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Summability Methods and Applications



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SUMMABILITY METHODS AND APPLICATIONS

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Preface

These are the lecture notes on summability methods, matrix transformations and their applications. They are based on courses taught by the authors in the master and Ph.D. programmes in mathematics at several universities in Germany, Serbia, South Africa, Turkey and the US. The material of the lecture notes could be covered in one semester in a four-hour per week course. Special emphasis is put on the application of summability methods and matrix transformations in fixed point theory.

The presented topics would also serve as a reference for further work, and could be used as a basis for seminar work, master and Ph.D. theses. The authors took care for the lecture notes to be self-contained and comprehensive. Only a solid background in real analysis is needed except at one place were an alternative optional proof is given for the Toeplitz theorem; it uses the uniform boundedness principle from functional analysis which is included without proof for the reader's convenience. Furthermore, some fundamentals of the Riemann–Stieltjes integral are needed in the proof of the regularity conditions for the Hausdorff summbility method and the solution of the related moment problem. The necessary results are included in an appendix.

Summability theory deals with a generalization of the concept of the convergence of sequences and series of real or complex numbers. One of the original ideas was to assign, in some way, a limit to divergent sequences or series. Methods of summability were also introduced for applications to problems in analysis such as the analytic continuation of power series and improvement of the rate of convergence of numerical series and to iteration processes in fixed point theory. These goals were achieved by considering a transform rather than the original sequence or series. This can be done in various different ways. Here we confine ourselves mainly to transformations by infinite matrices, in particular, to the most popular methods defined by Hausdorff matrices and their special cases, the Cesàro matrices of order $\alpha > -1$, the Hölder and Euler matrices, and Nörlund matrices. We also consider the Abel and Borel methods which are not given by a matrix. One section each is dedicated to these methods. We also study inclusion, growth, Mercerian and Tauberian theorems.

Finally, some results are proved concerning the connectedness of sets of limit points of matrix transforms of bounded complex sequences. These results are used in the application of matrix transforms and summability methods in fixed point theory, in particular, in the Mann iteration.

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1 Introduction

The classical summability theory deals with a generalization of the concept of the convergence of sequences and series of real or complex numbers. One of the original ideas was to assign, in some way, a limit to divergent sequences or series. Classical methods of summability were also introduced for applications to problems in analysis such as the analytic continuation of power series and improvement of the rate of convergence of numerical series. These goals were achieved by considering a transform rather than the original sequence or series. This can be done in various different ways. Here we confine ourselves mainly to transformations by infinite matrices, in particular, to the most popular methods defined by Hausdorff matrices and their special cases, the Cesàro matrices of order $\alpha > -1$, the Hölder and Euler matrices, and Nörlund matrices. We also consider the Abel and Borel methods which are not given by a matrix. We refer to [13, 21, 50, 61, 99, 112] for further reading on summability methods.

In the beginning, the idea was conceived that there should be a way to find sums for divergent series. One popular procedure was to formally put x = 1 in the power series expansion

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x} \quad (|x| < 1)$$

which lead to the *satisfying* result

(*)
$$1 - 1 + 1 - 1 \pm \dots = \frac{1}{2}.$$

Another natural approach is to study the *arithmetic means* of a sequence. If $x = (x_k)_{k=0}^{\infty}$ is a sequence of real or complex numbers then a new sequence $\sigma = \sigma(x) = (\sigma_n(x))_{n=0}^{\infty}$ is formed by the arithmetic means of the terms of the sequence x, namely

(1.1)
$$\sigma_n = \sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^n x_k \text{ for } n = 0, 1, \dots$$

Example 1.1. Let the sequence x be defined by

(a)
$$x_k = (-1)^k$$
, (b) $x_k = \frac{1}{2}(1 + (-1)^k)$.
(c) $x_k = k + 1$, (d) $x_k = (-1)^k (k + 1)$.

Each sequence x diverges, but $\lim_{n\to\infty} \sigma_n = 0$ for the sequence in (a), $\lim_{n\to\infty} \sigma_n = 1/2$ in (b), $\sigma_n \to \infty$ ($n \to \infty$) in (c), and $\lim_{n\to\infty} \sigma_{2n} = 1/2$ and $\lim_{n\to\infty} \sigma_{2n+1} = -1/2$ in (d).

Our first result states that the arithmetic means of a convergent sequence converge, and preserve the limit.

Theorem 1.2 (Cauchy). If $\lim_{k\to\infty} x_k = \xi$, then $\lim_{n\to\infty} \sigma_n = \xi$ for the arithmetic means (1.1) of the sequence $x = (x_k)_{k=0}^{\infty}$.

Proof. (i) First we assume $\xi = 0$. Then given $\varepsilon > 0$, there is a non-negative integer K_{ε} such that

(1.2)
$$|x_k| < \varepsilon/2 \text{ for all } k > K_{\varepsilon}.$$

Furthermore, since $1/(n+1) \to 0$ as $n \to \infty$, we can choose a non-negative integer $N = N(\varepsilon, K_{\varepsilon})$ such that

(1.3)
$$\frac{1}{n+1}\sum_{k=0}^{K_{\varepsilon}}|x_k| < \frac{\varepsilon}{2} \text{ for all } n > N.$$

Therefore, if n > N then we obtain from (1.2) and (1.3)

$$|\sigma_n| \leq \frac{1}{n+1} \sum_{k=0}^{K_{\varepsilon}} |x_k| + \frac{1}{n+1} \sum_{k=K_{\varepsilon}+1}^n |x_k| < \frac{\varepsilon}{2} + \frac{1}{n+1} \cdot \frac{\varepsilon}{2} \cdot \sum_{k=0}^n 1 = \varepsilon.$$

This completes the proof of Part (i).

(ii) Now we assume $\xi \neq 0$. We consider the sequence x' defined by $x'_k = x_k - \xi$ for $k = 0, 1, \ldots$. Then it follows by Part (i) that

$$\sigma_n' = \frac{1}{n+1} \sum_{k=0}^n x_k' \to 0 \quad (n \to \infty),$$

and so

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^n x_k = \frac{1}{n+1} \sum_{k=0}^n (x'_k + \xi) = \sigma'_n + \xi \to \xi \quad (n \to \infty).$$

The following notations will be used throughout. We write e and $e^{(n)}$ (n = 0, 1, ...) for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq 0$.

Definition 1.3. (a) The set of all sequences $x = (x_k)_{k=0}^{\infty}$ of complex numbers x_k is denoted by ω ; we write

$$c_{0} = \left\{ x \in \omega : \lim_{k \to \infty} x_{k} = 0 \right\},$$

$$c = \left\{ x \in \omega : x - \xi e \in c_{0} \text{ for some } \xi \in \mathbb{C} \right\}$$

$$\ell_{\infty} = \left\{ x \in \omega : \sup_{k} |x_{k}| < \infty \right\},$$

$$\ell_{1} = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_{k}| < \infty \right\}$$

for the sets of all *null*, *convergent* and *bounded sequences*, and the set of all *absolutely convergent series*, respectively.

(b) We define addition and multiplication of sequences by a scalar by

 $x + x' = (x_k + x'_k)_{k=0}^{\infty}$ and $\lambda x = (\lambda x_k)_{k=0}^{\infty}$ for all $x, x' \in \omega$ and all $\lambda \in \mathbb{C}$.

Remark 1.4. Obviously the sets ω , c_0 , c, ℓ_{∞} and ℓ_1 are linear spaces with the sum and product defined in Definition 1.3, ℓ_1 is a linear subspace of c_0 , c_0 is a linear subspace of c, c is a linear subspace of ℓ_{∞} , and ℓ_{∞} is a linear subspace of ω .

By x, y, z, \ldots , we always denote sequences of complex numbers x_k, y_k, z_k, \ldots . In this section, we also use the traditional notations $\sum a_k$ for series of complex numbers a_k , and $s_n = \sum_{k=0}^n a_k$ for their partial sums. All indices start from 0 unless stated otherwise.

Given an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers a_{nk} and a sequence $x = (x_k)_{k=0}^{\infty}$, we write $A_n = (a_{nk})_{k=0}^{\infty}$ and $A^{(k)} = (a_{nk})_{n=0}^{\infty}$ for the sequences in the n^{th} row and the k^{th} column of the matrix A, respectively $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ $(n = 0, 1, \ldots)$, each of the series being assumed convergent, and $Ax = (A_n x)_{n=0}^{\infty}$ for the sequence of the A transforms $A_n x$ of the sequence x.

We now turn to the problem how to assign a *sum* or a *limit* to a divergent series or a divergent sequence. Since it is obviously possible to assign a sum, for instance 0, to any divergent series, we abandon this quest and simply look for some type of function $L: S \to \mathbb{C}$ where S is some set of sequences. The function L will be required to have certain explicitly stated properties; for example, we usually require S to be a linear space which includes c and L to be linear and such that $L(x) = \lim_{k\to\infty} x_k$ whenever $x \in c$. Then if S contains a divergent sequence x, the number L(x) will be a *limit of a divergent sequence*.

Definition 1.5. Given an infinite matrix A, then the method of summability A is defined by y = Ax. The set

$$\omega_A = \{x \in \omega : Ax \text{ is defined}\}\$$

is called the *domain of* A; for any subset X of ω , we write

$$X_A = \{x \in \omega : Ax \in X\}$$

for the matrix domain of A in X, and in the special case X = c the set

$$c_A = \{x \in \omega : Ax \in c\}$$

is called the *convergence domain of* A.

If $x \in c_A$, then there is $\eta \in \mathbb{C}$ such that $\eta = \lim_A x = \lim_{n \to \infty} A_n x$, thus defining a map $\lim_A : c_A \to \mathbb{C}$. In this case, the sequence x is called summable A to η ; this is denoted by $x \to \eta(A)$. A series Σa_k is said to be summable A to η if the sequence of its partial sums is summable A to η ; this is denoted by $\Sigma a_k = \eta(A)$.

Remark 1.6. (a) Note that the same letter is used for a matrix and the method of summability defined by it.

(b) The notation X_A is consistent with the definition of ω_A ; $Ax \in X$ always implies that Ax exists, that is, $X_A \subset \omega_A$; A is linear on ω_A , and ω_A is a linear subspace of ω .

(c) By historical accident, sequences in c_A are called summable A instead of the more reasonable *limitable* A.

We shall be particularly interested in methods of summability that transform all convergent sequences into convergent sequences.

Definition 1.7. A method of summability A is called:

(a) conservative if $c \subset c_A$, that is, $Ax \in c$ whenever $x \in c$;

(b) multiplicative m if $\lim_{A} x = m \cdot \lim_{k \to \infty} x_k$ for all $x \in c$;

(c) *regular* if it is multiplicative 1.

(d) A real method of summability A, that is, a method of summability defined by a real matrix, is called *totally regular* if $x \to \xi$ implies $Ax \to \xi$ for all finite and infinite ξ .

Example 1.8. (a) The method I defined by the infinite identity matrix I with the rows $I_n = e^{(n)}$ for all n is totally regular since Ix = x; also $\omega_I = \omega$ and $c_I = c$.

(b) The method of the arithmetic means defined in (1.1) is regular by Theorem 1.2.

(c) Let Q be the matrix given by $Q_0 x = x_0$ and $Q_n x = (1/2)(x_{n-1} + x_n)$ for $n = 1, 2, \ldots$ Then Q is regular and sums the divergent sequence $((-1)^k)_{k=0}^{\infty}$; therefore c is a proper subset of c_Q .

(c) For A = 0, we have $c_A = \omega$, while at the opposite extreme one can construct a matrix A with $c_A = \{0\}$ by taking

$$Ax = (x_1, 0, x_1, x_2, 0, x_1, x_2, x_3, 0, \dots).$$

2 The Cesàro method of order 1

The Cesàro method of order 1 is one of the most important methods of summability.

Definition 2.1. The Cesàro method C_1 of order 1 is defined by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ where

$$a_{nk} = \begin{cases} \frac{1}{n+1} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

By $\sigma_n = \sigma_n(x)$, we denote the C_1 -means of the sequence $x = (x_k)_{k=0}^{\infty}$, that is, (1.1)

(2.1)
$$\sigma_n = \sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^n x_k \quad (n = 0, 1, \dots).$$

Remark 2.2. The equalities in (2.1) transform a sequence x into the sequence $\sigma = (\sigma_n)_{n=0}^{\infty}$; therefore this is referred to as the sequence-to-sequence-transformation for the C_1 method. If Σa_k is a series with partial sums $s_k = \sum_{j=0}^k a_j$ (k = 0, 1, ...), then (2.1) applied to the sequence $(s_k)_{k=0}^{\infty}$ yields

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k = \frac{1}{n+1} \sum_{k=0}^n \sum_{j=0}^k a_j = \frac{1}{n+1} \sum_{j=0}^n a_j \sum_{k=j}^n 1$$
$$= \frac{1}{n+1} \sum_{j=0}^n a_j (n+1-j) = \sum_{j=0}^n a_j \left(1 - \frac{j}{n+1}\right),$$

that is,

(2.2)
$$\sigma_n = \sum_{k=0}^n a_k \left(1 - \frac{k}{n+1} \right) \quad (n = 0, 1, \dots),$$

the series-to-sequence-transformation for the C_1 method. In future, we shall always consider the sequence-to-sequence transformations for methods of summability.

Theorem 2.3. The C_1 method is totally regular.

The proof is left as an exercise.

Now we prove a converse result which gives a necessary condition in (i) for the summability C_1 of a sequence; this result is analogous to the classical result that if a series $\sum a_k$ is convergent then $\lim_{k\to\infty} a_k = 0$.

Theorem 2.4. (a) We have

(2.3)
$$x \in c_{C_1} \text{ implies } \lim_{k \to \infty} \frac{x_k}{k} = 0 \text{ which is denoted by } x_k = o(k).$$

(b) Let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be an unbounded real sequence with $\lambda_k > 0$ for all k. Then there is a sequence $x \in c_{C_1}$ such that $x_k \neq o(k/\lambda_k)$. Hence the condition $x_k = o(k)$ in (2.3) of Part (a) is best possible.

Proof. (a) We assume $x \in c_{C_1}$. This implies $\lim_{n\to\infty} \sigma_n = \eta$ for some complex number η . By (2.1), we have $x_k = (k+1)\sigma_k - k\sigma_{k-1}$ for $k = 1, 2, \ldots$ This implies

$$\frac{x_k}{k} = \left(1 + \frac{1}{k}\right)\sigma_k - \sigma_{k-1} \to \eta - \eta = 0 \text{ as } k \to \infty.$$

Thus we have shown Part (a).

(b) Let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be a given unbounded real sequence with $\lambda_k > 0$ for all k. Then we can choose a sequence of indices k(i) such that $k(i+1) \ge k(i) + 2$ for all $i = 0, 1, \ldots$ and $\lambda_{k(i)} \nearrow \infty$ $(i \to \infty)$. Putting

$$\sigma_n = \begin{cases} \frac{1}{\sqrt{\lambda_{k(i)}}} & (n = k(i)) \\ 0 & (n \neq k(i)) \end{cases} \quad (i = 0, 1, \dots),$$

we conclude $\lim_{C_1} x = 0$ for the sequence x with $x_k = (k+1)\sigma_k - k\sigma_{k-1}$ (k = 1, 2, ...), and

$$\begin{aligned} \frac{\lambda_{k(i)}}{k(i)} x_{k(i)} &= \lambda_{k(i)} \left(1 + \frac{1}{k(i)} \right) \sigma_{k(i)} \\ &= \left(1 + \frac{1}{k(i)} \right) \sqrt{k(i)} \to \infty \text{ as } i \to \infty. \end{aligned}$$

Thus we have shown Part (b).

Remark 2.5. (a) It is obvious from Theorem 2.4(a) that the sequences in Example 1.1(c) and (d) are not summable C_1 .

(b) Theorem 2.4(a) shows that any sequence summable C_1 is of growth o(k). The converse, however, is not true in general. In fact, we shall later show that, given any regular method of summability A, there always exists a bounded sequence which is not summable A.

Remark 2.6. The C_1 method shows a phenomenon that is common to many methods of summability; it is the effect the so-called *dilution of series* may have on their summability. It is well known that the convergence or divergence of a series is not affected by adding zero terms; if one of the series

 $a_0 + a_1 + a_2 + \dots$ and $0 + 0 + \dots + a_0 + 0 + \dots + a_1 + 0 + \dots$

is convergent so is the other one, and the limit remains unchanged. In summability, however, such a change can alter the sum or even destroy the summability of a series altogether. For instance, the series

 $1 - 1 + 1 - 1 \pm \dots$ and $1 - 1 + 0 + 1 - 1 + 0 + \dots$

are summable C_1 to 1/2 and 1/3, respectively.

3 Hardy's Big O Tauberian Theorem

In this subsection we prove a *Tauberian theorem*.

A Tauberian theorem is one in which the convergence of a sequence is deduced from the convergence of some transform of the sequence together with some side conditions, so-called *Tauberian conditions*. The first such theorem was given by *A. Tauber*.

Theorem 3.1 (Hardy's Big O Tauberian Theorem).

If
$$\begin{cases} x \in c_{C_1} \\ and \\ x_n - x_{n-1} = O(1/n) \ (Tauberian \ condition) \end{cases}$$
 then $x \in c$.

Proof. Putting

(3.1)
$$\sigma_{n,k} = \frac{1}{k} \sum_{j=n}^{n+k-1} x_j \quad (k = 1, 2, \dots; n = 0, 1, \dots),$$

we conclude

(3.2)
$$\sigma_{n,k} = \frac{(n+k)\sigma_{n+k-1} - n\sigma_{n-1}}{k} = \left(1 + \frac{n}{k}\right)\sigma_{n+k-1} - \frac{n}{k}\sigma_{n-1}$$

If we let n and k tend to infinity through bounded values of n/k, then (3.2) defines a method A with $c_{C_1} \subset c_A$. For $\lim_{n\to\infty} \sigma_n = \eta$ implies that the right-hand side of (3.2) is $\eta + o(1)$. Putting $a_k = x_k - x_{k-1}$ ($k = 0, 1, \ldots; a_{-1} = 0$), we conclude from (3.2), using the series-to-sequence transformation given in (2.2),

$$\begin{aligned} \sigma_{n,k} &= \frac{1}{k} \left(\sum_{j=0}^{n+k-1} a_j (n+k-j) - \sum_{j=0}^{n-1} a_j (n-j) \right) \\ &= \frac{1}{k} \left(\sum_{j=0}^{n-1} a_j (n+k-j-(n-j)) + a_n \cdot k + \sum_{j=n+1}^{n+k-1} a_j (n+k-j) \right) \\ &= \frac{1}{k} \left(\sum_{j=0}^{n-1} a_j \cdot k + a_n \cdot k + \sum_{j=n+1}^{n+k-1} a_j (n+k-j) \right) \\ &= \sum_{j=0}^{n} a_j + \sum_{j=n+1}^{n+k-1} a_j \left(\frac{n+k-j}{k} \right), \end{aligned}$$

hence

$$\sigma_{n,k} = x_n + \sum_{j=n+1}^{n+k-1} a_j \left(\frac{n+k-j}{k}\right).$$

Let $\lim_{n\to\infty} \sigma_n = \eta$ and $a_n = O(1/n)$. Then, for some constant M > 0,

$$|\sigma_{n,k} - x_n| \le \sum_{j=n+1}^{n+k-1} |a_j| \frac{n+k-j}{k} \le \sum_{j=n+1}^{n+k-1} |a_j|$$
$$\le M \sum_{j=n+1}^{n+k-1} \frac{1}{j} \le M \frac{k-1}{n}.$$

Let $\varepsilon > 0$ be given. We put $k = [\varepsilon n] + 1$, where $[\alpha] = \max\{z \in \mathbb{Z} : z \leq \alpha\}$ for each $\alpha \in \mathbb{R}$. Then we have

$$|\sigma_{n,k} - x_n| \le \frac{M\varepsilon n}{n} = M\varepsilon.$$

Since $n/k \leq n/(n\varepsilon) = 1/\varepsilon$ is bounded, it follows that $\sigma_{n,k} \to \eta$, hence

$$\overline{\lim_{n \to \infty}} |x_n - \eta| \le M\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\lim_{n \to \infty} x_n = \eta$.

4 The Toeplitz theorem

We already mentioned in Section 1 that conservative methods are of special interest. The question naturally arises as to whether all conservative methods can be characterized. The affirmative answer was given in the famous *Toeplitz theorem* which establishes necessary and sufficient conditions for the entries of a matrix to define a conservative method of summability.

Definition 4.1. (a) Let X and Y be subsets of ω . Then (X, Y) denotes the class of all matrices A for which $X \subset Y_A$, that is, $A \in (X, Y)$ if and only if the series $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ converge for all $x \in X$ and for all n, and $Ax \in Y$ for all $x \in X$. (b) We write

(4.1)
$$||A|| = \sup_{n} \sum_{k=0}^{\infty} |a_{nk}| \text{ for every matrix } A$$

and $\Phi = \{A : ||A|| < \infty\}.$

Example 4.2. (a) We always have $A \in (\omega_A, \omega)$; furthermore $A \in (c, c)$ if and only if A is conservative.

(b) The matrix C_1 that defines the Cesàro method of order 1 satisfies $C_1 \in (c, c)$ (Theorem 1.2), and obviously $C_1 \in \Phi$.

(c) We have $e^{(n)} \in c_0$ for all n and $e \in c \setminus c_0$.

Now we study the famous Toeplitz theorem that characterizes the class (c, c). The difficult part of the proof was to establish the necessity of the row norm condition $||A|| < \infty$. The original proof used the classical method of the gliding hump. First we give the classical proof; at the end of this section we will prove the necessity of the row norm condition by using results from functional analysis.

Theorem 4.3 (Toeplitz, 1911). ([108]) (a) We have $A \in (c, c)$ if and only if the following three conditions hold

(i)
$$A \in \Phi$$
, (ii) $e^{(k)} \in c_A$ for $k = 0, 1, ...,$ (iii) $e \in c_A$.

(b) Let $A \in (c, c)$ and $x \in c$. Then putting

(4.2)
$$\alpha_k = \lim_A e^{(k)} \text{ for } k = 0, 1, \dots$$

(4.3)
$$\chi = \chi(A) = \lim_{A} e - \sum_{k=0}^{\infty} \alpha_k.$$

we have

(4.4)
$$\lim_{A} x = \chi \cdot \lim_{k \to \infty} x_k + \sum_{k=0}^{\infty} \alpha_k x_k$$

(c) A matrix A is regular if and only if the following three conditions hold

(i')
$$A \in \Phi$$
, (ii') $\alpha_k = 0$ $(k = 0, 1, ...)$, (iii') $\lim_A e = 1$.

Remark 4.4. (a) The difficult part of the proof is the necessity of the condition in (i). Here we give the classical proof by contradiction that uses the method of the gliding hump. Assuming $A \in (c, c)$ and that the condition in (i) is not satisfied, we will construct a sequence $x \in c_0$ such that $Ax \notin \ell_{\infty}$.

(b) The condition in (i) means that the row norms $||A_n|| = \sum_{k=0}^{\infty} |a_{nk}|$ exist for all n, and that the sequence $(||A_n||)_{n=0}^{\infty}$ of the row norms is bounded. Since

$$A_n e^{(k)} = \sum_{j=0}^{\infty} a_{nj} e_j^{(k)} = a_{nk} \text{ for all } n \text{ and all } k,$$

the condition in (ii) means that

$$\alpha_k = \lim_{A} e^{(k)} = \lim_{n \to \infty} a_{nk}$$
 exists for each k.

Hence the sequences $A^{(k)} = (a_{nk})_{n=0}^{\infty}$ in the columns of the matrix A are convergent. Finally, since $A_n e = \sum_{k=0}^{\infty} a_{nk}$ for all n, the condition in (iii) means that all row sums of the matrix A exist, and the sequence of the row sums converges.

Proof of Theorem 4.2. (a.1) First we prove the sufficiency of the conditions in (i), (ii) and (iii). We assume that the conditions in (i), (ii) and (iii) are satisfied. Let $x \in c$. Then $x \in \ell_{\infty}$, and so there is a constant M > 0 such that $\sup_k |x_k| \leq M$. We obtain from (i) for all n

$$\sum_{k=0}^{\infty} |a_{nk}| |x_k| \le \left(\sum_{k=0}^{\infty} |a_{nk}|\right) \sup_k |x_k| \le ||A|| \cdot M,$$

which implies $x \in \omega_A$. Furthermore, the condition in (ii) implies that the complex numbers α_k defined in (4.2) exist for all k. We have for each fixed integer m,

$$\sum_{k=0}^{m} |\alpha_k| = \lim_{n \to \infty} \sum_{k=0}^{m} |a_{nk}| \le ||A||,$$

hence

(4.5)
$$\sum_{k=0}^{\infty} |\alpha_k| \le ||A||, \text{ that is, } (\alpha_k)_{k=0}^{\infty} \in \ell_1,$$

(4.6)
$$\sum_{k=0}^{\infty} |\alpha_k x_k| < \infty \text{ for all } x \in c.$$

Given $x \in c_0$ and $\varepsilon > 0$, we can choose an integer $K = K_{\varepsilon}$ such that

(4.7)
$$|x_k| < \frac{\varepsilon}{4\|A\| + 1} \text{ for all } k > K_{\varepsilon},$$

and, by (ii), we can choose an integer $N = N_{\varepsilon}$ such that

(4.8)
$$\sum_{k=0}^{K} |a_{nk} - \alpha_k| |x_k| < \frac{\varepsilon}{2} \text{ for all } n > N.$$

Let n > N be given. Then (4.8), (4.7), (i) and (4.5) imply

$$(4.9) \quad \left| A_n x - \sum_{k=0}^{\infty} \alpha_k x_k \right| \leq \sum_{k=0}^{K} |a_{nk} - \alpha_k| |x_k| + \sum_{k=K+1}^{\infty} (|a_{nk}| + |\alpha_k|) |x_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4\|A\| + 1} \left(\sum_{k=0}^{\infty} |a_{nk}| + \sum_{k=0}^{\infty} |\alpha_k| \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4\|A\| + 1} (\|A\| + \|A\|) \leq \varepsilon,$$

hence $x \in c_A$ by (4.9). This implies $c_0 \subset c_A$. For $x \in c \setminus c_0$, we have $\xi = \lim_{k \to \infty} x_k \neq 0$ and consider the sequence $x' = x - \xi e$. Then $x' \in c_0$, and (4.9) implies $\lim_A x' = \sum_{k=0}^{\infty} \alpha_k x'_k$. We conclude from (iii) and the linearity of A on ω_A ,

(4.10)
$$A_n x = A_n x' + \xi A_n e \to \sum_{k=0}^{\infty} \alpha_k x'_k + \xi \lim_A e$$
$$= \xi (\lim_A e - \sum_{k=0}^{\infty} \alpha_k) + \sum_{k=0}^{\infty} \alpha_k x_k \quad (n \to \infty)$$

(Note that $\sum_{k=0}^{\infty} \alpha_k$ is convergent by (4.5).) This completes the proof of the sufficiency of the conditions (i), (ii) and (iii).

(a.2) Now we show the necessity of the conditions in (i), (ii) and (iii). We assume $A \in (c, c)$. It follows from $e^{(k)} \in c$ for all k and $e \in c$ that the conditions in (ii) and (iii) hold. To prove the necessity of (i), we first show

(4.11)
$$b_n = \sum_{k=0}^{\infty} |a_{nk}| < \infty \text{ for all } n = 0, 1, \dots$$

Assuming to the contrary $b_m = \infty$ for some integer m, we can choose an increasing sequence $(k(i))_{i=0}^{\infty}$ of integers k(i) with k(0) = 0 and

$$\sum_{k=k(i)}^{k(i+1)-1} |a_{mk}| > i+1 \text{ for all } i = 0, 1, \dots$$

We define the sequence x by

$$x_k = \frac{\operatorname{sgn}(a_{mk})}{i+1}$$
 $(k(i) \le k \le k(i+1) - 1; i = 0, 1, \dots),$

where, as usual, $\operatorname{sgn}(z)$ is defined by $\operatorname{sgn}(z) = |z|/z$ for $z \in \mathbb{C} \setminus \{0\}$ and $\operatorname{sgn}(0) = 0$. Then we obtain $x \in c_0$ and

$$\sum_{k=0}^{\infty} a_{mk} x_k = \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{k=k(i)}^{k(i+1)-1} |a_{mk}| > \sum_{i=0}^{\infty} 1 = \infty,$$

which is a contradiction to the assumption $A \in (c, c)$.

Now we show the necessity of the condition (i). We assume to the contrary that the condition is not satisfied, that is, $||A|| = \infty$, and construct a sequence $x \in c_0$ such that $Ax \notin \ell_{\infty}$, which contradicts the assumption $A \in (c, c)$. First we note, that since $||A|| = \infty$, given any real G > 0, we can choose an integer n such that $b_n > G$. We put

(4.12)
$$b_{n,m} = \sum_{k=0}^{m} |a_{nk}| \quad (m = 0, 1, ...),$$

(4.13)
$$\beta_n = \sum_{k=0}^n |\alpha_k| \quad (n = 0, 1, \dots) \text{ where } \alpha_k = \lim_{n \to \infty} a_{nk} \text{ by (ii)}$$

It follows from (4.11) that $\lim_{m\to\infty} b_{n,m} = b_n$ for every *n*. Furthermore, by (ii), we have $\lim_{n\to\infty} b_{n,m} = \beta_m$ for every *m*.

Let m_1 be an arbitrary integer. We recursively define two increasing sequences $(m(i))_{i=1}^{\infty}$ and $(n(j))_{j=1}^{\infty}$ of integers. Assuming that $m(1), m(2), \ldots, m(r)$ and $n(1), n(2), \ldots, n(r-1)$ have already been determined for some integer $r \ge 1$, then, since $b_n > G$ for arbitrary G and some n, we can choose an integer n(r) > n(r-1) such that

(4.14)
$$b_{n(r)} > 2r\beta_{m(r)} + r^2 + 2r + 2.$$

Since $\lim_{n\to\infty} b_{n,m(r)} = \beta_{m(r)}$, we can assume

(4.15)
$$b_{n(r),m(r)} < \beta_{m(r)} + 1.$$

Finally $\lim_{m\to\infty} b_{n(r),m} = b_{n(r)}$ implies that we can choose an integer m(r+1) > m(r) such that

(4.16)
$$|b_{n(r)} - b_{n(r),m(r+1)}| < 1.$$

It follows from (4.14), (4.15) and (4.16) that

$$\sum_{k=m(r)+1}^{m(r+1)} |a_{n(r),k}| = \sum_{k=0}^{\infty} |a_{n(r),k}| - \sum_{k=0}^{m(r)} |a_{n(r),k}| - \sum_{k=m(r+1)+1}^{\infty} |a_{n(r),k}| = b_{n(r)} - b_{n(r),m(r)} - (b_{n(r)} - b_{n(r),m(r+1)}) > 2r\beta_{m(r)} + r^2 + 2r + 2 - (\beta_{m(r)} + 1) - 1 > > r\beta_{m(r)} + r^2 + 2r,$$

that is,

(4.17)
$$\sum_{k=m(r)+1}^{m(r+1)} |a_{n(r),k}| > r\beta_{m(r)} + r^2 + 2r.$$

We define the sequence x by

$$x_k = \begin{cases} 0 & (0 \le k \le m(1)) \\ r^{-1} \operatorname{sgn}(a_{n(r),k}) & (m(r) + 1 \le k \le m(r+1)) \end{cases} \quad (r = 1, 2, \dots)$$

Then it follows that $|x_k| \leq 1$ for all k and $\lim_{k\to\infty} x_k = 0$, but on the other hand we have by (4.17), (4.15) and (4.16),

$$\begin{aligned} \left| \sum_{k=0}^{\infty} a_{n(r),k} x_k \right| \geq \\ & \left| \sum_{k=m(r)+1}^{m(r+1)} a_{n(r),k} x_k \right| - \sum_{k=0}^{m(r)} |a_{n(r),k}| |x_k| - \sum_{k=m(r+1)+1}^{\infty} |a_{n(r),k}| |x_k| \geq \\ & \frac{1}{r} \sum_{k=m(r)+1}^{m(r+1)} |a_{n(r),k}| - \sum_{k=0}^{m(r)} |a_{n(r),k}| - \sum_{k=m(r+1)+1}^{\infty} |a_{n(r),k}| > \\ & \beta_{m(r)} + r + 2 - (\beta_{m(r)} + 1) - 1 = r, \end{aligned}$$

hence $\sum_{k=0}^{\infty} a_{n(r),k} x_k \to \infty$ $(r \to \infty)$. This means that there is $x \in c_0$ with $Ax \notin c$, which is a contradiction to the assumption $A \in (c, c)$.

This completes the proof of the necessity of the conditions in (i), (ii) and (iii).

(b) We assume $A \in (c, c)$. Then the conditions (i), (ii) and (iii) hold by Part (a), and the conclusion follows from (4.10).

(c) This is an immediate consequence of Parts (a) and (b). \Box

We close this section with a functional analytic proof of the necessity of the row norm condition for conservative matrices using the *uniform boundedness principle* and the *Banach–Steinhaus closure theorem*, which we will state below without proof for the reader's convenience.

It is well known that c and ℓ_{∞} are Banach spaces with the supremum norm $\|\cdot\|_{\infty}$ defined by $\|x\|_{\infty} = \sup_{k} |x_{k}|$ for all sequences $x = (x_{k})_{k=0}^{\infty}$; also ℓ_{1} is a Banach space with the norm $\|\cdot\|_{1}$ defined by $\|x\|_{1} = \sum_{k=0}^{\infty} |x_{k}|$ for all $x = (x_{k})_{k=0}^{\infty} \in \ell_{1}$.

Theorem 4.5 (Uniform boundedness principle). ([110, Theorem 7.3.1]) Every pointwise bounded family \mathcal{F} of continuous linear functionals on a Banach space is uniformly bounded, that is, there exists a constant M such that $||f|| \leq M$ for all $f \in \mathcal{F}$.

Theorem 4.6 (Banach–Steinhaus closure theorem). ([110, Theorem 7.6.3] or [61, Corollary p. 115]) The limit function of a sequence of pointwise convergent linear functions from a Banach space into a normed space is continuous.

Corollary 4.7. If $A \in (c, c)$ then $A \in \Phi$.

Proof. We assume $A \in (c, c)$. Then the series $A_n x$ converge for all n and all $x \in c$ and $Ax \in c \subset \ell_{\infty}$. We observe that $A_n \in \ell_1$ by (4.11). For each n, we define the functional $f_n : c \to \mathbb{C}$ by $f_n x = A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ for all $x \in X$.

(i) First we show that f_n is a continuous linear functional on c for each $n \in \mathbb{N}_0$. We fix $n \in \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ be given. We define the functional $f_n^{[m]} : c \to \mathbb{C}$ by $f_n^{[m]}(x) = \sum_{k=0}^m a_{nk} x_k$ for all $x \in c$. Clearly, each functional $f_n^{[m]}$ is linear, and $A_n \in \ell_1$ implies

(4.18)
$$\left| f_n^{[m]}(x) \right| \le \sum_{k=0}^m |a_{nk}x_k| \le \left(\sum_{k=0}^\infty |a_{nk}| \right) \cdot \sup_k |x_k| = \|A_n\|_1 \cdot \|x\|_\infty < \infty,$$

hence each $f_n^{[m]}$ is continuous. Since $A_n x = \lim_{m \to \infty} f_n^{[m]}(x)$ for all $x \in c$, it follows from the Banach–Steinhaus closure theorem, Theorem 4.6, that f_n is a continuous linear functional.

This completes the proof of Part (i)

(ii) Now we show $A \in \Phi$. By Part (i), $(f_n)_{n=0}^{\infty}$ is a sequence of continuous linear functionals on c which is pointwise bounded, since $(f_n(x))_{n=0}^{\infty} = (A_n x)_{n=0}^{\infty} \in \ell_{\infty}$, and so $\sup_n ||f_n|| < \infty$ by the uniform boundedness principle, Theorem 4.5. It follows from (4.18) that

(4.19)
$$||f_n|| \le ||A_n||_1 \text{ for all } n$$

Let $m \in \mathbb{N}_0$ be given and $x = \sum_{k=0}^m \operatorname{sgn}(a_{nk})e^{(k)}$. Then we have $x \in c$, $||x||_{\infty} \leq 1$ and m

$$|f_n(x)| = \sum_{k=0}^m |a_{nk}| \le ||f_n|| \cdot ||x||_{\infty} \le ||f_n||.$$

Since $m \in \mathbb{N}_0$ was arbitrary, we obtain

(4.20)
$$||f_n|| \ge ||A_n||_1 \text{ for all } n$$

Now (4.19) and (4.20) yield $||f_n|| = ||A_n||_1$, and so $\sup_n ||f_n|| = \sup_n ||A_n||_1 < \infty$, that is, $A \in \Phi$.

The following results can easily be deduced from Theorem 4.3.

Corollary 4.8. We have

(a) $A \in (\ell_{\infty}, \ell_{\infty})$ if and only if $A \in \Phi$; (b) $(c_0, \ell_{\infty}) = (c, \ell_{\infty}) = (\ell_{\infty}, \ell_{\infty})$; (c) $A \in (c_0, c)$ if and only if $A \in \Phi$ and (4.21) $\lim_{m \to \infty} a_{nk} = \alpha_k \text{ exists for each } k;$

(d)
$$A \in (c_0, c_0)$$
 if and only if $A \in \Phi$ and (4.21) holds with $\alpha_k = 0$ for each k,
(e) $A \in (c, c_0)$ if and only if $A \in \Phi$, (4.21) holds with $\alpha_k = 0$ for each k and

(4.22)
$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 0.$$

Proof. The necessity of the conditions in each case follows similarly as in the proof of Theorem 4.3. In particular, $A \in \Phi$ implies $A \in (\ell_{\infty}, \ell_{\infty}) \subset (c, \ell_{\infty}) \subset (c_0, \ell_{\infty})$. Furthermore, we showed in the proof of Theorem 4.3 that if $A \notin \Phi$ then there exists a sequence $x \in c_0$ such that $Ax \notin \ell_{\infty}$. Thus $A \in (c_0, \ell_{\infty})$ implies $A \in \Phi$, and since $(\ell_{\infty}, \ell_{\infty}) \subset (c, \ell_{\infty}) \subset (c_0, \ell_{\infty})$, the condition $A \in \Phi$ is also necessary for $A \in (\ell_{\infty}, \ell_{\infty})$ and $A \in (c, \ell_{\infty})$. The necessity of the additional conditions in the remaining cases is trivial.

We apply Theorem 4.3 to obtain Theorem 1.2.

Example 4.9. The Cesàro matrix $C_1 = (c_{nk}^{(1)})_{n,k=0}^{\infty}$ is regular, since

$$\sum_{k=0}^{\infty} |c_{nk}^{(1)}| = \sum_{k=0}^{\infty} c_{nk}^{(1)} = \frac{1}{n+1} \sum_{k=1}^{n} 1 = 1 \text{ for all } n \in \mathbb{N}_0,$$

hence $C_1 \in \Phi$ and $\lim_{C_1} e = 1$. We also have

$$\lim_{C_1} e^{(k)} = \lim_{n \to \infty} \frac{1}{n+1} = 0 \text{ for all } k \in \mathbb{N}_0.$$

Thus the matrix C_1 satisfies the conditions (i'), (ii') and (iii') in Part (c) of Theorem 4.3, and thus is regular.

5 Coercive matrices

Now we characterize the classes (ℓ_{∞}, c_0) and (ℓ_{∞}, c) . No functional analytic proof seems to be known for these two cases. Instead the classical method of the gliding hump has to be used in the proof of the characterizations. Matrices in (ℓ_{∞}, c) are called *coercive*.

We need the following

Lemma 5.1. If $\sum_{k=0}^{\infty} |a_{nk}| < \infty$ for each n and $\sum_{k=0}^{\infty} |a_{nk}| \to 0 \ (n \to \infty)$, then $\sum_{k=0}^{\infty} |a_{nk}|$ is uniformly convergent in n.

Proof. Let $\varepsilon > 0$ be given. Since $\sum_{k=0}^{\infty} |a_{nk}| \to 0 \ (n \to \infty)$, there is $N \in \mathbb{N}_0$ such that $\sum_{k=0}^{\infty} |a_{nk}| < \varepsilon$ for all n > N. Since $\sum_{k=0}^{\infty} |a_{nk}| < \infty$ for each n with $0 \le n \le N$, there is an integer m(n) such that $\sum_{k=m(n)}^{\infty} |a_{nk}| < \varepsilon$. We choose $M = \max_{0 \le m \le N} m(n)$. Then we obtain $\sum_{k=m}^{\infty} |a_{nk}| < \varepsilon$ for all $m \ge M$ and for all n, and so $\sum_{k=0}^{\infty} |a_{nk}|$ is uniformly convergent in n.

Theorem 5.2 (Schur). We have (a) $A \in (\ell_{\infty}, c)$ if and only if

(5.1)
$$\sum_{k=0}^{\infty} |a_{nk}| \text{ converges uniformly in } n,$$

(5.2)
$$\lim_{n \to \infty} a_{nk} = \alpha_k \text{ for each } k;$$

(b)
$$A \in (\ell_{\infty}, c_0)$$
 if and only if condition (5.1) holds and

(5.3)
$$\lim_{n \to \infty} a_{nk} = 0 \text{ for each } k.$$

Proof. (a.1) First we show the sufficiency of the conditions in (5.1) and (5.2). So we assume that the conditions in (5.1) and (5.2) are satisfied. We show $A \in \Phi$. By (5.1), there is $k_0 \in \mathbb{N}_0$ such that

$$\sum_{k=k_0+1}^{\infty} |a_{nk}| < 1 \text{ for all } n = 0, 1, \dots,$$

and it follows from (5.2) that $(a_{nk})_{n=0}^{\infty} \in c \subset \ell_{\infty}$ for every $k \in \mathbb{N}_0$. Thus, for every k, there exists a constant M_k such that $|a_{nk}| < M_k$ for all $n = 0, 1, \ldots$. We put $M = 1 + \sum_{k=0}^{k_0} M_k$ and obtain

$$\sum_{k=0}^{\infty} |a_{nk}| \le \sum_{k=0}^{k_0} |a_{nk}| + \sum_{k=k_0}^{\infty} |a_{nk}| \le M \text{ for all } n,$$

that is $A \in \Phi$. Now $A \in \Phi$ and (5.2) imply $\alpha = (\alpha_k)_{k=0}^{\infty} \in \ell_1$ by (4.5) in Part (a.1) of the proof of Theorem 4.3, and so $\sum_{k=0}^{\infty} \alpha_k x_k$ converges for all $x \in \ell_{\infty}$. Furthermore, $x \in \ell_{\infty}$ and (5.1) together imply that $A_n x$ is absolutely and uniformly convergent in n, since

$$\sum_{k=0}^{\infty} |a_{nk}x_k| \le \left(\sum_{k=0}^{\infty} |a_{nk}|\right) \|x_k\|_{\infty} < \infty.$$

Therefore, we have

$$\lim_{A} x = \lim_{n \to \infty} A_n x = \sum_{k=0}^{\infty} \left(\lim_{n \to \infty} a_{nk} x_k \right) = \sum_{k=0}^{\infty} \alpha_k x_k,$$

hence $Ax \in c$. This shows the sufficiency of the conditions, and completes the proof of Part (a.1).

(a.2) Now we show the necessity of the conditions in (5.1) and (5.2). So we assume $A \in (\ell_{\infty}, c)$. It follows from $e^{(k)} \in \ell_{\infty}$ (k = 0, 1, ...) that, for each k, there exists a complex number α_k such that (5.2) holds. Furthermore $c \subset \ell_{\infty}$ implies $(\ell_{\infty}, c) \subset (c, c)$ and so $A \in \Phi$ by Part (a) of Theorem 4.3, and this and (5.2) imply $\alpha = (\alpha_k)_{k=0}^{\infty} \in \ell_1$ by (4.5) in Part (a.1) of the proof of Theorem 4.3. We define the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by $b_{nk} = a_{nk} - \alpha_k$ (n, k = 0, 1, ...), and obtain $B \in (\ell_{\infty}, c)$. We will show that this implies

(5.4)
$$\lim_{n \to \infty} \|B_n\|_1 = \lim_{n \to \infty} \sum_{k=0}^{\infty} |b_{nk}| = 0.$$

Then it will follow by Lemma 5.1 that $\sum_{k=0}^{\infty} |b_{nk}|$ converges uniformly in *n* whence $\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} |b_{nk} + \alpha_k|$ converges uniformly in *n*, which is condition (5.1).

To show that (5.4) must hold, we assume that it is not satisfied and construct a sequence $x \in \ell_{\infty}$ with $Bx \notin c$ which is a contradiction to $B \in (\ell_{\infty}, c)$. If $||B_n|_1 \neq 0$ $(n \to \infty)$, then there is a positive real c such that

$$\limsup_{n \to \infty} \|B_n\|_1 = c,$$

hence, for some subsequence $(n_j)_{j=0}^{\infty}$,

$$\lim_{j \to \infty} \left\| B_{n_j} \right\|_1 = c.$$

We omit the indices j, that is we assume without loss of generality

$$\lim_{n \to \infty} \|B_n\|_1 = c.$$

It follows from (5.2) that

(5.6)
$$\lim_{n \to \infty} b_{nk} = 0.$$

By (5.5) and (5.6), there is an integer n(1) such that

$$|||B_n||_1 - c| < \frac{c}{10}$$
 and $|b_{n(1),0}| < \frac{c}{10}$

Since $||B_{n(1)}||_1 < \infty$, we may choose an integer k(2) > 0 such that

$$\sum_{k=k(2)+1}^{\infty} |b_{n(1),k}| < \frac{c}{10},$$

and it follows that

$$\left|\sum_{k=0}^{k(2)} |b_{n(1),k}| - c\right| \le \left| \|B_{n(1)}\|_1 - c\right| + \sum_{k=k(2)+1}^{\infty} |b_{n(1),k}| + |b_{n(1),0}| < \frac{3c}{10}.$$

Now we choose an integer n(2) > n(1) such that

$$\sum_{k=0}^{k(2)} |b_{n(2),k}| < \frac{c}{10} \text{ and } |||B_{n(2)}||_1 - c| < \frac{c}{10},$$

and an integer k(3) > k(2) such that

$$\sum_{k=k(3)+1}^{\infty} |b_{n(2),k}| < \frac{c}{10}.$$

Again it follows that

$$\left|\sum_{k=k(2)+1}^{k(3)} |b_{n(2),k}| - c\right| < \frac{c}{10}.$$

Continuing in this way, we can determine sequences $(n(r))_{r=1}^{\infty}$ and $(k(r))_{r=1}^{\infty}$ of integers with $n(1) < n(2) < \ldots$ and $0 = k(1) < k(2) < \ldots$ such that

(5.7)
$$\sum_{k=0}^{k(r)} |b_{n(r),k}| < \frac{c}{10}, \quad \sum_{k=k(r+1)+1}^{\infty} |b_{n(r),k}| < \frac{c}{10}, \\ \left| \sum_{k=k(r)+1}^{k(r+1)} |b_{n(r),k}| - c \right| < \frac{3c}{10}.$$

Now we define the sequence x by

$$x_k = \begin{cases} 0 & (k=0) \\ (-1)^r \operatorname{sgn}(b_{n(r),k}) & (k(r)+1 \le k \le k(r+1)) \\ \end{cases} (r = 1, 2, \dots).$$

Then we obviously have $x \in \ell_{\infty}$ and $\sup_k |x_k| \leq 1$, and we conclude from (5.7)

$$\begin{split} |B_{n(r)}(x) - (-1)^r c| &\leq \sum_{k=0}^{k(r)} |b_{n(r),k}| |x_k| + \sum_{k=k(r+1)+1}^{\infty} |b_{n(r),k}| |x_k| + \\ &+ \left| \sum_{k=k(r)+1}^{k(r+1)} b_{n(r),k} x_k - (-1)^r c \right| \\ &\leq \sum_{k=0}^{k(r)} |b_{n(r),k}| + \sum_{k=k(r+1)+1}^{\infty} |b_{n(r),k}| + \\ &+ \left| (-1)^r \left(\sum_{k=k(r)+1}^{k(r+1)} |b_{n(r),k}| - c \right) \right| \\ &< \frac{c}{10} + \frac{c}{10} + \frac{3c}{10} = \frac{c}{2}. \end{split}$$

Consequently the sequence $(B_n(x))_{n=0}^{\infty}$ is not a Cauchy sequence and so not convergent. Thus if (5.4) is false then there is a sequence $x \in \ell_{\infty}$ such that $(B_n(x))_{n=0}^{\infty}$ is not convergent, which is a contradiction to $B(x) \in c$ for all $x \in \ell_{\infty}$. Therefore (5.4) must hold. This completes the proof of the necessity of the conditions, that is, of Part (a.2).

(b) Part (b) is proved in exactly the same way by putting $\alpha_k = 0$.

Remark 5.3. (a) Condition (5.1) in Theorem 5.2 may be replaced by either one of the conditions

(5.8)
$$\left\{\begin{array}{l} A_n = (a_{nk})_{k=0}^{\infty} \in \ell_1 \text{ for all } n, \quad \alpha = (\alpha_k)_{k=0}^{\infty} \in \ell_1 \\ \text{and } \lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| = 0 \end{array}\right\}$$

(5.9)
$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} |\alpha_k|, \text{ the series being convergent.}$$

(b) We have $A \in (\ell_{\infty}, c_0)$ if and only if

(5.10)
$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = 0$$

Proof. We assume that the condition in (5.2) holds. First we show that (5.1) implies (5.8). Since $\sum_{k=0}^{\infty} |a_{nk}|$ converges uniformly in n, each series $\sum_{k=0}^{\infty} |a_{nk}|$ must converge, that is $A_n \in \ell_1$ for all n. Furthermore, we saw at the beginning of the proof of Theorem 5.2(a) that the conditions in (5.1) and (5.2) imply $\alpha = (\alpha_k)_{k=0}^{\infty} \in \ell_1$. Finally, since the series $\sum_{k=0}^{\infty} |a_{nk}|$ is uniformly convergent in n, we conclude

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} \lim_{n \to \infty} |a_{nk}| = \sum_{k=0}^{\infty} |\alpha_k|$$

Now we show that (5.8) implies (5.9). This is an immediate consequence of

$$\left|\sum_{k=0}^{\infty} |a_{nk}| - \sum_{k=0}^{\infty} |\alpha_k|\right| \le \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \text{ for all } n = 0, 1, \dots$$

Finally we show that (5.9) implies (5.1). Let $\varepsilon > 0$ be given. It follows from (5.9) that there are integers n_1 and j_1 such that

$$\left|\sum_{k=0}^{\infty} |a_{nk}| - \sum_{k=0}^{\infty} |\alpha_k|\right| < \frac{\varepsilon}{3} \text{ for all } n \ge n_1 \text{ and } \sum_{k=j_1+1}^{\infty} |\alpha_k| < \frac{\varepsilon}{3}.$$

Furthermore, since $\lim_{n\to\infty} a_{nk} = \alpha_k$ for every k, there is $n_2 \in \mathbb{N}_0$ such that

$$\left|\sum_{k=0}^{j_0} |a_{nk}| - \sum_{k=0}^{j_1} |\alpha_k|\right| < \frac{\varepsilon}{3} \text{ for all } n \ge n_2.$$

We put $n_0 = \max\{n_1, n_2\}$. Then we have for all $j \ge j_1 + 1$ and for all $n \ge n_0$

$$\begin{split} \sum_{k=j}^{\infty} |a_{nk}| &\leq \sum_{k=j_1+1}^{\infty} |a_{nk}| \leq \left| \sum_{k=j_1+1}^{\infty} |a_{nk}| - \sum_{k=j_1+1}^{\infty} |\alpha_k| \right| + \sum_{k=j_1+1}^{\infty} |\alpha_k| \\ &\leq \left| \sum_{k=0}^{\infty} |a_{nk}| - \sum_{k=0}^{\infty} |\alpha_k| \right| + \left| \sum_{k=0}^{j_1} |a_{nk}| - \sum_{k=0}^{j_1} |\alpha_k| \right| + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}. \end{split}$$

Finally, for each n with $0 \le n \le n_0$, we can choose $\tilde{j}(n) \in \mathbb{N}_0$ such that $\sum_{k=j}^{\infty} |a_{nk}| < \varepsilon$ for all $j \ge \tilde{j}(n)$. We put $j_0 = \max\{j_1, \max\{\tilde{j}(n) : 0 \le n \le n_0\}\}$. Then we have

$$\sum_{k=j}^{\infty} |a_{nk}| < \varepsilon \text{ for all } j \ge j_0 \text{ and for all } n \in \mathbb{N}_0$$

that is condition (5.1) is satisfied.

As an immediate consequence of Theorems 4.3 and 5.2 we obtain the following famous result due to *Steinhaus*.

Remark 5.4 (Steinhaus). ([107, 60]) A regular matrix cannot sum all bounded sequences. For if there were a regular matrix A which sums all bounded sequences, then

$$\lim_{n \to \infty} a_{nk} = 0 \text{ and } \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1$$

by Theorem 4.3(c). On the other hand, $A \in (\ell_{\infty}, c)$ implies that $\sum_{k=0}^{\infty} |a_{nk}|$ is uniformly convergent in n by Theorem 5.2(a), hence

$$\sum_{k=0}^{\infty} a_{nk} \text{ converges uniformly in } n$$

But then it follows that

$$1 = \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} \lim_{n \to \infty} a_{nk} = 0,$$

a contradiction.

An interesting application of Schur's theorem is to show that weak and strong convergence coincide in ℓ_1 . We recall that a sequence (x_n) in a normed space $(X, \|\cdot\|)$ is said to be weakly convergent to a limit $x \in X$ if $f(x_n) \to f(x)$ $(n \to \infty)$ for all $f \in X^*$, where X^* denotes the space of all continuous linear functionals on X, and X^* has the norm $\|\cdot\|^*$ defined by $\|f\|^* = \sup\{|f(x)| : \|x\| \le 1\}$; it is said to be strongly convergent to a limit $x \in X$ if $\|x_n - x\| \to 0$ $(n \to \infty)$. Since

$$|f(x_n) - f(x)| = |f(x_n - x)| \le ||f||^* \cdot ||x_n - x||$$
 for all $f \in X^*$,

strong convergence implies weak convergence. The converse implication is not true in general. To see this, we consider the sequence $(e^{(n)})_{n=0}^{\infty}$ in $\ell_2 = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^2\}$ with the norm $\|\cdot\|_2$ given by $\|x\|_2 = (\sum_{k=0}^{\infty} |x_k|^2)^{1/2}$ for all $x \in \ell_2$. Let $f \in \ell_2^*$ be given, then it is well known that there is a sequence $a = (a_k)_{k=0}^{\infty} \in \ell_2$ such that $f(x) = \sum_{k=0}^{\infty} a_k x_k$ for all $x \in \ell_2$. It follows that $f(e^{(n)}) = a_n \to 0$ $(n \to \infty)$, and so the sequence $(e^{(n)})_{n=0}^{\infty}$ is weakly convergent to zero. But on the other hand, we have $\|e^{(m)} - e^{(n)}\|_2 = \sqrt{2}$ for all $m \neq n$, hence $(e^{(n)})_{n=0}^{\infty}$ is not a Cauchy sequence, and so not convergent in ℓ_2 . Therefore the sequence $(e^{(n)})_{n=0}^{\infty}$ is not strongly convergent.

In ℓ_1 , we have, however,

Theorem 5.5. Strong and weak convergence of sequences coincide in ℓ_1 .

Proof. We assume that the sequence $(x^{(n)})_{n=0}^{\infty}$ is weakly convergent to x in ℓ_1 , that is,

$$f(x^{(n)}) - f(x) \to 0 \ (n \to \infty)$$
 for each $f \in \ell_1^*$.

It is well known (for example [110, Example 6.4.2]) that to every $f \in \ell_1^*$ there corresponds a sequence $a \in \ell_\infty$ such that

$$f(y) = \sum_{k=0}^{\infty} a_k y_k$$
 for all $y \in \ell_1$.

We define the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by $b_{nk} = x_k^{(n)} - x_k$ (n, k = 0, 1, ...). Then we have for all $a \in \ell_{\infty}$

$$f(x^{(n)}) - f(x) = f(x^{(n)} - x) = \sum_{k=0}^{\infty} a_k (x_k^{(n)} - x_k)$$
$$= \sum_{k=0}^{\infty} b_{nk} a_k \to 0 \ (n \to \infty),$$

that is, $B \in (\ell_{\infty}, c_0)$. It follows from Theorem 5.2(b), that $\sum_{k=0}^{\infty} |b_{nk}|$ converges uniformly in n and $\lim_{n\to\infty} b_{nk} = 0$ for each k. Thus we have

$$\|x^{(n)} - x\|_1 = \sum_{k=0}^{\infty} |x_k^{(n)} - x_k| = \sum_{k=0}^{\infty} |b_{nk}| \to 0 \ (n \to \infty).$$

6 Inclusion and consistency theorems

In this section, we prove some *inclusion* and *consistency* theorems. First we need some results on the associativity of matrix multiplication, which, in general, is not associative for infinite matrices.

Definition 6.1. Let ϕ denote the set of all finite sequences, that is, of sequences that terminate in zeros.

- (a) A matrix A is said to be row finite if $A_n = (a_{nk})_{k=0}^{\infty} \in \phi$ (n = 0, 1, ...); it is called column finite if $A^k = (a_{nk})_{n=0}^{\infty} \in \phi$ (k = 0, 1, ...).
- (b) A matrix A is said to be *triangular* if $a_{nk} = 0$ for k > n (n = 0, 1, ...). A triangular matrix A is called a *triangle* if $a_{nn} \neq 0$ for all n.

Remark 6.2. Expressions such as

$$\sum_{n=0}^{\infty} t_n A_n x = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t_n a_{nk} x_k \text{ and } \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} t_n a_{nk} x_k$$

frequently arise in summability.

Given sequences $x = (x_n)_{n=0}^{\infty}, y = (y_n)_{n=0}^{\infty} \in \omega$, we write

$$x \bullet y = \sum_{n=0}^{\infty} x_n y_n.$$

Let $x, t \in \omega$ and A be an infinite matrix. We define the sequence b by

$$b_k = t \bullet A^k = \sum_{n=0}^{\infty} t_n a_{nk}$$
 for all $k = 0, 1, \dots$

We also have

$$t \bullet Ax = \sum_{n}^{\infty} t_n A_n x = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t_n a_{nk} x_k,$$
$$b \bullet x = \sum_{k=0}^{\infty} b_k x_k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} t_n a_{nk} x_k,$$

and $t \bullet Ax$ and $b \bullet x$ may be different, even if $t \in \ell_1$, A is a regular triangle, $x \in c_A$ and both numbers exists.

Example 6.3. We define the sequence t and the matrix A by $t_n = 2^{-n}$ and $A_n x = 2x_{n-1} - x_n$ (n = 0, 1, ...). Then we have

$$b_k = \sum_{n=0}^{\infty} t_n a_{nk} = t_k a_{kk} + t_{k+1} a_{k+1,k} = 2^{-k} \cdot (-1) + 2 \cdot 2^{-(k+1)} = 0 \text{ for all } k,$$

that is, b = 0. If we choose $x = ((-2)^k)_{k=0}^\infty$, then we have $A_0 x = 1$ and $A_n x = -2 \cdot 2^{k-1} + 2^k = 0$ for $n \ge 1$, that is, $Ax = e^{(0)}$, and so $t \bullet Ax = 1 \ne 0 = b \bullet x$.

The next result gives sufficient conditions for the multiplication of infinite matrices to be associative.

Theorem 6.4. Let $t, x \in \omega$, A be an infinite matrix and b be the sequence with $b_k = t \bullet A^k$ for $k = 0, 1, \ldots$. Then we have $t \bullet (Ax) = b \bullet x$ if

(i)
$$t \in \phi \text{ and } x \in \omega_A$$

or

(ii)
$$t \in \ell_1, A \in \Phi \text{ and } x \in \ell_\infty.$$

Proof. (i) Part (i) is obvious, involving only the adding of finitely many convergent series.

(ii) If the conditions in (ii) hold, then we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|t_n a_{nk} x_k| \le \left(\sum_{n=0}^{\infty}|t_n|\right) \|A\|\left(\sup_k |x_k|\right) < \infty,$$

and we may change the order of summation. Therefore it follows that

$$t \bullet Ax = \sum_{n=0}^{\infty} t_n A_n x = \sum_{n=0}^{\infty} t_n \left(\sum_{k=0}^{\infty} a_{nk} x_k \right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} t_n a_{nk} \right) x_k = \sum_{k=0}^{\infty} b_k x_k = b \bullet x.$$

We obtain as an immediate consequence of Theorem 6.4

Corollary 6.5. The set Φ and the set of row finite matrices have associative multiplication.

Proof. If $A \in \Phi$, then the rows of A are in ℓ_1 and the columns are in ℓ_{∞} , and so Theorem 6.4(ii) applies.

If A is row finite then condition of Theorem 6.4(i) holds for all x. \Box

Example **6.6.** (a) We have $B \circ A \neq B \cdot A$; indeed $B \circ A$ may not even be a matrix map.

Let $A_n x = x_n - x_{n-1}$ for n = 0, 1, 2... with the convention $x_{-1} = 0$. Then we have

$$e \bullet Ax = \sum_{n=0}^{\infty} A_n x = \lim_{m \to \infty} \sum_{n=0}^{m} (x_n - x_{n-1}) = \lim_{m \to \infty} x_m = \lim x \text{ for all } x \in c.$$

Now let B be the matrix with the rows $B_0 = e$ and $B_n = (0, 0, ...)$ for $n \ge 1$. Then we have

$$B(Ax) = (\lim x, 0, 0, \dots), \text{ in particular, } B \circ A \neq 0,$$

but $B \cdot A = 0$, since

$$(B \cdot A)_{nk} = \sum_{j=0}^{\infty} b_{nj} a_{jk} = b_{nk} - b_{n,k+1} = 0$$
 for all n and k .

To see that $B \circ A : c \to c$ is not given by a matrix it is sufficient to observe that it vanishes on ϕ ; the terms m_{nk} of a matrix M are determined by how it maps ϕ , since $m_{nk} = M_n e^{(k)}$ for all n and k. More is true: if $B \circ A$ were a matrix then it would have to be the matrix $B \cdot A$, since

$$B_n(A^{(k)}) = \sum_{j=0}^{\infty} A_j e^{(k)} = \sum_{j=0}^{\infty} b_{nj} \sum_{l=0}^{\infty} a_{jl} e_l^{(k)} = \sum_{j=0}^{\infty} b_{nj} a_{jk} = (B \cdot A)_{nk}$$

for all n and k, that is $B \circ A = B \cdot A$ on ϕ .

In cases like Corollary 6.5, in which (BA)x = B(Ax) for matrices A and B and sequences x in the space involved, we have $B \circ A = B \cdot A$ by definition.

(b) If matrix multiplication is associative, as in Corollary 6.5, a matrix has an inverse matrix (which is automatically unique) if and only if it has a unique right (or left) inverse.

A triangle A has a unique right inverse B: it may directly computed by mathematical induction, and is unique, since A is one to one. Moreover B is a triangle. Also BA = I, where I is the identity matrix, since A(I - BA) = 0 by Corollary 6.5; thus $B = A^{-1}$.

But A may have another left inverse. The inverse matrix B of the matrix A in Example 6.3 is given by

$$b_{nk} = \begin{cases} -2^k & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

Let C be the matrix with the rows $C_n = B_n + t$ for n = 0, 1, ..., where t is the sequence in Example 6.3. Then we have

$$(C \cdot A)_n = (B \cdot A)_n + b = e^{(n)}$$
 for all n ,

where $b = (b_k)_{k=0}^{\infty}$ is the sequence from Example 6.3 with $b_k = 0$ for all k. Thus we have $C \cdot A = I$.

Definition 6.7. If $c_B \supset c_A$ then B is said to be *stronger* than A, and A is said to be *weaker* than B; we denote this by $A \Rightarrow B$. If $c_B = c_A$ then A and B are called *equipotent*; we denote this by $A \Leftrightarrow B$.

Now we establish a test for the comparison of the strength of methods of summability given by triangles.

Theorem 6.8. Let A and B be triangles. Then B is stronger than A if and only if BA^{-1} is conservative.

Proof. We note that A^{-1} exists by Example 6.6(b), since A is a triangle. First we assume that B is stronger than A. Let $x \in c$ be given. Then it follows that $y = A^{-1}x \in c_A$, since $x \in c$ and $Ay = A(A^{-1}x) = x$ by Theorem 6.4(i). So we get $y \in c_A \subset c_B$, hence $By \in c$, but $By = (BA^{-1})x$. Therefore BA^{-1} is conservative.

Conversely we assume that BA^{-1} is conservative. Let $x \in c_A$ be given. Then we have $Ax \in c$ and, applying Theorem 6.4(i) again and using the assumption that BA^{-1} is conservative, we conclude $Bx = B(A^{-1}A)x = (BA^{-1})(Ax) \in c$, hence $x \in c_B$.

Definition 6.9. Two matrices A and B are called *consistent* if $\lim_A x = \lim_B x$ whenever $x \in c_A \cap c_B$. If B is stronger than A and consistent with A, then we write $B \supset A$. If $A \supset B \supset A$ then A and B are *equivalent*, denoted by $A \equiv B$.

Theorem 6.10. Let A and B be regular and row finite, and assume AB = BA. Then A and B are consistent. *Proof.* Let $x \in c_A \cap c_B$ be given. Applying Corollary 6.5, we obtain

$$\lim_{B} x = \lim_{n \to \infty} B_n x = \lim_{n \to \infty} A_n(Bx) = \lim_{n \to \infty} (AB)_n x$$
$$= \lim_{n \to \infty} (BA)_n x = \lim_{n \to \infty} B_n(Ax) = \lim_{n \to \infty} A_n x = \lim_{n \to \infty} x.$$

Theorem 6.11. Let A and B be triangles. Then $B \supset A$ if and only if $B \cdot A^{-1}$ is regular.

Proof. First we assume that A and B are triangles and $B \supset A$. Let $x \in c$ be given. Applying Corollary 6.5 and Theorem 6.8, we obtain, since $B \supset A$,

$$\lim BA^{-1}x = \lim_{B} A^{-1}x = \lim_{A} A^{-1}x = \lim_{A} x.$$

Conversely, we assume that BA^{-1} is regular. Then we have by Theorem 6.8 $c_B \supset c_A$, hence for all $x \in c_A \cap c_B = c_A$

$$\lim_{B} x = \lim_{B} \lim_{A} A^{-1}Ax = \lim_{B} (BA^{-1})Ax = \lim_{A} Ax = \lim_{A} Ax,$$

since BA^{-1} is regular.

7 The Cesàro methods of order greater than -1

In this section, we study the Cesàro methods C_{α} of order $\alpha > -1$ which are generalizations of the C_1 method. We use the traditional notations and write Σa_k for a series of complex numbers, and $s = (s_k)_{k=0}^{\infty}$ for the sequence of its partial sums.

Definition 7.1. Let $\delta \in \mathbb{R}$. Then the numbers

$$A_n^{\delta} = \binom{n+\delta}{n}$$
 for $n = 0, 1, \dots$

are called the n^{th} Cesàro coefficients of order δ . For $\alpha > -1$, the Cesàro method C_{α} of order α is defined by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ with

$$a_{nk} = \begin{cases} A_{n-k}^{\alpha-1} / A_n^{\alpha} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$
 $(n = 0, 1, ...).$

The $n^{th} C_{\alpha}$ mean of a sequence $s = (s_k)_{k=0}^{\infty}$ is defined by

$$\sigma_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_{n-k}^{\alpha-1} s_k \text{ for all } n = 0, 1, \dots,$$

and we write $s_n^{\alpha} = A_n^{\alpha} \sigma_n^{\alpha}$ for $n = 0, 1, \ldots$

The Cesàro coefficients have the properties in Lemma 7.2 below of which only those in (7.10) and (7.11) are not immediate consequences of their definition.

We recall for the result in (7.11) that Euler's Γ function Γ is defined by the improper integral for all x > 0 by

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} \cdot t^{x-1} dt.$$

It is also known from elementary analysis that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$ and the recursion formula $\Gamma(x+1) = x \cdot \Gamma(x)$ holds for all x > 0.

Lemma 7.2. The Cesàro coefficients have the following properties

- (7.1) $A_n^0 = 1 \text{ for } n = 0, 1, \dots;$
- (7.2) $A_0^{\alpha} = 1 \text{ for all } \alpha \in \mathbb{R};$

(7.3)
$$A_n^{\alpha} > 0 \text{ for all } \alpha > -1 \text{ and } n = 0, 1, \dots;$$

(7.4) $A_n^k = 0 \text{ for all } k \in -\mathbb{N} \text{ and } n = k, k+1, \dots;$

(7.5)
$$A_n^{\alpha}$$
 has fixed sign for all $\alpha < -1$, $n > -\alpha$ and $\alpha \notin \mathbb{Z}$;

(7.6)
$$A_n^{\alpha} = \frac{n+\alpha}{n} A_{n-1}^{\alpha} \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in \mathbb{R};$$

(7.7)
$$A_n^{\alpha} = \frac{n+\alpha}{\alpha} A_n^{\alpha-1} \text{ for all } n \in \mathbb{N}_0 \text{ and } \alpha \in \mathbb{R} \setminus \{0\};$$

(7.8)
$$A_n^{\alpha} \le A_{n+1}^{\alpha} \text{ for } n \in \mathbb{N}_0 \text{ and } \alpha > 0;$$

(7.9)
$$A_n^{\alpha} \ge A_{n+1}^{\alpha} \text{ for } n \in \mathbb{N}_0 \text{ and } -1 < \alpha < 0$$

Let $\alpha > -1$. Then there are constants K_1 and K_2 depending on α only such that

(7.10)
$$K_1(n+1)^{\alpha} \le A_n^{\alpha} \le K_2(n+1)^{\alpha} \text{ for } n = 0, 1, \dots$$

The Cesàro coefficients have the following asymptotic behaviour

(7.11)
$$\lim_{n \to \infty} \frac{A_n^{\alpha}}{(n+1)^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} \text{ for all } \alpha \notin -\mathbb{N},$$

where Γ denotes Euler's gamma function.

Proof. We have by definition:

- (7.1) $A_n^0 = \binom{n+0}{n} = 1$ for $n = 0, 1, \dots$ and
- (7.2) $A_0^{\alpha} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = 1$ for all $\alpha \in \mathbb{R}$.
- (7.3) Since $\alpha + 1 > 0$ for $\alpha > -1$, we obtain

$$A_n^{\alpha} = \frac{(n+\alpha)(n+\alpha-1)\cdots(\alpha+1)}{n!} > 0 \text{ for } n = 1, 2, \dots;$$

also $A_0^{\alpha} = 1 > 0$ by (7.2).

(7.4) Let $\alpha = -k, k \in \mathbb{N}$ and $n = k, k + 1, \ldots$ Then we have

$$A_n^{\alpha} = \binom{n-k}{n} = \frac{(n-k)(n-(k+1))\dots(-k+1)}{n!} = 0,$$

since one of the factors in the nominator is equal to 0.

(7.5) It follows for $-k < \alpha < -k+1$ and $n+\alpha > 0$ that

$$A_n^{\alpha} = \frac{(n+\alpha)\cdots(k+\alpha)(k-1+\alpha)\cdots(\alpha+1)}{n!}$$

where the factors $(n + \alpha), \ldots, (k + \alpha)$ are positive and the k factors $(k - 1 + \alpha), \ldots, (\alpha + 1)$ are negative.

It follows from the definition of the Cesàro coefficients that

(7.6)
$$A_n^{\alpha} = \frac{(n+\alpha)\cdots(\alpha+1)}{n!} = \frac{(n+\alpha)}{n} \cdot \frac{(n+\alpha-1)\cdots(\alpha+1)}{(n-1)!}$$

 $\frac{n+\alpha}{n} A_{n-1}^{\alpha}$ for $n = 1, 2, ...;$

(7.7)
$$A_n^{\alpha} = \frac{(n+\alpha)\cdots(\alpha+1)}{n!} = \frac{(n+\alpha)}{\alpha} \cdot \frac{(n+\alpha-1)\cdots(\alpha+1)\alpha}{n!}$$
$$\frac{n+\alpha}{\alpha} A_n^{\alpha-1} \text{ for } n = 0, 1, 2, \dots \text{ and } \alpha \neq 0.$$

(7.8) Since $(n + \alpha)/n > 1$ for $\alpha > 0$, (7.8) follows from (7.6).

(7.9) Since $(n + \alpha)/n < 1$ for $-1 < \alpha < 0$, (7.9) follows from (7.6).

(7.10) It is sufficient by (7.7) and (7.1) to consider the case of $0 < \alpha < 1$. Since $\log(1+x) \le x$ for all $x \ge 0$, we obtain

$$\log A_n^{\alpha} = \log \frac{(n+\alpha)\cdots(\alpha+1)}{n!} = \sum_{k=1}^n \log\left(1+\frac{\alpha}{k}\right) \le \alpha \cdot \sum_{k=1}^n \frac{1}{k}$$
$$\le \alpha \left(\log\left(n+1\right)\right) + \gamma, \text{ where } \gamma \text{ is Euler's constant.}$$

This implies

$$\left|\log A_n^{\alpha} - \alpha \log \left(n+1\right)\right| = \left|\log \frac{A_n^{\alpha}}{(n+1)^{\alpha}}\right| \le \alpha \cdot \gamma,$$

and so

$$e^{-\alpha\gamma} \le \frac{A_n^{\alpha}}{(n+1)^{\alpha}} \le e^{\alpha\gamma}$$

Putting $K_1 = e^{-\alpha \gamma}$ and $K_2 = e^{\alpha \gamma}$, we obtain the inequalities in (7.10).

(7.11 (i)) First we show

(7.12)
$$\lim_{n \to \infty} \frac{(n-1)! n^t}{t(t+1)\cdots(t+n-1)} = \Gamma(t) \text{ for } t > 0.$$

Since $\Gamma(t+n) = (t+n-1)\cdots t\Gamma(t)$ for all $n \in \mathbb{N}$ and t > 0, (7.12) is equivalent to

(7.13)
$$\lim_{n \to \infty} \frac{\Gamma(t+n)}{n^t \, \Gamma(n)} = 1.$$

Also since $\Gamma(t+1) = t\Gamma(t)$ and $\Gamma(n+1) = n!$, it suffices to show (7.13) for 0 < t < 1. So let 0 < t < 1. We put

$$I_1(n) = \int_0^n e^{-u} u^{t+n-1} du \text{ and } I_2(n) = \int_n^\infty e^{-u} u^{t+n-1} du \text{ for } n \in \mathbb{N},$$

so that $\Gamma(t+n) = I_1(n) + I_2(n)$. Since $u^t \le n^t$ and $u^{t-1} \ge n^{t-1}$ for $0 < u \le n$, we obtain

(7.14)
$$n^{t-1} \int_{0}^{n} e^{-u} u^{n} du \leq I_{1}(n) \leq n^{t} \int_{0}^{n} e^{-u} u^{n-1} du;$$

similarly we have

(7.15)
$$n^{t} \int_{n}^{\infty} e^{-u} u^{n-1} du \leq I_{2}(n) \leq n^{t-1} \int_{n}^{\infty} e^{-u} u^{n} du.$$

Integration by parts in (7.14) yields

$$\begin{split} n^{t-1} \left(-e^{-u} u^n \Big|_{u=0}^n + n \int_0^n e^{-u} u^{n-1} du \right) &= \\ &= n^t \int_0^n e^{-u} u^{n-1} du - n^{t+n-1} e^{-n} \le I_1(n) \le \\ &\le n^t \left(\frac{1}{n} e^{-u} u^n \Big|_{u=0}^n + \frac{1}{n} \int_0^n e^{-u} u^n du \right) = \\ &= n^{t-1} \int_0^n e^{-u} u^n du + n^{t+n-1} e^{-n}, \end{split}$$

Adding this to (7.15), we obtain

$$n^{t}\Gamma(n) - n^{t+n-1} e^{-n} \le \Gamma(t+n) \le n^{t-1}\Gamma(n+1) + n^{t+n-1} e^{-n}$$
$$= n^{t}\Gamma(n) + n^{t+n-1} e^{-n},$$

hence

$$1 - \frac{n^{n-1} \operatorname{e}^{-n}}{\Gamma(n)} \le \frac{\Gamma(t+n)}{n^t \Gamma(n)} \le 1 + \frac{n^{n-1} \operatorname{e}^{-n}}{\Gamma(n)},$$

or equivalently,

(7.16)
$$\left|\frac{\Gamma(t+n)}{n^t \Gamma(n)} - 1\right| \le \frac{n^{n-1} \operatorname{e}^{-n}}{\Gamma(n)} = \frac{n^n \operatorname{e}^{-n}}{n!}.$$

Now we show

(7.17)
$$\lim_{n \to \infty} \frac{n^n e^{-n}}{n!} = 0.$$

Let $m \in \mathbb{N}$ be given. Then we have for each $n \in \mathbb{N}$

$$e^n > \sum_{l=0}^m \frac{n^{\ell+n}}{(\ell+n)!},$$

hence

$$\frac{e^n n!}{n^n} > \sum_{l=0}^m \frac{n^\ell n!}{(\ell+n)!} = 1 + \sum_{l=1}^m \frac{n^\ell}{(n+1)(n+2)\dots(n+\ell)}$$
$$= 1 + \sum_{l=1}^m \frac{1}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)\dots\left(1 + \frac{\ell}{n}\right)} \to m+1 \ (n \to \infty).$$

Since $m \in \mathbb{N}$ was arbitrary, we have established (7.17). Now (7.17) and (7.16) yield (7.13) which is equivalent to (7.12).

This concludes Part (i) of the proof of (7.11).

(7.11 (ii)) We conclude from Part (7.11 (i)), putting $t = \alpha + 1 > 0$ for $\alpha > -1$

$$\frac{A_n^{\alpha}}{(n+1)^{\alpha}} = \frac{(n+\alpha)\cdots(\alpha+1)}{n!(n+1)^{\alpha}} = \frac{(n+t-1)\cdots t}{n^t(n-1)!} \cdot \frac{n^{\alpha}}{(n+1)^{\alpha}}$$
$$\to \frac{1}{\Gamma(t)} = \frac{1}{\Gamma(\alpha+1)} \ (n \to \infty).$$

Thus we have shown (7.11).

Remark 7.3. It is easy to see from the relations in Lemma 7.2 that $C_0 = I$, the identity matrix, that is, the C_0 method is the same as ordinary convergence, and, for $\alpha = 1$, the C_{α} method is the C_1 method of Section 2.

Many computations involving Cesàro methods are conveniently handled by the use of the binomial series which is studied in elementary analysis. We include the related theorem and its proof for the reader's convenience

Theorem 7.4. Let $\alpha \in \mathbb{R}$. Then we have

(a)
$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} {\alpha \choose n} x^n \text{ for } |x| < 1;$$

(b)
$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^{\alpha} x^n \text{ for } |x| < 1.$$

Proof. (a) For non-negative integers α , this reduces to the well-known binomial formula.

Let $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$. The function $f: (-1, \infty) \to \mathbb{R}$ defined by $f(x) = (1+x)^{\alpha}$ is of class C^{∞} on $(-1, \infty)$. We have for $\nu = 1, 2, \ldots$

$$f^{(\nu)}(x) = \alpha(\alpha - 1) \cdots (\alpha - \nu + 1)(1 + x)^{\alpha - \nu}, \quad \frac{f^{(0)}(0)}{0!} = 1, \quad \frac{f^{(\nu)}(0)}{\nu!} = \binom{\alpha}{\nu}.$$

The n^{th} Taylor polynomial is given by

$$T_n(x,0) = \sum_{\nu=0}^n {\alpha \choose \nu} x^{\nu} \text{ for } x \in (-1,\infty).$$

Cauchy's formula for the remainder term yields

$$R_n(x,0) = \frac{f^{(n+1)}(\Theta x)}{n!} (1-\Theta)^n x^{n+1}$$

= $\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} (1+\Theta x)^{\alpha-n-1} (1-\Theta)^n x^{n+1}$
= $\alpha {\binom{\alpha-1}{n}} x^{n+1} (1+\Theta x)^{\alpha-1} \left(\frac{1-\Theta}{1+\Theta x}\right)^n$
for some $\Theta \in (0,1)$.

Since $|1 + \Theta x| \ge 1 - \Theta |x| \ge 1 - \Theta$ for $x \in (-1, 1)$ and $\Theta \in (0, 1)$, there is a constant M independent of n such that

$$|R_n(x,0)| \le M \left| \binom{\alpha - 1}{n} \right| \cdot |x|^n.$$

We put

$$y_n = \binom{\alpha}{n} x^n \text{ for } x \neq 0,$$

and obtain

$$\lim_{n \to \infty} \frac{y_{n+1}}{y_n} = \lim_{n \to \infty} \frac{\alpha - n}{n+1} x = -x.$$

If $x \in (-1, 1)$, then there exists q such that |x| < q < 1, and so there is an $N \in \mathbb{N}$ such that

$$\left|\frac{y_{n+1}}{y_n}\right| \le q \text{ for all } n \ge N,$$

hence for n > N

$$|y_n| = \left|\frac{y_n}{y_{n-1}}\right| \cdot \left|\frac{y_{n-1}}{y_{n-2}}\right| \cdots \left|\frac{y_{N+1}}{y_N}\right| \cdot |y_N| \le q^{n-N} |y_N| = q^n \frac{|y_N|}{q^N},$$

and so $\lim_{n\to\infty} y_n = 0$. Finally, this implies $\lim_{n\to\infty} R_n(x,0) = 0$ on (-1,1). Thus we have shown Part (a).

(b) We conclude by Part (a) and the fact that

$$(-1)^{\nu} \binom{-\alpha-1}{\nu} = (-1)^{\nu} \frac{(-\alpha-1)(-\alpha-2)\cdots(-\alpha-1-\nu+1)}{\nu!}$$
$$= \frac{(\nu+\alpha)\cdots(\alpha+1)}{\nu!} = A^{\alpha}_{\nu},$$

that, for $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$ and |x| < 1,

$$\frac{1}{(1-x)^{\alpha+1}} = (1-x)^{-\alpha-1} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{-\alpha-1}{\nu} x^{\nu} = \sum_{\nu=0}^{\infty} A_{\nu}^{\alpha} x^{\nu}.$$

Thus we have established the identity in (b).

Remark 7.5. (a) Since

$$\lim_{n \to \infty} \frac{|y_{n+1}|}{|y_n|} = |x| < 1$$

in the proof of Part (a) of Theorem 7.5, we could have applied the ratio test to conclude the convergence of the series $\sum_{n=0}^{\infty} y_n$ which implies $\lim_{n\to\infty} y_n = 0$. In fact, the succeeding lines there give a proof of the ratio test.

(b) Using the identity in Part (b) of Theorem 7.5 and the Cauchy product of two power series, we obtain for sufficiently small |x|

$$\frac{1}{(1-x)^{\alpha+1}} \sum_{\nu=0}^{\infty} s_{\nu} x^{\nu} = \left(\sum_{\nu=0}^{\infty} A_{\nu}^{\alpha} x^{\nu}\right) \cdot \left(\sum_{\nu=0}^{\infty} s_{\nu} x^{\nu}\right)$$
$$= \sum_{\nu=0}^{\infty} \left(\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\alpha} s_{\mu}\right) x^{\nu} = \sum_{\nu=0}^{\infty} s_{\nu}^{(\alpha)} x^{\nu}.$$

The next important result gives a formula of the transformation between Cesàro means of different order.

Theorem 7.6. Let $\alpha > -1$ and $\alpha + \beta + 1 > -1$. Then we have for arbitrary sequences $s = (s_k)_{k=0}^{\infty}$

(7.18)
$$\sigma_n^{\alpha+\beta+1} = \frac{1}{A_n^{\alpha+\beta+1}} \sum_{k=0}^n A_{n-k}^{\beta} A_k^{\alpha} \sigma_k^{\alpha} \text{ for all } n = 0, 1, \dots$$

Proof. First we show

(7.19)
$$A_n^{\alpha+\beta+1} = \sum_{k=0}^n A_{n-k}^{\beta} A_k^{\alpha} \text{ for all } n = 0, 1, \dots$$

We obtain by Part (b) in Theorem 7.4 for |x| < 1,

$$\sum_{n=0}^{\infty} A_n^{\alpha+\beta+1} x^n = \frac{1}{(1-x)^{\alpha+\beta+2}} = \frac{1}{(1-x)^{\alpha+1}} \frac{1}{(1-x)^{\beta+1}}$$

$$= \left(\sum_{n=0}^{\infty} A_n^{\alpha} x^n\right) \left(\sum_{n=0}^{\infty} A_n^{\beta} x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n A_k^{\alpha} A_{n-k}^{\beta}\right) x^n.$$

Comparing coefficients, we obtain (7.19).

Now it follows from (7.19) that

$$\sum_{k=0}^{n} A_{n-k}^{\beta} A_{k}^{\alpha} \sigma_{k}^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\beta} \sum_{j=0}^{k} A_{k-j}^{\alpha-1} s_{j}$$
$$= \sum_{j=0}^{n} s_{j} \sum_{k=j}^{n} A_{n-k}^{\beta} A_{k-j}^{\alpha-1} = \sum_{j=0}^{n} s_{j} \sum_{k=0}^{n-j} A_{n-j-k}^{\beta} A_{k}^{\alpha-1}$$
$$= \sum_{j=0}^{n} A_{n-j}^{\beta+\alpha} s_{j} = A_{n}^{\alpha+\beta+1} \sigma_{n}^{\alpha+\beta+1},$$

and (7.18) is an immediate consequence.

Next we apply Theorem 7.6 to obtain an inverse formula for the C_{α} means.

Example 7.7 (Inverse formula for the C_{α} means). If we put $\beta = -(\alpha + 1)$ in (7.18) of Theorem 7.6, then we obtain

(7.20)
$$s_n = \sum_{k=0}^n A_{n-k}^{-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha} \quad (\alpha > -1; n = 0, 1, \dots),$$

which is an inverse formula for the C_{α} means.

The C_{α} methods become stronger with α increasing.

Theorem 7.8. Let $-1 < \alpha \leq \beta$. Then we have $C_{\alpha} \subset C_{\beta}$.

Proof. We may assume $\alpha < \beta$. We have by (7.18)

$$\sigma_n^\beta = \sigma_n^{\beta - \alpha - 1 + \alpha + 1} = \sum_{k=0}^\infty a_{nk} \sigma_k^\alpha$$

where

$$a_{nk} = \begin{cases} \frac{A_{n-k}^{\beta - \alpha - 1} A_k^{\alpha}}{A_n^{\beta}} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

Since $\beta - \alpha - 1 > -1$, it follows that $a_{nk} \ge 0$ for all n and k by (7.3) in Lemma 7.2. Therefore we have by (7.19)

$$\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} a_{nk} = 1 \text{ for all } n = 0, 1, \dots,$$
and conditions (i') and (iii') of Part (c) in Theorem 4.3 are satisfied.

Now we fix $k \in \mathbb{N}_0$. By (7.10) in Lemma 7.2, there are absolute constants K_1 and K_2 such that

$$K_1 \frac{(n-k+1)^{\beta-\alpha-1}(k+1)^{\alpha}}{(n+1)^{\beta}} \le a_{nk} \le K_2 \frac{(n-k+1)^{\beta-\alpha-1}(k+1)^{\alpha}}{(n+1)^{\beta}}$$

for all $n \ge k$. Obviously the terms on the left and the right tend to zero as $n \to \infty$, and so condition (ii') in Part (c) of Theorem 4.3 is also satisfied. Thus A is regular and the conclusion follows by Theorem 6.8.

Applying Theorem 7.8 with $\alpha = 0$ and $\beta = \alpha$, we obtain

Corollary 7.9. The C_{α} methods are regular for $\alpha > 0$.

Next we establish a Tauberian theorem by which the C_{α} summability for $-1 < \alpha < 0$ of a series $\sum a_k$ can be deduced from its convergence together with a condition on the growth of its terms; this is a result similar to Hardy's Big O Tauberian theorem, Theorem 3.1. First we need the following

Lemma 7.10 (Abel's summation by parts). Let $a = (a_k)_{k=0}^{\infty}$ and $b = (b_k)_{k=0}^{\infty}$ be arbitrary sequences, n and m be non-negative integers with $m \ge n$ and $B_m = \sum_{k=n}^{m} b_k$. Then we have

(7.21)
$$\sum_{k=n}^{m} a_k b_k = \sum_{k=n}^{m-1} (a_k - a_{k+1}) B_k + a_m B_m.$$

Proof. We have $b_k = B_k - B_{k-1}$ for $n \le k \le m$ where $B_{n-1} = 0$, hence

$$\sum_{k=n}^{m} a_k b_k = \sum_{k=n}^{m} a_k (B_k - B_{k-1}) = \sum_{k=n}^{m} a_k B_k - \sum_{k=n+1}^{m} a_k B_{k-1}$$
$$= \sum_{k=n}^{m} a_k B_k - \sum_{k=n}^{m-1} a_{k+1} B_k = \sum_{k=n}^{m-1} (a_k - a_{k+1}) B_k + a_m B_m.$$

Theorem 7.11. Let $\sum a_k = s$ and $a_k = 0(1/k)$. Then we have

$$\sum a_k = s(C_{\delta-1}) \text{ for } 0 < \delta < 1.$$

Proof. Since the Cesàro methods are linear, we may assume s = 0. Let $\Theta \in (0, 1)$. We put $N = [n\Theta] = \max\{m \in \mathbb{N}_0 : m \leq n\Theta\}$ and write

$$A_n^{\delta-1} s_n^{\delta-1} = \sum_{\nu=0}^n A_\nu^{\delta-1} a_{n-\nu} = \sum_{\nu=0}^{N-1} A_\nu^{\delta-1} a_{n-\nu} + \sum_{\nu=N}^n A_\nu^{\delta-1} a_{n-\nu}.$$

Then it follows that

$$S_1 = \sum_{\nu=0}^{N-1} A_{\nu}^{\delta-1} a_{n-\nu} = O(1/n) + \sum_{\nu=1}^{N-1} O(\nu^{\delta-1}) O((n-\nu)^{-1})$$

$$= O\left(\frac{N^{\delta}}{n-N}\right) = O\left(\frac{\Theta^{\delta}}{1-\Theta}n^{\delta-1}\right)$$

uniformly in Θ . Therefore, given $\varepsilon > 0$, we can choose a real $\Theta = \Theta(\varepsilon) \in (0, 1)$ such that $|S_1| < \varepsilon n^{\delta-1}$. Furthermore, we have $A_{\nu}^{\delta-1} - A_{\nu-1}^{\delta-1} = ((\delta-1)/\nu)A_{\nu-1}^{\delta-1} = O(\nu^{\delta-2})$ for $N \le \nu \le n$. Lemma 7.10 yields

$$S_{2} = \sum_{\nu=N}^{n} A_{\nu}^{\delta-1} a_{n-\nu} = \sum_{\nu=N}^{n-1} (A_{\nu}^{\delta-1} - A_{\nu-1}^{\delta-1}) \sum_{\mu=N}^{\nu} a_{n-\mu} + A_{n}^{\delta-1} \sum_{\nu=N}^{n} a_{n-\nu}$$
$$= o(1) \sum_{\nu=N}^{n-1} O(\nu^{\delta-2}) + o(1)O(n^{\delta-1}) = o(1)O(n^{\delta-1}).$$

Hence we have

$$\sigma_n^{\delta-1} = O(n^{1-\delta})o(1)O(n^{\delta-1}) = o(1),$$

and consequently $\sum a_k = O(C_{\delta-1}).$

Part (c) of the next theorem extends Part (a) of the growth theorem, Theorem 2.4, for the C_1 method.

Theorem 7.12. Let Σa_n be summable C_α for $\alpha > -1$. Then we have

(a)
$$s_n^{\beta} = \sum_{k=0}^n A_{n-k}^{\beta} a_k = o(n^{\alpha}) \text{ for } \beta \text{ with } -1 \le \beta < \alpha;$$

(b) $a_n = o(n^{\alpha}).$

(c) Let
$$\Sigma a_n$$
 be summable C_{α} for $\alpha > 0$. Then the partial sums s_n satisfy

$$s_n = o(n^{\alpha}).$$

Proof. (a) From

$$s_n^{\beta} = \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} s_k^{\alpha} = \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha},$$

we obtain

$$\frac{s_n^{\beta}}{A_n^{\alpha}} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha} \text{ for } n = 0, 1, \dots$$

We define the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ by

$$a_{nk} = \begin{cases} A_{n-k}^{\beta-\alpha-1} A_k^{\alpha} / A_n^{\alpha} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$
 $(n = 0, 1, \dots)$

and show that $A \in (c, c_0)$, that is, we show that the matrix A satisfies the conditions in Part (e) of Corollary 4.8.

(a.i) First we show that $A \in \Phi$. If $\alpha > 0$ then A_k^{α} is increasing, that is, $A_k^{\alpha}/A_n^{\alpha} \leq 1$, and so

$$\sum_{k=0}^{\infty} |a_{nk}| \le \sum_{k=0}^{n} |A_{n-k}^{\beta-\alpha-1}| \le K_2 \sum_{k=0}^{n} \frac{1}{(k+1)^{\alpha+1-\beta}} < K \quad (n=0,1,\dots)$$

for some absolute constants K_2 and K, since $\alpha + 1 - \beta > 1$. If $-1 < \alpha < 0$, then $A_0^{\beta - \alpha - 1} = 1$, and we obtain for k > 0

$$A_k^{\beta-\alpha-1} = \frac{(k+\beta-\alpha-1)\cdots(\beta-\alpha+1)(\beta-\alpha)}{k!}$$

where $\beta - \alpha + 1 > 0$ and $\beta - \alpha < 0$, hence

$$\sum_{k=0}^{n} |a_{nk}| = 1 - \sum_{k=0}^{n-1} \frac{A_{n-k}^{\beta-\alpha-1} A_k^{\alpha}}{A_n^{\alpha}} = 2 - \sum_{k=0}^{n} \frac{A_{n-k}^{\beta-\alpha-1} A_k^{\alpha}}{A_n^{\alpha}} = 2 - \frac{A_n^{\beta}}{A_n^{\alpha}} \le 2.$$

Thus $A \in \Phi$.

This completes Part (a.i) of the proof.

(a.ii) Next we show that

$$\lim_{n \to \infty} a_{nk} = 0 \text{ for each fixed } k,$$

that is, the condition in (4.21) holds with $\alpha_k = 0$ for all k. If $\beta - \alpha - 1 = -2, -3, \ldots$ then $A_{n-k}^{\beta-\alpha-1} = 0$ for all sufficiently large n, and otherwise

$$a_{nk} = O\left(\frac{(n-k+1)^{\beta-\alpha-1}}{(n+1)^{\alpha}}\right) \to 0 \ (n \to \infty)$$

since $\beta \leq \alpha$ and $\alpha > -1$ and A_k^{α} is constant. Thus $\lim_{n\to\infty} a_{nk} = 0$ for all k. This concludes Part (a.ii) of the proof.

(a.iii) Finally we show that the matrix A satisfies the condition in (4.22) of Corollary 4.8. We obtain by (7.19)

$$\sum_{k=0}^{\infty} a_{nk} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_{n-k}^{\beta-1} A_k^{\alpha} = \frac{A_n^{\beta}}{A_n^{\alpha}} \text{ for all } n,$$

and, since $\alpha > \beta$, this implies

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n \to \infty} \frac{A_n^{\beta}}{A_n^{\alpha}} = 0.$$

So the matrix A satisfies the condition in (4.22) of Corollary 4.8. This concludes Part (a.iii) of the proof.

Thus we have shown that $A \in (c, c_0)$. Therefore $\sigma_n^{\alpha} \to s \ (n \to \infty)$ implies $s_n^{\beta}/A_n^{\alpha} \to 0 \ (n \to \infty)$.

(b) This is Part (a) with $\beta = -1$ and $s_n^{-1} = a_n$ for all n.

(c) This is Part (a) with $\beta = 0$ and $s_n^0 = s_n$ for all n.

Remark 7.13. If $\alpha = 0$ then Theorem 7.12 yields the well-known fact that the convergence of Σa_n implies $a_n \to 0$ as $n \to \infty$. In the special case $\alpha = 1$, Part (c) of Theorem 7.12 reduces to Part (a) of Theorem 2.4.

Now we establish an equivalence result.

Theorem 7.14. We have $C_{\alpha} \equiv C_{\alpha-1}C_1 = C_1C_{\alpha-1}$ for $\alpha > 0$.

Proof. (i) First we show

$$(7.22) C_{\alpha-1}C_1 \subset C_{\alpha}.$$

Let $s = (s_k)_{k=0}^{\infty}$ be given and $\sigma^1 = (\sigma_n^1(s))_{n=0}^{\infty}$. Then it follows that

(7.23)
$$s_n^{\alpha}(s) = (n+\alpha)s_n^{\alpha-1}(\sigma^1) - (\alpha-1)s_n^{\alpha}(\sigma^1) \text{ for } n = 0, 1, \dots,$$

since for $\alpha = 1$

$$s_n^1(s) = (n+1)\sigma_n^1(s) = (n+1)s_n^0(\sigma),$$

and for $\alpha \neq 1$

$$s_{n}^{\alpha}(\sigma^{1}) = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \sigma_{k}^{1} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \frac{1}{k+1} \sum_{j=0}^{k} s_{j} = \sum_{j=0}^{n} s_{j} \sum_{k=j}^{n} \frac{A_{n-k}^{\alpha-1}}{k+1}$$
$$= \sum_{j=0}^{n} s_{j} \sum_{k=0}^{n-j} \frac{A_{n-j-k}^{\alpha-1}}{k+1+j} = \sum_{j=0}^{n} s_{j} \sum_{k=0}^{n-j} \frac{A_{k-1}^{\alpha}}{n-k+1},$$

hence

$$(n+\alpha)s_n^{\alpha-1}(\sigma^1) - (\alpha-1)s_n^{\alpha}(\sigma^1)$$

= $\sum_{k=0}^n s_k \sum_{j=0}^{n-k} \frac{(n+\alpha)A_j^{\alpha-2} - (\alpha-1)A_j^{\alpha-1}}{n-j+1}$
= $\sum_{k=0}^n s_k \sum_{j=0}^{n-k} \frac{A_j^{\alpha-2}}{n-j+1}(n+\alpha-(j+\alpha-1))$
= $\sum_{k=0}^n s_k \sum_{j=0}^{n-k} A_j^{\alpha-2} = \sum_{k=0}^n A_{n-k}^{\alpha-1}s_k = s_n^{\alpha}(s),$

and so (7.23) holds.

Now (7.23) implies for $\alpha > 0$

$$\begin{split} \sigma_n^{\alpha}(s) &= \frac{1}{A_n^{\alpha}}(n+\alpha)s_n^{\alpha-1}(\sigma^1) - \frac{1}{A_n^{\alpha}}(\alpha-1)s_n^{\alpha}(\sigma^1) \\ &= \frac{\alpha}{A_n^{\alpha-1}}s_n^{\alpha-1}(\sigma^1) - (\alpha-1)\sigma_n^{\alpha}(\sigma^1), \end{split}$$

hence

(7.24)
$$\sigma_n^{\alpha}(s) = \alpha \sigma_n^{\alpha-1}(\sigma^1) - (\alpha - 1)\sigma_n^{\alpha}(\sigma^1) \text{ for } n = 0, 1, \dots$$

Let $s_n \to s(C_{\alpha-1}C_1)$, then we obtain $\sigma_n^{\alpha-1}(\sigma^1) \to s \ (n \to \infty)$. We have from Theorem 7.8 $\sigma_n^{\alpha}(\sigma^1) \to s \ (n \to \infty)$, and (7.24) implies $\sigma_n^{\alpha}(s) \to \alpha s - (\alpha - 1)s = s \ (n \to \infty)$. Thus we have shown $C_{\alpha-1}C_1 \subset C_{\alpha}$. This completes Part (i) of the proof.

(ii) Next we show $C_{\alpha} \subset C_{\alpha-1}C_1$. Since $s_n^{\alpha} = \sum_{k=0}^n s_k^{\alpha-1}$, we have $s_n^{\alpha-1}(\sigma^1) = s_n^{\alpha}(\sigma^1) - s_{n-1}^{\alpha}(\sigma^1)$. Substituting this in (7.23), we obtain

$$\begin{split} s_n^{\alpha}(s) &= (n+\alpha)(s_n^{\alpha}(\sigma^1) - s_{n-1}^{\alpha}(\sigma^1)) - (\alpha-1)s_n^{\alpha}(\sigma^1) \\ &= (n+1)s_n^{\alpha}(\sigma^1) - (n+\alpha)s_{n-1}^{\alpha}(\sigma^1), \end{split}$$

and so, since $(n + \alpha)/A_n^{\alpha} = n/A_{n-1}^{\alpha}$,

$$\sigma_n^{\alpha}(s) = (n+1)\sigma_n^{\alpha}(\sigma^1) - \frac{n+\alpha}{A_n^{\alpha}}s_{n-1}^{\alpha}(\sigma^1)$$
$$= (n+1)\sigma_n^{\alpha}(\sigma^1) - n\sigma_{n-1}(\sigma^1) \text{ for } n = 0, 1, \dots.$$

This implies

$$\sum_{k=0}^{n} \sigma_{k}^{\alpha}(s) = (n+1)\sigma_{n}^{\alpha}(\sigma^{1}) \text{ for } n = 0, 1, \dots,$$

and consequently

(7.25)
$$\frac{1}{n+1} \sum_{k=0}^{n} \sigma_{k}^{\alpha}(s) = \sigma_{n}^{\alpha}(\sigma^{1}) \text{ for } n = 0, 1, \dots$$

The equations in (7.25) mean

(7.26)
$$C_1 C_\alpha = C_\alpha C_1$$
 as a matrix product.

Let $s_n \to s(C_\alpha)$ $(n \to \infty)$. Then we have $\sigma_n^{\alpha}(\sigma^1) \to s$ $(n \to \infty)$ by (7.26) and the regularity of C_1 . It follows from (7.24) that $\alpha \sigma_n^{\alpha-1}(\sigma^1) = \sigma_n^{\alpha}(s) + (\alpha - 1)\sigma_n^{\alpha}(\sigma^1) \to s + (\alpha - 1)s = \alpha s$, that is $s_n \to s(C_{\alpha-1}C_1)$ $(n \to \infty)$. This shows $C_\alpha \subset C_{\alpha-1}C_1$ and completes Part (ii) of the proof.

Remark 7.15. There is a generalization of Theorem 7.14, namely that

$$C_{\alpha+\beta} \equiv C_{\alpha}C_{\beta}.$$

The next two results are generalizations of Theorem 7.11.

Theorem 7.16. Let $\sum a_n = s(C_\alpha)$ for some $\alpha > -1$ and $a_n = O(1/n)$. Then the series $\sum a_n$ is convergent and summable $C_{\delta-1}$ for all $\delta > 0$.

Theorem 7.17. Let $a_n \in \mathbb{R}$ for $n = 0, 1, ..., \sum a_n = s(C_\alpha)$ for some $\alpha > -1$ and $na_n > -M$ (n = 0, 1, 2, ...) for some constant M. Then the series $\sum a_n$ is convergent.

Remark 7.18. We can simplify the proofs by a few preliminary observations. First, by Theorem 7.8, we may assume that α is an integer. Next we only need to prove that the series converges, since if its is convergent and satisfies $a_n = O(1/n)$, then it is summable $C_{\delta-1}$ by Theorem 7.11. Finally, we may assume $a_n \in \mathbb{R}$, for otherwise we may consider real and imaginary parts separately. Thus to establish Theorems 7.16 and 7.17, it is sufficient to prove Theorem 7.17 for integers α .

For the proof of Theorem 7.17, we need two results which are of some interest in themselves. We consider the series $\sum a_n$ and $\sum b_n$, and write $a = (a_n)$, $b = (na_n)$,

$$s_n^{\alpha}(a) = \sum_{\nu=0}^n A_{n-\nu}^{\alpha} a_{\nu}$$
 and $s_n^{\alpha}(b) = \sum_{\nu=0}^n A_{n-\nu}^{\alpha} b_{\nu}$ for $\alpha > -1$ and $n = 0, 1, \dots$

Theorem 7.19. (a) Let $\sum a_n$ be summable $C_{\beta+1}$ for some $\beta > -1$. Then $\sum a_n$ is summable C_{β} if and only if

$$s_n^\beta(b) = o(n^{\beta+1}).$$

(b) The series $\sum a_n$ is summable $C_{\beta+1}$ for some β where $\beta+1 > -1$ if and only if the series

$$\sum_{\nu=1}^{\infty} \frac{1}{A_{\nu}^{\beta+1}} \cdot \frac{s_{\nu}^{\beta}(b)}{\nu}$$

is convergent.

Proof. (i) First we prove

(7.27)
$$\frac{s_n^{\beta}(a)}{A_n^{\beta}} - \frac{s_n^{\beta+1}(a)}{A_n^{\beta+1}} = \frac{1}{A_n^{\beta+1}} \cdot \frac{s_n^{\beta}(b)}{\beta+1},$$

(7.28)
$$\frac{s_n^{\beta+1}(a)}{A_n^{\beta+1}} - \frac{s_{n-1}^{\beta+1}(a)}{A_{n-1}^{\beta+1}} = \frac{1}{A_n^{\beta+1}} \cdot \frac{s_n^{\beta}(b)}{n}.$$

We have by (7.6) and (7.7) in Lemma 7.2

$$\frac{s_n^{\beta}(a)}{A_n^{\beta}} - \frac{s_n^{\beta+1}(a)}{A_n^{\beta+1}} = \frac{1}{A_n^{\beta+1}} \left(\frac{A_n^{\beta+1}}{A_n^{\beta}} \cdot s_n^{\beta}(a) - s_n^{\beta+1}(a) \right)$$
$$= \frac{1}{A_n^{\beta+1}} \left(\frac{n+\beta+1}{\beta+1} \sum_{\nu=0}^n A_\nu^{\beta} a_{n-\nu} - \sum_{\nu=0}^n A_\nu^{\beta+1} a_{n-\nu} \right)$$
$$= \frac{1}{A_n^{\beta+1}} \sum_{\nu=0}^n \left(\frac{n+\beta+1}{\beta+1} - \frac{A_\nu^{\beta+1}}{A_\nu^{\beta}} \right) A_\nu^{\beta} a_{n-\nu}$$

$$= \frac{1}{A_n^{\beta+1}} \sum_{\nu=0}^n \left(\frac{n}{\beta+1} + 1 - \frac{\nu+\beta+1}{\beta+1} \right) A_\nu^\beta a_{n-\nu}$$

$$= \frac{1}{A_n^{\beta+1}} \sum_{\nu=0}^n \left(\frac{n-\nu}{\beta+1} \cdot a_{n-\nu} A_\nu^\beta \right) = \frac{1}{A_n^{\beta+1}} \cdot \frac{1}{\beta+1} \sum_{\nu=0}^n A_{n-\nu}^\beta \nu a_\nu$$

$$= \frac{1}{A_n^{\beta+1}} \cdot \frac{s_n^\beta(b)}{\beta+1},$$

that is, (7.27), and

$$\begin{split} \frac{s_n^{\beta+1}(a)}{A_n^{\beta+1}} &- \frac{s_{n-1}^{\beta+1}(a)}{A_{n-1}^{\beta+1}} = \frac{1}{A_n^{\beta+1}} \left(\sum_{\nu=0}^n A_\nu^{\beta+1} a_{n-\nu} - \frac{A_n^{\beta+1}}{A_{n-1}^{\beta+1}} \sum_{\nu=0}^{n-1} A_\nu^{\beta+1} a_{n-1-\nu} \right) \\ &= \frac{1}{A_n^{\beta+1}} \left(\sum_{\nu=0}^n A_\nu^{\beta} \frac{A_\nu^{\beta+1}}{A_\nu^{\beta}} a_{n-\nu} - \frac{n+\beta+1}{n} \sum_{\nu=1}^n A_{\nu-1}^{\beta+1} a_{n-\nu} \right) \\ &= \frac{1}{A_n^{\beta+1}} \left(\sum_{\nu=0}^n \frac{\nu+\beta+1}{\beta+1} \cdot A_\nu^{\beta} a_{n-\nu} - \frac{-\frac{n+\beta+1}{n} \sum_{\nu=1}^n \frac{A_\nu^{\beta+1}}{A_\nu^{\beta}} A_\nu^{\beta} a_{n-\nu} \right) \\ &= \frac{1}{A_n^{\beta+1}} \left(\sum_{\nu=0}^n \frac{\nu+\beta+1}{\beta+1} \cdot A_\nu^{\beta} a_{n-\nu} - \frac{-\frac{n+\beta+1}{n} \sum_{\nu=1}^n \frac{\nu}{\beta+1} A_\nu^{\beta} a_{n-\nu} \right) \\ &= \frac{1}{A_n^{\beta+1}} \left(\sum_{\nu=0}^n \frac{\nu+\beta+1}{\beta+1} \cdot A_\nu^{\beta} a_{n-\nu} - \frac{-\frac{n+\beta+1}{n} \sum_{\nu=1}^n \frac{\nu}{\beta+1} A_\nu^{\beta} a_{n-\nu} \right) \\ &= \frac{1}{A_n^{\beta+1}} \cdot \frac{1}{n(\beta+1)} \sum_{\nu=0}^n A_\nu^{\beta} a_{n-\nu} \times \\ &\times (n\nu+n\beta+n-n\nu-\beta\nu-\nu) \\ &= \frac{1}{A_n^{\beta+1}} \cdot \frac{1}{n(\beta+1)} \sum_{\nu=0}^n A_\nu^{\beta} a_{n-\nu} (\beta+1)(n-\nu) \\ &= \frac{1}{nA_n^{\beta+1}} \sum_{\nu=0}^n A_\nu^{\beta} (n-\nu) a_{n-\nu} = \frac{1}{nA_n^{\beta+1}} \sum_{\nu=0}^n A_{\nu-\nu}^{\beta} (\nu) a_\nu \\ &= \frac{1}{A_n^{\beta+1}} \cdot \frac{s_n^{\beta+1}(b)}{n}, \end{split}$$

that is, (7.27). This completes Part (i) of the proof.

(a) Part (a) of the theorem is an immediate consequence of (7.27).

(b) Summation of (7.28) yields

$$\sum_{\nu=1}^{N} \frac{s_{\nu}^{\beta}(b)}{\nu A_{\nu}^{\beta+1}} = \sum_{\nu=1}^{N} \left(\frac{s_{\nu}^{\beta+1}(a)}{A_{\nu}^{\beta+1}} - \frac{s_{\nu-1}^{\beta+1}(a)}{A_{\nu-1}^{\beta+1}} \right) = \frac{s_{N}^{\beta+1}(a)}{A_{N}^{\beta+1}} - s_{0}^{\beta+1}(a).$$

By definition, $s_0^{\beta+1}(a) = a_0$, hence we have

$$\frac{s_N^{\beta+1}(a)}{A_N^{\beta+1}} = a_0 + \sum_{\nu=1}^N \frac{s_\nu^\beta(b)}{\nu A_\nu^{\beta+1}},$$

and the conclusion is an immediate consequence.

Now we can prove Theorem 7.17.

Proof of Theorem 7.17. We may assume that $\alpha = \beta + 1$ is an integer and M = 1. If $s_n^{\beta} \neq o(n^{\beta+1})$, then there is a positive constant C such that one or the other of the inequalities

$$(7.29) s_n^\beta(b) > C n^{\beta+1}$$

or

$$(7.30) s_n^\beta(b) < -Cn^{\beta+1}$$

hold for infinitely many n. We assume that (7.29) holds for infinitely many values N of n.

If $\zeta > 1$ and $N \leq n \leq \zeta N$, then we have, since $b_0 = 0$,

(7.31)
$$s_{n}^{\beta}(b) - s_{N}^{\beta}(b) = \sum_{\nu=0}^{n} A_{n-\nu}^{\beta} b_{\nu} - \sum_{\nu=0}^{N} A_{N-\nu}^{\beta} b_{\nu}$$
$$= \sum_{\nu=1}^{N} \left(A_{n-\nu}^{\beta} - A_{N-\nu}^{\beta} \right) b_{\nu} + \sum_{\nu=N+1}^{n} A_{n-\nu}^{\beta} b_{\nu}$$

Since the coefficients of b_{ν} in both sums are positive and $b_{\nu} > -1$ for all ν by assumption, we have

$$\begin{split} s_n^{\beta}(b) - s_N^{\beta}(b) &> -\sum_{\nu=1}^N \left(A_{n-\nu}^{\beta} - A_{N-\nu}^{\beta} \right) - \sum_{\nu=N+1}^n A_{n-\nu}^{\beta} \\ &= -\sum_{\nu=1}^n A_{n-\nu}^{\beta} + \sum_{\nu=1}^N A_{N-\nu}^{\beta} = -\sum_{\nu=0}^{n-1} A_{n-1-\nu}^{\beta} + \sum_{\nu=0}^{N-1} A_{N-1-\nu}^{\beta} \\ &= -A_{n-1}^{\beta+1} + A_{N-1}^{\beta+1}. \end{split}$$

Now by (7.10) in Lemma 7.2, there are absolute constants K_1 and K_2 such that $A_{n-1}^{\beta+1} \leq K_1 n^{\beta+1}$ and $A_{N-1}^{\beta+1} \geq K_2 N^{\beta+1}$, and so, since $n \leq \zeta N$,

$$s_n^{\beta}(b) - s_N^{\beta}(b) > -K_1 n^{\beta+1} + K_2 N^{\beta+1} \ge - (K_1 \zeta^{\beta+1} - K_2) N^{\beta+1}$$
for $\zeta > 1$ and $N \le n \le \zeta N$.

Now we choose $\zeta > 1$ such that

$$s_n^{\beta}(b) - s_N^{\beta}(b) > -\frac{1}{2} \cdot CN^{\beta+1},$$

and obtain by (7.29)

$$s_n^{\beta}(b) = s_n^{\beta}(b) - s_N^{\beta}(b) + s_N^{\beta}(b) > -\frac{1}{2} \cdot CN^{\beta+1} + CN^{\beta+1}$$
$$= \frac{1}{2} \cdot CN^{\beta+1} \text{ for } N \le n \le \zeta N.$$

Since $A_n^{\beta+1} < K n^{\beta+1}$ for some absolute constant, putting C' = C/K, we obtain, since $\beta + 2 > 0$,

$$\sum_{n=N}^{\zeta N} \frac{s_n^{\beta}(b)}{nA_n^{\beta+1}} > \frac{1}{2} \cdot C' N^{\beta+1} \sum_{n=N}^{\zeta N} \frac{1}{n^{\beta+1}} \ge \frac{1}{2} \cdot C' N^{\beta+1} \cdot \frac{(\zeta-1)N}{(\zeta N)^{\beta+2}} = \frac{C'}{2} \cdot \frac{\zeta-1}{\zeta^{\beta+2}}.$$

But if this is true for infinitely many N, then the series

$$\sum_{\nu=1}^{\infty} \frac{s_{\nu}^{\beta}(b)}{\nu A_{\nu}^{\beta+1}} \text{ is divergent},$$

and $\sum a_n$ is not summable $C_{\beta+1}$ by Part (b) of Theorem 7.19. It follows that (7.29) cannot be true for infinitely many n.

A similar argument shows that (7.30) cannot be true for infinitely many n. Here we would use the range $\eta N \leq n \leq N$, where $\eta < 1$. Thus we must have $s_n^{\beta}(b) = o(n^{\beta+1})$. Then $\sum a_n$ is summable C_{β} by Part (a) of Theorem 7.19.

Repeating the argument $\beta + 1$ times, we see that the series $\sum a_n$ is convergent.

Remark 7.20. If we write s_n for the n^{th} partial sums of a series $\sum a_n$ and choose $\alpha = 1$ in Theorem 7.16, then we obtain that if the sequence (s_n) is summable C_1 and $s_n - s_{n+1} = O(1/n)$, then the sequence (s_n) is convergent, that is, Hardy's Big O Tauberian theorem, Theorem 3.1, is a special case of Theorem 7.16.

8 The Hölder methods for positive integers

In this section, we introduce the Hölder methods of order $n = 1, 2, \ldots$ This definition will be extended to real numbers $\alpha > 0$ at a later stage.

Definition 8.1. Let $H^1 = H = C_1$. Then the *Hölder matrix* H^n of order n for n = 1, 2, ... is defined as the n^{th} power of H.

Remark 8.2. Since $H^{n+1}(H^n)^{-1} = H$ and $H = C_1$ is conservative, it follows from Theorem 6.8 that $H^m \supset H^n$ if m > n. The inclusion is strict, since H sums the divergent sequence $((-1)^k)_{k=0}^{\infty}$ by Part (a) of Example 1.1.

The following Tauberian theorem holds.

Theorem 8.3. All Hölder matrices H^n (n = 1, 2, ...) are equivalent on ℓ_{∞} .

Proof. This means that if x is bounded and summable H^m then it is summable H^n to the same value.

First we assume that $x \in \ell_{\infty}$ is summable H^2 . We put y = H(x). Then it follows that $(n+1)(y_n - y_{n-1}) = x_n - y_{n-1}$ for $n = 0, 1, \ldots$. Since $x \in \ell_{\infty}$ implies $y = Hx \in \ell_{\infty}$, it follows that $((n+1)(y_n - y_{n-1}))_{n=0}^{\infty} \in \ell_{\infty}$. This and $Hy = H^2x \in c$ imply $y \in c$ by Hardy's Big O Tauberian theorem, Theorem 3.1. Therefore $\ell_{\infty} \cap c_{H^2} \subset c_H$. If $x \in \ell_{\infty}$ is summable H^{k+1} for some k > 1, we put $y = H^{k-1}x$. Then y is summable H^2 , hence summable H as just proved and so x is summable H^k . The limits are equal, as just mentioned.

9 The Euler methods of positive order

In this section, we study the Euler methods E_q or order q for positive real numbers q.

Definition 9.1. Let q > 0. The Euler method E_q of order q is defined by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ with

$$a_{nk} = \begin{cases} \frac{1}{(q+1)^n} \binom{n}{k} q^{n-k} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$
 $(n = 0, 1, \dots).$

The $n^{th} E_q$ mean s_n^q of a sequence $s = (s_k)_{k=0}^{\infty}$ is defined by

$$s_n^q = \frac{1}{(q+1)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} s_k$$
 for all $n = 0, 1, \dots$

It turns out that the Euler methods are regular.

Theorem 9.2. The Euler methods E_q are regular for all q > 0.

Proof. Since $a_{nk} \ge 0$ (n, k = 0, 1, ...) for q > 0, it follows that

$$\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} a_{nk} = \frac{1}{(q+1)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} = \frac{(q+1)^n}{(q+1)^n} = 1 \text{ for all } n,$$

and so conditions (i') and (iii') of Part (c) in Theorem 4.3 are satisfied.

Now we fix k. Since 0 < q/(q+1) < 1 there is a real $\rho > 0$ such that $q/(q+1) = 1/(1+\rho)$, and so

$$0 \le a_{nk} = \frac{1}{q^k} \binom{n}{k} \frac{q^n}{(q+1)^n} = \frac{1}{q^k} \binom{n}{k} \frac{1}{(1+\rho)^n} \le \frac{1}{q^k} \binom{n}{k} \cdot \frac{1}{\binom{n}{k+1}\rho^{k+1}}$$

Thus condition in (ii') of Part (c) in Theorem 4.3 is also satisfied and the statement follows from Theorem 6.8. $\hfill\square$

The next result gives a formula for the product of two Euler matrices.

Theorem 9.3. We have $E_pE_q = E_{(p+1)(q+1)-1}$ for all positive p and q. Proof. It is easy to see that

(9.1)
$$\binom{n}{\nu}\binom{n-\nu}{k-\nu} = \binom{n}{k}\binom{k}{\nu} \text{ for } 0 \le \nu \le k \le n; n = 0, 1, \dots$$

Applying (9.1), we obtain

$$\begin{split} s_n^p((s_k^q)) &= \frac{1}{(p+1)^n} \sum_{k=0}^n \binom{n}{k} p^{n-k} s_k^q \\ &= \frac{1}{(p+1)^n} \sum_{k=0}^n \binom{n}{k} \frac{p^{n-k}}{(q+1)^k} \sum_{\nu=0}^k \binom{k}{\nu} q^{k-\nu} s_\nu \\ &= \frac{1}{(p+1)^n} \sum_{\nu=0}^n s_\nu \sum_{k=\nu}^n \binom{n}{k} \binom{k}{\nu} q^{k-\nu} \frac{p^{n-k}}{(q+1)^k} \\ &= \frac{1}{(p+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} s_\nu \sum_{k=\nu}^n \binom{n-\nu}{k-\nu} p^{n-k} \frac{q^{k-\nu}}{(q+1)^k} \\ &= \frac{1}{(p+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} s_\nu \sum_{k=0}^{n-\nu} \binom{n-\nu}{k} p^{n-k-\nu} \frac{q^k}{(q+1)^{k+\nu}} \\ &= \frac{1}{(p+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} s_\nu \frac{1}{(q+1)^\nu} \sum_{k=0}^{n-\nu} \binom{n-\nu}{k} p^{n-\nu-k} \left(\frac{q}{q+1}\right)^k \\ &= \frac{1}{(p+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} s_\nu \frac{1}{(q+1)^\nu} \left(p + \frac{q}{q+1}\right)^{n-\nu} \\ &= \frac{1}{((p+1)(q+1))^n} \sum_{\nu=0}^n \binom{n}{\nu} s_\nu (p(q+1)+q)^{n-\nu} \\ &= \frac{1}{((p+1)(q+1))^n} \sum_{\nu=0}^n \binom{n}{\nu} (((p+1)(q+1)-1)^{n-\nu} s_\nu \\ &= s_n^{(p+1)(q+1)-1} ((s_k)) \text{ for all } n = 0, 1, \dots. \end{split}$$

Now we apply Theorem 9.3 to establish an *inverse formula* for the Euler means by expressing the sequence $s = (s_k)_{k=0}^{\infty}$ in terms of (s_n^q) .

Theorem 9.4 (Inverse formula for the Euler means). Let q > 0. Then we have

$$s_n = q^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (1+1/q)^k s_k^q \text{ for all } n = 0, 1, \dots$$

Proof. Since the computations in the proof of Theorem 9.3 are valid for all p > -1, we may put p = -q/(q+1) in Theorem 9.3. Then we have (p+1)(q+1) - 1 = 1 - 1 = 0 and

$$s_n = \frac{1}{\left(1 - \frac{q}{q+1}\right)^n} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \left(\frac{q}{q+1}\right)^{n-k} s_k^q$$
$$= q^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \left(1 + \frac{1}{q}\right)^k s_k^q \text{ for } n = 0, 1, \dots$$

The strength of Euler methods increases with q.

Theorem 9.5. Let q > q' > 0. Then we have $E_{q'} \subset E_q$.

Proof. Let 0 < q' < q. Then there is $\delta > 0$ such that $q = q' + \delta$. We put $\alpha = \delta/(q'+1) > 0$. Then it follows that $E_{\alpha}E_{q'} = E_{(\alpha+1)(q'+1)-1} = E_{\delta+q'} = E_q$ by Theorem 9.3. Since E_{α} is regular by Theorem 9.2, $s_n \to s(E_{q'})$ implies $s_n \to s(E_{\alpha}E_{q'})$, that is $s_n \to s(E_q)$.

The following result can be used in formal computations involving E_q means.

Remark 9.6. We define

$$x = \frac{z}{1+q-qz}.$$

Then the formal identity

$$\sum_{n=0}^{\infty} s_n^q z^{n+1} = (q+1) \sum_{n=0}^{\infty} s_n x^{n+1}$$

holds.

Proof. Since z = (1+q)x/(1+qx), we obtain for sufficiently small |x|

$$\begin{split} \sum_{n=0}^{\infty} s_n^q z^{n+1} &= \sum_{n=0}^{\infty} s_n^q \frac{x^{n+1}}{(1+qx)^{n+1}} (q+1)^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(1+qx)^{n+1}} (q+1) \sum_{\nu=0}^n \binom{n}{\nu} q^{n-\nu} s_{\nu} \\ &= (q+1) \sum_{\nu=0}^{\infty} s_{\nu} \sum_{n=\nu}^{\infty} \binom{n}{\nu} q^{n-\nu} \frac{x^{n+1}}{(1+qx)^{n+1}} \\ &= (q+1) \sum_{\nu=0}^{\infty} s_{\nu} \sum_{n=0}^{\infty} \binom{n+\nu}{\nu} q^n \frac{x^{n+1+\nu}}{(1+qx)^{n+1+\nu}} \\ &= (q+1) \sum_{\nu=0}^{\infty} s_{\nu} \frac{x^{\nu+1}}{(1+qx)^{\nu+1}} \sum_{n=0}^{\infty} A_n^{\nu} \left(\frac{qx}{1+qx}\right)^n \end{split}$$

$$= (q+1)\sum_{\nu=0}^{\infty} s_{\nu} \frac{x^{\nu+1}}{(1+qx)^{\nu+1}} \frac{1}{\left(1-\frac{qx}{1+qx}\right)^{\nu+1}}$$
$$= (q+1)\sum_{\nu=0}^{\infty} s_{\nu} \frac{x^{\nu+1}}{(1+qx)^{\nu+1}} (1+qx)^{\nu+1}$$
$$= (q+1)\sum_{\nu=0}^{\infty} s_{\nu} x^{\nu+1}.$$

Now we show that the Euler and Cesàro methods are incomparable.

Theorem 9.7. (a) Let q > 0 be given. Then we have $c_{E_q} \not\subset c_{C_\alpha}$ for all $\alpha > 0$. (b) Let $\alpha > 0$ be given. Then we have $c_{C_\alpha} \not\subset c_{E_q}$ for all q > 0.

Proof. (a) Let $\alpha, q > 0$. We have by Theorem 9.4

$$s_n = q^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \left(1 + \frac{1}{q}\right)^k s_k^q \text{ for all } n = 0, 1, \dots,$$

hence

$$\begin{aligned} \sigma_n^{\alpha} &= \frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_{n-k}^{\alpha-1} s_k \\ &= \frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_{n-k}^{\alpha-1} q^k \sum_{\nu=0}^k \binom{k}{\nu} (-1)^{k-\nu} \left(1 + \frac{1}{q}\right)^{\nu} s_{\nu}^q \\ &= \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^n \left((q+1)^{\nu} \sum_{k=\nu}^n \binom{k}{\nu} A_{n-k}^{\alpha-1} (-1)^{k-\nu} q^{k-\nu} \right) s_{\nu}^q \end{aligned}$$

for $n = 0, 1, \ldots$ We define the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ by

$$a_{nk} = \begin{cases} \frac{(q+1)^k}{A_n^{\alpha}} \sum_{\nu=k}^n \binom{\nu}{k} A_{n-\nu}^{\alpha-1} (-1)^{\nu-k} q^{\nu-k} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

for $n = 0, 1, \ldots$ Then we have

$$\sum_{k=0}^{\infty} |a_{nk}| \ge |a_{nn}| = \frac{(q+1)^n}{A_n^{\alpha}} \binom{n}{n} A_0^{\alpha-1} q^0$$
$$= \frac{(q+1)^n}{A_n^{\alpha}} \ge K \frac{(q+1)^n}{(n+1)^{\alpha}} \to \infty \quad (n \to \infty),$$

since q > 0. Thus condition (i) in Part (a) of Theorem 4.3 is not satisfied, and consequently the method A is not conservative by Theorem 6.8. This shows $c_{E_q} \not\subset c_{C_{\alpha}}$, and completes the proof of Part (a).

(b) Let $\alpha, q > 0$. Since $c_{C_{\beta}} \subset c_{C_{\alpha}}$ for $\beta \leq \alpha$ it suffices to show $c_{C_{\alpha}} \not\subset C_{E_q}$ for $0 < \alpha < 1$. We have by Example 7.7

$$s_n = \sum_{k=0}^n A_{n-k}^{-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha}$$
 for $n = 0, 1, ...,$

hence

$$s_{n}^{q} = \frac{1}{(q+1)^{n}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_{k}$$

= $\frac{1}{(q+1)^{n}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{\nu=0}^{k} A_{k-\nu}^{-\alpha-1} A_{\nu}^{\alpha} \sigma_{\nu}^{\alpha}$
= $\frac{1}{(q+1)^{n}} \sum_{\nu=0}^{n} A_{\nu}^{\alpha} \left(\sum_{k=\nu}^{n} \binom{n}{k} q^{n-k} A_{k-\nu}^{-\alpha-1} \right) \sigma_{\nu}^{\alpha}$

for all $n = 0, 1, \ldots$ We define the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ by

$$a_{nk} = \begin{cases} \frac{A_k^{\alpha}}{(q+1)^n} \sum_{\nu=k}^n \binom{n}{\nu} q^{n-\nu} A_{\nu-k}^{-\alpha-1} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

for $n = 0, 1, \ldots$ Since $0 < \alpha < 1$, we have $A_0^{-\alpha - 1} = 1$ and $A_{\nu}^{-\alpha - 1} < 0$ for $\nu \ge 1$, hence

$$\left|\sum_{\nu=0}^{n-k} \binom{n}{\nu+k} q^{n-\nu-k} A_{\nu}^{-\alpha-1}\right| = -\sum_{\nu=0}^{n-k} \binom{n}{\nu+k} q^{n-\nu-k} A_{\nu}^{-\alpha-1} + 2\binom{n}{k} q^{n-k}$$

for $0 \le k \le n$ and $n = 0, 1, \ldots$, so that

$$\sum_{k=0}^{\infty} |a_{nk}| = -\frac{1}{(q+1)^n} \sum_{k=0}^n A_k^{\alpha} \sum_{\nu=0}^{n-k} \binom{n}{\nu+k} q^{n-\nu-k} A_{\nu}^{-\alpha-1} + \frac{2}{(q+1)^n} \sum_{k=0}^n A_k^{\alpha} \binom{n}{k} q^{n-k}$$

$$= \Sigma_n^1 + \Sigma_n^2 \text{ for } n = 0, 1, \dots$$

We obtain

$$\Sigma_n^1 = -\frac{1}{(q+1)^n} \sum_{k=0}^n A_k^{\alpha} \sum_{\nu=k}^n \binom{n}{\nu} q^{n-\nu} A_{\nu-k}^{-\alpha-1}$$
$$= -\frac{1}{(q+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} q^{n-k} \sum_{k=0}^\nu A^{\alpha} A_{\nu-k}^{-\alpha-1}$$

$$= -\frac{1}{(q+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} \binom{n}{k} q^{n-k} A^0_{\nu} = -\frac{1}{(q+1)^n} \sum_{\nu=0}^n \binom{n}{\nu} q^{n-k}$$
$$= -\frac{(q+1)^n}{(q+1)^n} = -1 \text{ for } n = 0, 1, \dots.$$

Furthermore, there is a constant K > 0 depending on α only such that

$$\begin{split} \Sigma_n^2 &\geq K \frac{q^n}{(q+1)^n} \sum_{k=0}^n (k+1)^{\alpha} \binom{n}{k} q^{-k} \\ &= K \left(\frac{q}{q+1}\right)^n \sum_{k=0}^n (k+1)^{\alpha-1} (k+1) \binom{n}{k} q^{-k} \\ &\geq K (n+1)^{\alpha-1} \left(\frac{q}{q+1}\right)^n \sum_{k=0}^n (k+1) \binom{n}{k} q^{-k} \end{split}$$

for all $n = 0, 1, \ldots$ If we put

$$s_n(z) = \sum_{k=0}^n (k+1) \binom{n}{k} z^k$$
 for $z \in \mathbb{C}$ and $n = 0, 1, ...$

then it follows that

$$s_n(z) = \frac{d}{dz} \left(\sum_{k=0}^n \binom{n}{k} z^{k+1} \right) = \frac{d}{dz} \left(z(1+z)^n \right)$$
$$= (1+z)^n + nz(1+z)^{n-1}$$

and so

$$\sum_{k=0}^{n} (k+1) \binom{n}{k} q^{-k} = s_n (1/q) = \frac{(q+1)^n}{q^n} + n \frac{(q+1)^{n-1}}{q^n}$$

for all $n = 0, 1, \ldots$ Thus we obtain, since q > 0,

$$\begin{split} \Sigma_n^2 &\geq K(n+1)^{\alpha-1} \frac{q^n}{(q+1)^n} \left(\frac{(q+1)^n}{q^n} + n \frac{(q+1)^{n-1}}{q^n} \right) \\ &= K(n+1)^{\alpha-1} \left(1 + \frac{n}{1+q} \right) = K(n+1)^{\alpha-1} \frac{1+q+n}{1+q} \\ &\geq \frac{K}{q+1} (n+1)^{\alpha} \to \infty \quad (n \to \infty). \end{split}$$

Therefore we have

$$\sum_{k=0}^{\infty} |a_{nk}| \to \infty \ (n \to \infty),$$

that is, the condition (i) in Part (a) of Theorem 4.3 is not satisfied and so A is not conservative by Theorem 6.8. Thus we have shown $c_{C_{\alpha}} \not\subset c_{E_q}$. This completes the proof of Part (b).

We close this subsection with a Tauberian theorem for the E_1 method.

Theorem 9.8 (Tauberian theorem for E_1). If a sequence $(s_n)_{n=0}^{\infty}$ is summable E_1 to s and $a_n = s_n - s_{n-1} = o(1/\sqrt{n})$ then the sequence $(s_n)_{n=0}^{\infty}$ converges to s.

Proof. First we show

(9.2)
$$\frac{1}{2^n} \sum_{\nu=0}^n \binom{n}{\nu} (n-2\nu)^2 = n \text{ for all } n=0,1,\dots$$

To prove (9.2), we consider the sums

$$s_n^{(0)}(x) = \sum_{\nu=0}^n \binom{n}{\nu} x^{\nu} = (1+x)^n,$$

$$s_n^{(1)}(x) = \sum_{\nu=0}^n \binom{n}{\nu} (\nu+1) x^{\nu},$$

$$s_n^{(2)}(x) = \sum_{\nu=0}^n \binom{n}{\nu} (\nu+1) (\nu+2) x^{\nu}.$$

Then we obviously have $s_n^{(0)}(1) = 2^n$, $s_n^{(1)}(1) = 2^n + n2^{n-1}$ as in the proof of Theorem 9.7, and

$$s_n^{(2)}(x) = \frac{d^2}{dx^2} \left(x^2 \sum_{\nu=0}^n \binom{n}{\nu} x^\nu \right) = \frac{d^2}{dx^2} \left(x^2 (1+x)^n \right)$$
$$= 2(1+x)^n + 4nx(1+x)^{n-1} + n(n-1)x^2(1+x)^{n-2},$$

that is,

$$s_n^{(2)}(1) = 2^{n+1} + n2^{n+1} + n(n-1)2^{n-2}.$$

We observe that

$$(n-2\nu)^2 = n^2 - 4n\nu + 4\nu^2$$

= $n^2 - 4n(\nu+1) + 4n + 4(\nu+1)(\nu+2) - 4 \cdot 3\nu - 4 \cdot 2$
= $n^2 + 4n + 4 - 4(n+3)(\nu+1) + 4(\nu+1)(\nu+2),$

and so

$$\sum_{\nu=0}^{n} \binom{n}{\nu} (n-2\nu)^2 = (n^2+4n+4)s_n^{(0)}(1) - 4(n+3)s_n^{(1)}(1) + 4s_n^{(2)}(1)$$
$$= (n^2+4n+4)2^n - 4(n+3)(2^n+n2^{n-1}) + 4(2^{n+1}+n2^{n+1}+n(n-1)2^{n-2})$$
$$= n^2(2^n-2\cdot 2^n+2^n) + n(4\cdot 2^n-4\cdot 2^n-3\cdot 2^{n+1})$$
$$+ 4\cdot 2^{n+1} - 4\cdot 2^{n-2}) + 4\cdot 2^n - 12\cdot 2^{n+1} + 4\cdot 2^{n+1} = n2^n.$$

This shows (9.2).

Next we prove

(9.3)
$$\frac{1}{2^n} \sum_{\nu=0}^n \left| \binom{n}{\nu} - \binom{n}{\nu+1} \right| \sqrt{\nu+1} = \frac{1}{2^n} \sum_{\nu=0}^n \frac{|n-2\nu-1|}{\sqrt{\nu+1}}.$$

We write

$$b_{\nu} = \left| \binom{n}{\nu} - \binom{n}{\nu+1} \right| \sqrt{\nu+1} \text{ and } c_{\nu} = \binom{n}{\nu} \frac{|n-2\nu-1|}{\sqrt{\nu+1}}$$

for $\nu = 0, 1, ..., n$. Obviously we have $b_0 = c_0$ and $b_n = c_n$. If $1 \le \nu \le n - 1$ then we obtain

$$b_{\nu} = \binom{n}{\nu} \left| 1 - \frac{\binom{n}{\nu+1}}{\binom{n}{\nu}} \right| \sqrt{\nu+1}$$
$$= \binom{n}{\nu} \left| 1 - \frac{n \cdots (n-\nu)v!}{(\nu+1)!n \cdots (n-\nu+1)} \right| \sqrt{\nu+1}$$
$$= \binom{n}{\nu} \left| 1 - \frac{n-\nu}{\nu+1} \right| = \binom{n}{\nu} \left| \frac{2\nu+1-n}{\nu+1} \right| \sqrt{\nu+1}$$
$$= \binom{n}{\nu} \frac{|n-2\nu-1|}{\sqrt{\nu+1}} = c_{\nu}.$$

Now we show

(9.4)
$$\sum_{\nu=0}^{n} \binom{n}{\nu} \frac{|n-2\nu|}{\sqrt{\nu+1}} = O(2^{n}).$$

We write

$$s^{(1)}(n) = \sum_{|n-2\nu| \le \sqrt{n}} \binom{n}{\nu} \frac{|n-2\nu|}{\sqrt{\nu+1}}.$$

If $n \ge 2$ then $n - \sqrt{n} \ge n/2$, and so $|n - 2\nu| \le \sqrt{n}$ implies $2\nu \ge n - \sqrt{n} \ge n/2$ and $\nu + 1 \ge (n + 1)/4$, hence $1/\sqrt{\nu + 1} \le 2/\sqrt{n + 1}$. Therefore we have

(9.5)
$$s^{(1)}(n) \le \sum_{|n-2\nu| \le \sqrt{n}} \binom{n}{\nu} \frac{\sqrt{n}}{\sqrt{n+1}} \le 2\sum_{\nu=0}^{n} \binom{n}{\nu} = 2 \cdot 2^n \text{ for } n \ge 2.$$

We put

$$s^{(2)}(n) = \sum_{\sqrt{n} < |n-2\nu| \le \frac{3n}{n}} \binom{n}{\nu} \frac{|n-2\nu|}{\sqrt{\nu+1}}.$$

Now $|n-2\nu| \leq 3n/4$ implies $n-2\nu \leq 3n/4$, hence $\nu+1 \geq n/8$ and so $1/\sqrt{\nu+1} \leq \sqrt{8}/\sqrt{n}$. Since $|n-2\nu| \geq \sqrt{n}$, we also have $|n-2\nu|/\sqrt{n} \geq 1$ and so

$$\frac{|n-2\nu|}{\sqrt{\nu+1}} \leq \frac{\sqrt{8}|n-2\nu|}{\sqrt{n}} \leq \sqrt{8}\frac{|n-2\nu|^2}{n}.$$

Using (9.2), we obtain (9.6)

$$s^{(2)}(n) \le \sqrt{8} \sum_{\sqrt{n} < |n-2\nu| \le \frac{3n}{4}} \binom{n}{\nu} \frac{(n-2\nu)^2}{n} \le \sqrt{8} \sum_{\nu=0}^n \binom{n}{\nu} \frac{(n-2\nu)^2}{n} = \sqrt{8} \cdot 2^n.$$

We put

$$s^{(3)}(n) = \sum_{\frac{3n}{4} < |n-2\nu| \le n} \binom{n}{\nu} \frac{|n-2\nu|}{\sqrt{\nu+1}}.$$

If $\nu = 0$ then $|n - 2\nu|/\sqrt{\nu + 1} = n = |n - 2\nu|^2/n$. If $\nu \ge 1$ then $\nu + 1 \ge 16/9$ and so $1/\sqrt{\nu + 1} \le 3/4$. Now $3n/4 \le |n - 2\nu|$ implies

$$\frac{|n-2\nu|}{\sqrt{\nu+1}} \le \frac{3}{4}|n-2\nu| \le \frac{|n-2\nu|^2}{n},$$

and so again by (9.2)

(9.7)
$$s^{(3)}(n) \le \sum_{\frac{3n}{4} < |n-2\nu| \le n} {n \choose \nu} \frac{(n-2\nu)^2}{n} \le \sum_{\nu=0}^n {n \choose \nu} \frac{(n-2\nu)^2}{n} = 2^n.$$

Finally (9.4) follows from (9.5), (9.6) and (9.7).

Now we show that, given $\varepsilon > 0$, there is $N_{\varepsilon} \in \mathbb{N}_0$ such that $\sum_{k=0}^n |a_k| \le \varepsilon \sqrt{n}$ for all $n \ge N_{\varepsilon}$. Let $\varepsilon > 0$ be given. Since $a_k = o(1\sqrt{k})$, there is $k_0 \in \mathbb{N}_0$ such that $|a_k|\sqrt{k} \le \varepsilon$ for all $k \ge k_0$. Now we choose $N_{\varepsilon} \in \mathbb{N}_0$ so large that $1/\sqrt{n} \sum_{k=0}^{k_0} |a_k| < \varepsilon$ for all $n \ge N_{\varepsilon}$. Let $n \ge N_{\varepsilon}$. Then we have

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n}|a_{k}| \leq \frac{1}{\sqrt{n}}\left(\sum_{k=0}^{k_{0}}|a_{k}| + \sum_{k=k_{0}+1}^{n}|a_{k}|\right) < \varepsilon + \varepsilon \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$
$$\leq \varepsilon(1+C) \text{ for some absolute constant } C.$$

Now we show

(9.8)
$$|s_n - s_{\nu}| \frac{\sqrt{n}}{|n - \nu|} \to 0 \ (n \to \infty)$$
 uniformly in ν .

Let $\varepsilon > 0$ be given. Since $a_k \sqrt{k} \to \infty$ $(k \to \infty)$, there is $k_0 \in \mathbb{N}_0$ such that $|a_k|\sqrt{k} < \varepsilon$ for all $k \ge k_0$. We put $n_0 = 2k_0$. Then we have for all $n \ge n_0$ and for all $\nu \ge n/2$, if $\nu < n$

$$|s_n - s_\nu| \le \sum_{k=\nu+1}^n |a_k| < \sum_{k=\nu+1}^n \frac{\varepsilon}{\sqrt{k}} \le \sum_{k=\nu+1}^n \frac{\varepsilon}{\sqrt{\nu}} \le \frac{\sqrt{2}\varepsilon(n-\nu)}{\sqrt{n}}$$

and similarly, if $\nu \geq n$

$$|s_n - s_\nu| \le \frac{\varepsilon(\nu - n)}{\sqrt{n}},$$

 $_{\mathrm{thus}}$

$$|s_n - s_{\nu}| \le \frac{\sqrt{2}\varepsilon(n-\nu)}{\sqrt{n}}$$
 for all $n \ge n_0$ and all $\nu \ge \frac{n}{2}$.

Now we choose $N_{\varepsilon} \in \mathbb{N}_0$ with $N_{\varepsilon} > n_0$ such that $\sum_{k=0}^n |a_k| < \varepsilon \sqrt{n}$ for all $n \ge N_{\varepsilon}$. Let $n \ge 2N_{\varepsilon}$. Then we have for all $\nu \le n/2$

$$|s_n - s_{\nu}| \le \sum_{k=0}^n |a_k| + \sum_{k=0}^n |a_k| \le \varepsilon(\sqrt{n} + \sqrt{n/2})$$
$$\le \frac{2\varepsilon n}{\sqrt{n}} = \frac{4\varepsilon(n - n/2)}{\sqrt{n}} \le 4\varepsilon \frac{n - \nu}{\sqrt{n}}.$$

Thus we have shown, given $\varepsilon > 0$, there is $N_{\varepsilon} \in \mathbb{N}_0$ such that

$$|s_n - s_{\nu}| < 4\varepsilon \frac{|n - \nu|}{\sqrt{n}}$$
 for all $n \ge N_{\varepsilon}$ and all ν ,

that is (9.8) holds.

Finally we show $s_n \to s$ $(n \to \infty)$. Let $\varepsilon > 0$ be given. Since $s_n \to s(E_1)$ there is $n_0 \in \mathbb{N}_0$ such that $|\sigma_{2n} - s| < \varepsilon$ for all $n \ge n_0$ where $\sigma_{2n} = s_{2n}^1$. By (9.8), we there is $n_1 \in \mathbb{N}_0$ such that $|s_n - s_\nu| < \varepsilon |n - \nu| / \sqrt{n}$ for all $n \ge n_1$ and for all ν . Applying (9.4) with n replaced by 2n, we obtain that there is an absolute constant C such that $\sum_{\nu}^{2n} {2n \choose \nu} |2n - 2\nu| / \sqrt{\nu + 1} \le K \cdot 2^{2n}$ for all $n \in \mathbb{N}_0$. We put $N_{\varepsilon} = \max\{n_0, n_1\}$. Then we have for all $n \ge N_{\varepsilon}$

$$\begin{aligned} |s_n - s| &\leq |\sigma_{2n} - s_n| + |s - \sigma_{2n}| < \left| \frac{1}{2^{2n}} \sum_{\nu=0}^{2n} \binom{2n}{\nu} s_{\nu} - s_n \right| + \varepsilon \\ &\leq \frac{1}{2^{2n}} \sum_{\nu=0}^{2n} \binom{2n}{\nu} |s_{nu} - s_n| + \varepsilon < \frac{\varepsilon}{2^{2n}} \sum_{\nu=0}^{2n} \binom{2n}{\nu} \frac{|n - \nu|}{\sqrt{n}} + \varepsilon \\ &\leq \frac{\varepsilon}{2^{2n+1}} \sum_{\nu=0}^{2n} \binom{2n}{\nu} \frac{|2n - 2\nu|}{\sqrt{\nu + 1}} + \varepsilon < \varepsilon (K + 1). \end{aligned}$$

Therefore we have proved $s_n \to s \ (n \to \infty)$.

10 The Hausdorff methods

In this section we deal with the *Hausdorff methods*. They contain the Cesàro, Hölder and Euler methods as special cases.

Definition 10.1. Let $\mu = (\mu_n)_{n=0}^{\infty}$ be a complex sequence, $M = (m_{nk})_{n,k=0}^{\infty}$ be the diagonal matrix with $m_{nn} = \mu_n$ (n = 0, 1, ...), and D be the matrix with $d_{nk} = (-1)^k \binom{n}{k}$ for all n and k. The matrix $H(\mu) = DMD$ is called the Hausdorff matrix associated with the sequence μ , and $H(\mu)$ defines the Hausdorff method $H(\mu)$. When the sequence μ is the same throughout some discussion, we write $H = H(\mu)$, for short.

There is an explicit formula for the entries of Hausdorff matrices $H(\mu)$ which are triangles.

Remark 10.2. Since $\binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k}$ for $0 \le j \le k \le n$ and $(-1)^{j+2k} = (-1)^j$, we have, by definition, for $0 \le k \le n$

$$h_{nk} = h_{nk}(\mu) = \sum_{j=0}^{\infty} d_{nj} m_{jj} d_{jk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j} \binom{j}{k} \mu_{j}$$
$$= \binom{n}{k} \sum_{j=k}^{n} (-1)^{j+k} \binom{n-k}{j-k} \mu_{j} = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \mu_{j+k}$$

If we put $\Delta x_k = x_k - x_{k+1}$ (k = 0, 1, ...) for every sequence $x = (x_k)_{k=0}^{\infty}$ and $\Delta^m x_k = \Delta(\Delta^{m-1}x_k)$ for all integers $m \ge 2$, then it is easy to see that

$$\Delta^m x_k = \sum_{j=0}^m (-1)^j \binom{m}{j} x_{k+j} \quad (k = 0, 1, \dots),$$

and we obtain

(10.1)
$$h_{nk} = \begin{cases} 0 & (k > n) \\ \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \mu_{k+j} & (0 \le k \le n) \\ = \binom{n}{k} \Delta^{n-k} \mu_{k} & (n = 0, 1, \dots) \end{cases}$$

It is clear from (10.1) that every Hausdorff matrix H is triangular. Putting k = n in (10.1), we see that $h_{nn} = \mu_n$ for all n, hence H is a triangle if and only if $\mu_n \neq 0$ for all n.

Example 10.3. Let $\mu = e$. Then it follows that $\Delta^r \mu = 0$ for all $r \ge 1$, hence $h_{nn} = 1$ and $h_{nk} = 0$ for $k \ne n$ by (10.1). Therefore we have H(e) = I and

$$D^2 = DID = H(e) = I.$$

There is a simple formula for the product of two Hausdorff matrices.

Theorem 10.4. We have $H(\mu)H(\nu) = H(\mu\nu)$.

Proof. Since all matrices are row finite, multiplication is associative by Corollary 6.5 and it follows from Example 10.3 that

$$(DMD)(DND) = DM(DD)(ND) = DMIND = DMND = D(MN)D.$$

We obtain as an immediate consequence of Theorem 10.4.

Corollary 10.5. We have $(H(\mu))^r = H(\mu^r)$ for $r \in \mathbb{N}$. If $\mu_n \neq 0$ for all *n* then $(H(\mu))^{-1} = H(1/\mu)$ where $1/\mu = (1/\mu_n)_{n=0}^{\infty}$.

The next results concerns the consistency of Hausdorff matrices.

Theorem 10.6. All regular Hausdorff matrices are consistent.

Proof. Any two Hausdorff matrices commute by Theorem 10.4. Since Hausdorff matrices are row finite by Remark 10.2, the result for regular Hausdorff matrices follows from Theorem 6.10.

Example 10.7. We fix $t \in \mathbb{R}$ and put $\mu_n = t^n$ for $n = 0, 1, \ldots$. Then it follows that $\Delta \mu_n = t^n - t^{n+1} = (1-t)t^n$ for all n, hence $\Delta \mu = (1-t)\mu$, and consequently $\Delta^r \mu = (1-t)^r \mu$ for all $r = 1, 2, \ldots$. Therefore we have

$$h_{nk}(t) = h_{nk}(\mu) = \binom{n}{k} (1-t)^{n-k} t^k \quad (0 \le k \le n; n = 0, 1, \dots)$$

by (10.1). If $\mu_n = 1/(n+1)$ for all n then we get $\mu_n = \int_0^1 t^n dt$, and so

$$h_{nk}(\mu) = \int_{0}^{1} h_{nk}(t) dt = {n \choose k} \int_{0}^{1} (1-t)^{n-k} t^{k} dt$$
$$= \frac{1}{n+1} \quad (0 \le k \le n; n = 0, 1, \dots).$$

Thus we obtain $H(\mu) = C_1$ and the Cesàro matrix of order 1 is a Hausdorff matrix.

Example 10.8. The Hölder matrices are all Hausdorff matrices; indeed we have $H^k = H(\mu)$ with $\mu_n = (n+1)^{-k}$ for n = 0, 1, ... by Theorem 10.4 and Example 10.7.

There is a simple way to find out if a row finite matrix is a Hausdorff matrix.

Theorem 10.9. Let μ be a sequence with $\mu_m \neq \mu_n$ for $m \neq n$ and A be a row finite matrix. Then A is a Hausdorff matrix if and only if it commutes with $H(\mu)$.

Proof. If A is a Hausdorff matrix then we obtain $AH(\mu) = H(\mu)A$ by Theorem 10.4.

If $AH(\mu) = H(\mu)A$ then it follows that DAHD = DHAD, where $H = H(\mu)$. Substituting H = DMD and using $D^2 = I$, we obtain

$$DAHD = DADMDD = DADM = DHAD$$
$$= DDMDAD = MDAD,$$

that is, DADM = MDAD. Only a diagonal matrix can commute with M, for if MB = BM for some matrix B then

$$0 = \sum_{j=0}^{\infty} m_{nj} b_{jk} - \sum_{j=0}^{\infty} b_{nj} m_{jk} = \mu_n b_{nk} - b_{nk} \mu_k$$

= $b_{nk} (\mu_n - \mu_k)$ for all *n* and *k*,

and $\mu_n \neq \mu_k$ for $n \neq k$ implies $b_{nk} = 0$ for $n \neq k$, that is, B is a diagonal matrix. So DAD is a diagonal matrix, N = DAD say. Then we have

$$DND = D(DAD)D = D^2AD^2 = A,$$

and A is a Hausdorff matrix.

As an immediate consequence of Theorem 10.9 and Example 10.7, we obtain

Corollary 10.10. A row finite matrix A is a Hausdorff matrix if and only if it commutes with C_1 .

Since $C_1 C_{\alpha} = C_{\alpha} C_1$ by Theorem 7.14, we obtain from Corollary 10.10

Corollary 10.11. The Cesàro methods C_{α} are Hausdorff methods for $\alpha > 0$.

11 Conservative Hausdorff methods

Now we establish necessary and sufficient conditions for a Hausdorff method to be conservative. In view of the fact that Hausdorff matrices are given by a fixed matrix D and a sequence $\mu = (\mu)_{k=0}^{\infty}$, it is to be expected that conservative Hausdorff methods can be characterized by certain properties of the sequences μ .

First we need some important identities for the entries of Hausdorff matrices.

Lemma 11.1. Let $H = H(\mu)$. Then we have for all n = 0, 1, ...

m

$$(11.1) h_{nn} = \mu_n,$$

(11.2)
$$\sum_{k=0}^{n} h_{nk} = \mu_0,$$

(11.3)
$$h_{n0} = D_n \mu = \sum_{k=0}^n d_{nk} \mu_k$$

(11.4)
$$\sum_{k=0}^{m} h_{nk} - \sum_{k=0}^{m} h_{n+1,k} = \frac{m+1}{n+1} h_{n+1,m+1} \quad (m=0,1,\dots).$$

Proof. Identity (11.1) was proved in Remark 10.2. Furthermore (11.2) follows from

$$\sum_{k=0}^{\infty} h_{nk} = H_n e \text{ for all } n,$$
$$He = (DMD)e = DM(De) = DMe^{(0)} = D(\mu_0 e^{(0)}) = \mu_0 (De^{(0)}) = \mu_0 e^{(0)}$$

Putting k = 0 in (10.1), we obtain

$$h_{n0} = \binom{n}{0} \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \mu_{j} = \sum_{j=0}^{n} d_{nj} \mu_{j} = D_{n} \mu,$$

that is, (11.3).

It follows from (10.1) that

$$\frac{h_{n+1,k}}{\binom{n+1}{k}} = \Delta^{n+1-k}\mu_k = \Delta\left(\Delta^{n-k}\mu_k\right) = \Delta^{n-k}\mu_k - \Delta^{n-k}\mu_{k+1}$$
$$= \frac{h_{nk}}{\binom{n}{k}} - \Delta^{n+1-(k+1)}\mu_{k+1} = \frac{h_{nk}}{\binom{n}{k}} - \frac{h_{n+1,k+1}}{\binom{n+1}{k+1}}.$$

This implies

$$h_{nk} = \frac{\binom{n}{k}h_{n+1,k}}{\binom{n+1}{k}} + \frac{\binom{n}{k}h_{n+1,k+1}}{\binom{n+1}{k+1}} = \frac{n-k+1}{n+1}h_{n+1,k} + \frac{k+1}{n+1}h_{n+1,k+1},$$

hence

$$h_{nk} - h_{n+1,k} = \frac{1}{n+1} \left((k+1)h_{n+1,k+1} - k \cdot h_{n+1,k} \right),$$

and so

$$\sum_{k=0}^{m} h_{nk} - \sum_{k=0}^{m} h_{n+1,k} = \frac{1}{n+1} \sum_{k=0}^{m} ((k+1)h_{n+1,k+1} - k \cdot h_{n+1,k}) = \frac{m+1}{n+1} h_{n+1,m+1}. \quad \Box$$

It turns out that non-negative real Hausdorff methods are regular.

Theorem 11.2. Every Hausdorff method given by a non-negative real Hausdorff matrix H is conservative; all of its column limits are equal to zero except possibly the first.

Proof. It follows from (11.2) that the conditions in (i) and (iii) of Part (a) in Theorem 4.3 hold. Since $h_{n+1,m+1} \ge 0$ by assumption, $\sum_{k=0}^{m} h_{nk}$ is a non-negative decreasing function of n by (11.4). Hence $\lim_{n\to\infty} \sum_{k=0}^{m} h_{nk}$ exists for each fixed m. We put $y(n,m) = h_{n+1,m+1}$ for all n and m. Then we obtain from (11.4)

$$\sum_{n=0}^{r} \frac{y(n,m)}{n+1} = \frac{1}{m+1} \sum_{n=0}^{r} \sum_{k=0}^{m} (h_{nk} - h_{n+1,k}) = \frac{1}{m+1} \sum_{k=0}^{m} \left(\sum_{n=0}^{r} (h_{nk} - h_{n+1,k}) \right)$$
$$= \frac{1}{m+1} \sum_{k=0}^{m} (h_{0k} - h_{r+1,k}).$$

Since the last term in the identity above converges as $r \to \infty$, the series

$$\sum_{n=0}^{\infty} \frac{y(n,m)}{n+1}$$

converges. Furthermore, since $\lim_{n\to\infty} y(n,m)$ exists, it must be equal to zero. Consequently the condition in (ii) of Part (a) in Theorem 4.3 is also satisfied. (We note that $\lim_{n\to 0} y(n,m) = \lim_{n\to\infty} h_{n+1,m+1} = 0$ for all $m \ge 0$ is the second statement.) Example 11.3. Let $\mu = e^{(0)}$. Then the first column of $H(\mu)$ is equal to e and all other columns are zero.

The next notion is substantial for conservative Hausdorff methods.

Definition 11.4. A sequence μ is called *totally decreasing*, if the matrix $H(\mu)$ is non-negative.

Remark 11.5. By (10.1), a sequence μ is totally decreasing if and only if

 $\Delta^n \mu_k \ge 0$ for all n and k.

For n = 0 it says $\mu_k \ge 0$ for all k, for n = 1 it says $\mu_0 \ge \mu_1 \ge \mu_2 \ge \ldots$, and for n = 2 it says $\mu_0 - 2\mu_1 + \mu_2 \ge 0$, $\mu_1 - 2\mu_2 + \mu_3 \ge 0$, ... (a convexity condition). The sequence $(3, 2, 0, 0, \ldots)$ is not totally decreasing, since $3 - 2 \cdot 2 + 0 < 0$. The sequences in Examples 10.7 and 10.8 are totally decreasing.

It will turn out that a real Hausdorff method $H(\mu)$ is conservative if and only if the sequence μ is the difference of two totally decreasing sequences. We need the following Lemma to be able to prove this result.

Lemma 11.6. Let $\mu = (\mu_k)_{k=0}^{\infty}$ be a real sequence such that $H(\mu) \in \Phi$, and $D = (d_{nk})_{n,k=0}^{\infty}$ and A be the matrices with $d_{nk} = \Delta^n \mu_k$ and $a_{nk} = |d_{nk}|$ for all $n, k = 0, 1, \ldots$

(a) We put

$$f(m,n,k) = \sum_{j=0}^{m} {m \choose j} a_{n+m-j,k+j} \text{ for all } n,m,k = 0,1,\dots$$

Then f(m, n, k) is an increasing bounded function of m for all n and k.

(b) We put $g(n,k) = \lim_{m\to\infty} f(m,n,k)$ for all $n,k = 0,1,\ldots$ and define the sequence $\nu = (\nu_k)_{k=0}^{\infty}$ by $\nu_k = g(0,k)$ for $k = 0,1,\ldots$ Then we have

(11.5)
$$g(n,k) = \Delta^n \nu_k \text{ for all } n,k = 0,1,\ldots$$

Proof. (a) It follows from

$$d_{n+1,k} = \Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k) = \Delta^n \mu_k - \Delta^n \mu_{k+1} = d_{nk} - d_{n,k+1}$$

that $d_{nk} = d_{n+1,k} + d_{n,k+1}$. We define the operators L and R for all matrices $B = (b_{nk})_{n,k=0}^{\infty}$ by $(LB)_{nk} = b_{n+1,k}$ and $(RB)_{nk} = b_{n,k+1}$ (n, k = 0, 1, ...). Then we have D = LD + RD = (L+R)D which implies

(11.6)
$$D = (L+R)^m D = \sum_{j=0}^m \binom{m}{j} L^{m-j} R^j D \text{ for } m = 0, 1, \dots,$$

since LR = RL. Thus we can write

$$f(m, n, k) = ((L+R)^m A)_{n,k}$$
 for $m, n, k = 0, 1, \dots$

It follows that

$$a_{nk} = |d_{nk}| = |(LD)_{nk} + (RD)_{nk}| \le (L|D|)_{nk} + (R|D|)_{nk}$$

= ((L + R)A)_{nk} for all n, k = 0, 1,

Therefore we obtain

$$f(m, n, k) = ((L+R)^m A)_{nk} \le ((L+R)(L+R)^m A)_{nk}$$

= $((L+R)^{m+1} A)_{nk}$
= $f(m+1, n, k)$ for all $m, n, k = 0, 1, ...,$

that is, f(m, n, k) is increasing in m for all n and k. Furthermore

$$((L+R)^{n+k}A)_{00} = \sum_{j=0}^{n+k} \binom{n+k}{j} a_{n+k-j,j} \ge \binom{n+k}{k} a_{nk} \ge a_{nk} = (L^n R^k A)_{00}$$

implies

$$f(m, n, k) = ((L+R)^m A)_{nk} \le ((L+R)^m L^n R^k A)_{00}$$

$$\le ((L+R)^m (L+R)^{n+k} A)_{00} = ((L+R)^{m+n+k} A)_{00}$$

$$= f(m+n+k, 0, 0) \text{ for all } m, n, k = 0, 1, \dots$$

Finally, we obtain for all m

$$f(m,0,0) = \sum_{j=0}^{m} {m \choose j} a_{m-j,j} = \sum_{j=0}^{m} {m \choose j} |d_{m-j,j}|$$
$$= \sum_{j=0}^{m} {m \choose j} |\Delta^{m-j}\mu_j| = \sum_{j=0}^{m} |h_{mj}| \le ||H|| < \infty.$$

This completes the proof of Part (a).

(b) By Part (a), f(m, n, k) is an increasing bounded function in m for all n and k, and so $g(n, k) = \lim_{m \to \infty} f(m, n, k)$ exists for all n and k.

We fix k and prove identity (11.5) by induction with respect to $n \in \mathbb{N}_0$. First let n = 0. Then we have $g(0, k) = \nu_k = \Delta^0 \nu_k$ by the definition of the sequence ν . Now we assume that $g(n, k) = \Delta^n \nu_k$ holds for some $n \ge 0$ and each fixed k. We have

$$f(m+1, n, k) = ((L+R)^m (L+R)A)_{nk}$$

= $((L+R)^m A)_{n+1,k} + ((L+R)^m A)_{n,k+1}$
= $f(m, n+1, k) + f(m, n, k+1),$

hence g(n,k) = g(n+1,k) + g(n,k+1), and consequently by hypothesis

$$g(n+1,k) = g(n,k) - g(n,k+1) = \Delta^n \nu_k - \Delta^n \nu_{k+1} = \Delta^{n+1} \nu_k.$$

Theorem 11.7. Equivalent conditions for a real Hausdorff matrix $H = H(\mu)$ are

- (i) *H* is conservative
- (ii) $H \in \Phi$
- (iii) *H* is the difference of two non-negative Hausdorff matrices
- (iv) μ is the difference of two totally decreasing sequences.

Proof. Trivially, (i) implies (ii), (iii) and (iv) are equivalent by Remark 11.5, and (iii) implies (i) by Theorem 11.2. Therefore it is sufficient to show that (ii) implies (iv).

We assume that (ii) holds, that is, $H = H(\mu) \in \Phi$. Now (iv) will follow from the existence of a sequence ν such that $\Delta^n \nu \ge |\Delta^n \mu|$, since with $\alpha = 1/2(\nu + \mu)$ and $\beta = 1/2(\nu - \mu)$, we have $\mu = \alpha - \beta$.

Using the notations of Lemma 11.6, we define the sequence ν by

$$\nu_k = g(0,k)$$
 for $k = 0, 1, \dots$

Then we have by Lemma 11.6

$$\begin{aligned} |\Delta^{n}\mu_{k}| &= |d_{nk}| = \left| \sum_{r=0}^{m} \binom{m}{r} d_{n+m-r,k+r} \right| \leq \sum_{r=0}^{m} \binom{m}{r} |d_{n+m-r,k+r}| \\ &= \sum_{r=0}^{m} \binom{m}{r} a_{n+m-r,k+r} = f(m,n,k) \leq g(n,k) \\ &= \Delta^{n}\nu_{k} \text{ for all } n, k = 0, 1, \dots. \end{aligned}$$

12 The moment problem

Hausdorff matrices are closely related to the so-called *moment problem* in analysis which we are going to solve in this part.

Definition 12.1. A sequence μ is called *conservative* if it is the difference of two totally decreasing sequences.

Remark 12.2. By Theorem 11.7, a real Hausdorf matrix is conservative if and only if μ is conservative.

Now we turn to the representation of μ_n by integrals for conservative Hausdorff matrices.

Theorem 12.3. A real Hausdorff matrix $H(\mu)$ is conservative if and only if there exists a function $g \in bv[0,1]$, that is, a function of bounded variation on [0,1] (cf. Definition A.1), such that

(12.1)
$$\mu_n = \int_0^1 t^n \, dg(t) \text{ for all } n = 0, 1, \dots$$

The numbers μ_n defined by (12.1) are called moment constants, and the sequence μ is called moment sequence.

Proof. First we show the sufficiency of (12.1). By Theorem A.8, we may assume that the function g is increasing. Then we obtain as in Example 10.7

(12.2)
$$\Delta^n \mu_k = \int_0^1 t^k (1-t)^n \, dg(t) > 0.$$

Now we show the necessity of (12.1). The proof consists of four steps. First we construct a sequence $(g_m)_{m=0}^{\infty}$ of step functions and functions v(m, t, k) $(0 \le k \le m; m = 0, 1, ...)$ on the interval [0, 1] and then we show that $\mu_k = \int_0^1 v(m, t, k) \, dg_m(t)$ for $0 \le k \le m$ and m = 0, 1, ... In the third step we show that, for each fixed k, the functions v(m, t, k) converge uniformly to t^k on the interval [0, 1]. In the fourth step, we apply Helly's theorems (Theorems A.13 and B.7) to choose a subsequence $(g_{m(j)})_{j=0}^{\infty}$ of the sequence $(g_m)_{m=0}^{\infty}$ which converges to a function $g \in \text{bv}[0, 1]$ and such that $u_k = \int_0^1 t^k \, dg(t)$ holds for $k = 0, 1, \ldots$

(i) Construction of the functions v(m,t,k) and the sequence $(g_m)_{m=0}^{\infty}$ of step functions. It follows from (10.1) that

$$\mu_k = \sum_{r=0}^m \binom{m}{r} d_{m-r,k+r} = \sum_{r=0}^m \binom{m}{r} \Delta^{m-r} \mu_{k+r}$$
$$= \sum_{r=0}^m \frac{\binom{m}{r}}{\binom{m+k}{k+r}} h_{m+k,k+r} \text{ for } m = 0, 1, \dots$$

Replacing m by m-k, we have

$$\mu_k = \sum_{r=0}^{m-k} \frac{\binom{m-k}{r}}{\binom{m}{k+r}} h_{m,k+r} = \sum_{j=k}^m \frac{\binom{m-k}{j-k}}{\binom{m}{j}} h_{mj} \text{ for } m \ge k.$$

We put

$$u(m, j, k) = \begin{cases} \frac{\binom{m-k}{j-k}}{\binom{m}{j}} & (k \le j \le m) \\ 0 & (0 \le j < k). \end{cases}$$

Then we obtain

$$u(m, j, k) = \frac{(m-k)(m-k-1)\cdots(m-j+1)j!}{(j-k)!m(m-1)\cdots(m-j+1)}$$
$$= \frac{j(j-1)\cdots(j-(k-1))}{m(m-1)\cdots(m-(k-1))} = \prod_{r=0}^{k-1} \frac{j-r}{m-r},$$

and, since u(m, j, k) = 0 for $0 \le j < k$, we have

$$\mu_k = \sum_{j=0}^m u(m, j, k) h_{mj}$$

We put for $m \in \mathbb{N}_0$

$$v(m,t,k) = \begin{cases} \prod_{r=0}^{k-1} \frac{t-r/m}{1-r/m} & (k>0) \\ 1 & (k=0) \end{cases}$$

and

$$g_m(t) = \begin{cases} 0 & (t=0) \\ \sum_{j \le mt} h_{mj} & (0 < t \le 1). \end{cases}$$

This completes the proof of (i).

(ii) We show

(12.3)
$$\mu_k = \int_0^1 v(m, t, k) \, dg_m(t) \quad (m = 0, 1, \dots; k = 0, 1, \dots, m).$$

For each fixed m, the function g_m is constant in each interval $\left[\frac{i}{m}, \frac{i+1}{m}\right)$ $(i = 0, 1, \ldots, m-1)$ with a jump h_{mi} at $t = \frac{i}{m}$. Since the sequence μ is totally decreasing, it follows that $h_{mi} \ge 0$ by (10.1). Thus g_m is increasing and $g_m \in \text{bv}[0, 1]$ by Example A.2. Furthermore the total variation $\bigvee_0^1 g_m$ of the function g_m on the interval [0, 1] (cf. Definition A.1) is given by

(12.4)
$$\bigvee_{0}^{1} g_{m} = g_{m}(1) - g_{m}(0) = \sum_{j=0}^{m} h_{mj} \le ||H|| < \infty \text{ for each } m.$$

Since v(m, t, k) is a continuous function on the interval [0, 1] for each m and k, the integral

$$\int_{0}^{1} v(m,t,k) \, dg_m(t) = \sum_{i=0}^{m-1} \int_{\frac{i}{m}}^{\frac{i+1}{m}} v(m,t,k) \, dg_m(t)$$

exists. For each i with $0 \le i \le m-1$, let $(P_n^{(i)})$ be a sequence of partitions

$$P_n^{(i)} = \left\{ x_{0,n}^{(i)} = \frac{i}{m} < x_{1,n}^{(i)} < \dots < x_{n,n}^{(i)} = \frac{i+1}{m} \right\}$$

of the interval $[\frac{i}{m}, \frac{i+1}{m}]$, and let $\xi_{l,n}^{(i)} = x_{l+1,n}^{(i)}$ for $0 \le l \le n-1$. Then we obtain for $1 \le i \le m-1$

$$\begin{aligned} \sigma_{P_n^{(i)}}(v(m,\cdot,k),g_m;\xi_n^{(i)}) &= \sum_{l=0}^{n-1} v(m,x_{l+1,n}^{(i)},k)(g_m(x_{l+1,n}^{(i)}) - g_m(x_{l,n}^{(i)})) \\ &= v(m,x_{n,n}^{(i)},k)h_{m,i+1} \\ &= v(m,\frac{i+1}{m},k)h_{m,i+1} \end{aligned}$$

and for i = 0

$$\sigma_{P_n^{(0)}}(v(m,\cdot,k),g_m;\xi_n^{(0)}) = \sum_{l=0}^{n-1} v(m,x_{l+1,n}^{(0)},k)(g_m(x_{l+1,n}^{(0)}) - g_m(x_{l,n}^{(0)}))$$
$$= v(m,x_{1,1}^{(0)},k)h_{m0} + v(m,x_{n,n}^{(0)},k)h_{m1} = v(m,\frac{1}{n},k)h_{m0} + v(m,\frac{1}{m},k)h_{m1}.$$

Letting $\|P_n^{(i)}\| \to 0 \ (n \to \infty)$, we obtain

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} v(m,t,k) \, dg_m(t) = v(m,\frac{i+1}{m},k) h_{m,i+1} \text{ for } 1 \le i \le m-1,$$

and by the continuity of the function v(m, t, k)

$$\int_{0}^{\frac{1}{m}} v(m,t,k) \, dg_m(t) = v(m,0,k)h_{m0} + v(m,\frac{1}{m},k)h_{m1} \text{ for } i = 0.$$

Thus we have

$$\int_{0}^{1} v(m,t,k) \, dg_m(t) = v(m,0,k)h_{m0} + v(m,\frac{1}{m},k)h_{m1} + \sum_{i=2}^{m} v(m,\frac{i}{m},k)h_{mi}$$
$$= \sum_{j=0}^{m} v(m,\frac{j}{m},k)h_{mj},$$

and

$$v(m, \frac{j}{m}, k) = \prod_{r=0}^{k-1} \left(\frac{\frac{j}{m} - \frac{r}{m}}{1 - \frac{r}{m}} \right) = \prod_{r=0}^{k-1} \frac{j-r}{m-r} = u(m, j, k) \text{ for } k > 0$$

and by definition

$$v(m, \frac{j}{m}, 0) = 1 = u(m, j, 0)$$
 for $k = 0$.

Therefore it follows that

$$\int_{0}^{1} v(m,t,k) \, dg_m(t) = \sum_{j=0}^{m} u(m,j,k) h_{mj}$$

= μ_k for $m = 1, 2, \dots$ and $k = 0, 1, \dots, m$.

This proves (12.3) and concludes the proof of (ii).

(iii) We show that, for each fixed k, the functions v(m, t, k) converge uniformly on [0, 1] to t^k . We fix $k \ge 1$ and put

$$a(m,r,t) = \frac{t - \frac{r}{m}}{1 - \frac{r}{m}} \text{ for } 0 \le r < k \le m \text{ and } 0 \le t \le 1.$$

Since r < k and $0 \le t \le 1$, it follows that

$$\begin{aligned} |a(m,r,t) - t| &= \left| \frac{t - \frac{r}{m}}{1 - \frac{r}{m}} - t \right| + \left| \frac{mt - r}{m - r} - t \right| = \left| \frac{mt - r - mt + rt}{m - r} \right| \\ &= |rt - r| \frac{1}{m - r} = r \frac{1 - t}{m - r} \le \frac{k - 1}{m - k + 1}. \end{aligned}$$

Let $\varepsilon > 0$ be given. We choose

$$m_0 = \max\left\{2k - 2, (k - 1)\left(1 + \frac{2^k}{\varepsilon}\right)\right\}.$$

Then it follows that

$$\frac{k-1}{m_0-k+1} \le 1 \text{ and } 2^k \frac{k-1}{m_0-k+1} \le \varepsilon.$$

Let $m > m_0$ and $t \in [0, 1]$ be given. Then we have

$$\begin{aligned} v(m,t,k) &= \prod_{r=0}^{k-1} a(m,r,t) \le \left(t + \frac{k-1}{m-k+1}\right)^k \\ &= t^k + \sum_{j=1}^k \binom{k}{j} \left(\frac{k-1}{m-k+1}\right)^j t^{k-j} \\ &\le t^k + \frac{k-1}{m-k+1} \sum_{j=0}^k \binom{k}{j} < t^k + 2^k \frac{k-1}{m_0-k+1} \le t^k + \varepsilon, \end{aligned}$$

and similarly

$$v(m,t,k) \ge t^k - \sum_{j=1}^k \binom{k}{j} \left(\frac{k-1}{m-k+1}\right)^j t^{k-j} > t^k - \varepsilon.$$

Therefore we have

(12.5)
$$|v(m,t,k) - t^k| < \varepsilon \text{ for all } m > m_0 \text{ and for all } t \in [0,1],$$

that is, $v(m,t,k) \to t^k \ (m \to \infty)$ uniformly in t. This completes the proof of (iii).

(iv) We apply Theorems A.13 and B.7 to prove (12.1). We obtain from (12.5), (12.3) and (12.4)

$$\left| \mu_{k} - \int_{0}^{1} t^{k} dg_{m}(t) \right| = \left| \int_{0}^{1} (v(m, t, k) - t^{k}) dg(t) \right|$$
(12.6)
$$\leq \max_{t \in [0, 1]} |v(m, t, k) - t^{k}| \bigvee_{0}^{1} g_{m} \leq \varepsilon \|H\| \text{ for all } m > m_{0}.$$

By Theorem A.13, there is a subsequence $(g_{m(j)})$ of the sequence (g_m) with

$$g = \lim_{j \to \infty} g_{m(j)} \in \operatorname{bv}[0, 1].$$

If we let $m(j) \to \infty$ in (12.6) and apply Theorem B.7 then

$$\left|\mu_k - \int_0^1 t^k \, dg(t)\right| = \lim_{m(j) \to \infty} \left|\mu_k - \int_0^1 t^k \, dg_{m(j)}(t)\right| \le \varepsilon ||H||.$$

Since $\varepsilon > 0$ was arbitrary, (12.1) must hold.

Theorem 12.4. Let μ be conservative and $\mu_k = \int_0^1 t^k dg(t)$ for $k = 0, 1, \ldots$. Then the first column limit in $H(\mu)$ is g(0+) - g(0). Therefore $H(\mu)$ in m-multiplicative with m = g(1) - g(0) if an only if g is continuous at 0.

Proof. We may assume that g is increasing and that g(0) = 0. It follows from (12.2) that

$$h_{n0} = \int_{0}^{1} (1-t)^n dg(t) \ge \int_{0}^{\varepsilon} (1-t)^n dg(t) \ge (1-\varepsilon)^n g(\varepsilon) \to g(0+) \text{ as } \varepsilon \to 0.$$

Conversely we have

$$h_{n0} = \int_{0}^{\varepsilon} (1-t)^{n} dg(t) + \int_{\varepsilon}^{1} (1-t)^{n} dg(t) \le g(\varepsilon) + (1-\varepsilon)^{n} (g(1) - g(\varepsilon))$$

$$\to g(\varepsilon) \text{ as } n \to \infty.$$

This yields $\lim_{n\to\infty} h_{n0} \leq g(\varepsilon)$ for all $\varepsilon > 0$. The evaluation of *m* follows from (11.2) in Lemma 11.1.

Remark 12.5. Let $g \in bv[0,1]$. Then Theorem 12.4 suggests the following definitions. The sequence μ with $\mu_k = \int_0^1 t^k dg(t)$ (k = 0, 1, ...) is called *moment* sequence with respect to the function g. Without loss of generality, we may assume g(0) = 0. If g(1) = 1 and g(0+) = g(0) = 0 so that g is continuous at 0, then the sequence μ is called a regular moment sequence.

(a) A real Hausdorff method $H(\mu)$ is regular if and only if μ is a regular moment sequence.

(b) A real Hausdorff method $H(\mu)$ is conservative if and only if μ is a moment sequence.

13 The moment sequences for some matrices

Now we show that the Cesàro, Hölder and Euler methods are Hausdorff methods and determine their moment sequences.

We already know the following result (Corollary 10.11).

Theorem 13.1. The C_{α} methods are Hausdorff methods for $\alpha > 0$.

Now we determine the moment sequences for the Cesàro matrices.

Theorem 13.2. If $\alpha > 0$ then the moment sequence μ of the C_{α} matrix is given by

(13.1)
$$\mu_k = \alpha \int_0^1 t^k (1-t)^{\alpha-1} dt \text{ for } k = 0, 1, \dots$$

Proof. We know from Corollary 10.11 that C_{α} is a Hausdorff matrix $H(\mu)$ with

$$h_{mj} = \frac{A_{m-j}^{\alpha-1}}{A_m^{\alpha}} \quad (0 \le j \le m; m = 0, 1, \dots).$$

Although it would be easy to verify (13.1), we apply the constructive method of the proof of Theorem 12.3 to establish (13.1). Let $\alpha > 0$. We define the functions $g_m : [0,1] \to \mathbb{R}$ for $m = 0, 1, \ldots$ by

$$g_m(t) = \sum_{j \le [mt]} h_{mj} = \frac{1}{A_m^{\alpha}} \left(\sum_{j=0}^m A_{m-j}^{\alpha-1} - \sum_{j=[mt]+1}^m A_{m-j}^{\alpha-1} \right)$$
$$= 1 - \frac{A_{m-[mt]-1}^{\alpha}}{A_m^{\alpha}} = 1 - \frac{A_{m-[mt]-1}^{\alpha}}{(m-[mt])^{\alpha}} \frac{(m+1)^{\alpha}}{A_m^{\alpha}} \left(\frac{m-[mt]}{m+1} \right)^{\alpha}.$$

First we observe that by (7.11) in Lemma 7.2

$$\lim_{m \to \infty} \frac{A_{m-[mt]-1}^{\alpha}}{(m-[mt])^{\alpha}} \frac{(m+1)^{\alpha}}{A_m^{\alpha}} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} = 1 \text{ for } \alpha > 0.$$

Let $t \in (0, 1]$ be given. Then we have 0 < mt - 1 for all sufficiently large m, and so $0 < mt - 1 < [mt] \le mt$, hence

$$\begin{aligned} 0 &\leq \frac{m}{m+1}(1-t) = \frac{m-mt}{m+1} \leq \frac{m-[mt]}{m+1} \\ &\leq \frac{m-(mt-1)}{m+1} = 1 - \frac{mt}{m+1}, \end{aligned}$$

that is

$$\lim_{m \to \infty} \frac{m - [mt]}{m+1} = 1 - t.$$

Consequently it follows that

(13.2)
$$g(t) = \lim_{m \to \infty} g_m(t) = 1 - (1 - t)^{\alpha} \ (t \in (0, 1]) \text{ for } \alpha > 0,$$

and $g_m(0) = 0$ for all *m* implies g(0) = 0. Thus (13.2) holds for all $t \in [0, 1]$ and all $\alpha > 0$, and (13.1) follows from Theorem 12.3. Furthermore we have

$$h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dg(t)$$

$$= \binom{n}{k} \alpha \int_0^1 t^k (1-t)^{a+n-k-1} dt$$

$$= \binom{n}{k} \alpha \frac{k!}{(\alpha+n-k)\cdots(\alpha+n)}$$

$$= \frac{n\cdots(n-k+1)\alpha}{(\alpha+n)\cdots(\alpha+n-k)} = \frac{(n-k+\alpha-1)\cdots\alpha n!}{(n-k)!(\alpha+1)\cdots(n+\alpha)}$$

$$= \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots$$

Now we determine the moment sequences for the Hölder matrices.

Theorem 13.3. The Hölder matrices H^m (m = 1, 2...) are Hausdorff matrices $H(\mu)$ with the moment sequences μ given by

(13.3)
$$\mu_k = \frac{1}{(k+1)^m} = \frac{1}{\Gamma(m)} \int_0^1 t^k \, dg(t) \\ = \frac{1}{\Gamma(m)} \int_0^1 t^k \, (\log (1/t))^{m-1} \, dt \text{ for } k = 0, 1, \dots.$$

Proof. We already know from Example 10.8 that the Hölder matrices H^m (m = 0, 1, ...) are Hausdorff matrices $H(\mu)$ with $\mu_k = (k+1)^{-m}$ for k = 0, 1, ... We put

$$I(m) = \frac{1}{\Gamma(m)} \int_{0}^{1} t^{k} (\log (1/t)^{m-1}) dt \text{ for } m = 0, 1, \dots,$$

and substitute $u = \log(1/t)$ to obtain

$$I(m) = \frac{1}{\Gamma(m)} \int_{0}^{1} t^{k+1} (\log(1/t))^{m-1} \left(-\frac{dt}{t}\right) = -\frac{1}{\Gamma(m)} \int_{\infty}^{0} e^{-u(k+1)} u^{m-1} du.$$

Putting s = u(k+1), we conclude

$$I(m) = \frac{1}{\Gamma(m)} \int_{0}^{\infty} e^{-s} s^{m-1} \frac{1}{(k+1)^{m-1}} \frac{1}{k+1} ds$$
$$= \frac{1}{(k+1)^m} \frac{\Gamma(m)}{\Gamma(m)} = \frac{1}{(k+1)^m} \text{ for } m = 1, 2, \dots$$

Remark 13.4. The proof of Theorem 13.3 remains valid for all $\alpha = m > 0$. Therefore the definition of the Hölder methods H^{α} can be extended to all $\alpha > 0$ by (13.3).

Finally we determine the moment sequences for the Euler matrices.

Theorem 13.5. The Euler matrices E_q (q > 0) are Hausdorff matrices $H(\mu)$ with the moment sequences μ given by

(13.4)
$$\mu_k = \frac{1}{(q+1)^k} = \int_0^1 t^k \, dg(t) \text{ for } k = 0, 1, \dots,$$

where

(13.5)
$$g(t) = \begin{cases} 0 & \left(0 \le t < \frac{1}{q+1}\right) \\ 1 & \left(\frac{1}{q+1} \le t \le 1\right). \end{cases}$$

Proof. The integrals $\int_0^1 t^k dg(t)$ (k = 0, 1, ...) exist by Theorem B.3. Let $(P^{(n)})$ be a sequence of partitions $P^{(n)} = \{x_0^{(n)} = 0 < x_1^{(n)} < \cdots < x_{l(n)}^{(n)} \leq \frac{1}{q+1} < x_{l(n)+1}^{(n)} < \cdots < x_n^{(n)} = 1\}$ of the interval [0, 1] and $||P^{(n)}|| \to 0$ $(n \to \infty)$ and let $\xi_l^{(n)} \in [x_l, x_{l+1}]$ $(0 \le l \le n-1)$ with $\xi_{l(n)}^{(n)} = \frac{1}{q+1}$. Then we have for fixed $k \in \mathbb{N}_0$

$$\sigma_{P^{(n)}}(f,g;\xi^{(n)}) = \sum_{l=0}^{n-1} (\xi_l^{(n)})^k \left(g(x_{l+1}^{(n)}) - g(x_l^{(n)}) \right) = \left(\xi_{l(n)}^{(n)} \right)^k = \frac{1}{(q+1)^k}$$

Similarly we obtain

$$\Delta^{n-k}\mu_k = \int_0^1 t^k (1-t)^{n-k} \, dg(t) = \frac{1}{(q+1)^k} \left(1 - \frac{1}{q+1}\right)^{n-k} = \frac{q^{n-k}}{(q+1)^n},$$

hence

$$h_{nk} = \binom{n}{k} \frac{q^{n-k}}{(q+1)^n} \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots \square$$

14 Mercerian matrices

So far we were mainly interested in conservative methods. It is often useful to know when a matrix is equipotent with convergence, in which case the matrix is said to be *Mercerian*.

Definition 14.1. A matrix A is said to be *Mercerian* if $c_A = c$.

Applying Theorem 6.8 with B = I, we obtain

Theorem 14.2. A conservative triangle A is Mercerian if and only if A^{-1} is conservative.

We are going to establish a Mercerian theorem for Hausdorff matrices. Let a > 0 and $g(t) = t^a$ for $0 \le t \le 1$. Then we have

$$\mu_k = \int_0^1 t^k \, dg(t) = a \int_0^1 t^{k+a-1} \, dt = \frac{a}{a+k} \text{ for } k = 0, 1, \dots,$$

and the Hausdorff method $H(\mu)$ associated with this moment sequence is regular by Remark 12.5. If we put

$$\nu = \beta e + (1 - \beta)\mu \text{ for } \beta > 0,$$

then the Hausdorff method $H(\nu)$ is also regular, since

$$H(\nu) = \beta I + (1 - \beta)H(\mu)$$

by Example 10.3. The sequence ν is given by

$$\nu_k = \beta + (1 - \beta) \frac{a}{a+k} = \frac{\beta k + a}{a+k} \text{ for all } k = 0, 1, \dots$$

Theorem 14.3. Let b, c > 0 and the sequence μ be defined by

$$\mu_k = \frac{bk+1}{ck+1} \text{ for all } k = 0, 1, \dots$$

Then the Hausdorff matrix $H(\mu)$ is Mercerian.

Proof. We put $b = \beta/a$ and c = 1/a. Then we have

$$\mu_k = \frac{\frac{\beta}{a}k + 1}{\frac{1}{a}k + 1} = \frac{\beta k + a}{k + a} \text{ for all } k = 0, 1, \dots,$$

and $H(\mu)$ is regular. We put $a' = a/\beta$ and $\beta' = 1/\beta$ and obtain

$$\frac{1}{\mu_k} = \frac{k+a}{\beta k+a} = \frac{\frac{k}{\beta} + \frac{a}{\beta}}{k + \frac{a}{\beta}} = \frac{\beta' k + a'}{k + a'} \text{ for all } k = 0, 1, \dots,$$

and so $H(1/\mu)$ is also regular, but $H(1/\mu) = (H(\mu))^{-1}$ by Corollary 10.5.

We obtain as an immediate consequence of Theorems 14.3 and 6.8:

Corollary 14.4. Let a > 0 and the sequence $\mu(a)$ be defined by $\mu_k = (ak + 1)^{-1}$ for all $k = 0, 1, \ldots$. Then $H(\mu(a))$ and $H(\mu(b))$ are equivalent for all a, b > 0.

Proof. We have by Theorem 6.8 that

$$H(\mu(a)) \supset H(\mu(b))$$
 if and only if $H(\mu(a))H^{-1}(\mu(b)) = H(\mu(a)/\mu(b))$

is regular. Furthermore, since $\mu_k(a)/\mu_k(b) = (bk+1)/(ak+1)$ for all $k = 0, 1, \ldots$, the method $H(\mu(a)/\mu(b))$ is regular by Theorem 14.3.

We close this part with a result concerning the equivalence of Cesàro and Hölder methods.

Theorem 14.5. The Hölder methods H^m and the Cesàro methods C_m are equivalent for m = 1, 2, ...

Proof. Since $H^m = H(\mu)$ and $C_m = H(\nu)$ with

$$\mu_k = \frac{1}{(k+1)^m}$$
 and $\nu_k = \frac{1}{A_k^m}$ for $k = 0, 1, \dots,$

by Theorems 13.2 and 13.3, and we have

$$\frac{\mu_k}{\nu_k} = \frac{(k+m)\cdots(m+1)}{k!(k+1)^m} = \frac{1}{m!}\frac{(m+k)\cdots(k+1)}{(k+1)^m}$$
$$= \frac{1}{m!}\frac{(m+k)\cdots(k+2)}{(k+1)^{m-1}} = \frac{1}{m!}\prod_{r=2}^m\frac{k+r}{k+1} \text{ for all } k = 0, 1, \dots$$

Thus it follows that $H^m C_m^{-1}$ is the product of Hausdorff matrices $H(\lambda^{(r)})$ (r = 2, 3, ..., m), where the sequences $\lambda^{(r)}$ are defined by $\lambda_k^{(r)} = (k/r+1)/(k+1)$ and each matrix $H(\lambda)^{(r)}$ is Mercerian by Theorem 14.3. Thus $H^m \equiv C_m$.

Remark 14.6. The result of Theorem 14.5 holds for all real $\alpha > 0$. A proof can be found in [50, p. 264].

15 Nörlund matrices

In this section, we study *Nörlund matrices* which are generalizations of Cesàro matrices. It will turn out that Cesàro matrices are the only matrices that are both Hausdorff and Nörlund matrices.

Definition 15.1. (a) The convolution a * b of the sequences a and b is defined by

$$(a * b)_n = \sum_{k=0}^n a_k b_{n-k}$$
 for all $n = 0, 1, \dots$
The convolution of sequences obviously is commutative, a * b = b * a. (b) Let $p = (p_k)_{k=0}^{\infty}$ be a complex sequence with $p_0 = 1$ such that the sequence P = p * e satisfies $P_n = (p * e)_n \neq 0$ for all n. Then the Nörlund method (N, p) is defined by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ with

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

Hence we have

$$A_n x = \frac{(p * x)_n}{(p * e)_n} \text{ for arbitrary sequences } x = (x_k)_{k=0}^{\infty} \text{ and}$$
$$\sum_{k=0}^n a_{nk} = A_n e = 1 \text{ for all } n = 0, 1, \dots$$

The nth Nörlund mean t_n^p of a sequence $s = (s_k)_{k=0}^{\infty}$ is defined by

$$t_n^p = \frac{(p*s)_n}{(p*e)_n} = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_n \text{ for } n = 0, 1, \dots$$

Computations with Nörlund matrices are conveniently carried out by the use of formal power series

$$p(z) = \sum_{n=0}^{\infty} p_n z^n = (1-z)P(z)$$
 where $P(z) = \sum_{n=0}^{\infty} P_n z^n$.

For obvious reasons (N, p) is called a *polynomial matrix* if $p_n = 0$ for all sufficiently large n. Given a function p(z), a Nörlund matrix can be defined by the sequence of coefficients of its formal power series expansion.

Example 15.2. (a) Let p(z) = 1 for all z. Then we have $p = e^{(0)}$ and (N, p) = I. (b) Let $\alpha > 0$ and

$$p(z) = (1-z)^{-\alpha} = \sum_{n=0}^{\infty} {\binom{n+\alpha-1}{n}} z^n \text{ for } |z| < 1.$$

Then we have

$$P(z) = \frac{p(z)}{1-z} = \frac{1}{(1-z)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^{\alpha} z^n \text{ for } |z| < 1,$$

and $(N, p) = C_{\alpha}$.

The first result states that the Cesàro matrices are the only ones that are both Hausdorff and Nörlund matrices. **Theorem 15.3.** Let A be a Hausdorff and a Nörlund matrix. Then $A = C_{\alpha}$ for some α .

Proof. Let $A = (N, p) = H(\mu)$. We put $\alpha = p_1$. Then we have by (10.1)

$$\alpha \mu_n = \alpha a_{nn} = \frac{p_1}{P_n} = a_{n,n-1} = \binom{n}{n-1} \Delta^{n-(n-1)} \mu_{n-1} = n(\mu_{n-1} - \mu_n) \text{ for all } n,$$

that is, $\mu_n = n/(n+\alpha)\mu_{n-1}$. It follows by induction from $\mu_0 = a_{00} = 1$ that

$$\mu_n = \frac{1}{\binom{n+\alpha}{n}} = \frac{1}{A_n^{\alpha}} \text{ for all } n = 0, 1, \dots,$$

hence $A = C_{\alpha}$ by Theorem 13.2.

Theorem 15.4. Every polynomial matrix is regular.

Proof. Let A be a polynomial matrix. Then A consists of finitely many diagonals. Each column terminates in zeros; each row adds up to 1, and finally, for all sufficiently large n,

$$\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{m} \left| \frac{p_k}{P_m} \right|$$

where m is the smallest integer such that $p_k = 0$ for all k > m. Consequently we have $||A|| < \infty$. Therefore the conditions in (i'), (ii') and (iii') in Part (c) of Theorem 4.3 are satisfied and the matrix A is regular.

Now we give necessary and sufficient conditions for a Nörlund method to be conservative or regular.

Theorem 15.5. A Nörlund method (N, p) is conservative if and only if

(i)
$$\lim_{n \to \infty} \frac{p_n}{P_n} = \lambda \ exists$$

(ii) there is a constant M such that
$$\sum_{k=0}^{n} |p_k| \le M |P_n|$$
 for all $n = 0, 1, ...;$

it is regular if and only if the conditions in (i) and (ii) hold with $\lambda = 0$.

Proof. Let $A = (a_{nk})_{n,k=0}^{\infty}$ denote the matrix of the Nörlund method (N, p). Then we have

(15.1)
$$a_{n0} = \frac{p_n}{P_n} = 1 - \frac{P_{n-1}}{P_n} \text{ for all } n = 0, 1, \dots$$

1. First we show the necessity of the conditions in (i) and (ii).

If A is conservative, then $\lim_{n\to\infty} a_{n0} = \lambda$ exists, and (15.1) implies (i). The condition in (ii) is $||A|| < \infty$. This completes the proof of Part 1.

2. Now we show the sufficiency of the conditions in (i) and (ii). It remains to be shown that the condition in (i) implies the existence of

$$\alpha_k = \lim_{k \to \infty} a_{nk} \text{ for each } k.$$

It follows by induction and from (15.1) that

$$a_{n,k+1} = \frac{p_{n-k-1}}{P_n} = \frac{p_{n-k-1}}{P_{n-1}} \frac{P_{n-1}}{P_n}$$
$$= a_{n-1,k} \frac{P_{n-1}}{P_n} \to \alpha_k (1-\lambda) \quad (n \to \infty) \text{ for all } k.$$

Therefore the method (N, p) is conservative. We also have $\alpha_{k+1} = \alpha_k(1-\lambda)$, hence $\alpha_k = \alpha_0(1-\lambda)^k = \lambda(1-\lambda)^k$ for all k by (15.1). Hence the method (N, p) is regular if and only if $\lambda = 0$.

Corollary 15.6. Let $p_n \ge 0$ for all n. Then (N, p) is conservative if and only if $\lim_{n\to\infty} p_n/P_n$ exists, and regular if and only if $\lim_{n\to\infty} p_n/P_n = 0$.

Proof. Since $p_n \ge 0$ for all n, the condition in (ii) of Theorem 15.5 becomes redundant.

Corollary 15.7. Each of the following conditions is sufficient for (N, p) to be regular

(i)
$$p \in \ell_{\infty} \text{ and } p_n \ge 0 \text{ for all } n;$$

(ii)
$$p \in c_0 \text{ and } p_n \ge 0 \text{ for all } n;$$

(iii)
$$p \in \ell_1 \text{ and } \sum_{n=0}^{\infty} p_n \neq 0;$$

(iv)
$$p \in \phi$$
.

Proof. (iv) The condition in (iv) is sufficient by Theorem 15.4.

(iii) Now we assume that the conditions in (iii) are satisfied.

First $p \in \ell_1$ implies $p \in c_0$, and $l = \lim_{n \to \infty} P_n = \lim_{n \to \infty} \sum_{k=0}^n p_k \neq 0$, so we obtain

$$\lim_{n \to \infty} \frac{p_n}{P_n} = 0 \text{ and } \sum_{k=0}^{\infty} a_{nk} = 1 \text{ for all } n = 0, 1, \dots$$

Since $|P_n|>0$ for all n and |l|>0 together imply $m=\inf_n|P_n|>0,$ we have $M=(\sum_{k=0}^\infty |p_k|)/m<\infty$ and

$$\sum_{k=0}^{n} |p_k| \le \sum_{k=0}^{\infty} |p_k| \frac{|P_n|}{m} \le M |P_n| \text{ for all } n.$$

Thus the matrix (N, p) is regular by Theorem 15.5.

(ii) Now we observe that the condition in (ii) implies the condition in (i). We assume $p \in \ell_{\infty}$ and $p_n \ge 0$ for all n. Then $\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} a_{nk} = 1$. If $\sum_{k=0}^{\infty} p_k$ converges, then (iii) holds. If $\sum_{k=0}^{\infty} p_k$ does not converge, then $P_n \to \infty$ $(n \to \infty)$, and so $p_n/P_n \to 0$ $(n \to \infty)$, since $p \in \ell_{\infty}$.

In the remainder of this subsection, we assume $p_0 > 0$ and $p_n \ge 0$ for all n. First we prove a consistency theorem.

Theorem 15.8. Any two non-negative regular Nörlund methods (N, p) and (N, q) with $p_0, q_0 > 0$ are consistent.

Proof. We assume $s_n \to t(N,p)$ and $s_n \to t'(N,q)$. Let $r_n = (p * q)_n$ for $n = 0, 1, \ldots$ We show $(N,q) \subset (N,r)$. Writing

$$t_n^r = \frac{1}{R_n} \sum_{k=0}^n r_{n-k} s_k = \frac{(r*s)_n}{(r*e)_n} \text{ for } n = 0, 1, \dots,$$

we obtain, using the associativity of the convolution *,

$$(r*e)_n t_n^r = \frac{(r*e)_n (r*s)_n}{(r*e)_n} = (r*s)_n = ((p*q)*s)_n$$
$$= (p*(q*s))_n = (p*(q*e)t^q)_n = \sum_{\nu=0}^n p_{n-\nu} Q_\nu t_\nu^q,$$
$$(r*e)_n = ((p*q)*e)_n = (p*(q*e))_n = (p*Q)_n,$$

that is,

$$t_n^r = \frac{1}{(p * Q)_n} \sum_{\nu=0}^n p_{n-\nu} Q_\nu t_\nu^q$$
 for all $n = 0, 1, \dots$

We are going to show that the matrix A with

$$a_{nk} = \begin{cases} \frac{p_{n-k}Q_k}{(p*Q)_n} & (0 \le k \le n) \\ 0 & (k>n) \end{cases} \quad (n = 0, 1, \dots)$$

is regular. First we observe $a_{nk} \ge 0$ for all n and k, and so

$$\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{n} \frac{p_{n-k}Q_k}{(p*Q)_n} = \frac{(p*Q)_n}{(p*Q)_n} = 1 \text{ for } n = 0, 1, \dots,$$

hence conditions in (i') and (iii') in Part (c) of Theorem 4.3 are satisfied. Since (N, p) and (N, q) are regular, we conclude from Corollary 15.6

$$0 \le a_{nk} = \frac{p_{n-k}Q_k}{\sum_{j=0}^n p_j Q_{n-j}} \le \frac{p_{n-k}Q_k}{q_0 \sum_{j=0}^{n-k} p_j}$$

$$= \frac{p_{n-k}}{P_{n-k}} \frac{Q_k}{q_0} \to 0 \quad (n \to \infty) \text{ for each fixed } k$$

Therefore A is regular, and $s_n \to t'(N,q)$ implies $s_n \to t'(N,r)$. Similarly, interchanging the roles of the sequences p and q, we can show that $s_n \to t(N,p)$ implies $s_n \to t(N,r)$. Thus we have t = t'.

The next result establishes a relation between Nörlund summable series and power series.

Theorem 15.9. If (N, p) is a regular Nörlund method and the series $\sum a_n$ is summable (N, p) to s then the power series $\sum_{n=0}^{\infty} a_n x^n$ has positive radius of convergence and defines an analytic function a(x) which is regular for $0 \le x < 1$ and satisfies $\lim_{x\to 1^-} a(x) = s$.

Proof. We write

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$
, $P(x) = \sum_{n=0}^{\infty} P_n x^n$ and $T(x) = \sum_{n=0}^{\infty} P_n t_n^p x^n$

where

$$t_n^p = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k$$
 and $s_n = \sum_{k=0}^n a_k$ for $n = 0, 1, \dots$

Since (N, p) is regular, it follows from Corollary 15.6 that $p_n/P_n \to 0 \ (n \to \infty)$, and so

$$\frac{P_{n-1}}{P_n} = \frac{P_n - p_n}{P_n} \to 1 \quad (n \to \infty).$$

Thus the power series expansion of P(x) has radius of convergence 1, and p(x) = (1-x)P(x) for |x| < 1. Furthermore, $\sum a_n = s(N,p)$ implies $(t_n^p)_{n=0}^{\infty} \in \ell_{\infty}$, and so the power series expansion of T(x) also has radius of convergence 1. Since $p_0 > 0$ and $p_n \ge 0$ for all $n \ge 1$, it follows that p(x) > 0 and P(x) > 0 for $0 \le x < 1$. Therefore the function w(x) = T(x)/p(x) is regular at the origin and can be expanded in a power series

$$w(x) = \sum_{n=0}^{\infty} w_n x^n$$
 for $|x|$ small.

Now T(x) = w(x)p(x) implies

$$P_n t_n^p = (p * w)_n \quad (n = 0, 1, \dots),$$

 \mathbf{but}

$$P_n t_n^p = (p * s)_n \quad (n = 0, 1, ...)$$

hence $w_n = s_n$ for all *n*. Therefore $\sum_{n=0}^{\infty} s_n x^n$ and $\sum_{n=0}^{\infty} a_n x^n$ are regular at the origin,

$$a(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n = (1-x) \frac{T(x)}{p(x)} = \frac{T(x)}{P(x)}$$

and T(x) and P(x) are regular for |x| < 1. Hence a(x) is regular for |x| < 1 except for possible poles, none of which is in the interval (0, 1). Finally we have

$$a(x) = \frac{T(x)}{P(x)} = \frac{\sum_{n=0}^{\infty} P_n t_n^p x^n}{P(x)} = \sum_{n=0}^{\infty} c_n(x) t_n^p, \text{ where}$$
$$c_n(x) = \frac{P_n x^n}{P(x)} \text{ for all } n = 0, 1, \dots$$

We consider an arbitrary sequence $(x_n)_{n=0}^{\infty}$ with $0 < x_n \to 1-$ as n tends to ∞ . Then we have

$$a(x_n) = \sum_{k=0}^{\infty} c_{nk} t_k^p \text{ where } c_{nk} = \frac{P_k x_n^k}{P(x_n)} \text{ for } n, k = 0, 1, \dots$$

We are going to show that the matrix $C = (c_{nk})_{n,k=0}^{\infty}$ is regular. First it follows from $p_k \ge 0$ for all k and $x_n > 0$ for all n that $c_{nk} \ge 0$ for all n and k, hence

$$\sum_{k=0}^{\infty} |c_{nk}| = \sum_{k=0}^{\infty} c_{nk} = \frac{P(x_n)}{P(x_n)} = 1 \text{ for all } n = 0, 1, \dots$$

Now we fix k. Then it follows that

$$0 \le c_{nk} = \frac{P_k x_n^k}{p(x_n)} (1 - x_n) \le \frac{P_k x_n^k}{p_0} (1 - x_n) \to 0 \quad (n \to \infty).$$

Thus C is regular, and $t_n^p \to s \ (n \to \infty)$ implies $a(x_n) \to s \ (n \to \infty)$. Since the sequence $(x_n)_{n=0}^{\infty}$ was arbitrary, it follows that

$$\lim_{x \to 1-} a(x) = s.$$

Now we study the inclusion and equivalence of Nörlund methods. If (N, p) and (N, q) are regular Nörlund methods, then $p_n/P_n \to 0$, $q_n/Q_n \to 0$ as $n \to \infty$, and the power series

(15.2)
$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \ P(x) = \sum_{n=0}^{\infty} P_n x^n, q(x) = \sum_{n=0}^{\infty} q_n x^n \text{ and } Q_x = \sum_{n=0}^{\infty} Q_n x^n$$

are convergent for |x| < 1. Since $p_0, q_0 > 0$ and $p_n, q_n \ge 0$ for all n, the power series

(15.3)
$$k(x) = \sum_{n=0}^{\infty} k_n x^n = \frac{q(x)}{p(x)} = \frac{Q(x)}{P(x)}$$

(15.4)
$$l(x) = \sum_{n=0}^{\infty} l_n x^n = \frac{1}{k(x)}$$

are convergent for |x| small, and

$$(15.5) q = k * p, Q = k * P,$$

(15.6)
$$p = l * q \quad \text{and} \ P = l * Q.$$

The next result is an inclusion theorem for regular Nörlund methods.

Theorem 15.10. If (N, p) and (N, q) are regular Nörlund methods then $(N, p) \subset (N, q)$ if and only if there is a constant M independent of n such that

(15.7)
$$(|k|*P)_n \leq M \cdot Q_n \text{ for all } n = 0, 1, \dots \text{ where } |k| = (|k_n|)_{n=0}^{\infty}$$

(15.8)
$$\lim_{n \to \infty} \frac{k_n}{Q_n} = 0.$$

Proof. If $s(x) = \sum_{n=0}^{\infty} s_n x^n$ then we have

$$\sum_{n=0}^{\infty} Q_n t_n^q x^n = \sum_{n=0}^{\infty} (q * s)_n x^n = q(x)s(x),$$
$$\sum_{n=0}^{\infty} P_n t_n^p x^n = p(x)s(x) \text{ for } |x| \text{ small.}$$

It follows from (15.3) that

$$\sum_{n=0}^{\infty} Q_n t_n^q x^n = q(x)s(x) = k(x)p(x)s(x)$$
$$= \left(\sum_{n=0}^{\infty} k_n x^n\right) \left(\sum_{n=0}^{\infty} P_n t_n^p x^n\right) = \sum_{n=0}^{\infty} (k*(Pt^p))_n x^n,$$

and so

$$t_n^q = \sum_{\nu=0}^{\infty} a_{n\nu} t_{\nu}^p$$
 for $n = 0, 1, \dots$

where

$$a_{n\nu} = \begin{cases} \frac{k_{n-\nu}P_{\nu}}{Q_n} & (0 \le \nu \le n) \\ 0 & (\nu > n) \end{cases} \quad (n = 0, 1, \dots).$$

Now (15.7) is the condition in (i'), in Part (c) of Theorem 4.3. Furthermore, (15.5) implies

$$\sum_{\nu=0}^{\infty} a_{n\nu} = \frac{1}{Q_n} (k * P)_n = \frac{Q_n}{Q_n} = 1 \text{ for } n = 0, 1, \dots,$$

and the condition in (iii') in Part (c) of Theorem 4.3 is also satisfied. Since (N,q) is regular, it follows that for each fixed ν

$$\frac{Q_{n-\nu}}{Q_n} = \frac{Q_n - (q_n + \dots + q_{n-\nu+1})}{Q_n}$$

$$=1-\left(\frac{q_n}{Q_n}+\frac{q_{n-1}}{Q_n}+\cdots+\frac{q_{n-\nu+1}}{Q_n}\right)\to 1 \quad (n\to\infty),$$

and so

$$a_{n\nu} = \frac{k_{n-\nu}}{Q_n} P_{\nu} = \frac{k_{n-\nu}}{Q_{n-\nu}} \frac{Q_{n-\nu}}{Q_n} P_{\nu} \to 0 \quad (n \to \infty) \text{ for each fixed } \nu$$

if and only of (15.8) holds. Therefore the condition in (ii'), in Part (c) of Theorem 4.3 is also satisfied. $\hfill\square$

Remark 15.11. If $P_n \to \infty$ $(n \to \infty)$ then (15.8) is redundant in Theorem 15.10.

Proof. If $P_n \to \infty$ $(n \to \infty)$, then, given N > 0, we can choose r such that $P_r > N$. It follows from (15.7) that $N|k_{n-r}| \leq MQ_n$ and

$$0 \le \overline{\lim_{n \to \infty} \frac{|k_{n-r}|}{Q_{n-r}}} \le \frac{M}{N} \lim_{n \to \infty} \frac{Q_n}{Q_{n-r}} = \frac{M}{N},$$

and (15.8) follows from (15.7).

Theorem 15.12. If (N,q) is a regular Nörlund method with q_n increasing then $C_{\alpha} \subset (N,q)$ for $0 \leq \alpha \leq 1$.

Proof. By Theorem 7.8, it suffices to show $C_1 \subset (N,q)$. We have $(N,p) = C_1$ where p = e and $P_n = n + 1 \to \infty$ $(n \to \infty)$. Thus $p(x) = (1 - x)^{-1}$, k(x) = (1 - x)q(x), $k_0 = q_0$, $k_n = q_n - q_{n-1}$ for n > 0 and the condition in (15.7) becomes

$$\sum_{\nu=1}^{n} |q_{\nu} - q_{\nu-1}| (n - \nu + 1) + (n + 1)q_0 \le MQ_n.$$

Since the sequence $(q_n)_{n=0}^{\infty}$ is increasing, we have $q_{\nu} - q_{\nu-1} \ge 0$ ($\nu > 0$), hence

$$\sum_{\nu=1}^{n} (q_{\nu} - q_{\nu-1})(n - \nu + 1) + (n + 1)q_0 + \sum_{\nu=1}^{n} (q_{\nu} - q_{\nu-1}) \sum_{\mu=\nu}^{n} 1 + (n + 1)q_0$$
$$= \sum_{\mu=1}^{n} \sum_{\nu=1}^{\mu} (q_{\nu} - q_{\nu-1}) + (n + 1)q_0 = \sum_{\mu=1}^{n} (q_{\mu} - q_0) + (n + 1)q_0$$
$$= \sum_{\mu=1}^{n} q_{\mu} + q_0 = Q_n \text{ for } n = 0, 1, \dots$$

We close this section with an equivalence theorem for regular Nörlund methods.

Theorem 15.13. Let (N, p) and (N, q) be regular Nörlund methods. Then (N, p) and (N, q) are equivalent if and only if $k \in \ell_1$ and $l \in \ell_1$.

Proof. First we assume $(N,p) \equiv (N,q)$. It follows from $p_0, q_0 > 0$ that $k_0, l_0 > 0$, and $(N,p) \subset (N,q)$ implies $k_0P_n \leq MQ_n$ (n = 0, 1, ...) for some absolute constant M by (15.7). Thus P_n/Q_n is bounded. Similarly, $(N,q) \subset (N,p)$ implies that Q_n/P_n is bounded. It follows from (15.7) that

$$\frac{1}{P_n}(|k|*P)_r \leq \frac{MQ_n}{P_n} \text{ for } r < n.$$

We fix r and let $n \to \infty$, then

$$\sum_{\nu=0}^{r} |k_{\nu}| \le M \overline{\lim_{n \to \infty}} \frac{Q_n}{P_n},$$

and so $\sum_{n=0}^{\infty} |k_n| < \infty$. Similarly it can be shown that $\sum_{n=0}^{\infty} |l_n| < \infty$. Conversely we assume $k \in \ell_1$ and $l \in \ell_1$. Since $k \in \ell_1$ implies $k \in c_0$, we have $k_n/Q_n \to 0 \ (n \to \infty)$ and (15.8) holds. Furthermore

$$P_n = (Q * l)_n \le Q_n \sum_{n=0}^{\infty} |l_n|$$

implies

$$(P * |k|)_n \le Q_n \sum_{n=0}^{\infty} |l_n| \sum_{n=0}^{\infty} |k_n| \le MQ_n$$
 for all n ,

that is, (15.7) is satisfied. Therefore it follows by Theorem 15.10 that $(N, p) \subset (N, q)$. Similarly it can be shown that $(N, q) \subset (N, p)$.

16 The Abel method

Now we study the *Abel method*. This method is not defined by a matrix.

Definition 16.1. Let $(a_n)_{n=0}^{\infty}$ be a real sequence, s_n be the partial sums of the sequence $(a_n)_{n=0}^{\infty}$ and $\sum_{n=0}^{\infty} a_n x^n$ be convergent for |x| < 1. If

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n \to s \quad (x \to 1-),$$

then the series Σa_n and the sequence $(s_n)_{n=0}^{\infty}$ are said to be Abel summable to s. The corresponding method of summability is called the Abel method.

When no matrix A is involved, we may say summable A instead of Abel summable, and write $\Sigma a_n = s(A)$ when the series Σa_n is summable A to s.

Remark 16.2. It is obvious that if $\sum_{n=0}^{\infty} a_n x^n$ converges for |x| < 1 then we have

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{\sum_{n=0}^{\infty} x^n}.$$

We obtain as an immediate consequence of Abel's limit theorem

Theorem 16.3. The Abel method is regular.

Example 16.4. We have $\Sigma(-1)^n = 1/2(A)$. This takes us back to the *satisfying* result (*) mentioned in Section 1.

We need the following lemma to be able to prove a growth theorem for the Abel method.

Lemma 16.5. The inequality

(16.1)
$$\overline{\lim_{n \to \infty}} \left(|a_n|^{1/n} \right) \le 1$$

holds if and only if

(16.2)
$$a_n = O((1+\varepsilon)^n)$$
 for arbitrary $\varepsilon > 0$.

Proof. (i) First we assume that (16.1) is satisfied. Then given $\varepsilon > 0$ there is $N_{\varepsilon} \in \mathbb{N}_0$ such that $|a_n|^{1/n} \leq 1 + \varepsilon$ for all $n \geq N_{\varepsilon}$. This implies $|a_n| \leq (1 + \varepsilon)^n$ for all $n \geq N_{\varepsilon}$, and so there is a constant K_{ε} such that $|a_n| \leq K_{\varepsilon}(1 + \varepsilon)^n$ for all n and (16.2) is satisfied.

(ii) Conversely we assume that (16.2) is satisfied. Then we have $|a_n| \leq K(1+\varepsilon)^n$ $(n=0,1,\ldots)$ for some constant K, hence

$$|a_n|^{1/n} \le K^{1/n}(1+\varepsilon),$$

and (16.1) is satisfied.

Theorem 16.6. If the series Σa_n is summable A then the terms of the series satisfy

$$a_n = O(q^n)$$
 for arbitrary $q > 1$.

Proof. If Σa_n is summable A, then $\sum_{n=0}^{\infty} a_n x^n$ converges for |x| < 1, hence (16.1) holds by the Cauchy-Hadamard theorem for power series.

It turns out that the Abel method is stronger than any Cesàro method.

Theorem 16.7. We have $C_{\alpha} \subset A$ for all $\alpha > 0$.

Proof. We assume $\sum a_n = s(C_\alpha)$ and put $A(x) = \sum_{n=0}^{\infty} a_n x^n$. By Part (b) of Theorem 7.12, $\sum a_n = s(C_\alpha)$ implies $a_n = o(n^\alpha)$ and so the power series expansion of A(x) has radius of convergence 1. Let $(x_n)_{n=0}^{\infty}$ be an arbitrary sequence with $0 < x_n < 1$ for all n and $x_n \to 1 - (n \to \infty)$. Then it follows from

$$A(x) \cdot \frac{1}{(1-x)^{\alpha+1}} = \left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} A_n^{\alpha} x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n A_{n-k}^{\alpha} a_k\right) x^n$$

$$=\sum_{n=0}^{\infty}A_{n}^{\alpha}\sigma_{n}^{\alpha}x^{n}$$

that

$$A(x) = (1-x)^{\alpha+1} \sum_{n=0}^{\infty} s_n^{\alpha} x^n = (1-x)^{\alpha+1} \sum_{n=0}^{\infty} A_n^{\alpha} \sigma_n^{\alpha} x^n,$$
$$A(x_n) = (1-x_n)^{\alpha+1} \sum_{k=0}^{\infty} A_k^{\alpha} \sigma_k^{\alpha} x_n^k = \sum_{k=0}^{\infty} b_{nk} \sigma_k^{\alpha}$$

where $b_{nk} = (1 - x_n)^{\alpha + 1} A_k^{\alpha} x_n^k$ for all n and k. Since $0 < x_n < 1$ (n = 0, 1, ...), we have $b_{nk} \ge 0$ or all n and k, and

$$\sum_{k=0}^{\infty} |b_{nk}| = \sum_{k=0}^{\infty} b_{nk} = \sum_{k=0}^{\infty} (1 - x_n)^{\alpha + 1} A_k^{\alpha} x_n^k$$
$$= \frac{(1 - x_n)^{\alpha + 1}}{(1 - x_n)^{\alpha + 1}} = 1 \text{ for } n = 0, 1, \dots$$

We fix $k \in \mathbb{N}_0$. Then $b_{nk} = (1 - x_n)^{\alpha + 1} A_k^{\alpha} x_n^k \to 0 \ (n \to \infty)$, since $x_n \to 1$ and $\alpha > -1$. Therefore the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ satisfies the conditions of Part (c) of Theorem 4.3, and thus defines a regular method of summability.

The following growth theorem is a generalization to Abel summability of the well-known fact that if a series Σa_n converges then its terms converge to zero.

Theorem 16.8. If the series Σa_n is summable A then $a_n \to 0(A)$ as $n \to \infty$.

Proof. Let $\Sigma a_n = s(A)$. Then we have

$$\sum_{n=0}^{\infty} a_n x^n \to s \ (x \to 1-) \text{ or } (1-x) \sum_{n=0}^{\infty} a_n x^n \to 0 \ (x \to 1-),$$

hence $a_n = 0(A)$ by Definition 16.1.

Finally we prove a Tauberian theorem for the Abel method.

Theorem 16.9. If Σa_n is summable A to s and its terms satisfy $a_n = o(1/n)$ then the series Σa_n converges to s.

Proof. We put $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Let $\varepsilon > 0$ be given. Since $a_n = o(1/n)$, the C_1 method is regular, and $\Sigma a_n = s(A)$, we can choose $N_{\varepsilon} \in \mathbb{N}$ such that

(16.3)
$$|na_n| < \frac{\varepsilon}{3} \text{ for all } n > N_{\varepsilon},$$

(16.4)
$$\frac{1}{n+1}\sum_{k=0}^{n}k|a_k| = \sigma_n((k|a_k|)_{k=0}^{\infty}) < \frac{\varepsilon}{3} \text{ for all } n > N_{\varepsilon},$$

(16.5)
$$|A(1-1/n)-s| < \frac{\varepsilon}{3} \text{ for all } n > N_{\varepsilon}.$$

Thus, if $n > N_{\varepsilon}$, then we have

(16.6)
$$s_n - s = \sum_{k=0}^n a_k - s = \sum_{k=0}^\infty a_k x^k - s + \sum_{k=0}^n a_k (1 - x^k) - \sum_{k=n+1}^\infty a_k x^k.$$

Applying the mean value theorem of differentiation to $f(t) = t^k$ on the interval [x, 1], we obtain

(16.7)
$$1 - x^k \le k(1 - x).$$

Furthermore we have for k > n

(16.8)
$$|a_k| \le \frac{k|a_k|}{n} \le \frac{\varepsilon}{3n}.$$

Now (16.6), (16.7) and (16.8) imply

$$|s_n - s| \le |A(x) - s| + (1 - x) \sum_{k=0}^n |ka_k| + \frac{\varepsilon}{3n} \sum_{k=n+1}^\infty x^k$$
$$\le |A(x) - s| + (1 - x) \sum_{k=0}^n |ka_k| + \frac{\varepsilon}{3n} \frac{1}{1 - x}.$$

We put x = 1 - 1/n. Then we have by (16.5) and (16.4)

$$\begin{aligned} |s_n - s| &\leq |A(1 - 1/n) - s| + \frac{1}{n} \sum_{k=0}^n |ka_k| + \frac{\varepsilon}{3n} n\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for all } n > N_{\varepsilon}, \end{aligned}$$

hence $\lim_{n\to\infty} s_n = s$.

Remark 16.10. It can be shown ([99, Theorem III.21]) that an analogous more general result of the Tauberian theorem, Theorem 16.9, also holds true when the Tauberian condition $s_n = o(1/\sqrt{n})$ is replaced by $s_n = o(1/\sqrt{n})$. This result is due to Hardy and Littlewood [?].

17 The Borel method

In this section, we study the *Borel method*. This method is also not given by a matrix.

Definition 17.1. Let $(s_n)_{n=0}^{\infty}$ be a real sequence and $\sum_{n=0}^{\infty} (x^n/n!) s_n$ be convergent for all $x \in \mathbb{R}$. If

$$\sigma(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n \to s \quad (x \to \infty)$$

then the sequence $(s_n)_{n=0}^{\infty}$ is said to be *Borel summable to s*; this is denoted by $s_n \to s(B) \ (n \to \infty)$. The corresponding method of summability is called the *Borel method*.

First we observe that the Borel method is regular.

Theorem 17.2. The Borel method is regular.

Proof. We assume that $s_n \to s$ $(n \to \infty)$. Let $(x_n)_{n=0}^{\infty}$ be an arbitrary positive sequence with $x_n \to \infty$ as $n \to \infty$. Then we consider the sequence $b = (b_n)_{n=0}^{\infty}$ defined by

$$b_n = e^{-x_n} \sum_{k=0}^{\infty} \frac{x_n^k}{k!} s_k = \sum_{k=0}^{\infty} c_{nk} s_k$$
 for all $n = 0, 1, \dots$

where the matrix $C = (c_{nk})_{n,k=0}^{\infty}$ is defined by

$$c_{nk} = e^{-x_n} \frac{x_n^k}{k!}$$
 for all $n, k = 0, 1, \dots$

Since $x_n \ge 0$ for all n, we have

$$\sum_{k=0}^{\infty} |c_{nk}| = \sum_{k=0}^{\infty} c_{nk} = e^{-x_n} \sum_{k=0}^{\infty} \frac{x_n^k}{k!} = e^{-x_n} e^{x_n} = 1 \text{ for all } n = 0, 1, \dots,$$

and for fixed $k \in \mathbb{N}_0$

$$\lim_{n \to \infty} c_{nk} = \lim_{n \to \infty} \frac{\mathrm{e}^{-x_n} x_n^k}{k!} = 0.$$

Thus the conditions in Part (c) of Theorem 4.3 are satisfied and consequently the matrix C defines a regular method of summability. This implies $b_n \to s \ (n \to \infty)$. Since this holds for every positive sequence $(x_n)_{n=0}^{\infty}$ with $x_n \to \infty$ as $n \to \infty$, the Borel method is regular.

It turns out that the Borel method is stronger than any Euler method.

Theorem 17.3. We have $E_q \subset B$ for all q > 0.

Proof. It follows from Theorem 9.4 that

$$e^{-(q+1)x} \sum_{k=0}^{\infty} \frac{(x(q+1))^k}{k!} s_k^q = e^{-x} \left(\sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(qx)^{\nu}}{\nu!} \right) \left(\sum_{k=0}^{\infty} \frac{(x(q+1))^k}{k!} s_k^q \right)$$

$$= e^{-x} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \frac{x^{n-k}q^{n-k}}{(n-k)!} \frac{x^k(q+1)^k}{k!} s_k^q \right)$$
$$= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(q^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(\frac{q+1}{q} \right)^k s_k^q \right)$$
$$= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n,$$

that is,

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k = e^{-(q+1)x} \sum_{k=0}^{\infty} \frac{(x(q+1))^k}{k!} s_k^q.$$

As in the proof of Theorem 17.2, let $(x_n)_{n=0}^{\infty}$ be an arbitrary positive sequence with $x_n \to \infty$ as $n \to \infty$. We consider the matrix $C = (c_{nk})_{n,k=0}^{\infty}$ defined by

$$c_{nk} = e^{-(q+1)x_n} \frac{x_n^k (q+1)^k}{k!}$$
 for all $n, k = 0, 1, \dots$

Then we have $c_{nk} \ge 0$ for all n and k,

$$\sum_{k=0}^{\infty} |c_{nk}| = \sum_{k=0}^{\infty} c_{nk} = e^{-(q+1)x_n} \sum_{k=0}^{\infty} \frac{x_n^k (q+1)^k}{k!}$$
$$= e^{-(q+1)x_n} e^{(q+1)x_n} = 1 \text{ for all } n = 0, 1, \dots$$

and for each fixed $k \in \mathbb{N}_0$

$$\lim_{n \to \infty} c_{nk} = \lim_{n \to \infty} e^{-(q+1)x_n} \frac{x_n^k (q+1)^k}{k!} = 0.$$

Thus the conditions in Part (c) of Theorem 4.3 are satisfied and consequently the matrix C defines a regular method of summability. Now the conclusion follows by the same argument as that applied at the end of the proof of Theorem 17.2.

Now we prove a Tauberian theorem for the Borel method. The techniques of the proof are very similar to those applied in the proof of Theorem 9.8, the Tauberian theorem for the Euler method E_1 .

Theorem 17.4. If the sequence $(s_n)_{n=0}^{\infty}$ is Borel summable to s and $a_n = s_n - s_{n-1} = o(1/\sqrt{n})$ then the sequence $(s_n)_{n=0}^{\infty}$ converges to s.

Proof. Since the techniques applied in the proof are very similar to those in the proof of Theorem 9.8, we only outline the steps of the proof without going into the details again.

We observe that

$$S_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} x^2 = x^2 e^x,$$

$$S_{2}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} kx = \sum_{k=1}^{\infty} \frac{x^{k+1}}{(k-1)!} \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!} = S_{1}(x),$$

$$S_{3}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} k^{2} = \sum_{k=1}^{\infty} \frac{x^{k}k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} (k+1),$$

$$= x \sum_{k=0}^{\infty} \frac{x^{k}}{k!} + \sum_{k=1}^{\infty} \frac{x^{k+1}}{(k-1)!} = x e^{x} + S_{1}(x),$$

hence

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} (x-k)^2 e^{-x} \left(S_1(x) - 2S_2(x) + S_3(x)\right) = x.$$

(i) First we show

(17.1)
$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} = O(e^x).$$

As in the proof of Theorem 9.8, we have

(17.2)
$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} = \sum_{|k-x| \le \sqrt{x}} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} + \sum_{\sqrt{x} < |k-x| \le \frac{3x}{4}} \frac{x^k}{\sqrt{k+1}} \frac{|k-x|}{\sqrt{k+1}} + \sum_{\frac{3x}{4} < |k-x|} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}}.$$

If $x \ge 2$ then $x - \sqrt{x} \ge x/2$, and so $|k - x| \le \sqrt{x}$ implies $k \ge x - \sqrt{x} \ge x/2$, hence $\sqrt{x} \le \sqrt{2} \cdot \sqrt{k}$, and so

(17.3)
$$T_{1} = \sum_{|k-x| \le \sqrt{x}} \frac{x^{k}}{k!} \frac{|k-x|}{\sqrt{k+1}} \le \sum_{|k-x| \le \sqrt{x}} \frac{x^{k}}{k!} \frac{\sqrt{x}}{\sqrt{k+1}} \le \sqrt{2} \sum_{|k-x| \le \sqrt{x}} \frac{x^{k}}{k!} \cdot \sqrt{\frac{k}{k+1}} \le \sqrt{2} \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = \sqrt{2} \cdot e^{x}.$$

Furthermore $|k - x| \le 3x/4$ implies $x - k \le 3x/4$, hence $k \ge x/4$, that is, $k + 1 \ge 5x/4$, and so

$$\frac{1}{\sqrt{k+1}} \le \frac{2}{\sqrt{5} \cdot \sqrt{x}}.$$

Since $|k - x| \ge \sqrt{x}$, we also have $|k - x|/\sqrt{x} \ge 1$, hence

$$\frac{|k-x|}{\sqrt{k+1}} \le \frac{2}{\sqrt{5}} \cdot \frac{|k-x|}{\sqrt{x}} \le \frac{2}{\sqrt{5}} \cdot \frac{|k-x|^2}{x},$$

and so

(17.4)
$$T_2 = \sum_{\sqrt{x} < |k-x| \le \frac{3x}{4}} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} \le \frac{2}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \frac{|k-x|^2}{x}.$$

Finally, if k = 0 then, since x > 0,

$$\frac{|k-x|}{\sqrt{k+1}} = |x| = x = \frac{|k-x|^2}{x},$$

and if $k\geq 1$ then

$$k+1 \ge \frac{16}{9}$$
, hence $\frac{1}{\sqrt{k+1}} \le \frac{3}{9}$.

Thus, if |k - x| > 3x/4, then we obtain

$$\frac{|k-x|}{\sqrt{k+1}} \le \frac{3}{4} \cdot |k-x| = \frac{3x}{4} \cdot |k-1| \cdot \frac{1}{x} \le \frac{|k-x|^2}{x},$$

that is,

(17.5)
$$T_3 = \sum_{\frac{3x}{4} < |k-x|} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} \le \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \frac{|k-x|^2}{x}.$$

Now it follows from (17.2), (17.3), (17.4) and (17.5) that

(17.6)
$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} = T_1 + T_2 + T_3 = O(1) \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|k-x|^2}{x} \right).$$

We also have

$$\sum_{k=0}^{\infty} \frac{x^k k^2}{k!} \cdot \frac{1}{x} = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(k+1)x^k}{k!}$$
$$= \frac{d}{dx} \left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} \right) = \frac{d}{dx} (xe^x) = (x+1)e^x,$$
$$2\sum_{k=0}^{\infty} \frac{kx^{k+1}}{xk!} = 2\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} = -2\sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} = -2xe^x,$$

 and

$$\sum_{k=0}^{\infty} \frac{x^{k+2}}{xk!} = \sum_{k=1}^{\infty} \frac{x^{k+1}}{k!} = x \sum_{k=1}^{\infty} \frac{x^k}{k!} = x(e^x - 1),$$

that is,

(17.7)
$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \frac{|x-k|^2}{x} = (x+1)e^x - 2xe^x + x(e^x - 1) = e^x - x = O(e^x).$$

Hence it follows from (17.6) and (17.7) that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|k-x|}{\sqrt{k+1}} = O(e^x).$$

Thus we have shown (17.1).

(ii) Now we show that

(17.8)
$$|s_n - s_k| = o(1) \frac{|n-k|}{\sqrt{n}} \quad (n \to \infty) \text{ uniformly in } k.$$

 $(ii.\alpha)$ First we show that given $\varepsilon > 0$ there exists $N_{\varepsilon} \in \mathbb{N}_0$ such that

(17.9)
$$\sum_{k=0}^{n} |a_k| \le \varepsilon \sqrt{n} \text{ for all } n \ge N_{\varepsilon}.$$

Let $\varepsilon > 0$ be given. Since $a_k = o(1/\sqrt{k})$, there exists $k_0 \in \mathbb{N}_0$ such that

(17.10)
$$|a_k|\sqrt{k} < \varepsilon \text{ for all } k \ge k_0.$$

Now we choose $N_{\varepsilon} \in \mathbb{N}_0$ so large that we have

(17.11)
$$\frac{1}{\sqrt{n}}\sum_{k=0}^{k_0}|a_k| < \varepsilon \text{ for all } n \ge N_{\varepsilon}.$$

Let $n \ge N_{\varepsilon}$ be given. Then it follows from (17.10) and (17.11) that

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n}|a_{k}| = \frac{1}{\sqrt{n}}\left(\sum_{k=0}^{k_{0}}|a_{k}| + \sum_{k=k_{0}}^{n}|a_{k}|\right) < \varepsilon + \frac{\varepsilon}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$
$$\leq \varepsilon(1+C) \text{ for some absolute constant } C.$$

Thus we have shown (17.9).

 $(ii.\beta)$ Now we show

(17.12)
$$|s_n - s_k| \frac{\sqrt{n}}{|n-k|} \to 0 \ (n \to \infty)$$
 uniformly in k

Let $\varepsilon > 0$ be given. As above we choose $k_0 \in \mathbb{N}_0$ such that (17.10) holds. We put $n_0 = 2k_0$. Then we have for all $n \ge n_0$ and for all $\nu \ge n/2$, if $\nu < n$, by (17.10)

$$\begin{aligned} |s_n - s_\nu| &\leq |\sum_{k=\nu+1}^n |a_k| < \sum_{k=\nu+1}^n \frac{\varepsilon}{\sqrt{k}} \leq \varepsilon \sum_{k=\nu+1}^n \frac{1}{\sqrt{\nu}} \leq \frac{\varepsilon(n-\nu)}{\sqrt{\nu}} \\ &\leq \frac{\sqrt{2}\varepsilon(n-\nu)}{\sqrt{n}}, \end{aligned}$$

and, if $\nu \geq n$,

$$|s_n - s_\nu| \le |\sum_{k=n+1}^\nu |a_k| \le \varepsilon \sum_{k=n+1}^\nu \frac{1}{\sqrt{k}} \le \varepsilon \sum_{k=n+1}^\nu \frac{1}{\sqrt{n}} \le \frac{\varepsilon(\nu - n)}{\sqrt{n}}$$

Thus we have

$$|s_n - s_\nu| \le \frac{\sqrt{2\varepsilon}|n-\nu|}{\sqrt{n}}$$
 for all $n \ge n_0$ and for all $\nu \ge n/2$.

Now we choose $N_{\varepsilon} \in \mathbb{N}_0$ with $N_{\varepsilon} > n_0$ such that (17.9) holds for all $n \ge N_{\varepsilon}$. Let $n \ge 2N_{\varepsilon}$. Then we have for all $\nu \le n/2$ by (17.9)

$$\begin{aligned} |s_n - s_\nu| &\leq \sum_{k=0}^n |a_k| + \sum_{k=0}^\nu |a_\nu| < \varepsilon \left(\sqrt{n} + \sqrt{\frac{n}{2}}\right) \leq \frac{2\varepsilon n}{\sqrt{n}} \\ &\leq \frac{4\varepsilon (n - \frac{n}{2})}{\sqrt{n}} \leq \frac{4\varepsilon (n - \nu)}{\sqrt{n}}. \end{aligned}$$

Thus we have shown that given $\varepsilon > 0$, there exists $N_{\varepsilon} \in \mathbb{N}_0$ such that

$$|s_n - s_\nu| < \frac{4\varepsilon |n - \nu|}{\sqrt{n}}$$
 for all $n \ge N_\varepsilon$ and all ν ,

that is, (17.12) holds. This concludes Part (ii) of the proof.

Finally we have for x = n by (17.12)

$$\begin{aligned} |\sigma(x) - s_n| &= \left| e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k - s_n \right| \le e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} |s_k - s_n| \\ &\le e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} o(1) \frac{|n-k|}{\sqrt{n}} = o(1/\sqrt{n}) e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} |n-k|. \end{aligned}$$

Since $2x \le k$ implies $x \le k-x$, hence $(k-x)/x \ge 1$ and $(k-x)^2/x \ge k-x = |x-k|$, $\sigma_n(x) - s_n \to 0 \ (n \to \infty)$ follows from (17.7)

$$e^{-x} \sum_{k \ge 2x} \frac{x^k}{k!} |x-k| \le e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{|x-k|^2}{x} = O(1).$$

Remark 17.5. (a) Similarly it can be shown that $s_n \to s(B)$ and $a_n = O(1/\sqrt{n})$ imply $s_n = O(1)$.

(b) Since $E_1 \subset B$ by Theorem 17.3, Theorem 9.8 would follow from Theorem 17.4.

18 Limit points of sequences and their transforms

Here we apply some of the results of our previous sections to study sets of limit points of sequences and their transforms by methods of summability. Most of the results of this section can be found in the classical paper by *Barone* [11]

We denote by $\mathcal{L}(s)$ the set of all limit points of a complex sequence $s = (s_n)_{n=0}^{\infty}$.

First, examples are given which show that if no restrictions are placed on the complex sequence (s_n) , then the set $\mathcal{L}(s)$ need not be connected. Then sufficient conditions that $\mathcal{L}(s)$ be connected are given, and theorems are proved concerning transforms of bounded complex sequences and their sets of limit points. Furthermore it is determined whether the sets of limit points of the the Hölder, Cesàro, de la Vallée Poussin and Euler transforms of (s_n) are connected.

We recall that since $\mathcal{L}(s)$ is a closed set for each sequence (s_n) , to say that $\mathcal{L}(s)$ is *connected* means that $\mathcal{L}(s)$ cannot written as the disjoint union of two non-empty closed sets. To see that $\mathcal{L}(s)$ need not be connected, in general, we consider the following example.

Example 18.1. Let the sequences $(s_n^{(1)})$, $(s_n^{(2)})$ and $(s_n^{(3)})$ be defined as follows (18.1)

$$s_n^{(1)} = \begin{cases} 1 & (n = 2m) \\ i & (n = 2m + 1) \end{cases} \quad s_n^{(2)} = \begin{cases} 1 & (n = 3m) \\ i & (n = 3m + 1) \\ m + 2 & (n = 3m + 2) \end{cases} \text{ for } m = 0, 1, \dots,$$

and

$$(18.2) \quad (s_n^{(3)}): \begin{cases} 0, \frac{i}{10}, \frac{2i}{10}, \dots, i, \frac{1}{10} + i, \frac{2}{10} + i, \dots, 1 + i, 1 + \frac{9i}{10}, 1 + \frac{8i}{10}, \dots, 1, \\ 1 + \frac{i}{20}, 1 + \frac{2i}{20}, \dots, 1 + 2i, \frac{19}{20} + 2i, \frac{18}{20} + 2i, \dots, 2i, \frac{39i}{20}, \dots, 0, \\ \frac{i}{30}, \frac{2i}{30}, \dots, 3i, \frac{1}{30} + 3i, \frac{2}{30} + 3i, \\ & \dots, 1 + 3i, 1 + \frac{89i}{30}, 1 + \frac{88i}{30}, \dots, 1, \\ 1 + \frac{i}{40}, 1 + \frac{2i}{40}, \dots, 1 + 4i, \frac{39}{40} + 4i, \frac{38}{40} + 4i, \dots, \\ & \dots, 4i, \frac{159i}{40}, \frac{158i}{40} \dots, 0, \end{cases}$$

Then $\mathcal{L}(s^{(1)}) = \mathcal{L}(s^{(2)}) = \{1, i\}$, while $\mathcal{L}(s^{(3)})$ consists of all those points not below the real axis with real part 0 or 1. These sets are not connected.

The sequence $(s_n^{(1)})$ is bounded, while the sequences $(s_n^{(2)})$ and $(s_n^{(3)})$ are unbounded. The sequence $(s_n^{(3)})$ satisfies

(18.3)
$$\left(s_n^{(3)}\right) \in (c_0)_\Delta$$
, that is, $\Delta(s_n^{(3)}) \in c_0$,

where Δ denotes the operator of the (backward) differences defined for every sequence $(s_n)_{n=0}^{\infty}$ by

$$\Delta(s_n) = s_n - s_{n-1} \ (n = 0, 1...); \text{ where } s_{-1} = 0.$$

But neither $(s_n^{(1)})$ nor $(s_n^{(2)})$ has the property in (18.3).

First we establish sufficient conditions for $\mathcal{L}(s)$ to be connected.

Theorem 18.2. If a sequence $(s_n) \in \ell_{\infty}$ satisfies the condition in (18.3), then the set $\mathcal{L}(s)$ is connected.

Theorem 18.2 is a special case of the following result.

Theorem 18.3. If (s_n) is a compact sequence in a metric space (X, d) such that

(18.4)
$$\lim_{n \to \infty} d(s_n, s_{n-1}) = 0.$$

then the set $\mathcal{L}(s)$ is connected.

Proof. We write $\mathcal{L} = \mathcal{L}(s)$, for short, and assume that \mathcal{L} is not connected. Then $\mathcal{L}(s)$ can be written as the disjoint union of two sets S_1 and S_2 such that neither $S_1 = \emptyset$ nor $S_2 = \emptyset$. Since $S_1 \cap S_2 = \emptyset$, and S_1 and S_2 are closed and compact, $d(S_1, S_2) = \rho > O$. Also there exist $a_1 \in S_1$ and $a_2 \in S_2$ such that $d(a_1, a_2) = \rho$. Since $a_1, a_2 \in \mathcal{L}(s)$, there exist subsequences (s_{k_i}) and (s_{l_i}) of (s_n) such that

$$\lim_{j \to \infty} s_{k_j} = a_1, \ \lim_{j \to \infty} s_{l_j} = a_2 \text{ and } k_1 < l_1 < k_2 < l_2 \cdots.$$

Now there exists a positive integer N such that

(18.5)
$$d(s_{k_n}, a_1) < \frac{\rho}{4}, \ d(s_{l_n}, a_2) < \frac{\rho}{4} \text{ and } d(s_m, s_{m+1}) < \frac{\rho}{4} \text{ for all } k_n, l_n, m > N.$$

For each $k_n > N$ we consider the group

$$s_{k_n}, s_{k_n+1}, s_{k_n+2}, \ldots, s_{l_n}$$

of terms of the sequence (s_n) . It follows from (18.5) that

$$d(s_{k_n}, S_1) < \frac{\rho}{4}, \ d(s_{l_n}, S_2) < \frac{\rho}{4} \text{ and } d(s_m, s_{m+1}) < \frac{\rho}{4} \text{ for all } k_n, m > N.$$

Hence there must be some index p_n such that $k_n < k_n + p_n < l_n$ and

(18.6)
$$d(s_{k_n+p_n}, S_1) > \frac{\rho}{4} \text{ and } d(s_{k_n+p_n}, S_2) > \frac{\rho}{4} \text{ for } k_n > N.$$

This would mean that, for some elements, $d(s_{k_n+p_n}, s_{k_n+p_n+1}) > \rho/2$, and this is a contradiction.

We now have a subsequence $(s_{k_n+p_n})$ of the sequence (s_n) which satisfies (18.6). Since the sequence (s_n) is compact, this subsequence has a limit point, say c, such that

$$d(c, S_1) \ge \frac{\rho}{4}$$
 and $d(c, S_2) \ge \frac{\rho}{4}$.

Thus we have $c \in \mathcal{L}(s)$, but $c \notin S_1$ and $c \notin S_2$, and consequently $\mathcal{L}(s) \neq S_1 \cup S_2$. This is a contradiction. Therefore $\mathcal{L}(s)$ must be connected. \Box

Remark 18.4. The condition in (18.4) is not necessary for $\mathcal{L}(s)$ to be connected. To see this, we consider the sequence (s_n) with $s_n = e^{in}$ for all n. Then $\mathcal{L}(s)$ is connected by *Kronecker's density theorem* [58] or [14] for a simple constructive proof, but the condition in (18.4) is not satisfied. Now we consider theorems concerning matrix transforms and their sets of limit points. We recall that by the Toeplitz theorem, Part (c) of Theorem 4.3 a matrix transformation A is regular if and only if

(18.7)
$$\sup_{n}\sum_{k=0}^{\infty}|a_{nk}|<\infty,$$

(18.8)
$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

(18.9)
$$\lim_{n \to \infty} a_{nk} = 0 \text{ for all } k.$$

We will use the following well-known result without explicitly referring to it each time.

Proposition 18.5. ([21, Remark 22 (a), p. 22] or [111, 1.4.8]) Every triangle T has a unique inverse S which also is a triangle, and x = T(Sx) = S(Tx) for all $x \in \omega$.

The following general results are useful.

Lemma 18.6. Let $T = (t_{nk})_{n,k=0}^{\infty}$ be a triangle, $A = (a_{nk})_{n,k=0}^{\infty}$ be an arbitrary infinite matrix, and $C = (c_{nk}) = T \cdot A$, the product of the matrices T and A, that is,

$$c_{nk} = \sum_{j=0}^{n} t_{nj} a_{jk}$$
 for $n, k = 0, 1, \dots$

Then we have $A \in (X, Y_T)$ if and only if $C \in (X, Y)$.

Proof. First we assume $A \in (X, Y_T)$. Then the series $A_n x$ converge for all $x \in X$ and all n, hence $x \in \omega_A$. Since T is a triangle, we have $T_n \in \phi$ for all n, and it follows from Part (i) in Theorem 6.4

$$T_n(Ax) = \sum_{k=0}^{\infty} \left(\sum_{m=0}^n t_{nm} a_{mk} \right) x_k = C_n x \text{ for all } n \text{ and } x \in X,$$

that is, T(Ax) = Cx for all $x \in X$, and $Ax \in Y_T$ implies $Cx \in (X, Y)$. Hence we have $C \in (X, Y)$.

Now we assume $C \in (X, Y)$. Let S be inverse of T (Proposition 18.5). Then it follows by what we have just shown with A and T replaced by C and S, respectively, that

$$S_n(Cx) = (S \cdot C)_n x$$
 for all n and all $x \in X_n$

and clearly $S \cdot C = S \cdot (T \cdot A) = (S \cdot T) \cdot A = A$. Hence we have $T(Ax) = (T \cdot C)x = Cx \in Y$, that is, $Ax \in Y_T$ for all $x \in X$ and so $A \in (X, Y_T)$.

Theorem 18.7. Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an arbitrary infinite matrix and $s = (s_n) \in \ell_{\infty}$. If

(18.10)
$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{n,k} - a_{n-1,k}| = 0,$$

then the set $\mathcal{L}(As)$ is connected.

Proof. Since

 $(\Delta \cdot A)_{n,k} = a_{n,k} - a_{n-1,k}$ for all n and k, where $a_{-1,k} = 0$ for all k,

the condition in (18.10) implies by (5.10) in Part (b) of Remark 5.3 that $\Delta \cdot A \in (\ell_{\infty}, c_0)$, hence $A \in (\ell_{\infty}, (c_0)_{\Delta})$ by Lemma 18.6, that is, $\Delta(As) \in c_0$ for all sequences $s = (s_n) \in \ell_{\infty}$. Now the conclusion follows by Theorem 18.2.

From now on, let infinite matrices always be triangles, unless explicitly stated otherwise.

Theorem 18.8. Let $(s_n)_{n=0}^{\infty} \in \ell_{\infty}$ and the matrices A and B satisfy

(18.11)
$$\lim_{n \to \infty} \sum_{k=0}^{n} |a_{nk} - b_{nk}| = 0.$$

If $\mathcal{L}(As)$ is connected, so is $\mathcal{L}(Bs)$.

Proof. We show $\mathcal{L}(As) = \mathcal{L}(Bs)$ for all sequences $s = (s_n) \in \ell_{\infty}$. We may assume that at least one of the sets is not empty, $\mathcal{L}(Bs) \neq \emptyset$, say. If $\mathcal{L}(As) = \emptyset$ and $t \in \mathcal{L}(Bs)$, then there exists a subsequence $(B_{n_m}s)$ of the sequence (B_ns) such that $B_{n_m} \to t$ as $m \to \infty$, hence by (18.11)

$$\begin{split} |A_{n_m}s - t| &\le |B_{n_m}s - t| + |(B_{n_m} - A_{n_m})s| \\ &\le |B_{n_m}s - t| + \left(\sum_{n=0}^{n_m} |b_{n_m} - a_{n_m}|\right) \|s\|_{\infty} \to 0 \text{ as } m \to \infty, \end{split}$$

and so $t \in \mathcal{L}(As)$, which is a contradiction. So $\mathcal{L}(As) = \emptyset = \mathcal{L}(Bs)$. If $t \in \mathcal{L}(As)$ then $t \in \mathcal{L}(Bs)$ by the above argument with A and B interchanged, and the converse implication follows by the above argument.

Theorem 18.9. If the matrix A satisfies the condition

(18.12) there exists
$$P > 0$$
 such that $|a_{nn}| - \sum_{k=0}^{n-1} |a_{nk}| \ge P$ for all n ,

then the set $\mathcal{L}(As)$ of a sequence $(s_n) \in \ell_{\infty}$ need not be connected.

Proof. It is sufficient to show that a sequence (t_n) that has two limit points is the transform of a bounded sequence (s_n) . This may be done by showing that if (t_n) is bounded, then the sequence (s_n) obtained by the inverse transformation is also bounded. This, in turn, may be done by showing that if (s_n) is unbounded, then (t_n) is unbounded. For a given M > 0 there exists an n such that

$$|s_n| > M/P$$
 and $|s_k| < |s_n|$ for all $k < n$.

Then we have by (18.12)

$$\begin{aligned} |t_n| &= \left| \sum_{k=0}^n a_{nk} s_k \right| \ge |a_{nn}| \cdot |s_n| - \left| \sum_{k=0}^{n-1} a_{nk} s_k \right| \\ &\ge |a_{nn}| \cdot |s_n| - |s_n| \sum_{k=0}^{n-1} |a_{nk}| \\ &\ge |s_n| \left(|a_{nn}| - \sum_{k=0}^{n-1} |a_{nk}| \right) \ge M. \end{aligned}$$

Theorem 18.10. If the matrix A satisfies the conditions

(18.13) there exists a positive constant P such that $|a_{nn}| > P > 0$ for all n,

and

(18.14)
$$\frac{|a_{nk}|}{\sum_{p=k+1}^{n} |a_{np}|} \to \infty \ (n \to \infty) \ for \ each \ 0 \le k \le n-1,$$

then the set $\mathcal{L}(As)$ of a sequence $(s_n) \in \ell_{\infty}$ is connected.

Proof. If $s_n = 0$ for all n, then $t_n = A_n s = 0$ for all n.

If $s_n \neq 0$ for some *n*, then there exists *M* such that $|s_n| < M$ for all *n*. Let $k = \min\{n \in \mathbb{N} : s_n \neq 0\}$. Then we have by (18.13) and (18.14)

$$\begin{aligned} |t_n| &= \left| \sum_{p=k}^n a_{np} s_p \right| \ge |a_{nk}| \cdot |s_k| - M \sum_{p=k+1}^n |a_{np}| \\ &= \sum_{p=k+1}^n |a_{np}| \left(\frac{|a_{nk}| \cdot |s_k|}{\sum_{p=k+1}^n |a_{np}|} - M \right) \to \infty \text{ as } n \to \infty. \end{aligned}$$

Theorem 18.11. If the matrix A satisfies the condition

(18.15)
$$a_{nk} = f_n \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots \text{ and } \inf_n f_n = P > 0$$

then the set $\mathcal{L}(As)$ of a sequence $(s_n) \in \ell_{\infty}$ need not be connected.

Proof. Since $a_{nk} = f_n$ for $0 \le k \le n$ by (18.15), we have

$$t_n = A_n s = f_n \sum_{k=0}^n s_k$$
 for all n ,

hence

$$s_n = \frac{t_n}{f_n} - \frac{t_{n-1}}{f_{n-1}}$$
 for $n \ge 1$,

hence (s_n) is bounded whenever (t_n) is bounded, and the conclusion follows by Theorem 18.8.

Theorem 18.12. If the matrix A satisfies the conditions in (18.13),

(18.16)

there exsits $r \in \mathbb{C}$ such that $a_{nk} - ra_{n-1,k} = f_k$ for k < n and $n = 0, 1, \ldots$,

and

(18.17)

there exists
$$\rho \in (0,1)$$
 such that $|f_{n-1} - a_{n-1,n-1}| < \rho |a_{nn}|$ for all n ,

then the set $\mathcal{L}(As)$ of a sequence $(s_n) \in \ell_{\infty}$ need not be connected.

Proof. We assume that the sequence (s_n) is not bounded. Let M > 0 be given. Then there exists an n such that

$$|s_n| > \frac{M}{P(1-\rho)}$$
 and $|s_k| < |s_n|$ for all $k < n$.

Thus we have, writing again $t_n = A_n s$ for all n,

$$\begin{aligned} |t_n - (r+1)t_{n-1} + rt_{n-2}| &= \left| a_{nn} \left(s_n + (f_{n-1} - a_{n-1,n-1}) \frac{s_{n-1}}{a_{nn}} \right) \right| \\ &\geq |a_{nn}| \cdot \left(|s_n| - |s_{n-1}| \cdot \frac{|f_{n-1} - a_{n-1,n-1}|}{|a_{nn}|} \right) \\ &\geq |a_{nn}| \cdot (|s_n| - \rho|s_n|) > M. \end{aligned}$$

Therefore (t_n) is unbounded whenever (s_n) is unbounded.

Now we apply our results to study the connectedness of the sets of limit points of the Cesàro, Hölder, de la Valleée Poussin, and Euler transforms of bounded sequences.

As a first application, we study the sets $\mathcal{L}(C_{\alpha}s)$ of limit points of the Cesàro transforms C_{α} of order $\alpha > -1$ of bounded sequences. We remark that the more general case for Cesàro transforms of complex order with positive real part was considered in [11, Section 6].

First we deal with the case $\alpha > 0$.

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Theorem 18.13. The set $\mathcal{L}(C_{\alpha}s)$ of each sequence $s = (s_n) \in \ell_{\infty}$ is connected when $\alpha > 0$.

Proof. We recall that the entries a_{nk} of the triangle of the C_{α} transform for $\alpha > -1$ are given by Definition 7.1.

$$a_{nk} = \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}$$
 for $0 \le k \le n$ and $n = 0, 1, \dots$

We define the triangle $B = (b_{nk})_{n,k=0}^{\infty}$ by $B = \Delta(a_{nk})_{k=0}^{\infty}$, that is,

$$b_{nk} = \begin{cases} \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} - \frac{A_{n-k-1}^{\alpha-1}}{A_{n-1}^{\alpha}} & (0 \le k \le n-1) \\ \frac{1}{A_n^{\alpha}} & \text{for } n = 0, 1, \cdots . \end{cases}$$

First we observe that for $n \ge 2$ and $0 \le k \le n-1$ by (7.6) in Lemma 7.2

$$b_{nk} = \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \left(1 - \frac{A_n^{\alpha}}{A_{n-1}^{\alpha}} \cdot \frac{A_{n-k-1}^{\alpha-1}}{A_{n-k}^{\alpha-1}} \right) = \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \left(1 - \frac{(n+\alpha)(n-k)}{n(n-k+\alpha)} \right)$$
$$= \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \cdot \frac{1}{n(n-k+\alpha)} \left(n(n-k+\alpha) - (n+\alpha)(n-k) \right)$$
$$= \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \cdot \frac{n\alpha - \alpha(n-k)}{n(n-k+\alpha)} = \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} \cdot \frac{\alpha k}{n(n-k+\alpha)} \ge 0.$$

Since trivially $b_{nn} \ge 0$ for all n, we obtain for $n \ge 2$ by (7.19) in the proof of Theorem 7.6 and (7.2) in Lemma 7.2

$$\sum_{k=0}^{n} |b_{nk}| = \sum_{k=0}^{n} b_{nk} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n-1} A_{n-k}^{\alpha-1} - \frac{1}{A_{n-1}^{\alpha}} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} + \frac{1}{A_n^{\alpha}}$$
$$= \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} - \frac{A_{n-1}^{\alpha}}{A_{n-1}^{\alpha}} = \frac{A_n^{\alpha}}{A_n^{\alpha}} - 1 = 0.$$

Now the conclusion follows by Theorem 18.7.

Theorem 18.14. Let $-1 < \alpha \leq 0$. Then the set $\mathcal{L}(C_{\alpha}s)$ of a sequence $(s_n) \in \ell_{\infty}$ need not be connected.

Proof. If $\alpha = 0$, then the Cesàro transform is the identity, and the set $\mathcal{L}((s_n^{(1)}))$ for the sequence $(s_n^{(1)}) \in \ell_{\infty}$ in (18.1) of Example 18.1 is not connected.

Let $-1 < \alpha < 0$. We define the sequence (σ_n^{α}) by $\sigma_n^{\alpha} = (-1)^n$ which has $\{-1, 1\}$ as its set of limit points. We obtain by the inverse formula (7.20) in Example 7.7, since $A_k^{\alpha}, A_{n-k}^{-\alpha-1} \ge 0$ for all $0 \le k \le n$ and $n = 0, 1, \ldots$ and $A_n^0 = 1$ for all n by

(7.1) and (7.3) in Lemma 7.2, respectively, and applying the formula in (7.19) in the proof of Theorem 7.6 with $\beta = 0$ and (7.2) in Lemma 7.2

$$|s_n| = \left| \sum_{k=0}^n A_{n-k}^{-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha} \right| \le \sum_{k=0}^n |A_{n-k}^{-\alpha-1} A_k^{\alpha}| = \sum_{k=0}^n A_{n-k}^{-\alpha-1} A_k^{\alpha} = A_n^0 = 1 \text{ for all } n.$$

Now we consider the sets $\mathcal{L}(H^m s)$ for the Hölder transforms of order $m \in \mathbb{N}$ of bounded sequences $s = (s_n)$.

Theorem 18.15. The set $\mathcal{L}(H^{(m)}s)$ of each sequence $(s_n) \in \ell_{\infty}$ is connected for every $m \in \mathbb{N}$.

Proof. As we know from Theorem 8.3 that all Hölder methods $H^{(m)}$ are equivalent on ℓ_{∞} , we may assume m = 1. But the matrix of the $H^{(1)}$ transform is equal to that of the C_1 transform by definition, hence the conclusion follows by Theorem 18.13.

The de la Vallée Poussin method is usually defined by its series to sequence transformation. So it is useful to express the entries of the matrix B in terms of the matrix of the series to sequence transformation.

Lemma 18.16. Let the series to sequence transformation of a series $\sum_{k=0}^{\infty} u_k$ be given by

$$t_n = \sum_{k=0}^n d_{nk} u_k \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots$$

If A denotes the matrix of the corresponding sequence to sequence transformation and $B = (b_{nk})_{n,k}^{\infty}$ is the matrix with $b_{nk} = b_{nk} - b_{n-1,k}$ for all n and k, then we have

$$b_{nk} = \begin{cases} d_{nk} - d_{n,k+1} - d_{n-1,k} + d_{n-1,k+1} & (0 \le k \le n-2) \\ d_{n,n-1} - d_{nn} - d_{n-1,n-1} & (k = n-1) \\ d_{nn} & (k = n) \\ 0 & (k > n) \end{cases} \quad for \ n = 0, 1, \dots;$$

(here and elsewhere we use the convention that every term with a negative subscript is equal to zero).

Proof. Let $(s_n)_{n=0}^{\infty}$ denote the sequence of the partial sums $s_n = \sum_{k=0}^n u_k$ for $n = 0, 1, \ldots$ Then $u_k = s_k - s_{k-1}$ for $k = 0, 1, \ldots$ and we obtain

$$t_n = \sum_{k=0}^n d_{nk} u_k = \sum_{k=0}^n d_{nk} (s_k - s_{k-1}) = \sum_{k=0}^n d_{nk} s_k - \sum_{k=1}^n d_{nk} s_{k-1}$$
$$= \sum_{k=0}^{n-1} d_{nk} s_k + d_{nn} s_n - \sum_{k=0}^{n-1} d_{n,k+1} s_k = \sum_{k=0}^{n-1} (d_{nk} - d_{n,k+1}) s_k + d_{nn} \text{ for } n = 0, 1, \dots$$

Hence the sequence to sequence transformation is given by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ with

(18.19)
$$a_{nk} = \begin{cases} d_{nk} - d_{n,k+1} & (0 \le k \le n-1) \\ d_{nn} & (k=n) \end{cases} \text{ for } n = 0, 1, \dots$$

We obtain

$$b_{nk} = a_{nk} - a_{n-1,k} = \begin{cases} a_{nn} & \text{for } k = n \\ a_{n,k} - a_{n-1,k-1} = \\ d_{nk} - d_{n,k+1} - d_{n-1,k} + d_{n-1,k+1} & \text{for } 0 \le k \le n-1, \end{cases}$$

where $d_{n-1,n} = 0$ for k = n - 1, since $D = (d_{nk})_{n,k=0}^{\infty}$ is a triangle.

Now we consider the *de la Vallée Poussin* transformation defined by the matrix $D = (d_{nk})_{n,k=0}^{\infty}$ with

$$d_{nk} = \frac{(n!)^2}{(n-k)!(n+k)!}$$
 for $0 \le k \le n$ and $n = 0, 1, \dots$

for the series to sequence transformation.

First we observe that the de la Vallée Poussin method is regular.

Theorem 18.17. The de la Vallée Poussin method is regular.

Proof. We obtain for the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of the sequence to sequence transformation of the de la Vallée Poussin method by (18.19)

$$a_{nn} = d_{nn} = \frac{(n!)^2}{(2n!)} > 0$$

and $a_{nk} \ge 0$ for $0 \le k \le n-1$, since

$$\frac{d_{n,k+1}}{d_{nk}} = \frac{(n!)^2}{(n-k-1)!(n+k+1)!} \cdot \frac{(n-k)!(n+k)!}{(n)!^2} = \frac{n-k}{n+k+1} < 1.$$

Hence $a_{nk} > 0$ for all $k \leq n$, and so

(18.20)
$$\sum_{k=0}^{n} |a_{nk}| = \sum_{k=0}^{n} a_{nk} = \sum_{k=0}^{n-1} (d_{nk} - d_{n,k+1}) + d_{nn} = d_{n0} = 1,$$

that is, the conditions in (i') and (iii') in Part (c) of Theorem 4.3 are satisfied. We fix $k \in \mathbb{N}_0$. Then we have for all n > k

$$a_{nk} = d_{nk} - d_{n,k+1} = \frac{(n!)^2}{(n-k)!(n+k)!} - \frac{(n!)^2}{(n-k-1)!(n+k+1)!}$$
$$= \frac{(n!)^2}{(n-k)!(n+k+1)!} \cdot ((n+k+1) - (n-k))$$

$$= (2k+1) \cdot \frac{n!}{(n-k)!} \cdot \frac{n!}{(n+k+1)!} =$$

$$= (2k+1) \cdot \frac{n(n-1)\cdots(n-k+1)}{(n+k+1)(n+k)\cdots(n+1)}$$

$$= (2k+1) \cdot \frac{1}{n+k+1} \frac{n(n-1)\cdots(n-k+1)}{(n+k)\cdots(n+1)} \le \frac{2k+1}{n+k+1} \to 0 \text{ as } n \to \infty,$$

that is, the condition in (ii') in Part (c) of Theorem 4.3 is also satisfied. $\hfill\square$

Theorem 18.18. The set of limit points of the de la Vallée Poussin transform of each bounded sequence is connected.

Proof. (i) First, we show
(18.21)

$$b_{nk} = \begin{cases} \frac{((n-1)!)^2(2k+1)(k^2+k-n)}{(n-k)!(n+1+k)!} & (0 \le k < n-1) \\ \frac{((n-1)!)^2(n-2)}{2(2n-2)!} & (k=n-1) & \text{for } n = 0, 1, \dots, \\ \frac{(n!)^2}{(2n)!} & k = n \end{cases}$$

We apply (18.18) to the entries d_{nk} of the de la Vallée Poussin method and obtain for k = n

$$b_{nn} = d_{nn} = \frac{(n!)^2}{(2n)!},$$

for k = n - 1

$$\begin{split} b_{n,n-1} &= d_{n,n-1} - d_{nn} - d_{n-1,n-1} \\ &= \frac{(n!)^2}{(2n-1)!} - \frac{(n!)^2}{(2n)!} - \frac{((n-1)!)^2}{(2n-2)!} \\ &= \frac{((n-1)!)^2}{(2n)!} \left(2n \cdot n^2 - n^2 - 2n(2n-1) \right) \\ &= \frac{((n-1)!)^2n}{(2n)!} \left((2n-1)(n-2) \right) = \frac{((n-1)!)^2(2n-1)2n(n-2)}{2(2n)!} \\ &= \frac{((n-1))!^2(n-2)}{2(2n-2)!}, \end{split}$$

and for $k \leq n-2$

$$b_{nk} = d_{nk} - d_{n,k+1} - d_{n-1,k} + d_{n-1,k+1} = c_{nk} - c_{n-1,k}, \text{ where}$$

$$c_{nk} = d_{nk} - d_{n,k+1} = \frac{(n!)^2}{(n-k)!(n+k)!} - \frac{(n!)^2}{(n-k-1)!(n+k+1)!}$$

$$= \frac{(n!)^2}{(n-k)!(n+k+1)!} ((n+k+1) - (n-k))$$

$$= \frac{(n!)^2}{(n-k)!(n+k+1)!}(2k+1),$$

$$c_{n-1,k} = \frac{((n-1)!)^2}{(n-k-1)!(n+k)!}(2k+1)$$

and

$$b_{nk} = (2k+1) \cdot \left(\frac{(n!)^2}{(n-k)!(n+k+1)!} - \frac{((n-1)!)^2}{(n-k-1)!(n+k)!}\right)$$
$$= (2k+1) \cdot \frac{((n-1)!)^2}{(n-k)!(n+k+1)!} \left(n^2 - (n-k)(n+k+1)\right)$$
$$= \frac{((n-1)!)^2}{(n-k)!(n+k+1)!} \cdot (2k+1)(k^2+k-n).$$

Thus we have shown (18.21).

(ii) Now we show

(18.22)
$$\lim_{n \to \infty} \sum_{k=0}^{n} |b_{nk}| = 0.$$

It follows from (18.21) that $b_{nk} \leq 0$ for $k \leq (1/2)(-1 + \sqrt{1+4n})$. We put

$$m = \left[\frac{1}{2}\left(-1 + \sqrt{1+4n}\right)\right] = \max\left\{l \in \mathbb{N} : l \le \frac{1}{2}\left(-1 + \sqrt{1+4n}\right)\right\}.$$

Then $b_{nk} \leq 0$ for $k \leq m$ and $b_{nk} > 0$ for k > m. Since by (18.20)

$$\sum_{k=0}^{n} b_{nk} = \sum_{k=0}^{n} a_{nk} - \sum_{k=0}^{n-1} a_{n-1,k} = d_{n,0} - d_{n-1,0} = 0 \text{ for all } n \ge 1,$$

we obtain

$$\sum_{k=0}^{n} |b_{nk}| = \sum_{k=0}^{m} (-b_{nk}) + \sum_{k=m+1}^{n} b_{nk} = \sum_{k=0}^{m} (-b_{nk}) + \sum_{k=0}^{n} b_{nk} - \sum_{k=0}^{m} b_{nk}$$
$$= 2\sum_{k=0}^{m} (-b_{nk}) = 2\sum_{k=0}^{m} (a_{n-1,k} - a_{nk})$$
$$= 2\left(\sum_{k=0}^{m} (d_{n-1,k} - d_{n-1,k+1}) - \sum_{k=0}^{m} (d_{nk} - d_{n,k+1})\right)$$
$$= 2(d_{n-1,0} - d_{n-1,m+1} - d_{n,0} + d_{n,m+1}) = d_{n,m+1} - d_{n-1,m+1}$$
$$= 2\left(\frac{(n!)^2}{(n-m-1)!(n+m+1)!} - \frac{((n-1!))^2}{(n-m-2)!(n+m)!}\right)$$

$$= 2\left(\frac{((n-1!))^2}{(n-m-1)!(n+m+1)!}\left(n^2 - (n-m-1)(n+m+1)\right)\right)$$

= $2\left(\frac{(n-1)\cdots(n-m+1)}{(n+m-1)\cdots n}(m+1)^2\right)$
 $\leq \frac{1}{2n}\left(1 + \sqrt{1+4n}\right)^2 \to 0 \text{ as } n \to \infty.$

Finally, we consider the Euler transforms E_q for q > 0, given by the triangle $A = (a_{nk})_{k,n=0}^{\infty}$ with

$$a_{nk}(q) = \frac{1}{(q+1)^n} \binom{n}{k} q^{n-k}$$
 for $0 \le k \le n$ and $n = 0, 1, \dots$ (Definition 9.1).

We put r = 1/(q+1) for q > 0, that is, 0 < r < 1, and obtain q = 1/r - 1 = (1-r)/r, q+1 = 1/r and

$$a_{nk}(r) = r^n \binom{n}{k} \left(\frac{1-r}{r}\right)^{n-k} = \binom{n}{k} r^k (1-r)^{n-k} \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots$$

We write $E^{(r)}$ for the transform defined by the matrix A(r). Sine the methods E_q are regular for q > 0 by Theorem 9.2, so are the methods $E^{(r)}$ for 0 < r < 1.

In the proof of the next theorem, we need Stirling's well-known formula

(18.23)
$$\lim_{n \to \infty} \frac{n! \,\mathrm{e}^n}{n^n \sqrt{2\pi n}} = 1.$$

Theorem 18.19. The set $\mathcal{L}(E^{(r)}s)$ is connected for each sequence (s_n) when r > 0. *Proof.* (i) First we show

(18.24)
$$b_{nk} = \begin{cases} -r(1-r)^{n-1} & (k=0) \\ \binom{n-1}{k-1} r^k (1-r)^{n-k-1} \cdot \frac{k-rn}{k} & (1 \le k \le n-1) \text{ for } n = 0, 1, \dots, \\ r^n & (k=n) \end{cases}$$

Writing $a_{nk} = a_{nk}(r)$, for short, we obtain the entries b_{nk} of the matrix $B = \Delta(a_{nk})_{k=0}^{\infty}$ as follows:

$$b_{n0} = (1-r)^n - (1-r)^{n-1} = (1-r)^{n-1}(1-r-1)$$

= $-r(1-r)^{n-1} < 0$ for all $n \ge 2$,
 $b_{nn} = a_{nn} = r^n$,
 $b_{nk} = a_{nk} - a_{n-1,k} = \binom{n}{k} r^k (1-r)^{n-k} - \binom{n-1}{k} r^k (1-r)^{n-k-1}$
= $r^k (1-r)^{n-k-1} \binom{n-1}{k-1} c_{nk}$,

where

$$c_{nk} = \frac{\binom{n}{k}}{\binom{n-1}{k-1}}(1-r) - \frac{\binom{n-1}{k}}{\binom{n-1}{k-1}}.$$

It follows from

$$\frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n\cdots(n-k+1)}{k!} \cdot \frac{(k-1)!}{(n-1)\cdots(n-k+1)} = \frac{n}{k},$$
$$\frac{\binom{n-1}{k}}{\binom{n-1}{k-1}} = \frac{(n-1)\cdots(n-k)}{k!} \cdot \frac{(k-1)!}{(n-1)\cdots(n-k+1)} = \frac{n-k}{k},$$

that

$$c_{nk} = \frac{1}{k} \left(n(1-r) - (n-k) \right) = k - nr,$$

and so

$$b_{nk} = r^k (1-r)^{n-k-1} {\binom{n-1}{k-1}} \cdot \frac{k-rn}{k}$$
 for $1 \le k \le n-1$.

Thus we have established (18.24).

(ii) Now we show

(18.25)
$$S_n = \sum_{k=0}^n |b_{nk}|^2 \cdot a_{n-1,m} = \binom{n-1}{m} (1-r)^{n-1-m} r^m \text{ for } n = 0, 1, \dots,$$

where $m = [rn] = \max\{j \in \mathbb{N}_0 : j \leq rn\}$. It follows from (18.24) that $b_{nk} < 0$ if and only if $k \leq nr$. Since

$$\sum_{k=0}^{l} a_{lk} = \sum_{k=0}^{l} \binom{l}{k} (1-r)^{l-k} r^{k} = ((1-r)+r)^{l} = 1 \text{ for all } l,$$

we obtain $\sum_{k=0}^{n} b_{nk} = 0$. Therefore, we have

$$\sum_{k=0}^{n} |b_{nk}| = \sum_{k=0}^{m} (-b_{nk}) + \sum_{k=m+1}^{n} b_{nk} = \sum_{k=0}^{m} (-b_{nk}) + \sum_{k=0}^{n} b_{nk} + \sum_{k=0}^{m} (-b_{nk})$$
$$= 2 \cdot \sum_{k=0}^{m} (a_{n-1,k} - a_{nk})$$

It follows from

$$\frac{a_{nk}}{a_{n-1,k-1}} = \frac{\binom{n}{k}}{\binom{n-1}{k-1}} \cdot \frac{(1-r)^{n-k}r^k}{(1-r)^{n-1-(k-1)}r^{k-1}}$$
$$= r \cdot \frac{n \cdots (n-k+1)}{k!} \cdot \frac{(k-1)!}{(n-1) \cdots (n-1-(k-1)+1)}$$
$$= r \cdot \frac{n}{k} \ge 1 \text{ for } k \le rn$$

that

$$\sum_{k=0}^{n} |b_{nk}| = 2 \cdot \sum_{k=0}^{m} (a_{n-1,k} - a_{nk}) \le 2 \cdot \sum_{k=0}^{m} (a_{n-1,k} - a_{n,k-1})$$
$$= 2 \cdot a_{n-1,m} = \binom{n-1}{m} (1-r)^{n-1-m} r^{m}.$$

Thus we have established (18.25).

(iii) Now we show that there exist constants M and M' such that

(18.26)
$$S_n \leq \begin{cases} M \cdot \frac{(n-1)^{n-1/2}}{\left(n-\frac{1}{r}\right)^n} & 0 < r \le \frac{1}{2} \\ M' \cdot \frac{(n-1)^{n-1/2}}{\left(n-\frac{1}{1-r}\right)^n} & \frac{1}{2} < r < 1 \text{ for sufficiently large } n. \end{cases}$$

First let 0 < r < 1/2. Applying Stirling's formula (18.23), we obtain that there exists a constant K such that for all sufficiently large n

$$\binom{n-1}{m} = \frac{(n-1)!}{m!(n-m-1)!}$$

$$\leq K \cdot \frac{(n-1)^{n-1}\sqrt{n-1} e^{-(n-1)}}{m^m \sqrt{m} e^{-m}(n-m-1)^{n-m-1}\sqrt{n-m-1} e^{-(n-m-1)}}$$

$$= K \cdot \frac{(n-1)^{n-1/2}}{m^m \sqrt{m}(n-m-1)^{n-m-1}\sqrt{n-m-1}},$$

and so for all sufficiently large n

(18.27)
$$S'_n = S_n \cdot \frac{\left(n - \frac{1}{r}\right)^n}{(n-1)^{n-1/2}} \le K \cdot \frac{\left(n - \frac{1}{r}\right)^n \cdot (1-r)^{n-1-m} \cdot r^{m+1}}{m^m \sqrt{m(n-m-1)^{n-m-1}} \sqrt{n-m-1}}.$$

It follows from $nr - 1 \le m \le nr$ that

$$S'_n \le K \cdot \frac{\left(n - \frac{1}{r}\right)^n \cdot (1 - r)^{n-1-m} \cdot r^{m+1}}{(nr - 1)^{m+1/2} \cdot (n(1 - r) - 1)^{n-m-1/2}} = K \cdot \left(\frac{nr - 1}{(n(1 - r) - 1)}\right)^{n-m-1/2} \cdot \left(\frac{1 - r}{r}\right)^{n-1-m}$$
for all sufficiently large n .

Now $r \leq 1/2$ implies $(r-1)/r \leq 1$, $2nr \leq n$, hence $nr \leq n(1-r)$ and $n-1-m \geq n-1-nr = (n-r)-1 \geq n/2-1 \geq 0$ for $n \geq 2$. Thus there the last term is bounded, and we have established the first estimate in (18.26).

If 1/2 < r < 1, then 0 < r' = 1 - r < 1/2 and the second estimate in (18.26) follows with r replaced by r'. This we have established (18.26).

Finally if r < 1/2, then (n - 1/r) = n - 1 - (1 - r)/r and we obtain from (18.26)

$$S_n \le \frac{1}{\sqrt{n-1}} \cdot \left(\frac{1}{1 - \frac{1-r}{(n-1)r}}\right)^n$$
$$\lim_{n \to \infty} \left(\frac{1}{1 - \frac{1-r}{(n-1)r}}\right)^{n-1} \cdot \frac{1}{1 - \frac{1-r}{(n-1)r}} = e^{\frac{1-r}{r}}$$

implies $\lim_{n\to\infty} S_n = 0.$

Similarly we obtain $\lim_{n\to\infty} S_n = 0$ for 1/2 < r < 1. Now the conclusion follows from Theorem 18.7.

19 Matrix transformations and fixed point iterations

Matrix transformations play an important role in fixed point theory. We start with the definition of the concept of a fixed point.

Definition 19.1. Let X be a non-empty set and $f: X \to X$ be a function. Then $x_0 \in X$ is called a fixed point of f if $f(x_0) = x_0$. The set of all fixed points of f is denoted by F(f).

Fixed point theory is a major branch of nonlinear functional analysis because of its wide applicability. Numerous questions in physics, chemistry, biology, and economics lead to various nonlinear differential and integral equations.

We start our studies with Brouwer's famous fixed point theorem.

Theorem 19.2 (Brouwer's fixed point theorem). ([15])

Every continuous map from the closed unit ball of \mathbb{R}^n into itself has a fixed point.

One cannot expect uniqueness of the fixed point in Brouwer's theorem, in general.

An important generalization of Brouwer's fixed point theorem was obtained by Schauder.

Theorem 19.3 (Schauder's fixed point theorem). ([106])

Every continuous map from a non-empty, compact and convex subset C of a Banach space X into C has a fixed point.

Clearly the conditions in the hypothesis are preserved if the norm of X is replaced by an equivalent norm. Schauder's fixed point theorem can be used to prove Peano's existence theorem for the solution of systems of first order ordinary differential equations with initial conditions.

The continuous function $f : [0,1] \to [0,1]$ with f(x) = -x for $x \in [0,1]$ has a unique fixed point 0. The *Picard iteration sequence* with $(f^n(x_0))$ diverges for all initial values $x_0 \neq 0$.

The Mann iterations are more general than the Picard iterations, that is, the Picard iterations are special cases of the Mann iterations which Mann introduced in his paper [96] in 1953.

Let C be a convex compact subset of a Banach space X, and $T: C \to C$ be a continuous map. By Schauder's fixed point theorem [106], there exists at least one fixed point of the function T, that is, there exists $p \in C$ such that T(p) = p.

In 1953, Mann ([96]) studied the problem of constructing a sequence (x_n) in C which converges to a fixed point of T. Usually an arbitrary initial value $x_1 \in C$ is chosen, and then the sequence of successive iterations (x_n) of x_1 defined by

(19.1)
$$x_{n+1} = T(x_n)$$
 for $n = 1, 2, ...$

is considered. If this sequences converges, then its limit is a fixed point of the function T.

Definition 19.4 (Dotson [46], Hillam [51]). We assume that the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ satisfies the conditions

- $(A_1) a_{nk} \ge 0 ext{ for all } k \le n ext{ and } a_{nk} = 0 ext{ for } k > n;$
- (A₂) $\sum_{k=1}^{n} a_{nk} = 1$ for each $n \ge 1$;
- (A₃) $\lim_{n \to \infty} a_{nk} = 0$ for each $k \ge 1$.

We define the sequence (x_n) by $x_{n+1} = T(v_n)$, where

$$v_n = \sum_{k=1}^n a_{nk} x_k.$$

The sequence (x_n) is called the Mann iterative sequence, or simply, Mann iteration, and usually denoted by $M(x_1, A, T)$.

The conditions in (A_1) and (A_2) are necessary for $x_n, v_n \in C$. The matrix A in Definition 19.4 is said to be admissible. It is regular by the conditions in (i'), (ii') and (iii') in Part (c) of Theorem 4.3, lower triangular and has the following form

	1	0	0		0	0	
A =	a_{21}	a_{22}	0		0	0	
	•	•	•		•		
	a_{n1}	a_{n2}	• • •	a_{nn}	0	0	
	_ ·	•	•		•	•]	

Definition 19.5 (Hillam). ([51]) An infinite matrix A is said to be segmented if for n = 1, 2, ... the n^{th} and $(n + 1)^{\text{st}}$ rows are related as follows: (A_4) $a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk}$ for (k = 1, 2, ..., n; n = 1, 2, ...).

Definition 19.6 ([46]). The Mann iterative method $M(x_1, A, T)$ is referred to as the normal Mann iterative method if the matrix $A = [a_{nj}]$, besides the conditions in (A_1) , (A_2) , (A_3) and (A_4) , also satisfies

(A₅) either $a_{nn} = 1$ for all n, or $a_{nn} < 1$ for all n > 1.

Example 19.7. Let A be the Cesàro matrix of order 1 (Definition 2.1) In this case, the Mann method $M(x_1, A, T)$ is normal, and referred to as the *mean value method*, where the initial value is $x_1 \in C$ and

$$x_{n+1} = T(v_n)$$
 and $v_n = \frac{1}{n} \sum_{k=1}^n x_k$ for all $n = 1, 2, \dots$

We note

(19.2)
$$v_{n+1} - v_n = \frac{n \sum_{k=1}^{n+1} x_k - (n+1) \sum_{k=1}^n x_k}{(n+1)n} = \frac{T(v_n) - v_n}{n+1}.$$

In many special problems, the iterative method $M(x_1, A, T)$ converges even when the method $T^n x_1$ diverges.

Example 19.8. Let $C = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$, where $||\cdot||$ is the euclidean norm on $X = \mathbb{R}^2$. Furthermore, let A be the Cesàro matrix of order 1 and the function $T: C \to C$ be the rotation about the centre by the angle $\pi/4$. Then the Picard iteration $T^n(x_1)$ does not converge for any $x_1 \in C \setminus \{0\}$. Using Mann's method $M(x_1, A, T)$, the sequences (x_n) and (v_n) always converge (on a spiral) to the centre, independently of the choice of the initial value x_1 .

In his paper [46], Dotson proved the following theorem (see also Reinermann [101], Hillam [51] and Berinde[12]).

Theorem 19.9 (Dotson). ([46]) The following statements are true:

(a) The Mann method $M(x_1, A, T)$ is normal if and only if the matrix $A = (a_{nk})_{n,k=1}^{\infty}$ satisfies the conditions in (A_1) , (A_2) , (A_4) , (A_5) and (A'_3) , where

$$(A'_3) \qquad \qquad \sum_{n=1}^{\infty} a_{nn} \text{ is a divergent series.}$$

(b) The matrices $A = (a_{nk})_{n,k=1}^{\infty}$ (except for the identity matrix) in all normal Mann methods $M(x_1, A, T)$ are constructed as follows:

Let $0 \le c_n < 1$ for all n = 1, 2, ... and the series $\sum_{n=1}^{\infty} c_n$ be divergent. Then the matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is defined by

$$\begin{cases} a_{11} = 1, \ a_{1k} = 0 \ for \ k > 1; \\ a_{n+1,n+1} = c_n \ for \ n = 1, 2, \dots \\ a_{n+1,k} = a_{kk} \prod_{j=k}^n (1-c_j) \ for \ k = 1, 2, \dots, n \\ a_{n+1,k} = 0 \ for \ k > n+1 \ and \ n = 1, 2, \dots \end{cases}$$

(c) The sequence (v_n) in the normal Mann method $M(x_1, A, T)$ satisfies

(19.3)
$$v_{n+1} = (1 - c_n)v_n + c_n T(v_n)$$
 for $n = 1, 2, ...,$

where

(19.4)
$$c_n = a_{n+1,n+1}$$
 for all n

Proof. The statement in Part (a) follows from the following well-known result on infinite products, namely, that if $0 \le c_n < 1$ for all n, then $\lim_{n\to\infty} \prod_{k=1}^n (1-c_k) = 0$ if and only if the series $\sum_{k=1}^{\infty} c_k$ diverges.

To prove the statement in Part (b), we note that if the matrix A satisfies the conditions in (A1)–(A5), then it satisfies the condition in (b). It can be proved that if the matrix A satisfies the conditions in (b), where $c_n = a_{n+1,n+1}$ for all $n \in \mathbb{N}$, then it satisfies the conditions in (A1)–(A5).

The proof of Part (c) follows if we use the condition in (A4) and the definitions of the sequences (v_n) and (x_n) in Mann's method $M(x_1, A, T)$.

Example 19.10. For each λ with $0 \leq \lambda < 1$, let the infinite matrix $A_{\lambda} = (a_{nk})_{k=1,n=1}^{\infty}$ be defined by

$$\begin{cases} a_{n1} = \lambda^{n-1} \\ a_{nk} = \lambda^{n-k} (1-\lambda) \text{ for } k = 2, 3, \dots, n, \\ a_{nk} = 0 \text{ for } k > n \text{ and } n = 1, 2, 3, \dots, \end{cases}$$

where, for $\lambda = 0$, we put $a_{nn} = 1$ for all n. Hence A_0 is the infinite identity matrix. It can be shown that for each λ with $0 \leq \lambda < 1$, $M(x_1, A_{\lambda}, T)$ is a normal Mann method with $c_n = a_{n+1,n+1} = 1 - \lambda$ for all $n = 1, 2, 3 \dots$ Hence the sequence (v_n) in the normal Mann method $M(x_1, A_{\lambda}, T)$ is defined by

$$v_{n+1} = \lambda v_n + (1 - \lambda)T(v_n)$$
 for all n .

Let $S_{\lambda} = \lambda I + (1 - \lambda)T$ (where I is the identity map). Hence we have

$$v_{n+1} = S_{\lambda}(v_n) = S_{\lambda}^n(v_1) = S_{\lambda}^n(x_1)$$
 for all n .

We note that $S_0 = T$ and, in this case, the sequence (v_n) is obtained by Picard's iteration $(T^n(x_1))$. The sequence $(S_{1/2}^n(x_1))$ of Picard's iterations of the map $S_{1/2} = (1/2)(I+T)$ was studied by Krasnoselskii [57] and Edelstein [47], and the sequence $(S_{\lambda}^n(x_1))$ of Picard's iterations of the map S_{λ} for $0 < \lambda < 1$ was studied by Schäfer [105], Browder and Petryshyn [16], and Opial [98].

In the literature, mainly the normal Mann iterative method is studied. We continue with the next three results by Mann.

Theorem 19.11. ([96]) If one of the sequences (x_n) or (v_n) is convergent, then they both converge. In this case, they converge to the same limit point which is a fixed point of the function T.

Proof. Let $\lim_{n\to\infty} x_n = p$. Since A is a regular matrix, it follows that $\lim_{n\to\infty} v_n = p$. The continuity of the function T implies $\lim_{n\to\infty} T(v_n) = T(p)$, and from $T(v_n) = x_{n+1}$, it follows that T(p) = p. If we assume $\lim_{n\to\infty} v_n = q$, then $\lim_{n\to\infty} x_{n+1} = T(q)$, and the regularity of the matrix A implies $\lim_{n\to\infty} v_n = T(q)$. Hence we have T(q) = q.
If the sequences (x_n) and (v_n) are not convergent, then, since C is a compact set, each of the two sequences has at least two distinct limit points. We generalize the concept the set $\mathcal{L}(s)$ of limit points of a complex sequence at the beginning of Section 18 to sequences in arbitrary metric spaces.

Definition 19.12. Let $y = (y_n)$ be a sequence in the metric space (Y, d). Then $\mathcal{L}(y)$ is defined to be the set of all limit points of the sequence y, provided they exist.

Theorem 19.13. ([96]) If the matrix A satisfies the conditions in (A_1) , (A_2) , (A_3) and

(19.5)
$$\lim_{n \to \infty} \sum_{k=1}^{n} |a_{n+1,k} - a_{n,k}| = 0,$$

then $\mathcal{L}(x)$ and $\mathcal{L}(y)$ are closed and connected sets.

Proof. The set $\mathcal{L}(v)$ is closed and compact, and by (19.5), $\lim_{n\to\infty}(v_{n+1}-v_n)=0$. Hence the set $\mathcal{L}(v)$ is connected by [11, Theorem 4.3] (or Theorem 18.7). Since the function T is continuous and $\mathcal{L}(x) = T(\mathcal{L}(v))$, it follows that $\mathcal{L}(x)$ is a closed and connected set.

Theorem 19.14. ([96]) The set $\mathcal{L}(v)$ is a subset of $co(\mathcal{L}(x))$, where $co(\mathcal{L}(x))$ denotes the convex hull of the set $\mathcal{L}(x)$.

Proof. By Mazur's theorem [97], $\operatorname{co}(\mathcal{L}(x))$ is a closed set. All but finitely many terms of the sequence $x = (x_n)$ are elements of each open set that contains the set $\operatorname{co}(\mathcal{L}(x))$. Hence for all sufficiently large n, the terms v_n of the sequence $v = (v_n)$ are arbitrarily close to the the set $\mathcal{L}(x)$. Thus, the limit point of each convergent subsequence of the sequence v is an element of the set $\operatorname{co}(\mathcal{L}(x))$.

Now we consider the case when the Banach space is the real line \mathbb{R} , and the convex compact set C is a closed interval.

Theorem 19.15 (Mann). ([96]) Let $T : [a, b] \to [a, b]$ be a continuous map which has a unique fixed point $p \in [a, b]$ and A be the Cesàro matrix of order 1. Then Mann's sequence $M(x_1, A, T)$ converges to p for each $x_1 \in [a, b]$.

Proof. It follows from (19.2) that $v_{n+1} - v_n \to 0$ as $n \to \infty$. Since T is a continuous function and p is the unique fixed point of T, it follows that T(x) - x > 0 for x < p and T(x) - x < 0 for x > p. Hence, for each $\delta > 0$, there exists $\varepsilon > 0$ such that $|x - p| \ge \delta$ implies $|T(x) - x| \ge \varepsilon$. It follows from (19.2) that

$$v_{n+1} = v_1 + \sum_{k=1}^n \frac{T(v_k) - v_k}{k+1}.$$

Now from our previous considerations, we have $\lim_{n\to\infty} v_n = p$, and by Theorem 19.11, we obtain $\lim_{n\to\infty} x_n = p$.

In higher dimensional spaces, results similar to that of Theorem 19.15 have not been obtained.

Remark 19.16. In 1971, Franks and Marzec [48] showed that the condition of the uniqueness of the fixed point p in Theorem 19.15 is not necessary. In 1973, Hillam [51] extended those results to an arbitrary normal Mann method.

We note that any continuous function $f : [0,1] \to [0,1]$ has at least one fixed point by Brouwer's fixed point theorem.

Theorem 19.17 (Hillam). ([51]) Let C = [0, 1], $f : C \to C$ be a continuous map, the matrix A be defined by Theorem (19.9), and $\lim_{n\to\infty} c_n = 0$. Furthermore, let the iterative sequences $\tilde{x} = (\tilde{x}_n)$ and $x = (x_n)$ be generated as follows:

(19.6)
$$x_1 = \tilde{x}_1 \in [0, 1],$$

(19.7)
$$x_{n+1} = f(\tilde{x}_n) \text{ for } n = 1, 2, \dots$$

(19.8)
$$\tilde{x}_{n+1} = \sum_{k=1}^{n+1} a_{n+1,k} x_k \text{ for } n = 1, 2, \dots,$$

Then both sequences \tilde{x} and x converge to the same fixed point of f in the interval [0, 1].

Proof. It follows from (19.7), (19.8) and since A is segmented that

(19.9)
$$\tilde{x}_{n+1} = \tilde{x}_n + a_{n,n}(f(\tilde{x}_n) - \tilde{x}_n)$$
 for $n = 1, 2, \dots$

Since $\tilde{x}_n, f(\tilde{x}_n) \in [0, 1]$ for all n, we have

(19.10)
$$\tilde{x}_{n+1} - \tilde{x}_n \to 0 \text{ as } n \to \infty.$$

It suffices to prove that this sequence is convergent and its limit $\xi \in [0, 1]$ is a fixed point of the function f.

1. We prove that the sequence $\tilde{x} = (\tilde{x}_n)$ is convergent. The terms \tilde{x}_n of the sequence \tilde{x} are in [0, 1], and so \tilde{x} has at least one limit point. We assume that the sequence \tilde{x} has two distinct limit points ξ_1 and ξ_2 with $\xi_1 < \xi_2$.

1.a We are going to show that we have, by the assumption above, f(x) = x for all $x \in (\xi_1, \xi_2)$. Let $x^* \in (\xi_1, \xi_2)$. If $f(x^*) > x^*$, then, since f is a continuous function, there exists $\delta \in (0, (x^* - \xi_1)/2)$ such that $|x - x^*| < \delta$ implies f(x) > x. Hence $|\tilde{x}_n - x^*| < \delta$ implies $f(\tilde{x}_n) > \tilde{x}_n$. Thus we obtain from (19.9) that

(19.11)
$$|\tilde{x}_n - x^*| < \delta \text{ implies } \tilde{x}_{n+1} > \tilde{x}_n.$$

By (19.10), there exists N such that

(19.12)
$$|\tilde{x}_{n+1} - \tilde{x}_n| < \delta \text{ for } n = N, N+1, \dots$$

Since $\xi_2 > x^*$ is a limit point of the sequence \tilde{x} , we can choose N such that $\tilde{x}_N > \tilde{x}^*$. It follows from (19.11) and (19.12) that

$$\tilde{x}_n > x^* - \delta > \xi_1$$
 for $n = N, N + 1, \dots$

Thus ξ_1 is not a limit point of the sequence x, which contradicts our assumption. If $f(x^*) < x^*$, then, similarly as above, we obtain that ξ_2 is not a limit point of the sequence $\tilde{x} = (\tilde{x}_n)$, which again is a contradiction. Hence $f(x^*) = x^*$ for each $x^* \in (\xi_1, \xi_2)$.

1.b Let us prove that ξ_1 and ξ_2 are not limit points of the sequence $\tilde{x} = (\tilde{x}_n)$. We note that

(19.13)
$$\tilde{x}_n \notin (\xi_1, \xi_2)$$
 for $n = 1, 2, \dots$

If $f(\tilde{x}_n) = \tilde{x}_n$, then (19.9) implies $\tilde{x}_m = \tilde{x}_n$ for all m > n. So neither ξ_1 nor ξ_2 can be a limit point of the sequence (\tilde{x}_n) . Furthermore, (19.10) and (19.13) imply that there exists a natural number M such that $\tilde{x}_M \ge \xi_2$ for all n > M. Hence ξ is not a limit point of the sequence (\tilde{x}_n) . It follows from $\tilde{x}_M \le \xi_1$ that $\tilde{x}_n < \xi_1 < \xi_2$ for all n > M. Hence ξ_2 is not a limit point of the sequence \tilde{x} . Consequently the sequence \tilde{x} cannot have two distinct limit points, and so this sequence is convergent. We put $\lim_n \tilde{x}_n = \xi \in [0, 1]$.

2. We show $f(\xi) = \xi$. Since $\tilde{x}_n \to \xi$, we obtain by the continuity of f, $x_{n+1} = f(\tilde{x}_n) \to f(\xi)$. Since A is regular, the sequence $\tilde{x} = Ax$ converges to $f(\xi)$ and so $f(\xi) = \xi$.

We note that if $a_{nn} = 1/n$, then Theorem 19.15 is a special case of Theorem 19.17 .

In the next example, Hillam showed that the condition $\lim_{n\to\infty} c_n = 0$ in Theorem 19.17 is necessary for the sequences $\tilde{x} = (\tilde{x}_n)$ and $x = (x_n)$ to converge.

Example 19.18. ([51, Example 1.1]) Let $M \ge 1$ be given and A be an infinite triangular segmented matrix whose diagonal elements satisfy

$$a_{1,1} = 1$$
, $a_{nn} = 2/(M+1)$ for $n = 2, 3, 4, ...$

We define the function $f: [0,1] \to [0,1]$ by

(19.14)
$$f(x) = \begin{cases} 1 & 0 \le x < \frac{M-1}{2M} \\ \frac{M+1}{2} - Mx & \frac{M-1}{2M} \le x \le \frac{M+1}{2M} \\ 0 & \frac{M+1}{2M} < x \le 1. \end{cases}$$

Then f is a continuous function and has a unique fixed point at x = 1/2. We have by (19.9)

If

$$\tilde{x}_1 = x_1 = \frac{M+1}{2M},$$

then

(19.16)
$$\tilde{x}_n = \begin{cases} \frac{M-1}{2M}, & n \text{ odd} \\ \frac{M+1}{2M}, & n \text{ even.} \end{cases}$$

Thus the sequence \tilde{x} does not converge.

In [102], Rhoades conjectured the following.

Conjecture. Let $f : [a, b] \to [a, b]$ be a continuous function, A be a regular matrix which satisfies the conditions in (A_1) , (A_2) and (19.5). Then the iterative scheme defined by (19.6)-(19.8) converges to a fixed point of the function f.

In the next example, he showed that the assumption above does not hold if the condition in (19.5) is removed.

Example 19.19. Let A be be the identity matrix, [a, b] = [0, 1], f(x) = 1 - x and $x_1 = 0$.

Rhoades showed that the statement above is true for the large class of *weighted* means matrices.

The weighted means method is a triangular method of the matrix $A = (a_{nk})$ defined by $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \ge 0$ for n > 0, $P_n = \sum_{k=0}^n p_k$ and $P_n \to \infty$ as $n \to \infty$. Then the matrix A satisfies the condition in (19.15) if and only if $p_n/P_n \to 0$ as $n \to \infty$.

Theorem 19.20 (Rhoades). [104] Let A be the matrix of a regular weighted means method which satisfies the condition in (19.15). Let $f : [a, b] \rightarrow [a, b]$ be a continuous map. Then the iterative scheme (19.6)–(19.8) converges to a fixed point of the function f.

Proof. Without loss of generality, we may suppose that [a, b] = [0, 1]. Every regular weighted means method satisfies the conditions in (A_1) and (A_2) . By (19.8), we have interchanging the roles of x_n and \tilde{x}_n ,

(19.17)
$$\tilde{x}_{n+1} = \frac{p_{n+1}}{P_{n+1}} (f(\tilde{x}_n) - \tilde{x}_n) + \tilde{x}_n \text{ for all } n.$$

Since $\tilde{x}_n, f(\tilde{x}_n) \in [0, 1]$, it follows from (19.17) that

$$|\tilde{x}_{n+1} - \tilde{x}_n| \le \frac{p_{n+1}}{P_{n+1}} \to 0 \ (n \to \infty).$$

Now, by the proof of Theorem 19.17, the sequence $\tilde{x} = (\tilde{x}_n)$ is convergent.

We have to show that the sequence \tilde{x} converges to a fixed point of the function f. Let $z = \lim_{n \to \infty} \tilde{x}_n$. Then we have $\lim_{n \to \infty} f(\tilde{x}_n) = f(z)$. It follows from $x_{n+1} = f(\tilde{x}_n)$ for each $n \in \mathbb{N}$ that $\lim_{n \to \infty} x_n = f(z)$. Since A is a regular matrix, we obtain $z = \lim_{n \to \infty} \tilde{x}_n = \lim_{n \to \infty} A_n x = f(z)$.

In relation to Rhoades's conjecture, Hillam proved the next result.

Theorem 19.21 (Hillam). ([51, Proposition 7]) Let $f : [0,1] \to [0,1]$ be a continuous function, and let A denote the infinite regular lower triangular matrix satisfying the conditions (A₁), (A₂) and (19.5). Also let the sequences $\tilde{x} = (\tilde{x}_n) =$ and $x = (x_n)$ be generated by the formulae in (19.6)–(19.8). Then we have

(i) $\mathcal{L}(\tilde{x}) \subset \mathcal{L}(x);$

(ii) $\mathcal{L}(\tilde{x})$ and $\mathcal{L}(x)$ are closed and connected;

(iii) $\mathcal{L}(\tilde{x})$ contains at least one fixed point of f.

Proof. First we prove Part (ii). Since the sequence x is bounded, $\mathcal{L}(\tilde{x})$ is closed and connected by Barone's theorem, Theorem 18.7. Also $\mathcal{L}(x) = f(\mathcal{L}(\tilde{x}))$ is closed and connected by the continuity of f.

Now Part (i) follows from Theorem 19.14.

Now we prove (iii). If $\mathcal{L}(\tilde{x}) = \{x_0\}$, then x_0 is a fixed point of f by Theorem 19.11. So we now suppose that $\mathcal{L}(\tilde{x})$ contains more elements than one. Then we have by Part (ii) $\mathcal{L}(\tilde{x}) = [a, b]$ with a < b. We assume that $\mathcal{L}(\tilde{x})$ does not contain a fixed point of f. Without loss of generality let f(x) > x for all $x \in [a, b]$. Hence, $a \leq x < f(x)$ which implies that $a \notin \mathcal{L}(x)$, which contradicts Part (i).

The case f(x) < x follows similarly.

We close with the following remark.

Remark 19.22. In [51, Appendix 1], Hillam showed in a rather long example that Rhoades's conjecture is false, and that Part (iii) of Theorem 19.21 is *best possible*. Namely, he considered a special continuous function $f : [0, 1] \rightarrow [0, 1]$ with a unique fixed point x = 1/2. Then he showed for $\tilde{x}_1 = x_1 = 1$ that

$$\left[\frac{1}{4}, \frac{3}{4}\right] = \mathcal{L}((\tilde{x}_n)) \subset \mathcal{L}((x_n)) = [0, 1].$$

20 Applications in recent research

The most popular classical methods of summability studied in Sections 7–10 and 15 also play an important role in recent research.

As a first application, we mention the use of summability methods in fixed point theory beyond the results of Section 19. A summary of this topic can be found in the survey article [90] which includes results from the research papers [101, 102, 104, 103, 20].

The most important areas in modern summability, however, are the theories of matrix transformations, and, more recently of the study of compact bounded linear operators between BK spaces, which the topics in the previous sections are the absolutely essential basis for.

The famous theorems by Toeplitz and Schur (Theorems 4.3 and 5.2) that give necessary and sufficient conditions on the entries of an infinite matrix to map all convergent sequences into convergent sequences, and all bounded sequences into convergent sequences, respectively, give rise to the more general problem to characterize the classes (X, Y) of all infinite matrices that transform all sequences in a given sequence space X into a given sequence space Y. We presented the purely analytical method of the gliding hump in the proofs of Theorems 4.3 and 5.2. Modern summability uses functional analytical methods such as the uniform boundedness principle and the theories of FK, BK and AK spaces, which can successfully be applied in a great number of cases with the exception of characterizations of classes similar to that in Schur's theorem. We outlined the theory of FK, BK and AK spaces in [88] to the extend that enabled us to obtain the following known characterizations of matrix transformations between the classical sequence spaces ℓ_{∞} , c, c_0 and ℓ_p $(1 \le p < \infty)$ of bounded, convergent and null sequences and of absolutely p-summable series

Theorem 20.1. ([91, Theorem9.7.3]) Let $1 < p, r < \infty$, q = p/(p-1) and s = r/(r-1). Then the necessary and sufficient conditions for $A \in (X, Y)$ can be read from the following table

From X To Y	ℓ_{∞}	c_0	c	ℓ_1	ℓ_p
ℓ_{∞}	1.	2.	3.	4.	5.
c_0	6. (Thm 5.2 (a))	7.	8.	9.	10.
С	11. (Thm 5.2 (b))	12.	13.	14.	15.
ℓ_1	16.	17.	18.	19.	20.
ℓ_r	21.	22.	23.	24.	unknown

where

1., 2., 3. (1.1)
$$\sup_{n} \sum_{k=0}^{\infty} |a_{nk}| < \infty$$

4. (4.1) $\sup_{n,k} |a_{nk}| < \infty$
5. (5.1) $\sup_{n} \sum_{k=0}^{\infty} |a_{nk}|^{q} < \infty$
6. (6.1) $\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = 0$
7. (1.1) and (7.1), where (7.1) $\lim_{n \to \infty} a_{nk} = 0$ for every k
8. (1.1), (7.1) and (8.1), where (8.1) $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 0$
9. (4.1) and (7.1)
10. (5.1) and (7.1)
11. (11.1) and (11.2), where

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$$(11.1) \sum_{k=0}^{\infty} |a_{nk}| \text{ converges uniformly in } n$$

$$(11.2) \lim_{n \to \infty} a_{nk} = \alpha_k \text{ exists for every } k$$
12. (1.1) and (11.2)
13. (1.1), (11.2) and (13.1) where, (13.1) \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists}
14. (4.1) and (11.2)
15. (5.1) and (11.2)
16., 17., 18. (16.1), where (16.1)
$$\sup_{\substack{N \subseteq \mathbb{N}_0 \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} a_{nk} \right| \right) < \infty$$
19. (19.1)
$$\sup_{\substack{k \geq 0 \\ N \text{ finite}}} \sum_{n=0}^{\infty} |a_{nk}| < \infty$$
20. (20.1)
$$\sup_{\substack{N \subseteq \mathbb{N}_0 \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} a_{nk} \right|^q \right) < \infty$$
21., 22., 23. (21.1)
$$\sup_{\substack{k \leq \mathbb{N}_0 \\ K \text{ finite}}} \sum_{n=0}^{\infty} |a_{nk}|^r < \infty.$$

In view of the classical summability methods it is also of interest to characterize matrix transformations between the matrix domains in the classical sequence spaces, and the convergence domains of matrices (Definition 1.5). Since almost all classical matrix methods of summability are given by triangles T (Definition 6.1), it is of great interest to characterize matrix transformations between matrix domains of triangles in certain sequence spaces. We were able to reduce the characterizations of the classes $(X_T, Y_{\tilde{T}})$ for arbitrary triangles T and \tilde{T} , and FK spaces with AK as follows:

Theorem 20.2. ([62, Theorem 1]) Let X and Y be arbitrary subsets of ω and \tilde{T} be a triangle. Then $A \in (X, Y_{\tilde{T}})$ if and only if $C = \tilde{T}A \in (X, Y)$.

Theorem 20.3. ([88, Theorem 3.4]) Let X be an FK space with AK, T be a triangle, S be its inverse ([111, 1.4.8], [21, Remark 22 (a), p. 22]) and $R = S^t$, the transpose of S. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all n, where the matrices \hat{A} and $W^{(n)}$ are defined by

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj} s_{jk}$$
 for all n and k,

 $\mathbf{2}$

$$w_{mk}^{(n)} = \begin{cases} \sum_{j=m}^{\infty} a_{nj} s_{jk} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots).$$

Moreover, if $A \in (X_T, Y)$, then $\hat{A}x = A(Tx)$ for all $x \in X_T$.

Similar reductions also hold when $X = \ell_{\infty}$ and X = c.

Theorem 20.4. ([88, Remark 3.5])

- (a) The statement of Theorem 20.3 also holds for $X = \ell_{\infty}$.
- (b) Let Y be a linear subspace of ω . Then $A \in (c_T, Y)$ if and only if

$$\hat{A} \in (c_0, Y), \ W^{(n)} \in (c, c) \ for \ all \ n$$

and

$$\hat{A}e - (\alpha^{(n)})_{n=0}^{\infty} \in Y, \text{ where } \alpha^{(n)} = \lim_{m \to \infty} \sum_{k=0}^{m} w_{mk}^{(n)} \text{ for } n = 0, 1, \dots$$

Moreover, if $A \in (cT, Y)$, then

$$\hat{A}x = A(Tx) - \xi(\alpha^{(n)})_{n=0}^{\infty}$$
 for all $x \in c_T$, where $\xi = \lim_{k \to \infty} T_k x$.

For instance, Theorems 20.3 and 20.4 yield as an immediate consequence the characterizations of the classes (e_p^r, ℓ_∞) , (e_p^r, c_0) , (e_p^r, c) for $1 \le p \le \infty$ in [5], where $e_p^{(r)}$ is the matrix domain of the Euler matrix $E^{(r)} = (e_{nk}^r)_{n,k=0}^{\infty}$ (0 < r < 1) in ℓ_p with

$$e_{nk}^{(r)} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k & (0 \le k \le n) \\ 0 & (k > n) \end{cases}, \ (n = 0, 1, \dots).$$

We note that putting q = 1/r - 1, we obtain the Euler matrix E_q of Definition 9.1.

There are a great number of recent research papers by various authors that characterize classes of matrix transformations on special matrix domains of triangles in different sequence spaces.

Remark 20.5. We characterized matrix transformations on matrix domains of special triangles T in BK spaces, for instance, for $T = \Sigma$, the matrix of the partial sums, in [65], for $T = \Delta$, the matrix of the first oder differences in [65, 63], for $\Delta^{(m)}$, the matrix of the m^{th} order differences in [88, Section 3.4] and [82], the matrix domains of matrices of differences in the FK spaces

$$\ell(p) = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty \right\} \text{ and } c_0(p) = \left\{ x \in \omega : \lim_{k \to 0} |x_k|^{p_k} = 0 \right\}$$

for positive bounded sequences $p = (p_k)_{k=0}^{\infty}$ in [74, 75, 73], and matrix transformations on spaces of weighted means [88, Sections 3.5 and 3.6] and [77, 54, 23, 52]. Some other types of sequence spaces are those of strongly summable and bounded sequences, A-stongly convergent and bounded sequences, mixed norm and mixed paranorm spaces spaces. Results on the characterizations of matrix transformations on these spaces, some Banach algebras and applications to the solvability of infinite systems of linear equations in those spaces can be found in [88, Sections 3.7 and 3.8] and [53, 55, 54, 93, 18, 6, 7, 94, 95, 42, 79, 25, 26, 44, 45, 81, 72, 80, 36, 56, 32, 33] and in the survey paper [71].

There are also results by the authors concerning various related topics in summability such as statistical convergence in [30, 31], Hardy's inequality in [29]

Another interesting and more recent topic in modern summability is the characterization of classes of compact bounded linear operators between BK spaces by the use of the Hausdorff measure of noncompactness. This approach was first developed and applied on a large scale in 2000 in [85], where the necessary general theory of measures of noncompactness was outlined in Chapter 2, and the applicability in the characterization of compact operators between BK spaces was demonstrated in various cases of interest. Ever since a great number of papers on this subject have been published by various authors.

The theoretical background on measures of noncompactness can be found, for instance, in the monographs [1, 10, 109, 49] and text books [100, 85]. It is also worth mentioning the monograph which contains a comprehensive recent survey [9] on the use measures of noncompactness in fixed point theory and in the fields of differential and integral equation, and in particular, on the applications of the Hausdorff measure of noncompactness in the characterization of compact linear operators between BK spaces in [89] which is Chapter 3 in [9].

The first measure of noncompactness, denoted by α , was introduced by Kuratowski [59] in 1930. In 1955, G. Darbo [22] used the function α to prove his famous fixed point theorem which is a generalization of Schauder's fixed point theorem to continuous α -contractive self-mappings between non-empty convex, bounded and closed subsets of Banach spaces.

The Hausdorff or ball measure of noncompactness, denoted by χ , was first introduced by Goldenštein, Gohberg and Markus [113] in 1957, and later studied by Goldenštein and Markus [114] in 1965.

We recall that a measure of noncompactness in a set function ϕ on the class \mathcal{M}_X of all bounded sets in a complete metric space (X, d) into the set of non-negative real numbers which is regular, invariant under closure and semi-additive, that is, it satisfies the following conditions for all sets $Q, Q_1, Q_2 \in \mathcal{M}_X$

(i)
$$\phi(Q) = 0$$
 if and only if Q is relatively compact,
(ii) $\phi(\overline{Q}) = \phi(Q)$ and (iii) $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}.$

We mention one of the most important properties measures of noncompactness ϕ have, namely *Cantor's generalized intersection property*, which states that the intersection of a decreasing sequence (Q_n) of non-empty closed bounded subsets of a complete metric space with $\lim_{n\to\infty} \phi(Q_n) = 0$ is a non-empty compact set.

In the special case of Banach spaces X, some measures of noncompactness ψ

have some additional properties related to the linear structure of normed spaces such as sublinearity, absolute homogeneity, translation invariance and the invariance under the passage to the convex hull, that is, such a measure of noncompactness satisfies the following conditions for all sets $Q, Q_1, Q_2 \in \mathcal{M}_X$ and all scalars λ

(iv)
$$\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2)$$
, (v) $\psi(\lambda Q) = |\lambda|\psi(Q)$,
(vi) $\psi(x + Q) = \psi(Q)$ and (vii) $\psi(\operatorname{co}(Q)) = \psi(Q)$,

where co(Q) denotes the convex hull of the set Q. We remark that both the Kuratowski and Hausdorff measures satisfy Cantor's generalized intersection property and the invariance under passage to the convex hull, which are essential in the proofs of Darbo's fixed point theorem and its generalization, the Darbo–Sadovskiĭ fixed point theorem [115] of 1972.

We also note that the properties (i)-(vii) are included as axioms for measures of noncompactness in Banach spaces, for instance in [10, 1].

The Hausdorff measure of the closed unit ball in an infinite dimensional Banach space is well known and equal to 1 ([85, Theorem 2.12]).

The crucial result on the Hausdorff measure of noncompactness for our research is the following.

Theorem 20.6 (Goldenštein, Gohberg, Markus). ([113] [85, Theorem 2.23]) Let X be a Banach space with a Schauder basis (b_k) . Then the function $\mu : \mathcal{M}_X \to [0, \infty)$ defined by

$$\mu(Q) = \limsup_{n \to \infty} \left(\sup_{x \in Q} \left\| \mathcal{R}_n(x) \right\| \right)$$

with

$$\mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k \text{ for all } x = \sum_{k=0}^{\infty} \lambda_k x_k \in X$$

satisfies the following inequality

$$\frac{1}{L} \cdot \mu(Q) \le \chi(Q) \le \inf_n \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \le \mu(Q) \text{ for all } Q \in \mathcal{M}_X,$$

where $L = \limsup_{n \to \infty} \|\mathcal{R}_n\|$ is the basis constant.

We also need the concept of the measure of noncompactness of an operator and some useful results.

Definition 20.7. ([85, Definition 2.24]) Let ϕ_1 and ϕ_2 be measures of noncompactness on the Banach spaces X and Y, respectively. An operator $L: X \to Y$ is said to be (ϕ_2, ϕ_2) -bounded, if

$$L(Q) \in \mathcal{M}_Y$$
 for all $Q \in \mathcal{M}_X$

and there exists a real constant C with $C \geq 0$ such that

$$\phi_2(L(Q)) \le C \cdot \phi_1(Q) \text{ for all } \in \mathcal{M}_X.$$

If an operator L is (ϕ_1, ϕ_2) -bounded, then the number

$$||L||_{\phi_1,\phi_2} = \inf\{C \ge 0 : \phi_2(L(Q)) \le C \cdot \phi_1(Q) \text{ for all } \in \mathcal{M}_X\}$$

is called the (ϕ_1, ϕ_2) -measure of noncompactness of L, or simply measure of noncompactness of L. If $\phi_1 = \phi_2 = \phi$, then we write $\|L\|_{\phi} = \|L\|_{\phi,\phi}$, for short.

Theorem 20.8. ([85, Theorem 2.25]) Let X and Y be Banach spaces, L be a bounded linear operator from X into Y. Then

$$||L||_{\chi} = \chi(L(S_X)) = \chi(L(B_X)),$$

where $S_X = \{x \in X : ||x|| = 1\}$ and $B_X = \{x \in X : ||x|| < 1\}$ denote the unit sphere and open unit ball in X.

Theorem 20.9. ([85, Corollary 2.26]) Let X and Y be Banach spaces and L be a bounded linear operator from X into Y. Then

$$||L||_{\chi} = 0$$
 if and only if L is a compact operator

and

$$||L||_{\chi} \leq ||L||$$
, the usual operator norm of L.

An application of these results yields estimates or identities of the the Hausdorff measures of noncompactness of the matrix operators between the classical sequence spaces as in the table of Theorem 20.1 and the characterizations of the subclasses of compact matrix operators in the theorem with the single exception of the class of compact matrix operators from ℓ_1 into ℓ_{∞} .

We also obtained results on the Hausdorff measure and characterizations of compact matrix operators on the matrix domains of triangles in [85, Chapter 3], [3], in particular, between the spaces of sequences of m^{th} order differences in ℓ_{∞} , c and c_0 in [84], the spaces of null, convergent and bounded sequences of weighted means in [87, 78, 34, 44] and between spaces of sequences that are strongly bounded and convergent with index $p \geq 1$ by the Cesàro method of order one, and strongly μ convergent and bounded sequences in [83, 86, 64, 93, 92, 55, 93, 24, 42, 8, 41, 42, 69, 43, 2, 76], in mixed norm spaces [4, 45], in matrix domains of special triangles [27, 88, 37, 40, 77, 28], in matrix domains of general triangles [88, 44, 38, 35, 39, 68], and in mixed norm spaces [53, 45].

We also refer to the survey articles [17, 66, 67, 70, 71, 89] for further results.

Finally, results on the Hausdorff measures of noncompactness of general operators between certain BK spaces were obtained in [27, 2, 36, 45]. We mention that the characterization of the class of general compact bounded operators from the space of all convergent sequences into itself was applied to give a new proof of the classical result by Cohen and Dunford [19] that a regular matrix cannot be compact. The characterization of the class of general compact bounded operators from the space of all sequences, which are strongly C_1 summable with index $p \ge 1$, into the space of all convergent sequences was obtained in [2]. This characterization was used to prove a result similar to that of Cohen and Dunford, namely that those operators that preserve the limits cannot be compact. In the proof of Theorem 12.3, we applied two results from the theories of functions of bounded variations and the Riemann–Stieltjes integrals, namely Theorems A.13 and B.7. We list some of the basic results for functions of bounded variation and of Riemann–Stieltjes integrals in the first and second parts of the appendix.

A Functions of bounded variation

Throughout let [a, b] be a finite interval. Functions of *bounded variation* are functions which do not oscillate too much. They play an important role in the existence of *Riemann-Stieltjes integrals*.

Definition A.1. Let $f : [a, b] \to \mathbb{R}$ be a function and

$$P = \{x_0 = a < x_1 < \dots < x_n = b\}$$

be a partition of the interval [a, b]. We write

$$\bigvee(P;f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

The function f is said to be of bounded variation on [a, b] if

$$\bigvee_{a}^{b} f = \sup_{P} \bigvee(P; f) < \infty;$$

the class of all functions of bounded variation on [a, b] is denoted by bv[a, b]; $\bigvee_{a}^{b} f$ is called the *total variation of* f.

Monotone functions are of bounded variation.

Example A.2. If $f : [a, b] \to \mathbb{R}$ is monotone, then obviously $f \in bv[a, b]$.

Another class of functions of bounded variation is the class of functions that satisfy a *Lipschitz condition*.

Definition A.3. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy a *Lipschitz condition*, if there exists a constant M such that

$$|f(x) - f(y)| \le K|x - y| \text{ for all } x, y \in [a, b].$$

We say that f is of class LIP on [a, b] and denote this by $f \in \text{LIP}([a, b])$.

Example A.4. (a) If $f : [a, b] \to \mathbb{R}$ is differentiable on [a, b], and its derivative f' is bounded on [a, b], then $f \in \text{LIP}([a, b])$.

(b) If $f \in \text{LIP}([a, b])$, then $f \in \text{bv}[a, b]$.

Proof. (a) Let $x, y \in [a, b]$ be given. Since f' is bounded on [a, b], we can choose a constant M such that $|f'(t)| \leq M$ for all $t \in [a, b]$. The first mean value theorem of differentiation yield a $\xi \in (a, b)$ such that

$$|f(x) - f(y)| = |f'(\xi)| \cdot |x - y| \le M \cdot |x - y|,$$

whence $f \in \text{LIP}([a, b])$.

(b) Let $f \in \text{LIP}([a, b])$. Then there exists a constant M such that

$$|f(x) - f(y)| \le M \cdot |x - y|$$
 for all $x, y \in \text{LIP}([a, b])$.

Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of the interval [a, b]. Then we have

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \le M \sum_{k=0}^{n-1} |x_{k+1} - x_k| = M \cdot (b-a),$$

hence $\bigvee_{a}^{b} f \leq M \cdot (b-a)$, that is, $f \in bv[a, b]$.

The continuity of a function is neither a sufficient nor a necessary condition for it to be of bounded variation.

Example A.5. (a) Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin \frac{\pi}{x} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Then f is continuous, but $f \notin bv[0, 2]$.

(b) Let the $f:[1,1] \to \mathbb{R}$ be defined by f(x) = 0 for $x \in [-1,0]$ and f(x) = 1 for $x \in (0,1]$. Then f is discontinuous at x = 0, but $f \in bv[-1,1]$.

The next result states some properties of functions of bounded variation.

Theorem A.6. (a) If $f \in bv[a, b]$ then f is bounded on [a, b].

(b) Let $f, g \in bv[a, b]$ then $f \pm g \in bv[a, b]$ and $fg \in bv[a, b]$. If in addition $g(x) \ge \sigma > 0$ on [a, b] for some σ then $f/g \in bv[a, b]$.

(c) Let $f \in bv[a, b]$ and a < c < b. Then we have

$$\bigvee_{a}^{b} f = \bigvee_{a}^{c} f + \bigvee_{c}^{b} f.$$

Proof. (a) Let $f \in bv[a, b]$ and $x \in [a, b]$. Then we have

$$|f(x) - f(a)| + |f(b) - f(a)| \le \bigvee_{a}^{b} f \text{ and } |f(a) - f(b)| \le \bigvee_{a}^{b} f,$$

hence

$$|f(x)| \le \frac{1}{2} \cdot (|f(x) - f(a)| + |f(b) - f(x)| + |f(a) - f(b)| \le \bigvee_{a}^{b} f < \infty$$
 for all $x \in [a, b]$

so that f is bounded on [a, b].

(b) Let $P = \{x_0 < x_1 < \cdots < x_n = b\}$ be a partition of the interval [a, b] and $f, g \in bv[a, b]$. Then it follows that

$$\bigvee (P; f \pm g) = \sum_{k=0}^{n-1} |(f \pm g)(x_{k+1}) - (f \pm g)(x_k)|$$

$$\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

$$= \bigvee (P; f) + \bigvee (P; g) \leq \bigvee_{a}^{b} f + \bigvee_{a}^{b} g.$$

Since the partition P was arbitrary, we obtain

$$\bigvee_{a}^{b} (f \pm g) \le \bigvee_{a}^{b} f + \bigvee_{a}^{b} g.$$

Since $f,g\in \mathrm{bv}[a,b]$ implies that the functions f and g are bounded by Part (a), we have, putting

$$M_f = \sup_{x \in [a,b]} |f(x)|$$
 and $M_g = \sup_{x \in [a,b]} |g(x)|$,

$$\bigvee (P; f \cdot g) = \sum_{k=0}^{n-1} |(f \cdot g)(x_{k+1}) - (f \cdot g)(x_k)|$$

= $\sum_{k=0}^{n-1} |f(x_{k+1})(g(x_{k+1}) - g(x_k)) + g(x_k)(f(x_{k+1}) - f(x_k))|$
 $\leq M_f \cdot \bigvee (P; g) + M_g \cdot \bigvee (P; f) \leq M_f \cdot \bigvee_a^b g + M_g \cdot \bigvee_a^b f.$

Since the partition P was arbitrary, we obtain

$$\bigvee_{a}^{b} (f \cdot g) \le M_{f} \cdot \bigvee_{a}^{b} g + M_{g} \cdot \bigvee_{a}^{b} f,$$

hence $f \cdot g \in bv[a, b]$.

Finally, let $g \in bv[a, b]$ and $g(x) \ge \sigma > 0$ on [a, b] for some σ . Then we have

$$\bigvee \left(P; \frac{1}{g}\right) = \sum_{k=0}^{n-1} \left|\frac{1}{g(x_{k+1})} - \frac{1}{g(x_k)}\right| = \sum_{k=0}^{n-1} \frac{|g(x_{k+1}) - g(x_k)|}{|g(x_k)g(x_{k+1})|} \le \frac{1}{\sigma^2} \cdot \bigvee_a^b g.$$

Since the partition P was arbitrary, we obtain

$$\bigvee_{a}^{b} \left(\frac{1}{g}\right) \leq \frac{1}{\sigma^{2}} \cdot \bigvee_{a}^{b} g < \infty,$$

hence $1/g \in bv[a, b]$.

If $f, g \in bv[a, b]$ and $g(x) \ge \sigma > 0$ on [a, b] for some σ , then $1/g \in bv[a, b]$ by what we have just shown, and so $f/g = f \cdot (1/g) \in bv[a, b]$.

(c) It is obvious that

$$\bigvee_{a}^{c} f + \bigvee_{c}^{b} f \leq \bigvee_{a}^{b} f \text{ for any } c \in (a, b).$$

Let $\varepsilon > 0$ be given and

$$P_{\varepsilon} = \{x_0 = a < x_1 < \dots < x_{m-1} \le c < x_m < \dots < x_n = b\}$$

be a partition of the interval [a, b] such that

$$\bigvee_{a}^{b} f \leq \bigvee (P_{\varepsilon}; f) + \varepsilon.$$

Then we obtain

$$\bigvee_{a}^{b} f - \varepsilon \leq \sum_{k=0}^{m-2} |f(x_{k+1}) - f(x_{k})| + |f(x_{m-1}) - f(c)| + |f(c) - f(x_{m})| + \sum_{k=m}^{n-1} |f(x_{k+1}) - f(x_{k})|$$
$$\leq \bigvee_{a}^{c} f + \bigvee_{c}^{b} f.$$

Since $\varepsilon > 0$ was arbitrary, we also have

$$\bigvee_{a}^{b} f \leq \bigvee_{a}^{c} f + \bigvee_{c}^{b} f.$$

Remark A.7. (a) It follows from Part (c) of Theorem A.6 that if $f \in bv[a, b]$ then $f \in bv[a, c]$ and $f \in bv[c, b]$ for any c with a < c < b and the converse implication also holds true.

(b) If an interval can be split into finitely many subintervals such that f is monotone in each of these subintervals then f is of bounded variation on the whole interval.

The next result gives an important characterization of functions of bounded variation.

Theorem A.8. A function is of bounded variation on an interval if and only if it is the difference of two increasing functions.

Proof. (i) We assume that f is the difference of two increasing functions. Then it follows from Example A.2 and Part (b) of Theorem A.6 that $f \in bv[a, b]$.

(ii) Let $f \in bv[a, b]$ and $x \in [a, b]$. We define the function $\pi : [a, b] \to \mathbb{R}$ by

$$\pi(x) = \begin{cases} 0 & (x=a) \\ \bigvee_{a}^{x} f & (x \in (a,b]) \end{cases}$$

Then π is an increasing function by Part (c) of Theorem A.6. We define another function $\nu : [a, b] \to \mathbb{R}$ by

(A.1)
$$\nu(x) = \pi(x) - f(x) \text{ for all } x \in [a, b].$$

Let $a \leq x < y \leq b$. Then it follows from Part (c) of Theorem A.6 that

$$\nu(y) = \pi(y) - f(y) = \pi(x) + \bigvee_{x}^{y} f - f(y),$$

hence

$$\nu(y) - \nu(x) = \bigvee_{x}^{y} f - (f(y) - f(x)) \ge \bigvee_{x}^{y} f - |f(y) - f(x)| \ge \bigvee_{x}^{y} f - \bigvee_{x}^{y} f = 0.$$

Thus ν is an increasing function, and we have from (A.1) that $f = \pi - \nu$.

Remark A.9. (a) If $f \in bv[a, b]$ then the limits

$$f(x_0 + 0) = \lim_{x \to x_0^+} f(x)$$
 and $f(x_0 - 0) = \lim_{x \to x_0^-} f(x)$

exist for all $x_0 \in (a, b)$, and the set of points at which f is discontinuous is at most countable.

(b) If $f \in bv[a, b]$ then the derivative f' of f exists at almost every point of the interval [a, b] and f' is Lebesgue integrable on [a, b].

If $f \in bv[a, b]$ then, by Theorem A.8, there are two increasing functions φ and ψ such that $f = \varphi - \psi$. Let (x_k) denote the sequence of points at which one of the

functions φ or ψ is discontinuous $(a < x_k < b)$. We define the functions of jumps s_{φ} and s_{ψ} by

$$s_{\varphi}(x) = \begin{cases} 0 & (x = a) \\ \varphi(a+0) - \varphi(a) & \\ + \sum_{x_k < x} (\varphi(x_k+0) - \varphi(x_k - 0)) & (a < x \le b), \\ + \varphi(x) - \varphi(x - 0) & \end{cases}$$

and s_{ψ} similarly. Let $s(x) = s_{\varphi}(x) - s_{\psi}(x)$. Then $s \in \text{bv}[a, b]$, since s_{φ} and s_{ψ} are monotone functions. The function s is called *function of jumps of* f. The function f does not change if we take away from the sequence (x_k) all points of continuity of f. Therefore we may assume that (x_k) contains only points of discontinuity of f. It is obvious that the functions $\varphi - s_{\varphi}$ and $\psi - s_{\psi}$ are continuous and increasing. Therefore $g = f - s \in \text{bv}[a, b]$ and $g = f - s = \varphi - \psi - (s_{\varphi} - s_{\psi}) = \varphi - s_{\varphi} - (\psi - s_{\psi})$ is continuous. Thus the following theorem holds.

Theorem A.10. Every function $f \in bv[a, b]$ can be written as the sum of its function of jumps and a continuous function of bounded variation.

We need two lemmas to be able to prove Theorem A.13.

Lemma A.11. Let \mathcal{F} be an infinite family of functions $f : [a, b] \to \mathbb{R}$ such that there is a constant C with

(A.2)
$$|f(x)| \le C \text{ for all } f \in \mathcal{F} \text{ and for all } x \in [a, b].$$

Then, for any countable subset E of [a, b], there exists a sequence (f_n) of functions $f_n \in \mathcal{F}$ which converges at every point of E.

Proof. Let $E = \{x_k\}$ be a countable subset of [a, b]. We consider the set $M_1 = \{f(x_1) : f \in \mathcal{F}\}$. The set M_1 is bounded by (A.2). Hence there exists a convergent sequence

(A.3)
$$(f_n^{(1)}(x_1))$$
 in $M_1, y_1 = \lim_{n \to \infty} f_n^{(1)}(x_1)$, say

by the *Bolzano–Weierstrass theorem*. Now we consider the sequence $(f_n^{(1)}(x_2))$. This sequence is also bounded by (A.2) and again we can choose a convergent sequence

(A.4)
$$(f_n^{(2)}(x_2))$$
 with $y_2 = \lim_{n \to \infty} f_n^{(2)}(x_2)$.

Continuing in this way we can choose a countable set of convergent sequences

(A.5)
$$\begin{cases} (f_n^{(1)}(x_1)) & \text{such that} & y_1 = \lim_{n \to \infty} f_n^{(1)}(x_1) \\ (f_n^{(2)}(x_2)) & \text{such that} & y_2 = \lim_{n \to \infty} f_n^{(2)}(x_2) \\ \dots & \dots & \dots \\ (f_n^{(k)}(x_k)) & \text{such that} & y_k = \lim_{n \to \infty} f_n^{(k)}(x_k) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \end{pmatrix}$$

where each sequence of functions has been chosen from the preceding sequence without change of the order of terms. Now we consider the sequence on the diagonal in (A.5). For arbitrary fixed k, the sequence $(f_n^{(n)}(x_k))_{n\geq k}$ is a subsequence of the sequence $(f_n^{(k)}(x_k))$, hence convergent to y_k . Thus the sequence $(f_n^{(n)}(x))$ converges at every point $x \in E$.

Lemma A.12. Let \mathcal{F} be an infinite family of increasing functions $f : [a, b] \to \mathbb{R}$ satisfying (A.2). Then there is a sequence (f_n) of functions $f_n \in \mathcal{F}$ that converges at every point of the interval [a, b] to an increasing function φ .

Proof. Let $E = [a, b] \cap \mathbb{Q} \cup \{a\}$. By Lemma A.11, there is a sequence $F_0 = (f^{(n)})$ of functions $f^{(n)} \in \mathcal{F}$ such that $\lim_{n \to \infty} f^{(n)}(x_k)$ exists for all $x_k \in E$. We define the function ψ by

$$\psi(x_k) = \lim_{n \to \infty} f^{(n)}(x_k) \quad (x_k \in E).$$

Then ψ is defined on E and for $x_k, x_j \in E$ with $x_k < x_j$, we have $\psi(x_k) \leq \psi(x_j)$. We define ψ on $[a, b] \setminus E$ by

$$\psi(x) = \sup\{\psi(x_k) : x_k < x, \ x_k \in E\}.$$

Then ψ is an increasing function on \int_a^b and the set D of points of discontinuity of ψ is at most countable. At every point x_0 of continuity of ψ , we have

(A.6)
$$\lim_{n \to \infty} f^{(n)}(x_0) = \psi(x_0).$$

To prove (A.6), let $\varepsilon > 0$ be given. Then there are $x_k, x_j \in E$ such that $x_k < x_0 < x_j$ and

$$\psi(x_j) - \psi(x_k) < \varepsilon/2.$$

We fix x_j and x_k . Then there is an integer n_0 such that

$$|f^{(n)}(x_k) - \psi(x_k)| < \varepsilon/2 \text{ and } |f^{(n)}(x_j) - \psi(x_j)| < \varepsilon/2$$

for all $n > n_0$. Then we have for all $n > n_0$

$$\psi(x_0) - \varepsilon < f^{(n)}(x_0) \le f^{(n)}(x_j) < \psi(x_0) + \varepsilon,$$

hence (A.6) holds. Now we apply Lemma A.11 to the set \mathcal{F}_0 that consists of the functions of the sequence F_0 and to the countable set D to obtain a sequence (f_n) in \mathcal{F}_0 which converges on [a, b]. (Note that in points of convergence of the sequence $(f^{(n)})$ the subsequence (f_n) is also convergent). We define the function φ by

$$\varphi(x) = \lim_{n \to \infty} f_n(x).$$

Then ϕ is an increasing function.

Theorem A.13 (Helly). Let \mathcal{F} be an infinite family of functions $f : [a, b] \to \mathbb{R}$ with the property that there is a constant C such that

$$|f(x)| \leq C$$
 and $\bigvee_{a}^{b} f \leq C$ for all $f \in \mathcal{F}$.

Then it is possible to choose a sequence (f_n) of functions in \mathcal{F} that converges to a function $g \in bv[a, b]$ at every point of the interval [a, b].

Proof. For every $f \in \mathcal{F}$, we define the functions φ_f and ψ_f by

$$\varphi_f(x) = \bigvee_a^x f \text{ and } \psi_f(x) = \varphi_f(x) - f(x) \quad (x \in [a, b]).$$

Then the functions φ_f and ψ_f are increasing and

$$|\varphi_f(x)| \leq C$$
 and $|\psi_f(x)| \leq C$ for all $f \in \mathcal{F}$ on $[a, b]$.

We apply Lemma A.12 to the family $\{\varphi_f\}$ to obtain a convergent sequence (φ_k) with

$$\alpha(x) = \lim_{k \to \infty} \varphi_k(x) \text{ on } [a, b],$$

and then we apply Lemma A.12 to the family $\{\psi_k\}$ where $\psi_k(x) = \varphi_k(x) - f_k(x)$ to obtain a convergent subsequence $(\psi_{k(i)})$ such that

$$\beta(x) = \lim_{i \to \infty} \psi_{k(i)}(x) \text{ on } [a, b].$$

Then the sequence $(f_{k(i)})$ of functions in \mathcal{F} with $f_{k(i)}(x) = \varphi_{k(i)}(x) - \psi_{k(i)}(x)$ converges to

$$g(x) = \alpha(x) - \beta(x)$$
 on $[a, b]$ and $\varphi \in bv[a, b]$.

B The Riemann–Stieltjes integral

Riemann–Stieltjes integrals are a generalization of the Riemann integrals.

Definition B.1. Let $f, g : [a, b] \to \mathbb{R}$ be functions,

$$P = \{x_0 = a < x_1 < \dots < x_n = b\}$$

be a partition of the interval [a, b] and $\xi_k \in [x_k, x_{k+1}]$ for $k = 0, 1, \dots, n-1$. By

$$\sigma_P(f,g;\xi) = \sum_{k=0}^{n-1} f(\xi_k) (g(x_{k+1}) - g(x_k))$$

we denote the Riemann-Stieltjes sums of f with respect to g.

If $\sigma_P(f, g; \xi)$ converges to a limit $I \in \mathbb{R}$ as $||P|| = \max\{x_{k+1} - x_k : 0 \le k \le n-1\}$ tends to zero, where the limit I is independent of the choice of the partitions and

the points ξ_k , then I is called the Riemann-Stieltjes integral on [a, b] of the function f with respect to the function g. We write

$$I = \int_{a}^{b} f(x) \, dg(x)$$

First we list a few basic properties of the Riemann–Stieltjes integrals.

Remark B.2. (a) If g(x) = x on [a, b] then the Riemann-Stieltjes integral reduces to the Riemann integral.

(b) The following rules hold or the Riemann-Stieltjes integral

(B.1)
$$\int_{a}^{b} (f_1(x) + f_2(x)) dg(x) = \int_{a}^{b} f_1(x) dg(x) + \int_{a}^{b} f_2(x) dg(x),$$

(B.2)
$$\int_{a} f(x) d(g_1(x) + g_2(x)) = \int_{a} f(x) dg_1(x) + \int_{a} f(x) dg_2(x)$$

We have for $\alpha, \beta \in \mathbb{R}$

(B.3)
$$\int_{a}^{b} \alpha f(x) d(\beta g(x)) = \alpha \beta \int_{a}^{b} f(x) dg(x).$$

If the integrals $\int_a^b f(x) dg(x)$, $\int_a^c f(x) dg(x)$ and $\int_c^b f(x) dg(x)$ exist then

(B.4)
$$\int_{a}^{b} f(x) \, dg(x) = \int_{a}^{c} f(x) \, dg(x) + \int_{c}^{b} f(x) \, dg(x)$$

(c) The existence of the integral $\int_a^b f(x) dg(x)$ implies the existence of each of the integrals $\int_a^c f(x) dg(x)$ and $\int_c^b f(x) dg(x)$ for any c with a < c < b. The converse implication, however, is not true in general.

(d) Integration by parts The existence of either of the integrals $\int_a^b f(x) dg(x)$ or $\int_a^b g(x) df(x)$ implies the existence of the other integral and the following identity holds

(B.5)
$$\int_{a}^{b} f(x) dg(x) + \int_{a}^{b} g(x) df(x) = f(b)g(b) - f(a)g(a).$$

Proof. (a) Part (a) is obvious from the definition of the Riemann integral and Definition B.1.

(b) Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be an arbitrary partition of the interval [a, b] and $\xi_k \in [x_k, x_{k+1}]$ $(k = 0, 1, \dots, n-1)$ be arbitrary intermediate points.

(B.1) Then we obtain

$$\sigma_P(f_1 + f_2, g; \xi) = \sum_{k=0}^{n-1} (f_1(\xi_k) + f_2(\xi_k)) (g(x_{k+1}) - g(x_k))$$

= $\sum_{k=0}^{n-1} f_1(\xi_k) (g(x_{k+1}) - g(x_k)) +$
+ $\sum_{k=0}^{n-1} f_2(\xi_k) (g(x_{k+1}) - g(x_k))$
= $\sigma_P(f_1, g; \xi) + \sigma_P(f_2, g; \xi).$

The existence of

$$I_{j} = \lim_{\|P\|=0} \sigma_{P}(f_{j}, g; \xi) = \int_{a}^{b} f_{j}(x) \, dg(x) \text{ for } j = 1, 2$$

implies that of

$$I = \lim_{\|P\|=0} \sigma_P(f_1 + f_2, g; \xi) = \int_a^b (f_1(x) + f_2(x)) \, dg(x),$$

and we have $I = I_1 + I_2$.

(B.2) We have

$$\sigma_P(f, g_1 + g_2; \xi) = \sum_{k=0}^{n-1} f(\xi_k) \left(g_1(x_{k+1}) + g_2(x_{k+1}) - \left(g_1(x_k) + g_2(x_k) \right) \right)$$
$$= \sum_{k=0}^{n-1} f(\xi_k) \left(g_1(x_{k+1}) - g_1(x_k) \right) +$$
$$+ \sum_{k=0}^{n-1} f(\xi_k) \left(g_2(x_{k+1}) - g_2(x_k) \right)$$
$$= \sigma_P(f, g_1, \xi) + \sigma_P(f, g_2, \xi).$$

The existence of

$$I_j = \lim_{\|P\| \to 0} \sigma_P(f, g_j; \xi) \text{ for } j = 1, 2$$

implies that of

$$I = \lim_{\|P\|=0} \sigma_P(f, g_1 + g_2, \xi)$$

and we have $I = I_1 + I_2$. (B.3) We have

$$\sigma_P(\alpha f, \beta g; \xi) = \sum_{k=0}^{n-1} \alpha f(\xi_k) \left(\beta g(x_{k+1}) - \beta g(x_k)\right)$$
$$= \alpha \beta \sum_{k=0}^{n-1} f(\xi_k) \left(g(x_{k+1}) - g(x_k)\right) = \alpha \beta \sigma_P(f, g; \xi)$$

The existence of

$$I = \lim_{\|P\| \to 0} \sigma_P(f, g; \xi)$$

implies that of

$$I(\alpha,\beta) = \lim_{\|P\|\to 0} \sigma_P(\alpha f,\beta g;\xi),$$

and we have $I(\alpha, \beta) = \alpha \beta I$.

The next result gives sufficient conditions for the existence of the Riemann–Stieltjes integrals.

Theorem B.3. If f is a continuous function on the interval [a, b] and $g \in bv[a, b]$ then the integral $\int_a^b f(x) dg(x)$ exists.

Proof. We may assume by Theorem A.8 that the function g is increasing. Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of the interval [a, b]. We put

$$m_{k} = \inf\{f(x) : x \in [x_{k}, x_{k+1}]\} \text{ and}$$

$$M_{k} = \sup\{f(x) : x \in [x_{k}, x_{k+1}]\} \text{ for } k = 0, 1, \dots, n-1,$$

$$s = \sum_{k=0}^{n-1} m_{k} \left(g(x_{k+1}) - g(x_{k})\right) \text{ and } S = \sum_{k=0}^{n-1} M_{k} \left(g(x_{k+1}) - g(x_{k})\right),$$

and obtain

(B.6)
$$s \le \sigma_P(f, g; \xi) \le S_P.$$

If we add more points, s does not decrease, and S does not increase. Every sum s is less than or equal to any sum S. For let P_1 and P_2 be two different partitions of [a, b] with sums s_1 and S_1 , and s_2 and S_2 , respectively, then we consider the partition $P_3 = P_1 \cup P_2$ with sums s_2 and S_3 . Now

$$s_1 \leq s_3 \leq S_3 \leq S_2$$
 implies $s_1 \leq S_2$.

We put $I = \sup\{s\}$. Then we have for any partition P

$$s \leq I \leq S$$

and it follows from (B.6) that

$$|\sigma - I| \le S - s.$$

Let $\varepsilon > o$ be given. Since f is uniformly continuous on the compact interval [a, b], we can choose $\delta > 0$ such that

$$|f(x'') - f(x')| < \varepsilon \text{ for all } x'', x' \in [a, b] \text{ with } |x'' - x'| < \delta.$$

Then we have for all partitions P of the interval [a, b] with $||P|| < \delta$

$$M_k - m_k < \varepsilon$$
 for $k = 0, 1, ..., n - 1$,

hence

$$S - s < \varepsilon(g(b) - g(a))$$
, that is, $|\sigma_P(f, g; \xi) - I| < \varepsilon(g(b) - g(a))$.

The next result is useful for the evaluation of Riemann–Stieltjes integrals.

Theorem B.4. If f is a continuous function on the interval [a, b] and $g : [a, b] \to \mathbb{R}$ has a bounded, Riemann integrable derivative g' on [a, b] then

(B.7)
$$\int_{a}^{b} f(x) \, dg(x) = \int_{a}^{b} f(x) g'(x) \, dx$$

Proof. It follows from the hypotheses by Parts (a) and (b) of Example A.4 that $g \in bv[a, b]$, and consequently the integral $\int_a^b f(x) dg(x)$ exists by Theorem B.3. On the other hand, since $f \cdot g'$ is bounded and Riemann integrable by the hypothesis, the integral on the right hand side in (B.7) exists as well. Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of the interval [a, b]. By the mean value theorem of differentiation, there exists a number $\bar{x}_k \in (x_k, x_{k+1})$ for each $k = 0, 1, \ldots, n-1$ such that

$$g(x_{k+1}) - g(x_k) = g'(\bar{x}_k)(x_{k+1} - x_k)$$
 for $k = 0, 1, \dots, n-1$,

and so

$$\sigma_P(f,g,\bar{x}) = \sum_{k=0}^{n-1} f(\bar{x}_k) \left(g(x_{k+1}) - g(x_k) \right) = \sum_{k=0}^{n-1} f(\bar{x}_k) \cdot g'(\bar{x}_k) (x_{k+1} - x_k).$$

Letting $||P|| \to 0$, we obtain (B.7)).

Now we give an estimate for the absolute value of a Riemann–Stieltjes integral. **Theorem B.5.** Let f be a continuous function on the interval [a,b], $g \in bv[a,b]$ and $M = max\{|f(x)|; x \in [a,b]\}$. Then we have

(B.8)
$$\left| \int_{a}^{b} f(x) \, dg(x) \right| \leq M \cdot \bigvee_{a}^{b} g$$

Proof. Let $P = \{x_0 = a < x_1 < \cdots x_n = b\}$ be any partition of the interval [a, b] and $\xi_k \in [x_k, x_{k+1}]$ $(k = 0, 1, \dots, n-1)$ be arbitrary intermediate points. Then we have

$$|\sigma_P(f,g;\xi)| = \left|\sum_{k=0}^{n-1} f(\xi_k) \left(g(x_{k+1}) - g(x_k)\right)\right| \le M \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)| \le M \cdot \bigvee_a^b g.$$

By Theorem B.3, the integral $\int_a^b f(x) dg(x)$ exists. Letting $||P|| \to \infty$, we obtain (B.8).

The next result gives a sufficient condition for the interchange of the limit and the integral; it is analogous to the result for Riemann integrals.

Theorem B.6. Let $g \in bv[a, b]$ and (f_n) be a sequence of functions $f_n \in C[a, b]$ that converges uniformly on the interval [a, b] to a function f. Then we have

$$\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dg(x) = \int_{a}^{b} f(x) \, dg(x).$$

Proof. We put $M_n = \max\{|f_n - f(x)| : x \in [a, b]\}$ for $n = 0, 1, \ldots$ (We observe that f is continuous on [a, b] being the limit function of a uniformly convergent sequence of continuous functions.) It follows from (B.8) that

$$\left|\int_{a}^{b} f_n(x) dg(x) - \int_{a}^{b} f(x) dg(x)\right| \le M_n \bigvee_{a}^{b} g \ (n = 0, 1, \dots).$$

The uniform convergence to f on \int_a^b of the sequence $(f_n)_{n=0}^{\infty}$ implies $M_n \to 0$ $(n \to \infty)$, and the statement of the theorem is an immediate consequence. \Box

Theorem B.7 (E. Helly). Let f be a continuous function on the interval [a, b], (g_n) be a sequence of functions $g_n \in bv[a, b]$ with

$$g(x) = \lim_{n \to \infty} g_n(x)$$
 for all $x \in [a, b]$.

If there is an absolute constant C such that $\bigvee_a^b g_n \leq C$ for all n = 0, 1, ..., then we have

(B.9)
$$\lim_{n \to \infty} \int_{a}^{b} f(x) \, dg_n(x) = \int_{a}^{b} f(x) \, dg(x)$$

Proof. First we show $g \in bv[a, b]$. Let $P = \{x_0 = a < x_1 < \cdots < x_m = b\}$ be a partition of the interval [a, b]. Then it follows that

$$\sum_{k=0}^{m-1} |g_n(x_{k+1}) - g_n(x_k)| \le C \text{ for all } n,$$

and $n \to \infty$ yields

$$\sum_{k=0}^{m-1} |g(x_{k+1}) - g(x_k)| \le C \text{ for all } n.$$

Since P was an arbitrary partition of the interval [a, b], we conclude

(B.10)
$$\bigvee_{a}^{b} g \le C.$$

Now we show that (B.9) holds.

Since f is continuous on the interval [a, b] and $g, g_n \in bv[a, b]$ for all n, the integrals $\int_a^b f(x) dg(x)$ and $\int_a^b f(x) dg_n(x)$ exist for all n by Theorem B.3. Since the function f is uniformly continuous on the compact interval [a, b], given $\varepsilon > 0$, we can choose a partition P of the interval [a, b] such that

$$\sup\{|f(x') - f(x'')| : x', x'' \in [x_k, x_{k+1}] \ (k = 0, 1, \dots, n-1)\} < \frac{\varepsilon}{3(C+1)}.$$

Then it follows that

$$\int_{a}^{b} f(x) dg(x) = \sum_{k=0}^{m-1} \int_{x_{k}}^{x_{k+1}} f(x) dg(x)$$
$$= \sum_{k=0}^{m-1} \int_{x_{k}}^{x_{k+1}} (f(x) - f(x_{k})) dg(x) + \sum_{k=0}^{m-1} f(x_{k}) \int_{x_{k}}^{x_{k+1}} dg(x).$$

Since

$$\int_{x_k}^{x_{k+1}} dg(x) = g(x_{k+1}) - g(x_k) \text{ and } |f(x) - f(x_k)| < \frac{\varepsilon}{3(C+1)}$$

for $x \in [x_k, x_{k+1}]$ $(k = 0, 1, \dots, m-1),$

it follows from Theorem B.5 that

$$\left|\int_{x_k}^{x_{k+1}} (f(x) - f(x_k)) dg(x)\right| \le \frac{\varepsilon}{3(C+1)} \bigvee_{x_k}^{x_{k+1}} g,$$

hence, by Part (c) of Theorem A.6,

$$\left|\sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} (f(x) - f(x_k)) \, dg(x)\right| < \frac{\varepsilon}{3(C+1)} \bigvee_a^b g \le \frac{\varepsilon}{3}$$

Therefore there is a number $\Theta \in [-1,1]$ such that

$$\int_{a}^{b} f(x) \, dg(x) = \sum_{k=0}^{m-1} f(x_k) (g(x_{k+1}) - g(x_k)) + \Theta \frac{\varepsilon}{3}.$$

Similarly, to every $n = 0, 1, \ldots$, there is a number $\Theta_n \in [-1, 1]$ such that

$$\int_{a}^{b} f(x) \, dg_n(x) = \sum_{k=0}^{m-1} f(x_k) (g_n(x_{k+1}) - g_n(x_k)) + \Theta_n \frac{\varepsilon}{3}.$$

For sufficiently large n, we have

$$\left|\sum_{k=0}^{m-1} f(x_k)(g_n(x_{k+1}) - g_n(x_k)) - \sum_{k=0}^{m-1} f(x_k)(g(x_{k+1}) - g(x_k))\right| < \frac{\varepsilon}{3},$$

hence

$$\left| \int_{a}^{b} f(x) \, dg_n(x) - \int_{a}^{b} f(x) \, dg(x) \right| < \varepsilon. \qquad \Box$$

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