [**6**,**95**,**424**] and [**210**] for some results ensuring the uniqueness of solutions of problem $Z_{\alpha,R}(A,B)$ $(Z_{\alpha,L}(A,B))$. Having things set up in such a way, we will face some very strange pecularities of hypercyclic degenerate equations later (cf. Example 2.10.15). Even in the case that $B = I$, we do not assume in Definition 2.10.1 the existence of a global (g_α, g_β)-regularized C-resolvent family for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$, which has been the general framework for the analysis of hypercyclic abstract PDEs so far ($\beta \geqslant 1$, $C \in L(E)$ injective).

We continue by observing that, for any arbitrarily large number $\alpha > 0$, we can simply construct a Banach space E and closed linear operators A and B on E such that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are both topologically mixing. For example, let E be the weighted l^1-space $l^1_r := \{(x_k)_{k\in\mathbb{N}} : x_k \in \mathbb{C},\ \sum_{k=1}^{\infty} r_k|x_k| < \infty\}$, normed by $\|(x_k)_{k\in\mathbb{N}}\| := \sum_{k=1}^{\infty} r_k|x_k|,\ (x_k)_{k\in\mathbb{N}} \in l^1_r$; here $(r_k)_{k\in\mathbb{N}}$ denotes any sequence of positive real numbers satisfying that there exists $M > 0$ such that $r_k r_{k+1}^{-1} \leqslant M$ for all $k \in \mathbb{N}$. Take any $B \in L(E)$ and define after that $A \in L(E)$ by $A(x_k)_{k\in\mathbb{N}} := (Bx_{k+1})_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}} \in l^1_r$; then, by [**301**, Theorem 2.3], the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are both topologically mixing. We leave to the interested reader details about construction of hypercyclic and topologically mixing degenerate (multi-term) fractional differential equations on weighted l^1-spaces, involving operators that are certain functions of unilateral backward shift operators (cf. [**143**, **470**, Section 5] and [**301**] for further information).

Let $\tilde{E} = E$, let HC_R (HC_L) designate the set of all hypercyclic vectors for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$, and let $\mathcal{O} = (O_n)_{n\in\mathbb{N}}$ be an open base of the topology of E ($O_n \neq \emptyset$, $n \in \mathbb{N}$). Assuming that the strong solutions of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are unique, it can be simply proved that the equality $HC_{R,L} = \bigcap_{n\in\mathbb{N}} \bigcup_{t\geqslant 0} T_{R,L}(t)^{-1}(O_n)$ holds and $\bigcup_{t\geqslant 0} T_{R,L}(t)^{-1}(O_n)$ is a dense subset of E for all $n \in \mathbb{N}$. Regrettably, we cannot conclude from the above that the topological transitivity of $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$ implies the existence of hypercyclic vectors for $(\mathrm{DFP})_R$ $((\mathrm{DFP})_L)$.

We need the following extension of [**300**, Lemma 2.1].

LEMMA 2.10.2. *Suppose* $\alpha > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, $x \in D(A) \cap D(B)$ *and* $Ax = \lambda Bx$. *Then the function* $u(t;x) := E_\alpha(\lambda t^\alpha)x$, $t \geqslant 0$ *satisfies* $u, Au, Bu \in C([0,\infty) : E)$, *as well as the equalities*

$$(g_\alpha * Au(\cdot;x))(t) = B(u(t;x) - x) = Bu(t;x) - Bx, \quad t \geqslant 0.$$

Furthermore, the function $u(\cdot;x)$ *is a strong solution of problem* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*).*

PROOF. The proof is very similar to that of [**300**, Lemma 2.1]; it is only worth noticing here that a direct computation combined with the equality $\mathbf{D}_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$, $t \geqslant 0$ (see e.g. [**61**, (1.25)]) implies that the function $t \mapsto u(t;x)$, $t \geqslant 0$ is a strong solution of problem $(\mathrm{DFP})_L$. $\square$

In the following theorem, we rethink the Desch–Schappacher–Webb and Banasiak–Moszyński criteria for chaos of strongly continuous semigroups.

THEOREM 2.10.3. *Suppose that* $\alpha \in (0,2)$ *and* Ω *is an open connected subset of* $\mathbb{C}$ *which satisfies* $\Omega \cap (-\infty, 0] = \emptyset$ *and* $\Omega \cap i\mathbb{R} \neq \emptyset$. *Let* $f \colon \Omega^\alpha \to E$ *be an analytic mapping such that* $f(\lambda^\alpha) \in \mathrm{Kern}(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$, *and let* $\tilde{E} := \overline{span\{f(\lambda^\alpha) : \lambda \in \Omega\}}$. *Then the problems* $(\mathrm{DFP})_R$ *and* $(\mathrm{DFP})_L$ *are* $\tilde{E}$*-topologically mixing; furthermore, if* $Af(\lambda^\alpha) \in \tilde{E}$ *for all* $\lambda \in \Omega$, *then the problems* $(\mathrm{DFP})_R^{\tilde{E}}$ *and* $(\mathrm{DFP})_L^{\tilde{E}}$, *obtained by replacing the operators* A *and* B *in* $(\mathrm{DFP})_R$ *and* $(\mathrm{DFP})_L$ *with the operators* $A_{|\tilde{E}}$ *and* $B_{|\tilde{E}}$, *respectively, are topologically mixing in the Banach space* $\tilde{E}$.

PROOF. Without loss of generality, we may assume that $\Omega \cap i(0,\infty) \neq \emptyset$. Observe the following: If Ω_0 is an arbitrary non-empty subset of Ω which admits a cluster point in Ω, then the (weak) analyticity of mapping $\lambda \mapsto f(\lambda^\alpha) \in \tilde{E}$, $\lambda \in \Omega$ implies that $\Psi(\Omega_0) := span\{f(\lambda^\alpha) : \lambda \in \Omega_0\}$ is dense in the Banach space $\tilde{E}$. Further on, it is clear that there exist numbers $\lambda_0 \in \Omega \cap i(0,\infty)$ and $\delta > 0$ such that any of the sets $\Omega_{0,+} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta,\ \arg(\lambda) \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})\}$ and $\Omega_{0,-} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta,\ \arg(\lambda) \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)\}$ admits a cluster point in Ω as well as that $\arg(\lambda^\alpha t^\alpha) < \alpha\pi/2$, $\lambda \in \Omega_{0,+}$ and $\arg(-\lambda^\alpha t^\alpha) \in \pi - \alpha\pi/2$, $\lambda \in \Omega_{0,-}$. Due to Theorem 1.5.1 (cf. also (40)–42), we have:

$$E_\alpha(\lambda^\alpha t^\alpha) \to \infty,\ t \to \infty,\ \lambda \in \Omega_{0,+} \ \text{ and } \ E_\alpha(\lambda^\alpha t^\alpha) \to 0,\ t \to \infty,\ \lambda \in \Omega_{0,-}. \tag{221}$$

Suppose that $y \in \Psi(\Omega_{0,-})$, $z \in \Psi(\Omega_{0,+})$, $\varepsilon > 0$, $y = \sum_{i=1}^n \beta_i f(\lambda_i^\alpha)$, $z = \sum_{j=1}^m \gamma_j f(\tilde{\lambda_j}^\alpha)$, α_i, $\beta_j \in \mathbb{C}$, $\lambda_i \in \Omega_{0,-}$ and $\tilde{\lambda_j} \in \Omega_{0,+}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Using (221), we get that there exists a number $t_0(z) > 0$ such that $E_\alpha(\tilde{\lambda_j}^\alpha t^\alpha) \neq 0$, $t \geqslant t_0(z)$. Put $z_t := \sum_{j=1}^m \frac{\gamma_j}{E_\alpha(\tilde{\lambda_j}^\alpha t^\alpha)} f(\tilde{\lambda_j}^\alpha)$ and $x_t := y + z_t$, $t \geqslant t_0(z)$. Owing to Lemma 2.10.2, we have $\{x_t,\ y,\ z_t\} \subseteq Z_\alpha(A,B) \cap \tilde{E}$, $t \geqslant t_0(z)$; denote by $u(\cdot;x_t)$, $u(\cdot;y)$ and $u(\cdot;z_t)$ the corresponding strong solutions ($t \geqslant t_0(z)$). Keeping in mind (221) and Lemma 2.10.2, we obtain that there exists a number $t(y,z,\varepsilon) > t_0(z)$ such that: $\|u(t;y)\| < \varepsilon$, $u(t;z_t) = z$ and $\|z_t\| < \varepsilon$, $t \geqslant t(y,z,\varepsilon)$. Hence, $\|x_t - y\| = \|z_t\| < \varepsilon$ and $\|u(t;x_t) - z\| = \|u(t;y)\| < \varepsilon$, $t \geqslant t(y,z,\varepsilon)$, whence it easily follows without any substantial difficulties that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are $\tilde{E}$-topologically mixing. The rest of the proof is simple and therefore omitted. $\square$

REMARK 2.10.4. It is worth noting that Theorem 2.10.3 does not continue to hold in the case that $\alpha = 2$. Then we can pass to the theory of degenerate equations of first order on product spaces and derive, in such a way, some conclusions on the subspace topologically mixing (chaotic) properties of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$. In order to better explain these facts, suppose that $\emptyset \neq \Omega$ is an open connected subset of $\mathbb{C}$ which intersects the imaginary axis, and that $f : \Omega^2 \to E$ is an analytic mapping satisfying $Af(\lambda^2) = \lambda^2 Bf(\lambda^2)$, $\lambda \in \Omega$. Define the operators $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ and $\mathcal{B}'$ on $E \times E$ by $\mathcal{A}(x,y) := (y, Ax)$, $(x,y) \in E \times D(A)$, $\mathcal{A}'(x,y) := (By, Ax)$, $(x,y) \in D(A) \times D(B)$, $\mathcal{B}(x,y) := (x, By)$, $(x,y) \in E \times D(B)$ and $\mathcal{B}'(x,y) := (Bx, By)$, $(x,y) \in D(B) \times D(B)$. Then $\mathcal{A}(f(\lambda^2), \lambda f(\lambda^2)) = \lambda \mathcal{B}(f(\lambda^2), \lambda f(\lambda^2))$, $\lambda \in \Omega$ and $\mathcal{A}'(f(\lambda^2), \lambda f(\lambda^2)) = \lambda \mathcal{B}'(f(\lambda^2), \lambda f(\lambda^2))$, $\lambda \in \Omega$. Making use of Theorem 2.10.3 with $\alpha = 1$, and the first of these equalities (resp., the second one), we can prove the validity of some results about (subspace) topologically mixing (chaotic) properties of the problem $(\mathrm{DFP})_L$, with the initial condition $u'(0) = 0$ replaced by $u'(0) = y$ (resp., the problem $(\mathrm{DFP})_R$, with the initial condition $(Bu)'(0) = 0$ replaced by $(Bu)'(0) = By$); for more details, see [**292**, Lemma 3.2.33, Theorem 3.2.34, Remark 3.2.35].

Observe that the conclusions stated in [**300**, Remark 1(i),(iii)] can be reformulated for degenerate fractional differential equations. Now we want to illustrate Theorem 2.10.3 with some examples.

EXAMPLE 2.10.5. Suppose that $0 < \alpha < 2$, $n \in \mathbb{N} \setminus \{1\}$ and $E := L^p([0,\infty))$ $(1 \leqslant p < \infty)$. Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $z \in \mathbb{C}$ be a complex polynomial of degree n $(a_j \in \mathbb{C},\ 0 \leqslant j \leqslant n)$, and let $Q(z) = \sum_{j=0}^{m} b_j z^j$, $z \in \mathbb{C}$ be a non-zero complex polynomial of degree $m \leqslant n-1$ $(b_j \in \mathbb{C},\ 0 \leqslant j \leqslant m)$. Suppose, further, that there exists $z_0 \in \mathbb{C}_- \equiv \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ such that $P(z_0) = 0$ and $P'(z_0)Q(z_0) \neq 0$. Set $Af := \sum_{j=0}^{n} a_j f^{(j)}$ and $Bf := \sum_{j=0}^{m} b_j f^{(j)}$ with maximal distributional domain; then A and B are closed linear operators on E. The mapping $z \mapsto (P/Q)(z)$ is analytic in a neighborhood of point z_0 so that there exists an open connected subset U of $\mathbb{C}$ containing zero, satisfying additionally that the inverse mapping $(P/Q)^{-1} : U \to (P/Q)^{-1}(U)$ is well defined, analytic and that $(P/Q)^{-1}(U) \subseteq \mathbb{C}_-$. Let Ω be any open connected subset of $\mathbb{C} \setminus (-\infty, 0]$ such that $\Omega^\alpha \subseteq U$. Set $\tilde{E} := \overline{span\{\exp(\cdot (P/Q)^{-1}(\lambda^\alpha)) : \lambda \in \Omega\}}$. Then it readily follows from Theorem 2.10.3 that the corresponding problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, with A and B replaced respectively by $A_{|\tilde{E}}$ and $B_{|\tilde{E}}$, are topologically mixing in the Banach space $\tilde{E}$.

EXAMPLE 2.10.6. A great number of differential operators generating strongly continuous semigroups whose (subspace) chaoticity has been proved by applying the Desch–Schappacher–Webb or Banasiak–Moszyński criterion (cf. [**238**], [**292**, Chapter 3] and references cited there for corresponding examples) can be used for construction of degenerate topologically mixing fractional PDEs, in many different ways. Generally speaking, let $\mathcal{A}$ be a closed linear operator on E satisfying that there exist an open connected subset $\emptyset \neq \Lambda$ of $\mathbb{C}$ and an analytic mapping $g : \Lambda \to E \smallsetminus \{0\}$ such that $\mathcal{A}g(\nu) = \nu g(\nu)$, $\nu \in \Lambda$. Suppose that $P(z)$ and $Q(z)$ are non-zero complex polynomials, and $0 < \alpha < 2$. Define $S := \{z \in \mathbb{C} : P(z) = 0\}$, $\Lambda' := \Lambda \smallsetminus S$, $A := Q(\mathcal{A})$, $B := P(\mathcal{A})$ (with the exception of Example 2.10.15 below, this will be a kind of prototype for construction of topologically mixing degenerate fractional differential equations in the sequel of this section) and suppose that

$$\frac{Q}{P}(\Lambda') \cap \{te^{\pm i\alpha\pi/2} : t \geqslant 0\} \neq \emptyset. \tag{222}$$

Then Theorem 2.10.3 implies that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are both $\tilde{E}$-topologically mixing, with $\tilde{E} := \overline{span\{g(\nu) : \nu \in \Lambda\}}$. Now we shall briefly describe how one can employ this result in the study of topologically mixing properties of the linearized Benney–Luke equation

$$(z_0 - \Delta)\mathbf{D}_t^\alpha u = \alpha_0 \Delta u - \beta_0 \Delta^2 u \quad (0 < \alpha < 2,\ \alpha_0 > 0,\ \beta_0 > 0,\ z_0 \in \mathbb{C}), \tag{223}$$

on symmetric spaces of non-compact type, Damek–Ricci, Riemannian symmetric or Heckman–Opdam root spaces (cf. [**35**, **259**, **457**, **474**] and Example 2.10.11 below). In the integer case $\alpha = 1$, the equation (223) is also chaotic (cf. Subsection 2.3) and, as already mentioned, this equation is important in evolution modelling of some problems appearing in the theory of liquid filtration [**199**, **485**, p. 6]. Here $\mathcal{A} = \Delta$, $P(z) = z_0 - z$ and $Q(z) = \alpha_0 z - \beta_0 z^2$ ($z \in \mathbb{C}$). Observing that the equality $Ag(\nu) = e^{i\alpha\pi/2}tBg(\nu)$ holds provided that $t \in \mathbb{R}$ and $\Lambda \ni \nu = (2\beta_0)^{-1}(\alpha_0 + e^{i\alpha\pi/2}t - \sqrt{(\alpha_0 + e^{i\alpha\pi/2}t)^2 - 4\beta_0 e^{i\alpha\pi/2}tz_0})$, as well as that

$$\lim_{t\to+\infty} (2\beta_0)^{-1}\Big(\alpha_0 + e^{i\alpha\pi/2}t - \sqrt{(\alpha_0 + e^{i\alpha\pi/2}t)^2 - 4\beta_0 e^{i\alpha\pi/2}tz_0}\Big) = z_0,$$

it readily follows that the condition (222) holds provided that $z_0 \in \Lambda$. Notice that, in this case, $z_0 \in \sigma_p(\mathcal{A})$ so that the operator B is not injective.

We close this subsection by stating the following extension of [**143**, Theorem 2.3] and [**85**, Theorem 1.2].

THEOREM 2.10.7. *Suppose that $F \in \{R, L\}$, $\alpha > 0$ and $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive reals tending to $+\infty$. If the set $E_{0,F}$, which consists of those elements $y \in Z_F(A, B) \cap \tilde{E}$ for which there exists a strong solution $t \mapsto u(t; y)$, $t \geqslant 0$ of problem* $(\mathrm{DFP})_F$ *such that $\lim_{n\to\infty} u(t_n; y) = 0$, is dense in $\tilde{E}$, and if the set $E_{\infty,F}$, which consists of those elements $z \in Z_F(A, B) \cap \tilde{E}$ for which there exist a null sequence $(\omega_n)_{n\in\mathbb{N}} \in Z_F(A, B) \cap \tilde{E}$ and a sequence $(u_n(\cdot; \omega_n))_{n\in\mathbb{N}}$ of strong solutions of problem* $(\mathrm{DFP})_F$ *such that $\lim_{n\to\infty} u(t_n; \omega_n) = z$, is also dense in $\tilde{E}$, then the problem* $(\mathrm{DFP})_F$ *is $\tilde{E}$-topologically transitive.*

PROOF. The proof of theorem is quite similar to the proofs of above-mentioned theorems. Let $y,\ z \in \tilde{E}$ and $\varepsilon > 0$ be given in advance. Since $E_{0,F}$ and $E_{\infty,F}$ are both dense in $\tilde{E}$, there exist $y_0 \in E_{0,F} \cap \tilde{E}$ and $z_0 \in E_{\infty,F} \cap \tilde{E}$ such that $\|y - y_0\| < \varepsilon/2$ and $\|z - z_0\| < \varepsilon/3$. By definition of sets $E_{0,F}$ and $E_{\infty,F}$, we have that there exists a strong solution $u(\cdot; y_0)$ of problem $(\mathrm{DFP})_F$ such that

$\lim_{n\to+\infty} u(t_n;y_0)=0$ and there exist a null sequence $(\omega_n)_{n\in\mathbb{N}}\in Z_F(A,B)\cap\tilde{E}$ and a sequence $(u_n(\cdot;\omega_n))_{n\in\mathbb{N}}$ of strong solutions of problem $(\mathrm{DFP})_F$ such that $\lim_{n\to\infty} u(t_n;\omega_n)=z_0$. Let $n_0\in\mathbb{N}$ satisfy that $\|\omega_{n_0}\|<\varepsilon/2$, $\|z_0-u(t_{n_0};\omega_{n_0})\|<\varepsilon/3$ and $\|u(t_{n_0};y_0)\|<\varepsilon/3$. Put $x:=y_0+\omega_{n_0}$ and $t:=t_{n_0}$. Then

$$\|y-x\|\leqslant\|y-y_0\|+\|\omega_{n_0}\|\leqslant\frac{\varepsilon}{2}+\|\omega_{n_0}\|<\varepsilon,$$

and

$$\begin{aligned}\|z-u(t;x)\|&=\|z-u(t;y_0)-u(t;\omega_{n_0})\|\\&\leqslant\|z-z_0\|+\|z_0-u(t;\omega_{n_0})\|+\|u(t;y_0)\|<3\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

This completes the proof. □

2.10.2. Hypercyclic and topologically mixing properties of certain classes of degenerate abstract multi-term fractional differential equations. In this subsection, we shall assume that $n\in\mathbb{N}\setminus\{1\}$, $A_0=A,A_1,\ldots,A_{n-1},A_n=B$ are closed linear operators on E, $0\leqslant\alpha_1<\cdots<\alpha_{n-1}<\alpha_n$ and $0\leqslant\alpha<\alpha_n$. Fix a number $i\in\mathbb{N}^0_{m_n-1}$. Let us remind ourselves that $\alpha_0=\alpha$, $m_j=\lceil\alpha_j\rceil$ $(j\in\mathbb{N}^n_0)$, $D_i=\{j\in\mathbb{N}_{n-1}:m_j-1\geqslant i\}$ and $\mathcal{D}_i=\{j\in\mathbb{N}^0_{n-1}:m_j-1\geqslant i\}$. Recall that $T_{j,L}u(t)=A_j\mathbf{D}_t^{\alpha_j}u(t)$, $T_{j,R}u(t)=\mathbf{D}_t^{\alpha_j}A_ju(t)$ and $T_ju(t)$ denotes exactly one of terms $T_{j,L}u(t)$ or $T_{j,R}u(t)$ $(t\geqslant0,\ j\in\mathbb{N}^0_{n-1})$. In order to avoid the confusion with the existence of solutions of problem [(90)-(91)], already described in Section 2.2, we shall consider henceforth (the only exception is Remark 2.10.10(i) below) the hypercyclic and topologically mixing properties of the following degenerate abstract multi-term fractional problem:

$$\begin{aligned}&B\mathbf{D}_t^{\alpha_n}u(t)+\sum_{j=1}^{n-1}T_ju(t)=T_0u(t),\quad t\geqslant0;\\&u^{(i)}(0)=x,\quad u^{(k)}(0)=0,\ k\in\mathbb{N}^0_{m_n-1}\setminus\{i\}.\end{aligned}\tag{224}$$

This is, clearly, a problem of kind [(90)-(91)] because we can always get the term $T_0u(t)$ on the left hand side of (224).

Definition 2.10.8. Let $\tilde{E}$ be a closed linear subspace of E. Then it is said that the equation (224) is:

(i) $\tilde{E}$-hypercyclic iff there exist an element $x\in\tilde{E}$ and a strong solution $t\mapsto u_i(t;x)$, $t\geqslant0$ of (224) such that $\{u_i(t;x):t\geqslant0\}$ is a dense subset of $\tilde{E}$; such an element is called a $\tilde{E}$-hypercyclic vector of (224);
(ii) $\tilde{E}$-topologically transitive iff for every $y,z\in\tilde{E}$ and for every $\varepsilon>0$, there exist an element $x\in\tilde{E}$, a strong solution $t\mapsto u_i(t;x)$, $t\geqslant0$ of (224) and a number $t\geqslant0$ such that $\|y-x\|<\varepsilon$ and $\|z-u_i(t;x)\|<\varepsilon$;
(iii) $\tilde{E}$-topologically mixing iff for every $y,z\in\tilde{E}$ and for every $\varepsilon>0$, there exists $t_0\geqslant0$ such that, for every $t\geqslant t_0$, there exist an element $x_t\in\tilde{E}$ and a strong solution $t\mapsto u_i(t;x_t)$, $t\geqslant0$ of (224), with x replaced by x_t, such that $\|y-x_t\|<\varepsilon$ and $\|z-u_i(t;x_t)\|<\varepsilon$.

In the case $\tilde{E} = E$, it is also said that a $\tilde{E}$-hypercyclic vector of (224) is a hypercyclic vector of (224) and(eq224 is topologically transitive, resp. topologically mixing.

The following extension of [**302**, Theorem 2.3] is crucially important in the analysis of topologically mixing degenerate multi-term fractional problems.

THEOREM 2.10.9. *Suppose that* $\emptyset \neq \Omega$ *is an open connected subset of* $\mathbb{C} \setminus \{0\}$, $f : \Omega \to E \setminus \{0\}$ *is an analytic function, and* $f_j : \Omega \to \mathbb{C} \setminus \{0\}$ *is a scalar-valued function* $(1 \leqslant j \leqslant n)$ *so that*

$$(225) \qquad Af(\lambda) = f_n(\lambda)Bf(\lambda) = f_j(\lambda)A_jf(\lambda), \quad \lambda \in \Omega,\ 1 \leqslant j \leqslant n.$$

Suppose, further, that Ω_+ *and* Ω_- *are two non-empty subsets of* Ω, *and each of them admits a cluster point in* Ω. *Define* $\tilde{E} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$,

$$(226) \quad H_i(\lambda,t) := \mathcal{L}^{-1}\Biggl(\frac{z^{\alpha_n-i-1}+\sum_{j\in D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j-i-1}-\chi_{\mathcal{D}_i}(0)f_n(\lambda)z^{\alpha-i-1}}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\Biggr)(t),$$

and

$$F_i(\lambda,t) := H_i(\lambda,t)f(\lambda),$$

for any $t \geqslant 0$ *and* $\lambda \in \Omega$. *If*

$$(227) \qquad \lim_{t\to+\infty}|H_i(\lambda,t)| = +\infty, \quad \lambda \in \Omega_+ \quad \textit{and} \quad \lim_{t\to+\infty}H_i(\lambda,t) = 0, \quad \lambda \in \Omega_-,$$

then (224) *is* $\tilde{E}$*-topologically mixing; furthermore, if* $Af(\lambda)$, $Bf(\lambda)$ *and* $A_jf(\lambda)$ *belongs to* $\tilde{E}$ *for all* $\lambda \in \Omega$, *then* (224) *is* $\tilde{E}$*-topologically mixing with the operators* $A_1, \ldots, A_{n-1}$, A *and* B *replaced respectively by* $A_{1|\tilde{E}}, \ldots, A_{n-1|\tilde{E}}$, $A_{|\tilde{E}}$ *and* $B_{|\tilde{E}}$.

PROOF. First of all, let us observe that the (weak) analyticity of mapping $\lambda \mapsto f(\lambda) \in \tilde{E}$, $\lambda \in \Omega$ implies that the linear span of set $\{f(\lambda) : \lambda \in \Omega'\}$ is dense in $\tilde{E}$ for each non-empty subset Ω' of Ω which admits a cluster point in Ω. Using the analytical properties of Laplace transform stated in Theorem 1.4.10, it can be easily proved that for each $\lambda \in \Omega$ and for each real number $\nu \leqslant \alpha_n - 1$, the function

$$t \mapsto \mathcal{L}^{-1}\Biggl(\frac{z^{\nu}}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\Biggr)(t), \quad t \geqslant 0,$$

is well defined, continuous for $t \geqslant 0$ and exponentially bounded (furthermore, this function can be analytically extended to the sector $\Sigma_{\pi/2}$). Hence, the function $t \mapsto H_i(\lambda,t)$, $t \geqslant 0$ has the above-mentioned properties ($\lambda \in \Omega$). Keeping in mind the proof of [**300**, Theorem 2.3] and (225)–(227), the only non-trivial thing that should be explained is the fact that the function $u_i(t; f(\lambda)) := F_i(\lambda,t)$, $t \geqslant 0$ is a strong solution of (224) with $x = f(\lambda)$ ($\lambda \in \Omega$). By performing the Laplace transform, it can be simply verified with the help of formula (39) that the above holds if we prove that the Caputo fractional derivative $\mathbf{D}_t^{\zeta}F_i(\lambda,t)$ is well defined for any $t \geqslant 0$, $\lambda \in \Omega$ and $\zeta \in [0,\alpha_n]$. For this, observe first that there exists a sufficiently large number $\omega > 0$ such that the function $z \mapsto z^{\zeta}\widetilde{F_i(\lambda;t)}(z) - \chi_{[i+1,\infty)}(\lceil\zeta\rceil)z^{\zeta-i-1}f(\lambda)$ is identically equal to

$$\frac{z^{\alpha_n+\zeta-i-1} + \sum_{j\in D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\zeta-i-1} - \chi_{\mathcal{D}_i}(0)f_n(\lambda)z^{\alpha+\zeta-i-1}}{z^{\alpha_n} + \sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j} - f_n(\lambda)z^{\alpha}}f(\lambda),$$

for $\operatorname{Re} z > \omega$, $\lceil\zeta\rceil < i-1$, $\lambda \in \Omega$, and

$$\frac{-\sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\zeta-i-1} - f_n(\lambda)z^{\alpha+\zeta-i-1}(\chi_{\mathcal{D}_i}(0)-1)}{z^{\alpha_n} + \sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j} - f_n(\lambda)z^{\alpha}}f(\lambda),$$

for $\operatorname{Re} z > \omega$, $\lceil\zeta\rceil \geqslant i-1$ and $\lambda \in \Omega$. By the foregoing, for every $\lambda \in \Omega$ and $\zeta \in [0, \alpha_n]$, there exists a continuous exponentially bounded E-valued mapping $t \mapsto G_i(\lambda;t;\zeta)$, $t \geqslant 0$ such that

$$\widetilde{G_i(\lambda;t;\zeta)}(z) = z^{\zeta}\widetilde{F_i(\lambda;t)}(z) - \chi_{[i+1,\infty)}(\lceil\zeta\rceil)z^{\zeta-i-1}f(\lambda), \quad \operatorname{Re} z > \omega.$$

Applying the Laplace transform, it can be easily checked that, for every $\lambda \in \Omega$ and $\zeta \in [0, \alpha_n]$, one has:

$$(228)\qquad (g_{m_n} * G_i(\lambda;\cdot,\alpha_n))(t) = (g_{m_n-\alpha_n} * [F_i(\lambda,\cdot) - g_{i+1}(\cdot)f(\lambda)])(t), \quad t \geqslant 0,$$

$$(229)\qquad G_i(\lambda;t;\zeta) = (g_{\alpha_n-\zeta} * G_i(\lambda;\cdot;\alpha_n))(t), \quad t \geqslant 0, \text{ if } \lceil\zeta\rceil \geqslant i-1,$$

and

$$(230)\qquad (g_{\alpha_n-\zeta} * G_i(\lambda;\cdot;\alpha_n))(t) + g_{i+1-\lceil\zeta\rceil}(t)f(\lambda) = G_i(\lambda;t;\zeta), \quad t \geqslant 0, \text{ if } \lceil\zeta\rceil < i-1.$$

Making use of (228), we get that $F_i(\lambda,\cdot) \in C^{m_n-1}([0,\infty) : E)$ for all $\lambda \in \Omega$. Consider first the case $\lceil\zeta\rceil < i-1$. Convoluting (228) with $g_{1+\lceil\zeta\rceil-\zeta+\alpha_n-m_n}(t)$, and using after that the definition of Caputo fractional derivatives and (230), we get that $\mathbf{D}_t^{\zeta}F_i(\lambda,t) = G_i(\lambda;t;\zeta)$, $t \geqslant 0$, $\lambda \in \Omega$. Suppose finally that $\lceil\zeta\rceil \geqslant i-1$. Then the previous equality continues to hold, which can be seen by using the same arguments, with the equation (230) replaced by (229). $\square$

REMARK 2.10.10. (i) Consider the equation (224) with term $B\mathbf{D}_t^{\alpha_n}u(t)$ replaced by $\mathbf{D}_t^{\alpha_n}Bu(t)$, and endowed with the same initial conditions (therefore, the choice of initial conditions is not the same as in Section 2.2). Denote by (224)' the above described Cauchy problem. A function $u \in C^{m_n-1}([0,\infty) : E)$ is said to be a strong solution of problem (224)' iff $u^{(i)}(0) = x$, $u^{(k)}(0) = 0$, $k \in \mathbb{N}_{m_n-1}^0 \smallsetminus \{i\}$, the terms $\mathbf{D}_t^{\alpha_n}Bu(t)$ and $T_ju(t)$ are well defined and continuous for any $t \geqslant 0$, $j \in \mathbb{N}_{n-1}^0$, and $\mathbf{D}_t^{\alpha_n}Bu(t) + \sum_{j=1}^{n-1}T_ju(t) = T_0u(t)$, $t \geqslant 0$. As in Definition 2.10.8, it will be said that the problem (224)' is $\tilde{E}$-topologically mixing iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exist an element $x_t \in \tilde{E}$ and a strong solution $t \mapsto u_i(t;x_t)$, $t \geqslant 0$ of (224)', with x replaced by x_t, such that $||y - x_t|| < \varepsilon$ and $||z - u_i(t;x_t)|| < \varepsilon$. Then the assertion of Theorem 2.10.9 continues to hold for problem (224)' because $B\mathbf{D}_t^{\alpha_n}F_i(\lambda;t) = \mathbf{D}_t^{\alpha_n}BF_i(\lambda;t)$, $t \geqslant 0$, $\lambda \in \Omega$.

(ii) In the present state of our knowledge, we do not know how the equalities appearing in (227) can be proved without a direct calculation of function $H_i(\lambda, t)$.

Although formulated for degenerate equations, it should be noted that Theorem 2.10.9 can be also used for construction of new important examples of non-degenerate multi-term fractional differential equations:

EXAMPLE 2.10.11. Let $0 < \gamma \leqslant 1$, $a > 0$, $p > 2$, and let X be a symmetric space of non-compact type and rank one. Our intention is to prove some results on topologically mixing properties of the following fractional analog of strongly damped Klein–Gordon equation (see e.g. [**433**, p. 276])

$$\mathbf{D}_t^{2\gamma}u(t) + 2a\Delta_{X,p}^{\natural}\mathbf{D}_t^{\gamma}u(t) + (\Delta_{X,p}^{\natural} - b)u(t) = 0, \quad t > 0. \tag{231}$$

Let P_p be the parabolic domain defined in [**259**]; then we know that $int(P_p) \subseteq \sigma_p(\Delta_{X,p}^{\natural})$. Consider first the equation (231) with $1/2 < \gamma \leqslant 1$ and $i = 1$. Let b be any positive real number such that $b - P_p$ intersects the imaginary axis. Then we have $B = I$, $f_2(\lambda) = \lambda$, $f_1(\lambda) = \lambda/(2a(b-\lambda))$, $\lambda \in \Omega := b - int(P_p)$ and

$$\widetilde{H_1(\lambda, t)}(z) = \frac{z^{2\gamma-2} - 2a(b-\lambda)z^{\gamma-2}}{z^{2\gamma} - 2a(b-\lambda)z^{\gamma} - \lambda}, \quad \lambda \in \Omega,\ t \geqslant 0.$$

Observing that $z^{2\gamma} - 2a(b-\lambda)z^{\gamma} - \lambda = (z^{\gamma} - r_1(\lambda))(z^{\gamma} - r_2(\lambda))$, $z \in \mathbb{C}$, $\lambda \in \Omega$, where

$$r_{1,2}(\lambda) = a(\lambda - b) \pm \sqrt{a^2(\lambda-b)^2 + \lambda}, \quad \lambda \in \Omega,$$

it can be simply verified with the help of formula (43) that, for every $t \geqslant 0$ and for every $\lambda \in \Omega$ such that $a^2(\lambda - b)^2 + \lambda \neq 0$,

$$\begin{aligned} H_1(\lambda, t) = {} & \frac{t^{1-\gamma}}{2\sqrt{a^2(\lambda-b)^2+\lambda}}\big[E_{\gamma,2-\gamma}(r_1(\lambda)t^{\gamma}) - E_{\gamma,2-\gamma}(r_2(\lambda)t^{\gamma})\big] \\ & - \frac{ta(\lambda-b)}{\sqrt{a^2(\lambda-b)^2+\lambda}}\big[E_{\gamma,2}(r_1(\lambda)t^{\gamma}) - E_{\gamma,2}(r_2(\lambda)t^{\gamma})\big]. \end{aligned}$$

This implies by Theorem 1.5.1 that there exist sufficiently small numbers $\varepsilon > 0$, $x_+ > 0$ and a sufficiently large negative number $x_- < 0$ such that (227) holds with $\Omega_+ := \{\lambda \in \Omega : |\lambda - x_+| < \varepsilon\}$ and $\Omega_- := \{\lambda \in \Omega : |\lambda - x_-| < \varepsilon\}$. By Theorem 2.10.9, it readily follows that the equation (231) admits a topologically mixing solution; using the consideration given in the next example, one can simply prove that the same holds if $0 < \gamma \leqslant 1$ and $i = 0$. Observe, finally, that for certain values of complex parameters $b,\ d \in \mathbb{C}$, the method described above proves the existence of topologically mixing solutions of the following fractional analog of vibrating beam type equation

$$\mathbf{D}_t^{2\gamma}u(t) + 2(aD + bI)\mathbf{D}_t^{\gamma}u(t) + (D + dI)u(t) = 0, \quad t > 0,$$

where D denotes the square of operator $\Delta_{X,p}^{\natural}$.

At the end of this subsection, we shall present an application of Theorem 2.10.9 in the study of hypercyclic and topologically mixing properties of degenerate multi-term fractional problems involving the bounded perturbations of one-dimensional Ornstein–Uhlenbeck operators [**115**, **427**]. Observe that we can similarly construct corresponding examples of degenerate equations with multi-dimensional Ornstein–Uhlenbeck operators or with $n=3$ (cf. [**302**, Example 1, 3.-4.]).

EXAMPLE 2.10.12. Suppose $\alpha \in (0,1]$, $E := L^2(\mathbb{R})$, $c_1 > c > \frac{b}{2} > 0$, $P(z)$ is a non-zero complex polynomial, S is the set which consists of all roots of $P(z)$, $\Omega := \{\lambda \in \mathbb{C} : \lambda \notin S,\ \lambda \neq c-c_1,\ \operatorname{Re}\lambda < c-\frac{b}{2}\}$ and $\mathcal{A}_c u := u'' + bxu' + cu$ is the bounded perturbation of the one-dimensional Ornstein–Uhlenbeck operator acting with domain $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$. Define $g_1 \colon \Omega \to E$ and $g_2 \colon \Omega \to E$ by

$$g_1(\lambda) := \mathcal{F}^{-1}\big(e^{-\frac{\xi^2}{2b}}\xi|\xi|^{-(2+\frac{\lambda-c}{b})}\big)(\cdot), \quad \lambda \in \Omega$$

and

$$g_2(\lambda) := \mathcal{F}^{-1}\big(e^{-\frac{\xi^2}{2b}}|\xi|^{-(1+\frac{\lambda-c}{b})}\big)(\cdot), \quad \lambda \in \Omega.$$

Then the mapping $g_i \colon \Omega \to E$ is analytic for $i=1,2$, and for every open connected subset Ω' of Ω which admits a cluster point in Ω, one has $E = \overline{span\{g_i(\lambda) : \lambda \in \Omega',\ i=1,2\}}$; cf. [**300**, Example 2.5(iii)]. We analyze the topologically mixing properties of the equation

$$(232)\quad \Big(\frac{\partial^2}{\partial x^2} + bx\frac{\partial}{\partial x} + c_1\Big)\mathbf{D}_t^{2\alpha}u(t,x) + P\Big(\frac{\partial^2}{\partial x^2} + bx\frac{\partial}{\partial x}\Big)\mathbf{D}_t^{\alpha}u(t,x) = (c-c_1)u(t),$$

endowed with the initial conditions described in (224); here $n=2$, $\alpha_j = j\alpha$ ($j = 0,1,2$), $A = (c-c_1)I$, $B = \mathcal{A}_c + A$, $A_1 = P(\mathcal{A}_c - cI)$, $f_1(\lambda) = (c-c_1)/P(\lambda - c)$ and $f_2(\lambda) = (c-c_1)/(\lambda-(c-c_1))$ ($\lambda \in \Omega$). Put

$$r_{1,2}(\lambda) := \frac{1}{2}\Bigg[-\frac{P(\lambda-c)}{\lambda-(c-c_1)} \pm \sqrt{\Big(\frac{P(\lambda-c)}{\lambda-(c-c_1)}\Big)^2 + 4\frac{c-c_1}{\lambda-(c-c_1)}}\Bigg], \quad \lambda \in \Omega,$$

and denote by S' the set which consists of all roots of polynomial $z \mapsto (P(z-c))^2 + 4(c-c_1)(z-(c-c_1))$, $z \in \mathbb{C}$. Then

$$\widetilde{H_0(\lambda,t)}(z) = \frac{z^{2\alpha-1} + \frac{P(\lambda-c)}{\lambda-(c-c_1)}z^{\alpha-1}}{z^{2\alpha} + \frac{P(\lambda-c)}{\lambda-(c-c_1)}z^{\alpha} - \frac{c-c_1}{\lambda-(c-c_1)}}, \quad \lambda \in \Omega,\ t \geqslant 0,$$

and, in the case that $1/2 < \alpha \leqslant 1$,

$$\widetilde{H_1(\lambda,t)}(z) = \frac{z^{2\alpha-2}}{z^{2\alpha} + \frac{P(\lambda-c)}{\lambda-(c-c_1)}z^{\alpha} - \frac{c-c_1}{\lambda-(c-c_1)}}, \quad \lambda \in \Omega,\ t \geqslant 0,$$

which simply implies by the formula (43) that, for every $\lambda \in \Omega \smallsetminus S'$ and $t \geqslant 0$,

$$H_0(\lambda,t) = \frac{\big(r_1(\lambda) + \frac{P(\lambda-c)}{\lambda-(c-c_1)}\big)e^{r_1(\lambda)t}}{\sqrt{\big(\frac{P(\lambda-c)}{\lambda-(c-c_1)}\big)^2 + 4\frac{c-c_1}{\lambda-(c-c_1)}}} - \frac{\big(r_2(\lambda) + \frac{P(\lambda-c)}{\lambda-(c-c_1)}\big)e^{r_2(\lambda)t}}{\sqrt{\big(\frac{P(\lambda-c)}{\lambda-(c-c_1)}\big)^2 + 4\frac{c-c_1}{\lambda-(c-c_1)}}},$$

if $\alpha = 1$,

$$H_0(\lambda, t) = \frac{t^{-\alpha}}{\sqrt{\left(\frac{P(\lambda-c)}{\lambda-(c-c_1)}\right)^2 + 4\frac{c-c_1}{\lambda-(c-c_1)}}}\left[E_{\alpha,1-\alpha}(r_1(\lambda)t^\alpha) - E_{\alpha,1-\alpha}(r_2(\lambda)t^\alpha)\right] + \frac{P(\lambda-c)}{\lambda-(c-c_1)}\left[E_\alpha(r_1(\lambda)t^\alpha) - E_\alpha(r_2(\lambda)t^\alpha)\right],$$

if $0 < \alpha < 1$, and

$$H_1(\lambda, t) = \frac{t^{1-\alpha}}{\sqrt{\left(\frac{P(\lambda-c)}{\lambda-(c-c_1)}\right)^2 + 4\frac{c-c_1}{\lambda-(c-c_1)}}}\left[E_{\alpha,2-\alpha}(r_1(\lambda)t^\alpha) - E_{\alpha,2-\alpha}(r_2(\lambda)t^\alpha)\right],$$

if $1/2 < \alpha \leqslant 1$. Consider now the case in which $P(x) \in \mathbb{R}[x]$ and $P(\xi) > 0$ for some $\xi \in (-c_1, (-b)/2)$; albeit the last condition might seem strange, it is worth noting here that the second eqaulity in (227) does not hold provided that $\alpha = 1$ and $P(x) = -x - c_1$, $x \in \mathbb{R}$. Then there exist a sufficiently small number $\varepsilon > 0$ and a sufficiently large negative number $x_+ < 0$ such that, for every $\lambda' \in \Omega_+ \equiv \{\lambda \in \Omega : |\lambda - x_+| < \varepsilon\}$, we have $r_1(\lambda') > 0$ and $r_2(\lambda') < 0$. On the other hand, it can be simply verified that there exists $x_- \in (c - c_1, c - (b/2))$ such that, for every $\lambda' \in \Omega_- \equiv \{\lambda \in \Omega : |\lambda - x_-| < \varepsilon\}$, we have $r_1(\lambda') < 0$ and $r_2(\lambda') < 0$. Keeping in mind Theorem 1.5.1, Theorem 2.10.9 and the fact that the conclusions stated in [**302**, Remark 1;3.] can be extended to degenerate (multi-term) fractional differential equations, we have that (227) holds with sets Ω_+ and Ω_-. An endevour should be made for finding some other classes of complex polynomials $P(z)$ for which the problem (232) admits a topologically mixing solution.

2.10.3. Hypercyclic and topologically mixing properties of abstract degenerate first and second order equations. The main purpose of this subsection is to provide the basic information about hypercyclic and topologically mixing properties of abstract degenerate first and second order equations. We start by stating the following proposition, the proof is a consequence of a straightforward computation and therefore omitted (cf. [**292**, Proposition 3.2.19] for the case $\alpha = 2$).

PROPOSITION 2.10.13. (i) *Suppose that* $\alpha = 1$, $x \in Z_{1,R}(A,B)$ ($x \in Z_{1,L}(A,B)$) *and the function* $t \mapsto u(t;x)$, $t \geqslant 0$ *is a strong solution of* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*). Then, for every* $s \geqslant 0$, $u(s;x) \in Z_{1,R}(A,B)$ ($u(s;x) \in Z_{1,L}(A,B)$) *and a strong solution of* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*), with initial condition* x *replaced by* $u(s;x)$*, is given by* $u(t;u(s;x)) := u(t+s;x)$, $t \geqslant 0$.

(ii) *Suppose that* $\alpha = 2$, $x \in Z_{2,R}(A,B)$ ($x \in Z_{2,L}(A,B)$) *and the function* $t \mapsto u(t;x)$, $t \geqslant 0$ *is a strong solution of* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*). Then, for every* $s \geqslant 0$, $u(s;x) \in Z_{2,R}(A,B)$ ($u(s;x) \in Z_{2,L}(A,B)$) *and a strong solution of* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*), with initial condition* x *replaced by* $u(s;x)$*, is given by* $u(t;u(s;x)) := \frac{1}{2}[u(t+s;x) + u(|t-s|;x)]$, $t \geqslant 0$.

If $\alpha \in \{1,2\}$, $F \in \{R, L\}$, $x \in Z_{\alpha,F}(A,B)$, the strong solutions of $(\mathrm{DFP})_F$ are unique, $t > 0$ and $u(t;x) = x$, then Proposition 2.10.13 implies that $u(nt;x) = x$,

$n \in \mathbb{N}$, so that the notion of a periodic point of problem $(\mathrm{DFP})_F$ is meaningful for such class of degenerate first and second order equations. In contrast to this, if the strong solutions of problem $(\mathrm{DFP})_F$ are not unique, the supposition $u(t;x) = x$ for some $t > 0$ does not imply the validity of equality $u(nt;x) = x$, $n \in \mathbb{N}$, in general (cf. the final part of Example 2.10.15), so that Definition 2.10.14(i) below can be viewed only as an attempt to define satisfactorily the notion of (subspace) chaoticity of problem $(\mathrm{DFP})_F$. Compared with the classical Devaney definition of chaos, we assume here the (subspace) topological transitivity of problem $(\mathrm{DFP})_F$ in place of its hypercyclicity.

Recall that the symbol $E \oplus E$ ($\tilde{E} \oplus \tilde{E}$) denotes the direct sum of Banach spaces E and E ($\tilde{E}$ and $\tilde{E}$). Define $D(A \oplus A) := D(A) \times D(A)$ ($D(B \oplus B) := D(B) \times D(B)$) and $A \oplus A(x,y) := (Ax, Ay)$, $x, y \in D(A)$ ($B \oplus B(x,y) := (Bx, By)$, $x, y \in D(B)$). Then $A \oplus A$ and $B \oplus B$ are closed linear operators on $E \oplus E$.

Definition 2.10.14. Let $\alpha \in \{1,2\}$, and let $\tilde{E}$ be a closed linear subspace of E. Then it is said that the problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is:

- (i) $\tilde{E}$-chaotic, if $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is $\tilde{E}$-topologically transitive and the set of $\tilde{E}$-periodic points of $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$), i.e., the set of those elements $x \in Z_{\alpha,R}(A,B) \cap \tilde{E}$ ($x \in Z_{\alpha,L}(A,B) \cap \tilde{E}$) for which there exist a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$), and a positive real number $t > 0$ such that $u(t;x) = x$, is dense in $\tilde{E}$,
- (ii) $\tilde{E}$-weakly mixing, if the problem $(\mathrm{DFP})_R^{\oplus}$ ($(\mathrm{DFP})_L^{\oplus}$), obtained by replacing the operators A and B in problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) with the operators $A \oplus A$ and $B \oplus B$, respectively, is ($\tilde{E} \oplus \tilde{E}$)-hypercyclic in $E \oplus E$.

Recall that we have already introduced the notions of (subspace) hypercyclicity, topological transitivity and topological mixing property for problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) in Definition 2.10.8. If $\alpha = 1$ and the assumptions of Theorem 2.10.3 hold, then it can be easily seen (see e.g. the proof of [**143**, Theorem 3.1]) that the set of those elements $x \in Z_{\alpha,R}(A,B) \cap \tilde{E}$ ($x \in Z_{\alpha,L}(A,B) \cap \tilde{E}$) for which there exist a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$), and a positive real number $t > 0$ such that $u(nt;x) = x$ for all $n \in \mathbb{N}$, is dense in $\tilde{E}$ (no matter whether the strong solutions of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) are unique or not); in particular, the set of $\tilde{E}$-periodic points of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is dense in $\tilde{E}$, and $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is $\tilde{E}$-chaotic.

In the following illustrative example, we shall present some new principal features of hypercyclic degenerate equations in comparision with non-degenerate ones (cf. also the paragraph directly after Definition 2.10.1).

Example 2.10.15. Let $I := [0,\infty)$, $1 \leqslant p < \infty$, let $\rho : I \to (0,\infty)$ be an admissible weight function on I, and let $E := L^p_\rho(I,\mathbb{C})$ (cf. [**143**, Definition 4.1, Definition 4.3]). Then the set $\mathcal{D}_{(0,\infty)}$ is dense in E. Consider the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ with $\alpha = 1$ and the closed linear operators $Bu(x) := u'(x) + u(x)$, $x \geqslant 0$ and $Au(x) := u''(x) - u(x)$, $x \geqslant 0$, acting with their maximal distributional domains. We will prove that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are topologically mixing, independently of the choice of $\rho(\cdot)$. In order to do that, suppose that

$y, z \in \mathcal{D}_{(0,\infty)}$ and $\varepsilon > 0$ are given in advance; let $y, z \in \mathcal{D}_{(0,l)}$ for some $l > 0$. Then there exists a C^∞ strictly decaying function $H\colon (-\infty, l] \to (0,\infty)$ such that

$$H(l-x) \geqslant \rho(x+l), \quad x \geqslant 0. \tag{233}$$

Let $G\colon \mathbb{R} \to (0,\infty)$ by any C^∞-function such that

$$G(x) \leqslant 1, \quad x \geqslant l+1 \qquad \text{and} \qquad G(x) = e^{-2x}(1+H(x))^{1/p}, \quad x \leqslant 0. \tag{234}$$

Then a simple computation shows that, for every $\varphi \in \mathcal{D}_{(0,\infty)}$, the function

$$u(t; G(x)\varphi(x)) := e^{-t}G(x-t)\varphi(x+t), \quad x \geqslant 0, \ t \geqslant 0, \tag{235}$$

is a strong solution of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, with $u(0)(x) = G(x)\varphi(x)$, $x \geqslant 0$. Define, for every $t \geqslant 0$, the function $z_t\colon I \to \mathbb{C}$ by

$$z_t(x) := 0 \text{ for } x < t \text{ and } z_t(x) := e^t \frac{u(x-t)}{G(x-2t)} \text{ for } x \geqslant t.$$

Then $z_t \in \mathcal{D}_{(0,\infty)}$ and $u(t; G(x)z_t(x)) = z(x)$, $x \geqslant 0$ $(t \geqslant 0)$. The existence of positive real number $t_0 \geqslant l+1$ such that the requirements of part (iii) of Definition 2.10.1 hold simply follows if we prove that $\lim_{t\to+\infty} G(x)z_t(x) = 0$ in E (then, for every $t \geqslant t_0$, we can put $x_t(\cdot) := y(\cdot) + G(\cdot)z_t(\cdot)$, observe only that $u(t; G(x)(y(x)/G(x))) = 0$ for $t \geqslant l$). This can be proved by using (233)–(234), [**143**, Lemma 4.2] and the following calculus (with a suitable constant $M_1 > 0$ depending only on l and $\rho(\cdot)$):

$$\begin{aligned}
\|G(x)z_t(x)\|^p &= e^{tp}\int_t^{t+l} \left|\frac{z(x-t)G(x)}{G(x-2t)}\right|^p \rho(x)dx \\
&\leqslant M_1 e^{tp}\rho(t+l)\int_0^l \left|\frac{z(x)G(x+t)}{G(x-t)}\right|^p dx \\
&\leqslant M_1 e^{-tp}\rho(t+l)\int_0^l \frac{|z(x)|^p e^{2xp}}{1+H(x-t)}dx \\
&\leqslant M_1 e^{-tp}\int_0^l |z(x)|^p e^{2xp}dx, \quad t \geqslant l+1.
\end{aligned}$$

The above conclusions continue to hold, with almost the same proof, if $I = \mathbb{R}$, $\rho\colon I \to (0,\infty)$ is an admissible weight function on I, and $E := L^p_\rho(I,\mathbb{C})$. Strictly speaking, for any $\varepsilon > 0$ and $y, z \in \mathcal{D}_{(-l,l)}$ given in advance, we can simply construct a C^∞ function $G\colon \mathbb{R} \to (0,\infty)$ such that $G(x) \sim e^{ax}$, $x \to -\infty$ for some $a > \max(0, (\omega - p)/2p)$, and

$$G(x+t) \leqslant G(t-l) \leqslant \rho(t+l)^{(-1)/p}e^{-at-t^2}, \quad x \geqslant -l, \ t \geqslant 0;$$

here we assume that $\omega \geqslant 0$ and there exists $M \geqslant 1$ such that $\rho(x) \leqslant Me^{\omega t}\rho(x+t)$, $x, t \in \mathbb{R}$. Arguing as in the case $I = [0,\infty)$, it can be easily seen that there exists a finite constant $M_2 \geqslant 1$ such that

$$\int_{-\infty}^{\infty} \left|e^{-t}G(x-t)\frac{y}{G}(x+t)\right|^p \rho(x)dx$$

$$\leqslant Me^{(\omega-p)t}\Big\|\frac{y}{G}(x)\Big\|^p \sup_{x\in[-l-2t,l-2t]} G(x)^p \to 0 \text{ as } t\to+\infty,$$

and

$$\begin{aligned}\Big\|G(x)e^t\frac{u(x-t)}{G(x-2t)}\Big\|^p &= \int_{t-l}^{t+l}\Big|G(x)e^t\frac{u(x-t)}{G(x-2t)}\Big|^p\rho(x)dx\\ &\leqslant M_2e^{tp}\rho(t+l)\int_{-l}^{l}|u(x)|^p\Big|\frac{G(x+t)}{G(x-t)}\Big|^p dx\\ &\leqslant M_2e^{tp-t^2}\int_{-l}^{l}|u(x)|^p e^{-ax}dx\to 0 \text{ as } t\to+\infty.\end{aligned}$$

Then we can prove in a routine manner that the corresponding problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are topologically mixing. It is also worth noting that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are chaotic in the sense of Definition 2.10.14(i), we shall only outline the main details needed for the proof of this fact in the case that $I=\mathbb{R}$. Clearly, it suffices to show that, for every $l>0$, $\varepsilon>0$, $\varphi\in\mathcal{D}_{(-l,l)}$ and $P>2l$ given in advance, we can find a function $v_P\in E$ such that $\|v_P-\varphi\|<\varepsilon$ and $u(P;v_P(x))=v_P(x)$, $x\in\mathbb{R}$; cf. (235). Let $G\colon\mathbb{R}\to(0,\infty)$ be any C^∞ function such that $G(x)=1$, $x\in[-l-P,l]$. Using the calculation similar to that already employed in the proof of [**416**, Theorem 2], we have that there exists a finite constant $M_3\geqslant1$ such that the function

$$v_{P,G}(x):=\varphi(x)+\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}e^{nP}\frac{G(x)G(x-P)}{G(x-nP)G(x-(n+1)P)}\varphi(x-nP),\quad x\geqslant0,$$

satisfies $u(P;v_{P,G}(x))=v_{P,G}(x)$, $x\in\mathbb{R}$ and

$$\|v_{P,G}-\varphi\|^p\leqslant M_3\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\rho(nP+l)e^{nP}\int_{-l}^{l}|G(x+nP)G(x+(n-1)P)|^p\varphi(x)dx.$$

The term appearing on the right hand side of the above inequality will not exceed ε if we choose $G(\cdot)$ sufficiently small on intervals $[-l+nP,l+nP]$ $(n\in\mathbb{N})$ and $[-l-nP,l-nP]$ $(n\in\mathbb{N}\smallsetminus\{1\})$. The problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are also weakly mixing; we leave the proof to the reader.

The assertion of [**163**, Theorem 2.1] in which S. El Mourchid has investigated the connection between the imaginary point spectrum and hypercyclicity of strongly continuous semigroups cannot be reformulated for degenerate fractional equations. The following theorem shows that the above-mentioned result can be extended to linear Sobolev equations of first order:

THEOREM 2.10.16. *Assume that* $\alpha=1$, $\omega_1,\omega_2\in\mathbb{R}\cup\{-\infty,\infty\}$, $\omega_1<\omega_2$, $t_0>0$ *and* $k\in\mathbb{N}$. *Let* $f_j\colon(\omega_1,\omega_2)\to E$ *be a Bochner integrable function with the property that, for every* $j=1,\dots,k$, *we have* $Af_j(s)=isBf_j(s)$ *for a.e.* $s\in(\omega_1,\omega_2)$. *Put* $\psi_{r,j}:=\int_{\omega_1}^{\omega_2}e^{irs}f_j(s)ds$, $r\in\mathbb{R}$, $1\leqslant j\leqslant k$.

(i) *Assume that* $span\{f_j(s):s\in(\omega_1,\omega_2)\smallsetminus\Omega,\ 1\leqslant j\leqslant k\}$ *is dense in* E *for every subset* Ω *of* (ω_1,ω_2) *with zero measure. Let the following condition hold*

(PS): *The functions* $s \mapsto Bf_j(s)$, $s \in (\omega_1, \omega_2)$ *and* $s \mapsto sBf_j(s)$, $s \in (\omega_1, \omega_2)$ ($s \mapsto sf_j(s)$, $s \in (\omega_1, \omega_2)$ *and* $s \mapsto sBf_j(s)$, $s \in (\omega_1, \omega_2)$) *are Bochner integrable for* $1 \leqslant j \leqslant k$.

Then the problem $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*) is topologically mixing.*

(ii) *Put* $\tilde{E} := \overline{span\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}}$. *If* (PS) *holds, then the problem* $(\mathrm{DFP})_R$ *(*$(\mathrm{DFP})_L$*) is* $\tilde{E}$*-topologically mixing.*

SKETCH OF PROOF. The proof of theorem is quite similar to those of [**163**, Theorem 2.1] and [**291**, Theorem 3.1.42(i)], and we shall only outline here that the validity of condition (PS) implies by direct calculations that $span\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}$ is contained in $Z_{1,R}(A,B)$ ($Z_{1,L}(A,B)$), as well as that $u(t;\psi_{r,j}) = \psi_{r+t,j}$, $t \geqslant 0$ provided $r \in \mathbb{R}$ and $1 \leqslant j \leqslant k$.

For more details about Theorem 2.10.16, we refer the reader to [**163**, Remark 2.2–Remark 2.4] and [**291**, Remark 3.1.43]; observe only that the uniqueness of strong solutions of $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) implies that the operator $T_R(t_0;\cdot)$ ($T_L(t_0;\cdot)$) is topologically mixing, provided that (i) holds, and that the part of operator $T_R(t_0;\cdot)$ ($T_L(t_0;\cdot)$) in $\tilde{E}$ is topologically mixing in the Banach space $\tilde{E}$, provided that (ii) holds. We shall illustrate Theorem 2.10.16 with the following simple modification of Example 2.10.5.

EXAMPLE 2.10.17. Let $E := BUC(\mathbb{R})$ or $E := C_b(\mathbb{R})$, and let $\alpha = 1$, $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$. Suppose that $P(z)$ and $Q(z)$ are non-zero complex polynomials such that, for every $s \in (\omega_1, \omega_2)$, all zeroes of polynomial $z \mapsto R_s(z) \equiv P(z) - isQ(z)$, $z \in \mathbb{C}$ lie on the imaginary axis. Denote by $ig_1(s), \ldots, ig_k(s)$ the zeroes of $R_s(z)$. Then there exists a positive function $s \mapsto h(s)$, $s \in \mathbb{R}$ such that the condition (PS) stated in the formulation of Theorem 2.10.16 holds with the operators $A := P(d/dx)$ and $B := Q(d/dx)$, acting with maximal distributional domains, and with functions $f_j(s)x = h(s)^{-1}e^{ig_j(s)x}$ ($s \in (\omega_1, \omega_2)$, $1 \leqslant j \leqslant k$, $x \in \mathbb{R}$). Define $\tilde{E}$ as in the formulation of part (ii) of Theorem 2.10.16. Then the corresponding problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are both $\tilde{E}$-topologically mixing (it should be observed that the uniqueness of strong solutions of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ holds in a great number of particular choices of polynomials $P(z)$ and $Q(z)$ satisfying the above requirements, see e.g. [**6**] and [**306**, Theorem 4.2, Remark 4.4]).

In order to formulate The Hypercyclicity Criterion for degenerate abstract differential equations of first order, we need to introduce some additional notions. If $x \in Z_{1,R}(A,B)$ ($x \in Z_{1,L}(A,B)$), then we denote by $S_{R,x}$ ($S_{L,x}$) the set which consists of all strong solutions of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$). Then a linear mapping $\mathcal{T}_R \colon Z_{1,R}(A,B) \to C([0,\infty) : E)$ ($\mathcal{T}_L \colon Z_{1,R}(A,B) \to C([0,\infty) : E)$) is said to be regular if $\mathcal{T}_R(x) \in S_{R,x}$, $x \in Z_{1,R}(A,B)$ ($\mathcal{T}_L(x) \in S_{L,x}$, $x \in Z_{1,L}(A,B)$) and (cf. Proposition 2.10.13(i))

$$\mathcal{T}_F(\mathcal{T}_F(x)(t))(s) = \mathcal{T}_F(x)(t+s), \quad x \in Z_{1,F}(A,B),\ t,s \geqslant 0,\ F \in \{R,L\}.$$

If $\mathcal{T}_R \colon Z_{1,R}(A,B) \to C([0,\infty) : E)$ ($\mathcal{T}_L \colon Z_{1,R}(A,B) \to C([0,\infty) : E)$) is a regular mapping and $t \geqslant 0$, then we define the linear mapping $T_{R,t,\mathcal{T}} \colon Z_{1,R}(A,B) \to$

$Z_{1,R}(A,B)$ ($T_{L,t,\mathcal{T}}\colon Z_{1,L}(A,B) \to Z_{1,L}(A,B)$) by $T_{R,t,\mathcal{T}}(x) := (\mathcal{T}_R(x))(t)$, $x \in Z_{1,R}(A,B)$ ($T_{L,t,\mathcal{T}}(x) := (\mathcal{T}_L(x))(t)$, $x \in Z_{1,L}(A,B)$); notice that in the case of uniqueness of strong solutions of $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$), one has $T_{F,t,\mathcal{T}}(x) = T_F(t;x)$, $t \geqslant 0$, $x \in Z_{1,F}(A,B)$, $F \in \{R,L\}$. A closed linear subspace $\tilde{E}$ of E is said to be $\mathcal{T}_R$-admissible ($\mathcal{T}_L$-admissible) if

$$T_{F,t,\mathcal{T}}(Z_{1,F}(A,B) \cap \tilde{E}) \subseteq Z_{1,F}(A,B) \cap \tilde{E} \text{ for } F = R \ (F = L).$$

Observing that for any regular mapping $\mathcal{T}_F(\cdot)$ we have

$$T_{F,t,\mathcal{T}_F}(T_{F,s,\mathcal{T}_F}(x)) = T_{F,t+s,\mathcal{T}_F}(x), \quad x \in Z_{1,F}(A,B),\ t,s \geqslant 0,\ F \in \{R,L\},$$

one can repeat verbatim the proof of [**291**, Theorem 3.1.34] so as to verify that the following theorem holds true.

THEOREM 2.10.18 (The Hypercyclicity Criterion for Degenerate First Order Equations). *Let $F \in \{R,L\}$, let $\mathcal{T}_F\colon Z_{1,F}(A,B) \to C([0,\infty) : E)$ be regular, and let $\tilde{E}$ be a $\mathcal{T}_F$-admissible closed linear subspace of E. Assume that there exist subsets $\overline{Y_1}, \overline{Y_2} \subseteq Z_{1,F}(A,B) \cap \tilde{E}$, both dense in $\tilde{E}$, a mapping $\bar{S}\colon \overline{Y_1} \to \overline{Y_1}$ and a bounded linear operator D in $\tilde{E}$ such that:*

(i) $T_{F,s,\mathcal{T}_F}(\bar{S}y) = y$, $y \in \overline{Y_1}$,
(ii) $\lim_{n\to\infty} \bar{S}^n y = 0$, $y \in \overline{Y_1}$,
(iii) $\lim_{n\to\infty} T_{F,n,\mathcal{T}_F}(\omega) = 0$, $\omega \in \overline{Y_2}$,
(iv) $R(D)$ *is dense in* $\tilde{E}$,
(v) $R(D) \subseteq Z_{1,F}(A,B) \cap \tilde{E}$, $T_{F,n,\mathcal{T}_F} D \in L(\tilde{E})$, $n \in \mathbb{N}$ *and*
(vi) $DT_{F,1,\mathcal{T}_F}(x) = T_{F,1,\mathcal{T}_F}(Dx)$, $x \in Z_{1,F}(A,B) \cap \tilde{E}$.

Then the abstract Cauchy problem $(\mathrm{DFP})_F$ *is both* $(\tilde{E}\oplus\tilde{E})$*-hypercyclic and* $(\tilde{E}\oplus\tilde{E})$*-topologically transitive; in particular,* $(\mathrm{DFP})_F$ *is* $\tilde{E}$*-weakly mixing.*

Before proceeding further, it should be noticed that Theorem 2.10.18 clarifies only sufficient conditions for the $(\tilde{E} \oplus \tilde{E})$-hypercyclicity and $(\tilde{E} \oplus \tilde{E})$-topological transitivity of the abstract Cauchy problem $(\mathrm{DFP})_F$; because of that, we can hardly name this theorem as criterion. Observe, however, that there exist much more important results named as criteria, like the Desch–Schappacher–Webb or Banasiak–Moszyński criterion, which clarify only sufficient conditions, here concretly, for the (subspace) chaoticity of strongly continuous semigroups.

In some concrete situations, the assumptions of Theorem 2.10.18 can be relaxed; for example, if $B = I$, $\tilde{E} = E$ and if A generates a global C-regularized semigroup, with $R(C)$ being dense in E, then the conditions (iv)–(vi) automatically hold (cf. [**291**, Subsection 3.1.4] and [**137**, Theorem 3.4]). Our impossibility to define a mild solution of problem $(\mathrm{DFP})_L$ in a proper way, or to conclude that there exists a mild solution of problem $(\mathrm{DFP})_R$ for any $x \in E$, has a series of obvious unpleasant consequences concerning extending [**292**, Theorems 3.1.13-3.1.14, 3.1.16; Proposition 3.1.17] to weakly mixing degenerate Cauchy problems of first order. In contrast to the above, [**85**, Corollary 1.3, Theorem 1.4] (cf. also [**292**, Theorem 3.2.26(ii)–(iii)]) can be rephrased for degenerate second order problems:

THEOREM 2.10.19. *Let $\tilde{E}$ be a closed linear subspace of E, let $\alpha = 2$, and let $F \in \{R, L\}$.*

(i) *Suppose that $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive reals tending to $+\infty$. Denote by $X_{1,\tilde{E}}$ the set which consists of those elements $x \in Z_F(A,B) \cap \tilde{E}$ for which there exists a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem* $(\mathrm{DFP})_F$ *such that $u(0;x) = x$ and $\lim_{n\to\infty} u(t_n;x) = \lim_{n\to\infty} u(2t_n;x) = 0$. If $X_{1,\tilde{E}}$ is dense in $\tilde{E}$, then the problem* $(\mathrm{DFP})_F$ *is $\tilde{E}$-topologically transitive.*

(ii) *Denote by $X'_{1,\tilde{E}}$ the set which consists of those elements $x \in Z_F(A,B)\cap\tilde{E}$ for which there exists a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem* $(\mathrm{DFP})_F$ *such that $u(0;x) = x$ and $\lim_{t\to+\infty} u(t;x) = 0$. If $X'_{1,\tilde{E}}$ is dense in $\tilde{E}$, then the problem* $(\mathrm{DFP})_F$ *is $\tilde{E}$-topologically mixing.*

PROOF. Denote by $E_{0,F}$ the set which consists of those elements $y \in Z_F(A,B)\cap \tilde{E}$ for which there exists a strong solution $t \mapsto u(t;y)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_F$ such that $\lim_{n\to\infty} u(t_n;y) = 0$, and by $E_{\infty,F}$ the set which consists of those elements $z \in Z_F(A,B) \cap \tilde{E}$ for which there exist a null sequence $(\omega_n)_{n\in\mathbb{N}} \in Z_F(A,B) \cap \tilde{E}$ and a sequence $(u_n(\cdot;\omega_n))_{n\in\mathbb{N}}$ of strong solutions of problem $(\mathrm{DFP})_F$ such that $\lim_{n\to\infty} u(t_n;\omega_n) = z$. Then we have the obvious inclusion $X_{1,\tilde{E}} \subseteq E_{0,F}$, so that $E_{0,F}$ is dense in $\tilde{E}$. Now we will prove that $X_{1,\tilde{E}} \subseteq E_{\infty,F}$. In order to do that, suppose $z \in X_{1,\tilde{E}}$. Set $\omega_n := 2u(t_n;z)$, $n \in \mathbb{N}$. Then $\lim_{n\to+\infty} \omega_n = 0$, and Proposition 2.10.13(ii) implies that $\omega_n \in Z_F(A,B)$, $n \in \mathbb{N}$; the corresponding strong solution is given by $u_n(s;\omega_n) := u(s+t_n;z) + u(|t_n - s|;z)$, $s \geqslant 0$, $n \in \mathbb{N}$. The prescribed assumptions imply that $u_n(t_n;\omega_n) = u(2t_n;z) + z$, $n \in \mathbb{N}$, so that $\lim_{n\to+\infty} u(t_n : \omega_n) = z$. Therefore, $E_{\infty,F}$ is also dense in $\tilde{E}$ and the proof of (i) follows from an application of Theorem 2.10.7. The proof of (ii) is quite similar and therefore omitted. □

In order to illustrate Theorem 2.10.19(ii), suppose that $I := \mathbb{R}$, $\rho\colon I \to (0,\infty)$ is an admissible weight function on I, and $E := L^p_\rho(I,\mathbb{C})$. Let the operators $(Bu)(x) := u''(x) + 2u'(x) + u(x)$, $x \geqslant 0$ and $(Au)(x) := u^{(iv)}(x) - 2u''(x) + u(x)$, $x \geqslant 0$ act with their maximal distributional domains. Then, for every C^∞ function $G\colon \mathbb{R} \to (0,\infty)$ and for every $\varphi \in \mathcal{D}$, the function

$$u(t;G(x)\varphi(x)) := \tfrac{1}{2}[e^{-t}G(x-t)\varphi(x+t) + e^{t}G(x+t)\varphi(x-t)], \quad x \in \mathbb{R},\ t \geqslant 0,$$

is a strong solution of problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) with $\alpha = 2$, and $u(0;G(x)\varphi(x)) = G(x)\varphi(x)$, $x \geqslant 0$. Arguing as in Example 2.10.15 we can prove that $\mathcal{D} \subseteq X'_{1,\tilde{E}}$. Hence, the problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is topologically mixing by Theorem 2.10.19(ii).

For further information on hypercyclicity and topologically mixing property of non-degenerate differential equations of first and second order, we refer the reader to the work of T. Kalmes [**264**, **266**]. Li–Yorke chaotic properties of abstract non-degenerate differential equations of first order has recently been analyzed in [**328**]. As pointed out there, we are not in a position to apply the method from our previous research of distributionally chaotic properties of linear operators [**112**] (cf. also

Section 2.11 and [**68**]) in the analysis of Li–Yorke chaotic properties of abstract non-degenerate fractional differential equations. For more more details about Li–Yorke chaotic and distributionally chaotic properties of abstract degenerate differential equations, we refer the reader to [**294**].

2.10.4. $\mathcal{D}$-Hypercyclic and $\mathcal{D}$-topologically mixing properties of degenerate fractional differential equations. In this subsection, we shall work in the setting of separable infinite-dimensional Fréchet spaces over the field $\mathbb{C}$. Let E be such a space; the use of symbol $\mathcal{D}$ is clear from the context.

In [**237**], K.-G. Grosse-Erdmann and S. G. Kim have proposed the way of computing the orbit of a pair (x, y) under the action of a bilinear mapping $B\colon E \times E \to E$, with E being a separable Banach space. After that, the notion of bihypercyclicity of mapping B has been introduced. Several interesting examples of bihypercyclic bilinear mappings have been presented in [**237**], showing also that every separable Banach space supports a bihypercyclic bilinear mapping as well as that every separable Banach space E supports a bihypercyclic symmetric bilinear mapping whenever E supports a non-injective hypercyclic operator. In the setting of infinite-dimensional separable Fréchet spaces, a slightly different way of computing the orbit of a pair (x, y) under the action of bilinear mapping B has been proposed by J. Bès and J. A. Conejero in [**70**, Definition 1]. The notion of orbit of a tuple $(x_1, x_2, \ldots, x_n) \in E^N$ under the action of an N-linear operator $M : E^n \to E$ and the notion of supercyclicity of the operator M have been introduced in the same definition, while the notion of N-linear Devaney chaos of M has been introduced in [**70**, Definition 18] ($N \geqslant 2$). In [**70**, Theorem 5, Theorem 8], it has been proved that every separable infinite-dimensional Fréchet space E supports, for any integer $N \geqslant 2$, an N-linear operator having a residual set of supercyclic vectors as well as that, for any integer $N \geqslant 2$, there exists an N-linear operator on the space $\omega = \mathbb{K}^{\mathbb{N}}$ (endowed with the product topology) that supports a dense N-linear orbit. The existence of hypercyclic N-linear operators on the Fréchet space $\mathrm{H}(\mathbb{C})$ has been investigated in [**70**, Section 4].

Following the approaches used in [**237**] and [**70**], we define the orbits $\mathrm{Orb}(S; (B_i)_{1\leqslant i\leqslant l})$ and $\mathrm{Orb}(S; (M_i)_{1\leqslant i\leqslant l})$ for any non-empty subset S of E^N and any mappings $B_i : (E^N)^{b_i} \to E^N$, $M_i : E^N \to E$ ($1 \leqslant i \leqslant l$); before going any further, it would be worthwhile to note that, in our analysis, these mappings need not be (separately) linear or continuous. Having this done, we have an open door (after a necessary patching up with some technicalities concerning the well-posedness of problem [(236)–(237)] below) to introduce the notions of $\mathfrak{D}$-hypercyclicity and $\mathfrak{D}$-topologically mixing property of degenerate abstract multi-term fractional problems (for more details, cf. Definition 2.10.20). In Theorem 2.10.22 and Theorem 2.10.23, we reformulate [**237**, Theorem 2] in our context, and prove the conjugacy lemma for abstract degenerate multi-term fractional differential equations. The main objective in Theorem 2.10.24 is to clarify the kind of Desch–Schappacher–Webb and Banasiak–Moszyński criteria ([**48**, **122**, **143**]) for $\mathfrak{D}$-topologically mixing of certain classes of abstract degenerate higher-order differential equations with integer order derivatives. As explained in Remark 2.10.25(iii), Theorem 2.10.24 cannot be so easily transmitted to abstract degenerate differential equations with Caputo fractional derivatives.

Let $n \in \mathbb{N} \setminus \{1\}$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$, $f \in C([0,\infty) : E)$, and let $A_0, A_1, \ldots, A_{n-1}, B$ be closed linear operators on E. Denote $m_i = \lceil \alpha_i \rceil$, $i \in \mathbb{N}_n^0$, $A_0 = A$, $A_n = B$, $T_{i,L}u(t) = A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geqslant 0$, $i \in \mathbb{N}_n^0$ and $\alpha_i > 0$, $T_{i,R}u(t) = \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geqslant 0$ and $i \in \mathbb{N}_n^0$. Henceforth we shall always assume that, for every $t \geqslant 0$ and $i \in \mathbb{N}_n^0$, $T_i u(t)$ denotes either $T_{i,L}u(t)$ or $T_{i,R}u(t)$. Denote, with a little abuse of notation,

$$P_\lambda = \lambda^{\alpha_n} B + \sum_{i=0}^{n-1} \lambda^{\alpha_i} A_i, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

$\mathcal{I} = \{i \in \mathbb{N}_n^0 : \alpha_i > 0$ and $T_{i,L}u(t)$ appears on the left hand side of (236)$\}$, $Q = \max \mathcal{I}$, if $\mathcal{I} \neq \emptyset$ and $Q = m_Q = 0$, if $\mathcal{I} = \emptyset$. In this subsection, we introduce and further analyze the notions of $\mathcal{D}$-hypercyclicity and $\mathcal{D}$-topologically mixing property of the following homogeneous degenerate abstract multi-term problem:

$$\sum_{i=0}^{n} T_i u(t) = 0, \quad t \geqslant 0, \tag{236}$$

equipped with the following initial conditions (cf. also Section 2.2; the problem is the same but the summation of $T_i u(t)$ is taken over all i going from 1 to $n-1$):

$$u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant m_Q - 1 \text{ and } (A_i u)^{(j)}(0) = u_{i,j} \text{ if } m_i - 1 \geqslant j \geqslant m_Q. \tag{237}$$

If $T_n u(t) = T_{n,L}u(t)$, then (237) reads as follows:

$$u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant m_n - 1.$$

Although the introduced notions seem to be new even for non-degenerate abstract differential equations of first order [**445**], we shall focus our attention almost completely on degenerate multi-term problems.

In Subsection 2.10.1, we have investigated the hypercyclicity and topologically mixing property of the equations $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ with $x_0 = x$ and $x_1 = \cdots = x_{m-1} = 0$, as well as the problem

$$B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=0}^{n-1} T_i u(t) = 0, \quad t \geqslant 0; \ u^{(j)}(0) = u_j, \ j = 0, \ldots, m_n - 1, \tag{238}$$

provided that there exists an index $i \in \mathbb{N}_{m_n-1}^0$ such that $u_j = 0$, $j \in \mathbb{N}_{m_n-1}^0 \setminus \{i\}$. Here we continue the study of hypercyclicity and topologically mixing property of problems (238) and $(\mathrm{DFP})_R$ by allowing that there exist two or more non-zero components of the tuple $(u_0, \ldots, u_{m_n-1})$ (i.e., the tuple $(Bu_0, \ldots, Bu_{m-1})$ in the case of consideration of problem $(\mathrm{DFP})_R$). The analysis of $\mathcal{D}$-hypercyclicity and $\mathcal{D}$-topologically mixing property of problem [(236)–(237)] is very intricate in general case and, with the exception of some minor facts and results concerning the existence and uniqueness of solutions, the most general abstract form of problem [(236)–(237)] will not be considered in this subsection any longer (cf. Subsection

2.11.2 for some results on distributionally chaotic properties of this problem). The notion of a strong solution of problem [(236)–(237)] will be understood in the sense of Definition 2.3.1.

For any $p \in \mathbb{N}$ and $r \in \mathbb{N}_p$, we define $\mathrm{Proj}_{r,p} : E^p \to E$ by $\mathrm{Proj}_{r,p}(x_1, \ldots, x_p) := x_r$, $\vec{x} = (x_1, \ldots, x_p) \in E^p$. Denote by $\mathfrak{T}$ the exact number of initial values subjected to the equation [(236)–(237)]; in other words, $\mathfrak{T}$ is the sum of number m_Q and the cardinality of set consisting of those pairs $(i, j) \in \mathbb{N}_n \times \mathbb{N}^0_{m_n-1}$ for which $m_i - 1 \geqslant j \geqslant m_Q$. To make this thing more precise, suppose that $\{i_1, \ldots, i_s\} = \{i \in \mathbb{N}_n : m_i - 1 \geqslant m_Q\}$ and $i_1 < \cdots < i_s$. Then the set of all initial values appearing in (237) is given by $\{u_0, \ldots, u_{m_Q-1}; u_{i_1,m_Q}, \ldots, u_{i_1,m_{i_1}-1}; \ldots; u_{i_s,m_Q}, \ldots, u_{i_s,m_{i_s}-1}\}$ $= \{(u_j)_{0 \leqslant j \leqslant m_Q-1}; (u_{i_{s'},j})_{1 \leqslant s' \leqslant s, m_Q \leqslant j \leqslant m_{i_{s'}}-1}\}$ so that $\mathfrak{T} = m_{i_1} + \cdots + m_{i_s} + (1 - s)m_Q$. Denote by $\mathfrak{Z}$ ($\mathfrak{Z}_{uniq}$) the set of all tuples of initial values $\vec{x} = ((u_j)_{0 \leqslant j \leqslant m_Q-1};$ $(u_{i_{s'},j})_{1 \leqslant s' \leqslant s, m_Q \leqslant j \leqslant m_{i_{s'}}-1}) \in E^{\mathfrak{T}}$ for which there exists a (unique) strong solution of problem [(236)–(237)]. Then $\mathfrak{Z}$ is a linear subspace of $E^{\mathfrak{T}}$ and $\mathfrak{Z}_{uniq} \subseteq \mathfrak{Z}$. The equality $\mathfrak{Z} = \mathfrak{Z}_{uniq}$ holds iff the zero function is a unique strong solution of the problem [(236)–(237)] with the initial value $\vec{x} = \vec{0}$. For any $\vec{x} \in \mathfrak{Z}$, we denote by $\mathfrak{S}(\vec{x})$ the set consisting of all strong solutions of problem [(236)–(237)] with the initial value $\vec{x}$. In the remainder of this section, we shall only explore the problems (238) and $(\mathrm{DFP})_R$; observe that the problem $(\mathrm{DFP})_L$ is a very special case of problem (238) and that $\mathfrak{T} = m_n$ for problem (238), and $\mathfrak{T} = m$ for problem $(\mathrm{DFP})_R$. By (PQ) we denote either (238) or $(\mathrm{DFP})_R$. We shall always assume henceforth that $\emptyset \neq W \subseteq \mathbb{N}_{\mathfrak{T}}$, $\hat{E}_i$ is a linear subspace of E $(i \in W)$, $\tilde{E}$, $\check{E}$ are linear subspaces of $E^{\mathfrak{T}}$, as well as that $\vec{\beta} := (\beta_0, \beta_1, \ldots, \beta_{\mathfrak{T}-1}) \in [0, \alpha_n]^{\mathfrak{T}}$, $l \in \mathbb{N}$, $\emptyset \neq S \subseteq E^{\mathfrak{T}}$, $B_i : (E^{\mathfrak{T}})^{b_i} \to E^{\mathfrak{T}}$ and $M_i : E^{\mathfrak{T}} \to E$ are given mappings ($b_i \in \mathbb{N}$ for $1 \leqslant i \leqslant l$). Set $\mathfrak{B} := (\tilde{E}, \check{E}, S, (B_i)_{1 \leqslant i \leqslant l}, \{\hat{E}_i : i \in W\}, \vec{\beta})$ and $\mathfrak{M} := (\tilde{E}, \check{E}, S, (M_i)_{1 \leqslant i \leqslant l}, \{\hat{E}_i : i \in W\}, \vec{\beta})$. Let $\mathfrak{P} : \mathfrak{Z} \to P(\cup_{\vec{x} \in \mathfrak{Z}} \mathfrak{S}(\vec{x}))$ be a fixed mapping satisfying $\emptyset \neq \mathfrak{P}(\vec{x}) \subseteq \mathfrak{S}(\vec{x})$, $\vec{x} \in \mathfrak{Z}$. Following K.-G. Grosse-Erdmann-S. G. Kim [**237**, pp. 701-702], we introduce the set $\mathbb{U}_p(S)$ $(p \in \mathbb{N}_0)$ recursively by $\mathbb{U}_0(S) := S$, $\mathbb{U}_{p+1}(S) := \mathbb{U}_p(S) \cup \{B_i(\vec{x_1}, \ldots, \vec{x_{b_i}}) : 1 \leqslant i \leqslant l,\ \vec{x_1}, \ldots, \vec{x_{b_i}} \in \mathbb{U}_p(S)\}$. If $\mathfrak{T} \geqslant 2$, then we introduce the set $\mathbf{U}_p(S)$ $(p \in \mathbb{N}_0)$ following the approach of J. Bès-J. A. Conejero [**70**, pp. 2-3]: $\mathbf{U}_0(S) =: S$, $\mathbf{U}_{p+1}(S) := \mathbf{U}_p(S) \cup \{(x_2, x_3, \ldots, x_{\mathfrak{T}}, M_i(x_1, x_2, \ldots, x_{\mathfrak{T}})) : 1 \leqslant i \leqslant l,\ (x_1, x_2, \ldots, x_{\mathfrak{T}}) \in \mathbf{U}_p(S)\}$; if $\mathfrak{T} = 1$, then $\mathbf{U}_p(S) := S$, $p \in \mathbb{N}_0$. Define

$$\mathrm{Orb}(S; (B_i)_{1 \leqslant i \leqslant l}) := \bigcup_{p \in \mathbb{N}_0} \mathbb{U}_p(S), \quad \mathrm{Orb}(S; (M_i)_{1 \leqslant i \leqslant l}) := \bigcup_{p \in \mathbb{N}_0} \mathbf{U}_p(S),$$

and denote by $\mathcal{M}_{\mathfrak{B}}$ ($\mathcal{M}_{\mathfrak{M}}$) the set consisting of those tuples $\vec{x} \in \mathrm{Orb}(S; (B_i)_{1 \leqslant i \leqslant l}) \cap \mathfrak{Z}$ ($\vec{x} \in \mathrm{Orb}(S; (M_i)_{1 \leqslant i \leqslant l}) \cap \mathfrak{Z}$) for which $\mathrm{Proj}_{i,\mathfrak{T}}(\vec{x}) \in \hat{E}_i$, $i \in W$. In the sequel, we shall denote by D_i ($\mathfrak{D}$) either B_i or M_i ($\mathfrak{B}$ or $\mathfrak{M}$) and, in the case that $l = 1$, we shall also write $\mathrm{Orb}(S; B_1)$, $\mathrm{Orb}(S; M_1)$ and $\mathrm{Orb}(S; D_1)$ in place of $\mathrm{Orb}(S; (B_i)_{1 \leqslant i \leqslant l})$, $\mathrm{Orb}(S; (M_i)_{1 \leqslant i \leqslant l})$ and $\mathrm{Orb}(S; (D_i)_{1 \leqslant i \leqslant l})$, respectively. A similar terminological agreement will be used in the case that the set W is a singleton.

Motivated by some results from the theory of abstract higher-order differential equations with integer order derivatives, obtained by the usual converting of higher-order equations into first order matrix differential equations by introducing the derivative, the second derivative, ... , the $(n-1)$th derivative of the unknown E-valued function as a part of a new enlarged unknown E^n-valued function (cf. [**541**, pp. 79-83], [**199**, Section 5.7], [**210**, Theorem 5.6.3] and Theorem 2.10.24 below for further information), we would like to propose the following definition (concerning the abstract multi-term differential equations with Caputo fractional derivatives, we do not yet recognize the ideal option for work).

Definition 2.10.20. The abstract Cauchy problem (238) is said to be:

(i) $(\mathfrak{D},\mathfrak{P})$-hypercyclic iff there exist a tuple $\vec{x} \in \mathcal{M}_{\mathfrak{D}} \cap \tilde{E}$ and a function $u(\cdot;\vec{x}) \in \mathfrak{P}(\vec{x})$ such that $\{((\mathbf{D}_s^{\beta_0}u(s;\vec{x}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x}))_{s=t},\ldots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x}))_{s=t}) : t \geqslant 0\}$ is a dense subset of $\check{E}$; such a vector is called a $(\mathfrak{D},\mathfrak{P})$-hypercyclic vector of problem (238).

(ii) $\mathfrak{D}$-hypercyclic iff it is $(\mathfrak{D},\mathfrak{S})$-hypercyclic; any $(\mathfrak{D},\mathfrak{S})$-hypercyclic vector of problem (238) will be also called a $\mathfrak{D}$-hypercyclic vector of problem (238).

(iii) $\mathfrak{D}_{\mathfrak{P}}$-topologically transitive iff for every open non-empty subsets U and V of $E^{\mathfrak{T}}$ satisfying that $U\cap\tilde{E} \neq \emptyset$ and $V\cap\check{E} \neq \emptyset$, there exist a tuple $\vec{x} \in \mathcal{M}_{\mathfrak{D}}$, a function $u(\cdot;\vec{x}) \in \mathfrak{P}(\vec{x})$ and a number $t \geqslant 0$ such that $\vec{x} \in U \cap \tilde{E}$ and $((\mathbf{D}_s^{\beta_0}u(s;\vec{x}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x}))_{s=t},\ldots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x}))_{s=t}) \in V \cap \check{E}$.

(iv) $\mathfrak{D}$-topologically transitive iff it is $\mathfrak{D}_{\mathfrak{S}}$-topologically transitive.

(v) $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing iff for every open non-empty subsets U and V of $E^{\mathfrak{T}}$ satisfying that $U \cap \tilde{E} \neq \emptyset$ and $V \cap \check{E} \neq \emptyset$, there exists a number $t_0 \geqslant 0$ such that, for every number $t \geqslant t_0$, there exist a tuple $\vec{x_t} \in \mathcal{M}_{\mathfrak{D}}$ and a function $u(\cdot;\vec{x_t}) \in \mathfrak{P}(\vec{x_t})$ such that $\vec{x_t} \in U \cap \tilde{E}$ and $((\mathbf{D}_s^{\beta_0}u(s;\vec{x_t}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x_t}))_{s=t},\ldots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x_t}))_{s=t}) \in V \cap \check{E}$.

(vi) $\mathfrak{D}$-topologically mixing iff it is $\mathfrak{D}_{\mathfrak{S}}$-topologically mixing.

If $\mathfrak{Q}(\vec{x})$ is any non-empty subset consisting of solutions of problem $(\mathrm{DFP})_R$ with the initial value $\vec{x}$, then we denote by $\mathfrak{Q}_s(\vec{x})$ the set $\mathfrak{Q}(\vec{x}) \cap C^{m-1}([0,\infty) : E)$ $(\vec{x} \in \mathfrak{Z})$. We introduce the notions of $(\mathfrak{D},\mathfrak{P}_s)$-hypercyclicity, $\mathfrak{D}_{\mathfrak{P}_s}$-topological transitivity and $\mathfrak{D}_{\mathfrak{P}_s}$-topologically mixing property of problem $(\mathrm{DFP})_R$ in the same way as in Definition 2.10.20, with the sets $\mathfrak{P}(\vec{x})$ and $\mathfrak{P}(\vec{x_t})$ replaced respectively by $\mathfrak{P}_s(\vec{x})$ and $\mathfrak{P}_s(\vec{x_t})$. Finally, we say that the problem $(\mathrm{DFP})_R$ is $\mathfrak{D}$-hypercyclic ($\mathfrak{D}$-topologically transitive, $\mathfrak{D}$-topologically mixing) iff it is $\mathfrak{D}_{\mathfrak{S}_s}$-hypercyclic ($\mathfrak{D}_{\mathfrak{S}_s}$-topologically transitive, $\mathfrak{D}_{\mathfrak{S}_s}$-topologically mixing).

Remark 2.10.21. (i) We have presented only one way for computing the orbit $\mathrm{Orb}(S;(D_i)_{1\leqslant i\leqslant l})$. In the case that $\mathfrak{T} \geqslant 2$, the orbit $\mathrm{Orb}(S;(D_i)_{1\leqslant i\leqslant l})$ can have a very unpleasant form and it is very difficult to say, in general, whether there exists an element of $\mathrm{Orb}(S;(D_i)_{1\leqslant i\leqslant l})$ that is a $(\mathfrak{D},\mathfrak{P})$-hypercyclic vector of problem (238). On the other hand, in the definition of $\mathcal{M}_{\mathfrak{D}}$ we can take any non-empty subset S' of $E^{\mathfrak{T}}$ instead of $\mathrm{Orb}(S;(D_i)_{1\leqslant i\leqslant l})$; but, this is only a very special case of our definition with $\mathfrak{D} = \mathfrak{B}$, $l = 1$, $b_1 = 1$ and $B_1 : E^{\mathfrak{T}} \to E^{\mathfrak{T}}$ being the identity mapping. It is also worth noting that the continuous version of Herrero-Bourdon theorem [**238**, Theorem 7.17, pp. 190-191] suggests us to define the set $\mathcal{M}_{\mathfrak{D}}$ as the union of those vectors $\vec{x} \in span\{\mathrm{Orb}(S;(D_i)_{1\leqslant i\leqslant l})\} \cap \mathfrak{Z}$

for which $\mathrm{Proj}_{i,\mathfrak{T}}(\vec{x}) \in \hat{E}_i$, $i \in W$. If we define $\mathcal{M}_{\mathfrak{D}}$ in such a way, then the assertion of Theorem 2.10.23 below continues to hold, the assertion of Theorem 2.10.22 continues to hold with the mapping $\mathfrak{P}'(\cdot) = c\mathfrak{P}(\cdot/c)$ replaced by $\mathfrak{P}(\cdot)$, while the assertion of Theorem 2.10.24 continues to hold if we assume that $\{\vec{x_\lambda} : \lambda \in \Omega\} \subseteq \mathrm{Orb}(S; (D_i)_{1\leqslant i\leqslant l})$. Observe also that the various notions of hypercyclicity and topologically mixing of abstract degenerate equations introduced in Definition 2.10.1 and Definition 2.10.8 are special cases of the notion introduced in Definition 2.10.20, and that [**301**, Theorem 2.4] can be restated in our context.

(ii) Let $0 \leqslant \beta \leqslant \alpha < 2$, and let the requirements of Theorem 2.10.3 hold (in (iii) and in the sequel of (ii) of this remark, we will use the almost same terminology as in Subsection 2.10.1; the only exception will be the notation used to denote the space $\tilde{E}$). Applying Theorem 1.5.1, we get that

$$\lim_{t\to+\infty} \frac{t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha)}{E_\alpha(\lambda^\alpha t^\alpha)} = \lambda^{\beta-\alpha}, \quad \lambda \in \mathbb{C}_+$$

and

$$\lim_{t\to+\infty} t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha) = 0, \quad \lambda \in \Omega_{0,-}.$$

Using the identity

$$\mathbf{D}_t^\beta E_\alpha(\lambda^\alpha t^\alpha) = \lambda^\alpha t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha), \quad t \geqslant 0,\ \lambda \in \mathbb{C} \smallsetminus (-\infty, 0],$$

which can be shown directly, we may conclude by a careful inspection of the proof of Theorem 2.10.3 that the problem $(\mathrm{DFP})_L$ is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, provided that $\vec{\beta} = (\beta, \beta)$, $W = \{1\}$, $\hat{E}_1 = \overline{span\{f(\lambda^\alpha) : \lambda \in \Omega\}}$, $\tilde{E} = \hat{E}_1 \times \{0\}$, $\check{E} = \{(z, z) : z \in \hat{E}_1\}$, $\hat{E}_1 \times \{0\} \subseteq \mathrm{Orb}(S; (D_i)_{1\leqslant i\leqslant l})$ and $\mathfrak{P}((\sum_{i=1}^m \alpha_i f(\lambda_i^\alpha), 0)) = \{\sum_{i=1}^m \alpha_i E_\alpha(\cdot^\alpha \lambda_i^\alpha) f(\lambda_i^\alpha)\}$ ($m \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $\lambda_i \in \Omega$ for $1 \leqslant i \leqslant m$); this, in turn, implies that the problem $(\mathrm{DFP})_R$ is $\mathfrak{D}_{\mathfrak{P}_s}$-topologicially mixing (the only thing worth noticing here is that, given y and z as in the proof of Theorem 2.10.3, the vector $\vec{x_t}$ can be chosen to be $(y + \sum_{j=1}^m \frac{\gamma_j}{\tilde{\lambda_j}^\beta E_\alpha(\tilde{\lambda_j}^\alpha t^\alpha)} f(\tilde{\lambda_j}^\alpha), 0)$, $t > 0$ sufficiently large). The above is a slight improvement of above-mentioned result, which can be applied in the analysis of fractional analogues of the linearized Boussinesq equation $(\sigma^2\Delta - 1)u_{tt} + \gamma^2\Delta u = 0$ on symmetric spaces of non-compact type, as well. Observe, finally, that Definition 2.10.20, Definition 2.10.1 and Definition 2.10.8 have some advantages over [**300**, Definition 2.2] and [**302**, Definition 2.2]. For example, an application of Theorem 2.10.3 shows that the abstract Cauchy problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, with $1 < \alpha < 2$, $E = L^2(\mathbb{R})$, $B = I$ and $A = \mathcal{A}_c$ being the bounded perturbation of the one-dimensional Ornstein–Uhlenbeck operator from [**300**, Example 2.5(iii)], are both topologically mixing in the sense of Definition 2.10.1; the topologically mixing property of corresponding problems in the sense of [**300**, Definition 2.2] can be proved only in the case that $0 < \alpha \leqslant 1$, cf. [**115**] and [**300**].

(iii) Consider the situation of Theorem 2.10.9 with the second equality in (227) replaced by

$$\lim_{t\to+\infty} \mathbf{D}_t^{\beta_j} H_i(\lambda,t)=0,\quad \lambda\in\Omega_-,\ 0\leqslant j\leqslant \mathfrak{T}-1,$$

and with the first equality in (227) replaced by

$$\lim_{t\to+\infty} |F(\lambda,t)|=+\infty,\quad \lambda\in\Omega_+,\ 0\leqslant j\leqslant \mathfrak{T}-1,$$

where $a>0$ and $F:\Omega_+\times(a,+\infty)\to\mathbb{C}$ is a certain function. Set $E_0:=\overline{span\{f(\lambda):\lambda\in\Omega\}}$. If we suppose additionally that there exist complex numbers $G_{\beta_0},\ldots,G_{\beta_{\mathfrak{T}-1}}$ such that

$$\lim_{t\to+\infty}\frac{\mathbf{D}_t^{\beta_j} H_i(\lambda,t)}{F(\lambda,t)}=G_{\beta_j},\quad \lambda\in\Omega_+,\ 0\leqslant j\leqslant \mathfrak{T}-1,$$

then the proof of Theorem 2.10.9 shows that the abstract Cauchy problem (238) is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, provided that $W=\{i\}$, $\hat{E}_i=E_0$, $\tilde{E}=\{\vec{x}\in E^{\mathfrak{T}}:\mathrm{Proj}_{i,\mathfrak{T}}(\vec{x})\in E_0$ and $\mathrm{Proj}_{j,\mathfrak{T}}(\vec{x})=0$ for $j\in\mathbb{N}_{\mathfrak{T}}\setminus\{i\}\}$, $\check{E}=\{(G_{\beta_0}z,\ldots,G_{\beta_{\mathfrak{T}-1}}z):z\in\hat{E}_1\}$, $\tilde{E}\subseteq \mathrm{Orb}(S;(D_j)_{1\leqslant j\leqslant l})$ and $\mathfrak{P}((\sum_{j=1}^m\alpha_j f(0,\ldots,\lambda_j,\ldots,0))=\{\sum_{j=1}^m\alpha_j H_i(\lambda_j,\cdot)f(\lambda_j)\}$ ($m\in\mathbb{N}$, $\alpha_j\in\mathbb{C}$, $\lambda_j\in\Omega$ for $1\leqslant j\leqslant m$), where λ_j appears on i-th place starting from zero (there exists a great number of concrete examples in which the above conditions hold with $\vec{\beta}$ being the constant multiple of $(1,1,\ldots,1)$, see e.g. our analysis of topologically mixing properties of strongly damped Klein–Gordon equation in Example 2.10.11; we refer the reader to Theorem 2.10.24 and Example 2.10.26 for the case in which $\vec{\beta}$ is not of the form described above). Also, it should be noticed that the comments from (ii) and (iii) can be formulated in the light of [**300**, Remark 1(iii)] and [**302**, Remark 1;3.], as well as that the proof of Theorem 2.10.9 implies that the term $\mathbf{D}_t^{\beta_l}H_i(\lambda,t)$ is equal to

$$\mathcal{L}^{-1}\left(\frac{z^{\alpha_n+\beta_l-i-1}+\sum_{j\in D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\beta_l-i-1}-\chi_{\mathcal{D}_i}(0)f_n(\lambda)z^{\alpha+\beta_l-i-1}}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\right)(t),$$

for $t\geqslant 0$, $\lambda\in\Omega$, $l\in\mathbb{N}^0_{\mathfrak{T}-1}$, $\lceil\beta_l\rceil<i-1$, and

$$\mathcal{L}^{-1}\left(\frac{-\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\beta_l-i-1}-f_n(\lambda)z^{\alpha+\beta_l-i-1}(\chi_{\mathcal{D}_i}(0)-1)}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\right),$$

for $t\geqslant 0$, $\lambda\in\Omega$, $l\in\mathbb{N}^0_{\mathfrak{T}-1}$, $\lceil\beta_l\rceil\geqslant i-1$.

(iv) As indicated in Subsection 2.10.1, it is much better to introduce the notions of $\mathfrak{D}$-hypercyclicity, $\mathfrak{D}$-topological transitivity and $\mathfrak{D}$-topologically mixing property of problem (PQ) with the set $\mathfrak{Z}$ than with $\mathfrak{Z}_{uniq}$ (the choice of strong solutions in Definition 2.10.20 is almost inevitable). Consider now, for the sake of brevity, the abstract Cauchy problem (238). If $\mathfrak{Z}=\mathfrak{Z}_{uniq}$, then we define the operator $T(t):\mathcal{M}_{\mathfrak{D}}\to E^{\mathfrak{T}}$ by $T(t)\vec{x}:=$

$((\mathbf{D}_s^{\beta_0} u(s;\vec{x}))_{s=t}, (\mathbf{D}_s^{\beta_1} u(s;\vec{x}))_{s=t}, \ldots, (\mathbf{D}_s^{\beta_{\mathfrak{T}}-1} u(s;\vec{x}))_{s=t})$ $(t \geqslant 0)$, where $u(\cdot;\vec{x})$ denotes the unique strong solution of problem (238) with the initial value $\vec{x}$. Let $(O_n)_{n\in\mathbb{N}}$ be an open base of the topology of $E^{\mathfrak{T}}$ $(O_n \neq \emptyset$, $n \in \mathbb{N})$. If $\tilde{E} = \check{E} = E^{\mathfrak{T}}$ and if we denote by $HC_{\mathfrak{D}}$ the set which consists of all $\mathfrak{D}$-hypercyclic vectors of problem (238), then we have the obvious equality $HC_{\mathfrak{D}} = \bigcap_{n\in\mathbb{N}} \bigcup_{t\geqslant 0} T(t)^{-1}(O_n)$; cf. also [**237**, Theorem 1], [**70**, Proposition 4] and Subsection 2.10.1. Unfortunately, we cannot conclude from the above that $\mathfrak{D}$-topological transitivity of problem (238) implies its $\mathfrak{D}$-hypercyclicity (in this place, it is worth noting that there exists a continuous linear operator on the space $\varphi = \bigoplus_{n\in\mathbb{N}} \mathbb{K}$ that is topologically transitive but not hypercyclic [**84**, Theorem 2.2], so that the connections between $\mathfrak{D}$-hypercyclicity and $\mathfrak{D}$-topological transitivity seem to be more intricate in non-metrizable locally convex spaces).

It is worth noticing that the assertion of [**237**, Theorem 2] admits an adequate reformulation in our context. Before we state the corresponding theorem, it would be very helpful to introduce the sets $[(B_i)_{1\leqslant i\leqslant l}]_p(S)$ and $[(M_i)_{1\leqslant i\leqslant l}]_p(S)$ $(p \in \mathbb{N}_0)$ recursively by $[(B_i)_{1\leqslant i\leqslant l}]_0(S) := [(M_i)_{1\leqslant i\leqslant l}]_0(S) := S$, $[(B_i)_{1\leqslant i\leqslant l}]_{p+1}(S) := \{B_i(\vec{x_1}, \ldots, \vec{x_{b_i}}) : 1 \leqslant i \leqslant l, \vec{x_1} \in [(B_i)_{1\leqslant i\leqslant l}]_{j_1}(S), \ldots, \vec{x_{b_i}} \in [(B_i)_{1\leqslant i\leqslant l}]_{j_{b_i}}(S)$ for some numbers $j_1, \ldots, j_{b_i} \in \mathbb{N}_p$ with $j_1 + \cdots + j_{b_i} = p\}$, $[(M_i)_{1\leqslant i\leqslant l}]_{p+1}(S) := \mathbf{U}_{p+1}(S) \setminus \mathbf{U}_p(S)$, $p \in \mathbb{N}_0$. Then the set $[(B_i)_{1\leqslant i\leqslant l}]_p(S)$ $([(M_i)_{1\leqslant i\leqslant l}]_p(S))$ contains all the elements from $\mathrm{Orb}(S; (B_i)_{1\leqslant i\leqslant l})$ $(\mathrm{Orb}(S; (M_i)_{1\leqslant i\leqslant l}))$ obtained by n applications of operators $B_1, \ldots, B_l$ $(M_1, \ldots, M_l)$, totally counted, and the following holds:

$$\mathrm{Orb}(S; (B_i)_{1\leqslant i\leqslant l}) = \bigcup_{p\in\mathbb{N}_0} [(B_i)_{1\leqslant i\leqslant l}]_p(S),$$

$$\mathrm{Orb}(S; (M_i)_{1\leqslant i\leqslant l}) = \bigcup_{p\in\mathbb{N}_0} [(M_i)_{1\leqslant i\leqslant l}]_p(S). \tag{239}$$

Suppose now that $c \in \mathbb{K} \setminus \{0\}$ as well as $B_i' : (E^{\mathfrak{T}})^{b_i} \to E^{\mathfrak{T}}$ and $M_i' : E^{\mathfrak{T}} \to E$ satisfy

$$B_i'(c\vec{x_1}, \ldots, c\vec{x_{b_i}}) = cB_i(\vec{x_1}, \ldots, \vec{x_{b_i}}),$$

provided $\vec{x_1}, \ldots, \vec{x_{b_i}} \in E^{\mathfrak{T}}$, $1 \leqslant i \leqslant l$, and

$$(cx_2, \ldots, cx_{\mathfrak{T}}, M_i'(cx_1, cx_2, \ldots, cx_{\mathfrak{T}})) = c(x_2, \ldots, x_{\mathfrak{T}}, M_i(x_1, x_2, \ldots, x_{\mathfrak{T}})),$$

provided that $(x_1, x_2, \ldots, x_{\mathfrak{T}}) \in E^{\mathfrak{T}}$, $1 \leqslant i \leqslant l$. Define $S_c := \{c\vec{x} : \vec{x} \in S\}$. Then we can inductively prove that $[(B_i')_{1\leqslant i\leqslant l}]_p(S_c) = \{c\vec{x} : \vec{x} \in [(B_i)_{1\leqslant i\leqslant l}]_p(S)\}$ and $[(M_i')_{1\leqslant i\leqslant l}]_p(S_c) = \{c\vec{x} : \vec{x} \in [(M_i)_{1\leqslant i\leqslant l}]_p(S)\}$ for all $p \in \mathbb{N}_0$, so that (239) implies

$$\{c\vec{x} : \vec{x} \in \mathrm{Orb}(S; (B_i)_{1\leqslant i\leqslant l})\} = \mathrm{Orb}(S_c; (B_i')_{1\leqslant i\leqslant l})$$

and

$$\{c\vec{x} : \vec{x} \in \mathrm{Orb}(S; (M_i)_{1\leqslant i\leqslant l})\} = \mathrm{Orb}(S_c; (M_i')_{1\leqslant i\leqslant l}).$$

Now it is very simple to prove the following

THEOREM 2.10.22. *Set* $\mathfrak{D}' := (\tilde{E}, \check{E}, S_c, (D_i')_{1 \leqslant i \leqslant l}, \{\hat{E}_i : i \in W\}, \vec{\beta})$ *and* $\mathfrak{P}' : \mathfrak{Z} \to P(\cup_{\vec{x} \in \mathfrak{Z}} \mathfrak{S}(\vec{x}))$, *by* $\mathfrak{P}'(\vec{x}) := c\mathfrak{P}(\vec{x}/c)$, $\vec{x} \in \mathfrak{Z}$. *Then the abstract Cauchy problem* (238), *resp.* $(\mathrm{DFP})_R$, *is* $\mathfrak{D}$*-hypercyclic* ($(\mathfrak{D}, \mathfrak{P})$*-hypercyclic,* $\mathfrak{D}_{\mathfrak{P}}$*-topologically transitive, resp.* $\mathfrak{D}_{\mathfrak{P}_s}$*-topologically transitive,* $\mathfrak{D}$*-topologically transitive,* $\mathfrak{D}_{\mathfrak{P}}$*-topologically mixing, resp.* $\mathfrak{D}_{\mathfrak{P}_s}$*-topologically mixing,* $\mathfrak{D}$*-topologically mixing) iff the abstract Cauchy problem* (238), *resp.* $(\mathrm{DFP})_R$, *is* $\mathfrak{D}'$*-hypercyclic* ($(\mathfrak{D}', \mathfrak{P}')$*-hypercyclic,* $\mathfrak{D}'_{\mathfrak{P}'}$*-topologically transitive, resp.* $\mathfrak{D}_{\mathfrak{P}'_s}$*-topologically transitive,* $\mathfrak{D}'$*-topologically transitive,* $\mathfrak{D}'_{\mathfrak{P}'}$*-topologically mixing, resp.* $\mathfrak{D}'_{\mathfrak{P}'_s}$*-topologically mixing,* $\mathfrak{D}'$*-topologically mixing).*

Suppose now that X is another Fréchet space over the field of $\mathbb{C}$ and $\phi\colon X \to E$ is a linear topological homeomorphism. Then the mapping $\phi^{\mathfrak{T}}\colon X^{\mathfrak{T}} \to E^{\mathfrak{T}}$, defined in the very obvious way, is a linear topological homeomorphism between the spaces $X^{\mathfrak{T}}$ and $E^{\mathfrak{T}}$. Define $S_\phi := (\phi^{\mathfrak{T}})^{-1}(S)$ and the closed linear operators A_i^X on X by $D(A_i^X) := \phi^{-1}(D(A_i))$ and $A_i^X x = y$ iff $A_i(\phi x) = \phi y$ $(0 \leqslant i \leqslant n)$. For any E-valued function $t \mapsto u(t)$, $t \geqslant 0$ we define the X-valued function $t \mapsto u_\phi(t)$, $t \geqslant 0$ by $u_\phi(t) := \phi^{-1}(u(t))$, $t \geqslant 0$. Then it is readily seen that the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for $t \geqslant 0$ iff the Caputo fractional derivative $\mathbf{D}_t^\alpha u_\phi(t)$ is defined for $t \geqslant 0$; if this is the case, we have $\mathbf{D}_t^\alpha u_\phi(t) = \phi^{-1}(\mathbf{D}_t^\alpha u(t))$, $t \geqslant 0$. Using this fact, we can simply prove that the function $t \mapsto u(t)$, $t \geqslant 0$ is a strong solution of problem (PQ) with the initial value $\vec{x} = (x_1, \ldots, x_{\mathfrak{T}}) \in E^{\mathfrak{T}}$ iff the function $t \mapsto u_\phi(t)$, $t \geqslant 0$ is a strong solution of problem $(\mathrm{PQ})_\phi$ with the initial value $\vec{x}^\phi := (\phi^{-1}(x_1), \ldots, \phi^{-1}(x_{\mathfrak{T}})) \in X^{\mathfrak{T}}$, where the abstract Cauchy problem $(\mathrm{PQ})_\phi$ is defined by replacing all the operators A_i in the problem (PQ) with the operators A_i^X $(0 \leqslant i \leqslant n)$. If we denote by $\mathfrak{Z}^\phi$ $(\mathfrak{Z}^\phi_{uniq})$ the set consisting of those tuples $\vec{x}^\phi \in X^{\mathfrak{T}}$ for which there exists a (unique) strong solution of the problem $(\mathrm{PQ})_\phi$, then the above implies $\mathfrak{Z}^\phi = (\phi^{\mathfrak{T}})^{-1}\mathfrak{Z}$ $(\mathfrak{Z}^\phi_{uniq} = (\phi^{\mathfrak{T}})^{-1}\mathfrak{Z}_{uniq})$.

Define the mappings $B_{i,\phi} : (X^{\mathfrak{T}})^{b_i} \to X^{\mathfrak{T}}$ and $M_{i,\phi} : X^{\mathfrak{T}} \to X$ by

$$B_{i,\phi}(\vec{x_1}, \ldots, \vec{x_{b_i}}) := (\phi^{\mathfrak{T}})^{-1}(B_i(\phi^{\mathfrak{T}}\vec{x_1}, \ldots, \phi^{\mathfrak{T}}\vec{x_{b_i}})) \text{ and } M_{i,\phi}(\vec{x}) := \phi^{-1}(M_i(\phi^{\mathfrak{T}}\vec{x})),$$

for any $\vec{x_1}, \ldots, \vec{x_{b_i}}, \vec{x} \in X^{\mathfrak{T}}$, $1 \leqslant i \leqslant l$, as well as the mappings $\mathfrak{P}_\phi : \mathfrak{Z}^\phi \to P(\{\mathfrak{S}(\vec{x}^\phi) : \vec{x}^\phi \in \mathfrak{Z}^\phi\})$ and $(\mathfrak{P}_\phi)_s : \mathfrak{Z}^\phi \to P(\{\mathfrak{S}(\vec{x}^\phi) : \vec{x}^\phi \in \mathfrak{Z}^\phi\})$ by $\mathfrak{P}_\phi((\phi^{\mathfrak{T}})^{-1}\vec{x}) := \{u_\phi(\cdot) : u(\cdot) \in \mathfrak{P}(\vec{x})\}$ and $(\mathfrak{P}_\phi)_s((\phi^{\mathfrak{T}})^{-1}\vec{x}) := \{u_\phi(\cdot) : u(\cdot) \in \mathfrak{P}_s(\vec{x})\}$ $(\vec{x} \in \mathfrak{Z})$, respectively. Set

$$\mathfrak{D}_\phi := ((\phi^{\mathfrak{T}})^{-1}(\tilde{E}), (\phi^{\mathfrak{T}})^{-1}(\check{E}), S_\phi, (D_{i,\phi})_{1 \leqslant i \leqslant l}, \{\phi^{-1}(\hat{E}_i) : i \in W\}, \vec{\beta}).$$

Having in mind the proof of [**237**, Theorem 3], we can show that

$$\phi^{\mathfrak{T}}(\mathrm{Orb}(S_\phi; (D_{i,\phi})_{1 \leqslant i \leqslant l})) = \mathrm{Orb}(S; (D_i)_{1 \leqslant i \leqslant l}).$$

Now it is quite simple to prove the following conjugacy lemma for abstract degenerate multi-term fractional differential equations (cf. [**263**, Lemma 1.4] for a pioneering result in this direction):

THEOREM 2.10.23. *The abstract Cauchy problem* (238), *resp.* $(\mathrm{DFP})_R$, *is* $\mathfrak{D}$*-hypercyclic* ($(\mathfrak{D}, \mathfrak{P})$*-hypercyclic,* $\mathfrak{D}_{\mathfrak{P}}$*-topologically transitive, resp.* $\mathfrak{D}_{\mathfrak{P}_s}$*-topologically transitive,* $\mathfrak{D}$*-topologically transitive,* $\mathfrak{D}_{\mathfrak{P}}$*-topologically mixing, resp.* $\mathfrak{D}_{\mathfrak{P}_s}$*-topologi-*

mathcally mixing, $\mathfrak{D}$-topologically mixing) iff the abstract Cauchy problem $(238)_\phi$, *resp.* $(\mathrm{DFP})_{R,\phi}$, *is $\mathfrak{D}_\phi$-hypercyclic ($(\mathfrak{D}_\phi, \mathfrak{P}_\phi)$-hypercyclic, $\mathfrak{D}_{\phi_{\mathfrak{P}_\phi}}$-topologically transitive, resp. $\mathfrak{D}_{\phi_{(\mathfrak{P}_\phi)_s}}$-topologically transitive, $\mathfrak{D}_\phi$-topologically transitive, $\mathfrak{D}_{\phi_{\mathfrak{P}_\phi}}$-topologically mixing, resp. $\mathfrak{D}_{\phi_{(\mathfrak{P}_\phi)_s}}$-topologically mixing, $\mathfrak{D}_\phi$-topologically mixing).*

We continue by stating the following theorem.

THEOREM 2.10.24. *Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, and let $f \colon \Omega \to E$ be an analytic mapping satisfying that*

$$P_\lambda f(\lambda) = \Bigl(\lambda^{\alpha_n} B + \sum_{i=0}^{n-1} \lambda^{\alpha_i} A_i\Bigr) f(\lambda) = 0, \quad \lambda \in \Omega. \tag{240}$$

Set $\vec{x_\lambda} := [f(\lambda)\ \lambda f(\lambda)\ \ldots\ \lambda^{n-1} f(\lambda)]^T$ ($\lambda \in \Omega$), $E_0 := span\{\vec{x_\lambda} : \lambda \in \Omega\}$, $\tilde{E} := \check{E} := \overline{E_0}$, $\vec{\beta} := (0, 1, \ldots, n-1)$, $W := \mathbb{N}_n$ and $\hat{E}_i := span\{f(\lambda) : \lambda \in \Omega\}$, $i \in W$. Let $\emptyset \neq S \subseteq E^n$ be such that $E_0 \subseteq \mathrm{Orb}(S; (D_i)_{1 \leqslant i \leqslant l})$. Then $\vec{x_\lambda} \in \mathfrak{M}_{\mathfrak{D}}$, $\lambda \in \Omega$ and the abstract Cauchy problem (238) *is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing provided that $\sum_{j=1}^q e^{\lambda_j \cdot} f(\lambda_j) \in \mathfrak{P}(\sum_{j=1}^q \vec{x_{\lambda_j}})$ for any $\sum_{j=1}^q \vec{x_{\lambda_j}} \in E_0$ ($q \in \mathbb{N}$; $\lambda_j \in \Omega$, $1 \leqslant j \leqslant q$).*

PROOF. We shall content ourselves with sketching it. Consider the operator matrices

$$\mathcal{A} := \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & I \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{n-1} \end{bmatrix}$$

and

$$\mathcal{B} := \begin{bmatrix} I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \cdot\cdot I & 0 \\ 0 & 0 & 0 & \ldots & B \end{bmatrix},$$

acting on E^n with their maximal domains. Then the operator matrix $\mathcal{A}$ is closable, the operator matrix $\mathcal{B}$ is closed and, due to (240), $\bar{\mathcal{A}}\vec{x_\lambda} = \lambda \mathcal{B} \vec{x_\lambda}$, $\lambda \in \Omega$. Furthermore, if we suppose that Ω_0 is an arbitrary open connected subset of Ω which admits a cluster point in Ω, then the linear span of the set $\{\vec{x_\lambda} : \lambda \in \Omega_0\}$ is dense in $\tilde{E}$. Now the statement follows similarly as in the proof of Theorem 2.10.1. $\square$

REMARK 2.10.25. (i) The assertions of Theorem 2.10.1 and Theorem 2.10.9 continue to hold, with appropriate modifications, in the setting of separable sequentially complete locally convex spaces. The conclusion stated in Theorem 2.10.24 remains true if we consider the equation (238) with the same initial conditions and with the term $B\frac{d^n}{dt^n}u(t)$ replaced by $\frac{d^n}{dt^n}Bu(t)$ (cf. [**308**, Remark 12(i)]).

(ii) Suppose $\alpha_1 \in (0,1)$ and $\alpha_i = i\alpha_1$, $i \in \mathbb{N}_n$. Keeping in mind the proofs of Theorem 2.10.24 and [**308**, Theorem 5], we can deduce some results about $\mathfrak{D}$-topologically mixing properties of the problem

$$
\begin{gathered}
B(\mathbf{D}_t^{\alpha_1})^n u(t) + \sum_{i=0}^{n-1} A_i (\mathbf{D}_t^{\alpha_1})^i u(t) = 0, \quad t \geqslant 0, \\
((\mathbf{D}_t^{\alpha_1})^j u(t))_{t=0} = u_j, \quad j = 0, \dots, n-1,
\end{gathered} \tag{241}
$$

and its analogues obtained by replacing, optionally, some of the terms $B(\mathbf{D}_t^{\alpha_1})^n u(t)$ and $A_i(\mathbf{D}_t^{\alpha_1})^i u(t)$ by $(\mathbf{D}_t^{\alpha_1})^n Bu(t)$ and $(\mathbf{D}_t^{\alpha_1})^i A_i u(t)$, respectively $(0 \leqslant i \leqslant n-1)$. The case $\alpha_1 \in (1,2)$ can be considered quite similarly.

(iii) It should be emphasized that Theorem 2.10.24 cannot be so simply reformulated in the case that there exists an index $i \in \mathbb{N}_n$ such that $\alpha_i \notin \mathbb{N}$. In actual fact, probably the only way to exploit (240) is to find analytic functions $F_i \colon \Omega \to \mathbb{C}$ $(0 \leqslant i \leqslant m_n - 1)$ such that the equation (238), equipped with the initial conditions $u^{(i)}(0) = F_i(\lambda) f(\lambda)$, $0 \leqslant i \leqslant m_n - 1$, has a strong solution of the form $u(t;\lambda) = G(\lambda,t) f(\lambda)$, $t \geqslant 0$ $(\lambda \in \Omega)$, where

$$
\lambda^{-\alpha_n} \mathbf{D}_t^{\alpha_n} G(\lambda,t) = \cdots = \lambda^{-\alpha_1} \mathbf{D}_t^{\alpha_1} G(\lambda,t) = G(\lambda,t), \quad t \geqslant 0 \ \ (\lambda \in \Omega). \tag{242}
$$

By [**153**, Theorem 7.2], the validity of (242) would imply that for each $t \geqslant 0$, $\lambda \in \Omega$ and $i \in \mathbb{N}_n$, we have:

$$
G(\lambda,t) = F_0(\lambda) E_{\alpha_i}(\lambda^{\alpha_i} t^{\alpha_i}) + \sum_{k=1}^{m_i-1} F_k(\lambda) \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} E_{\alpha_i}(\lambda^{\alpha_i} s^{\alpha_i})\, ds,
$$

i.e., that for each $t \geqslant 0$, $\lambda \in \Omega$ and $i \in \mathbb{N}_n$, we have:

$$
G(\lambda,t) = \sum_{l=0}^{\infty} \sum_{k=0}^{m_i-1} \lambda^{\alpha_i l} F_k(\lambda) \frac{t^{\alpha_i l + k}}{\Gamma(\alpha_i l + k + 1)}.
$$

The function $t \mapsto G(\lambda,t) - F_0(\lambda)$, $t \geqslant 0$ behaves asymptotically like $\lambda^{\alpha_1} F_0(\lambda) g_{\alpha_1+1}(t)$ as $t \to 0+$, so that the number α_1 cannot be an element of the interval $(0,1)$ (to see this, consider the asymptotic behaviour of function $t \mapsto G(\lambda,t) - F_0(\lambda)$, $t \geqslant 0$ as $t \to 0+$, with the number α_1 replaced by α_2). Considering the asymptotic behaviour of function $t \mapsto G(\lambda,t) - F_0(\lambda) - tF_1(\lambda)$, $t \geqslant 0$ ($t \mapsto G(\lambda,t) - F_0(\lambda) - tF_1(\lambda) - (t^2/2)F_2(\lambda)$, $t \geqslant 0$; ...) as $t \to 0+$, we obtain similarly that α_1 cannot be an element of the interval $(1,2)$ ($(2,3)$; ...); consequently, $\alpha_1 \in \mathbb{N}$. Ignoring the first order α_1, and repeating the same procedure with the order α_2, we get that $\alpha_2 \in \mathbb{N}$. A similar line of reasoning shows that $\alpha_3, \dots, \alpha_n \in \mathbb{N}$.

(iv) Hypercyclic and topologically mixing properties of higher-order non-degenerate differential equations with integer order derivatives have been studied in a series of recent papers by using the usual reduction into first order matrix differential equations (cf. [**114, 118**], [**292**, Section 3.2] and the references cited there). It should be observed that we can prove a slight extension of this theorem by using the analyses from [**300**, Remark 1(iii)] and [**302**, Remark 1;3.] (cf. also Remark 2.10.21(iii) and Example 2.10.26(i) below).

(v) In Theorem 2.10.16, we have reconsidered the well known assertion of S. El Mourchid [**163**, Theorem 2.1] concerning the connection between the imaginary point spectrum and hypercyclicity of strongly continuous semigroups. On the basis of this result, we can state some new facts about $\mathcal{D}$-topologically mixing properties of problem (238) considered in Theorem 2.10.24, provided that the equation (240) holds for all values of complex parameter λ belonging to some subinterval of imaginary axis.

We close this section by providing some illustrative examples.

EXAMPLE 2.10.26. (i) Consider the equation (238) with $\alpha_i = i$, $i \in \mathbb{N}_n$ and with the operator $A_0 = A$ replaced by $-A$. Although it may seem contrary, Theorem 2.10.24 is not so easily comparable to Theorem 2.10.9 in this case. For example, in the situation of Example 2.10.12 with $P(z) = -z$ and $\alpha = 1$, the assumptions of Theorem 2.10.9 are satisfied with $E := L^2(\mathbb{R})$, $c_1 > c > \frac{b}{2} > 0$, $A := (c-c_1)I$, $B := \mathcal{A}_c - A$, $A_1 := -\mathcal{A}_c + cI$, where the operator $\mathcal{A}_c$ is defined by $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$ and $\mathcal{A}_c u := u'' + bxu' + cu$, $u \in D(\mathcal{A}_c)$, $\Omega := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ \lambda \neq c - c_1,\ \operatorname{Re}\lambda < c - \frac{b}{2}\}$, $f(\lambda) := g_1(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}\xi|\xi|^{-(2+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ or $f(\lambda) := g_2(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}|\xi|^{-(1+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$, $f_1(\lambda) := (c-c_1)/(c-\lambda)$ and $f_2(\lambda) := (c-c_1)/(\lambda - (c-c_1))$ $(\lambda \in \Omega)$. In particular, there is no open connected subset Ω' of Ω satisfying $\Omega' \cap i\mathbb{R} \neq \emptyset$ and $(\lambda^2/(f_2(\lambda)) + \lambda/(f_1(\lambda)) - 1)Af(\lambda) = 0$, $\lambda \in \Omega'$, i.e., the equation (240) does not hold with this choice of $f(\lambda)$. This is quite predictable because the equation (240), with the set Ω and the function $f(\cdot)$ replaced respectively by $'\Omega$ and $'f(\cdot)$ therein (and in our further analysis, for the sake of consistency of notation), is equivalent to say that $(\lambda^2-\lambda)\mathcal{A}_c\,'f(\lambda) = (\lambda^2(c-c_1) - \lambda c + (c-c_1))'f(\lambda)$, $\lambda \in\ '\Omega$. Denote by Λ the set of all complex numbers $z \in i\mathbb{R} \setminus \{0\}$ for which there exists $\delta(z) > 0$ such that $\{0,1\} \cap L(z,\delta(z)) = \emptyset$, as well as that for each $\lambda \in L(z,\delta(z))$ we have $\operatorname{Re}(c - c_1 - \frac{c_1}{\lambda-1} + \frac{c-c_1}{\lambda^2-\lambda}) < c - \frac{b}{2}$. Recalling that $\{z \in \mathbb{C} : \operatorname{Re} z < c - \frac{b}{2}\} \subseteq \sigma_p(\mathcal{A}_c)$, it readily follows that Theorem 2.10.24 can be also applied here with $'\Omega := \bigcup_{z\in\Lambda} L(z,\delta(z))$ and $'f_i(\lambda) := g_i(c - c_1 - \frac{c_1}{\lambda-1} + \frac{c-c_1}{\lambda^2-\lambda})$, $\lambda \in\ '\Omega$ $(i = 1,2)$, producing slightly different results from those obtained by applying Theorem 2.10.9 (with $\hat{E} = \check{E} = \overline{span\{['f_i(\lambda)\ \lambda\,'f_i(\lambda)]^T : \lambda \in\ '\Omega,\ i = 1,2\}}$, the subspace of E^2 whose first and second projection equals to E; cf. [**115**] and [**300**]). On the other hand, there exists a great number of very simple (non-)degenerate equations where we can apply Theorem 2.10.24 but not Theorem 2.10.9. Consider, for example, the equation $u'''(t) + (c_2 - \mathcal{A}_c)u'(t) + c_1u(t) = 0$, $t \geqslant 0$, where $c_1 \in \mathbb{C} \setminus \{0\}$ and $c_2 \in \mathbb{C}$. The analysis taken up in [**302**, Remark 1(vi)] (cf. also [**292**, Theorem 3.3.9, Remark 3.3.10(v)]), with $c_3 = 0$, shows that there do not exist an open connected subset Ω_- of $\mathbb{C}$

and an index $i \in \{0, 1, 2\}$ such that the second equality in (227) holds. Contrary to this, there exist $t > 0$ and $\varepsilon > 0$ such that the equation (240) holds with $\Omega = L(it, \varepsilon)$.

(ii) Suppose Ω is an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, $f \colon \Omega \to E$ is an analytic mapping, $g \colon \Omega \to \mathbb{C} \smallsetminus \{0\}$ is a scalar-valued mapping and $Af(\lambda) = g(\lambda)f(\lambda)$, $\lambda \in \Omega$. Let $P_j(z)$ be non-zero complex polynomials ($j \in \mathbb{N}_n^0$), and let

$$\lambda^n P_n(g(\lambda)) + \sum_{j=0}^{n-1} \lambda^j P_j(g(\lambda)) = 0, \quad \lambda \in \Omega.$$

Then the equation (240) holds with $B := P_n(A)$ and $A_j := P_j(A)$, $j \in \mathbb{N}_{n-1}^0$. If, additionally, the preassumption $\sum_{j=0}^{n-1} \langle x_j^*, \lambda^j f(\lambda) \rangle = 0$, $\lambda \in \Omega$ for some continuous linear functionals $x_j^* \in E^*$ given in advance ($j \in \mathbb{N}_{n-1}^0$) implies $x_j^* = 0$ for all $j \in \mathbb{N}_{n-1}^0$, then the space E_0 from the formulation of Theorem 2.10.24 equals to E^n (cf. [**114**, Theorem 3.1] for a concrete example of this type with $n = 3$). Some applications of Theorem 2.10.24 in Fréchet function spaces can be given following the analysis from [**292**, Example 3.1.29].

2.11. The existence of distributional chaos in abstract degenerate fractional differential equations

The main aim of this section is to investigate a class of distributionally chaotic abstract degenerate (multi-term) fractional differential equations. We assume that X is an infinite-dimensional Fréchet space over the field $\mathbb{C}$ and the topology of X is induced by the fundamental system $(p_n)_{n \in \mathbb{N}}$ of increasing seminorms. Let us recall that the translation invariant metric $d : X \times X \to [0, \infty)$, defined by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}, \quad x, y \in X, \tag{243}$$

satisfies the following properties:

$$d(x+u, y+v) \leqslant d(x, y) + d(u, v), \quad x, y, u, v \in X,$$

$$d(cx, cy) \leqslant (|c| + 1) d(x, y), \quad c \in \mathbb{C}, x, y \in X, \tag{244}$$

and

$$d(\alpha x, \beta x) \geqslant \frac{|\alpha - \beta|}{1 + |\alpha - \beta|} d(0, x), \quad x \in X, \ \alpha, \beta \in \mathbb{C}. \tag{245}$$

Given $\varepsilon > 0$ in advance, set $L(0, \varepsilon) := \{x \in X : d(x, 0) < \varepsilon\}$. By Y we denote another Fréchet space over the field $\mathbb{C}$; we assume that the topology of Y is induced by the fundamental system $(p_n^Y)_{n \in \mathbb{N}}$ of increasing seminorms. We define the translation invariant metric $d_Y \colon Y \times Y \to [0, \infty)$ by replacing $p_n(\cdot)$ with $p_n^Y(\cdot)$ in (243). In the case that $(X, \| \cdot \|)$ or $(Y, \| \cdot \|_Y)$ is a Banach space, then it will be assumed that the distance of two elements $x, y \in X$ ($x, y \in Y$) is given by $d(x, y) := \|x - y\|$

($d_Y(x, y) := \|x - y\|_Y$). With this terminological change, our structural results clarified in this section continue to hold in the case that X or Y is a Banach space.

We will split our exposition into two parts. In Subsection 2.11.1, we consider distributionally chaotic properties of linear (not necessarily continuous) operators and their sequences. Subsection 2.11.2 is devoted to te study of distributionally chaotic properties of abstract degenerate (multi-term) fractional differential equations.

2.11.1. Distributional chaos for single operators. The main purpose of this section is to investigate the basic distributionally chaotic properties of a sequence $(T_k)_{k\in\mathbb{N}}$ of linear mappings between the spaces X and Y. We start with the following definition.

Definition 2.11.1. Suppose that, for every $k \in \mathbb{N}$, $T_k \colon D(T_k) \to Y$ is a linear (not necessarily continuous) operator and $\tilde{X}$ is a closed linear subspace of X. Then we say that the sequence $(T_k)_{k\in\mathbb{N}}$ is $\tilde{X}$-distributionally chaotic iff there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x, y \in S$ of distinct points we have that

$$\overline{dens}(\{k \in \mathbb{N} : d_Y(T_k x, T_k y) \geqslant \sigma\}) = 1 \text{ and}$$
$$\overline{dens}(\{k \in \mathbb{N} : d_Y(T_k x, T_k y) < \varepsilon\}) = 1.$$

The sequence $(T_k)_{k\in\mathbb{N}}$ is said to be densely $\tilde{X}$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. A linear operator $T : D(T) \to Y$ is said to be (densely) $\tilde{X}$-distributionally chaotic iff the sequence $(T_k \equiv T^k)_{k\in\mathbb{N}}$ is. The set S is said to be $\sigma_{\tilde{X}}$-scrambled set (σ-scrambled set in the case that $\tilde{X} = X$) of the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T).

The above notions are clearly equivalent in the case that $\tilde{X} = X$ and then we also say that the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T) is distributionally chaotic. Of course, it is of interest to know the minimal linear subspace $\tilde{X}$ for which the sequence $(T_k)_{k\in\mathbb{N}}$ is $\tilde{X}$-distributionally chaotic because it is then $\hat{X}$-distributionally chaotic for any other linear subspace $\hat{X}$ of X such that $\tilde{X} \subseteq \hat{X}$. Observe that there exist some important cases in which the sequence of linear mappings $(T_{k|\tilde{X}})$, acting between the spaces $\tilde{X}$ and $Y = \tilde{X}$, is distributionally chaotic; see e.g. the proof of implication (i) $\Rightarrow$ (ii) of [**68**, Theorem 12]. On the other hand, we are always trying to find the maximal possible subspace $\tilde{X}$ of X for which the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T) is densely $\tilde{X}$-distributionally chaotic.

Definition 2.11.2. Suppose that, for every $k \in \mathbb{N}$, $T_k \colon D(T_k) \to Y$ is a linear (not necessarily continuous) operator, $\tilde{X}$ is a closed linear subspace of X, $x \in \bigcap_{k=1}^{\infty} D(T_k)$ and $m \in \mathbb{N}$. Then we say that:

(i) the orbit of x under $(T_k)_{k\in\mathbb{N}}$, i.e. the set $\{T_k x : k \in \mathbb{N}\}$, is distributionally near to 0 iff there exists $A \subseteq \mathbb{N}$ such that $\overline{dens}(A) = 1$ and $\lim_{k\in A, k\to\infty} T_k x = 0$,

(ii) the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is distributionally m-unbounded iff there exists $B \subseteq \mathbb{N}$ such that $\overline{dens}(B) = 1$ and $\lim_{k\in B,k\to\infty} p_m^Y(T_k x) = \infty$; the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is said to be distributionally unbounded iff there exists $q \in \mathbb{N}$ such that this orbit is distributionally q-unbounded (if Y is a Banach space, this simply means that $\lim_{k\in B,k\to\infty} \|T_k x\|_Y = \infty$),

(iii) x is a $\tilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$ (distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$, in the case that $\tilde{X} = X$) iff $x \in \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ and the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is both distributionally near to 0 and distributionally unbounded.

If $T\colon D(T) \to Y$ is a linear operator and $x \in D_\infty(T)$, then we say that the orbit $\{T^k x : k \in \mathbb{N}\}$ is distributionally near to 0 (distributionally unbounded) iff the sequence $(T_k \equiv T^k)_{k\in\mathbb{N}}$ is distributionally near to 0 (distributionally unbounded); x is said to be a $\tilde{X}$-distributionally irregular vector for T (distributionally irregular vector for T, in the case that $\tilde{X} = X$) iff x is $\tilde{X}$-distributionally irregular vector for the sequence $(T_k \equiv T^k)_{k\in\mathbb{N}}$ (distributionally irregular vector for the sequence $(T_k \equiv T^k)_{k\in\mathbb{N}}$, in the case that $\tilde{X} = X$).

Suppose that $X' \subseteq \tilde{X}$ is a linear manifold. Then we say that X' is $\tilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ (distributionally irregular manifold in the case that $\tilde{X} = X$) iff any element $x \in (X' \cap \bigcap_{k=1}^{\infty} D(T_k)) \setminus \{0\}$ is a $\tilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$; the notion of a $(\tilde{X}$-)distributionally irregular manifold for a linear operator $T\colon D(T) \to Y$ is defined similarly. Following [**68**, Definition 14], it will be said that X' is a uniformly $\tilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ (uniformly distributionally irregular manifold in the case that $\tilde{X} = X$) iff there exists $m \in \mathbb{N}$ such that the orbit of each vector $x \in (X' \cap \bigcap_{k=1}^{\infty} D(T_k)) \setminus \{0\}$ under $(T_k)_{k\in\mathbb{N}}$ is both distributionally m-unbounded and distributionally near to 0. If so, then (243)–245 implies that X' is 2^{-m}-scrambled set for the sequence $(T_k)_{k\in\mathbb{N}}$. It can be simply verified that the following holds: If $x \in \tilde{X} \cap \bigcap_{k=1}^{\infty} D(T_k)$ is a $\tilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$, then $X' \equiv span\{x\}$ is a uniformly $\tilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$. If X' is dense in $\tilde{X}$, then we deal with the notions of dense $(\tilde{X}$-)distributionally irregular manifolds, dense uniformly $(\tilde{X}$-)distributionally irregular manifolds, etc.

If $(T_k)_{k\in\mathbb{N}}$ and $\tilde{X}$ are given in advance, then we define the linear mappings $\mathcal{T}_k\colon D(\mathcal{T}_k) \to Y$ by $D(\mathcal{T}_k) := D(T_k) \cap \tilde{X}$ and $\mathcal{T}_k x := T_k x$, $x \in D(\mathcal{T}_k)$ $(k \in \mathbb{N})$. Then $(\mathcal{T}_k)_{k\in\mathbb{N}}$ is a sequence of linear mappings between the Fréchet spaces $\tilde{X}$ and Y. In some way, the next proposition enables us to reduce our further research in this subsection to the case in which $\tilde{X} = X$. An uncomplicated proof is left to the reader.

Proposition 2.11.3. (i) *The sequence $(T_k)_{k\in\mathbb{N}}$ is $\tilde{X}$-distributionally chaotic iff the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$ is distributionally chaotic.*

(ii) *A vector x is a $\tilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$ iff x is a distributionally irregular vector for the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$.*

(iii) *A linear manifold X' is a (uniformly) $\tilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ iff X' is a (uniformly) distributionally irregular manifold for the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$.*

We continue by stating the following theorem.

THEOREM 2.11.4. (i) *Suppose that $(T_k)_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$ and X_0 is a dense linear subspace of X, satisfying that for each $x \in X_0$ there exists a set $A_x \subseteq \mathbb{N}$ such that $\overline{dens}(A_x) = 1$ and $\lim_{k\in A_x} T_k x = 0$. If there exist a zero sequence (y_k) in X, a number $\varepsilon > 0$ and a strictly increasing sequence (N_k) in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$ and some $m \in \mathbb{N}$,*

$$\operatorname{card}(\{1 \leqslant j \leqslant N_k : p_m(T_j y_k) > \varepsilon\}) \geqslant N_k(1 - k^{-1}),$$

(for every $k \in \mathbb{N}$, $\operatorname{card}(\{1 \leqslant j \leqslant N_k : \|T_j y_k\|_Y > \varepsilon\}) \geqslant N_k(1 - k^{-1})$, in the case that Y is a Banach space), then there exists a distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$, and particularly, $(T_k)_{k\in\mathbb{N}}$ is distributionally chaotic.

(ii) *Suppose that X is separable, $(T_k)_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$, X_0 is a dense linear subspace of X, as well as:*

(a) *$\lim_{k\to\infty} T_k x = 0$, $x \in X_0$, and*

(b) *there exists $x \in X$ such that its orbit under $(T_k)_{k\in\mathbb{N}}$ is distributionally unbounded.*

Then there exists a dense uniformly distributionally irregular manifold for $(T_k)_{k\in\mathbb{N}}$, and particularly, $(T_k)_{k\in\mathbb{N}}$ is densely distributionally chaotic.

PROOF. We will only outline the most relevant details of the proof. In the case that Y is a Fréchet space, the assertion (i) can be simply proved by replacing the operator T^j with the operator T_j ($j \in \mathbb{N}$) throughout the proofs of [**68**, Propositions 7 and 9]. Assuming that Y is a Banach space, the required assertion follows from the above by endowing Y with the following increasing family of seminorms $p_n^Y(y) := n\|y\|_Y$ ($n \in \mathbb{N}$, $y \in Y$), which turns the space Y into a linearly and topologically homeomorphic Fréchet space. A careful inspection of the proof of [**68**, Theorem 15] shows that the assertion (ii) holds provided that X and Y are Fréchet spaces, and $p_m^Y(T_i x) \leqslant p_{i+m}(x)$, $x \in X$, $i, m \in \mathbb{N}$; the only thing we need to do is replace any operator T^i appearing in the proof of afore-mentioned theorem with the operator T_i ($i \in \mathbb{N}$). Observe further that we can always construct a fundamental system $(p_n'(\cdot))_{n\in\mathbb{N}}$ of increasing seminorms on the space X, inducing the same topology, so that $p_m^Y(T_i x) \leqslant p_{i+m}'(x)$, $x \in X$, $i, m \in \mathbb{N}$. In such a way, we may conclude that the assertion [**68**, Theorem 15] holds provided that X and Y are Fréchet spaces. In the case that X or Y is a Banach space, then we can 'renorm' it, as described above, and use after that the obtained result for Fréchet spaces. $\square$

Suppose now that $T\colon D(T) \subseteq X \to X$ is a linear mapping, $C \in L(X)$ is an injective mapping, as well as

$$R(C) \subseteq D_\infty(T) \text{ and } T^n C \in L(X) \text{ for all } n \in \mathbb{N}. \tag{246}$$

Then (246) implies that, for every $n \in \mathbb{N}$, the mapping $T_n \colon R(C) \to X$ defined by $T_n(Cx) := T^n Cx$, $x \in X$, $n \in \mathbb{N}$ is an element of the space $L([R(C)], X)$. The next theorem follows almost immediately from Theorem 2.11.4.

THEOREM 2.11.5. *Let the above conditions hold.*

(i) *Suppose that X_0 is a dense linear subspace of X, satisfying that for each $x \in X_0$ there exists a set $A_x \subseteq \mathbb{N}$ such that $\overline{dens}(A_x) = 1$ and $\lim_{k \in A_x, k \to \infty} T^k Cx = 0$. If there exist a zero sequence (z_k) in X, a number $\varepsilon > 0$ and a strictly increasing sequence (N_k) in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$ and some $m \in \mathbb{N}$,*

$$\operatorname{card}(\{1 \leqslant j \leqslant N_k : p_m(T^j Cz_k) > \varepsilon\}) \geqslant N_k(1 - k^{-1}),$$

(for every $k \in \mathbb{N}$, $\operatorname{card}(\{1 \leqslant j \leqslant N_k : \|T^j Cz_k\|_Y > \varepsilon\}) \geqslant N_k(1 - k^{-1})$, in the case that Y is a Banach space), then there exists a distributionally vector $x \in R(C)$ for the operator T. In particular, T is distributionally chaotic and σ-scrambled set S of T can be chosen to be a linear submanifold of $R(C)$.

(ii) *Suppose that X is separable, X_0 is a dense linear subspace of X, as well as:*

(a) $\lim_{k \to \infty} T^k Cx = 0$, *$x \in X_0$, and*

(b) *there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that $\overline{dens}(B) = 1$, and $\lim_{k \to \infty, k \in B} p_m(T^k Cx) = \infty$, resp. $\lim_{k \to \infty, k \in B} \|T^k Cx\| = \infty$ if X is a Banach space.*

Then there exists a uniformly distributionally irregular manifold W for the operator T, and particularly, T is distributionally chaotic. Furthermore, if $R(C)$ is dense in X, then W can be chosen to be dense in X and T is densely distributionally chaotic.

EXAMPLE 2.11.6. Suppose that X is separable, $D(A)$ and $R(C)$ are dense in X, $CA \subseteq AC$, $z_0 \in \mathbb{C} \smallsetminus \{0\}$, $\beta \geqslant -1$, $d \in (0,1]$, $m \in (0,1)$, $\varepsilon \in (0,1]$, $\gamma > -1$ and the following conditions hold:

(§) $P_{z_0,\beta,\varepsilon,m} := e^{i\arg(z_0)}(|z_0| + (P_{\beta,\varepsilon,m} \cup B_d)) \subseteq \rho_C(A)$, $(\varepsilon, m(1+\varepsilon)^{-\beta}) \in \partial B_d$,

(§§) the family $\{(1+|\lambda|)^{-\gamma}(\lambda - A)^{-1}C : \lambda \in P_{z_0,\beta,\varepsilon,m}\} \subseteq L(X)$ is equicontinuous, and

(§§§) the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\lambda \in P_{z_0,\beta,\varepsilon,m}$ is continuous for every fixed element $x \in E$.

Let $\Gamma_1(z_0,\beta,\varepsilon,m) = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi \geqslant \varepsilon,\ \eta = -m(1+\xi)^{-\beta}\}$, $\Gamma_2(z_0,\beta,\varepsilon,m) = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi^2 + \eta^2 = d^2,\ \xi \leqslant \varepsilon\}$ and $\Gamma_3(z_0,\beta,\varepsilon,m) = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi \geqslant \varepsilon,\ \eta = m(1+\xi)^{-\beta}\}$. We assume that the curve $\Gamma(z_0,\beta,\varepsilon,m) = \bigcup_{i=1}^{3}(e^{-i\arg(z_0)}\Gamma_i(z_0,\beta,\varepsilon,m) - |z_0|)$ is oriented so that $\operatorname{Im}\lambda$ decreases along $(e^{-i\arg(z_0)}\Gamma_1(z_0,\beta,\varepsilon,m) - |z_0|)$; since there is no risk for confusion, we also write Γ for $\Gamma(z_0,\beta,\varepsilon,m)$. Let $b \in (0,1/2)$ be fixed, set $\delta_b := \arctan(\cos \pi b)$ and $A_0 := e^{-i\arg(z_0)}A - |z_0|$. Define, for every $z \in \Sigma_{\delta_b}$,

$$T_b(z)x := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}(\lambda - A_0)^{-1}Cx\, d\lambda, \quad x \in X.$$

Then $T_b(z) \in L(X)$ is injective and has dense range in X for any $z \in \Sigma_{\delta_b}$ (see e.g. the proofs of [**296**, Theorem 3.15-Theorem 3.16]). Furthermore, $T_b(z)A \subseteq AT_b(z)$, $z \in \Sigma_{\delta_b}$ and the condition (246) holds with T and C replaced respectively by A and $T_b(z)$ ($z \in \Sigma_{\delta_b}$). If there exist a dense subset X_0 of X and a number $\lambda \in \sigma_p(A)$ such that $\lim_{k\to\infty} A^k x = 0$, $x \in X_0$ and $|\lambda| > 1$, then Theorem 2.11.5(ii) implies that the operator zA^n is densely distributionally chaotic for any $n \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| = 1$. Now we will illustrate this result by some examples.

(i) [**143**] Let $a, b, c > 0$, $c < \frac{b^2}{2a} < 1$, $X := L^2([0, \infty))$ and

$$\Lambda := \Big\{\lambda \in \mathbb{C} : \Big|\lambda - \Big(c - \frac{b^2}{4a}\Big)\Big| \leqslant \frac{b^2}{4a},\ \operatorname{Im}\lambda \neq 0 \text{ if } \operatorname{Re}\lambda \leqslant c - \frac{b^2}{4a}\Big\}.$$

Consider the operator $-A$ defined by $D(-A) := \{f \in W^{2,2}([0, \infty)) : f(0) = 0\}$ and $-Au := au_{xx} + bu_x + cu$, $u \in D(A)$. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a non-constant complex polynomial such that $a_n > 0$ and

$$P(-\Lambda) \cap \{z \in \mathbb{C} : |z| = 1\} \neq \emptyset.$$

Then $-P(A)$ generates an analytic C_0-semigroup of angle $\pi/2$, $P(-\Lambda) \subseteq \sigma_p(P(A))$ and it is not difficult to prove that the operator $P(A)$ is densely distributionally chaotic.

(ii) [**115, 292, 427**] This example has already appeared in our previous examinations from Section 2.10. Suppose $X := L^2(\mathbb{R})$, $c > b/2 > 0$, $\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < c - b/2\}$ and $\mathcal{A}_c u := u'' + 2bxu' + cu$ is the bounded perturbation of the one-dimensional Ornstein–Uhlenbeck operator acting with domain $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$. Then $\mathcal{A}_c$ generates a strongly continuous semigroup, $\Omega \subseteq \sigma_p(\mathcal{A}_c)$, and for any open connected subset Ω' of Ω which admits a cluster point in Ω, we have $E = \overline{span\{g_i(\lambda) : \lambda \in \Omega',\ i = 1, 2\}}$, where $g_1(\lambda)$ and $g_2(\lambda)$ are defined as before. This simply implies that the operator $\mathcal{A}_c$ is densely distributionally chaotic, this is also the property which holds for the multi-dimensional Ornstein–Uhlenbeck operators from [**115**, Section 4].

(iii) [**168**] Suppose $r > 0$, $\sigma > 0$, $\nu = \sigma/\sqrt{2}$, $\gamma = r/\mu - \mu$, $s > 1$, $s\nu > 1$ and $\tau \geqslant 0$. Set

$$Y^{s,\tau} := \Big\{u \in C((0, \infty)) : \lim_{x\to 0} \frac{u(x)}{1 + x^{-\tau}} = \lim_{x\to\infty} \frac{u(x)}{1 + x^s} = 0\Big\}.$$

Then $Y^{s,\tau}$, equipped with the norm

$$\|u\|_{s,\tau} := \sup_{x>0} \Big|\frac{u(x)}{(1 + x^{-\tau})(1 + x^s)}\Big|, \quad u \in Y^{s,\tau},$$

becomes a separable Banach space. Let $D_\mu := \nu x d/dx$, with maximal domain in $Y^{s,\tau}$, and let the Black–Scholes operator $\mathcal{B}$ be defined by $\mathcal{B} := D_\nu^2 + \gamma D_\mu - r$. H. Emamirad, G. R. Goldstein and J. A. Goldstein have proved in [**168**] that the operator $\mathcal{B}$ generates a chaotic strongly continuous semigroup (it can be easily seen that the Black–Scholes semigroup is densely distributionally chaotic, as well; see the next subsection for the notion). By [**168**, Lemma 3.3], the proof of [**168**, Lemma 3.5] (cf. especially the Figure 1 in the abovementioned paper, in the $Ox'y'$ coordinate system, with $x' = x/\nu$ and $y' = y/\nu$) and the previous examination, it readily follows that the operator $\mathcal{B}$ is densely distributionally chaotic.

(iv) [**137**] Assume that ω_1, ω_2, V_{ω_2,ω_1}, $Q(z)$, $Q(B)$, N and X possess the same meaning as in [**137**, Section 5], and

$$Q(\mathrm{int}(V_{\omega_2,\omega_1})) \cap \{z \in \mathbb{C} : |z| = 1\} \neq \emptyset.$$

Then it can be proved with the help of Theorem 2.11.5(ii) that the unbounded operator $Q(B)$ is densely distributionally chaotic; the Devaney chaoticity of $Q(B)$ can be proved in a similar fashion.

(v) Making use of Theorem 2.11.5(ii), we can also prove that certain polynomials of the Laplace–Beltrami operator $\Delta^{\natural}_{X,p}$, acting on the symmetric spaces of non-compact type, Damek–Ricci or Heckman–Opdam root spaces ([**35**, **259**, **474**]), are densely distributionally chaotic.

It is not difficult to see that the operators considered in the previous example are also chaotic (in the sense of [**137**, Definition 2.1]). Here we would like to mention in passing that Q. Menet [**425**] has recently solved [**68**, Problem 37] by constructing a linear continuous operator T acting on a classical Banach function space that is both chaotic and not distributionally chaotic. Motivated by the research of J. Bes, C. K. Chan and S. M. Seubert [**69**], where the chaotic behaviour of the abstract Laplace operator Δ has been analyzed, we would like to propose the following problem:

PROBLEM DC. Suppose $1 \leqslant p < \infty$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open (possibly unbounded) set, and the operator Δ acts on $L^p(\Omega)$ with maximal distributional domain and without any boundary conditions. Is it true that Δ is densely distributionally chaotic?

The proof of following slight extension of [**430**, Theorem 3] relies on use of K. Ball's planck theorems [**430**, Theorems 1 and 2] (cf. also [**419**, Proposition 9.1(a)]):

PROPOSITION 2.11.7. *Suppose that X and Y are Banach spaces, and $T_k \in L(X, Y)$ for each $k \in \mathbb{N}$. If either*

(i) $\sum_{k=1}^{\infty} \frac{1}{\|T_k\|} < \infty$

or

(ii) *X is a complex Hilbert space and* $\sum_{k=1}^{\infty} \frac{1}{\|T_k\|^2} < \infty$,

then there exists $x \in X$ such that $\lim_{k\to\infty} \|T_k x\|_Y = \infty$.

Taking into account the assertion of Proposition 2.11.7 and the fact that $[R(C)]$ is a Banach (complex Hilbert space) provided that X is, we may conclude that the following holds (cf. [**68**, Theorem 16] and [**66**, Corollary 30]): If $B \subseteq \mathbb{N}$, $\overline{dens}(B) = 1$, the assumptions of Theorem 2.11.5(i) and (ii)(a) are valid, and exactly one of the following two conditions holds:

(i) X is a Banach space and $\sum_{k\in B} \frac{1}{\|T^k C\|} < \infty$,

or

(ii) X is a complex Hilbert space and $\sum_{k\in B}\frac{1}{\|T^kC\|^2}<\infty$,

then there exists $x \in X$ such that $\lim_{k\to\infty,k\in B}\|T^kCx\| = \infty$, i.e., the condition stated in Theorem 2.11.5(ii)(b) automatically holds.

Before we move ourselves to the next subsection, we want to mention distributionally chaotic properties of multivalued linear operators as an interesting theme for our researchers.

2.11.2. Distributionally chaotic properties of abstract degenerate fractional differential equations. The following continuous version of Theorem 2.11.5(ii) will be essentially utilized in this subsection.

THEOREM 2.11.8. *Suppose that X is separable, X_0 is a dense linear subspace of X, $(T(t))_{t\geqslant 0} \subseteq L(X, Y)$ is a strongly continuous operator family, as well as:*

(a) $\lim_{t\to\infty} T(t)x = 0$, $x \in X_0$,

(b) *there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq [0,\infty)$ such that $\overline{Dens}(B) = 1$, and $\lim_{t\to\infty,t\in B} p_m(T(t)x) = \infty$, resp. $\lim_{t\to\infty,t\in B}\|T(t)x\| = \infty$ if X is a Banach space.*

Then there exist a dense linear subspace S of X and a number $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x, y \in S$ of distinct points we have

$$\overline{Dens}(\{t \geqslant 0 : d_Y(T(t)x, T(t)y) \geqslant \sigma\}) = 1$$

and

$$\overline{Dens}(\{t \geqslant 0 : d_Y(T(t)x, T(t)y) < \varepsilon\}) = 1.$$

PROOF. The proof is very similar to those of [**68**, Theorem 15] and Theorem 2.11.5(ii). Consider first the case in which X and Y are Frechét spaces. If so, the family $(T(t))_{t\geqslant 0} \subseteq L(X, Y)$ is locally equicontinuous because it is strongly continuous and X is barreled ([**296**]). Hence, for every $l, n \in \mathbb{N}$, there exist $c_{l,n} > 0$ and $a_{l,n} \in \mathbb{N}$ such that $p_l^Y(T(t)x) \leqslant c_{l,n}p_{a_{l,n}}(x)$, $x \in X$, $t \in [0, n]$. Suppose, for the time being, that:

$$p_k^Y(T(t)x) \leqslant p_{k+\lceil t\rceil}(x), \quad x \in X,\ t \geqslant 0,\ k \in \mathbb{N}. \tag{247}$$

Without loss of generality, we may assume that $m = 1$. Then one can find a sequence $(x_k)_{k\in\mathbb{N}}$ in X_0 such that $p_k(x_k) \leqslant 1$, $k \in \mathbb{N}$ and a strictly increasing sequence of positive real numbers $(t_k)_{k\in\mathbb{N}}$ tending to infinity such that:

$$\overline{Dens}(\{1 \leqslant t \leqslant t_k : p_1(T(t)x_k) > k2^k\}) \geqslant t_k(1 - k^{-2})$$

and

$$\overline{Dens}(\{1 \leqslant t \leqslant t_k : p_k(T(t)x_l) < k^{-1}\}) \geqslant t_k(1 - k^{-2}), \quad l = 1, \dots, k-1.$$

Let $(r_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ so that:

$$r_{j+1} \geqslant 1 + r_j + \lceil t_{r_j+1}\rceil, \quad j \in \mathbb{N}.$$

Due to the proof of [**68**, Theorem 15], we obtain the existence of a dense linear subspace S of X such that, for every $x \in S$, there exist two sets $A_x,\ B_x \subseteq [0,\infty)$ such that $\overline{Dens}(A) = \overline{Dens}(B) = 1$, $\lim_{t\to\infty,t\in A_x} T(t)x = 0$ and $\lim_{t\to\infty,t\in B_x} p_1(T(t)x)$

$= \infty$. Now the final conlusion of theorem follows as in the discrete case. Introducing recursively the following fundamental system of increasing seminorms $p_n'(\cdot)$ $(n \in \mathbb{N})$ on X:

$$\begin{aligned}
&p_1'(x) \equiv p_1(x), \quad x \in X,\\
&p_2'(x) \equiv p_1'(x) + c_{1,1}p_{a_{1,1}}(x) + p_2(x), \quad x \in X,\\
&\cdots\\
&p_{n+1}'(x) \equiv p_n'(x) + c_{1,n}p_{a_{1,n}}(x) + \cdots + c_{n,1}p_{a_{n,1}}(x) + p_{n+1}(x), \quad x \in X,\\
&\cdots,
\end{aligned}$$

we may assume without loss of generality that (247) holds, hence the assertion is proved in the case that X and Y are Frechét spaces. If X or Y is a Banach space, then we can 'renorm' it, as it has been explained in the proof of Theorem 2.11.5(i), and use after that the above result to finish the proof of theorem. $\square$

Suppose that $n \in \mathbb{N} \smallsetminus \{1\}$, $A_1, \ldots, A_{n-1}, A_n = B$ are closed linear operators on X and $0 \leqslant \alpha_1 < \cdots < \alpha_n$. In this section, we scrutinize distributionally chaotic properties of abstract degenerate fractional Cauchy problem [(90)-(91)], which will be simply denoted by (ACP) here. We focus special attention on distributionally chaotic solutions of the fractional Sobolev equations (DFP)$_R$ and (DFP)$_L$, as well. Along with the problems (DFP)$_R$ and (DFP)$_L$, we consider the associated abstract integral equation (this is, clearly, a special case of problem (52) with $a(t) = g_\alpha(t)$):

$$Bu(t) = f(t) + \int_0^t g_\alpha(t-s)Au(s)ds, \quad t \geqslant 0, \tag{248}$$

where $f \in C([0,\infty) : X)$. Henceforth (DFP) denotes either (DFP)$_R$ or (DFP)$_L$. By a mild solution of the problem (DFP)$_R$ we mean any continuous function $t \mapsto u(t)$, $t \geqslant 0$ such that the mapping $t \mapsto Bu(t)$, $t \geqslant 0$ is continuous and $A(g_\alpha * u)(t) = Bu(t) - \sum_{k=0}^{\lceil \alpha \rceil -1} g_{k+1}(t)Bx_k$, $t \geqslant 0$. The set of all vectors $\vec{x} = (Bx_0, Bx_1, \ldots, Bx_{\lceil \alpha \rceil -1})$ for which there exists a mild solution of problem (DFP)$_R$ will be denoted by $Z_{\alpha,R}^{mild}(A,B)$. Denote by $\mathfrak{T}$ the exact number of initial values subjected to the problem (ACP); that is, $\mathfrak{T}$ is the sum of number m_Q and the cardinality of set consisting of those pairs $(i,j) \in \mathbb{N}_n \times \mathbb{N}^0_{m_n-1}$ for which $m_i - 1 \geqslant j \geqslant m_Q$. By $\mathfrak{Z}$ $(\mathfrak{Z}_{uniq})$ we denote the set of all tuples of initial values $\vec{x} = ((u_j)_{0 \leqslant j \leqslant m_Q-1}; (u_{i_{s'},j})_{1 \leqslant s' \leqslant s, m_Q \leqslant j \leqslant m_{i_{s'}}-1}) \in X^{\mathfrak{T}}$ for which there exists a (unique) strong solution of problem (ACP).

The notion of (subspace) distributional chaoticity of problem (ACP) is introduced in the following definition.

Definition 2.11.9. Let $\tilde{X}$ be a closed linear subspace of $X^{\mathfrak{T}}$. Then it is said that the abstract Cauchy problem (ACP) is $\tilde{X}$-distributionally chaotic iff there are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $\vec{x}$, $\vec{y} \in S$ of distinct tuples we have that there exist strong solutions $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ and $t \mapsto u(t;\vec{y})$, $t \geqslant 0$ of problem (ACP) with the property that

$$\overline{Dens}(\{t \geqslant 0 : d(u(t;\vec{x}), u(t;\vec{y})) \geqslant \sigma\}) = 1 \text{ and}$$

$$\overline{Dens}(\{t \geqslant 0 : d(u(t;\vec{x}), u(t;\vec{y})) < \varepsilon\}) = 1.$$

If we can choose S to be dense in $\tilde{X}$, then we also say that the problem (ACP) is densely $\tilde{X}$-distributionally chaotic (S is called a $\sigma_{\tilde{X}}$-scrambled set). In the case that $\tilde{X} = X$, it is also said that the problem (ACP) is (densely) distributionally chaotic; S is then called a σ-scrambled set.

The notion introduced in Definition 2.11.9 can be slightly modified for the problem $(\mathrm{DFP})_R$ by requiring that, for every two distinct tuples vectors of $\sigma_{\tilde{X}}$-scrambled set $S \subseteq \tilde{X} \cap Z^{mild}_{\alpha,R}(A,B)$, there exist mild solutions $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ and $t \mapsto u(t;\vec{y})$, $t \geqslant 0$ of problem $(\mathrm{DFP})_R$ obeying the properties prescribed. We will not follow this approach henceforth.

DEFINITION 2.11.10. Let $n \in \mathbb{N}$, let $\tilde{X}$ be a closed linear subspace of $X^{\mathfrak{T}}$, and let $\vec{x} \in \tilde{X} \cap \mathfrak{Z}$. Then it is said that the vector $\vec{x}$ is:

(i) $\tilde{X}$-(ACP)-distributionally near to 0 iff there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ of problem (ACP) such that
$$\overline{Dens}(Z) = 1 \text{ and } \lim_{t\in Z, t\to+\infty} u(t;\vec{x}) = 0;$$

(ii) $\tilde{X}$-(ACP)-distributionally n-unbounded iff there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ of problem (ACP) such that
$$\overline{Dens}(Z) = 1 \text{ and } \lim_{t\in Z, t\to+\infty} p_n(u(t;\vec{x})) = +\infty;$$
$\vec{x}$ is said to be $\tilde{X}$-(ACP)-distributionally unbounded iff there exists $q \in \mathbb{N}$ such that $\vec{x}$ is $\tilde{X}$-(ACP)-distributionally q-unbounded (if $(X, \|\cdot\|)$ is a Banach space, this simply means that there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ of problem (ACP) such that $\overline{Dens}(Z) = 1$ and $\lim_{t\in Z, t\to+\infty} \|u(t;\vec{x})\| = +\infty$);

(iii) a $\tilde{X}$-(ACP)-distributionally irregular vector iff there exist an integer $q \in \mathbb{N}$, two subsets B_0, B_∞ of $[0,\infty)$ with $\overline{Dens}(B_0) = \overline{Dens}(B_\infty) = 1$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ of problem (ACP) such that

$$\lim_{t\in B_0, t\to+\infty} u(t;\vec{x}) = 0 \text{ and } \lim_{t\in B_\infty, t\to+\infty} p_q(u(t;\vec{x})) = +\infty. \tag{249}$$

In the case that $\tilde{X} = X^{\mathfrak{T}}$, then we also say that $\vec{x}$ is (ACP)-distributionally near to 0, resp., (ACP)-distributionally n-unbounded, (ACP)-distributionally unbounded; a $\tilde{X}$-distributionally irregular vector for (ACP) is then called a distributionally irregular vector for (ACP).

Suppose that $X' \subseteq \tilde{X} \cap \mathfrak{Z}$ is a linear manifold. Then we say that X' is a $\tilde{X}$-distributionally irregular manifold for (ACP) (distributionally irregular manifold for (ACP), in the case that $\tilde{X} = X^{\mathfrak{T}}$) iff any element $x \in X' \smallsetminus \{0\}$ is $\tilde{X}$-distributionally irregular vector for (ACP). Further on, we say that X' is a uniformly $\tilde{X}$-distributionally irregular manifold for (ACP) (uniformly distributionally irregular manifold for (ACP), in the case that $\tilde{X} = X^{\mathfrak{T}}$) iff there exists $q \in \mathbb{N}$ such that, for every $\vec{x} \in X' \smallsetminus \{0\}$, there exist two subsets B_0, B_∞ of $[0,\infty)$ with

$\overline{Dens}(B_0) = \overline{Dens}(B_\infty) = 1$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geqslant 0$ of problem (ACP) such that (249) holds. It can be simply verified with the help of translation invariance of metric $d(\cdot,\cdot)$ and inequalities (244)–(245) that the following holds: If $0 \neq \vec{x} \in \tilde{X} \cap \mathfrak{Z}$ is a $\tilde{X}$-distributionally irregular vector for (ACP), then $X' \equiv span\{\vec{x}\}$ is a uniformly $\tilde{X}$-distributionally irregular manifold for (ACP).

Remark 2.11.11. (i) If $\vec{x}$ is a $\tilde{X}$-distributionally irregular vector for (ACP), then $\vec{x}$ is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded. The converse statement holds provided that strong solutions of problem (ACP) are unique. If this is not the case and $\vec{x} \neq 0$ is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded, then we can prove the following (cf.Definition 2.11.9): There are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ (S is, in fact, equal to $span\{\vec{x}\}$) and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $\vec{x}, \vec{y} \in S$ of distinct vectors we have that there exist strong solutions $t \mapsto u_i(t;\vec{x})$, $t \geqslant 0$ and $t \mapsto u_i(t;\vec{y})$, $t \geqslant 0$ ($i = 1, 2$) of problem (ACP) with the property that

$$\overline{Dens}(\{t \geqslant 0 : d(u_1(t;\vec{x}), u_1(t;\vec{y})) \geqslant \sigma\}) = 1 \text{ and}$$
$$\overline{Dens}(\{t \geqslant 0 : d(u_2(t;\vec{x}), u_2(t;\vec{y})) < \varepsilon\}) = 1.$$

If this is the case, we say that the problem (ACP) is quasi $\tilde{X}$-distributionally chaotic (quasi distributionally chaotic, provided that $\tilde{X} = X^{\mathfrak{T}}$). The set S is called quasi $\sigma_{\tilde{X}}$-scrambled set (quasi σ-scrambled set, provided that $\tilde{X} = X^{\mathfrak{T}}$).

(ii) Suppose that (ACP) is $\tilde{X}$-distributionally chaotic and S is the corresponding $\sigma_{\tilde{X}}$-scrambled set. Then, for every two distinct vectors $\vec{x}$, $\vec{y} \in S$, $\vec{x} - \vec{y}$ is a $\tilde{X}$-distributionally vector for (ACP).

(iii) Suppose that (ACP) is quasi $\tilde{X}$-distributionally chaotic and S is the corresponding quasi $\sigma_{\tilde{X}}$-scrambled set. Then, for every two distinct vectors $\vec{x}, \vec{y} \in S$, $\vec{x} - \vec{y}$ is a quasi $\tilde{X}$-distributionally vector for (ACP), i.e., $\vec{x} - \vec{y}$ is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded.

It is worth noting that the non-triviality of subspace $\bigcap_{i=1}^n N(A_i)$ in X immediately implies that the problem (ACP) is distributionally chaotic:

Example 2.11.12. Suppose that $0 \neq x \in \bigcap_{i=1}^n N(A_i)$. We can always find a sequence $(a_n)_{n\in\mathbb{N}_0}$ of non-negative real numbers and a scalar-valued function $f \in C^\infty([0,\infty))$ such that $a_0 = 0$, $a_n > a_{n-1}+2$, $n \in \mathbb{N}$, $\lim_{n\to+\infty}(a_{n-1}-1)(a_n-1)^{-1} = 0$, $f(t) = 0$ for $t \in \bigcup_{n\in\mathbb{N}}[a_{2n-1}, a_{2n}]$ and $f(t) = 2n$ for $t \in \bigcup_{n\in\mathbb{N}_0}[a_{2n}+1, a_{2n+1}-1]$. Since the sets $B_0 := \bigcup_{n\in\mathbb{N}}[a_{2n-1}, a_{2n}]$ and $B_\infty := \bigcup_{n\in\mathbb{N}_0}[a_{2n}+1, a_{2n+1}-1]$ have the upper densities equal to 1, it is very simple to verify that the function $u(t;\vec{x}) := f(t)x$, $t \geqslant 0$ is a strong solution of problem (ACP) with the initial value $\vec{x} = ((u_j \equiv f^{(j)}(0)x)_{0\leqslant j\leqslant m_Q-1}; (u_{i_{s'},j} \equiv 0)_{1\leqslant s'\leqslant s, m_Q\leqslant j\leqslant m_{i_{s'}}-1}) \in X^{\mathfrak{T}}$, as well as that $\vec{x}$ is a distributionally irregular vector for (ACP). In particular, $\vec{x} = \vec{0}$ can be a distributionally irregular vector for (ACP).

It is not clear whether there exists a quasi distributionally chaotic problem (ACP) that is not distributionally chaotic. In the sequel, we will consider only the classical notion of (subspace) distributional chaoticity of problem (ACP).

The notions of exponentially equicontinuous (g_α, C)-regularized resolvent family for (248) and exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B will be crucially important in our work. We know the following facts about exponentially equicontinuous (g_α, C)-regularized resolvent families (cf. Section 2.1–Section 2.2 for more details).

LEMMA 2.11.13. (i) *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family for* (248)*. Suppose that the following condition holds:*

(P)" *There exists a number $\omega_1 > \omega$ such that, for every $x \in X$, there exists a function $h(\lambda; x) \in LT{-}X$ such that $h(\lambda; x) = \lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx$, provided $\operatorname{Re}\lambda > \omega_1$.*

Let $x_0, \ldots, x_{\lceil\alpha\rceil-1} \in D(A) \cap D(B)$. Then the function $u(t; (BCx_0, \ldots, BCx_{\lceil\alpha\rceil-1})) := \sum_{j=0}^{\lceil\alpha\rceil-1} \int_0^t g_j(t-s)R(s)x_j\, ds$, $t \geqslant 0$ is a unique strong solution of $(\mathrm{DFP})_R$*, with the initial values Bx_j replaced by BCx_j $(0 \leqslant j \leqslant \lceil\alpha\rceil - 1)$.*

(ii) *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B. Then for each $x_0, \ldots, x_{\lceil\alpha\rceil-1} \in D(A) \cap D(B)$ the function $u(t; (Cx_0, \ldots, Cx_{\lceil\alpha\rceil-1})) := \sum_{j=0}^{\lceil\alpha\rceil-1} \int_0^t g_j(t-s)R(s) Bx_j\, ds$, $t \geqslant 0$ is a unique strong solution of problem* $(\mathrm{DFP})_L$*, with the initial values x_j replaced by Cx_j $(0 \leqslant j \leqslant \lceil\alpha\rceil - 1)$.*

In the following theorem, we will consider the subspace distributionally chaotic properties of problem $(\mathrm{DFP})_R$.

THEOREM 2.11.14. *Suppose that $\alpha > 0$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family for* (248)*,* (P)" *holds and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Suppose, further, that the following conditions hold:*

- *Let F_i be a separable complex Fréchet space, let $F_i \subseteq D(A) \cap D(B)$, and let F_i be continuously embedded in X $(i \in \mathcal{V})$.*
- *Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(CBf_i) \leqslant c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$.*
- *For each $i \in \mathcal{V}$ there exists a dense subset F_i^0 of F_i satisfying that $\lim_{t\to+\infty}(g_i * R(\cdot)f_i)(t) = 0$, $f_i \in F_i^0$.*
- *Let there exist $\vec{f_\infty} = (f_{0,\infty}, \ldots, f_{\lceil\alpha\rceil-1,\infty}) \in F$, $m \in \mathbb{N}$ and a set $D \subseteq [0,\infty)$ such that $\overline{Dens}(D) = 1$, and*

$\lim_{t\to+\infty,t\in D} p_m(\sum_{i\in\mathcal{V}}(g_i * R(\cdot)f_{i,\infty})(t)) = +\infty$, *resp.* $\lim_{t\to+\infty,t\in D} \|\sum_{i\in\mathcal{V}}(g_i * R(\cdot)f_{i,\infty})(t)\| = +\infty$ *if* $(X, \|\cdot\|)$ *is a Banach space.*

Then we have that the problem $(\mathrm{DFP})_R$ *is densely* $\overline{\{(CBf_0,\ldots,CBf_{\lceil\alpha\rceil-1}) : \vec{f}=(f_0,\ldots,f_{\lceil\alpha\rceil-1})\in F\}}^{X^{\lceil\alpha\rceil}}$*-distributionally chaotic.*

Proof. It is clear that F is an infinite-dimensional separable complex Fréchet space. Define $V(t)\vec{f} := \sum_{i=0}^{\lceil\alpha\rceil-1}(g_i * R(\cdot)f_i)(t)$, $t \geqslant 0$ ($\vec{f} = (f_0,\ldots,f_{\lceil\alpha\rceil-1}) \in F$) and $F_0 := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i^0$, where $G_i^0 := F_i^0$, if $i \in \mathcal{V}$, and $G_i^0 := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$. Then F_0 is dense in F and $(V(t))_{t\geqslant 0} \subseteq L(F,X)$ is a strongly continuous operator family. An application of Theorem 2.11.8 yields that there exist a dense linear subspace S of F and a number $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $\vec{f'}, \vec{f''} \in S$ of distinct vectors we have

$$\overline{Dens}(\{t \geqslant 0 : d(V(t)\vec{f'}, V(t)\vec{f''}) \geqslant \sigma\}) = 1 \text{ and}$$
$$\overline{Dens}(\{t \geqslant 0 : d(V(t)\vec{f'}, V(t)\vec{f''}) < \varepsilon\}) = 1.$$

Suppose that $CBf_i = 0$ for all $i \in \mathcal{V}$ and $f_i \in F_i$. Then (54) and the uniqueness theorem for the Laplace transform together imply that $R(t)f_i = 0$ for all $i \in \mathcal{V}$ and $f_i \in F_i$, which contradicts the existence of m-distributionally unbounded vector $\vec{f_\infty}$ from F. Hence, there exist $i \in \mathcal{V}$ and $f_i \in F_i$ such that $CBf_i \neq 0$. Using this fact and the continuity of mapping $CB\colon F_i \to X$ for each $i \in \mathcal{V}$, we can simply verify that $\{(CBf_0,\ldots,CBf_{\lceil\alpha\rceil-1}) : \vec{f} = (f_0,\ldots,f_{\lceil\alpha\rceil-1}) \in S\}$ is a non-trivial subspace of $X^{\lceil\alpha\rceil}$. Now the final conclusion simply follows by using the continuity of mappings $CB\colon F_i \to X$ ($i \in \mathcal{V}$) once more, and Lemma 2.11.13(i). $\square$

Similarly, by using Theorem 2.11.8 and Lemma 2.11.13(ii), we can prove the following theorem on subspace distributional chaoticity of problem $(\mathrm{DFP})_L$.

Theorem 2.11.15. *Suppose that* $\alpha > 0$, $C \in L(X)$ *is injective,* $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geqslant 0}$ *is an exponentially equicontinuous* (g_α, C)*-regularized resolvent family generated by* A, B, *and* $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. *Suppose, further, that the following conditions hold:*

- *Let* F_i *be a separable complex Fréchet space, and let* $F_i \subseteq D(A) \cap D(B)$ $(i \in \mathcal{V})$.
- *Suppose that for each* $n \in \mathbb{N}$ *and* $i \in \mathcal{V}$ *there exist a number* $c_{n,i} > 0$ *and a continuous seminorm* $q_{n,i}(\cdot)$ *on* F_i *so that* $p_n(Bf_i) + p_n(Cf_i) \leqslant c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. *Set* $G_i := F_i$, *if* $i \in \mathcal{V}$, $G_i := \{0\}$, *if* $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$, *and* $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$.
- *Suppose, further, that for each* $i \in \mathcal{V}$ *there exists a dense subset* F_i^0 *of* F_i *satisfying that* $\lim_{t\to+\infty}(g_i * R(\cdot)Bf_i)(t) = 0$, $f_i \in F_i^0$.
- *Let there exist* $\vec{f} = (f_0,\ldots,f_{\lceil\alpha\rceil-1}) \in F$, $m \in \mathbb{N}$ *and a set* $D \subseteq [0,\infty)$ *such that* $\overline{Dens}(D) = 1$, *and* $\lim_{t\to+\infty,t\in D} p_m(\sum_{i\in\mathcal{V}}(g_i * R(\cdot)Bf_i)(t)) = +\infty$, *resp.* $\lim_{t\to+\infty,t\in D} \|\sum_{i\in\mathcal{V}}(g_i * R(\cdot)Bf_i)(t)\| = +\infty$ *if* $(X,\|\cdot\|)$ *is a Banach space.*

Then we have that the problem $(\mathrm{DFP})_L$ *is densely* $\overline{\{(Cf_0,\dots,Cf_{\lceil\alpha\rceil-1}):\vec{f}=(f_0,\dots,f_{\lceil\alpha\rceil-1})\in F\}}^{X^{\lceil\alpha\rceil}}$*-distributionally chaotic.*

REMARK 2.11.16. (i) Suppose that $l\in\mathbb{N}_0$. Then the increasing family of seminorms $p_{n,l}^{A,B,C}(\cdot):=p_n(C^{-l}\cdot)+p_n(C^{-l}A\cdot)+p_n(C^{-l}B\cdot)$ $(n\in\mathbb{N})$ turns $C^l(D(A)\cap D(B))$ into a Fréchet space, which will be denoted by $[D(A)\cap D(B)]_C^l$ in the sequel. In the concrete situation of Theorem 2.11.14 or Theorem 2.11.15, F_i can be chosen to be some of closed linear subspaces of $[D(A)\cap D(B)]_C^l$ that is separable for the topology induced from $[D(A)\cap D(B)]_C^l$. If X is separable and $C=I$, then for each number $\lambda>\omega$ the mapping $(\lambda^\alpha B-A)^{-1}\colon X\to[D(A)\cap D(B)]$ $(\equiv[D(A)\cap D(B)]_I^0)$ is a linear topological isomorphism and, in this case, $[D(A)\cap D(B)]$ and F_i will be separable $(i\in\mathcal{V})$.

(ii) If we suppose additionally that $R(t)B\subseteq BR(t)$, $t\geqslant 0$ in the formulation of Theorem 2.11.15, then we do not need to assume that for each $n\in\mathbb{N}$ and $i\in\mathcal{V}$ there exist a number $c_{n,i}>0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(Bf_i)\leqslant cq_{n,i}(f_i)$, $f_i\in F_i$ (because, in this case, the operator $V_L(t)\vec{\cdot}=\sum_{i=0}^{\lceil\alpha\rceil-1}(g_i*R(\cdot)B\cdot_i)(t)$, $t\geqslant 0$ $(\vec{\cdot}=(\cdot_0,\dots,\cdot_{\lceil\alpha\rceil-1})\in F)$ belongs to the space $L(F,X)$ and $(V_L(t))_{t\geqslant 0}$ is a strongly continuous operator family in $L(F,X)$).

Keeping in mind Theorem 2.10.3, it is very natural to raise the following issue:

PROBLEM 1. Suppose that $\alpha\in(0,2)$ and Ω is an open connected subset of $\mathbb{C}$ which satisfies $\Omega\cap(-\infty,0]=\emptyset$ and $\Omega\cap i\mathbb{R}\neq\emptyset$. Let $f\colon\Omega^\alpha\to X$ be an analytic mapping such that $f(\lambda^\alpha)\in N(A-\lambda^\alpha B)\smallsetminus\{0\}$, $\lambda\in\Omega$. Does there exist a closed linear subspace X' of $X^{\lceil\alpha\rceil}$ such that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are (densely) X'-distributionally chaotic?

The method proposed in the proofs of [**112**, Theorem 4.1] and its discrete precursor [**68**, Theorem 15] cannot be applied here and, because of that, one has to follow new paths capable of moving us towards a solution of this problem. Unfortunately, we will present only some partial answers to Problem 1 by assuming that the strong solutions of problem (DFP) are governed by an exponentially equicontinuous (g_α,C)-regularized resolvent family for (248) (in the case of problem $(\mathrm{DFP})_R$) or an exponentially equicontinuous (g_α,C)-regularized resolvent family generated by A, B (in the case of problem $(\mathrm{DFP})_L$).

We start by stating the following result.

THEOREM 2.11.17. (i) *Suppose that* $0<\alpha<2$, $C\in L(X)$ *is injective,* $CA\subseteq AC$, $CB\subseteq BC$, $(R(t))_{t\geqslant 0}$ *is an exponentially equicontinuous* (g_α,C)*-regularized resolvent family for* (248), (P)" *holds and* $\emptyset\neq\mathcal{V}\subseteq\mathbb{N}^0_{\lceil\alpha\rceil-1}$. *Suppose, further, that the following conditions hold:*

– *Let* F_i *be a separable complex Fréchet space, let* $F_i\subseteq D(A)\cap D(B)$, *and let* F_i *be continuously embedded in* X $(i\in\mathcal{V})$.
– *Suppose that for each* $n\in\mathbb{N}$ *and* $i\in\mathcal{V}$ *there exist a number* $c_{n,i}>0$ *and a continuous seminorm* $q_{n,i}(\cdot)$ *on* F_i *so that* $p_n(CBf_i)\leqslant$

$c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$.

- *Let $H_i : \Omega^\alpha \to F_i$ be an analytic mapping such that $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \smallsetminus \{0\}$, $\lambda \in \Omega$ ($i \in \mathcal{V}$).*

Set $F_i' := \overline{span\{H_i(\lambda^\alpha) : \lambda \in \Omega\}}^{F_i}$ ($i \in \mathcal{V}$), $F_i' := \{0\}$ ($i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$) and $F' := \prod_{i=0}^{\lceil\alpha\rceil-1} F_i'$. Then the problem $(\mathrm{DFP})_R$ *is densely $\overline{\{(CBf_0', \dots, CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \dots, f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.*

(ii) *Suppose that $0 < \alpha < 2$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B, and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Suppose, further, that the following conditions hold:*

- *Let F_i be a separable complex Fréchet space, and let $F_i \subseteq D(A) \cap D(B)$ ($i \in \mathcal{V}$).*
- *Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(Bf_i) + p_n(Cf_i) \leqslant c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$.*
- *Let $H_i : \Omega^\alpha \to F_i$ be an analytic mapping such that $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \smallsetminus \{0\}$, $\lambda \in \Omega$ ($i \in \mathcal{V}$).*

Set $F_i' := \overline{span\{H_i(\lambda^\alpha) : \lambda \in \Omega\}}^{F_i}$ ($i \in \mathcal{V}$), $F_i' := \{0\}$ ($i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$) and $F' := \prod_{i=0}^{\lceil\alpha\rceil-1} F_i'$. Then the problem $(\mathrm{DFP})_L$ *is densely $\overline{\{(Cf_0', \dots, Cf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \dots, f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.*

Proof. Suppose that Ω_0 is an arbitrary open connected subset of Ω which admits a cluster point in Ω. Then the (weak) analyticity of mapping $\lambda \mapsto H_i(\lambda^\alpha) \in F_i$, $\lambda \in \Omega$ implies that $\Psi(\Omega_0, i) := \mathrm{span}\{H_i(\lambda^\alpha) : \lambda \in \Omega_0\}$ is dense in the Fréchet space F_i'; in particular, $(F_i')_0 := \Psi(\Omega \cap \mathbb{C}_-, i)$ is dense in F_i' ($i \in \mathcal{V}$). The rest of proof is almost the same in cases (i) and (ii), so that we will consider only (i). Since $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \smallsetminus \{0\}$, $\lambda \in \Omega$, we can apply the uniqueness theorem for Laplace transform, (43) and (54) in order to see that $R(t)H_i(\lambda^\alpha) = E_\alpha(t^\alpha\lambda^\alpha)CH_i(\lambda^\alpha)$, $t \geqslant 0$, $\lambda \in \Omega$ and that $(g_i * E_\alpha(\cdot^\alpha\lambda^\alpha))(t) = t^i E_{\alpha,i+1}(t^\alpha\lambda^\alpha)$, $t \geqslant 0$, $i \in \mathbb{N}_0$ ($i \in \mathcal{V}$). Now the claimed assertion follows from an application of Theorem 2.11.14 and the asymptotic expansion formulae (40)–(42). $\square$

Remark 2.11.18. Suppose that the requirements of Problem 1 hold, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, X is separable, $l \in \mathbb{N}_0$ and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$.

(i) Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family for (248), and let (P)" hold. Then the closed graph theorem implies that $(\lambda^\alpha B - A)^{-1}C \in L(X)$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega_1$. Suppose that $\mathrm{Re}\,\lambda_0 > \omega_1$, $(\lambda^\alpha B - A)^{-1}CA \subseteq A(\lambda^\alpha B - A)^{-1}C$ and

$(\lambda^\alpha B - A)^{-1}CB \subseteq B(\lambda^\alpha B - A)^{-1}C$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$. Set

$$X_l := \overline{span\{C^l(\lambda_0^\alpha B - A)^{-1}Cf(\lambda^\alpha) : \lambda \in \Omega\}}^{[D(A)\cap D(B)]_C^l}.$$

Then the mapping $G\colon X \to [D(A) \cap D(B)]_C^l$, given by $G(x) := C^l(\lambda_0^\alpha B - A)^{-1}Cx$, $x \in X$, is continuous and the space $\overline{G(X)}^{[D(A)\cap D(B)]_C^l}$ is separable. Set $F_i := [D(A) \cap D(B)]_C^l$ $(i \in \mathcal{V})$. Define G_i and the space F as in the formulation of Theorem 2.11.17(i). Then, for every $i \in \mathcal{V}$, the mapping $H_i\colon \Omega^\alpha \to F_i$, given by $H_i(\lambda^\alpha) := C^l(\lambda_0^\alpha B - A)^{-1}Cf(\lambda^\alpha)$, $\lambda \in \Omega$, is analytic $(i \in \mathcal{V})$. Define now F_i' and F' as in the formulation of Theorem 2.11.17(i). Applying Theorem 2.11.17(i), we get that $(\mathrm{DFP})_R$ is densely $\overline{\{(CBf_0', \ldots, CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \ldots, f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic (observe that $C^l(X_m) = X_{m+l}$ and $C(X_l) \subseteq X_l$ for all $l,\ m \in \mathbb{N}_0$, as well as that $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_l \supseteq \ldots$ and $\{(CBf_0', \ldots, CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \ldots, f_{\lceil\alpha\rceil-1}') \in F'\} = \{(x_0, \ldots, x_{\lceil\alpha\rceil-1}) \in X^{\lceil\alpha\rceil} : x_i = 0$ for $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V}$, and $x_i \in B(X_{l+1})$ for $i \in \mathcal{V}\}$; in the sequel, we will use the abbreviation $X^{\lceil\alpha\rceil}_{B,l,\mathcal{V}}$ to denote the above set).

(ii) Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B, obeying additionally that $(\lambda^\alpha B - A)^{-1}C$ commutes with A and B for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. Then Theorem 2.11.17(ii) and the above consideration imply that the problem $(\mathrm{DFP})_L$ is densely $\overline{\{(x_0, \ldots, x_{\lceil\alpha\rceil-1}) \in X^{\lceil\alpha\rceil} : x_i = 0,\ i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \smallsetminus \mathcal{V};\ x_i \in X_{l+1},\ i \in \mathcal{V}\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic. We will denote the above set simply by $X^{\lceil\alpha\rceil}_{l,\mathcal{V}}$.

Now we will present an illustrative example of application of obtained results.

Example 2.11.19. Suppose that $0 < \alpha < 2$, $\cos(\pi/\alpha) \leqslant 0$, $l \in \mathbb{N}_0$, $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$, $1 \leqslant p < \infty$, $\omega \geqslant 0$, $P_1(z)$ and $P_2(z)$ are non-zero complex polynomials, $N_1 = dg(P_1(z))$, $N_2 = dg(P_2(z))$, $P_2(x) \neq 0$ for all $x \in \mathbb{R}$, $\beta > \frac{1}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ and (70) holds. Then there exist numbers $z_0 \in \mathbb{C}$ and $r \geqslant 0$ such that:

$$P_1(-iz_0) = re^{\pm i\alpha\pi/2}P_2(-iz_0), \quad P_2(z_0) \neq 0$$

and

$$P_1(-i\cdot)'(z_0)P_2(-iz_0) - P_1(z_0)P_2(-i\cdot)'(z_0) \neq 0.$$

Let $a > 0$ be such that $|\operatorname{Re}(z_0)| < a/p$. Set $\rho(x) := e^{-a|x|}$, $x \in \mathbb{R}$,

$$L^p_\rho(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C} \mid f(\cdot) \text{ is measurable, } \int_{\mathbb{R}} |f(x)|^p\rho(x)dx < \infty \right\}$$

and $\|f\| := (\int_{\mathbb{R}} |f(x)|^p\rho(x)dx)^{1/p}$; equipped with this norm, $X := L^p_\rho(\mathbb{R})$ becomes an infinite-dimensional separable complex Banach space. It is well known that the operator $-iA_0$, defined by

$$D(-iA_0) := \{f \in X \mid f(\cdot) \text{ is loc. abs. continuous, } f' \in X\}, \ (-iA_0)f := f',$$

is the generator of a C_0-group on X (cf. [**143**, Theorem 4.9]). Therefore, we can define the closed linear operators $A := \overline{P_1(A_0)}$ and $B := \overline{P_2(A_0)}$ on X by using the functional calculus for bounded commuting C_0-groups; recall that these operators are densely defined, and B is injective. Arguing as in Example 2.10.5, we obtain that there exist an open connected subset Ω of $\mathbb{C} \smallsetminus (-\infty, 0]$ intersecting the imaginary axis and an open connected neighborhood W of point z_0, contained in the vertical strip $\{z \in \mathbb{C} : |\operatorname{Re}(z)| < a/p\}$, such that the mapping $(P_1(-i\cdot)/P_2(-i\cdot))^{-1} \colon \Omega^\alpha \to W$ is well defined, analytic and bijective. Set

$$f(\lambda^\alpha) := e^{(P_1(-i\cdot)/P_2(-i\cdot))^{-1}(\lambda^\alpha)\cdot}, \quad \lambda \in \Omega$$

and, for every $t \geqslant 0$,

$$R_\alpha(t) := \Big(E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(1+|x|^2)^{-\beta/2}\Big)(A_0), \quad G_\alpha(t) := \overline{P_2(A_0)}^{-1} R_\alpha(t).$$

Then $(R_\alpha(t))_{t\geqslant 0} \subseteq L(X)$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family for (248), (P)" holds, $(G_\alpha(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family generated by A, B, the mapping $f \colon \Omega^\alpha \to X$ is analytic and $Af(\lambda^\alpha) = \lambda^\alpha B f(\lambda^\alpha)$, $\lambda \in \Omega$. Furthermore, there exists $\omega_1 > \omega$ such that $(\lambda^\alpha B - A)^{-1}C$ commutes with A and B for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$; here $C = R_\alpha(0)$. Now it readily follows that the problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is densely $X^{\lceil\alpha\rceil}_{B,l,\mathcal{V}}$-distributionally chaotic(densely $X^{\lceil\alpha\rceil}_{l,\mathcal{V}}$-distributionally chaotic); unfortunately, in the present situation, we do not know to say anything about the optimality of this result. Before passing on to some concrete examples where the established conclusions can be applied, it should be noticed that we can prove a similar result provided that the state space is chosen to be the Banach space $C_{0,\rho}(\mathbb{R})$ (cf. [**143**, Definition 4.3]) or the Fréchet space $X' := \{f \in C^\infty(\mathbb{R}) : f^{(n)} \in L^p_\rho(\mathbb{R}) \text{ for all } n \in \mathbb{N}_0\}$, equipped with the following family of seminorms $p_n(f) := \sum_{j=0}^n \|f^{(j)}\|_{L^p_\rho(\mathbb{R})}$, $n \in \mathbb{N}_0$; in this case, the operators $A_{|X'}$ and $B_{|X'}$ are linear and continuous on X', $(C_{|X'})^{-1} \in L(X')$, as well as $((C_{|X'})^{-1}R_\alpha(t)_{|X'})_{t\geqslant 0} \subseteq L(X')$ is a global exponentially equicontinuous $(g_\alpha, I_{X'})$-regularized resolvent family for (248), (P)" holds in our concrete situation, and $((C_{|X'})^{-1}G_\alpha(t)_{|X'})_{t\geqslant 0} \subseteq L(X', [D(B_{|X'})])$ is a global exponentially equicontinuous $(g_\alpha, I_{X'})$-regularized resolvent family generated by $A_{|X'}$, $B_{|X'}$):

(i) Assuming that $P_1(z) = -\alpha_0 z^2 - \beta_0 z^4$ and $P_2(z) = \gamma_0 + z^2$, where α_0, β_0, γ_0 are positive real numbers, we are in a position to clarify some results on subspace distributionally chaoticity of fractional linearized Benney–Luke equation

$$(\gamma_0 - \Delta)\mathbf{D}_t^\alpha u = \alpha_0 \Delta u - \beta_0 \Delta^2 u.$$

(ii) Assuming that $P_1(z) = z^2$ and $P_2(z) = -\eta z^2 - 1$, where $\eta > 0$, we are in a position to clarify some results on subspace distributionally chaoticity of the fractional Barenblatt–Zheltov–Kochina equation

$$(\eta\Delta - 1)\mathbf{D}_t^{\alpha}u(t) + \Delta u = 0 \quad (\eta > 0).$$

In the remainder of subsection, we will always assume that

$$T_n u(t) = B\mathbf{D}_t^{\alpha_n}u(t) = T_{n,L}u(t).$$

Then it is evident that the abstract degenerate Cauchy problem

$$B\mathbf{D}_t^{\alpha_n}u(t) + \sum_{i=1}^{n-1} T_i u(t) = 0; \quad u^{(k)}(0) = u_k,\ 0 \leqslant k \leqslant m_n - 1 \tag{250}$$

is a special subcase of problem (ACP). The Caputo fractional derivative $\mathbf{D}_t^{\alpha_n}u(t)$ is defined for any strong solution $t \mapsto u(t)$, $t \geqslant 0$ of problem (250) and this, in turn, implies that we can define the Caputo fractional derivative $\mathbf{D}_t^{\zeta}u(t)$ for any number $\zeta \in [0, \alpha_n]$. Motivated by our results from Subsection 2.10.4, we introduce the following notion:

Definition 2.11.20. Let $\tilde{X}$ be a closed linear subspace of X^{m_n}, let $k \in \mathbb{N}$, and let $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_k) \in [0, \alpha_n]^k$. Then it is said that the abstract Cauchy problem (250) is $(\tilde{X}, \vec{\beta})$-distributionally chaotic iff there are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $\vec{x}, \vec{y} \in S$ of distinct tuples we have that there exist strong solutions $t \mapsto u(t; \vec{x})$, $t \geqslant 0$ and $t \mapsto u(t; \vec{y})$, $t \geqslant 0$ of problem (250) with the property that

$$\overline{Dens}\Bigg(\Bigg\{t \geqslant 0 : \sum_{i=1}^{k} d(\mathbf{D}_t^{\beta_i}u(t;\vec{x}), \mathbf{D}_t^{\beta_i}u(t;\vec{y})) \geqslant \sigma\Bigg\}\Bigg) = 1 \text{ and}$$

$$\overline{Dens}\Bigg(\Bigg\{t \geqslant 0 : \sum_{i=1}^{k} d(\mathbf{D}_t^{\beta_i}u(t;\vec{x}), \mathbf{D}_t^{\beta_i}u(t;\vec{y})) < \varepsilon\Bigg\}\Bigg) = 1.$$

As before, if we can choose S to be dense in $\tilde{X}$, then we say that the problem (250) is densely $(\tilde{X}, \vec{\beta})$-distributionally chaotic (S is called a $(\sigma_{\tilde{X}}, \vec{\beta})$-scrambled set). In the case $\tilde{X} = X^{m_n}$, it is also said that the problem (250) is (densely) $\vec{\beta}$-distributionally chaotic; S is then called a $(\sigma, \vec{\beta})$-scrambled set.

The classical definitions of $\tilde{X}$-distributional chaos of problem 250 can be obtained by plugging $\vec{\beta} = (0, 0, \ldots, 0) \in [0, \alpha_n]^k$ in Definition 2.11.20; we can also define some other notions of $\tilde{X}$-distributional chaos of problem 250 by replacing, optionally, some of terms $\mathbf{D}_t^{\beta_i}u(t;\vec{x})$ and $\mathbf{D}_t^{\beta_i}u(t;\vec{y})$ in Definition 2.11.20 with $\mathbf{D}_t^{\beta_i}A_i'u(t;\vec{x})$ or $A_i''\mathbf{D}_t^{\beta_i}u(t;\vec{x})$, and $\mathbf{D}_t^{\beta_i}B_i'u(t;\vec{y})$ or $B_i''\mathbf{D}_t^{\beta_i}u(t;\vec{y})$, respectively, where A_i', A_i'', B_i', B_i'' are closed linear operators on X $(1 \leqslant i \leqslant k)$.

Now we would like to present an illustrative example from **[113]**.

Example 2.11.21. The study of various hypercyclic, chaotic and topologically mixing properties of the viscous van Wijngaarden–Eringen equation:

$$(1 - a_0^2 u_{xx})u_{tt} = (\mathrm{Re}_b)^{-1}u_{xxt} + u_{xx}, \tag{251}$$

which corresponds to the linearized version of the equation that models the acoustic planar propagation in bubbly liquids, has recently been carried out by J. A. Conejero, C. Lizama and M. Murillo-Arcila in **[113]**; here, $a_0 > 0$ denotes the dimensionless bubble radius and $\mathrm{Re}_b > 0$ is a Reynolds number. The state space in their analysis is X_ρ, which consists of real analytic functions of Herzog type

$$X_\rho := \left\{ f : \mathbb{R} \to \mathbb{C} \ ; \ f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, \ x \in \mathbb{R} \text{ for some } (a_n)_{n\geqslant 0} \in c_0(\mathbb{N}_0) \right\},$$

which is an isomorphic copy of the sequence space $c_0(\mathbb{N}_0)$. More precisely, it has been proved that the bounded matricial operator

$$A := \begin{bmatrix} O & I \\ -(1 - a_0^2 u_{xx})^{-1} u_{xx} & (\mathrm{Re}_b)^{-1}(1 - a_0^2 u_{xx})^{-1} u_{xx} \end{bmatrix}$$

generates a strongly continuous semigroup on X_ρ^2 satisfying the assumptions of Desch–Schappacher–Webb criterion, provided $a_0 < 1$, $\sqrt{5}/6 < a_0 \mathrm{Re}_b < 1/2$ and $\rho > r_0 a_0^{-1}/(2^{-1} a_0^{-2} (\mathrm{Re}_b)^{-1} - 3r_0)$ $(r_0 := 4^{-1} a_0^{-2} (\mathrm{Re}_b)^{-1} (1 - 4a_0^2 \mathrm{Re}_b^2)^{1/2})$. This immediately implies that the abstract degenerate second order Cauchy problem (251) is densely $(0,1)$-distributionally chaotic (cf. Definition 2.11.20 and Theorem 2.10.24).

The following problem is in a close connection with the last mentioned theorem:

PROBLEM 2. Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying (240). Does there exist a tuple $\vec{\beta} \in [0, \alpha_n]^k$ and a closed linear subspace X' of X^n such that the problem $(\mathrm{ACP})_{B,n}$ is (densely) $(X', \vec{\beta})$-distributionally chaotic?

Now we are going to enquire into the basic distributionally chaotic properties of the following special subcase of problem (250):

$$(252) \qquad B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = 0; \quad u^{(k)}(0) = u_k, \ 0 \leqslant k \leqslant m_n - 1.$$

In our analysis, we are primarily concerned with exploiting Theorem 2.11.8 and, because of that, we need to assume that strong solutions of (252) are governed by some known degenerate resolvent families for (252); besides of that, it is very important to know whether strong solutions of (252) are unique or not. Here we will focus our attention on the use of (C_1, C_2)-existence and uniqueness families for (252), with both operators $C_1 \in L(X)$ and $C_2 \in L(X)$ being injective. Let us recall that for $0 \leqslant i \leqslant m_n - 1$ we define the sets $D_i := \{ j \in \mathbb{N}_{n-1} : m_j - 1 \geqslant i \}$, $D_i' := \mathbb{N}_{n-1} \setminus D_i$ and

$$\mathbf{D}_i := \left\{ u_i \in \bigcap_{j \in D_i'} D(A_j) : A_j u_i \in R(C_1), \ j \in D_i' \right\}.$$

The existence of a C_2-uniqueness family for (252) implies the uniqueness of strong solutions of this problem, while the existence of a C_1-existence family for (252) implies the following assertion, already known from our previous examinations (cf. Subsection 2.3.5 for more details):

LEMMA 2.11.22. *Suppose that $(E(t))_{t\geqslant 0}$ is a C_1-existence family for* (252), *and $u_i \in \mathbf{D}_i$ for $0 \leqslant i \leqslant m_n - 1$. Define, for every $t \geqslant 0$,*

$$(253)\quad u(t) := \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in \mathbb{N}_{n-1}\smallsetminus D_i} (g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)})(t)C_1^{-1}A_j u_i.$$

Then the function $t \mapsto u(t)$, $t \geqslant 0$ is a strong solution of (252).

The following theorem can be proved by using Theorem 2.11.8 and Lemma 2.11.22.

THEOREM 2.11.23. *Suppose that $C_1 \in L(X)$ is injective and $(E(t))_{t\geqslant 0}$ is a C_1-existence family for* (252). *Let F be a separable complex Fréchet space, let $F \subseteq \mathbf{D}_0 \times \mathbf{D}_1 \times \cdots \times \mathbf{D}_{m_n-1}$, and let F be continuously embedded in X^{m_n}. Define $V(t) : F \to X$ by $V(t)\vec{u} := \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\smallsetminus D_i}(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)})(t)C_1^{-1}A_j u_i$ ($t \geqslant 0$, $\vec{u} = (u_0, u_1, \ldots, u_{m_n-1}) \in F$; cf.* (253)*). Suppose that $V(t) \in L(F, X)$ for all $t \geqslant 0$, as well as that F_0 is a dense subset of F satisfying that $\lim_{t\to+\infty} V(t)\vec{x} = 0$, $\vec{x} \in F_0$. Let there exist $\vec{x} \in F$, $m \in \mathbb{N}$ and a set $B \subseteq [0,\infty)$ such that $\overline{Dens}(B) = 1$, and $\lim_{t\to+\infty,t\in B} p_m(V(t)\vec{x}) = +\infty$, resp. $\lim_{t\to+\infty,t\in B} \|V(t)\vec{x}\|_X = +\infty$ if $(X, \|\cdot\|_X)$ is a Banach space. Then the problem* (252) *is densely $\bar{F}^{X^{m_n}}$-distributionally chaotic.*

Now we will apply Theorem 2.11.23 in the study of distributionally chaotic properties of fractional analogues of the viscous van Wijngaarden-Eringen equation.

EXAMPLE 2.11.24. Suppose $1/2 < \alpha \leqslant 1$ and $p > 2$. Consider the following fractional degenerate multi-term problem:

$$(254)\quad (1 + a_0^2\Delta^{\natural}_{X,p})\mathbf{D}_t^{2\alpha}u(t,x) + (\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}\mathbf{D}_t^{\alpha}u(t,x) + \Delta^{\natural}_{X,p}u(t,x) = 0,\ t \geqslant 0;$$
$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

on a symmetric space X of non-compact type and rank one. Let P_p be the parabolic domain defined in [**259**]; then we know that $int(P_p) \subseteq \sigma_p(\Delta^{\natural}_{X,p})$. In our concrete situation, we have that $n = 3$, $\alpha_3 = 2\alpha$, $\alpha_2 = \alpha$, $\alpha_1 = 0$, $B = (1 + a_0^2\Delta^{\natural}_{X,p})$, $A_2 = (\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}$, $A_1 = \Delta^{\natural}_{X,p}$ and $\mathbf{P}_\lambda = 1 + (a_0^2 + \lambda^{-\alpha} + \lambda^{-2\alpha})\Delta^{\natural}_{X,p}$ for $\mathrm{Re}\,\lambda > 0$. Then it is clear that $z(\lambda) := (a_0^2 + \lambda^{-\alpha} + \lambda^{-2\alpha})^{-1} \to a_0^{-2}$ for $|\lambda| \to \infty$, as well as that

$$\lambda^{-1}\mathbf{P}_\lambda^{-1} = \lambda^{-1}z(\lambda)(z(\lambda) + \Delta^{\natural}_{X,p})^{-1}, \quad \mathrm{Re}\,\lambda > 0 \text{ suff. large.}$$

Taking into account Theorem 1.4.10 and the fact that the operator $-\Delta^{\natural}_{X,p}$ generates an analytic strongly continuous semigroup on X, we may conclude from the above that $\lambda^{-1}\mathbf{P}_\lambda^{-1} \in LT-L(X)$. Since $\lambda^{-1}z(\lambda)I \in LT-L(X)$ (cf. the proof of Theorem 2.10.9), we can apply the resolvent equation and Theorem 1.4.10 in order to see that

$$\lambda^{-2\alpha-1}\Delta^{\natural}_{X,p}z(\lambda)(z(\lambda) + \Delta^{\natural}_{X,p})^{-1} \in LT - L(X),$$

$$\lambda^{-\alpha-1}(\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}z(\lambda)(z(\lambda)+\Delta^{\natural}_{X,p})^{-1} \in LT-L(X)$$

and

$$\lambda^{-1}(1+a_0^2\Delta^{\natural}_{X,p})z(\lambda)(z(\lambda)+\Delta^{\natural}_{X,p})^{-1} \in LT-L(X).$$

By Theorem 2.3.33(i)/(b), we have that there exists an exponentially bounded I-existence family $(E(t))_{t\geqslant 0}$ for (254). It is not difficult to see with the help of Theorem 2.3.33(ii) that $(E(t))_{t\geqslant 0}$ is likewise an exponentially bounded I-uniqueness family for (254), so that the strong solutions of (254) are unique. Furthermore, we have that $\mathbf{D}_i = D(\Delta^{\natural}_{X,p})$ for $i=0,1$. Let $f\colon int(P_p) \to X \setminus \{0\}$ be an analytic mapping satisfying that $\Delta^{\natural}_{X,p}f(\lambda)=\lambda f(\lambda)$, $\lambda \in int(P_p)$. Using the proof of Theorem 2.10.9, we get that the function $t \mapsto u(t;(f(\lambda),f(\lambda')))$, $t\geqslant 0$, given by

$$u(t;(f(\lambda),f(\lambda'))) := H_0(\lambda,t)f(\lambda)+H_1(\lambda',t)f(\lambda'),\quad t\geqslant 0 \quad (\lambda,\lambda' \in int(P_p)),$$

where

$$H_0(\lambda,t) := \mathcal{L}^{-1}\Bigl(\frac{z^{2\alpha-1}-(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1}z^{\alpha-1}}{z^{2\alpha}-(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1}z^{\alpha}-(\lambda-a_0^2)^{-1}}\Bigr)(t),\quad t\geqslant 0,$$

and

$$H_1(\lambda',t) := \mathcal{L}^{-1}\Bigl(\frac{z^{2\alpha-1}}{z^{2\alpha}-(\mathrm{Re}_b)^{-1}(\lambda'-a_0^2)^{-1}z^{\alpha}-(\lambda'-a_0^2)^{-1}}\Bigr)(t),\quad t\geqslant 0,$$

is a unique strong solution of (254) with $u(0,\cdot)=f(\lambda)$ and $u_t(0,\cdot)=f(\lambda')$. Direct computations show that

$$H_0(\lambda,t) = \frac{r_1(\lambda)-(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1}}{\sqrt{D_\lambda}}e^{r_1(\lambda)t} - \frac{r_2(\lambda)-(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1}}{\sqrt{D_\lambda}}e^{r_2(\lambda)t}$$

if $\alpha=1$,

$$\begin{aligned} H_0(\lambda,t) = {} & \frac{t^{-\alpha}}{\sqrt{D_\lambda}}[E_{\alpha,1-\alpha}(r_1(\lambda)t^\alpha)-E_{\alpha,1-\alpha}(r_2(\lambda)t^\alpha)] \\ & - \frac{(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1}}{\sqrt{D_\lambda}}[E_\alpha(r_1(\lambda)t^\alpha)-E_\alpha(r_2(\lambda)t^\alpha)],\quad t>0, \end{aligned}$$

if $0<\alpha<1$, and

$$H_1(\lambda',t) = \frac{t^{1-\alpha}}{\sqrt{D_\lambda}}[E_{\alpha,2-\alpha}(r_1(\lambda')t^\alpha)-E_{\alpha,2-\alpha}(r_2(\lambda')t^\alpha)],\quad t>0,$$

where

$$r_{1,2}(\lambda) := \frac{(\mathrm{Re}_b)^{-1}(\lambda-a_0^2)^{-1} \pm \sqrt{(\mathrm{Re}_b)^{-2}(\lambda-a_0^2)^{-2}+4(\lambda-a_0^2)^{-1}}}{2}$$

and

$$D_\lambda := (\mathrm{Re}_b)^{-2}(\lambda-a_0^2)^{-2}+4(\lambda-a_0^2)^{-1}.$$

Set $F := [D(\Delta^{\natural}_{X,p})] \times [D(\Delta^{\natural}_{X,p})]$. Then F is a separable infinite-dimensional complex Banach space. Define $V(t)$ as in the formulation of Theorem 2.11.23, with $C_1 = I$; then it is clear that $V(t) \in L(F,X)$ for all $t\geqslant 0$. From the uniqueness of strong solutions of (254), it readily follows that $V(t)(f(\lambda),f(\lambda')) = u(t;(f(\lambda),f(\lambda')))$, $t\geqslant 0$ $(\lambda,\lambda' \in int(P_p))$. Using the asymptotic expansion formulae (40)–(42) and a simple analysis, we obtain that there exist a sufficiently small number $\varepsilon > 0$ and two sufficiently large negative numbers $x_- < 0$ and $x'_- < 0$ such that the first requirement in Theorem 2.11.23 holds with $F_0 = span\{(1 + \Delta^{\natural}_{X,p})^{-1} f(\lambda) : \lambda \in L(x_-, \varepsilon)\} \times span\{(1 + \Delta^{\natural}_{X,p})^{-1} f(\lambda') : \lambda' \in L(x'_-, \varepsilon)\}$.

It is clear that there exists a great number of concrete situations (consider, for example, the case in which $a_0 \to 0+$ and $\mathrm{Re}_b \to +\infty$) in which there exists a number $\lambda_0 \in int(P_p)$ such that

$$r_1(\lambda_0) \in \Sigma_{\gamma\pi/2}.$$

If this is the case, the vector $(f(\lambda_0), 0)$ is distributionally unbounded and the problem (254) is densely distributionally chaotic; observe also that the problem (254) is densely $(X \times \{0\})$-distributionally chaotic and $(\{0\} \times X)$-distributionally chaotic by Theorem 2.11.23. The same holds for the problem (254)' obtained by interchanging the terms $(\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}\mathbf{D}^{\alpha}_t u(t,x)$ and $(\mathrm{Re}_b)^{-1}\mathbf{D}^{\alpha}_t\Delta^{\natural}_{X,p} u(t,x)$ in (254); this follows directly from Definition 2.11.9 and the fact that the mapping $t \mapsto u(t; (f(\lambda), f(\lambda')))$, $t \geqslant 0$, defined above, is still a strong solution of (254)' for $\lambda, \lambda' \in int(P_p)$. Observe, finally, that all established conclusions for the problems (254) and (254)' continue to hold if we replace the operator $\Delta^{\natural}_{X,p}$ and the state space X in our analysis with the operator $(\Delta^{\natural}_{X,p})_\infty$ and the Fréchet space $[D_\infty(\Delta^{\natural}_{X,p})]$, respectively.

In the previous example, we have employed some ideas contained in the proof of Theorem 2.10.9. Assuming that the requirements of this theorem hold, we can pose the problem of existence of a closed linear subspace X' of X^{m_n}, an integer $k \in \mathbb{N}$ and a tuple $\vec{\beta} \in [0, \alpha_n]^k$ such that the problem (224) is (densely) $(X', \vec{\beta})$-distributionally chaotic; a similar question can be posed for the problem (250). We close this section with the observation that, in the present situation, we can give only some partial answers to the problems addressed above by using Theorem 2.11.8 as an essential tool in the consideration. For more details about various generalizations of distributional chaos and applications to the abstract PDEs, the reader may consult [**294**].

2.12. Appendix and notes

In his fundamental paper [**507**] (1994), G. A. Sviridyuk succeeded in developing the phase space method for the Sobolev equations of first order and some classes of incomplete higher-order Sobolev equations. The notions of a relatively σ-bounded operator, a relatively p-sectorial operator, a chain of associated vectors and a group of solution operators have been introduced in the Banach space setting. Concerning applications, mention should be made of the author's analysis of abstract linearized Oskolkov system [**442, 443**], which models the dynamics of an incompressible viscoelastic fluid of Kelvin-Voigt type. We refer the reader to the monograph [**509**] for a comprehensive survey of results of G. A. Sviridyuk and his students obtained in the period 1994–2003. Here, we will only insribe the most important results established in the papers [**515**] by G. A. Sviridyuk–A. A. Zamyshlyaeva and [**204**] by V. E. Fedorov, and say a few words about optimal control problems of abstract degenerate differential equations.

In [**515**], the authors have investigated the phase space of homogeneous linear higher-order equation

$$Au^{(n)}(t) = B_{n-1}u^{(n-1)}(t) + \cdots + B_0u(t), \quad t \in \mathbb{R}, \tag{255}$$

where $n \in \mathbb{N} \setminus \{1\}$, A and $B_{n-1}, \ldots, B_0$ are closed linear operators acting between the Banach spaces X and Y. Let us recall that a set $\mathcal{P} \subseteq X$ is said to be the phase space of equation (255) iff the following two conditions are satisfied:

(i) If $u(t)$ is a strong solution of (255), then $u(t) \in \mathcal{P}$ for all $t \geqslant 0$;
(ii) For any $x_k \in \mathcal{P}$ $(0 \leqslant k \leqslant n-1)$, there exists a unique strong solution of equation (255) with the initial conditions $u^{(k)}(0) = x_k$ $(0 \leqslant k \leqslant n-1)$.

Set $\vec{B} := (B_{n-1}, \ldots, B_0)$. Then we say that the operator pencil $\vec{B}$ is polynomially A-bounded iff there exists a finite number $a > 0$ such that $(\lambda^n A - \lambda^{n-1}B_{n-1} - \cdots - B_0)^{-1} \in L(Y, X)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| > a$ (cf. [**161**, **225**, **248**, **408**, **426**] and [**463**] for further information concerning operator pencils). If the condition [**515**, (A), p. 271] holds and the operator pencil $\vec{B}$ is polynomially A-bounded, then the point ∞ can be a removable singular point of the A-resolvent of the pencil $\vec{B}$, a pole of order $p \in \mathbb{N}$ of the A-resolvent of the pencil $\vec{B}$ and, finally, an essentially singular point of the A-resolvent of the pencil $\vec{B}$. In the first two cases, the phase space of equation (255) coincides with the range of projection

$$P := \frac{1}{2\pi i} \int_\gamma \lambda^{n-1}(\lambda^n A - \lambda^{n-1}B_{n-1} - \cdots - B_0)^{-1} A \, d\lambda,$$

where $\gamma = \{\lambda \in \mathbb{C} : |\lambda| = R > a\}$; cf. [**515**, Definition 3.2, Theorem 3.1]. If the point ∞ is an essentially singular point of the A-resolvent of the pencil $\vec{B}$, then the strong solutions of equation (255) need not be unique and the phase space $\mathcal{P}$ of equation (255) is very difficult to be profiled. The main result of paper is [**515**, Theorem 4.1], in which the authors proved the existence and uniqueness of strong solutions of equation (255) provided that the point ∞ is not an essentially singular point of the A-resolvent of the pencil $\vec{B}$, and the above-mentioned condition (A) holds. Applications have been made to the abstract Boussinesq-Love equation in finite domains, which has been already considered in Example 2.3.48 and Example 2.3.53 in the case that the condition (A) is violated.

In [**204**], the problem of existence of an exponentially bounded solution semigroup strongly holomorphic in a sector is studied for a Sobolev-type linear equation $Lu'(t) = Mu(t)$, where $L \in L(X, Y)$ is non-injective operator and M is a closed densely defined operator acting between the sequentially complete locally convex spaces X and Y. The author has introduced the notion of a (L, p)-sectoriality of the operator M as follows: The operator M is said to be (L, p)-sectorial iff the following holds:

(i) there exist $\omega \in \mathbb{R}$ and $\theta \in (0, \pi)$ such that $\omega + \Sigma_\theta \subseteq \rho^L(M)$;

(ii) the operator families

$$\left\{ R^L_{(\lambda,p)}(M) \prod_{k=0}^{p} (\lambda_k - a) : \lambda = (\lambda_0, \ldots, \lambda_p) \in (\omega + \Sigma_\theta)^{p+1} \right\} \subseteq L(X)$$

and

$$\left\{ L^L_{(\lambda,p)}(M) \prod_{k=0}^{p} (\lambda_k - a) : \lambda = (\lambda_0, \ldots, \lambda_p) \in (\omega + \Sigma_\theta)^{p+1} \right\} \subseteq L(Y)$$

are equicontinuous.

Here, for any $p \in \mathbb{N}_0$ and $\lambda = (\lambda_0, \ldots, \lambda_p) \in (\rho^L(M))^{p+1}$,

$$R^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} R^L_\lambda(M) \text{ and } L^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} L^L_\lambda(M).$$

After that, the author introduces the notion of a strong (L, p)-sectoriality of the operator M from the right (left) and proves the existence of limit $\lim_{t\to 0+} U(t)$ ($\lim_{t\to 0+} V(t)$) in $L(X)$ ($L(Y)$) for a strongly holomorphic solution semigroup $(U(t))_{t>0} \subseteq L(X)$ ($(V(t))_{t>0} \subseteq L(Y)$) in this case; cf. [**204**, Theorem 2.1]. Under the general assumption on (L, p)-sectoriality of operator M, semireflexivity of spaces X and Y plays an important role since it allows one to decompose the spaces X and Y in a certain way; observe, however, that we do not need this assumption in the case that the operator M is a strongly (L, p)-sectorial from the right (left); cf. [**204**, Theorem 5.1, Proposition 5.1] for more details. An interesting application has been obtained in the analysis of periodic solutions of certain degenerate Cauchy problems of first order [**204**, Theorem 9.2].

Concerning control of abstract degenerate differential equations with integer order derivatives, the reader may consult the recent monograph [**451**] by M. V. Plekhanova and V. E. Fedorov. Applications have been given in the analysis of optimal control of Benny–Luke equation, equation of transition processes in semiconductors

$$\begin{gathered}
(\lambda - \Delta)\omega_t(x, t) = \alpha\omega(x, t) + y(x, t), \quad (x, t) \in \Omega \times (0, T), \\
\omega(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\
\omega(x, 0) = u(x), \quad x \in \Omega,
\end{gathered}$$

the systems of linearized quasi-stationary phase-field equations

$$\begin{gathered}
z(x, 0) = z_0(x), \quad x \in \Omega, \\
\nu \frac{\partial}{\partial n} z(x, t) + (1 - \nu) z(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\
\nu \frac{\partial}{\partial n} \Theta(x, t) + (1 - \nu)\Theta(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\
\frac{\partial}{\partial t} z(x, t) = \Delta z(x, t) - \Delta\Theta(x, t) + f(x, t), \quad (x, t) \in \Omega \times (0, T), \\
\Delta\Theta(x, t) - \beta\Theta(x, t) + z(x, t) + g(x, t), \quad (x, t) \in \Omega \times (0, T),
\end{gathered}$$

the systems of linearized systems of Boussinesq equations

$$z(s,0) = z_0(s), \quad r(s,0) = r_0(s), \quad \Theta(s,0) = \Theta_0(s), \qquad s \in \Omega,$$
$$z(s,t) = 0, \quad \Theta(s,t) = 0, \qquad (s,t) \in \partial\Omega \times (0,T),$$
$$\frac{\partial z(s,t)}{\partial t} = \nu\Delta z(s,t) - r(s,t) - \alpha\Theta(s,t)e_3 + u(s,t), \quad (s,t) \in \Omega \times (0,T),$$
$$\nabla \cdot z = 0, \quad (s,t) \in \Omega \times (0,T),$$
$$\frac{\partial \Theta(s,t)}{\partial t} = \beta\Delta\Theta(s,t) + z_3(s,t) + \omega(s,t), \quad (s,t) \in \Omega \times (0,T),$$

as well as the systems of linearized Oskolkov and Sobolev equations. Concerning control of abstract fractional degenerate equations, we have been able to locate only one research paper [**450**], written by M. V. Plekhanova.

Now we would like to introduce the notion of an (a,k)-regularized C-resolvent family of solving operators and the phase space of certain classes of abstract degenerate Volterra integral equations (cf. also Section 2.8). We assume that the closed graph theorem holds for mappings from Y into Y, and for mappings from Y into X.

Consider the following abstract degenerate Cauchy problem (cf. (52)):

$$Lu(t) = f(t) + \int_0^t a(t-s)Mu(s)\,ds, \quad t \in [0,\tau), \tag{256}$$

where $t \mapsto f(t) \in Y$, $t \in [0,\tau)$ is a continuous mapping and $a \in L^1_{loc}([0,\tau))$, $a \neq 0$. Let $\lambda \in \rho^L(M)$. Then, along with the equation (256), we consider the equation

$$(\lambda L - M)^{-1}Lu(t) = (\lambda L - M)^{-1}f(t) + \int_0^t a(t-s)(\lambda L - M)^{-1}Mu(s)\,ds, \quad t \in [0,\tau), \tag{257}$$

and, after the substitution $v(t) = (\lambda L - M)u(t)$, $t \in [0,\tau)$, the equation

$$L(\lambda L - M)^{-1}v(t) = f(t) + \int_0^t a(t-s)M(\lambda L - M)^{-1}v(s)\,ds, \quad t \in [0,\tau). \tag{258}$$

Both of these equations can be rewritten in the form of abstract degenerate Cauchy problem (52):

- (257): Take $A = (\lambda L - M)^{-1}M$, $B = (\lambda L - M)^{-1}L$ and $g(t) = (\lambda L - M)^{-1}f(t)$. Then $B \in L(X)$, A is a linear operator acting on X but A is not closed, in general. This obviously hinders the analysis of problem (257).
- (258): Take $A = M(\lambda L - M)^{-1}$, $B = L(\lambda L - M)^{-1}$ and $g(t) = f(t)$. Then the situation is much simpler because $A \in L(Y)$ and $B \in L(Y)$.

For ease the presentation, we need to introduce the following technical definition: Let X and Y be SCLCSs, let $0 < \tau \leqslant \infty$, and let $A\colon D(A) \subseteq X \to Y$ and $B\colon D(B) \subseteq X \to Y$ be linear (not necessarily closed) operators. Consider the following abstract degenerate Cauchy problem (52). Then a function $u \in C([0,\tau) : X)$ is said to be a (mild) solution of (52) iff $(a * u)(t) \in D(A)$,

$t \in [0,\tau)$, $A(a * u)(t) = Bu(t) - f(t)$, $t \in [0,\tau)$ and the mapping $t \mapsto Bu(t)$, $t \in [0,\tau)$ is continuous.

Put $Z := X$, in the case of examination of problem (257), and $Z := Y$, in the case of examination of problem (258). In order to explain our next steps, suppose that any of these two problems is reduced to the problem (52), and that the single-valued operator $B^{-1}A$ generates a (local) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)} \subseteq L(Z)$, obeying the property [**292**, (22)] with A and x replaced therein with $B^{-1}A$ and u, respectively. Then we have

$$R(t)u = k(t)Cu + B^{-1}A\int_0^t a(t-s)R(s)u\,ds, \quad t \in [0,\tau),\ u \in Z,\ \text{i.e.,}$$

$$BR(t)u = k(t)BCu + A\int_0^t a(t-s)R(s)u\,ds, \quad t \in [0,\tau),\ u \in Z,$$

so that for each $u \in Z$ the function $t \mapsto R(t)u$, $t \in [0,\tau)$ is a mild solution of (52) with $f(t) = k(t)BCu$, $t \in [0,\tau)$. For the sake of simplicity and better understanding, in the remainder of this analysis we will consider only the global case $\tau = \infty$. Our results from [**292**, Section 2.8] motivate us to introduce the following definition (cf. [**509**, Definitions 2.3.1, 2.8.2, 3.2.1 and 4.4.1] for the semigroup case $(a(t) \equiv 1)$):

Definition 2.12.1. Let Z be an SCLCS, let $B \in L(Z)$, and let $A\colon D(A) \subseteq Z \to Z$ be a linear operator acting on Z. Suppose that $C \in L(Z)$ is injective, $a \in L^1_{loc}([0,\infty))$, $a \neq 0$ and $k \in C([0,\infty))$. Set $\mathcal{G} := \{f \in C([0,\infty) : Z) : (\exists w \in Z) f(t) = k(t)Bw \text{ for all } t \geqslant 0\}$.

(i) By a (global) (a,k)-regularized C-resolvent family of solving operators for the problem (52), with $f \in \mathcal{G}$, we mean any strongly continuous operator family $(R(t))_{t\geqslant 0} \subseteq L(Z)$ satisfying the following two conditions:
  (a) $R(s)(a*R)(t)u-(a*R)(s)R(t)u = k(s)(a*R)(t)Cu-k(t)(a*R)(s)Cu$, $t,s \geqslant 0$, $u \in Z$.
  (b) For each $u \in Z$ the function $t \mapsto R(t)u$, $t \geqslant 0$ is a mild solution of problem (52) with $f(t) = k(t)BCu$, $t \geqslant 0$.

  We say that $(R(t))_{t\geqslant 0}$ is exponentially equicontinuous (locally equicontinuous) iff there exists $\omega \geqslant 0$ such that the operator family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(Z)$ is equicontinuous (iff for each $T > 0$ the operator family $\{R(t) : t \in [0,T]\} \subseteq L(Z)$ is equicontinuous).

(ii) It is said that a global (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of solving operators for the problem (52), with $f \in \mathcal{G}$, is analytic of angle $\theta \in (0,\pi]$, iff there exists a function $\mathbf{R}\colon \Sigma_\theta \to L(Z)$ which satisfies that, for every $u \in Z$, the mapping $z \mapsto \mathbf{R}(z)u$, $z \in \Sigma_\theta$ is analytic as well as that $\mathbf{R}(t) = R(t)$, $t > 0$. It is said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of solving operators for the problem (52), with $f \in \mathcal{G}$, of angle θ, resp. equicontinuous analytic (a,k)-regularized C-resolvent family of solving operators for the problem (52), with $f \in \mathcal{G}$, of angle θ, iff for every $\gamma \in (0,\theta)$, there exists $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma \operatorname{Re} z}\mathbf{R}(z) : z \in \Sigma_\gamma\} \subseteq L(Z)$ is equicontinuous.

(iii) It is said that a global (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of solving operators for the problem (52), with $f \in \mathcal{G}$, is entire iff there exists a function $\mathbf{R}\colon \mathbb{C} \to L(Z)$ which satisfies that, for every $u \in Z$, the mapping $z \mapsto \mathbf{R}(z)u$, $z \in \mathbb{C}$ is entire (since there is no risk for confusion, we will identify $R(\cdot)$ and $\mathbf{R}(\cdot)$ in the sequel).

Observe that the notion of an analytic (a,k)-regularized C-resolvent family of solving operators for the problem (52), with $f \in \mathcal{G}$, is introduced in [**204**, Definition 2.1] in a slightly different manner, provided that $a(t) \equiv 1$, $k(t) \equiv 1$ and $C = I$. It is also worth noting that we do not require the validity of limit equality $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)u = R(0)u$ ($\gamma \in (0,\theta)$, $u \in Z$) in (b).

The notion of phase space of problem (52) is introduced in the following definition.

Definition 2.12.2. Let $B \in L(Z)$, let A be a linear operator on Z, and let $\tau = \infty$. Then a set $\mathcal{P} \subseteq Z$ is called the phase space of equation (52), with $f \in \mathcal{G}$, iff the following holds:

(i) any mild solution $t \mapsto u(t)$, $t \geqslant 0$ of (52), with $f(t) = k(t)Bw$ ($t \geqslant 0$) belongs to $\mathcal{P}$, that is, $u(t) \in \mathcal{P}$, $t \geqslant 0$;

(ii) for any $w \in \mathcal{P}$ there exists a unique mild solution $t \mapsto u(t)$, $t \geqslant 0$ of (52), with $f(t) = k(t)Bw$, $t \geqslant 0$.

It is very interesting and unpleasant question to profile the phase space of Eq. (52). In connection with this problem, we would like to mention that the method proposed in [**515**] does not work for abstract degenerate multi-term fractional problems with Caputo derivatives. We have found the problem of characterizing the phase space of such equations (defined similarly as for higher-order Cauchy problems with integer order derivatives) very serious and difficult to be answered, as well.

We continue by stating the following generalization of [**204**, Theorem 2.1].

Theorem 2.12.3. *Suppose that the operator M is (L,p)-sectorial and $0 < \alpha < 2\theta\pi^{-1}$. Let the space Z and the operators A, B be defined as in* (257)*, resp.* (258)*, and let the class $\mathcal{G}$ be defined as in* Definition 2.12.1*, with $k(t) \equiv 1$. Then there exists an exponentially equicontinuous, analytic (g_α, I)-regularized resolvent family $(R(t))_{t\geqslant 0}$ for* (52)*, with $a(t) = g_\alpha(t)$ and $f \in \mathcal{G}$, of angle* $\min(\theta\alpha^{-1} - \pi 2^{-1}, \pi 2^{-1})$.

Proof. Without loss of generality, we may assume that the operator M is (L,p)-sectorial with the constant $\omega \geqslant 0$. Let $0 < \varepsilon < \gamma' < \gamma$ and $\delta > 0$. Then there exists a sufficiently large number $\omega' > \omega$ such that the operator families $\{(\lambda - \omega')\lambda^{\alpha-1} R^L_{\lambda^\alpha}(M) : \lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\gamma'}\} \subseteq L(Z)$ and $\{(\lambda - \omega')\lambda^{\alpha-1} L^L_{\lambda^\alpha}(M) : \lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\gamma'}\} \subseteq L(Z)$ are equicontinuous. Let the oriented contour Γ be as in the second part of proof of [**27**, Theorem 2.6.1], with the numbers ω and γ replaced therein with the numbers ω' and γ', respectively. Denote by Ω the open region on the right of Γ. Then the mappings $\lambda \mapsto \lambda^{\alpha-1} R^L_{\lambda^\alpha}(M)$, $\lambda \in \Omega$ and

$\lambda \mapsto \lambda^{\alpha-1}L^L_{\lambda^\alpha}(M)$, $\lambda \in \Omega$ are strongly analytic (and strongly continuous on $\bar{\Omega}$). Define, for every $u \in Z$ and $z \in \Sigma_{\gamma'-\varepsilon}$,

$$R(z)u := \frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda^{\alpha-1}R^L_{\lambda^\alpha}(M)u\,d\lambda,$$

in the case of consideration of problem (257), and

$$R(z)u := \frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda^{\alpha-1}L^L_{\lambda^\alpha}(M)u\,d\lambda,$$

in the case of consideration of problem (258). Then the proof of [**27**, Theorem 2.6.1] shows that for each $p \in \circledast_Z$ there exist $q \in \circledast_Z$ and $c > 0$ such that $p(R(z)u) \leqslant cq(u)$, $u \in Z$, $z \in \Sigma_{\gamma'-\varepsilon}$, as well as that $\mathcal{L}(R(t)u)(\lambda) = \lambda^{\alpha-1}R^L_{\lambda^\alpha}(M)u$, $u \in Z$, in the case of consideration of problem (257), and $\mathcal{L}(R(t)x)(\lambda) = \lambda^{\alpha-1}L^L_{\lambda^\alpha}(M)u$, $u \in Z$, in the case of consideration of problem (258). The proof of (b) follows similarly as in the proof of [**204**, Theorem 2.1], while the equality (a) from Definition 2.12.1 follows by taking the Laplace transform twice, with respect to variables t and s separately, and using the resolvent equations. This completes the proof of theorem. $\square$

If the conclusions of Theorem 2.12.3 hold, then for each $\varepsilon \in (0, \gamma)$ there exists $\omega_\varepsilon \geqslant \omega$ such that the set $\{z^k e^{-\omega z}R^{(k)}(z) : z \in \Sigma_{\gamma-\varepsilon,\ k\in\mathbb{N}_0}\}$ is bounded; cf. [**27**, p. 86] and [**296**, Theorem 3.4(i)]. Unfortunately, Theorem 2.12.3 is only a partial result and it is an open problem to reconsider the remaining results from [**204**] in fractional case.

Stochastic degenerate Cauchy problems. In this monograph, we analyze only degenerate deterministic Volterra integro-differential equations. Stochastic Cauchy problems and stochastic differential equations occupy a great deal of mathematical territory nowdays; for further information concerning degenerate case, we recommend for the reader [**98**, **194**, **511**, **552**] and [**554**].

Semilinear degenerate Cauchy problems. In Subsection 2.2.5, we have considered semilinear degenerate fractional Cauchy problems with abstract differential operators. For more details about abstract semilinear degenerate Cauchy problems with integer order derivatives, we refer the reader to [**59**, **83**, **89**, **94**, **192**, **208**, **209**, **267**, **268**, **319**, **468**, **469**, **505**] and [**508**]. Semilinear degenerate Cauchy inclusions will be investigated in Subsection 3.5.1-Subsection 3.5.2.

Nonlinear degenerate Cauchy problems. In this monograph, we are focused entirely on the linear theory of abstract degenerate Volterra integro-differential equations. Concerning quasilinear and purely nonlinear degenerate equations of Sobolev type, mention should be made of monographs A. Favini, G. Marinoschi [**190**, **491**]by R. Showalter, [**503**] by A. G. Sveshnikov, A. B. Al'shin, M. O. Korpusov, Yu. D. Pletner, [**19**] by A. B. Al'shin, M. O. Korpusov, A. G. Sveshnikov, [**216**] by V. E. Fedorov, A. V. Nagumanova, as well as the papers [**88**, **193**, **201**, **368**, **369**, **400**, **443**, **486**, **492**, **493**] and [**525**]. Nonlinear Sobolev equations arise naturally in many problems in various fields of mathematical physics (cf. [**19**] for further information); here are some examples:

(a) The Camassa–Holm equation (1993)

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

is important in modeling of undirectional propagation of shalow-water waves over a flat bottom.

(b) The Benjamin–Bona–Mahony equation (or BBM equation)

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

is also known as the regularized long-wave equation (RLWE). This equation was investigated by T. B. Benjamin, J. L. Bona and J. J. Mahony (1972) as an improvement of the Korteweg–de Vries equation (KdV equation) for modeling long surface gravity waves of small amplitude. Strictly speaking, BBM equation was introduced by D. H. Peregrine, in 1966, for the purpose of study of undular bores.

(c) The Rosenau–Burgers equation

$$u_t + u_{xxxxt} - \alpha u_{xx} + (u^{p+1}/p+1)_x = 0 \quad (p > 0)$$

is important in modeling of bore propagation. In the case that $\alpha = 0$, the equation was introduced by P. Rosenau (1988) for treating the dynamics of dense discrete systems in order to overcome the shortenings by the KdV equation.

(d) The semiconductor equation (2005)

$$\frac{\partial}{\partial t}(\Delta u - u) + \Delta u + \alpha u^3 = 0, \quad \alpha \in \mathbb{R}$$

models nonstationary processes in crystalline semiconductors.

(e) The Lonngern wave equation (1975)

$$\frac{\partial^2}{\partial t^2}(u_{xx} - \alpha u + \beta u^2) + u_{xx} = 0 \quad (\alpha > 0,\ \beta > 0)$$

describes electric signals in telegraph lines.

We also want to outline that the pioneering works of L. A. Lyusternik, S. G. Krein, N. A. Sidorov and B. V. Loginov (cf. [**496**, **497**] for further information), in which the authors had analyzed bifurcating solutions of nonlinear parameter-dependent singular and degenerate differential equations, influenced many authors to investigate mathematical modelings based on the Showalter-Sidorov problem (cf. [**206**, **214**, **272**, **511**, **514**, **551**] and references cited therein for further information).

Abstract degenerate integro-differential equations with delay. The analysis of abstract degenerate integro-differential equations with delay has received much attention recently. For the basic references in this direction, we refer the reader to L. V. Borel, V. E. Fedorov [**86**], C. Bu [**91**], A. Favini, L. Vlasenko [**198**], V. E. Fedorov, E. A. Omel'chenko [**217**, **218**], C. Lizama, R. Ponce [**397**], and to the doctoral dissertation of R. A. A. Cuello [**123**]. Concerning non-degenerate case, the monographs by S. Abbas, M. Benchohra [**4**], whose twelfth chapter also contains some information on functional evolution inclusions, N. V. Azbelev, V. P.

Maksimov, L. F. Rakhmatullina [**39**], J. K. Hale, S. M. Verduyn Lunel [**246**], Y. Kuang [**362**] and J. Wu [**540**] are of crucial importance.

In his doctoral dissertation [**123**], R. A. A. Cuello has investigated the abstract, degenerate, functional second order differential equations in the vector-valued Lebesgue, Besov and Triebel–Lizorkin function spaces. Special focus is placed on following degenerate functional differential equations

$$
\begin{aligned}
(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds
&= \gamma_\infty u(t) + Au(t) + \frac{d}{dt}\int_{-\infty}^{t} a(t-s)Au(s)ds \\
&\quad + b_\infty Bu(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \quad t \in [0, 2\pi],
\end{aligned}
$$

equipped with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, and

$$
\begin{aligned}
(Mu)''(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds
&= \gamma_\infty u(t) + Au(t) + \frac{d}{dt}\int_{-\infty}^{t} a(t-s)Au(s)ds \\
&\quad + b_\infty Bu(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \quad t \in [0, 2\pi],
\end{aligned}
$$

equipped with periodic boundary conditions $\Lambda u(0) = \Lambda u(2\pi)$, $(Mu)(0) = (Mu)(2\pi)$ and $(Mu')(0) = (Mu')(2\pi)$. Here, A, B, Λ and M are closed linear operators on a complex Banach space E, satisfying that $D(A) \cap D(B) \subseteq D(\Lambda) \cap D(M)$, $a, b, c \in L^1([0,\infty))$, $f(\cdot)$ is an E-valued function defined on $[0, 2\pi]$, and γ_∞, b_∞ are certain complex constants. The author has applied abstract results in the analysis of the following abstract degenerate functional Cauchy problem

$$
\begin{aligned}
\frac{\partial}{\partial t}(m(x)u_t(x,t)) - \Delta u_t(t,x)
&= \Delta u(t,x) + \int_{-\infty}^{t} a(t-s)\Delta u(s,x)ds + f(t,x), \quad (t,x) \in [0,2\pi] \times \Omega,
\end{aligned}
$$

$$
u(t,x) = \Big(\frac{\partial}{\partial t}u(t,x)\Big)_{t=0} = 0, \quad (t,x) \in [0,2\pi] \times \partial\Omega,
$$

$$
u(0,x) = u(2\pi,x),\ m(x)\Big(\frac{\partial}{\partial t}u(t,x)\Big)_{t=0} = m(x)\Big(\frac{\partial}{\partial t}u(t,x)\Big)_{t=2\pi},\ x \in \Omega,\ t \in [0,2\pi],
$$

where Ω is an open domain in $\mathbb{R}^n$ and Δ is the Dirichlet Laplacian (see also the paper [**24**] by R. A. A. Cuello and V. Keyantuo). The fractional analogues of above equations have not been analyzed by now.

Abstract degenerate Cauchy problems of first order: semigroup theory approach. In [**517**], B. Thaller and S. Thaller have continued the investigations of R. Showalter [**494, 495**] concerning the well-posedness of abstract degenerate Cauchy problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ of first order. The results of [**517**], formulated in the setting of Hilbert spaces, have been reconsidered and substantially improved by the authors in the paper [**518**], whose main ideas are described as follows.

Suppose that A and B are two linear operators acting between the Banach spaces X and Y. In [**518**, Section 2], the authors have assumed that there is a linear subspace $D_X \subseteq D(A) \cap D(B)$ such that for all $\lambda > \omega$, where $\omega \in \mathbb{R}$, the restriction of operator $A - \lambda B$ to D_X is invertible and with values in $R(B)$. Assuming certain Hille–Yosida type conditions on the operators $(\lambda B - A)^{-1} B$ and $B(\lambda B - A)^{-1}$ $(\lambda > \omega)$, the authors have applied the results from the theory of non-degenerate strongly continuous semigroups for proving the existence and uniqueness of strict solutions of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ (cf. [**518**, Definition 1.1, Theorem 3.1–Theorem 3.2]). Using the notion of joint closedness of operators A and B [**475**], the authors have considered the so-called invariant subspaces for the time evolution in Section 4. In Section 5, it has been assumed that the operator B is closed. Then $N(B)$ is a closed subspace in X and the authors have enquired into the possibility to consider the inital degenerate Cauchy problem by passing to the corresponding non-degenerate Cauchy problem in the Banach space $X/N(B)$. Section 6 considers the case in which X and Y are Hilbert spaces.

Singular abstract Cauchy problems. In their fundamental monograph [**95**], R. Caroll and R. Showalter have primarily dealt with singular or degenerate equations of the following two forms:

$$A(t)u_{tt} + B(t)u_t + C(t)u = g, \quad t > 0 \tag{259}$$

and

$$(A(\cdot)u)_{tt} + (B(\cdot)u)_t + C(\cdot)u = g, \quad t > 0, \tag{260}$$

where $A(t)$, $B(t)$ and $C(t)$ are three families of possibly nonlinear differential operators acting on an SCLCS E. According to R. Caroll and R. Showalter, the equation (259) is called singular if at least one of the operator coefficients tends to infinity as $t \to 0+$, while the equation (259) is called degenerate if some of the operator coefficients tends to zero as $t \to 0+$; the same notions are accepted for the equation (260))=. Roughly speaking, if these equations can be solved with respect to the highest order derivative, then we call them of parabolic or hyperbolic type; if this is not the case, then they are of Sobolev type (as the reader can easily observe, in this monograph we do not distinguish, in some heuristical manner, Sobolev and degenerate type equations). There is an enormous literature on singular abstract Cauchy problems with integer order derivatives, especially on the famous Euler–Poisson–Darboux equation. Concerning purely fractional case, there exists only a few research papers containing very limited material about this important class of equations (cf.B. Baeumer, M. M. Meerschaert, E. Nane [**41**], V. Keyantuo, C. Lizama [**275**], M. M. Meerschaert, E. Nane, P. Vellaisamy [**418**] and

E. Nane [**432**]; even for the above-mentioned Euler–Poisson–Darboux equation, there is no significant reference covering the fractional evolution case).

Abstract degenerate differential equations and empathy theory. In [**476**], N. Sauer has investigated the abstract Cauchy problem

$$\frac{d}{dt}(Bu(t)) = Au(t), \quad t > 0; \lim_{t \to 0+} Bu(t) = y \in Y,$$

where $A\colon D \to Y$ and $B\colon D \to Y$ are two linear operators defined on a common domain $\mathbf{D}$ contained in a Banach space X and Y is another Banach space. For this purpose, he defines the notion of an empathy, that is any double family of evolution operators $\langle (S(t))_{t>0}, (E(t))_{t>0}\rangle$ satisfying the following properties:

(i) $E(t) \in L(Y)$ and $S(t) \in L(Y, X)$ for all $t > 0$;
(ii) $S(t + s) = S(t)E(s)$ for arbitrary $t, s > 0$ (the empathy relation);
(iii) There exists $\xi > 0$ such that $P(\xi)$ is invertible;
(iv) For every $y \in Y$ and $\lambda > 0$, we have $e^{-\lambda \cdot}E(\cdot)y \in L^1((0, \infty) : Y)$ and $e^{-\lambda \cdot}S(\cdot)y \in L^1((0, \infty) : X)$.

It is well known that, for any empathy $\langle (S(t))_{t>0}, (E(t))_{t>0}\rangle$, the following holds:

(a) $(E(t))_{t>0}$ is a semigroup;
(b) $(S(t))_{t>0}$ and $(E(t))_{t>0}$ are strongly continuous and the norms $\|S(t)\|$, $\|E(t)\|$ are locally uniformly bounded on $(0, \infty)$.
(c) For any number $\zeta > 0$, the operator $P(\zeta)$ is invertible.

Define $R(\lambda)y := \int_0^\infty e^{-\lambda t}E(t)y\,dt$ and $P(\lambda)y := \int_0^\infty e^{-\lambda t}E(t)y\,dt$ ($\lambda > 0$, $y \in Y$). Then we have:

(d) $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) = (\mu - \lambda)R(\mu)R(\lambda)$ and $P(\lambda) - P(\mu) = (\mu - \lambda)P(\lambda)R(\mu) = (\mu - \lambda)P(\mu)R(\lambda)$ for $\lambda, \mu > 0$.
(e) $N(R(\lambda)) = N(R(\mu)) := N_E$, $R(R(\lambda)) = R(R(\mu)) := \mathbf{D}_E \subseteq Y$ and $R(P(\lambda)) = R(P(\mu)) := \mathbf{D} \subseteq Y$ for any $\lambda, \mu > 0$;
(f) $S(t)[\mathbf{D}_E] \subseteq \mathbf{D}$ and $E(t)[\mathbf{D}_E] \subseteq \mathbf{D}$ for $t > 0$;
(g) For any $y \in \mathbf{D}_E$, we have $\lim_{t\to 0} E(t)y = y$;
(h) There exists a linear operator $C_0\colon \mathbf{D}_E \to \mathbf{D}$ such that $\lim_{t\to 0} S(t)y = C_0 y$ for any $y \in \mathbf{D}_E$;
(i) For any $\lambda, \mu > 0$, we have that $A_\lambda = A_\mu$ and $B_\lambda = B_\mu$, where $A_\lambda := [\lambda R(\lambda) - I]P^{-1}(\lambda)\colon \mathbf{D} \to Y$ and $B_\lambda := R(\lambda)P^{-1}(\lambda) : \mathbf{D} \to \mathbf{D}_E$.

Set $A := A_1$ and $B := B_1$. Then the pair $\langle A, B\rangle$ is said to be the generator of the empathy $\langle (S(t))_{t>0}, (E(t))_{t>0}\rangle$.

It is clear that applications of empathy theory to abstract degenerate Volterra integro-differential equations are very limited. Mention should be made of an intriguing application of [**476**, Theorem 8.2] to the following system of PDEs of first order

$$\rho(x)v_t(x, t) = f_x(x, t), \quad \sigma f_t(x, t) = v_x(x, t), \tag{261}$$

for which it is well-known that plays an important role in modeling of logitudinal vibrations in an elastic bar of length l and linear density $\rho = \rho(x)$, $0 \leqslant x \leqslant l$.

In (261), v denotes the velocity, f the internal force and $\sigma > 0$ the reciprocal of Hooke's constant.

For further information on empathy theory and its applications to abstract differential equations, we refer the reader to the recent doctoral dissertation of W.-S. Lee [**374**]. Up to now, we do not have any reference dealing with some fractional analogues of empathy. Before we move ourselves to the next chapter, we want also to recommend the monograph [**473**] by A. M. Samoilenko, M. I. Shkil' and V. P. Yakovets' for the basic results on linear degenerate systems of ordinary differential equations.

CHAPTER 3

MULTIVALUED LINEAR OPERATORS APPROACH

The main purpose of this chapter is to study various classes of abstract degenerate Volterra integro-differential equations by using the multivalued linear operators approach. Applying suitable substitutions, we put these equations into the form of abstract degenerate Volterra integro-differential inclusions with multivalued linear operators and examine after that the possibility of getting well-posedness results by making use of some known results and methods from the corresponding theory of non-degenerate equations (observe, however, that this approach has some drawbacks because we inevitably lose some valuable information on the well-posedness of initial problems). Despite of this, abstract degenerate Volterra integro-differential inclusions with multivalued linear operators deserve special attention and analysis.

We use the standard terminology throughout this chapter. By $E, X, Y, \ldots$ we denote Hausdorff sequentially complete locally convex spaces over the field of complex numbers. By $\circledast_X$ ($\circledast$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of X. Any place where we need the pivot space to be a Banach (Fréchet) space will be especially emphasized.

3.1. Abstract degenerate Volterra integro-differential inclusions

Let $0 < \tau \leqslant \infty$, $\alpha > 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{F}\colon [0,\tau) \to P(Y)$, and let $\mathcal{A}\colon X \to P(Y)$, $\mathcal{B}\colon X \to P(Y)$ be two given mappings (possibly non-linear). Suppose that B is a single-valued linear operator. As mentioned in the introductory part, the main aim of this section is to analyze the following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A}\int_0^t a(t-s)u(s)\,ds + \mathcal{F}(t), \quad t \in [0,\tau), \tag{262}$$

as well as the following fractional Sobolev inclusions:

$$(\mathrm{DFP})_{\mathbf{R}}: \begin{cases} \mathbf{D}_t^{\alpha} Bu(t) \in \mathcal{A}u(t) + \mathcal{F}(t), & t \geqslant 0, \\ (Bu)^{(j)}(0) = Bx_j, & 0 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

and

$$(\mathrm{DFP})_{\mathbf{L}}: \begin{cases} \mathcal{B}\mathbf{D}_t^{\alpha} u(t) \subseteq \mathcal{A}u(t) + \mathcal{F}(t), & t \geqslant 0, \\ u^{(j)}(0) = x_j, & 0 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

In the following general definition, we introduce various types of solutions to the abstract degenerate inclusions (262), $(\mathrm{DFP})_{\mathbf{R}}$ and $(\mathrm{DFP})_{\mathbf{L}}$.

Definition 3.1.1. (i) A function $u \in C([0,\tau) : X)$ is said to be a pre-solution of (262) iff $(a*u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,\tau)$, as well as (262) holds. By a solution of (262), we mean any pre-solution $u(\cdot)$ of (262) satisfying additionally that there exist functions $u_{\mathcal{B}} \in C([0,\tau) : Y)$ and $u_{a,\mathcal{A}} \in C([0,\tau) : Y)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds$ for $t \in [0,\tau)$, as well as

$$u_{\mathcal{B}}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

Strong solution of (262) is any function $u \in C([0,\tau) : X)$ satisfying that there exist two continuous functions $u_{\mathcal{B}} \in C([0,\tau) : Y)$ and $u_{\mathcal{A}} \in C([0,\tau) : Y)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for all $t \in [0,\tau)$, and

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

(ii) Let $B = \mathcal{B}$ be single-valued. By a p-solution of $(\mathrm{DFP})_{\mathbf{R}}$, we mean any X-valued function $t \mapsto u(t)$, $t \geqslant 0$ such that the term $t \mapsto \mathbf{D}_t^{\alpha}Bu(t)$, $t \geqslant 0$ is well-defined, $u(t) \in D(\mathcal{A})$ for $t \geqslant 0$, and the requirements of $(\mathrm{DFP})_{\mathbf{R}}$ hold; a pre-solution of $(\mathrm{DFP})_{\mathbf{R}}$ is any p-solution of $(\mathrm{DFP})_{R}$ that is continuous for $t \geqslant 0$. Finally, a solution of $(\mathrm{DFP})_{\mathbf{R}}$ is any pre-solution $u(\cdot)$ of $(\mathrm{DFP})_{\mathbf{R}}$ satisfying additionally that there exists a function $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for $t \geqslant 0$, and $\mathbf{D}_t^{\alpha}Bu(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geqslant 0$.

(iii) By a pre-solution of $(\mathrm{DFP})_{\mathbf{L}}$, we mean any continuous X-valued function $t \mapsto u(t)$, $t \geqslant 0$ such that the term $t \mapsto \mathbf{D}_t^{\alpha}u(t)$, $t \geqslant 0$ is well defined and continuous, as well as that $\mathbf{D}_t^{\alpha}u(t) \in D(\mathcal{B})$ and $u(t) \in D(\mathcal{A})$ for $t \geqslant 0$, and that the requirements of $(\mathrm{DFP})_{\mathbf{L}}$ hold; a solution of $(\mathrm{DFP})_{\mathbf{L}}$ is any pre-solution $u(\cdot)$ of $(\mathrm{DFP})_{\mathbf{L}}$ satisfying additionally that there exist functions $u_{\alpha,\mathcal{B}} \in C([0,\infty) : Y)$ and $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\alpha,\mathcal{B}}(t) \in \mathcal{B}\mathbf{D}_t^{\alpha}u(t)$ and $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for $t \geqslant 0$, as well as that $u_{\alpha,\mathcal{B}}(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geqslant 0$.

Before proceeding further, we want to observe that the existence of solutions to (262), $(\mathrm{DFP})_{\mathbf{R}}$ or $(\mathrm{DFP})_{\mathbf{L}}$ immediately implies that $\sec_c(\mathcal{F}) \neq \emptyset$, as well as that any strong solution of (262) is already a solution of (262), provided that $\mathcal{A}$ and $\mathcal{B}$ are MLOs with $\mathcal{A}$ being closed; this can be simply verified with the help of Theorem 1.2.3. The notion of a (pre-)solution of problems $(\mathrm{DFP})_{\mathbf{R}}$ and $(\mathrm{DFP})_{\mathbf{L}}$ can be similarly defined on any finite interval $[0,\tau)$ or $[0,\tau]$, where $0 < \tau < \infty$, and extends so the notion of a strict solution of problem (E) given on pp. 33–34 of **[199]** ($\mathcal{B} = I$, $\alpha = 1$, $\mathcal{F}(t) = f(t)$ is continuous single-valued). We refer the reader to **[211]** and **[306]** for some results about the wellposedness of some special cases of problem $(\mathrm{DFP})_{\mathbf{R}}$. In our further work, it will be assumed that $\mathcal{A}$ and $\mathcal{B}$ are multivalued linear operators. Observe that we cannot consider the qualitative properties of solutions of problems (262), $(\mathrm{DFP})_{\mathbf{R}}$ or $(\mathrm{DFP})_{\mathbf{L}}$ in full generality by a simple passing to the multivalued linear operators $\mathcal{B}^{-1}\mathcal{A}$ or $\mathcal{A}\mathcal{B}^{-1}$ (see e.g. the definition of a solution of (262)). Concerning this question, we have the following remark.

Remark 3.1.2. (cf. also Section 2.1) Suppose that $0 < \tau \leqslant \infty$, $\alpha > 0$, as well as that $A\colon D(A) \subseteq X \to Y$ and $B\colon D(B) \subseteq X \to Y$ are two single-valued linear operators. Then $B^{-1}A$ is an MLO in X, and AB^{-1} is an MLO in Y.

(i) Suppose that $u(\cdot)$ is a pre-solution (or, equivalently, solution) of problem (262) with $\mathcal{B} = I_X$, $\mathcal{A} = B^{-1}A$ and $\mathcal{F} = f : [0,\tau) \to D(B)$ being single-valued. Then $u \in C([0,\tau) : X)$ and $Bu(t) = A(a * u)(t) + Bf(t)$, $t \in [0,\tau)$. If, in addition to this, $B \in L(X,Y)$ and $u(\cdot)$ is a strong solution of problem (262) with the above requirements being satisfied, then the mappings $t \mapsto Au(t)$, $t \in [0,\tau)$ and $t \mapsto Bu(t)$, $t \in [0,\tau)$ are continuous, and $(a * Au)(t) = Bu(t) - Bf(t)$, $t \in [0,\tau)$.

(ii) Suppose that $v(\cdot)$ is a pre-solution (solution) of problem $(\mathrm{DFP})_{\mathbf{R}}$ with $\mathcal{B} = I_Y$, $\mathcal{A} = AB^{-1}$, $\mathcal{F} = f : [0,\tau) \to Y$ being single-valued, and $v_j = Bx_j$ $(0 \leqslant j \leqslant \lceil \alpha \rceil - 1)$. Let $B^{-1} \in L(Y,X)$. Then the function $u(t) := B^{-1}v(t)$, $t \geqslant 0$ is a pre-solution (solution) of problem $(\mathrm{DFP})_{\mathbf{R}}$ with $\mathcal{B} = B$ and $\mathcal{A} = A$.

(iii) Suppose that $\mathcal{F} = f : [0,\tau) \to D(B)$ is single-valued and $u(\cdot)$ is a pre-solution of problem $(\mathrm{DFP})_{\mathbf{L}}$ with $\mathcal{B} = I_X$ and $\mathcal{A} = B^{-1}A$. Then $u(\cdot)$ is a pre-solution of problem $(\mathrm{DFP})_{\mathbf{L}}$ with $\mathcal{B} = B$, $\mathcal{A} = A$ and $\mathcal{F}(t) = Bf(t)$, $t \in [0,\tau)$. If, in addition to this, $B \in L(X,Y)$ and $u(\cdot)$ is a solution of problem $(\mathrm{DFP})_{\mathbf{L}}$ with the above requirements being satisfied, then $u(\cdot)$ is a solution of problem $(\mathrm{DFP})_{\mathbf{L}}$ with $\mathcal{B} = B$, $\mathcal{A} = A$ and $\mathcal{F}(t) = Bf(t)$, $t \in [0,\tau)$.

(iv) Suppose that $u\colon [0,\infty) \to D(A) \cap D(B)$. Then $u(\cdot)$ is a p-solution of problem $(\mathrm{DFP})_{\mathbf{R}}$ with $\mathcal{B} = B$ and $\mathcal{A} = A$ iff $v = Bu(\cdot)$ is a pre-solution of problem

$$\begin{cases} \mathbf{D}_t^{\alpha} v(t) \in AB^{-1}v(t) + \mathcal{F}(t), & t \geqslant 0, \\ v^{(j)}(0) = Bx_j, & 0 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

(v) Suppose that $C_Y \in L(Y)$ is injective and the closed graph theorem holds for the mappings from Y into Y. Then we define the set $\rho_{C_Y}^{B}(A) := \{\lambda \in \mathbb{C} : \lambda B - A \text{ is injective and } (\lambda B - A)^{-1}C_Y \in L(Y)\}$. It can be simply checked that $\rho_{C_Y}^{B}(A) \subseteq \rho_{C_Y}(AB^{-1})$, as well as that

$$(\lambda - AB^{-1})^{-1}C_Y = B(\lambda B - A)^{-1}C_Y, \quad \lambda \in \rho_{C_Y}^{B}(A). \tag{263}$$

This is an extension of [**199**, Theorem 1.14] and holds even in the case that the operator C_Y does not commute with AB^{-1}, when we define the C_Y-resolvent set of the operator $\lambda - AB^{-1}$ in the same way as before. Observe also that the assumption $D(A) \subseteq D(B)$, which has been used in [**199**, Section 1.6], is not necessary for the validity of (263).

(vi) Suppose that $X = Y$, $C \in L(X)$ is injective, $B \in L(X)$, $CA \subseteq AC$ and $CB \subseteq BC$. Define the set $\rho_C^B(A)$ as above. Then we have $\rho_C^B(A) \subseteq \rho_C(B^{-1}A)$ and

$$(\lambda - B^{-1}A)^{-1}Cx = (\lambda B - A)^{-1}CBx, \quad x \in X.$$

Furthermore, if $C = I$, $X \neq Y$ and $B \in L(X,Y)$, then $\rho^B(A) \subseteq \rho(B^{-1}A)$ and the previous equality holds.

Consider now the case in which the operator $\mathcal{A}$ is closed, the operator $\mathcal{B} = B$ is single-valued and the function $\mathcal{F}(t) = f(t)$ is Y-continuous at each point $t \geqslant 0$. Then any pre-solution $u(\cdot)$ of problem $(\mathrm{DFP})_{\mathbf{R}}$ is already a solution of this problem, and Theorem 1.2.3 in combination with the identity (38) implies that

$$Bu(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)Bx_k - (g_\alpha * f)(t) \in \mathcal{A}(g_\alpha * u)(t), \quad t \geqslant 0.$$

Suppose, conversely, that there exists a function $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$, $t \geqslant 0$ and

$$Bu(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)Bx_k - (g_\alpha * f)(t) = (g_\alpha * u_{\mathcal{A}})(t), \quad t \geqslant 0.$$

Then it can be simply verified that $u(\cdot)$ is a solution of problem $(\mathrm{DFP})_{\mathbf{R}}$; it is noteworthy that we do not need the assumption on closedness of $\mathcal{A}$ in this direction. Even in the case that $\mathcal{A} = A$ is a closed single-valued linear operator, a corresponding statement for the problem $(\mathrm{DFP})_{\mathbf{L}}$ cannot be proved. Suppose, finally, that the operators $\mathcal{A}$ and $\mathcal{B}$ are closed, $u(\cdot)$ is a solution of problem $(\mathrm{DFP})_{\mathbf{R}}$, the function $\mathcal{F}(t) = f(t)$ is Y-continuous at each point $t \geqslant 0$, as well as the functions $u_{\alpha,\mathcal{B}} \in C([0,\infty) : Y)$ and $u_{\mathcal{A}} \in C([0,\infty) : Y)$ satisfy the requirements stated in Definition 3.1.1(iii). Using again Theorem 1.2.3 and the identity (38), it readily follows that

$$\begin{aligned}\mathcal{B}\Bigg[u(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)x_k\Bigg] &\ni (g_\alpha * u_{\alpha,\mathcal{B}})(t)\\ &= (g_\alpha * u_{\mathcal{A}})(t) + (g_\alpha * f)(t) \in \mathcal{A}(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \geqslant 0.\end{aligned}$$

The proof of following important theorem can be deduced by using Theorem 1.2.3, Theorem 1.4.2[(iv),(vi)], Theorem 1.4.4 and the argumentation already seen in the proof of [**280**, Theorem 3.1] (cf. also [**285**, Fundamental Lemma 3.1]); observe that we do not use the assumption on the exponential boundedness of function $u(t)$ here. After formulation, we will only include the most relevant details needed for the proof of implication (iii) $\Rightarrow$ (iv).

THEOREM 3.1.3. *Suppose that* $\mathcal{A}\colon X \to P(Y)$ *and* $\mathcal{B}\colon X \to P(Y)$ *are* MLOs, *as well as that* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed. Assume, further, that* $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\mathrm{abs}(|a|) < \infty$, $u \in C([0,\infty) : X)$, $u \in (\mathrm{P1}) - X$, *as well as that* $u(t) \in D(\mathcal{B})$, $t \geqslant 0$, $a * u \in C([0,\infty) : X_{\mathcal{A}})$, $a * u \in (\mathrm{P1}) - X_{\mathcal{A}}$, $\mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{B}u) < \infty$, $\mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{F}) < \infty$, *and* $\omega > \max(0, \omega_X(u), \mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{B}u), \mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{F}), \mathrm{abs}_{X_{\mathcal{A}}}(a*u))$. *Consider the following assertions:*

(i) $u(\cdot)$ *is a solution of* (262) *with* $\tau = \infty$.
(ii) $u(\cdot)$ *is a pre-solution of* (262) *with* $\tau = \infty$.

(iii) *For any section* $u_{\mathcal{B}} \in \sec(\mathcal{B}u)$ *there is a section* $f \in \sec(\mathcal{F})$ *such that*

$$\widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda) \in \tilde{a}(\lambda)\mathcal{A}\tilde{u}(\lambda), \quad \operatorname{Re}\lambda > \omega, \ \tilde{a}(\lambda) \neq 0.$$

(iv) *For any section* $u_{\mathcal{B}} \in \sec(\mathcal{B}u)$ *there is a section* $f \in \sec(\mathcal{F})$ *such that*

$$\widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda) \in \tilde{a}(\lambda)\mathcal{A}\tilde{u}(\lambda), \quad \lambda \in \mathbb{N}, \ \lambda > \omega, \ \tilde{a}(\lambda) \neq 0. \tag{264}$$

(v) *For any section* $u_{\mathcal{B}} \in \sec(\mathcal{B}u)$ *there is a section* $f \in \sec(\mathcal{F})$ *such that*

$$(1 * u_{\mathcal{B}})(t) - (1 * f)(t) \in \mathcal{A}(1 * a * u)(t), \quad t \geqslant 0. \tag{265}$$

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). *Furthermore, if* $\mathcal{B} = B$ *is single-valued,* $Bu \in C([0,\infty) : Y_{\mathcal{A}})$ *and* $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ *is single-valued, then the above is equivalent.*

Sketch of proof for (iv) $\Rightarrow$ (v). Suppose that for any section $u_{\mathcal{B}} \in \sec(\mathcal{B}u)$ there is a section $f \in \sec(\mathcal{F})$ such that (264) holds. Let a number $\lambda \in \mathbb{N}$ with $\lambda > \omega$ and $\tilde{a}(\lambda) = 0$ be temporarily fixed. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in (λ,∞) such that $\tilde{a}(\lambda_n) \neq 0$ and $\lim_{n\to+\infty} \lambda_n = \lambda$. Since $(\tilde{a}(\lambda_n)\tilde{u}(\lambda_n), \widetilde{u_{\mathcal{B}}}(\lambda_n) - \tilde{f}(\lambda_n)) \in \mathcal{A}$, $n \in \mathbb{N}$, i.e., $(\widetilde{a * u}(\lambda_n), \widetilde{u_{\mathcal{B}}}(\lambda_n) - \tilde{f}(\lambda_n)) \in \mathcal{A}$, $n \in \mathbb{N}$, and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, it readily folows that $(\widetilde{a * u}(\lambda), \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$; in other words, $(0, \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$. By the foregoing, we have that $(\widetilde{a * u}(\lambda), \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$ for all $\lambda \in \mathbb{N}$ with $\lambda > \omega$. Using Theorem 1.2.3, we get that $\int_0^\infty e^{-\lambda t}(u_{\mathcal{B}} - f)^{[2]}(t)dt \in \mathcal{A}\int_0^\infty e^{-\lambda t}(a * u)^{[2]}(t)dt$ ($\lambda \in \mathbb{N}$, $\lambda > \omega$) and now we can apply Theorem 1.4.4, along with the $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closedness of $\mathcal{A}$, in order to see that $u_{\mathcal{B}}^{[2]}(t) - f^{[2]}(t) \in \mathcal{A}(a * u)^{[2]}(t)$, $t \geqslant 0$. This simply implies (265). $\square$

Remark 3.1.4. Observe that we do not require any type of closedness of the operator $\mathcal{B}$ in the formulation of Theorem 3.1.3. Even in the case that $X = Y$ and $\mathcal{B} = B = I$, we cannot differentiate the equation (265) once more without making an additional assumption that $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ is single-valued (cf. [**280**, l. -1, p. 173; l. 1–03, p. 174], where the author has made a small mistake in the consideration; in actual fact, the equation [**280**, (3.1)] has to be valid for some $f \in \sec_c(\mathcal{F})$ in order for the proof of implication (iii) $\Rightarrow$ (i) of [**280**, Theorem 3.1] to work).

If Ω is a non-empty open subset of $\mathbb{C}$ and $G\colon \Omega \to X$ is an analytic mapping that it is not identically equal to the zero function, then we can simply prove that for each zero λ_0 of $G(\cdot)$ there exists a uniquely determined natural number $n \in \mathbb{N}$ such that $G^{(j)}(\lambda_0) = 0$ for $0 \leqslant j \leqslant n-1$ and $G^{(n)}(\lambda_0) \neq 0$. Owing to this fact, we can repeat almost verbatim the arguments given in the proof of [**280**, Theorem 3.2] to verify the validity of the following Ljubich uniqueness type theorem:

Theorem 3.1.5. *Suppose* $\mathcal{A}\colon X \to P(Y)$ *is an* MLO, $\mathcal{B} = B\colon D(B) \subseteq X \to Y$ *is a single-valued linear operator,* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed and* B *is* $X_B \times Y_B$*-closed, where* $Y_{\mathcal{A}} \hookrightarrow Y_B$. *Assume, further, that* $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\operatorname{abs}(|a|) < \infty$, $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ *is single-valued,* $\operatorname{abs}_{Y_{\mathcal{A}}}(f) < \infty$, *and there exist a sequence*

$(\lambda_k)_{k\in\mathbb{N}}$ *of complex numbers and a number* $\omega > \operatorname{abs}(|a|)$ *such that* $\lim_{k\to\infty} \operatorname{Re}\lambda_k = +\infty$, $\tilde{a}(\lambda_k) \neq 0$, $k \in \mathbb{N}$, *and*

$$\frac{1}{\tilde{a}(\lambda_k)} Bx \notin \mathcal{A}x, \quad k \in \mathbb{N},\ 0 \neq x \in D(\mathcal{A}) \cap D(B).$$

Then there exists a unique pre-solution of (262), *with* $\tau = \infty$, *satisfying that* $u \in (\mathrm{P1}) - X_B$, $u(t) \in D(B)$, $t \geqslant 0$, $Bu \in C([0,\infty) : Y_{\mathcal{A}})$, $a * u \in C([0,\infty) : X_{\mathcal{A}})$, $a * u \in (\mathrm{P1}) - X_{\mathcal{A}}$ *and* $\operatorname{abs}_{Y_{\mathcal{A}}}(Bu) < \infty$.

In the following extension of [**292**, Theorem 2.1.34], we will prove one more Ljubich's uniqueness criterium for abstract Cauchy problems with multivalued linear operators (cf. also [**285**, Theorem 3.5] and [**292**, Theorem 2.10.44]).

THEOREM 3.1.6. *Suppose* $\alpha > 0$, $\lambda > 0$, $\mathcal{A}$ *is an* MLO *in* X, $\{(n\lambda)^{\alpha} : n \in \mathbb{N}\} \subseteq \rho_C(\mathcal{A})$ *and, for every* $\sigma > 0$ *and* $x \in X$, *we have*

$$\lim_{n\to\infty} \frac{((n\lambda)^{\alpha} - \mathcal{A})^{-1}Cx}{e^{n\lambda\sigma}} = 0.$$

Then, for every $x_0, \ldots, x_{\lceil\alpha\rceil-1} \in X$, *there exists at most one pre-solution of the initial value problem* $(\mathrm{DFP})_{\mathbf{R}}$ *with* $\mathcal{B} = I$.

PROOF. It suffices to show that the zero function is the only pre-solution of the problem $(\mathrm{DFP})_{\mathbf{R}}$ with $\mathcal{B} = I$ and the initial values $x_0, \ldots, x_{\lceil\alpha\rceil-1}$ chosen to be zeroes. Let $u(\cdot)$ be a pre-solution of such a problem. Set $z_n(t) := ((n\lambda)^{\alpha} - \mathcal{A})^{-1}Cu(t)$, $t \geqslant 0$, $n \in \mathbb{N}$. Then it can be easily checked with the help of Theorem 1.2.4(i) that $z_n(\cdot)$ is a solution of the initial value problem:

$$\begin{cases} z_n \in C^{\lceil\alpha\rceil}((0,\infty) : X) \cap C^{\lceil\alpha\rceil-1}([0,\infty) : X), \\ \mathbf{D}_t^{\alpha} z_n(t) = (n\lambda)^{\alpha} z_n(t) - Cu(t), \quad t > 0, \\ z_n^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant \lceil\alpha\rceil - 1. \end{cases}$$

This implies $z_n(t) = -(u * \cdot^{\alpha-1}E_{\alpha,\alpha}((n\lambda)^{\alpha}\cdot^{\alpha-1}))(t)$, $t \geqslant 0$, $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} e^{-n\lambda\sigma} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}((n\lambda)^{\alpha}s^{\alpha})Cu(t-s)\,ds = 0 \quad (t > 0,\ \sigma > 0).$$

Now we can argue as in the second part of proof of [**292**, Theorem 2.1.34] so as to conclude that $u(t) = 0$, $t \geqslant 0$ (in the case that $\alpha \in \mathbb{N}$, the assertion can be proved by passing to the theory of abstract Cauchy problems of first order since [**292**, Lemma 2.1.33(i)] admits an extension to multivalued linear operators). $\square$

REMARK 3.1.7. Observe that, in the formulation of Theorem 3.1.6, we do not require any type of closedness of the operator $\mathcal{A}$.

The following theorem is very similar to [**61**, Theorem 3.1, Theorem 3.3] and [**292**, Theorem 2.4.2]. Because of its importance, we will include the most relevant details of proof.

THEOREM 3.1.8 (Subordination principle for abstract time-fractional inclusions). *Suppose that* $0 < \alpha < \beta$, $\gamma = \alpha/\beta$, $\mathcal{A}\colon X \to P(Y)$ *is an* MLO, $\mathcal{B} = B\colon D(B) \subseteq X \to Y$ *is a single-valued linear operator,* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed and*

B is $X_B \times Y_B$-closed, where $X_{\mathcal{A}} \hookrightarrow X_B$ and $Y_{\mathcal{A}} \hookrightarrow Y_B$. Assume, further, that $f_\beta \in LT_{or} - Y_{\mathcal{A}}$ is single-valued and there exists a pre-solution (or, equivalently, solution) $u(t) := u_\beta(t)$ of (262), with $\tau = \infty$, $a(t) = g_\beta(t)$ and $\mathcal{F} = f_\beta$, satisfying that $u_\beta \in LT_{or} - X_B$, $Bu_\beta \in LT_{or} - Y_{\mathcal{A}}$, $g_\beta * u_\beta \in LT_{or} - X_{\mathcal{A}}$ and that for each seminorm $p \in \circledast_{X_B}$ there exists $\omega_p \geqslant 0$ such that $p(u_\beta(t)) = O(e^{\omega_p t})$, $t \geqslant 0$, $p \in \circledast_{X_B}$. Define

$$u_\alpha(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})u_\beta(s)ds,\ t > 0 \text{ and } u_\alpha(0) := u_\beta(0)$$

and

$$f_\alpha(t)x := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})f_\beta(s)ds,\ t > 0 \text{ and } f_\alpha(0) := f_\beta(0).$$

Then $u_\alpha(t)$ is a solution of (262), with $\tau = \infty$, $a(t) = g_\alpha(t)$ and $\mathcal{F}(t) = f_\alpha(t) \in LT_{or} - Y_{\mathcal{A}}$, satisfying additionally that $u_\alpha \in LT_{or} - X_B$, $Bu_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and

$$p(u_\alpha(t)) = O(\exp(\omega_p^{1/\gamma}t)), \quad p \in \circledast_{X_B},\ t \geqslant 0. \tag{266}$$

Let $p \in \circledast_{X_B}$ be fixed. Then the condition

$$p(u_\beta(t)) = O((1 + t^{\xi_p})e^{\omega_p t}) \text{ for some } \xi_p \geqslant 0, \tag{267}$$

resp.,

$$p(u_\beta(t)) = O(t^{\xi_p}e^{\omega_p t}), \quad t \geqslant 0 \tag{268}$$

implies that

$$p(u_\alpha(t)) = O((1 + t^{\xi_p\gamma})(1 + \omega_p t^{\xi_p(1-\gamma)})\exp(\omega_p^{1/\gamma}t)), \quad t \geqslant 0, \tag{269}$$

resp.,

$$p(u_\alpha(t)) = O(t^{\xi_p\gamma}(1 + \omega_p t^{\xi_p(1-\gamma)})\exp(\omega_p^{1/\gamma}t)), \quad t \geqslant 0. \tag{270}$$

Furthermore, the following holds:

- (i) *The mapping $t \mapsto u_\alpha(t)$, $t > 0$ admits an extension to $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ and the mapping $z \mapsto u_\alpha(z)$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ is analytic.*
- (ii) *Let $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$. If, for every $p \in \circledast$, one has $\omega_p = 0$, then for each $\theta \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$ the following holds: $\lim_{z\to 0, z\in\overline{\Sigma_\theta}} u_\alpha(z) = u_\alpha(0)$.*
- (iii) *If $\omega_p > 0$ for some $p \in \circledast$, then for each $\theta \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2}))$ the following holds: $\lim_{z\to 0, z\in\overline{\Sigma_\theta}} u_\alpha(z) = u_\alpha(0)$.*

PROOF. The proofs of (i)–(iii) follows similarly as in that of [**61**, Theorem 3.3], while the proof that the condition (267), resp. (268), implies (269), resp. (270), follows similarly as in that of [**292**, Theorem 2.4.2]. Furthermore, it can be easily seen that the estimate (266) holds for solution $u_\alpha(\cdot)$. By Theorem 3.1.3, we should only show that $u_\alpha \in LT_{or} - X_B$, $Bu_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $f_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and

$$\widetilde{Bu_\alpha}(\lambda) - \widetilde{f_\alpha}(\lambda) \in \lambda^{-\alpha}\mathcal{A}\widetilde{u_\alpha}(\lambda), \quad \lambda > \omega \text{ suff. large.} \tag{271}$$

Since $u_\beta \in LT_{or} - X_B$, the proof of [**61**, Theorem 3.1] immediately implies that $u_\alpha \in LT_{or} - X_B$, as well as that $\widetilde{u_\alpha}(\lambda) = \lambda^{\gamma-1}\widetilde{u_\beta}(\lambda^\beta)$, $\lambda > \omega$ suff. large. Similarly, we have that $f_\alpha \in LT_{or} - Y_{\mathcal{A}}$ and $\tilde{f_\alpha}(\lambda) = \lambda^{\gamma-1}\tilde{f_\beta}(\lambda^\beta)$, $\lambda > \omega$ suff. large. Keeping in mind that $X_{\mathcal{A}} \hookrightarrow X_B$ and $g_\beta * u_\beta \in LT_{or} - X_{\mathcal{A}}$, we can prove that $(g_\alpha * u_\alpha)(t) = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})(g_\beta * u_\beta)(s)ds$, $t > 0$ by performing the Laplace transform (the convergence of last integral is taken for the topology of $X_{\mathcal{A}}$). This simply implies that $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and $(\mathcal{L}(g_\alpha * u_\alpha))(\lambda) = \lambda^{\gamma-1}(\mathcal{L}(g_\beta * u_\beta))(\lambda^\gamma)$, $\lambda > \omega$ suff. large. Since $Y_{\mathcal{A}} \hookrightarrow Y_B$, a similar line of reasoning shows that $Bu_\alpha(t) = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})(Bu_\beta)(s)ds$, $t > 0$ (the convergence of this integral is taken for the topology of $Y_{\mathcal{A}}$) and $\widetilde{Bu_\alpha}(\lambda) = B\widetilde{u_\alpha}(\lambda)$, $\lambda > \omega$ suff. large. The proof of (271) now follows from a simple computation. ☐

We can similarly prove the following subordination principles for abstract degenerate Volterra inclusions in locally convex spaces (cf. [**459**, Section 4] and [**292**, Theorem 2.1.8, Theorem 2.8.7] for more details concerning non-degenerate case and, especially, the case in which $b(t) = g_1(t)$ or $b(t) = g_2(t)$).

Theorem 3.1.9. *Let $b(t)$ and $c(t)$ satisfy* (P1), *let $\int_0^\infty e^{-\beta t}|b(t)|dt < \infty$ for some $\beta \geqslant 0$, and let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\beta}, \quad \alpha = 0 \text{ otherwise.}$$

Suppose that $\operatorname{abs}(|a|) < \infty$, $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda > \alpha$, $\mathcal{A}\colon X \to P(Y)$ *is an* MLO, *$\mathcal{B} = B\colon D(B) \subseteq X \to Y$ is a single-valued linear operator, $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed and B is $X_B \times Y_B$-closed, where $X_{\mathcal{A}} \hookrightarrow X_B$ and $Y_{\mathcal{A}} \hookrightarrow Y_B$. Assume, further, that $f_\beta \in LT_{or} - Y_{\mathcal{A}}$ is single-valued and there exists a pre-solution (or, equivalently, solution) $u(t) := u_b(t)$ of* (262), *with $\tau = \infty$, $a(t)$ replaced with $b(t)$ therein, and $\mathcal{F} = f_b$, satisfying that $u_b \in LT_{or} - X_B$, $Bu_b \in LT_{or} - Y_{\mathcal{A}}$, $b * u_b \in LT_{or} - X_{\mathcal{A}}$ and the family $\{e^{-\omega_b t}u_b(t) : t \geqslant 0\}$ is bounded in X_B ($\omega_b \geqslant 0$). Assume, further, that $c(t)$ is completely positive and there exists a function $f_a \in LT_{or} - Y_{\mathcal{A}}$ satisfying*

$$\widetilde{f_a}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\widetilde{f_b}\Big(\frac{1}{\tilde{c}(\lambda)}\Big), \quad \lambda > \omega_0,\ \widetilde{f_b}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0, \text{ for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\omega_b}, \quad \omega_a = 0 \text{ otherwise.}$$

Then, for every $r \in (0, 1]$, there exists a solution $u(t) := u_{a,r}(t)$ of (262), *with $\tau = \infty$, $a(t)$ and $\mathcal{F} = f_r := g_r * f_a$, satisfying that $u_{a,r} \in LT_{or} - X_B$, $Bu_{a,r} \in LT_{or} - Y_{\mathcal{A}}$, $a * u_{a,r} \in LT_{or} - X_{\mathcal{A}}$ and the set $\{e^{-\omega_a t}u_{a,r}(t) : t \geqslant 0\}$ is bounded in X_B, if $\omega_b = 0$ or $\omega_b\tilde{c}(0) \neq 1$, resp., the set $\{e^{-\varepsilon t}u_{a,r}(t) : t \geqslant 0\}$ is bounded in X_B for any $\varepsilon > 0$, if $\omega_b > 0$ and $\omega_b\tilde{c}(0) = 1$. Furthermore, the function $t \mapsto u_{a,r}(t) \in X_B$, $t \geqslant 0$ is locally Hölder continuous with the exponent $r \in (0, 1]$.*

Remark 3.1.10. (i) In Theorem 3.1.8 and Theorem 3.1.9, we have only proved the existence of a solution of the subordinated inclusion. The uniqueness of solutions can be proved, for example, by using Theorem 3.1.5, Theorem 3.1.6 or Theorem 2.3.6.

(ii) In Theorem 3.1.9, we have faced ourselves with a loss of regularity for solutions of the subordinated problem. Even in the case that $X = Y$ and $B = I$, it is not so simple to prove the existence of a solution of problem (262), with $\tau = \infty$, $a(t)$ and $\mathcal{F} = f_a$, without imposing some additional unfavorable conditions. In the next section, we will introduce various types of solution operator families for the abstract Volterra inclusion (262) and there we will reconsider the problem of loss of regularity for solutions of subordinated problem once more (cf. Theorem 3.2.7).

3.2. Multivalued linear operators as subgenerators of various types of (a, k)-regularized C-resolvent operator families

In [**292**, Section 2.8], the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness families has been introduced and analyzed within the theory of abstract non-degenerate Volterra equations. The main aim of this section is to consider multivalued linear operators in locally convex spaces as subgenerators of (a, k)-regularized (C_1, C_2)-existence and uniqueness families, as well as to consider in more detail the class of (a, k)-regularized C-resolvent families. Unless specified otherwise, we assume that $0 < \tau \leqslant \infty$, $k \in C([0, \tau))$, $k \neq 0$, $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, $\mathcal{A}\colon X \to P(X)$ is an MLO, $C_1 \in L(Y, X)$, $C_2 \in L(X)$ is injective, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. The following definition is an extension of [**292**, Definition 2.8.2] ($X = Y$, A is a closed single-valued linear operator on X) and [**482**, Definition 3.5] ($X = Y$, $C = C_1$, $a(t) = k(t) = 1$).

Definition 3.2.1. Suppose $0 < \tau \leqslant \infty$, $k \in C([0, \tau))$, $k \neq 0$, $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, $\mathcal{A}\colon X \to P(X)$ is an MLO, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(Y, X) \times L(X)$ iff the mappings $t \mapsto R_1(t)y$, $t \geqslant 0$ and $t \mapsto R_2(t)x$, $t \in [0, \tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, as well as the following conditions hold:

$$\Big(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y\Big) \in \mathcal{A}, \quad t \in [0, \tau), \ y \in Y \text{ and} \tag{272}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \text{ whenever } t \in [0, \tau) \text{ and } (x, y) \in \mathcal{A}. \tag{273}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a, k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ iff (272) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a, k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ iff (273) holds.

As an immediate consequence of definition, we have that $R(R_1(0) - k(0)C_1) \subseteq \mathcal{A}0$. Furthermore, if $a(t)$ is a kernel on $[0, \tau)$, then $R_2(t)\mathcal{A}$ is single-valued for any $t \in [0, \tau)$ and $R_2(t)y = 0$ for any $y \in \mathcal{A}0$ and $t \in [0, \tau)$. Now we will extend the definition of an (a, k)-regularized C-resolvent family subgenerated by a single-valued linear operator (cf. [**292**, Definition 2.1.1]).

DEFINITION 3.2.2. Suppose that $0 < \tau \leqslant \infty$, $k \in C([0, \tau))$, $k \neq 0$, $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, $\mathcal{A}\colon X \to P(X)$ is an MLO, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a, k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ iff $(R(t))_{t\in[0,\tau)}$ is a mild (a, k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0, \tau))$.

An (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous iff, for every $t \in (0, \tau)$, the family $\{R(s) : s \in [0, t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geqslant 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that the family $\{e^{-\omega t}R(t) : t \geqslant 0\}$ is equicontinuous; the infimum of such numbers is said to be the exponential type of $(R(t))_{t\geqslant 0}$. The above notion can be simply understood for the classes of mild (a, k)-regularized C_1-existence families and mild (a, k)-regularized C_2-uniqueness families; a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(Y, X) \times L(X)$ is said to be locally equicontinuous (exponentially equicontinuous, provided that $\tau = \infty$) iff both operator families $(R_1(t))_{t\geqslant 0}$ and $(R_2(t))_{t\geqslant 0}$ are. It would take too long to consider the notion of q-exponential equicontinuity for the classes of mild (a, k)-regularized C_1-existence families and mild (a, k)-regularized C_2-uniqueness families (cf. [**292**, Section 2.4] for more details about non-degenerate case). If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geqslant 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a, C)-resolvent family; 0-times integrated (a, C)-resolvent family is further abbreviated to (a, C)-resolvent family. We will accept a similar terminology for the classes of mild (a, k)-regularized C_1-existence families and mild (a, k)-regularized C_2-uniqueness families; in the case of consideration of convoluted C-semigroups, it will be always assumed that the condition (272) holds with $a(t) = 1$ and the operator C_1 replaced by C. Let us mention in passing that the operator semigroups generated by multivalued linear operators have been analyzed by A. G. Baskakov in [**57**].

The following proposition can be simply proved with the help of Theorem 1.2.3 and Theorem 1.1.4(ii).

PROPOSITION 3.2.3. *Suppose that $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(Y, X) \times L(X)$ is a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$ and $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a, k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$. Let $b \in L^1_{loc}([0, \tau))$ be such that $a * b \neq 0$ in $L^1([0, \tau))$ and $k * b \neq 0$ in $C([0, \tau))$. Then $((b * R_2)(t))_{t\geqslant 0}$ is a mild $(a, k * b)$-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$. Furthermore, the following holds:*

(i) *Let $\mathcal{A}$ be $X^1_{\mathcal{A}} \times X^2_{\mathcal{A}}$-closed. Suppose that, for every $y \in Y$, the mapping $t \mapsto (a * R_1)(t)y$, $t \in [0, \tau)$ is continuous in $X^1_{\mathcal{A}}$ and the mapping $t \mapsto R_1(t)y$, $t \in [0, \tau)$ is continuous in $X^2_{\mathcal{A}}$. Then $((b * R_1)(t))_{t\geqslant 0}$ is a mild $(a, k * b)$-regularized C_1-existence family with a subgenerator $\mathcal{A}$.*

(ii) *Let $\mathcal{A}$ be $X^1_{\mathcal{A}} \times X^2_{\mathcal{A}}$-closed. Suppose that, for every $x \in D(\mathcal{A})$ and $y \in R(\mathcal{A})$, the mapping $t \mapsto R(t)x$, $t \in [0,\tau)$ is continuous in $X^1_{\mathcal{A}}$ and the mapping $t \mapsto R(t)y$, $t \in [0,\tau)$ is continuous in $X^2_{\mathcal{A}}$. Then $((b * R)(t))_{t\geqslant 0}$ is an $(a, k * b)$-regularized C-regularized family with a subgenerator $\mathcal{A}$.*

Although the parts (i) and (ii) of the above proposition have been stated for $X^1_{\mathcal{A}} \times X^2_{\mathcal{A}}$-closed subgenerators, the most important case in our further study will be that in which $X^1_{\mathcal{A}} = X^2_{\mathcal{A}} = X$. This is primarily caused by the following fact: Let $\mathcal{A}$ be a subgenerator of a mild (a,k)-regularized C_1-existence family (mild (a,k)-regularized C_2-uniqueness family; mild (a,k)-regularized C-resolvent family) $(R_1(t))_{t\in[0,\tau)}$ $((R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)})$. Then $\bar{\mathcal{A}}$ is likewise a subgenerator of $(R_1(t))_{t\in[0,\tau)}$ $((R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)}$, provided in addition that $(R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous).

Suppose that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Arguing as in non-degenerate case (cf. the paragraph directly preceding [**292**, Definition 2.8.3]), we may conclude that

$$
\begin{aligned}
(274)\quad & (a * R_2)(s)R_1(t)y - R_2(s)(a * R_1)(t)y \\
&\qquad = k(t)(a * R_2)(s)C_1 y - k(s)C_2(a * R_1)(t)y, \quad t \in [0,\tau),\ y \in Y.
\end{aligned}
$$

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)})$ is defined by

$$
\mathcal{A}_{int} := \left\{(x,y) \in X \times X : R_2(t)x - k(t)C_2 x = \int_0^t a(t-s)R_2(s)y\,ds,\ t \in [0,\tau)\right\};
$$

we define the integral generator of an (a,k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$ in the same way as above. The integral generator $\mathcal{A}_{int}$ is an MLO in X which is, in fact, the maximal subgenerator of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ with respect to the set inclusion; furthermore, the assumption $R_2(t)C_2 = C_2R_2(t)$, $t \in [0,\tau)$ implies that $C_2^{-1}\mathcal{A}_{int}C_2 = \mathcal{A}_{int}$ so that $C^{-1}\mathcal{A}_{int}C = \mathcal{A}_{int}$ for resolvent families. The local equicontinuity of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ immediately implies that $\mathcal{A}_{int}$ is closed. Observe that, in the above definition of integral generator, we do not require that the function $a(t)$ is a kernel on $[0,\tau)$, as in non-degenerate case. In the case of resolvent families, the following holds:

(i) Suppose that $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous and $\mathcal{A}$ is a closed subgenerator of $(R(t))_{t\in[0,\tau)}$. Then

$$
(275)\qquad \left(\int_0^t a(t-s)R(s)x\,ds, R(t)x - k(t)Cx\right) \in \mathcal{A}, \quad t \in [0,\tau),\ x \in \overline{D(\mathcal{A})}.
$$

(ii) If $\mathcal{A}$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$, then $C^{-1}\mathcal{A}C$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$, too.

(iii) Suppose that $a(t)$ is a kernel on $[0,\tau)$, $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of $(R(t))_{t\in[0,\tau)}$, and $x \in D(\mathcal{A}) \cap D(\mathcal{B})$. Then $R(t)(y-z) = 0$, $t \in [0,\tau)$ for each $y \in \mathcal{A}x$ and $z \in \mathcal{B}x$.

(iv) Let $\mathcal{A}$ be a subgenerator of $(R(t))_{t\in[0,\tau)}$, and let $\lambda \in \rho_C(\mathcal{A})$ ($\lambda \in \rho(\mathcal{A})$). Suppose that $x \in X$, $y = (\lambda-\mathcal{A})^{-1}Cx$ ($y = (\lambda-\mathcal{A})^{-1}x$) and $z \in \mathcal{A}y$. Then Theorem 1.2.4(i) implies that $\lambda(\lambda-\mathcal{A})^{-1}Cx - Cx \in \mathcal{A}(\lambda-\mathcal{A})^{-1}Cx = \mathcal{A}y$ ($\lambda(\lambda-\mathcal{A})^{-1}x - x \in \mathcal{A}(\lambda-\mathcal{A})^{-1}x = \mathcal{A}y$), so that $R(t)y - k(t)Cy \in \mathcal{A}\int_0^t a(t-s)R(s)[\lambda(\lambda-\mathcal{A})^{-1}Cx - Cx]\,ds = \mathcal{A}\{\lambda(\lambda-\mathcal{A})^{-1}C\int_0^t a(t-s)R(s)x\,ds - \int_0^t a(t-s)R(s)Cx\,ds\}$, $t \in [0,\tau)$ and $\int_0^t a(t-s)R(s)Cx\,ds \in D(\mathcal{A})$, $t \in [0,\tau)$; from this, we may conclude that $R(t)Cx - k(t)C^2x \in (\lambda-\mathcal{A})\mathcal{A}(\lambda-\mathcal{A})^{-1}C\int_0^t a(t-s)R(s)x\,ds$, $t \in [0,\tau)$; similarly, we have that $\int_0^t a(t-s)R(s)x\,ds \in D(\mathcal{A})$ and $R(t)x - k(t)Cx \in (\lambda-\mathcal{A})\mathcal{A}(\lambda-\mathcal{A})^{-1}\int_0^t a(t-s)R(s)x\,ds$, $t \in [0,\tau)$, provided that $\lambda \in \rho(\mathcal{A})$.

The following extensions of [**292**, Theorem 2.8.5, Theorem 2.1.5] are stated without proofs.

THEOREM 3.2.4. *Suppose $\mathcal{A}$ is a closed* MLO *in X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, C_2 is injective, $\omega_0 \geqslant 0$ and $\omega \geqslant \max(\omega_0, \operatorname{abs}(|a|), \operatorname{abs}(k))$.*

(i) *Let $(R_1(t), R_2(t))_{t\geqslant 0} \subseteq L(Y,X) \times L(X)$ be strongly continuous, and let the family $\{e^{-\omega t}R_i(t) : t \geqslant 0\}$ be equicontinuous for $i = 1,2$.*

(a) *Suppose $(R_1(t), R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective, $R(C_1) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)\mathcal{A})$,*

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C_1 y = \int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y, \tag{276}$$

$$\Big\{\frac{1}{\tilde{a}(z)} : \operatorname{Re} z > \omega,\ \tilde{k}(z)\tilde{a}(z) \neq 0\Big\} \subseteq \rho_{C_1}(\mathcal{A}) \tag{277}$$

and

$$\tilde{k}(\lambda)C_2 x = \int_0^\infty e^{-\lambda t}[R_2(t)x - (a * R_2)(t)y]dt, \quad \textit{whenever } (x,y) \in \mathcal{A}. \tag{278}$$

Here, $\rho_{C_1}(\mathcal{A})$ is defined in the obvious way.

(b) *Let (277) hold, and let (276) and (278) hold for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Then $(R_1(t), R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$.*

(ii) *Let $(R_1(t))_{t\geqslant 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_1(t) : t \geqslant 0\}$ be equicontinuous. Then $(R_1(t))_{t\geqslant 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator $\mathcal{A}$ iff for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, one has $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$ and*

$$\tilde{k}(\lambda)C_1 y \in (I - \tilde{a}(\lambda)\mathcal{A})\int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y.$$

(iii) *Let* $(R_2(t))_{t\geqslant 0}$ *be strongly continuous, and let the family* $\{e^{-\omega t}R_2(t) : t \geqslant 0\}$ *be equicontinuous. Then* $(R_2(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* C_2*-uniqueness family with a subgenerator* $\mathcal{A}$ *iff for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*, the operator* $I - \tilde{a}(\lambda)\mathcal{A}$ *is injective and* (278) *holds.*

THEOREM 3.2.5. *Let* $(R(t))_{t\geqslant 0} \subseteq L(X)$ *be a strongly continuous operator family such that there exists* $\omega \geqslant 0$ *satisfying that the family* $\{e^{-\omega t}R(t) : t \geqslant 0\}$ *is equicontinuous, and let* $\omega_0 > \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$*. Suppose that* $\mathcal{A}$ *is a closed* MLO *in* X *and* $C\mathcal{A} \subseteq \mathcal{A}C$*.*

(i) *Assume that* $\mathcal{A}$ *is a subgenerator of the global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (272) *for all* $x = y \in X$*. Then, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*, the operator* $I - \tilde{a}(\lambda)\mathcal{A}$ *is injective,* $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$*, as well as*

$$\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt,\ x \in X,\ \operatorname{Re}\lambda > \omega_0,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0, \tag{279}$$

$$\left\{\frac{1}{\tilde{a}(\lambda)} : \operatorname{Re}\lambda > \omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(\mathcal{A}) \tag{280}$$

and $R(s)R(t) = R(t)R(s)$*,* $t, s \geqslant 0$*.*

(ii) *Assume* (279)–(280)*. Then* $\mathcal{A}$ *is a subgenerator of the global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (272) *for all* $x = y \in X$ *and* $R(s)R(t) = R(t)R(s)$*,* $t, s \geqslant 0$*.*

Before proceeding further, we would like to observe that the proof of [**27**, Proposition 4.1.3] implies that, for every strongly continuous, exponentially equicontinuous operator family $(S(t))_{t\geqslant 0} \subseteq L(E)$, the following Abel ergodicity property

$$\lim_{\lambda\to+\infty} \lambda \int_0^\infty e^{-\lambda t}S(t)x\,dt = S(0)x, \quad x \in E$$

holds true.

REMARK 3.2.6. (i) Suppose that $(R(t))_{t\geqslant 0}$ is a degenerate exponentially equicontinuous (a,k)-regularized C-resolvent family in the sense of [**306**, Definition 2.2], and $B \in L(X)$. Using Remark 1.2.1(iv)/(a), Remark 3.1.2(iv) and Theorem 3.2.5(ii), it can be easily seen that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family with a closed subgenerator $B^{-1}A$.

(ii) Suppose that $n \in \mathbb{N}$, X and Y are Banach spaces, $A\colon D(A) \subseteq X \to Y$ is closed, $B \in L(X,Y)$ and $(V(t))_{t\geqslant 0} \subseteq L(X)$ is a degenerate exponentially bounded n-times integrated semigroup generated by linear operators A, B, in the sense of [**424**, Definition 1.5.3]. Then the arguments mentioned above show that $(V(t))_{t\geqslant 0}$ is an exponentially bounded n-times integrated (g_1, I)-regularized family (semigroup) with a closed subgenerator $B^{-1}A$.

(iii) Let $n \in \mathbb{N}_0$. Due to Theorem 3.2.5(ii), the notion of an exponentially bounded (a,k)-regularized C-resolvent family extends the notion of a degenerate exponentially bounded n-times integrated semigroup generated by an MLO, introduced in [**424**, Definition 1.6.6, Definition 1.6.8].

(iv) Suppose that $(R(t))_{t\geqslant 0} \subseteq L(X,[D(B)])$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B, in the sense of [**307**, Definition 2.5]. Then [**307**, Theorem 2.3(i)] in combination with Remark 3.1.2(v) and Theorem 3.2.5(ii) implies that $(BR(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by $\overline{B^{-1}A}$ (recall that $B^{-1}A$ is closed provided that $C = I$).

The proof of following extension of [**292**, Theorem 2.1.8(i), Theorem 2.8.7(i)] is standard and therefore omitted; we can similarly reformulate Theorem 3.1.8 and [**292**, Proposition 2.1.16] for the class of mild (a,k)-regularized (C_1,C_2)-existence and uniqueness families ((a,k)-regularized C-resolvent families). Here it is only worth noting that the existence of a mild (a,k_1)-regularized C_1-existence family $(R_{0,1}(t))_{t\geqslant 0}$ in the second part of theorem is not automatically guaranteed by the denseness of $\mathcal{A}$ (even in the case that the operator $\mathcal{A} = A$ is single-valued, it seems that the condition $C_1A \subseteq AC_1$ is indispensable for such a mild existence family to exist).

THEOREM 3.2.7. *Suppose $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $\mathcal{A}$ is a closed* MLO *in X, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Let $b(t)$ and $c(t)$ satisfy* (P1), *let $\int_0^\infty e^{-\beta t}|b(t)|dt < \infty$ for some $\beta \geqslant 0$, and let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\beta},\ \alpha = 0 \text{ otherwise.}$$

Suppose that $\operatorname{abs}(|a|) < \infty$ *and $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda \geqslant \alpha$. Let $\mathcal{A}$ be a subgenerator of a (b,k)-regularized C_1-existence family $(R_1(t))_{t\geqslant 0}$ ((b,k)-regularized C_2-uniqueness family $(R_2(t))_{t\geqslant 0}$; (b,k)-regularized C-resolvent family $(R_0(t))_{t\geqslant 0}$ with the property that* (272) *holds for $R_1(\cdot)$ replaced with $R_0(\cdot)$ and each $x = y \in X$) satisfying that the family $\{e^{-\omega_b t}R_1(t) : t \geqslant 0\}$ ($\{e^{-\omega_b t}R_2(t) : t \geqslant 0\}$; $\{e^{-\omega_b t}R(t) : t \geqslant 0\}$) is equicontinuous for some $\omega_b \geqslant 0$. Assume, further, that $c(t)$ is completely positive and there exists a scalar-valued continuous kernel $k_1(t)$ satisfying* (P1) *and*

$$\widetilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big), \quad \lambda > \omega_0,\ \tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0, \text{ for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\omega_b}, \quad \omega_a = 0 \text{ otherwise.}$$

Then, for every $r \in (0,1]$, $\mathcal{A}$ is a subgenerator of a global $(a,k_1 * g_r)$-regularized C_1-existence family $(R_{r,1}(t))_{t\geqslant 0}$ ($(a,k_1 * g_r)$-regularized C_2-uniqueness family $(R_{r,2}(t))_{t\geqslant 0}$; $(a,k_1 * g_r)$-regularized C-resolvent family $(R_{r,0}(t))_{t\geqslant 0}$ with the property that (272) *holds for $R_1(\cdot)$ replaced with $R_{r,0}(\cdot)$ and each $x = y \in X$) such that the family $\{e^{-\omega_a t}R_{r,i}(t) : t \geqslant 0\}$ is equicontinuous and that the mapping $t \mapsto R_{r,i}(t)$, $t \geqslant 0$ is locally Hölder continuous with exponent r, if $\omega_b = 0$ or*

$\omega_b\tilde{c}(0) \neq 1$ $(i = 0, 1, 2)$, resp., for every $\varepsilon > 0$, there exists $M_\varepsilon \geqslant 1$ such that the family $\{e^{-\varepsilon t}R_{r,i}(t) : t \geqslant 0\}$ is equicontinuous and that the mapping $t \mapsto R_{r,i}(t)$, $t \geqslant 0$ is locally Hölder continuous with exponent r, if $\omega_b > 0$ and $\omega_b\tilde{c}(0) = 1$ $(i = 0, 1, 2)$. Furthermore, if $\mathcal{A}$ is densely defined, then $\mathcal{A}$ is a subgenerator of a global (a, k_1)-regularized C_2-uniqueness family $(R_{0,2}(t))_{t\geqslant 0}$ ((a, k_1)-regularized C-resolvent family $(R_{0,0}(t))_{t\geqslant 0}$ with the property that (272) holds for $R_1(\cdot)$ replaced with $R_{0,0}(\cdot)$ and each $x = y \in X$) such that the family $\{e^{-\omega_a t}R_i(t) : t \geqslant 0\}$ is equicontinuous, resp., for every $\varepsilon > 0$, the family $\{e^{-\varepsilon t}R_i(t) : t \geqslant 0\}$ is equicontinuous $(i = 1, 2)$.

Let $(R_1(t), R_2(t))_{t\in[0,\tau)}$ be a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then it is not complicated to see that the function $t \mapsto R_1(t)y$, $t \in [0, \tau)$ $(y \in Y)$, resp. $t \mapsto R_2(t)x$, $t \in [0, \tau)$ $(x \in D(\mathcal{A}))$, is a solution of problem (262) with $\mathcal{B} = I$ and $f(t) = k(t)C_1y$, $t \in [0, \tau)$, resp. a strong solution of (262) with $\mathcal{B} = I$ and $f(t) = k(t)C_2x$, $t \in [0, \tau)$, provided additionally in the last case that $R_2(t)x \in D(\mathcal{A})$, $t \in [0, \tau)$ and $R_2(t)\mathcal{A}x \subseteq \mathcal{A}R_2(t)x$, $t \in [0, \tau)$. Furthermore, it is very simple to transmit the assertions of [**292**, Proposition 2.8.8, Proposition 2.8.9] to mild (a, k)-regularized (C_1, C_2)-existence and uniqueness families subgenerated by multivalued linear operators:

PROPOSITION 3.2.8. (i) *Suppose that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$, as well as that $(R_2(t))_{t\in[0,\tau)}$ is locally equicontinuous and the functions $a(t)$ and $k(t)$ are kernels on $[0, \tau)$. Then $C_2R_1(t) = R_2(t)C_1$, $t \in [0, \tau)$.*

(ii) *Suppose that $(R_2(t))_{t\in[0,\tau)}$ is a locally equicontinuous mild (a, k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$. Then every strong solution $u(t)$ of (262) with $\mathcal{B} = I$ and $\mathcal{F} = f \in C([0, \tau) : X)$ satisfies*

$$(R_2 * f)(t) = (kC_2 * u)(t), \quad 0 \leqslant t < \tau. \tag{281}$$

Furthermore, the problem (262) has at most one pre-solution provided, in addition, that the functions $a(t)$ and $k(t)$ are kernels on $[0, \tau)$ and the function $\mathcal{F}(t)$ is single-valued.

The first part of following theorem is an extension of [**292**, Theorem 2.1.28(ii)] and its validity can be verified with the help of proof of [**395**, Theorem 2.7], Lemma 1.2.2 and Theorem 1.2.3; the second part of theorem is an extension of [**292**, Proposition 2.1.31] and can be shown by the arguments contained in the proof of [**438**, Theorem 2.5], along with Lemma 1.2.2.

THEOREM 3.2.9. (i) *Suppose that $(R(t))_{t\in[0,\tau)}$ is a locally equicontinuous (a, k)-regularized C-resolvent family generated by $\mathcal{A}$, the equation (272) holds for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein with $R(\cdot)$ and C, respectively, $k(t)$ is a kernel on $[0, \tau)$, u, $f \in C([0, \tau) : X)$, and (281) holds with $R_2(\cdot)$ and C_2 replaced therein with $R(\cdot)$ and C, respectively. Then $u(t)$ is a solution of the abstract Volterra inclusion (262) with $\mathcal{B} = I$ and $\mathcal{F} = f$.*

(ii) *Suppose that the functions $a(t)$ and $k(t)$ are kernels on $[0, \tau)$, and $\mathcal{A}$ is a closed* MLO *in X. Consider the following assertions:*

(a) *$\mathcal{A}$ is a subgenerator of a locally equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying the equation* (272) *for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein by $R(\cdot)$ and C, respectively.*

(b) *For every $x \in X$, there exists a unique solution of* (262) *with $\mathcal{B} = I$ and $\mathcal{F}(t) = f(t) = k(t)Cx$, $t \in [0,\tau)$.*

Then (a) $\Rightarrow$ (b). *If, in addition, X is a Fréchet space, then the above are equivalent.*

It is noteworthy that some additional conditions ensure the validity of implication (b) $\Rightarrow$ (a) in complete locally convex spaces. We will explain this fact for the problem $(\text{DFP})_{\mathbf{L}}$, where after integration we have $a(t) = g_\alpha(t)$. Assume that there exists a unique solution of problem $(\text{DFP})_{\mathbf{L}}$ with $\mathcal{B} = I$, $\mathcal{F}(t) \equiv 0$, $x_0 \in C(D(\mathcal{A}))$ and $x_j = 0$, $1 \leqslant j \leqslant \lceil\alpha\rceil - 1$. If, in addition to this, X is complete, $\mathcal{A}$ is closed, $C\mathcal{A} \subseteq \mathcal{A}C$ and for each seminorm $p \in \circledast$ and $T > 0$ there exist $q \in \circledast$ and $c > 0$ such that $p(u(t;Cx)) \leqslant cq(x)$, $x \in D(\mathcal{A})$, $t \in [0,T]$, then the arguments used in non-degenerate case show that $\mathcal{A}$ is a subgenerator of a locally equicontinuous (g_α, C)-resolvent family $(R_\alpha(t))_{t\geqslant 0}$. The proof of following complex characterization theorem for (a,k)-regularized C-resolvent families is left to the reader as an easy exercise.

THEOREM 3.2.10. *Let $\omega_0 > \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$, and let $\mathcal{A}$ be a closed* MLO *in X. Assume that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective and $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)\mathcal{A})$. If there exists a function $\Upsilon \colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to L(X)$ which satisfies:*

(i) *$\Upsilon(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$,*

(ii) *the mapping $\lambda \mapsto \Upsilon(\lambda)x$, $\operatorname{Re}\lambda > \omega_0$ is analytic for every fixed $x \in X$,*

(iii) *there exists $r \geqslant -1$ such that the family $\{\lambda^{-r}\Upsilon(\lambda) : \operatorname{Re}\lambda > \omega_0\} \subseteq L(X)$ is equicontinuous,*

then, for every $\alpha > 1$, $\mathcal{A}$ is a subgenerator of a global $(a, k * g_{\alpha+r})$-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$ which satisfies that the family $\{e^{-\omega_0 t}R_\alpha(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous. Furthermore, $(R_\alpha(t))_{t\geqslant 0}$ is a mild $(a, k * g_{\alpha+r})$-regularized C-existence family having $\mathcal{A}$ as subgenerator.

In the first part of following example, we will briefly explain how one can use multiplication operators for construction of local integrated semigroups generated by multivalued operators; in the second part of example, we will apply the complex characterization theorem for proving the existence of a very specific exponentially equicontinuous, convoluted fractional resolvent family.

EXAMPLE 3.2.11. (i) (cf. also [**28**, Example 4.4(c)]) Suppose that $1 \leqslant p \leqslant \infty$, $X := L^p(1,\infty)$, $1 < a < b < \infty$, $J := [a,b]$, $m_b(x) := \chi_J(x)$ and $m_a(x) := x + ie^x$ $(x > 1)$. Consider the multiplication operators $A \colon D(A) \to X$ and $B \in L(X)$, where $D(A) := \{f(x) \in X : (x+ie^x)f(x) \in X\}$, $Af(x) := (x + ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ $(x > 1$, $f \in X)$. Then it is very simple to prove that, for every $\alpha \in (0,1)$, the resolvent set of the multivalued linear operator $\mathcal{A} := B^{-1}A$ contains the exponential region $E(\alpha,1) := \{x+iy : x \geqslant 1,\ |y| \leqslant e^{\alpha x}\}$, as well as that $(\lambda - \mathcal{A})^{-1}f(x) = (\lambda B - A)^{-1}Bf(x) = m_b(x)f(x)/\lambda m_b(x) - m_a(x)$ for $x > 1$, $f \in X$. Furthermore, the operator $\mathcal{A}$ generates a local once integrated semigroup $(S_1(t))_{t\in[0,1]}$, given by

$$(S_1(t)f)(x) = \begin{cases} (x+ie^x)^{-1}[e^{t(x+ie^x)} - 1]f(x), & t \in [0,1],\ x \notin J,\ f \in X, \\ 0, \quad t \in [0,1],\ x \in J,\ f \in X. \end{cases}$$

(ii) Put $X := \{f \in C^\infty([0,\infty)) : \lim_{x\to+\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$ and $||f||_k := \sum_{j=0}^{k} \sup_{x\geqslant 0} |f^{(j)}(x)|$, $f \in X$, $k \in \mathbb{N}_0$. Then the topology induced by these norms turns X into a Fréchet space (cf. also [**292**, Example 2.4.6(ii)]). Let $\alpha \in (0,1)$ and $J = [a,b] \subseteq [0,\infty)$ be such that $\overline{\Sigma_{\alpha\pi/2}} \cap \{x+ie^x : x \in J\} = \emptyset$, and let $m_b \in C^\infty([0,\infty))$ satisfy $0 \leqslant m_b(x) \leqslant 1$, $x \geqslant 0$, $m_b(x) = 1$, $x \notin J$ and $m_b(x) = 0$, $x \in [a+\varepsilon, b-\varepsilon]$ for some $\varepsilon > 0$. As in the first part of this example, we use the multiplication operators $A\colon D(A) \to X$ and $B \in L(X)$, where $D(A) = \{f(x) \in E : (x+ie^x)f(x) \in X\}$, $Af(x) := (x+ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ $(x \geqslant 0,\ f \in X)$. In a recent research study with S. Pilipović and D. Velinov [**354**], we have shown that A cannot be the generator of any local integrated semigroup in X, as well as that A generates an ultradistribution semigroup of Beurling class. Set $\mathcal{A} := B^{-1}A$. We will prove that there exists a sufficiently large number $\omega > 0$ such that for each $s > 1$ and $d > 0$ the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \operatorname{Re}\lambda > \omega,\ \lambda \in \Sigma_{\alpha\pi/2}\} \subseteq L(X)$ is equicontinuous, which immediately implies by Theorem 3.2.10 that $\mathcal{A}$ generates an exponentially equicontinuous $(g_\alpha, \mathcal{L}^{-1}(e^{-d|\lambda|^{\alpha/s}}))$-regularized resolvent family. It is clear that the resolvent of $\mathcal{A}$ will be given by $(\lambda-\mathcal{A})^{-1}f(x) = (\lambda B-A)^{-1}Bf(x) = m_b(x)f(x)/\lambda m_b(x)-m_a(x)$ for $x \geqslant 0$, $f \in X$. Since $m_b(x)f(x)/\lambda m_b(x) - m_a(x) = 1/\lambda - (x+ie^x)$ for $x \notin J$, our first task will be to estimate the derivatives of function $1/\lambda - (\cdot + ie^{\cdot})$ outside the interval J. In order to do that, observe first that any complex number $\lambda \in \mathbb{C} \smallsetminus S$, where $S := \{x+ie^x : x \geqslant 0\}$, belongs to the resolvent set of A and

$$(\lambda - A)^{-1}f(x) = \frac{f(x)}{\lambda - (x+ie^x)}, \quad \lambda \in \mathbb{C} \smallsetminus S,\ x \geqslant 0.$$

Fix, after that, numbers $s > 1$, $d > 0$, $a > 0$, $b > 1$ satisfying that $x - \ln(((x-b)/a)^s + 1) \geqslant 1$, $x \geqslant b$. Set $\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant a|\operatorname{Im}\lambda|^{1/s} + b\}$ and denote by Γ the upwards oriented boundary of the region Ω. Inductively, we can prove that for each number $n \in \mathbb{N}$ there exist complex polynomials $P_j(z) = \sum_{l=0}^{j} a_{j,l}z^l$ $(1 \leqslant j \leqslant n)$ such that $\operatorname{dg}(P_j) = j$, $|a_{j,l}| \leqslant (n+1)!$ $(1 \leqslant j \leqslant n,\ 0 \leqslant l \leqslant j)$ and

$$\frac{d^n}{dx^n}(\lambda - (x+ie^x))^{-1} = \sum_{j=1}^{n} (\lambda - (x+ie^x))^{-j-1}P_j(e^x), \quad x \geqslant 0,\ \lambda \in \mathbb{C} \smallsetminus S. \tag{282}$$

Suppose $\lambda \in \Omega$ and $x \geqslant 0$. If $|\operatorname{Im}\lambda - e^x| \geqslant 1$, then we have the following estimate

$$\begin{aligned}(283)\qquad \frac{e^{2jx}}{(\operatorname{Re}\lambda - x)^{2k} + (\operatorname{Im}\lambda - e^x)^{2k}} &\leqslant \frac{e^{2jx}}{(\operatorname{Im}\lambda - e^x)^{2k}} \\ &\leqslant 2^{2j}(1 + |\operatorname{Im}\lambda|)^{2j}, \quad k \in \mathbb{N}_0,\ 0 \leqslant j < k.\end{aligned}$$

If $|\operatorname{Im}\lambda - e^x| < 1$, then $\operatorname{Im}\lambda > 0$, $0 \leqslant x < \ln(\operatorname{Im}\lambda + 1)$, and

$$\begin{aligned}(284)\qquad \frac{e^{2jx}}{(\operatorname{Re}\lambda - x)^{2k} + (\operatorname{Im}\lambda - e^x)^{2k}} &\leqslant \frac{e^{2jx}}{(\operatorname{Re}\lambda - x)^{2k}} \\ \leqslant \frac{(\operatorname{Im}\lambda + 1)^{2j}}{\operatorname{Re}\lambda - \ln(((\operatorname{Re}\lambda - b)/a)^s + 1)} &\leqslant (\operatorname{Im}\lambda + 1)^{2j}, \quad k \in \mathbb{N}_0,\ 0 \leqslant j < k.\end{aligned}$$

Let $\omega' > 0$ be such that $\{\lambda \in \Sigma_{\alpha\pi/2} : \operatorname{Re}\lambda > \omega'\} \subseteq \Omega$. Combining (282)–(284), it can be simply proved that for each number $n \in \mathbb{N}$ there exists a finite constant $c_n > 0$ such that

$$(285)\qquad \sum_{k=0}^{n} \sup_{x \geqslant 0, x \notin J} \left| \frac{d^n}{dx^n} (\lambda - (x + ie^x))^{-1} \right| \leqslant c_n e^{d|\lambda|^{1/s}}, \quad \lambda \in \Sigma_{\alpha\pi/2},\ \operatorname{Re}\lambda > \omega'.$$

We can similarly prove an estimate of type (285) for the derivatives of function $(\lambda m_b(x) - (x + ie^x))^{-1}$ on the interval J, which is well-defined for $\lambda \in \Sigma_{\alpha\pi/2}$ due to our assumption $0 \leqslant m_b(x) \leqslant 1$, $x \geqslant 0$ and the condition $\overline{\Sigma_{\alpha\pi/2}} \cap \{x + ie^x : x \in J\} = \emptyset$. In actual fact, induction shows that for each number $n \in \mathbb{N}$ there exist numbers $a_{j,l_1,\ldots,l_s}$ such that $|a_{j,l_1,\ldots,l_s}| \leqslant (n + 1)!$ $(1 \leqslant j \leqslant n,\ 0 \leqslant l \leqslant j)$ and

$$\begin{aligned}(286)\qquad \frac{d^n}{dx^n}(\lambda m_b(x) - (x + ie^x))^{-1} &= \sum_{j=1}^{n} (\lambda m_b(x) - (x + ie^x))^{-j-1} \\ \times \sum_{l=0}^{j} a_{j,l_1,\ldots,l_s} \prod_{l_1 m_1 + \cdots + l_s m_s = n} &(\lambda m_b^{(l_j)}(x) - m_a^{(l_j)}(x))^{m_j}, \quad x \in J,\ \lambda \in \Sigma_{\alpha\pi/2}.\end{aligned}$$

Since $d :=\operatorname{dist}(\overline{\Sigma_{\alpha\pi/2}}, \{x + ie^x : x \in J\})$ is a positive real number and $|(\lambda m_b^{(l_j)}(x) - m_a^{(l_j)}(x))^{m_j}| \leqslant c^{m_j}|\lambda|^{m_j}$ for all $\lambda \in \Sigma_{\alpha\pi/2}$ with $\operatorname{Re}\lambda > \omega$, where the number $\omega > \omega'$ is sufficiently large, (286) shows that for each number $n \in \mathbb{N}$ there exists a finite number $c_n' > 0$ such that

$$(287)\qquad \sum_{k=0}^{n} \sup_{x \geqslant 0, x \in J} \left| \frac{d^n}{dx^n} (\lambda m_b(x) - (x + ie^x))^{-1} \right| \leqslant c_n' e^{d|\lambda|^{1/s}}, \quad \lambda \in \Sigma_{\alpha\pi/2},\ \operatorname{Re}\lambda > \omega.$$

By (285) and (287), we have that the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \lambda \in \Sigma_{\alpha\pi/2},\ \operatorname{Re}\lambda > \omega\} \subseteq L(X)$ is equicontinuous, as claimed.

Now we would like to tell something more about the importance of condition $k(0) \neq 0$ in part (ii) of subsequent theorem. If all the necessary requirements hold, the arguments contained in the proof of [**285**, Theorem 3.6] imply the existence of a global $(a, k * g_1)$-regularized C-resolvent family $(R_1(t))_{t\geqslant 0}$ subgenerated by $\mathcal{A}$, which additionally satisfies that for each $t \geqslant 0$ the operator $R_1(t)\mathcal{A}$ is single-valued on $D(\mathcal{A})$. Then it is necessary to differentiate the equality $R_1(t)x - (k * g_1)(t)Cx = \int_0^t a(t-s)R_1(s)\mathcal{A}x\,ds$, $t \geqslant 0$, $x \in D(\mathcal{A})$ and employ the fact that $(\frac{d}{dt}R_1(t)x)_{t=0} = k(0)Cx$ $(x \in \overline{D(\mathcal{A})})$ (cf. the proof of [**285**, Theorem 3.6], as well as the proofs of [**199**, Proposition 2.1] and [**292**, Proposition 2.1.7]) in order to see that the function $R\colon D(R) \equiv \{\tilde{a}(\lambda)^{-1} : \lambda > b,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \to L(\overline{D(\mathcal{A})})$, given by $R(\tilde{a}(\lambda)^{-1}) := (\tilde{a}(\lambda)^{-1} - \mathcal{A})^{-1}C$, $\lambda \in D(R)$, is a C-pseudoresolvent in the sense of [**384**, Definition 3.1], satisfying additionally that $N(R(\lambda)) = \{0\}$, $\lambda \in D(R)$. Only after that, we can use [**384**, Theorem 3.4] with a view to prove the existence of a single-valued linear operator A, with domain and range contained in $\overline{D(\mathcal{A})}$, which satisfies the properties required in (ii): this examination shows the full importance of concepts introduced in Definition 3.2.1 and Definition 3.2.2 in integrated and convoluted case $k(0) = 0$. Keeping in mind Theorem 1.2.4(i) as well as the proofs of [**285**, Theorem 3.6] and [**292**, Theorem 1.2.6], the remaining parts of following theorem can be deduced, more or less, as in non-degenerate case.

THEOREM 3.2.12. *Suppose* $\omega \in \mathbb{R}$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, $\mathcal{A}$ *is a closed* MLO *in* X, $\lambda_0 \in \rho_C(\mathcal{A})$, $b \geqslant \max(0, \omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$,

$$\Big\{\frac{1}{\tilde{a}(\lambda)} : \lambda > b,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\Big\} \subseteq \rho_C(\mathcal{A}),$$

the function $H\colon D(H) \equiv \{\lambda > b : \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \to L(X)$, *given by* $H(\lambda)x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$, $x \in X$, $\lambda \in D(H)$, *satisfies that the mapping* $\lambda \mapsto H(\lambda)x$, $\lambda \in D(H)$ *is infinitely differentiable for every fixed* $x \in X$ *and, for every* $p \in \circledast$, *there exist* $c_p > 0$ *and* $r_p \in \circledast$ *such that:*

$$p\Big(l!^{-1}(\lambda-\omega)^{l+1}\frac{d^l}{d\lambda^l}H(\lambda)x\Big) \leqslant c_p r_p(x), \quad x \in X,\ \lambda \in D(H),\ l \in \mathbb{N}_0. \tag{288}$$

Then, for every $r \in (0,1]$, *the operator* $\mathcal{A}$ *is a subgenerator of a global* $(a, k * g_r)$-*regularized* C-*resolvent family* $(R_r(t))_{t\geqslant 0}$ *satisfying that, for every* $p \in \circledast$,

$$p(R_r(t+h)x - R_r(t)x) \leqslant \frac{2c_p r_p(x)}{r\Gamma(r)}\max(e^{\omega(t+h)}, 1)h^r, \quad t \geqslant 0,\ h > 0,\ x \in X,$$

and that, for every $p \in \circledast$ *and* $B \in \mathcal{B}$, *the mapping* $t \mapsto p_B(R_r(t))$, $t \geqslant 0$ *is locally Hölder continuous with exponent* r; *furthermore,* $(R_r(t))_{t\geqslant 0}$ *is a mild* $(a, k * g_r)$-*regularized* C-*existence family having* $\mathcal{A}$ *as subgenerator, and the following holds:*

- (i) *Suppose that* $\mathcal{A}$ *is densely defined. Then* $\mathcal{A}$ *is a subgenerator of a global* (a, k)-*regularized* C-*resolvent family* $(R(t))_{t\geqslant 0} \subseteq L(X)$ *satisfying that the family* $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(X)$ *is equicontinuous. Furthermore,* $(R(t))_{t\geqslant 0}$ *is a mild* (a, k)-*regularized* C-*existence family having* $\mathcal{A}$ *as subgenerator.*
- (ii) *Suppose that* $k(0) \neq 0$. *Then the operator* $C' := C_{|\overline{D(\mathcal{A})}} \in L(\overline{D(\mathcal{A})})$ *is injective,* $\mathcal{A}0$ *is a closed subspace of* X, $\overline{D(\mathcal{A})} \cap \mathcal{A}0 = \{0\}$, *and we have*

the following: Define the operator $A\colon D(A) \subseteq \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$ *by*

$$D(A) := \{x \in \overline{D(\mathcal{A})} : Cx = (\lambda_0 - \mathcal{A})^{-1}Cy \text{ for some } y \in \overline{D(\mathcal{A})}\}$$

and

$$Ax := C^{-1}\mathcal{A}Cx, \quad x \in D(A).$$

Then A *is a well-defined single-valued closed linear operator in* $\overline{D(\mathcal{A})}$, *and moreover,* A *is the integral generator of a global* (a,k)*-regularized* C'*-resolvent family* $(S(t))_{t\geqslant 0} \subseteq L(\overline{D(\mathcal{A})})$ *satisfying that the family* $\{e^{-\omega t}S(t) : t \geqslant 0\} \subseteq L(\overline{D(\mathcal{A})})$ *is equicontinuous,* $A\int_0^t a(t-s)S(s)x\,ds = S(t)x - k(t)Cx$, $t \in [0,\tau)$, $x \in \overline{D(\mathcal{A})}$ *and* $R_1(t)x = \int_0^t S(s)x\,ds$, $t \geqslant 0$, $x \in \overline{D(\mathcal{A})}$.

In the following extension of [**395**, Proposition 2.5] and [**292**, Proposition 2.1.4(ii)], we will revisit the condition $k(0) \neq 0$ from Theorem 3.2.12 once more. A straightforward proof is omitted.

PROPOSITION 3.2.13. *Let* $\mathcal{A}$ *be a closed subgenerator of a mild* (a,k)*-regularized* C_1*-resolvent family* $(R_1(t))_{t\in[0,\tau)}$ *(mild* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$*;* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$*). If* $k(t)$ *is absolutely continuous and* $k(0) \neq 0$*, then* $\mathcal{A}$ *is a subgenerator of a mild* (a,g_1)*-regularized* C_1*-resolvent family* $(R_1(t))_{t\in[0,\tau)}$ *(mild* (a,g_1)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$*;* (a,g_1)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$*).*

Now we would like to present some illustrative applications of results obtained so far.

EXAMPLE 3.2.14. Let $\alpha \in (0,1)$.

(i) [**199**] Consider the following time-fractional analogue of homogeneous counterpart of problem [**199**, Example 2.1, (2.18)]:

$$(P)_{m,\alpha} : \begin{cases} \mathbf{D}_t^{\alpha}[m(x)v_\alpha(t,x)] = -\frac{\partial}{\partial x}v_\alpha(t,x), & t \geqslant 0,\ x \in \mathbb{R}; \\ m(x)v_\alpha(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Let $X = Y := L^2(\mathbb{R})$, and let the operator $A := -d/dx$ act on X with its maximal distributional domain $H^1(\mathbb{R})$.

(a) Suppose first that $(Bf)(x) := \chi_{(-\infty,a)\cap(b,\infty)}(x)f(x)$, $x \in \mathbb{R}$ $(f \in X)$, where $-\infty < a < b < \infty$. Then $B \in L(X)$, $B = B^*$, $B^2 = B$ and $(P)_{m,\alpha}$ is formulated in X in the following abstract form

$$(P)'_{m,\alpha} : \begin{cases} B^*\mathbf{D}_t^{\alpha}Bv_\alpha(t) = \mathbf{D}_t^{\alpha}Bv_\alpha(t) = Av_\alpha(t), & t \geqslant 0; \\ Bv_\alpha(0) = u_0. \end{cases}$$

Further on, the multivalued linear operator $\mathcal{A} := (B^*)^{-1}AB^{-1}$ is maximal dissipative in the sense of [**199**, Definition, p. 35] and $\|(\lambda - \mathcal{A})^{-1}\| \leqslant \lambda^{-1}$, $\lambda > 0$. By the foregoing, we know that the operator $\mathcal{A}$ is single-valued on $\overline{D(\mathcal{A})}$; with a little abuse of notation, we will denote by $T \subseteq \mathcal{A}$ the single-valued linear operator which generates a bounded strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on $\overline{D(\mathcal{A})}$ (cf.

Theorem 3.2.12(ii), where we have denoted this operator by A). Using [**285**, Theorem 3.6(a)] and the consideration from the paragraph directly preceding the formulation of [**199**, Theorem 2.8], it readily follows that $D(T) = D(\mathcal{A})$. Suppose now that $u_0 = Bv_0$, where $v_0 \in D(A)$ and $Av_0 \in R(B^*)$, i.e., that $u_0 \in D(\mathcal{A}) = D(T)$ (cf. the proof of [**199**, Theorem 2.10]). Due to [**199**, Theorem 2.8, Theorem 2.10], the problem $(P)'_{m,1}$, with $\alpha = 1$, has a unique solution $v_1(t)$ satisfying $Bv_1(t) = T(t)u_0$; moreover,

$$(d/dt)Bv_1(t) = B^*(d/dt)Bv_1(t) \\ = Av_1(t) = (d/dt)T(t)u_0 = T(t)Tu_0, \quad t \geqslant 0. \tag{289}$$

Since the condition [**199**, (2.14)] holds, we get that there exists $\lambda_0 > 0$ such that $(\lambda_0 B - A)^{-1} \in L(X)$; hence, $v_1(\cdot) = (\lambda_0 B - A)^{-1}(\lambda_0 B - A)v_1(\cdot) \in C([0,\infty) : X)$ is bounded, as well as $(d/dt)Bv_1(t)$, $Bv_1(t)$ and $Av_1(t)$ are continuous and bounded for $t \geqslant 0$. Define $v_\alpha(t) := \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})v_1(s)\,ds,\ t > 0$ and $v_\alpha(0) := v_1(0)$. Using Theorem 3.1.8 and the arguments contained in its proof, it readily follows that the function $v_\alpha(\cdot)$ is a bounded solution of problem $(P)'_{m,\alpha}$, satisfying in addition that the functions $t \mapsto v_\alpha(\cdot)$, $t > 0$ and $t \mapsto Av_\alpha(\cdot)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$. The uniqueness of solutions of problem $(P)'_{m,\alpha}$ can be proved with the help of Theorem 3.1.6.

(b) Suppose now that $(Bf)(x) := \chi_{(a,\infty)}(x)f(x)$, $x \in \mathbb{R}$ ($f \in X$), where $-\infty < a < \infty$. Then $B \in L(X)$, $B = B^*$, $B^2 = B$ and the conclusions established in part (a) of this example, ending with the equation (289), continue to hold. In our concrete situation, we have the validity of condition [**199**, (2.11)] but not the condition [**199**, (2.14)], in general. Define $f_\alpha(t) := \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})Bv_1(s)\,ds,\ t > 0$, $f_\alpha(0) := Bv_1(0) = u_0$, $h_\alpha(t) := \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})Av_1(s)\,ds,\ t > 0$ and $h_\alpha(0) := Av_1(0)$. By the foregoing, we have that $f_\alpha,\ h_\alpha \in C([0,\infty) : X)$ are bounded and $\mathbf{D}_t^\alpha f_\alpha(t) = h_\alpha(t)$, $t \geqslant 0$, which simply implies $Bh_\alpha(t) = h_\alpha(t)$, $t \geqslant 0$. By (289), we have that $Av_1(t) = T(t)Tu_0 \in B^{-1}[Av_1(t)]$ and $BAv_1(t) = Av_1(t)$ ($t \geqslant 0$), whence we may conclude that $Av_1(t) \in \mathcal{A}[Bv_1(t)]$ ($t \geqslant 0$). Since $\mathcal{A}$ is closed, an application of Theorem 1.2.3 yields that $h_\alpha(t) = Bh_\alpha(t) \in AB^{-1}f_\alpha(t)$ ($t \geqslant 0$); consequently, the function $t \mapsto f_\alpha(t)$, $t \geqslant 0$ is a pre-solution of problem $(\mathrm{DFP})_R$ with $B \equiv I$, $\mathcal{F}(t) \equiv 0$ and, by Remark 3.1.2(iv), the problem $(P)'_{m,\alpha}$ has a bounded p-solution $v_\alpha(\cdot)$ satisfying, in addition, that the functions $t \mapsto Bv_\alpha(\cdot)$, $t > 0$ and $t \mapsto Av_\alpha(\cdot)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$. The uniqueness follows again from an essential application of Theorem 3.1.6.

(ii) [**280, 285**] Here we would like to observe, without going into full details, that we can similarly prove some results on the existence and uniqueness of analytical solutions of the abstract Volterra equation

$$\frac{\partial}{\partial r}v_\alpha(t,r)=a(r)\int_0^t g_\alpha(t-s)v_\alpha(s,r)ds+f(t,r),\quad t\geqslant 0,\ r\in[0,1],$$

on the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$, where $a\in C^1[0,1]$ and the mapping $t\mapsto f(t,\cdot)$, $t\geqslant 0$ is continuous and exponentially bounded with the values in the Banach space $C[0,1]$ (cf. [**280**, Example 1] and Theorem 3.1.8); using Theorem 3.1.9 instead of Theorem 3.1.8, we can consider the well-posedness in $C[0,1]$ for the equation

$$\frac{\partial}{\partial r}v_c(t,r)=a(r)\int_0^t c(t-s)v_c(s,r)ds+f(t,r),\quad t\geqslant 0,\ r\in[0,1],$$

where $c(\cdot)$ is a completely positive function.

(iii) Fractional Maxwell's equations have gained much attention in recent years (see e.g. [**128**, **253**, **404**, **516**, **562**] and references cited therein for more details on the subject). Here we want to briefly explain how we can use the analysis of A. Favini and A. Yagi [**199**, Example 2.2] for proving the existence and uniqueness of analytical solutions of certain classes of inhomogeneous abstract time-fractional Maxwell's equations in $\mathbb{R}^3$; the time-fractional analogues of Poisson-wave equations (see e.g. [**199**, Example 2.3, Example 6.23]) will be considered somewhere else.

Consider the following abstract time-fractional Maxwell's equations:

$$\mathrm{rot}E=-\mathbf{D}_t^\alpha B,\quad \mathrm{rot}H=\mathbf{D}_t^\alpha D+J \tag{290}$$

in $\mathbb{R}^3$, where E (resp. H) denotes the electric (resp. magnetic) field intensity, B (resp. D) denotes the electric (resp. magnetic) flux density, and where J is the current density. It is assumed that the medium which fills the space $\mathbb{R}^3$ is linear but possibly anisotropic and nonhomogeneous, which means that $D=\varepsilon E$, $B=\mu H$ and $J=\sigma E+J'$ with some 3×3 real matrices $\varepsilon(x)$, $\mu(x)$, $\sigma(x)$ ($x\in\mathbb{R}^3$) and J' being a given forced current density. Let any component of $\varepsilon(x)$, $\mu(x)$, $\sigma(x)$ be a bounded, measurable function in $\mathbb{R}^3$, let the conditions [**199**, (2.23)–(2.25)] hold, and let $f(t)=-(J'(\cdot,t)\ 0)^T$. Then we can formulate the problem (290) in the following abstract form

$$(P)_1:\begin{cases}B^*\mathbf{D}_t^\alpha Bv_1(t)=Av_1(t)+f(t),\quad t\geqslant 0;\\ Bv_1(0)=u_0,\end{cases}$$

in the space $X:=\{L^2(\mathbb{R}^3)\}^6$, using the bounded self-adjoint operator B of multiplication by $\sqrt{c(x)}$ acting in X, and A being the closed linear operator in X given by [**199**, (2.27)]. In our concrete situation, the conditions [**199**, (2.10) and (2.14)] hold, so that the assumptions $f\in C^2([0,\infty):X)$ and $u_0=Bv_0$ for some $v_0\in D(A)$ satisfying $Av_0+f(0)\in R(B^*)$ ensure by [**199**, Corollary 2.11] that the problem $(P)_1$ has a unique strict solution $v_1(\cdot)$ in the sense of equation [**199**, (2.13)]. Suppose, additionally, that the function $f''(t)$ is exponentially bounded.

Then we can use [**199**, Theorem 2.5], the proof of [**199**, Corollary 2.11] and the arguments from part (i)/(a) of this example in order to see that the solution $v_1 \in C([0,\infty) : X)$ is exponentially bounded, as well as that $H(t) := (d/dt)Bv_1(t)$, $Bv_1(t)$ and $Av_1(t)$ are continuous and exponentially bounded for $t \geqslant 0$. Define $v_\alpha(t)$ and $f_\alpha(t)$ as before, $H_\alpha(t) := \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})H(s)\,ds$, $t > 0$ and $H_\alpha(0) := H(0)$. Performing the Laplace transform, it can be simply verifed that $(g_{1-\alpha} * (Bv_\alpha - u_0))(t) = \int_0^t H_\alpha(s)\,ds$, $t \geqslant 0$, so that $\mathbf{D}_t^\alpha Bv_\alpha(t)$ exists and equals to $H_\alpha(t)$. On the other hand, we have $B^*Bv_1(t) = A(g_1 * v_1)(t) + B^*u_0 + \int_0^t f(s)\,ds$, $t \geqslant 0$, so that $B^*Bv_\alpha(t) = A(g_\alpha * v_\alpha)(t) + B^*u_0 + \int_0^t f_\alpha(s)\,ds$, $t \geqslant 0$ by Theorem 3.1.8. This implies $\mathbf{D}_t^\alpha B^*Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t)$ and, since $\mathbf{D}_t^\alpha Bv_\alpha(t)$ exists, $B^*\mathbf{D}_t^\alpha Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t)$, $t \geqslant 0$. Clearly, $Bv_\alpha(0) = u_0$ so that $v_\alpha \in C([0,\infty) : X)$ is an exponentially bounded solution of problem

$$(P)_\alpha : \begin{cases} B^*\mathbf{D}_t^\alpha Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t), & t \geqslant 0; \\ Bv_\alpha(0) = u_0, \end{cases}$$

that is analytically extensible on the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$ and satisfies, in addition, that the mapping $Av_\alpha \in C([0,\infty) : X)$ is exponentially bounded and analytically extensible on the same sector, as well. The uniqueness of solutions of problem (P_α) follows from Theorem 3.1.6.

We end this example with the observation that Theorem 3.1.8 and Theorem 3.1.9 can be successfully applied in the analysis of a large class of abstract degenerate Volterra integro-differential equations that are subordinated, in a certain sense, to degenerate differential equations of first and second order for which we know that are well posed ([**199**, **205**, **210**, **475**, **509**, **517**, **518**]).

Concerning the adjoint type theorems, it should be noticed that the assertions of [**292**, Theorem 2.1.12(i)/(ii); Theorem 2.1.13] continue to hold for (a,k)-regularized C-resolvent families subgenerated by closed multivalued linear operators. Furthermore, it is not necessary to assume that the operator $\mathcal{A}$ is densely defined if analyzing [**292**, Theorem 2.1.12(i)].

Suppose now that $\mathcal{A}$ is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $n \in \mathbb{N}$ and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leqslant j \leqslant n$. Then we can prove inductively that, for every $t \in [0,\tau)$,

$$R(t)x = k(t)Cx_0 + \sum_{j=1}^{n-1}(a^{*,j} * k)(t)Cx_j + (a^{*,n} * R(\cdot)x_n)(t). \tag{291}$$

Keeping in mind the identity (291), Theorem 1.2.3 and Proposition 3.2.8(ii), it is almost straightforward to transfer the assertion of [**292**, Proposition 2.1.32] to degenerate case:

PROPOSITION 3.2.15. (i) *Suppose* $\alpha \in (0,\infty) \smallsetminus \mathbb{N}$, $x \in D(\mathcal{A})$ *as well as* $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\tau) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \in [0,\tau)$ *and* $\mathcal{A}$ *is a closed subgenerator of a* (g_α, C)*-regularized resolvent family* $(R(t))_{t\in[0,\tau)}$. *Set*

$v(t) := (g_{\lceil\alpha\rceil-\alpha} * f)(t)$, $t \in [0,\tau)$. If $v \in C^{\lceil\alpha\rceil-1}([0,\tau) : X)$ and $v^{(k)}(0) = 0$ for $1 \leqslant k \leqslant \lceil\alpha\rceil - 2$, then the function $u(t) := R(t)x + (R * C^{-1}f)(t)$, $t \in [0,\tau)$ is a unique solution of the following abstract time-fractional inclusion:

$$\begin{cases} u \in C^{\lceil\alpha\rceil}((0,\tau) : X) \cap C^{\lceil\alpha\rceil-1}([0,\tau) : X), \\ \mathbf{D}_t^{\alpha} u(t) \in \mathcal{A}u(t) + \frac{d^{\lceil\alpha\rceil-1}}{dt^{\lceil\alpha\rceil-1}}(g_{\lceil\alpha\rceil-\alpha} * f)(t), \quad t \in [0,\tau), \\ u(0) = Cx, \ u^{(k)}(0) = 0, \quad 1 \leqslant k \leqslant \lceil\alpha\rceil - 1. \end{cases}$$

(ii) *Suppose $r \geqslant 0$, $n \in \mathbb{N} \setminus \{1\}$, $x_0 = x$, $f_0(\cdot) = f(\cdot)$, $x_j \in \mathcal{A}x_{j-1}$ for $1 \leqslant j \leqslant n$, $f_j(t) \in \mathcal{A}f_{j-1}(t)$ for $t \in [0,\tau)$ and $1 \leqslant j \leqslant n$, $f_j \in C([0,\tau) : X)$ for $0 \leqslant j \leqslant n$, and $\mathcal{A}$ is a closed subgenerator of a $(g_{1/n}, g_{r+1})$-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. Then the function $v(t) := R(t)x + (R * C^{-1}f)(t)x$, $t \in [0,\tau)$ is a unique solution of the following abstract time-fractional inclusion:*

$$\begin{cases} v \in C^1((0,\tau) : X) \cap C([0,\tau) : X), \\ v'(t) \in \mathcal{A}v(t) + \sum_{j=1}^{n-1} g_{(j/n)+r}(t)Cx_j + \sum_{j=0}^{n-1} (g_{(j/n)+r} * f_j)(t) \\ \qquad\qquad + \frac{d}{dt} g_{r+1}(t)Cx, \quad t \in (0,\tau), \\ v(0) = g_{r+1}(0)Cx. \end{cases}$$

Furthermore, $v \in C^1([0,\tau) : X)$ provided that $r \geqslant 1$ or $x = 0$ and $r \geqslant 0$.

3.2.1. Differential and analytical properties of (a,k)-regularized C-resolvent families. The main structural characterizations of differential and analytical (a,k)-regularized C-resolvent families generated by single-valued linear operators continue to hold in our framework (cf. [**75**, **76**, **199**, Chapter III] and [**182**] for some references on infinitely differentiable semigroups generated by MLOs).

The notions of various types of analyticity of degenerate (a,k)-regularized C-resolvent families are introduced in the following definition.

Definition 3.2.16. (cf. [**292**, Definition 2.2.1] for non-degenerate case)

(i) Suppose that $\mathcal{A}$ is an MLO in X. Let $\alpha \in (0,\pi]$, and let $(R(t))_{t\geqslant 0}$ be an (a,k)-regularized C-resolvent family which do have $\mathcal{A}$ as a subgenerator. Then it is said that $(R(t))_{t\geqslant 0}$ is an analytic (a,k)-regularized C-resolvent family of angle α, if there exists a function $\mathbf{R} : \Sigma_\alpha \to L(X)$ which satisfies that, for every $x \in X$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\alpha$ is analytic as well as that:
   (a) $\mathbf{R}(t) = R(t)$, $t > 0$ and
   (b) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = R(0)x$ for all $\gamma \in (0,\alpha)$ and $x \in X$.

(ii) Let $(R(t))_{t\geqslant 0}$ be an analytic (a,k)-regularized C-resolvent family of angle $\alpha \in (0,\pi]$. Then it is said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle α, resp. equicontinuous analytic (a,k)-regularized C-resolvent family of angle α, if for every $\gamma \in (0,\alpha)$, there exists $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that the family $\{e^{-\omega_\gamma \operatorname{Re} z}\mathbf{R}(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $R(\cdot)$ and $\mathbf{R}(\cdot)$.

In the following example, we will treat a time-fractional analogue of the linearized fractional Benney–Luke equation in L^2-spaces and there we will meet some interesting examples of exponentially bounded, analytic fractional resolvent families of bounded operators whose angle of analyticity can be strictly greater than $\pi/2$; in our approach, we do not use neither multivalued linear operators nor relatively p-radial operators [**199**, **509**]. The method employed by G. A. Sviridyuk and V. E. Fedorov [**509**] for the usually considered Benney–Luke equation of first order can be very hepful for achieving the final conclusions stated in (i)–(ii), as well as for the concrete choice of the state space X_0 below (cf. also Example 2.3.49 and Example 2.3.53).

Example 3.2.17. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary, and Δ is the Dirichlet Laplacian in $X := L^2(\Omega)$, acting with domain $H^2(\Omega) \cap H_0^1(\Omega)$. By $\{\lambda_k\}$ $[= \sigma(\Delta)]$ we denote the eigenvalues of Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leqslant -\lambda_2 \ldots \leqslant -\lambda_k \leqslant \cdots \to +\infty$ as $k \to \infty$) numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal eigenfunctions. Then, for every $\zeta > 0$, we define the spectral fractional power $C_\zeta \in L(X)$ of $-\Delta$ by $C_\zeta \cdot := (-\Delta)^{-(\zeta)/2}\cdot := \sum_{k\geqslant 1}\langle \cdot, \phi_k\rangle(-\lambda_k)^{-(\zeta/2)}\phi_k$ (cf. [**484**] for more details). Then C_ζ is injective and $R(C_\zeta) =: D((-\Delta)^{\zeta/2}) = \{f \in L^2(\Omega) : \sum_{k\geqslant 1}|\langle f, \phi_k\rangle|^2(-\lambda_k)^\zeta < \infty\}$. Let $\lambda \in \sigma(\Delta)$, let $0 < \eta \leqslant 2$, and let $\alpha, \beta > 0$. Consider the following time-fractional analogue of the linearized Benney–Luke equation:

$$(P)_{\eta,f} : \begin{cases} (\lambda - \Delta)\mathbf{D}_t^\eta u(t,x) = (\alpha\Delta - \beta\Delta^2)u(t,x) + f(t,x), \quad t \geqslant 0, \ x \in \Omega, \\ \left(\frac{\partial^k}{\partial t^k}u(t,x)\right)_{t=0} = u_k(x), \quad x \in \Omega, \ 0 \leqslant k \leqslant \lceil \eta \rceil - 1, \\ u(t,x) = \Delta u(t,x) = 0, \quad t \geqslant 0, \ x \in \partial\Omega. \end{cases}$$

Denote by X_0 the vector space of those functions from X that are orthogonal to the eigenfunctions $\phi_k(\cdot)$ for $\lambda_k = \lambda$. Then X_0 is a closed subspace of X, and therefore, becomes the Banach space equipped with the topology inherited by the X-norm (cf. [**509**, Example 5.3.1, Theorem 5.3.2] for the case $\eta = 1$). On the other hand, the operators $A := \alpha\Delta - \beta\Delta^2$ and $B := \lambda - \Delta$, acting with maximal domains, are closed in $L^2(\Omega)$. Set $\theta := \min((\pi/\eta) - (\pi/2), \pi)$. Using the Parseval equality and the asymptotic expansion formulae for Mittag–Leffler functions, we can simply prove that the operator family $(T_\eta(z))_{z\in\Sigma_\theta\cup\{0\}} \subseteq L(X_0)$, given by

$$t \mapsto T_\eta(z)\cdot := \sum_{k|\lambda_k\neq\lambda} E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}z^\eta\Big)\langle \cdot, \phi_k\rangle\phi_k, \quad z \in \Sigma_\theta \cup \{0\},$$

is well-defined, provided $\eta \in (0,2)$. If $\eta = 2$, then we define $(T_2(t))_{t\geqslant 0} \subseteq L(X_0)$ in the same way as above; since $E_2(z^2) = \cosh(z)$, we have that, for every $t \geqslant 0$,

$$T_2(t)\cdot = \frac{1}{2}\sum_{k|\lambda_k\neq\lambda}[e^{it((\beta\lambda^2 - \alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}} + e^{-it((\beta\lambda^2 - \alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}}]\langle \cdot, \phi_k\rangle\phi_k$$

and that $(T_2(t))_{t\geqslant 0}$ is bounded in the uniform operator norm. Differentiating $T_2(t)$ term by term, it can be easily seen that the mapping $t \mapsto T_2(t)f$, $t \geqslant 0$ is continuously differentiable for any $f \in D((-\Delta)^{1/2}) \cap X_0$, and therefore, continuous. Since $D((-\Delta)^{1/2}) \cap X_0$ is dense in X_0 and $(T_2(t))_{t\geqslant 0}$ is bounded, we have that $(T_2(t))_{t\geqslant 0}$ is strongly continuous. Now we can proceed as in the proof of Theorem 3.1.8 in order to see that, for every $\eta \in (0,2)$, $(T_\eta(t))_{t\geqslant 0}$ is an exponentially bounded, analytic (g_η, I)-regularized resolvent family of angle θ. A straightforward computation shows that, for every $\eta \in (0,2]$, the integral generator $\mathcal{A}$ of $(T_\eta(t))_{t\geqslant 0}$ is a closed single-valued operator in X_0, given by $\mathcal{A} = \{(f,g) \in X_0 \times X_0 : (\lambda - \lambda_k)\langle g, \phi_k\rangle = (\alpha\lambda_k - \beta\lambda_k^2)\langle f, \phi_k\rangle$ for all $k \in \mathbb{N}$ with $\lambda_k \neq \lambda\}$; in particular, $\mathcal{A}$ is an extension of the operator $B^{-1}A_{|X_0}$. It is also clear that $(T_\eta(t))_{t\geqslant 0}$ is a mild (g_η, I)-existence family generated by $\mathcal{A}$. Keeping in mind the identity [**61**, (1.25)], we can directly compute that the homogeneous counterpart of problem $(P)_{\eta,f} \equiv (P)_{\eta,0}$, with $x_j = 0$ for $1 \leqslant j \leqslant \lceil\zeta\rceil - 1$, has an exponentially bounded pre-solution $u_{h,0}(t) = T_\eta(t)x_0$, $t \geqslant 0$ for any $x_k \in D(A) \cap X_0$ $(0 \leqslant k \leqslant \lceil\eta\rceil - 1)$, which seems to be an optimal result in the case that $\eta \leqslant 1$. Concerning the homogeneous counterpart of problem $(P)_{\eta,0}$ with $x_0 = 0$, its solution $u_{h,1}(t)$ has to be found in the form $u_{h,1}(t) = \int_0^t T_\eta(s)x_1\,ds$, $t \geqslant 0$. Consider first the case $\eta \in (1,2)$. Then for each $k \in \mathbb{N}$ with $\lambda_k \neq \lambda$, we have

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big[g_{2-\eta} * \Big(E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\cdot^\eta\Big) - 1\Big)\Big](t) \\
&\qquad = \mathbf{D}_t^\eta E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}t^\eta\Big) = \frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}t^\eta\Big), \quad t \geqslant 0.
\end{aligned}$$

On the other hand, expanding the function $E_\eta(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\cdot^\eta) - 1$ in a power series we get that

$$\frac{d}{dt}\Big[g_{2-\eta} * \Big(E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\cdot^\eta\Big) - 1\Big)\Big](t) = t\sum_{n=0}^{\infty}\frac{(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}t^\eta)^{n+1}t^{n\eta}}{\Gamma(n\eta + 2)}, \quad t \geqslant 0.$$

The previous two equalities together imply that $\frac{d}{dt}[g_{2-\eta} * (E_\eta(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\cdot^\eta) - 1)](t) = \frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\int_0^t E_\eta(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}s^\eta)ds$, $t \geqslant 0$ and

$$\begin{aligned}
\mathbf{D}_t^\eta[g_1 * T_\eta(\cdot)x_1](t) &= \sum_{k|\lambda_k \neq \lambda}\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}\int_0^t E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}s^\eta\Big)ds\langle x_1, \phi_k\rangle\phi_k \\
&= \sum_{k|\lambda_k \neq \lambda}\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}tE_{\eta,2}\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k}t^\eta\Big)\langle x_1, \phi_k\rangle\phi_k, \quad t \geqslant 0.
\end{aligned}$$

Using again the asymptotic expansion formulae for Mittag-Leffler functions, we obtain that the above series converges for any $x_1 \in X_0$ and belongs to $D(B)$ provided, in addition, that $x_1 \in D(B) \cap X_0$. In this case, the equality $B\mathbf{D}_t^\eta u_{h,1}(t) = Au_{h,1}(t)$, $t \geqslant 0$ readily follows, so that the function $u_h(t) := u_{h,0}(t) + u_{h,1}(t)$, $t \geqslant 0$ is a pre-solution of problem $(\mathrm{DFP})_L$ provided that $x_0 \in D(A) \cap X_0$ and $x_1 \in D(B) \cap X_0$ (with $X = Y = L^2(\Omega)$ in Definition 3.1.1(iii)); furthermore, the mappings $t \mapsto u_h(t) \in L^2(\Omega)$, $t > 0$ and $t \mapsto Bu_h(t) \in L^2(\Omega)$, $t > 0$ can be analytically extended to the sector Σ_θ. The situation is slightly different in the case that $\eta = 2$; then a simple calculus shows that, formally,

$$Bu''_{h,1}(t) = Au_{h,1}(t) = \frac{1}{2} \sum_{k|\lambda_k \neq \lambda} [i((\beta\lambda^2 - \alpha\lambda_k)/(\lambda - \lambda_k))^{1/2} e^{it((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}}$$
$$- i((\beta\lambda^2 - \alpha\lambda_k)/(\lambda - \lambda_k))^{1/2} e^{-it((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}}]\langle x_1, \phi_k\rangle\phi_k, \quad t \geqslant 0.$$

Hence, the function $u_h(t) := u_{h,0}(t) + u_{h,1}(t)$, $t \geqslant 0$ is a pre-solution of problem $(\mathrm{DFP})_L$ with $x_0 \in D(A) \cap X_0$ and $x_1 \in D((-\Delta)^{3/2}) \cap X_0$. The range of any pre-solution of problem $(P)_{\eta,0}$ must be contained in X_0, so that the uniqueness of solutions of problem $(P)_{\eta,f}$ follows from its linearity and Proposition 3.2.8(ii).

Before dealing with the inhomogeneous problem $(P)_{\eta,f}$, we would like to observe that the assumptions $(x, y) \in \mathcal{A}$ and $x \in D(A)$ imply $(x, y) \in B^{-1}A_{|X_0}$. Keeping in mind this remark, Theorem 1.2.3, as well as the fact that the assertion of [**459**, Proposition 2.1(iii)] admits a reformulation in our framework, we can simply prove that for any function $h \in W^{1,1}_{loc}([0,\infty) : X_0)$ satisfying that

$$t \mapsto \sum_{k|\lambda_k \neq \lambda} (\alpha\lambda_k - \beta\lambda_k^2)\Big\langle \frac{d}{dt}(g_\eta * h)(t), \phi_k\Big\rangle\phi_k \in L^1_{loc}([0,\infty) : X_0), \tag{292}$$

the function $u_{Bh}(t) := \int_0^t T_\eta(t-s)\frac{d}{ds}(g_\eta * h)ds$, $t \geqslant 0$ is a solution of problem $(P)_{\eta,Bh}$. On the other hand, the operator B annihilates any function from $span\{\phi_k : k|\lambda = \lambda_k\}$ so that the function $t \mapsto \sum_{k|\lambda_k=\lambda} \frac{\langle f(t),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}\phi_k$, $t \geqslant 0$ is a pre-solution of problem $(P)_{\eta,\sum_{k|\lambda_k=\lambda}\langle f(\cdot),\phi_k\rangle\phi_k}$, provided that the following condition holds:

(Q) : $\mathbf{D}_t^\eta\langle f(t), \phi_k\rangle$ exists in $L^2(\Omega)$ for $k|\lambda = \lambda_k$, $\langle x_0, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $\langle x_1, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $1 < \eta \leqslant 2$, $\langle x_0, \phi_k\rangle = \frac{\langle f(0),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}$ for $k|\lambda = \lambda_k$, and $\langle x_1, \phi_k\rangle = \frac{\langle f'(0),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}$ for $k|\lambda = \lambda_k$, $1 < \eta \leqslant 2$.

Summa summarum, we have the following:

(i) $0 < \eta < 2$: Suppose that $x_0 \in D(A) \cap X_0$, $x_1 \in D(B) \cap X_0$, if $\eta > 1$, $\sum_{k|\lambda_k\neq\lambda} \frac{\langle f(\cdot),\phi_k\rangle}{\lambda-\lambda_k}\phi_k = h \in W^{1,1}_{loc}([0,\infty) : X_0)$ satisfies (292), and the condition (Q) holds. Then there exists a unique pre-solution of problem $(P)_{\eta,f}$.

(ii) $\eta = 2$: Suppose $x_1 \in D((-\Delta)^{3/2}) \cap X_0$ and the remaining assumptions from (i) hold. Then there exists a unique pre-solution of problem $(P)_{\eta,f}$.

Observe also that our results on the well-posedness of fractional analogue of the Benney–Luke equation, based on a very simple approach, are completely new provided that $\eta > 1$, as well as that we have obtained some new results on the well-posedness of the inhomogeneous Cauchy problem $P_{\eta,f}$ in the case that $\eta < 1$ (cf. [**210**, Theorem 4.2] for the first result in this direction).

The following theorem can be deduced by making use of the argumentation contained in the proof of [**295**, Theorem 2.16]. Here we would like to observe that the equality $R_{\lambda,\mu} = 0$, stated on [**295**, p. 12, l. 4], can be proved by taking the Laplace transform of term appearing on [**295**, p. 12, l. 1-2] in variable μ, and by using the strong analyticity of mapping $\lambda \mapsto F(\lambda) \in L(X)$, $\lambda \in N$, along with the equality $R_{\lambda,\mu} = 0$ for $\operatorname{Re}\lambda > \omega$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$ (the repeated use of identity [**295**, (2.30)] on [**295**, p. 12, l. 4] is wrong and makes a circulus vitiosus):

THEOREM 3.2.18 (cf. [**292**, Theorem 2.2.4] for non-degenerate case). *Suppose that $\alpha \in (0, \pi/2]$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, and $\tilde{k}(\lambda)$ can be analytically continued to a function $g\colon \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$, where $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$. Suppose, further, that $\mathcal{A}$ is a closed subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle α satisfying that the family $\{e^{-\omega z}R(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ is equicontinuous for all angles $\gamma \in (0,\alpha)$, as well as that the equation (272) holds for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein by $R(\cdot)$ and C, respectively. Set*

$$N := \{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha} : g(\lambda) \neq 0\}.$$

Then N is an open connected subset of $\mathbb{C}$. Assume that there exists an analytic function $\hat{a} : N \to \mathbb{C}$ such that $\hat{a}(\lambda) = \tilde{a}(\lambda)$, $\operatorname{Re}\lambda > \omega$. Then the operator $I - \hat{a}(\lambda)\mathcal{A}$ is injective for every $\lambda \in N$, $R(C) \subseteq R(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)$ for every $\lambda \in N_1 := \{\lambda \in N : \hat{a}(\lambda) \neq 0\}$, the operator $(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)^{-1}C \in L(X)$ is single-valued ($\lambda \in N_1$), the family

$$\{(\lambda - \omega)g(\lambda)(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)^{-1}C : \lambda \in \ N_1 \cap (\omega + \Sigma_{\frac{\pi}{2}+\gamma_1})\} \subseteq L(X)$$

is equicontinuous for every angle $\gamma_1 \in (0,\alpha)$, the mapping

$$\lambda \mapsto (I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)^{-1}Cx, \quad \lambda \in N_1 \text{ is analytic for every } x \in X,$$

and

$$\lim_{\lambda\to+\infty, \tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx = R(0)x, \quad x \in X.$$

Keeping in mind Lemma 1.2.2, Theorem 1.2.3 and Theorem 3.2.5, we can repeat almost verbatim the proof of [**292**, Theorem 2.2.5] in order to see that the following result holds.

THEOREM 3.2.19. *Assume that $\mathcal{A}$ is a closed* MLO *in X, $C\mathcal{A} \subseteq \mathcal{A}C$, $\alpha \in (0, \pi/2]$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$ and $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$. Assume, further, that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective with $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$. If there exist a function $q : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ and an operator $D \in L(X)$ such that, for every $x \in X$, the mapping $\lambda \mapsto q(\lambda)x$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}$ is analytic as well as that:*

$$q(\lambda)x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx, \quad \operatorname{Re}\lambda > \omega, \ \tilde{k}(\lambda) \neq 0, \ x \in X,$$

the family $\{(\lambda - \omega)q(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}\} \subseteq L(X)$ is equicontinuous for $\gamma \in (0,\alpha)$, and

$$\lim_{\lambda\to+\infty} \lambda q(\lambda)x = Dx, \quad x \in X, \ \text{if } \overline{D(\mathcal{A})} \neq X,$$

then $\mathcal{A}$ is a subgenerator of an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle α satisfying that $R(z)\mathcal{A} \subseteq \mathcal{A}R(z)$, $z \in$

Σ_{α}, the family $\{e^{-\omega z}R(z) : z \in \Sigma_{\gamma}\} \subseteq L(X)$ is equicontinuous for all angles $\gamma \in (0, \alpha)$, as well as that the equation (272) holds for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein by $R(\cdot)$ and C, respectively.

The classes of exponentially equicontinuous, analytic (a, k)-regularized C_1-existence families and (a, k)-regularized C_2-uniqueness families can be introduced and analyzed, as well. For the sequel, we need the following notion.

DEFINITION 3.2.20. Let $X = Y$, and let $\mathcal{A}$ be a subgenerator of a C_1-existence family $(R_1(t))_{t\geqslant 0}$ (cf. Definition 3.2.1(i) with $a(t) \equiv 1$ and $k(t) \equiv 1$). Then $(R_1(t))_{t\geqslant 0}$ is said to be entire iff, for every $x \in X$, the mapping $t \mapsto R_1(t)x$, $t \geqslant 0$ can be analytically extended to the whole complex plane.

Using the arguments contained in the proof of [**296**, Theorem 3.15], we can deduce the following result.

THEOREM 3.2.21. *Suppose $r \geqslant 0$, $\theta \in (0, \pi/2)$, $\mathcal{A}$ is a closed* MLO *and $-\mathcal{A}$ is a subgenerator of an exponentially equicontinuous, analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant 0}$ of angle θ. Then there exists an operator $C_1 \in L(X)$ such that $\mathcal{A}$ is a subgenerator of an entire C_1-existence family in X.*

REMARK 3.2.22. (i) It should be observed that we do not require the injectivity of operator C_1 here. The operators $T_{\alpha}(z)$ and $S_{\alpha,z_0}(z)$, appearing in the proof of [**296**, Theorem 3.15], annulates on the subspace $\mathcal{A}0$.

(ii) Theorem 3.2.21 is closely linked with the assertions of [**298**, Theorem 2.1, Theorem 2.2]. These results can be extended to abstract degenerate fractional differential inclusions, as well.

EXAMPLE 3.2.23. In a great number of research papers, many authors have investigated infinitely differentiable semigroups generated by multivalued linear operators of form AB^{-1} or $B^{-1}A$, where the operators A and B satisfy the condition [**199**, (3.14)], or its slight modification, with certain real constants $0 < \beta \leqslant \alpha \leqslant 1$, $\gamma \in \mathbb{R}$ and $c,\ C > 0$ (in our notation, we have $A = L$ and $B = M$). The validity of this condition with $\alpha = 1$ (see e.g. [**199**, Example 3.3, 3.6]) immediately implies by Theorem 3.2.19 and Remark 3.1.2(v) that the operator AB^{-1} generates an exponentially bounded, analytic σ-times integrated semigroup of angle $\Sigma_{\operatorname{arcctan}(1/c)}$, provided that $\sigma > 1-\beta$; in the concrete situation of [**199**, Example 3.4, 3.5], the above holds with the operator AB^{-1} replaced by $B^{-1}A$. Unfortunately, this fact is not sufficiently enough for taking up a fairly complete study of the abstract degenerate Cauchy problems that are subordinated to those appearing in the above-mentioned examples; later on, we will construct the corresponding subordination fractional operator families with removable singularities at zero and analyze their basic structural properties (cf. the proof of [**61**, Theorem 3.1]). On the other hand, from the point of view of possible applications of Theorem 3.2.21, it is very important to know that the operators AB^{-1} or $B^{-1}A$ generate exponentially bounded, analytic integrated semigroups. This enables us to consider the abstract degenerate Cauchy problems that are backward to those appearing in [**199**, Example 3.3-Example 3.6].

For example, we can treat the following modification of the backward Poisson heat equation in the space $L^p(\Omega)$:

$$(P)_b : \begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = -\Delta v + bv, \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$. Let B be the multiplication in $L^p(\Omega)$ with $m(x)$, and let $A = \Delta - b$ act with the Dirichlet boundary conditions. Then Theorem 3.2.21 implies that there exists an operator $C_1 \in L(L^p(\Omega))$ such that $\mathcal{A} = -AB^{-1}$ is a subgenerator of an entire C_1-existence family; hence, for every $u_0 \in R(C_1)$, the problem (P) has a unique solution $t \mapsto u(t)$, $t \geqslant 0$ which can be extended entirely to the whole complex plane. Furthermore, the set of all initial values u_0 for which there exists a solution of problem $(P)_b$ is dense in $L^p(\Omega)$ provided that there exists a constant $d > 0$ such that $|m(x)| \geqslant d$ a.e. $x \in \Omega$.

Example 3.2.24. It is clear that the examples presented in [**199**, Chapter III] can serve one for examination of a wide class of abstract degenerate relaxation equations which are not subordinated to the problems of first order (cf. Subsection 3.5.2 for the continuation): Suppose that the condition [**199**, (3.1)] holds with certain real constants $0 < \beta \leqslant \alpha \leqslant 1$, $c, M > 0$, as well as that $\theta \in (\pi/2, 0)$, $\zeta \in (0,1)$ and $\frac{\pi}{2} > \pi - \arctan\frac{1}{c} + \theta > \frac{1}{2}\pi\zeta$. Then $\Sigma_{\pi-\arctan\frac{1}{c}+\theta} \subseteq \rho(e^{i\theta}\mathcal{A})$ and, in general, $\rho(e^{i\theta}\mathcal{A})$ does not contain any right half plane. An application of Theorem 3.2.19 shows that the operator $e^{i\theta}\mathcal{A}$ generates an exponentially bounded, analytic (g_ζ, g_{r+1})-regularized resolvent family of angle $\theta' := \min((\pi - \arctan(1/c) + \theta - (\pi\zeta/2))/\zeta, \pi/2)$, where $r > \zeta(1-\beta)$, if $\mathcal{A}$ is not densely defined, and $r = \zeta(1-\beta)$, otherwise.

Suppose now that $x \in E$, $1 - \zeta > \eta > 1 - \zeta\beta$, $\delta > 0$, $0 < \gamma < \theta'$, $t > 0$ is fixed temporarily, $\Gamma_1 := \{re^{i((\pi/2)+\gamma)} : r \geqslant t^{-1}\} \cup \{t^{-1}e^{i\theta} : \theta \in [0, (\pi/2) + \gamma]\}$, $\Gamma_2 := \{re^{-i((\pi/2)+\gamma)} : r \geqslant t^{-1}\} \cup \{t^{-1}e^{i\theta} : \theta \in [-(\pi/2) - \gamma, 0]\}$ and $\Gamma := \Gamma_1 \cup \Gamma_2$ is oriented counterclockwise. Define $u(0) := 0$ and

$$u(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-\eta}(\lambda^\zeta - e^{i\theta}\mathcal{A})^{-1}x\,d\lambda.$$

Arguing as in [**27**, Theorem 2.6.1, Theorem 2.6.4], it readily follows that $u \in C([0,\infty) : E)$, $\|u(t)\| = O(t^{\eta+\zeta\beta-1})$, $t \geqslant 0$ and the mapping $t \mapsto u(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\theta'}$. Keeping in mind Theorem 1.2.3 and Theorem 1.2.4(i), we obtain that there exists a continuous section $t \mapsto u_{\mathcal{A},\theta,\zeta}(t)$, $t > 0$ of the multivalued mapping $t \mapsto e^{i\theta}\mathcal{A}(g_\zeta * u)(t)$, $t > 0$, with the meaning clear, such that

$$u(t) = u_{\mathcal{A},\theta,\zeta}(t) + g_{\eta+\zeta}(t)x, \quad t > 0.$$

Observe, finally, that the Riemann–Liouville fractional derivative $D_t^\zeta u(t)$ need not be defined here.

Let us recall that for any sequence $(M_n)_{n\in\mathbb{N}_0}$ of positive real numbers satisfying $M_0 = 1$, $(M.1)$, $(M.2)$ and $(M.3)'$, we define the function

$$\omega_L(t) := \sum_{n=0}^{\infty} \frac{t^n}{M_n}, \quad t \geqslant 0.$$

The most important results concerning differential properties of non-degenerate (a, k)-regularized C-resolvent families remain true, with almost minimal reformulations, in our new setting. The proofs of following extensions of [**292**, Theorem 2.2.15, Theorem 2.2.17] are omitted.

THEOREM 3.2.25. *Suppose that $\mathcal{A}$ is a closed* MLO *in X,* $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, *$r \geqslant -1$ and there exists $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$ such that, for every $z \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\}$, we have that the operator $I - \tilde{a}(z)\mathcal{A}$ is injective and $R(C) \subseteq R(I - \tilde{a}(z)\mathcal{A})$. If, additionally, for every $\sigma > 0$, there exist $C_\sigma > 0$ and an open neighborhood $\Omega_{\sigma,\omega}$ of the region*

$$\Lambda_{\sigma,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\},$$

and a function $h_\sigma : \Omega_{\sigma,\omega} \to L(X)$ such that, for every $x \in X$, the mapping $\lambda \mapsto h_\sigma(\lambda)x$, $\lambda \in \Omega_{\sigma,\omega}$ is analytic as well as that $h_\sigma(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\operatorname{Re}\lambda > \omega$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, and the family $\{|\lambda|^{-r}h_\sigma(\lambda) : \lambda \in \Lambda_{\sigma,\omega}\}$ is equicontinuous, then, for every $\zeta > 1$, $\mathcal{A}$ is a subgenerator of an exponentially equicontinuous $(a, k * g_{\zeta+r})$-regularized C-resolvent family $(R_\zeta(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto R_\zeta(t)$, $t > 0$ is infinitely differentiable in $L(X)$.

THEOREM 3.2.26. *Let $(M_n)_{n\in\mathbb{N}_0}$ satisfy (M.1), (M.2) and (M.3)'.*

(i) *Suppose that* $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, *$\mathcal{A}$ is a closed subgenerator of a (local) (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $\omega > \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$ and $m \in \mathbb{N}$. Denote, for every $\varepsilon \in (0, 1)$ and a corresponding $K_\varepsilon > 0$,*

$$F_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -\ln \omega_L(K_\varepsilon|\operatorname{Im}\lambda|) + \omega\}.$$

Assume that, for every $\varepsilon \in (0, 1)$, there exist $K_\varepsilon > 0$, an open neighborhood $O_{\varepsilon,\omega}$ of the region $G_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : \operatorname{Re}\lambda \leqslant \omega\}$, a mapping $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ and analytic mappings $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ such that:

(a) *$f_\varepsilon(\lambda) = \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega$; $g_\varepsilon(\lambda) = \tilde{a}(\lambda)$, $\operatorname{Re}\lambda > \omega$,*

(b) *for every $\lambda \in F_{\varepsilon,\omega}$, the operator $I - g_\varepsilon(\lambda)\mathcal{A}$ is injective and $R(C) \subseteq R(I - g_\varepsilon(\lambda)\mathcal{A})$,*

(c) *for every $x \in X$, the mapping $\lambda \mapsto h_\varepsilon(\lambda)x$, $\lambda \in G_{\varepsilon,\omega}$ is analytic, $h_\varepsilon(\lambda) = f_\varepsilon(\lambda)(I - g_\varepsilon(\lambda)\mathcal{A})^{-1}C$, $\lambda \in G_{\varepsilon,\omega}$,*

(d) *the family $\{(1 + |\lambda|)^{-m}e^{-\varepsilon|\operatorname{Re}\lambda|}h_\varepsilon(\lambda) : \lambda \in F_{\varepsilon,\omega},\ \operatorname{Re}\lambda \leqslant \omega\} \subseteq L(X)$ is equicontinuous and the family $\{(1 + |\lambda|)^{-m}h_\varepsilon(\lambda) : \lambda \in \mathbb{C},\ \operatorname{Re}\lambda \geqslant \omega\} \subseteq L(X)$ is equicontinuous.*

Then the mapping $t \mapsto R(t)$, $t \in (0,\tau)$ *is infinitely differentiable in* $L(X)$ *and, for every compact set* $K \subseteq (0,\tau)$, *there exists* $h_K > 0$ *such that the set* $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{M_n} : t \in K,\ n \in \mathbb{N}_0\}$ *is equicontinuous.*

(ii) *Suppose that* $\mathrm{abs}(k) < \infty$, $\mathrm{abs}(|a|) < \infty$, $\mathcal{A}$ *is a closed subgenerator of a* (*local*) (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$, $\omega > \max(0, \mathrm{abs}(k), \mathrm{abs}(|a|))$ *and* $m \in \mathbb{N}$. *Denote, for every* $\varepsilon \in (0,1)$, $\rho \in [1,\infty)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geqslant -K_\varepsilon |\mathrm{Im}\,\lambda|^{1/\rho} + \omega\}.$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $K_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geqslant \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega,\rho} : \mathrm{Re}\,\lambda \leqslant \omega\}$, *a mapping* $h_\varepsilon : O_{\varepsilon,\omega} \to L(X)$ *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *such that the conditions* (i)(a)–(d) *of this theorem hold with* $F_{\varepsilon,\omega}$, *resp.* $G_{\varepsilon,\omega}$, *replaced by* $F_{\varepsilon,\omega,\rho}$, *resp.* $G_{\varepsilon,\omega,\rho}$. *Then the mapping* $t \mapsto R(t)$, $t \in (0,\tau)$ *is infinitely differentiable in* $L(X)$ *and, for every compact set* $K \subseteq (0,\tau)$, *there exists* $h_K > 0$ *such that the set* $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{n!^\rho} : t \in K,\ n \in \mathbb{N}_0\}$ *is equicontinuous.*

Let us recall that the case $\rho = 1$ in Theorem 3.2.26 is very important because it gives a sufficient condition for an (a,k)-regularized C-resolvent family to be real analytic.

Following J. Prüss [**459**, Definition 3.1, p. 68] and the author [**292**, Definition 2.1.23], it will be said that the abstract Volterra inclusion (262) with $\mathcal{B} = I$ (denoted henceforth by (VI)) is (kC)-parabolic iff the following holds:

(i) $|a|(t)$ and $k(t)$ satisfy (P1) and there exist meromorphic extensions of the functions $\tilde{a}(\lambda)$ and $\tilde{k}(\lambda)$ on $\mathbb{C}_+$, denoted by $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$. Let N be the subset of $\mathbb{C}_+$ which consists of all zeroes and possible poles of $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$.

(ii) There exists $M \geqslant 1$ such that, for every $\lambda \in \mathbb{C}_+ \smallsetminus N$, $1/\hat{a}(\lambda) \in \rho_C(\mathcal{A})$ and $||\hat{k}(\lambda)(I - \hat{a}(\lambda)\mathcal{A})^{-1}C|| \leqslant M/|\lambda|$.

If $k(t) \equiv 1$, resp. $C = I$, then it is also said that (VI) is C-parabolic, resp. k-parabolic.

For the sequel, we need to repeat the following well-known notion. Suppose that $n \in \mathbb{N}$, $|a|(t)$ satisfies (P1) and $\mathrm{abs}(a) = 0$. Following [**459**, Definition 3.3, p. 69], we say that $a(t)$ is n-regular iff there exists $c > 0$ such that

$$|\lambda^m \hat{a}^{(m)}(\lambda)| \leqslant c|\hat{a}(\lambda)|, \quad \lambda \in \mathbb{C}_+,\ 1 \leqslant m \leqslant n.$$

Set $a^{(-1)}(t) := \int_0^t a(s)ds$, $t \geqslant 0$ and suppose that $a(t)$ and $b(t)$ are n-regular for some $n \in \mathbb{N}$. Then we know that $\hat{a}(\lambda) \neq 0$, $\lambda \in \mathbb{C}_+$, as well as that $(a * b)(t)$ and $a^{(-1)}(t)$ are n-regular, and $a'(t)$ is n-regular provided that $\mathrm{abs}(a') = 0$. Furthermore, the n-regularity of $a(t)$ is equivalent to say that there exists $c' > 0$ such that

$$|(\lambda^m \hat{a}(\lambda))^{(m)}| \leqslant c'|\hat{a}(\lambda)|, \quad \lambda \in \mathbb{C}_+,\ 1 \leqslant m \leqslant n.$$

In the case that $\arg(\hat{a}(\lambda)) \neq \pi$, $\lambda \in \mathbb{C}_+$, the n-regularity of $a(t)$ is also equivalent with the existence of a constant $c'' > 0$ such that

$$|\lambda^m (\ln \hat{a}(\lambda))^{(m)}| \leqslant c'', \quad \lambda \in \mathbb{C}_+, \ 1 \leqslant m \leqslant n.$$

The assertion of [**292**, Theorem 2.1.24] continues to hold in the degenerate case:

THEOREM 3.2.27. *Assume $n \in \mathbb{N}$, $a(t)$ is n-regular, $(E, \|\cdot\|)$ is a Banach space, $\mathcal{A}$ is a closed* MLO *in E,* (VI) *is C-parabolic, and the mapping $\lambda \mapsto (I-\tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\lambda \in \mathbb{C}_+$ is continuous. Denote by D_t^{ζ} the Riemann–Liouville fractional derivative of order $\zeta > 0$. Then, for every $\alpha \in (0,1]$, $\mathcal{A}$ is a subgenerator of an $(a, g_{\alpha+1})$-regularized C^2-resolvent family $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\sup_{h>0,t\geqslant 0} h^{-\alpha}||S_\alpha(t+h) - S_\alpha(t)|| < \infty$, $D_t^{\alpha} S_\alpha(t)C^{k-1} \in C^{k-1}((0,\infty) : L(E))$, $1 \leqslant k \leqslant n$ as well as:*

$$\|t^j D_t^j D_t^{\alpha} S_\alpha(t)C^{k-1}\| \leqslant M, \quad t \geqslant 0, \ 1 \leqslant k \leqslant n, \ 0 \leqslant j \leqslant k-1, \tag{293}$$

$$\begin{aligned} \|t^k D_t^{k-1} D_t^{\alpha} S_\alpha(t)C^{k-1} &- s^k D_s^{k-1} D_s^{\alpha} S_\alpha(s)C^{k-1}\| \\ &\leqslant M|t-s|\Big(1+\ln\frac{t}{t-s}\Big), \quad 0 \leqslant s < t < \infty, \ 1 \leqslant k \leqslant n, \end{aligned} \tag{294}$$

and, for every $T > 0$, $\varepsilon > 0$ and $k \in \mathbb{N}_n$, there exists $M_{T,k}^{\varepsilon} > 0$ such that

$$\begin{aligned} \|t^k D_t^{k-1} D_t^{\alpha} S_\alpha(t)C^{k-1} &- s^k D_s^{k-1} D_s^{\alpha} S_\alpha(s)C^{k-1}\| \\ &\leqslant M_{T,k}^{\varepsilon}(t-s)^{1-\varepsilon}, \quad 0 \leqslant s < t \leqslant T, \ 1 \leqslant k \leqslant n. \end{aligned} \tag{295}$$

Furthermore, if $\mathcal{A}$ is densely defined, then $\mathcal{A}$ is a subgenerator of a bounded (a, C^2)-regularized resolvent family $(S(t))_{t\geqslant 0}$ which satisfies $S(t)C^{k-1} \in C^{k-1}((0,\infty) : L(E))$, $1 \leqslant k \leqslant n$ and (293)–(295) *with $D_t^{\alpha} S_\alpha(t)C^{k-1}$ replaced by $S(t)C^{k-1}$ $(1 \leqslant k \leqslant n)$ therein.*

The representation formula [**459**, (3.41), p. 81] and the assertions of [**459**, Corollary 3.2-Corollary 3.3, pp. 74–75] can be extended to exponentially bounded (a, C)-regularized resolvent families subgenerated by multivalued linear operators, as well. For more details about parabolicity of problem (VI) in non-degenerate case, we refer the reader to [**459**, Chapter I, Section 3].

In the remainder of this section, we consider the abstract degenerate Volterra inclusion

$$0 \in \mathcal{B}u(t) + \sum_{j=0}^{n-1} \mathcal{A}_j(a_j * u)(t) + \mathcal{F}(t), \quad t \in [0,\tau). \tag{296}$$

Here, $n \in \mathbb{N}$, $0 < \tau \leqslant \infty$, $\mathcal{F}\colon [0,\tau) \to P(Y)$, $a_0, \ldots, a_{n-1} \in L^1_{loc}([0,\tau))$, and $\mathcal{A} \equiv \mathcal{A}_0, \ldots, \mathcal{A}_{n-1}, \mathcal{B} \equiv \mathcal{A}_n$ are multivalued linear operators acting between the sequentially complete locally convex spaces X and Y. Set $a_n(t) :=$ δ-distribution and $\tilde{\delta} := 1$. In the following definition, we introduce the notion of a $\mathcal{V}$-(pre)-solution of inclusion (296); see Introduction.

DEFINITION 3.2.28. Suppose $\mathcal{V} \subseteq \mathbb{N}_n^0$.

(i) A function $u \in C([0,\tau) : X)$ is said to be a $\mathcal{V}$-pre-solution of problem (296) iff there exist functions $u_j \in C([0,\tau) : Y)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ such that $u_j(t) \in \mathcal{A}_j u(t)$, $t \in [0,\tau)$, $j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$, $(a_j * u)(t) \in D(\mathcal{A}_j)$, $t \in [0,\tau)$, $j \in \mathcal{V}$ and

$$0 \in \sum_{j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}} (a_j * u_j)(t) + \sum_{j \in \mathcal{V}} \mathcal{A}_j (a_j * u)(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

(ii) A function $u \in C([0,\tau) : X)$ is said to be a $\mathcal{V}$-solution of problem (296) iff there exist functions $u_j \in C([0,\tau) : Y)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ and $u_{j,\mathcal{A}_j} \in C([0,\tau) : Y)$ $(j \in \mathcal{V})$ such that $u_j(t) \in \mathcal{A}_j u(t)$, $t \in [0,\tau)$, $j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$, $u_{j,\mathcal{A}_j}(t) \in \mathcal{A}_j(a_j * u)(t)$, $t \in [0,\tau)$, $j \in \mathcal{V}$ and

$$0 \in \sum_{j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}} (a_j * u_j)(t) + \sum_{j \in \mathcal{V}} u_{j,\mathcal{A}_j}(t) + \mathcal{F}(t), \quad t \in [0,\tau). \tag{297}$$

Suppose that $\mathcal{V} \subseteq \mathbb{N}_n^0$ and $u(t)$ is a $\mathcal{V}$-solution of problem (296). Then it is clear that $\mathrm{sec}_c(\mathcal{F}) \neq \emptyset$. Furthermore, if $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathbb{N}_n^0$, then we can simply prove with the help of Theorem 1.2.3 that any $\mathcal{V}_1$-(pre)-solution of problem (296) is a $\mathcal{V}_2$-(pre)-solution of (296), provided that the operators $\mathcal{A}_j$ are closed for $j \in \mathcal{V}_2 \smallsetminus \mathcal{V}_1$. If $\mathcal{V} = \emptyset$, then the notion of a pre-solution of (296) coincides with that of solution of (296); if this is the case, then any solution of (296) is also said to be a strong solution of (296). Here it is worth noting that the assumption $u \in C([0,\tau) : X)$ in Definition 3.2.28(i) is slightly redundant in the case that $\mathcal{V} = \emptyset$ because then we do not have a problem with defining the convolutions $(a_j * u)(t)$ for $j \in \mathcal{V}$. Assuming $\mathcal{V} = \emptyset$ and all the remaining assumptions from Definition 3.2.28(i) hold, $u(t)$ will be called a p-strong solution of problem (296). In the case that $n = 1$, the most important examples of p-strong solutions of problem (296) with $a_0(t) = g_\alpha(t)$, where $\alpha > 0$, are obtained by integrating α-times p-solutions of abstract degenerate fractional problem $(\mathrm{DFP})_R$.

Remark 3.2.29. Suppose that $A_0, \ldots, A_{n-1}, B$ are single-valued linear operators between the spaces X and Y, and the mapping $f\colon [0,\tau) \to D(B)$ is given in advance. Consider the following degenerate Volterra integral equation:

$$0 = Bu(t) + \sum_{j=0}^{n-1} A_j(a_j * u)(t) + Bf(t), \quad t \in [0,\tau), \tag{298}$$

which is, unquestionably, the most important subcase of (296). Along with this equation, we examine the accompanied problem of type (296):

$$0 \in u(t) + \sum_{j=0}^{n-1} B^{-1}A_j(a_j * u)(t) + f(t), \quad t \in [0,\tau), \tag{299}$$

with the multivalued linear operators $\mathcal{A}_0 := B^{-1}A_0, \ldots, \mathcal{A}_{n-1} := B^{-1}A_{n-1}, \mathcal{B} := I$ in X. Let $\mathcal{V} \subseteq \mathbb{N}_n^0$. Then the following holds:

(i) If $\mathcal{V} = \mathbb{N}_n^0$, then any $\mathcal{V}$-pre-solution of problem (299) is a $\mathcal{V}$-pre-solution of problem (298).

(ii) If $B \in L(X,Y)$, then any $\mathcal{V}$-(pre)-solution of problem (299) is a $\mathcal{V}$-(pre)-solution of problem (298).

REMARK 3.2.30. Suppose that $\tau = \infty$, $|a_j|(t)$ $(0 \leqslant j \leqslant n-1)$ satisfy (P1), and the MLOs $\mathcal{A}_j$ are closed for $0 \leqslant j \leqslant n$. Let $u \in C([0,\tau) : X)$, $u_j \in C([0,\tau) : Y)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$, and $u_{j,\mathcal{A}_j} \in C([0,\tau) : Y)$ $(j \in \mathcal{V})$ be Laplace transformable functions. Suppose that there exists a real number $\omega \geqslant \max(0, \operatorname{abs}(|a_j|))$ $(0 \leqslant j \leqslant n-1)$ such that $\widetilde{u_j}(\lambda) \in \mathcal{A}_j \tilde{u}(\lambda)$, $\lambda > \omega$, $j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$ and $\widetilde{u_{j,\mathcal{A}_j}}(\lambda) \in \mathcal{A}_j(\widetilde{a_j}(\lambda)\tilde{u}(\lambda))$, $\lambda > \omega$, $j \in \mathcal{V}$. Then an application of Proposition 1.4.7 yields that $u_j(t) \in \mathcal{A}_j u(t)$, $t \geqslant 0$, $j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$ and $u_{j,\mathcal{A}_j}(t) \in \mathcal{A}_j(a_j * u)(t)$, $t \geqslant 0$, $j \in \mathcal{V}$. If, in addition to this, there exists a continuous section $f \in \sec_c(\mathcal{F})$ that is Laplace transformable (not necessarily exponentially bounded) and

$$\sum_{j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}} \widetilde{a_j}(\lambda)\widetilde{u_j}(\lambda) + \sum_{j \in \mathcal{V}} \widetilde{u_{j,\mathcal{A}_j}}(\lambda) + \tilde{f}(\lambda) = 0, \quad \lambda > \omega,$$

then the uniqueness theorem for Laplace transform implies that $u(t)$ is a $\mathcal{V}$-solution of problem (296).

The proofs of following subordination principles for $\mathcal{V}$-solutions of problem (296) are standard and therefore omitted; it is worth noting that we can formulate similar results provided that the multivalued linear operators employed in our analysis are relatively closed.

THEOREM 3.2.31. *Suppose that $\tau = \infty$, $\gamma \in (0,1)$, $\omega > 0$, $|a_j|(t)$ and $|b_j|(t)$ $(0 \leqslant j \leqslant n-1)$ satisfy* (P1), *$b_n(t) :=$ δ-distribution and*

$$\widetilde{a_j}(\lambda^\gamma) := \widetilde{b_j}(\lambda), \quad \lambda > \omega.$$

Suppose that the MLOs *$\mathcal{A}_j$ are closed for $0 \leqslant j \leqslant n$. Let $u \in LT_{or} - X$ be a $\mathcal{V}$-solution of problem* (296)*, with $\mathcal{F} = f \in LT_{or} - Y$ being single-valued, and the functions $u_j \in C([0,\tau) : Y)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$, $u_{j,\mathcal{A}_j} \in C([0,\tau) : Y)$ $(j \in \mathcal{V})$, appearing in* (297)*, being Laplace transformable and exponentially bounded. Define*

$$u_\gamma(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})u(s)\,ds, \quad t > 0,\ u_\gamma(0) := u(0),$$

$$u_{j,\gamma}(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})u_j(s)\,ds, \quad t > 0,\ u_{j,\gamma}(0) := u_j(0),\ j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V},$$

$$u_{j,\mathcal{A}_j,\gamma}(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})u_{j,\mathcal{A}_j}(s)\,ds, \quad t > 0,\ u_{j,\mathcal{A}_j,\gamma}(0) := u_{j,\mathcal{A}_j}(0),\ j \in \mathcal{V},$$

$$f_\gamma(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})f(s)\,ds, \quad t > 0,\ f_\gamma(0) := f(0).$$

Then the function $u_\gamma(t)$ is an exponentially bounded $\mathcal{V}$-solution of problem (296)*, with the functions $a_j(t)$ replaced by $b_j(t)$ $(0 \leqslant j \leqslant n-1)$ and the inhomogeneity $\mathcal{F}(t)$ replaced by $f_\gamma(t)$. Furthermore, the functions $u_{j,\gamma} \in C([0,\tau) : Y)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ and $u_{j,\mathcal{A}_j,\gamma} \in C([0,\tau) : Y)$ $(j \in \mathcal{V})$ satisfy the requirements from* Definition 3.2.28(ii).

THEOREM 3.2.32. *Let $|a_j|(t)$, $|b_j|(t)$ and $c(t)$ satisfy* (P1), *let $\beta \geqslant 0$ be such that $\int_0^\infty e^{-\beta t}|b_j(t)|dt < \infty$, $j \in \mathbb{N}_{n-1}^0$, and let $\mathcal{A}_j$ be closed for $0 \leqslant j \leqslant n$; $b_n(t) := \delta$-distribution. Set*

$$\alpha := \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\beta},\ \alpha := 0 \text{ otherwise.}$$

Let $u^b \in LT_{or} - X$ be a $\mathcal{V}$-solution of problem (296), *with the function $\mathcal{F} = f^b \in LT_{or} - Y$ being single-valued, and let the set $\{e^{-\omega_b t}u_b(t) : t \geqslant 0\}$ be bounded in X ($\omega_b \geqslant 0$). Let $u_j^b \in LT_{or} - Y$ ($j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$) and $u_{j,\mathcal{A}_j}^b \in LT_{or} - Y$ ($j \in \mathcal{V}$) be the functions fulfilling the requirements from* Definition 3.2.28(ii). *Suppose that $c(t)$ is completely positive and*

$$\widetilde{b_j}(1/\tilde{c}(\lambda)) = \widetilde{a_j}(\lambda), \quad j \in \mathbb{N}_{n-1}^0,\ \lambda > \alpha. \tag{300}$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)dt > \frac{1}{\omega_b},\ \omega_a = 0 \text{ otherwise.}$$

Then, for every $r \in (0,1]$, there exist locally Hölder continuous (with the exponent $r \in (0,1]$) functions $u^{a,r} \in LT_{or}-X$, $u_j^{a,r} \in LT_{or}-Y$ ($j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$), $u_{j,\mathcal{A}_j}^{a,r} \in LT_{or}-Y$ ($j \in \mathcal{V}$) and $f^a \in LT_{or} - Y$ such that

$$\widetilde{u^{a,r}}(\lambda) = \frac{1}{\lambda^{1+r}\tilde{c}(\lambda)}\widetilde{u^b}(1/\tilde{c}(\lambda)), \quad \lambda > \alpha,$$

$$\widetilde{u_j^{a,r}}(\lambda) = \frac{1}{\lambda^{1+r}\tilde{c}(\lambda)}\widetilde{u_j^b}(1/\tilde{c}(\lambda)), \quad j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V},\ \lambda > \alpha,$$

$$\widetilde{u_{j,\mathcal{A}_j}^{a,r}}(\lambda) = \frac{1}{\lambda^{1+r}\tilde{c}(\lambda)}\widetilde{u_{j,\mathcal{A}_j}^b}(1/\tilde{c}(\lambda)), \quad j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V},\ \lambda > \alpha$$

and

$$\widetilde{f^{a,r}}(\lambda) = \frac{1}{\lambda^{1+r}\tilde{c}(\lambda)}\widetilde{f^b}(1/\tilde{c}(\lambda)) \text{ for } \lambda > \alpha.$$

Furthermore, for every $r \in (0,1]$, $u^{a,r}(t)$ is a $\mathcal{V}$-solution of problem (296), *with the function $\mathcal{F} = f^{a,r}$, as well as the functions $u_j^{a,r}(t)$ ($j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}$) and $u_{j,\mathcal{A}_j}^{a,r}(t)$ ($j \in \mathcal{V}$) satisfy the requirements from* Definition 3.2.28(ii), *and the set $\{e^{-\omega_a t}u^{a,r}(t) : t \geqslant 0\}$ is bounded in X, if $\omega_b = 0$ or $\omega_b\tilde{c}(0) \neq 1$, resp., the set $\{e^{-\varepsilon t}u^{a,r}(t) : t \geqslant 0\}$ is bounded in X for any $\varepsilon > 0$, if $\omega_b > 0$ and $\omega_b\tilde{c}(0) = 1$.*

It is worth noting that the mapping $t \mapsto u_\gamma(t)$, $t > 0$ (cf. Theorem 3.2.31) admits an extension to $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ and the mapping $z \mapsto u_\gamma(z)$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ is analytic. Furthermore, the solution $u_\gamma(t)$ possesses some extra properties clarified so many times before.

Assume that Z is a sequentially complete locally convex space. We would like to propose the following general definition.

DEFINITION 3.2.33. Suppose that $k \in C([0,\tau))$, $C_1 \in L(Z,Y)$ and $(E(t))_{t\in[0,\tau)} \subseteq L(Z,X)$ is a strongly continuous operator family.

(i) It is said that $(E(t))_{t\in[0,\tau)}$ is a pre-$(k, C_1, \mathcal{V})$-existence family for (296) iff for each $z \in Z$ there exist continuous sections $u_{j,z} \in \mathrm{sec}_c(\mathcal{A}_j E(\cdot)z)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ such that $(a_j * E(\cdot)z)(t) \in D(\mathcal{A}_j)$, $t \in [0,\tau)$, $j \in \mathcal{V}$, $z \in Z$ and

$$0 \in \sum_{j\in\mathbb{N}_n^0 \smallsetminus \mathcal{V}} (a_j * u_{j,z})(t) + \sum_{j\in\mathcal{V}} \mathcal{A}_j(a_j * E(\cdot)z)(t) + k(t)C_1 z, \quad t \in [0,\tau), \ z \in Z.$$

(ii) It is said that $(E(t))_{t\in[0,\tau)}$ is a $(k, C_1, \mathcal{V})$-existence family for (296) iff for each $z \in Z$ there exist continuous sections $u_{j,z} \in \mathrm{sec}_c(\mathcal{A}_j E(\cdot)z)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ and $u_{j,\mathcal{A}_j,z} \in \mathrm{sec}_c(\mathcal{A}_j(a_j * E(\cdot)z))$ such that

$$\sum_{j\in\mathbb{N}_n^0 \smallsetminus \mathcal{V}} (a_j * u_{j,z})(t) + \sum_{j\in\mathcal{V}} u_{j,\mathcal{A}_j,z}(t) + k(t)C_1 z = 0, \quad t \in [0,\tau), \ z \in Z. \tag{301}$$

If $\mathrm{card}(\mathcal{V}) \leqslant 1$, then any pre-$(k, C_1, \mathcal{V})$-existence family for (296) is automatically a $(k, C_1, \mathcal{V})$-existence family for (296). Moreover, if $(E(t))_{t\in[0,\tau)}$ is a (pre-) $(k, C_1, \mathcal{V})$-existence family for (296), then it is clear that, for every $z \in Z$, the mapping $u_z(t) := E(t)z$, $t \in [0,\tau)$ is a $\mathcal{V}$(-pre)-solution of problem (296) with $\mathcal{F}(t) := k(t)C_1 z$, $t \in [0,\tau)$. In our analyses of uniqueness of solutions of problem (296), it will be crucial for us to assume that $X = Y$. We will use the following definition.

Definition 3.2.34. Suppose that $k \in C([0,\tau))$, $C_2 \in L(X)$ is injective, $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is strongly continuous and $X = Y$. Then we say that $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family for (296) iff

$$U(t)x_n + \sum_{j=0}^{n-1} (a_j * U(\cdot)x_j) + k(t)C_2 x = 0, \quad t \in [0,\tau),$$

whenever $x \in \bigcap_{j=0}^{n} D(\mathcal{A}_j)$ and $x_j \in \mathcal{A}_j x$ $(0 \leqslant j \leqslant n)$.

The validity of following proposition can be simply verified.

Proposition 3.2.35. *Suppose that $k \in C([0,\tau))$, $k(t)$ is a kernel on $[0,\tau)$, $C_2 \in L(X)$ is injective, $(U(t))_{t\in[0,\tau)}$ is a locally equicontinuous k-regularized C_2-uniqueness family for* (296), *$X = Y$ and the function $\mathcal{F} = f\colon [0,\tau) \to Y$ is single-valued. Then the problem* (296) *has at most one strong solution.*

Observe that, in the formulation of Proposition 3.2.35, we do not require the closedness of multivalued linear operators $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}, \mathcal{B}$. Using Proposition 3.2.35, we can simply prove that there exists a unique $\mathcal{V}$-solution of problem (296), provided the closedness of all operators $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}, \mathcal{B}$ and the validity of conditions from Proposition 3.2.35.

The class of exponentially equicontinuous k-regularized C_2-uniqueness families for (123) can be simply characterized by using the vector-valued Laplace transform:

Theorem 3.2.36. *Let $k(t)$ and $|a_j|(t)$ $(0 \leqslant j \leqslant n-1)$ satisfy* (P1). *Suppose that $\tau = \infty$, $k \in C([0,\infty))$, $C_2 \in L(X)$ is injective, $(U(t))_{t\geqslant 0}$ is strongly continuous and the operator family $\{e^{-\omega t}U(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous for some* *real number* $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a_j|))$ $(0 \leqslant j \leqslant n-1)$. *Then* $(U(t))_{t\geqslant 0}$ *is a* k*-regularized* C_2*-uniqueness family for* (296) *iff the following holds:*

$$\int_0^\infty e^{-\lambda t} U(t)\Biggl[x_n + \sum_{j=0}^{n-1} \widetilde{a_j}(\lambda) x_j\Biggr] dt = -\tilde{k}(\lambda) C_2 x, \quad \operatorname{Re} \lambda > \omega,$$

whenever $x \in \bigcap_{j=0}^{n} D(\mathcal{A}_j)$ *and* $x_j \in \mathcal{A}_j x$ $(0 \leqslant j \leqslant n)$.

The proof of following subordination principle for k-regularized C_2-uniqueness families is omitted.

THEOREM 3.2.37. *Suppose that* $\tau = \infty$, $k, k_1 \in C([0,\infty))$, $C_2 \in L(X)$ *is injective,* $c(t)$ *is completely positive,* $c(t)$, $k(t)$, $k_1(t)$, $|a_0|(t), \ldots, |a_{n-1}|(t)$, $|b_0|(t), \ldots,$ $|b_{n-1}|(t)$ *satisfy* (P1), *and* $(U(t))_{t\geqslant 0}$ *is an exponentially equicontinuous* k*-regularized* C_2*-uniqueness family for problem* (296), *with the functions* $a_j(t)$ *replaced by functions* $b_j(t)$ *therein* $(0 \leqslant j \leqslant n)$. *Let* $\omega_0 > 0$ *be such that, for every* $\lambda > \omega_0$ *with* $\tilde{c}(\lambda) \neq 0$ *and* $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, (300) *holds. Then for each* $r \in (0, 1]$ *there exists a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(k_1 * g_r)$*-regularized* C_2*-uniqueness family for* (296).

On the other hand, the class of exponentially equicontinuous pre-$(k, C_1, \mathcal{V})$-existence families for (296) cannot be so simply characterized with the help of vector-valued Laplace transform. As the next theorem shows, this is not the case with the class of exponentially equicontinuous $(k, C_1, \mathcal{V})$-existence families for (296):

THEOREM 3.2.38. *Suppose that* $\tau = \infty$, $k \in C([0,\infty))$, $k(t)$ *and* $|a_j|(t)$ $(0 \leqslant j \leqslant n-1)$ *satisfy* (P1) *and* $(E(t))_{t\geqslant 0}$ *is a* $(k, C_1, \mathcal{V})$*-existence family for* (296). *Let* $\mathcal{A}_j$ *be closed for* $0 \leqslant j \leqslant n$. *Suppose that there exists a real number* $\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(|a_j|))$ $(0 \leqslant j \leqslant n-1)$ *such that the operator family* $\{e^{-\omega t} E(t) : t \geqslant 0\} \subseteq L(Z, X)$ *is equicontinuous, as well as that the continuous sections* $u_{j,z} \in \sec_c(\mathcal{A}_j E(\cdot) z)$ $(j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V})$ *and* $u_{j,\mathcal{A}_j,z} \in \sec_c(\mathcal{A}_j(a_j * E(\cdot)z))$, *appearing in* (301), *are exponentially bounded with the exponential growth bound less than or equal to* ω (*and the meaning clear*). *Set*

$$\mathcal{P}_\lambda := \sum_{j=0}^{n} \widetilde{a_j}(\lambda) \mathcal{A}_j, \quad \operatorname{Re} \lambda > \omega.$$

Then, for every $\lambda \in \mathbb{C}$ *with* $\operatorname{Re} \lambda > \omega$ *and* $\widetilde{a_j}(\lambda) \neq 0$ $(j \in \mathcal{V})$, *we have*

$$\tilde{k}(\lambda) C_1 z \in \mathcal{P}_\lambda \widetilde{E}(\lambda) z, \quad z \in Z.$$

PROOF. Performing the Laplace transform, we get:

$$\sum_{j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}} \widetilde{a_j}(\lambda) \widetilde{u_{j,z}}(\lambda) + \sum_{j \in \mathcal{V}} \widetilde{u_{j,\mathcal{A}_j,z}}(\lambda) + \tilde{k}(\lambda) C_1 z = 0, \quad \operatorname{Re} \lambda > \omega, \ z \in Z. \tag{302}$$

Since the operators $\mathcal{A}_j$ are closed for $0 \leqslant j \leqslant n$, we can apply Theorem 1.2.3 in order to see that

$$\widetilde{u_{j,z}}(\lambda) \in \mathcal{A}_j \widetilde{E}(\lambda) z \quad (j \in \mathbb{N}_n^0 \smallsetminus \mathcal{V}, \ \operatorname{Re} \lambda > \omega)$$

and

$$\widetilde{u_{j,\mathcal{A}_j,z}}(\lambda) \in \mathcal{A}_j \widetilde{(a_j * E(\cdot)z)}(\lambda) = \mathcal{A}_j[\tilde{a_j}(\lambda)\widetilde{E}(\lambda)z] \quad (j \in \mathcal{V},\ \operatorname{Re}\lambda > \omega).$$

Substituting this in (302) and using the fact that $\mathcal{A}_j[\tilde{a_j}(\lambda)\widetilde{E}(\lambda)z] = \tilde{a_j}(\lambda)\mathcal{A}_j\widetilde{E}(\lambda)z$ $(j \in \mathcal{V},\ \operatorname{Re}\lambda > \omega,\ \tilde{a_j}(\lambda) \neq 0)$, the result immediately follows. $\square$

The assertion of Theorem 2.3.33(i)/(b) can be extended to multivalued linear operators by assuming some very restrictive additional conditions. Contrary to this, the assertion of Theorem 2.3.32(i) admits a very simple reformulation in our context:

THEOREM 3.2.39. *Suppose that* $0 \leqslant \alpha_0 < \cdots < \alpha_n$ *and* $\mathcal{V} = \mathbb{N}_n^0$. *Let* $(E(t))_{t\in[0,\tau)}$ *be a* $(g_1, C_1, \mathcal{V})$*-existence family for* (296), *with* $a_j(t) := g_{\alpha_n-\alpha_j}(t)$ $(0 \leqslant j \leqslant n-1)$, *and let* $0 < T < \tau$. *Suppose, further, that* $\mathcal{A}_j$ *is closed for* $0 \leqslant j \leqslant n$. *Set* $m_i := \lceil \alpha_i \rceil$, $i \in \mathbb{N}_n^0$ *and, for every* $i \in \mathbb{N}_{m_n-1}^0$, $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geqslant i\}$, $D_i' := \{j \in \mathbb{N}_{n-1}^0 : m_j - 1 \geqslant i\}$, $D_i'' := \mathbb{N}_{n-1}^0 \smallsetminus D_i'$ *and*

$$\mathbf{D}_i := \Big\{u_i \in \bigcap_{j\in D_i''} D(\mathcal{A}_j) : \mathcal{A}_j u_i \cap R(C_1) \neq \emptyset,\ j \in D_i''\Big\}.$$

If $u_i \in \mathbf{D}_i$ *for* $0 \leqslant i \leqslant m_n-1$, *then we define* $u(t)$ *by* (119) *with the term* $E^{(m_n1-i)}(\cdot)$ *replaced by* $E(\cdot)$ *therein, and with elements* $v_{i,j} \in Z$ *satisfying* $\mathcal{A}_j u_i \cap C_1 v_{i,j} \neq \emptyset$ *for* $0 \leqslant j \leqslant n-1$. *Then the Caputo fractional derivative* $\mathbf{D}_t^{\zeta}u(t)$ *is defined for any number* $\zeta \in [0, \alpha_n]$ *and, for every* $j \in \mathbb{N}_n^0$, *there exists a continuous section* $u_{j,\alpha_j}(t)$ *of* $\mathcal{A}_j\mathbf{D}_t^{\alpha_j}(t)$ *such that*

$$\sum_{j=0}^{n} u_{j,\alpha_j}(t) = 0, \quad 0 \leqslant t \leqslant T.$$

Before we briefly describe how we can provide some illustrative examples of results obtained so far, it is worth noting that the assertion of Theorem 2.3.34, concerning the inhomogeneous problem (296) with single-valued linear operators $\mathcal{A}_j = A_j$ $(0 \leqslant j \leqslant n)$, cannot be extended to the multivalued linear operators case. Possible applications of Theorem 3.2.39 to matrix differential equations with Caputo fractional derivatives will be analyzed somewhere else.

1. In [**199**, Example 6.1], the authors have considered a damped Poisson-wave equation in the space $X := L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open domain with smooth boundary. Let $m(x) \in L^\infty(\Omega)$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, let Δ be the Dirichlet–Laplacian in $L^2(\Omega)$, acting with domain $H_0^1(\Omega) \cap H^2(\Omega)$, and let $A(x; D)$ be a second order linear differential operator on Ω with coefficients continuous on $\bar{\Omega}$. Using the analysis from the above-mentioned example (cf. also the proof of [**199**, Theorem 6.1]), it readily follows from Theorem 3.2.36 that there exists $\zeta > 0$ such that there exists an exponentially bounded g_ζ-regularized $\Delta(1-\Delta)^{-1}$-uniqueness family for problem (296), with $n = 2$, $\mathcal{B} = B \in L(X)$ being the scalar multiplication with $m(x)$, $\mathcal{A}_0 := -\Delta$, $\mathcal{A}_1 := A(x; D) := A$, $a_0(t) := g_1(t)$ and $a_1(t) := g_2(t)$. Applying Theorem 3.2.32 and Theorem 3.2.37, we obtain that for a

large class of inhomogeneities $f(t,x)$ there exists a unique solution of the abstract degenerate Volterra integral equation:

$$m(x)u(t,x) - \int_0^t c(t-s)\Delta u(s,x)ds \\ + A\int_0^t (c*c)(t-s)u(s,x)ds = f(t,x), \quad t \geqslant 0, \ x \in \Omega,$$

where $c(t)$ is a completely positive function. Observe, finally, that the subordination principles clarified in Theorem 3.2.31-Theorem 3.2.32 and Theorem 3.2.37 can be applied to many other problems from [**199**, Chapter VI].

2. Suppose that $\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous (g_δ, g_σ)-regularized C-resolvent family $(R_\delta(t))_{t\geqslant 0}$ $(0 < \delta \leqslant 2, \sigma \geqslant 1)$. Then the analyses contained in [**292**, Example 2.10.30(i)] and Remark 3.2.29, along with Theorem 3.2.36, can be used for proving some results on the existence and uniqueness of solutions of the abstract degenerate Volterra inclusion:

$$0 \in u(t) + \sum_{j=1}^{n-1} c_j (g_{\alpha_n - \alpha_j} * u)(t) - \mathcal{A}(g_{\alpha_n} * u)(t) + f(t), \quad t \geqslant 0,$$

where $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_n < \delta$ and $c_j \in \mathbb{C}$ $(1 \leqslant j \leqslant n-1)$.

3.2.2. Non-injectivity of regularizing operators C_2 and C. In this subsection, we consider multivalued linear operators as subgenerators of mild (a,k)-regularized (C_1,C_2)-resolvent operator families and (a,k)-regularized C-resolvent operator families. We use the same notion and notation as before but now we allow that the operators C_2 and C are possibly non-injective (see Definition 3.2.1–Definition 3.2.2 and Definition 3.2.34). Without any doubt, this choice has some obvious displeasing consequences on the uniqueness of corresponding abstract Volterra integro-differential inclusions (see e.g. Proposition 3.2.8(ii), Theorem 3.2.9(ii) and Proposition 3.2.35).

We assume that X and Y are two SCLCSs, $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A}\colon X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, C, $C_2 \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$. We define the integral generator $\mathcal{A}_{int}$ of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$; (a,k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$) in the same way as before. Then we have that $\mathcal{A}_{int} \subseteq C_2^{-1}\mathcal{A}_{int}C_2$ ($\mathcal{A}_{int} \subseteq C^{-1}\mathcal{A}_{int}C$) is still the maximal subgenerator of $(R_2(t))_{t\in[0,\tau)}$ ($(R(t))_{t\in[0,\tau)}$) with respect to the set inclusion and the local equicontinuity of $(R_2(t))_{t\in[0,\tau)}$ ($(R(t))_{t\in[0,\tau)}$) implies that $\mathcal{A}_{int}$ is closed; as the next illustrative example shows, $C^{-1}\mathcal{A}_{int}C$ need not be a subgenerator of $(R(t))_{t\in[0,\tau)}$ and the inclusion $C^{-1}\mathcal{A}_{int}C \subseteq \mathcal{A}_{int}$ is not true for resolvent operator families, in general:

EXAMPLE 3.2.40. Let $T(t) := C \in L(X)$ for all $t \geqslant 0$. Then $(T(t))_{t\geqslant 0}$ is a global C-regularized semigroup ($a(t) = k(t) = 1$) with the integral generator $\mathcal{A}_{int} = X \times N(C)$ and any MLO $\mathcal{A}$ satisfying $R(C) \times \{0\} \subseteq \mathcal{A}$ and $R(\mathcal{A}) \subseteq N(C)$ is a subgenerator of $(T(t))_{t\geqslant 0}$. In particular, $\mathcal{A} = R(C) \times N(C)$ is a subgenerator of $(T(t))_{t\geqslant 0}$, $C^{-1}\mathcal{A}C = C^{-1}\mathcal{A}_{int}C = X \times N(C^2)$ which is, in general, a proper extension of $\mathcal{A}_{int}$, and not a subgenerator of $(T(t))_{t\geqslant 0}$ provided that there exists an element $x \in X$ such that $C^2x = 0$ and $Cx \neq 0$. Observe, finally, that $(T(t))_{t\geqslant 0}$ is exponentially equicontinuous and $N(C) \subseteq (\lambda - \mathcal{A}_{int})^{-1}C0$ for all $\lambda > 0$, so that the operator $(\lambda - \mathcal{A}_{int})^{-1}C$ need not be single-valued in the case that C is not injective.

Suppose that $a(t)$ is a kernel on $[0, \tau)$, $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of an (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and $x \in D(\mathcal{A}) \cap D(\mathcal{B})$. Then $R(t)(y - z) = 0$, $t \in [0, \tau)$ for each $y \in \mathcal{A}x$ and $z \in \mathcal{B}x$. Furthermore, the local equicontinuity of $(R(t))_{t\in[0,\tau)}$ and the closedness of $\mathcal{A}$ imply that the inclusion (275) continues to hold without injectivity of C being assumed.

In the following definition, we introduce the notion of an (a, k, C)-subgenerator of any strongly continuous operator family $(Z(t))_{t\in[0,\tau)} \subseteq L(X)$. This definition extends the corresponding ones introduced by C.-C. Kuo [**366**, **367**, Definition 2.4] in the setting of Banach spaces, where it has also been assumed that the operator $\mathcal{A} = A$ is linear and single-valued.

DEFINITION 3.2.41. Let $0 < \tau \leqslant \infty$, $C \in L(X)$, $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, $k \in C([0, \tau))$ and $k \neq 0$. Suppose that $(Z(t))_{t\in[0,\tau)} \subseteq L(X)$ is a strongly continuous operator family. By an (a, k, C)-subgenerator of $(Z(t))_{t\in[0,\tau)}$ we mean any MLO $\mathcal{A}$ in X satisfying the following two conditions:

(i) $Z(t)x - k(t)Cx = \int_0^t a(t - s)Z(s)y\,ds$, whenever $t \in [0, \tau)$ and $y \in \mathcal{A}x$.

(ii) For all $x \in X$ and $t \in [0, \tau)$, we have $\int_0^t a(t - s)Z(s)x\,ds \in D(\mathcal{A})$ and $Z(t)x - k(t)Cx \in \mathcal{A}\int_0^t a(t - s)Z(s)x\,ds$.

The (a, k, C)-integral generator $\mathcal{A}_{int}$ of $(Z(t))_{t\in[0,\tau)}$ (integral generator, if there is no risk for confusion) is defined by

$$\mathcal{A}_{int} := \left\{(x, y) \in X \times X : Z(t)x - k(t)Cx = \int_0^t a(t-s)Z(s)y\,ds \text{ for all } t \in [0, \tau)\right\}.$$

If $\mathcal{A}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, then it is clear that $(Z(t))_{t\in[0,\tau)}$ is a mild (a, k)-regularized (C, C)-existence and uniqueness family which do have $\mathcal{A}$ as subgenerator. Since we have not assumed that $\mathcal{A}$ commutes with C or $(Z(t))_{t\in[0,\tau)}$, it does not follow automatically from Definition 3.2.41 that $(Z(t))_{t\in[0,\tau)}$ is an (a, k)-regularized C-resolvent family with subgenerator $\mathcal{A}$. By $\chi(Z)$ we denote the set consisting of all subgenerators of $(Z(t))_{t\in[0,\tau)}$. The local equicontinuity of $(Z(t))_{t\in[0,\tau)}$ yields that for each subgenerator $\mathcal{A} \in \chi(Z)$ we have $\bar{\mathcal{A}} \in \chi(Z)$. The set $\chi(Z)$ can have infinitely many elements; if $\mathcal{A} \in \chi(Z)$, then $\mathcal{A} \subseteq \mathcal{A}_{int}$ (cf. [**386**, Example 4.10, 4.11]; in these examples, the partially ordered set $(\chi_{sv}(Z), \subseteq)$, where $\chi_{sv}(Z)$ denotes the set consisting of all single-valued linear subgenerators of $(Z(t))_{t\in[0,\tau)}$, does not have the greatest element) and, if $\chi(Z)$ is finite, then it need not be a singleton [**291**]. In general, the set $\chi(Z)$ can be empty and the integral generator of $(Z(t))_{t\in[0,\tau)}$ need not be a subgenerator of $(Z(t))_{t\in[0,\tau)}$ in the case that $\tau<\infty$:

EXAMPLE 3.2.42. [**385**] Let $0<\tau<\infty$ and let $U\colon[\tau/2,\tau)\to L(X)$ be a strongly continuous function such that $U(\tau/2)=0$ and $U(t)$ is injective for all $t\in(\tau/2,\tau)$. Define $T:[0,\tau)\to L(X\times X)$by $T(t)(x\ y)^T:=(0\ y)^T$ for $t\in[0,\tau/2)$, $x,y\in X$ and $T(t)(x\ y)^T:=(U(t)x\ y)^T$ for $t\in[\tau/2,\tau)$, $x,y\in X$. Set $C:=T(0)$. Then C is not injective, $(T(t))_{t\in[0,\tau)}\subseteq L(X\times X)$ is a non-degenerate local C-regularized semigroup (cf. the next subsection for the notion) and it can be easily seen that the violation of condition

$$U(t)\int_{\tau/2}^{s}U(r)x\,dr=\int_{\tau/2}^{t}U(r)U(s)x\,dr,\quad x\in E,\ t,\ s\in[\tau/2,\tau)$$

implies that the set $\chi(T)$ is empty (in the MLO sense) as well as that the integral generator $\mathcal{A}_{int}$ of $(T(t))_{t\in[0,\tau)}$ is not a subgenerator of $(T(t))_{t\in[0,\tau)}$. We can similarly construct an example of a non-degenerate local C-regularized cosine function $(C(t))_{t\in[0,\tau)}$, with C being not injective, so that $(C(t))_{t\in[0,\tau)}$ does not have any subgenerator in the MLO sense [**487**]. Observe, finally, that the notion introduced in [**386**, Definition 4.3], [**531**, Definition 2.1] and [**291**, Remark 2.1.8(i)] cannot be used for proving the nonemptiness of set $\chi(Z)$ in the degenerate case.

If $\mathcal{A}$ and $\mathcal{B}$ are subgenerators of $(Z(t))_{t\in[0,\tau)}$, then for any complex numbers α,β such that $\alpha+\beta=1$ we have that $\alpha\mathcal{A}+\beta\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$. Set $\mathcal{A}\wedge B:=(1/2)\mathcal{A}+(1/2)\mathcal{B}$. We define the operator $\mathcal{A}\vee_0\mathcal{B}$ by $D(\mathcal{A}\vee_0\mathcal{B})$:=span$[D(\mathcal{A})\cup D(\mathcal{B})]$ and

$$\mathcal{A}\vee_0\mathcal{B}(ax+by):=aAx+bBy,\quad x\in D(\mathcal{A}),\ y\in D(\mathcal{B}),\ a,b\in\mathbb{C};$$

$\mathcal{A}\vee\mathcal{B}:=\overline{\mathcal{A}\vee_0\mathcal{B}}$. Then $\mathcal{A}\vee_0\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, and $\mathcal{A}\vee\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, provided that $(Z(t))_{t\in[0,\tau)}$ is locally equicontinuous. In the case of non-degenerate K-convoluted C-semigroups, C injective, it is well known that the set $\chi(Z)$, equipped with the operations $\wedge$ and $\vee$, forms a complete Boolean lattice [**531**], [**291**, Remark 2.1.8(ii)-(iii)]. We will not discuss the properties of $(\chi(Z),\wedge,\vee)$ in general case.

If $\mathcal{A}$ is a closed, $\mathcal{A}\in\chi(Z)$, $0\in\operatorname{supp}(a)$ and $y\in\mathcal{A}x$, then we have $(\int_0^t a(t-s)Z(s)x\,ds, Z(t)x-k(t)Cx)\in\mathcal{A}$, $t\in[0,\tau)$, i.e.,

$$\Bigl(\int_0^t a(t-s)Z(s)x\,ds,\int_0^t a(t-s)Z(s)y\,ds\Bigr)\in\mathcal{A},\quad t\in[0,\tau).\tag{303}$$

Suppose now that $\tau_0\in(0,\tau)$. By [**291**, Theorem 3.4.40], there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1[0,\tau_0]$ such that $(a*f_n)(t)\to g_1(t)$ in $L^1[0,\tau_0]$. Then the closedness of $\mathcal{A}$ along with (303) shows that $(\int_0^t g_1(t-s)Z(s)x\,ds,\int_0^t g_1(t-s)Z(s)y\,ds)\in\mathcal{A}$, $t\in[0,\tau_0]$. After differentiation, we get that $(Z(t)x,Z(t)y)\in\mathcal{A}$, $t\in[0,\tau_0]$ and since τ_0 was arbitrary, we have that $Z(t)\mathcal{A}\subseteq\mathcal{A}Z(t)$, $t\in[0,\tau)$ for any closed subgenerator $\mathcal{A}$ of $(Z(t))_{t\in[0,\tau)}$. If this is the case and $Z(t)C=CZ(t)$, $t\in[0,\tau)$, then $C^{-1}\mathcal{A}C$ also commutes with $Z(t)$: Suppose that $(x,y)\in C^{-1}\mathcal{A}C$. Then $Cy\in\mathcal{A}Cx$, $CZ(t)x=Z(t)Cx\in D(\mathcal{A})$, $t\in[0,\tau)$ and $CZ(t)y=Z(t)Cy\in$

$Z(t)\mathcal{A}Cx \subseteq \mathcal{A}CZ(t)x = \mathcal{A}Z(t)Cx$, $t \in [0,\tau)$ so that $Z(t)y \in C^{-1}\mathcal{A}CZ(t)x$, $t \in [0,\tau)$ and $Z(t)[C^{-1}\mathcal{A}C] \subseteq [C^{-1}\mathcal{A}C]Z(t)$, $t \in [0,\tau)$. Suppose again that $\mathcal{A}$ is a closed subgenerator of $(Z(t))_{t\in[0,\tau)}$, $0 \in \operatorname{supp}(a)$ and $y \in \mathcal{A}x$. Then $(\int_0^t a(t-s)Z(s)y\,ds, Z(t)y - k(t)Cy) = (Z(t)x - k(t)Cx, Z(t)y - k(t)Cy) \in \mathcal{A}$, $t \in [0,\tau)$. Since $(Z(t)x, Z(t)y) \in \mathcal{A}$, $t \in [0,\tau)$, the above easily implies that $(Cx, Cy) \in \mathcal{A}$ so that $C\mathcal{A} \subseteq \mathcal{A}C$, i.e., $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$. Now we proceed by repeating some parts of the proof of [**291**, Proposition 2.1.6(i)]. Let $(x,y) \in \mathcal{A}_{int}$. As above, we have $(\int_0^t a(t-s)Z(s)x\,ds, \int_0^t a(t-s)Z(s)y\,ds) \in \mathcal{A}$, $t \in [0,\tau)$ and $(Z(t)x, Z(t)y) \in \bar{\mathcal{A}} = \mathcal{A}$, $t \in [0,\tau)$. This implies $Z(t)y \in \mathcal{A}Z(t)x = \mathcal{A}[\Theta(t)Cx + \int_0^t a(t-s)Z(s)y\,ds]$, $t \in [0,\tau)$ and, since $\int_0^t a(t-s)Z(s)y\,ds \in D(\mathcal{A})$ for $t \in [0,\tau)$, $Cx \in D(\mathcal{A})$ as well as $0 \in \mathcal{A}[\Theta(t)Cx + \int_0^t a(t-s)Z(s)y\,ds - \int_0^t a(t-s)Z(s)y\,ds] - \Theta(t)Cy$, $t \in [0,\tau)$. Hence, $Cy \in \mathcal{A}Cx$ and $\mathcal{A}_{int} \subseteq C^{-1}\mathcal{A}C$. If, additionally, the operator C is injective and $Z(t)C = CZ(t)$, $t \in [0,\tau)$, then we can simply verify that $C^{-1}\mathcal{A}C$ is likewise a closed subgenerator of $(W(t))_{t\in[0,\tau)}$, so that $\mathcal{A}_{int} = C^{-1}\mathcal{A}C$ by previously proved inclusion $\hat{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and the fact that $\mathcal{A}_{int}$ extends any subgenerator from $\chi(W)$. Let $\mathcal{A}$ and $\mathcal{B}$ be two subgenerators of $(Z(t))_{t\in[0,\tau)}$, let $\mathcal{B}$ be closed, and let $a(t)$ kernel on $[0,\tau)$. Suppose that $y \in \mathcal{A}x$. Then $(\int_0^t a(t-s)Z(s)y\,ds, Z(t)y - k(t)Cy) = (Z(t)x - k(t)Cx, Z(t)y - k(t)Cy) \in \mathcal{B}$, $t \in [0,\tau)$, which implies by Theorem 1.2.3 that $((a*Z)(t)x - (a*k)(t)Cx, (a*Z)(t)y - (a*k)(t)Cy) \in \mathcal{B}$, $t \in [0,\tau)$. Since $(a*Z)(t)x \in D(\mathcal{B})$, $t \in [0,\tau)$, the above implies that $Cx \in D(\mathcal{B})$. Hence, $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$. We continue by observing that Proposition 3.2.3, Proposition 3.2.8, Proposition 3.2.13, the equation (274) and assertions clarified in the paragraph directly after Theorem 3.2.7 continue to hold without any terminological changes. If $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is strongly continuous and (274) holds, then it can be easily seen that the integral generator $\mathcal{A}_{int}$ of $(R_2(t))_{t\in[0,\tau)}$ is a subgenerator of a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)}$. As Example 3.2.42 shows, this is no longer true if (274) holds only for $0 \leqslant t, s, t+s < \tau$. Now we will state the following proposition.

PROPOSITION 3.2.43. *Suppose that $\mathcal{A}$ is a closed* MLO, $0 < \tau \leqslant \infty$, $a \in L^1_{loc}([0,\tau))$, $a*a \neq 0$ *in* $L^1_{loc}([0,\tau))$, $k \in C([0,\tau))$ *and* $k \neq 0$. *If $\pm\mathcal{A}$ are subgenerators of mild (a,k)-regularized C_1-existence families $(R_{1,\pm}(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized C_2-uniqueness families $(R_{2,\pm}(t))_{t\in[0,\tau)}$; (a,k)-regularized C-resolvent families $(R_\pm(t))_{t\in[0,\tau)}$), then $\mathcal{A}^2$ is a subgenerator of a mild $(a*a,k)$-regularized C_1-existence family $(R_1(t) \equiv (1/2)R_1(t) + (1/2)R_{1,-}(t))_{t\in[0,\tau)}$ (mild $(a*a,k)$-regularized C_2-uniqueness family $(R_2(t) \equiv (1/2)R_2(t) + (1/2)R_{2,-}(t))_{t\in[0,\tau)}$; mild $(a*a,k)$-regularized C-resolvent family $(R(t) \equiv (1/2)R_+(t) + (1/2)R_-(t))_{t\in[0,\tau)}$).*

PROOF. We will prove the proposition only for mild (a,k)-regularized C_1-existence families. Let $x \in E$ and $t \in [0,\tau)$ be fixed. Then $\frac{1}{2}[R_{1,+}(t)x - R_{1,-}(t)x] = \frac{1}{2}[R_{1,+}(t)x - k(t)C_1x] - [R_{1,-}(t)x - k(t)C_1x] \in \frac{1}{2}\mathcal{A}(a*R_{1,+}(\cdot)x)(t) + \frac{1}{2}\mathcal{A}(a*R_{1,-}(\cdot)x)(t) = \mathcal{A}(a*R_1(\cdot)x)(t)$. Applying Theorem 1.2.3, we get that $\frac{1}{2}(a*[R_{1,+}(\cdot)x - R_{1,-}(\cdot)x])(t) \in \mathcal{A}(a*a*R_1(\cdot)x)(t)$. Since $\pm\mathcal{A}$ are subgenerators of mild (a,k)-regularized C_1-existence families $(R_{1,\pm}(t))_{t\in[0,\tau)}$, the above inclusion implies

$(a * a * R_1(\cdot)x)(t) \in D(\mathcal{A}^2)$ and $\frac{1}{2}([R_{1,+}(t)x - k(t)C_1x] + [R_{1,-}(t)x - k(t)C_1x]) = R_1(t)x - k(t)C_1x \in \mathcal{A}^2(a * a * R_1(\cdot)x)(t)$, as required. $\square$

The following analogues of Theorem 3.2.4[(i),(iii)] and Theorem 3.2.5 hold true.

THEOREM 3.2.44. *Suppose that $\mathcal{A}$ is a closed* MLO *in X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, $|a(t)|$ and $k(t)$ satisfy* (P1), *as well as that $(R_1(t), R_2(t))_{t\geqslant 0} \subseteq L(Y,X) \times L(X)$ is strongly continuous. Let $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$ be such that the operator family $\{e^{-\omega t}R_i(t) : t \geqslant 0\}$ is equicontinuous for $i = 1, 2$. Then the following holds:*

(i) *$(R_1(t), R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$ iff for every $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda > \omega$ *and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, we have $R(C_1) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)\mathcal{A})$,*

$$\int_0^\infty e^{-\lambda t}R_1(t)y\,dt \in \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C_1y, \quad y \in Y, \tag{304}$$

and

$$\tilde{k}(\lambda)C_2x = \int_0^\infty e^{-\lambda t}[R_2(t)x - (a * R_2)(t)y]dt, \text{ whenever } (x,y) \in \mathcal{A}. \tag{305}$$

(ii) *$(R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$ iff* (305) *holds for* $\operatorname{Re}\lambda > \omega$.

THEOREM 3.2.45. *Suppose that $\mathcal{A}$ is a closed* MLO *in X, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $|a(t)|$ and $k(t)$ satisfy* (P1), *as well as that $(R(t))_{t\geqslant 0} \subseteq L(X)$ is strongly continuous and commutes with C on X. Let $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$ be such that the operator family $\{e^{-\omega t}R(t) : t \geqslant 0\}$ is equicontinuous. Then $(R(t))_{t\geqslant 0}$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ iff for every $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda > \omega$ *and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, we have $R(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)\mathcal{A})$,* (304) *holds with $R_1(\cdot)$, C_1 and Y, y replaced with $R(\cdot)$, C and X, x therein, as well as* (305) *holds with $R_2(\cdot)$ and C_2 replaced with $R(\cdot)$ and C therein.*

Keeping in mind Theorem 3.2.45 and [**292**, Theorem 1.2.2], it is very simple to prove the following complex characterization theorem (cf. Theorem 3.2.10):

THEOREM 3.2.46. *Suppose that $\mathcal{A}$ is a closed* MLO *in X, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $|a(t)|$ and $k(t)$ satisfiy* (P1), *$\omega_0 > \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$ and, for every $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda > \omega_0$ *and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, we have $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$. If there exists a function $\Upsilon\colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to L(X)$ which satisfies:*

(a) *$\Upsilon(\lambda)x \in \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$ for* $\operatorname{Re}\lambda > \omega_0$, *$\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, $x \in X$,*
(b) *the mapping $\lambda \mapsto \Upsilon(\lambda)x$,* $\operatorname{Re}\lambda > \omega_0$ *is analytic for every fixed $x \in X$,*
(c) *there exists $r \geqslant -1$ such that the family $\{\lambda^{-r}\Upsilon(\lambda) :$* $\operatorname{Re}\lambda > \omega_0\} \subseteq L(X)$ *is equicontinuous,*
(d) *$\Upsilon(\lambda)x - \tilde{a}(\lambda)\Upsilon(\lambda)y = \tilde{k}(\lambda)Cx$ for* $\operatorname{Re}\lambda > \omega_0$, *$(x,y) \in \mathcal{A}$, and*
(e) *$\Upsilon(\lambda)Cx = C\Upsilon(\lambda)x$ for* $\operatorname{Re}\lambda > \omega_0$, *$x \in X$,*

then, for every $\alpha > 1$, $\mathcal{A}$ is a subgenerator of a global $(a, k * g_{\alpha+r})$-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$ which satisfies that the family $\{e^{-\omega_0 t}R_\alpha(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous.

The real representation theorem for generation of degenerate (a, k)-regularized C-resolvent families can be also formulated but the assertion of Theorem 3.2.12(ii) is not attainable in the case that the operator C is not injective. The assertions of Theorem 3.2.7, Proposition 3.2.36 and Proposition 3.2.37 continue to hold with minimal terminological changes. Since the identity (291) holds for degenerate (a, k)-regularized C-resolvent families, with C being not injective, Proposition 3.2.15 can be reformulated without substantial difficulties, as well, but we cannot prove the uniqueness of solutions of corresponding abstract time-fractional inclusions.

As already mentioned, the adjoint type theorems [**292**, Theorem 2.1.12(i)/(ii); Theorem 2.1.13] continue to hold for (a, k)-regularized C-regularized families subgenerated by closed multivalued linear operators and it is not necessary to assume that the operator $\mathcal{A}$ is densely defined in the case of consideration of [**292**, Theorem 2.1.12(i)]. All this remains true if the operator C is not injective, when we also do not need to assume that $R(C)$ is dense in X.

If C is not injective, then we introduce the notion of (exponential equicontinuous) analyticity of degenerate (a, k)-regularized C-resolvent families in the same way as in Definition 3.2.16. Then Theorem 3.2.18 does not admit a satisfactory reformulation in our new frame. On the other hand, the assertion of Theorem 3.2.19 can be rephrased by taking into consideration the conditions (d)–(e) from Theorem 3.2.46. Differential properties of degenerate (a, k)-regularized C-resolvent families clarified in Theorem 3.2.25–Theorem 3.2.26 continue to hold after a reformulation of the same type.

3.2.3. Degenerate K-convoluted C-semigroups and degenerate K-convoluted C-cosine functions in locally convex spaces. It is well known that the notions of a degenerate (local) K-convoluted C-semigroup and a degenerate (local) K-convoluted C-cosine function in locally convex space can be introduced in a slightly different manner, by using the convoluted versions of semigroup property and d'Alambert functional equation. The main aim of this subsection is to consider the classes of degenerate K-convoluted C-semigroups and degenerate K-convoluted C-cosine functions in locally convex spaces by following this approach (cf. also [**291**, **296**, **366**, **367**, **386**] for similar concepts).

The underlying sequentially complete locally convex space will be denoted by E. In this subsection, we will not require the injectiveness of regularized operator $C \in L(E)$. If $0 < \tau \leqslant \infty$ and $(W(t))_{t\in[0,\tau)} \subseteq L(E)$ is strongly continuous, then we denote, as before, $W^{[n]}(t)x = \int_0^t g_n(t-s)W(s)x\,ds$, $x \in E$, $t, s \in [0, \tau)$, $n \in \mathbb{N}$.

DEFINITION 3.2.47. Let $0 \neq K \in L^1_{loc}([0, \tau))$. A strongly continuous operator family $(S_K(t))_{t\in[0,\tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) K-convoluted C-semigroup iff the following holds:

(i) $S_K(t)C = CS_K(t)$, $t \in [0, \tau)$, and
(ii) For all $x \in E$ and $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$S_K(t)S_K(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)S_K(r)Cx\,dr.$$

DEFINITION 3.2.48. Let $0 \neq K \in L^1_{loc}([0, \tau))$. A strongly continuous operator family $(C_K(t))_{t\in[0,\tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) K-convoluted C-cosine function iff the following holds:

(i) $C_K(t)C = CC_K(t)$, $t \in [0, \tau)$, and

(ii) For all $x \in E$ and $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$2C_K(t)C_K(s)x = \left(\int_t^{t+s} - \int_0^s \right) K(t+s-r)C_K(r)Cx\,dr + \int_{t-s}^t K(r-t+s)C_K(r)Cx\,dr + \int_0^s K(r+t-s)C_K(r)Cx\,dr, \quad t \geqslant s;$$

$$2C_K(t)C_K(s)x = \left(\int_s^{t+s} - \int_0^t \right) K(t+s-r)C_K(r)Cx\,dr + \int_{s-t}^s K(r+t-s)C_K(r)Cx\,dr + \int_0^t K(r-t+s)C_K(r)Cx\,dr, \quad t < s.$$

By a (local) C-regularized semigroup, resp., (local) C-regularized cosine function, we mean any strongly continuous operator family $(S(t))_{t\in[0,\tau)} \subseteq L(E)$, resp., $(C(t))_{t\in[0,\tau)} \subseteq L(E)$, satisfying that $S(t)C = CS(t)$, $t \in [0, \tau)$ and $S(t + s)C = S(t)S(s)$ for all $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, resp., $C(t)C = CC(t)$, $t \in [0, \tau)$ and $2C(t)C(s) = C(t + s)C + C(|t - s|)C$ for all $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$.

The notions of local equicontinuity and exponential equicontinuity will be taken in the usual way. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geqslant 0$, then it is also said that $(S_K(t))_{t\in[0,\tau)}$ is an α-times integrated C-semigroup; 0-times integrated semigroup is nothing else but C-regularized semigroup. The above notions can be simply understood for the class of K-convoluted C-cosine functions.

Set $\Theta(t) := \int_0^t K(s)ds$, $t \in [0, \tau)$. For a K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, resp., K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, we define its (integral) generator $\hat{\mathcal{A}}$ by graph

$$\hat{\mathcal{A}} := \left\{ (x, y) \in E \times E : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0, \tau) \right\}, \text{ resp.,}$$

$$\hat{\mathcal{A}} := \left\{ (x, y) \in E \times E : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y\,ds,\ t \in [0, \tau) \right\};$$

with $\Theta(t) \equiv 1$, we obtain the definition of integral generator of a C-semigroup (C-cosine function).

Denote by $(W(t))_{t\in[0,\tau)}$ any of the above considered operator families, and $a(t) \equiv 1$ ($a(t) \equiv t$) in the case of consideration semigroups (cosine functions). In what follows, we will refer to $(W(t))_{t\in[0,\tau)}$ as a (local) (a, Θ)-regularized C-resolvent family. It is worth noting that the functional equality of $(W(t))_{t\in[0,\tau)}$ and its strong continuity together imply that $W(t)W(s) = W(s)W(t)$ for all $t,\ s \in [0, \tau)$ with $t + s \in [0, \tau]$; in general case, it is not true that $W(t)W(s) = W(s)W(t)$ for $\tau < t + s < 2\tau$ (cf. **[385]** and Example 3.2.42). We will accept the following notion of non-degeneracy: $(W(t))_{t\in[0,\tau)}$ is said to be non-degenerate iff the assumption

$W(t)x = 0$ for all $t \in [0, \tau)$ implies $x = 0$ (cf. [**386**, p. 2] for more details on the subject).

It is clear that the integral generator $\hat{\mathcal{A}}$ of $(W(t))_{t\in[0,\tau)}$ is an MLO in E and that the local equicontinuity of $(W(t))_{t\in[0,\tau)}$ implies that $\hat{\mathcal{A}}$ is closed. Furthermore, we have that $\hat{\mathcal{A}} \subseteq C^{-1}\hat{\mathcal{A}}C$ in the MLO sense, and that $\hat{\mathcal{A}} = C^{-1}\hat{\mathcal{A}}C$ provided additionally that the operator C is injective. A subgenerator of $(W(t))_{t\in[0,\tau)}$ is nothing else but an (a, Θ, C)-subgenerator of $(W(t))_{t\in[0,\tau)}$ (see Definition 3.2.41).

In [**386**, Example 4.10], for each $\alpha > 0$ it has been constructed an example of a global degenerate α-times integrated C-semigroup with infinitely many single-valued bounded subgenerators. This example shows that the equivalence relation $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$, as well as any of the equalities $Ax = Bx$, $x \in D(A)\cap D(B)$ and $\rho(A) = \emptyset$, $A \neq \hat{\mathcal{A}}$ does not hold for subgenerators in the degenerate case (cf. [**291**, Proposition 2.1.6(ii)–(iii),(viii)], and [**291**, Proposition 2.1.16] for cosine operator functions case). Furthermore, the subgenerators from this example do not have the same eigenvalues, in general, so that the assertion of [**291**, Proposition 2.1.6(v)] does not hold in the degenerate case, as well; the same example shows that the equality $C^{-1}\mathcal{A}C = C^{-1}\mathcal{B}C$ (in the MLO sense) is not generally true for subgenerators of degenerate integrated C-semigroups (cf. [**291**, Proposition 2.1.6(ii)]).

For the sequel, we need the following useful extension of [**366**,**367**, Lemma 2.1]:

LEMMA 3.2.49. *Let $0 < \tau \leqslant \infty$, and let $(W(t))_{t\in[0,\tau)}$ be a strongly continuous operator family which commutes with C. Then the following is equivalent:*

(i) *$(W(t))_{t\in[0,\tau)}$ is an (a, Θ)-regularized C-resolvent family.*

(ii) *For any complex non-zero polynomial $P(z)$ and for any $a_0 \in \mathbb{C}$, $(W^{P,a_0}(t) \equiv (P * W)(t) + a_0W(t))_{t\in[0,\tau)}$ is an $(a, P * \Theta + a_0\Theta)$-regularized C-resolvent family.*

(iii) *There exist a complex non-zero polynomial $P(z)$ and a number $a_0 \in \mathbb{C}$ such that $(W^{P,a_0}(t))_{t\in[0,\tau)}$ is an $(a, P * \Theta + a_0\Theta)$-regularized C-resolvent family.*

PROOF. We will prove the lemma in the case that $(W(t))_{t\in[0,\tau)} = (S_K(t))_{t\in[0,\tau)}$ or $(W(t))_{t\in[0,\tau)} = (C_K(t))_{t\in[0,\tau)}$ for some $K \in L^1_{loc}([0, \tau))$, $K \neq 0$. We will prove the implication (i) $\Rightarrow$ (ii) by induction on degree of $P(z)$. Consider first the semigroup case. If $\mathrm{dg}(P) = 0$, then there exists a number $a_1 \in \mathbb{C}$ such that $P(z) \equiv a_1$ and, since $(S_\Theta(t) \equiv S_K^{[1]}(t))_{t\in[0,\tau)}$ is a Θ-convoluted C-semigroup by the proof of [**366**, Lemma 2.1], it suffices to show that, for every $x \in E$ and for every $t,\ s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have:

$$\begin{aligned}
&S_\Theta(t)S_K(s) + S_K(t)S_\Theta(s)\\
&\quad = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right]\{\Theta(t + s - r)S_K(r)Cx + K(t + s - r)S_\Theta(r)Cx\}dr.
\end{aligned}$$

Let such x, t, s be fixed. Since $S_K(s)S_\Theta(t)x = S_K(t)S_\Theta(s)x$, the identities [**366**, (2.2) and (2.4)] yield

$$
\begin{aligned}
(306)\quad & \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)S_\Theta(r)Cx\,dr \\
& \qquad = S_K(t)S_\Theta(s)x + \Theta(s)S_\Theta(t)Cx = S_K(s)S_\Theta(t)x + \Theta(t)S_\Theta(s)Cx
\end{aligned}
$$

and

$$
\begin{aligned}
(307)\quad & \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] \Theta(t+s-r)S_K(r)Cx\,dr \\
= & \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)S_\Theta(r)Cx\,dr - \Theta(s)S_\Theta(t)Cx - \Theta(t)S_\Theta(s)Cx \\
& = S_K(s)S_\Theta(t)x + \Theta(t)S_\Theta(s)Cx - \Theta(s)S_\Theta(t)Cx - \Theta(t)S_\Theta(s)Cx \\
& \qquad = S_K(s)S_\Theta(t)x - \Theta(s)S_\Theta(t)Cx.
\end{aligned}
$$

The claimed assertion follows by adding (306) and (307). Suppose now that (ii) holds for any complex non-zero polynomial $P(z)$ of degree strictly less than $n \in \mathbb{N}$, any number $a_0 \in \mathbb{C}$ and any K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Let us prove that (ii) holds for arbitrary complex non-zero polynomial $P(z) = a_{n+1}g_{n+1}(z) + a_n g_n(z) + \cdots + a_1 g_1(z)$ and arbitrary number $a_0 \in \mathbb{C}$. Then we can always find complex numbers $A_0,\ A_1,\ B_1, \ldots, B_n$ such that: $a_{n+1} = A_1 B_n$, $a_j = A_1 B_{j-1} + A_0 B_j$ for $1 \leqslant j \leqslant n$ and $a_0 = A_0 B_0$, so that $W^{P,a_0}(\cdot) = S_K^{P,a_0}(\cdot) = A_1(g_1 * S_{K_1})(\cdot) + A_0 S_{K_1}(\cdot) = S_{K_1}^{A_1,A_0}(\cdot)$, where $S_{K_1}(\cdot) = S_K^{P_1,B_0}(\cdot)$ with $P_1(z) = B_n g_n(z) + B_{n-1}g_{n-1}(z) + \cdots + B_1 g_1(z)$. Hence, (ii) follows from its validity for constant polynomials and induction hypothesis. The implication (ii) $\Rightarrow$ (iii) is trivial and the implication (ii) $\Rightarrow$ (iii) holds on account of the proof of [**366**, Lemma 2.1], with $P(z) = 1$ and $a_0 = 0$. The proof for K-convoluted C-cosine functions is almost the same and here it is only worth pointing out how one can prove that part (ii) holds for constant complex polynomials. Let $(C_K(t))_{t\in[0,\tau)}$ be a K-convoluted C-cosine function, let $P(z) \equiv a_1 \in \mathbb{C}$, and let $a_0 \in \mathbb{C}$. It is very simple to prove that (ii) holds provided $a_0 = 0$, so that $(C_\Theta(t) = C_K^{[1]}(t))_{t\in[0,\tau)}$ is a Θ-convoluted C-cosine function. In the remnant of proof, we assume that $t, s \in [0,\tau)$, $t+s < \tau$ and $t \geqslant s$ (the case $t < s$ is analogous). It suffices to show that

$$
\begin{aligned}
(308)\quad & 2[C_\Theta(t)C_K(s)x + C_K(t)C_\Theta(s)x] \\
& = \left(\int_t^{t+s} - \int_0^s\right)\{\Theta(t+s-r)C_K(r)Cx + K(t+s-r)C_\Theta(r)Cx\}dr \\
& \qquad + \int_{t-s}^t \{\Theta(r-t+s)C_K(r)Cx + K(r-t+s)C_\Theta(r)Cx\}dr \\
& \qquad\qquad + \int_0^s \{\Theta(r+t-s)C_K(r)Cx + K(r+t-s)C_\Theta(r)Cx\}dr.
\end{aligned}
$$

Since $(C_\Theta(t))_{t\in[0,\tau)}$ is a Θ-convoluted C-cosine function, the final part of proof of [**291**, Theorem 2.1.13] shows that (see e.g. the equations [**291**, (36)–(41)])

$$
(309)\quad 2C_K(t)C_\Theta(s)x = 2\Theta(t)C_\Theta(s)Cx
$$

$$+\Bigl(\int_t^{t+s}-\int_0^s\Bigr)\Theta(t+s-r)C_K(r)Cx\,dr$$
$$+\int_{t-s}^t\Theta(r-t+s)C_K(r)Cx\,dr-\int_0^s\Theta(r+t-s)C_K(r)Cx\,dr.$$

Using the equality $2C_\Theta(t)C_K(s)x=2(d/ds)C_\Theta(t)C_\theta(s)x$ and the composition property for $(C_\Theta(t))_{t\in[0,\tau)}$, a straightforward computation shows that the value of term $2C_\Theta(t)C_K(s)x$, appearing on the left hand side of (308), is equal to R_1-R, where R_1, resp., R, is the term on the right hand side of (308), resp., (309). The proof of the lemma is thereby complete. $\square$

Inspecting the proofs of [**366**, **367**, Theorem 2.2] yields the following:

THEOREM 3.2.50. *Let $0<\tau\leqslant\infty$, and let $(W(t))_{t\in[0,\tau)}$ be a strongly continuous operator family which commutes with C.*

(i) *If $(W(t))_{t\in[0,\tau)}$ is an (a,Θ)-regularized C-resolvent family, then*

(310) $(a*W)(t)[W(s)-\Theta(s)C]=[W(t)-\Theta(t)C](a*W)(s)$ *for* $0\leqslant t,s,\ \ t+s<\tau.$

(ii) *Suppose that $(W(t))_{t\in[0,\tau)}$ is locally equicontinuous and* (310) *holds. Then $(W(t))_{t\in[0,\tau)}$ is an (a,Θ)-regularized C-resolvent family.*

By Theorem 3.2.50(i), we have that the integral generator $\hat{\mathcal{A}}$ of a global (a,Θ)-regularized C-resolvent family $(W(t))_{t\geqslant0}$ is always its subgenerator (as already seen, this statement is not true in local case).

The following theorem is a slight extension of [**366**, **367**, Theorem 2.5].

THEOREM 3.2.51. *Let $0<\tau\leqslant\infty$ and $C\in L(E)$. Suppose that $a(t)\equiv1$ or $a(t)\equiv t$, $\mathcal{A}$ is an* MLO *and $(Z(t))_{t\in[0,\tau)}\subseteq L(E)$ is a strongly continuous operator family which commutes with C. Let $(Z(t))_{t\in[0,\tau)}$ be locally equicontinuous. Then the following holds:*

(i) *If $\mathcal{A}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, then $(Z(t))_{t\in[0,\tau)}$ is an (a,Θ)-regularized C-resolvent family.*
(ii) *Suppose that* (i) *holds with $\mathcal{A}=A$ being single-valued and linear, as well as that C is injective. Then $(Z(t))_{t\in[0,\tau)}$ is non-degenerate.*

PROOF. Let $x\in E$ be fixed. Then (ii) of Definition 3.2.41 yields that $\int_0^s a(s-r)Z(r)x\,dr\in D(\mathcal{A})$ and $Z(s)x-\Theta(s)Cx\in\mathcal{A}\int_0^s a(s-r)Z(r)x\,dr$. By (i) of Definition 3.2.41, we have

$$[Z(t)-\Theta(t)C]\int_0^s a(s-r)Z(r)x\,dr=\int_0^t a(t-r)Z(r)[Z(s)x-\Theta(s)Cx]dr$$
$$=(a*Z)(t)[Z(s)x-\Theta(s)Cx],$$

for $0\leqslant t,\,s<\tau$. Now we can apply Theorem 3.2.50(ii) in order to see that (i) holds true. The proof of (ii) is simple and therefore omitted. $\square$

Now it is quite simple to construct an example of a global degenerate strongly continuous semigroup (i.e., I-regularized semigroup) which do not have any linear subgenerator:

EXAMPLE 3.2.52. [**199**] Suppose that $\mathcal{A}$ is a non single-valued MLO which satisfies the Hille–Yosida condition [**199**, (H-Y), p. 28] on a Banach space E. By [**199**, Theorem 2.4], $\mathcal{A}$ is the integral generator (a subgenerator) of a global strongly continuous semigroup $(T(t))_{t\geqslant 0}$ which vanishes on the closed subspace $\mathcal{A}0$ of E. An application of Theorem 3.2.51(ii) shows that $(T(t))_{t\geqslant 0}$ has no linear, single-valued subgenerator.

It should also be observed that a global degenerate strongly continuous semigroup can have infinitely many subgenerators: Let $P \in L(E)$, $P^2 = P$ and $T(t) := P$, $t \geqslant 0$. Then it can be simply verified that, for every linear subspace V of $N(P)$, $\mathcal{A} = N(I-P) \times V$ is a subgenerator of $(T(t))_{t\geqslant 0}$. The arguments used in the proofs of [**366**, Lemma 2.8] and [**367**, Lemma 2.9] enable one to deduce the following lemma:

LEMMA 3.2.53. *Let $0 < \tau \leqslant \infty$, $x \in E$, $0 \in \operatorname{supp}(\Theta)$ and $(W(t))_{t\in[0,\tau)}$ be an (a,Θ)-regularized C-resolvent family. Then the existence of a number $\tau_0 \in (0,\tau)$ such that $W(t)x = 0$, $t \in [0,\tau_0)$ implies $CW(t)x = 0$, $t \in [0,\tau)$.*

Keeping in mind Lemma 3.2.53, it is quite simple to prove the following extension of [**366**, Theorem 2.9] and [**367**, Theorem 2.10]:

THEOREM 3.2.54. *Suppose that $0 < \tau \leqslant \infty$, $C \in L(E)$ is injective, $0 \in\operatorname{supp}(\Theta)$ and $(W(t))_{t\in[0,\tau)}$ is an (a,Θ)-regularized C-resolvent family. Then $(W(t))_{t\in[0,\tau)}$ is non-degenerate iff the integral generator $\hat{\mathcal{A}}$ of $(W(t))_{t\in[0,\tau)}$ is its subgenerator.*

Before proceeding further, we would like to mention that the examination from Example 3.2.52 shows that the existence of a subgenerator of a (local) C-regularized semigroup in the MLO sense does not imply its non-degeneracy, even supposing that $C = I$. The reader with a little experience will succeed in transferring [**366**, Theorem 2.13] and [**367**, Theorem 2.14] to locally equicontinuous K-convoluted C-semigroups and K-convoluted C-cosine functions in locally convex spaces.

The following proposition extends the assertions of [**291**, Proposition 2.1.3] and [**366**, **367**, Proposition 2.3] (for locally equicontinuous operator families, this proposition follows almost immediately from Theorem 3.2.49).

PROPOSITION 3.2.55. *Let $0 < \tau \leqslant \infty$, and let $(W(t))_{t\in[0,\tau)}$ be an (a,Θ)-regularized C-resolvent family. Suppose that $H \in L^1_{loc}([0,\tau))$ and $H *_0 K \neq 0$ in $L^1_{loc}([0,\tau))$. Set $W_H(t)x := \int_0^t H(t-s)W(s)x\,ds$, $x \in E$, $t \in [0,\tau)$. Then $(W_H(t))_{t\in[0,\tau)}$ is an $(a, H *_0 \Theta)$-regularized C-resolvent family.*

PROOF. We will include all details of proof for semigroups, in purely convoluted case; the proof in all other cases can be given by applying the same trick. It is clear that $(W_H(t) = S_{K,H}(t))_{t\in[0,\tau)} \subseteq L(E)$ is a strongly continuous operator family which commutes with C. Therefore, it suffices to show that, for every $x \in E$ and $t, s \in [0,\tau)$ with $t+s \in [0,\tau)$, the following holds:

$$S_{H,K}(t)S_{H,K}(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right](H*K)(t+s-r)S_{H,K}(r)Cx\,dr,$$

i.e., that for each functional $x^* \in E^*$ we have:

$$\int_0^t \int_0^s H(t-r)H(s-\sigma)\Big[\int_0^{r+\sigma} - \int_0^r - \int_0^\sigma\Big] K(r+\sigma-v)\langle x^*, S_K(v)Cx\rangle\, dv\, d\sigma\, dr \tag{311}$$
$$= \Big[\int_0^{t+s} - \int_0^t - \int_0^s\Big] \int_0^r \int_0^{t+s-r} K(t+s-r-v)$$
$$\times H(v)H(r-\sigma)\langle x^*, S_K(\sigma)Cx\rangle\, dv\, d\sigma\, dr.$$

By Lemma 3.2.49, the above equality holds for all non-zero complex polynomials $H(\cdot)$ so that the final conclusion follows by applying Stone–Weierstrass theorem and the dominated convergence theorem in (311). $\square$

REMARK 3.2.56. (i) Suppose that $\mathcal{A}$ is a closed subgenerator of $(W(t))_{t\in[0,\tau)}$. Then we can simply prove with the help of Theorem 1.2.3 that $\mathcal{A}$ is a subgenerator of $(W_H(t))_{t\in[0,\tau)}$.

(ii) Suppose that $\mathcal{A}$ ($\mathcal{A}_H$) is a closed subgenerator of $(W(t))_{t\in[0,\tau)}$ ($(W_H(t))_{t\in[0,\tau)}$). Then $\mathcal{A} \subseteq \mathcal{A}_H$, with the equality if $H(t)$ is a kernel on $[0,\tau)$.

Now we will extend the assertion of [**291**, Theorem 2.1.11] to degenerate operator families.

THEOREM 3.2.57. *Suppose that $\mathcal{A}$ is a closed* MLO *in E, $0 < \tau \leqslant \infty$, $K \in L^1_{loc}([0,\tau))$, $K \neq 0$ and $(C_K(t))_{t\in[0,\tau)}$ is a strongly continuous operator family which commutes with C. Set*

$$S_\Theta(t) := \begin{pmatrix} \int_0^t C_K(s)ds & \int_0^t (t-s)C_K(s)ds \\ C_K(t) - \Theta(t)C & \int_0^t C_K(s)ds \end{pmatrix}, \quad 0 \leqslant t < \tau$$

and $\mathcal{C}(x\ y)^T := (Cx\ Cy)^T$ $(x, y \in E)$. Then we have:

(i) *The following assertions are equivalent:*
  (a) *$(C_K(t))_{t\in[0,\tau)}$ is a K-convoluted C-cosine function on E.*
  (b) *$(S_\Theta(t))_{t\in[0,\tau)}$ is a Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ on $E\times E$.*

Suppose that the equivalence relation (a) $\Leftrightarrow$ (b) *in* (i) *holds. Then we have:*

(ii) *$\mathcal{A}$ is a subgenerator of $(C_K(t))_{t\in[0,\tau)}$ iff $\mathcal{B} := \left(\begin{smallmatrix} 0 & I \\ \mathcal{A} & 0 \end{smallmatrix}\right)$ is a subgenerator of $(S_\Theta(t))_{t\in[0,\tau)}$.*

(iii) *Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be the integral generators of $(C_K(t))_{t\in[0,\tau)}$ and $(S_\Theta(t))_{t\in[0,\tau)}$, respectively. Then the inclusion $\left(\begin{smallmatrix} 0 & I \\ \hat{\mathcal{A}} & 0 \end{smallmatrix}\right) \subseteq \hat{\mathcal{B}}$ holds true. Furthermore, if $(C_K(t))_{t\in[0,\tau)}$ is non-degenerate, then $\left(\begin{smallmatrix} 0 & I \\ \hat{\mathcal{A}} & 0 \end{smallmatrix}\right) = \hat{\mathcal{B}}$.*

PROOF. Suppose that (a) holds. Then it is clear that $(S_\Theta(t))_{t\in[0,\tau)}$ is a strongly continuous operator family in $L(E\times E)$ which commutes with $\mathcal{C}$. Therefore, it suffices to show that the semigroup property holds for $(S_\Theta(t))_{t\in[0,\tau)}$, i.e., that the following holds for $0 \leqslant t, s, t+s < \tau$ and $x \in E$:

$$C_\Theta(t)C_\Theta(s)x + C_\Theta^{[1]}(t)[C_K(s) - \Theta(s)C]x \tag{312}$$

$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) C_\Theta(\sigma) Cx \, d\sigma,$$

$$(313) \quad C_\Theta(t) C_\Theta^{[1]}(s)x + C_\Theta^{[1]}(t) C_\Theta(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) C_\Theta^{[1]}(\sigma) Cx \, d\sigma,$$

$$(314) \quad [C_K(t) - \Theta(t)C]C_\Theta(s)x + C_\Theta(t)[C_K(s) - \Theta(s)C]x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) \{C_K(\sigma)Cx - \Theta(s)C^2x\} \, d\sigma$$

and

$$(315) \quad [C_K(t) - \Theta(t)C]C_\Theta^{[1]}(s)x + C_\Theta(t)C_\Theta(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) C_\Theta(\sigma) Cx \, d\sigma.$$

The proofs of equations (312)–(315) and implication (b) $\Rightarrow$ (a) below will be given only in the case that $s \leqslant t$; the case $s > t$ can be considered similarly. First of all, we will prove (314). By [**291**, Lemma 2.1.12], we have that

$$\left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) \Theta(s) \, d\sigma = 0.$$

Therefore, we need to prove that

$$(316) \quad [C_K(t) - \Theta(t)C]C_\Theta(s)x + C_\Theta(t)[C_K(s) - \Theta(s)C]x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) C_K(\sigma) Cx \, d\sigma.$$

In order to do that, observe that the partial integration and Newton-Leibniz formula imply:

$$(317) \quad \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-\sigma) C_\Theta(\sigma) Cx \, d\sigma = \Theta(s) C_\Theta(t) Cx + \Theta(t) C_\Theta(s) Cx + \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma) C_\Theta(\sigma) Cx \, d\sigma,$$

$$(318) \quad \int_{t-s}^t \{\Theta(r-t+s) C_K(r) Cx + K(r-t+s) C_\Theta(r) Cx\} dr = \Theta(s) C_\Theta(t) Cx$$

and

$$(319) \quad \int_0^s \{\Theta(r+t-s) C_K(r) Cx + K(r+t-s) C_\Theta(r) Cx\} dr = \Theta(t) C_\Theta(s) Cx.$$

Inserting (317)–(319) in (308) and dividing after that both sides of obtained equation with two, we immediately get (316). Consider now the equation (312). The both sides equal zero for $t = 0$ and it suffices therefore to show that their derivatives in variable t are equal, i.e., that:

$$C_K(t)C_\Theta(s)x + C_\Theta(t)[C_K(s) - \Theta(s)C]x \\ = \frac{d}{dt}\Biggl[\int_0^{t+s} - \int_0^t - \int_0^s\Biggr]\Theta(t+s-\sigma)C_\Theta(\sigma)Cx\,d\sigma.$$

This follows from (314) and the usual limit procedure. The equation (315) is a consequence of (312) and Theorem 3.2.50(i), so that it remains to be proved (312). Here we can clearly deal with the function $\Theta(t)$ replaced in all places with the function $K(t)$: then (312) follows immediately from (314) by applying the partial intgration. Hence, (b) holds. The implication (b) $\Rightarrow$ (a) can be proved as follows. By Lemma 3.2.49, we may assume without loss of generality that $K(t)$ is locally absolutely continuous on $[0, \tau)$. Since (314) holds, [**291**, Lemma 2.1.12] implies that the equation [**291**, (26)] holds true. Due to the proof of [**291**, Theorem 2.1.13] (cf. [**291**, (27)-(35)]) shows that the composition property holds for $(C_K(t))_{t\in[0,\tau)}$. This completes the proof of (i). The proofs of (ii) and (iii) follow from simple calculations and therefore omitted. $\square$

Remark 3.2.58. (i) Let $\mathcal{A}$ be an MLO in E, and let $\mathcal{B}$ be defined as in (i). Then $\mathcal{A}$ is closed iff $\mathcal{B}$ is closed. Furthermore, $\bigl(\begin{smallmatrix} 0 & I \\ C^{-1}\mathcal{A}C & 0 \end{smallmatrix}\bigr) \subseteq \mathcal{C}^{-1}\mathcal{B}\mathcal{C}$, with the equality in the case that the operator C is injective.

(ii) Theorem 3.2.57 can be simply reformulated for C-regularized cosine functions on E and induced once integrated $\mathcal{C}$-semigroups on the product space $E \times E$. In non-degenerate case, a similar assertion is also known for $(a * a, k)$-regularized C-resolvent families on E and induced $(a, a * k)$-regularized $\mathcal{C}$-resolvent families on the product space $E \times E$ (cf. [**291**, Theorem 2.1.27(xiv)]). We leave to the interested reader as an interesting problem to find an appropriate analogue of the last mentioned result for degenerate regularized resolvent families.

The extension type theorems for non-degenerate integrated semigroups has been considered for the first time by W. Arendt, O. El-Mennaoui and V. Keyantuo [**28**], who have proved that a closed linear operator A generates a local $(2n)$-times integrated semigroup on the interval $[0, 2\tau)$, provided that A generates a local n-times integrated semigroup on the interval $[0, \tau)$ ($n \in \mathbb{N}$, $0 < \tau < \infty$). Immediately after that, I. Ciorănescu and G. Lumer [**110**] have extended their result to the class of local K-convoluted C-semigroups (see e.g. [**291**, Theorem 2.1.9] for a precise formulation). On the other hand, S. W. Wang and M. C. Gao [**532**] have investigated automatic extensions of non-degenerate local C-regularized semigroups and non-degenerate local C-regularized cosine functions. By all means, the method established in [**532**] cannot be used for proving extension type theorems for degenerate C-regularized semigroups and degenerate C-regularized cosine functions which do not have subgenerators. The situation is much more simpler if degenerate operator families under our consideration have subgenerators. Keeping in mind

Theorem 1.2.3 and elementary properties of multivalued linear operators, the following two theorems can be proved in almost the same way as in the single-valued linear case (cf. V. Keyantuo, P. J. Miana, L. Sánchez-Lajusticia [**278**, Theorem 4.4], P. J. Miana, V. Poblete [**429**, Theorem 3.3], and [**291**, Theorem 2.1.9, Corollary 2.1.10, Theorem 2.1.14, Corollary 2.1.15] for some special cases of two last mentioned results):

THEOREM 3.2.59. *Suppose that $\mathcal{A}$ is a closed* MLO. *Let $n \in \mathbb{N}$, $0 < \tau < \infty$, $0 < \tau_0 < \tau$, $K \in L^1_{loc}([0,(n+1)\tau))$, $K \neq 0$, and let $(S_K(t))_{t\in[0,\tau)}$ be a local $K_{|[0,\tau)}$-convoluted C-semigroup with a subgenerator $\mathcal{A}$. Define recursively the family of operators $(S_{K,n+1}(t))_{t\in[0,(n+1)\tau_0]}$ by $S_{K,n+1}(t)x := \int_0^t K(t-s)S_{K,n}(s)Cx\,ds$, $x \in E$, for $t \in [0, n\tau_0]$ and*

$$S_{K,n+1}(t)x := S_{K,n}(n\tau_0)S_K(t-n\tau_0)x + \int_0^{n\tau_0} K(t-s)S_{K,n}(s)Cx\,ds + \int_0^{t-n\tau_0} K^{*,n}(t-s)S_K(s)Cx\,ds$$

for $x \in E$ and $t \in (n\tau_0,(n+1)\tau_0]$. Then $(S_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)}$ is a local $(K^{*,n+1})_{|[0,(n+1)\tau_0)}$-convoluted C^{n+1}-semigroup with a subgenerator $\mathcal{A}$.

THEOREM 3.2.60. *Suppose that $\mathcal{A}$ is a closed* MLO. *Let $n \in \mathbb{N}$, $0 < \tau < \infty$, $0 < \tau_0 < \tau$, $K \in L^1_{loc}([0,(n+1)\tau))$, $K \neq 0$, and let $(C_K(t))_{t\in[0,\tau)}$ be a local $K_{|[0,\tau)}$-convoluted C-cosine function with a subgenerator $\mathcal{A}$. Define recursively the family of operators $(C_{K,n+1}(t))_{t\in[0,(n+1)\tau_0]}$ by $C_{K,n+1}(t)x := \int_0^t K(t-s)C_{K,n}(s)Cx\,ds$, $x \in E$, for $t \in [0, n\tau_0]$ and*

$$C_{K,n+1}(t)x := 2C_{K,n}(n\tau_0)C_K(t-n\tau_0)x + \int_0^{n\tau_0} K(t-s)C_{K,n}(s)Cx\,ds + \int_0^{t-n\tau_0} K^{*,n}(t-s)C_K(s)Cx\,ds - \int_{2n\tau_0-t}^{n\tau_0} K(t+s-2n\tau_0)C_{K,n}(s)Cx\,ds - \int_0^{t-n\tau_0} K(s-t+2n\tau_0)C_K(s)x\,ds$$

for $x \in E$ and $t \in (n\tau_0,(n+1)\tau_0]$. Then $(C_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)}$ is a local $(K^{*,n+1})_{|[0,(n+1)\tau_0)}$-convoluted C^{n+1}-cosine function with a subgenerator $\mathcal{A}$.

REMARK 3.2.61. Consider the situation of Theorem 3.2.59 (Theorem 3.2.60). Let $\hat{\mathcal{A}}$ be the integral generator of $(S_K(t))_{t\in[0,\tau)}$ $((C_K(t))_{t\in[0,\tau)})$. Then it is a very undesirable question to precisely profile the integral generator $\hat{\mathcal{A}}_{n+1}$ of $(S_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)}$ $((C_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)})$ in general case. We will prove that the integral generator of $(S_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)}$ $((C_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)})$ is $\hat{\mathcal{A}}$, provided that the operator C is injective (observe that we do not require the condition $0 \in$supp(K) here). The analysis is quite similar in both cases and we will consider only K-convoluted C-semigroups. Since $\mathcal{A}$ is a subgenerator of $(S_{K,n+1}(t))_{t\in[0,(n+1)\tau_0)}$, and $\hat{\mathcal{A}}$ is the integral generator of $(S_K(t))_{t\in[0,\tau)}$, the foregoing arguments yield that $\hat{\mathcal{A}}_{n+1} = C^{-(n+1)}\mathcal{A}C^{n+1}$ and $\hat{\mathcal{A}} = C^{-1}\mathcal{A}C = C^{-1}\hat{\mathcal{A}}C$.

Inductively, $C^{-l}\hat{\mathcal{A}}C^{l}=\hat{\mathcal{A}}$, $l\in\mathbb{N}$ so that

$$\hat{\mathcal{A}}_{n+1}=C^{-(n+1)}\mathcal{A}C^{n+1}=C^{-n}[C^{-1}\mathcal{A}C]C^{n}=C^{-n}\hat{\mathcal{A}}C^{n}=\hat{\mathcal{A}}.$$

The statements clarifed in Theorem 3.2.59–Theorem 3.2.60 and Remark 3.2.61 can be simply reformulated for the classes of local C-regularized semigroups and local C-regularized cosine functions. It is also worth noting that the assertions of [**291**, Proposition 2.3.3, Proposition 2.3.4] [**291**, Proposition 2.3.8(i)–(ii)] can be reword for abstract degenerate inclusions of first order (second order) by replacing the sequence $x_0=x$, $Ax=x_1,\ldots,A^kx=x_k$ in their formulations with an arbitrary sequence $(x_j)_{0\leqslant j\leqslant k}$ satisfying $x_j\in\mathcal{A}x_{j-1}$ $(1\leqslant j\leqslant k)$. Involving into consideration the conditions (d)–(e) from Theorem 3.2.46, one can simply prove the analogues of [**291**, Proposition 2.4.2. Corollary 2.4.3, Theorem 2.4.5, Theorem 2.4.8, Corollary 2.4.9] for degenerate analytic K-convoluted C-semigroups in locally convex spaces.

We close this subsection by rethinking our structural results from [**291**, Section 2.7] for degenerate K-convoluted C-semigroups subgenerated by multivalued linear operators. The subsequent theorem is very similar to [**291**, Theorem 2.7.1].

THEOREM 3.2.62. (i) *Suppose that E is Banach space, $M>0$, $\beta\geqslant 0$, $|K(t)|\leqslant Me^{\beta t}$, $t\geqslant 0$, $(S_K(t))_{t\in[0,\tau)}$ is a (local) K-convoluted semigroup with a closed subgenerator $\mathcal{A}$ and, for every $\varepsilon>0$, there exist $\varepsilon_0\in(0,\tau\varepsilon)$ and $T_\varepsilon>0$ such that $1/|\tilde{K}(\lambda)|\leqslant T_\varepsilon e^{\varepsilon_0|\lambda|}$, $\operatorname{Re}\lambda>\beta$, $\tilde{K}(\lambda)\neq 0$. Then, for every $\varepsilon>0$, there exist $\bar{C}_\varepsilon>0$ and $\bar{K}_\varepsilon>0$ such that, for every λ which belongs to the following set*

$$\Omega^1_\varepsilon:=\{\lambda\in\mathbb{C}:\tilde{K}(\lambda)\neq 0,\quad \operatorname{Re}\lambda>\beta,\ \operatorname{Re}\lambda\geqslant\varepsilon|\lambda|+\bar{C}_\varepsilon\},$$

there exists an operator $F(\lambda)\in L(E)$ so that $F(\lambda)\mathcal{A}\subseteq\mathcal{A}F(\lambda)$, $\lambda\in\Omega^1_\varepsilon$, $F(\lambda)x\in(\lambda-\mathcal{A})^{-1}x$, $\lambda\in\Omega^1_\varepsilon$, $x\in E$, $F(\lambda)x-x=F(\lambda)y$, whenever $\lambda\in\Omega^1_\varepsilon$ and $(x,y)\in\mathcal{A}$,

$$\|F(\lambda)\|\leqslant\bar{K}_\varepsilon e^{\varepsilon_0|\lambda|},\quad \lambda\in\Omega^1_\varepsilon,\ \tilde{K}(\lambda)\neq 0,$$

and that the mapping $\lambda\mapsto F(\lambda)\in L(E)$, $\lambda\in\Omega^1_\varepsilon$ is analytic.

(ii) *Suppose that E is Banach space, $\alpha>0$, $M>0$, $\beta\geqslant 0$, $\Phi\colon\mathbb{C}\to[0,\infty)$, $|K(t)|\leqslant Me^{\beta t}$, $t\geqslant 0$, $(S_K(t))_{t\in[0,\tau)}$ is a local K-convoluted semigroup with a closed subgenerator $\mathcal{A}$ and $1/|\tilde{K}(\lambda)|\leqslant e^{\Phi(\alpha\lambda)}$, $\operatorname{Re}\lambda>\beta$, $\tilde{K}(\lambda)\neq 0$. Then, for every $t\in(0,\tau)$, there exist $\beta(t)>0$ and $M(t)>0$ such that, for every λ which belongs to the following set*

$$\Lambda_{t,\alpha,\beta(t)}:=\Big\{\lambda\in\mathbb{C}:\tilde{K}(\lambda)\neq 0,\ \operatorname{Re}\lambda\geqslant\frac{\Phi(\alpha\lambda)}{t}+\beta(t)\Big\},$$

there exists an operator $F(\lambda)\in L(E)$ so that $F(\lambda)\mathcal{A}\subseteq\mathcal{A}F(\lambda)$, $\lambda\in\Lambda_{t,\alpha,\beta(t)}$, $F(\lambda)x\in(\lambda-\mathcal{A})^{-1}x$, $\lambda\in\Lambda_{t,\alpha,\beta(t)}$, $x\in E$, $F(\lambda)x-x=F(\lambda)y$, whenever $\lambda\in\Lambda_{t,\alpha,\beta(t)}$ and $(x,y)\in\mathcal{A}$,

$$\|F(\lambda)\|\leqslant M(t)e^{\Phi(\alpha\lambda)},\quad \lambda\in\Lambda_{t,\alpha,\beta(t)},\ \tilde{K}(\lambda)\neq 0$$

and that the mapping $\lambda\mapsto F(\lambda)\in L(E)$, $\lambda\in\Lambda_{t,\alpha,\beta(t)}$ is analytic. Furthermore, the existence of a sequence (t_n) in $[0,\tau)$ satisfying $\lim_{n\to\infty}t_n=$

τ and $\sup_{n\in\mathbb{N}}\ln\|S_K(t_n)\|<\infty$ implies that there exist $\beta'>0$ and $M'>0$ such that the above holds with the region $\Lambda_{t,\alpha,\beta(t)}$ replaced by $\Lambda_{\tau,\alpha,\beta'}$ and the number $M(t)$ replaced by M'.

THEOREM 3.2.63. *Suppose that $K(t)$ satisfies* (P1), *$\mathcal{A}$ is a closed* MLO, *$r_0\geqslant\max(0,\mathrm{abs}(K))$, $\Phi\colon[r_0,\infty)\to[0,\infty)$ is a continuously differentiable, strictly increasing mapping, $\lim_{t\to\infty}\Phi(t)=+\infty$, $\Phi'(\cdot)$ is bounded on $[r_0,\infty)$ and there exist $\alpha>0$, $\gamma>0$ and $\beta>r_0$ such that, for every λ which belongs to the following set*

$$\Psi_{\alpha,\beta,\gamma}:=\Bigl\{\lambda\in\mathbb{C}:\mathrm{Re}\,\lambda\geqslant\frac{\Phi(\alpha|\,\mathrm{Im}\,\lambda|)}{\gamma}+\beta\Bigr\}$$

there exists an operator $F(\lambda)\in L(E)$ so that $F(\lambda)\mathcal{A}\subseteq\mathcal{A}F(\lambda)$, $\lambda\in\Psi_{\alpha,\beta,\gamma}$, $F(\lambda)x\in(\lambda-\mathcal{A})^{-1}Cx$, $\lambda\in\Psi_{\alpha,\beta,\gamma}$, $x\in E$, $F(\lambda)C=CF(\lambda)$, $\lambda\in\Psi_{\alpha,\beta,\gamma}$, $F(\lambda)x-Cx=F(\lambda)y$, whenever $\lambda\in\Psi_{\alpha,\beta,\gamma}$ and $(x,y)\in\mathcal{A}$, and that the mapping $\lambda\mapsto F(\lambda)x$ is analytic on $\Omega_{\alpha,\beta,\gamma}$ and continuous on $\Gamma_{\alpha,\beta,\gamma}$, where $\Gamma_{\alpha,\beta,\gamma}$ denotes the upwards oriented boundary of $\Psi_{\alpha,\beta,\gamma}$ and $\Omega_{\alpha,\beta,\gamma}$ the open region which lies to the right of $\Gamma_{\alpha,\beta,\gamma}$. Let the following conditions hold.

(i) *There exists $\sigma>0$ such that the operator family $\{e^{\Phi(\sigma|\lambda|)}F(\lambda):\lambda\in\overline{\Omega_{\alpha,\beta,\gamma}}\}\subseteq L(E)$ is equicontinuous.*
(ii) *There exists a function $m\colon[0,\infty)\to(0,\infty)$ such that $m(s)=1$, $s\in[0,1]$ and that, for every $s>1$, there exists an $r_s>r_0$ so that $\frac{\Phi(t)}{\Phi(st)}\geqslant m(s)$, $t\geqslant r_s$.*
(iii) $\lim_{t\to\infty}te^{-\Phi(\sigma t)}=0$.
(iv) $(\exists a\geqslant 0)(\exists r_a'>r_0)(\forall t>r_a')\frac{\ln t}{\Phi(t)}\geqslant a$.

Then $\mathcal{A}$ is a subgenerator of a local K-convoluted C-semigroup on $[0,a+m(\frac{\alpha}{\sigma\gamma}))$.

We can similarly formulate analogues of [**291**, Theorem 2.7.2(iii)–(iv)] for local integrated (C-)semigroups and [**291**, Theorem 2.7.3] for local K-convoluted C-cosine functions.

The proof of [**296**, Theorem 3.15] (cf. also Theorem 3.2.21 and Example 3.2.23, where we have studied the entire solutions of backward heat Poisson equation) essentially shows that the existence of numbers $r\geqslant 0$ and $\theta\in(0,\pi/2)$ and an injective operator $C\in L(E)$ such that $-\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous, analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant 0}$ of angle θ implies that there exists an operator $C_1\in L(E)$ such that $\mathcal{A}$ is a subgenerator of an entire C_1-regularized semigroup in E. Using the usual matrix reduction, we can apply this result in the analysis of problem

$$\frac{d}{dt}(Cu')-Bu'+Au(t)=F(t),\quad 0<t\leqslant T,$$
$$u(0)=u_0,\quad Cu'(0)=Cu_1,$$

provided that the conditions from [**199**, Section 6.1] hold. Multivalued matricial operators on product spaces can also serve one for construction of exponentially bounded degenerate integrated semigroups; see e.g. [**292**, Example 3.2.24].

Now we will provide two more illustrative examples of application of our abstract results.

EXAMPLE 3.2.64. Let $E := l_2(\mathbb{C})$ be the Hilbert space consisted of all square-summable complex sequences equipped with the norm $\|x\| := \|\langle x_1, x_2, \ldots, x_n, \ldots \rangle\| := (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$, $x \in E$. Let $(b_n)_{n\in\mathbb{N}}$ be any real sequence with the property that $0 \leqslant b_n \leqslant 1$, $n \in \mathbb{N}$ and let $a_n = n + (n^{-2}e^{-2n} - n^2)^{1/2}$, $n \in \mathbb{N}$. Define, for every $\langle x_1, x_2, \ldots, x_n, \ldots \rangle \in E$ and $t \in [0, 1)$,

$$T(t)\langle x_1, x_2, \ldots, x_n, \ldots \rangle := \langle b_1 e^{ta_1} x_1, b_2 e^{ta_2} x_2, \ldots, b_n e^{ta_n} x_n, \ldots \rangle,$$

as well as $C := T(0)$, $I := \{j \in \mathbb{N} : b_j \neq 0\}$ and $D \in L(E)$ by $D\langle x_1, x_2, \ldots, x_n, \ldots \rangle := \langle y_1, y_2, \ldots, y_n, \ldots \rangle$, where $y_j = a_j x_j$ for $j \in I$ and $y_j = 0$ for $j \notin I$. Then $(T(t))_{t\in[0,1)}$ is a local C-regularized semigroup with the integral generator

$$\hat{\mathcal{A}} := \{(\langle x_1, x_2, \ldots, x_n, \ldots \rangle, \langle y_1, y_2, \ldots, y_n, \ldots \rangle) : y_j = a_j x_j \text{ for all } j \in I\}$$

and any linear subspace $\mathcal{A}$ of $E \times E$ satisfying $E \times R(D) \subseteq \mathcal{A} \subseteq \hat{\mathcal{A}}$ is a subgenerator of $(T(t))_{t\in[0,1)}$.

EXAMPLE 3.2.65. Put $E := \{f \in C^{\infty}([0, \infty)) : \lim_{x\to+\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$ and $||f||_k := \sum_{j=0}^{k} \sup_{x\geqslant 0} |f^{(j)}(x)|$, $f \in X$, $k \in \mathbb{N}_0$. This calibration induces a Fréchet topology on E. Let $J = [a, b] \subseteq (0, \infty)$, and let $m_b \in C^{\infty}([0, \infty))$ satisfy $0 \leqslant m_b(x) \leqslant 1$, $x \geqslant 0$, $m_b(x) = 1$, $x \notin J$ and $m_b(x) = 0$, $x \in [a + \varepsilon, b - \varepsilon]$ for some $\varepsilon > 0$. Consider the multiplication operators $A : D(A) \to E$ and $B \in L(E)$, where $D(A) = \{f(x) \in E : (-1 - x + ie^x)f(x) \in E\}$, $Af(x) := (-1 - x + ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ $(x \geqslant 0,\ f \in E)$. Set $\mathcal{A} := B^{-1}A$. A similar line of reasoning as in Example 3.2.11(ii) shows that, for every $s > 1$, $d > 0$ and $\omega > 0$, the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \operatorname{Re}\lambda > \omega\} \subseteq L(E)$ is equicontinuous. Now we can apply Theorem 3.2.46 in order to see that $\mathcal{A}$ is the integral generator of an exponentially equicontinuous $\mathcal{L}^{-1}(e^{-d|\lambda|^{1/s}})$-convoluted semigroup on E. On the other hand, there do not exist numbers $n \in \mathbb{N}$ and $\tau > 0$ such that $\mathcal{A}$ is the integral generator of a local n-times integrated semigroup $(S_n(t))_{t\in[0,\tau)}$ on E. In actual fact, $(S_n(t))_{t\in[0,\tau)}$ must be given by the following formula

$$\begin{aligned}(S_n(t)f)(x) := \Big[\frac{e^{t(-1-x+ie^x)}}{(-1 - x + ie^x)^n} - \frac{t^{n-1}}{(n-1)!}\frac{1}{-1 - x + ie^x} - \ldots \\ - \frac{t}{(-1 - x + ie^x)^{n-1}} - \frac{1}{(-1 - x + ie^x)^n}\Big] f(x),\end{aligned}$$

for any $f \in E$, $x \geqslant 0$ and $t \in [0, \tau)$. This immediately implies that for each $t \in (0, 1)$ there exists $f_t \in E$ such that $\|S_n(t)f_t\|_{n+1} = +\infty$, which is a contradiction.

We close this subsection by enquiring into the basic properties of stationary dense multivalued linear operators, subgenerators of integrated semigroups and subgenerators of integrated cosine functions. The following definition has been introduced by P. C. Kunstmann **[363]** in single-valued case.

DEFINITION 3.2.66. A multivalued linear operator $\mathcal{A}$ is said to be stationary dense iff

$$n(\mathcal{A}) := \inf\{k \in \mathbb{N}_0 : D(\mathcal{A}^m) \subseteq \overline{D(\mathcal{A}^{m+1})} \text{ for all } m \geqslant k\} < \infty.$$

The number $n(\mathcal{A})$ is called the stationarity of $\mathcal{A}$.

REMARK 3.2.67. (i) Since the translations of a multivalued linear operator $\mathcal{A}$ have the same domains of non-negative integer powers as $\mathcal{A}$ as, we have that for each complex number λ the stationarity of $\lambda - \mathcal{A}$ is the same as that of $\mathcal{A}$.

(ii) Let us recall that a densely defined single-valued linear operator A need not be stationary dense (see e.g. [**363**, Example 1.3]) and that the validity of additional condition $\rho(A) \neq \emptyset$ implies that A must be stationary dense with $n(A) = 0$. Let us prove that the last statement continues to hold in multivalued linear setting. More precisely, let $\mathcal{A}$ be a densely defined MLO and $\rho(\mathcal{A}) \neq \emptyset$. Then $\mathcal{A}$ is stationary dense and $n(\mathcal{A}) = 0$. In order to see this, it suffices to assume that $0 \in \rho(\mathcal{A})$; cf. (i). Observe that, for every $x \in E$, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $D(\mathcal{A})$ such that $\lim_{k\to+\infty} x_k = x$. This implies that $\lim_{k\to+\infty} \mathcal{A}^{-n}x_k = \lim_{k\to+\infty}(\mathcal{A}^{-1})^n x_k = (\mathcal{A}^{-1})^n x = \mathcal{A}^{-n}x$ in E, so that $D(\mathcal{A}^{n+1})$ is dense in $R(\mathcal{A}^{-n}) = D(\mathcal{A}^n)$ for all $n \in \mathbb{N}_0$.

(iii) Let $\mathcal{A}$ be an MLO with $\rho(\mathcal{A}) \neq \emptyset$. Then part (i) and an elementary argumentation show that $n(\mathcal{A}) = \inf\{k \in \mathbb{N}_0 : D(\mathcal{A}^k) \subseteq \overline{D(\mathcal{A}^{k+1})}\}$.

Suppose that $0 < \tau \leqslant \infty$, $n \in \mathbb{N}_0$ and $\mathcal{A}$ is a closed MLO satisfying that the abstract Cauchy inclusion

$$(\mathrm{ACI})_1 : \begin{cases} u'(t) \in \mathcal{A}u(t), & t \in [0,\tau), \\ u(0) = x \end{cases}$$

has a mild solution (that is any continuous function $u(\cdot;x) \in C([0,\tau) : E)$ such that $u(0;x) = x$ and $u(t;x) - x \in \mathcal{A}\int_0^t u(s;x)\,ds$, $t \in [0,\tau)$) for all initial values $x \in D(\mathcal{A}^n)$. Let an element $x \in D(\mathcal{A}^n)$ be fixed, and let $u(\cdot;x)$ be a mild solution of $(\mathrm{ACI})_1$ (observe that we do not require the uniqueness of mild solutions here). Using Theorem 1.2.3 and partial integration, we can inductively prove that

$$(-1)^k \int_0^\tau \varphi^{(k)}(t)u(t;x)\,dt \in \mathcal{A}^k \int_0^\tau \varphi(t)u(t;x)\,dt, \quad \varphi \in \mathcal{D}_{[0,\tau)},\ k \in \mathbb{N}.$$

This, in particular, implies $u(t;x) \in \overline{D_\infty(\mathcal{A})}$, $t \in [0,\tau)$, so that $D(\mathcal{A}^n) \subseteq \overline{D_\infty(\mathcal{A})}$ and $n(\mathcal{A}) \leqslant n$. This is an extension of [**363**, Lemma 1.7] to multivalued linear case.

The following proposition is well-known in non-degenerate case [**291**, **363**].

PROPOSITION 3.2.68. *Let $n \in \mathbb{N}_0$ and $0 < \tau \leqslant \infty$.*

(i) *Suppose that $\mathcal{A}$ is a closed subgenerator of a (local) n-times integrated existence family $(S_n(t))_{t\in[0,\tau)}$. Then $n(\mathcal{A}) \leqslant n+1$, and the validity of condition $S_n(0) = \delta_{0,n}I$ implies $n(\mathcal{A}) \leqslant n$.*

(ii) *Suppose that $\mathcal{A}$ is a closed subgenerator of a (local) n-times integrated cosine existence family $(C_n(t))_{t\in[0,\tau)}$. Then $n(\mathcal{A}) \leqslant \lfloor \frac{n+2}{2} \rfloor$, and the validity of condition $C_n(0) = \delta_{0,n}I$ implies $n(\mathcal{A}) \leqslant \lfloor \frac{n+1}{2} \rfloor$.*

PROOF. (i): Suppose first that $S_n(0) = \delta_{0,n}I$. Then it suffices to show that the abstract Cauchy inclusion $(\mathrm{ACI})_1$ has a unique mild solution for all initial values $x \in D(\mathcal{A}^n)$. This follows from a simple computation showing that, for every $x \in D(\mathcal{A}^n)$, the function

$$u(t;x) = S_n(t)y_n + \frac{t^{n-1}}{(n-1)!}y_{n-1} + \cdots + ty_1 + x, \quad t \in [0,\tau)$$

is a mild solution of $(\mathrm{ACI})_1$, where the sequence $(y_j)_{1\leqslant j\leqslant n}$ is chosen so that $(x,y_1) \in \mathcal{A}$, $(y_1,y_2) \in \mathcal{A}, \ldots, (y_{n-1},y_n) \in \mathcal{A}$. The rest of (i) is a consequence of the fact that $(S_n^{[1]}(t))_{t\in[0,\tau)}$ is an $(n+1)$-times integrated semigroup with a subgenerator $\mathcal{A}$ and $S_n^{[1]}(0) = 0$. (ii): The proof of (ii) can be given by using the fact that

$$S_{n+1}(t) \equiv \begin{pmatrix} \int_0^t C_n(s)ds & \int_0^t (t-s)C_n(s)ds \\ C_n(t) - g_{n+1}(t) & \int_0^t C_n(s)ds \end{pmatrix}, \quad 0 \leqslant t < \tau$$

is an $(n+1)$-times integrated cosine existence family with a subgenerator $\mathcal{B} = \left(\begin{smallmatrix} 0 & I \\ \mathcal{A} & 0 \end{smallmatrix}\right)$ (cf. Theorem 3.2.57 and the proof of [**291**, Theorem 2.1.11(i)]) and the fact that

$$n(\mathcal{A}) = 2n(\mathcal{B}),$$

which can be verified by reasoning similarly as in the proof of [**291**, Lemma 2.1.22]. $\square$

REMARK 3.2.69. The estimates $n(\mathcal{A}) \leqslant n+1$ and $n(\mathcal{A}) \leqslant \lfloor \frac{n+2}{2} \rfloor$ obtained above cannot be refined in the degenerate case. We will explain this in the case that $n = 0$. Suppose $P \in L(E)$ and $P^2 = P$. Then $(T(t) \equiv P)_{t\geqslant 0}$ $((C(t) \equiv P)_{t\geqslant 0})$ is a strongly continuous semigroup (strongly continuous cosine operator function) with a closed subgenerator $\mathcal{A} = N(I-P) \times \{0\}$, which satisfies $n(\mathcal{A}) = 1$ provided that $N(I-P)$ is not densely defined in E (consider, for example, the matricial operator $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$).

It is also worth noting that the following extension of [**363**, Lemma 1.5] holds in our framework.

PROPOSITION 3.2.70. *Suppose that $\mathcal{A}$ is a closed* MLO, *$\alpha \geqslant -1$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers satisfying that $\lim_{n\to+\infty} |\lambda_n| = +\infty$ and the family $\{(1+|\lambda_n|)^{-\alpha}R(\lambda_n : \mathcal{A})\ ;\ n \in \mathbb{N}\} \subseteq L(E)$ is equicontinuous. Then $\mathcal{A}$ is stationary dense and $n(\mathcal{A}) \leqslant \lfloor \alpha \rfloor + 2$.*

PROOF. Without loss of generality, we may assume that $|\lambda_n| \geqslant 1$, $n \in \mathbb{N}$ and $(|\lambda_n|)_{n\in\mathbb{N}}$ is strictly increasing. Let $k \geqslant \lfloor \alpha \rfloor + 2$. We will have to prove that $D(\mathcal{A}^k) \subseteq \overline{D(\mathcal{A}^{k+1})}$. So, let $x \in D(\mathcal{A}^k)$. Then there exists a sequence $(y_j)_{1\leqslant j\leqslant k}$ such that $(x,y_1) \in \mathcal{A}$, $(y_1,y_2) \in \mathcal{A}, \ldots, (y_{k-1},y_k) \in \mathcal{A}$. Then $\lambda_n R(\lambda_n : \mathcal{A})x \in D(\mathcal{A}^{k+1})$, $n \in \mathbb{N}$ and it can be easily seen with the help of Theorem 1.2.4(i) and Theorem 1.2.8(ii) that, for every $n \in \mathbb{N}$, we have

$$\lambda_n R(\lambda_n : \mathcal{A})x - x = R(\lambda_n : \mathcal{A})y_1 \to 0, \quad n \to +\infty.$$

Hence, $D(\mathcal{A}^k) \subseteq \overline{D(\mathcal{A}^{k+1})}$ as claimed. $\square$

3.3. Degenerate C-distribution semigroups and degenerate C-ultradistribution semigroups in locally convex spaces

In our recent joint research study with S. Pilipović and D. Velinov [**354**], we have introduced and systematically analyzed the classes of C-distribution semigroups and C-ultradistribution semigroups in locally convex spaces (cf. [**65**,**99**,**109**, **177**, **282**, **289**, **291**, **351**, **364**, **365**, **391**, **424**, **488**, **530**] and references cited therein for the current state of theory). The main aim of this section is to present the most important things about the classes of degenerate C-distribution semigroups and degenerate C-ultradistribution semigroups in barreled sequentially complete locally convex spaces. We consider multivalued linear operators as infinitesimal generators of such semigroups and allow the regularizing operator C to be non-injective (cf. [**58**, **282**, **406**] and [**423**, **424**] for the primary source of information on degenerate distribution semigroups in Banach spaces). In contrast to the analyses from [**424**, Section 2.2] and [**58**, Section 3], we do not use any decomposition of the state space E. Throughout this section, we assume that $C \in L(E)$ is not necessarily injective operator as well as that $(M_p)_{p\in\mathbb{N}_0}$ is a sequence of positive real numbers satisfying $(M.1)$, $(M.2)$ and $(M.3)'$. Any employment of the condition $(M.3)$ will be explicitely emphasized. Since E is barreled, the uniform boundedness principle [**419**, p. 273] implies that each $\mathcal{G} \in \mathcal{D}'(L(E))$ ($\mathcal{G} \in \mathcal{D}'^{*}(L(E))$) is boundedly equicontinuous, i.e., that for every $p \in \circledast$ and for every bounded subset B of $\mathcal{D}$ ($\mathcal{D}^*$), there exist $c > 0$ and $q \in \circledast$ such that $p(\mathcal{G}(\varphi)x) \leqslant cq(x)$, $\varphi \in B$, $x \in E$.

Definition 3.3.1. Let $\mathcal{G} \in \mathcal{D}'_0(L(E))$ ($\mathcal{G} \in \mathcal{D}'^{*}_0(L(E))$) satisfy $C\mathcal{G} = \mathcal{G}C$. Then it is said that $\mathcal{G}$ is a pre-(C-DS) (pre-(C-UDS) of $*$-class) iff the following holds:

$$\text{(C.S.1)}\qquad \mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi), \quad \varphi,\ \psi \in \mathcal{D}\ \ (\varphi,\ \psi \in \mathcal{D}^*).$$

If, additionally,

$$\text{(C.S.2)}\qquad \mathcal{N}(\mathcal{G}) := \bigcap_{\varphi\in\mathcal{D}_0} N(\mathcal{G}(\varphi)) = \{0\} \quad \Bigg(\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi\in\mathcal{D}_0^*} N(\mathcal{G}(\varphi)) = \{0\}\Bigg),$$

then $\mathcal{G}$ is called a C-distribution semigroup (C-ultradistribution semigroup of $*$-class), (C-DS) ((C-UDS) of $*$-class) in short. A pre-(C-DS) $\mathcal{G}$ is called dense iff

$$\text{(C.S.3)}\ \ \mathcal{R}(\mathcal{G}) := \bigcup_{\varphi\in\mathcal{D}_0} R(\mathcal{G}(\varphi)) \text{ is dense in } E \ \Bigg(\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi\in\mathcal{D}_0^*} R(\mathcal{G}(\varphi)) \text{ is dense in } E\Bigg).$$

The notion of a dense pre-(C-UDS) $\mathcal{G}$ of $*$-class (and the set $\mathcal{R}(\mathcal{G})$) is defined similarly.

If $C = I$, then we also write pre-(DS), pre-(UDS), (DS), (UDS), respectively, instead of pre-(C-DS), pre-(C-UDS), (C-DS), (C-UDS).

Suppose that $\mathcal{G}$ is a pre-(C-DS) (pre-(C-UDS) of $*$-class). Then $\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi)$ for all $\varphi,\ \psi \in \mathcal{D}$ ($\varphi, \psi \in \mathcal{D}^*$), and $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

The structural characterization of a pre-(C-DS) $\mathcal{G}$ (pre-(C-UDS) $\mathcal{G}$ of $*$-class) on its kernel space $\mathcal{N}(\mathcal{G})$ is described in the following theorem (cf. [**291**, Proposition 3.1.1] and the proofs of [**364**, Lemma 2.2], [**291**, Proposition 3.5.4]).

THEOREM 3.3.2. (i) *Let $\mathcal{G}$ be a pre-*(C-DS)*, and let the space $L(\mathcal{N}(\mathcal{G}))$ be admissible. Then, with $N = \mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N ($G_1 = \mathcal{G}_{|N}$), we have: There exists an integer $m \in \mathbb{N}$ for which there exist unique operators $T_0, T_1, \ldots, T_m \in L(\mathcal{N}(\mathcal{G}))$ commuting with C so that $G_1 = \sum_{j=0}^{m} \delta^{(j)} \otimes T_j$, $T_iC^i = (-1)^i T_0^{i+1}$, $0 \leqslant i \leqslant m-1$ and $T_0T_m = T_0^{m+2} = 0$.*

(ii) *Let (M_p) satisfy (M.3), let $\mathcal{G}$ be a pre-*(C-UDS) *of $*$-class, and let the space $\mathcal{N}(\mathcal{G})$ be barreled. Then, with $N = \mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N ($G_1 = \mathcal{G}_{|N}$), we have: There exists a unique set of operators $(T_j)_{j\in\mathbb{N}_0}$ in $L(\mathcal{N}(\mathcal{G}))$ commuting with C so that $G_1 = \sum_{j=0}^{\infty} \delta^{(j)} \otimes T_j$, $T_jC^j = (-1)^j T_0^{j+1}$, $j \in \mathbb{N}$ and the set $\{M_jT_jL^j : j \in \mathbb{N}_0\}$ is bounded in $L(\mathcal{N}(\mathcal{G}))$, for some $L > 0$ in the Beurling case, resp. for every $L > 0$ in the Roumieu case.*

Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$), and let $T \in \mathcal{E}_0'$ ($T \in \mathcal{E}_0'^{*}$), i.e., T is a scalar-valued distribution (ultradistribution of $*$-class) with compact support contained in $[0, \infty)$. Define

$$G(T) := \{(x, y) \in E \times E : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0 \ \ (\varphi \in \mathcal{D}_0^*)\}.$$

Then it can be easily seen that $G(T)$ is a closed MLO; furthermore, if $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$) satisfy (C.S.2), then $G(T)$ is a closed linear operator. Assuming that the regularizing operator C is injective, definition of $G(T)$ can be equivalently introduced by replacing the set $\mathcal{D}_0$ ($\mathcal{D}_0^*$) with the set $\mathcal{D}_{[0,\varepsilon)}$ ($\mathcal{D}^*_{[0,\varepsilon)}$) for any $\varepsilon > 0$. In general case, for every $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$), we have $\psi_+ := \psi \mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'$ ($\mathcal{E}_0'^{*}$), where $\mathbf{1}_{[0,\infty)}$ stands for the characteristic function of $[0, \infty)$, so that the definition of $G(\psi_+)$ is clear. We define the (infinitesimal) generator of a pre-(C-DS) $\mathcal{G}$ by $\mathcal{A} := G(-\delta')$ (cf. [**354**] for more details about non-degenerate case, and [**58**, Definition 3.4] and [**282**] for some other approaches used in the degenerate case). Then $\mathcal{N}(\mathcal{G}) \times \mathcal{N}(\mathcal{G}) \subseteq \mathcal{A}$ and $\mathcal{N}(\mathcal{G}) = \mathcal{A}0$, which simply implies that $\mathcal{A}$ is single-valued iff (C.S.2) holds. If this is the case, then we also have that the operator C must be injective: Suppose that $Cx = 0$ for some $x \in E$. By (C.S.1), we get that $\mathcal{G}(\varphi)\mathcal{G}(\psi)x = 0$, φ, $\psi \in \mathcal{D}$. In particular, $\mathcal{G}(\psi)x \in \mathcal{N}(\mathcal{G}) = \{0\}$ so that $\mathcal{G}(\psi)x = 0$, $\psi \in \mathcal{D}$. Hence, $x \in \mathcal{N}(\mathcal{G}) = \{0\}$ and therefore $x = 0$. Further on, if $\mathcal{G}$ is a pre-(C-DS) (pre-(C-UDS) of $*$-class), $T \in \mathcal{E}_0'$ ($T \in \mathcal{E}_0'^{*}$) and $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$), then $\mathcal{G}(\varphi)G(T) \subseteq G(T)\mathcal{G}(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$. If $\mathcal{G}$ is a pre-(C-DS) (pre-(C-UDS) of $*$-class) and φ, $\psi \in \mathcal{D}$ (φ, $\psi \in \mathcal{D}^*$), then the assumption $\varphi(t) = \psi(t)$, $t \geqslant 0$, implies $\mathcal{G}(\varphi) = \mathcal{G}(\psi)$. As in the Banach space case, we can prove the following (cf. [**291**, Proposition 3.1.3, Lemma 3.1.6]): Suppose that $\mathcal{G}$ is a pre-(C-DS) (pre-(C-UDS) of $*$-class). Then $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}$, $x \in E$ ($\psi \in \mathcal{D}^*$, $x \in E$) and $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$, while $C^{-1}\mathcal{A}C = \mathcal{A}$ provided that C is injective. Furthermore, the following holds:

PROPOSITION 3.3.3. *Let $\mathcal{G}$ be a pre-*(C-DS) *(pre-*(C-UDS) *of $*$-class), S, $T \in \mathcal{E}_0'$ (S, $T \in \mathcal{E}_0'^{*}$), $\varphi \in \mathcal{D}_0$ ($\varphi \in \mathcal{D}_0^*$), $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$) and $x \in E$. Then we have:*

(i) $(\mathcal{G}(\varphi)x,\ \mathcal{G}(\overbrace{T*\cdots*T}^{m}*\varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.
(ii) $G(S)G(T) \subseteq G(S*T)$ *with* $D(G(S)G(T)) = D(G(S*T)) \cap D(G(T))$, *and* $G(S) + G(T) \subseteq G(S+T)$.
(iii) $(\mathcal{G}(\psi)x,\ \mathcal{G}(-\psi')x - \psi(0)Cx) \in G(-\delta')$.
(iv) *If* $\mathcal{G}$ *is dense, then its generator is densely defined.*

The assertions (ii)-(vi) of [**291**, Proposition 3.1.2] can be reformulated for pre-(C-DS)'s (pre-(C-UDS)'s of $*$-class) in locally convex spaces; here it is only worth noting that the reflexivity of state space E implies that the spaces E^* and $E^{**} = E$ are both barreled and sequentially complete:

PROPOSITION 3.3.4. *Let* $\mathcal{G}$ *be a pre-*(C-DS) *(pre-*(C-UDS) *of* $*$*-class). Then the following holds:*

(i) $C(\overline{\langle \mathcal{R}(\mathcal{G}) \rangle}) \subseteq \overline{\mathcal{R}(\mathcal{G})}$, *where* $\langle \mathcal{R}(\mathcal{G}) \rangle$ *denotes the linear span of* $\mathcal{R}(\mathcal{G})$.
(ii) *Assume* $\mathcal{G}$ *is not dense and* $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. *Put* $R := \overline{\mathcal{R}(\mathcal{G})}$ *and* $H := \mathcal{G}_{|R}$. *Then* H *is a dense pre-*(C_1-DS) *(pre-*(C_1-UDS) *of* $*$*-class) on* R *with* $C_1 = C_{|R}$.
(iii) *The dual* $\mathcal{G}(\cdot)^*$ *is a pre-*(C^*-DS) *(pre-*(C^*-UDS) *of* $*$*-class) on* E^* *and* $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.
(iv) *If* E *is reflexive, then* $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.
(v) *The* $\mathcal{G}^*$ *is a* (C^*-DS) *((*C^*-UDS) *of* $*$*-class) in* E^* *iff* $\mathcal{G}$ *is a dense pre-*(C-DS) *(pre-*(C-UDS) *of* $*$*-class). If* E *is reflexive, then* $\mathcal{G}^*$ *is a dense pre-*(C^*-DS) *(pre-*(C^*-UDS) *of* $*$*-class) in* E^* *iff* $\mathcal{G}$ *is a* (C-DS) *((*C-UDS) *of* $*$*-class).*

The following proposition has been considered for the first time by J. Kisyński in [**282**, Proposition 2] (E Banach space, $C = I$).

PROPOSITION 3.3.5. *Suppose that* $\mathcal{G} \in \mathcal{D}_0'(L(E))$ $(\mathcal{G} \in \mathcal{D}_0'^*(L(E)))$ *and* $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ $(\varphi \in \mathcal{D}^*)$. *Then* $\mathcal{G}$ *is a pre-*(C-DS) *(pre-*(C-UDS) *of* $*$*-class) iff*

$$\mathcal{G}(\varphi')\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}(\psi') = \psi(0)\mathcal{G}(\varphi)C - \varphi(0)\mathcal{G}(\psi)C, \quad \varphi, \psi \in \mathcal{D} \ (\varphi, \psi \in \mathcal{D}^*). \tag{320}$$

PROOF. If $\mathcal{G}$ satisfies (C.S.1), then (320) follows immediately from (C.S.1) and the equality $\varphi' *_0 \psi - \varphi *_0 \psi' = \psi(0)\varphi - \varphi(0)\psi$, $\varphi, \psi \in \mathcal{D}$ $(\varphi, \psi \in \mathcal{D}^*)$. Suppose now that (320) holds, $\varphi, \psi \in \mathcal{D}$ $(\varphi, \psi \in \mathcal{D}^*)$, $a > 0$ and $\operatorname{supp}(\psi) \subseteq (-\infty, a]$. Since $\mathcal{G} \in \mathcal{D}_0'(L(E))$ $(\mathcal{G} \in \mathcal{D}_0'^*(L(E)))$, and the function $t \mapsto \int_0^a [\varphi(t-s)\psi(s) - \varphi(-s)\psi(t+s)]\,ds$, $t \in \mathbb{R}$ belongs to $\mathcal{D}$ $(\mathcal{D}^*)$ with $(\varphi *_0 \psi)(t) = \int_0^a [\varphi(t-s)\psi(s) - \varphi(-s)\psi(t+s)]\,ds$, $t \geqslant 0$, we have

$$\begin{aligned}
\mathcal{G}(\varphi *_0 \psi)Cx &= \mathcal{G}\int_0^a [\varphi(\cdot - s)\psi(s) - \varphi(-s)\psi(\cdot + s)]Cx\,ds \\
&= \int_0^a [\psi(s)\mathcal{G}(\varphi(\cdot - s))Cx - \varphi(-s)\mathcal{G}(\psi(\cdot + s))Cx]ds \\
&= \int_0^a [\mathcal{G}(\varphi'(\cdot - s))\mathcal{G}(\psi(\cdot + s))x - \mathcal{G}(\varphi(\cdot - s))\mathcal{G}(\psi'(\cdot + s))x]ds \qquad (321)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^a \frac{d}{ds}[\mathcal{G}(\varphi(\cdot - s))\mathcal{G}(\psi(\cdot + s))x]ds \qquad (322)\\
&= \mathcal{G}(\varphi)\mathcal{G}(\psi)x - \mathcal{G}(\varphi(\cdot - a))\mathcal{G}(\psi(\cdot + a))x\\
&= \mathcal{G}(\varphi)\mathcal{G}(\psi)x - \mathcal{G}(\varphi(\cdot - a))0x = \mathcal{G}(\varphi)\mathcal{G}(\psi)x,
\end{aligned}
$$

for any $x \in E$ and $\varphi, \psi \in \mathcal{D}$ ($\varphi, \psi \in \mathcal{D}^*$), where (321) follows from an application of (320), and (322) from an elementary argumentation involving the continuity of $\mathcal{G}$ as well as the facts that for each function $\zeta \in \mathcal{D}$ ($\zeta \in \mathcal{D}^*$) we have that $\lim_{h\to 0}(\tau_h\zeta) = \zeta$ in $\mathcal{D}$ ($\mathcal{D}^*$), $\lim_{h\to 0} \frac{1}{h}(\tau_h\zeta - \zeta) = \zeta'$ in $\mathcal{D}$ ($\mathcal{D}^*$) and the set $\{\tau_h\zeta : |h| \leqslant 1\}$ is bounded in $\mathcal{D}$ ($\mathcal{D}^*$). The proof of proposition is complete. $\square$

In [**351**], we have recently proved that every (C-DS) ((C-UDS) of $*$-class) in locally convex space is uniquely determined by its generator. Contrary to the single-valued case, different pre-(C-DS)'s (pre-(C-UDS)'s of $*$-class) can have the same generator. To see this, we can employ [**364**, Example 2.3]: Let $C = I$, E is a Banach space and $T \in L(E)$ is nilpotent of order $n \geqslant 2$. Then the pre-(C-DS)'s $\mathcal{G}_1(\cdot) \equiv \sum_{i=0}^{n-2} \cdot^{(i)}(0)T^{i+1}$ and $\mathcal{G}_2(\cdot) \equiv 0$ have the same generator $\mathcal{A} \equiv E \times E$. In Theorem 3.3.6 and Theorem 3.3.8, we clarify connections between degenerate C-distribution semigroups and degenerate local integrated C-semigroups. For the proof of first theorem, we need some preliminaries from our previous research study of distribution cosine functions (see e.g. [**291**, Section 3.4]): Let $\eta \in \mathcal{D}_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty} \eta(t)dt = 1$. Then, for every fixed $\varphi \in \mathcal{D}$, we define $I(\varphi)$ as follows

$$
I(\varphi)(x) := \int_{-\infty}^{x} \left[\varphi(t) - \eta(t)\int_{-\infty}^{\infty} \varphi(u)du\right]dt, \quad x \in \mathbb{R}. \tag{323}
$$

It can be simply verified that, for every $\varphi \in \mathcal{D}$ and $n \in \mathbb{N}$, we have $I(\varphi) \in \mathcal{D}$, $I^n(\varphi^{(n)}) = \varphi$, $\frac{d}{dx}I(\varphi)(x) = \varphi(x) - \eta(x)\int_{-\infty}^{\infty} \varphi(u)du$, $x \in \mathbb{R}$ as well as that, for every $\varphi \in \mathcal{D}_{[a,b]}$ ($-\infty < a < b < \infty$), we have: $\operatorname{supp}(I(\varphi)) \subseteq [\min(-2, a), \max(-1, b)]$. This simply implies that, for every $\tau > 2$, $-1 < b < \tau$ and for every m, $n \in \mathbb{N}$ with $m \leqslant n$, we have:

$$
I^n(\mathcal{D}_{(-\tau,b]}) \subseteq \mathcal{D}_{(-\tau,b]} \text{ and } \frac{d^m}{dx^m}I^n(\varphi)(x) = I^{m-n}\varphi(x), \quad \varphi \in \mathcal{D}, \ x \geqslant 0, \tag{324}
$$

where $I^0\varphi := \varphi$, $\varphi \in \mathcal{D}$. Now we are ready to show the following extension of [**364**, Proposition 4.3 a)], given here with a different proof.

THEOREM 3.3.6. *Let $\mathcal{G}$ be a pre-*(C-DS) *generated by $\mathcal{A}$, and let $\mathcal{G}$ be of finite order. Then, for every $\tau > 0$, there exist a number $n_\tau \in \mathbb{N}$ and a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$ such that*

$$
\mathcal{G}(\varphi)x = (-1)^{n_\tau} \int_0^\infty \varphi^{(n_\tau)}(s)S_{n_\tau}(s)x\, ds, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)}, \ x \in E. \tag{325}
$$

Furthermore, $(S_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated C-existence family with a subgenerator $\mathcal{A}$, and the admissibility of space $L(\mathcal{N}(\mathcal{G}))$ implies that $S_{n_\tau}(t)x = 0$, $t \in [0, \tau)$ for some $x \in \mathcal{N}(\mathcal{G})$ iff $T_ix = 0$ for $0 \leqslant i \leqslant n_\tau - 1$; see Theorem 3.3.2(i) *with $m \geqslant n_\tau - 1$.*

PROOF. Let $\tau > 2$ and $\rho \in \mathcal{D}_{[0,1]}$ with $\int \rho \, dm = 1$ be fixed. Set $\rho_n(\cdot) := n\rho(n\cdot)$, $n \in \mathbb{N}$. Then, for every $t \in [0,\tau)$, the sequence $\rho_n^t(\cdot) := \rho_n(\cdot - t)$ converges to δ_t as $n \to +\infty$ (in the space of scalar-valued distributions). Since $\mathcal{G}$ is of finite order, we know that there exist a number $n_\tau \in \mathbb{N}$ and a strongly continuous operator family $(S_{n_\tau}(t))_{t\in[0,\tau)} \subseteq L(E)$ such that (325) holds good. We will first prove that $(S_{n_\tau}(t))_{t\in[0,\tau)}$ is a local n_τ-times integrated C-existence family which commutes with C and do have $\mathcal{A}$ as a subgenerator. In order to do that, observe that the commutation of $\mathcal{G}(\cdot)$ and C yields

$$\int_0^\infty \varphi^{(n_\tau)}(s)CS_{n_\tau}(s)x\,ds = \int_0^\infty \varphi^{(n_\tau)}(s)S_{n_\tau}(s)Cx\,ds, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E.$$

Plugging $\varphi = I^{n_\tau}(\rho_n^t)$ in this expression (cf. also (324)), we get that

$$\int_0^\infty \rho_n^t(s)CS_{n_\tau}(s)x\,ds = \int_0^\infty \rho_n^t(s)S_{n_\tau}(s)Cx\,ds, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E,\ t \in [0,\tau).$$

Letting $n \to +\infty$ we obtain $CS_{n_\tau}(t)x = S_{n_\tau}(t)Cx$, $x \in E$, $t \in [0,\tau)$. Now we will prove that the condition (B) holds with the number α replaced with the number n_τ therein. By Proposition 3.3.3(iii), we have $(\mathcal{G}(\varphi)x, \mathcal{G}(-\varphi')x - \varphi(0)Cx) \in \mathcal{A}$, $\varphi \in \mathcal{D}$, $x \in E$. Applying integration by parts and multiplying with $(-1)^{n_\tau+1}$ after that, the above implies

$$\biggl(\int_0^\infty \varphi^{(n_\tau+1)}(s)\int_0^s S_{n_\tau}(r)x\,dr\,ds, \int_0^\infty \varphi^{(n_\tau+1)}(s)S_{n_\tau}(s)x\,ds + (-1)^{n_\tau}\varphi(0)Cx\biggr) \in \mathcal{A},$$

for any $\varphi \in \mathcal{D}_{(-\tau,\tau)}$ and $x \in E$. Plugging $\varphi = I^{n_\tau+1}(\rho_n^t)$ in this expression, we get that

$$\biggl(\int_0^\infty \rho_n^t(s)\int_0^s S_{n_\tau}(r)x\,dr\,ds, \int_0^\infty \rho_n^t(s)S_{n_\tau}(s)x\,ds + (-1)^{n_\tau}I^{n_\tau+1}(\rho_n^t)(0)Cx\biggr) \in \mathcal{A}, \tag{326}$$

for any $t \in [0,\tau)$ and $x \in E$. Let us prove that

$$\lim_{n\to+\infty} I^{n_\tau+1}(\rho_n^t)(x) = (-1)^{n_\tau+1}g_{n_\tau+1}(t-x), \quad t \in [0,\tau),\ 0 \leqslant x \leqslant t. \tag{327}$$

Let $t \in [0,\tau)$ and $x \in [0,t]$ be fixed. Then a straightforward integral calculation shows that

$$I^{n_\tau+1}(\varphi)(x) = (-1)^{n_\tau+1}\int_x^\infty \int_{x_{n_\tau}}^\infty \int_{x_{n_\tau-1}}^\infty \cdots \int_{x_2}^\infty \varphi(x_1)dx_1dx_2\dots dx_{n_\tau+1}$$

for any $\varphi \in \mathcal{D}$. For $\varphi = I^{n_\tau+1}(\rho_n^t)$, we have

$$\begin{aligned} I^{n_\tau+1}(\rho_n^t)(0) &= (-1)^{n_\tau+1}\int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_{x_2}^{t+(1/n)} \\ &\quad \times \rho_n^t(x_1)dx_1dx_2\dots dx_{n_\tau+1} \\ &= (-1)^{n_\tau+1}\int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_{x_3}^{t+(1/n)} \\ &\quad \times \biggl[1 - \int_0^{nx_2-nt}\rho(x_1)dx_1\biggr]dx_2\dots dx_{n_\tau+1} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n_\tau+1}\int_x^{t+(1/n)}\int_{x_{n_\tau}}^{t+(1/n)}\int_{x_{n_\tau-1}}^{t+(1/n)}\cdots\int_{x_3}^{t+(1/n)} \\
&\quad\times dx_2\dots dx_{n_\tau+1} \\
&\qquad - (-1)^{n_\tau+1}\int_x^{t+(1/n)}\int_{x_{n_\tau}}^{t+(1/n)}\int_{x_{n_\tau-1}}^{t+(1/n)}\cdots\int_{t}^{t+(1/n)} \\
&\quad\times \int_0^{nx_2-nt}\rho(x_1)dx_1dx_2\dots dx_{n_\tau+1} \\
&:= (-1)^{n_\tau+1}[I_1(t,x,n)-I_2(t,x,n)],\quad t\in[0,\tau).
\end{aligned}
$$

Since

$$
\int_t^{t+(1/n)}\int_0^{nx_2-nt}\rho(x_1)dx_1dx_2 \leqslant 1/n,\quad t\in[0,\tau),\ n\in\mathbb{N},
$$

we have that $\lim_{n\to+\infty} I_2(t,x,n)=0$, $t\in[0,\tau)$. Clearly,

$$
\lim_{n\to+\infty} I_1(t,x,n)=\int_x^t\int_{x_{n_\tau}}^t\int_{x_{n_\tau-1}}^t\cdots\int_{x_3}^t dx_2\dots dx_{n_\tau+1}=g_{n_\tau+1}(t-x).
$$

This gives (327). Keeping in mind this equality and letting $n\to+\infty$ in (326), we obtain (B). It remains to be proved the semigroup property of $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Toward this end, let us recall that

$$
(328)\quad (\varphi*_0\psi)^{(n_\tau)}(u)=(\varphi^{(n_\tau)}*_0\psi)(u)+\sum_{j=0}^{n_\tau-1}\varphi^{(j)}(0)\psi^{(n_\tau-1-j)}(u),\ \ \varphi,\psi\in\mathcal{D},\ u\in\mathbb{R}.
$$

Fix $x\in E$ and $t,\ s\in[0,\tau)$ with $t+s\in[0,\tau)$. Using (328), (C.S.1) and the foregoing arguments, we get that, for every $m,n\in\mathbb{N}$ sufficiently large:

$$
\begin{aligned}
&\int_0^t\int_0^s\rho_n^t(u)\rho_m^s(v)S_{n_\tau}(u)S_{n_\tau}(v)x\,du\,dv \\
&= (-1)^{n_\tau}\int_0^{t+s}\Bigl[(\rho_n^t *_0 I^{n_\tau}(\rho_m^s))(u)+\sum_{j=0}^{n_\tau-1}I^{n_\tau-j}(\rho_n^t)(0)I^{j+1}(\rho_m^s)(u)\Bigr]S_{n_\tau}(u)Cx\,du.
\end{aligned}
$$

Letting $n\to+\infty$, we obtain with the help of (327) that

$$
\begin{aligned}
&\int_0^s\rho_m^s(v)S_{n_\tau}(t)S_{n_\tau}(v)x\,dv \\
&\qquad= (-1)^{n_\tau}\lim_{n\to+\infty}\int_0^{t+s}\Bigl[\Bigl(\rho_n^t *_0 I^{n_\tau}(\rho_m^s)\Bigr)(u) \\
&\qquad\quad+\sum_{j=0}^{n_\tau-1}I^{n_\tau-j}(\rho_n^t)(0)I^{j+1}(\rho_m^s)(u)\Bigr]S_{n_\tau}(u)Cx\,du \\
&\qquad= (-1)^{n_\tau}\int_0^t\Bigl[\sum_{j=0}^{n_\tau-1}(-1)^{n_\tau-j}g_{n_\tau-j}(t)I^{j+1}(\rho_m^s)(u)\Bigr]S_{n_\tau}(u)Cx\,du
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{n_\tau}\int_t^{t+s}\Biggl[I^{n_\tau}(\rho_m^s)(u-t)\\
&+\sum_{j=0}^{n_\tau-1}(-1)^{n_\tau-j}g_{n_\tau-j}(t)I^{j+1}(\rho_m^s)(u)\Biggr]S_{n_\tau}(u)Cx\,du\\
=&\sum_{j=0}^{n_\tau-1}(-1)^j g_{n_\tau-j}(t)\int_0^s I^{j+1}(\rho_m^s)(u)S_{n_\tau}(u)Cx\,du\\
&+(-1)^{n_\tau}\int_t^{t+s} I^{n_\tau}(\rho_m^s)(u-t)S_{n_\tau}(u)Cx\,du.
\end{aligned}
$$

The semigroup property now easily follows by letting $m\to+\infty$ in the above expression, with the help of (327) and the identity

$$
\sum_{j=0}^{n_\tau-1} g_{n_\tau-j}(t)g_{j+1}(s-u)=g_{n_\tau}(t+s-u),\quad u>0.
$$

The rest essentially follows as in the proof of [**364**, Proposition 4.3 a)]. $\square$

REMARK 3.3.7. (i) We have already seen that $\mathcal{G}(\cdot)\equiv 0$ is a degenerate pre-distribution semigroup with the generator $\mathcal{A}\equiv E\times E$. Then, for every $\tau>0$ and for every number $n_\tau\in\mathbb{N}$, there exists only one local n_τ-times integrated semigroup $(S_{n_\tau}(t)\equiv 0)_{t\in[0,\tau)}$ so that (325) holds. It is clear that the condition (B) holds and that condition (A) does not hold here. Denote by $\mathcal{A}_\tau$ the integral generator of $(S_{n_\tau}(t)\equiv 0)_{t\in[0,\tau)}$. Then $\mathcal{A}_\tau=\{0\}\times E$ is strictly contained in the integral generator $\mathcal{A}$ of $\mathcal{G}$. Furthermore, if $C\neq 0$, then there do not exist $\tau>0$ and $n_\tau\in\mathbb{N}$ such that $\mathcal{A}$ is the integral generator (subgenerator) of a local n_τ-times integrated C-semigroup.

(ii) A similar line of reasoning as in the final part of the proof of [**291**, Theorem 3.1.9] shows that for each $(x,y)\in\mathcal{A}$ there exist elements $x_0,x_1,\ldots,x_{n_\tau}$ in E such that

$$
S_{n_\tau}(t)x-g_{n_\tau+1}(t)Cx-\int_0^t S_{n_\tau}(s)y\,ds=\sum_{j=0}^{n_\tau} g_{j+1}(t)x_j,\quad t\in[0,\tau)
$$

and $x_j\in\mathcal{A}x_{j-1}$ for $1\leqslant j\leqslant n_\tau$. In purely mutivalued case, it is not clear how we can prove that $x_j=0$ for $0\leqslant j\leqslant n_\tau$ without imposing some additional displeasing conditions; if $\mathcal{A}$ is single-valued, then it can be easily seen that $x_j=0$ for $0\leqslant j\leqslant n_\tau$ so that $(S_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated C-semigroup with a subgenerator $\mathcal{A}$.

(iii) Using dualization, we can simply reformulate the second equality appearing on the second line after the equation [**364**, (11)] in our context.

The proof of subsequent theorem is very similar to that of [**291**, Theorem 3.1.8].

THEOREM 3.3.8. *Suppose that there exists a sequence* $((p_k,\tau_k))_{k\in\mathbb{N}_0}$ *in* $\mathbb{N}_0\times(0,\infty)$ *such that* $\lim_{k\to\infty}\tau_k=\infty$, $(p_k)_{k\in\mathbb{N}_0}$ *and* $(\tau_k)_{k\in\mathbb{N}_0}$ *are strictly increasing, as* *well as that for each $k \in \mathbb{N}_0$ there exists a local p_k-times integrated C-semigroup $(S_{p_k}(t))_{t\in[0,\tau_k)}$ on E so that*

$$S_{p_m}(t)x = (g_{p_m-p_k} *_0 S_{p_k}(\cdot)x)(t), \quad x \in E,\ t \in [0,\tau_k), \tag{329}$$

provided $k < m$. Define

$$\mathcal{G}(\varphi)x := (-1)^{p_k} \int_0^\infty \varphi^{(p_k)}(t) S_{p_k}(t)x\, dt, \quad \varphi \in \mathcal{D}_{(-\infty,\tau_k)},\ x \in E,\ k \in \mathbb{N}_0.$$

Then $\mathcal{G}$ is well-defined and $\mathcal{G}$ is a pre-(C-DS).

REMARK 3.3.9.
- (i) Denote by $\mathcal{A}_k$ the integral generator of $(S_{p_k}(t))_{t\in[0,\tau_k)}$ $(k \in \mathbb{N}_0)$. Then $\mathcal{A}_k \subseteq \mathcal{A}_m$ for $k > m$ and $\bigcap_{k\in\mathbb{N}_0} \mathcal{A}_k \subseteq \mathcal{A}$, where $\mathcal{A}$ is the integral generator of $\mathcal{G}$. Even in the case that $C = I$, $\bigcup_{k\in\mathbb{N}_0} \mathcal{A}_k$ can be a proper subset of $\mathcal{A}$.
- (ii) Suppose that $\mathcal{A}$ is a subgenerator of $(S_{p_k}(t))_{t\in[0,\tau_k)}$ for all $k \in \mathbb{N}_0$. Then (329) automatically holds.
- (iii) In the case that $C = I$, then it suffices to suppose that there exists an MLO $\mathcal{A}$ such that $\mathcal{A}$ is a subgenerator of a local p-times integrated semigroup $(S_p(t))_{t\in[0,\tau)}$ for some $p \in \mathbb{N}$ and $\tau > 0$ [**329**].

Suppose that $\alpha \in (0,\infty) \smallsetminus \mathbb{N}$ and $f \in C([0,\infty) : E)$. Set $f_{n-\alpha}(t) := (g_{n-\alpha} * f)(t)$, $t \geqslant 0$. Making use of the dominated convergence theorem, and the change of variables $s \mapsto s - t$, we get that

$$\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} \varphi(s)ds = \int_0^\infty g_{n-\alpha}(s)\varphi^{(n)}(t+s)ds, \quad t \geqslant 0,\ \varphi \in \mathcal{D}.$$

Hence,

$$\begin{aligned}\int_0^\infty W_+^\alpha \varphi(t) f(t)dt &= (-1)^n \int_0^\infty g_{n-\alpha}(s)\varphi^{(n)}(t+s) f(t)ds\, dt \\ &= (-1)^n \int_0^\infty \int_0^t \varphi^{(n)}(t) g_{n-\alpha}(s) f(t-s)ds\, dt \\ &= (-1)^n \int_0^\infty \varphi^{(n)}(t) f_{n-\alpha}(t)dt, \quad \varphi \in \mathcal{D},\end{aligned}$$

where W_+^α denotes the Weyl fractional derivative. Therefore, if $\mathcal{A}$ is the integral generator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geqslant 0}$ on E, then we have that:

$$\int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x\, dt = (-1)^n \int_0^\infty \varphi^{(n)}(t) S_n(t)x\, dt, \quad x \in E,\ \varphi \in \mathcal{D},$$

with $(S_n(t))_{t\geqslant 0}$ being the global n-times integrated C-semigroup generated by $\mathcal{A}$. Keeping in mind this equality and the proof of [**291**, Theorem 3.1.8], we can deduce the following:

THEOREM 3.3.10. *Assume that $\alpha \geqslant 0$ and $\mathcal{A}$ is the integral generator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geqslant 0}$ on E. Set*

$$\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x\, dt, \quad x \in E,\ \varphi \in \mathcal{D}.$$

Then $\mathcal{G}$ is a pre-(C-DS) *whose integral generator contains $\mathcal{A}$.*

We will accept the following definition of an exponential pre-(C-DS) (pre-(C-UDS) of $*$-class).

DEFINITION 3.3.11. Let $\mathcal{G}$ be a pre-(C-DS) (pre-(C-UDS) of $*$-class). Then $\mathcal{G}$ is said to be an exponential pre-(C-DS) (pre-(C-UDS) of $*$-class) iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E))$ ($e^{-\omega t}\mathcal{G} \in \mathcal{S}'^*(L(E))$). We use the shorthand pre-(C-EDS) (pre-(C-EUDS) of $*$-class) to denote an exponential pre-(C-DS) (pre-(C-UDS) of $*$-class).

We have the following fundamental result:

THEOREM 3.3.12. *Assume that $\alpha \geqslant 0$ and $\mathcal{A}$ generates an exponentially equicontinuous α-times integrated C-semigroup $(S_\alpha(t))_{t\geqslant 0}$. Define $\mathcal{G}$ through $\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-*(C-EDS) *whose integral generator contains $\mathcal{A}$.*

REMARK 3.3.13. (i) Suppose that $\mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A}$, $\omega \in \mathbb{R}$ and $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E))$. Suppose, further, that there exist a non-negative integer n and a continuous function $V : \mathbb{R} \to L(E)$ satisfying that
$$\langle e^{-\omega t}\mathcal{G}, \varphi\rangle = (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t)V(t)dt, \quad \varphi \in \mathcal{D},$$
and that there exists a number $r \geqslant 0$ such that the operator family $\{(1+t^r)^{-1}V(t) : t \geqslant 0\} \subseteq L(E)$ is equicontinuous. Since $e^{-\omega \cdot}\mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A}-\omega$, the proof of Theorem 3.3.6 shows that $(V(t))_{t\geqslant 0}$ is an exponentially equicontinuous n-times integrated C-semigroup; by Theorem 3.3.12, the integral generator $\hat{\mathcal{A}}^\omega$ of $(V(t))_{t\geqslant 0}$ is contained in $\mathcal{A} - \omega$. Define
$$S_n(t)x := e^{\omega t}V(t)x + \int_0^t \sum_{k=1}^{\infty} \binom{n}{k} \frac{(-1)^k \omega^k (t-s)^{k-1}}{(k-1)!} e^{\omega s}V(s)x\,ds.$$
Arguing as in the proof of [**291**, Theorem 2.5.1, Theorem 2.5.3], we can prove that the MLO $\hat{\mathcal{A}}^\omega + \omega$ ($\subseteq \mathcal{A}$) is the integral generator of an exponentially equicontinuous n-times integrated C-semigroup $(S_n(t))_{t\geqslant 0}$.

(ii) The conclusions from Theorem 3.3.12 and the first part of this remark can be reword for the classes of q-exponentially equicontinuous integrated C-semigroups and q-exponentially equicontinuous pre-(C-DS)'s; cf. [**354**] for the notion.

REMARK 3.3.14. Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E))$), $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$) and $\mathcal{A}$ is a closed MLO on E satisfying that $\mathcal{G}(\varphi)\mathcal{A} \subseteq \mathcal{A}\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$) and

$$\mathcal{G}(-\varphi')x - \varphi(0)Cx \in \mathcal{A}\mathcal{G}(\varphi)x, \quad x \in E,\ \varphi \in \mathcal{D}\ \ (\varphi \in \mathcal{D}^*). \tag{330}$$

In [**354**], we have proved the following:

(i) If $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ satisfies (C.S.1).
(ii) If $\mathcal{G}$ satisfies (C.S.2) holds, C is injective and $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ is a (C-DS) ((C-UDS) of $*$-class) generated by $C^{-1}AC$.
(iii) Consider the distribution case. If E is admissible and $\mathcal{A} = A$ is single-valued, then the condition (C.S.2) automatically holds for $\mathcal{G}$.

As we have already seen, the conclusion from (ii) immediately implies that $\mathcal{A} = A$ must be single-valued and that the operator C must be injective.

Concerning the assertion (i), its validity is not true in multivalued case: Let $C = I$, let $\mathcal{A} \equiv E \times E$, and let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E))$) be arbitrarily chosen. Then $\mathcal{G}$ commutes with $\mathcal{A}$ and (330) holds but $\mathcal{G}$ need not satisfy (C.S.1).

Concerning the assertion (iii) in multivalued case, we can prove that the admissibility of state space E implies that for each $x \in \mathcal{N}(\mathcal{G})$ there exist an integer $k \in \mathbb{N}$ and a finite sequence $(y_i)_{0 \leqslant i \leqslant k-1}$ in $D(\mathcal{A})$ such that $y_i \in \mathcal{A}y_{i+1}$ $(0 \leqslant i \leqslant k-1)$ and $Cx \in \mathcal{A}y_0 \subseteq \mathcal{A}^{k+2}0$.

Now we will revisit some conditions introduced by J. L. Lions [**391**] in our new framework. Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E))$) and $\mathcal{G}$ commutes with C. We analyze the following conditions for $\mathcal{G}$:

(d_1) $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}_0$ (φ, $\psi \in \mathcal{D}_0^*$),
(d_3) $\mathcal{R}(\mathcal{G})$ is dense in E,
(d_4) for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0,\infty) : E)$ so that $u_x(0) = Cx$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)dt$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$),
(d_5) $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}$, $x \in E$ ($\psi \in \mathcal{D}^*$, $x \in E$).

Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E))$) is a pre-(C-DS) (pre-(C-UDS) of $*$-class). Then it is clear that $\mathcal{G}$ satisfies (d_1), our previous examinations shows that $\mathcal{G}$ satisfies (d_5); by the proof of [**291**, Proposition 3.1.24], we have that $\mathcal{G}$ also satisfies (d_4). On the other hand, it is well known that (d_1), (d_4) and (C.S.2) taken together do not imply (C.S.1), even if $C = I$; see e.g. [**291**, Remark 3.1.20]. Furthermore, let (d_1), (d_3) and (d_4) hold. Then (d_5) holds, as well. In order to see this, fix $x \in \mathcal{R}(\mathcal{G})$ and $\varphi \in \mathcal{D}$; then it suffices to show that $(Cx, \mathcal{G}(\varphi)x) \in G(\varphi_+)$. Suppose that (ρ_n) is a regularizing sequence and $u_x(t)$ is a function appearing in the formulation of the property (d_4). Due to the proof of [**291**, Proposition 3.1.19], we have that, for every $\eta \in \mathcal{D}_0$,

$$\begin{aligned}
\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}((\varphi_+ * \rho_n) * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi_+ * \rho_n)x \\
&= \mathcal{G}(\eta)\int_0^\infty (\varphi_+ * \rho_n)(t)u_x(t)dt \\
&\to \mathcal{G}(\eta)\int_0^\infty \varphi(t)u_x(t)dt = \mathcal{G}(\eta)\mathcal{G}(\varphi)x,\ n \to \infty; \\
\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}(\varphi_+ * \eta * \rho_n)Cx \to \mathcal{G}(\varphi_+ * \eta)Cx,\ n \to \infty.
\end{aligned}$$

Hence, $\mathcal{G}(\varphi_+ * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi)x$ and (d_5) holds, as claimed. On the other hand, (d_1) is a very simple consequence of (d_5); to verify this, observe that for each $\varphi \in \mathcal{D}_0$ and $\psi \in \mathcal{D}$ we have $\psi_+ * \varphi = \psi *_0 \varphi = \varphi *_0 \psi$, so that ($d_5$) is equivalent to say that

$\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$ $(\varphi \in \mathcal{D}_0,\ \psi \in \mathcal{D})$. In particular,

$$\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi), \quad \varphi \in \mathcal{D}_0,\ \psi \in \mathcal{D}. \tag{331}$$

Suppose now that (d_5) holds. Let $\varphi \in \mathcal{D}_0$ and $\psi, \eta \in \mathcal{D}$. Observing that $\psi_+ * \eta_+ * \varphi = (\psi *_0 \eta)_+ * \varphi$, we have (cf. also [**364**, Remark 3.13]):

$$\begin{aligned} \mathcal{G}(\varphi)\mathcal{G}(\eta)\mathcal{G}(\psi) &= C\mathcal{G}(\eta_+ * \varphi)\mathcal{G}(\psi) \\ &= C\mathcal{G}(\psi_+ * \eta_+ * \varphi) = C\mathcal{G}((\psi *_0 \eta)_+ * \varphi)C \\ &= C\mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta) = \mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta)C. \end{aligned} \tag{332}$$

By (331)–(332), we get

$$\mathcal{G}(\eta)\mathcal{G}(\psi)\mathcal{G}(\varphi) = \mathcal{G}(\psi *_0 \eta)C\mathcal{G}(\varphi). \tag{333}$$

Due to (331)–(333), we have the following:

(i) (d_5) and (d_3) together imply (C.S.1); in particular, (d_1), (d_3) and (d_4) together imply (C.S.1). This is an extension of [**291**, Proposition 3.1.19].

(ii) (d_5) and (d_2) together imply that $\mathcal{G}$ is a (C-DS) ((C-UDS) of $*$-class); in particular, $\mathcal{A} = A$ must be single-valued and C must be injective.

On the other hand, (d_5) does not imply (C.S.1) even supposing that $C = I$. A simple counterexample is $\mathcal{G} \in \mathcal{D}_0'(L(E))$ $(\mathcal{G} \in \mathcal{D}_0'^{*}(L(E)))$ given by $\mathcal{G}(\varphi)x := \varphi(0)x$, $x \in E$, $\varphi \in \mathcal{D}$ $(\varphi \in \mathcal{D}^*)$.

We have already mentioned that the exponential region $E(a,b)$ has been defined for the first time by W. Arendt, O. El-Mennaoui and V. Keyantuo in [**28**]:

$$E(a,b) := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\} \quad (a,b>0).$$

Set

$$\hat{\varphi}(\lambda) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)\,dt, \quad \lambda \in \mathbb{C},\ \varphi \in \mathcal{D} \quad (\varphi \in \mathcal{D}^*).$$

Now we are able to state the following theorem:

THEOREM 3.3.15. *Let $a > 0$, $b > 0$ and $\alpha > 0$. Suppose that $\mathcal{A}$ is a closed* MLO *and, for every λ which belongs to the set $E(a,b)$, there exists an operator $F(\lambda) \in L(E)$ so that $F(\lambda)\mathcal{A} \subseteq \mathcal{A}F(\lambda)$, $\lambda \in E(a,b)$, $F(\lambda)x \in (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in E(a,b)$, $x \in E$, $F(\lambda)C = CF(\lambda)$, $\lambda \in E(a,b)$, $F(\lambda)x - Cx = F(\lambda)y$, whenever $\lambda \in E(a,b)$ and $(x,y) \in \mathcal{A}$, and that the mapping $\lambda \mapsto F(\lambda)x$ is analytic on $\Omega_{a,b}$ and continuous on $\Gamma_{a,b}$, where $\Gamma_{a,b}$ denotes the upwards oriented boundary of $E(a,b)$ and $\Omega_{a,b}$ the open region which lies to the right of $\Gamma_{a,b}$. Let the operator family $\{(1+|\lambda|)^{-\alpha}F(\lambda) : \lambda \in E(a,b)\} \subseteq L(E)$ be equicontinuous. Define*

$$\mathcal{G}(\varphi)x := (-i)\int_{\Gamma_{a,b}} \hat{\varphi}(\lambda)F(\lambda)x\,d\lambda, \quad x \in E,\ \varphi \in \mathcal{D}.$$

Then $\mathcal{G}$ is a pre-(C-DS) *generated by an extension of $\mathcal{A}$.*

PROOF. Arguing as in non-degenerate case [**354**], we can prove that $\mathcal{G} \in \mathcal{D}_0'(L(E))$, as well as that $\mathcal{G}$ commutes with C and $\mathcal{A}$. The prescribed assumptions imply by [**329**, Theorem 3.23] (cf. also [**291**, Theorem 2.7.2(iv)]) that for each $n \in \mathbb{N}$ with $n > \alpha + 1$ the MLO $\mathcal{A}$ subgenerates a local n-times integrated C-semigroup $(S_n(t))_{t\in[0,a(n-\alpha-1))}$. It is straightforward to prove [**354**] that

$$\mathcal{G}(\varphi)x = (-1)^n \int_{-\infty}^{\tau} \varphi^{(n)}(t)S_n(t)x\,dt, \quad x \in E,\ \varphi \in \mathcal{D}_{(-\infty,a(n-\alpha-1))}.$$

Now the conclusion directly follows from Theorem 3.3.8 and Remark 3.3.9 (i)–(ii). □

REMARK 3.3.16. (i) If C is injective, $\mathcal{A} = A$ is single-valued, $\rho_C(A) \subseteq E(a,b)$ and $F(\lambda) = (\lambda - \mathcal{A})^{-1}C$, $\lambda \in E(a,b)$, then $\mathcal{G}$ is a (C-DS) generated by $C^{-1}AC$ [**354**]. Even in the case that $C = I$, the integral generator $\mathcal{A}$ of $\mathcal{G}$, in multivalued case, can strictly contain $C^{-1}\mathcal{A}C$; see Remark 3.3.7(i).

(ii) Let us briefly consider the ultradistribution case. Suppose that there exist $l > 0$, $\beta > 0$ and $k > 0$, in the Beurling case, resp., for every $l > 0$ there exists $\beta_l > 0$, in the Roumieu case, such that the assumptions of Theorem 3.3.8 hold with the exponential region $E(a,b)$ replaced with the region $\Omega^{(M_p)}_{l,\beta} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant M(l|\lambda|) + \beta\}$, resp. $\Omega^{\{M_p\}}_{l,\beta_l} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant M(l|\lambda|) + \beta_l\}$. Define $\mathcal{G}$ similarly as above. Then $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$, $\mathcal{G}$ commutes with C and $\mathcal{A}$, and (330) holds. But, in the present situation, we do not know whether $\mathcal{G}$ has to satisfy (C.S.1) in the degenerate case. This is a new open problem we would like to address to our readers.

Concerning smoothing properties of C-(ultra)distribution semigroups in locally convex spaces, we will only present, in Theorem 3.3.18, how one can solve the second part of the fourth question raised at the end of paper [**330**] in the affirmative. For this purpose, we need the following definition (C will be injective and $\mathcal{A} = A$ will be single-valued).

DEFINITION 3.3.17 (cf. [**330**, Definition 2.3] for the Banach space case). Suppose that $\mathcal{G}$ is a C-distribution semigroup (C-ultradistribution semigroup of $*$-class), $j \in \mathbb{N}$ and $\alpha \in (0, \frac{\pi}{2}]$. Then it is said that $\mathcal{G}$ is an (infinitely, j-times) differentiable C-distribution semigroup (C-ultradistribution semigroup of $*$-class), resp. an analytic C-distribution semigroup of angle α (analytic C-ultradistribution semigroup of $*$-class and angle α), iff $G(\delta_t)C \in L(E)$, $t > 0$ and the mapping $t \mapsto G(\delta_t)C \in L(E)$, $t > 0$ is (infinitely, j-times) differentiable, resp. iff $G(\delta_t)C \in L(E)$, $t > 0$ and the mapping $t \mapsto G(\delta_t)C \in L(E)$, $t > 0$ can be analytically extended to the sector Σ_α (since no confusion seems likely, we shall denote the extension to the sector Σ_α by the same symbol); $\mathcal{G}$ is called real analytic iff $G(\delta_t)C \in L(E)$, $t > 0$ and if for every $t > 0$ there exist a number $c_t > 0$ and an analytic function $f : \{z \in \mathbb{C}\colon |z - t| < c_t\} \to L(E)$ such that $f(s) = G(\delta_s)C$, $s \in (t - c_t, t + c_t)$.

Suppose that $\mathcal{G}$ is a differentiable (C-DS) ((C-UDS) of $*$-class) generated by A. Let $x \in E$ and $\varphi \in \mathcal{D}_{(0,\infty)}$ ($\varphi \in \mathcal{D}^*_{(0,\infty)}$) be fixed. Then

$$\mathcal{G}(\psi)G(\delta_t)Cx = G(\delta_t)\mathcal{G}(\psi)Cx = \mathcal{G}(\tau_t\psi)Cx, \quad t > 0,\ \psi \in \mathcal{D}_0\ \ (\psi \in \mathcal{D}^*_0). \tag{334}$$

By the proof of [**291**, Proposition 3.1.24], we have

$$\mathcal{G}(\varphi)\mathcal{G}(\psi)x = \int_0^\infty \varphi(t)\mathcal{G}(\tau_t\psi)Cx\,dt, \quad \psi \in \mathcal{D}_0 \ \ (\psi \in \mathcal{D}_0^*). \tag{335}$$

Owing to (334)–(335), we get that $\mathcal{G}(\psi)\mathcal{G}(\varphi)x = \mathcal{G}(\psi)\int_0^\infty \varphi(t)G(\delta_t)Cx\,dt$. Due to (C.S.2), it follows that

$$\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)G(\delta_t)Cx\,dt, \quad x \in E,\ \varphi \in \mathcal{D}_{(0,\infty)} \ \ (\varphi \in \mathcal{D}^*_{(0,\infty)}). \tag{336}$$

Since $A\mathcal{G}(\varphi)x = \mathcal{G}(-\varphi')x$, $x \in E$, $\varphi \in \mathcal{D}_{(0,\infty)}$ $(\varphi \in \mathcal{D}^*_{(0,\infty)})$, we obtain from (336) that

$$A\int_0^\infty \varphi(t)G(\delta_t)Cx\,dt = \int_0^\infty \varphi(t)\frac{d}{dt}G(\delta_t)Cx\,dt, \quad x \in E,\ \varphi \in \mathcal{D}_{(0,\infty)} \ \ (\varphi \in \mathcal{D}^*_{(0,\infty)}).$$

Choosing a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_{(0,\infty)}$ $(\mathcal{D}^*_{(0,\infty)})$ converging to δ_{t_0} in the sense of (ultra-)distributions, and using the closedness of A, we obtain from the previous equality that

$$AG(\delta_{t_0})Cx = \Big(\frac{d}{dt}G(\delta_t)Cx\Big)_{t=t_0}, \quad x \in E,\ t_0 > 0. \tag{337}$$

Now we are able to formulate the following theorem.

THEOREM 3.3.18. *Suppose that $\mathcal{G}$ is a differentiable distribution semigroup (ultradistribution semigroup of $*$-class) generated by A. Then $\mathcal{G}$ is infinitely differentiable.*

PROOF. Put $T(t) := G(\delta_t)$, $t > 0$. Combining Proposition 3.3.3(ii) and the equality $\delta_{t+s} = \delta_t * \delta_s$ $(t,\ s \geqslant 0)$, it readily follows that $(T(t))_{t>0} \subseteq L(E)$ is a semigroup, i.e., that $T(t+s) = T(t)T(s)$, $t,s > 0$. It suffices to show that, for every $n \in \mathbb{N}$, the following holds:

(a) The mapping $t \mapsto T(t) \in L(E)$, $t > 0$ is n-times differentiable and $T^{(n)}(t) = A^nT(t) \in L(E)$, $t > 0$.

We will prove the validity of assertion (a) by induction on n (cf. also [**445**, Lemma 4.2, p. 52]). Clearly, the differentiability of $\mathcal{G}$ taken together with (337) implies that (a) holds for $n = 1$. Suppose now that (a) holds for some $n \in \mathbb{N}$; let us prove that (a) holds with the number n replaced by $n+1$. Let the number $t > 0$ be fixed, and let $0 < s < t$. Then the induction hypothesis implies

$$T^{(n)}(t') = A^nT(t') = A^nT(t'-s)T(s) = T(t'-s)A^nT(s) \in L(E), \quad t' > s.$$

Hence,

$$\begin{aligned} T^{(n+1)}(t) &= \lim_{h\to 0}\frac{T^{(n)}(t+h) - T^{(n)}(t)}{h} \\ &= \lim_{h\to 0}\frac{T(t+h-s) - T(t-s)}{h}A^nT(s) = T'(t-s)A^nT(s) \in L(E). \end{aligned}$$

This shows that the mapping $t \mapsto T(t) \in L(E)$, $t > 0$ is $(n+1)$-times differentiable and $T^{(n+1)}(t) = T'(t-s)A^nT(s) \in L(E)$. It remains to be proved that

$T^{(n+1)}(t) = A^{n+1}T(t)$. But, this simply follows from the following equalities involving (337): $T'(t-s)A^nT(s) = [AT(t-s)]A^nT(s) = A[T(t-s)A^nT(s)] = A[A^nT(t)] = A^{n+1}T(t)$. $\square$

In [**330**], we have essentially used some results from the theory of distribution semigroups for the purpose of characterizing differentiability of non-degenerate integrated semigroups. Unfortunately, these methods are inapplicable in the degenerate case.

3.4. Degenerate C-distribution cosine functions and degenerate C-ultradistribution cosine functions in locally convex spaces

In this section, we assume that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$, $(M.1)$, $(M.2)$ and $(M.3)'$. We need some preliminaries concerning the first antiderivative of a vector-valued (ultra)distribution; cf. also the previous section for distributional case. Let $\eta \in \mathcal{D}_{[-2,-1]}$ ($\eta \in \mathcal{D}^*_{[-2,-1]}$) be a fixed test function satisfying $\int_{-\infty}^{\infty} \eta(t)dt = 1$. Then, for every fixed $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$), we define $I(\varphi)$ by (323). It can be simply verified that, for every $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$) and $n \in \mathbb{N}$, we have $I(\varphi) \in \mathcal{D}$ ($I(\varphi) \in \mathcal{D}^*$), $I^n(\varphi^{(n)}) = \varphi$, $\frac{d}{dx}I(\varphi)(x) = \varphi(x) - \eta(x)\int_{-\infty}^{\infty}\varphi(u)du$, $x \in \mathbb{R}$ as well as that, for every $\varphi \in \mathcal{D}_{[a,b]}$ ($\varphi \in \mathcal{D}^*_{[a,b]}$), where $-\infty < a < b < \infty$, we have: $\operatorname{supp}(I(\varphi)) \subseteq [\min(-2,a), \max(-1,b)]$. This simply implies that, for every $\tau > 2$, $-1 < b < \tau$ and for every $m, n \in \mathbb{N}$ with $m \leqslant n$, (324) holds in both cases, distributional and ultradistributional. Define now G^{-1} by $G^{-1}(\varphi) := -G(I(\varphi))$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$). In distributional case, it is well known that $G^{-1} \in \mathcal{D}'(L(E))$ and $(G^{-1})' = G$; more precisely, $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi)$, $\varphi \in \mathcal{D}$. Since, for every $h > 0$ and for every non-empty compact subset K of $\mathbb{R}$, we have that the convergence $\varphi_n \to \varphi$, $n \to \infty$ in $\mathcal{D}_K^{M_p,h}$ implies the convergence $I(\varphi_n) \to I(\varphi)$, $n \to \infty$ in $\mathcal{D}_{K'}^{M_p,h}$, where $K' = [\min(-2, \inf(K)), \max(-1, \sup(K))]$, the same holds in ultradistributional case. It is not problematic to verify that, in both cases, distributional and ultradistributional, the implication $\operatorname{supp}(G) \subseteq [0,\infty) \Rightarrow \operatorname{supp}(G^{-1}) \subseteq [0,\infty)$ holds true.

Throughout this section, we assume that E is a barreled SCLCS and $C \in L(E)$ is not necessarily injective operator. We introduce the notions of pre-(C-DCF) and (C-DCF) (pre-(C-UDCF) of $*$-class and (C-UDCF) of $*$-class) as follows:

Definition 3.4.1. An element $\mathbf{G} \in \mathcal{D}_0'(L(E))$ ($\mathbf{G} \in \mathcal{D}_0'^*(L(E))$) is called a pre-(C-DCF) (pre-(C-UDCF) of $*$-class) iff $\mathbf{G}(\varphi)C = C\mathbf{G}(\varphi)$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$) and

$$(CCF_1): \quad \mathbf{G}^{-1}(\varphi *_0 \psi)C = \mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi) + \mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi), \quad \varphi, \psi \in \mathcal{D} \ \ (\varphi, \psi \in \mathcal{D}^*);$$

if, additionally,

$$(CCF_2): \qquad x = y = 0 \ \text{ iff } \ \mathbf{G}(\varphi)x + \mathbf{G}^{-1}(\varphi)y = 0, \quad \varphi \in \mathcal{D}_0 \ \ (\varphi \in \mathcal{D}_0^*),$$

then $\mathbf{G}$ is called a C-distribution cosine function (C-ultradistribution cosine function of $*$-class), inshort (C-DCF) ((C-UDCF) of $*$-class). A pre-(C-DCF) (pre-(C-UDCF) of $*$-class) $\mathbf{G}$ is called dense iff the set $\mathcal{R}(\mathbf{G}) := \bigcup_{\varphi \in \mathcal{D}_0} R(\mathbf{G}(\varphi))$ ($\mathcal{R}(\mathbf{G}) := \bigcup_{\varphi \in \mathcal{D}_0^*} R(\mathbf{G}(\varphi))$) is dense in E.

It is clear that (CCF_2) implies $\mathcal{N}(\mathbf{G}) := \bigcap_{\varphi\in\mathcal{D}_0} N(\mathbf{G}(\varphi)) = \{0\}$ and $\bigcap_{\varphi\in\mathcal{D}_0} N(\mathbf{G}^{-1}(\varphi)) = \{0\}$, and that the assumption $\mathbf{G} \in \mathcal{D}'_0(L(E))$ implies $\mathbf{G}(\varphi) = 0$, $\varphi \in \mathcal{D}_{(-\infty,0]}$. For $\psi \in \mathcal{D}$, we set $\psi_+(t) := \psi(t)H(t)$, $t \in \mathbb{R}$, where $H(t)$ denotes the Heaviside function. Then $\psi_+ \in \mathcal{E}'_0$, $\psi \in \mathcal{D}$ and $\varphi * \psi_+ \in \mathcal{D}_0$ for any $\varphi \in \mathcal{D}_0$. The above holds in ultradistributional case, as well.

The following proposition is essential and can be deduced by making use of the arguments given in the proof of [**291**, Proposition 3.4.3].

PROPOSITION 3.4.2. *Let* $\mathbf{G} \in \mathcal{D}'_0(L(E))$ $(\mathbf{G} \in \mathcal{D}'^*_0(L(E)))$ *and* $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. *Then* $\mathbf{G}$ *is a pre-*(C-DCF) *in* E (*pre-*(C-UDCF) *of* $*$*-class in* E) *iff*

$$\mathcal{G} \equiv \begin{pmatrix} \mathbf{G} & \mathbf{G}^{-1} \\ \mathbf{G}' - \delta \otimes C & \mathbf{G} \end{pmatrix}$$

is a pre-($\mathcal{C}$-DS) *in* $E \oplus E$ *(pre-*($\mathcal{C}$-UDS) *of* $*$*-class in* $E \oplus E$*), where*

$$\mathcal{C} \equiv \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Moreover, $\mathcal{G}$ *is a* ($\mathcal{C}$-DS) (($\mathcal{C}$-UDS) *of* $*$*-class*) *iff* $\mathbf{G}$ *is a pre-*(C-DCF) (*pre-*(C-UDCF) *of* $*$*-class*) *which satisfies* (CCF_2).

Using Proposition 3.4.2, the proof of [**291**, Proposition 3.4.4], as well as the fact that any pre-(C-DS) (pre-(C-UDS) of $*$-class) satisfies (d_5), and the fact that, for every $\mathbf{G} \in \mathcal{D}'_0(L(E))$ $(\mathbf{G} \in \mathcal{D}'^*_0(L(E)))$ commuting with C, the validity of conditions (d_5) and (C.S.2) implies that $\mathbf{G}$ is a (C-DS) ((C-UDS) of $*$-class), we can prove the following generalization of [**292**, Proposition 3.2.4(ii)].

PROPOSITION 3.4.3. *Let* $\mathbf{G} \in \mathcal{D}'_0(L(E))$ $(\mathbf{G} \in \mathcal{D}'^*_0(L(E)))$ *and* $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. *Then the following holds:*

(i) *If* $\mathbf{G}$ *is a pre-*(C-DCF) (*pre-*(C-UDCF) *of* $*$*-class*)*, then*

$$\mathbf{G}^{-1}(\varphi * \psi_+)C = \mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi) + \mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi), \quad \varphi \in \mathcal{D}_0,\ \psi \in \mathcal{D}. \tag{338}$$

(ii) *If* (CCF_2) *and* (338) *hold, then* $\mathbf{G}$ *is a* (C-DCF) ((C-UDCF) *of* $*$*-class*).

If $\mathbf{G}$ is a pre-(C-DCF) (pre-(C-UDCF) of $*$-class), then we can almost directly prove that the dual $\mathbf{G}(\cdot)^*$ is a pre-(C^*-DCF) (pre-(C^*-UDCF) of $*$-class) on E^* satisfying $\mathcal{N}(\mathbf{G}^*) = \overline{\mathcal{R}(\mathbf{G})}^{\circ}$, and that the reflexivity of E additionally implies that $\mathcal{N}(\mathbf{G}) = \overline{\mathcal{R}(\mathbf{G}^*)}^{\circ}$.

Making use of Proposition 3.4.2 and Proposition 3.3.5, we can achieve the following proposition.

PROPOSITION 3.4.4. *Suppose that* $\mathbf{G}\in\mathcal{D}'_0(L(E))$ $(\mathbf{G}\in\mathcal{D}'^*_0(L(E)))$ *and* $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. *Then* $\mathbf{G}$ *is a pre-*(C-DCF) (*pre-*(C-UDCF) *of* $*$*-class*) *iff for every* $\varphi, \psi \in \mathcal{D}$ $(\varphi, \psi \in \mathcal{D}^*)$*, we have:*

$$\mathbf{G}^{-1}(\varphi)\mathbf{G}'(\psi) - \mathbf{G}'(\varphi)\mathbf{G}^{-1}(\psi) = \psi(0)\mathbf{G}^{-1}(\varphi)C - \varphi(0)\mathbf{G}^{-1}(\psi)C.$$

Assume $\mathbf{G}$ is a pre-(C-DCF) (pre-(C-UDCF) of $*$-class). Then we define the (integral) generator $\mathbf{A}$ of $\mathbf{G}$ by

$$\mathbf{A} := \{(x, y) \in E \oplus E : \mathbf{G}^{-1}(\varphi'')x = \mathbf{G}^{-1}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\}.$$

Then $\mathbf{A}$ is a closed MLO and it can be easily seen that $\mathbf{A} \subseteq C^{-1}\mathbf{A}C$, with the equality in the case that the operator C is injective. If (CCF_2) holds, then it is clear that $\mathbf{A} = A$ is a closed single-valued linear operator.

We can almost straightforwardly extend the assertion of [**291**, Lemma 3.4.7]:

LEMMA 3.4.5. *Let* $\mathbf{A}$ *be the generator of a pre-*(C-DCF) (*pre-*(C-UDCF) *of* $*$-*class*) $\mathbf{G}$. *Then* $\mathcal{A} \subseteq \mathcal{B}$, *where* $\mathcal{A} \equiv \left(\begin{smallmatrix} 0 & I \\ \mathbf{A} & 0 \end{smallmatrix}\right)$ *and* $\mathcal{B}$ *is the generator of* $\mathcal{G}$. *Furthermore,* $(x, y) \in \mathbf{A} \Leftrightarrow \left(\binom{x}{0}, \binom{0}{y}\right) \in \mathcal{B}$ *and* $\mathcal{B}$ *is single-valued iff* $\mathbf{G}$ *is a* (C-DCF) ((C-UDCF) *of* $*$-*class*).

It is worth observing that

$$\left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right) \in \mathcal{B}, \quad x \in E. \tag{339}$$

Suppose now that $\mathbf{G}$ is a ($\mathbf{C}$-DCF) (($\mathbf{C}$-UDCF) of $*$-class) generated by $\mathbf{A}$. Then Proposition 3.4.2 yields that $\mathcal{G}$ is a (C-DS) ((C-UDS) of $*$-class). This implies that the integral generator $\mathcal{B}$ of $\mathcal{G}$ is single-valued and the operator $\mathcal{C}$ is injective. Therefore, the integral generator $\mathbf{A}$ of $\mathbf{G}$ is single-valued and the operator C is injective.

It is not clear whether the assumption that the integral generator $\mathbf{A}$ of a pre-($\mathbf{C}$-DCF) (pre-($\mathbf{C}$-UDCF) of $*$-class) $\mathbf{G}$ is single-valued implies (CCF_2) for $\mathbf{G}$. Let $\mathbf{G}$ be a pre-(C-DCF) (pre-(C-UDCF) of $*$-class) generated by $\mathbf{A}$. Due to the proofs of [**291**, Proposition 3.4.8] and [**291**, Proposition 3.4.30], we obtain that the following holds:

(a) Let $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$) and $x, y \in E$. Then $(\mathbf{G}(\psi)x, y) \in \mathbf{A}$ iff
$$\mathbf{G}(\psi'')x + \psi'(0)Cx - y \in \bigcap_{\varphi \in \mathcal{D}_0} N(\mathbf{G}^{-1}(\varphi)) \quad \Bigg(\in \bigcap_{\varphi \in \mathcal{D}_0^*} N(\mathbf{G}^{-1}(\varphi))\Bigg).$$

(b) $(\mathbf{G}(\psi)x,\ \mathbf{G}(\psi'')x + \psi'(0)Cx) \in \mathbf{A}$, $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$), $x \in E$.

(c) $(\mathbf{G}^{-1}(\psi)x, -\mathbf{G}(\psi')x - \psi(0)Cx) \in \mathbf{A}$, $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$), $x \in E$.

(d) $\mathbf{G}(\varphi *_0 \psi)Cx - \mathbf{G}(\varphi)\mathbf{G}(\psi)x \in \mathbf{A}\mathbf{G}^{-1}(\varphi)\mathbf{G}^{-1}(\psi)x$, $\varphi, \psi \in \mathcal{D}$ ($\varphi, \psi \in \mathcal{D}^*$), $x \in E$.

If $\mathbf{G}$ is a ($\mathbf{C}$-DCF) (($\mathbf{C}$-UDCF) of $*$-class) generated by $\mathbf{A}$, then the operators $\mathcal{B}$ and $\mathbf{A}$ are single-valued; then a similar line of reasoning as in the proof of [**291**, Proposition 3.4.8(iii)-(iv)] shows that, for every $\psi \in \mathcal{D}$ ($\psi \in \mathcal{D}^*$), we have $\mathbf{G}(\psi)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}(\psi)$ and $\mathbf{G}^{-1}(\psi)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}^{-1}(\psi)$. It is not clear whether the above inclusions hold for pre-($\mathbf{C}$-DCF)'s (pre-($\mathbf{C}$-UDCF)'s of $*$-class) in general case.

THEOREM 3.4.6. *Suppose that* $\mathbf{G} \in \mathcal{D}_0'(L(E))$ ($\mathbf{G} \in \mathcal{D}_0'^*(L(E))$), $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$, *and* $\mathbf{A}$ *is a closed* MLO *on* E *satisfying that* $\mathbf{G}(\cdot)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}(\cdot)$ *and*

$$\mathbf{G}(\varphi'')x + \varphi'(0)Cx \in \mathbf{A}\mathbf{G}(\varphi)x, \quad x \in E,\ \varphi \in \mathcal{D} \ \ (\varphi \in \mathcal{D}^*). \tag{340}$$

Then the following holds:

(i) *If* $\mathbf{A} = A$ *is single-valued, then* $\mathbf{G}$ *is a pre-*(C-DCF) (*pre-*(C-UDCF) *of* $*$-*class*).

(ii) *If* $\mathbf{G}$ *satisfies* (CCF_2), C *is injective and* $\mathbf{A} = A$ *is single-valued, then* $\mathbf{G}$ *is a* (C-DCF) ((C-UDCF) *of* $*$*-class) generated by* $C^{-1}AC$.

(iii) *Consider the distribution case. If* E *is admissible, then the condition* (CCF_2) *automatically holds for* $\mathbf{G}$.

PROOF. We will only outline the most important details of proof. It can be simply proved that $\mathcal{G} \in \mathcal{D}_0'(L(E \oplus E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E \oplus E))$), $\mathcal{G}(\cdot)\mathcal{C} = \mathcal{C}\mathcal{G}(\cdot)$, and that $\mathcal{A}$ is a closed MLO in $E \oplus E$. Furthermore, $\mathcal{G}(\cdot)\mathcal{A} \subseteq \mathcal{A}\mathcal{G}(\cdot)$ and

$$\mathcal{G}(-\varphi')(x\ y)^T - \varphi(0)\mathcal{C}(x\ y)^T \in \mathcal{A}\mathcal{G}(\varphi)(x\ y)^T, \quad x, y \in E,\ \varphi \in \mathcal{D}\ \ (\varphi \in \mathcal{D}^*).$$

By Remark 3.3.14, it readily follows that $\mathcal{G}$ is a pre-($\mathcal{C}$-DS) in $E \oplus E$ so that (i) follows immediately from Proposition 3.4.2. In order to prove (ii), notice that $\mathcal{G}$ satisfies (C.S.2) and that the analysis from Remark 3.3.14 implies that $\mathcal{G}$ is a pre-($\mathcal{C}$-DS) in $E \oplus E$ generated by $\mathcal{C}^{-1}\mathcal{A}\mathcal{C} = \left(\begin{smallmatrix} 0 & I \\ C^{-1}AC & 0 \end{smallmatrix}\right)$. Now the part (ii) simply follows from Proposition 3.4.2 and Lemma 3.4.5. Observing that the admissibility of E implies the admissibility of $E \oplus E$, the proof of (iii) can be deduced similarly. $\square$

REMARK 3.4.7. Concerning the assertion (i), its validity is not true in multivalued case: Let $C = I$, let $\mathbf{A} \equiv E \times E$, and let $\mathbf{G} \in \mathcal{D}_0'(L(E))$ ($\mathcal{G} \in \mathcal{D}_0'^*(L(E))$) be arbitrarily chosen. Then $\mathbf{G}$ commutes with $\mathbf{A}$ and (340) holds but $\mathbf{G}$ need not satisfy (CCF_1).

REMARK 3.4.8. Let $\mathbf{G} \in \mathcal{D}_0'(L(E))$ ($\mathbf{G} \in \mathcal{D}_0'^*(L(E))$) and $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. Suppose that $\mathcal{A} = A$ is single-valued and C is injective. If $\mathbf{G}$ is a (C-DCF) in E ((C-UDCF) of $*$-class in E), then $\mathcal{B}$ is single-valued and we can proceed as in the proof of [**291**, Proposition 3.4.8(iii)] so as to conclude that $\mathbf{G}(\cdot)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}(\cdot)$. Combining this fact with Proposition 3.4.2, the consideration from Remark 3.3.14 and with the proof of Theorem 3.4.6, we get that $\mathbf{G}$ is a (C-DCF) in E ((C-UDCF) of $*$-class in E) generated by $\mathbf{A}$ iff $\mathcal{G}$ is a ($\mathcal{C}$-DS) in $E \oplus E$ (($\mathcal{C}$-UDS) of $*$-class in $E \oplus E$) generated by $\mathcal{A}$. This is an extension of [**291**, Theorem 3.2.8(ii)]. In the degenerate case, the integral generator of $\mathcal{G}$ can strictly contain $\mathcal{A}$. In order to verify this, let E be an arbitrary Banach space, let $P \in L(E)$, and let $P^2 = P$. Define $\mathbf{G}_P(\varphi)x := \int_0^\infty \varphi(t)dt\, Px$, $x \in E$, $\varphi \in \mathcal{D}$. Then $\mathbf{G}_P^{-1}(\varphi)x = \int_0^\infty t\varphi(t)dt\, Px$, $x \in E$, $\varphi \in \mathcal{D}$, $\mathbf{G}_P$ is a pre-(DCF) in E, and

$$\{x, y\} \subseteq N(P) \Leftrightarrow \mathbf{G}_P(\varphi)x + \mathbf{G}_P^{-1}(\varphi)y = 0 \text{ for all } \varphi \in \mathcal{D}_0;$$

see [**291**, Example 3.4.46]. A simple computation shows that the integral generator of $\mathbf{G}_P$ is the MLO $\mathbf{A} = E \times N(P)$. Furthermore, $([x\ y]^T, [z\ u]^T) \in \mathcal{A}$ iff $y = z$ and $u \in N(P)$, while $([x\ y]^T, [z\ u]^T) \in \mathcal{B}$ iff $y - z \in N(P)$ and $u \in N(P)$. Hence, $\mathcal{B}$ strictly contains $\mathcal{A}$.

REMARK 3.4.9. Suppose that $\mathcal{A} = A$ is single-valued and C is injective. Since any ($\mathcal{C}$-DS) in $E \oplus E$ (($\mathcal{C}$-UDS) of $*$-class in $E \oplus E$) is uniquely determined by its generator, the conclusion established in Remark 3.4.8 shows that there exists at most one (C-DCF) in E ((C-UDCF) of $*$-class in E) generated by A. Even in the case that E is a Banach space and $C = I$, this is no longer true in the degenerate case. To see this, let E be an arbitrary Banach space, let $P_1 \in L(E)$, $P_1^2 = P_1$,

$P_2 \in L(E)$, $P_2^2 = P_2$, $N(P_1) = N(P_2)$ and $P_1 \neq P_2$; cf. the previous remark. Then pre-(DCF)'s $\mathbf{G}_{P_1}$ and $\mathbf{G}_{P_2}$ are different but have the same integral generator. We can choose, for example, the matricial operators

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

We continue by stating the following theorem.

THEOREM 3.4.10. *Let $a > 0$, $b > 0$ and $\alpha > 0$. Suppose that $\mathbf{A}$ is a closed* MLO *and, for every λ which belongs to the set $E(a,b)$, there exists an operator $H(\lambda) \in L(E)$ so that $H(\lambda)\mathbf{A} \subseteq \mathbf{A}H(\lambda)$, $\lambda \in E(a,b)$, $\lambda H(\lambda)x - Cx \in \mathbf{A}[H(\lambda)x/\lambda]$, $\lambda \in E(a,b)$, $x \in E$, $H(\lambda)C = CH(\lambda)$, $\lambda \in E(a,b)$, $\lambda H(\lambda)x - Cx = H(\lambda)y/\lambda$, whenever $\lambda \in E(a,b)$ and $(x,y) \in \mathbf{A}$, and that the mapping $\lambda \mapsto H(\lambda)$ is strongly analytic on $\Omega_{a,b}$ and strongly continuous on $\Gamma_{a,b}$, where $\Gamma_{a,b}$ denotes the upwards oriented boundary of $E(a,b)$ and $\Omega_{a,b}$ the open region which lies to the right of $\Gamma_{a,b}$. Let the operator family $\{(1+|\lambda|)^{-\alpha}H(\lambda) : \lambda \in E(a,b)\} \subseteq L(E)$ be equicontinuous. Set*

$$\mathbf{G}(\varphi)x := (-i)\int_{\Gamma_{a,b}} \hat{\varphi}(\lambda)H(\lambda)x\,d\lambda, \quad x \in E,\ \varphi \in \mathcal{D}.$$

Then $\mathbf{G}$ is a pre-(C-DCF) *generated by an extension of $\mathbf{A}$.*

PROOF. Set

$$F(\lambda) := \begin{bmatrix} H(\lambda) & H(\lambda)/\lambda \\ \lambda H(\lambda) - C & H(\lambda) \end{bmatrix}, \quad \lambda \in E(a,b)$$

and

$$\mathcal{G}(\varphi)(x\ y)^T := (-i)\int_{\Gamma_{a,b}} \hat{\varphi}(\lambda)F(\lambda)(x\ y)^T\,d\lambda, \quad x,y \in E,\ \varphi \in \mathcal{D}.$$

The prescribed assumptions imply that the function $F(\cdot)$ has the properties necessary for applying Theorem 3.3.15. Furthermore, $\operatorname{supp}(\mathbf{G}) \subseteq [0,\infty)$, $\mathbf{G}$ commutes with C and, by the proof of [**292**, Theorem 3.2.6], we have that

$$\mathcal{G} = \begin{bmatrix} \mathbf{G} & \mathbf{G}^{-1} \\ \mathbf{G}' - \delta \otimes C & \mathbf{G} \end{bmatrix}.$$

By Theorem 3.3.15, we get that $\mathcal{G}$ is a pre-($\mathcal{C}$-DS) generated by an extension of $\mathcal{A}$. Due to Proposition 3.4.2 and Lemma 3.4.5, we obtain that $\mathbf{G}$ is a pre-(C-DCF) generated by an extension of $\mathbf{A}$, as claimed. $\square$

REMARK 3.4.11. (i) Suppose that C is injective, $\mathbf{A} = A$ is single-valued, $\rho_C(A) \subseteq E^2(a,b) \equiv \{\lambda^2 : \lambda \in E(a,b)\}$ and $H(\lambda) = \lambda(\lambda^2 - \mathcal{A})^{-1}C$, $\lambda \in E^2(a,b)$. Then $\mathcal{G}$ is a (C-DCF) generated by $C^{-1}AC$. Even in the case that $C = I$, the integral generator $\mathbf{A}$ of $\mathbf{G}$, in multivalued case, can strictly contain $C^{-1}\mathbf{A}C$.

(ii) Let $\mathbf{A}$ be a closed MLO, let C be injective and commute with $\mathbf{A}$, and let $\rho_C(\mathbf{A}) \subseteq E^2(a,b)$. Then the choice $H(\lambda) = \lambda(\lambda^2 - \mathbf{A})^{-1}C$, $\lambda \in E(a,b)$ is always possible.

(iii) In ultradistributional case, it is necessary to replace the exponential region $E(a,b)$ from the formulation of Theorem 3.4.10 with a corresponding ultra-logarithmic region. Define the operator $\mathbf{G}(\varphi)$ similarly as above. In non-degenerate case ($\mathbf{A} = A$ single-valued, C injective), it can be proved that $\mathbf{G}(\varphi)$ is a pre-(C-UDCF) generated by an extension of $\mathbf{A}$; unfortunately, we do not know then whether $\mathbf{G}$ has to satisfy (CCF_1) in the degenerate case.

Since any pre-(C-DS) (pre-(C-UDS) of $*$-class) satisfies the condition (d_3), we want to observe that the assertion of [**292**, Theorem 3.2.11(i)] can be formulated in our context. In non-degenerate case ($\mathbf{A} = A$ single-valued, C injective), the assertion of [**292**, Theorem 3.2.11(ii)] can be simply reformulated for (C-DCF)'s of finite order, when we can also prove the equivalence of the statements (a)–(f) clarified on p. 380 of [**292**].

The analysis of degenerate almost C-(ultra)distribution cosine functions is without the scope of this book. For more details, see [**428**] and [**291**, Subsection 3.4.5] and [**292**, pp. 380–384].

In the remainder of this section, we investigate relations between degenerate C-distribution cosine functions and degenerate integrated C-cosine functions. We start by stating the following result.

THEOREM 3.4.12. *Let $\mathbf{G}$ be a pre-*(C-DCF) *generated by $\mathbf{A}$, and let $\mathbf{G}$ be of finite order. Then, for every $\tau > 0$, there exist a number $n_\tau \in \mathbb{N}$ and a local n_τ-times integrated C-cosine function $(C_{n_\tau}(t))_{t\in[0,\tau)}$ such that*

$$\mathcal{G}(\varphi) = (-1)^{n_\tau} \int_0^\infty \varphi^{(n_\tau)}(t) C_{n_\tau}(t)\,dt, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)}. \tag{341}$$

Furthermore, $(C_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated C-cosine existence family with a subgenerator $\mathbf{A}$.

PROOF. Let $\mathcal{G}$ and $\mathcal{C}$ be as in the formulation of Proposition 3.4.2, and let $\mathcal{A}$ be the MLO defined in Lemma 3.4.5. Then $\mathcal{G}$ is a pre-($\mathcal{C}$-DS) in $E \oplus E$ generated by a closed MLO $\mathcal{B}$ which contains $\mathcal{A}$. Since $\mathbf{G}$ is of finite order, we know that, for every $\tau > 0$, there exist a number $n_\tau \in \mathbb{N}$ and a continuous mapping $C_{n_\tau} : [0,\tau) \to L(E)$ such that (341) holds true. Define

$$S_{n_\tau+1}(t) := \begin{pmatrix} \int_0^t C_{n_\tau}(s)\,ds & \int_0^t (t-s) C_{n_\tau}(s)\,ds \\ C_{n_\tau}(t) - g_{n_\tau+1}(t)C & \int_0^t C_{n_\tau}(s)\,ds \end{pmatrix}, \quad 0 \leqslant t < \tau.$$

Then $S_{n_\tau+1} : [0,\tau) \to L(E \oplus E)$ is continuous and

$$\mathcal{G}(\varphi) = (-1)^{n_\tau+1} \int_0^\infty \varphi^{(n_\tau+1)}(t) S_{n_\tau+1}(t)\,dt, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)}.$$

This immediately implies that $(S_{n_\tau+1}(t))_{t\in[0,\tau)}$ is an $(n_\tau+1)$-times integrated $\mathcal{C}$-integrated semigroup with a subgenerator $\mathcal{B}$. Due to Theorem 3.2.57, we have that $(C_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated $\mathcal{C}$-times integrated cosine function so that it remains to be proved that $(C_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated $\mathcal{C}$-cosine existence family with subgenerator $\mathbf{A}$, i.e., that $(\int_0^t (t-s)C_{n_\tau}(s)x\,ds, C_{n_\tau}(t)x - g_{n_\tau+1}(t)Cx) \in \mathbf{A}$ for all $t \in [0,\tau)$ and $x \in E$. This is equivalent to say that

$$\left(\begin{pmatrix} \int_0^t (t-s)C_{n_\tau}(s)x\,ds \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \int_0^t (t-s)C_{n_\tau}(s)x\,ds \end{pmatrix} \right) \in \mathcal{B}, \quad x \in E,\ t \in [0,\tau),$$

which simply follows from the inclusion (339) and the fact that $(S_{n_\tau+1}(t))_{t\in[0,\tau)}$ is an $(n_\tau+1)$-times integrated $\mathcal{C}$-integrated existence family with a subgenerator $\mathcal{B}$. The proof of the theorem is thereby complete. □

REMARK 3.4.13. (i) It is not clear how we can reconsider Theorem 3.3.6 and Theorem 3.4.12 in ultradistributional case.

(ii) If $\mathbf{A} = A$ is single-valued, then $\mathcal{A}$ is single-valued, as well. If so, then $(S_{n_\tau+1}(t))_{t\in[0,\tau)}$ is an $(n_\tau+1)$-times integrated $\mathcal{C}$-integrated semigroup with a subgenerator $\mathcal{A}$, which implies by Theorem 3.2.57(ii) that $(C_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated C-cosine function with a subgenerator A.

(iii) If the assumptions of Theorem 3.4.12 hold, then $\mathbf{G}(\varphi)\mathbf{G}(\psi) = \mathbf{G}(\psi)\mathbf{G}(\varphi)$, $\varphi, \psi \in \mathcal{D}$ (in the Banach space setting, this gives the affirmative answer to the question raised on p. 769 of [**331**]). As a simple consequence, we have that, for every $\psi \in \mathcal{D}$, we have $\mathbf{G}(\psi)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}(\psi)$ and $\mathbf{G}^{-1}(\psi)\mathbf{A} \subseteq \mathbf{A}\mathbf{G}^{-1}(\psi)$.

(iv) Suppose that $C = I$ and $\mathcal{A}$ ($\mathbf{A}$) is the integral generator of a pre-(DS) $\mathcal{G}$ (pre-(DCF) $\mathbf{G}$) of finite order. Then Proposition 3.2.68 combined with Theorem 3.3.6 and Theorem 3.4.12 shows that $\mathcal{A}$ ($\mathbf{A}$) is stationary dense. Unfortunately, we have already seen in some examples that the resolvent set of $\mathcal{A}$ ($\mathbf{A}$) can be empty (compare to [**363**, Theorem 3.5] in semigroup case).

Making use of Theorem 3.2.57 and Proposition 3.4.2, we can simply prove the following analogue of Theorem 3.3.8 for degenerate differential equations of second order. This is an extension of [**291**, Theorem 3.2.5(iii)].

THEOREM 3.4.14. *Assume that there exists a sequence* $((p_k,\tau_k))_{k\in\mathbb{N}_0}$ *in* $\mathbb{N}_0 \times (0,\infty)$ *such that* $\lim_{k\to\infty}\tau_k = \infty$, $(p_k)_{k\in\mathbb{N}_0}$ *and* $(\tau_k)_{k\in\mathbb{N}_0}$ *are strictly increasing, as well as for each* $k \in \mathbb{N}_0$ *there exists a local* p_k*-times integrated* C*-cosine function* $(C_{p_k}(t))_{t\in[0,\tau_k)}$ *on* E *satisfying that*

$$C_{p_m}(t)x = (g_{p_m-p_k} * C_{p_k}(\cdot)x)(t), \quad x \in E,\ t \in [0,\tau_k), \tag{342}$$

provided $k < m$. *Define*

$$\mathbf{G}(\varphi)x := (-1)^{p_k} \int_0^\infty \varphi^{(p_k)}(t)C_{p_k}(t)x\,dt, \quad \varphi \in \mathcal{D}_{(-\infty,\tau_k)},\ x \in E,\ k \in \mathbb{N}_0.$$

Then $\mathbf{G}$ *is well-defined and* $\mathbf{G}$ *is a pre-*(C-DCF).

As in the case of degenerate C-distribution semigroups, we have the following remarks and comments on Theorem 3.4.14.

REMARK 3.4.15. (i) Let $\mathbf{A}_k$ be the integral generator of $(C_{p_k}(t))_{t\in[0,\tau_k)}$ $(k \in \mathbb{N}_0)$. Then $\mathbf{A}_k \subseteq \mathbf{A}_m$ for $k > m$ and $\bigcap_{k\in\mathbb{N}_0} \mathbf{A}_k \subseteq \mathbf{A}$, where $\mathbf{A}$ is the integral generator of $\mathbf{G}$. Even in the case that $C = I$, $\bigcup_{k\in\mathbb{N}_0} \mathbf{A}_k$ can be a proper subset of $\mathbf{A}$.

(ii) Suppose that $\mathbf{A}$ is a subgenerator of $(C_{p_k}(t))_{t\in[0,\tau_k)}$ for all $k \in \mathbb{N}_0$. Then (342) automatically holds.

(iii) If $C = I$, then it suffices to suppose that there exists an MLO $\mathbf{A}$ subgenerating a local p-times integrated cosine function $(C_p(t))_{t\in[0,\tau)}$ for some $p \in \mathbb{N}$ and $\tau > 0$ [**329**].

Proposition 3.4.2 enables us to simply introduce the notion of an exponential pre-(C-DCF) in E (exponential pre-(C-UDCF) of $*$-class in E):

DEFINITION 3.4.16. Let $\mathbf{G}$ be a pre-(C-DCF) (pre-(C-UDCF) of $*$-class). Then $\mathbf{G}$ is said to be an exponential pre-(C-DCF) (pre-(C-UDCF) of $*$-class) iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E \oplus E))$ $(e^{-\omega t}\mathcal{G} \in \mathcal{S}'^*(L(E \oplus E)))$. We use the shorthand pre-(C-EDCF) (pre-(C-EUDCF) of $*$-class) to denote an exponential pre-(C-DCF) (pre-(C-UDCF) of $*$-class).

It can be simply verified that a pre-(C-DCF) (pre-(C-UDCF) of $*$-class) $\mathbf{G}$ is exponential iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\mathbf{G}^{-1} \in \mathcal{S}'(L(E))$ $(e^{-\omega t}\mathbf{G}^{-1} \in \mathcal{S}'^*(L(E)))$; see e.g. [**292**, Theorem 3.2.10(i)] and the proof of [**291**, Proposition 3.4.21].

By the foregoing, we have the following result.

THEOREM 3.4.17. *Assume that $\alpha \geqslant 0$ and $\mathbf{A}$ is the integral generator of a global α-times integrated C-cosine function $(C_\alpha(t))_{t\geqslant 0}$ on E. Set*

$$\mathbf{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) C_\alpha(t)x\,dt, \quad x \in E,\ \varphi \in \mathcal{D}.$$

Then $\mathbf{G}$ is a pre-(C-DCF) *whose integral generator contains $\mathbf{A}$. Furthermore, if $(C_\alpha(t))_{t\geqslant 0}$ is exponentially equicontinuous, then $\mathbf{G}$ is exponential.*

REMARK 3.4.18. It is clear that $\mathbf{G}(\cdot) \equiv 0$ is a degenerate pre-distribution cosine function with the generator $\mathcal{A} \equiv E \times E$, as well as that, for every $\tau > 0$ and for every integer $n_\tau \in \mathbb{N}$, there exists only one local n_τ-times integrated cosine function $(C_{n_\tau}(t) \equiv 0)_{t\in[0,\tau)}$ satisfying (341). Then condition (B)' holds and condition (A)' does not hold here. Designate by $\mathbf{A}_\tau$ the integral generator of $(C_{n_\tau}(t) \equiv 0)_{t\in[0,\tau)}$. Then $\mathbf{A}_\tau = \{0\} \times E$ is strictly contained in the integral generator $\mathbf{A}$ of $\mathbf{G}$. Furthermore, if $C \neq 0$, then there do not exist numbers $\tau > 0$ and $n_\tau \in \mathbb{N}$ such that $\mathbf{A}$ generates (subgenerates) a local n_τ-times integrated C-cosine function.

The notion of a q-exponential pre-(C-DCF) (pre-(C-UDCF) of $*$-class) can be also introduced and further analyzed. For the sake of brevity, we shall skip all related details concerning this topic here. It is also worth noting that the assertions of [**291**, Theorem 3.6.13, Theorem 3.6.14] can be simply reformulated for non-degenerate ultradistribution sines in locally convex spaces. For more details concerning the semigroup case, the reader may consult [**355**].

We would like to round off this section by drawing the readers' attention on some instructive examples of (pre-)ultradistribution sines in Fréchet and Banach function spaces.

EXAMPLE 3.4.19. (i) Several times before, we have dealt with the Fréchet space $E = \{f \in C^{\infty}([0,\infty)) : \lim_{x\to+\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$, equipped with the family of norms $\|f\|_k := \sum_{j=0}^{k} \sup_{x\geqslant 0} |f^{(j)}(x)|$, $f \in E$ $(k \in \mathbb{N}_0)$. Suppose $c_0 > 0$, $\beta > 0$, $s > 1$, $\bar{a} > 0$ and $M_p := p!^s$. Define the operator A by $D(A) := \{u \in E : c_0 u'(0) = \beta u(0)\}$ and $Au := c_0 u''$. Then, for every two sufficiently small number $\varepsilon > 0$, $\varepsilon' > 0$ and for every integer $k \in \mathbb{N}_0$, there exist constants $c(\varepsilon,\varepsilon') > 0$ and $c(k,\varepsilon,\varepsilon') > 0$ such that

$$\|(\lambda - A)^{-1} f\|_k \leqslant c(k,\varepsilon,\varepsilon') e^{c(\varepsilon,\varepsilon')|\lambda|^{\varepsilon'}} \|f\|_k, \quad f \in E, \ \lambda \in \Sigma_{\pi-\varepsilon}. \tag{343}$$

Suppose now that $P(z)$ is a non-constant complex polynomial of degree $k \in \mathbb{N}$ satisfying that there exists a positive real number $a > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > a$, all zeroes of polynomial $z \mapsto P(z) - \lambda$, $z \in \mathbb{C}$ belong to $\mathbb{C} \smallsetminus (-\infty, 0]$. Let $\bar{a} > a$. Then it can be easily seen with the help of (343) that, for every two sufficiently small number $\varepsilon > 0$, $\varepsilon' > 0$ and for every integer $k \in \mathbb{N}_0$, there exist constants $c(\varepsilon,\varepsilon') > 0$ and $c(k,\varepsilon,\varepsilon') > 0$ such that

$$\|(\lambda - P(A))^{-1} f\|_k \leqslant c(k,\varepsilon,\varepsilon') e^{c(\varepsilon,\varepsilon')|\lambda|^{\varepsilon'}} \|f\|_k, \quad f \in E, \ \operatorname{Re}\lambda > \bar{a}.$$

Set

$$\mathbf{G}_P(\varphi) f := (-i) \int_{\bar{a}-i\infty}^{\bar{a}+i\infty} \lambda \hat{\varphi}(\lambda)(\lambda^2 - P(A))^{-1} f \, d\lambda, \quad f \in E, \ \varphi \in \mathcal{D}^{(M_p)}.$$

Then $\mathbf{G}_P$ is an exponential pre-(EUDCF) of (M_p)-class generated by $P(A)$, and it is very difficult to answer whether the condition (CCF_2) holds for $\mathbf{G}_P$, in general.

(ii) In this part, we use the notation from [**27**, Chapter 8]. Let $p \in [1,\infty)$, $m > 0$, $\rho \in [0,1]$, $r > 0$, and let $a \in S^m_{\rho,0}$ satisfies (H_r). Suppose that $E = L^p(\mathbb{R}^n)$ or $E = C_0(\mathbb{R}^n)$ (in the last case, we assume $p = \infty$), $0 \leqslant l \leqslant n$, $A := \operatorname{Op}_E(a)$ and the following inequality

$$n\Big|\frac{1}{2} - \frac{1}{p}\Big|\Big(\frac{m - r - \rho + 1}{r}\Big) < 1 \tag{344}$$

holds. Let us recall that if $a(\cdot)$ is an elliptic polynomial of order m, then (344) holds with $m = r$ and $\rho = 1$. Suppose that there exists a sequence (M_p) satisfying $(M.1)$, $(M.2)$ and $(M.3)'$, as well as that $a(\mathbb{R}^n) \cap \Lambda^2_{l,\zeta,\eta} = \emptyset$ for some constants $l \geqslant 1$, $\zeta > 0$ and $\eta \in \mathbb{R}$. Here,

$$\Lambda_{l,\zeta,\eta} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \zeta M(l|\operatorname{Im}\lambda|) + \eta\} \text{ and } \Lambda^2_{l,\zeta,\eta} = \{\lambda^2 : \lambda \in \Lambda_{l,\zeta,\eta}\}.$$

Put $\mathbb{N}^l_0 := \{\eta \in \mathbb{N}^n_0 : \eta_{l+1} = \cdots = \eta_n = 0\}$ and $E_l := \{f \in E : f^{(\eta)} \in E \text{ for all } \eta \in \mathbb{N}^l_0\}$. Then the calibration $(q_\eta(f) := \|f^{(\eta)}\|_E, \ f \in E_l; \ \eta \in \mathbb{N}^l_0)$ induces a Fréchet topology on E_l [**541**]. Define the operator A_l on E_l

by $D(A_l) := \{f \in E_l : \mathrm{Op}_E(a)f \in E_l\}$ and $A_l f := \mathrm{Op}_E(a)f$ $(f \in D(A_l))$. Then we know that there exist numbers $\eta' \geqslant \eta$, $N \in \mathbb{N}$ and $M \geqslant 1$ such that $\Lambda^2_{l,\zeta,\eta'} \subseteq \rho(A_l)$ and for each $\eta \in \mathbb{N}_0^l$ we have

$$q_\eta(R(\lambda : A_l)f) \leqslant M(1+|\lambda|)^N q_\eta(f), \quad \lambda \in \Lambda^2_{l,\zeta,\eta'},\ f \in E_l.$$

Keeping in mind Theorem 3.4.10 and Remark 3.4.11, we get that A_l generates an ultradistribution cosine function of (M_p)-class in E_l. This implies (see e.g. [**291**, Theorem 3.6.14] for the Banach space case) that the abstract Cauchy problem

$$(ACP_2) : \begin{cases} u \in C^\infty([0,\infty) : E_l) \cap C([0,\infty) : [D(A_l)]), \\ u_{tt}(t,x) = A_l u(t,x), \quad t \geqslant 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),\ x \in \mathbb{R}^n \end{cases}$$

has a unique solution for any $u_0, u_1 \in E^{(M_p)}(A_l)$, where $E^{(M_p)}(A_l)$ is the abstract Beurling space of operator A_l; furthermore, for every compact set $K \subseteq [0,\infty)$ and for every $n \in \mathbb{N}$ and $h > 0$, the solution u of (ACP_2) satisfies

$$\sup_{t \in K,\ p \in \mathbb{N}_0} \frac{h^p}{M_p} \Big(\Big\| \frac{d^p}{dt^p} u(t) \Big\|_n + \Big\| \frac{d^{p+1}}{dt^{p+1}} u(t) \Big\|_n \Big) < \infty.$$

Multiplication operators in L^p-spaces generating degenerate locally integrated cosine functions can be simply constructed following the method proposed in Example 3.2.11 and [**291**, Example 3.4.44]. These examples can serve as examples of non-exponential pre-(DCF)'s in Banach spaces. If the condition (PW) clarified below holds, then it can be simply proved that there exists a continuous linear operator C such that $\mathcal{A}$ generates a global once integrated C-cosine function that is not exponentially bounded, in general. This example can be used for construction of non-exponential pre-(C-DCF)'s in Banach spaces.

3.5. Subordinated fractional resolvent families with removable singularities at zero

In this section, we assume that $(E, \|\cdot\|)$ is a complex Banach space and the following condition holds (cf. also [**199**, (P), p. 47] with $\alpha = 1$):

(PW) There exist finite constants $c, M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -c(|\operatorname{Im}\lambda| + 1)\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1+|\lambda|)^{-\beta}, \quad \lambda \in \Psi.$$

Our intention is to analyze the fractional resolvent families with removable singularities at zero, which are subordinated to infinitely differentiable semigroups generated by the multivalued linear operators satisfying the condition (PW). We will prove proper extensions of [**529**, Theorem 3.3(ii), Theorem 3.4(iii), Theorem 4.1], given in single-valued case.

Define the Yosida approximation $A_n \in L(E)$ of the operator $\mathcal{A}$ by $A_n := n[nR(n : \mathcal{A}) - I]$ ($n \in \mathbb{N}$). Then there exist numbers $c' \in (0, c)$, $M' > 0$ and $n_0 \in \mathbb{N}$ such that $\Psi_{c'} \subseteq \rho(A_n)$ for $n \geqslant n_0$, as well as

$$\|R(\lambda : A_n)\| \leqslant M'(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Psi_{c'},\ n \geqslant n_0.$$

Unfortunately, this fact cannot be essentially employed in the analysis of abstract degenerate fractional differential equations involving the multivalued linear operators satisfying the condition (PW) or condition (QP), and a great number of results presented in [**199**, Section 3.2-Section 3.5] is not attainable in our framework.

Before proceeding any further, we need to slightly modify the definition of Caputo fractional derivatives of order $\gamma \in (0, 1)$. In this section, we will use the following notion: Let $0 < T < \infty$. The Caputo fractional derivative $\mathbf{D}_t^{\gamma} u(t)$ (we will not change the terminology we have used so far) is defined for those functions $u \colon [0, T] \to E$ for which $u_{|(0,T]}(\cdot) \in C((0, T] : E)$, $u(\cdot) - u(0) \in L^1((0, T) : E)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0, T) : E)$, by

$$\mathbf{D}_t^{\gamma} u(t) = \frac{d}{dt}[g_{1-\gamma} * (u(\cdot) - u(0))](t), \quad t \in (0, T].$$

Let the contour $\Gamma := \{\lambda = -c(|\eta| + 1) + i\eta : \eta \in \mathbb{R}\}$ be oriented so that $\operatorname{Im} \lambda$ increases along Γ. Set $T(0) := I$ and

$$T(t)x := \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} (\lambda - \mathcal{A})^{-1} x \, d\lambda, \quad t > 0,\ x \in E.$$

Then $(T(t))_{t \geqslant 0} \subseteq L(E)$ is a semigroup on E, and we have the following estimate

$$\|T(t)\| = O(t^{\beta - 1}), \quad t > 0; \tag{345}$$

furthermore, for every $\theta \in (0, 1)$,

$$\|T(t)\|_{L(E, E^{\theta}_{\mathcal{A}})} = O(t^{\beta - \theta - 1}), \quad t > 0. \tag{346}$$

Concerning the strong continuity of $(T(t))_{t \geqslant 0}$ at zero, it is necessary to remind ourselves of the fact that [**199**]:

(CW) $T(t)x \to x$, $t \to 0+$ for any $x \in E$ belonging to the space $D((-\mathcal{A})^{\theta})$ with $\theta > 1 - \beta$ ($x \in E^{\theta}_{\mathcal{A}}$ with $1 > \theta > 1 - \beta$).

It is very simple to prove that

$$R(\lambda : \mathcal{A})x = \int_0^{\infty} e^{-\lambda t} T(t)x \, dt, \quad \operatorname{Re} \lambda > 0,\ x \in E. \tag{347}$$

From now on, we assume that $0 < \gamma < 1$. Set, for every $\nu \in (-\beta, \infty)$,

$$T_{\gamma,\nu}(t)x := t^{-\gamma} \int_0^{\infty} s^{\nu} \Phi_{\gamma}(st^{-\gamma}) T(s)x \, ds, \quad t > 0,\ x \in E \text{ and } T_{\gamma,0}(0) := I. \tag{348}$$

Since

$$T_{\gamma,\nu}(t)x = t^{\gamma\nu} \int_0^{\infty} s^{\nu} \Phi_{\gamma}(s) T(st^{\gamma})x \, ds, \quad t > 0,\ x \in E, \tag{349}$$

the estimates (345)–(346) combined with (a3) imply that the integral which defines the operator $T_{\gamma,\nu}(t)$ is absolutely convergent as well as

$$\|T_{\gamma,\nu}(t)\| = O(t^{\gamma(\nu+\beta-1)}), \quad t > 0.$$

Moreover, for every $\theta \in (0,1)$ and $\nu > \theta - \beta$,

$$\|T_{\gamma,\nu}(t)\|_{L(E,E^\theta_{\mathcal{A}})} = O(t^{\gamma(\nu+\beta-\theta-1)}), \quad t > 0.$$

Further on, (349) taken together with (a3) implies that, for every $\nu > -\beta$,

$$\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^\infty s^\nu \Phi_\gamma(s)[T(st^\gamma)x - x]ds, \quad t > 0, \ x \in E. \tag{350}$$

Using the dominated convergence theorem, (a3), (345), (350) and (CW), we can deduce the following:

(b1) $\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x \to \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, $t \to 0+$ provided that $\theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$, or that $1 > \theta > 1-\beta$ and $x \in E^\theta_{\mathcal{A}}$ $(\nu > -\beta)$.

Taking into account the proof of [**61**, Theorem 3.1] and (347), we get

(b2) $\int_0^\infty e^{-\lambda t} T_{\gamma,0}(t)x\,dt = \lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma t}T(t)x\,dt = \lambda^{\gamma-1}(\lambda^\gamma - \mathcal{A})^{-1}x$, $\operatorname{Re}\lambda > 0$, $x \in E$.

Owing to [**199**, Theorem 3.5], (a3) and (350), we have

(b3) $\|\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(t^{\gamma(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]})$, $t > 0$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$ and $\|\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(t^{\gamma(\beta+\theta-1)}\|x\|_{E^\theta_{\mathcal{A}}})$, $t > 0$, provided $1 > \theta > 1-\beta$, $x \in E^\theta_{\mathcal{A}}$ $(\nu > -\beta)$.

Set $\xi := \min((1/\gamma - 1)\pi/2, \pi)$. It is worth noting that the proof of [**61**, Theorem 3.3(i)–(ii)] implies that, for every $\nu > -\beta$, the mapping $t \mapsto T_{\gamma,\nu}(t)x$, $t > 0$ can be analytically extended to the sector Σ_ξ (we will denote this extension by the same symbol) and that, for every $\theta \in (0,1)$, $\varepsilon \in (0,\xi)$ and $\nu > -\beta$,

(b4) $\|T_{\gamma,\nu}(z)\| = O(|z|^{\gamma(\nu+\beta-1)})$, $z \in \Sigma_{\xi-\varepsilon}$.

Moreover, for every $\theta \in (0,1)$, $\varepsilon \in (0,\xi)$ and $\nu > \theta - \beta$,

(b5) $\|T_{\gamma,\nu}(z)\|_{L(E,E^\theta_{\mathcal{A}})} = O(|z|^{\gamma(\nu+\beta-1-\theta)})$, $z \in \Sigma_{\xi-\varepsilon}$.

Keeping in mind (b4)–(b5) and the Cauchy integral formula, we can prove that, for every $\theta \in (0,1)$, $\varepsilon \in (0,\xi)$, $\nu > -\beta$ and $n \in \mathbb{N}$,

(b4)' $\|(d^n/dz^n)T_{\gamma,\nu}(z)\| = O(|z|^{\gamma(\nu+\beta-1)-n})$, $z \in \Sigma_{\xi-\varepsilon}$,

as well as that, for every $\theta \in (0,1)$, $\varepsilon \in (0,\xi)$, $\nu > \theta - \beta$ and $n \in \mathbb{N}$,

(b5)' $\|(d^n/dz^n)T_{\gamma,\nu}(z)\|_{L(E,E^\theta_{\mathcal{A}})} = O(|z|^{\gamma(\nu+\beta-1-\theta)-n})$, $z \in \Sigma_{\xi-\varepsilon}$.

In the case that $\varepsilon \in (0,\xi)$ and $z \in \Sigma_{\xi-\varepsilon}$, then the uniqueness theorem for analytic functions, (a3) and the asymptotic expansion formula for the Wright functions (cf. also the first part of proof of [**61**, Theorem 3.3]) together imply that $\int_0^\infty z^{-\gamma(1+\nu)}s^\nu\Phi_\gamma(sz^{-\gamma})ds = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$; hence,

$$\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^\infty s^\nu\Phi_\gamma(se^{i\varphi})[T(s|z|^\gamma)x - x]ds,$$

where $\varphi = -\gamma \arg(z)$. Keeping in mind this identity, (C), [**199**, Theorem 3.5] and the proof of [**61**, Theorem 3.3], we can deduce the following extension of [**529**, Theorem 3.4(i)] and the properties (b1), (b3):

(b1)' Suppose that $\varepsilon \in (0,\xi)$ and $\delta = \xi - \varepsilon$. Then $\lim_{z\to 0, z\in\Sigma_\delta} \frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x = \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, provided that $\theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$, or that $1 > \theta > 1-\beta$ and $x \in E^\theta_{\mathcal{A}}$ $(\nu > -\beta)$.

(b3)' Suppose that $\varepsilon \in (0,\xi)$ and $\delta = \xi - \varepsilon$. Then $\|\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(|z|^{\gamma(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]})$, $z \in \Sigma_\delta$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$, and $\|\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(|z|^{\gamma(\beta+\theta-1)}\|x\|_{E^\theta_{\mathcal{A}}})$, $z \in \Sigma_\delta$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$.

Remark 3.5.1. As already observed, the angle of analyticity of considered operator families can be increased depending on the concrete value of constant $c > 0$ from the condition (PW). Here we will not discuss this question in more detail.

Following E. Bazhlekova [**61**] and R.-N. Wang, D.-H. Chen, T.-J. Xiao [**529**], we define

$$S_\gamma(z) := T_{\gamma,0}(z) \text{ and } P_\gamma(z) := \gamma T_{\gamma,1}(z)/z^\gamma, \quad z \in \Sigma_\xi;$$

cf. the proof of [**529**, Theorem 3.1], where the corresponding operators have been denoted by $\mathcal{S}_\gamma(z)$ and $\mathcal{P}_\gamma(z)$. The analysis contained in the proof of property (b4) enables one to see that the estimate [**529**, (3.1)] holds on closed subsectors of $\Sigma_{\pi/2-\omega}$ (cf. the formulation of [**529**, Theorem 3.1]). It is clear that the first statement of [**529**, Theorem 3.2] holds since the operators $\mathcal{S}_\gamma(z)$ and $\mathcal{P}_\gamma(z)$ depend analytically, in the uniform operator topology, on the complex parameter z belonging to an appropriate sector containing $(0,\infty)$. Furthermore, due to (b4)'-(b5)', we have that for each $\varepsilon \in (0,\xi)$ the following holds: $\|(d/dz)T_{\gamma,\nu}(z)\| = O(|z|^{\gamma(\nu+\beta-1)-1})$, $z \in \Sigma_{\xi-\varepsilon}$ and $\|(d/dz)P_\gamma(z)\| = O(|z|^{\gamma(\nu+\beta-1)-1})$, $z \in \Sigma_{\xi-\varepsilon}$. By the Darboux inequality, it readily follows that, for every $R > 0$, the mappings $z \mapsto S_\gamma(z) \in L(E)$, $z \in \Sigma_{\xi-\varepsilon} \smallsetminus B_R$ and $z \mapsto P_\gamma(z) \in L(E)$, $z \in \Sigma_{\xi-\varepsilon} \smallsetminus B_R$ are uniformly continuous. Arguing in such a way, we have proved an extension of the second statement in [**529**, Theorem 3.2] for degenerate fractional differential equations.

It is clear that $T_{\gamma,\nu}(z)x = z^{-\gamma}\int_0^\infty s^\nu \Phi_\gamma(sz^{-\gamma})T(s)x\,ds$, $z \in \Sigma_\xi$, $x \in E$ and $s \mapsto \frac{1}{2\pi i}\int_\Gamma (-\lambda)^\theta e^{s\lambda}(\lambda-\mathcal{A})^{-1}\cdot d\lambda$ is a bounded linear section of the operator $(-\mathcal{A})^\theta T(s)$ for $\theta > 1-\beta$ and $s \geqslant 0$ (cf. [**199**, Proposition 3.2, pp. 48–49]). Along with Theorem 1.2.3, the above implies

$$P_{\gamma,\theta}(z)x := \frac{\gamma z^{-2\gamma}}{2\pi i}\int_0^\infty s\Phi_\gamma(sz^{-\gamma})\Big[\int_\Gamma (-\lambda)^\theta e^{s\lambda}(\lambda-\mathcal{A})^{-1}x\,d\lambda\Big]ds \in (-\mathcal{A})^\theta P_\gamma(z)x \tag{351}$$

for all $z \in \Sigma_\xi$ and $x \in E$, as well as that $(P_{\gamma,\theta}(z))_{z\in\Sigma_\xi} \subseteq L(E)$ for $\theta > 1-\beta$. By the foregoing, we have

$$\|P_{\gamma,\theta}(z)\| = O(|z|^{\gamma(\beta-\theta-1)}), \quad z \in \Sigma_{\xi-\varepsilon} \quad (\varepsilon \in (0,\xi)). \tag{352}$$

Differentiating (349), it is not difficult to prove that

$$\frac{d}{dz}S_\gamma(z)x = \frac{\gamma z^{-\gamma-1}}{2\pi i}\int_0^\infty s\Phi_\gamma(sz^{-\gamma})T'(s)x\,ds = -z^{\gamma-1}P_{\gamma,1}(z)x, \quad z \in \Sigma_\xi,\ x \in E. \tag{353}$$

Applying (351) with $\theta = 1$, and (353), we get that:

$$\frac{d}{dz}S_\gamma(z)x = -z^{\gamma-1}P_{\gamma,1}(z)x \in z^{\gamma-1}\mathcal{A}P_\gamma(z)x, \quad z \in \Sigma_\xi,\ x \in E. \tag{354}$$

Further on, Theorem 1.2.3 and Theorem 1.2.4(i) can serve one to prove that the assumption $(x, y) \in \mathcal{A}$ implies $S_\gamma(z)y \in \mathcal{A}S_\gamma(z)x$ and $P_\gamma(z)y \in \mathcal{A}P_\gamma(z)x$ $(z \in \Sigma_\xi)$, so that the mapping $t \mapsto \frac{d}{dt}S_\gamma(t)x$, $t > 0$ is locally integrable for any $x \in D(\mathcal{A})$ by (354). Keeping in mind (b4), we have proved a proper extension of [**529**, Theorem 3.3] to degenerate fractional differential equations. Before proceeding further, we would like to point out that, for every $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$, the mapping $z \mapsto \frac{d}{dz}S_\gamma(z)x = \frac{d}{dz}[S_\gamma(z)x - x]$ is bounded by $|z|^{\gamma(\beta+\theta-1)-1}$ on subsectors of Σ_ξ $(1 > \theta > 1 - \beta)$; this follows from the Cauchy integral formula and the property (b3)' with $\nu = 0$. In particular, the mapping $t \mapsto \frac{d}{dt}S_\gamma(t)x$, $t > 0$ is locally integrable for any $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$, where $1 > \theta > 1 - \beta$. Suppose that $(x, y) \in \mathcal{A}$. Then an elementary application of Cauchy formula, combined with Theorem 1.2.4(i) and definition of $T(\cdot)$, implies that $T(s)y = T'(s)x$, $s > 0$. Having in mind (348) with $\nu = 1$, and definition of $P_\gamma(\cdot)$, it readily follows that $P_\gamma(z)y = -P_{\gamma,1}(z)x$, $z \in \Sigma_\xi$; therefore, $\frac{d}{dz}S_\gamma(z)x = z^{\gamma-1}P_\gamma(z)y$, provided $z \in \Sigma_\xi$ and $(x, y) \in \mathcal{A}$. Under such a circumstance, we obtain after integration that $S_\gamma(z)x - x = \int_0^z \lambda^{\gamma-1}P_\gamma(\lambda)y\,d\lambda$. This extends the assertion of [**529**, Theorem 3.4(ii)].

Suppose again that $(x, y) \in \mathcal{A}$. Performing the Laplace transform, we obtain with the help of (b2) and Theorem 1.2.4(i) that $(g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t) = \int_0^t S_\gamma(s)y\,ds$, $t \geqslant 0$. This immediately implies that $\mathbf{D}_t^\gamma S_\gamma(t)x = S_\gamma(t)y \in \mathcal{A}S_\gamma(t)x$, $t > 0$, which extends the assertion of [**529**, Theorem 3.4(iii)]. The original proof of this result, much more complicated than ours, is based on the use of functional calculus for almost sectorial operators established by F. Periago and B. Straub in [**447**] (as announced earlier, it is very difficult to develop a similar calculus for almost sectorial multivalued linear operators). Furthermore, we want to observe that this result is not optimal. In actual fact, let $1 > \theta > 1 - \beta$ and let $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$ be fixed. Then the mapping $t \mapsto F(t) := (g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t)$, $t \geqslant 0$ is continuous and its restriction on $(0, \infty)$ can be analytically extended to the sector Σ_ξ, with the estimate $\|F(z)\| = O(|z|^{\gamma(\beta+\theta-2)+1})$ on any proper subsector of Σ_ξ (cf. (b3)'). By the Cauchy integral formula, we obtain that $\|F'(z)\| = O(|z|^{\gamma(\beta+\theta-2)})$ on proper subsectors of Σ_ξ. In particular, the Caputo fractional derivative $\mathbf{D}_t^\gamma S_\gamma(t)x$ is defined. On the other hand, Theorem 1.2.3 in combination with [**199**, Proposition 3.2, 3.4] implies that

$$t \mapsto F_\gamma(t)x := \frac{1}{2\pi i}\int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})\left[\int_\Gamma \lambda e^{\lambda s}R(\lambda : \mathcal{A})x\,d\lambda\right]ds, \quad t > 0$$

is a continuous section of the multivalued mapping $\mathcal{A}S_\gamma(t)x$, $t > 0$, with the meaning clear. Then $T(t)x - x = \int_0^t T'(s)x\,ds$, $t \geqslant 0$ and $T'(t)x = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda :$

$\mathcal{A})x\,d\lambda$, $t>0$, which simply implies by (347) that

$$\int_0^\infty e^{-zt}T'(t)x\,dt = zR(z:\mathcal{A})x - x \tag{355}$$
$$= \int_0^\infty e^{-zt}\left[\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda:\mathcal{A})x\,d\lambda\right]dz, \quad z>0.$$

Using Fubini theorem, definition of $F_\gamma(\cdot)$ and the identity [**61**, (3.10)], we get that

$$\int_0^\infty e^{-zt}F_\gamma(t)x\,dz = z^{\gamma-1}\int_0^\infty e^{-z^\gamma t}\left[\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda:\mathcal{A})x\,d\lambda\right]dz, \quad z>0,$$

which clearly implies with the help of (355) that:

$$\int_0^\infty e^{-zt}\int_0^t F_\gamma(s)x\,ds\,dz = z^{\gamma-2}\int_0^\infty e^{-z^\gamma t}\left[\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda:\mathcal{A})x\,d\lambda\right]dz$$
$$= z^{\gamma-2}[z^\gamma R(z^\gamma:\mathcal{A})x - x], \quad z>0.$$

Using this equation, (b2) and the uniqueness theorem for Laplace transform, it readily follows that

$$(g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t) = \int_0^t F_\gamma(s)x\,ds, \quad t \geqslant 0.$$

Now it becomes clear that:

$$\mathbf{D}_t^\gamma S_\gamma(t)x = F_\gamma(t)x \in \mathcal{A}S_\gamma(t)x, \quad t>0,\ x\in D((-\mathcal{A})^\theta)\cap E_{\mathcal{A}}^\theta \ (1>\theta>1-\beta). \tag{356}$$

The identity [**61**, (3.10)] almost immediately implies that

$$\int_0^\infty e^{-\lambda t}t^{-\gamma-1}\Phi_\gamma(st^{-\gamma})dt = \frac{1}{\gamma s}e^{-\lambda^\gamma s}, \quad s>0,\ \lambda>0.$$

Keeping in mind this equality and (347), we get that

$$\int_0^\infty\int_0^\infty \gamma sT(s)[e^{-\lambda t}t^{-\gamma-1}\Phi_\gamma(st^{-\gamma})dt]x\,ds = (\lambda^\gamma-\mathcal{A})^{-1}x, \quad \lambda>0,\ x\in E.$$

Using Fubini theorem and definition of $T_{\gamma,1}(\cdot)$, the above yields

$$\int_0^\infty e^{-\lambda t}t^{\gamma-1}P_\gamma(t)x\,dt = (\lambda^\gamma-\mathcal{A})^{-1}x, \quad \lambda>0,\ x\in E.$$

By (b2) and the uniqueness theorem for Laplace transform, we obtain finally the following generalization of [**529**, Theorem 3.4(iv)]:

$$S_\gamma(t)x = (g_{1-\gamma} * [\cdot^{\gamma-1}P_\gamma(\cdot)x])(t), \quad t>0,\ x\in E. \tag{357}$$

This identity continues to hold on sector Σ_ξ. Arguing as in non-degenerate case (cf. [**529**, Lemma 3.1, Theorem 3.5]), we can prove that the compactness of $R(\lambda:\mathcal{A})$ for some $\lambda\in\rho(\mathcal{A})$ implies the compactness of operators $S_\gamma(t)$ and $P_\gamma(t)$ for all $t>0$.

The consideration from [**529**, Lemma 4.1] is completely meaningful for abstract degenerate fractional differential equations and gives rise us to introduce the following definition (cf. [**529**, Definition 4.1, Definition 4.2] and compare to [**199**, Definition, p. 53]):

DEFINITION 3.5.2. Let $T \in (0, \infty)$ and $f \in L^1((0,T) : E)$. Consider the following abstract degenerate fractional inclusion:

$$(\mathrm{DFP})_f : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), & t \in (0,T], \\ u(0) = u_0. \end{cases}$$

(i) By a mild solution of $(\mathrm{DFP})_f$, we mean a function

$$u(t) = S_\gamma(t)u_0 + \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\,ds, \quad t \in (0,T].$$

(ii) By a classical solution of $(\mathrm{DFP})_f$, we mean any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^{\gamma} u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^{\gamma} u(t) - f(t) \in \mathcal{A}u(t)$ for $t \in (0,T]$.

A mild solution $u(t)$ of problem $(\mathrm{DFP})_f$ is automatically continuous on $(0,T]$. If $x \in D((-\mathcal{A})^\theta) \cap E_{\mathcal{A}}^\theta$, where $1 > \theta > 1-\beta$, then (356) implies that the mapping $u(t) = S_\gamma(t)x$ is a classical solution of $(\mathrm{DFP})_f$, with $f \equiv 0$. The following theorem is an important extension of [**529**, Theorem 4.1], even for non-degenerate fractional differential equations with almost sectorial operators.

THEOREM 3.5.3. *Suppose that $T \in (0,\infty)$, $1 \geqslant \theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$, resp. $1 > \theta > 1-\beta$ and $x \in E_{\mathcal{A}}^\theta$, as well as that there exist constants $\sigma > \gamma(1-\beta)$ and $M \geqslant 1$ such that*

$$\|f(t) - f(s)\| \leqslant M|t-s|^\sigma, \quad 0 < t,\ s \leqslant T. \tag{358}$$

Let $1 \geqslant \theta > 1-\beta$, resp. $1 > \theta > 1-\beta$, and let

$$f \in L^\infty((0,T) : [D((-\mathcal{A})^\theta)]), \ \textit{resp.}\ f \in L^\infty((0,T) : E_{\mathcal{A}}^\theta).$$

Then there exists a unique classical solution of problem $(\mathrm{DFP})_f$.

PROOF. We will prove the theorem only in the case that $1 > \theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$ (cf. also Theorem 3.5.12 below). The uniqueness of classical solutions of problem $(\mathrm{DFP})_f$ is an immediate consequence of Theorem 3.1.6. By the foregoing arguments, it suffices to show that the function

$$\omega(t) := \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\,ds, \quad 0 \leqslant t \leqslant T,$$

enjoys the following properties:

(i) $\omega(t)$ is continuous at the point $t = 0$;
(ii) $\mathbf{D}_t^{\gamma}\omega(t) = \omega_1(t) := \int_0^t S_\gamma'(t-s)f(s)\,ds + f(t)$, $0 < t \leqslant T$ and $\omega_1(t)$ is continuous on $(0,T]$;
(iii) $\omega_2(t) := \omega_1(t) - f(t) = \int_0^t S_\gamma'(t-s)f(s)\,ds \in \mathcal{A}\omega(t)$, $0 < t \leqslant T$.

The statement (i) follows from the Hölder continuity of $f(\cdot)$ (cf. (358)) and a simple computation involving the estimate $\|P_\gamma(t)\| = O(t^{\gamma(\beta-1)})$, $t > 0$. For the proof of (ii), observe that (353) in combination with contour representation of $T'(\cdot)$

and [**199**, Proposition 3.2, Proposition 3.4] implies that there exist constants $C_\theta > 0$ and $C'_\theta > 0$ such that, for every $0 < s \leqslant T$ and $0 < \omega \leqslant T$,

$$
\begin{aligned}
(359) \qquad \|S'_\gamma(\omega)f(s)\| &= \left\| \frac{\gamma}{2\pi i} \int_0^\infty v\omega^{\gamma-1}\Phi_\gamma(v)T'(v\omega^\gamma)f(s)dv \right\| \\
&\leqslant C_\theta \frac{\gamma}{2\pi} \|f(s)\|_{[D((-\mathcal{A})^\theta)]} \int_0^\infty v\omega^{\gamma-1}\Phi_\gamma(v)(v\omega^\gamma)^{\beta+\theta-2}dv \\
&= C_\theta \frac{\gamma}{2\pi} \|f(s)\|_{[D((-\mathcal{A})^\theta)]}\omega^{\gamma(\beta+\theta-1)-1} \int_0^\infty v^{\beta+\theta-1}\Phi_\gamma(v)dv \\
&= C'_\theta \|f(s)\|_{[D((-\mathcal{A})^\theta)]}\omega^{\gamma(\beta+\theta-1)-1}.
\end{aligned}
$$

Using this estimate with $\omega = t - s$, where $0 < s < t \leqslant T$, and integrating the obtained estimate along the interval $[0, T]$ in variable s, we get that there exists a constant $C''_\theta > 0$ such that, for every $0 < t \leqslant T$,

$$
\begin{aligned}
(360) \quad \left\| \int_0^t S'_\gamma(t-s)f(s)ds \right\| &\leqslant C''_\theta \int_0^t (t-s)^{\gamma(\beta+\theta-1)-1}\|f(s)\|_{[D((-\mathcal{A})^\theta)]}ds \\
&\leqslant \frac{C''_\theta}{\gamma(\beta+\theta-1)} t^{\gamma(\beta+\theta-1)}\|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}.
\end{aligned}
$$

Let $h > 0$ and let $h \leqslant T - t$ for some fixed time $t \in (0, T)$. Making use of (359) and dominated convergence theorem, we obtain that

$$
(361) \qquad \lim_{h\to 0+} \int_0^t \frac{S_\gamma(t+h-s) - S_\gamma(t-s)}{h} f(s)ds = \int_0^t S'_\gamma(t-s)f(s)ds;
$$

here it is only worth noting that (359) and the mean value theorem together imply that, for every $s \in (0, t)$,

$$
\begin{aligned}
\left\| \frac{S_\gamma(t+h-s) - S_\gamma(t-s)}{h} f(s) \right\| &\leqslant \frac{1}{h} \int_{t-s}^{t+h-s} \|S'_\gamma(r)f(s)\|dr \\
&\leqslant \frac{\|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}}{h} \int_{t-s}^{t+h-s} r^{\gamma(\beta+\theta-1)-1}dr \\
&\leqslant \text{Const.}\ \left[(t-s)^{\gamma(\beta+\theta-1)-1} + (t+1-s)^{\gamma(\beta+\theta-1)-1}\right].
\end{aligned}
$$

Having in mind the estimate $\|S_\gamma(t)\| = O(t^{\gamma(\beta-1)})$, $t > 0$, the strong continuity of operator family $S_\gamma(\cdot)$ on $D((-\mathcal{A})^\theta)$ and the Hölder continuity of $f(\cdot)$, we can repeat almost verbatim the arguments from the corresponding part of proof of [**529**, Theorem 4.1] in order to see that

$$
(362) \qquad \lim_{h\to 0+} \frac{1}{h} \int_t^{t+h} S_\gamma(t+h-s)f(s)ds = f(t).
$$

Due to (361)–(362), we have that the mapping $t \mapsto \int_0^t S_\gamma(t-s)f(s)ds$, $0 < t < T$ is differentiable from the right; we can similarly prove the differentiability of this mapping from the left, for $0 < t \leqslant T$, so that

$$
\frac{d}{dt} \int_0^t S_\gamma(t-s)f(s)ds = \int_0^t S'_\gamma(t-s)f(s)ds + f(t), \quad 0 < t \leqslant T.
$$

Now it is not difficult to prove with the help of (357) and (360) that $\mathbf{D}_t^{\gamma}\omega(t)$ exists and equals to $\omega_1(t)$, as claimed. Now we will prove that the mapping $t \mapsto \int_0^t S'_\gamma(t-s)f(s)ds$, $0 < t \leqslant T$ is continuous (observe here that this mapping is continuous for $t = 0$; cf. (360)). As in the proof of [**529**, Theorem 4.1], we have $\int_0^t S'_\gamma(t-s)f(s)ds = I_1(t) + I_2(t)$, where $I_1(t) := \int_0^t S'_\gamma(t-s)[f(s) - f(t)]ds$ and $I_2(t) := \int_0^t S'_\gamma(t-s)f(t)ds$. By (b3), we have that $I_2(t+h) \to I_2(t)$ as $h \to 0$, for $0 < t \leqslant T$ and the meaning clear. To complete the whole proof, it suffices to show that the mapping $I_1(t) := \int_0^t S'_\gamma(t-s)[f(s) - f(t)]ds$, $0 < t \leqslant T$ is continuous. For the sake of brevity, we will only prove that the above mapping is continuous from the right, for $0 < t < T$. Suppose, as above, $h > 0$ and $h \leqslant T - t$. Then

$$I_1(t+h) - I_1(t) = h_1(t) + h_2(t) + h_3(t),$$

where

$$h_1(t) := \int_0^t (S'_\gamma(t+h-s) - S'_\gamma(t-s))[f(s) - f(t)]ds,$$

$$h_2(t) := \int_0^t S'_\gamma(t+h-s)[f(t) - f(t+h)]ds$$

and

$$h_3(t) := \int_t^{t+h} S'_\gamma(t+h-s)[f(s) - f(t+h)]ds.$$

We can prove that $h_1(t) \to 0$ as $h \to 0+$ by means of the dominated convergence theorem and the following estimates (cf. (359)–(360)):

$$\begin{aligned}\left\| \int_0^t S'_\gamma(t+h-s)[f(s)-f(t)]ds \right\| &\leqslant \text{Const. } (t+h-s)^{\gamma(\beta+\theta-1)-1} \|f(s)-f(t)\|_{[D((-\mathcal{A})^\theta)]} \\ &\leqslant \text{Const. } \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}\big[(t-s)^{\gamma(\beta+\theta-1)-1} + (t+1-s)^{\gamma(\beta+\theta-1)-1}\big]\end{aligned}$$

and

$$\left\| \int_0^t S'_\gamma(t-s)[f(s) - f(t)]ds \right\| \leqslant \frac{2C''_\theta}{\gamma(\beta+\theta-1)} t^{\gamma(\beta+\theta-1)} \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}.$$

On the other hand, we may conclude that $h_2(t) \to 0$ as $h \to 0+$ by using the estimate $\|S'_\gamma(t)\| = O(t^{\gamma(\beta-1)-1})$, $t > 0$, the Hölder continuity of $f(\cdot)$ and our standing assumption $\sigma > \gamma(1-\beta)$:

$$\begin{aligned}\|h_2(t)\| &\leqslant \text{Const.} \int_0^t (t+h-s)^{\gamma(\beta-1)-1} h^\sigma ds \\ &\leqslant \text{Const. } h^\sigma \big[(t+h)^{\gamma(\beta-1)} - h^{\gamma(\beta-1)}\big] \to 0 \text{ as } h \to 0+ .\end{aligned}$$

Finally, an application of (359) yields:

$$\begin{aligned}\|h_3(t)\| &\leqslant \text{Const.} \int_t^{t+h} (t+h-s)^{\gamma(\beta+\theta-1)-1} \|f(s) - f(t+h)\|_{[D((-\mathcal{A})^\theta)]} ds \\ &\leqslant \text{Const. } \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])} h^{\gamma(\beta+\theta-1)} \to 0 \text{ as } h \to 0+ .\end{aligned}$$

This proves (ii). The proof of (iii) follows by applying (354), (360) and Theorem 1.2.3. □

Remark 3.5.4. It is clear that the validity of condition (358) implies that the sequence $(f_n(t))_{n\in\mathbb{N}} \subseteq C([0,T] : E)$, where $f_n(t) := f(t)$ for $t \in [1/n, T]$ and $f_n(t) := f(1/n)$ for $t \in [0, 1/n]$, is a Cauchy sequence in $C([0,T] : E)$ and therefore convergent. Hence, there exists $\lim_{t\to 0+} f(t)$ in E and $f(t)$ can be extended to a Hölder continuous function from the space $C^{\sigma}([0,T] : E)$. This implies that the Caputo fractional derivative $\mathbf{D}_t^{\gamma}\omega(t)$ (cf. (ii)) is defined in the strong sense and that (ii) holds, in fact, for $0 \leqslant t \leqslant T$.

It is clear that Theorem 3.5.3 can be applied in the analysis of a large class of relaxation degenerate differential equations that are subordinated to degenerate differential equations of first order considered in [**199**, Section 3.7]. For example, we can prove some new results on the following inhomogeneous fractional Poisson heat equation in the space $L^p(\Omega)$:

$$(P)_\gamma : \begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = \Delta v - bv + f(t,x), \quad t \geqslant 0, \ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^{\infty}(\Omega)$, $1 < p < \infty$ and $0 < \gamma < 1$; cf. also Example 3.5.13(ii) below.

3.5.1. Semilinear degenerate Cauchy inclusions. Assume that the condition (PW) holds. In this section, we treat the following semilinear degenerate fractional Cauchy inclusion:

$$(\mathrm{DFP})_{f,s,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma}u(t) \in \mathcal{A}u(t) + f(t,u(t)), \quad t \in (0,T], \\ u(0) = u_0, \end{cases}$$

where $T \in (0,\infty)$ and $0 < \gamma \leqslant 1$. Suppose first that $0 < \gamma < 1$. By a mild solution $u(t)$ of problem $(\mathrm{DFP})_{f,s,\gamma}$, we mean any function $u \in C((0,T] : E)$ such that

$$u(t) = S_\gamma(t)u_0 + \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s,u(s))\,ds, \quad t \in (0,T].$$

As in linear case, a classical solution of $(\mathrm{DFP})_f$ is any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^{\gamma}u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^{\gamma}u(t) - f(t,u(t)) \in \mathcal{A}u(t)$ for $t \in (0,T]$. In [**529**, Theorem 5.1, Theorem 5.3, Corollary 5.1], the authors have applied various types of fixed point theorems in the study of existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{f,s,\gamma}$, provided that the operator $\mathcal{A}$ is single-valued, linear and almost sectorial. In contrast to the assertions of [**529**, Theorem 5.2, Theorem 5.4], the above-mentioned results can be immediately extended to semilinear degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\gamma}$. This is also the case with the assertion of [**310**, Theorem 2.1].

In [**378**, Theorem 3.1], F. Li has proved the existence of mild solutions for a class of delay semilinear fractional differential equations. The extension of this result to degenerate equations is immediate, as well.

Following T. Dlotko [**156**], we can similarly define the notions of mild and classical solutions of semilinear degenerate Cauchy inclusion $(\mathrm{DFP})_{f,s,1}$ of first order ($\gamma = 1$): any function $u \in C((0,T] : E)$ such that

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in (0,T]$$

is said to be a mild solution of problem $(\mathrm{DFP})_{f,s,1}$. By a classical solution, we mean any function $u \in C([0,T] : E) \cap C^1((0,T] : E)$ such that $u(t) \in D(\mathcal{A})$, $t \in (0,T]$, $u(0) = u_0$ and $u'(t) \in \mathcal{A}u(t) + f(t,u(t))$, $t \in (0,T]$. The extensions of [**156**, Theorem 1, Proposition 2] for semilinear degenerate Cauchy inclusions of first order can be simply proved.

In the remainder of this section, we will reconsider the assertions of [**446**, Theorem 3.1, Theorem 3.2] for semilinear degenerate Cauchy inclusions; cf. [**445**, Theorem 1.4, p. 185] for probably the first result in this direction.

As the next two theorems show, Theorem 3.1 and Theorem 3.2 of [**446**] can be fully generalized to semilinear degenerate Cauchy inclusions of first order.

THEOREM 3.5.5. *Let $T > 0$, and let $\gamma = 1$. Suppose that the mapping $f : [0,T] \times E \to E$ is continuous in t on $[0,T]$ and for each $t_0 > 0$ and $K > 0$ there exists a constant $L(t_0,K) > 0$ such that $\|f(t,x) - f(t,y)\| \leqslant L(t_0,K)\|x-y\|$, provided $0 < t < t_0$, $x, y \in E$ and $\|x\|, \|y\| \leqslant K$. Denote by $\mathbf{\Omega}$ the domain of continuity of semigroup $(T(t))_{t \geqslant 0}$; that is, $\mathbf{\Omega} = \{x \in E : \lim_{t \to 0+} T(t)x = x\}$. Then, for every $u_0 \in \mathbf{\Omega}$, there exist a number $\tau_{\max} = \tau_{\max}(u_0) > 0$ and a unique mild solution $u \in C([0,\tau_{\max}) : E)$ of problem $(\mathrm{DFP})_{f,s,1}$. If*

(i) *$f(t,x) \in D(\mathcal{A})$ for all $t > 0$ and $x \in \mathbf{\Omega}$;*

(ii) *for each $t_0 > 0$ and $K > 0$ there exists a constant $C = C(t_0,K) > 0$ such that*

$$\|f(t,x)\|_{[D(\mathcal{A})]} \leqslant C \text{ for all } x \in \mathbf{\Omega} \text{ with } \|x\| \leqslant C \text{ and } 0 < t < t_0; \tag{363}$$

(iii) *there exists a function $g \in L^{\infty}_{loc}((0,\infty) : \mathbb{R})$ such that*

$$\|f(t,x)\| \leqslant g(t)\|x\| \text{ a.e. } t \geqslant 0 \text{ and } x \in \mathbf{\Omega},$$

then $\tau_{\max} = \infty$.

PROOF. The proof is almost the same as that of [**446**, Theorem 3.1]; here we only want to observe that the term

$$\int_0^t \|[T(\tau_{\max} - s) - T(\tau_{\max} - s)]f(s,u(s))\|ds,$$

appearing on [**446**, p. 418, l. 11], can be estimated with the help of mean value theorem, (363) and [**199**, Proposition 3.2, 3.4], as follows:

$$\int_0^t \|[T(\tau_{\max} - s) - T(\tau_{\max} - s)]f(s,u(s))\|ds \leqslant (\tau_{\max} - t)C_1 C t^{\beta}/\beta,$$

where C is the constant from (363) and C_1 is the constant from the formulation of [**199**, Proposition 3.4]. $\square$

For the sequel, we need the following equality (see e.g. [**181**, (3.3)]):

$$\mathcal{A}^{-\theta}x = \frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(s+\mathcal{A})^{-1}x\,ds, \quad 1>\theta>1-\beta,\ x\in E. \tag{364}$$

Suppose that $y \in (-\mathcal{A})^\theta x$, where $1>\theta>1-\beta$. Then (364) and the obvious equality $(s-\mathcal{A})^{-1}T(t)y = T(t)(s-\mathcal{A})^{-1}y$, $t, s>0$ together imply

$$\begin{aligned}(-\mathcal{A})^{-\theta}T(t)y &= \frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(s-\mathcal{A})^{-1}T(t)y\,ds\\ &= T(t)\frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(s-\mathcal{A})^{-1}y\,ds\\ &= T(t)(-\mathcal{A})^{-\theta}y = T(t)x, \quad t>0.\end{aligned}$$

Hence,

$$T(t)(-\mathcal{A})^\theta \subseteq (-\mathcal{A})^\theta T(t), \quad t>0,\ 1>\theta>1-\beta. \tag{365}$$

Owing to (365), we can estimate the term $\|u(t;u_0)-u(t;u_1)\|_{[D((-\mathcal{A})^\theta)]}$ (cf. line 11 of Step 2, p. 420, the proof of [**446**, Theorem 3.2]) as in non-degenerate case; furthermore, on the same page of proof, we can use [**199**, Theorem 3.5] ([**199**, Proposition 3.2]) in place of [**447**, Theorem 3.9(vii)] [**447**, Theorem 3.9(iii)]. Keeping in mind these observations, we can formulate the following extension of [**446**, Theorem 3.2] for abstract degenerate Cauchy inclusions of first order.

THEOREM 3.5.6. *Let $T>0$, let $\gamma=1$, and let condition* (H) *hold. Suppose that $\beta>\theta>1-\beta$ and $0<t<\tau_{\max}(u_0)$. Then there exist $r>0$ and $K>0$ such that the assumption $u_1\in B_{\theta,r}(u_0) := \{u\in D((-\mathcal{A})^\theta) : \|u-u_0\|_{[D((-\mathcal{A})^\theta)]}\leqslant r\}$ implies that there exists a unique mild solution $u(t;u_1)\in C([0,\tau_{\max}(u_1)) : E)$ of problem* $(\mathrm{DFP})_{f,s,1}$ *with $\tau_{\max}(u_1)\geqslant\tau$. Moreover,*

$$\|u(t;u_0)-u(t;u_1)\| \leqslant K\|u_0-u_1\|_{[D((-\mathcal{A})^\theta)]}, \quad 0\leqslant t\leqslant\tau$$

and, for every $\varepsilon\in(0,\tau)$, there exists a constant $C_\varepsilon>0$ such that

$$\|u(t;u_0)-u(t;u_1)\|_{[D((-\mathcal{A})^\theta)]} \leqslant C_\varepsilon\|u_0-u_1\|_{[D((-\mathcal{A})^\theta)]}, \quad \varepsilon\leqslant t\leqslant\tau.$$

The situation is much more complicated if we consider abstract degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\gamma}$ of order $\gamma\in(0,1)$. Concerning [**446**, Theorem 3.1], we would like to point out that we cannot use, in fractional case, the well-known procedure for construction of a mild solution of problem $(\mathrm{DFP})_{f,s,\gamma}$ defined in a maximal time interval (see e.g. the integral equation [**447**, (8), p. 417]). The best we can do is prove the local existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{f,s,\gamma}$, as it has been explained in [**529**, Remark 4.1].

Concerning [**446**, Theorem 3.2], we can prove the following:

THEOREM 3.5.7. *Let $\gamma\in(0,1)$, and let condition* (H) *hold. Suppose that $\beta>\theta>1-\beta$. Then there exist $r>0$, $\tau>0$ and $K>0$ such that, for every $u_1\in B_{\theta,r}(u_0)$, there exists a unique mild solution $u(t;u_1)\in C([0,\tau] : E)$ of problem* $(\mathrm{DFP})_{f,s,\gamma}$. *Moreover,*

$$\|u(t;u_0)-u(t;u_1)\| \leqslant K\|u_0-u_1\|_{[D((-\mathcal{A})^\theta)]}, \quad 0\leqslant t\leqslant\tau$$

and there exists a constant $C > 0$ such that, for every $\varepsilon \in (0, \tau)$, we have

$$\|u(t; u_0) - u(t; u_1)\|_{[D((-\mathcal{A})^\theta)]} \leqslant C\varepsilon^{\gamma(\beta-1)}\|u_0 - u_1\|_{[D((-\mathcal{A})^\theta)]}, \quad \varepsilon \leqslant t \leqslant \tau.$$

PROOF. We will only outline the most relevant details of proof in the degenerate case.

1. Line 6 of Step 1, p. 419, the proof of [**446**, Theorem 3.2]: Due to (359), we have that
$$\begin{aligned}\|S_\gamma(t)x - x\| &= \left\| \int_0^t S'_\gamma(s)x\, ds \right\| \\ &\leqslant C'_\theta t^{\gamma(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]}, \quad t > 0;\end{aligned}$$
therefore, $\lim_{t\to 0+} \|S_\gamma(t)u_1 - u_1\| = 0$, uniformly on the ball $B_{\theta,r}(u_0)$.
2. Line 1, [**446**, p. 420]: Here we may apply (b3) in order to get the existence of a constant $c_\theta > 0$ such that
$$\|S_\gamma(t)(u_1 - u_0)\| \leqslant c_\theta t^{\gamma(\beta+\theta-1)}\|u_1 - u_0\|_{[D((-\mathcal{A})^\theta)]}, \quad t > 0.$$
3. By (365), Theorem 1.2.3 and definition of $S_\gamma(\cdot)$, we have that
$$S_\gamma(t)(-\mathcal{A})^\theta \subseteq (-\mathcal{A})^\theta S_\gamma(t), \quad t > 0,\ 1 > \theta > 1 - \beta.$$
From this, we may conclude that

$$\|S_\gamma(t)x\|_{[D((-\mathcal{A})^\theta)]} \leqslant \|S_\gamma(t)\|\|x\|_{[D((-\mathcal{A})^\theta)]} = O(t^{\gamma(\beta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]}), \ t > 0. \tag{366}$$

On the other hand, Theorem 1.2.3 and (351) together imply that

$$\begin{aligned}&\frac{\gamma}{2\pi i}\int_0^t (t-s)^{-\gamma-1} \\ &\times \int_0^\infty r\Phi_\gamma(r(t-s)^{-\gamma})\left[\int_\Gamma (-\lambda)^\theta e^{r\lambda}(\lambda - \mathcal{A})^{-1} f(s, u(s; u_1))d\lambda\right] dr\, ds \\ &\qquad \in (-\mathcal{A})^\theta \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s, u(s; u_1))ds, \quad t > 0,\ 0 < s \leqslant \tau,\end{aligned} \tag{367}$$

since $\beta > \theta$ and the norm of integrand in the first line does not exceed $(t - s)^{\gamma(\beta-\theta-1)}\|f(s, u(s; u_1))\|$ by (352). Hence, $u(t; u_1) \in D((-\mathcal{A})^\theta)$ for all $t \in [0, \tau]$ and $u_1 \in B_{\theta,r}(u_0)$. For a fixed element $u_1 \in B_{\theta,r}(u_0)$, the continuity of mapping $t \mapsto u(t; u_1) \in [D((-\mathcal{A})^\theta)]$, $t \in (0, \tau]$ follows from (366), the analyticity of $S_\gamma(\cdot)$, the expression (367) and the dominated convergence theorem. To finish the proof, we can repeat verbatim the corresponding part of proof of [**446**, Theorem 3.2]. ☐

Observe, finally, that Theorem 3.5.6 and Theorem 3.5.7 continue to hold if we consider the space $E^\theta_{\mathcal{A}}$ in place of $[D((-\mathcal{A})^\theta)]$. These theorems, as well some other theorems previously considered in this section, require the condition $\beta > 1/2$, which seems to be restrictive in the degenerate case (in numerous examples from [**199**, Chapter III], the condition (PW) holds with $\beta = 1/2$). For example, in the case of consideration of semilinear analogues of problem $(P)_\gamma$, Theorem 3.5.6 and Theorem 3.5.7 can be applied provided the additional condition [**199**, (3.42)] on the function

$m(x)$, which ensures us to get the better exponent $\beta = 1/(2-\rho)$ in (PW), with $0 < \rho \leqslant 1$.

3.5.2. Purely fractional case. In this subsection, we will briefly explain how we can extend a great part of our results established in the previous part of this section by investigating the abstract degenerate fractional Cauchy inclusions involving the multivalued linear operators satisfying the following condition (cf. [**325**] for more details):

(QP): There exist finite numbers $0 < \beta \leqslant 1$, $0 < d \leqslant 1$, $M > 0$ and $0 < \eta' < \eta'' < 1$ such that

$$\Psi := \{\lambda \in \mathbb{C} : |\lambda| \leqslant d \text{ or } \lambda \in \overline{\Sigma_{\pi\eta''/2}}\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1+|\lambda|)^{-\beta}, \quad \lambda \in \Psi.$$

Set $\delta := \min(\pi/2(\eta''-\eta')/\eta', \pi/2)$. Let $\delta' \in (0,\delta)$, let $0 < \varepsilon < \delta'$ be arbitrarily chosen, and let

$$T_{\eta',r}(z)x := \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda z}\lambda^r(\lambda^{\eta'} - \mathcal{A})^{-1}x\,d\lambda, \quad x \in E,\ r \in \mathbb{R},\ z \in \Sigma_{\delta'-\varepsilon},$$

where Γ_ω is oriented counterclockwise and consists of $\Gamma_\pm := \{te^{i((\pi/2)+\delta')} : t \geqslant \omega\}$ and $\Gamma_0 := \{\omega e^{i\zeta} : |\zeta| \leqslant (\pi/2)+\delta'\}$. Observe that the Cauchy formula implies that the definition of $T_{\eta',r}(z)$ is independent of $\omega > 0$. Arguing as in the proof of [**27**, Theorem 2.6.1], with $\omega = 1/|z|$, we get that for each $\theta' \in (0,\theta)$ and $r \in \mathbb{R}$, the following holds:

$$\|T_{\eta',r}(z)\| = O(|z|^{\eta'\beta-r-1}), \quad z \in \Sigma_{\theta'}$$

and

$$\int_0^\infty e^{-\lambda t}T_{\eta',r}(t)x\,dt = \lambda^r(\lambda^{\eta'}-\mathcal{A})^{-1}x, \quad x \in E,\ \lambda > 0,\ \text{provided } \eta'\beta > r; \tag{368}$$

$(T_{\eta',\eta'-1}(t))_{t>0}$ and $(T_{\eta',0}(t))_{t>0}$ will be the most important operator families.

First of all, we need to reconsider the assertions from [**199**, Section 3.1] in the fractional case. In the following proposition, we will prove a fractional analogue of the second inequality from [**199**, Proposition 3.2].

PROPOSITION 3.5.8. *Suppose that $0 < \theta < 1$ and $\theta \leqslant \beta$. Then we have $R(T_{\eta',\eta'-1}(t)) \subseteq E^\theta_{\mathcal{A}}$, $t > 0$. Furthermore, for every $\theta \in (0,1)$, there exists a constant $C_\theta > 0$ such that*

$$\sup_{s>0} s^\theta\|sR(s:\mathcal{A})T_{\eta',\eta'-1}(t)x - T_{\eta',\eta'-1}(t)x\| \leqslant C_\theta t^{\eta'(\beta-\theta-1)}\|x\|, \quad t > 0,\ x \in E.$$

PROOF. Let $t > 0$ and $s > 0$ be fixed, and let $\omega > 0$ be such that $\omega^\eta < s$. By Lemma 1.2.4(ii) and a simple computation, we have that, for every $x \in E$,

$$\begin{aligned} &sR(s:\mathcal{A})T_{\eta',\eta'-1}(t)x - T_{\eta',\eta'-1}(t)x \\ &\quad = \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'}-s}R(s:\mathcal{A})x\,d\lambda - \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{2\eta'-1}}{\lambda^{\eta'}-s}R(\lambda^{\eta'}:\mathcal{A})x\,d\lambda. \end{aligned} \tag{369}$$

The Cauchy theorem yields

$$\frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'}-s}R(s:\mathcal{A})x\,d\lambda$$
$$=\int_0^\infty e^{-vt}v^{\eta'-1}\Big[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'}v^{\eta'}-s}-\frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'}v^{\eta'}-s}\Big]sR(s:\mathcal{A})x\,dv,\quad x\in E.$$

It is clear that there exists a constant $a>0$ such that $|e^{\pm i\pi\eta'}v^{\eta'}-s|\geqslant a(v^{\eta'}+s)$, $v>0$. Using this fact and the inequality

$$\theta t+(1-\theta)s\geqslant t^\theta s^{1-\theta},\quad t,\ s\geqslant 0,\ \theta\in(0,1), \tag{370}$$

we get that there exists a constant $C_{\theta,1}>0$, independent of $s>0$, such that:

$$\Big\|s^\theta\int_0^\infty e^{-vt}v^{\eta'-1}\Big[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'}v^{\eta'}-s}-\frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'}v^{\eta'}-s}\Big]sR(s:\mathcal{A})x\,dv\Big\|$$
$$\leqslant 2Ma^{-1}\|x\|\int_0^\infty e^{-vt}v^{\eta'-1}\frac{s^{1-\beta+\theta}}{v^{\eta'}+s}dv$$
$$\leqslant C_{\theta,1}\|x\|\int_0^\infty e^{-vt}v^{\eta'-1}\frac{s^{1-\beta+\theta}}{v^{\eta'(\beta-\theta)}s^{1-\beta+\theta}}dv$$
$$=C_{\theta,1}\|x\|\Gamma(\eta'(1-\beta-\theta))t^{\eta'(\beta-\theta-1)},\quad t>0.$$

Now we will estimate the second term in (369) multiplied with s^θ. It suffices to consider the following two cases: $s>t^{-\eta'}$ and $s<t^{-\eta'}$. Suppose first that $s>t^{-\eta'}$. Then there exists a constant $b>0$ such that $|\lambda^{\eta'}-s|\geqslant b(|\lambda|^{\eta'}+s)$, $\lambda\in\Gamma_{1/t}$. Using Cauchy theorem and (370), we get that there exist two constants $C_{\theta,2},\ C_{\theta,3}>0$, independent of $s>0$, such that:

$$\Big\|\frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{2\eta'-1}s^\theta}{\lambda^{\eta'}-s}R(\lambda^{\eta'}:\mathcal{A})x\,d\lambda\Big\|=\Big\|\frac{1}{2\pi i}\int_{\Gamma_{1/t}} e^{\lambda t}\frac{\lambda^{2\eta'-1}s^\theta}{\lambda^{\eta'}-s}R(\lambda^{\eta'}:\mathcal{A})x\,d\lambda\Big\|$$
$$\leqslant\frac{1}{2\pi b}\int_{\Gamma_{1/t}} e^{\operatorname{Re}\lambda t}\frac{|\lambda|^{2\eta'-1-\eta'\beta}s^\theta}{|\lambda|^{\eta'}+s}|d\lambda|$$
$$\leqslant C_{\theta,2}\int_{\Gamma_{1/t}} e^{\operatorname{Re}\lambda t}\frac{|\lambda|^{2\eta'-1-\eta'\beta}s^\theta}{|\lambda|^{\eta'(1-\theta)}s^\theta}|d\lambda|$$
$$=C_{\theta,2}\int_{\Gamma_{1/t}} e^{\operatorname{Re}\lambda t}|\lambda|^{\eta'(1+\theta-\beta)-1}|d\lambda|$$
$$\leqslant C_{\theta,3}t^{\eta'(\beta-\theta-1)},\quad t>0,$$

where the last estimate follows from the calculation contained in the proof of [**27**, Theorem 2.6.1]. If $s<t^{-\eta'}$, then the equation (369) continues to hold with the number ω replaced with $1/t$ therein. In the newly arisen situation, the residue theorem shows that

$$\frac{s^\theta}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'}-s}R(s:\mathcal{A})x\,d\lambda \tag{371}$$

$$= \int_0^\infty e^{-vt} s^\theta v^{\eta'-1} \Big[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'} v^{\eta'} - s} - \frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'} v^{\eta'} - s} \Big] sR(s:\mathcal{A})x\, dv$$
$$+ 2\pi i s^\theta \operatorname{Res}_{\lambda = s^{1/\eta'}} \Big[\frac{e^{\lambda t}\lambda^{\eta'-1}}{\lambda^{\eta'} - s} sR(s:\mathcal{A})x \Big]$$
$$= \int_0^\infty e^{-vt} s^\theta v^{\eta'-1} \Big[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'} v^{\eta'} - s} - \frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'} v^{\eta'} - s} \Big] sR(s:\mathcal{A})x\, dv$$
$$+ \frac{s^\theta}{\eta'} e^{ts^{1/\eta'}} sR(s:\mathcal{A})x, \quad x \in E.$$

We can estimate the first summand in (371) and the term

$$\Big\| \frac{1}{2\pi i} \int_{\Gamma_\omega} e^{\lambda t} \frac{\lambda^{2\eta'-1} s^\theta}{\lambda^{\eta'} - s} R(\lambda^{\eta'}:\mathcal{A})x\, d\lambda \Big\| = \Big\| \frac{1}{2\pi i} \int_{\Gamma_{1/t}} e^{\lambda t} \frac{\lambda^{2\eta'-1} s^\theta}{\lambda^{\eta'} - s} R(\lambda^{\eta'}:\mathcal{A})x\, d\lambda \Big\|$$

as in the case that $s > t^{-\eta'}$, with the same final estimate. For the second summand in (371), we have the following estimates:

$$\|s^\theta e^{ts^{1/\eta'}} sR(s:\mathcal{A})x/\eta'\| \leqslant M\|x\| s^{1+\theta-\beta} e^1/\eta' \leqslant M\|x\| t^{-\eta'(1+\theta-\beta)} e/\eta', \quad t > 0.$$

The proof of the theorem is thereby complete. □

Let Γ' be the integral contour used in the definition of fractional power $(-\mathcal{A})^\theta$, $\theta > 1-\beta$ (cf. Subsection 1.2.1 for more details and the notation used). Denote by Φ the open region on the right of Γ'.

We need the following useful lemma.

LEMMA 3.5.9. (i) *Suppose that $1-\beta < \theta \leqslant 1$. Then there exists a constant $C_\theta > 0$ such that*

$$\|\lambda R(\lambda:\mathcal{A})x - x\| \leqslant C_\theta |\lambda|^{(1-\beta-\theta)} \|x\|_{[D((-\mathcal{A})^\theta)]}, \ \lambda \in \overline{\Sigma_{\eta''\pi/2}}, \ x \in D((-\mathcal{A})^\theta). \tag{372}$$

(ii) *Suppose that $1-\beta < \theta < 1$. Then there exists a constant $C_\theta > 0$ such that*

$$\|\lambda R(\lambda:\mathcal{A})x - x\| \leqslant C_\theta |\lambda|^{(1-\beta-\theta)} \|x\|_{E^\theta_{\mathcal{A}}}, \quad \lambda \in \overline{\Sigma_{\eta''\pi/2}}, \ x \in E^\theta_{\mathcal{A}}. \tag{373}$$

PROOF. Suppose first that $\theta = 1$. Then Lemma 1.2.4(i) implies that for any $(x,y) \in \mathcal{A}$ one has

$$\|\lambda R(\lambda:\mathcal{A})x - x\| = \|R(\lambda:\mathcal{A})y\| \leqslant M(1+|\lambda|)^{-\beta}, \quad \lambda \in \overline{\Sigma_{\eta''\pi/2}}.$$

Taking the infimum, we immediately obtain (372). Let $1-\beta < \theta < 1$. Then the function

$$\lambda \mapsto H(\lambda) := \lambda^{(\beta+\theta-1)}[\lambda R(\lambda:\mathcal{A})x - x], \quad \lambda \in \overline{\Sigma_{\eta''\pi/2}}$$

is continuous on $\overline{\Sigma_{\eta''\pi/2}}$, holomorphic on $\Sigma_{\eta''\pi/2}$ and

$$\|H(\lambda)\| \leqslant M|\lambda|^{(\beta+\theta-1)}[M|\lambda|^{1-\beta} + 1]\|x\|, \quad \lambda \in \overline{\Sigma_{\eta''\pi/2}}, \ x \in E.$$

Let $R_1 > 0$ be sufficiently large, obeying the properties that $|z+\lambda| \geqslant 1$ for $z \in \Gamma'$, $\lambda = Re^{\pm i\pi\eta''/2}$ and $-\lambda = -Re^{\pm i\pi\eta''/2} \in \Phi$ $(R \geqslant R_1)$. Put $\Gamma_{R,\pm} := \{Re^{\pm i\pi\eta''/2} : R \geqslant R_1\}$. Then it is clear that there exists a constant $a > 0$ such that $a^{-1}(|z| +$

$|\lambda|) \geqslant |z+\lambda|$, $z \in \Gamma'$, $\lambda \in \Gamma_{R,\pm}$. By the Phragmén–Lindelöf type theorem [**27**, Theorem 3.9.8, p. 179], it suffices to show that the estimates (372) and (373) hold for $\lambda \in \Gamma_{R,\pm}$, with an appropriately chosen constant $C_\theta > 0$ independent of $\lambda \in \overline{\Sigma_{\eta''\pi/2}}$ and $x \in D((-\mathcal{A})^\theta))$ ($x \in E^\theta_{\mathcal{A}}$). Suppose first that $x \in D((-\mathcal{A})^\theta)$ and $y \in (-\mathcal{A})^\theta x$ is arbitrarily chosen. Then $x = (-\mathcal{A})^{-\theta} y = \frac{1}{2\pi i}\int_{\Gamma'} z^{-\theta} R(z : -\mathcal{A}) y\, dz$ and it is not difficult to prove with the help of Lemma 1.2.4(ii) and the residue theorem that

$$\lambda^{\eta'} R(\lambda^{\eta'} : \mathcal{A})x - x = \frac{(-1)}{2\pi i}\int_{\Gamma'} \frac{z^{1-\theta}}{z+\lambda} R(z : -\mathcal{A}) y\, dz + (-\lambda)^{-\theta} y, \quad \lambda \in \Gamma_{R,\pm}. \tag{374}$$

Keeping in mind the parametrization of Γ', (374) and the arbitrariness of y, we get that for each $\lambda \in \Gamma_{R,\pm}$ the following holds:

$$\begin{aligned}\|\lambda R(\lambda : \mathcal{A})x - x\| &- |\lambda|^{-\theta}\|x\|_{[D((-\mathcal{A})^\theta)]} \\ &\leqslant a\|x\|_{[D((-\mathcal{A})^\theta)]}\int_{-\infty}^{c} \frac{|v|^{1-\theta-\beta}}{|v|+|\lambda|}(1+\beta^2(4M^2)^{-1}(c-v)^{2\beta-2})^{1/2} dv.\end{aligned}$$

For the estimation of this integral, we divide the path of integration into three segments: $(-\infty, 0]$, $[0, c/2]$ and $[c/2, c]$. We have

$$\begin{aligned}&\Big(\int_{-\infty}^{0} + \int_0^{c/2}\Big)\frac{|v|^{1-\theta-\beta}}{|v|+|\lambda|}(1+\beta^2(4M^2)^{-1}(c-v)^{2\beta-2})^{1/2} dv \\ &\qquad\leqslant 2(1+\beta^2(4M^2)^{-1}(c/2)^{2\beta-2})^{1/2}\int_0^\infty \frac{v^{1-\theta-\beta}}{v+|\lambda|} dv \\ &\qquad= 2(1+\beta^2(4M^2)^{-1}(c/2)^{2\beta-2})^{1/2}|\lambda|^{1-\theta-\beta}\int_0^\infty \frac{v^{1-\theta-\beta}}{v+1} dv \\ &\qquad= 2(1+\beta^2(4M^2)^{-1}(c/2)^{2\beta-2})^{1/2}|\lambda|^{1-\theta-\beta}\frac{(-\pi)}{\sin \pi(\theta+\beta)}, \quad \lambda \in \Gamma_{R,\pm}.\end{aligned}$$

The integral over segment $[c/2, c]$ can be majorized by using the inequality

$$\frac{|v|^{1-\theta-\beta}}{|v|+|\lambda|} \leqslant \frac{|v|^{1-\theta-\beta}}{|v|^{2-\theta-\beta}|\lambda|^{\theta+\beta-1}}, \quad \lambda \in \Gamma_{R,\pm},\ v \in [c/2, c],$$

giving the same final estimate. This completes the proof of (i). In order to prove (ii), fix an element $x \in E^\theta_{\mathcal{A}}$. Let us observe that there exists a finite constant $c_\theta > 0$, independent of $x \in E^\theta_{\mathcal{A}}$, such that, for every $\lambda \in \Gamma_{R,\pm}$,

$$\begin{aligned}c_\theta|\lambda|^{1-\beta-\theta}\|x\|_{E^\theta_{\mathcal{A}}} &\geqslant \big\|(|\lambda|-\lambda)(\lambda-\mathcal{A})^{-1}[(|\lambda|-\mathcal{A})^{-1}x - x]\big\| \\ &= \Big\|\Big(1-\frac{|\lambda|}{\lambda}\Big)[(|\lambda|-\mathcal{A})^{-1}x - x] + (\lambda-\mathcal{A})^{-1}x \\ &\qquad + \Big[\Big(1-\frac{|\lambda|}{\lambda}\Big)x - (|\lambda|-\mathcal{A})^{-1}x\Big]\Big\|,\end{aligned}$$

where the equality follows from Lemma 1.2.4(ii) and a simple computation. This implies that, for every $\lambda \in \Gamma_{R,\pm}$,

$$\Big|1-\frac{|\lambda|}{\lambda}\Big|\|(|\lambda|-\mathcal{A})^{-1}x - x\|$$

$$
\begin{aligned}
&\leqslant c_\theta|\lambda|^{1-\beta-\theta}\|x\|_{E^\theta_{\mathcal{A}}} + \|(\lambda-\mathcal{A})^{-1}x\| + \Big\|\Big[\Big(1-\frac{|\lambda|}{\lambda}\Big)x - (|\lambda|-\mathcal{A})^{-1}x\Big]\Big\| \\
&\leqslant c_\theta|\lambda|^{(1-\beta-\theta)}\|x\|_{E^\theta_{\mathcal{A}}} + M|\lambda|^{-\beta}\|x\| \\
&\quad + \Big\|\Big(1-\frac{|\lambda|}{\lambda}\Big)x - (|\lambda|-\mathcal{A})^{-1}\Big(1-\frac{|\lambda|}{\lambda}\Big)x\Big\| + \|(|\lambda|-\mathcal{A})^{-1}x\| \\
&\leqslant c_\theta|\lambda|^{1-\beta-\theta}\|x\|_{E^\theta_{\mathcal{A}}} + M|\lambda|^{-\beta}\|x\| + 2|\lambda|^{-\theta}\|x\|_{E^\theta_{\mathcal{A}}} + M|\lambda|^{-\beta}\|x\|.
\end{aligned}
$$

Taking into account this estimate, the proof of (ii) is completed through a routine argument. □

REMARK 3.5.10 ([**181**]). (i) The operator $(-\mathcal{A})^n$, defined as fractional power, coincides with the usual power $(-\mathcal{A})^n$ $(n \in \mathbb{N})$.

(ii) The space $[D(\mathcal{A})]$ is continuously embedded in $[D((-\mathcal{A})^\theta)]$ provided that $\beta > 1/2$ and $1-\beta < \theta < \beta$.

Now we are ready to prove the following generalization of [**199**, Theorem 3.5] for degenerate fractional differential equations.

THEOREM 3.5.11. *Let $\delta' \in (0,\delta)$.*

(i) *Suppose that $1-\beta < \theta \leqslant 1$. Then there exists a constant $C_{\theta,\delta'} > 0$ such that*

$$\|T_{\eta',\eta'-1}(z)x - x\| \leqslant C_{\theta,\delta'}|z|^{\eta'(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]},\quad z \in \Sigma_{\delta'},\ x \in D((-\mathcal{A})^\theta). \tag{375}$$

(ii) *Suppose that $1-\beta < \theta < 1$. Then there exists a constant $C_{\theta,\delta'} > 0$ such that*

$$\|T_{\eta',\eta'-1}(z)x - x\| \leqslant C_{\theta,\delta'}|z|^{\eta'(\beta+\theta-1)}\|x\|_{E^\theta_{\mathcal{A}}},\quad z \in \Sigma_{\delta'},\ x \in E^\theta_{\mathcal{A}}.$$

PROOF. Let $\delta'' \in (\delta',\delta)$, and let $0 < \varepsilon < \delta - \delta''$ be arbitrarily chosen. Then it is clear that

$$T_{\eta',\eta'-1}(z)x - x = \frac{1}{2\pi i}\int_{\Gamma_{1/|z|}} \frac{e^{\lambda z}}{\lambda}[\lambda^{\eta'}R(\lambda^{\eta'} : \mathcal{A})x - x]d\lambda,\quad z > 0,\ x \in E.$$

Now the result follows from Lemma 3.5.9 and the calculus contained in the proof of [**27**, Theorem 2.6.1]. □

Suppose now that $\theta > 1-\beta$. It is clear that there exists a sufficiently small number $t_0 > 0$ such that, for every $t \in (0,t_0]$ and $\lambda \in \Gamma_{1/t}$, we have $\lambda^{\eta'} \in \Phi$. Then Lemma 1.2.4(ii) and Fubini theorem together imply that, for every $t \in (0,t_0]$,

$$
\begin{aligned}
(-\mathcal{A})^{-\theta}T_{\eta',\eta'-1}(t)x &= \frac{(-1)}{(2\pi i)^2}\int_{\Gamma_{1/t}} e^{\lambda t}\lambda^{\eta'-1}(\lambda^{\eta'}-\mathcal{A})^{-1}\Big[\int_{\Gamma'}\frac{dz}{z^\theta(\lambda^{\eta'}-z)}\Big]d\lambda \\
&\quad + \frac{1}{(2\pi i)^2}\int_{\Gamma_{1/t}} e^{\lambda t}\lambda^{\eta'-1}(\lambda^{\eta'}-\mathcal{A})^{-1}\Big[\int_{\Gamma'}\frac{(z-\mathcal{A})^{-1}x}{z^\theta(\lambda^{\eta'}-z)}dz\Big]d\lambda.
\end{aligned}
$$

Applying the residue theorem on the first integral and Fubini theorem on the second one, we get from the above and definition of $(-\mathcal{A})^{-\theta}$ that, for every $t \in (0,t_0]$,

$$(-\mathcal{A})^{-\theta}T_{\eta',\eta'-1}(t)x = \frac{1}{2\pi i}\int_{\Gamma_{1/t}} e^{\lambda t}\lambda^{\eta'-1-\eta'\theta}(\lambda^{\eta'}-\mathcal{A})^{-1}x\,d\lambda + (-\mathcal{A})^{-\theta}x$$

$$+\frac{1}{2\pi i}\int_{\Gamma'} z^{1-\theta}(z-\mathcal{A})^{-1}\Biggl[\int_{\Gamma_{1/t}}\frac{e^{\lambda t}}{\lambda(\lambda^{\eta'}-z)}\,d\lambda\Biggr]dz$$
$$:= I_1(t)+(-\mathcal{A})^{-\theta}x+I_3(t).$$

Making use of dominated convergence theorem, we immediately obtain that $I_1(t)\to 0$ as $t\to 0+$. Let $1>\zeta>2-\theta-\beta$. Then (370) implies

$$\int_{\Gamma_{1/t}}\frac{|d\lambda|}{|\lambda||\lambda^{\eta'}-z|}\leqslant a\int_{\Gamma_{1/t}}\frac{|d\lambda|}{|\lambda|(|\lambda|^{\eta'}+|z|)}\leqslant a\int_{\Gamma_{1/t}}\frac{|d\lambda|}{|\lambda||\lambda|^{\eta'(1-\zeta)}|z|^{\zeta}},\tag{376}$$

where $a>0$ is a constant independent of $t\in(0,t_0]$, $\lambda\in\Gamma_{1/t}$ and $z\in\Gamma'$. With (376) in view, we may apply the dominated convergence theorem in order to see that $I_3(t)\to 0$ as $t\to 0+$. Keeping in mind Theorem 3.5.11(ii) and the commutation of operators $(-\mathcal{A})^{-\theta}$ and $T_{\eta',\eta'-1}(t)$, we obtain from the above that the following holds:

(CS) $T_{\eta',\eta'-1}(t)x\to x$, $t\to 0+$ for any $x\in E$ belonging to the space $D((-\mathcal{A})^{\theta})$ with $\theta>1-\beta$ ($x\in E^{\theta}_{\mathcal{A}}$ with $1>\theta>1-\beta$).

In the remainder of this subsection, we assume that $0<\gamma<1$. For the simplicity of notation, set $\mathcal{T}_{\eta'}(t):=T_{\eta',\eta'-1}(t)$, $t>0$. Define $\eta:=\gamma\eta'$ and, for every $\nu>-1-\eta'(\beta-1)$,

$$\mathcal{T}^{\nu}_{\eta',\gamma}(t)x:=t^{-\gamma}\int_0^{\infty}s^{\nu}\Phi_{\gamma}(st^{-\gamma})\mathcal{T}_{\eta'}(s)x\,ds,\quad t>0,\ x\in E \text{ and } \mathcal{T}^{0}_{\eta',\gamma}(0):=I.$$

Then it is clear that:

$$\mathcal{T}^{\nu}_{\eta',\gamma}(t)x=t^{\gamma\nu}\int_0^{\infty}s^{\nu}\Phi_{\gamma}(s)\mathcal{T}_{\eta'}(st^{\gamma})x\,ds,\quad t>0,\ x\in E.$$

Keeping in mind the results proved in this subsection, it is quite simple to deduce the following:

(B0) The operator $\mathcal{T}^{\nu}_{\eta',\gamma}(t)$ is absolutely convergent and

$$\|\mathcal{T}^{\nu}_{\eta',\gamma}(t)\|=O(t^{\gamma(\nu+\eta'(\beta-1))}),\quad t>0.$$

For every $\nu>-1-\eta'(\beta-1)$, we have

$$\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(t)}{t^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x=\int_0^{\infty}s^{\nu}\Phi_{\gamma}(s)[\mathcal{T}_{\eta'}(st^{\gamma})x-x]ds,\quad t>0,\ x\in E.$$

(B1) $\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(t)}{t^{\gamma\nu}}x\to\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, $t\to 0+$ provided that $\theta>1-\beta$ and $x\in D((-\mathcal{A})^{\theta})$, or that $1>\theta>1-\beta$ and $x\in E^{\theta}_{\mathcal{A}}$ ($\nu>-1-\eta'(\beta-1)$).

(B1)' Suppose that $\varepsilon\in(0,\xi)$ and $\delta=\xi-\varepsilon$. Then $\lim_{z\to 0,z\in\Sigma_{\delta}}\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(z)}{z^{\gamma\nu}}x=\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, provided that $\theta>1-\beta$ and $x\in D((-\mathcal{A})^{\theta})$, or that $1>\theta>1-\beta$ and $x\in E^{\theta}_{\mathcal{A}}$ ($\nu>-\beta$).

(B2) $\int_0^{\infty}e^{-\lambda t}\mathcal{T}^{0}_{\eta',\gamma}(t)x\,dt=\lambda^{\gamma-1}\int_0^{\infty}e^{-\lambda^{\gamma}t}\mathcal{T}_{\eta'}(t)x\,dt=\lambda^{\eta-1}(\lambda^{\eta}-\mathcal{A})^{-1}x$, $\operatorname{Re}\lambda>0$, $x\in E$.

(B3) $\|\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(t)}{t^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\|=O(t^{\eta(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^{\theta})]})$, $t>0$, provided $1>\theta>1-\beta$, $x\in D((-\mathcal{A})^{\theta})$ and $\|\frac{\mathcal{T}^{\nu}_{\eta',\gamma}}{t^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\|=O(t^{\eta(\beta+\theta-1)}\|x\|_{E^{\theta}_{\mathcal{A}}})$, $t>0$, provided $1>\theta>1-\beta$, $x\in E^{\theta}_{\mathcal{A}}$ $(\nu>-1-\eta'(\beta-1))$.

(B3)' Suppose that $\varepsilon\in(0,\xi)$, $\delta=\xi-\varepsilon$ and $\nu>-1-\eta'(\beta-1)$. Then $\|\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(z)}{z^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\|=O(|z|^{\eta(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^{\theta})]})$, $z\in\Sigma_{\delta}$, provided $1>\theta>1-\beta$, $x\in D((-\mathcal{A})^{\theta})$, and $\|\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(z)}{z^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\|=O(|z|^{\eta(\beta+\theta-1)}\|x\|_{E^{\theta}_{\mathcal{A}}})$, $z\in\Sigma_{\delta}$, provided $1>\theta>1-\beta$, $x\in E^{\theta}_{\mathcal{A}}$.

(B4) For every $\nu>-1-\eta'(\beta-1)$, the mapping $t\mapsto\mathcal{T}^{\nu}_{\eta',\gamma}(t)x$, $t>0$ can be analytically extended to the sector Σ_{ξ} (we will denote this extension by the same symbol) and, for every $\theta\in(0,1)$, $\varepsilon\in(0,\xi)$ and $\nu>-1-\eta'(\beta-1)$, $\|\mathcal{T}^{\nu}_{\eta',\gamma}(z)\|=O(|z|^{\gamma(\nu+\eta'(\beta-1))})$, $z\in\Sigma_{\xi-\varepsilon}$.

(B4)' For every $\theta\in(0,1)$, $\varepsilon\in(0,\xi)$, $\nu>-1-\eta'(\beta-1)$ and $n\in\mathbb{N}$, $\|(d^n/dz^n)\mathcal{T}^{\nu}_{\eta',\gamma}(z)\|=O(|z|^{\gamma(\nu+\eta'(\beta-1))-n})$, $z\in\Sigma_{\xi-\varepsilon}$.

(B5) For every $\theta\in(0,1)$, $\varepsilon\in(0,\xi)$ and $\nu>-1-\eta'(\beta-\theta-1)$,

$$\sup_{s>0}s^{\theta}\Big\|sR(s:\mathcal{A})\mathcal{T}^{\nu}_{\eta',\gamma}(z)x-\mathcal{T}^{\nu}_{\eta',\gamma}(z)x\Big\|=O(|z|^{\gamma(\nu+\eta'(\beta-\theta-1))}),\quad z\in\Sigma_{\xi-\varepsilon}.$$

(B5)' For every $\theta\in(0,1)$, $\varepsilon\in(0,\xi)$, $\nu>-1-\eta'(\beta-\theta-1)$ and $n\in\mathbb{N}$,

$$\sup_{s>0}s^{\theta}\|sR(s:\mathcal{A})(d^n/dz^n)\mathcal{T}^{\nu}_{\eta',\gamma}(z)x-(d^n/dz^n)\mathcal{T}^{\nu}_{\eta',\gamma}(z)x\|=O(|z|^{\gamma(\nu+\eta'(\beta-\theta-1))-n}),$$

for $z\in\Sigma_{\xi-\varepsilon}$. The angle of analyticity ξ of operator family $(\mathcal{T}^{\nu}_{\eta',\gamma}(t))_{t>0}$ is not optimal and can be improved by using the examinations from the proof of [**296**, Theorem 3.10]. We will not go into further details concerning this question here.

(B6) We have

$$\frac{\mathcal{T}^{\nu}_{\eta',\gamma}(z)}{z^{\gamma\nu}}x-\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x=\int_0^{\infty}s^{\nu}\Phi_{\gamma}(se^{i\varphi})[\mathcal{T}_{\eta'}(s|z|^{\gamma})x-x]ds,\quad x\in E,$$

where $\varphi=-\gamma\arg(z)$.

Now it is time to distinguish the following operator families:

$$\mathcal{S}_{\eta}(z):=\mathcal{T}^{0}_{\eta',\gamma}(z)\text{ and }\mathcal{P}_{\eta}(z):=z^{1-\eta}\frac{d}{dz}\int_0^z g_{\eta}(z-s)\mathcal{S}_{\eta}(s)ds\quad(z\in\Sigma_{\xi}).$$

Clearly, $\mathcal{S}_{\eta}(z)$ and $\mathcal{P}_{\eta}(z)$ depend analytically on parameter z in the uniform operator topology. It is not difficult to prove that, for every $\varepsilon\in(0,\xi)$, we have

$$\|\mathcal{S}_{\eta}(z)\|+\|\mathcal{P}_{\eta}(z)\|=O(|z|^{\eta(\beta-1)}),\quad z\in\Sigma_{\xi-\varepsilon},$$

as well as that $\|(d/dz)\mathcal{P}_{\eta}(z)\|=O(|z|^{\eta(\beta-1)-1})$, $z\in\Sigma_{\xi-\varepsilon}$. Furthermore, we have the following:

(B7) For every $R>0$, the mappings $z\mapsto\mathcal{S}_{\eta}(z)\in L(E)$, $z\in\Sigma_{\xi-\varepsilon}\setminus B_R$ and $z\mapsto\mathcal{P}_{\eta}(z)\in L(E)$, $z\in\Sigma_{\xi-\varepsilon}\setminus B_R$ are uniformly continuous.

(B8) We have

$$\mathcal{S}_\eta(z)x = \Big(g_{1-\eta} * [\cdot^{\eta-1}\mathcal{P}_\gamma(\cdot)x]\Big)(z), \quad z \in \Sigma_\xi,\ x \in E,$$

$$\int_0^\infty e^{-\lambda t} t^{\eta-1} P_\eta(t)x\,dt = (\lambda^\eta - \mathcal{A})^{-1}x, \quad \lambda > 0,\ x \in E$$

and

$$\mathcal{P}_\gamma(t)x = t^{-\eta} \int_0^\infty \gamma s \Phi_\gamma(st^{-\gamma}) T_{\eta',0}(s)x\,ds, \quad t > 0,\ x \in E.$$

The last equality continues to hold on subsectors of Σ_ξ.

(B9) We have

$$\frac{d}{dz}\mathcal{S}_\eta(z)x \in z^{\eta-1}\mathcal{A}\mathcal{P}_\eta(z)x, \quad z \in \Sigma_\xi,\ x \in E.$$

(B10) Suppose now that $(x, y) \in \mathcal{A}$. Then $\mathcal{S}_\eta(z)x - x = \int_0^z \lambda^{\eta-1}\mathcal{P}_\eta(\lambda)y\,d\lambda$, $z \in \Sigma_\xi$ and

$$\frac{d}{dz}\mathcal{S}_\eta(z)x = z^{\eta-1}\mathcal{P}_\eta(z)y, \quad z \in \Sigma_\xi;$$

the mapping $t \mapsto \frac{d}{dt}\mathcal{S}_\eta(t)x$, $t > 0$ is locally integrable.

(B11) Let $1 > \theta > 1 - \beta$, and let $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$. Then the mapping $t \mapsto F(t) := (g_{1-\eta} * [\mathcal{S}_\eta(\cdot)x - x])(t)$ is continuous for $t \geqslant 0$ and can be analytically extended from the positive real axis to the sector Σ_ξ, as well as that $\mathbf{D}^\eta_t \mathcal{S}_\eta(t)x = F'(t) \in \mathcal{A}\mathcal{S}_\eta(t)x$, $t > 0$.

(B12) The mapping $t \mapsto \frac{d}{dt}\mathcal{S}_\eta(t)x$, $t > 0$ is locally integrable for any $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$.

The following extension of Theorem 3.5.3 holds true:

THEOREM 3.5.12. *Suppose that $T \in (0, \infty)$, $1 \geqslant \theta > 1 - \beta$ and $x \in D((-\mathcal{A})^\theta)$, resp. $1 > \theta > 1 - \beta$ and $x \in E^\theta_{\mathcal{A}}$, as well as that there exist constants $\sigma > \eta(1 - \beta)$ and $M \geqslant 1$ such that* (358) *holds. Let $1 \geqslant \theta > 1 - \beta$, resp. $1 > \theta > 1 - \beta$, and let*

$$f \in L^\infty((0, T) : [D((-\mathcal{A})^\theta)]), \ \textit{resp.}\ f \in L^\infty((0, T) : E^\theta_{\mathcal{A}}).$$

Then there exists a unique classical solution of problem $(\mathrm{DFP})_f$.

PROOF. We will only provide the most relevant points of proof provided that $1 \geqslant \theta > 1 - \beta$ and $x \in D((-\mathcal{A})^\theta)$. Let $\delta' \in (0, \delta)$ be fixed. Then (375) and the Cauchy integral formula together imply:

$$\begin{aligned}\|(d/dz)\mathcal{T}_{\eta'}(z)x\| &= \|(d/dz)[\mathcal{T}_{\eta'}(z)x - x]\| \\ &\leqslant C_{\theta,\delta'}|z|^{\eta'(\beta+\theta-1)-1}\|x\|_{[D((-\mathcal{A})^\theta)]}, \quad z \in \Sigma_{\delta'},\ x \in D((-\mathcal{A})^\theta).\end{aligned}$$

Using this estimate, we can prove that there exists a constant $C_\theta > 0$ such that, for every $0 < s \leqslant T$ and $0 < \omega \leqslant T$,

$$\|S'_\gamma(\omega)f(s)\| \leqslant C_\theta \|f(s)\|_{[D((-\mathcal{A})^\theta)]}\,\omega^{\eta(\beta+\theta-1)-1}.$$

The rest of proof is almost the same as that of Theorem 3.5.3. $\square$

Therefore, it makes sense to consider the mild solutions of following semilinear degenerate fractional Cauchy inclusion:

$$(\mathrm{DFP})_{f,s,\eta} : \begin{cases} \mathbf{D}_t^{\eta} u(t) \in \mathcal{A}u(t) + f(t, u(t)), & t \in (0, T], \\ u(0) = u_0, \end{cases}$$

where $T \in (0, \infty)$. As before, a mild solution $u(t) := u(t; u_0)$ of problem $(\mathrm{DFP})_{f,s,\eta}$ is any function $u \in C((0, T] : E)$ such that

$$u(t) = \mathcal{S}_\eta(t)u_0 + \int_0^t (t-s)^{\eta-1}\mathcal{P}_\eta(t-s)f(s, u(s))\,ds, \quad t \in (0, T].$$

Here we would like to observe that it is not clear how we can prove an extension of Theorem 3.5.7 in the case that the operator family $(\mathcal{S}_\eta(t))_{t>0}$ is not subordinated to a degenerate semigroup. With the exception of this result, all other results from Subsection 3.5.1 continue to hold in our new setting and we will only say a few words about the assertion of [**529**, Theorem 5.3], where the authors have investigated the existence of mild solutions of semilinear degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\eta}$, provided that the resolvent of $\mathcal{A}$ is compact. The operator family $(\mathcal{S}_\eta(t))_{t>0}$ is then subordinated to a semigroup $(T(t))_{t>0}$ which do have a removable singularity at zero, and the compactness of operators $\mathcal{S}_\eta(t)$ and $\mathcal{P}_\eta(t)$ for $t > 0$ (cf. [**529**, Lemma 3.1, Theorem 3.5]) has been proved by following a method based on the use of semigroup property of $(T(t))_{t>0}$. In purely fractional case, we can argue as follows. Recall that the set consisting of all compact operators on E is a closed linear subspace of $L(E)$ forming a two-sided ideal in $L(E)$. Since (368) and (B2) hold in the uniform operator topology, we can apply Lemma 1.2.4(iii), the Post–Widder inversion formula [**27**, Theorem 1.7.7] and the formulae [**61**, (2.16)–(2.17)] in order to see that the operator $T_{\eta',r}(t)$ is compact for $\eta'\beta > r$, $t > 0$ and the operator $\mathcal{S}_\eta(t)$ is compact for $t > 0$. Keeping in mind the third equality in (B8), we obtain that the operator $\mathcal{P}_\eta(t)$ is compact for $t > 0$, as well. Now we can reformulate [**529**, Theorem 5.3] by means of the following approximation in Step 3 of its proof:

$$\Gamma^{\eta}_{\varepsilon,\delta}(t) := \mathcal{S}_\eta(t) + \int_0^{t-\varepsilon} (t-s)^{2\gamma-1} \int_\delta^\infty \gamma\tau\Phi_\gamma(\tau)T_{\eta',0}(\tau(t-s)^\gamma)\,d\tau\,ds,$$

for $t \in (0, T]$, $\delta > 0$, $0 < \varepsilon < t$ and $u \in \Omega_r$; cf. [**529**] for the notion. Assuming the condition of type [**529**, (H2)], we can prove the estimate

$$\begin{aligned} &\left\| \mathcal{S}_\eta(t)u_0 + \int_0^t (t-s)^{\eta-1}\mathcal{P}_\eta(t-s)f(s,u(s))\,ds - \Gamma^{\eta}_{\varepsilon,\delta}(t) \right\| \\ &\quad \leqslant \mathrm{Const.}\left(\int_0^{t-\varepsilon} (t-s)^{q[\beta\eta+\gamma-1]}\,ds \right)^{1/q} \|m_r\|_{L^p(0,T)} \int_0^\infty \tau^{\eta'\beta}\Phi_\gamma(\tau)\,d\tau \\ &\qquad + \mathrm{Const.}\left(\int_{t-\varepsilon}^{t} (t-s)^{q[\beta\eta+\gamma-1]}\,ds \right)^{1/q} \|m_r\|_{L^p(0,T)} \int_0^\infty \tau^{\eta'\beta}\Phi_\gamma(\tau)\,d\tau \end{aligned}$$

for $p > 1$ and $q = p/p - 1$. Now it is quite simple to reformulate [**529**, Theorem 5.3] in our context.

EXAMPLE 3.5.13. (i) **[527]** Suppose that $\alpha \in (0,1)$, $m \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^{4m} and $E := C^{\alpha}(\bar{\Omega})$. Let us consider the operator $A\colon D(A) \subseteq C^{\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega})$ given by

$$Au(x) := \sum_{|\beta| \leqslant 2m} a_\beta(x) D^\beta u(x) \text{ for all } x \in \bar{\Omega}$$

with domain $D(A) := \{u \in C^{2m+\alpha}(\bar{\Omega}) : D^\beta u_{|\partial\Omega} = 0 \text{ for all } |\beta| \leqslant m-1\}$. Here, $\beta \in \mathbb{N}_0^n$, $|\beta| = \sum_{i=1}^n \beta_j$, $D^\beta = \prod_{i=1}^n (\frac{1}{i}\frac{\partial}{\partial x_i})^{\beta_i}$, and we assume that $a_\beta\colon \bar{\Omega} \to \mathbb{C}$ satisfy the following:

(i) $a_\beta(x) \in \mathbb{R}$ for all $x \in \bar{\Omega}$ and $|\beta| = 2m$.
(ii) $a_\beta \in C^\alpha(\bar{\Omega})$ for all $|\beta| \leqslant 2m$, and
(iii) there is a constant $M > 0$ such that

$$M^{-1}|\xi|^{2m} \leqslant \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \leqslant M|\xi|^{2m} \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in \bar{\Omega}.$$

Then it is well known that there exists a sufficiently large number $\sigma > 0$ such that the operator $-A_\sigma \equiv -(A+\sigma)$ satisfies $\Sigma_\omega \cup \{0\} \subseteq \rho(-A_\sigma)$ with some $\omega \in (\frac{\pi}{2}, \pi)$ and

$$\|R(\lambda : -A_\sigma)\| = O(|\lambda|^{\frac{\alpha}{2m}-1}), \quad \lambda \in \Sigma_\omega. \tag{377}$$

Let us recall that A is not densely defined and the exponent $\frac{\alpha}{2m} - 1$ in (377) is sharp. Define $A_{\sigma,\delta} := e^{i(\pi/2\pm\delta)}A_\sigma$. Suppose that $\omega - (\pi/2) < \delta < \omega - \eta(\pi/2)$, $1 \geqslant \theta > \alpha/2m$, $u_0 \in D((-A_{\sigma,\delta})^\theta)$, $\sigma > \eta\alpha/2m$, (358) holds and $f \in L^\infty((0,T) : [D((-\mathcal{A})^\theta)])$. Then the condition (QP) holds for each number $\eta'' \in (\eta, 1)$ such that $\omega - (\pi/2) < \delta < \omega - \eta''(\pi/2)$. Applying Theorem 3.5.12, we obtain that the abstract fractional Cauchy problem

$$\begin{cases} \mathbf{D}_t^\eta u(t,x) = A_{\sigma,\delta}u(t,x) + f(t,x), & t \in (0,T], \\ u(0) = u_0, \end{cases}$$

has a unique classical solution, which is analytically extendable to the sector Σ_ϑ provided that $f(t,x) \equiv 0$ ($\vartheta \equiv \min((\frac{(2/\pi)(\omega-\delta)}{\eta} - 1)\pi/2, \pi)$).

(ii) **[199]** Consider now the following modification of inhomogeneous fractional Poisson heat equation in the space $L^p(\Omega)$:

$$(P)_\eta^\delta : \begin{cases} \mathbf{D}_t^\eta[m(x)v(t,x)] = e^{\pm i\delta}(\Delta - b)v(t,x) + f(t,x), & t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $0 < \eta < 1$. Let the operator $A := \Delta - b$ act on E with the Dirichlet boundary conditions, and let B be the multiplication operator by the function $m(x)$. As it has been proved in [**199**, Example 3.6], there exist an appropriate angle $\omega \in (\frac{\pi}{2}, \pi)$ and a number $d > 0$ such that the multivalued linear operator $\mathcal{A} := AB^{-1}$ satisfies $\Psi_{d,\omega} = \{\lambda \in \mathbb{C} : |\lambda| \leqslant d \text{ or } \lambda \in \overline{\Sigma_\omega}\} \subseteq \rho(\mathcal{A})$ and $\|R(\lambda : \mathcal{A})\| \leqslant$

$M(1+|\lambda|)^{-1/p}$, $\lambda \in \Psi_{d,\omega}$; here it is worth noting that the validity of additional condition [**199**, (3.42)] on the function $m(x)$ enables us to get the better exponent β in (QP), provided that $p > 2$. Henceforth we consider the general case. Suppose, as in part (i), that $\omega - (\pi/2) < \delta < \omega - \eta(\pi/2)$, $1 \geqslant \theta > 1 - 1/p$, $u_0 \in D((-e^{\pm i\delta}\mathcal{A})^{\theta})$, $\sigma > \eta(1-1/p)$, (358) holds and $f \in L^{\infty}((0,T) : [D((-e^{\pm i\delta}\mathcal{A})^{\theta})])$. Then Theorem 3.5.12 implies that the abstract Cauchy problem $(P)^{\delta}_{\eta}$ has a unique solution $t \mapsto v(t,\cdot)$, $t \in (0,T]$, i.e., any function $v(t,\cdot)$ satisfying that $Bv(t,\cdot) \in C([0,T] : E)$, the Caputo fractional derivative $\mathbf{D}_t^{\eta} Bv(t,\cdot)$ is well-defined and belongs to the space $C((0,T] : E)$, $Bv(t,\cdot) \in C((0,T] : E)$, $m(x)v(0,x) = u_0(x)$, $x \in \Omega$ and $(P)^{\delta}_{\eta}$ holds identically.

Observe that the trick used in previous example can be also applied in the analysis of limit problems of fractional diffusion equations in complex systems on the so-called dumbbell domains ([**32**, **529**]), as well as in the analysis of a large class of abstract degenerate fractional differential inclusions involving the rotations of multivalued linear operators considered in [**199**, Section 3.7]. By [**447**, Proposition 3.6] ([**411**, Corollary 5.6]), fractional powers of almost sectorial operators (sectorial multivalued linear operators) satisfy, under some assumptions, the condition (QP) and can therefore be used for providing certain applications of our abstract results. Suitable translations of generators of fractionally integrated semigroups with corresponding growth order satisfy the condition (QP) with $\eta'' = 1$ and $0 < \beta < 1$, as well (cf. [**446**, Example 3.3]).

3.6. Hypercyclic and topologically mixing properties of abstract degenerate (multi-term) time-fractional inclusions

The main aim of this section, which is divided into four subsections, is to provide the basic information on hypercyclic and topologically mixing properties of abstract degenerate (multi-term) time-fractional inclusions.

By E we denote a separable infinite-dimensional Fréchet space over the field of complex numbers. We assume that the topology of E is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms. Let us recall that the translation invariant metric $d\colon E \times E \to [0,\infty)$ is defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x, y \in E.$$

By $\mathcal{A}$ and $\sigma_p(\mathcal{A})$ we denote a multivalued linear operator in E and its point spectrum, respectively.

3.6.1. Hypercyclic and topologically mixing properties of problem $(DFP)_{\alpha,\mathcal{A}}$. In this subsection, we will consider hypercyclic and topologically mixing properties of the following abstract degenerate time-fractional inclusion:

$$(\mathrm{DFP})_{\alpha,\mathcal{A}} : \begin{cases} \mathbf{D}_t^{\alpha} u(t) \in \mathcal{A}u(t), \quad t \geqslant 0, \\ u(0) = x;\ u^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

Let us recall that by a (strong) solution of $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ we mean any continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ such that the term $t \mapsto \mathbf{D}_t^{\alpha} u(t)$, $t \geqslant 0$ is well defined and continuous, as well as that $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ holds. It is worth noting here that we do not require a priori the closedness of the operator $\mathcal{A}$ henceforth. Denote by $Z_{\alpha}(\mathcal{A})$ the set which consists of those vectors $x \in E$ for which there exists a solution of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$. Then $Z_{\alpha}(\mathcal{A})$ is a linear subspace of E. The following is an extension of [**300**, Lemma 2.1] to multivalued linear operator case. The proof is almost straightforward after pointing out that $\mathbf{D}_t^{\alpha} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha})$, $t \geqslant 0$, $\lambda \in \mathbb{C}$, $\alpha > 0$.

LEMMA 3.6.1. *Suppose $\alpha > 0$, $\lambda \in \mathbb{C}$, $x \in E$ and $\lambda x \in \mathcal{A}x$. Then $x \in Z_{\alpha}(\mathcal{A})$ and one solution of $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is given by $u(t) \equiv u(t;x) = E_{\alpha}(\lambda t^{\alpha})x$, $t \geqslant 0$.*

The notion of a (subspace-)hypercyclicity, (subspace-)topological transitivity and (subspace-)topologically mixing property of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ are introduced in the following definition.

DEFINITION 3.6.2. Let $\alpha > 0$, and let $\tilde{E}$ be a closed linear subspace of E. Then it is said that:

(i) an element $x \in Z_{\alpha}(\mathcal{A}) \cap \tilde{E}$ is a $\tilde{E}$-hypercyclic vector for $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ iff there exists a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ with the property that the set $\{u(t;x) : t \geqslant 0\}$ is a dense subset of $\tilde{E}$.

Furthermore, we say that the abstract Cauchy problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is:

(ii) $\tilde{E}$-topologically transitive iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exist $x \in Z_{\alpha}(\mathcal{A}) \cap \tilde{E}$, a strong solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ and $t \geqslant 0$ such that $d(x,y) < \varepsilon$ and $d(u(t;x), z) < \varepsilon$;

(iii) $\tilde{E}$-topologically mixing iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exist $x_t \in Z_{\alpha}(\mathcal{A}) \cap \tilde{E}$ and a strong solution $t \mapsto u(t;x_t)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ such that $d(x_t, y) < \varepsilon$ and $d(u(t;x_t), z) < \varepsilon$.

In the case $\tilde{E} = E$, it is also said that a $\tilde{E}$-hypercyclic vector for $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is a hypercyclic vector for $(\mathrm{DFP})_{\alpha,\mathcal{A}}$, and that $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is topologically transitive, resp. topologically mixing.

In the following theorem, we will reformulate the Desch–Schappacher–Webb and Banasiak–Moszyński criteria for the abstract time-fractional inclusion $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ (cf. [**300**, Theorem 2.3] and its proof):

THEOREM 3.6.3. *Assume $\alpha \in (0,2)$ and there exists an open connected subset Ω of $\mathbb{C}$ which satisfies $\Omega \cap (-\infty, 0] = \emptyset$, $\Omega^{\alpha} := \{\lambda^{\alpha} : \lambda \in \Omega\} \subseteq \sigma_p(\mathcal{A})$ and $\Omega \cap i\mathbb{R} \neq \emptyset$. Let $f \colon \Omega^{\alpha} \to E$ be an analytic mapping such that $f(\lambda^{\alpha}) \in N(\mathcal{A} - \lambda^{\alpha}) \smallsetminus \{0\}$, $\lambda \in \Omega$ and let $\tilde{E} := \overline{span\{f(\lambda^{\alpha}) : \lambda \in \Omega\}}$. Then the abstract degenerate inclusion $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is $\tilde{E}$-topologically mixing.*

The assertion of Theorem 2.10.7 can be restated for multivalued linear operators, as well:

THEOREM 3.6.4. *Suppose that $\alpha > 0$ and $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive reals tending to $+\infty$. If the set $E_{0,\alpha}$, which consists of those elements $y \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$ for which there exists a strong solution $t \mapsto u(t;y)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ such that $\lim_{n\to\infty} u(t_n;y) = 0$, is dense in $\tilde{E}$, and if the set $E_{\infty,\alpha}$, which consists of those elements $z \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$ for which there exist a null sequence $(\omega_n)_{n\in\mathbb{N}} \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$ and a sequence $(u_n(\cdot;\omega_n))_{n\in\mathbb{N}}$ of strong solutions of problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ such that $\lim_{n\to\infty} u(t_n;\omega_n) = z$, is also dense in $\tilde{E}$, then the problem $(\mathrm{DFP})_{\alpha,\mathcal{A}}$ is $\tilde{E}$-topologically transitive.*

3.6.2. Hypercyclic and topologically mixing properties of abstract degenerate Cauchy problems of first and second order. Concerning linear dynamical properties, we have already seen that the abstract degenerate Cauchy problems of first and second order have numerous peculiarities compared with the abstract degenerate fractional Cauchy problems. The main aim of this subsection is to investigate some of these peculiarities for abstract degenerate inclusions of first and second order.

We start by stating the following simple proposition, which has been already considered in a slightly different context.

PROPOSITION 3.6.5. (i) *Suppose that $\alpha = 1$, $x \in Z_1(\mathcal{A})$ and the function $t \mapsto u(t;x)$, $t \geqslant 0$ is a solution of problem $(\mathrm{DFP})_{1,\mathcal{A}}$. Then, for every $s \geqslant 0$, $u(s;x) \in Z_1(\mathcal{A})$ and a solution of $(\mathrm{DFP})_{1,\mathcal{A}}$, with initial condition x replaced by $u(s;x)$, is given by $u(t;u(s;x)) := u(t+s;x)$, $t \geqslant 0$.*

(ii) *Suppose that $\alpha = 2$, $x \in Z_2(\mathcal{A})$ and the function $t \mapsto u(t;x)$, $t \geqslant 0$ is a solution of $(\mathrm{DFP})_{2,\mathcal{A}}$. Then, for every $s \geqslant 0$, $u(s;x) \in Z_2(\mathcal{A})$ and a solution of $(\mathrm{DFP})_{2,\mathcal{A}}$, with initial condition x replaced by $u(s;x)$, is given by $u(t;u(s;x)) := \frac{1}{2}[u(t+s;x) + u(|t-s|;x)]$, $t \geqslant 0$.*

Now we would like to state a result similar to that of S. El Mourchid [**163**, Theorem 2.1]:

THEOREM 3.6.6. *Assume that $\alpha = 1$, $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$, $t_0 > 0$ and $k \in \mathbb{N}$. Let $f_j \colon (\omega_1, \omega_2) \to E$ be integrable, and let for each $j = 1, \ldots, k$ we have $f_j(s) \in is\mathcal{A}f_j(s)$ for a.e. $s \in (\omega_1, \omega_2)$. Put $\psi_{r,j} := \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\,ds$, $r \in \mathbb{R}$, $1 \leqslant j \leqslant k$. Put $\tilde{E} := \overline{span\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}}$. If the operator $\mathcal{A}$ is closed, then the problem $(\mathrm{DFP})_{1,\mathcal{A}}$ is $\tilde{E}$-topologically mixing.*

The assertion of Theorem 2.10.19 can be formulated for multivalued linear operators, as well:

THEOREM 3.6.7. *Let $\tilde{E}$ be a closed linear subspace of E, and let $\alpha = 2$.*

(i) *Suppose that $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive reals tending to $+\infty$. Denote by $X_{1,\tilde{E}}$ the set which consists of those elements $x \in Z_2(\mathcal{A}) \cap \tilde{E}$ for which there exists a solution $t \mapsto u(t;x)$, $t \geqslant 0$ of problem $(\mathrm{DFP})_{2,\mathcal{A}}$ such that $u(0;x) = x$ and $\lim_{n\to\infty} u(t_n;x) = \lim_{n\to\infty} u(2t_n;x) = 0$. If $X_{1,\tilde{E}}$ is dense in $\tilde{E}$, then the problem $(\mathrm{DFP})_{2,\mathcal{A}}$ is $\tilde{E}$-topologically transitive.*

(ii) *Denote by* $X'_{1,\tilde{E}}$ *the set which consists of those elements* $x \in Z_2(\mathcal{A}) \cap \tilde{E}$ *for which there exists a strong solution* $t \mapsto u(t;x)$, $t \geqslant 0$ *of problem* $(\mathrm{DFP})_{2,\mathcal{A}}$ *such that* $u(0;x) = x$ *and* $\lim_{t\to+\infty} u(t;x) = 0$. *If* $X'_{1,\tilde{E}}$ *is dense in* $\tilde{E}$, *then the problem* $(\mathrm{DFP})_{2,\mathcal{A}}$ *is* $\tilde{E}$*-topologically mixing.*

As commented before, Theorem 3.6.3 is no longer true in the case that $\alpha = 2$. If so, then we can pass to the equation of first order with the multivalued linear operator $\left[\begin{smallmatrix} 0 & I \\ \mathcal{A} & 0 \end{smallmatrix}\right]$ and the vector $(f(\lambda^2)\,\lambda f(\lambda^2))^T$, for λ belonging to an open connected subset Ω of $\mathbb{C}$ intersecting the imaginary axis, and apply Theorem 3.6.3, with $\alpha = 1$, after that (cf. Theorem 3.6.12 below for more details).

The interested reader is encouraged to formulate a version of The Hypercyclicity Criterion for degenerate first order inclusions (cf. Theorem 2.10.18).

3.6.3. Hypercyclic and topologically mixing properties of certain classes of abstract degenerate multi-term fractional differential inclusions. In this subsection, we assume that $n \in \mathbb{N} \setminus \{1\}$, $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$, $\mathcal{A}$ and $\mathcal{B}$ are multivalued linear operators on E (not necessarily closed), $0 \leqslant \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n$ and $0 \leqslant \alpha < \alpha_n$. Fix a number $i \in \mathbb{N}^0_{m_n-1}$. Denote $\mathcal{A}_0 = \mathcal{A}$, $\alpha_0 = \alpha$, $m_j = \lceil \alpha_j \rceil$ ($j \in \mathbb{N}^n_0$), $D_i = \{j \in \mathbb{N}_{n-1} : m_j - 1 \geqslant i\}$ and $\mathcal{D}_i = \{j \in \mathbb{N}^0_{n-1} : m_j - 1 \geqslant i\}$. In Subsection 3.6.3 and Subsection 3.6.4, we will inquire into the hypercyclic and topologically mixing properties of the following abstract degenerate multi-term fractional inclusion:

$$0 \in \mathcal{B}\mathbf{D}_t^{\alpha_n} u(t) + \sum_{j=1}^{n-1} \mathcal{A}_j \mathbf{D}_t^{\alpha_j} u(t) - \mathcal{A}\mathbf{D}_t^{\alpha} u(t), \quad t \geqslant 0; \tag{378}$$
$$u^{(k)}(0) = x_k, \quad k \in \mathbb{N}^0_{m_n-1}.$$

In this subsection, we will consider the case in which:

$$u^{(i)}(0) = x_i = x \text{ and } u^{(k)}(0) = x_k = 0, \quad k \in \mathbb{N}^0_{m_n-1} \setminus \{i\}. \tag{379}$$

Because no confusion seems likely, we will denote such a degenerate inclusion by the same symbol (378). Recall, a (strong) solution of (378) is any continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ such that the Caputo fractional derivative $\mathbf{D}_t^{\alpha_n} u(t)$ is well-defined, as well as that the initial conditions in (378) hold and there exist continuous sections $a_j(t) \in \sec_c(\mathcal{A}_j \mathbf{D}_t^{\alpha_j} u(t))$ ($0 \leqslant j \leqslant n$, $t \geqslant 0$) such that

$$0 = \sum_{j=1}^{n} a_j(t) - a_0(t), \quad t \geqslant 0.$$

If (379) holds, then $u(t)$ will be also denoted by $u_i(t) \equiv u_i(t;x)$. We will use the following definition.

Definition 3.6.8. Let $\tilde{E}$ be a closed linear subspace of E. Then it is said that the equation (378) is:

(i) $\tilde{E}$-hypercyclic iff there exist an element $x \in \tilde{E}$ and a strong solution $t \mapsto u_i(t;x)$, $t \geqslant 0$ of (378) such that $\{u_i(t;x) : t \geqslant 0\}$ is a dense subset of $\tilde{E}$; such an element is called a $\tilde{E}$-hypercyclic vector of (378);

(ii) $\tilde{E}$-topologically transitive iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exist an element $x \in \tilde{E}$, a strong solution $t \mapsto u_i(t; x)$, $t \geqslant 0$ of (378) and a number $t \geqslant 0$ such that $d(x, y) < \varepsilon$ and $d(u_i(t; x), z) < \varepsilon$;

(iii) $\tilde{E}$-topologically mixing iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exist an element $x_t \in \tilde{E}$ and a strong solution $t \mapsto u_i(t; x_t)$, $t \geqslant 0$ of (378), with x replaced by x_t, such that $d(x_t, y) < \varepsilon$ and $d(u_i(t; x_t), z) < \varepsilon$.

In the case $\tilde{E} = E$, it is also said that a $\tilde{E}$-hypercyclic vector of (378) is a hypercyclic vector of (378) and that (378) is topologically transitive, resp. topologically mixing.

The assertion of Theorem 2.10.9 can be extended to multivalued linear operators as follows:

THEOREM 3.6.9. *Suppose that $\emptyset \neq \Omega$ is an open connected subset of $\mathbb{C} \smallsetminus \{0\}$, $f\colon \Omega \to E \smallsetminus \{0\}$ is an analytic function, $f_j\colon \Omega \to \mathbb{C} \smallsetminus \{0\}$ is a scalar-valued function $(1 \leqslant j \leqslant n)$, $g\colon \Omega \to E$ satisfies $g(\lambda) \in \mathcal{A} f(\lambda)$, $\lambda \in \Omega$ and*

$$g(\lambda) \in f_n(\lambda)\mathcal{B} f(\lambda); \quad g(\lambda) \in f_j(\lambda)\mathcal{A}_j f(\lambda), \qquad \lambda \in \Omega,\ 1 \leqslant j \leqslant n-1.$$

Suppose, further, that Ω_+ and Ω_- are two non-empty subsets of Ω, and each of them admits a cluster point in Ω. Define $\tilde{E} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$, $H_i(\lambda, t)$ by (226), and $F_i(\lambda, t) := H_i(\lambda, t) f(\lambda)$, for any $t \geqslant 0$ and $\lambda \in \Omega$. If

$$\lim_{t \to +\infty} |H_i(\lambda, t)| = +\infty, \quad \lambda \in \Omega_+ \quad \text{and} \quad \lim_{t \to +\infty} H_i(\lambda, t) = 0, \quad \lambda \in \Omega_-,$$

then (378) is $\tilde{E}$-topologically mixing. Furthermore, there exist continuous sections $a_{j,i}(\lambda, t) \in \sec_c(\mathcal{A}_j F_i(\lambda, t))$ such that the terms $\mathbf{D}_t^{\alpha_j} a_{j,i}(\lambda, t)$ are well-defined $(0 \leqslant j \leqslant n,\ t \geqslant 0,\ \lambda \in \Omega)$ and

$$0 = \sum_{j=1}^{n} \mathbf{D}_t^{\alpha_j} a_{j,i}(\lambda, t) - \mathbf{D}_t^{\alpha} a_{0,i}(\lambda, t), \quad t \geqslant 0,\ \lambda \in \Omega.$$

3.6.4. $\mathcal{D}$-Hypercyclic and $\mathcal{D}$-topologically mixing properties of abstract degenerate multi-term fractional differential inclusions. Now we will briefly explain how we can, following the method proposed in Subsection 2.10.4, slightly generalize the notion introduced in the previous three subsections. For the sake of simplicity, we will not consider here the orbits of multilinear mappings.

Denote by $\mathfrak{Z}$ ($\mathfrak{Z}_{uniq}$) the set of all tuples of initial values $\vec{x} = (x_0, x_1, \ldots, x_{m_n - 1}) \in E^{m_n}$ for which there exists a (unique) strong solution of problem (378). Then $\mathfrak{Z}$ is a linear subspace of E^{m_n} and $\mathfrak{Z}_{uniq} \subseteq \mathfrak{Z}$. For any $\vec{x} \in \mathfrak{Z}$, we denote by $\mathfrak{S}(\vec{x})$ the set consisting of all strong solutions of problem (378) with the initial value $\vec{x}$. Assume that $\mathfrak{P}\colon \mathfrak{Z} \to P(\cup_{\vec{x} \in \mathfrak{Z}} \mathfrak{S}(\vec{x}))$ is a fixed mapping satisfying $\emptyset \neq \mathfrak{P}(\vec{x}) \subseteq \mathfrak{S}(\vec{x})$, $\vec{x} \in \mathfrak{Z}$. Let $\emptyset \neq W \subseteq \mathbb{N}_{m_n}$, let $\hat{E}_i$ be a linear subspace of E $(i \in W)$, and let $\tilde{E}$, $\check{E}$ be linear subspaces of E^{m_n}. Suppose that the tuple $\vec{\beta} = (\beta_0, \beta_1, \ldots, \beta_{m_n - 1}) \in [0, \alpha_n]^{m_n}$ is fixed. Set, with a little abuse of notation in comparision with Subsection 2.10.4,

$$\mathfrak{D} := (\tilde{E}, \check{E}, \{\hat{E}_i : i \in W\}, \vec{\beta}).$$

Denote by $\mathcal{M}_{\mathfrak{D}}$ the set consisting of those tuples $\vec{x} \in \mathfrak{Z}$ for which $\mathrm{Proj}_{i,m_n}(\vec{x}) \in \hat{E}_i$, $i \in W$.

Definition 3.6.10. (cf. also Definition 2.10.20) The abstract Cauchy problem (378) is said to be:

(i) $(\mathfrak{D},\mathfrak{P})$-hypercyclic iff there exist a tuple $\vec{x} \in \mathcal{M}_{\mathfrak{D}} \cap \tilde{E}$ and a function $u(\cdot;\vec{x}) \in \mathfrak{P}(\vec{x})$ such that

$$\mathfrak{G} := \big\{ \big((\mathbf{D}_s^{\beta_0} u(s;\vec{x}))_{s=t}, (\mathbf{D}_s^{\beta_1} u(s;\vec{x}))_{s=t}, \ldots, (\mathbf{D}_s^{\beta_{m_n-1}} u(s;\vec{x}))_{s=t} \big) : t \geqslant 0 \big\}$$

is a dense subset of $\check{E}$; such a vector is called a $(\mathfrak{D},\mathfrak{P})$-hypercyclic vector of problem (378).

(ii) $\mathfrak{D}$-hypercyclic iff it is $(\mathfrak{D},\mathfrak{S})$-hypercyclic; any $(\mathfrak{D},\mathfrak{S})$-hypercyclic vector of problem (378) will be also called a $\mathfrak{D}$-hypercyclic vector of problem (378).

(iii) $\mathfrak{D}_{\mathfrak{P}}$-topologically transitive iff for every pair of open non-empty subsets U and V of E^{m_n} satisfying $U \cap \tilde{E} \neq \emptyset$ and $V \cap \check{E} \neq \emptyset$, there exist a tuple $\vec{x} \in \mathcal{M}_{\mathfrak{D}}$, a function $u(\cdot;\vec{x}) \in \mathfrak{P}(\vec{x})$ and a number $t \geqslant 0$ such that $\vec{x} \in U \cap \tilde{E}$ and

$$\big((\mathbf{D}_s^{\beta_0} u(s;\vec{x}))_{s=t}, (\mathbf{D}_s^{\beta_1} u(s;\vec{x}))_{s=t}, \ldots, (\mathbf{D}_s^{\beta_{m_n-1}} u(s;\vec{x}))_{s=t} \big) \in V \cap \check{E}. \tag{380}$$

(iv) $\mathfrak{D}$-topologically transitive iff it is $\mathfrak{D}_{\mathfrak{S}}$-topologically transitive.

(v) $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing iff for every pair of open non-empty subsets U and V of E^{m_n} satisfying $U \cap \tilde{E} \neq \emptyset$ and $V \cap \check{E} \neq \emptyset$, there exists a number $t_0 \geqslant 0$ such that, for every number $t \geqslant t_0$, there exist a tuple $\vec{x_t} \in \mathcal{M}_{\mathfrak{D}}$ and a function $u(\cdot;\vec{x_t}) \in \mathfrak{P}(\vec{x_t})$ such that $\vec{x_t} \in U \cap \tilde{E}$ and (380) holds with vector $\vec{x}$ replaced by $\vec{x_t}$ therein.

(vi) $\mathfrak{D}$-topologically mixing iff it is $\mathfrak{D}_{\mathfrak{S}}$-topologically mixing.

Remark 3.6.11. Let $0 \leqslant \beta \leqslant \alpha < 2$, and let the requirements of Theorem 3.6.3 hold (here the notation used to denote the space $\tilde{E}$ is slightly different from that used in the formulation of above-mentioned theorem). Then the consideration from Remark 2.10.21(ii) shows that the problem $(\mathrm{DFP})_\alpha$ is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, provided that $\vec{\beta} = (\beta,\beta)$, $W = \{1\}$, $\hat{E}_1 = \overline{span\{f(\lambda^\alpha) : \lambda \in \Omega\}}$, $\tilde{E} = \hat{E}_1 \times \{0\}$, $\check{E} = \{(z,z) : z \in \hat{E}_1\}$ and $\mathfrak{P}((\sum_{i=1}^m \alpha_i f(\lambda_i^\alpha), 0)) = \{\sum_{i=1}^m \alpha_i E_\alpha(\cdot^\alpha \lambda_i^\alpha) f(\lambda_i^\alpha)\}$ ($m \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $\lambda_i \in \Omega$ for $1 \leqslant i \leqslant m$). By assuming some extra conditions, a similar assertion can be proved for a general problem (378) (cf. Remark 2.10.21(iii)).

The conjugacy lemma stated in Theorem 2.10.23 admits a very simple reformulation in our context. Using the proof of Theorem 2.10.24 and the usual matrix conversion of abstract higher-order differential equations with integer order derivatives into the first order matrix differential equation, we can simply verify the validity of following analogue of Theorem 2.10.24.

Theorem 3.6.12. *Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, and let $f\colon \Omega \to E$ be an analytic mapping satisfying*

$$0 \in \Big(\lambda^{\alpha_n} \mathcal{B} + \sum_{i=1}^{n-1} \lambda^{\alpha_i} \mathcal{A}_i - \mathcal{A} \Big) f(\lambda), \quad \lambda \in \Omega.$$

Set $\vec{x_\lambda} := [f(\lambda)\ \lambda f(\lambda)\ \ldots\ \lambda^{n-1}f(\lambda)]^T$ $(\lambda \in \Omega)$*,* $E_0 := span\{\vec{x_\lambda} : \lambda \in \Omega\}$*,* $\tilde{E} := \check{E} := \overline{E_0}$*,* $\vec{\beta} := (0, 1, \ldots, n-1)$*,* $W := \mathbb{N}_n$ *and* $\hat{E}_i := span\{f(\lambda) : \lambda \in \Omega\}$*,* $i \in W$*. Then* $\vec{x_\lambda} \in \mathfrak{M}_{\mathfrak{D}}$*,* $\lambda \in \Omega$ *and the abstract Cauchy problem* (378) *is* $\mathfrak{D}_{\mathfrak{P}}$*-topologically mixing provided that* $\sum_{j=1}^{q} e^{\lambda_j \cdot} f(\lambda_j) \in \mathfrak{P}(\sum_{j=1}^{q} \vec{x_{\lambda_j}})$ *for any* $\sum_{j=1}^{q} \vec{x_{\lambda_j}} \in E_0$ $(q \in \mathbb{N}$*;* $\lambda_j \in \Omega,\ 1 \leqslant j \leqslant q)$*.*

As observed in Remark 2.10.25(iii), Theorem 3.6.12 cannot be so simply reformulated for the abstract degenerate multi-term inclusion (378), provided that there exists an index $i \in \mathbb{N}_n$ such that $\alpha_i \notin \mathbb{N}$. Examples already given in [**292**, Chapter 3] and Section 2.10 can serve for illustration of our theoretical results. Now we would like to present some new elaborate examples in support of Theorem 3.6.3, Theorem 3.6.9 and Theorem 3.6.12.

Example 3.6.13. (i) Suppose that $E := C^2(\mathbb{R})$ is equipped with the usual Fréchet topology, $0 < \alpha < 2$, $m \in C(\mathbb{R})$ and $m(x) > 0$, $x \in \mathbb{R}$. For any $\lambda \in \mathbb{C}$, we denote by $\{f_\lambda^1(x), f_\lambda^2(x)\}$ the fundamental set of solutions of ordinary differential equation $y'' = \lambda m(x)y$. Using the elementary theory of linear ordinary differential equations, and direct computation of matrix exponential

$$e^{x\left[\begin{smallmatrix} 0 & 1 \\ -\lambda m(x) & 0 \end{smallmatrix}\right]}, \quad x \in \mathbb{R},\ \lambda \in \mathbb{C} \smallsetminus (-\infty, 0],$$

we can simply prove that for any arbitrarily chosen open connected subset Ω of $\mathbb{C} \smallsetminus (-\infty, 0]$ satisfying that $\Omega \cap \{e^{\pm it\alpha/2} : t \geqslant 0\} \neq \emptyset$, the mappings $\lambda \mapsto f_\lambda^1(x) \in E$ and $\lambda \mapsto f_\lambda^2(x) \in E$ are analytic. Let Ω be such a set. Denote $\tilde{E} := \overline{span\{f_\lambda^i(x) : \lambda \in \Omega, i = 1, 2\}}$. Then we can apply Theorem 3.6.3 in order to see that the abstract time-fractional Poisson heat equation:

$$\mathbf{D}_t^\alpha[m(x)u(t,x)] = \Delta u(t,x), \quad t \geqslant 0, \quad x \in \mathbb{R};$$
$$m(x)u(0,x) = \phi(x); \ \Big(\frac{\partial}{\partial t}[m(x)u(t,x)]\Big)_{t=0} = 0, \text{ if } \alpha > 1,$$

is $\tilde{E}$-topologically mixing, with the meaning clear.

(ii) Suppose that $n = 3$, $\frac{1}{3} < a < \frac{1}{2}$, $\alpha_3 = 3a$, $\alpha_2 = 2a$, $\alpha_1 = 0$, $\alpha = a$, $c_1 < 0$, $c_2 > 0$ and $i = 1$. Then the analysis given in [**292**, Example 3.3.12(iii)], in combination with Theorem 3.6.9, enables one to deduce some results on topologically mixing properties of the following abstract degenerate multi-term inclusion:

$$0 \in \mathbf{D}_t^{3a}u(t) + c_2\mathbf{D}_t^{2a}u(t) + c_1\mathbf{D}_t^{a}u(t) - \mathcal{A}u(t), \quad t > 0,$$
$$u(0) = 0, \quad u'(0) = x, \quad u''(0) = 0,$$

where $\mathcal{A}$ is an MLO and satisfies certain conditions.

(iii) Suppose that $\mathbf{A}$ is an MLO, Ω is an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, $f \colon \Omega \to E$ is an analytic mapping, $\lambda f(\lambda) \in \mathbf{A}f(\lambda)$,

$\lambda \in \Omega$, $P_i(z)$ is a non-zero complex polynomial ($0 \leqslant i \leqslant n$) and

$$z^n P_n(z) + \sum_{i=1}^{n-1} z^i P_i(z) - P_0(z) \equiv 0. \tag{381}$$

Set $\mathcal{A}_i := P_i(\mathbf{A})$, $i \in \mathbb{N}_n^0$. Then for any non-zero complex polynomial $P(z)$ we have $P(\lambda)f(\lambda) \in P(\mathbf{A})f(\lambda)$, $\lambda \in \Omega$ so that (381) implies

$$0 \in \left(\lambda^n \mathcal{B} + \sum_{i=1}^{n-1} \lambda^i \mathcal{A}_i - \mathcal{A}\right) f(\lambda), \quad \lambda \in \Omega.$$

Hence, Theorem 3.6.12 is susceptible to applications.

3.7. Perturbation results for abstract degenerate Volterra integro-differential equations

We start this section by observing that the following simple lemma holds for multivalued linear operators in locally convex spaces.

Lemma 3.7.1. *Let $\mathcal{A}$ be an* MLO *in E, and let $B \in L(E)$. If $\lambda \in \rho(\mathcal{A})$ and $1 \in \rho(B(\lambda - \mathcal{A})^{-1})$, then $\lambda \in \rho(\mathcal{A} + B)$ and*

$$(\lambda - (\mathcal{A} + B))^{-1} = (\lambda - \mathcal{A})^{-1}(1 - B(\lambda - \mathcal{A})^{-1})^{-1}.$$

Proof. Clearly,

$$\begin{aligned}(\lambda - \mathcal{A})^{-1}(1 - B(\lambda - \mathcal{A})^{-1})^{-1} &= ((1 - B(\lambda - \mathcal{A})^{-1})(\lambda - \mathcal{A}))^{-1} \\ &= (\lambda - \mathcal{A} - B(\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A}))^{-1} \\ &\supseteq (\lambda - \mathcal{A} - B)^{-1}.\end{aligned}$$

Therefore, it suffices to show that

$$x \in (\lambda - \mathcal{A} - B)(\lambda - \mathcal{A})^{-1}(1 - B(\lambda - \mathcal{A})^{-1})^{-1}x, \quad x \in E.$$

But, this is an immediate consequence of the fact that $x = (1 - B(\lambda - \mathcal{A})^{-1})^{-1}x - B(\lambda - \mathcal{A})^{-1}(1 - B(\lambda - \mathcal{A})^{-1})^{-1}x$, $x \in E$. $\square$

Keeping in mind Lemma 3.7.1, the identity [**199**, (1.2)] and the argumentation already given in non-degenerate case, the assertions of [**292**, Theorem 2.6.18-Theorem 2.6.19] can be reformulated for (a, k)-regularized resolvent families in Banach spaces, more or less, without substantial difficulties. The situation is simpler with the assertions of [**349**, Theorem 4.1, Corollary 4.5], which can be almost straightforwardly reformulated for certain classes of K-convoluted semigroups generated by mutivalued linear operators.

The main problem in transferring [**292**, Theorem 2.6.3] to (a, k)-regularized resolvent families subgenerated by mutivalued linear operators lies in the fact that it is not clear how one can prove that the operator $I - (\mathcal{A} + B)/\lambda^\alpha$, appearing in the final part of the proof of this theorem, is injective for $\operatorname{Re}\lambda > 0$ suff. large and $\tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0$ (cf. also [**292**, Theorem 2.6.5, Corollary 2.6.6-Corollary 2.6.9] for further information on this type of bounded commuting perturbations). Nevertheless, the following illustrative example shows that there exist some situations when we can directly apply [**292**, Theorem 2.6.3] (here, concretely, one of its most important consequences, [**292**, Corollary 2.6.6]) in the study of perturbation properties of some well-known degenerate equations of mathematical physics and their fractional analogues:

EXAMPLE 3.7.2. Assume that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on $E = L^p(\mathbb{R}^n)$, for some $1 \leqslant p < \infty$ (possible applications can be given in $L^p(\mathbb{R}^n)_l$-type spaces, as well; cf. [**541**]). Set $\mathbf{A} := (A_1, \ldots, A_n)$. Suppose now that $P_1(x)$ and $P_2(x)$ are two non-zero complex polynomials in n variables and $1 \leqslant \alpha < 2$; put $N_1 := dg(P_1(x))$ and $N_2 := dg(P_2(x))$. Let $\omega \geqslant 0$, $N \in \mathbb{N}$, $r \in (0, N]$, let $Q(x)$ be an r-coercive complex polynomial of degree N, $a \in \mathbb{C} \smallsetminus Q(\mathbb{R}^n)$ and $\gamma = \frac{n}{r}\left|\frac{1}{p} - \frac{1}{2}\right| \max\left(N, \frac{N_1+N_2}{\min(1,\alpha)}\right)$. Suppose that $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, $P_2(x)$ is an elliptic polynomial, and (70) holds. Then [**27**, Corollary 8.3.4] yields that $\overline{P_2(\mathbf{A})}^{-1} \in L(E)$ (the violation of this condition has some obvious unpleasant consequences on the existence and uniqueness of solutions of perturbed problems); hence, $\overline{P_1(\mathbf{A})}\;\overline{P_2(\mathbf{A})}^{-1}$ is a closed linear operator in E. Set

$$R_\alpha(t) := \Big(E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a - Q(x))^{-\gamma}\Big)(\mathbf{A}), \quad t \geqslant 0.$$

By the foregoing, we have that $(R_\alpha(t))_{t \geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family generated by $\overline{P_1(\mathbf{A})}\;\overline{P_2(\mathbf{A})}^{-1}$. Set $Df(x) := \int_{-\infty}^{\infty} \psi(x-y)f(y)dy$, $f \in E$, where $\psi \in L^1(\mathbb{R}^n)$. Then $D \in L(E)$ and commutes with $\overline{P_1(\mathbf{A})}\;\overline{P_2(\mathbf{A})}^{-1}$. Applying [**292**, Corollary 2.6.6], we get that the operator $\overline{P_1(\mathbf{A})}\;\overline{P_2(\mathbf{A})}^{-1} + D$ generates an exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family, which can be applied in the study of the following perturbation of the abstract fractional Barenblatt–Zheltov–Kochina equation

$$(\eta\Delta - 1)\mathbf{D}_t^\alpha u(t) + \Delta u = \int_{-\infty}^{\infty} \psi(x-y)(\eta\Delta - 1)u(t,y)dy \quad (\eta > 0),$$

equipped with the usual initial conditions. We can similarly treat the following perturbation of abstract Boussinesq equation of second order

$$(\sigma^2\Delta - 1)u_{tt} + \gamma^2\Delta u = \int_{-\infty}^{\infty} \psi(x-y)(\sigma^2\Delta - 1)u(t,y)dy \quad (\sigma > 0,\ \gamma > 0).$$

We shall present one more example in support of use of perturbation theory for abstract non-degenerate differential equations (a similar approach works in the analysis of analytical solutions of perturbed abstract fractional Barenblatt–Zheltov–Kochina equations in finite domains):

EXAMPLE 3.7.3. In Example 3.2.17, we have already considered the following fractional analogue of Benney–Luke equation:

$$(P)_{\eta,f} : \begin{cases} (\lambda - \Delta)\mathbf{D}_t^\eta u(t,x) = (\alpha\Delta - \beta\Delta^2)u(t,x) + f(t,x), & t \geqslant 0,\ x \in \Omega, \\ \big(\frac{\partial^k}{\partial t^k} u(t,x)\big)_{t=0} = u_k(x), \quad x \in \Omega,\ 0 \leqslant k \leqslant \lceil \eta \rceil - 1, \\ u(t,x) = \Delta u(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \end{cases}$$

where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary, Δ is the Dirichlet Laplacian in $E = L^2(\Omega)$, acting with domain $H^2(\Omega) \cap H_0^1(\Omega)$, $\lambda \in \sigma(\Delta)$, $0 < \eta < 2$ and $\alpha, \beta > 0$. Denote by $\{\lambda_k\}$ $[= \sigma(\Delta)]$ the eigenvalues of Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leqslant -\lambda_2 \ldots \leqslant -\lambda_k \leqslant \cdots \to +\infty$ as $k \to \infty$) numbered in nonascending order with regard to multiplicities; by $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal eigenfunctions. Let E_0 be the closed subspace of E consisting of those functions from E that are orthogonal to the eigenfunctions $\phi_k(\cdot)$ for $\lambda_k = \lambda$. Define the closed single-valued linear operator $\mathbf{A}$ in E_0 by its graph: $\mathbf{A} = \{(f, g) \in E_0 \times E_0 : (\lambda - \lambda_k)\langle g, \phi_k\rangle = (\alpha\lambda_k - \beta\lambda_k^2)\langle f, \phi_k\rangle$ for all $k \in \mathbb{N}$ with $\lambda_k \neq \lambda\}$. Then the operator $\mathbf{A}$ generates an exponentially bounded, analytic (g_η, I)-regularized resolvent family of angle $\theta \equiv \min((\pi/\eta) - (\pi/2), \pi/2)$. Suppose that B is a closed linear operator in E satisfying that there exists a number $a > 0$ such that for all sufficiently small numbers $b > 0$ we have $D(\mathbf{A}) \subseteq D(B)$ and $\|Bf\| \leqslant a\|f\| + b\|\mathbf{A}f\|$, $f \in D(\mathbf{A})$. Applying [**61**, Theorem 2.25] and our previous analyses, we get that the problem $(P)_{\eta,B,f}$, obtained by replacing the term $f(t, x)$ on the right-hand side of the first equation of problem $(P)_{\eta,f}$ by $(\lambda - \Delta)Bu(t, x) + f(t, x)$, has a unique solution provided that $x_0 \in D(\Delta^2) \cap E_0$, $x_1 \in D(\Delta) \cap E_0$, if $\eta > 1$, $\sum_{k|\lambda_k \neq \lambda} \frac{\langle f(\cdot), \phi_k\rangle}{\lambda - \lambda_k}\phi_k = h \in W_{loc}^{1,1}([0, \infty) : E_0)$ satisfies

$$t \mapsto \sum_{k|\lambda_k \neq \lambda} (\alpha\lambda_k - \beta\lambda_k^2)\Big\langle \frac{d}{dt}(g_\eta * h)(t), \phi_k\Big\rangle\phi_k \in L_{loc}^1([0, \infty) : E_0),$$

$B\phi_k = 0$ for $\lambda = \lambda_k$, and the condition (Q) holds. Finally, we would like to observe that V. E. Fedorov and O. A. Ruzakova have analyzed in [**219**, Section 5], by using a completely different method, perturbations of degenerate differential equations of first order involving polynomials of elliptic selfadjoint operators, as well as that V. E. Fedorov and L. V. Borel have analyzed in [**207**] a class of loaded degenerate integro-differential equations proving particularly some results on the modifed Benney–Luke equation of the form

$$(\lambda - \Delta)u_t(t, x) = (\alpha\Delta - \beta\Delta^2)u(t, x) + \int_0^T \int_\Omega k(x, y, t, s)u(y, s)dy\, d\mu(s), \quad t \in [0, T],$$

where $\mu\colon [0, T] \to \mathbb{R}$ is a function of bounded variation. Observe that the analysis contained in Example 3.2.17 together with [**291**, Theorem A.12] enables one to look into the well-posedness of the following integral equation

$$(\lambda - \Delta)u_t(t, x) = (\alpha\Delta - \beta\Delta^2)(g_\eta * u)(t, x) + (\lambda - \Delta)\int_0^t B(t - s)u(s)ds, \quad t \geqslant 0,$$

where $B(\cdot)$ satisfies certain properties.

The assertion of [**292**, Theorem 2.6.11] admits an extension in our context. We will give some details of the proof immediately after considering Theorem 3.7.5.

THEOREM 3.7.4. *Suppose $M > 0$, $\omega \geqslant 0$, the functions $|a|(t)$ and $k(t)$ satisfy* (P1)*, as well as $\mathcal{A}$ is a densely defined, closed subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t \geqslant 0}$ which satisfies that, for every seminorm $p \in \circledast$, we have $p(R(t)x) \leqslant Me^{\omega t}p(x)$, $x \in E$, $t \geqslant 0$. Suppose, further, $C^{-1}B \in L(E)$,*

$BCx = CBx$, $x \in D(\mathcal{A})$, there exist a locally integrable function $b(t)$ and a number $\omega_0 \geqslant \omega$ such that $|b|(t)$ satisfies (P1) *and $\tilde{b}(\lambda) = \frac{\tilde{a}(\lambda)}{\tilde{k}(\lambda)}$, $\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$. Let $\mu > \omega_0$ and $\gamma \in [0,1)$ be such that*

$$(382) \quad \int_0^\infty e^{-\mu t} p\Big(C^{-1}B \int_0^t b(t-s)R(s)x\,ds\Big)dt \leqslant \gamma p(x), \quad x \in D(\mathcal{A}),\ p \in \circledast.$$

Then the operator $\mathcal{A}+B$ is a closed subgenerator of an (a,k)-regularized C-resolvent family $(R_B(t))_{t\geqslant 0}$ which satisfies $p(R_B(t)x) \leqslant \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geqslant 0$, $p \in \circledast$ and

$$R_B(t)x = R(t)x + \int_0^t R_B(t-r)C^{-1}B \int_0^r b(r-s)R(s)x\,ds\,dr, \quad t \geqslant 0,\ x \in D(\mathcal{A}).$$

Furthermore, the equation (272) *holds with $R(t)$ replaced by $R_B(t)$ therein.*

As observed in [**292**, Theorem 2.6.12], in many cases we do not have the existence of a function $b(t)$ and a complex number z such that $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z$, $\operatorname{Re}\lambda > \omega_1$, $\tilde{k}(\lambda) \neq 0$ (in this case, Theorem 3.7.4 continues to hold with some natural adaptations). The above-mentioned theorem admits an extension in our context, as well. Before we formulate this extension, let us only outline a few relevant details needed for its proof. First of all, suppose that $\mathcal{A}$ is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $l \in \mathbb{N}$ and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leqslant j \leqslant l$. Then we have already seen that, for every $t \in [0,\tau)$,

$$R(t)x_0 = k(t)Cx_0 + \sum_{j=1}^{l-1}(a^{*,j} * k)(t)Cx_j + (a^{*,l} * R(\cdot)x_l)(t).$$

In the case that $\tau = \infty$ and the Laplace transform can be applied, the above equation implies that, for certain values of complex parameter λ, we have:

$$\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}x_0 = \tilde{k}(\lambda)Cx_0 + \sum_{j=1}^{l-1}\tilde{a}(\lambda)^j\tilde{k}(\lambda)Cx_j + \tilde{a}(\lambda)^l\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}x_l.$$

If we define the operator family $(S(t))_{t\geqslant 0}$ as explained below, then the previous equation implies that the identity [**292**, (180)] continues to hold with the single-valued operator A replaced by the MLO $\mathcal{A}$, provided in addition that the number λ in this equation satisfies $\tilde{a}(\lambda) \neq 0$. Furthermore, the identities [**292**, (181), (183)] also hold, and the assumption $y \in (I-\tilde{a}(\lambda)(\mathcal{A}+B))x$ implies on account of Lemma 1.2.4 and the validity of identity [**292**, (180)] that $\widetilde{R_B}(\lambda)y = (I-\tilde{S}(\lambda))^{-1}\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}Cy = \tilde{k}(\lambda)Cx$ for $\operatorname{Re}\lambda > 0$ suff. large and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Owing to the condition (i) in Theorem 3.7.5, we have that the operator $\mathcal{A}+B$ is closed and commutes with C. The representation $(I-\tilde{S}(\lambda))^{-1} = \sum_{n=0}^{\infty}[(\frac{1}{\tilde{a}(\lambda)}-\mathcal{A})^{-1}CC^{-1}B]^n$ implies along with the closedness of $\mathcal{A}$ that

$$\tilde{k}(\lambda)Cx \in \Big(\frac{1}{\tilde{a}(\lambda)} - (\mathcal{A}+B)\Big)(I-\tilde{S}(\lambda))^{-1}\tilde{k}(\lambda)\Big(\frac{1}{\tilde{a}(\lambda)} - \mathcal{A}\Big)^{-1}Cx, \quad x \in E,$$

and $R(C) \subseteq R(I-\tilde{a}(\lambda)(\mathcal{A}+B))$ for $\operatorname{Re}\lambda > 0$ suff. large and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Now it is clear that the Laplace transform identity [**292**, (182)] holds with the operator

$A+B$ replaced by $\mathcal{A}+B$, provided in addition that the number λ in this equation satisfies $\tilde{a}(\lambda) \neq 0$. After that, we can apply Laplace transform. The following holds:

THEOREM 3.7.5. *Suppose* $M,\ M_1 > 0$, $\omega \geqslant 0$, $l \in \mathbb{N}$ *and* $\mathcal{A}$ *is a closed subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *such that* $p(R(t)x) \leqslant Me^{\omega t}p(x)$, $x \in E$, $t \geqslant 0$, $p \in \circledast$ *and* (272) *holds. Let* $|a|(t)$ *and* $k(t)$ *satisfy* (P1), *and let the following conditions hold:*

(i) $BCx = CBx$, $x \in D(\mathcal{A})$; *if* $x = x_0 \in \overline{D(\mathcal{A})}$, *then* $C^{-1}Bx \in D(\mathcal{A}^l)$ *and there exists a sequence* $(x_j)_{1\leqslant j\leqslant l}$ *such that* $x_j \in \mathcal{A}x_{j-1}$ *for* $1 \leqslant j \leqslant l$, *as well as that:*

$$p(Cx_j) \leqslant M_1 p(x), \quad x \in \overline{D(\mathcal{A})},\ p \in \circledast,\ 0 \leqslant j \leqslant l-1,\ \textit{and}$$
$$p(x_l) \leqslant M_1 p(x), \quad x \in \overline{D(\mathcal{A})},\ p \in \circledast.$$

(ii) *There exist a locally integrable function* $b(t)$ *and a complex number* z *such that* $|b|(t)$ *satisfies* (P1) *and*

$$\tilde{a}(\lambda)^{l+1}/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z, \quad \operatorname{Re}\lambda > \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k)),\ \tilde{k}(\lambda) \neq 0.$$

(iii) $\lim_{\lambda\to+\infty}\int_0^\infty e^{-\lambda t}|a(t)|dt = 0$ *and* $\lim_{\lambda\to+\infty}\int_0^\infty e^{-\lambda t}|b(t)|dt = 0$.

Define, for every $x = x_0 \in \overline{D(\mathcal{A})}$ *and* $t \geqslant 0$,

$$S(t)x := \sum_{j=0}^{l-1} a^{*,j+1}(t)Cx_j + \int_0^t b(t-s)R(s)x_l ds + zR(t)x_l,$$

where $(x_j)_{1\leqslant j\leqslant l}$ *is an arbitrary sequence satisfying the assumptions prescribed in* (i). *Then, for every* $x \in E$, *there exists a unique solution of the integral equation*

$$R_B(t)x = R(t)x + (S * R_B)(t)x, \quad t \geqslant 0; \tag{383}$$

furthermore, $(R_B(t))_{t\geqslant 0}$ *is an* (a,k)*-regularized* C*-resolvent family with a closed subgenerator* $\mathcal{A}+B$, *there exist* $\mu \geqslant \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$ *and* $\gamma \in [0,1)$ *such that* $p(R_B(t)x) \leqslant \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geqslant 0$, $p \in \circledast$ *and* (272) *holds with* $R(t)$ *replaced by* $R_B(t)$ *therein.*

REMARK 3.7.6. It is worth noting that Theorem 3.7.4 continues to hold, with appropriate changes, in the case that B is not necessarily bounded operator from $\overline{D(\mathcal{A})}$ into E. More precisely, suppose that E is complete, B is a closed linear operator in E, and the requirements of Theorem 3.7.4 hold with the condition $C^{-1}B \in L(E)$ replaced by that $D(\mathcal{A}) \subseteq D(C^{-1}B)$ and the mapping $t \mapsto C^{-1}B(b * R)(t)x$, $t \geqslant 0$ is well-defined, continuous and Laplace transformable for all $x \in D(\mathcal{A})$. Then the final conclusions established in Theorem 3.7.4 remain valid; here, it is only worth noting that the closedness of the operator $\mathcal{A}+B$ can be proved (cf. [**292**, Remark 2.6.13] for more details, especially, the condition ($\natural$) therein) by using the inclusion $(I - \tilde{S}(\lambda))^{-1}(I - \tilde{a}(\lambda)(\mathcal{A}+B))^{-1}x \subseteq (I - \tilde{a}(\lambda)\mathcal{A})x$, $x \in D(\mathcal{A})$.

REMARK 3.7.7. The method proposed in the proofs of [**459**, Theorem 1.2, Theorem 2.3] and [**292**, Theorem 2.6.13] enables one to deduce some results on the well-posedness of perturbed abstract Volterra inclusion:

$$u(t) \in f(t) + (a + a * k)(t) * \mathcal{A}u(t) + (b * u)(t), \quad t \in [0, \tau), \tag{384}$$

provided that $\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous (a, k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$, b, $k \in L^1_{loc}([0, \infty))$ and $f \in C([0, \infty))$. The starting point is the observation that the regularized resolvent families for (384) satisfy the integral equations like [**459**, (1.28)] or (383).

Now it is quite easy to formulate the following extension of [**292**, Corollary 2.6.15].

COROLLARY 3.7.8. *Suppose $M, M_1 > 0$, $\omega \geqslant 0$, $\alpha > 0$, $\beta \geqslant 0$, $\mathcal{A}$ is a closed subgenerator of a $(g_\alpha, g_{\alpha\beta+1})$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying $p(R(t)x) \leqslant Me^{\omega t}p(x)$, $x \in E$, $t \geqslant 0$, $p \in ⊛$ and* (272) *holds with $a(t) = g_\alpha(t)$ and $k(t) = g_{\alpha\beta+1}(t)$. Assume exactly one of the following conditions:*

- (i) *$\alpha - 1 - \alpha\beta \geqslant 0$, $BCx = CBx$, $x \in D(\mathcal{A})$, and* (a) $\vee$ (b)*, where:*
  - (a) *$p(C^{-1}Bx) \leqslant M_1 p(x)$, $x \in \overline{D(\mathcal{A})}$, $p \in ⊛$.*
  - (b) *E is complete,* (382) *holds, $D(\mathcal{A}) \subseteq D(C^{-1}B)$, as well as the mapping $t \mapsto C^{-1}B(b * R)(t)x$, $t \geqslant 0$ is well-defined, continuous and Laplace transformable for all $x \in D(\mathcal{A})$.*
- (ii) *$\alpha - 1 - \alpha\beta < 0$, $BCx = CBx$, $x \in D(\mathcal{A})$, $l = \lceil \frac{\alpha\beta+1-\alpha}{\alpha} \rceil$ and* (i) *of* Theorem 3.7.5 *holds.*

Then there exist $\mu > \omega$ and $\gamma \in [0, 1)$ such that $\mathcal{A} + B$ is a closed subgenerator of a $(g_\alpha, g_{\alpha\beta+1})$-regularized C-resolvent family $(R_B(t))_{t\geqslant 0}$ satisfying $p(R_B(t)x) \leqslant \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geqslant 0$, $p \in ⊛$, and (272) *holds with $R(t)$ replaced by $R_B(t)$ therein, with $a(t) = g_\alpha(t)$ and $k(t) = g_{\alpha\beta+1}(t)$.*

Observe that the local Hölder continuity is an example of the property that is stable under perturbations described in the previous three assertions (cf. [**292**, Remark 2.6.14] for more details, and [**292**, Remark 2.6.16] for inheritance of analytical properties under perturbations described in Corollary 3.7.8).

Now we would like to provide an illustrative application of non-degenerate version of Theorem 3.7.4 (possibilities for work exist even in the case that $C \neq I$ and E is not a Banach space).

EXAMPLE 3.7.9. In a great number of research papers, C. Lizama and his coauthors have analyzed possibilities to apply Theorem 1.4.12(i) in the qualitative analysis of abstract non-degenerate Volterra integro-differential equations. The main goal of this example is to show how the above-mentioned result, combined with Theorem 3.7.4, can be used for proving some sufficient conditions for generation of specific classes of (a, k)-regularized resolvent families. We continue the analysis of R. Ponce and M. Warma [**456**] here.

Suppose that $(E, \| \cdot \|)$ is a Banach space, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geqslant 0$, $0 < \zeta \leqslant 1$ and $\omega \in \mathbb{R}$. Let any of the following two conditions hold:

(i) $\alpha > 0$, $\mathcal{A}$ is an MLO satisfying

$$\omega + \Sigma_{\zeta\pi/2} \subseteq \rho(\mathcal{A}) \text{ and } \|R(\lambda : \mathcal{A})\| = O(|\lambda - \omega|^{-1}), \quad \lambda \in \omega + \Sigma_{\zeta\pi/2}. \tag{385}$$

(ii) $\alpha < 0$, $\alpha + \beta^{\zeta} \geqslant |\alpha|$ and $\mathcal{A}$ is an MLO satisfying (385).

Then it is well-known that the operator $\mathcal{A}_{|\overline{D(\mathcal{A})}}$ is single-valued, linear and densely defined in the Banach space $\overline{D(\mathcal{A})}$, as well as that (385) holds with the operator $\mathcal{A}$ replaced with the operator $\mathcal{A}_{|\overline{D(\mathcal{A})}}$; see e.g. [**444**, Lemma 4.1]. Set $a(t) := 1 + (1 * k)(t)$, where $k(t) := \alpha e^{-\beta t} g_{\zeta}(t)$. Owing to the proof of [**456**, Theorem 2.1], we get that $\mathcal{A}_{|\overline{D(\mathcal{A})}}$ generates an exponentially bounded $(a, 1)$-regularized resolvent family $(S_{\omega}(t))_{t\geqslant 0}$ in $\overline{D(\mathcal{A})}$, provided that $\omega = 0$. In the general case $\omega \neq 0$, Theorem 3.7.4 and decomposition $\mathcal{A}_{|\overline{D(\mathcal{A})}} = (\mathcal{A}_{|\overline{D(\mathcal{A})}} - \omega I_{|\overline{D(\mathcal{A})}}) + \omega I_{|\overline{D(\mathcal{A})}}$ show that $\mathcal{A}_{|\overline{D(\mathcal{A})}}$ generates an exponentially bounded $(a, 1)$-regularized resolvent family $(S(t))_{t\geqslant 0}$ in $\overline{D(\mathcal{A})}$, as well. This extends the assertion of [**456**, Corollary 2.2], and can be applied in the analysis of Poisson heat equation with memory, in the space $H^{-1}(\Omega)$; see e.g. [**199**, Example 3.3]. The proof of [**456**, Theorem 2.3] works in degenerate case and we may conclude the following: Let $\alpha \neq 0$, $\beta \geqslant 0$, $0 < \zeta < \tilde{\zeta} \leqslant 1$, $\omega < 0$ and $\beta + \omega \leqslant 0$. If (i) holds with the number ζ replaced with the number $\tilde{\zeta}$ therein, then $\|S(t)\| = O(e^{-\beta t})$, $t \geqslant 0$; if (ii) holds with the number ζ replaced with the number $\tilde{\zeta}$ therein, then $\|S(t)\| = O((1 + \alpha\omega t^{\zeta+1})e^{-(\beta-(\alpha\omega)^{1/(\zeta+1)})t})$, $t \geqslant 0$.

The proof of following result is very similar to that of [**292**, Theorem 2.6.22] and, because of that, we will skip it.

THEOREM 3.7.10. *Let $k(t)$ and $|a|(t)$ satisfy* (P1). *Suppose $\delta \in (0, \pi/2]$, $\omega \geqslant \max(0, \mathrm{abs}(|a|), \mathrm{abs}(k))$, there exist analytic functions $\hat{k} \colon \omega + \Sigma_{\frac{\pi}{2}+\delta} \to \mathbb{C}$ and $\hat{a} \colon \omega + \Sigma_{\frac{\pi}{2}+\delta} \to \mathbb{C}$ such that $\hat{k}(\lambda) = \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega$, $\hat{a}(\lambda) = \tilde{a}(\lambda)$, $\operatorname{Re}\lambda > \omega$ and $\hat{k}(\lambda)\hat{a}(\lambda) \neq 0$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\delta}$. Let $\mathcal{A}$ be a closed subgenerator of an analytic (a, k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle δ, and let* (272) *hold. Suppose that, for every $\eta \in (0, \delta)$, there exists $c_{\eta} > 0$ such that*

$$p(e^{-\omega \operatorname{Re} z} R(z)x) \leqslant c_{\eta} p(x), \quad x \in E,\ p \in \circledast,\ z \in \Sigma_{\eta},$$

as well as $b, c \geqslant 0$, B is a linear operator satisfying $D(C^{-1}\mathcal{A}C) \subseteq D(B)$, $BCx = CBx$, $x \in D(C^{-1}\mathcal{A}C)$ and

$$p(C^{-1}Bx) \leqslant bp(y) + cp(x), \text{ whenever } (x, y) \in C^{-1}\mathcal{A}C, \quad p \in \circledast.$$

Assume that at least one of the following conditions holds:

(i) *$\mathcal{A}$ is densely defined, the numbers b and c are sufficiently small, there exists $|C|_{\circledast} > 0$ such that $p(Cx) \leqslant |C|_{\circledast} p(x)$, $x \in E$, $p \in \circledast$ and, for every $\eta \in (0, \delta)$, there exists $\omega_{\eta} \geqslant \omega$ such that $|\hat{k}(\lambda)^{-1}| = O(|\lambda|)$, $\lambda \in \omega_{\eta} + \Sigma_{\frac{\pi}{2}+\eta}$ and $|\hat{a}(\lambda)/\hat{k}(\lambda)| = O(|\lambda|)$, $\lambda \in \omega_{\eta} + \Sigma_{\frac{\pi}{2}+\eta}$.*

(ii) *$\mathcal{A}$ is densely defined, the number b is sufficiently small, there exists $|C|_{\circledast} > 0$ such that $p(Cx) \leqslant |C|_{\circledast} p(x)$, $x \in E$, $p \in \circledast$ and, for every $\eta \in (0, \delta)$, there exists $\omega_{\eta} \geqslant \omega$ such that $|\hat{k}(\lambda)^{-1}| = O(|\lambda|)$, $\lambda \in \omega_{\eta} + \Sigma_{\frac{\pi}{2}+\eta}$ and $\hat{a}(\lambda)/(\lambda\hat{k}(\lambda)) \to 0$, $|\lambda| \to \infty$, $\lambda \in \omega_{\eta} + \Sigma_{\frac{\pi}{2}+\eta}$.*

(iii) $\mathcal{A}$ *is densely defined, the number* c *is sufficiently small,* $b = 0$ *and, for every* $\eta \in (0, \delta)$*, there exists* $\omega_\eta \geqslant \omega$ *such that* $|\hat{a}(\lambda)/\hat{k}(\lambda)| = O(|\lambda|)$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

(iv) $b = 0$ *and, for every* $\eta \in (0, \delta)$*, there exists* $\omega_\eta \geqslant \omega$ *such that* $\hat{a}(\lambda)/(\lambda\hat{k}(\lambda)) \to 0$, $|\lambda| \to \infty$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

Then $C^{-1}(C^{-1}\mathcal{A}C + B)C$ *is the integral generator of an exponentially equicontinuous, analytic* (a, k)*-regularized* C*-resolvent family* $(R_B(t))_{t\geqslant 0}$ *of angle* δ*, which satisfies* $R_B(z)[C^{-1}(C^{-1}\mathcal{A}C + B)C] \subseteq [C^{-1}(C^{-1}\mathcal{A}C + B)C]R_B(z)$, $z \in \Sigma_\delta$ *and the following condition:*

$$\forall \eta \in (0, \delta)\ \exists \omega'_\eta > 0\ \exists M_\eta > 0\ \forall p \in \circledast :$$
$$p(R_B(z)x) \leqslant M_\eta e^{\omega'_\eta \operatorname{Re} z} p(x), \quad x \in E,\ z \in \Sigma_\eta.$$

Furthermore, in cases (iii) *and* (iv)*, the above remains true with the operator* $C^{-1}(C^{-1}\mathcal{A}C + B)C$ *replaced by* $C^{-1}\mathcal{A}C + B$.

In this book, we will not discuss possibilities to generalize results on rank 1-perturbations [**26**] and time-dependent perturbations [**547**] to (a, k)-regularized C-resolvent families subgenerated by mutivalued linear operators; for more details about non-degenerate case, we refer the reader to [**292**, Lemma 2.6.26–Theorem 2.6.33] and [**292**, Theorem 2.6.34, Corollary 2.6.35–Corollary 2.6.38, Theorem 2.6.40, Corollary 2.6.42–Corollary 2.6.45]. In the following theorem, we will extend the assertions of time-dependent perturbations [**61**, Theorem 2.26] and [**292**, Theorem 2.6.46(i)] to multivalued linear operators. The proof is very similar to that of Theorem 3.7.13 below and, because of that, we will skip it.

THEOREM 3.7.11. *Suppose* $\alpha \geqslant 1$, $M \geqslant 1$, $\omega \geqslant 0$ *and* $\mathcal{A}$ *is a closed subgenerator of a* (*local*) (g_α, C)*-regularized resolvent family* $(S_\alpha(t))_{t\in[0,\tau)}$ *satisfying* $p(S_\alpha(t)x) \leqslant Me^{\omega t}p(x)$, $t \in [0, \tau)$, $x \in E$, $p \in \circledast$ *and* (272) *with* $R(t)$ *and* $a(t)$ *replaced by* $S_\alpha(t)$ *and* $g_\alpha(t)$*, respectively. Let* $(B(t))_{t\in[0,\tau)} \subseteq L(E)$, $\mathrm{R}(B(t)) \subseteq \mathrm{R}(C)$, $t \in [0, \tau)$ *and* $C^{-1}B(\cdot) \in C([0, \tau) : L(E))$. *Assume that* $t \mapsto C^{-1}f(t)$, $t \in [0, \tau)$ *is a locally integrable* E*-valued mapping such that the mapping* $t \mapsto (d/dt)C^{-1}f(t)$ *is defined for a.e.* $t \in [0, \tau)$ *and locally integrable on* $[0, \tau)$ (*in the sense of* [**410**, Definition 4.4.3]). *Then there exists a unique solution of the integral Volterra inclusion:*

$$u(t, f) \in f(t) + \mathcal{A}\int_0^t g_\alpha(t-s)u(s, f)\,ds + \int_0^t g_\alpha(t-s)B(s)u(s, f)\,ds, \quad t \in [0, \tau); \tag{386}$$

here, by a solution of (386) *we mean any continuous function* $u \in C([0, \tau) : E)$ *such that there exists a continuous section* $u_{\mathcal{A},\alpha,f}(t) \in \sec_c(\mathcal{A}\int_0^t g_\alpha(t-s)u(s, f)\,ds)$ *for* $t \in [0, \tau)$*, with the property that* $u(t, f) = f(t) + u_{\mathcal{A},\alpha,f}(t) + \int_0^t g_\alpha(t-s)B(s)u(s, f)\,ds$, $t \in [0, \tau)$. *The solution* $u(t, f)$ *is given by* $u(t, f) := \sum_{n=0}^{\infty} S_{\alpha,n}(t)$, $t \in [0, \tau)$*, where we define* $S_{\alpha,n}(t)$ *recursively by*

$$S_{\alpha,0}(t) := S_\alpha(t)C^{-1}f(0) + \int_0^t S_\alpha(t-s)(C^{-1}f)'(s)\,ds, \quad t \in [0, \tau) \tag{387}$$

and

$$(388)\quad S_{\alpha,n}(t) := \int_0^t \int_0^{t-\sigma} g_{\alpha-1}(t-\sigma-s)S_\alpha(s)C^{-1}B(\sigma)S_{\alpha,n-1}(\sigma)ds\,d\sigma, \quad t \in [0,\tau).$$

Denote, for every $T \in (0,\tau)$ *and* $p \in \circledast$, $K_{T,p} := \max_{t\in[0,T]} p(C^{-1}B(t))$ *and* $F_{T,p} := p(C^{-1}f(0)) + \int_0^T e^{-\omega s}p((C^{-1}f)'(s))ds$. *Then, for every* $p \in \circledast$, *we have:*

$$p(u(t,f)) \leqslant Me^{\omega t}E_\alpha(MK_{T,p}t^\alpha)F_{T,p}, \quad t \in [0,T]$$

and

$$p(u(t,f) - S_{\alpha,0}(t)) \leqslant Me^{\omega t}(E_\alpha(MK_{T,p}t^\alpha) - 1)F_{T,p}, \quad t \in [0,T].$$

Motivated by the research of A. Favini and A. Yagi [**199**, Chapter III], we introduce the following definition (for the sake of convenience, we shall work only in Banach spaces).

Definition 3.7.12. Suppose that $(E, \|\cdot\|)$ is a Banach space, $\alpha > 0$, $\zeta \in (0,1)$, $0 < \tau \leqslant \infty$, $\mathcal{A}$ is an MLO in E, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in(0,\tau)} \subseteq L(E)$ is a (g_α, C)-regularized resolvent family of growth order ζ, with a subgenerator $\mathcal{A}$, iff the family $\{t^\zeta R(t) : t \in (0,\tau)\} \subseteq L(E)$ is bounded, as well as that $R(t)C = CR(t)$, $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in (0,\tau))$ and

$$\int_0^t g_\alpha(t-s)R(s)y\,ds = R(t)x - Cx, \text{ whenever } t \in (0,\tau) \text{ and } (x,y) \in \mathcal{A}.$$

It directly follows from definition that, for every $\nu > \zeta$, the operator family $((g_\nu * R)(t))_{t\in[0,\tau)}$ is a g_ν-times integrated (g_α, C)-regularized resolvent family with a subgenerator $\mathcal{A}$. Consider now the abstract integral Volterra inclusion (386) in which we are finding solutions defined on the finite time interval $(0,\tau)$, with $B(\cdot) \in C([0,\tau) : L(E))$ and $f \in C((0,\tau) : E)$. By a solution of (386) on $(0,\tau)$ we mean any continuous function $u \in C((0,\tau) : E)$ such that the mapping $t \mapsto u(t,f)$, $t \in (0,\tau)$ is locally integrable at the point $t = 0$ and there exists a continuous section $u_{\mathcal{A},\alpha,f}(t) \in \sec_c(\mathcal{A}\int_0^t g_\alpha(t-s)u(s,f)ds)$ for $t \in (0,\tau)$, with the property that $u(t,f) = f(t) + u_{\mathcal{A},\alpha,f}(t) + \int_0^t g_\alpha(t-s)B(s)u(s,f)ds$, $t \in (0,\tau)$. The subsequent theorem is very similar to [**61**, Theorem 2.26] and [**292**, Theorem 2.6.46(i)]. For the sake of clarity, we will include the proof.

Theorem 3.7.13. *Suppose* $\alpha \geqslant 1$, $M \geqslant 1$, $\omega \geqslant 0$ *and* $\mathcal{A}$ *is a closed subgenerator of a (local)* (g_α, C)*-regularized resolvent family* $(S_\alpha(t))_{t\in(0,\tau)}$ *of growth order* $\zeta \in (0,1)$, *satisfying that* $\|g_{\zeta+1}(t)S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \in (0,\tau)$ *and that* (272) *holds for* $t \in (0,\tau)$ *with* $R(t)$ *and* $a(t)$ *replaced by* $S_\alpha(t)$ *and* $g_\alpha(t)$, *respectively. Let* $(B(t))_{t\in[0,\tau)} \subseteq L(E)$, $\mathrm{R}(B(t)) \subseteq \mathrm{R}(C)$, $t \in [0,\tau)$ *and* $C^{-1}B(\cdot) \in C([0,\tau) : L(E))$. *Assume that* $t \mapsto C^{-1}f(t)$, $t \in [0,\tau)$ *is a continuous* E*-valued mapping such that the mapping* $t \mapsto S_{\alpha,0}(t)$, *defined by* (387), *is a solution of problem*

$$v(t,f) \in f(t) + \mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)ds, \quad t \in (0,\tau)$$

and satisfies $\|g_{\zeta+1}(t)S_{\alpha,0}(t)\| \leqslant Me^{\omega t}$, $t \in (0,\tau)$ (*cf.* [**199**, Theorem 3.7-Theorem 3.13] *for more details*). *Then there exists a unique solution* $u(t,f)$ *of the abstract integral Volterra inclusion* (386) *on the interval* $(0,\tau)$. *Moreover, the solution* $u(t,f)$ *is given by* $u(t,f) := \sum_{n=0}^{\infty} S_{\alpha,n}(t)$, $t \in (0,\tau)$, *where we define* $S_{\alpha,n}(t)$ *for* $t \in (0,\tau)$ *recursively by* (388). *Denote, for every* $T \in (0,\tau)$, $K_T := \max_{t\in[0,T]} \|C^{-1}B(t)\|$. *Then there exists a constant* $c_{\alpha,\gamma} > 0$ *such that:*

$$\|u(t,f)\| \leqslant c_{\alpha,\gamma} e^{\omega t} t^{-\eta} E_{\alpha-\zeta,1-\zeta}(MK_T t^{\alpha-\zeta}), \quad t \in (0,T] \tag{389}$$

and

$$\|u(t,f) - S_{\alpha,0}(t)\| \leqslant c_{\alpha,\gamma} e^{\omega t} t^{-\eta}(E_{\alpha-\zeta,1-\zeta}(MK_T t^{\alpha-\zeta}) - 1), \quad t \in (0,T]. \tag{390}$$

Proof. It is very simple to prove that there exists a constant $c_{\alpha,\gamma} > 0$ such that:

$$\|S_{\alpha,n}(t)\| \leqslant M^{n+1}K_T^n e^{\omega t}(g_{\eta+1}(t))^{-1}\frac{t^{(\alpha-\eta)n}}{\Gamma((\alpha-\eta)n+1-\eta)}, \quad t \in (0,T],\ n \in \mathbb{N}_0,$$

which implies that the series $\sum_{n=0}^{\infty} S_{\alpha,n}(t)$ converges uniformly on compact subsets of $[\varepsilon,T]$ and (389)–(390) hold ($0 < \varepsilon < T$). Clearly, $u(t,f) = S_{\alpha,0}(t) + \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)C^{-1}B(s)u(s,f)\,ds$, $t \in [0,T]$. With the help of Theorem 1.2.3, this implies

$$\begin{aligned}
u(t,f) &\in f(t) + \mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)\,ds + [g_{\alpha-1} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f)](t)\\
&\in f(t) + \mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)\,ds + \mathcal{A}\int_0^t g_\alpha(t-s)(g_{\alpha-1} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f))(s)\,ds\\
&\qquad + [g_{\alpha-1} * C * C^{-1}B(\cdot)u(\cdot,f)](t)\\
&= f(t) + \mathcal{A}\int_0^t g_\alpha(t-s)u(s,f)\,ds + \int_0^t g_\alpha(t-s)B(s)u(s,f)\,ds, \quad t \in (0,\tau).
\end{aligned}$$

Therefore, $u(t,x)$ is a solution of (386). Since the variation of parameters formula holds in our framework, the uniqueness of solutions follows similarly as in the proof of [**61**, Theorem 2.26]. $\square$

Now we will present an illustrative example of application of the above result.

Example 3.7.14. (i) It is clear that Theorem 3.7.13 can be applied in the analysis of a great number of the abstract degenerate Cauchy problems of first order appearing in [**199**, Chapter III] (applications can be also made to some time-oscillation degenerate equations for which the range of possible values of corresponding Caputo fractional derivative depends directly on the value of constant $c > 0$ in condition [**199**, (P), p. 47], provided that $\alpha = 1$ in (P)). For example, we can consider the following time-dependent perturbation of the Poisson heat equation in the space $E = L^p(\Omega)$:

$$(P)_{t-d} : \begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = \Delta v(t,x) + bv(t,x) + m(x)B(t)v(t,x), \quad t \geqslant 0,\ x \in \Omega;\\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega,\\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $B \in C([0,\infty) : L(E))$.

(ii) Suppose that A, B and C are three closed linear operators in E, $D(B) \subseteq D(A) \cap D(C)$, $B^{-1} \in L(E)$ and the conditions [**199**, (6.4)–(6.5)] hold with certain numbers $c > 0$ and $0 < \beta \leqslant \alpha \leqslant 1$. In [**199**, Chapter VI], the second order differential equation

$$\frac{d}{dt}(Cu'(t)) + Bu'(t) + Au(t) = f(t), \quad t > 0,$$

has been considered by the usual converting into the first order matricial system

$$\frac{d}{dt}Mz(t) = Lz(t) + F(t), \quad t > 0,$$

where

$$M = \begin{bmatrix} I & O \\ O & C \end{bmatrix}, \ L = \begin{bmatrix} O & I \\ -A & -B \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \quad (t > 0).$$

Owing to the proof of [**199**, Theorem 6.1], we get that the multivalued linear operator $L_{[D(B)]\times E}(M_{[D(B)]\times E})^{-1}$ generates a (g_1, I)-regularized resolvent family $(S_1(t))_{t>0}$ of growth order $\zeta = ((1-\beta)/\alpha)$ in the pivot space $[D(B)] \times E$, satisfying additionally that there exists $\omega \geqslant 0$ with the property that $\|g_{\zeta+1}(t)S_1(t)\| \leqslant Me^{\omega t}$, $t > 0$. Assuming that the mappings $t \mapsto B_{1,3}(t) \in L([D(B)])$, $t \geqslant 0$ and $t \mapsto B_{2,4}(t) \in L(E)$, $t \geqslant 0$ are continuous, Theorem 3.7.13 is susceptible to applications so that we are in a position to consider the-wellposedness of the following system of equations:

$$u_1'(t) = u_2(t) + B_1(t)u_1(t) + B_2(t)u_2(t) + f_1(t), \quad t > 0;$$
$$\frac{d}{dt}(Cu_2(t)) = -Au_1(t) - Bu_2(t) + B_3(t)u_1(t) + B_4(t)u_2(t) + f_2(t), \quad t > 0.$$

Many concrete examples of applications can be found in [**199**, Section 6.2].

Evidently, Theorem 3.7.13 can be applied only in the case that $\alpha \geqslant 1$. As already seen, the case $0 < \alpha < 1$ is much more sofisticated from the theoretical and practical point of view. We close this section with the observation that H. K. Avad and A. V. Glushak [**38**] have analyzed the perturbed time-fractional problem

$$\begin{aligned} &D_t^\alpha u(t) = Au(t) + F(t, B(t)u(t)), \quad t > 0, \\ &\lim_{t\to 0+} (g_{1-\alpha} * u)(t) = u_0, \end{aligned}$$

where A is a closed linear operator acting on a Banach space E, $F(\cdot,\cdot)$ and $B(\cdot)$ possess certain properties, and $0 < \alpha < 1$. Some of their results, for example [**38**, Theorem 3.1], can be simply reword for abstract time-relaxation inclusions.

Finally, we would like to note that perturbation results for multivalued linear operators have been also examined by R. Cross, A. Favini and Y. Yakubov in [**121**].

3.8. Approximation and convergence of degenerate (a, k)-regularized C-resolvent families

The state space in this section will be denoted by E. If $(A_l)_{l\in\mathbb{N}_0}$ $(((R_l(t))_{t\geqslant 0})_{l\in\mathbb{N}_0})$ is a sequence of closed linear operators on E (strongly continuous operator families in $L(E)$), then we also write A $((R(t))_{t\geqslant 0})$ in place of A_0 $((R_0(t))_{t\geqslant 0})$.

Making use of Theorem 1.4.11 and Theorem 3.2.5, we can simply prove an extension of [**326**, Theorem 2.3] for (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators:

THEOREM 3.8.1. *Assume that, for every $n \in \mathbb{N}_0$, $|a_n|(t)$ and $k_n(t)$ satisfy* (P1) *and $\mathcal{A}_n$ is a closed subgenerator of an (a_n, k_n)-regularized C_n-resolvent family $(R_n(t))_{t\geqslant 0}$ which satisfies* (272) *with $a(t)$, $R(t)$ and $k(t)$ replaced respectively by $a_n(t)$, $R_n(t)$ and $k_n(t)$ $(n \in \mathbb{N}_0)$. Assume further that there exists a real number $\omega \geqslant \sup_{n\in\mathbb{N}_0} \max(0, \operatorname{abs}(|a_n|), \operatorname{abs}(k_n))$ such that, for every $p \in ⊛$, there exist $c_p > 0$ and $r_p \in ⊛$ with*

$$p(e^{-\omega t}R_n(t)x) \leqslant c_p r_p(x), \quad t \geqslant 0,\ x \in E,\ n \in \mathbb{N}_0. \tag{391}$$

Let $\lambda_0 \geqslant \omega$. Put $\mathfrak{T} := \{\lambda > \lambda_0 : \widetilde{a_n}(\lambda)\widetilde{k_n}(\lambda) \neq 0 \text{ for all } n \in \mathbb{N}_0\}$. Then the following assertions are equivalent.

(i) $\lim_{n\to\infty} \widetilde{k_n}(\lambda)(I-\widetilde{a_n}(\lambda)\mathcal{A}_n)^{-1}C_n x = \tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}Cx$, $\lambda \in \mathfrak{T}$, $x \in E$ *and the sequence $(R_n(t)x)_n$ is equicontinuous at each point $t \geqslant 0$ $(x \in E)$.*

(ii) $\lim_{n\to\infty} R_n(t)x = R(t)x$, $t \geqslant 0$, $x \in E$, *uniformly on compacts of* $[0,\infty)$.

Keeping in mind Lemma 1.2.4(i) and Theorem 3.2.5, it is almost straightforward to formulate an extension of [**326**, Theorem 2.4] for (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators, as well; the only thing worth noting is that, in the proof of last mentioned theorem, we can replace the sequence $(A_nH_n(\lambda')x)_{n\in\mathbb{N}}$ with $(\widetilde{k_n}(\lambda')\widetilde{a_n}(\lambda')^{-1}[\widetilde{a_n}(\lambda')^{-1}(\widetilde{a_n}(\lambda')^{-1} - \mathcal{A}_n)^{-1}C_nx - C_nx])_{n\in\mathbb{N}}$:

THEOREM 3.8.2. *Assume that, for every $n \in \mathbb{N}_0$, $|a_n|(t)$ and $k_n(t)$ satisfy* (P1) *and $\mathcal{A}_n$ is a closed subgenerator of an (a_n, k_n)-regularized C_n-resolvent family $(R_n(t))_{t\geqslant 0}$ which satisfies* (272) *with $a(t)$, $R(t)$ and $k(t)$ replaced respectively by $a_n(t)$, $R_n(t)$ and $k_n(t)$ $(n \in \mathbb{N}_0)$. Assume further that there exists a real number $\omega \geqslant \sup_{n\in\mathbb{N}_0} \max(0, \operatorname{abs}(|a_n|), \operatorname{abs}(k_n))$ such that, for every seminorm $p \in ⊛$, there exist a number $c_p > 0$ and a seminorm $r_p \in ⊛$ such that* (391) *holds. Let $\lambda_0 \geqslant \omega$, and let $\mathfrak{T}$ be defined as above. Assume that the following conditions hold:*

(i) *The sequence $(k_n(t))_n$ is equicontinuous at each point $t \geqslant 0$.*

(ii) *For every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in E, one has $\sup_{n\in\mathbb{N}} p(C_nx_n) < \infty$.*

(iii) *There exists $\lambda' \in \mathfrak{T}$ such that $\mathrm{R}((\frac{1}{\tilde{a}(\lambda')} - \mathcal{A})^{-1}C)$ is dense in E as well as that the sequences $(\widetilde{k_n}(\lambda')\widetilde{a_n}(\lambda')^{-1})_{n\in\mathbb{N}}$ and $(\widetilde{a_n}(\lambda')^{-1})_{n\in\mathbb{N}}$ are bounded.*

(iv) *For every $\varepsilon > 0$ and $t \geqslant 0$, there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that*

$$\int_0^{\min(t,s)} |a_n(\max(t,s)-r)-a_n(\min(t,s)-r)|\,dr + \int_{\min(t,s)}^{\max(t,s)} |a_n(\max(t,s)-r)|\,dr < \varepsilon,$$

provided $|t-s|<\delta$, $s\geqslant 0$ *and* $n\geqslant n_0$.

Then

$$\lim_{n\to\infty}\widetilde{k_n}(\lambda)(I-\widetilde{a_n}(\lambda)\mathcal{A}_n)^{-1}C_nx=\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}Cx,\quad \lambda\in\mathfrak{T}_0,\ x\in E$$

is equivalent to say that $\lim_{n\to\infty}R_n(t)x=R(t)x$, $t\geqslant 0$, $x\in E$, *uniformly on compacts of* $[0,\infty)$.

The conclusions from [**326**, Remark 2.5(ii)] can be formulated in our context; the same holds for the parts (i) and (iii) of this remark. Since subordination principles established in [**61**] and [**459**] hold in our framework, Theorem 3.8.2 can be used for proving the following extension of [**326**, Theorem 2.6]:

THEOREM 3.8.3. *Suppose* $\alpha>0$, $\beta\geqslant 1$, $\mathcal{A}$ *is a closed subgenerator of an exponentially equicontinuous* (g_α,g_β)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (272) *with* $a(t)=g_\alpha(t)$ *and* $k(t)=g_\beta(t)$, *and* $R(C)$ *as well as* $D(\mathcal{A})$ *are dense in* E. *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive real numbers with* $\lim_{n\to\infty}\alpha_n=\alpha$, *and let* $\gamma_n=\alpha_n/\alpha$ $(n\in\mathbb{N})$. *Then, for every* $n\in\mathbb{N}$, *the operator* $\mathcal{A}$ *is a subgenerator of an exponentially equicontinuous* $(g_{\alpha_n},g_{1+\gamma_n(\beta-1)})$*-regularized* C*-resolvent family* $(R_n(t))_{t\geqslant 0}$ *satisfying* (272) *with* $a(t)=g_{\alpha_n}(t)$, $k(t)=g_{1+\gamma_n(\beta-1)}(t)$ *and* $(R(t))_{t\geqslant 0}$ *replaced by* $(R_n(t))_{t\geqslant 0}$. *Furthermore,* $\lim_{n\to\infty}R_n(t)x=R(t)x$, $t\geqslant 0$, $x\in E$, *uniformly on compacts of* $[0,\infty)$.

It is very technical to extend the assertion of [**326**, Theorem 2.7] to (a,k)-regularized C-resolvent families subgenerated by multivalued linear operators.

THEOREM 3.8.4. *Assume that, for every* $n\in\mathbb{N}_0$, $|a_n|(t)$ *and* $k_n(t)$ *satisfy* (P1) *and* $\mathcal{A}$ *is a closed subgenerator of an* (a_n,k_n)*-regularized* C_n*-resolvent family* $(R_n(t))_{t\geqslant 0}$ *which satisfies* (272) *with* $a(t)$, $R(t)$ *and* $k(t)$ *replaced respectively by* $a_n(t)$, $R_n(t)$ *and* $k_n(t)$ $(n\in\mathbb{N}_0)$. *Denote by* $a_{n,k}(t)$ *the* k*-th convolution power of the function* $a_n(t)$ $(k\in\mathbb{N})$. *Assume further that there exists a real number* $\omega\geqslant\sup_{n\in\mathbb{N}_0}\max(0,\mathrm{abs}(|a_n|),\mathrm{abs}(k_n))$ *such that, for every seminorm* $p\in\circledast$, *there exist a number* $c_p>0$ *and a seminorm* $r_p\in\circledast$ *such that* (391) *holds. Let* $\lambda_0\geqslant\omega$. *Suppose that* $l\in\mathbb{N}$ *and the following holds:*

(i) $\lim_{n\to\infty}\widetilde{k_n}(\lambda)(I-\widetilde{a_n}(\lambda)\mathcal{A})^{-1}C_nx=\tilde{k}(\lambda)(I-\tilde{a}(\lambda)\mathcal{A})^{-1}Cx$, $\lambda\in\mathfrak{T}$, $x\in D(A^l)$.

(ii) *The sequences* $(k_n(t))_n$, $((a_n*k_n)(t))_n,\ldots,$ *and* $((a_{n,l-1}*k_n)(t))_n$ *are equicontinuous at each point* $t\geqslant 0$.

(iii) *The sequence* $(C_nx)_n$ *is bounded for any* $x\in D(\mathcal{A}^l)$.

(iv) *The condition* (iv) *of* Theorem 3.8.2 *holds with the function* $a_n(t)$ *replaced by* $a_{n,l}(t)$.

Then, for every $x\in\overline{D(\mathcal{A}^l)}$, *one has* $\lim_{n\to\infty}R_n(t)x=R(t)x$, $t\geqslant 0$, *uniformly on compacts of* $[0,\infty)$.

Using the Hille–Yosida theorem for degenerate (a,k)-regularized C-resolvent families and arguing as in the proof of [**326**, Theorem 2.8], we can prove the following:

THEOREM 3.8.5. *Assume that, for every* $n \in \mathbb{N}_0$, $|a_n|(t)$ *and* $k_n(t)$ *satisfy* (P1), $\mathcal{A}_n$ *is a closed* MLO, *and* $\lambda_0 > \omega \geqslant \sup_{n\in\mathbb{N}_0} \max(0, \operatorname{abs}(|a_n|), \operatorname{abs}(k_n))$. *Assume that* $\lim_{n\to\infty} \widetilde{a_n}(\lambda) = \tilde{a}(\lambda)$, $\lambda \in \mathfrak{T}$ *and* $\lim_{n\to\infty} \widetilde{k_n}(\lambda) = \tilde{k}(\lambda)$, $\lambda \in \mathfrak{T}$. *Suppose that* $L(E) \ni \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\lambda \in \mathfrak{T}$, *and for every* $n \in \mathbb{N}$, $\mathcal{A}_n$ *is a subgenerator of an* (a_n, k_n)*-regularized* C_n*-resolvent family* $(R_n(t))_{t\geqslant 0}$ *which satisfies* (272) *with* $a(t)$, $R(t)$ *and* $k(t)$ *replaced respectively by* $a_n(t)$, $R_n(t)$ *and* $k_n(t)$. *Let* (391) *hold for* $t \geqslant 0$, $x \in E$ *and* $n \in \mathbb{N}$, *and let*

$$\lim_{n\to\infty} \widetilde{k_n}(\lambda)(I - \widetilde{a_n}(\lambda)\mathcal{A}_n)^{-1}C_n x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx, \quad x \in E, \ \lambda \in \mathfrak{T}.$$

Suppose, further, that for each $\lambda \in \mathfrak{T}$ *there exists an open ball* $\Omega_\lambda \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > \lambda_0\}$, *with center at point* λ *and radius* $2\varepsilon_\lambda > 0$, *so that* $\widetilde{a_n}(z)\widetilde{k_n}(z) \neq 0$, $z \in \Omega_\lambda$, $n \in \mathbb{N}_0$. *Then the following holds:*

- (i) *For each* $r \in (0, 1]$, $\mathcal{A}$ *is a subgenerator of a global* $(a, k * g_r)$*-regularized* C*-resolvent family* $(R_r(t))_{t\geqslant 0}$ *satisfying* (272) *as well as that, for every seminorm* $p \in \circledast$,

$$p(R_r(t+h)x - R_r(t)x) \leqslant \frac{2c_p r_p(x)}{r\Gamma(r)} \max(e^{\omega(t+h)}, 1)h^r, \quad t \geqslant 0, \ h > 0, \ x \in E,$$

  and that, for every seminorm p *and for every bounded set* $B \in \mathcal{B}$, *the mapping* $t \mapsto p_B(R_r(t))$, $t \geqslant 0$ *is locally Hölder continuous with exponent* r.
- (ii) *If* $\mathcal{A}$ *is densely defined, then* $\mathcal{A}$ *is a subgenerator of a global* (a, k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (272) *and that the family* $\{e^{-\omega t}R(t) : t \geqslant 0\}$ *is equicontinuous.*

Suppose that $\mathcal{A}$ is an MLO, $C\mathcal{A} \subseteq \mathcal{A}C$ and $\rho_C(\mathcal{A}) \neq \emptyset$. Then, for every $\lambda \in \rho_C(\mathcal{A})$, we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1}C)$, which implies that the operator $(\lambda I - \mathcal{A})^{-1}C$ is injective iff $\mathcal{A}$ is single-valued. Although the resolvent equation holds in our framework, the non-injectivity of operator $(\lambda I - \mathcal{A})^{-1}C$ in multivalued case does not permit us to state a satisfactory extension of [**326**, Corollary 2.10] for degenerate resolvent families. The assertion of [**326**, Proposition 2.11(i)] can be formulated in our context (cf. Lemma 1.2.4), which is not the case with the assertions of [**326**, Proposition 2.11(ii)] and [**326**, Proposition 2.12].

The interested reader will want to prove some results on approximation and convergence of degenerate (A, k)-regularized C-(pseudo)resolvent families introduced in Section 2.9 (cf. [**326**, Theorem 2.16, Theorem 2.17] for non-degenerate case), as well as degenerate (a, k)-regularized C-resolvent families introduced in Sections 2.2–2.3.

Now we would like to mention that the assertions of [**61**, Theorem 2.21, Theorem 2.23] continue to hold in the degenerate case (if $C \neq I$ or E is not a Banach space, then the consideration is left to the interested reader):

REMARK 3.8.6. Let $\alpha > 0$, $(E, \|\cdot\|)$ be a Banach space, let the numbers $b^\alpha_{k,n}$ be defined by (393), and let $\mathcal{A}$ be a closed subgenerator of an exponentially bounded (g_α, I)-regularized resolvent family $(S_\alpha(t))_{t\geqslant 0}$ satisfying $S_\alpha(t)x -$

$g_{\alpha+1}(t)x \in \mathcal{A}\int_0^t S_\alpha(s)x\,ds$, $t \geqslant 0$, $x \in E$. Then

$$S_\alpha(t)x = \lim_{n\to\infty} \frac{1}{(n-m)!} \sum_{k=1}^{n-m+1} b^\alpha_{k,n-m+1}(I-(t/n)^\alpha \mathcal{A})^{-k}x, \quad x \in E,\ t \geqslant 0,$$

and the convergence is uniform on compacts of $[0,\infty)$. The only non-trivial thing worth explaining here is that the formula [**61**, (2.43)] holds in multivalued linear case, with the equality replaced with the inclusion $\ni$; this can be proved by induction, Theorem 1.2.4(i) and elementary properties of multivalued linear operators (although the resolvent $(\zeta^\alpha - \mathcal{A})^{-1}$ is not necessarily injective for $\zeta > 0$ suff. large, we can apply it on both sides of the equality [**61**, (2.43)], i.e, the inductive hypothesis, and employ after that the equality $(\zeta^\alpha - \mathcal{A})(\zeta^\alpha - \mathcal{A})^{-1}\mathcal{A}F^{(j+1)}(\lambda)x = \mathcal{A}F^{(j+1)}(\lambda)x$, $x \in E$ in order to see that [**61**, (2.43)] holds with j replaced by $j+1$ therein). This is an extension of [**61**, Theorem 2.21]. Concerning [**61**, Theorem 2.23], it is only worth pointing out that the equality $S_\alpha(t)x - S_\alpha(s)x = \int_0^t P_\alpha(\tau)y\,d\tau - \int_0^s P_\alpha(\tau)y\,d\tau$, where $(x,y) \in \mathcal{A}$ and $P_\alpha(t)y := (g_{\alpha-1} * S_\alpha(\cdot)y)(t)$, $t > 0$ can be proved by taking the Laplace transform in both variables, t and s, and applying after that Theorem 1.2.4(i): Let $\alpha \geqslant 1$, let $x \in D(\mathcal{A})$, and let $(x,y) \in \mathcal{A}$. Then

$$\bigg\| S_\alpha(t)x - \lim_{n\to\infty} \frac{1}{(n-m)!} \sum_{k=1}^{n-m+1} b^\alpha_{k,n-m+1} \times (I-(t/n)^\alpha \mathcal{A})^{-k}x \bigg\| = O(n^{(-1)/2}), \quad n \to +\infty.$$

Furthermore, if $(S_\alpha(t))_{t\geqslant 0}$ is bounded, then there exists a constant $M_\alpha > 0$ such that

$$\bigg\| S_\alpha(t)x - \lim_{n\to\infty} \frac{1}{(n-m)!} \sum_{k=1}^{n-m+1} b^\alpha_{k,n-m+1} \times (I-(t/n)^\alpha \mathcal{A})^{-k}x \bigg\| \leqslant M_\alpha n^{(-1)/2} t^\alpha \|y\|, \quad n \in \mathbb{N}.$$

3.8.1. Laguerre expansions of degenerate (a,k)-regularized C-resolvent families. In this subsection, we shall present the basic results about Laguerre expansions of degenerate (a,k)-regularized C-resolvent families in locally convex spaces (cf. the recent paper by L. Abadias and P. J. Miana [**2**] for C_0-semigroup case).

We start by recalling that Rodrigues' formula gives the following representation of generalized Laguerre polynomials

$$L_n^\alpha(t) \equiv e^t \frac{t^{-\alpha}}{n!} \frac{d^n}{dt^n}(e^{-t}t^{n+\alpha}), \quad t \in \mathbb{R} \ \ (n \in \mathbb{N}_0,\ \alpha > -1).$$

If $\alpha \notin -\mathbb{N}$ and $n \in \mathbb{N}_0$, then we define

$$l_n^\alpha(t) \equiv \frac{1}{\Gamma(n+\alpha+1)} \frac{d^n}{dt^n}(e^{-t}t^{n+\alpha}), \quad t > 0.$$

The reader may consult [**2**, Proposition 2.6] for the most important properties of functions $l_n^\alpha(t)$ ($\alpha \notin -\mathbb{N}$, $n \in \mathbb{N}_0$). For example, it is well known that

$$l_n^\alpha(t) \sim g_{\alpha+1}(t), \quad t \to 0+; \qquad l_n^\alpha(t) \sim (-1)^n e^{-t} g_{n+\alpha+1}(t), \quad t \to +\infty,$$

and

$$\frac{d^k}{dt^k} l_n^\alpha(t) = l_{n+k}^{\alpha-k}(t), \quad t > 0, \ k \in \mathbb{N}_0.$$

The following theorem can be deduced by using the argumentation contained in the proof of [**2**, Theorem 3.3], stated in the Banach space case (cf. also [**373**, Theorem 3, Section 4.23]).

THEOREM 3.8.7. *Suppose that $f\colon (0,\infty) \to E$ is a differentiable mapping, $\alpha > -1$ and for each seminorm $p \in ⊛$ we have $\int_0^\infty e^{-t} t^\alpha p(f(t))^2 dt < \infty$. Then*

$$f(t) = \sum_{n=0}^\infty \frac{n! L_n^\alpha(t)}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-s} s^\alpha L_n^\alpha(s) f(s)\, ds, \quad t > 0.$$

Since

$$\int_0^\infty e^{-s} s^\alpha L_n^\alpha(s) f(s)\, ds = \int_0^\infty \frac{d^n}{ds^n}\left(e^{-s} s^{n+\alpha}\right) \frac{f(s)}{n!} ds$$
$$= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (n+\alpha) \ldots (n+\alpha-(k-1)) \int_0^\infty e^{-s} s^{n+\alpha-k} \frac{f(s)}{n!} ds$$

and

$$\frac{(n+\alpha)\ldots(n+\alpha-(k-1))}{\Gamma(n+\alpha+1)} = \frac{1}{\Gamma(n+\alpha+1-k)},$$

for any $n, k \in \mathbb{N}_0$ with $k \leqslant n$ and $\alpha > -1$, we immediately obtain the following corollary of Theorem 3.8.7.

COROLLARY 3.8.8. *Suppose that $f\colon (0,\infty) \to E$ is a differentiable mapping, $\alpha > -1$ and for each seminorm $p \in ⊛$ we have $\int_0^\infty e^{-s} s^\alpha p(f(s))^2 ds < \infty$. Then, for every $t > 0$, the following equality holds:*

$$f(t) = \sum_{n=0}^\infty \sum_{k=0}^n \frac{L_n^\alpha(t)(-1)^{n-k}\binom{n}{k}}{\Gamma(n+\alpha+1-k)} \int_0^\infty e^{-s} s^{n+\alpha-k} f(s)\, ds. \tag{392}$$

Before proceeding further, let us observe that the last formula can be rewritten in the following equivalent way:

$$f(t) = \sum_{n=0}^\infty L_n^\alpha(t) \int_0^\infty l_n^\alpha(s) f(s)\, ds.$$

Suppose now that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family with a closed subgenerator $-\mathcal{A}$, the functions $k(t)$ and $|a|(t)$ satisfy (P1), the family $\{e^{-\omega t} R(t) : t \geqslant 0\}$ is equicontinuous for some $\omega \geqslant 0$,

$\omega_0 \equiv \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k)) < 1/2$ and $\alpha > -1$. If, in addition, $\tilde{k}(1)\tilde{a}(1) \neq 0$, then Theorem 3.2.5 implies that, for every $\alpha \in \mathbb{N}_0$ and $x \in E$,

$$\int_0^\infty e^{-s}s^{n+\alpha-k}R(s)x\,ds = (-1)^{n+\alpha-k}\Big(\frac{d^{n+\alpha-k}}{d\lambda^{n+\alpha-k}}(\mathcal{L}R(\cdot)x)(\lambda)\Big)_{\lambda=1}$$
$$= (-1)^{n+\alpha-k}\Big(\frac{d^{n+\alpha-k}}{d\lambda^{n+\alpha-k}}\Big[\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}+\mathcal{A}\Big)^{-1}Cx\Big]\Big)_{\lambda=1};$$

then one can use the product rule, the identity $(d^n/d\lambda^n)(\lambda+\mathcal{A})^{-1}Cx = (-1)^n n!(\lambda+\mathcal{A})^{-n-1}Cx$, $n \in \mathbb{N}_0$, $x \in E$, as well as the well known Faà di Bruno's formula

$$\frac{d^n}{dx^n}f(g(x)) = \sum \frac{n!}{m_1!m_2!\dots m_n!}f^{(m_1+m_2+\cdots+m_n)}(g(x))\prod_{j=1}^n\Big(\frac{g^{(j)}(x)}{j!}\Big)^{m_j},$$

where the summation is taken over those multi-indices $(m_1, m_2, \dots, m_n) \in \mathbb{N}_0^n$ for which $m_1 + 2m_2 + \cdots + nm_n = n$, in order to express the right hand side of (392), with $f(t) = R(t)x$, $t > 0$ in terms of subgenerator $-\mathcal{A}$ (notice, however, that it is very difficult to express the value of $\int_0^\infty e^{-s}s^{n+\alpha-k}R(s)x\,ds$ in terms of $-\mathcal{A}$, if $\alpha \notin \mathbb{N}_0$ and $n \in \mathbb{N}_0$). At any rate, the obtained representation formula is very complicated, practically almost irrelevant, but can be simplified in some cases; for example, if $k(t) = 1$ and $a(t) = g_\vartheta(t)$ for some $\vartheta > 0$, then we have

$$\frac{d^n}{d\lambda^n}\Big[\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}+\mathcal{A}\Big)^{-1}Cx\Big] = \frac{d^n}{d\lambda^n}[\lambda^{\vartheta-1}(\lambda^\vartheta+\mathcal{A})^{-1}Cx]$$
$$= (-1)^n\lambda^{-(n+1)}\sum_{k=1}^{n+1}b^\vartheta_{k,n+1}\lambda^{\vartheta k}(\lambda^\vartheta+\mathcal{A})^{-k}Cx, \quad n \in \mathbb{N}_0,\ x \in E,\ \operatorname{Re}\lambda > \omega_0,$$

where the numbers $b^\vartheta_{k,n+1}$ are given by the following recurrence relations:

$$\begin{aligned} &b^\vartheta_{1,1} = 1,\\ &b^\vartheta_{k,n} = (n-1-k\vartheta)b^\vartheta_{k,n-1} + \vartheta b^\vartheta_{k-1,n-1}, \quad 1 \leqslant k \leqslant n,\ n = 2, 3, \dots,\\ &b^\vartheta_{k,n} = 0, \quad k > n,\ n = 1, 2, \dots, \end{aligned} \tag{393}$$

cf. the formulae [**61**, (2.16)–(2.17)].

3.8.2. Laguerre expansions of solutions to abstract non-degenerate differential equations of first order. The main aim of this subsection, which can be viewed of some independent interest, is to show how Laguerre expansions can be elegantly used for proving some representation formulae for solutions of abstract non-degenerate differential equations of first order whose solutions are governed by fractionally integrated semigroups and exponential ultradistribution semigroups of Beurling class. All operators considered in this subsection will be single-valued, and all subgenerators of (fractionally integrated) C-semigroups under examination will be closed.

Suppose first that the operator $-A$ subgenerates the global C-regularized semigroup $(T(t))_{t\geqslant 0}$ satisfying that there exists $\omega \geqslant 0$ such that the family $\{e^{-\omega t}T(t) : t \geqslant 0\}$ is equicontinuous. Let $\omega' > \omega - (1/2)$. Then, for every $z \in \mathbb{C}$, the operator

$-A+z$ is a subgenerator of the global C-regularized semigroup $(e^{zt}T(t))_{t\geqslant 0}$ and we can construct the complex powers of the operator $A+\omega'+1$ following the method proposed in Section 1.1. Any of these powers is closed and injective.

Let $\theta \in (\pi/2,\pi)$, and let $d' \in (0,1)$ be sufficiently small. Then, for every $\beta \in \mathbb{C}_+$, we have defined the operator $(A+\omega'+1)_C^{-\beta}$ by

$$(A+\omega'+1)_C^{-\beta}x := \frac{1}{2\pi i}\int_{\Gamma_{\theta,d'}} z^{-\beta}(z-(A+\omega'+1))^{-1}Cx\,dz, \quad x\in E,$$

where $\Gamma_{\theta,d'} = \partial(\Sigma_{\omega'} \setminus B_{d'})$ denotes the boundary of $\Sigma_{\omega'} \setminus B_{d'}$, oriented in such a way that $\operatorname{Im} z$ increases along $\Gamma_{\omega',d'}$. If $\operatorname{Re}\beta > 1$, then it can be easily seen with the help of resolvent equation and Cauchy formula that, for every $x \in E$,

$$\begin{aligned}(A+\omega'+1)(A+\omega'+1)_C^{-\beta}x &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,d'}} z^{-\beta}(A+\omega'+1)(z-(A+\omega'+1))^{-1}Cx\,dz\\ &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,d'}} z^{-\beta}\Big[-Cx+z(z-(A+\omega'+1))^{-1}Cx\Big]dz\\ &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,d'}} z^{-\beta+1}(z-(A+\omega'+1))^{-1}Cx\,dz\\ &= (A+\omega'+1)_C^{-(\beta-1)}x.\end{aligned}$$

Inductively, we obtain that

$$(A+\omega'+1)^n(A+\omega'+1)_C^{-\beta}x = (A+\omega'+1)_C^{-(\beta-n)}x, \quad \operatorname{Re}\beta > n,\ x\in E\ (n\in\mathbb{N}_0),$$

i.e., by definition of powers $(A+\omega'+1)_{-\beta}$ and $(A+\omega'+1)_{-(\beta-n)}$,

$$(A+\omega'+1)^n(A+\omega'+1)_{-\beta}Cx = (A+\omega'+1)_{-(\beta-n)}Cx, \quad \operatorname{Re}\beta > n,\ x\in E\ \ (n\in\mathbb{N}_0). \tag{394}$$

By [**292**, Lemma 2.9.73(iii)], we have

$$(A+\omega'+1)_{-\beta}Cx = \int_0^\infty \frac{s^{\beta-1}}{\Gamma(\beta)}e^{-(\omega'+1)s}T(s)x\,ds, \quad x\in E,\ \beta>0. \tag{395}$$

Furthermore, by (394), we have that for each $n \in \mathbb{N}_0$ and $\alpha > -1$, $(A+\omega'+1)_{-(n+\alpha+1)}Cx \in D((A+\omega'+1)^n) = D(A^n)$ $(x\in E)$. Applying the binomial formula and (394), we get

$$\begin{aligned}&(A+\omega')^n(A+\omega'+1)_{-(n+\alpha+1)}Cx\\ &\quad= ((A+\omega'+1)-1)^n(A+\omega'+1)_{-(n+\alpha+1)}Cx\\ &\quad= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(A+\omega'+1)_{-(n+\alpha-k+1)}Cx, \quad x\in E.\end{aligned} \tag{396}$$

Observing finally that for each $x \in D(A)$ the mapping $t \mapsto T(t)x$, $t \geqslant 0$ is continuously differentiable with $(d/dt)T(t)x = T(t)Ax$, $t \geqslant 0$, Corollary 3.8.8 combined with the equations (395)–(396) immediately implies the following:

THEOREM 3.8.9. *Suppose that the operator $-A$ is a subgenerator of the global C-regularized semigroup $(T(t))_{t\geqslant 0}$ satisfying that there exists $\omega \geqslant 0$ such that the* *family* $\{e^{-\omega t}T(t) : t \geqslant 0\}$ *is equicontinuous. Let* $\omega' > \omega - (1/2)$*. Then, for every* $x \in D(A)$ *and* $\alpha > -1$*, we have that*

$$T(t)x = e^{\omega' t} \sum_{n=0}^{\infty} L_n^{\alpha}(t)(A+\omega')^n (A+\omega'+1)_{-n-\alpha-1}Cx, \quad t > 0.$$

It is worth noting that Theorem 3.8.9 enables one to consider Laguerre expansions of certain classes of semigroups that are strongly continuous for $t > 0$, like semigroups of class $(C_{(k)})$ and semigroups of growth order $r > 0$ (cf. [**291**, Theorems 1.2.15, 1.2.17 and 1.2.19]), which as a further consequence has that we can approximate by Laguerre polynomials solutions of incomplete abstract Cauchy problems, in general with modified Liouville right-sided time-fractional derivatives (cf. [**292**, Theorems 2.9.58, 2.9.60]). For the sake of brevity, we will present only two applications of Theorem 3.8.9. The first one is connected with Laguerre expansions of solutions of first order abstract Cauchy problems associated with generators of fractionally integrated semigroups in locally convex spaces (cf. [**291**, Subsection 2.9.7] for more details). The second one is connected with Laguerre expansions of solutions of first order abstract Cauchy problems whose solutions are governed by exponential ultradistribution semigroups of Beurling class (cf. [**291**, Theorem 3.6.9]):

(i) Assume that $\zeta \in (0,\infty) \setminus \mathbb{N}$, $\omega \geqslant 0$, $1 \geqslant \sigma' > \sigma > 0$, as well as that $-A$ is the integral generator of an ζ-times integrated semigroup $(S_\zeta(t))_{t\geqslant 0}$ satisfying that the family $\{e^{-\omega t}S_\zeta(t) : t \geqslant 0\}$ is equicontinuous. Suppose $\varepsilon > 0$, $\lfloor \zeta \rfloor = \lfloor \zeta + \varepsilon \rfloor$ and $\alpha > -1$. If $\alpha \notin \mathbb{N}_0$, then the right hand side of (392), with $f(t)$ replaced by $e^{-\omega t}S_\zeta(t)x$, cannot be so simply evaluate in terms of A. Nevertheless, for every $\gamma \in (0, \pi/2)$, there exists $d \in (0,1]$ such that $\Sigma(\gamma,d) \subseteq \rho(-A-\omega-\sigma)$ and the family $\{(1+|\lambda|)^{1-\zeta}(\lambda+A+\omega+\sigma)^{-1} : \lambda \in \Sigma(\gamma,d)\}$ is equicontinuous. Set $C_\zeta := (A+\omega+\sigma)^{-1-\lfloor \zeta \rfloor}$. Then it is not difficult to prove that the operator $A+\omega+\sigma$ is C_ζ-sectorial of angle $\pi/2$ and that the condition [**103**, (H)] holds with $d = \sigma/2$. Therefore, for every $z \in \mathbb{C}$, we can construct the power $(A+\omega+\sigma)_z$ following the method proposed in [**103**], with the operator C replaced by C_ζ. This power is injective, and belongs to $L(E)$ provided that $\operatorname{Re} z < -\zeta$. Furthermore, we know that $-A$ is the integral generator of a global $(A+\omega+\sigma)_{-(\zeta+\varepsilon)}$-regularized semigroup $(T(t))_{t\geqslant 0}$ and the family $\{e^{-(\omega+\sigma')t}T(t) : t \geqslant 0\}$ is equicontinuous (see e.g. [**291**, Theorem 2.3.26]). Applying Theorem 3.8.9, we get that for each $t > 0$ and $x \in D(A)$,

$$\begin{aligned} (397)\quad & e^{-(\omega+\sigma)t}T(t)x \\ &= \sum_{n=0}^{\infty} L_n^{\alpha}(t)(A+\omega+\sigma)^n(A+\omega+\sigma+1)_{-n-\alpha-1}(A+\omega+\sigma)_{-(\zeta+\varepsilon)}x; \end{aligned}$$

here we define the power $(A+\omega+\sigma+1)_{-n-\alpha-1}$ $(n \in \mathbb{N}_0)$ by using the construction from [**103**]; note that the operator $A+\omega+\sigma+1$ is C-sectorial of angle $\pi/2$ with $C \equiv (A+\omega+\sigma)_{-(\zeta+\varepsilon)}$. Perhaps the most important consequence of (397) is given as follows. Let $x \in D((A+\omega+\sigma)_{1+\zeta+\varepsilon})$.

Then $x \in D((A+\omega+\sigma)_{\zeta+\varepsilon})$, $x_0 := (A+\omega+\sigma)_{\zeta+\varepsilon}x \in D(A+\omega+\sigma)$ and $x = (A+\omega+\sigma)_{-(\zeta+\varepsilon)}x_0$. Furthermore, the function $t \mapsto u(t) \equiv T(t)x_0$, $t \geqslant 0$ is the classical solution of the abstract Cauchy problem

$$(ACP_1) : u'(t) = Au(t), \quad t \geqslant 0, \ u(0) = x.$$

By (397), it readily follows that, for every $t > 0$,

$$u(t) = e^{(\omega+\sigma)t} \sum_{n=0}^{\infty} L_n^{\alpha}(t)(A+\omega+\sigma)^n (A+\omega+\sigma+1)_{-n-\alpha-1}x.$$

(ii) Suppose that the sequence $(M_p)_{p\in\mathbb{N}_0}$ of positive real numbers satisfies the conditions $(M.1)$, $(M.2)$ and $(M.3)$, $M_0 = 1$, $(E, \|\cdot\|)$ is a Banach space and $-A$ generates an exponential ultradistribution semigroup of (M_p)-class. Then the abstract Cauchy problem

$$u \in C^{\infty}([0,\infty) : E) \cap C([0,\infty) : [D(A)]); \quad u'(t) + Au(t) = 0, \qquad t \geqslant 0, \ u(0) = x,$$

has a unique solution for all $x \in E^{(M_p)}(A)$, there exist an injective operator $C \in L(E)$ and a number $\omega \in \mathbb{R}$ such that $E^{(M_p)}(A) \subseteq C(D_\infty(A))$ and that $-A$ generates an exponentially bounded C-regularized semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\| = O(e^{\omega t})$, $t \geqslant 0$ (cf. [**291**, Theorems 3.6.8, 3.6.9; Lemma 3.6.5]). If $x \in E^{(M_p)}(A)$ and $\omega' > \omega - (1/2)$, then it is clear that $u(t) = T(t)C^{-1}x$, $t \geqslant 0$ and

$$u(t) = e^{\omega' t} \sum_{n=0}^{\infty} L_n^{\alpha}(t)(A+\omega')^n (A+\omega'+1)_{-n-\alpha-1}x, \quad t > 0.$$

In [**3**], L. Abadias and P. J. Miana have recently analyzed the Hermite expansions of non-degenerate C_0-groups and cosine operator functions in Banach spaces. The interested reader may try to reconsider the results from [**3**] for some other classes of (non-)degenerate resolvent operator families.

For futher information on approximation and convergence of degenerate differential equations, we refer the reader to [**50**, **199**, **370**] and references cited in the introductory part.

3.9. The existence and uniqueness of solutions of abstract incomplete differential inclusions

The state space in this section will be denoted by X. Let $C \in L(X)$ be not necessarily injective, and let $\mathcal{A}$ be a multivalued linear operator commuting with C. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as in the case that the operator C is injective; $\rho_C(\mathcal{A})$ is the union of those complex numbers $\lambda \in \mathbb{C}$ for which $R(C) \subseteq R(\lambda - \mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X. The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$). In our previous work, we have faced ourselves with some situations in which it is indispensable to assume that the operator $(\lambda - \mathcal{A})^{-1}C$ is not single-valued. This will not be the case in this section.

It is incredibly important to observe the following:

REMARK 3.9.1. Theorem 1.2.4, Proposition 1.2.6(i) and Theorem 1.2.8 continue to hold even if the injectivity of C is disregarded.

Concerning the analytical properties of C-resolvents, we have the following:

PROPOSITION 3.9.2. *Let $\emptyset \neq \Omega \subseteq \rho_C(\mathcal{A})$ be open, and let $x \in X$.*

(i) *Suppose that $R(C)$ is dense in X. Then the local boundedness of the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ implies its analyticity. Furthermore, if X is barreled, then the local boundedness of the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$, $\lambda \in \Omega$ implies its analyticity.*

(ii) *Suppose that $R(C)$ is dense in X and $\mathcal{A}$ is closed. Then we have $R(C) \subseteq$ $\mathrm{R}((\lambda-\mathcal{A})^n)$, $n \in \mathbb{N}$ and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda-\mathcal{A})^{-1}Cx = (-1)^{n-1}(n-1)!(\lambda-\mathcal{A})^{-n}Cx, \quad n \in \mathbb{N}. \tag{398}$$

Furthermore, if X is barreled, then $R(C) \subseteq R((\lambda-\mathcal{A})^n)$, $n \in \mathbb{N}$ and

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda-\mathcal{A})^{-1}C = (-1)^{n-1}(n-1)!(\lambda-\mathcal{A})^{-n}C \in L(X), \quad n \in \mathbb{N}. \tag{399}$$

REMARK 3.9.3. Let $\emptyset \neq \Omega \subseteq \rho_C(\mathcal{A})$ be open, and let $x \in X$. Suppose that $\mathcal{A}$ is closed. Then $\operatorname{card}((\lambda-\mathcal{A})^{-n}Cx) \leqslant 1$, $\lambda \in \rho_C(\mathcal{A})$, $n \in \mathbb{N}$, $x \in X$. This can be proved by induction, observing that $(\lambda-\mathcal{A})^{-1}0$ is a singleton ($\lambda \in \rho_C(\mathcal{A})$) as well as that for each $y \in (\lambda-\mathcal{A})^{-(n+1)}Cx$ ($\lambda \in \rho_C(\mathcal{A})$, $n \in \mathbb{N}$, $x \in X$) we have

$$\begin{aligned}(\lambda-\mathcal{A})^{-(n+1)}Cx &= y + (\lambda-\mathcal{A})^{-(n+1)}C0 \\ &= y + (\lambda-\mathcal{A})^{-1}(\lambda-\mathcal{A})^{-n}C0 = y + (\lambda-\mathcal{A})^{-1}0 = \{y\}.\end{aligned}$$

Taking into account the proof of [**138**, Corollary 2.8], this is a crucial thing in showing that (398) holds, and that (399) holds, provided in addition that X is barreled.

3.9.1. Complex powers of multivalued linear operators with polynomially bounded C-resolvent. In this subsection, we deal with the following conditions:

(H)$_0$: Let $C \in L(X)$ be not necessarily injective, let $\mathcal{A}$ be closed, and let $C\mathcal{A} \subseteq \mathcal{A}C$. There exist real numbers $d \in (0,1]$, $c \in (0,1)$, $\varepsilon \in (0,1]$ and $\alpha \geqslant -1$ such that $P_{\alpha,\varepsilon,c} \cup B_d \subseteq \rho_C(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-\alpha}(\lambda-\mathcal{A})^{-1}C : \lambda \in P_{\alpha,\varepsilon,c} \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$ is strongly analytic on $\operatorname{int}(P_{\alpha,\varepsilon,c} \cup B_d)$ and strongly continuous on $\partial(P_{\alpha,\varepsilon,c} \cup B_d)$

and

(HS)$_0$: Let $C \in L(X)$ be not necessarily injective, let $\mathcal{A}$ be closed, and let $C\mathcal{A} \subseteq \mathcal{A}C$. There exist real numbers $d \in (0,1]$, $\vartheta \in (0,\pi/2)$ and $\alpha \geqslant -1$ such that $\Sigma_\vartheta \cup B_d \subseteq \rho_C(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-\alpha}(\lambda-\mathcal{A})^{-1}C : \lambda \in \Sigma_\vartheta \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$ is strongly analytic on $\operatorname{int}(\Sigma_\vartheta \cup B_d)$ and strongly continuous on $\partial(\Sigma_\vartheta \cup B_d)$.

Here, $B_d = \{z \in \mathbb{C} : |z| \leqslant d\}$ and $P_{\alpha,\varepsilon,c} = \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta \in \mathbb{R},\ \ |\eta| \leqslant c(1+\xi)^{-\alpha}\}$. Assume that the condition $(\mathrm{H})_0$, resp. $(\mathrm{HS})_0$, holds. Without loss of generality, we may assume that there exists a number $\lambda_0 \in \operatorname{int}(\rho_C(\mathcal{A})) \smallsetminus (P_{\alpha,\varepsilon,c} \cup B_d)$, resp., $\lambda_0 \in \operatorname{int}(\rho_C(\mathcal{A})) \smallsetminus (\Sigma_\vartheta \cup B_d)$. Then we can prove inductively (cf. also Theorem 1.2.8(i)) that, for every $z \in \rho_C(\mathcal{A}) \smallsetminus \{\lambda_0\}$:

$$(400)\quad (z-\mathcal{A})^{-1}C(\lambda_0-\mathcal{A})^{-k}Cx = \frac{(-1)^k}{(z-\lambda_0)^k}(z-\mathcal{A})^{-1}C^2x + \sum_{i=1}^{k} \frac{(-1)^{k-i}(\lambda_0-\mathcal{A})^{-i}C^2x}{(z-\lambda_0)^{k+1-i}}.$$

Strictly speaking, for $k = 1$ this is the usual resolvent equation. Suppose that (400) holds for all natural numbers $\leqslant k$. Then (398) shows that

$$\begin{aligned}
(z-\mathcal{A})^{-1}C(\lambda_0-\mathcal{A})^{-(k+1)}Cx &= (z-\mathcal{A})^{-1}C\frac{(-1)^k}{k!}\Big(\frac{d^k}{d\lambda^k}(\lambda-\mathcal{A})^{-1}Cx\Big)_{\lambda=\lambda_0}\\
&= (z-\mathcal{A})^{-1}C\frac{(-1)^k}{k!}\Big(\frac{d}{d\lambda}\Big[(-1)^{k-1}(k-1)!(\lambda-\mathcal{A})^{-k}Cx\Big]\Big)_{\lambda=\lambda_0}\\
&= \frac{(-1)}{k}\Big(\frac{d}{d\lambda}\Big[(z-\mathcal{A})^{-1}C(\lambda-\mathcal{A})^{-k}Cx\Big]\Big)_{\lambda=\lambda_0}
\end{aligned}$$

and we can employ the inductive hypothesis and a simple computation in order to see that (400) holds with k replaced with $k+1$ therein. Set

$$(401)\qquad C_1 := C(\lambda_0-\mathcal{A})^{-\lfloor \alpha+2\rfloor}C,\ \text{ if } \alpha > -1, \text{ and } C_1 := C,\ \text{ if } \alpha = -1.$$

Then Theorem 1.2.4(i) implies by iteration that C_1 commutes with $\mathcal{A}$. Furthermore, the validity of $(\mathrm{H})_0$, resp. $(\mathrm{HS})_0$, implies by Theorem 1.2.8(i) that the following holds:

(H): There exist real numbers $d \in (0,1]$, $c \in (0,1)$ and $\varepsilon \in (0,1]$ such that $P_{\alpha,\varepsilon,c} \cup B_d \subseteq \rho_{C_1}(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-1}(\lambda-\mathcal{A})^{-1}C_1 : \lambda \in P_{\alpha,\varepsilon,c} \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C_1$ is strongly analytic on $\operatorname{int}(P_{\alpha,\varepsilon,c} \cup B_d)$ and strongly continuous on $\partial(P_{\alpha,\varepsilon,c} \cup B_d)$,

resp.,

(HS): There exist real numbers $d \in (0,1]$ and $\vartheta \in (0,\pi/2)$ such that $\Sigma_\vartheta \cup B_d \subseteq \rho_{C_1}(\mathcal{A})$, the operator family $\{(1+|\lambda|)^{-1}(\lambda-\mathcal{A})^{-1}C_1 : \lambda \in \Sigma_\vartheta \cup B_d\} \subseteq L(X)$ is equicontinuous, the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C_1$ is strongly analytic on $\operatorname{int}(\Sigma_\vartheta \cup B_d)$ and strongly continuous on $\partial(\Sigma_\vartheta \cup B_d)$.

So, the condition (H), resp. (HS), holds and $C_1\mathcal{A} \subseteq \mathcal{A}C_1$; in particular, $-\mathcal{A}$ is C_1-positive, resp. $-\mathcal{A}$, is C_1-sectorial of angle $\pi-\vartheta$ and $B_d \subseteq \rho_{C_1}(-\mathcal{A})$. Once this is done, the further analyses may begin. Put $\Gamma_1(\alpha,\varepsilon,c,d) := \{\xi+i\eta : \xi \leqslant -\varepsilon,\ \eta = -c(1+|\xi|)^{-\alpha}\}$, $\Gamma_2(\alpha,\varepsilon,c,d) := \{\xi+i\eta : \xi^2+\eta^2 = d^2,\ \xi \geqslant -\varepsilon\}$ and $\Gamma_3(\alpha,\varepsilon,c,d) := \{\xi+i\eta : \xi \leqslant -\varepsilon,\ \eta = c(1+|\xi|)^{-\alpha}\}$. The curve $\Gamma(\alpha,\varepsilon,c,d) := \Gamma_1(\alpha,\varepsilon,c,d) \cup \Gamma_2(\alpha,\varepsilon,c,d) \cup \Gamma_3(\alpha,\varepsilon,c,d)$ is oriented so that $\operatorname{Im}(\lambda)$ increases along $\Gamma_2(\alpha,\varepsilon,c,d)$ and that $\operatorname{Im}(\lambda)$ decreases along $\Gamma_1(\alpha,\varepsilon,c,d)$ and $\Gamma_3(\alpha,\varepsilon,c,d)$. Since there is no risk for confusion, we also write Γ for $\Gamma(\alpha,\varepsilon,c,d)$. We similarly define the curves $\Gamma_{1,S}(\vartheta,d)$, $\Gamma_{2,S}(\vartheta,d)$, $\Gamma_{3,S}(\vartheta,d)$ and $\Gamma_S(\vartheta,d)$ for $\vartheta \in (0,\pi/2)$ and $d \in (0,1]$.

Define

$$f_{C_1}(\mathcal{A})x := \frac{1}{2\pi i}\int_\Gamma f(z)(z+\mathcal{A})^{-1}C_1x\,dz, \quad x \in X,$$

where $f(z)$ is a holomorphic function on an open neighborhood $\Omega_{\alpha,\varepsilon,c,d}$ of $-(P_{\alpha,\varepsilon,c}\cup B_d) \smallsetminus (-\infty,0]$ and the estimate

$$|f(z)| \leqslant M|z|^{-s}, \quad z \in \Omega_{\alpha,\varepsilon,c,d}$$

holds for some positive number $s > 0$. Denote by $\mathcal{H}$ the class consisting of such functions. Then an application of Cauchy's theorem shows that the definition of $f_{C_1}(\mathcal{A})$ does not depend on a particular choice of curve$\Gamma(\alpha,\varepsilon,c,d)$ (with the meaning clear). Furthermore, a standard calculus involving the Cauchy theorem, the Fubini theorem and Theorem 1.2.4(ii) shows that

$$f_{C_1}(\mathcal{A})g_{C_1}(\mathcal{A}) = (fg)_{C_1}(\mathcal{A})C_1, \quad f,g,fg \in \mathcal{H}. \tag{402}$$

Given $b \in \mathbb{C}$ with $\operatorname{Re} b > 0$, set $(-\mathcal{A})_{C_1}^{-b} := (z^{-b})_{C_1}(\mathcal{A})$ and $(-\mathcal{A})_{C_1}^{-0} := C_1$. By Remark 3.9.3(ii) and the residue theorem, we get $(-\mathcal{A})_{C_1}^{-n} = (-\mathcal{A})^{-n}C_1$ $(n \in \mathbb{N})$; moreover, $(-\mathcal{A})_{C_1}^{-b}C_1 = C_1(-\mathcal{A})_{C_1}^{-b}$ $(\operatorname{Re} b > 0)$, the mapping $b \mapsto (-\mathcal{A})_{C_1}^{-b}x$, $\operatorname{Re} b > 0$ is analytic for every fixed $x \in X$, and the following holds:

$$\frac{d}{db}(-\mathcal{A})_{C_1}^{-b}x = \frac{(-1)}{2\pi i}\int_\Gamma (\ln z)z^{-b}(z+\mathcal{A})^{-1}C_1x\,dz, \quad x \in X,\ \operatorname{Re} b > 0.$$

Applying the equality (402) once more, we get that

$$(-\mathcal{A})_{C_1}^{-b_1}(-\mathcal{A})_{C_1}^{-b_2} = (-\mathcal{A})_{C_1}^{-(b_1+b_2)}C_1, \quad \operatorname{Re} b_1, \operatorname{Re} b_2 > 0.$$

It is very simple to prove that

$$(-\mathcal{A})_{C_1}^{-b}x = -\frac{\sin \pi b}{\pi}\int_0^\infty \lambda^{-b}(\lambda-\mathcal{A})^{-1}C_1x\,d\lambda, \quad 0 < \operatorname{Re} b < 1,\ x \in X,$$

so that the family $\{(-\mathcal{A})_{C_1}^{-b} : 0 < b < 1\}$ is equicontinuous. Define now the powers with negative imaginary part of exponent by

$$(-\mathcal{A})_{-b} := C_1^{-1}(-\mathcal{A})_{C_1}^{-b}, \quad \operatorname{Re} b > 0.$$

Then $(-\mathcal{A})_{-b}$ is a closed MLO and $(-\mathcal{A})_{-n} = C_1^{-1}(-\mathcal{A})^{-n}C_1$ $(n \in \mathbb{N})$. We define the powers with positive imaginary part of exponent by

$$(-\mathcal{A})_b := ((-\mathcal{A})_{-b})^{-1} = ((-\mathcal{A})_{C_1}^{-b})^{-1}C_1, \quad \operatorname{Re} b > 0.$$

Clearly, $(-\mathcal{A})_n = C_1^{-1}(-\mathcal{A})^nC_1$ for every $n \in \mathbb{N}$, and $(-\mathcal{A})_b$ is a closed MLO due to the fact that $(-\mathcal{A})_{-b}$ is a closed MLO $(b \in \mathbb{C}_+)$. Following [**410**, Definition 7.1.2] and our previous analyses of non-degenerate case [**103**], we introduce the purely imaginary powers of $-\mathcal{A}$ as follows: Let $\tau \in \mathbb{R} \smallsetminus \{0\}$. Then the power $(-\mathcal{A})_{i\tau}$ is defined by

$$(-\mathcal{A})_{i\tau} := C_1^{-2}(1-\mathcal{A})_2(-\mathcal{A})_{-1}(-\mathcal{A})_{1+i\tau}(1-\mathcal{A})_{-2}C_1^2,$$

where $(1-\mathcal{A})_2 = C_1^{-1}(1-\mathcal{A})^2C_1$ and $(1-\mathcal{A})_{-2} = C_1^{-1}(1-\mathcal{A})^{-2}C_1$. We will later see (cf. (S.4)) that $C_1(D(\mathcal{A}^2)) \subseteq D((-\mathcal{A})_{1+i\tau})$, so that the closedness of $(-\mathcal{A})_{i\tau}$ follows from a simple calculation involving the closedness of $(-\mathcal{A})_{1+i\tau}$. Further on, the Cauchy integral formula and (398) together imply that the operator family $\{\lambda^k(\lambda-\mathcal{A})^{-k}C_1 : \lambda>0\} \subseteq L(X)$ is equicontinuous for all $k\in\mathbb{N}$. If $y\in\mathcal{A}^k x$ for some $k\in\mathbb{N}$ and $x\in D(\mathcal{A}^k)$, then there exists a sequence $(y_j)_{1\leqslant j\leqslant k}$ in X such that $y_k=y$ and $(x,y_1)\in\mathcal{A}$, $(y_1,y_2)\in\mathcal{A},\ldots,$ $(y_{k-1},y_k)\in\mathcal{A}$. Then we can inductively prove with the help of Theorem 1.2.4(i) that

$$\lambda^k(\lambda-\mathcal{A})^{-k}C_1x=C_1x+\sum_{j=1}^{k}\binom{k}{j}(\lambda-\mathcal{A})^{-j}C_1y_j,\quad \lambda>0,$$

which implies that $\lim_{\lambda\to+\infty}\lambda^k(\lambda-\mathcal{A})^{-k}C_1x=C_1x$, $k\in\mathbb{N}$, $x\in D(\mathcal{A}^k)$; cf. [**103**, Lemma 2.7]. The assertion of [**103**, Lemma 2.5] also holds in our framework.

Now we will revisit multivalued analogues of some statements established in [**103**, Theorem 2.8, Theorem 2.10, Lemma 2.14].

(S.1) Suppose $\operatorname{Re} b\neq 0$. Then it is checked at once that $(-\mathcal{A})_b\subseteq C_1^{-1}(-\mathcal{A})_bC_1$, with the equality in the case that the operator C_1 is injective.

(S.2) Suppose $\operatorname{Re} b_1<0$ and $\operatorname{Re} b_2<0$. Then $R(C_1)\subseteq D((-\mathcal{A})_z)$, $\operatorname{Re} z<0$,

$$\begin{aligned}(-\mathcal{A})_{C_1}^{b_1+b_2}x &\in C_1^{-1}(-\mathcal{A})_{C_1}^{b_1+b_2}C_1x=C_1^{-1}(-\mathcal{A})_{C_1}^{b_1}(-\mathcal{A})_{C_1}^{b_2}x\\ &\subseteq C_1^{-1}(-\mathcal{A})_{C_1}^{b_1}C_1^{-1}(-\mathcal{A})_{C_1}^{b_2}C_1x=(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}C_1x,\quad x\in X.\end{aligned}$$

This, in turn, implies $(-\mathcal{A})_{C_1}^{b_1+b_2}\subseteq(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}C_1$, $C_1^{-1}(-\mathcal{A})_{C_1}^{b_1+b_2}\subseteq C_1^{-1}(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}C_1$ and

$$(-\mathcal{A})_{b_1+b_2}\subseteq C_1^{-1}(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}C_1. \tag{403}$$

Let $y\in(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}x$. Thus, $C_1y\in(-\mathcal{A})_{C_1}^{b_1}C_1^{-1}(-\mathcal{A})_{C_1}^{b_2}x$. This yields the existence of an element $u\in C_1^{-1}(-\mathcal{A})_{C_1}^{b_2}x$ such that $C_1z=(-\mathcal{A})_{C_1}^{b_2}x$ and $C_1y=(-\mathcal{A})_{C_1}^{b_1}u$. So, $C_1^2y=(-\mathcal{A})_{C_1}^{b_1}C_1u=(-\mathcal{A})_{C_1}^{b_1}(-\mathcal{A})_{C_1}^{b_2}C_1x$, $C_1y\in C_1^{-1}(-\mathcal{A})_{C_1}^{b_1+b_2}C_1x=(-\mathcal{A})_{b_1+b_2}C_1x$ and $y\in C_1^{-1}(-\mathcal{A})_{b_1+b_2}C_1x$. Hence,

$$(-\mathcal{A})_{b_1}(-\mathcal{A})_{b_2}\subseteq C_1^{-1}(-\mathcal{A})_{b_1+b_2}C_1. \tag{404}$$

(S.3) Suppose now that $\operatorname{Re} b_1>0$ and $\operatorname{Re} b_2>0$. Using the equations (403)–(404) with b_1 and b_2 replaced respectively by $-b_1$ and $-b_2$ therein, and taking the inverses after that, it readily follows from (S.2) that (403)–(404) holds in this case.

(S.4) Repeating almost verbatim the corresponding parts of the proof of [**103**, Theorem 2.8(ii.2)], we can deduce the following: Suppose that $\operatorname{Re} b>0$ and $k=\lceil\operatorname{Re} b\rceil$, resp. $k=\lceil\operatorname{Re} b\rceil+1$, provided that $\operatorname{Re} b\notin\mathbb{N}$, resp. $\operatorname{Re} b\in\mathbb{N}$. Let $x=C_1y$ for some $y\in D(\mathcal{A}^k)$. Then there exists a sequence $(y_j)_{1\leqslant j\leqslant k}$ in X such that $(y,y_1)\in\mathcal{A}$, $(y_1,y_2)\in\mathcal{A},\ldots,$ $(y_{k-1},y_k)\in\mathcal{A}$. Furthermore, $C_1(D(\mathcal{A}^k))\subseteq D((-\mathcal{A})_b)$ and, for every such a sequence, we have

$$\frac{1}{2\pi i}\int_\Gamma z^{b-\lfloor\operatorname{Re} b\rfloor-1}(z+\mathcal{A})^{-1}C_1y_k\,dz\in(-\mathcal{A})_bx.$$

(S.5) The assertion of [**103**, Theorem 2.8(iii)] is not really interested in multivalued case because $(-\mathcal{A})_b x$ is not singleton, in general.

(S.6) Let $\tau \in \mathbb{R}$. Then a straightforward calculation involving (S.1) shows that $(-\mathcal{A})_{i\tau} \subseteq C_1^{-1}(-\mathcal{A})_{i\tau}C_1$. The equality $(-\mathcal{A})_{i\tau} = C_1^{-1}(-\mathcal{A})_{i\tau}C_1$ can be also trivially verified provided that the operator C_1 is injective.

(S.7) Let $x = C_1 y$ for some $y \in D(\mathcal{A})$, and let $\tau \in \mathbb{R}$. Keeping in mind (400), (402), (S.4), the residue theorem and Theorem 1.2.3, we can prove as in the single-valued linear case that:

$$\frac{1}{2\pi i}\int_\Gamma z^{-1+i\tau}\frac{z}{z+1}(z+\mathcal{A})^{-1}C_1^2 x\,dz \in (1-\mathcal{A})(-\mathcal{A})_{-1}(-\mathcal{A})_{1+i\tau}(1-\mathcal{A})_{-2}C_1^2 x.$$

Let $u \in (1-\mathcal{A})y$. Using Theorem 1.2.3 and Theorem 1.2.4(i), we get from the above that

$$\begin{aligned} C_1^{-3}(1-\mathcal{A})C_1&\left[\frac{1}{2\pi i}\int_\Gamma z^{-1+i\tau}\frac{z}{z+1}(z+\mathcal{A})^{-1}C_1^2 x\,dz\right] \\ &= \frac{1}{2\pi i}\int_\Gamma z^{-1+i\tau}\frac{z}{z+1}(z+\mathcal{A})^{-1}C_1 u\,dz \in (-\mathcal{A})_{i\tau}x, \end{aligned}$$

so that $C_1(D(\mathcal{A})) \subseteq D((-\mathcal{A})_{i\tau})$. Unfortunately, a great number of important properties of purely imaginary powers established in [**103**, Theorem 2.10] does not continue to hold in multivalued linear case.

(S.8) Let $n \in \mathbb{N}_0$, let $b \in \mathbb{C}$ and let $\operatorname{Re} b \in (0, n+1) \smallsetminus \mathbb{N}$. Set $(1-b)(2-b)\ldots(n-b) := 1$ for $n = 0$. Then, for every $x \in X$, we have

$$C_1^n(-\mathcal{A})_{C_1}^{-b}x = \frac{(-1)^n n!}{(1-b)(2-b)\ldots(n-b)}\frac{\sin\pi(n-b)}{\pi}\int_0^\infty t^{n-b}(t-\mathcal{A})^{-(n+1)}C_1^{n+1}x\,dt.$$

This can be shown following the lines of the proof of [**171**, Theorem 5.27, p. 138].

3.9.2. Abstract incomplete differential inclusions. Throughout this subsection, we assume that the condition $(\mathrm{HS})_0$ holds. Define C_1 through (401). Then (HS) holds and we can define the fractional powers of $-\mathcal{A}$ as it has been done above.

In our previous work, we have used the function

$$\begin{aligned} f_t(\lambda) &= \frac{1}{\pi}e^{-t\lambda^\gamma\cos\pi\gamma}\sin(t\lambda^\gamma\sin\pi\gamma) \\ &= \frac{1}{2\pi i}(e^{-t\lambda^\gamma e^{-i\pi\gamma}} - e^{-t\lambda^\gamma e^{i\pi\gamma}}), \quad t>0,\ \lambda>0, \end{aligned}$$

exploited by A. V. Balakrishnan in [**44**]. As already seen, this function enjoys the following properties:

Q2. $\int_0^\infty \lambda^n f_t(\lambda)d\lambda = 0$, $n \in \mathbb{N}_0$, $t>0$.

Q3. Let $m \geqslant -1$. Then the improper integral $\int_0^\infty \lambda^n f_t(\lambda)(\lambda-\mathcal{A})^{-1}C_1\cdot d\lambda$ is absolutely convergent and defines a bounded linear operator on X ($n \in \mathbb{N}_0$).

Put now, for $0 < \gamma < 1/2$,

$$S_\gamma(t)x := \int_0^\infty f_t(\lambda)(\lambda - \mathcal{A})^{-1}C_1x\,d\lambda, \quad t > 0,\ x \in X.$$

Then $S_\gamma(t) \in L(X)$, $t > 0$ and the following holds:

LEMMA 3.9.4. *We have*

$$S_\gamma(t) = (e^{-tz^\gamma})_{C_1}(\mathcal{A}), \quad t > 0,\ 0 < \gamma < 1/2. \tag{405}$$

Furthermore, $S_\gamma(t)$ can be defined by (405) *for all $t \in \Sigma_{(\pi/2)-\gamma\pi}$, and the mapping $t \mapsto S_\gamma(t)$, $t \in \Sigma_{(\pi/2)-\gamma\pi}$ is strongly analytic* ($0 < \gamma < 1/2$)*.*

PROOF. Observe that, for every $t = t_1 + it_2 \in \Sigma_{(\pi/2)-\gamma\pi}$ and $z \in \mathbb{C} \setminus \{0\}$, we have

$$|e^{-tz^\gamma}| \leqslant e^{-|z|^\gamma t_1 \cos(\gamma \arg(z))[1-|\tan(\arg(t))|\tan(\gamma\arg(z))]}.$$

Keeping this estimate in mind, it is very simple to deform the path of integration $\Gamma_S(\vartheta, d)$ into the negative real axis, showing that for each $t \in \Sigma_{(\pi/2)-\gamma\pi}$ and $x \in X$ we have:

$$\frac{1}{2\pi i}\int_{\Gamma_S(\vartheta,d)} e^{-t\lambda^\gamma}(\lambda + \mathcal{A})^{-1}C_1x\,d\lambda$$
$$= \frac{1}{2\pi i}\int_0^\infty (e^{-t\lambda^\gamma e^{-i\pi\gamma}} - e^{-t\lambda^\gamma e^{i\pi\gamma}})(\lambda - \mathcal{A})^{-1}C_1x\,d\lambda.$$

The rest of proof is left to the interested reader. □

Set

$$\varphi_\gamma := (\pi/2) - \gamma(\pi - \vartheta), \quad \text{for } 0 < \gamma \leqslant 1/2.$$

THEOREM 3.9.5. *Put $S_\gamma(0) := C_1$, $S_{\gamma,\zeta}(t)x := \int_0^t g_\zeta(t-s)S_\gamma(s)x\,ds$, $x \in X$, $t \in \Sigma_{(\pi/2)-\gamma\pi}$ ($\zeta > 0$), and $S_{\gamma,0}(t) := S_\gamma(t)$, $t \in \Sigma_{(\pi/2)-\gamma\pi}$. Then the family $\{S_\gamma(t) : t > 0\}$ is equicontinuous, and there exist strongly analytic operator families $(\mathbf{S}_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$ and $(\mathbf{S}_{\gamma,\zeta}(t))_{t\in\Sigma_{\varphi_\gamma}}$ such that $\mathbf{S}_\gamma(t) = S_\gamma(t)$, $t > 0$ and $\mathbf{S}_{\gamma,\zeta}(t) = S_{\gamma,\zeta}(t)$, $t > 0$. Furthermore, the following holds:*

- (i) $\mathbf{S}_\gamma(t_1)\mathbf{S}_\gamma(t_2) = \mathbf{S}_\gamma(t_1 + t_2)C_1$ *for all* $t_1, t_2 \in \Sigma_{\varphi_\gamma}$.
- (ii) *We have* $\lim_{t\to 0, t\in\Sigma_{\varphi_\gamma - \varepsilon}} \mathbf{S}_\gamma(t)x = C_1x$, $x \in \overline{D(\mathcal{A})}$, $\varepsilon \in (0, \varphi_\gamma)$.
- (iii) $\mathbf{S}_\gamma(z)(-\mathcal{A})_\nu \subseteq (-\mathcal{A})_\nu\mathbf{S}_\gamma(z)$, $z \in \Sigma_{\varphi_\gamma}$, $\nu \in \mathbb{C}_+$.
- (iv) *If $D(\mathcal{A})$ is dense in X, then $(S_\gamma(t))_{t\geqslant 0}$ is an equicontinuous analytic C_1-regularized semigroup of angle φ_γ. Moreover, $(S_\gamma(t))_{t\geqslant 0}$ is a C_1-regularized existence family with a subgenerator $-(-\mathcal{A})_\gamma$ and the supposition $(x, y) \in -(-\mathcal{A})_\gamma$ implies $(C_1x, C_1y) \in \hat{\mathcal{A}}_\gamma$, where $\hat{\mathcal{A}}_\gamma$ is the integral generator of $(S_\gamma(t))_{t\geqslant 0}$; otherwise, for every $\zeta > 0$, $(S_{\gamma,\zeta}(t))_{t\geqslant 0}$ is an exponentially equicontinuous, analytic ζ-times integrated C_1-regularized semigroup, $(S_{\gamma,\zeta}(t))_{t\geqslant 0}$ is a ζ-times integrated C_1-existence family with a subgenerator $-(-\mathcal{A})_\gamma$ and the supposition $(x, y) \in -(-\mathcal{A})_\gamma$ implies $(C_1x, C_1y) \in \hat{\mathcal{A}}_\gamma$.*

(v) *For every $x \in X, t \in \Sigma_{(\pi/2)-\gamma\pi}$ and $n \in \mathbb{N}$, we have*

$$\left(S_\gamma(t)x, -\int_0^\infty \lambda^n f_t(\lambda)(\lambda - \mathcal{A})^{-1}C_1 x\, d\lambda\right) \in \mathcal{A}^n. \tag{406}$$

(vi) *Suppose $\beta > 0$. Denote by $\Omega_{\theta,\gamma}$, resp. Ψ_γ, the continuity set of $(S_\gamma(te^{i\theta}))_{t>0}$, resp. $(S_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$. Then, for every $x \in \Omega_{\theta,\gamma}$, the incomplete abstract Cauchy inclusion*

$$(FP_\beta): \begin{cases} u \in C^\infty((0,\infty) : X), \\ D_-^\beta u(t) \in e^{i\theta\beta}(-\mathcal{A})_{\gamma\beta}u(t), \quad t > 0, \\ \lim_{t\to 0+} u(t) = C_1 x, \\ \textit{the set } \{u(t) : t > 0\} \textit{ is bounded in } X, \end{cases}$$

has a solution $u(t) = S_\gamma(te^{i\theta})x$, $t > 0$, which can be analytically extended to the sector $\Sigma_{\varphi_\gamma - |\theta|}$. If, additionally, $x \in \Psi_\gamma$, then for every $\delta \in (0, \varphi_\gamma)$ and $j \in \mathbb{N}_0$, we have that the set $\{z^j u^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded in X.

Proof. The proof of (i) for real parameters $t_1, t_2 > 0$ follows almost directly from definition of $S_\gamma(\cdot)$, by applying (402); (v) is an easy consequence of Theorem 1.2.3, Theorem 1.2.4(i) and the property Q2. A very simple proof of (iii) is omitted. Set, for $|\theta| < \vartheta$ and $0 < \gamma < 1/2$,

$$S_{\theta,\gamma}(t)x := \int_0^\infty f_{t,\gamma}(\lambda)(\lambda - e^{i\theta}\mathcal{A})^{-1}C_1 x\, d\lambda, \quad x \in X,\ t \in \Sigma_{(\pi/2)-\gamma\pi}.$$

Let $\theta_1 \in (0, \vartheta)$ and $\theta_2 \in (-\vartheta, 0)$. Define

$$\mathbf{S}_\gamma(t)x := \begin{cases} S_\gamma(t)x,\ t \in \Sigma_{(\pi/2)-\gamma\pi}, \\ S_{\theta_1,\gamma}(te^{-i\gamma\theta_1}),\ \text{if } t \in e^{i\gamma\theta_1}\Sigma_{(\pi/2)-\gamma\pi}, \\ S_{\theta_2,\gamma}(te^{-i\gamma\theta_2}),\ \text{if } t \in e^{i\gamma\theta_2}\Sigma_{(\pi/2)-\gamma\pi}. \end{cases}$$

Then an elementary application of Cauchy formula shows that the operator family $(\mathbf{S}_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$ is well defined; furthermore, $(\mathbf{S}_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$ is strongly analytic and equicontinuous on any proper subsector of Σ_{φ_γ} (cf. also the proof of [**291**, Theorem 2.9.48]). Using Theorem 1.2.4(i), we get that $\lim_{\lambda\to+\infty}[\lambda(\lambda - \mathcal{A})^{-1}C_1 x - \lambda(\lambda + 1)^{-1}C_1 x] = 0$ as $\lambda \to +\infty$ ($x \in D(\mathcal{A})$). Taking into account this equality and the proof of [**410**, Theorem 5.5.1(iv), p. 130], we get that $\lim_{t\to 0+} S_\gamma(t)x = C_1 x$, $x \in D(\mathcal{A})$. Now the proofs of (i)-(iii) can be straightforwardly completed.

We will prove (iv) provided that $D(\mathcal{A})$ is dense in E. It is clear that $(S_\gamma(t))_{t\geqslant 0}$ is an equicontinuous analytic C_1-regularized semigroup $(S_\gamma(t))_{t\geqslant 0}$ of angle φ_γ. Since, for every $t > 0$ and $x \in X$,

$$C_1(-z^{-\gamma}e^{-tz^\gamma} + z^{-\gamma})_{C_1}(\mathcal{A})x = -(z^{-\gamma})_{C_1}(\mathcal{A})[(e^{-tz^\gamma})_{C_1}(\mathcal{A})x - C_1 x],$$

we have

$$C_1 S_{\gamma,1}(t)x = -(-\mathcal{A})_{C_1}^{-\gamma}[S_\gamma(t)x - C_1 x], \quad t \geqslant 0,\ x \in X.$$

This clearly implies that $(S_{\gamma,1}(t)x, S_{\gamma,1}(t)x - C_1 x) \in -(-\mathcal{A})_\gamma$, $t \geqslant 0$, $x \in X$, so that $(S_{\gamma,\zeta}(t))_{t\geqslant 0}$ is a ζ-times integrated C_1-existence family with a subgenerator $-(-\mathcal{A})_\gamma$. The supposition $(x, y) \in -(-\mathcal{A})_\gamma$ implies $C_1 x = -(-\mathcal{A})_{C_1}^{-\gamma}y$ and we can similarly prove that $(C_1x, C_1y) \in \hat{\mathcal{A}}_\gamma$. Arguing as in the proof of [**103**, Theorem 3.5(i)/(b)'], we get that, for every $x \in X$ and $t > 0$, the following equality holds, with $z = te^{i\theta} \in \Sigma_{(\pi/2)-\gamma\pi}$,

$$(407)\quad D_-^\beta S_\gamma(te^{i\theta})x = \frac{e^{i\theta\beta}}{2\pi i}\int_0^\infty \lambda^{\gamma\beta}[e^{-i\gamma\beta\pi}e^{-z\lambda^\gamma e^{-i\pi\gamma}} - e^{i\gamma\beta\pi}e^{-z\lambda^\gamma e^{i\pi\gamma}}](\lambda-\mathcal{A})^{-1}C_1x\,d\lambda.$$

Deforming the path of integration $\Gamma_S(\vartheta, d)$ into the negative real axis, as it has been done in the proof of Lemma 3.9.4, we get

$$(408)\quad (\cdot^{\gamma\beta}e^{-z\cdot^\gamma})_{C_1}(\mathcal{A}) = \frac{1}{2\pi i}\int_0^\infty \lambda^{\gamma\beta}[e^{-i\gamma\beta\pi}e^{-z\lambda^\gamma e^{-i\pi\gamma}} - e^{i\gamma\beta\pi}e^{-z\lambda^\gamma e^{i\pi\gamma}}](\lambda-\mathcal{A})^{-1}C_1x\,d\lambda.$$

Since

$$C_1(e^{-z\cdot^\gamma})_{C_1}(\mathcal{A}) = (\cdot^{-\gamma\beta})_{C_1}(\mathcal{A})(\cdot^{-\gamma\beta}e^{-z\cdot^\gamma})_{C_1}(\mathcal{A}),$$

(407)–(408) immediately implies that

$$(e^{-i\theta\beta}D_-^\beta S_\gamma(te^{i\theta})x, S_\gamma(te^{i\theta})x) \in C_1^{-1}(-\mathcal{A})^{\gamma\beta}_{C_1}, \quad t > 0,\ x \in X,$$

i.e.,

$$(S_\gamma(te^{i\theta})x, e^{-i\theta\beta}D_-^\beta S_\gamma(te^{i\theta})x) \in (-\mathcal{A})_{\gamma\beta}, \quad t > 0,\ x \in X.$$

The proof of (vi) now can be completed routinely. $\square$

REMARK 3.9.6. (i) If $l = \beta\gamma \in \mathbb{N}$, then the operator $(-\mathcal{A})_{\gamma\beta}$ in the formulation of problem $(\mathrm{FP})_\beta$ can be replaced with the operator $(-\mathcal{A})^l$ therein; cf. (406).

(ii) Suppose that the operator C_1 is injective. Then we can simply prove that $(S_{\gamma,\zeta}(t))_{t\geqslant 0}$ is a ζ-times integrated C_1-semigroup with a subgenerator $-(-\mathcal{A})_\gamma$, which implies that the integral generator of $(S_{\gamma,\zeta}(t))_{t\geqslant 0}$ is $-C_1^{-1}(-\mathcal{A})_\gamma C_1 = -(-\mathcal{A})_\gamma$. A similar statement holds in the case that $\gamma = 1/2$, which is further discussed in the following theorem.

THEOREM 3.9.7. *The limit contained in the expression*

$$(409)\qquad S_{1/2}(t)x := \frac{1}{\pi}\lim_{N\to\infty}\int_0^N \sin(t\sqrt{\lambda})(\lambda-\mathcal{A})^{-1}C_1x\,d\lambda, \quad t > 0,$$

exists in $L(X)$ for every $x \in X$. Put $S_{1/2}(0) := C_1$. Then the family $\{S_{1/2}(t) : t > 0\}$ is equicontinuous, there exists a strongly analytic operator family $(\mathbf{S}_{1/2}(t))_{t\in\Sigma_{\varphi_{1/2}}}$ such that $\mathbf{S}_{1/2}(t) = S_{1/2}(t)$, $t > 0$ and the following holds:

(i) $\mathbf{S}_{1/2}(t)\mathbf{S}_{1/2}(s) = \mathbf{S}_{1/2}(t+s)C_1$ *for all* $t, s \in \Sigma_{\varphi_{1/2}}$.
(ii) $\lim_{t\to 0, t\in\Sigma_{\varphi_{1/2}-\varepsilon}} \mathbf{S}_{1/2}(t)x = C_1x$, $x \in \overline{D(\mathcal{A})}$, $\varepsilon \in (0, \varphi_{1/2})$.
(iii) $\mathbf{S}_{1/2}(t)(-\mathcal{A})_\nu \subseteq (-\mathcal{A})_\nu\mathbf{S}_{1/2}(t)$, $t \in \Sigma_{\varphi_{1/2}}$, $\nu \in \mathbb{C}_+$.
(iv) *If $D(\mathcal{A})$ is dense in X, then $(S_{1/2}(t))_{t\geqslant 0}$ is an equicontinuous analytic C_1-regularized semigroup of angle φ_γ. Furthermore, $(S_{1/2}(t))_{t\geqslant 0}$ is a C_1-regularized existence family with a subgenerator $-(-\mathcal{A})_{1/2}$ and the supposition $(x, y) \in -(-\mathcal{A})_{1/2}$ implies $(C_1x, C_1y) \in \hat{\mathcal{A}}_{1/2}$, where $\hat{\mathcal{A}}_{1/2}$ is the integral generator of $(S_{1/2}(t))_{t\geqslant 0}$; otherwise, for every $\zeta > 0$,*

$(S_{1/2,\zeta}(t))_{t\geqslant 0}$ is an exponentially equicontinuous, analytic ζ-times integrated C_1-regularized semigroup, $(S_{1/2,\zeta}(t))_{t\geqslant 0}$ is a ζ-times integrated C_1-existence family with a subgenerator $-(-\mathcal{A})_{1/2}$ and the supposition $(x,y)\in -(-\mathcal{A})_{1/2}$ implies $(C_1x, C_1y)\in \hat{\mathcal{A}}_{1/2}$.

(v) *Then $R(S_{1/2}(t))\subseteq D_\infty(\mathcal{A})$, $t>0$ and, for every $x\in \overline{D(\mathcal{A})}$, the incomplete abstract Cauchy problem*

$$(P_2):\begin{cases} u\in C^\infty((0,\infty):X),\\ u''(t)\in -\mathcal{A}u(t), \quad t>0,\\ \lim_{t\to 0+}u(t)=C_1x,\\ \textit{the set } \{u(t):t>0\} \textit{ is bounded in } X,\end{cases}$$

has a solution $u(t)=S_{1/2}(t)x$, $t>0$. Moreover, the mapping $t\mapsto u(t)$, $t>0$ can be analytically extended to the sector $\Sigma_{\varphi_{1/2}}$ and, for every $\delta\in(0,\varphi_{1/2})$ and $j\in\mathbb{N}_0$, we have that the set $\{z^j u^{(j)}(z):z\in\Sigma_\delta\}$ is bounded in X.

Proof. First of all, observe that $\varphi_{1/2}=\vartheta/2$. Applying the partial integration, (398) and the equicontinuity of family $\{\lambda^2(\lambda-\mathcal{A})^{-2}C_1:\lambda>0\}$, we obtain that the limit contained in (409) exists and equals

$$S_{1/2}(t)x=\int_0^\infty f(\lambda,t)(\lambda-\mathcal{A})^{-2}C_1x\,d\lambda,\quad t>0,\ x\in X,$$

where $f(\lambda,t)=2\pi^{-1}t^{-2}[\sin(t\sqrt{\lambda})-t\sqrt{\lambda}\cos(t\sqrt{\lambda})]$ for $\lambda>0$ and $t>0$. As in the single-valued case, the change of variables $x=t\sqrt{\lambda}$ shows that the operator family $\{S_{1/2}(t):t>0\}$ is both equicontinuous and strongly continuous. Let $(x,y)\in\mathcal{A}$. Then Theorem 1.2.4(i) shows that

$$\begin{aligned} S_{1/2}(t)x-C_1x&=\frac{1}{\pi}\lim_{N\to\infty}\int_0^N \sin(t\sqrt{\lambda})((\lambda+A)^{-1}C_1x-\lambda^{-1}C_1x)d\lambda\\ &=\frac{1}{\pi}\lim_{N\to\infty}\int_0^N\frac{\sin(t\sqrt{\lambda})}{\lambda}(\lambda-\mathcal{A})^{-1}C_1y\,d\lambda.\end{aligned}$$

Keeping in mind the last equality and the equicontinuity of family $\{S_{1/2}(t):t\geqslant 0\}$, we get that $\lim_{t\to 0}S_{1/2}(t)x=C_1x$ for all $x\in\overline{D(\mathcal{A})}$. Now we proceed as in the proof of Theorem 2.7.5. Let $0<\delta'<\delta<\vartheta/2$, $1/2>\gamma_0>\delta/\vartheta$ and $\theta\in(-\vartheta,(-\delta)/\gamma_0)$. Then, for every $\gamma\in(\gamma_0,1/2)$, we have $\theta\in(-\vartheta,(-\delta)/\gamma)$ and $\gamma>\delta/\vartheta$. Let $\varepsilon\in(0,(\pi-\vartheta)/2)$ be sufficiently small. Define, for every $\gamma\in(\gamma_0,1/2)$ and $x\in X$,

$$F_\gamma(\lambda)x:=\begin{cases}\frac{e^{i\theta\gamma}\sin\gamma\pi}{\pi}\int_0^\infty\frac{v^\gamma(v-e^{i\theta}\mathcal{A})^{-1}C_1x\,dv}{(\lambda e^{i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}, & \text{if } \arg(\lambda)\in(-\varepsilon,(\pi/2)+\delta),\\ \frac{e^{-i\theta\gamma}\sin\gamma\pi}{\pi}\int_0^\infty\frac{v^\gamma(v-e^{-i\theta}\mathcal{A})^{-1}C_1x\,dv}{(\lambda e^{-i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}, & \text{if } \arg(\lambda)\in(-(\pi/2)-\delta,\varepsilon).\end{cases}$$

If $x\in X$ and $\arg(\lambda)\in(-\varepsilon,(\pi/2)+\delta)$, resp., $\arg(\lambda)\in(-(\pi/2)-\delta,\varepsilon)$, then

$$\int_0^\infty e^{-\lambda e^{i\theta\gamma}t}S_{\theta,\gamma}(t)x\,dt=\frac{\sin\gamma\pi}{\pi}\int_0^\infty\frac{v^\gamma(v-e^{i\theta}\mathcal{A})^{-1}C_1x}{(\lambda e^{i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}dv,\tag{410}$$

resp.,

$$\int_0^\infty e^{-\lambda e^{-i\theta\gamma}t} S_{-\theta,\gamma}(t)x\,dt = \frac{\sin\gamma\pi}{\pi}\int_0^\infty \frac{v^\gamma(v+e^{-i\theta}\mathcal{A})^{-1}C_1x}{(\lambda e^{-i\theta\gamma}+v^\gamma\cos\pi\gamma)^2+v^{2\gamma}\sin^2\gamma\pi}\,dv. \tag{411}$$

Furthermore,

$$e^{i\theta\gamma}\int_0^\infty e^{-\lambda e^{i\theta\gamma}t} S_{\theta,\gamma}(t)x\,dt = e^{-i\theta\gamma}\int_0^\infty e^{-\lambda e^{-i\theta\gamma}t} S_{-\theta,\gamma}(t)x\,dt, \quad \lambda\in\Sigma_\varepsilon. \tag{412}$$

By (410)–(412), we deduce that the function $\lambda\mapsto F_\gamma(\lambda)x$, $\lambda\in\Sigma_{(\pi/2)+\delta}$ is well defined, analytic and bounded by $\mathrm{Const}_{\delta'}|\lambda|^{-1}$ on sector $\Sigma_{(\pi/2)+\delta'}$ $(x\in X)$, as well as

$$S_\gamma(z)x = \frac{1}{2\pi i}\int_{\Gamma_{\delta',z}} e^{\lambda z}F_\gamma(\lambda)x\,d\lambda, \quad x\in X,\ z\in\Sigma_{\delta'},\ \gamma\in(\gamma_0,1/2), \tag{413}$$

where $\Gamma_{\delta',z}:=\Gamma_{\delta',z,1}\cup\Gamma_{\delta',z,2}$, $\Gamma_{\delta',z,1}:=\{re^{i((\pi/2)+\delta')}:r\geqslant|z|^{-1}\}\cup\{|z|^{-1}e^{i\vartheta}:\vartheta\in[0,(\pi/2)+\delta']\}$ and $\Gamma_{\delta',z,2}:=\{re^{-i((\pi/2)+\delta')}:r\geqslant|z|^{-1}\}\cup\{|z|^{-1}e^{i\vartheta}:\vartheta\in[-(\pi/2)-\delta',0]\}$ are oriented counterclockwise. The dominated convergence theorem shows that, for every $x\in X$ and $z\in\Sigma_{\delta'}$,

$$\begin{aligned}\lim_{\gamma\to\frac12-}S_\gamma(z)x &= \frac{e^{i\theta/2}}{2\pi^2 i}\int_{\Gamma_{\delta',z,1}} e^{\lambda z}\int_0^\infty \frac{v^{1/2}(v-e^{i\theta}\mathcal{A})^{-1}C_1x}{\lambda^2e^{i\theta}+v}\,dv\,d\lambda\\ &\quad+\frac{e^{-i\theta/2}}{2\pi^2 i}\int_{\Gamma_{\delta',z,2}} e^{\lambda z}\int_0^\infty \frac{v^{1/2}(v-e^{-i\theta}\mathcal{A})^{-1}C_1x}{\lambda^2e^{-i\theta}+v}\,dv\,d\lambda\\ &:=\mathbf{S}_{1/2}(z)x.\end{aligned}$$

Define $F_{1/2}(\lambda)$ by replacing the number γ with the number $1/2$ in definition of $F_\gamma(\lambda)$. Then, for every $x\in X$, the function $\lambda\mapsto F_{1/2}(\lambda)x$, $\lambda\in\Sigma_{(\pi/2)+\delta}$ is well defined and analytic on $\Sigma_{(\pi/2)+\delta}$; furthermore, for each $q\in\circledast$ there exists $r_q\in\circledast$ such that $q(F_{1/2}(\lambda)x)\leqslant r_q(x)\,\mathrm{Const}_{\delta'}|\lambda|^{-1}$, $\lambda\in\Sigma_{(\pi/2)+\delta'}$, $x\in X$ [**317**]. Define $(\mathbf{S}_{1/2}(z))_{z\in\Sigma_{\vartheta/2}}\subseteq L(X)$ by $\mathbf{S}_{1/2}(z)x:=\lim_{\gamma\to\frac12-}S_\gamma(z)x$, $z\in\Sigma_{\vartheta/2}$, $x\in X$; this operator family is equicontinuous on any proper subsector of $\Sigma_{\vartheta/2}$ and satisfies additionally that the mapping $z\mapsto\mathbf{S}_{1/2}(z)x$, $z\in\Sigma_{\vartheta/2}$ is analytic for all $x\in X$. Letting $\gamma\to\frac12-$ in (413), it is not difficult to prove that

$$\mathbf{S}_{\frac12}(z)x = \frac{1}{2\pi i}\int_{\Gamma_{\delta',z}} e^{\lambda z}F_{\frac12}(\lambda)x\,d\lambda, \quad x\in X,\ z\in\Sigma_{\delta'},$$

so that the proof of [**27**, Theorem 2.6.1] implies

$$\int_0^\infty e^{-\lambda t}\mathbf{S}_{\frac12}(t)x\,dt = F_{\frac12}(\lambda)x, \quad x\in X,\ \lambda>0. \tag{414}$$

On the other hand, by the proof of [**410**, Theorem 5.5.2, p. 133] we have

$$\int_0^\infty e^{-\lambda t}S_{\frac12}(t)x\,dt = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\nu}}{\lambda^2+\nu}(\nu-\mathcal{A})^{-1}C_1x\,d\nu = F_{\frac12}(\lambda)x, \quad x\in X,\ \lambda>0. \tag{415}$$

Using the uniqueness theorem for the Laplace transform, we obtain from (414)–(415) that $\mathbf{S}_{1/2}(t)=S_{1/2}(t)$, $t>0$. Now the proofs of (i)–(iii) become standard and therefore omitted.

For simplicity, we assume that $\mathcal{A}$ is densely defined in (iv). Then the only non-trivial thing that should be proved is that the supposition $(x,y) \in -(-\mathcal{A})_{1/2}$ implies $(C_1x, C_1y) \in \hat{\mathcal{A}}_{1/2}$. So, let $(x,y) \in -(-\mathcal{A})_{1/2}$, i.e., $C_1x = -(-\mathcal{A})_{C_1}^{-1/2}y$. A similar line of reasoning as in the proof of identity [**103**, (51), p. 489] shows that

$$C_1 \int_0^\infty e^{-\lambda t} S_\gamma(t)y\,dt = C_1(-\mathcal{A})_{C_1}^{-\gamma}y - \lambda \int_0^\infty e^{-\lambda t} S_\gamma(t)y\,dt, \quad \lambda > 0,\ \gamma \in (0,1/2).$$

Taking the limits of both sides of previous equality when $\gamma \to 1/2-$, we get that

$$C_1 \int_0^\infty e^{-\lambda t} S_{1/2}(t)y\,dt = C_1(-\mathcal{A})_{C_1}^{-1/2}y - \lambda \int_0^\infty e^{-\lambda t} S_{1/2}(t)y\,dt, \quad \lambda > 0.$$

Then the uniqueness theorem for Laplace transform simply implies that

$$S_{1/2}(t)C_1x - C_1^2x = \int_0^t S_{1/2}(s)C_1y\,ds, \quad t \geqslant 0,$$

as claimed.

Now we will prove (v) by slightly modifying the corresponding part of proof of Theorem 2.7.5(i). In order to do that, we will first show that for each $x \in X$ we have $S''_{1/2}(t)x \in -\mathcal{A}S_{1/2}(t)x$, $t > 0$. Fix temporarily an element $x \in X$. Owing to Theorem 3.9.5(v) and (407), cf. also Remark 3.9.6(i), we have that

$$D_-^{\frac{1}{\gamma}} S'_\gamma(t)x \in -\mathcal{A}S_\gamma(t)x, \quad t > 0,$$

i.e.,

$$\frac{d^2}{dt^2} \int_0^\infty g_{3-\frac{1}{\gamma}}(s)S'_\gamma(t+s)x\,ds \in \mathcal{A}S_\gamma(t)x, \quad t > 0,\ \gamma \in (\gamma_0, 1/2).$$

Therefore,

$$\int_0^\infty g_{3-\frac{1}{\gamma}}(s)S'''_\gamma(t+s)x\,ds \in \mathcal{A}S_\gamma(t)x, \quad t > 0,\ \gamma \in (\gamma_0, 1/2).$$

Applying the partial integration, we get

$$\int_0^\infty g_{4-\frac{1}{\gamma}}(s)S^{(iv)}_\gamma(t+s)x\,ds \in -\mathcal{A}S_\gamma(t)x, \quad t > 0,\ \gamma \in (\gamma_0, 1/2).$$

The dominated convergence theorem yields by letting $\gamma \to 1/2-$ that

$$\int_0^\infty sS^{(iv)}_{1/2}(t+s)x\,ds \in -\mathcal{A}S_{1/2}(t)x, \quad t > 0,$$

which clearly implies after an application of integration by parts that $S''_{1/2}(t)x \in -\mathcal{A}S_{1/2}(t)x$, $t > 0$, as claimed. By (ii), the function $u(t) = S_{1/2}(t)x$, $t > 0$ is a solution of problem (P_2) for $x \in \overline{D(\mathcal{A})}$. Furthermore, we obtain by induction that $S^{(2n)}_{1/2}(t)x \in (-1)^n\mathcal{A}^nS_{1/2}(t)x$, $t > 0$, $n \in \mathbb{N}$, $x \in X$, so that $R(S_{1/2}(t)) \subseteq D_\infty(\mathcal{A})$, $t > 0$. The proof of the theorem is thereby complete. $\square$

Examples of exponentially bounded integrated semigroups generated by multivalued linear operators can be found in [**199**, Section 5.3, Section 5.8] (cf. also Example 3.2.11(i)). These example can serve one to provide possible applications of Theorem 3.9.5 and Theorem 3.9.7.

EXAMPLE 3.9.8. Assume that $M_p = p!^s$ for some $s > 1$. Set $\omega(z) := \prod_{i=1}^{\infty}(1 + \frac{iz}{p^s})$, $z \in \mathbb{C}$. Suppose that there exist constants $l > 0$ and $\omega > 0$ satisfying that $RHP_\omega \equiv \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subseteq \rho(\mathcal{A})$ and the operator family $\{e^{-M(l|\lambda|)}R(\lambda : \mathcal{A}) \,|\, \lambda \in RHP_\omega\} \subseteq L(X)$ is equicontinuous (cf. [**291**, **292**] for a great number of such examples with $\mathcal{A}$ being single-valued, and Example 3.2.65 for purely multivalued linear case, with X being a Fréchet space); here, $M(\cdot)$ denotes the associated function of sequence (M_p). Let $\bar{\omega} > \omega$. Then there exists a sufficiently large number $n \in \mathbb{N}$ such that the expression

$$S(t) := \frac{1}{2\pi i}\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty} e^{\lambda t}\frac{R(\lambda : \mathcal{A})}{\omega^n(i\lambda)}d\lambda, \quad t \geqslant 0$$

defines an exponentially equicontinuous $C \equiv S(0)$-regularized semigroup $(S(t))_{t\geqslant 0}$ with a subgenerator $\mathcal{A}$ (cf. Theorem 1.2.3, Theorem 1.2.4(i) and the proof of [**291**, Theorem 3.6.4]). It is not difficult to see that $(\lambda - \mathcal{A})^{-1}Cf = \int_0^\infty e^{-\lambda t}S(t)f\,dt$, $\operatorname{Re}\lambda > \bar{\omega}$, $f \in X$, so that Theorem 3.9.5 and Theorem 3.9.7 can be applied with the operator $\mathcal{A}$ replaced with the operator $\mathcal{A} - \bar{\omega}$ therein. Observe that, even in single-valued linear case, the operator C need not be injective because our assumptions do not imply that $\mathcal{A} = A$ generates an ultradistribution semigroup of Beurling class.

In this book, we will not discuss the generation of degenerate fractional regularized resolvent families by the negatives of constructed fractional powers. For more details, cf. [**103**, Section 3] and [**291**, Remark 2.9.49].

At the end, we would like to observe that the assertions of Theorem 2.7.3–Theorem 2.7.5 can be formulated in the multivalued linear operators setting. For applications, the most important is the following case: X is a Banach space, $\Sigma_\vartheta \cup B_d \subseteq \rho(\mathcal{A})$, there exist finite numbers $M_1 \geqslant 1$ and $\nu \in (0, 1]$ such that (13) holds with the operator $\mathcal{A}$ and number β replaced with $-\mathcal{A}$ and ν therein; see [**199**, Chapter III, Chapter VI] for a great number of concrete examples. Define the operators $S_\gamma(\cdot)$ as before. Then we may conclude the following:

(i) Suppose that $\beta\gamma > 1 - \nu$. Then $(S_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$ is an analytic semigroup of growth order $\frac{1-\nu}{\gamma}$. Denote by $\Omega_{\theta,\gamma}$, resp. Ψ_γ, the continuity set of $(S_\gamma(te^{i\theta}))_{t>0}$, resp. $(S_\gamma(t))_{t\in\Sigma_{\varphi_\gamma}}$. Then $\overline{D(\mathcal{A})} \subseteq \Psi_\gamma$ and, for every $x \in \Omega_{\theta,\gamma}$, the incomplete abstract Cauchy inclusion (FP_β), with $C_1 = I$, has a solution $u(t) = S_\gamma(te^{i\theta})x$, $t > 0$, which can be analytically extended to the sector $\Sigma_{\varphi_\gamma - |\theta|}$. If, additionally, $x \in \Psi_\gamma$, then for every $\delta \in (0, \varphi_\gamma)$ and $j \in \mathbb{N}_0$, we have that the set $\{z^j u^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded in X.
(ii) Suppose that $1/2 < \nu < 1$. Then the incomplete abstract Cauchy problem (P_2), with $C_1 = I$, has a solution $u(t)$, $t > 0$ for all $x \in D(\mathcal{A})$. Moreover, the mapping $t \mapsto u(t)$, $t > 0$ can be analytically extended to the sector

$\Sigma_{\varphi_{1/2}}$ and, for every $\delta \in (0, \varphi_{1/2})$ and $j \in \mathbb{N}_0$, we have that the set $\{z^j(1+|z|^{2\nu-2})^{-1}u^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded in X.

As announced before, it is very non-trivial to find some necessary and sufficient conditions ensuring the uniqueness of solutions of problems (FP_β) and (P_2).

3.10. Inverse generator problem

The main purpose of this section is to analyze the inverse generator problem for abstract degenerate Volterra integro-differential equations in sequentially complete locally convex spaces. More specifically, we consider the problem of generation of mild (a,k)-regularized (C_1, C_2)-existence and uniqueness families and (a,k)-regularized C-resolvent families by the inverses of closed multivalued linear operators. As before, by X and Y we denote two Hausdorff sequentially complete locally convex spaces over the field of complex numbers. The injectiveness of operators C, C_1, C_2, if needed, will be explicitly emphasized.

We start by stating the following useful result:

PROPOSITION 3.10.1. *Suppose that* $C \in L(X)$, $\lambda \in \mathbb{C} \setminus \{0\}$, $\mathcal{A}$ *is an* MLO *and* $\lambda^{-1} \in \rho_C(\mathcal{A})$. *Then we have* $\lambda \in \rho_C(\mathcal{A}^{-1})$ *and*

$$(\lambda - \mathcal{A}^{-1})^{-1}C = \lambda^{-1}[C - \lambda^{-1}(\lambda^{-1} - \mathcal{A})^{-1}C].$$

PROOF. Suppose $x \in X$. Then a simple computation involving the definition of inverse of an MLO shows that

$$(Cx, \lambda^{-1}[Cx - \lambda^{-1}(\lambda^{-1} - \mathcal{A})^{-1}Cx]) \in (\lambda - \mathcal{A}^{-1})^{-1}$$

iff

$$-Cx + \lambda^{-1}(\lambda^{-1} - \mathcal{A})^{-1}Cx \in \mathcal{A}(\lambda^{-1} - \mathcal{A})^{-1}Cx,$$

which is true due to Theorem 1.2.4(i). It suffices to prove that the operator $(\lambda - \mathcal{A}^{-1})^{-1}C$ is single-valued. If we suppose that $\{y, z\} \subseteq (\lambda - \mathcal{A}^{-1})^{-1}Cx$, then we have $\lambda y - Cx \in \mathcal{A}^{-1}y$ and $\lambda z - Cx \in \mathcal{A}^{-1}z$. Hence, $y \in \mathcal{A}[\lambda y - Cx]$ and $z \in \mathcal{A}[\lambda z - Cx]$. This simply implies $Cx \in (\lambda^{-1} - \mathcal{A})^{-1}C[\lambda Cx - \lambda y]$ and $Cx \in (\lambda^{-1} - \mathcal{A})^{-1}C[\lambda Cx - \lambda z]$. Since the operator $(\lambda^{-1} - \mathcal{A})^{-1}C$ is single-valued, we simply get from the above that $y = z = \lambda^{-1}[Cx - \lambda^{-1}(\lambda^{-1} - \mathcal{A})^{-1}Cx]$. $\square$

Now we will state a general result which gives the necessary and sufficient conditions for a multivalued linear operator $\mathcal{A}^{-1}$ to be a subgenerator of an exponentially equicontinuous mild (a,k)-regularized C_1-existence family or an exponentially equicontinuous mild (a,k)-regularized C_2-uniqueness family.

PROPOSITION 3.10.2. *Suppose that* $\mathcal{A}$ *is a closed* MLO *in* X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, $|a(t)|$ *and* $k(t)$ *satisfy* (P1), *as well as that* $(R_1(t), R_2(t))_{t \geqslant 0} \subseteq L(Y,X) \times L(X)$ *is strongly continuous. Let* $\omega \geqslant \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$ *be such that the operator family* $\{e^{-\omega t}R_i(t) : t \geqslant 0\}$ *is equicontinuous for* $i = 1, 2$. *Then the following holds:*

(i) $(R_1(t),R_2(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* (C_1,C_2)*-existence and uniqueness family with a subgenerator* $\mathcal{A}^{-1}$ *iff for every* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$*, we have* $R(C_1)\subseteq \mathrm{R}(\tilde{a}(\lambda)-\mathcal{A})$*,*

$$\tilde{a}(\lambda)\tilde{k}(\lambda)C_1y\in(\tilde{a}(\lambda)-\mathcal{A})\int_0^\infty e^{-\lambda t}[k(t)C_1y-R_1(t)y]dt,\quad y\in Y \tag{416}$$

and

$$\tilde{k}(\lambda)C_2x=\int_0^\infty e^{-\lambda t}[R_2(t)x-(a*R_2)(t)y]dt,\quad \textit{whenever } (y,x)\in\mathcal{A}. \tag{417}$$

(ii) $(R_1(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* C_1*-existence family with a subgenerator* $\mathcal{A}^{-1}$ *iff for every* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$*, we have* $R(C_1)\subseteq R(\tilde{a}(\lambda)-\mathcal{A})$ *and* (416)*.*

(iii) $(R_2(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* C_2*-uniqueness family with a subgenerator* $\mathcal{A}^{-1}$ *iff* (417) *holds for* $\operatorname{Re}\lambda>\omega$*.*

PROOF. It is clear that $\mathcal{A}^{-1}$ is a closed MLO. The part (iii) follows immediately from Lemma 3.2.44 and definition of $\mathcal{A}^{-1}$. For the rest, it suffices to prove (ii). Suppose first that, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$, we have $R(C_1)\subseteq R(\tilde{a}(\lambda)-\mathcal{A})$ and (416). Then a simple computation gives that for such values of parameter λ we have

$$-\int_0^\infty e^{-\lambda t}(a*R_1)(t)y\,dt\in\mathcal{A}\int_0^\infty e^{-\lambda t}[k(t)C_1y-R_1(t)y]dt,\quad y\in Y,$$

$R(C_1)\subseteq R(I-\tilde{a}(\lambda)\mathcal{A}^{-1})$ and

$$\tilde{k}(\lambda)C_1y\in(I-\tilde{a}(\lambda)\mathcal{A}^{-1})\int_0^\infty e^{-\lambda t}R_1(t)y\,dt,\quad y\in Y. \tag{418}$$

By Lemma 3.2.44(ii), we get that $(R_1(t))_{t\geqslant 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator $\mathcal{A}^{-1}$. For the converse, we can apply Lemma 3.2.44(ii) again so as to conclude that, for every $y\in Y$ and for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$, we have $R(C_1)\subseteq R(I-\tilde{a}(\lambda)\mathcal{A}^{-1})$ and (418). As above, this simply implies $R(C_1)\subseteq R(\tilde{a}(\lambda)-\mathcal{A})$ and (416). $\square$

Using Lemma 3.2.45 and a similar argumentation, we can prove the following:

PROPOSITION 3.10.3. *Suppose that* $\mathcal{A}$ *is a closed* MLO *in* X*,* $C\in L(X)$*,* $C\mathcal{A}\subseteq\mathcal{A}C$*,* $|a(t)|$ *and* $k(t)$ *satisfy* (P1)*, as well as that* $(R(t))_{t\geqslant 0}\subseteq L(X)$ *is strongly continuous and commutes with* C *on* X*. Let* $\omega\geqslant\max(0,\operatorname{abs}(|a|),\operatorname{abs}(k))$ *be such that the operator family* $\{e^{-\omega t}R(t):t\geqslant 0\}$ *is equicontinuous. Then* $(R(t))_{t\geqslant 0}$ *is an* (a,k)*-regularized* C*-resolvent family with a subgenerator* $\mathcal{A}^{-1}$ *iff for every* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$*, we have* $R(C)\subseteq\mathrm{R}(\tilde{a}(\lambda)-\mathcal{A})$*,* (416) *holds with* $R_1(\cdot)$*,* C_1 *and* Y*,* y *replaced with* $R(\cdot)$*,* C *and* X*,* x *therein, as well as* (417) *holds with* $R_2(\cdot)$ *and* C_2 *replaced with* $R(\cdot)$ *and* C *therein.*

The complex characterization theorem ensuring the existence of an exponentially equicontinuous degenerate (a,k)-regularized C-resolvent families, in combination with Proposition 3.10.3, enables one to formulate the subsequent result, which provides sufficient conditions for the operator $\mathcal{A}^{-1}$ to be a subgenerator of an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family:

Proposition 3.10.4. *Assume that $\mathcal{A}$ is a closed* MLO *in X, $C\mathcal{A} \subseteq \mathcal{A}C$, $\alpha \in (0,\pi/2]$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$ and $\omega \geqslant \max(0,\operatorname{abs}(k),\operatorname{abs}(|a|))$. Assume, further, that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda) \neq 0$, we have $R(C) \subseteq R(\tilde{a}(\lambda) - \mathcal{A})$ as well as that there exist a function $\Upsilon \colon \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ and an operator $D' \in L(X)$ such that, for every $x \in X$, the mapping $\lambda \mapsto \Upsilon(\lambda)x$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}$ is analytic as well as that:*

- (i) *There exists a function $\mathbf{k} : \Sigma_\alpha \cup \{0\} \to \mathbb{C}$ which is analytic on Σ_α, continuous on any closed subsector $\overline{\Sigma_\gamma}$ $(0 < \gamma < \alpha)$ and which additionally satisfies that $\sup_{z\in\Sigma_\gamma} |e^{-\omega z}\mathbf{k}(z)| < \infty$ $(0 < \gamma < \alpha)$ and $\mathbf{k}(t) = k(t)$ for all $t \geqslant 0$;*
- (ii) *$\Upsilon(\lambda)x \in \tilde{a}(\lambda)(\tilde{a}(\lambda) - A)^{-1}Cx$ for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, $\tilde{a}(\lambda) \neq 0$;*
- (iii) *$\Upsilon(\lambda)Cx = C\Upsilon(\lambda)x$ for $\operatorname{Re}\lambda > \omega$, $x \in X$;*
- (iv) *$\tilde{a}(\lambda)\Upsilon(\lambda)x - \Upsilon(\lambda)y = \tilde{a}(\lambda)Cx$, provided $\operatorname{Re}\lambda > \omega$ and $(x,y) \in \mathcal{A}$;*
- (v) *$\lim_{\lambda\to+\infty} \Upsilon(\lambda)x = D'x$, $x \in X$, if $\overline{R(\mathcal{A})} \neq X$.*

Then the function $\lambda \mapsto \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega$ has an analytic extension $\lambda \mapsto \hat{k}(\lambda)$, $\lambda \in \omega + \Sigma_{(\pi/2)+\alpha}$ satisfying that $\sup_{\omega+\Sigma_{(\pi/2)+\gamma}} |(\lambda - \omega)\hat{k}(\lambda)| < \infty$ for $0 < \gamma < \alpha$ and $\lim_{\lambda\to+\infty} \lambda\hat{k}(\lambda) = k(0)$. If, additionally,

- (vi) *the family $\{(\lambda - \omega)\hat{k}(\lambda)\Upsilon(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}\} \subseteq L(X)$ is equicontinuous for all $\gamma \in (0,\alpha)$,*

then $\mathcal{A}^{-1}$ is a subgenerator of an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle α satisfying that $R(z)\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}R(z)$, $z \in \Sigma_\alpha$, the family $\{e^{-\omega z}R(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ is equicontinuous for all angles $\gamma \in (0,\alpha)$, as well as that the equation (272) holds for each $y = x \in X$, with $\mathcal{A}$, $R_1(\cdot)$ and C_1 replaced therein by $\mathcal{A}^{-1}$, $R(\cdot)$ and C, respectively.

Proof. It is clear that $\mathcal{A}^{-1}$ is a closed MLO in X and $C\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}C$. As already seen multiple times before, condition (i) yields that the function $\lambda \mapsto \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega$ has an analytic extension $\lambda \mapsto \hat{k}(\lambda)$, $\lambda \in \omega + \Sigma_{(\pi/2)+\alpha}$ satisfying that $\sup_{\omega+\Sigma_{(\pi/2)+\gamma}} |(\lambda-\omega)\hat{k}(\lambda)| < \infty$ for $0 < \gamma < \alpha$ and $\lim_{\lambda\to+\infty} \lambda\hat{k}(\lambda) = k(0)$. Define $q(\lambda) := \hat{k}(\lambda)C - \hat{k}(\lambda)\Upsilon(\lambda)$, $\lambda \in \omega + \Sigma_{(\pi/2)+\alpha}$ and $D := k(0)C - k(0)D'$. Then $q(\cdot)$ is analytic and a simple computation involving condition (ii) shows that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda) \neq 0$, we have $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A}^{-1})$ with $Cx \in (I - \tilde{a}(\lambda)\mathcal{A}^{-1})[Cx - \Upsilon(\lambda)x]$, $x \in X$. The remainder of proof follows from an elementary argumentation and corresponding statement for the operator $\mathcal{A}$. $\square$

The most intriguing case in which the assumptions of Proposition 3.10.4 hold is: $\omega = 0$, $a(t) = g_\alpha(t)$ for some number $\alpha \in (0,2)$, $k(t) = 1$, $C \in L(X)$ is injective and satisfies that $C^{-1}\mathcal{A}C = \mathcal{A}$ is the integral generator of an equicontinuous analytic (g_α, g_1)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$ of angle $\gamma \in (0, \min(\pi/2, (\pi/\alpha) - (\pi/2))]$. Then we can make a choice in which $\Upsilon(\lambda) = \lambda^\alpha(\lambda^\alpha - \mathcal{A})^{-1}C$ and $D' = S(0)$, providing thus a proper extension of [**383**, Theorem 4.1(i)] and [**135**, Proposition 1]. Since subordination principles can be formulated for (a,k)-regularized C-resolvent families subgenerated by MLOs (C need not be injective, in general), we can simply prove an extension of [**383**, Theorem 4.1(ii)] for (g_α, g_β)-regularized C-resolvent families.

In connection with Proposition 3.10.4 and [**383**, Theorem 4.1], we would like to propose the following:

PROPOSITION 3.10.5. *Suppose that $\mathcal{A}$ is a closed* MLO, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $\beta \geqslant 0$, $\alpha \in (0,2)$, $a(t) = g_\alpha(t)$, $k(t) = g_{\beta+1}(t)$ *and* $\gamma \in (0, \min(\pi/2, (\pi/\alpha)-(\pi/2))]$. *Suppose, further, that* $\Sigma_{((\pi/2)+\gamma)\alpha} \subseteq \rho_C(\mathcal{A})$, *the mapping* $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in \Sigma_{((\pi/2)+\gamma)\alpha}$ *is analytic* ($x \in X$) *and the following two conditions are satisfied:*

(i) *For every* $\gamma' \in (0,\gamma)$, *there exists a finite constant* $M_{\gamma'} > 0$ *such that the operator family* $\{\lambda^{\alpha+\beta}(\lambda^\alpha - \mathcal{A})^{-1}C : \lambda \in \Sigma_{(\pi/2)+\gamma'},\ |\lambda| \leqslant 1\} \subseteq L(X)$ *is equicontinuous.*

(ii) *If* $\overline{R(\mathcal{A})} \neq X$, *then there exists* $D' \in L(X)$ *such that* $\lim_{\lambda\to 0+} \lambda(\lambda - \mathcal{A})^{-1}Cx = D'x$, $x \in X$.

Then the operator $\mathcal{A}^{-1}$ *is a subgenerator of an exponentially equicontinuous, analytic* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *of angle* γ *satisfying that* $R(z)\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}R(z)$, $z \in \Sigma_\gamma$ *and the family* $\{e^{-\omega z}R(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ *is equicontinuous for every real numbers* $\omega > 0$ *and* $\gamma \in (0,\gamma')$. *Moreover, the equation* (272) *holds for each* $y = x \in X$, *with* $\mathcal{A}$, $R_1(\cdot)$ *and* C_1 *replaced therein by* $\mathcal{A}^{-1}$, $R(\cdot)$ *and* C, *respectively.*

PROOF. As above, we have that $\mathcal{A}^{-1}$ is a closed MLO in X and $C\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}C$. It is clear that the function $k(t)$ satisfies condition (i) from Proposition 3.10.4. If $\omega > 0$, then we can apply Proposition 3.10.4 with the function $\Upsilon : \omega + \Sigma_{\frac{\pi}{2}+\gamma} \to L(X)$ defined by $\Upsilon(\lambda) := \lambda^{-\alpha}(\lambda^{-\alpha} - \mathcal{A})^{-1}C$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}$. In actual fact, conditions (ii)-(iv) of Proposition 3.10.4 clearly hold; condition (v) of Proposition 3.10.4 holds because of assumption (ii) of this proposition, while condition (vi) of Proposition 3.10.4 follows from condition (i) of this proposition and a simple computation with a new variable $z = 1/\lambda$. $\square$

REMARK 3.10.6. It is worth noting that the behaviour of function $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \Sigma_{((\pi/2)+\gamma)\alpha}$ at the point $\lambda = \infty$ does not play any role for applying Proposition 3.10.5.

It seems that Proposition 3.10.5 is not considered elsewhere, even for the abstract non-degenerate differential equations of first order. We will only present an illustrative application of Proposition 3.10.5 to the abstract degenerate differential equations of fractional order:

EXAMPLE 3.10.7. Let us recall that, for every two linear single-valued operators A and B, we have $(AB^{-1})^{-1} = BA^{-1}$ and $(B^{-1}A)^{-1} = A^{-1}B$ in the MLO sense. Consider now, for the sake of illustration, the fractional Poisson heat equation $(P)_b$ in the space $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$; then B is the multiplication in $L^p(\Omega)$ with $m(x)$, and $A = \Delta - b$ acts with the Dirichlet boundary conditions (see also [**199**, Example 3.6]). Let $\mathcal{A} := AB^{-1}$; then for a suitable chosen number $b > 0$, we have the existence of an angle $\theta \in (\pi/2, \pi)$ and a finite number $M > 0$ such that

$$\|(\lambda - \mathcal{A})^{-1}\| \leqslant M|\lambda|^{(-1)/p}, \quad \lambda \in \Sigma_\theta. \tag{419}$$

Let $\alpha \in [1, 2\theta/\pi)$. By Proposition 3.10.5, with $D' = 0$ and $\beta = 0$, we get that $\mathcal{A}^{-1}$ generates an analytic (g_α, g_1)-regularized resolvent family $(R(t))_{t\geqslant 0}$ of angle $\gamma \in (0, \min((\theta/\alpha) - (\pi/2), \pi/2)]$, satisfying that the operator family $\{e^{-\omega z}R(z) : z \in \Sigma_{\gamma'}\} \subseteq L(X)$ is bounded for every $\omega > 0$ and $\gamma' \in (0, \gamma)$. Moreover, let $0 < \varepsilon < \gamma' < \gamma$. Then we have the following integral representation

$$R(z)x = x - \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda z}\frac{(\lambda^{-\alpha} - \mathcal{A})^{-1}x}{\lambda^{\alpha+1}}\,d\lambda, \quad x \in X,\ z \in \Sigma_{\gamma'-\varepsilon},$$

where the contour Γ is defined in the proof of [**27**, Theorem 2.6.1] (we only need to replace the number γ with the number γ' therein). Using the estimate (419) and the integral computation contained in the proof of afore-mentioned theorem, after letting $\omega \to 0+$ we get that $(R(t))_{t\geqslant 0}$ is an equicontinuous analytic (g_α, g_1)-regularized resolvent family of angle γ. Hence, we can analyze the well-posedness of the reversed fractional Poisson heat equation in the space $L^p(\Omega)$:

$$(P)_{b;r} : \begin{cases} \mathbf{D}_t^\alpha[(\Delta - b)v(t,x)] = m(x)v(t,x), \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ (\Delta - b)v(0,x) = v_0(x),\ \left(\frac{d}{dt}[(\Delta - b)v(t,x)]\right)_{t=0} = v_1(x), \quad x \in \Omega. \end{cases}$$

For possible applications to abstract degenerate second-order differential equations, we refer the reader to [**199**, Example 6.1] and [**293**, Example 3.10.10].

It is worth noticing that the existence and behaviour of C-resolvent of a multi-valued linear operator $\mathcal{A}$ around zero is most important for the generation of certain classes of (a,k)-regularized C-resolvent families, C-(ultra)distribution semigroups and C-(ultra)distribution cosine functions by the inverse operator $\mathcal{A}^{-1}$. More to the point, the existence of C-resolvent of $\mathcal{A}$ at the point $\lambda = +\infty$ does not play any role for the generation of C-(ultra)distribution semigroups and C-(ultra)distribution cosine functions by the inverse operator $\mathcal{A}^{-1}$; in the following example, we will explain this fact only for C-distribution semigroups (a similar statement holds for the generation of locally defined fractional C-resolvent families):

EXAMPLE 3.10.8. Suppose that $a > 0$, $b > 0$ and recall that $E(a,b) := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\}$. It can be easily seen that the set $1/E(a,b) := \{1/\lambda : \lambda \in E(a,b)\}$ is a relatively compact subset of $\mathbb{C}$, as well as that $1/E(a,b)$ is contained in the strip $\{\lambda \in \mathbb{C} : 0 < \operatorname{Re}\lambda < 1/b\}$. Let $\mathcal{A}$ be a closed MLO commuting with the operator $C \in L(X)$, and let there exist $n \in \mathbb{N}$ such that the operator family $\{\lambda^n(\lambda - \mathcal{A})^{-1}C : \lambda \in 1/E(a,b)\} \subseteq L(X)$ is equicontinuous. If we suppose additionally that the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$ is analytic on $1/\Omega_{a,b}$ and continuous on $1/\Gamma_{a,b}$, with the meaning clear, where $\Gamma_{a,b}$ denotes the upwards oriented boundary of $E(a,b)$ and $\Omega_{a,b}$ the open region which lies to the right of $\Gamma_{a,b}$, then an extension of the operator $\mathcal{A}^{-1}$ generates a C-distribution semigroup $\mathcal{G}$. Moreover, if C is injective and $\mathcal{A}^{-1}$ is single valued, then the operator $C^{-1}\mathcal{A}^{-1}C$ is the integral generator of $\mathcal{G}$. On the other hand, for the generation of exponential C-distribution semigroups by an extension of the operator $\mathcal{A}^{-1}$ one has to assume that the operator family $\{\lambda^n(\lambda-\mathcal{A})^{-1}C : 0 < \operatorname{Re}\lambda < c\} \subseteq L(X)$ is equicontinuous for some real number $c > 0$ and integer $n \in \mathbb{N}$. Finally, let $\alpha \in (0,2)$ and $\omega > 0$. Denote by Ω the unbounded region lying between the boundary of sector $\Sigma_{\alpha\pi/2}$ and the curve $\{\lambda^{-\alpha} : \operatorname{Re}\lambda = \omega\}$. If $\Omega \subseteq \rho_C(\mathcal{A})$ and the family $\{\lambda^n(\lambda-\mathcal{A})^{-1}C : \lambda \in \Omega\} \subseteq L(X)$ is equicontinuous for some integer $n \in \mathbb{N}$, then there exists a positive real number $\beta > 0$ such that the operator $\mathcal{A}$ is a subgenerator of a global $(g_\alpha, g_{\beta+1})$-regularized C-resolvent operator family $(R(t))_{t\geqslant 0}$ satisfying that the operator family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous.

In the following theorem, we reconsider the statement of [**383**, Theorem 4.1(i)] for subgenerators of degenerate $(g_\alpha, g_{\beta+1})$-regularized C-resolvent families, where $\alpha \in (0,2]$, $\beta \geqslant 0$ and the operator $C \in L(X)$ is possibly non-injective. More to the point, here the subgenerator $\mathcal{A}$ is not necessarily injective or single-valued (the interested reader may try to extend the statements of [**383**, Theorem 4.1(ii), Corollary 4.1, Corollary 4.2, Corollary 4.3], as well), which will be crucial for applications carried out in Example 3.10.19 below:

THEOREM 3.10.9. *Suppose that $\alpha \in (0,2]$, $\beta \geqslant 0$ and a closed* MLO *$\mathcal{A}$ is a subgenerator of an exponentially equicontinuous $(g_\alpha, g_{\beta+1})$-regularized C-resolvent family $(S(t))_{t\geqslant 0}$ such that the operator family $\{t^{-\beta}S(t) : t > 0\} \subseteq L(X)$ is equicontinuous. Then, for every number $\gamma > \beta + (1/2)$, the operator $\mathcal{A}^{-1}$ is a subgenerator of an $(g_\alpha, g_{\gamma+1})$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying that the operator family $\{t^{-\gamma}R(t) : t > 0\} \subseteq L(X)$ is equicontinuous.*

PROOF. Define

$$R(t)x := g_{\gamma+1}(t)Cx - t^{1+\beta+\gamma}\int_0^\infty J_{1+\beta+\gamma}(2\sqrt{st})s^{-\frac{1+\beta+\gamma}{2}}S(s)x\,ds, \quad t > 0,\ x \in X.$$

Arguing as in the proof of [**383**, Theorem 4.1(i)], we get that $(R(t))_{t\geqslant 0} \subseteq L(X)$ is strongly continuous as well as that the operator family $\{t^{-\gamma}R(t) : t > 0\} \subseteq L(X)$ is equicontinuous and

$$\int_0^\infty e^{-\lambda t}R(t)x\,dt = \lambda^{-(1+\gamma)}Cx - \lambda^{-(2+\beta+\gamma)}\int_0^\infty e^{-s/\lambda}S(s)x\,ds, \quad \operatorname{Re}\lambda > 0,\ x \in X. \tag{420}$$

Further on, $\mathcal{A}^{-1}$ is a closed MLO and, by Lemma 3.2.45, we have

$$\frac{Cx}{\lambda^{\beta+1}} \in \Big(I - \frac{\mathcal{A}}{\lambda^\alpha}\Big)\int_0^\infty e^{-\lambda t}S(t)x\,dt, \quad \operatorname{Re}\lambda > 0,\ x \in X \tag{421}$$

and

$$\frac{Cy}{\lambda^{\beta+1}} = \int_0^\infty e^{-\lambda t}S(t)y\,dt - \frac{1}{\lambda^\alpha}\int_0^\infty e^{-\lambda t}S(t)x\,dt, \tag{422}$$

provided $\operatorname{Re}\lambda > 0$ and $(x,y) \in X$. Having in mind Lemma 3.2.45 and (420), it suffices to show that

$$(423)\quad \frac{Cx}{\lambda^{1+\gamma}} \in \lambda^{-(1+\gamma)}Cx - \lambda^{-(2+\beta+\gamma)} \int_0^\infty e^{-s/\lambda} S(s)x\,ds - \lambda^{-\alpha}\mathcal{A}^{-1}\Bigl[\lambda^{-(1+\gamma)}Cx - \lambda^{-(2+\beta+\gamma)} \int_0^\infty e^{-s/\lambda} S(s)x\,ds\Bigr], \quad \operatorname{Re}\lambda > 0,\ x \in X$$

and

$$(424)\quad \frac{Cx}{\lambda^{1+\gamma}} = \lambda^{-(1+\gamma)}Cx - \lambda^{-(2+\beta+\gamma)} \int_0^\infty e^{-s/\lambda} S(s)x\,ds - \lambda^{-\alpha}\Bigl[\lambda^{-(1+\gamma)}Cx - \lambda^{-(2+\beta+\gamma)} \int_0^\infty e^{-s/\lambda} S(s)y\,ds\Bigr],$$

provided $\operatorname{Re}\lambda > 0$ and $(y, x) \in \mathcal{A}$. Keeping in mind the definition of operator $\mathcal{A}^{-1}$, the equation (423) follows almost immediately from (421), while the equation (424) follows almost immediately from (422), with the number λ replaced therein with the number $1/\lambda$. □

REMARK 3.10.10. (i) Keeping in mind the proof of [**383**, Theorem 4.1(i)], Theorem 3.10.9 and Lemma 3.2.44(ii), the above result can be simply reformulated for the classes of exponentially equicontinuous mild $(g_\alpha, g_{\beta+1})$-regularized C_1-existence families and exponentially equicontinuous mild $(g_\alpha, g_{\beta+1})$-regularized C_2-uniqueness families. It is also worth noticing that the representation formula obtained in [**383**] with the help of (47) is motivated by earlier results of R. deLaubenfels established in [**134**]. In all these results, the Laplace transform identities for various Bessel type functions play a crucial role.

(ii) For applications, it will be crucial to reconsider and extend the conclusions obtained in [**383**, Remark 4.2] for abstract degenerate fractional differential equations. Suppose that the operator family $\{(1+t^\delta)^{-1}S(t) : t > 0\} \subseteq L(X)$ is equicontinuous for some number $\delta \geqslant 0$ and all remaining assumptions in Theorem 3.10.9 hold. Then, for every non-negative real number $\gamma > 2\delta + (1/2) - \beta$, the operator $\mathcal{A}^{-1}$ is a subgenerator of an $(g_\alpha, g_{\gamma+1})$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$. Moreover, a simple calculation shows that the operator family $\{t^{-\gamma}(1+t^{\beta-\delta}+t^\beta)^{-1}R(t) : t > 0\} \subseteq L(X)$ is equicontinuous.

It is crucial to formulate the following proper generalization of Theorem 3.10.9 (where $\omega_0' = 0$, $f(\lambda) = 1/\lambda$, $a(t) = b(t) = g_\alpha(t)$, $k(t) = g_{\beta+1}(t)$, $k_1(t) = g_{\gamma+1}(t)$, $g(t) = g_{\gamma-\beta}(t)$ and $S_0(t) = t^{1+\beta+\gamma}\int_0^\infty J_{1+\beta+\gamma}(2\sqrt{st})s^{-\frac{1+\beta+\gamma}{2}}S(s)x\,ds$, $t > 0$, $x \in X$; see also Remark 3.10.10) for various classes of (a, k)-regularized C-resolvent families:

THEOREM 3.10.11. *Suppose that $\mathcal{A}$ is a closed* MLO *in X, $C, C_2 \in L(X)$, $C_1 \in L(Y, X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $\omega_0' \geqslant \max(\operatorname{abs}(|a|), \operatorname{abs}(k), 0)$, the functions $b(t)$ and $k_1(t)$ satisfy* (P1) *with $\omega_0 \geqslant \max(0, \operatorname{abs}(|b|))$, the function $k_1(t)$ is continuous for $t \geqslant 0$ and $|k_1(t)| = O(e^{\omega_0 t}P(t))$ for $t \geqslant 0$, where $P(t) = \sum_{j=0}^{l} a_j t^{\zeta_j}$, $t \geqslant 0$ ($l \in \mathbb{N}$, $a_j \geqslant 0$ and $\zeta_j \geqslant 0$ for $1 \leqslant j \leqslant l$). Let $f \colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda >$*

$\omega_0'\}$ and $G\colon \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to \mathbb{C}$ be two given functions, let $\tilde{a}(\lambda) \neq 0$ for $\operatorname{Re}\lambda > \omega_0'$, and let

$$\tilde{b}(\lambda) = \frac{1}{\tilde{a}(f(\lambda))} \quad \textit{and} \quad \widetilde{k_1}(\lambda) = G(\lambda)\tilde{k}(f(\lambda)), \quad \operatorname{Re}\lambda > \omega_0. \tag{425}$$

(i) *Suppose, further, $\mathcal{A}$ is a subgenerator of a global mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\geqslant 0} \subseteq L(Y,X)$ such that the operator family $\{e^{-\omega t}R_1(t) : t \geqslant 0\} \subseteq L(Y,X)$ is equicontinuous for each number $\omega > \omega_0'$, as well as there exists a strongly continuous operator family $(S_0(t))_{t\geqslant 0} \subseteq L(Y,X)$ such that the family $\{e^{-\omega_0 t}S_0(t) : t \geqslant 0\} \subseteq L(Y,X)$ is equicontinuous and*

$$\int_0^\infty e^{-\lambda t}S_0(t)y\,dt = G(\lambda)\int_0^\infty e^{-sf(\lambda)}R_1(s)y\,ds, \quad y \in Y,\ \operatorname{Re}\lambda > \omega_0. \tag{426}$$

Then the operator $\mathcal{A}^{-1}$ is a subgenerator of a global mild (b,k_1)-regularized C_1-existence family $(S_1(t) \equiv k_1(t)C_1 - S_0(t))_{t\geqslant 0} \subseteq L(Y,X)$ and the operator family $\{e^{-\omega_0 t}(P(t))^{-1}S_1(t) : t > 0\} \subseteq L(Y,X)$ is equicontinuous.

(ii) *Suppose, further, $\mathcal{A}$ is a subgenerator of a global mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\geqslant 0} \subseteq L(X)$ such that the operator family $\{e^{-\omega t}R_2(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous for each number $\omega > \omega_0'$, as well as there exists a strongly continuous operator family $(S_0(t))_{t\geqslant 0} \subseteq L(X)$ such that the family $\{e^{-\omega_0 t}S_0(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous and (426) holds with $y = x \in X$ and $R_1(\cdot)$ replaced by $R_2(\cdot)$ therein. Then the operator $\mathcal{A}^{-1}$ is a subgenerator of a global mild (b,k_1)-regularized C_2-uniqueness family $(S_2(t) \equiv k_1(t)C_2 - S_0(t))_{t\geqslant 0} \subseteq L(X)$ and the operator family $\{e^{-\omega_0 t}(P(t))^{-1}S_2(t) : t > 0\} \subseteq L(X)$ is equicontinuous.*

(iii) *Suppose, further, $\mathcal{A}$ is a subgenerator of a global (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0} \subseteq L(X)$ such that the operator family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous for each number $\omega > \omega_0'$, as well as there exists a strongly continuous operator family $(S_0(t))_{t\geqslant 0} \subseteq L(X)$ such that the family $\{e^{-\omega_0 t}S_0(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous and (426) holds with $y = x \in X$ and $R_1(\cdot)$ replaced by $R(\cdot)$ therein. Then the operator $\mathcal{A}^{-1}$ is a subgenerator of a global mild (b,k_1)-regularized C-resolvent family $(S(t) \equiv k_1(t)C - S_0(t))_{t\geqslant 0} \subseteq L(X)$ and the operator family $\{e^{-\omega_0 t}(P(t))^{-1}S(t) : t > 0\} \subseteq L(X)$ is equicontinuous.*

Proof. We will prove only (i). Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{b}(\lambda)\widetilde{k_1}(\lambda) \neq 0$ be given. Due to Proposition 3.10.2(ii), it suffices to show that $R(C_1) \subseteq R(\tilde{b}(\lambda) - \mathcal{A})$ and

$$\tilde{b}(\lambda)\widetilde{k_1}(\lambda)C_1 y \in (\tilde{b}(\lambda) - \mathcal{A})\int_0^\infty e^{-\lambda t}[k_1(t)C_1 y - S_1(t)y]dt, \quad y \in Y. \tag{427}$$

But, our assumption (425) implies $\tilde{a}(f(\lambda))\tilde{k}(f(\lambda)) \neq 0$. Since $\operatorname{Re}(f(\lambda)) > \omega_0'$, Lemma 3.2.44(ii) yields that $R(C_1) \subseteq R([\tilde{a}(f(\lambda))]^{-1} - \mathcal{A})$ and

$$\widetilde{R_1}(f(\lambda))y \in \frac{\tilde{k}(f(\lambda))}{\tilde{a}(f(\lambda))}([\tilde{a}(f(\lambda))]^{-1} - \mathcal{A})^{-1}C_1 y, \quad y \in Y.$$

This simply gives (427) after a simple calculation involving the equations (425) and (426). □

REMARK 3.10.12. In the existing literature concerning the inverse generator problem, the authors have investigated only the following special case: $\omega_0' = 0$, $f(\lambda) = 1/\lambda$ and $a(t) = b(t) = g_\alpha(t)$, for some number $\alpha \in (0, 2)$. Even if $f(\lambda) = 1/\lambda$, the equality $a(t) = b(t) = g_\alpha(t)$, where $\alpha \in (0, 2)$, is not necessary for applying Theorem 3.10.11. For example, suppose that $P(\lambda) = \sum_{j=0}^{n} a_j \lambda^{\zeta_j}$ and $Q(\lambda) = \sum_{j=0}^{m} b_j \lambda^{\eta_j}$ for some non-negative real numbers ζ_j $(0 = \zeta_0 \leqslant \zeta_1 \leqslant \ldots \leqslant \zeta_n)$, η_j $(0 = \eta_0 \leqslant \eta_1 \leqslant \ldots \leqslant \eta_m)$ and complex numbers a_j $(0 \leqslant j \leqslant n)$, b_j $(0 \leqslant j \leqslant m)$ such that $b_0 = 0$, $a_0 a_n b_m \neq 0$, $\eta_m > \zeta_n$ and $P(\lambda)Q(\lambda) \neq 0$ for $\operatorname{Re} \lambda > 0$ [we can take, for example, $P(\lambda) = \lambda + 2$ and $Q(\lambda) = \lambda^3$]. If $a(t) = \mathcal{L}^{-1}(P(\lambda)/Q(\lambda))(t)$, $t \geqslant 0$, then we can prove that $\operatorname{abs}(|a|) = 0$. Furthermore, we can prove that there exists a function $b(t)$ satisfying $\operatorname{abs}(|b|) = 0$ and $\tilde{b}(\lambda) = 1/\tilde{a}(1/\lambda) = Q(1/\lambda)/P(1/\lambda)$, $\operatorname{Re} \lambda > 0$. So, if the second equality in (425) holds with the functions $k(t)$ and $k_1(t)$ being continuous for $t \geqslant 0$, then the most simplest case in which Theorem 3.10.11(iii) is applicable is that case in which $X := \mathbb{C}$, $C := I$, $\mathcal{A} := 0$ and $R(t) := k(t)I$, $t \geqslant 0$, when $S_0(t) = k_1(t)I$, $t \geqslant 0$.

REMARK 3.10.13. In a great deal of concrete situations, it is almost impossible to represent $S_0(\cdot)$ in terms of $R_1(\cdot)$ directly, so that the use of complex characterization theorem for the Laplace transform is sometimes unavoidable.

We continue by stating the following corollary of Theorem 3.10.11 (the case in which $\sigma = -1$ and $a = b = \beta \geqslant 0$ has been already considered in Theorem 3.10.9 and remarks following it):

THEOREM 3.10.14. *Suppose that $\alpha \in (0, 2)$, $\sigma \in (-1, 0)$, $\beta \geqslant 0$, $\mathcal{A}$ is a closed* MLO *in X, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $\mathcal{A}$ is a subgenerator of an exponentially equicontinuous $(g_\alpha, g_{\beta+1})$-regularized C_1-resolvent family $(R(t))_{t \geqslant 0}$ such that the operator family $\{(t^a + t^b)^{-1} R(t) : t > 0\} \subseteq L(X)$ is equicontinuous for two real numbers a, b such that $-1 < a \leqslant b$. Let $\eta > 1 + b$ and $\eta \geqslant 1 + \beta$. Define*

$$F(t) := t^{|\sigma|(\eta-\beta-1)} + \left(t^{|\sigma|(\eta-b-1)} \chi_{(0,1]}(t) + t^{|\sigma|(\eta-a-1)} \chi_{[1,\infty)}(t)\right), \quad t > 0.$$

Then the operator $\mathcal{A}^{-1}$ is a subgenerator of a global $(g_{\alpha|\sigma|}, g_{1+|\sigma|(\eta-\beta-1)})$-regularized C-resolvent family $(S(t))_{t \geqslant 0} \subseteq L(X)$ and the operator family $\{[F(t)]^{-1} S(t) : t > 0\} \subseteq L(X)$ is equicontinuous.

PROOF. We will apply Theorem 3.10.11(iii) with $\omega_0 = \omega_0' = 0$, $f(\lambda) = \lambda^\sigma$, $a(t) = g_\alpha(t)$, $b(t) = g_{|\sigma|\alpha}(t)$, $k(t) = g_{\beta+1}(t)$, $k_1(t) = g_{|\sigma|(\eta-\beta-1)\beta+1}(t)$ and $G(\lambda) = \lambda^{-1+\sigma\eta}$. Set

$$S_0(t)x := \int_0^\infty t^{-\sigma\eta} \phi(-\sigma, 1 - \sigma\eta; -st^{-\sigma}) R(s)x \, ds, \quad t \geqslant 0, \ x \in X.$$

Using (46), the assumption $-1 < \sigma < 0$ and the dominated convergence theorem, it readily follows that the mapping $t \mapsto S_0(t)x$, $t > 0$ is continuous for every fixed element $x \in X$. Since $S_0(0) = 0$ and $[F(\cdot) - \cdot^{|\sigma|(\eta-\beta-1)}](0) = 0$, it suffices to show that the operator family $\{[F(t)]^{-1}S(t) : t > 0\} \subseteq L(X)$ is equicontinuous as well as that

$$\int_0^\infty e^{-\lambda t} S_0(t)x\,dt = \lambda^{-1+\sigma\eta} \int_0^\infty e^{-sf(\lambda)} R(s)x\,ds, \quad x \in X,\ \operatorname{Re}\lambda > 0. \tag{428}$$

The asymptotic formula (46) and the fact that the operator family $\{(t^a+t^b)^{-1}R(t) : t > 0\} \subseteq L(X)$ is equicontinuous together imply that for each seminorm $p \in ⊛$ there exist a finite real constant $m > 0$ and a seminorm $q \in ⊛$ such that

$$\begin{aligned} p(S_0(t)x) &\leqslant mq(x)t^{-\sigma\eta} \int_0^\infty \exp(-m(st^{-\sigma})^{1/(1+\sigma)})(s^a + s^b)ds \\ &= mq(x)t^{\sigma(1-\eta)} \int_0^\infty \exp(-mr^{1/(1+\sigma)})(t^{\sigma a}r^a + t^{\sigma b}r^b)dr. \end{aligned}$$

Observe that the above integral converges due to our assumption $-1 < a \leqslant b$, which also implies $\sigma a \geqslant \sigma b$ and the equicontinuity of operator family $\{[F(t)]^{-1}S(t) : t > 0\}$. Moreover, by the equation (45) and the Fubini theorem, we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t} S_0(t)x\,dt &= \int_0^\infty R(s)x \cdot \int_0^\infty e^{-\lambda t} t^{-\sigma\eta} \phi(-\sigma, 1-\sigma\eta; -st^{-\sigma})dt\,ds \\ &= \lambda^{-1+\sigma\eta} \int_0^\infty e^{-sf(\lambda)} R(s)x\,ds, \quad x \in X,\ \operatorname{Re}\lambda > 0. \end{aligned}$$

This implies (428) and completes the proof of theorem. □

As in Remark 3.10.10, it is worth noting that we can formulate the above result for the classes of exponentially equicontinuous mild $(g_\alpha, g_{\beta+1})$-regularized C_1-existence families and exponentially equicontinuous mild $(g_\alpha, g_{\beta+1})$-regularized C_2-uniqueness families (concerning the inverse generator problem, it is our duty to say that we have not been able to find certain applications with these classes of solution operator families). It is also worth noting the following:

Remark 3.10.15. Let the requirements of Theorem 3.10.14 hold with $a = 0$. Then the subordination principle for degenerate (a, k)-regularized C-resolvent families shows that the operator $\mathcal{A}$ is a subgenerator of a global $(g_{\alpha|\sigma|}, g_{1+|\sigma|\beta})$-regularized C-resolvent family $(W(t))_{t\geqslant 0}$ satisfying that the operator family $\{(1 + t^{b|\sigma|})^{-1}W(t) : t \geqslant 0\} \subseteq L(X)$ is equicontinuous. Arguing as in [**383**, Remark 4.2], with the function $f(\lambda) = 1/\lambda$, we get that the operator $\mathcal{A}^{-1}$ is a subgenerator of a global $(g_{\alpha|\sigma|}, g_{1+\gamma})$-regularized C-resolvent family $(W(t))_{t\geqslant 0}$ provided $\gamma \geqslant 0$ and $\gamma > 2b|\sigma| + (1/2) - \beta|\sigma|$. The integration rate obtained here with the function $f(\lambda) = \lambda^\sigma$ is better provided that $|\sigma|(\eta - 2b - 1) < 1/2$.

Before we move ourselves to Subsection 3.10.1, we will analyze the Poisson wave type equation in the space $L^2(\Omega)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded domain with smooth boundary (see [**199**, Example 2.3]):

Example 3.10.16. Let $X := H_0^1(\Omega) \times L^2(\Omega)$ and $m \in L^\infty(\Omega)$. Consider the bounded linear operator

$$M := \begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix},$$

in X, and an unbounded linear operator

$$L := \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix},$$

in X, with domain $D(L) := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$. Then we know that the MLO $\mathcal{A} := M^{-1}LM^{-1} - I$ satisfies $(0, \infty) \subseteq \rho(\mathcal{A})$ and $\|R(\lambda : \mathcal{A})\| \leqslant 1/\lambda$, $\lambda > 0$. An application of Theorem 3.2.12 gives that for each number $r > 0$ the MLO $\mathcal{A} - I$ generates a global (g_1, g_{1+r})-resolvent family $(S_r(t))_{t\geqslant 0}$ such that $\|S_r(t)\| = O(t^r)$, $t \geqslant 0$. Suppose $\sigma \in (-1, 0)$, $r > 0$ and $\eta > 1 + r$. Then, due to Theorem 3.10.14, the operator $(\mathcal{A} - I)^{-1}$ is the integral generator of an $(g_{|\sigma|}, g_{1+|\sigma|(\eta-r-1)})$-resolvent family $(R_r(t))_{t\geqslant 0}$ such that $\|R_r(t)\| = O(t^{|\sigma|(\eta-r-1)})$, $t \geqslant 0$. Suppose that $(u_0\, v_0)^T \in X$, $(u_1\, v_1)^T \in R(\mathcal{A} - I)$, $(f_1(\cdot)\, f_2(\cdot))^T \in C([0, \infty) : X)$,

$$\begin{aligned} &(g_{|\sigma|} * f_1)(t) + u_0 = g_{1+|\sigma|(\eta-r-1)}(t)u_1 \text{ and} \\ &(g_{|\sigma|} * f_2)(t) + v_0 = g_{1+|\sigma|(\eta-r-1)}(t)v_1, \end{aligned} \tag{429}$$

for any $t \geqslant 0$. Since $D_t^{|\sigma|}(u(t)\, v(t))^T \in (\mathcal{A} - I)^{-1}(u(t)\, v(t))^T + (f_1(t)\, f_2(t))^T$, $t \geqslant 0$ is equivalent with $(u(t)\, v(t))^T \in (\mathcal{A} - I)[D_t^{|\sigma|}(u(t)\, v(t))^T - (f_1(t)\, f_2(t))^T]$, $t \geqslant 0$, after a simple computation involving the condition (429), we get that the function $t \mapsto (u(t)\, v(t))^T \equiv R_r(t)(u_1\, v_1)^T$, $t \geqslant 0$ is a unique strong solution of the following system:

$$\begin{aligned} m(x)[u(t,x) + D_t^{|\sigma|}u(t,x) - f_1(t,x)] &= D_t^{|\sigma|}v(t,x) - f_2(t,x), \\ m(x)[v(t,x) + D_t^{|\sigma|}v(t,x) - f_2(t,x)] &= \Delta[D_t^{|\sigma|}u(t,x) - f_1(t,x)]; \\ u(0,x) = u_0(x),\ v(0,x) &= v_0(x), \quad x \in \Omega. \end{aligned}$$

3.10.1. Applications to degenerate time-fractional equations with abstract differential operators. Assume that $n \in \mathbb{N}$ and $iA_j, 1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space X. Assume, further, that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials in n variables, and $0 < \alpha < 2$; set $N_1 := dg(P_1(x))$, $N_2 := dg(P_2(x))$ and $m := \lceil \alpha \rceil$. In Section 2.2.3, we have analyzed the generation of some specific classes of (g_α, C)-regularized resolvent families associated with the following fractional degenerate abstract Cauchy problem

$$\text{(DFP)} : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t) + f(t), & t \geqslant 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, & 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

provided that $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and there exists a non-negative real number $\omega \geqslant 0$ such that (70) holds, where $0^{1/\alpha} := 0$. Let us recall that our assumptions imply that the operator $\overline{P_2(A)}$ is injective; here we want to note that the operator $\overline{P_2(A)}$ need not be invertible, in general (see e.g. [**27**, Remark 8.3.5]).

In the remainder of this subsection, we will focus our attention on the case $\omega = 0$. If $0 \notin P_1(\mathbb{R}^n)$ and $\omega = 0$, then we have $\sup_{x\in\mathbb{R}^n} \mathrm{Re}((P_2(x)/P_1(x))^{1/\alpha}) \leqslant 0$, so that the well-posedness of the reverse fractional degenerate abstract Cauchy problem

$$(\mathrm{DFP})_r: \begin{cases} \mathbf{D}_t^\alpha \overline{P_1(A)}u(t) = \overline{P_2(A)}u(t) + f(t), & t \geqslant 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, & 1 \leqslant j \leqslant \lceil \alpha \rceil - 1 \end{cases}$$

can be analyzed as in Section 2.2.3 [**306**]. But, the real problems occur if $0 \in P_1(\mathbb{R}^n)$, when the methods established in [**306**] are inapplicable. Our main result concerning the well-posedness of problem $(\mathrm{DFP})_r$ is stated as follows:

THEOREM 3.10.17. *Suppose $0 < \alpha < 2$, $\sigma \in (-1, 0)$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $N \in \mathbb{N}$ and $r \in (0, N]$. Let $Q(x)$ be an r-coercive complex polynomial of degree N, $a \in \mathbb{C} \setminus Q(\mathbb{R}^n)$, $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ (resp. $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}| \max(N, \frac{N_1+N_2}{\min(1,\alpha)})$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and (70) holds with $\omega = 0$. Set*

$$\mathcal{B} \equiv \overline{\overline{P_2(A)} \cdot \overline{P_1(A)}^{-1}},$$

$C := ((a - Q(x))^{-\gamma})(A)$, $\delta := \max(1, \alpha)n/2$, if $E \neq L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$, and $\delta := \max(1, \alpha)n|(1/p) - (1/2)|$, if $E = L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. Then $C \in L(X)$ is injective and the following holds:

- (i) *For each positive real number $\gamma > 2\delta + (1/2)$ the multivalued linear operator $\mathcal{B}$ is a subgenerator of a global exponentially bounded $(g_\alpha, g_{\gamma+1})$-regularized C-resolvent family $(R_\alpha(t))_{t \geqslant 0}$ satisfying that the operator family $\{[t^\gamma(1 + t^{\beta-\delta} + t^\beta)]^{-1} R_\alpha(t) : t > 0\} \subseteq L(X)$ is equicontinuous.*
- (ii) *For each positive real number $\eta > 1 + \delta$, the multivalued linear operator $\mathcal{B}$ is a subgenerator of a global exponentially bounded $(g_{\alpha|\sigma|}, g_{|\sigma|(\eta-1)})$-regularized C-resolvent family $(R_\alpha(t))_{t \geqslant 0}$ satisfying that the operator family $\{[F(t)]^{-1} R_\alpha(t) : t > 0\} \subseteq L(X)$ is equicontinuous, where*

$$F(t) = t^{|\sigma|(\eta-1)} + (t^{|\sigma|(\eta-\delta-1)} \chi_{(0,1]}(t) + t^{|\sigma|(\eta-1)} \chi_{[1,\infty)}(t)), \quad t > 0.$$

PROOF. We will prove only (i). Set

$$S_\alpha(t) := \Big(E_\alpha\Big(t^\alpha \frac{P_1(x)}{P_2(x)}\Big)(a - Q(x))^{-\gamma}\Big)(A), \quad t \geqslant 0.$$

Then we have that the operator $C \in L(X)$ is injective as well as

$$\lambda^{\alpha-1}(\lambda^\alpha - \overline{P_1(A)} \cdot \overline{P_2(A)}^{-1})^{-1} Cx = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\, dt, \quad \operatorname{Re} \lambda > 0, \ x \in X.$$

On the other hand, for any two closed linear operators A and B in X the following holds:

$$\overline{AB^{-1}}^{-1} = \overline{BA^{-1}}. \tag{430}$$

Using (430) and Theorem 1.2.4(i), the above implies that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$, we have $R(C) \subseteq R(I - \lambda^{-\alpha}\mathcal{B}^{-1})$, (304) holds with $R_1(\cdot)$, C_1, $\mathcal{A}$ and Y, y replaced with $R(\cdot)$, C, $\mathcal{B}^{-1}$ and X, x therein, as well as (305) holds with $R_2(\cdot)$, $\mathcal{A}$ and C_2 replaced with $S_\alpha(\cdot)$, $\mathcal{B}^{-1}$ and C therein. Due to Lemma 3.2.45, we get that

$(S_\alpha(t))_{t\geqslant 0} \subseteq L(X)$ is a global exponentially bounded (g_α, C)-regularized resolvent family with a subgenerator $\mathcal{B}^{-1}$. Moreover, we have

$$\|S_\alpha(t)\| \leqslant M(1+t^{\max(1,\alpha)n/2}), \quad t \geqslant 0, \text{resp.},$$

$$\|S_\alpha(t)\| \leqslant M(1+t^{\max(1,\alpha)n|\frac{1}{p}-\frac{1}{2}|}), \quad t \geqslant 0. \tag{431}$$

Using (431), Theorem 3.10.9 and Remark 3.10.10(iii), the required assertion easily follows. □

REMARK 3.10.18. (i) As before, we can choose the regularizing operator C in a slightly different manner and refine the obtained conclusions by imposing some additional conditions on behaviour of the rational function $P_1(x)/P_2(x)$.

(ii) The operator $\overline{P_2(A)} \cdot \overline{P_1(A)}^{-1}$ is closed provided that at least one of the operators $\overline{P_1(A)}$ or $\overline{P_2(A)}$ is invertible, when we have $\mathcal{B} = \overline{P_2(A)} \cdot \overline{P_1(A)}^{-1}$.

(iii) Consider the statement (i). The foregoing arguments also imply

$$\lambda^{\alpha-\gamma-1}\overline{P_1(A)}\big(\lambda^\alpha \overline{P_1(A)} - \overline{P_2(A)}\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}R_\alpha(t)x\,dt, \quad \operatorname{Re}\lambda > 0,\ x \in X,$$

so that for each positive real number $\gamma > 2\delta + (1/2)$ we have that $(R_\alpha(t))_{t\geqslant 0}$ is an exponentially bounded $(g_\alpha, g_{\gamma+1})$-regularized C-resolvent family for the abstract degenerate Cauchy problem

$$\overline{P_1(A)}u(t) = f(t) + \int_0^t a(t-s)\overline{P_2(A)}u(s)ds, \quad t \geqslant 0.$$

Due to Theorem 2.2.8, we have that for each $x \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$ the function $u(t) = R_\alpha(t)x$, $t \geqslant 0$ is a unique strong solution of (52) with $f(t) = g_{1+\gamma}(t)C\overline{P_1(A)}x$, $t \geqslant 0$. Moreover, we know that $(\overline{P_2(A)}^{-1}S_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by $\overline{P_2(A)}$, $\overline{P_1(A)}$. Unfortunately, the operator $\overline{P_1(A)}$ need not be injective and the above fact cannot be used for the construction of $(g_\alpha, g_{1+\gamma})$-regularized C-resolvent family generated by $\overline{P_1(A)}$, $\overline{P_2(A)}$ so that it is not clear how we can consider the well-posedness of problem

$$(\mathrm{DFP})_{L,r}: \begin{cases} \overline{P_1(A)}\mathbf{D}_t^\alpha u(t) = \overline{P_2(A)}u(t) + f(t), & t \geqslant 0,\\ u(0) = Cx; \quad u^{(j)}(0) = 0, & 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

in general. Similar conclusions can be formulated for the statement (ii).

It is worth noting that Theorem 3.10.17 can be simply reformulated in E_l-type spaces. In the following application of Theorem 3.10.17, we consider the case in which $P_2(x) \equiv 1$:

EXAMPLE 3.10.19. (i) Suppose $0 < \alpha < 2$. Let E be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and let $0 \leqslant l \leqslant n$. Let the operator $\mathbf{T}_1\langle\cdot\rangle$, the Fréchet space $X := E_l$, the set $\mathbb{N}_0^l$ and the seminorms $q_\eta(\cdot)$ $(\mathbb{N}_0^l)$ possess the same meaning as before, let $a_\eta \in \mathbb{C}$, $0 \leqslant$

$|\eta| \leqslant N$ and let $P(D)f := \sum_{|\eta|\leqslant N} a_\eta D^\eta f$, with its maximal distributional domain. Suppose that

$$\sup_{x\in\mathbb{R}^n} \operatorname{Re}(P(x)^{1/\alpha}) \leqslant 0.$$

Define $S_\alpha(t) =: \mathbf{T_1}\langle E_\alpha(t^\alpha P(x))(a - P(x))^{-\gamma}\rangle$, $t \geqslant 0$. Then $C := S_\alpha(0) \in L(X)$ is injective, $(S_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family with the integral generator $P(D)$, and

$$q_\eta(S_\alpha(t)f) \leqslant M(1 + t^{\max(1,\alpha)n/2})q_\eta(f), \quad t \geqslant 0,\ f \in E_l,\ \eta \in \mathbb{N}_0^l, \text{ resp.,}$$

$$q_\eta(S_\alpha(t)f) \leqslant M(1 + t^{\max(1,\alpha)n|\frac{1}{p}-\frac{1}{2}|})q_\eta(f), \quad t \geqslant 0,\ f \in E_l,\ \eta \in \mathbb{N}_0^l, \tag{432}$$

with M being independent of $f \in X$ and $\eta \in \mathbb{N}_0^l$; see also [**292**, Remark 2.5.5]. Set $\delta := \max(1,\alpha)n/2$, if $E \neq L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$, and $\delta := \max(1,\alpha)n|(1/p) - (1/2)|$, if $E = L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$. By Theorem 3.10.9 and Remark 3.10.10(ii), we have that for each positive real number $\gamma > 2\delta + (1/2)$, the multivalued linear operator $P(D)^{-1}$ is a subgenerator of an exponentially equicontinuous $(g_\alpha, g_{\gamma+1})$-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$; moreover, since $C^{-1}P(D)C = P(D)$, we have $C^{-1}P(D)^{-1}C = P(D)^{-1}$ and $P(D)^{-1}$ is therefore the integral generator of $(R_\alpha(t))_{t\geqslant 0}$. In our concrete situation, we can employ (432) in order to see that

$$q_\eta(R_\alpha(t)f) \leqslant t^\gamma(1 + t^{\beta-\delta} + t^\beta)q_\eta(f), \quad t \geqslant 0,\ f \in E_l,\ \eta \in \mathbb{N}_0^l.$$

The established conclusions can be applied in many different directions and here we will present only an application of Proposition 3.2.15(ii): Suppose $m \in \mathbb{N} \smallsetminus \{1\}$, $x_0 = x$, $f_0(\cdot) = f(\cdot)$, $P(D)x_j = x_{j-1}$ for $1 \leqslant j \leqslant m$, $P(D)f_j(t) = f_{j-1}(t)$ for $t \geqslant 0$ and $1 \leqslant j \leqslant m$, $f_j \in C([0,\infty) : X)$ for $0 \leqslant j \leqslant m$, and $\alpha = 1/m$. Then the function $v(t) := R_\alpha(t)x + (R_\alpha * C^{-1}f)(t)x$, $t \geqslant 0$ is a unique solution of the following abstract time-fractional equation:

$$\begin{cases} v \in C^1((0,\infty) : X) \cap C([0,\infty) : X), \\ v(t) = P(D)\Big[v_t(t,x) - \sum_{j=1}^{m-1} g_{(j/m)+r}(t)Cx_j \\ \qquad\qquad - \sum_{j=0}^{m-1}(g_{(j/m)+\gamma} * f_j)(t) - g_\gamma(t)Cx\Big], \quad t > 0, \\ v(0) = 0. \end{cases}$$

Furthermore, $v \in C^1([0,\infty) : X)$ provided that $\gamma \geqslant 1$ or $x = 0$.

(ii) It is worth noting that the operator $P(D)$ need not be injective. For example, if $l = 0$, $n = 1$ and $P(x) = -x^2 + ix$, $x \in \mathbb{R}$, then the state space is $X = BUC(\mathbb{R})$ but the operator $P(D)\cdot = \cdot'' + \cdot'$ is not injective since $P(D)f = 0$ for all constant functions $f(\cdot)$; in particular, the operator $P(D)^{-1}$ is not single-valued and $P(D)^{-1} \notin L(X)$. On the other hand, the injectiveness of operator $P(D)$ holds in many concrete situations; for example, we have already shown that the Laplacian Δ in $L^p(\mathbb{R}^n)$ is injective as well as that the assumption $0 \notin P(\mathbb{R}^n)$ implies that the operator

$P(D)$ is injective. The Korteweg–De Vries operator $P(D)\cdot = \cdot''' + \cdot'$ in $L^p(\mathbb{R})$ is injective ($1 \leqslant p < \infty$), as well.

We close the section with the following example:

EXAMPLE 3.10.20. Suppose that $P_1(x) = -|x|^2$ and $P_2(x) = \sum_{|\eta|\leqslant Q} a_\eta x^\eta$ ($x \in \mathbb{R}^n$), $0 \notin P_2(\mathbb{R}^n)$ and (70) holds with $\omega = 0$. By Theorem 3.10.17, there exist a non-negative real number $\gamma \geqslant 0$ and an injective operator $C \in L(X)$ such that the operator $\mathcal{B}$ is a subgenerator of a global polynomially bounded $(g_\alpha, g_{1+\gamma})$-regularized C-resolvent family $(R_\alpha(t))_{t\geqslant 0}$ (cf. also the conclusions from Remark 3.10.18(ii)–(iii)), so that we can analyze the existence and uniqueness of strong (mild) solutions of the following fractional degenerate Cauchy problem of order $\alpha \in (0, 2)$:

$$(PR)_\alpha : \begin{cases} \mathbf{D}_t^\alpha u_{xx}(t,x) = \sum_{|\eta|\leqslant Q} a_\eta D^\eta u(t,x) + f(t,x), & t \geqslant 0,\ x \in \mathbb{R}^n, \\ u(0,x) = C\phi(x);\ u_t(0,x) = C\psi(x) \text{ if } \alpha > 1. \end{cases}$$

Without going into full details, we will only note that in the case $\alpha \in (1, 2)$, the validity of certain conditions on the initial values $\phi(x)$, $\psi(x)$ and the inhomogenity $f(t,x)$ yield that the unique strong solution of $(PR)_\alpha$ is given by

$$u(t,x) := \mathbf{D}_t^{1+\gamma}\left[R_\alpha(t)\phi(x) + \int_0^t R_\alpha(s)\psi(x)\,ds + (g_{\alpha-1} * R_\alpha * f)(t,x)\right],\ \ t \geqslant 0,\ x \in \mathbb{R}^n.$$

For more details, see [**387**, Theorem 13] and [**388**, Lemma 4.1].

3.11. Quasi-asymptotically almost periodic functions and applications

Throughout this section, we assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are two complex Banach spaces. By $\|T\|_{L(X,Y)}$ we denote the norm of a continuous linear mapping $T \in L(X, Y)$. As explained in [**336**], the notion of quasi-asymptotical almost periodicity is very important in the study of qualitative properties of the infinite convolution product

$$\mathbf{F}(t) := \int_{-\infty}^t R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{433}$$

where $f : \mathbb{R} \to X$ is a Weyl-p-almost periodic function satisfying certain extra conditions and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family having a certain growth order at zero and infinity.

The class of q-aap. functions is maybe unique in the existing literature with regards to its invariance under the action of infinite convolution product (433), for the functions defined on $\mathbb{R}$, and its invariance under the action of finite convolution product

$$F(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0, \tag{434}$$

for the functions defined on $[0, \infty)$; all that we need is the uniform integrability of solution operator family $(R(t))_{t>0}$, i.e., the condition $\int_0^\infty \|R(s)\|_{L(X,Y)}\,ds < \infty$

(Proposition 3.11.33, Proposition 3.11.34). Similar statements hold for Stepanov classes of q-aap. functions, where we use a slightly stronger condition

$$\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$$

($1/p+1/q = 1$; see Proposition 3.11.35 and Proposition 3.11.36). It is clear that the results of this section are susceptible to applications to a wide class of inhomogenous abstract Volterra integro-differential equations and inclusions; basically, application is possible at any place where the variation of parameters formula or some of its generalizations plays a role.

Let $I = \mathbb{R}$ or $I = [0, \infty)$ in the sequel. By $C_b(I : X)$, $C_0(I : X)$ and $BUC(I : X)$ we denote the vector spaces consisting of all bounded continuous functions $f : I \to X$, all bounded continuous functions $f : I \to X$ such that $\lim_{|t|\to+\infty} \|f(t)\| = 0$ and all bounded uniformly continuous functions $f : I \to X$, respectively.

3.11.1. Asymptotically almost periodic type functions, asymptotically almost automorphic type functions and their generalizations. Let $f : I \to X$ be continuous. Given $\varepsilon > 0$, we call $\tau > 0$ an ε-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$, $t \in I$. By $\vartheta(f, \varepsilon)$ we denote the set consisted of all ε-periods for $f(\cdot)$. It is said that $f(\cdot)$ is almost periodic (ap.) iff for each $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \varepsilon)$. The space consisted of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$.

The class of asymptotically almost periodic functions was introduced by M. Fréchet in 1941, for the case that $I = [0, \infty)$ (more details about aap. functions with values in Banach spaces can be found in [**100, 144, 240**] and references cited therein). If $I = \mathbb{R}$, there are several non-equivalent notions of an aap. function. Here we follow the approach of C. Zhang [**557**]:

A function $f \in C_b(I : X)$ is said to be asymptotically almost periodic iff for every $\varepsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I of length l contains, at least, one number τ such that $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$ provided $|t|, |t+\tau| \geqslant M$. The space consisting of all aap. functions from I into X is denoted by $AAP(I : X)$. For a function $f \in C_b(I : X)$, the following statements are equivalent (see [**465**] for the case that $I = [0, \infty)$ and [**557**, Theorem 2.6] for the case that $I = \mathbb{R}$):

(i) $f \in AAP(I : X)$.
(ii) There exist uniquely determined functions $g \in AP(I : X)$ and $\phi \in C_0(I : X)$ such that $f = g + \phi$.

Unless stated otherwise, in the sequel we will always assume that $1 \leqslant p < \infty$. Let $l > 0$, and let $f,\ g \in L^p_{loc}(I : X)$. We define the Stepanov 'metric' by

$$D^p_{S_l}[f(\cdot), g(\cdot)] := \sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t) - g(t)\|^p dt \right]^{1/p}.$$

The Stepanov 'norm' of $f(\cdot)$ is defined by $\|f\|_{S^p_l} := D^p_{S_l}[f(\cdot),0]$. It is said that a function $f \in L^p_{loc}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff $\|f\|_{S^p} := \sup_{t\in I}(\int_t^{t+1}\|f(s)\|^p\,ds)^{1/p} < \infty$. The space $L^p_S(I : X)$ consisted of all S^p-bounded functions becomes a Banach space equipped with the above norm. We say that a function $f \in L^p_S(I : X)$ is Stepanov p-almost periodic, S^p-ap. shortly, iff the function $\hat{f}\colon I \to L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in I$, $s \in [0,1]$ is ap.. It is said that $f \in L^p_S(I : X)$ is asymptotically Stepanov p-almost periodic, S^p-aap. shortly, iff $\hat{f}\colon I \to L^p([0,1] : X)$ is aap.. By $APS^p(I : X)$ and $AAPS^p(I : X)$ we denote the spaces consisted of all S^p-ap. functions $I \mapsto X$ and S^p-aap. functions $I \mapsto X$, respectively.

Let $1 \leqslant p < \infty$, $l > 0$, and $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. Recall that, for every two numbers $l_1, l_2 > 0$, there exist two positive real constants $k_1, k_2 > 0$ independent of f, g, such that

$$k_1 D^p_{S_{l_1}}[f(\cdot),g(\cdot)] \leqslant D^p_{S_{l_2}}[f(\cdot),g(\cdot)] \leqslant k_2 D^p_{S_{l_1}}[f(\cdot),g(\cdot)].$$

The symbol $S^p_0([0,\infty) : X)$ stands for the vector space consisting of all functions $q \in L^p_{loc}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$. If $1 \leqslant p < q < \infty$ and $f(\cdot)$ is (asymptotically) Stepanov q-almost periodic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost periodic. Therefore, the (asymptotic) Stepanov p-almost periodicity of $f(\cdot)$ for some $p \in [1,\infty)$ implies the (asymptotical) Stepanov p-almost periodicity of $f(\cdot)$. It is a well-known fact that if $f(\cdot)$ is an almost periodic (respectively, asymptotically almost periodic) function then $f(\cdot)$ is also S^p-almost periodic (respectively, asymptotically S^p-almost periodic) for $1 \leqslant p < \infty$. And in general, the converse statement is false.

The notion of an (equi-)Weyl almost periodic function is given as follows (see **[22, 293]** and references cited therein for more details on the subject):

Definition 3.11.1. Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I : X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl-p-almost periodic, $f \in e-W^p_{ap}(I : X)$ for short, iff for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\|f(t+\tau)-f(t)\|^p dt\right]^{1/p} \leqslant \varepsilon, \text{ i.e., } D^p_{S_l}[f(\cdot+\tau),f(\cdot)] \leqslant \varepsilon.$$

(ii) We say that the function $f(\cdot)$ is Weyl-p-almost periodic, $f \in W^p_{ap}(I : X)$ for short, iff for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\|f(t+\tau)-f(t)\|^p\,dt\right]^{1/p} \leqslant \varepsilon, \text{ i.e., } \lim_{l\to\infty} D^p_{S_l}[f(\cdot+\tau),f(\cdot)] \leqslant \varepsilon.$$

We also need the definition of an asymptotically almost automorphic function defined on the interval I. For beginning, let us recall that a continuous function $f\colon \mathbb{R} \to X$ is said to be almost automorphic (aa., for short) iff for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g\colon \mathbb{R} \to X$ such that $\lim_{n\to\infty} f(t+a_n) = g(t)$ and $\lim_{n\to\infty} g(t-a_n) = f(t)$, pointwise for $t \in \mathbb{R}$. Any aa. function $f(\cdot)$ needs to be bounded and the following supremum formula holds (see e.g. [**293**, Lemma 3.9.9]):

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} \|f(x)\| = \sup_{x\geqslant a} \|f(x)\| \quad \text{for any number} \quad a \in \mathbb{R}.$$

We will use the following notion (see also [**155**, Definition 2.3]):

Definition 3.11.2. (i) A bounded continuous function $f\colon \mathbb{R} \to X$ is said to be asymptotically almost automorphic iff there exist two functions $h \in AA(\mathbb{R} : X)$ and $q \in C_0(\mathbb{R} : X)$ such that $f = h + q$ on $\mathbb{R}$.

(ii) A bounded continuous function $f\colon [0,\infty) \to X$ is said to be asymptotically almost automorphic iff there exist two functions $h \in AA(\mathbb{R} : X)$ and $q \in C_0([0,\infty) : X)$ such that $f = h + q$ on $[0,\infty)$.

It is well known that any (asymptotically) almost periodic function defined on the interval I is (asymptotically) almost automorphic as well as that the converse statement does not hold in general. Following G. M. N'Guérékata and A. Pankov [**242**], a function $f \in L^p_{loc}(\mathbb{R} : X)$ is called Stepanov p-almost automorphic, S^p-aa. for short, iff for asymptotically every real sequence (a_n), there exist a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \|f(a_{n_k}+s) - g(s)\|^p ds = 0 \text{ and } \lim_{k\to\infty} \int_t^{t+1} \|g(s-a_{n_k}) - f(s)\|^p ds = 0$$

for each $t \in \mathbb{R}$. By $AAS^p(\mathbb{R} : X)$ we denote the vector space consisting of all S^p-aa. functions from $\mathbb{R}$ to X.

The following definition seems to be new in case $I = \mathbb{R}$:

Definition 3.11.3. (i) An S^p-bounded function $f : \mathbb{R} \to X$ is said to be asymptotically Stepanov p-almost automorphic, S^p-aaa. for short, iff there exist two functions $h \in AAS^p(\mathbb{R} : X)$ and an S^p-bounded function $q\colon \mathbb{R} \to X$ such that $\hat{q} \in C_0(\mathbb{R} : L^p([0,1] : X))$ and $f = h + q$ a.e. on $\mathbb{R}$.

(ii) An S^p-bounded function $f\colon [0,\infty) \to X$ is said to be asymptotically Stepanov p-almost automorphic iff there exist two functions $h \in AAS^p(\mathbb{R} : X)$ and an S^p-bounded function $q\colon [0,\infty) \to X$ such that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$ and $f = h + q$ a.e. on $[0,\infty)$.

By $AAAS^p(I : X)$ we denote the vector space consisting of all asymptotically S^p-almost automorphic functions $I \to X$.

It can be easily verified that the (asymptotical) S^p-almost automorphy of $f(\cdot)$ implies the (asymptotical) almost automorphy of the mapping $\hat{f}\colon I \to L^p([0,1] : X)$ defined above. Any (asymptotically) S^p-almost periodic function $f\colon I \mapsto X$ has to be (asymptotically) S^p-almost automorphic, while the converse statement does not hold in general.

Denote by $C_\omega(I : X)$ the space consisting of all continuous ω-periodic functions $g\colon I \to X$. The following two definitions as well Definition 3.11.6 seem to be new in case $I = \mathbb{R}$, likewise (see H. R. Henríquez et al. [**251**], H. R. Henríquez [**250**], W. Dimbour, S. M. Manou-Abi [**154**] for case $I = [0,\infty)$):

DEFINITION 3.11.4. Let $\omega \in I$. Then we say that a bounded continuous function $f\colon I \to X$ is S-asymptotically ω-periodic iff $\lim_{|t|\to\infty} \|f(t+\omega) - f(t)\| = 0$. Denote by $SAP_\omega(I : X)$ the space consisting of all such functions.

DEFINITION 3.11.5. Let $\omega \in I$. A bounded continuous function $f\colon I \to X$ is said to be asymptotically ω-almost periodic iff there exist a function $g \in C_\omega(I : X)$ and a function $q \in C_0(I : X)$ such that $f(t) = g(t) + q(t)$ for all $t \in I$. Denote by $AP_\omega(I : X)$ the vector space consisting of all such functions.

It is straightforward to see that $AP_\omega(I : X) \subseteq SAP_\omega(I : X)$ and the inclusion is strict. We will also work with the class of Stepanov S^p-asymptotically ω-periodic functions.

DEFINITION 3.11.6. Let $\omega \in I$. A Stepanov p-bounded function $f(\cdot)$ is said to be Stepanov p-asymptotically ω-periodic iff

$$\lim_{|t|\to\infty} \int_t^{t+1} \|f(s+\omega) - f(s)\|^p ds = 0.$$

Denote by $S^pSAP_\omega(I : X)$ the space consisting of all such functions.

We have that $SAP_\omega(I : X) \subseteq S^pSAP_\omega(I : X)$ and the inclusion is strict.

3.11.2. Evolution systems and Green's functions. The following definition is well known in the existing literature:

DEFINITION 3.11.7. A family $\{U(t,s) : t \geqslant s,\ t,\ s \in \mathbb{R}\} \subseteq L(X)$ is said to be an evolution system iff the following statements hold:

(a) $U(s,s) = I$, $U(t,s) = U(t,r)U(r,s)$ for $t \geqslant r \geqslant s$ and $t, r, s \in \mathbb{R}$,
(b) $\{(\tau,s) \in \mathbb{R}^2 : \tau > s\} \ni (t,s) \mapsto U(t,s)x$ is continuous for any fixed element $x \in X$.

In the sequel, it will be always assumed that the family $A(\cdot)$ satisfies the following condition introduced by P. Acquistapace and B. Terreni in [**5**] (with $\omega = 0$):

(H1): There is a number $\omega \geqslant 0$ such that the family of closed linear operators $A(t)$, $t \in \mathbb{R}$ on X satisfies $\overline{\Sigma_\phi} \subseteq \rho(A(t) - \omega)$,

$$\|R(\lambda : A(t) - \omega)\| = O((1+|\lambda|)^{-1}), \quad t \in \mathbb{R},\ \lambda \in \overline{\Sigma_\phi}, \text{ and}$$

$$\|(A(t) - \omega)R(\lambda : A(t) - \omega)[R(\omega : A(t)) - R(\omega : A(s))]\| = O(|t-s|^\mu |\lambda|^{-\nu}),$$

for any $t, s \in \mathbb{R}$, $\lambda \in \overline{\Sigma_\phi}$, where $\phi \in (\pi/2, \pi)$, $0 < \mu,\ \nu \leqslant 1$ and $\mu + \nu > 1$.

Then we know that there exists an evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$, satisfying that $\|U(t,s)\| = O(1)$ for $t \geqslant s$, as well as a great deal of other conditions [**5**]. Besides (H1), we will also assume that the following condition holds:

(H2): The evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$ is hyperbolic (or, equivalently, has exponential dichotomy), i.e., there exist a family of projections $(P(t))_{t\in\mathbb{R}} \subseteq L(X)$, being uniformly bounded and strongly continuous in t, and constants $M', \omega > 0$ such that (a)-(c) holds with $Q := I - P$ and $Q(\cdot) := I - P(\cdot)$, where I stands for the identity operator on X and:

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geqslant s$,

(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geqslant s$ (here we define $U_Q(s,t) = U_Q(t,s)^{-1}$),

(c) $\|U(t,s)P(s)\| \leqslant M'e^{-\omega(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leqslant M'e^{-\omega(t-s)}$ for all $t \geqslant s$.

It is said that $U(\cdot,\cdot)$ is exponentially stable iff the choice $P(t) = I$ for all $t \in \mathbb{R}$ can be made; $U(\cdot,\cdot)$ is said to be (bounded) exponentially bounded iff there exist two finite real constants $M > 0$ and $(\omega = 0)$ $\omega \in \mathbb{R}$ such that $\|U(t,s)P(s)\| \leqslant Me^{-\omega(t-s)}$ for all $t \geqslant s$. The associated Green's function $\Gamma(\cdot,\cdot)$ is defined through

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geqslant s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}. \end{cases}$$

Let M' be the constant appearing in (H2). Then

$$\|\Gamma(t,s)\| \leqslant M'e^{-\omega|t-s|}, \quad t,\ s \in \mathbb{R} \tag{435}$$

and the function

$$u(t) := \int_{-\infty}^{+\infty} \Gamma(t,s)f(s)ds, \quad t \in \mathbb{R} \tag{436}$$

is a unique bounded continuous function on $\mathbb{R}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \quad t \geqslant s;$$

cf. [**407**]. In the sequel, it will be said that $u(\cdot)$ is a mild solution of the abstract Cauchy problem (449).

Let $f\colon [0,\infty) \to X$ be continuous. By a mild solution of the abstract Cauchy problem (450) we mean the function

$$u(t) := U(t,0)x + \int_0^t U(t,s)f(s)ds, \quad t \geqslant 0. \tag{437}$$

For more details on the subject, we refer the reader to [**407**, Section 5].

We will also consider the following semilinear Cauchy problems:

$$u'(t) = A(t)u(t) + F(t,u(t)), \quad t \in \mathbb{R} \tag{438}$$

and

$$u'(t) = A(t)u(t) + F(t,u(t)), \quad t > 0;\ u(0) = x. \tag{439}$$

Definition 3.11.8. (i) A function $u \in C_b(\mathbb{R} : X)$ is said to be a mild solution of (438) iff

$$u(t) = \int_{-\infty}^{+\infty} \Gamma(t,s)F(s,u(s))ds, \quad t \in \mathbb{R}.$$

(ii) A function $u \in C_b([0,\infty) : X)$ is said to be a mild solution of (439) iff

$$u(t) = U(t,0)x + \int_0^t U(t,s)F(s,u(s))ds, \quad t \geqslant 0.$$

3.11.3. Quasi-asymptotically almost periodic functions and their generalizations. We start by recalling the following definition [**336**]:

Definition 3.11.9. Suppose that $I = [0, \infty)$ or $I = \mathbb{R}$. Then we say that a bounded continuous function $f : I \to X$ is quasi-asymptotically almost periodic iff for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t+\tau) - f(t)\| \leqslant \varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau). \tag{440}$$

Denote by $Q - AAP(I : X)$ the set consisting of all quasi-asymptotically almost periodic functions from I into X.

In order to avoid unnecessary repeating, we will use the shorthand

(S): "there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number".

Remark 3.11.10. It is not relevant whether we will write (440) or

$$\|f(t+\tau) - f(t)\| \leqslant \varepsilon, \text{ provided } t \in I,\ |t| \geqslant M(\varepsilon, \tau) \text{ and } |t+\tau| \geqslant M(\varepsilon, \tau).$$

Using this observation, it can be easily seen that the class of aap. functions is contained in the class of q-aap. functions (the number M depends only on ε and not on τ for aap. functions). The converse statement is not true, however: Let $I = \mathbb{R}$ and let $f(\cdot)$ be any bounded scalar-valued continuous function such that $f(t) = 1$ for all $t \geqslant 0$ and $f(t) = 0$ for all $t \leqslant -1$. Then $f(\cdot)$ is q-aap., not aap. and not equi-Weyl-p-ap. for any $p \in [1, \infty)$ [**22**,**336**]. Applying Theorem 3.11.13(i) below we can see that $f(\cdot)$ is not aaa., as well.

We continue by providing an illustrative example and one more remark.

Example 3.11.11. Suppose that $f \in C^1(I : X) \cap C_b(I : X)$ and $f' \in C_0(I : X)$. Then $f \in Q - AAP(I : X)$. In order to see this, it suffices to apply the Lagrange mean value theorem as well as to choose, in Definition 3.11.9, $L(\varepsilon) > 0$ arbitrarily and any $\tau \neq 0$ from an arbitrary interval I' of length $L(\varepsilon)$. Then, for this $\varepsilon > 0$ and $\tau \in I'$, there exists a sufficiently large $M(\varepsilon, \tau) > 0$ such that $[t, t+\tau] \subseteq \{s \in I : |s| \geqslant M_0(\varepsilon, \tau)\}$ for $|t| \geqslant M(\varepsilon, \tau)$, where $M_0(\varepsilon, \tau)$ is already chosen so that $\|f'(\xi)\| \leqslant \varepsilon/|\tau|$ for $|\xi| \geqslant M_0(\varepsilon, \tau)$; then we have

$$\|f(t+\tau) - f(t)\| \leqslant |\tau| \sup_{\xi \in [t, t+\tau]} \|f'(\xi)\| \leqslant \varepsilon, \quad |t| \geqslant M(\varepsilon, \tau).$$

It is worth noticing that there exists a function $f(\cdot)$ that is not aap. and satisfies the above properties; a typical example is given by $f(t) := \sin(\ln(1+t))$, $t \geqslant 0$ (see also [**466**, Example 4.1, Theorem 4.2]).

Remark 3.11.12. In our joint research paper [**341**] with D. Velinov, we have recently introduced and analyzed the class of (asymptotically) almost anti-periodic functions. The notion of quasi-asymptotically almost anti-periodicity (q-aanp., for short) can be introduced in the following way: A bounded continuous $f : I \to X$ is called quasi-asymptotically almost anti-periodic iff for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t+\tau) + f(t)\| \leqslant \varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau). \tag{441}$$

Suppose that $t, t+\tau \in I$ as well as that $|t| \geqslant M(\varepsilon, \tau) + |\tau|$. Then $|t+\tau| \geqslant M(\varepsilon, \tau)$ and applying (441) twice, we get that

$$\begin{aligned} \|f(t+2\tau) - f(t)\| &= \|[f(t+2\tau) + f(t+\tau)] - [f(t+\tau) + f(t)]\| \\ &\leqslant \|f(t+2\tau) + f(t+\tau)\| + \|f(t+\tau) + f(t)\| \leqslant 2\varepsilon. \end{aligned}$$

Hence, any q-aanp. function is automatically q-aap.. Further analysis of q-aanp. functions and their Stepanov generalizations are without scope of this book.

The space $C_b(I : X) \setminus Q - AAP(I : X)$ is sufficiently large; it is clearly non-empty because it is very plainly to construct an example of an infinite-differentiable bounded function $f : I \to \mathbb{C}$ such that for each number $\tau \in I$ there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in I with the properties that $\lim_{n \to \infty} |t_n| = \infty$ and $|f(t_n + \tau) - f(t_n)| \geqslant 1$ for all $n \in \mathbb{N}$. Furthermore, we have the following:

THEOREM 3.11.13. (i) *$AAA(I : X) \cap Q - AAP(I : X) = AAP(I : X)$ and $[AAA(I : X) \setminus AAP(I : X)] \cap Q - AAP(I : X) = \emptyset$.*
(ii) *$AA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X) = AP(\mathbb{R} : X)$.*

PROOF. For the sake of brevity, we will consider only the case that $I = \mathbb{R}$. It is clear that $AAP(\mathbb{R} : X) \subseteq AAA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X)$. To prove the converse inclusion, suppose that $f \in AAA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X)$. Then there exist two functions $h \in AA(\mathbb{R} : X)$ and $q \in C_0(\mathbb{R} : X)$ such that $f = h + q$ on $\mathbb{R}$ and for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon, \tau) > 0$ such that

$$\|[h(t+\tau) - h(t)] + [q(t+\tau) - q(t)]\| \leqslant \varepsilon, \text{ provided } t \in \mathbb{R} \text{ and } |t| \geqslant M(\varepsilon, \tau). \tag{442}$$

Fix a number $\varepsilon > 0$ and suppose that the real number τ satisfies (442) for $|t| \geqslant M(\varepsilon, \tau)$. Since $q \in C_0(\mathbb{R} : X)$, we have that there exists a finite number $M_1(\varepsilon, \tau) \geqslant M(\varepsilon, \tau)$ such that

$$\|h(t+\tau) - h(t)\| \leqslant 2\varepsilon, \text{ provided } t \in \mathbb{R} \text{ and } |t| \geqslant M_1(\varepsilon, \tau). \tag{443}$$

Define the function $H : \mathbb{R} \to X$ by $H(t) := h(t+\tau) - h(t)$, $t \in \mathbb{R}$. Since the space $AA(\mathbb{R} : X)$ is translation invariant, we have $H \in AA(\mathbb{R} : X)$. Applying supremum formula and (443), we get

$$\sup_{t \in \mathbb{R}} \|H(t)\| = \sup_{t \geqslant M_1(\varepsilon, \tau)} \|H(t)\| = \sup_{t \geqslant M_1(\varepsilon, \tau)} \|h(t+\tau) - h(t)\| \leqslant 2\varepsilon.$$

Hence, $\|h(t+\tau) - h(t)\| \leqslant 2\varepsilon$ for all $t \in \mathbb{R}$ and $h(\cdot)$ is ap. by definition. Hence, $AAP(\mathbb{R} : X) = AAA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X)$, which immediately implies the second equality in (i). The proof of (ii) follows from the above arguments, as well. $\square$

It is expected that the range of a function $f \in Q - AAA(I : X) \cap BUC(I : X)$ need not be relatively compact, as in the case of aap. functions. In the following example, we will explain this fact in case $I = [0, \infty)$:

EXAMPLE 3.11.14. Let $X := c_0(\mathbb{C})$. Although the final conclusions presented here holds for the function $f(\cdot)$ considered in [**251**, Example 3.1], we will prove that the range of function

$$f(t) := \Big(\frac{4n^2t^2}{(t^2+n^2)^2} \Big)_{n\in\mathbb{N}}, \quad t \geqslant 0$$

is not relatively compact in X. Using a similar analysis as in the afore-mentioned example, we get that the function $f(\cdot)$ is bounded and uniformly continuous with the estimate $\|f(t+s)-f(t)\| \leqslant 8s$, $t, s > 0$ holding true. Further on, for each $t > 0$ and $\tau \geqslant 0$, we have:

$$\|f(t+\tau)-f(t)\| \leqslant \sup_{n\in\mathbb{N}} \frac{4n^2[(t+\tau)^2+\tau^2]}{(t^2+n^2)^2((t+\tau)^2+n^2)^2} \leqslant t^{-4} + 4\frac{\tau^2}{t^2}, \quad t > 0,\ \tau \geqslant 0.$$

By [**251**, Remark 3.1, Proposition 3.3], it readily follows that the range of $f(\cdot)$ is not relatively compact as well as that there is no number $\tau > 0$ such that $f(\cdot)$ is τ-normal on compact subsets; see [**251**, Definition 3.2] for the notion.

Now we will prove that the existence of a number $\omega \in I$ such that $f\colon I \to X$ is S-asymptotically ω-periodic implies that $f(\cdot)$ is qaap.:

PROPOSITION 3.11.15. *We have* $SAP_\omega(I : X) \subseteq Q - AAP(I : X)$.

PROOF. For $\varepsilon > 0$ given in advance, we can take $L(\varepsilon) = 2\omega$. Then any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains a number $\tau = n\omega$ for some $n \in \mathbb{N}$. For this n and ε, there exists a finite number $M(\varepsilon, n) > 0$ such that $\|f(t+\omega)-f(t)\| \leqslant \varepsilon/n\omega$ for $|t| \geqslant M(\varepsilon, n)$. Then the final conclusion follows from the estimates

$$\|f(t+n\omega)-f(t)\| \leqslant \sum_{k=0}^{n-1} \|f(t+(k+1)\omega)-f(t+k\omega)\| \leqslant \sum_{k=0}^{n-1} \frac{\varepsilon}{n\omega} = \varepsilon/\omega,$$

provided $|t| \geqslant M(\varepsilon, n) + n\omega$. □

In [**548**, Example 17], R. Xie and C. Zhang have constructed an example of a function $f \in SAP_2([0,\infty) : X)$ that is not uniformly continuous. By the above proposition, the function $f(\cdot)$ is q-aap. and not uniformly continuous.

The following simple proposition, already known in the case that $I = [0,\infty)$, can be deduced by using the arguments contained in the proof of [**78**, Proposition 3.6]. An alternative proof can be given by using Theorem 3.11.13, Proposition 3.11.15 and an easy reformulation of [**251**, Lemma 3.1] in case $I = \mathbb{R}$:

PROPOSITION 3.11.16. *Let* $\omega \in I$.

(i) *Suppose that* $f \in SAP_\omega(I : X) \cap AAA(I : X)$. *Then* $f \in AP_\omega(I : X)$.
(ii) *Suppose that* $f \in SAP_\omega(I : X) \cap AA(I : X)$. *Then* $f \in C_\omega(I : X)$.

Now we will introduce the Stepanov generalization of q-aap. functions:

DEFINITION 3.11.17. Let $f \in L^p_S(I : X)$. Then it is said $f(\cdot)$ is Stepanov p-quasi-asymptotically almost periodic (S^p-qaap., for short) iff for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon,\tau)>0$ such that

$$\int_t^{t+1}\|f(s+\tau)-f(s)\|^p ds \leqslant \varepsilon^p, \text{ provided } t\in I \text{ and } |t|\geqslant M(\varepsilon,\tau). \tag{444}$$

Denote by $S^pQ-AAP(I:X)$ the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from I into X.

Using only definition, it readily follows that $Q-AAP(I:X)\subseteq S^pQ-AAP(I:X)$; it is clear that this inclusion can be strict since the function $f(t):=\chi_{[-1,\infty)}(t)$, $t\in\mathbb{R}$ is in class $S^pQ-AAP(\mathbb{R}:X)$ but not in class $Q-AAP(\mathbb{R}:X)$ because $f(\cdot)$ is not continuous. Furthermore, it follows immediately from definition that any S^p-aap. function is S^p-qaap. so that $AAPS^p(I:X)\subseteq S^pQ-AAP(I:X)$; this inclusion can be also strict since the continuous function $f(\cdot)$ from Remark 3.11.10 is not S^p-aap.. Furthermore, if $1\leqslant p<p'<\infty$, then $S^{p'}Q-AAP(I:X)\subseteq S^pQ-AAP(I:X)$ and for any function $f\in L^p_S(I:X)$, we have that $f(\cdot)$ is S^pq-aap. iff the function $\hat{f}\colon I\to L^p([0,1]:X)$ is q-aap.. Using this fact and Theorem 3.11.13, we can simply verify the validity of following result:

THEOREM 3.11.18. (i) *$S^pAAA(I:X)\cap S^pQ-AAP(I:X)=S^pAAP(I:X)$ and $[S^pAAA(I:X)\setminus S^pAAP(I:X)]\cap S^pQ-AAP(I:X)=\emptyset$.*
(ii) *$S^pAA(\mathbb{R}:X)\cap S^pQ-AAP(\mathbb{R}:X)=S^pAP(\mathbb{R}:X)$.*

The proof of following result is very similar to that of Proposition 3.11.15; therefore, it is omitted:

PROPOSITION 3.11.19. *We have $S^pSAP_\omega(I:X)\subseteq S^pQ-AAP(I:X)$.*

After introduction of Definition 3.11.17, it seems reasonable to ask whether we can analyze Weyl and Besicovitch generalizations of q-aap. functions. The following result says that the space $S^pQ-AAP(I:X)$ is contained in $W^p_{ap}(I:X)$ and that the above question is uninteresting (we have already seen that $S^pQ-AAP(I:X)\nsubseteq e-W^p_{ap}(I:X)$):

PROPOSITION 3.11.20. *We have $S^pQ-AAP(I:X)\subseteq W^p_{ap}(I:X)$.*

PROOF. Let $\varepsilon>0$ be given. Then (S) holds with a number $M(\varepsilon,\tau)>0$ such that (444) is satisfied. We need to estimate the term

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\|f(t+\tau)-f(t)\|^p dt\right]^{1/p} \tag{445}$$

as $l\to+\infty$. There exist four possibilities:

1. $|x|\geqslant M(\varepsilon,\tau)$ and $|x+l|\geqslant M(\varepsilon,\tau)$.
2. $|x|\geqslant M(\varepsilon,\tau)$ and $|x+l|\leqslant M(\varepsilon,\tau)$.
3. $|x|\leqslant M(\varepsilon,\tau)$ and $|x+l|\geqslant M(\varepsilon,\tau)$.
4. $|x|\leqslant M(\varepsilon,\tau)$ and $|x+l|\leqslant M(\varepsilon,\tau)$.

Let us consider the first case. For $x\geqslant 0$, we have

$$\frac{1}{l}\int_x^{x+l}\|f(t+\tau)-f(t)\|^p dt$$

$$\leqslant \frac{1}{l}\Biggl(\int_x^{x+1}\|f(t+\tau)-f(t)\|^p dt+\cdots+\int_{x+\lfloor l\rfloor}^{x+l}\|f(t+\tau)-f(t)\|^p dt\Biggr)\leqslant \frac{1}{l}l\varepsilon^p.$$

If $I=\mathbb{R}$ and $x\leqslant 0$, then $x\leqslant -M(\varepsilon,\tau)$, $x+l\geqslant M(\varepsilon,\tau)$ and arguing as above we get

$$\begin{aligned}
\frac{1}{l}\int_x^{x+l}&\|f(t+\tau)-f(t)\|^p dt\\
&\leqslant \frac{1}{l}\int_x^{-M(\varepsilon,\tau)}\|f(t+\tau)-f(t)\|^p dt\\
&+\frac{1}{l}\int_{-M(\varepsilon,\tau)}^{M(\varepsilon,\tau)}\|f(t+\tau)-f(t)\|^p dt+\frac{1}{l}\int_{M(\varepsilon,\tau)}^{x+l}\|f(t+\tau)-f(t)\|^p dt\\
&\leqslant \frac{1}{l}\int_x^{-M(\varepsilon,\tau)}\|f(t+\tau)-f(t)\|^p dt\\
&+\frac{2^{p-1}(M(\varepsilon,\tau)+2)}{l}\|f\|_{S^p}^p+\frac{1}{l}\int_{M(\varepsilon,\tau)}^{l-M(\varepsilon,\tau)}\|f(t+\tau)-f(t)\|^p dt\\
&\leqslant \frac{\varepsilon^p}{l}(-M(\varepsilon,\tau)-x)+\frac{2^{p-1}(M(\varepsilon,\tau)+2)}{l}\|f\|_{S^p}^p+\frac{\varepsilon^p}{l}(l-2M(\varepsilon,\tau))\\
&\leqslant 2\varepsilon^p+\frac{2^{p-1}(M(\varepsilon,\tau)+2)}{l}\|f\|_{S^p}^p.
\end{aligned}$$

This implies the existence of a sufficiently large number $l(\varepsilon,\tau)>0$ such that the term in (445) is not greater than ε for any $l\geqslant l(\varepsilon,\tau)$. The analysis of cases 2.–4. is analogous and therefore omitted. □

Proposition 3.11.20 is motivated by the old results of A. S. Kovanko [**357**, Théorème I, Théorème II], where the notion of asymptotical almost periodicity has been taken in a slightly different manner. It is also worth noting that the inclusion $S^pQ-AAP(I:X)\subseteq W^p_{ap}(I:X)$ is strict because the space $W^p_{ap}(I:X)$ contains certain Stepanov unbounded functions (see e.g. [**22**, Example 4.28] with $I=\mathbb{R}$ and $p=1$).

Further on, arguing as in the proofs of structural results of [**74**, pp. 3–4], we may deduce the following:

THEOREM 3.11.21. *Let $f\colon I\to X$ be q-aap. (S^p-qaap.). Then we have:*

- (i) *$cf(\cdot)$ is q-aap. (S^p-qaap.) for any $c\in\mathbb{C}$.*
- (ii) *If $X=\mathbb{C}$ and $\inf_{x\in I}|f(x)|=m>0$, then $1/f(\cdot)$ is q-aap. (S^p-qaap.).*
- (iii) *If $(g_n\colon I\to X)_{n\in\mathbb{N}}$ is a sequence of q-aap. functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g\colon I\to X$, then $g(\cdot)$ is q-aap..*
- (iv) *If $(g_n\colon I\to X)_{n\in\mathbb{N}}$ is a sequence of S^p-qaap. functions and $(g_n)_{n\in\mathbb{N}}$ converges to a function $g\colon I\to X$ in the space $L^p_S(I:X)$, then $g(\cdot)$ is S^p-qaap..*
- (v) *The functions $f(\cdot+a)$ and $f(b\,\cdot)$ are likewise q-aap. (S^p-qaap.), where $a\in I$ and $b\in I\setminus\{0\}$.*

Concerning the pointwise products of (Stepanov) scalar-valued q-aap. functions and (Stepanov) vector-valued q-aap. functions, the following classes play an important role:

DEFINITION 3.11.22. By $Q_h - AAP(I : X)$ ($S^pQ_h - AAP(I : X)$) we denote the class consisting of all q-aaa. (S^p-qaap.) $f\colon I \to X$ satisfying that for each $\varepsilon > 0$ and $\tau \in I$ there exists a finite number $M(\varepsilon, \tau) > 0$ such that (440) ((444)) holds true.

The functions from Example 3.11.11 belong to the class $Q_h - AAP(I : X)$. It is also worth noting that the class $S^pQ_h - AAP(I : X)$ contains all functions that are S-asymptotically ω-periodic in the Stepanov sense for any number $\omega > 0$. Let $f\colon I \to X$ and $g\colon I \to \mathbb{C}$ be given. Using the elementary definitions and inequality

$$\|fg(t+\tau) - fg(t)\| \leqslant |g(t+\tau)|\|f(t+\tau) - f(t)\| + \|f(t)\||g(t+\tau) - g(t)|, \quad t, \tau \in I,$$

it readily follows the validity of following proposition:

PROPOSITION 3.11.23. *Let the functions $f\colon I \to X$ and $g : I \to \mathbb{C}$ be bounded. If $f \in Q_h - AAP(I : X)$ and $g \in Q - AAP(I : \mathbb{C})$ ($f \in S^pQ_h - AAP(I : X)$ and $g \in S^pQ - AAP(I : \mathbb{C})$) or $f \in Q - AAP(I : X)$ and $g \in Q_h - AAP(I : \mathbb{C})$ ($f \in S^pQ - AAP(I : X)$ and $g \in S^pQ_h - AAP(I : \mathbb{C})$), then we have $fg \in Q - AAP(I : X)$ ($fg \in S^pQ - AAP(I : X)$).*

The conclusion established in the above proposition cannot be deduced if the functions $f(\cdot)$ and $g(\cdot)$ belong to the classes of qaap. functions or Stepanov qaap. functions, as the next instructive examples show:

EXAMPLE 3.11.24. We have that $AP(\mathbb{R} : \mathbb{C}) \cdot SAP_2(\mathbb{R} : \mathbb{C})$ is not a subset of $Q - AAP(\mathbb{R} : \mathbb{C})$ and, in particular, $Q - AAP(\mathbb{R} : \mathbb{C}) \cdot Q - AAP(\mathbb{R} : \mathbb{C})$ is not a subset of $Q - AAP(\mathbb{R} : \mathbb{C})$. A typical example of function belonging to the space $[AP(\mathbb{R} : \mathbb{C}) \cdot SAP_2(\mathbb{R} : \mathbb{C})] \cap S^pAAP(\mathbb{R} : \mathbb{C})$ but not to the space $Q - AAP(\mathbb{R} : \mathbb{C})$ is the function $\cos(\sqrt{2}\pi \cdot)f(\cdot)$, which can be verified as for the function $fg(\cdot)$ considered below, but much simpler.

Assume that $\alpha, \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. Then the functions

$$g_0(t) = \sin\Big(\frac{1}{2 + \cos \alpha t + \cos \beta t}\Big), \quad t \in \mathbb{R}$$

and

$$g(t) = \cos\Big(\frac{1}{2 + \cos \alpha t + \cos \beta t}\Big), \quad t \in \mathbb{R}$$

are Stepanov p-ap. but not ap.. These functions are bounded continuous, not uniformly continuous and cannot be q-aap. because they are also aa. (see Theorem 3.11.13 and [**293**] for more details). Consider now the case that $\alpha = \pi$ and $\beta = \sqrt{2}\pi$ for function $g(\cdot)$. Let the function $f(\cdot)$ be defined on the real line by zero outside the non-negative real axis and by $f(t) := f_{\overline{\{1/n+1\}}}(t)$, $t \geqslant 0$, where the function $f_{\overline{\{1/n+1\}}}(\cdot)$ has the same meaning in [**548**, Example 17]. That is, we define $f_{\overline{\{1/n+1\}}}(\cdot)$ by $f_{\overline{\{1/n+1\}}}(t) := 0$ for $t \in \{0, 2, 2n + 1 - \frac{1}{n+1}, 2n + 1 + \frac{1}{n+1} : n \in \mathbb{N}\}$,

$f_{\overline{\{1/n+1\}}}(t) := 1$ for $t \in 2\mathbb{N}+1$ and linearly outside. Assume that for each $\varepsilon > 0$ (S) holds with $I = [0, \infty)$ and a number $M(\varepsilon, \tau) > 0$ satisfying (440). If $\tau \notin 2\mathbb{N}$, then there exist two integers $k \in \mathbb{N}_0$, $n_0 \in \mathbb{N}$ and a real number $d \in (0, 2)$ such that $\tau = 2k + d$ and $d > 2/1 + n$ for all $n \geqslant n_0$. Take now any number $s = 2n + 1$, where $n \in \mathbb{N}$ is chosen so that $2n + 1 \geqslant M(\varepsilon, \tau)$ and

$$\Big|\cos\Big(\frac{1}{1+\cos\sqrt{2}(2n+1)\pi}\Big)\Big| > \frac{4\varepsilon}{3}. \tag{446}$$

The existence of such a number can be easily shown. Then $f(2n+1) \geqslant 3/4$ and $2n + 1 + \tau \in [2(n+k) + \frac{1}{n+k+1}, 2(n+k+1) + 1 - \frac{1}{n+k+1}]$ due to the inequality $d > 2/1 + n$. In combination with (446), the above yields $fg(2n + 1 + \tau) = 0$ and $|fg(2n + 1)| > \varepsilon$, which is a contradiction since (440) holds with $t = 2n + 1$. If $\tau = 2k$ for some $k \in \mathbb{N}$, then there exists a sufficiently large $s_0(\varepsilon) > 0$ such that

$$\begin{aligned} |fg(s+2k) - fg(s)| &\geqslant |f(s+2k)||g(s+2k) - g(s)| - |f(s+2k) - f(s)| \\ &\geqslant |f(s+2k)||g(s+2k) - g(s)| - \frac{\varepsilon}{2},\ s \geqslant s_0(\varepsilon). \end{aligned}$$

If $n \in \mathbb{N}$ is arbitrary and $s = 2n + 1 \geqslant s_0(\varepsilon)$, the above estimate yields

$$|fg(2n+1+2k) - fg(2n+1)| \geqslant \frac{3}{4}|g(2n+1+2k) - g(2n+1)| - \frac{\varepsilon}{2}. \tag{447}$$

Further on, it is very elementary to prove that there exists a strictly increasing sequence $(2a_{l,k} + 1)_{l\in\mathbb{N}}$ of odd integers such that the inequality

$$\Big|\cos\Big(\frac{1}{1+\cos\sqrt{2}(2a_{l,k}+1+2k)\pi}\Big) - \cos\Big(\frac{1}{1+\cos\sqrt{2}(2a_{l,k}+1)\pi}\Big)\Big| \geqslant \frac{8\varepsilon}{3}$$

holds provided $l \in \mathbb{N}$.

In combination with (447), we get that (440) does not hold with $s = 2n + 1$. Hence, $fg \notin Q - AAP(\mathbb{R} : \mathbb{C})$ while in the meantime $g \in S^pQ - AAP(\mathbb{R} : \mathbb{C}) \smallsetminus Q - AAP(\mathbb{R} : \mathbb{C})$ and $f \in SAP_2(\mathbb{R} : \mathbb{C})$. Since the function $f(\cdot)$ is Stepanov p-vanishing, i.e, $\lim_{t\to+\infty} \int_t^{t+1} |f(s)|^p ds = 0$, it can be easily seen that $fg \in S^pAAP([0, \infty) : X)$, as well.

Example 3.11.25. It is very simple to illustrate that the pointwise product of two essentially bounded functions from $S^pQ - AAP(I : \mathbb{C})$ need not belong to the same class. We will show this only in the case that $I = [0, \infty)$ by giving an example of a Stepanov p-ap. function $g(\cdot)$ and a function $f \in SAP_4([0, \infty) : \mathbb{C})$ such that $fg \notin S^pQ - AAP([0, \infty) : \mathbb{C})$. To see this, put $g(t) := \operatorname{sign}(\sin t)$, $t \geqslant 0$, where $\operatorname{sign}(0) := 0$. Then it is well known that $g(\cdot)$ is Stepanov p-ap. function; see e.g. **[293]**. We construct $f(\cdot)$ in the following way: Define $f(t) := 0$ for $t \in \{0, 43/10, 46/10, 4n + \frac{1}{2} - \frac{1}{4n+1}, 4n + \frac{3}{2} + \frac{1}{4n+1} : n \in \mathbb{N}\}$, $f(t) := 1$ for $t \in \cup_{n\in\mathbb{N}}[4n + 1/2, 4n + 3/2]$ and linearly outside. Then it can be easily seen that $f \in SAP_4([0, \infty) : \mathbb{C})$. Furthermore, the function $fg(\cdot)$ does not belong to the class $S^pQ - AAP([0, \infty) : \mathbb{C})$ because if we suppose the contrary, then we can always take the segment $[4n + 1/2, 4n + 3/2] \subseteq [M(\varepsilon, \tau), \infty)$ sufficiently large, with the meaning clear, and the condition (444) will be always violated with $t = 4n + 1/2$.

This can be seen by considering separately two possible cases: $\tau \in 4\mathbb{N}$ and $\tau \notin 4\mathbb{N}$. In the first case, we have the existence of a number $k \in \mathbb{N}$ such that $\tau = 4k$. Then

$$\int_{4n+\frac{1}{2}}^{4n+\frac{3}{2}} |\operatorname{sign}(\sin(s+\tau))f(s+\tau) - \operatorname{sign}(\sin s)f(s)|^p ds$$
$$= \int_{4n+\frac{1}{2}}^{4n+\frac{3}{2}} |\operatorname{sign}(\sin(s+\tau)) - \operatorname{sign}(\sin s)|^p ds$$

for all $n \in \mathbb{N}$. Further on, observe that

$$\sin(s+\tau) - \sin s = 2\sin 2k \cos(s+2k), \quad s \in \mathbb{R}.$$

If $\sin 2k > 0$, the terms $\sin(s+\tau)$ and $\sin s$ will have different signs for all $s \in [4n+1/2, 4n+3/2]$ provided that there exists a natural number $m \in \mathbb{N}$ such that $4n+2k+1/2$ and $4n+2k+3/2$ belong to the set $(\pi/2+2m\pi, 3\pi/2+2m\pi)$. This could happen for arbitrarily large values of $n \in \mathbb{N}$, so that (444) does not hold. The examination is similar provided that $\sin 2k < 0$. In the second case, let $\tau = 4m + \tau_0$ for some $m \in \mathbb{N}_0$ and $\tau_0 \in (0,4)$. Since the integer multiples of π get arbitrarily close to the integers, there is a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $[4n_k+1/2, 4n_k+3/2] \subseteq \cup_{l\in\mathbb{N}}(2l\pi, (2l+1)\pi)$. For $n = n_k \geqslant M(\varepsilon,\tau)$, we have $\operatorname{sign}(\sin s) = 1$ for all $s \in [4n_k+1/2, 4n_k+3/2]$ and we can use the estimate

$$\int_{4n_k+\frac{1}{2}}^{4n_k+\frac{3}{2}} |\operatorname{sign}(\sin(s+\tau))f(s+\tau) - \operatorname{sign}(\sin s)|^p ds$$
$$\geqslant 2^{p-1}\int_{4n_k+\frac{1}{2}}^{4n_k+\frac{3}{2}} [1 - |f(s+\tau)|^p] ds$$
$$= 2^{p-1}\left[1 - \int_{4n_k+\tau+\frac{1}{2}}^{4n_k+\tau+\frac{3}{2}} |f(s)|^p ds\right].$$

to conclude that there exist a zero sequence $(a_k)_{k\in\mathbb{N}}$ and a positive constant $c(\tau_0) \in (0,1)$ such that

$$\int_{4(n_k+m)+\tau_0+1/2}^{4(n_k+m)+\tau_0+3/2} |f(s)|^p ds > c(\tau_0) + a_k$$

for all $k \in \mathbb{N}$ such that $n_k \geqslant M(\varepsilon,\tau)$, the violation of (444) becomes apparent.

It is clear that the sets $Q_h - AAP(I : X)$ and $S^pQ_h - AAP(I : X)$ equipped with the usual operations of pointwise sums and products with scalars form vector spaces. This is no longer true for the spaces $Q{-}AAP(I : X)$ and $S^pQ{-}AAP(I : X)$, as the following example shows:

EXAMPLE 3.11.26. For the sake of simplicity, we will consider only the case that $I = [0,\infty)$. Let $f(\cdot)$ and $g(\cdot)$ be as in the former example. Repeating the same argumentation (in the case that $\tau \notin 4\mathbb{N}$, then we can use the estimate $|\operatorname{sign}(\sin(s+\tau)) + f(s+\tau) - \operatorname{sign}(\sin s) - f(s)| = |\operatorname{sign}(\sin(s+\tau)) + f(s+\tau) - 2| \geqslant 1 - |f(s+\tau)|$ for all $s \in [4n_k+1/2, 4n_k+3/2] \subseteq \cup_{l\in\mathbb{N}}(2l\pi, (2l+1)\pi)$), we get that $f+g \notin$

$S^pQ-AAP([0,\infty):\mathbb{C})$, so that $S^pQ-AAP([0,\infty):\mathbb{C})+S^pQ-AAP([0,\infty):\mathbb{C})$ is not contained in $S^pQ-AAP([0,\infty):\mathbb{C})$. Now we will prove by a simple indirect proof that $AP([0,\infty):\mathbb{C})+SAP_4([0,\infty):\mathbb{C})$ does not make a subclass of the class $S^pQ-AAP([0,\infty):\mathbb{C})$ so that $Q-AAP([0,\infty):X)$ cannot be a vector space, as announced above. There exists a sequence $(g_n)_{n\in\mathbb{N}}$ converging to $g(\cdot)$ in $L^p_S([0,\infty):\mathbb{C})$. This implies that g_n+f converges to $g+f$ in $L^p_S([0,\infty):\mathbb{C})$ as $n\to\infty$. Due to Theorem 3.11.21(iv), there exists a number $n_0\in\mathbb{N}$ such that $g_{n_0}+f$ does not belong to the class $S^pQ-AAP([0,\infty):\mathbb{C})$; it is almost trivial to construct a sequence $(g_n)_{n\in\mathbb{N}}$ in $C_{2\pi}([0,\infty):\mathbb{C})$ converging to $g(\cdot)$ in $L^p_S([0,\infty):\mathbb{C})$, so that a function from $C_{2\pi}([0,\infty):\mathbb{C})+SAP_4([0,\infty):\mathbb{C})$ need not belong to the class $S^pQ-AAP([0,\infty):\mathbb{C})$, as well. The interested reader may try to provide some concrete examples here.

The compactness in the spaces of (equi-)Weyl-p-almost periodic functions has been analyzed in [**358**,**359**] with the help of Lusternik type theorems. It is without scope this book to analyze similar problems for the space of q-aap. functions and its Stepanov generalizations.

3.11.4. Quasi-asymptotically almost periodic functions depending on two parameters and composition principles. The main aim of this subsection is to investigate q-aap. functions depending on two parameters and compositions of q-aap. functions. We start with the folowing definition:

DEFINITION 3.11.27. Suppose that $F: I\times X\to Y$ is a continuous function. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically almost periodic, uniformly on bounded subsets of X, iff for each $\varepsilon>0$ (S) holds with a number $M(\varepsilon,\tau)>0$ such that for each bounded subset B of X we have:

$$\|F(t+\tau,x)-F(t,x)\|_Y\leqslant\varepsilon,\ \text{provided } t\in I,\ x\in B \text{ and } |t|\geqslant M(\varepsilon,\tau).$$

Denote by $Q-AAP(I\times X:Y)$ the set consisting of all quasi-asymptotically almost periodic functions from $I\times X$ into Y.

Arguing as in the proofs of [**144**, Theorem 3.30, Theorem 3.31], we may deduce the following results about compositions of q-aap. functions:

THEOREM 3.11.28. *Suppose that $F\in Q-AAP(I\times X:Y)$ and $f\in Q-AAP(I:X)$. If there exists a finite number $L>0$ such that*

$$\|F(t,x)-F(t,y)\|_Y\leqslant L\|x-y\|,\quad x,y\in X,\ t\in I, \tag{448}$$

then the function $t\mapsto F(t,f(t))$, $t\in I$ belongs to the class $Q-AAP(I:Y)$.

THEOREM 3.11.29. *Suppose that $F\in Q-AAP(I\times X:Y)$ and $f\in Q-AAP(I:X)$. If the function $x\mapsto F(t,x)$, $t\in I$ is uniformly continuous on every bounded subset $B\subseteq X$ uniformly for $t\in I$, then the function $t\mapsto F(t,f(t))$, $t\in I$ belongs to the class $Q-AAP(I:Y)$.*

The class of S^p-qaap. functions depending on two parameters is introduced in the following definition:

DEFINITION 3.11.30. Suppose that a function $F\colon I \times X \to Y$ satisfies that for each $x \in X$ the function $t \mapsto F(t,x)$, $t \in I$ is Stepanov p-bounded. Then we say that $F(\cdot,\cdot)$ is Stepanov p-quasi-asymptotically almost periodic, uniformly on bounded subsets of X, iff for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon,\tau) > 0$ such that for each bounded subset B of X we have:

$$\int_t^{t+1} \|F(s+\tau,x) - F(s,x)\|^p \, ds \leqslant \varepsilon^p, \text{ provided } t \in I,\ x \in B \text{ and } |t| \geqslant M(\varepsilon,\tau).$$

Denote by $S^pQ - AAP(I \times X : Y)$ the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from $I \times X$ into Y.

In [**335**, Definition 3.1], we have recently introduced the class $e - W_{ap,K}(I \times X, X)$ consisting of equi-Weyl-p-ap. functions, uniformly with respect to compact subsets of X. The class $e - W_{ap,K}(I \times X, Y)$ with two different pivot spaces X and Y as well as the class $(e-)W_{ap,B}(I \times X, Y)$ consisting of all (equi-)Weyl-p-ap. functions, uniformly with respect to bounded subsets of X, can be introduced in a similar way. Following the method proposed in the proof of Proposition 3.11.20, we can show then $S^pQ - AAP(I \times X : Y) \subseteq W_{ap,B}(I \times X, Y)$. The following composition principles can be deduced in exactly the same way as it has been done in the proofs of [**399**, Lemma 2.1, Theorem 2.2]:

THEOREM 3.11.31. *Suppose that the following conditions hold:*

(i) $F \in S^pQ - AAP(I \times X : Y)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *and a function* $L_F \in L_S^r(I)$ *such that:*
$$\|F(t,x) - F(t,y)\| \leqslant L_F(t)\|x-y\|_Y, \quad t \in I,\ x,y \in Y;$$
(ii) $x \in S^pQ - AAP(I : X)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{x(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* X*; here,* $m(\cdot)$ *denotes the Lebesgue measure.*

Then $q := pr/p + r \in [1,p)$ *and* $F(\cdot, x(\cdot)) \in S^qQ - AAP(I : Y)$.

THEOREM 3.11.32. *Suppose that the following conditions hold:*

(i) $F \in S^pQ - AAP(I \times X : Y)$ *with* $p \geqslant 1$, $L > 0$ *and*
$$\|F(t,x) - F(t,y)\| \leqslant L\|x-y\|, \quad t \in I,\ x,y \in X.$$
(ii) $x \in S^pQ - AAP(I : X)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K = \{x(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y.

Then $F(\cdot, x(\cdot)) \in S^qQ - AAP(I : Y)$.

3.11.5. Invariance of quasi-asymptotical almost periodicity under the action of convolution products. Concerning the invariance of quasi-asymptotical almost periodicity under the action of finite convolution product, we have the following result:

PROPOSITION 3.11.33. *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\| ds < \infty$. *If* $f \in Q - AAP([0,\infty) : X)$, *then the function* $F(\cdot)$, *defined through* (434), *belongs to the class* $Q - AAP([0,\infty) : Y)$.

PROOF. Without loss of generality, we may assume that $X = Y$. It is clear that $\|F(t)\| = \| \int_0^t R(s)f(t-s)ds\| \leqslant \|f\|_\infty \int_0^\infty \|R(s)\|ds$, $t \geqslant 0$ so that $F(\cdot)$ is bounded. Since

$$\|F(t) - F(t')\| \leqslant \int_0^\infty \|R(s)\|\|f(t-s) - f(t'-s)\|ds$$

for any t, $t' \geqslant 0$, the continuity of $F(t)$ for $t \geqslant 0$ follows from the boundedness of $f(\cdot)$ and the dominated convergence theorem. Let $\varepsilon > 0$ be given. Then (S) holds with a number $M(\varepsilon, \tau) > 0$ such that (440) holds. On the other hand, the condition $\int_0^\infty \|R(s)\|ds < \infty$ implies $\lim_{t\to+\infty} \int_t^\infty \|R(s)\|ds = 0$ so that there exists a finite number $M_0(\varepsilon) > 0$ such that $\int_t^\infty \|R(s)\|ds < \varepsilon$ for any $t \geqslant M_0(\varepsilon)$. Let $t \geqslant M(\varepsilon, \tau) + M_0(\varepsilon)$. Then we have

$$\begin{aligned}
&\|F(t+\tau) - F(t)\| \\
&= \left\| \int_0^t R(s)[f(t+\tau-s) - f(t-s)]ds + \int_t^{t+\tau} R(s)f(t+\tau-s)ds \right\| \\
&\leqslant \int_0^t \|R(s)\|\|f(t+\tau-s) - f(t-s)\|ds + \|f\|_\infty \int_t^{t+\tau} \|R(s)\|ds \\
&\leqslant \int_0^{t-M(\varepsilon,\tau)} \|R(s)\|\|f(t+\tau-s) - f(t-s)\|ds \\
&+ \int_{t-M(\varepsilon,\tau)}^t \|R(s)\|\|f(t+\tau-s) - f(t-s)\|ds + \varepsilon\|f\|_\infty \\
&\leqslant \varepsilon \int_0^\infty \|R(s)\|ds + 2\|f\|_\infty\varepsilon + \varepsilon\|f\|_\infty,
\end{aligned}$$

which completes the proof in a routine manner. □

The situation is quite similar for the infinite convolution product:

PROPOSITION 3.11.34. *Suppose that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and $\int_0^\infty \|R(s)\|ds < \infty$. If $f \in Q - AAP(\mathbb{R} : X)$, then the function $\mathbf{F}(t)$, defined through* (433)*, belongs to the class $Q - AAP(\mathbb{R} : Y)$.*

PROOF. Without loss of generality, we may assume that $X = Y$. The boundedness and continuity of $\mathbf{F}(\cdot)$ can be proved as in the former proposition. To prove that $\mathbf{F}(\cdot)$ satisfies the remaining requirement from definition of quasi-asymptotical almost periodicity, fix a number $\varepsilon > 0$. By definition, (S) holds with a number $M(\varepsilon, \tau) > 0$ satisfying (440). On the other hand, the condition $\int_0^\infty \|R(s)\|ds < \infty$ implies $\lim_{t\to+\infty} \int_t^\infty \|R(s)\|ds = 0$ so that there exists a finite number $M_0(\varepsilon) > 0$ such that $\int_t^\infty \|R(s)\|ds < \varepsilon$ for any $t \geqslant M_0(\varepsilon)$. Let $|t| \geqslant M(\varepsilon, \tau) + M_0(\varepsilon)$. If $t \leqslant -M(\varepsilon, \tau) - M_0(\varepsilon)$, then $t \leqslant -M(\varepsilon, \tau)$, $|t - s| \geqslant M(\varepsilon, \tau)$ for all $s \geqslant 0$, and

$$\|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| \leqslant \int_0^\infty \|R(s)\|\|f(t+\tau-s) - f(t-s)\|ds \leqslant \varepsilon \int_0^\infty \|R(s)\|ds.$$

If $t \geqslant M(\varepsilon,\tau)+M_0(\varepsilon)$, then $t-M(\varepsilon,\tau) \geqslant M_0(\varepsilon)$ so that $\int_{t-M(\varepsilon,\tau)}^{\infty}\|R(s)\|ds<\varepsilon$; furthermore, in this case we have:

$$\begin{aligned}\|\mathbf{F}(t+\tau)-\mathbf{F}(t)\| &\leqslant \int_0^{\infty}\|R(s)\|\|f(t+\tau-s)-f(t-s)\|ds\\ &\leqslant \varepsilon\Bigl(\int_0^{t-M(\varepsilon,\tau)}+\int_{t+M(\varepsilon,\tau)}^{\infty}\Bigr)\|R(s)\|ds+2\|f\|_{\infty}\int_{t-M(\varepsilon,\tau)}^{t+M(\varepsilon,\tau)}\|R(s)\|ds\\ &\leqslant 2\varepsilon\int_0^{\infty}\|R(s)\|ds+2\|f\|_{\infty}\int_{t-M(\varepsilon,\tau)}^{\infty}\|R(s)\|ds\\ &\leqslant 2\varepsilon\int_0^{\infty}\|R(s)\|ds+2\|f\|_{\infty}\varepsilon,\end{aligned}$$

which completes the proof of proposition. □

Suppose that $1/p+1/q=1$ and $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}<\infty$. This condition implies $\int_0^{\infty}\|R(s)\|ds<\infty$ and can be used for the examination of q-aap. properties of convolution products with Stepanov S^p-qaap. inhomogenities $f(\cdot)$. Keeping in mind the proofs of Proposition 3.11.33 and Proposition 3.11.34, as well as the proofs of Proposition 2.6.11 and Proposition 3.5.3 in **[293]**, the following results can be deduced:

Proposition 3.11.35. *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}<\infty$. *If* $f\in S^pQ-AAP([0,\infty):X)$, *then the function* $F(\cdot)$, *defined by* (434), *belongs to the class* $Q-AAP([0,\infty):Y)$.

Proposition 3.11.36. *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}<\infty$. *If* $f\in S^pQ-AAP(\mathbb{R}:X)$, *then the function* $\mathbf{F}(\cdot)$, *defined by* (433), *belongs to the class* $Q-AAP(\mathbb{R}:Y)$.

For the sake of completeness, we will include the proofs (the preassumption $X=Y$ can be made):

Proof of Proposition 3.11.35. It is easy to see that

$$\|F(t)\|\leqslant\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}\|f\|_{S^p},\quad t\geqslant 0.$$

The continuity of $F(t)$ for $t\geqslant 0$ can be proved as follows. Let $t,\ t'\geqslant 0$ and $|t-t'|\leqslant 1$. Then we have

$$\|F(t)-F(t')\|\leqslant\int_0^t\|R(s)\|\|f(t-s)-f(t'-s)\|ds+\int_t^{t'}\|R(s)\|\|f(t'-s)\|ds$$

$$\leqslant\sum_{k=0}^{\lfloor t\rfloor}\|R(\cdot)\|_{L^q[k,k+1]}\|f(t-\cdot)-f(t'-\cdot)\|_{L^p[k,k+1]}+\|f\|_{S^p}\|R(\cdot)\|_{L^q[\min(t,t'),\max(t,t')]}.$$

Since $f \in L^p_{loc}([0,\infty) : X)$, we have $\lim_{t' \to t} \|f(t-\cdot) - f(t'-\cdot)\|_{L^p[k,k+1]} = 0$ for $k = 0, \dots, \lfloor t \rfloor$. Clearly, $\lim_{t' \to t} \|R(\cdot)\|_{L^q[\min(t,t'),\max(t,t')]} = 0$ so that the function $F(\cdot)$ is continuous at point t. Let $\varepsilon > 0$ be given. Then (S) holds with a number $M(\varepsilon,\tau) > 0$ satisfying (444). Furthermore, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that $\sum_{k=k_0(\varepsilon)}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \varepsilon$. For any $t \geqslant M(\varepsilon,\tau) + k_0(\varepsilon) + 1$, we have

$$\begin{aligned}
\|F(t+\tau) - F(t)\| &\leqslant \sum_{k=0}^{\lfloor t \rfloor} \|R(\cdot)\|_{L^q[k,k+1]} \|f(t+\tau-\cdot) - f(t-\cdot)\|_{L^p[k,k+1]} \\
&\leqslant \varepsilon \sum_{k=0}^{k_0(\varepsilon)} \|R(\cdot)\|_{L^q[k,k+1]} + 2^{p-1}\|f\|_{S^p} \sum_{k=k_0(\varepsilon)}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \\
&\leqslant \varepsilon \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} + 2\|f\|_{S^p}\varepsilon,
\end{aligned}$$

finishing the proof. □

Proof of Proposition 3.11.36. The boundedness and continuity of function $\mathbf{F}(\cdot)$ can be shown as in the proof of [**293**, Proposition 3.5.3]. Let $\varepsilon > 0$ be given. Then (S) holds with a number $M(\varepsilon,\tau) > 0$ satisfying (444). As above, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that $\sum_{k=k_0(\varepsilon)}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \varepsilon$. Let $t \in \mathbb{R}$ be such that $|t| \geqslant M(\varepsilon,\tau) + k_0(\varepsilon) + 1$. Then we have

$$\|F(t+\tau) - F(t)\| \leqslant \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \|f(\cdot+\tau) - f(\cdot)\|_{L^p[t-(k+1),t-k]}.$$

If $t \leqslant -M(\varepsilon,\tau)$, then $[t-(k+1), t-k] \subseteq (-\infty, -M(\varepsilon,\tau)]$ for any $k \in \mathbb{N}_0$ and the above estimate immediately implies $\|F(t+\tau) - F(t)\| \leqslant \varepsilon \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]}$. If $t \geqslant M(\varepsilon,\tau) + k_0(\varepsilon) + 1$, then $\lfloor t - M(\varepsilon,\tau) \rfloor \geqslant k_0(\varepsilon)$ so that

$$\begin{aligned}
\|F(t+\tau) - F(t)\| \leqslant &\sum_{k=0}^{\lfloor t-M(\varepsilon,\tau) \rfloor} \|R(\cdot)\|_{L^q[k,k+1]} \|f(\cdot+\tau) - f(\cdot)\|_{L^p[t-(k+1),t-k]} \\
&+ \sum_{k=\lfloor t-M(\varepsilon,\tau) \rfloor}^{\lceil t+M(\varepsilon,\tau) \rceil} \|R(\cdot)\|_{L^q[k,k+1]} \|f(\cdot+\tau) - f(\cdot)\|_{L^p[t-(k+1),t-k]} \\
&+ \sum_{k=\lceil t+M(\varepsilon,\tau) \rceil}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \|f(\cdot+\tau) - f(\cdot)\|_{L^p[t-(k+1),t-k]} \\
\leqslant &\varepsilon \left(\sum_{k=0}^{\lfloor t-M(\varepsilon,\tau) \rfloor} + \sum_{k=\lceil t+M(\varepsilon,\tau) \rceil}^{\infty} \right) \|R(\cdot)\|_{L^q[k,k+1]} + 2\|f\|_{S^p}\varepsilon \\
\leqslant &2\varepsilon \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} + 2\|f\|_{S^p}\varepsilon,
\end{aligned}$$

finishing the proof. □

3.11.6. Applications to abstract nonautonomous differential equations of first order. Throughout this subsection, it will be always assumed that the operator family $A(\cdot)$ satisfies the condition (H1) and the evolution system $U(\cdot, \cdot)$ generated by $A(\cdot)$ is hyperbolic, i.e., the condition (H2) holds true.

We analyze here the following abstract quasi-linear differential equations of first order:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{449}$$

$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x \tag{450}$$

and their semilinear analogues. In our recent research studies [**338**, **339**], the author has considered the existence and uniqueness of generalized almost periodic properties of mild solutions of (449)–(450) and their semilinear analogues. In the formulations and proofs of all structural results from [**338**, Section 2] and [**339**, Section 3], the essential boundedness of forcing term $f(\cdot)$ has been required as well as certain additional conditions on the generalized almost periodicity of $f(\cdot)$. In contrast to this, in the formulation of the following result, we require the Stepanov p-boundedness of function $f(\cdot)$ for some exponent $p \in [1, \infty)$; by $\mathcal{F}$ we denote a general function space consisted of continuous functions from $[0, \infty)$ into X.

THEOREM 3.11.37. *Let $I = [0, \infty)$, $1/p + 1/q = 1$ and $f \in S^pQ - AAP(\mathbb{R} : X)$. If $x \in P(0)X \cap \overline{D(A(0))}$, the function $t \mapsto \int_0^t U(t,s)Q(s)f(s)ds$, $t \geqslant 0$ belongs to the class $\mathcal{F}$ and for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon, \tau) > 0$ such that*

$$\sum_{k=0}^{\infty} \|\Gamma(t+\tau, t+\tau-\cdot) - \Gamma(t, t-\cdot)\|_{L^q[k,k+1]} \leqslant \varepsilon, \quad \text{provided } t \geqslant M(\varepsilon, \tau), \tag{451}$$

then there exists a unique mild solution $u(\cdot)$ of (450) and this solution belongs to the class $Q - AAP([0, \infty) : X) + \mathcal{F}$.

PROOF. Since $x \in P(0)X \cap \overline{D(A(0))}$, the mapping $t \mapsto U(t, 0)x$, $t > 0$ is continuous, exponentially decaying and satisfies $\lim_{t\to 0+} U(t, 0)x = x$ [**407**]. The continuity of function $u(t) = U(t, 0)x + \int_0^t U(t, s)f(s)ds$, $t \geqslant 0$, given by (437), can be deduced as in the proof of [**338**, Theorem 2.1], since any of the considered terms in the corresponding part of proof of above-mentioned result can be majorized in a similar way, by using the S^p-boundedness of function $f(\cdot)$ and the Hölder inequality. Clearly, $u(t) = \int_0^t \Gamma(t, s)f(s)ds + \int_0^t U(t, s)Q(s)f(s)ds$, $t \geqslant 0$ and, by our preassumption made, it suffices to show that the function $t \mapsto \int_0^t \Gamma(t, s)f(s)ds$, $t \geqslant 0$ belongs to the class $Q - AAP([0, \infty) : X)$. Applying the Hölder inequality, the estimate (435) and the S^p-boundedness of function $f(\cdot)$, we get that there exists a finite positive constant $M'' > 0$ such that

$$\begin{aligned}
\|u(t)\| &\leqslant M'\|f\|_{S^p} \sum_{k=0}^{\infty} \|e^{-\omega|\cdot|}\|_{L^q[t-(k+1),t-k]} \\
&\leqslant M'\|f\|_{S^p} \sum_{k=0}^{\infty} \|e^{-\omega|\cdot|}\|_{L^\infty[t-(k+1),t-k]}
\end{aligned}$$

$$\leqslant M'\|f\|_{S^p}\sum_{k=0}^{\infty}[e^{-\omega|t-k|}+e^{-\omega|t-k-1|}]$$
$$\leqslant M'\|f\|_{S^p}e^{\omega t}\sum_{k=0}^{\infty}[e^{-\omega k}+e^{-\omega(k+1)}]\leqslant M''e^{\omega|t|},$$

for any $t\geqslant 0$. Fix a number $\varepsilon>0$. By definition, (S) holds with a number $M(\varepsilon,\tau)>0$ satisfying (444). Keeping in mind the estimate (451), the final conclusion follows from the computation

$$\|u(t+\tau)-u(t)\|\leqslant\int_0^t\|\Gamma(t+\tau,t+\tau-s)-\Gamma(t,t-s)\|\|f(t-s)\|ds$$
$$+\int_t^{t+\tau}\|\Gamma(t+\tau,t+\tau-s)\|\|f(t+\tau-s)-f(t-s)\|ds$$
$$\leqslant\|f\|_{S^p}\sum_{k=0}^{\lfloor t\rfloor}\|\Gamma(t+\tau,t+\tau-\cdot)-\Gamma(t,t-\cdot)\|_{L^q[k,k+1]}+2\|f\|_{S^p}\sum_{k=\lfloor t\rfloor}^{\lceil t+\tau\rceil}\|e^{-\omega|\cdot|}\|_{L^q[k,k+1]}$$
$$\leqslant\|f\|_{S^p}\sum_{k=0}^{\infty}\|\Gamma(t+\tau,t+\tau-\cdot)-\Gamma(t,t-\cdot)\|_{L^q[k,k+1]}+2\|f\|_{S^p}\sum_{k=\lfloor t\rfloor}^{\infty}\|e^{-\omega|\cdot|}\|_{L^q[k,k+1]}$$

and the obvious equality $\lim_{t\to\infty}\sum_{k=\lfloor t\rfloor}^{\infty}\|e^{-\omega|\cdot|}\|_{L^q[k,k+1]}=0$. $\square$

REMARK 3.11.38. It can be simply shown that (451) implies

$$\int_0^{+\infty}\|\Gamma(t+\tau,t+\tau-s)-\Gamma(t,t-s)\|ds\leqslant\varepsilon,\quad t\geqslant M(\varepsilon,\tau).\tag{452}$$

If we assume that $f\in L^{\infty}([0,\infty):X)$ in place of $f\in L^p_S([0,\infty):X)$, then the validity of (452) in place of (451) implies that $u(\cdot)$ is q-aap..

Concerning the abstract Cauchy problem (449), we have the following result:

THEOREM 3.11.39. *Let $I=\mathbb{R}$, $1/p+1/q=1$ and $f\in S^pQ-AAP(\mathbb{R}:X)$. If for each $\varepsilon>0$ (S) holds with a number $M(\varepsilon,\tau)>0$ such that*

$$\sum_{k\in\mathbb{Z}}\|\Gamma(t+\tau,t+\tau-\cdot)-\Gamma(t,t-\cdot)\|_{L^q[k,k+1]}\leqslant\varepsilon,\ \textit{provided } t\in\mathbb{R}\textit{ and }|t|\geqslant M(\varepsilon,\tau),\tag{453}$$

then there exists a unique mild solution $u(\cdot)$ of (449) and this solution is q-aap..

PROOF. As in the proof of Theorem 3.11.37, we can deduce that the function $u(t)=\int_{-\infty}^{+\infty}\Gamma(t,s)f(s)ds$, $t\in\mathbb{R}$, defined by (436), is bounded. The continuity of $u(\cdot)$ can be shown following the lines of proof of [**338**, Theorem 2.1]. Assume now that $\varepsilon>0$ is a given number. Then (S) holds with a number $M(\varepsilon,\tau)>0$ satisfying (444). It is clear that, for every $t\in\mathbb{R}$, we have:

$$\|u(t+\tau)-u(t)\|\leqslant\int_{-\infty}^{t}e^{-\omega|s|}\|f(t+\tau-s)-f(t-s)\|ds$$
$$+\int_t^{\infty}e^{-\omega|s|}\|f(t+\tau-s)-f(t-s)\|ds$$

$$
\begin{aligned}
&+ \int_{-\infty}^{\infty} \|\Gamma(t+\tau, t+\tau-s) - \Gamma(t, t-s)\| \|f(t-s)\| ds \\
&\leqslant \int_{-\infty}^{t} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds \\
&+ \int_{t}^{\infty} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds \\
&+ \|f\|_{S^p} \sum_{k\in\mathbb{Z}} \|\Gamma(t+\tau, t+\tau-\cdot) - \Gamma(t, t-\cdot)\|_{L^q[k,k+1]}.
\end{aligned}
$$

Keeping in mind (453), we get that:

$$
\begin{aligned}
\|u(t+\tau) - u(t)\| \leqslant &\int_{-\infty}^{t} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds \\
&+ \int_{t}^{\infty} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds + \|f\|_{S^p} \varepsilon,
\end{aligned}
$$

provided $|t| \geqslant M(\varepsilon, \tau)$. By the proof of Proposition 3.11.36, it follows the existence of a finite real number $M_1(\varepsilon, \tau) > 0$ such that

$$
\int_{-\infty}^{t} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds < \varepsilon, \text{ provided } |t| \geqslant M_1(\varepsilon, \tau).
$$

On the other hand, there exists an integer $k_0(\varepsilon) \in \mathbb{N}$ such that $e^{-\omega k} \leqslant \varepsilon$ for all $k \geqslant k_0(\varepsilon)$. Let $|t| \geqslant 2M(\varepsilon, \tau) + 1 + k_0(\varepsilon)$. For the second addend, we can use the following calculus involving the Hölder inequality, after dividing the interval of integration $(-\infty, 0]$ into two subintervals $(-\infty, -M(\varepsilon, \tau)]$ and $[-M(\varepsilon, \tau), 0]$:

$$
\begin{aligned}
&\int_{t}^{\infty} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds \\
&\quad \leqslant \varepsilon \sum_{k=0}^{\infty} \|e^{-\omega|\cdot|}\|_{L^q[t+M(\varepsilon,\tau)+k, t+M(\varepsilon,\tau)+k+1]} \\
&\quad \leqslant + \int_{-M(\varepsilon,\tau)}^{0} e^{-\omega|s|} \|f(t+\tau-s) - f(t-s)\| ds \\
&\quad \leqslant \varepsilon e^{-\omega|t+M(\varepsilon,\tau)|} \sum_{k=0}^{\infty} [e^{-\omega k} + e^{-\omega(k+1)}] + 2\|f\|_{S^p} \sum_{k=0}^{\lfloor M(\varepsilon,\tau) \rfloor} \left\| e^{-\omega|\cdot|} \right\|_{L^q[t+k, t+k+1]} \\
&\quad \leqslant \varepsilon e^{-\omega k_0(\varepsilon)} \sum_{k=0}^{\infty} [e^{-\omega k} + e^{-\omega(k+1)}] + 2\|f\|_{S^p} \sum_{k=0}^{\lfloor M(\varepsilon,\tau) \rfloor} [e^{-\omega|t-(k+1)|} + e^{-\omega|t-k|}] \\
&\quad \leqslant \varepsilon e^{-\omega k_0(\varepsilon)} \sum_{k=0}^{\infty} [e^{-\omega k} + e^{-\omega(k+1)}] + 2\|f\|_{S^p} (1 + M(\varepsilon, \tau)) e^{-\omega(M(\varepsilon,\tau) + k_0(\varepsilon))} \\
&\quad \leqslant \varepsilon e^{-\omega k_0(\varepsilon)} \sum_{k=0}^{\infty} [e^{-\omega k} + e^{-\omega(k+1)}] + 2\|f\|_{S^p} (1 + \omega) e^{-\omega k_0(\varepsilon)},
\end{aligned}
$$

which completes the proof. $\square$

REMARK 3.11.40. The condition (453) implies

$$\int_{-\infty}^{+\infty} \|\Gamma(t+\tau,t+\tau-s)-\Gamma(t,t-s)\|ds \leqslant \varepsilon, \text{ provided } t \in \mathbb{R} \text{ and } |t| \geqslant M(\varepsilon,\tau). \tag{454}$$

If we assume that $f \in L^{\infty}(\mathbb{R} : X)$ in place of $f \in L_S^p(\mathbb{R} : X)$, then the validity of (454) in place of (453) implies that $u(\cdot)$ is q-aap..

REMARK 3.11.41. It is worth noting that Theorem 3.11.37 and Remark 3.11.38, as well as Theorem 3.11.39 and Remark 3.11.40, continue to hold in the case that the operator family $(A(t))_{t\in\mathbb{R}}$ generates an exponentially stable evolution family $(U(t,s))_{t\geqslant s}$ in the sense of [**145**, Definition 3.1]; in this case, the condition (H1) need not be satisfied and the condition (H2) holds with $P(t) = I$ and $Q(t) = 0$, $t \in \mathbb{R}$; $\Gamma(t,s) \equiv U(t,s)$.

3.11.7. Semilinear Cauchy problems. In this subsection, we consider the existence and uniqueness of q-aap. solutions of the abstract Cauchy problems (438) and (439). We first state the following result about the abstract Cauchy problem (439):

THEOREM 3.11.42. *Let $I = [0,\infty)$, the evolution system $U(\cdot,\cdot)$ be exponentially stable, let $x \in P(0)X \cap \overline{D(A(0))}$ and let $F \in Q-AAP([0,\infty) \times X : X)$. Suppose that for each $\varepsilon > 0$ (S) holds with a number $M(\varepsilon,\tau) > 0$ satisfying (452). If there exists a finite number $L \in (0,\omega/M')$ such that (448) holds, then there exists a unique mild solution $u(\cdot)$ of (439) belonging to the class $Q-AAP([0,\infty) : X)$.*

PROOF. As before, the mapping $t \mapsto U(t,0)x$, $t > 0$ is continuous, exponentially decaying and satisfies $\lim_{t\to 0+} U(t,0)x = x$. Let $\mathcal{P}\colon Q-AAP([0,\infty) : X) \to Q-AAP([0,\infty) : X)$ be defined through

$$\mathcal{P}f(t) := U(t,0)x + \int_0^t U(t,s)F(s,f(s))ds, \quad t \geqslant 0.$$

We will first show that the mapping $\mathcal{P}$ is well defined. Since $C_0([0,\infty) : X) + Q-AAP([0,\infty) : X) = Q-AAP([0,\infty) : X)$ and $Q-AAP([0,\infty) : X)$ is a complete metric space by Theorem 3.11.21(iii), it suffices to show that the mapping $t \mapsto \int_0^t U(t,s)F(s,f(s))ds$, $t \geqslant 0$ belongs to the class $Q-AAP([0,\infty) : X)$. Due to Theorem 3.11.28, the function $F(\cdot,f(\cdot))$ is in class $Q-AAP([0,\infty) : X)$; since $Q(t) = 0$ for all $t \in \mathbb{R}$, the prescribed assumption on the condition (452) yields that Theorem 3.11.37 (see also Remark 3.11.38) can be applied, showing that the mapping $t \mapsto \int_0^t U(t,s)F(s,f(s))ds$, $t \geqslant 0$ belongs to the class $Q-AAP([0,\infty) : X)$. Furthermore, the condition $L \in (0,\omega/M')$ implies after a simple calculation involving (435) and (448) that $\mathcal{P}(\cdot)$ is a contraction, so that the final conclusion simply follows by applying the Banach contraction principle. $\square$

We can similarly prove the following result on the abstract Cauchy problem (438):

THEOREM 3.11.43. *Let $I = \mathbb{R}$, the evolution system $U(\cdot,\cdot)$ be exponentially stable and $F \in Q-AAP(\mathbb{R} \times X : X)$. Suppose that for each $\varepsilon > 0$ (S) holds* *with a number* $M(\varepsilon,\tau)>0$ *satisfying* (454). *If there exists a finite number* $L\in(0,\omega/2M')$ *such that* (448) *holds, then there exists a unique mild solution* $u(\cdot)$ *of* (438) *belonging to the class* $Q-AAP(\mathbb{R}:X)$.

As mentioned before, the semilinear Cauchy problems with S^p-qaap. forcing term $F(\cdot,\cdot)$ cannot be so easily considered because the range of function $x(\cdot)$, appearing in the formulations of Theorem 3.11.31 and Theorem 3.11.32, need not be relatively compact.

We close the section by providing an illustrative example.

EXAMPLE 3.11.44. Let $X:=L^2[0,\pi]$ and Δ denote the Dirichlet Laplacian in X, acting with the domain $H^2[0,\pi]\cap H_0^1[0,\pi]$; then we know that Δ generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X, satisfying the estimate $\|T(t)\|\leqslant e^{-t}$, $t\geqslant 0$. Of concern is the following problem

$$u_t(t,x)=u_{xx}(t,x)+q(t,x)u(t,x)+f(t,x),\quad t\geqslant 0,\ x\in[0,\pi]; \tag{455}$$

$$u(0)=u(\pi)=0,\ u(0,x)=u_0(x)\in X, \tag{456}$$

where $q\colon\mathbb{R}\times[0,\pi]\to\mathbb{R}$ is a jointly continuous function satisfying that $q(t,x)\leqslant-\gamma_0$, $(t,x)\in\mathbb{R}\times[0,\pi]$, for some number $\gamma_0>0$. Define

$$A(t)\varphi:=\Delta\varphi+q(t,\cdot)\varphi,\quad \varphi\in D(A(t)):=D(\Delta)=H^2[0,\pi]\cap H_0^1[0,\pi],\qquad t\in\mathbb{R}.$$

Then $(A(t))_{t\in\mathbb{R}}$ generates an exponentially stable evolution family $(U(t,s))_{t\geqslant s}$ in the sense of [**145**, Definition 3.1], which is given by

$$U(t,s)\varphi:=T(t-s)e^{\int_s^t q(r,\cdot)\,dr}\varphi,\quad t\geqslant s.$$

It is clear that we can rewrite the initial value problem (455)–(456) in the following form:

$$u'(t)=A(t)u(t)+f(t),\quad t\geqslant 0;\ u(0)=u_0.$$

Hence, Theorem 3.11.37, resp. Theorem 3.11.39 (Theorem 3.11.42, resp. Theorem 3.11.43), can be applied provided that for each $\varepsilon>0$ (S) holds with a number $M(\varepsilon,\tau)>0$ such that the following condition holds:

$$\sum_{k=0}^{\infty}\Bigl\|e^{-|\cdot|}\sup_{x\in[0,\pi]}\bigl|e^{\int_{t+\tau-\cdot}^{t+\tau}q(r,x)\,dr}-e^{\int_{t-\cdot}^{t}q(r,x)\,dr}\bigr|\Bigr\|_{L^q[k,k+1]}<\varepsilon,\quad t\geqslant M(\varepsilon,\tau), \tag{457}$$

resp.

$$\sum_{k\in\mathbb{Z}}\Bigl\|e^{-|\cdot|}\sup_{x\in[0,\pi]}\bigl|e^{\int_{t+\tau-\cdot}^{t+\tau}q(r,x)\,dr}-e^{\int_{t-\cdot}^{t}q(r,x)\,dr}\bigr|\Bigr\|_{L^q[k,k+1]}<\varepsilon,\quad |t|\geqslant M(\varepsilon,\tau). \tag{458}$$

The conditions (457) and (458) hold for a wide class of functions $q(\cdot,\cdot)$ and we will prove here that this condition particularly holds for the function $q(t,x):=-\gamma_0-3t^2-f(x)$, $t\geqslant 0$, $x\in[0,\pi]$, where $f\colon[0,\infty)\to[0,\infty)$ is a continuous function (see also [**338**, Example 3.1], where we have analyzed the same choice). In our concrete situation, we have

$$\sup_{x\in[0,\pi]}\bigl|e^{\int_{t+\tau-s}^{t+\tau}q(r,x)\,dr}-e^{\int_{t-s}^{t}q(r,x)\,dr}\bigr|$$
$$\leqslant \text{Const.}\cdot e^{-s^3}\bigl|e^{3s(t+\tau)(s-t-\tau)}-e^{3st(s-t)}\bigr|,\ t,s,\tau\geqslant 0.$$

Using this estimate and the Lagrange mean value theorem, it readily follows that:

$$\begin{aligned}
&\Big\| e^{-|\cdot|} \sup_{x\in[0,\pi]} \big| e^{\int_{t+\tau-\cdot}^{t+\tau} q(r,x)\,dr} - e^{\int_{t-\cdot}^{t} q(r,x)\,dr} \big| \Big\|_{L^{\infty}[k,k+1]} \\
&\leqslant \text{Const.} \cdot |\tau| \sup_{s\in[k,k+1]} \big[e^{3st(s-t)} + e^{3s(t+\tau)(s-t-\tau)}\big] \cdot \big[3(k+1)^2 + 6(k+1)(t+\tau)\big] \\
&\leqslant \text{Const.} \cdot |\tau| \big[e^{3kt(k-t)} + e^{3(k+1)t(k+1-t)} + e^{3k(t+\tau)(k-t-\tau)} + e^{3(k+1)(t+\tau)(k+1-t-\tau)}\big] \\
&\quad \cdot \big[3(k+1)^2 + 6(k+1)(t+\tau)\big], \quad t,\tau \geqslant 0,\ k \in \mathbb{N}_0.
\end{aligned}$$

Let $3/4 < c < 1$. Then $3ts(s-ct) \leqslant 3s^3/4c$ for all $t,\ s \geqslant 0$ and therefore we can continue the calculation as follows:

$$\begin{aligned}
&\leqslant \text{Const.} \cdot |\tau| \big[e^{3kt(k-ct)} e^{-3ct^2} + e^{3(k+1)t(k+1-ct)} e^{-3ct^2} \\
&+ e^{3k(t+\tau)(k-c(t+\tau))} e^{-3c(t+\tau)^2} + e^{3(k+1)(t+\tau)(k+1-c(t+\tau))} e^{-3c(t+\tau)^2}\big] \\
&\cdot \big[3(k+1)^2 + 6(k+1)(t+\tau)\big] \\
&\leqslant \text{Const.} \cdot |\tau| \big[e^{3k^3/4c} e^{-3ct^2} + e^{3(k+1)^3/4c} e^{-3ct^2} \\
&+ e^{3k^3/4c} e^{-3c(t+\tau)^2} + e^{3(k+1)^3/4c} e^{-3c(t+\tau)^2}\big] \\
&\cdot \big[3(k+1)^2 + 6(k+1)(t+\tau)\big] \\
&\leqslant \text{Const.} \cdot |\tau| e^{3(k+1)^3/4c} e^{-3ct^2} \big[3(k+1)^2 + 6(k+1)(t+\tau)\big], \quad t,\tau \geqslant 0.
\end{aligned}$$

Since $3/4c < 1$, the series in (457) is convergent with $q = \infty$ and has a sum which does not exceed Const. $\cdot |\tau| e^{-3ct^2}(1+t+\tau)$, $t,\ \tau \geqslant 0$. At the end, it suffices to observe that for each $\varepsilon > 0$ and $\tau \geqslant 0$ there exists a finite number $M(\varepsilon,\tau) > 0$ such that $|\tau| e^{-3ct^2}(1+t+\tau) < \varepsilon$ for any $t \geqslant M(\varepsilon,\tau)$. This shows that Theorem 3.11.37 can be applied with any exponent $p \in [1,\infty)$.

3.12. Almost periodic and asymptotically almost periodic type solutions with variable exponents $L^{p(x)}$

As in the previous one, in this section we will assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are two non-trivial complex Banach spaces.

We will use the following notion of Caputo fractional derivatives of order $\gamma \in (0,1)$. If $u\colon [0,\infty) \to X$ satisfies, for every $T > 0$, $u \in C((0,T] : X)$, $u(\cdot) - u(0) \in L^1((0,T) : X)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0,T) : X)$, then we define its Caputo fractional derivative by,

$$\mathbf{D}_t^{\gamma} u(t) = \frac{d}{dt}\big[g_{1-\gamma} * (u(\cdot) - u(0))\big](t), \quad t \in (0,T].$$

The Weyl fractional derivative W_+^{α} of order $\alpha > 0$ is sometimes also called the Weyl-Liouville fractional derivative of order α. This fractional derivative can be applied to a wide class of functions containing the class of rapidly decreasing functions. In this section, we will use the following definition of the Weyl–Liouville fractional derivative $D_{t,+}^{\gamma} u(t)$ of order $\gamma \in (0,1)$: The Weyl–Liouville fractional derivative $D_{t,+}^{\gamma} u(t)$ of order γ is defined for those continuous functions $u\colon \mathbb{R} \to$

X such that $t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)ds, \quad t \in \mathbb{R}.$$

Set $\mathbf{D}_t^1 u(t) := (d/dt)u(t)$ and $D_{t,+}^1 u(t) := -(d/dt)u(t)$.

3.12.1. Lebesgue spaces with variable exponents $L^{p(x)}$. Let $\emptyset \neq \Omega \subseteq \mathbb{R}$ be a nonempty subset and let $M(\Omega : X)$ stand for we the collection of all measurable functions $f\colon \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the vector space of all Lebesgue measurable functions $p\colon \Omega \to [1,\infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, set

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geqslant 0, \quad 1 \leqslant p(x) < \infty, \\ 0, & 0 \leqslant t \leqslant 1, \quad p(x) = \infty, \\ \infty, & t > 1, \quad p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_{\Omega} \varphi_{p(x)}(\|f(x)\|)dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent as follows,

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}$$

equivalently

$$L^{p(x)}(\Omega : X) = \{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \};$$

see, e.g., [**152**, p. 73].

For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ in the following manner:

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\{\lambda > 0 : \rho(f/\lambda) \leqslant 1\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g. [**152**, Theorem 3.2.7] for scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geqslant 1$ is a constant function. For any $p \in M(\Omega)$, we set

$$p^- := \operatorname{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{esssup}_{x\in\Omega} p(x).$$

Define

$$C_+(\Omega) := \{ p \in M(\Omega) : 1 < p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a.e. } x \in \Omega \}$$

and

$$D_+(\Omega) := \{ p \in M(\Omega) : 1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a.e. } x \in \Omega \}.$$

For $p \in D_+([0,1])$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lesbesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that

$$L^{p(x)}(\Omega : X) = \{ f \in M(\Omega : X) : \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty \}.$$

Furthermore, if $p \in C_+(\Omega)$, then $L^{p(x)}(\Omega : X)$ is uniformly convex and thus reflexive [**176**].

We will use the following lemma (see, e.g., [**152**, Lemma 3.2.20, (3.2.22); Corollary 3.3.4; p. 77] for scalar-valued case):

LEMMA 3.12.1. (i) *Let $p, q, r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leqslant 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) *Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leqslant p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$.*

(iii) *Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leqslant \|g\| \leqslant \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leqslant \|f\|_{p(x)}$.*

For additional details upon Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to the following sources: [**150**, **152**, **176**] and [**436**].

3.12.2. Generalized almost periodic and generalized asymptotically almost periodic functions in Lebesgue spaces with variable exponents $L^{p(x)}$. The following notion of Stepanov $p(x)$-boundedness differs from the one introduced by T. Diagana and M. Zitane in [**150**, Definition 3.10] and [**151**, Definition 4.5], where the authors have used the condition $p \in C_+(\mathbb{R})$:

DEFINITION 3.12.2. Let $p \in \mathcal{P}([0, 1])$ and let $I = \mathbb{R}$ or $I = [0, \infty)$. A function $f \in M(I : X)$ is said to be Stepanov $p(x)$-bounded (or $S^{p(x)}$-bounded), if $f(\cdot + t) \in L^{p(x)}([0, 1] : X)$ for all $t \in I$, and $\sup_{t \in I} \|f(\cdot + t)\|_{p(x)} < \infty$, that is,

$$\|f\|_{S^{p(x)}} := \sup_{t \in I} \inf \left\{ \lambda > 0 : \int_0^1 \varphi_{p(x)}\Big(\frac{\|f(x + t)\|}{\lambda}\Big) dx \leqslant 1 \right\} < \infty.$$

The collection of such functions will be denoted by $L_S^{p(x)}(I : X)$.

From Definition 3.12.2 it follows that the space $L_S^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in L_S^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in L_S^{p(x)}(I : X)$. This is not the case with the notion introduced by T. Diagana and M. Zitane [**150**, **151**], since there the space $L_S^{p(x)}(I : X)$ may or may not be translation invariant depending on $p(x)$. Furthermore, let us note that the notion introduced in these papers is meaningful even in the case that $p \in \mathcal{P}(\mathbb{R})$. We introduce the concept of (asymptotic) $S^{p(x)}$-almost periodicity as follows:

DEFINITION 3.12.3. (i) Let $p \in \mathcal{P}([0, 1])$ and let $I = \mathbb{R}$ or $I = [0, \infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be Stepanov $p(x)$-almost periodic (or Stepanov $p(x)$-a.p.), if the function $\hat{f} \colon I \to L^{p(x)}([0, 1] : X)$ is almost periodic. The collection of such functions will be denoted by $APS^{p(x)}(I : X)$.

(ii) Let $p \in \mathcal{P}([0,1])$ and let $I = [0,\infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be asymptotically Stepanov $p(x)$-almost periodic (or asymptotically Stepanov $p(x)$-a.p.), if the function $\hat{f} \colon I \to L^{p(x)}([0,1] : X)$ is asymptotically almost periodic. The collection of such functions will be denoted by $AAPS^{p(x)}(I : X)$. The abbreviation $S_0^{p(x)}([0,\infty) : X)$ will be used to denote the set of all functions $q \in L_S^{p(x)}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^{p(x)}([0,1] : X))$.

As in the case of Stepanov $p(x)$-boundedness, the space $APS^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in APS^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in APS^{p(x)}(I : X)$. A similar statement holds for the space $AAPS^{p(x)}([0,\infty) : X)$. It is clear that the notions of (asymptotic) Stepanov $p(x)$-boundedness and (asymptotic) Stepanov $p(x)$-almost periodicity are equivalent with those introduced in the previous section, provided that $p(x) \equiv p \geqslant 1$ is a constant function.

Equipped with the norm $\| \cdot \|_{S^{p(x)}}$, the space $L_S^{p(x)}(I : X)$ consisting of all S^p-bounded functions is a Banach space, which is continuously embedded in $L_S^1(I : X)$, for any $p \in \mathcal{P}([0,1])$. Furthermore, it can be easily shown that $APS^{p(x)}(I : X)$ ($AAPS^{p(x)}(I : X)$ with $I = [0,\infty)$) is a closed subspace of $L_S^{p(x)}(I : X)$ and therefore is Banach space itself, for any $p \in \mathcal{P}([0,1])$.

If $p \in \mathcal{P}([0,1])$, then Lemma 3.12.1(ii) yields $L^{p(x)}([0,1] : X) \hookrightarrow L^1([0,1] : X)$, where the symbol $\hookrightarrow$ stands for a “continuous embedding", so that $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$, as well.

We have

PROPOSITION 3.12.4. *Suppose $p \in \mathcal{P}([0,1])$. Then the following continuous embedding hold,*

(i) $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$; *and*
(ii) $APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ *and* $AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^1([0,\infty) : X)$.

Similarly,

PROPOSITION 3.12.5. *Suppose $p \in D_+([0,1])$ and $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ for a.e. $x \in [0,1]$. Then the following continuous embedding hold,*

(i) $L_S^{p^+}(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^{p^-}(I : X)$; *and*
(ii) $APS^{p^+}(I : X) \hookrightarrow APS^{p(x)}(I : X) \hookrightarrow APS^{p^-}(I : X)$ *and* $AAPS^{p^+}([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^{p^-}([0,\infty) : X)$.

Now we will prove that any almost periodic function is $S^{p(x)}$-almost periodic, for any $p \in \mathcal{P}([0,1])$.

PROPOSITION 3.12.6. *Let $p \in \mathcal{P}([0,1])$, and let $f \colon I \to X$ be almost periodic. Then $f(\cdot)$ is $S^{p(x)}$-almost periodic.*

PROOF. To prove that $f(\cdot)$ is $S^{p(x)}$-bounded and $\|f\|_{L_S^{p(x)}} \leqslant \|f\|_\infty$, it suffices to show that, for every $t \in \mathbb{R}$, we have:

$$[\|f\|_\infty, \infty) \subseteq \Big\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\Big(\frac{\|f(x+t)\|}{\lambda}\Big)dx \leqslant 1\Big\}. \tag{459}$$

For $\lambda \geqslant \|f\|_\infty$, we have $\|f(x+t)\|/\lambda \leqslant 1$, $t \in I$. It can be simply perceived that, in this case,

$$\varphi_{p(x)}\Big(\frac{\|f(x+t)\|}{\lambda}\Big) \leqslant 1, \quad t \in I,$$

so that the integrand does not exceed 1; as a matter of fact, by definition of $\varphi_{p(x)}(\cdot)$, we only need to observe that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. Hence, (459) holds. Using the uniform continuity of $f(\cdot)$ and a similar argumentation, we can show that the function $\hat{f} \colon I \to L^{p(x)}([0,1] : X)$ is uniform continuous. For direct proof of almost periodicity of function $\hat{f} \colon I \to L^{p(x)}([0,1] : X)$, we can argue as follows. For $\varepsilon > 0$ given as above, there is a finite number $l > 0$ such that any subinterval I' of I of length l contains a number $\tau \in I'$ such that $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$, $t \in I$. It suffices to observe that, for this $\varepsilon > 0$, we can choose the same length $l > 0$ and the same ε-almost period τ from I' ensuring the validity of inequality $\|\hat{f}(t+\tau+\cdot) - \hat{f}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon$, $t \in I$: in order to see that the last inequality holds true, we only need to prove that, for every $t \in I$, we have

$$[\varepsilon, \infty) \subseteq \Big\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\Big(\frac{\|f(t+\tau+x) - f(t+x)\|}{\lambda}\Big)dx \leqslant 1\Big\}.$$

Indeed, if $\lambda \geqslant \varepsilon$, then $\|f(t+\tau+x) - f(t+x)\|/\lambda \leqslant 1$, $t \in I$ and the integrand cannot exceed 1: this simply follows from definition of $\varphi_{p(x)}(\cdot)$ and observation that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+\tau+x) - f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. The proof of the proposition is thereby complete. $\square$

We can similarly prove the following proposition:

PROPOSITION 3.12.7. *Let $p \in \mathcal{P}([0,1])$, and let $f \colon [0,\infty) \to X$ be asymptotically almost periodic. Then $f(\cdot)$ is asymptotically $S^{p(x)}$-almost periodic.*

Taking into account Proposition 3.12.4(ii) and the method employed in the proof of Proposition 3.12.6, we can state the following:

PROPOSITION 3.12.8. *Assume that $p \in \mathcal{P}([0,1])$ and $f \in L_S^{p(x)}(I : X)$. Then the following holds:*

- (i) $L^\infty(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$.
- (ii) $AP(I : X) \hookrightarrow APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ *and* $AAP([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^1([0,\infty) : X)$.
- (iii) *The continuity (uniform continuity) of $f(\cdot)$ implies continuity (uniform continuity) of $\hat{f}(\cdot)$.*

In general case, we have the following:

PROPOSITION 3.12.9. *Assume that* $p,\ q \in \mathcal{P}([0,1])$ *and* $p \leqslant q$ *a.e. on* $[0,1]$. *Then we have:*

(i) $L_S^{q(x)}(I : X) \hookrightarrow L_S^{p(x)}(I : X)$.

(ii) $APS^{q(x)}(I : X) \hookrightarrow APS^{p(x)}(I : X)$ *and* $AAPS^{q(x)}([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X)$.

(iii) *If* $p \in D_+([0,1])$, *then*

$$L^\infty(I : X) \cap APS^{p(x)}(I : X) = L^\infty(I : X) \cap APS^1(I : X)$$

and

$$L^\infty([0,\infty) : X) \cap AAPS^{p(x)}([0,\infty) : X) = L^\infty([0,\infty) : X) \cap AAPS^1([0,\infty) : X).$$

PROOF. We will prove only (iii) for almost periodicity. Keeping in mind Proposition 3.12.5(ii), it suffices to assume that $p(x) \equiv p > 1$. Then, clearly, $L^\infty(I : X) \cap APS^p(I : X) \subseteq L^\infty(I : X) \cap APS^1(I : X)$ and it remains to be proved the opposite inclusion. So, let $f \in L^\infty(I : X) \cap APS^1(I : X)$. The required conclusion is a consequence of elementary definitions and following simple calculation, which is valid for any $t, \tau \in \mathbb{R}$:

$$\begin{aligned}\left[\int_t^{t+1} \|f(\tau+s) - f(s)\|^p ds\right]^{1/p} & \\ &\leqslant \left[\int_t^{t+1} (2\|f\|_\infty)^{p-1} \|f(\tau+s) - f(s)\| ds\right]^{1/p} \\ &= (2\|f\|_\infty)^{(p-1)/p} \left[\int_t^{t+1} \|f(\tau+s) - f(s)\| ds\right]^{1/p}. \quad \square\end{aligned}$$

REMARK 3.12.10. It is well known that $APS^{p(x)}(I : X)$ can be strictly contained in $APS^1(I : X)$, even in the case that $p(x) \equiv p > 1$ is a constant function. For example, H. Bohr and E. Følner have proved that, for any given number $p > 1$, we can construct a Stepanov almost periodic function defined on the whole real axis that is not Stepanov p-almost periodic (see [**81**, Example, p. 70]). The same example shows that $AAPS^p([0,\infty) : X)$ can be strictly contained in $AAPS^1([0,\infty) : X)$ for $p > 1$ (see e.g. [**249**, Lemma 1]).

REMARK 3.12.11. Proposition 3.12.6 and Proposition 3.12.7 can be simply deduced by using Proposition 3.12.9(ii) and the equalities $AP(I : X) = APS^\infty(I : X) \cap C(I : X)$, $AAP([0,\infty) : X) = AAPS^\infty([0,\infty) : X) \cap C([0,\infty) : X)$, which can be proved almost trivially.

Now we would like to present the following illustrative example:

EXAMPLE 3.12.12. Define $\mathrm{sign}(0) := 0$. Then, for every almost periodic function $f : \mathbb{R} \to \mathbb{R}$, we have that the function $F(\cdot) := \mathrm{sign}(f(\cdot))$ is Stepanov 1-almost periodic [**377**]. Since $F \in L^\infty(\mathbb{R})$, Proposition 3.12.9(iii) yields that the function $F(\cdot)$ is Stepanov p-almost periodic for any $p \geqslant 1$, while Proposition 3.12.8(i) yields that the function $F(\cdot)$ is Stepanov $p(x)$-bounded for any $p \in \mathcal{P}([0,1])$. Due to

Proposition 3.12.5(ii), we have $F \in APS^{p(x)}(\mathbb{R} : \mathbb{C})$ for any $p \in D_+([0,1])$. Consider now the case that $f(x) := \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ and $p(x) := 1 - \ln x$, $x \in [0,1]$. We will prove that $F \notin APS^{p(x)}(\mathbb{R} : \mathbb{C})$. Speaking-matter-of-factly, it is sufficient to show that, for every $\lambda \in (0, 2/e)$ and for every $l > 0$, we can find an interval $I \subseteq \mathbb{R}$ of length $l > 0$ such that, for every $\tau \in I$, there exists $t \in \mathbb{R}$ such that

$$\int_0^1 \Bigl(\frac{1}{\lambda}\Bigr)^{1-\ln x} \Bigl| \operatorname{sign}[\sin(x+t+\tau) + \sin \sqrt{2}(x+t+\tau)] \\ - \operatorname{sign}[\sin(x+t) + \sin \sqrt{2}(x+t)] \Bigr|^{1-\ln x} dx = \infty.$$

Let $\lambda \in (0, 2/e)$ and $l > 0$ be given. Take arbitrarily any interval $I \subseteq \mathbb{R} \setminus \{0\}$ of length l and after that take arbitrarily any number $\tau \in I$. Since $(1/\lambda)^{1-\ln x} \geqslant 1/x$, $x \in [0,1]$ and $1 - \ln x \geqslant 1$, $x \in [0,1]$, a continuity argument shows that it is enough to prove the existence of a number $t \in \mathbb{R}$ such that

$$[\sin(t+\tau) + \sin \sqrt{2}(t+\tau)] \cdot [\sin t + \sin \sqrt{2}t] < 0. \tag{460}$$

If $\sin \tau + \sin \sqrt{2}\tau > 0$ ($\sin \tau + \sin \sqrt{2}\tau < 0$), then we can take $t \sim 0-$ ($t \sim 0+$). Hence, we assume henceforward $\sin \tau + \sin \sqrt{2}\tau = 0$ and $\tau \neq 0$. There exist two possibilities:

$$\tau \in \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \setminus \{0\} \quad \text{or} \quad \tau \in \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}.$$

In the first case, take $t_0 = \frac{\pi}{\sqrt{2}-1}$. Then an elementary argumentation shows that $\tau + t_0 \notin \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \cup \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}$ so that $\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) \neq 0$. If $\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) > 0$ ($\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) < 0$), then for t satisfying (460) we can take any number belonging to a small left/right interval around t_0 for which $\sin t + \sin \sqrt{2}t < 0$ ($\sin t + \sin \sqrt{2}t > 0$). In the second case, there exists an integer $m \in \mathbb{Z}$ such that $\tau = \frac{(2m+1)\pi}{\sqrt{2}-1}$ and we can take $t_0 = \frac{(-2m+1)\pi}{\sqrt{2}-1}$. Then $\tau + t_0 = \frac{2\pi}{\sqrt{2}-1}$ and $\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) \neq 0$, so that we can use a trick similar to that used in the first case. Let us only mention in passing that, with the notion introduced in [**149**], the function $F(\cdot)$ cannot be $S^{p(x)}$-almost automorphic, as well.

The situation is quite different if we consider the case that $f(x) := \sin x$, $x \in \mathbb{R}$. Then $F(\cdot)$ is Stepanov $p(x)$-almost periodic for any $p \in \mathcal{P}([0,1])$. Speaking-matter-of-factly, it can be easily shown that the mapping $\hat{F} : \mathbb{R} \to L^{p(x)}[0,1]$ is continuous and $\|F(t+\tau+\cdot) - F(t+\cdot)\|_{L^{p(x)}[0,1]} = 0$ for all $t \in \mathbb{R}$ and $\tau \in 2\pi\mathbb{Z}$. This, in turn, implies the claimed statement.

Keeping in mind the proofs of Proposition 3.12.6, [**149**, Proposition 3.5] and [**249**, Lemma 1], we can clarify the following result:

Proposition 3.12.13. *Suppose that $p \in \mathcal{P}([0,1])$ and $f : [0,\infty) \to X$ is an asymptotically $S^{p(x)}$-almost periodic function. Then there are uniquely determined $S^{p(x)}$-bounded functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:*

(i) *g is $S^{p(x)}$-almost periodic,*
(ii) *$\hat{q}$ belongs to the class $C_0([0,\infty) : L^{p(x)}([0,1] : X))$,*

(iii) $f(t) = g(t) + q(t)$ *for all* $t \geqslant 0$.

Moreover, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ *of positive reals such that* $\lim_{n\to\infty} t_n = \infty$ *and* $g(t) = \lim_{n\to\infty} f(t+t_n)$ *a.e.* $t \geqslant 0$.

REMARK 3.12.14. The definition of an (equi-)Weyl $p(x)$-almost periodic function (see e.g. **[293]** for the case that $p(x) \equiv p \in [1, \infty)$) can be introduced as follows: Suppose $I = \mathbb{R}$ or $I = [0, \infty)$. Let $p \in \mathcal{P}(I)$ and $f(\cdot + \tau) \in L^{p(x)}(K : X)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$p(x)$-almost periodic, $f \in e - W_{ap}^{p(x)}(I : X)$ for short, iff for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that
$$\sup_{t\in I}[l^{(-1)/p(t)} \|f(\cdot + \tau) - f(\cdot)\|_{L^{p(x)}[t,t+l]}] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$p(x)$-almost periodic, $f \in W_{ap}^{p(x)}(I : X)$ for short, iff for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that
$$\lim_{l\to\infty} \sup_{t\in I}[l^{(-1)/p(t)} \|f(\cdot + \tau) - f(\cdot)\|_{L^{p(x)}[t,t+l]}] \leqslant \varepsilon.$$

The notion of (equi-)Weyl $p(x)$-almost periodicity as well as the corresponding notion for Besicovitch classes of almost periodic functions will not attract our attention here. We will also skip all details concerning asymptotical $p(x)$-almost periodicity for Weyl and Besicovitch classes.

3.12.3. Generalized two-parameter almost periodic type functions and composition principles. Assume that $(Y, \|\cdot\|_Y)$ is a complex Banach space, as well as that $I = \mathbb{R}$ or $I = [0, \infty)$. By $C_0([0, \infty) \times Y : X)$ we denote the space consisting of all continuous functions $h : [0, \infty) \times Y \to X$ such that $\lim_{t\to\infty} h(t, y) = 0$ uniformly for y in any compact subset of Y. A continuous function $f : I \times Y \to X$ is said to be uniformly continuous on bounded sets, uniformly for $t \in I$ iff for every $\varepsilon > 0$ and every bounded subset K of Y there exists a number $\delta_{\varepsilon,K} > 0$ such that $\|f(t, x) - f(t, y)\| \leqslant \varepsilon$ for all $t \in I$ and all $x, y \in K$ satisfying that $\|x - y\| \leqslant \delta_{\varepsilon,K}$. If $f : I \times Y \to X$, set $\hat{f}(t, y) := f(t + \cdot, y)$, $t \geqslant 0$, $y \in Y$. We need to recall the following well-known definition (see e.g. **[293]** for more details):

DEFINITION 3.12.15. Let $1 \leqslant p < \infty$.

(i) A function $f : I \times Y \to X$ is said to be almost periodic iff $f(\cdot, \cdot)$ is bounded, continuous as well as for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exists $l(\varepsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\varepsilon, K)$ contains a number τ with the property that $\|f(t + \tau, y) - f(t, y)\| \leqslant \varepsilon$ for all $t \in I$, $y \in K$. The collection of such functions will be denoted by $AP(I \times Y : X)$.

(ii) A function $f : [0, \infty) \times Y \to X$ is said to be asymptotically almost periodic iff it is bounded continuous and admits a decomposition $f = g + q$, where $g \in AP([0,\infty) \times Y : X)$ and $q \in C_0([0,\infty) \times Y : X)$. Denote by $AAP([0,\infty) \times Y : X)$ the vector space consisting of all such functions.

The notion of (asymptotical) Stepanov $p(x)$-almost periodicity for the functions depending on two parameters is introduced as follows:

DEFINITION 3.12.16. Let $p \in \mathcal{P}([0,1])$.

(i) A function $f\colon I \times Y \to X$ is called Stepanov $p(x)$-almost periodic, $S^{p(x)}$-almost periodic for short, iff $\hat{f} : I \times Y \to L^{p(x)}([0,1] : X)$ is almost periodic. The vector space consisting of all such functions will be denoted by $APS^{p(x)}(I \times Y : X)$.
(ii) A function $f\colon [0,\infty) \times Y \to X$ is said to be asymptotically $S^{p(x)}$-almost periodic iff $\hat{f} : [0,\infty) \times Y \to L^{p(x)}([0,1] : X)$ is asymptotically almost periodic. The vector space consisting of all such functions will be denoted by $AAPS^{p(x)}([0,\infty) \times Y : X)$.

The proof of following proposition is very similar to the proof of [**293**, Lemma 2.2.6] and therefore omitted.

PROPOSITION 3.12.17. *Let $p \in \mathcal{P}([0,1])$. Suppose that $f\colon [0,\infty) \times Y \to X$ is an asymptotically $S^{p(x)}$-almost periodic function. Then there are two functions $g\colon \mathbb{R} \times Y \to X$ and $q\colon [0,\infty) \times Y \to X$ satisfying that for each $y \in Y$ the functions $g(\cdot, y)$ and $q(\cdot, y)$ are Stepanov $p(x)$-bounded, as well as that the following holds:*

(i) $\hat{g}\colon \mathbb{R} \times Y \to L^{p(x)}([0,1] : X)$ is almost periodic,
(ii) $\hat{q} \in C_0([0,\infty) \times Y : L^{p(x)}([0,1] : X))$,
(iii) $f(t,y) = g(t,y) + q(t,y)$ for all $t \geqslant 0$ and $y \in Y$.

Moreover, for every compact set $K \subseteq Y$, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty} t_n = \infty$ and $g(t,y) = \lim_{n\to\infty} f(t + t_n, y)$ for all $y \in Y$ and a.e. $t \geqslant 0$.

In [**293**, Theorem 2.7.1, Theorem 2.7.2], we have slightly improved the important composition principle atributed to W. Long, S.-H. Ding [**399**, Theorem 2.2]. Further refinements for $S^{p(x)}$-almost periodicity can be deduced similarly, with appealing to Lemma 3.12.1(i)–(iii) and the arguments employed in the proof of [**399**, Theorem 2.2]:

THEOREM 3.12.18. *Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) $f \in APS^{p(x)}(I \times Y : X)$ and there exist a function $r \in \mathcal{P}([0,1])$ such that $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ and a function $L_f \in L_S^{r(x)}(I)$ such that:

$$\|f(t,x) - f(t,y)\| \leqslant L_f(t)\|x - y\|_Y, \quad t \in I,\ x,\ y \in Y; \tag{461}$$

(ii) $u \in APS^{p(x)}(I : Y)$, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{u(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y; here, $m(\cdot)$ denotes the Lebesgue measure.

Define $q \in \mathcal{P}([0,1])$ *by* $q(x) := p(x)r(x)/p(x) + r(x)$*, if* $x \in [0,1]$ *and* $r(x) < \infty$*,* $q(x) := p(x)$*, if* $x \in [0,1]$ *and* $r(x) = \infty$*. Then* $q(x) \in [1, p(x))$ *for* $x \in [0,1]$*,* $r(x) < \infty$ *and* $f(\cdot, u(\cdot)) \in APS^{q(x)}(I : X)$*.*

Concerning asymptotical two-parameter Stepanov $p(x)$-almost periodicity, we can deduce the following composition principles with $X = Y$; the proof is very similar to those of [**293**, Proposition 2.7.3, Proposition 2.7.4] established in the case of constant functions p, q, r:

PROPOSITION 3.12.19. *Let* $I = [0, \infty)$*, and let* $p \in \mathcal{P}([0,1])$*. Suppose that the following conditions hold:*

- (i) $g \in APS^{p(x)}(I \times X : X)$*, there exist a function* $r \in \mathcal{P}([0,1])$ *such that* $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ *and a function* $L_g \in L_S^{r(x)}(I)$ *such that* (461) *holds with the function* $f(\cdot,\cdot)$*replaced by the function* $g(\cdot,\cdot)$ *therein.*
- (ii) $v \in APS^{p(x)}(I : X)$*, and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K = \{v(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* X*.*
- (iii) $f(t,x) = g(t,x) + q(t,x)$ *for all* $t \geqslant 0$ *and* $x \in X$*, where* $\hat{q} \in C_0([0,\infty) \times X : L^{q(x)}([0,1] : X))$ *with* $q(\cdot)$ *defined as above;*
- (iv) $u(t) = v(t) + \omega(t)$ *for all* $t \geqslant 0$*, where* $\hat{\omega} \in C_0([0,\infty) : L^{p(x)}([0,1] : X))$*.*
- (v) *There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{u(t) : t \in I \setminus E'\}$ *is relatively compact in* X*.*

Then $f(\cdot, u(\cdot)) \in AAPS^{q(x)}(I : X)$*.*

3.12.4. Generalized (asymptotical) almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$: action of convolution products. Throughout this section, we assume that $p \in \mathcal{P}([0,1])$ and a multivalued linear operator $\mathcal{A}$ fulfills the condition (P). We will first investigate infinite convolution products. The results obtained can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following abstract Cauchy differential inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + g(t), \quad t \in \mathbb{R}$$

and the following abstract Cauchy relaxation differential inclusion

$$D_{t,+}^{\gamma} u(t) \in -\mathcal{A}u(t) + g(t), \quad t \in \mathbb{R}, \tag{462}$$

where $D_{t,+}^{\gamma}$ denotes the Weyl–Liouville fractional derivative of order $\gamma \in (0,1)$ and $g\colon \mathbb{R} \times X \to X$ satisfies certain assumptions; see [**293**] for further information in this direction. Keeping in mind composition principles clarified in the previous section, it is almost straightforward to reformulate some known results concerning semilinear analogues of the above inclusions (see e.g. [**293**, Theorem 2.7.6–Theorem 2.7.9; Theorem 2.9.10–Theorem 2.9.11; Theorem 2.9.17–Theorem 2.9.18]); because of that, this question will not be examined here for the sake of brevity.

We start by stating the following generalization of [**337**, Proposition 2.11] (the reflexion at zero keeps the spaces of Stepanov p-almost periodic functions unchanged, which may or may not be the case with the spaces of Stepanov $p(x)$-almost periodic functions):

PROPOSITION 3.12.20. *Suppose that* $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. *If* $\breve{g} \colon \mathbb{R} \to X$ *is* $S^{p(x)}$*-almost periodic, then the function* $G \colon \mathbb{R} \to Y$*, given by* (433) *with the function* $f(\cdot)$ *replaced by the function* $g(\cdot)$ *therein, is well-defined and almost periodic.*

PROOF. Without loss of generality, we may assume that $X = Y$. It is clear that, for every $t \in \mathbb{R}$, we have that $G(t) = \int_0^\infty R(s)g(t-s)ds$ and that the last integral is absolutely convergent due to Lemma 3.12.1(i) and $S^{p(x)}$-boundedness of function $\breve{g}(\cdot)$:

$$\begin{aligned}\int_0^\infty \|R(s)\|\|g(t-s)\|ds &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\|\|g(t-s)\|ds \\ &= \sum_{k=0}^{\infty} \int_0^1 \|R(s+k)\|\|g(t-s-k)\|ds \\ &\leqslant 2\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}([0,1]:X)} \|g(t-k-\cdot)\|_{L^{p(x)}([0,1]:X)} \\ &\leqslant 2M \sup_{t\in\mathbb{R}} \|\breve{g}(\cdot - t)\|_{L^{p(x)}([0,1]:X)},\end{aligned}$$

for any $t \in \mathbb{R}$. Let a number $\varepsilon > 0$ be fixed. Then there is a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\|\breve{g}(t-\tau+\cdot) - \breve{g}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon$, $t \in \mathbb{R}$. Invoking Lemma 3.12.1(i) and this fact, we get

$$\begin{aligned}\|G(t+\tau) - G(t)\| &\leqslant \int_0^\infty \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|dr \\ &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|dr \\ &= \sum_{k=0}^{\infty} \int_0^1 \|R(r+k)\| \cdot \|g(t+\tau-r-k) - g(t-r-k)\|dr \\ &\leqslant 2\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} \|g(t+\tau-\cdot-k) - g(t-\cdot-k)\|_{L^{p(x)}[0,1]} \\ &= 2\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} \|\breve{g}(\cdot-t-\tau+k) - \breve{g}(\cdot-t+k)\|_{L^{p(x)}[0,1]} \\ &\leqslant 2\varepsilon \sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} = 2M\varepsilon, \quad t \in \mathbb{R},\end{aligned}$$

which clearly implies that the set of all ε-periods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. It remains to be proved the uniform continuity of $G(\cdot)$. Since $\hat{\breve{g}}(\cdot)$ is uniformly continuous, we have the existence of a number $\delta \in (0,1)$ such that

$$\|\breve{g}(\cdot - t') - \breve{g}(\cdot - t)\|_{L^{p(x)}[0,1]} < \varepsilon, \text{ provided } t,\ t' \in \mathbb{R} \text{ and } |t-t'| < \delta. \tag{463}$$

For any $\delta' \in (0,\delta)$, the above computation with $\tau = \delta' = t' - t$ and (463) together imply that, for every $t \in \mathbb{R}$,

$$\begin{aligned}\|G(t+\delta') - G(t)\| &\leqslant 2\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} \|\check{g}(\cdot - t' + k) - \check{g}(\cdot - t + k)\|_{L^{p(x)}[0,1]} \\ &\leqslant 2\varepsilon \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} = 2M\varepsilon.\end{aligned}$$

This completes the proof of proposition. □

EXAMPLE 3.12.21. (i) Suppose that $\beta \in (0,1)$ and $(R(t))_{t>0} = (T(t))_{t>0}$ is a degenerate semigroup generated by $\mathcal{A}$. Let us recall that there exists a finite constant $M > 0$ such that $\|T(t)\| \leqslant Mt^{\beta-1}$, $t \in (0,1]$ and $\|T(t)\| \leqslant Me^{-ct}$, $t \geqslant 1$. Let $p_0 > 1$ be such that

$$\frac{p_0}{p_0 - 1}(\beta - 1) \leqslant -1,$$

let $p \in \mathcal{P}([0,1])$, and let $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Assume that we have constructed a function $\check{g} \in APS^{p(x)}(\mathbb{R} : X)$ such that $\check{g} \notin APS^{p}(\mathbb{R} : X)$ for all $p \geqslant p_0$ (Question: Could we manipulate here somehow with the construction established in [**81**, Example, p. 70]?). Then, in our concrete situation, [**337**, Proposition 2.11] cannot be applied since

$$\frac{p}{p-1}(\beta - 1) \leqslant -1, \quad p \in [1, p_0).$$

Now we will briefly explain that $\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$, showing that Proposition 3.12.20 is applicable. Strictly speaking, for $k = 0$, $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$ by our assumption, while, for $k \geqslant 1$, it can be simply shown that $\|R(\cdot+k)\|_{L^{q(x)}[0,1]} \leqslant Me^{-ck}$ so that $\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$, as claimed.

(ii) By a mild solution of (462), we mean the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)g(s)\,ds$, $t \in \mathbb{R}$. Let $p \in \mathcal{P}([0,1])$, and let $\|R_\gamma(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Then, for $k \geqslant 1$, we have $\|R_\gamma(\cdot+k)\|_{L^{q(x)}[0,1]} \leqslant M_2 k^{-1-\gamma}$. Hence, $\sum_{k=0}^{\infty} \|R_\gamma(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$ and we can apply Proposition 3.12.20.

In the following proposition, whose proof is very similar to that of [**149**, Proposition 3.12], we state some invariance properties of generalized asymptotical almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$ under the action of finite convolution products (see also [**293**, Proposition 2.7.5, Lemma 2.9.3] for similar results). This proposition generalizes [**337**, Proposition 2.13] provided that $p > 1$ in its formulation.

PROPOSITION 3.12.22. *Suppose that $p \in \mathcal{P}([0,1])$, $q \in D_+([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that, for every $t \geqslant 0$, we have that $m_t := \sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(x)}[0,1]} < \infty$. Suppose, further, that $\check{g}\colon \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, $q \in L_S^{p(x)}([0,\infty) : X)$ and $f(t) = g(t) + q(t)$, $t \geqslant 0$. Let $r_1, r_2 \in \mathcal{P}([0,1])$ and the following holds:*

(i) *For every $t \geqslant 0$, the mapping $x \mapsto \int_0^{t+x} R(t+x-s)q(s)ds$, $x \in [0,1]$ belongs to the space $L^{r_1(x)}([0,1]:X)$ and we have*

$$\lim_{t\to+\infty} \left\| \int_0^{t+x} R(t+x-s)q(s)ds \right\|_{L^{r_1(x)}[0,1]} = 0.$$

(ii) *For every $t \geqslant 0$, the mapping $x \mapsto m_{t+x}$, $x \in [0,1]$ belongs to the space $L^{r_2(x)}[0,1]$ and we have*

$$\lim_{t\to+\infty} \|m_{t+x}\|_{L^{r_2(x)}[0,1]} = 0.$$

Then the function $H(\cdot)$, given by

$$H(t) := \int_0^t R(t-s)f(s)ds, \quad t \geqslant 0,$$

is well defined, bounded and belongs to the class $APS^{p(x)}(\mathbb{R}:X) + S_0^{r_1(x)}([0,\infty):X) + S_0^{r_2(x)}([0,\infty):X)$, with the meaning clear.

REMARK 3.12.23. In [**337**, Remark 2.14], we have examined the conditions under which the function $H(\cdot)$ defined above is asymptotically almost periodic, provided that the function $g(\cdot)$ is S^p-almost periodic for some $p \in [1,\infty)$. The interested reader may try to analyze similar problems with function $\breve{g}(\cdot)$ being $S^{p(x)}$-almost periodic for some $p \in \mathcal{P}([0,1])$.

3.12.5. Some applications. Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\Omega$ and let $1 < p < \infty$. Among other things, one can make use of Proposition 3.12.22 to establish the existence and uniqueness of asymptotically $S^{p(x)}$-almost automorphic solutions to the damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(m(x)\dfrac{\partial u}{\partial t}\Big) + (2\omega m(x) - \Delta)\dfrac{\partial u}{\partial t} + (A(x;D) - \omega\Delta \\ \qquad\qquad\qquad\qquad + \omega^2 m(x))u(x,t) = f(x,t), \quad t \geqslant 0,\ x \in \Omega; \\ u = \dfrac{\partial u}{\partial t} = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x),\ m(x)\Big[\Big(\dfrac{\partial u}{\partial t}\Big)(x,0) + \omega u_0\Big] = m(x)u_1(x), \quad x \in \Omega, \end{cases}$$

where $m(x) \in L^\infty(\Omega)$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting on its maximal domain, $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second-order linear differential operator on Ω with continuous coefficients on $\bar{\Omega}$, see, e.g., [**199**, Example 6.1] and [**293**] for further details.

Notice that we can also consider the existence and uniqueness of asymptotically $S^{p(x)}$-almost periodic solutions to the following fractional damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \mathbf{D}_t^\gamma(m(x)\mathbf{D}_t^\gamma u) + (2\omega m(x) - \Delta)\mathbf{D}_t^\gamma u + (A(x;D) \\ \qquad\qquad - \omega\Delta + \omega^2 m(x))u(x,t) = f(x,t), \quad t \geqslant 0,\ x \in \Omega; \\ u = \mathbf{D}_t^\gamma u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x),\ m(x)[\mathbf{D}_t^\gamma u(x,0) + \omega u_0] = m(x)u_1(x), \quad x \in \Omega. \end{cases}$$

Additionally, it is also clear that Proposition 3.12.20 can be used to study the existence and uniqueness of almost periodic solutions of the following abstract integral inclusion

$$u(t) \in \mathcal{A}\int_{-\infty}^t a(t-s)u(s)ds + g(t), \quad t \in \mathbb{R}$$

where $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\check{g}\colon \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic and $\mathcal{A}$ is a closed multivalued linear operator on X, see, e.g., [**293**].

3.13. Appendix and notes

Without any doubt, the most important monograph which obeys the multivalued linear operators approach to abstract degenerate differential equations is [**199**], written by A. Favini and A. Yagi. The fundamental part of this monograph is Chapter III, where the authors have considered the generation of infinitely differentiable semigroups by multivalued linear operators, provided that there exist finite constants c, $M > 0$ and $\beta \in (0,1]$ such that the condition (PW) holds (cf. Section 3.5 for more details). As already mentioned, a great part of results from [**199**, Section 3.2-Section 3.5] is not attainable in fractional case.

We start by sketching briefly the most relevant details about fractional powers and interpolation theory for multivalued linear operators satisfying the condition (PW); cf. [**181**]. Let $\mathcal{A}$ be such an MLO, and let $(T(t))_{t>0}$ be a semigroup generated by $\mathcal{A}$ (the pivot space is one of Banach's and will be denoted by E). Then the fractional power $(-\mathcal{A})^\theta$ is defined for $|\theta| > 1-\beta$ and it is very difficult to tell what would be the fractional power $(-\mathcal{A})^\theta$ in the case that $|\theta| \leqslant 1-\beta$ (we can provide a substantially larger information base about fractional powers in the classical case $\beta = 1$), without using some regularizing techniques established in the previous section. The spaces $(E; D(\mathcal{A}))_{\gamma,p}$ and $E_{\mathcal{A}}^{\gamma,p}$ are defined as follows. If X is a Banach space, $g : (0,\infty) \to X$ is an X-valued strongly measurable function and $q \in [1,\infty)$, set $\|g\|_{L_q^*(X)} := (\int_0^\infty \|g(t)\|_X^q \frac{dt}{t})^{1/q}$ and $\|g\|_{L_\infty^*(X)} := \operatorname{ess\,sup}_{t>0} \|g(t)\|_X$. Suppose that either $p_0, p_1 \in [1,\infty)$ or $p_0 = p_1 = \infty$; then for any $\gamma \in (0,1)$, we set $p = ((1-\gamma)p_0^{-1} + \gamma p_1^{-1})^{-1}$ if $p_0, p_1 \in [1,\infty)$, and $p = \infty$ if $p_0 = p_1 = \infty$. The real interpolation space $(E; D(\mathcal{A}))_{\gamma,p}$ between E and $D(\mathcal{A})$ is defined by

$$\begin{aligned}(E; D(\mathcal{A}))_{\gamma,p} := \big\{x \in E : x = v_0(t) + v_1(t) \text{ for some } t > 0,\ \text{where} \\ v_0 \in C((0,\infty) : E),\ v_1 \in C((0,\infty) : [D(\mathcal{A})]), \\ \|\cdot^\gamma\, v_0(\cdot)\|_{L_{p_0}^*(E)} + \|\cdot^{\gamma-1}\, v_1(\cdot)\|_{L_{p_1}^*([D(\mathcal{A})])} < \infty\big\}.\end{aligned}$$

Equipped with the norm

$$\|x\|_{(E;D(\mathcal{A}))_{\gamma,p}} := \inf_{v_0,v_1} \big[\| \cdot^{\gamma} v_0(\cdot)\|_{L^*_{p_0}(E)} + \| \cdot^{\gamma-1} v_1(\cdot)\|_{L^*_{p_1}([D(\mathcal{A})])}\big],$$

where the infimum is taken over all possible representations of x having the above form, $(E;D(\mathcal{A}))_{\gamma,p}$ becomes a Banach space. It is well known that there exists a constant $c_1(\gamma,p) > 0$ such that the interpolation inequality $\|x\|_{(E;D(\mathcal{A}))_{\gamma,p}} \leqslant c_1(\gamma,p)\|x\|^{1-\gamma}\|x\|^{\gamma}_{[D(\mathcal{A})]}$, $x \in D(\mathcal{A})$ holds, as well as that $[D(\mathcal{A})] \hookrightarrow (E;D(\mathcal{A}))_{\gamma,p} \hookrightarrow E$. The Banach space $E^{\gamma,p}_{\mathcal{A}}$ ($\gamma \in (0,1)$, $p \in [1,\infty]$) is defined as follows:

$$E^{\gamma,p}_{\mathcal{A}} := \big\{x \in E \ ; \ [x]_{E^{\gamma,p}_{\mathcal{A}}} := \|t^{\gamma}[t(t-A)^{-1}x - x]\|_{L^*_p(E)} < \infty,$$
$$\|x\|_{E^{\gamma,p}_{\mathcal{A}}} := \|x\| + [x]_{E^{\gamma,p}_{\mathcal{A}}}\big\}.$$

Observe that we have already used the space $E^{\gamma}_{\mathcal{A}} = E^{\gamma,\infty}_{\mathcal{A}}$ in our previous work.

The following holds:

(a) $[D((-\mathcal{A})^{\theta_1})] \hookrightarrow [D((-\mathcal{A})^{\theta_2})]$ for $1-\beta < \theta_2 < \theta_1 + \beta - 1$;

(b) If $\gamma \in (0,1)$ and $1 \leqslant p_1 \leqslant p \leqslant p_2 \leqslant \infty$, then
$$[D(\mathcal{A})] \hookrightarrow (E;D(\mathcal{A}))_{\gamma,p_1} \hookrightarrow (E;D(\mathcal{A}))_{\gamma,p} \hookrightarrow (E;D(\mathcal{A}))_{\gamma,p_2} \hookrightarrow E;$$
furthermore, if $0 < \gamma_2 < \gamma_1 < 1$, then
$$(E;D(\mathcal{A}))_{\gamma_1,\infty} \hookrightarrow (E;D(\mathcal{A}))_{\gamma_2,1};$$

(c) $E^{\gamma,p}_{\mathcal{A}} \cap \mathcal{A}0 = \{0\}$ for $\gamma \in (0,1)$, $p \in [1,\infty]$;

(d) $E^{\gamma,p}_{\mathcal{A}} \hookrightarrow (E;D(\mathcal{A}))_{\gamma,p}$ for $\gamma \in (0,1)$, $p \in [1,\infty]$;

(e) $(E;D(\mathcal{A}))_{\gamma,p} \hookrightarrow E^{\gamma+\beta-1,p}_{\mathcal{A}}$ for $\gamma \in (1-\beta,1)$, $p \in [1,\infty]$ and $E^{\gamma,1}_{\mathcal{A}} \hookrightarrow [D((-\mathcal{A})^{\theta})]$ for $\gamma \in (1-\beta,1)$;

(f) $(T(t))_{t>0}$ is strongly continuous on $(E;D(\mathcal{A}))_{\gamma,p}$ and $E^{\gamma,p}_{\mathcal{A}}$ for $\gamma \in (1-\beta,1)$, $p \in [1,\infty]$;

(g) Define $T(0) := I$ and let $X \in \{E^{\gamma,p}_{\mathcal{A}}, (E;D(\mathcal{A}))_{\gamma,p}\}$ ($\gamma \in (0,1)$, $p \in [1,\infty]$). Then for each $\gamma \in (2-\alpha-\beta,1)$ there exists a constant $c > 0$ such that
$$\|T(t) - T(s)\|_{L(X,E)} \leqslant c(t-s)^{(\alpha+\beta+\gamma-2)/\alpha} \text{ for } t > s \geqslant 0.$$

For further information about fractional powers and interpolation spaces of multivalued linear operators, as well as their applications in the qualitative analysis of abstract degenerate differential equations of first order, we refer the reader to the paper [**180**] by A. Favaron.

Concerning the monograph [**199**], we only want to say a few new words about the most important results from [**199**, Chapter VI], which cannot be so simply reconsidered in fractional case, as well. The first section of this chapter is devoted to the study of following initial value problem of parabolic type:

$$\frac{d}{dt}(Cu') + Bu' + Au = f(t), \quad t \in (0,T]; \ u(0) = u_0, \ Cu'(0) = Cu_1, \tag{464}$$

where A, B, C are three closed linear operators acting on E, $D(B) \subseteq D(A)$ and B has a bounded inverse. By a solution of (464) we mean any function $u \in C^1((0,T] : E)$ such that $u(t) \in D(A)$, $t \in (0,T]$, $u'(t) \in D(B) \cap D(C)$, $t \in (0,T]$, $Cu' \in C^1((0,T] : E)$, $Bu' \in C((0,T] : E)$, (464) holds on $(0,T]$ and $\lim_{t\to 0+} u(t) = u_0$,

$\lim_{t\to 0+} Cu'(t) = Cu_1$. As already mentioned in Example 3.7.14(ii), the problem (464) can be rewritten in the following matricial form

$$\frac{d}{dt}Mz(t) = Lz(t) + F(t), \quad t \in (0,T],$$

where

$$M = \begin{bmatrix} I & O \\ O & C \end{bmatrix}, \; L = \begin{bmatrix} O & I \\ -A & -B \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \quad (t \in (0,T]).$$

The underlying Banach space is chosen to be $[D(B)] \times E$. Using this approach, the authors have proved that there exists a unique solution of (464) provided the validity of following conditions (cf. [**199**, Theorem 6.1]):

(i) There exist constants $c > 0$, $\alpha \in (0,1]$ and $\beta \in (0,\alpha]$ such that

$$\Theta_{\alpha,c} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant c(1 + |\operatorname{Im}\lambda|)^{\alpha}\} \subseteq \rho_C(B)$$

and

$$\|C(\lambda C + B)^{-1}\| = O((1 + |\lambda|)^{-\beta}), \quad \lambda \in -\Theta_{\alpha,c};$$

(ii) $u_0,\ u_1 \in D(B)$, $2\alpha + \beta > 2$, $(2 - \alpha - \beta)/\alpha < \sigma \leqslant 1$, and

(iii) $f \in C^{\sigma}([0,T] : E)$.

Under some additional conditions, in [**199**, Theorem 6.2] it has been shown the Hölder continuity of functions Bu' and $(Cu')'$, where u is a solution of (464).

Set $U := AB^{-1}$ and $\mathcal{A} := BC^{-1}$. The conditions

(a) $$\|B(\lambda B + A)^{-1}\| = O((1 + |\lambda|)^{-\gamma}), \quad \lambda \in -\Theta_{\alpha,c},$$
where $0 < \gamma \leqslant 1$, and

(b) $$\|U(\lambda + \mathcal{A})^{-1}\| = O((1 + |\lambda|)^{-\delta}), \quad \lambda \in -\Theta_{\alpha,c},\ |\lambda| \text{ suff. large},$$
where $\delta > 0$,

play a crucial role in [**199**, Theorem 6.3–Theorem 6.4]. The leading feature of [**199**, Theorem 6.7] is to extend these assertions to the following higher-order initial value problem:

$$\begin{cases} \frac{d}{dt}(A_n u^{(n-1)}) + \sum_{j=0}^{n-1} A_j u^{(j)} = f(t), \quad t \in (0,T], \\ u^{(j)}(0) = u_j, \quad 0 \leqslant j \leqslant \lceil \alpha \rceil - 2, \\ A_n u^{(n-1)}(0) = A_n u_{n-1}, \end{cases}$$

by using an idea of M. K. Balaev [**43**]. In [**199**, Section 6.2], the authors have presented a great number of concrete applications of results from Section 6.1, while in [**199**, Section 6.3], the complete second order degenerate Cauchy problem

$$\begin{cases} Cu''(t) + Bu'(t) + Au(t) = f(t), \quad t \in (0,T], \\ u(0) = u_0,\ u'(0) = u_1, \end{cases}$$

has been considered in a Hilbert space H. The abstract results have been applied in the analysis of Poisson wave equation

$$\begin{cases} m_1(x)\frac{\partial^2 u}{\partial t^2}(x,t) + m_2(x)\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \Sigma, \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}u(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

in $L^2(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $m_1(x)$, $m_2(x)$ are two given nonnegative continuous functions on $\bar{\Omega}$. We would like to address the problem of qualitative analysis of some fractional analogues of Poisson wave equation to our readers.

Section 6.5 is devoted to the study of initial value problem

$$\begin{cases} -\frac{d^2}{dt^2}(Bu) + Au = f(t), & 0 \leqslant t \leqslant 1, \\ Bu(0) = Bu(1) = 0, \end{cases}$$

provided that there exist constants $c > 0$, $\alpha \in (0,1]$ and $\beta \in (0,\alpha]$ such that $\Theta_{\alpha,c} \subseteq \rho(AB^{-1})$ and $\|B(\lambda B - A)^{-1}\| = O((1+|\lambda|)^{-\beta})$, $\lambda \in \Theta_{\alpha,c}$. Finally, the abstract Cauchy problem

$$\begin{cases} u''(t) = t^m Bu(t), & t \geqslant 0, \\ u(0) = u_0, \ \frac{du}{dt}(0) = u_1, \end{cases}$$

where $m > 0$ and B is a closed densely defined operator in a Banach space E satisfying that there exists a finite constant $M > 0$ such that $\|(\lambda + B)^{-1}\| \leqslant M\lambda^{-1}$, $\lambda > 0$ has been investigated in Section 6.6.

Semilinear degenerate Cauchy inclusions. Concerning semilinear degenerate differential inclusions of first order, we can warmly recommend the reading of monograph [**268**] by M. Kamenskii, V. Obukhovskii and P. Zecca. In what follows, we shall briefly describe the most important results from the paper [**437**].

Suppose that E is a real reflexive Banach space, $F\colon [0,T] \times E \to P(E)$ is a given multimap, as well as that A and B are two single-valued linear operators on E satisfying that $D(A) \subseteq D(B)$ and $B(D(A)) \subseteq R(A)$. In [**437**], V. Obukhovskii and P. Zecca have analyzed the semilinear differential equation

$$\frac{d}{dt}[Bu(t)] = Au(t) + F(t, Bu(t)), \quad t \in [0,T] \ ; \ Bu(0) = y_0 \in \overline{B(D(A))} \tag{465}$$

by using the change of variables $v(t) = Bu(t)$ and passing after that to the corresponding abstract degenerate semilinear Cauchy inclusion with the multivalued linear operator $\mathcal{A} = AB^{-1}$. They have assumed that the operator $\mathcal{A}$ satisfies the Hille–Yosida condition, so that $\mathcal{A}$ generates a strongly continuous semigroup $(U(t))_{t\geqslant 0}$ on E. A function $u(t)$ is said to be a mild solution of (465) iff there exists a measurable section $f(t)$ of the multivalued mapping $t \mapsto F(t, Bu(t))$, $t \in [0,T]$ such that:

$$Bu(t) = U(t)Bu(0) + \int_0^t U(t-s)f(s)\,ds, \quad t \in [0,T].$$

Denote $P_0(E) := \{S \subseteq E : S \neq \emptyset\}$, $K(E) := \{S \in P_0(E) : S \text{ is compact}\}$ and $Kv(E) := \{S \in K(E) : S \text{ is convex}\}$. Following [**437**, Definition 2.12], we say that a multimap $\mathbf{F}\colon X \to P_0(Y)$, where X and Y are real Banach spaces, is:

(i) upper semicontinuous (u.s.c.) iff $\mathbf{F}^{-1}(V) = \{x \in X : \mathbf{F}(x) \subseteq V\}$ is an open subset of X for every open set $V \subseteq Y$;
(ii) lower semicontinuous (l.s.c.) iff $\mathbf{F}^{-1}(W)$ is a closed subset of X for every closed set $W \subseteq Y$.

Recall that the Hausdorff measure of noncompactness of a non-empty subset Ω of E is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

In [**437**, Theorem 3.1], the authors have proved that the set Σ consisting of all mild solutions of problem (465) is nonempty as well as that the set $B\Sigma = \{v \in C([0,T] : E) \mid v(t) = Bu(t),\ u \in \Sigma\}$ is compact in $C([0,T] : E)$, provided the following conditions:

(T1) $F(\cdot,\cdot)$ has nonempty, compact, and convex values;
(T2) For every $x \in E$, the multimapping $F(\cdot,x)\colon [0,T] \to Kv(E)$ has a strongly measurable selection;
(T3) The multimapping $F(t,\cdot) : E \to Kv(E)$ is u.s.c. for a.e. $t \in [0,T]$;
(T4) There exists a function $\alpha \in L^1([0,T])$ such that $\alpha(t) \geqslant 0$ for a.e. $t \in [0,T]$, and

$$\|F(t,x)\| := \sup\{\|z\| : z \in F(t,x)\} \leqslant \alpha(t)(1 + \|x\|) \text{ for a.e. } t \in [0,T];$$

(T5) There exists a function $k \in L^1([0,T])$ such that $k(t) \geqslant 0$ for a.e. $t \in [0,T]$, and

$$\chi(F(t,D)) \leqslant k(t)\chi(D) \text{ for a.e. } t \in [0,T] \text{ and every bounded set } D \subseteq E.$$

The conditions (T2) and (T3) are so-called the upper Carathéodory conditions. In [**437**, Theorem 3.2], the condition (T1) has been replaced by the following one:

(T6) There exists a sequence of disjoint compact sets $\{I_n\}$, $I_n \subseteq [0,T]$, such that $m([0,T] \smallsetminus I) = 0$, where $I = \bigcup_n I_n$, and the restriction of $F(\cdot,\cdot)$ on each set $I_n \times E$ is l.s.c.

If we assume the separability of space E and (T4)-(T6), then there exists at least one mild solution of problem (465). Boundary value problems have been analyzed in [**437**, Subsection 3.2].

Now we will present the most important results established in the research study [**221**] by M. Fuhrman.

Sums of generators of analytic semigroups: multivalued linear operators approach. Approximation and interpolation results for multivalued linear operators have been essentially utilized by M. Fuhrman in proving a few remarkable perturbation results for generators of analytic semigroups [**221**]. Let us recall that the generator of an analytic semigroup is any closed (not necessarily densely defined) single-valued operator A satisfying that there exist numbers $\omega \in \mathbb{R}$, $\alpha \in (0,\pi/2)$ and $M > 0$ such that $\omega + \Sigma_{\pi-\alpha} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant M/|\lambda - \omega|$, $\lambda \in \omega + \Sigma_{\pi-\alpha}$. Let B be another linear operator with domain and range contained in E. Looking for conditions under the operator $A + B$ is again a generator of an analytic semigroup is a famous problem in the perturbation theory of linear operators.

The author employs the following two conditions:

(H1.S) A and B are linear operators in E and there exist numbers $c > 0$ and $\theta \in (0, \pi/2)$ such that

$$\|(z-A)^{-1}\| + \|(z-B)^{-1}\| \leqslant c/|z|, \quad z \in \Sigma_{\pi-\theta}.$$

(H2.DPG.S) For every $v \in \Sigma_{\pi-\theta}$, we have $(A-v)^{-1}(D(B)) \subseteq D(B)$ and there exist numbers $c > 0$, α, β such that $-1 \leqslant \alpha < \beta \leqslant 1$ and

$$\|[B;(A-v)^{-1}](B-z)^{-1}\| \leqslant c|v|^{\alpha-1}|z|^{-\beta}, \quad v,\ z \in \Sigma_{\pi-\theta}$$

(here, by $[P;Q]$ we denote the commutator $PQ - QP$ of two linear operators in E).

The condition (H2.DPG.S) has its origins in the pioneering paper by G. da Prato, P. Grisvard [**127**].

If the conditions (H1.S) and (H2.DPG.S) are satisfied with $D(A)$ and $D(B)$ being dense in E, then the operator $A+B$ is closable and its closure $\overline{A+B}$ generates a strongly continuous analytic semigroup. On the other hand, the operator $A+B$ need not be closable if some of domains $D(A)$ or $D(B)$ is not dense in E. Since any multivalued linear operator is closable, in this case we can consider the operator $\overline{A+B}$ in the MLO sense. This is the starting point of analysis conducted in [**221**].

Using the Yosida approximations of operators A and B, M. Fuhrman has proved the following results:

(i) Assume that (H1.S) and (H2.DPG.S) hold, as well as that $D(B)$ is dense in E. Then $\overline{A+B}$ generates an analytic semigroup in the MLO sense.

(ii) Assume that (H1.S) and (H2.DPG.S) hold. Then there exists a linear relation $\mathcal{A}$ which extends the operator $A+B$ and generates an analytic semigroup in the MLO sense. Furthermore, we have

$$D(A) \cap D(B) \subseteq D(\mathcal{A}) \subseteq (E,[D(A)])_{\nu,p} \cap (E,[D(B)])_{\nu,p} \quad (\nu \in (0,1),\ p \in [1,\infty]).$$

If we assume some additional conditions, then there exists $\nu_0 \in (0,1)$ such that for each $\nu \in (0,\nu_0)$ we have the equality

$$(E,[D(\mathcal{A})])_{\nu,p} = (E,[D(A)])_{\nu,p} \cap (E,[D(B)])_{\nu,p} \quad (p \in [1,\infty]).$$

Maximal time regularity for abstract degenerate Volterra integro-differential equations. The study of maximal time and space regularity for abstract (degenerate) differential equations is still an active field of research. Here we will present the main results of research study [**183**] by A. Favaron and A. Favini (cf. also [**182**]) regarding the question of maximal time regularity for the following abstract degenerate Volterra integro-differential equation

$$\frac{d}{dt}(Mv(t)) = [\lambda_0 M + L]v(t) + \int_0^t k(t-s)L_1 v(s)\,ds \\ + h(t)y + f(t), \quad t \in [0,T];\ Mv(0) = Mv_0, \tag{466}$$

where L, M, L_1 are three closed linear operators with domains and ranges contained in a complex Banach space X, $\lambda_0 \in \mathbb{C}$, $D(L) \subseteq D(M) \cap D(L_1)$, $v_0 \in D(M)$, $y \in X$, $f\colon [0,T] \to X$ and $h,\ k\colon [0,T] \to \mathbb{C}$. The authors have shown that the Hölder continuity of mappings h, k, f in time, combined with some extra assumptions on the operator $(\lambda_0 M + L)$ and values of $f(0)$, y, implies that there exists a unique global strict solutions of (466) and that the derivative $(Mv)'$ has the same Hölder exponent as the mappings h and k as. The obtained result is in full accordance with that already established in the monograph [**199**] in the case that $(\lambda_0, h, k) = (0, 0, 0)$, which provides a certain loss of regularity of derivative $(Mv)'$ with respect to f. The authors obey the multivalued linear operators approach to (466), pointing out that the substitution $v_1 = Mv(t)$ cannot be directly applied. An application of abstract results to a concrete degenerate integro-differential equation, arising in modeling direct and inverse problems of heat conduction for materials with memory, has been provided. The time-relaxation analogue of (466), obtained by replacing the term $\frac{d}{dt}(Mv(t))$ with $\mathbf{D}_t^{\alpha}(Mv(t))$ $(0 < \alpha < 1)$ is much more complicated for the analysis and we would like to propose the question of time regularity for such an abstract degenerate integro-differential equation.

Abstract fractional differential inclusions with Riemann–Liouville derivatives. The study of abstract fractional inclusions with Riemann–Liouville derivatives has not been received much attention in this chapter. We have already explored, in the second chapter, the abstract multi-term fractional differential equations of type (SC1), (SC2) or (SC3), pointing out that the case (SC3) is very sofisticated. Here it is worth noting that A. V. Glushak has investigated, in a series of his research papers [**227, 230**], the abstract fractional differential equations with Riemann–Liouville derivatives of orders $\alpha, \beta \in (0, 1)$, accompanied with the initial conditions that is not of type (SC3). For example, the abstract fractional Cauchy problems like

$$D_t^{\alpha}(t^k D_t^{\beta} u(t)) = t^{\gamma} Au(t) + f(t),\ t > 0;$$
$$\lim_{t\to 0+} J_t^{1-\beta} u(t) = u_0,\ \lim_{t\to 0+} J_t^{1-\alpha}(t^k D_t^{\beta} u(t)) = 0,$$
$$D_t^{\alpha} D_t^{\beta} u(t) = Au(t) + f(t),\ t > 0;\ \lim_{t\to 0+} J_t^{1-\beta} u(t) = u_0,\ \lim_{t\to 0+} J_t^{1-\alpha} D_t^{\beta} u(t) = 0,$$

and

$$D_t^{\alpha} u(t) = Au(t) + f(t),\ t > 0;\ \lim_{t\to 0+} J_t^{1-\alpha} u(t) = u_0$$

have been investigated. The structural results obtained by A. V. Glushak hold, with insignificant modifications, in the multivalued linear operators framework.

And, at the very end of monograph, we shall say a few words about a singular perturbation problem for abstract Volterra integro-differential inclusions.

Singular perturbation theory for abstract Volterra integro-differential inclusions. When talking about abstract Volterra integro-differential inclusions, it is almost inevitable to say some basic things about singular perturbation theory. As already marked in [**292**], singular perturbations of various kinds of abstract non-degenerate Volterra integro-differential equations have been investigated, among many other research papers, in [**80, 130, 141, 142, 170, 178, 179, 223, 235, 273, 283, 389, 392, 393, 398**] and [**500**]. Due primarily to the time limits, we have not been able to edit the fundamental info about this intriguing topic here. We only want to say that it is very simply and straightforwardly, after creating a stable theory of degenerate resolvent operator families in this chapter, to clarify some results about the singular perturbation problem for abstract Volterra integro-differential inclusions. For example, C. Lizama and H. Prado [**398**] have investigated the singular differential problem

$$\varepsilon^2 u_\varepsilon''(t) + u_\varepsilon'(t) = Au_\varepsilon(t) + (K * u_\varepsilon)(t) + f_\varepsilon(t), \quad t \geqslant 0, \ \varepsilon > 0$$

for the abstract integro-differential equation

$$\omega'(t) = A\omega(t) + (K * \omega)(t) + f(t), \quad t \geqslant 0,$$

when $\varepsilon \to 0+$. The main results of this research study, [**398**, Theorem 3.6, Theorem 3.8], admit very simple reformulations for corresponding abstract integro-differential inclusions obtained by replacing the closed single-valued linear operator A in the above equations with a closed multivalued linear operator $\mathcal{A}$. Details can be left to the interested readers.

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The theory of linear degenerate Volterra integro-differential equations is still very undeveloped. The main purpose of this monograph is to provide an overview of the recent research on abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations in sequentially complete locally convex spaces. Some of our results seem to be new even for abstract non-degenerate differential equations in Banach spaces.

Fractional calculus and fractional differential equations are rapidly growing fields of research, having invaluable importance in modeling of various problems appearing in applied and theoretical sciences. The present monograph is probably the first research monograph that is specifically dedicated to the abstract degenerate (multi-term) fractional differential equations and the abstract degenerate Volterra integro-differential equations.

The book is not compulsively detailed and there are numerous important topics not analyzed here. Selection is based on the personal views and aspirations of the author, with the special attention paid to the applications of various types of convoluted or C-regularized families of solution operators in the analysis of abstract degenerate integro-differential equations.

Mathematical Institute
of the Serbian Academy
of Sciences and Arts