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Marko Kostić

Dedicated to my mother

Contents

PREFACE

The book provides a detailed introduction to the scope of main methods of the theory of ill-posed abstract Cauchy problems which has been rapidly developing over the last two decades, starting presumably with [**3**], [**82**] and [**61**]. The main purpose of the book is to enable the reader to acquire the most important strategies for dealing with various classes of generalized semigroups and cosine functions in a Banach space setting.

The material is divided into three individual chapters. The introductive chapter is mainly the review of the basic tools and concepts which will be utilized in the remaining part of the book. The reader with a little experience should move through the first chapter fairly quickly. The second is devoted to the extensive study of (exponentially bounded) convoluted C-(semi)groups and cosine functions and their relations with abstract Cauchy problems. The justification for concentrating to this topic is based upon the fact that several structural properties of various kinds of integrated C-semigroups and cosine functions have not been fully cleared so far. We discuss composition properties, automatic extension, analyticity, perturbations and spectral properties of subgenerators of convoluted C-semigroups and cosine functions. In the third section, we systematically analyze (ultra-)distribution (semi-)groups, their differential and analytical properties, distribution cosine functions, $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-groups and regularization of ultradistribution semigroups and sines. We recollect the basic properties of (a,k)-regularized C-resolvent families and ill-posed hyperbolic Volterra equations of nonscalar type. In addition, a comprehensive survey of the vast literature related to the subject of the book is given.

In terms of prerequisites, the present book assumes that the reader has a vague familiarity with the content of functions of one complex variable, the basic Banach space and Lebesgue integration theory. Most of the subject matter, as regards to difficulty, is intended to be accessible to a graduate in Mathematics reader.

The author would like to express his sincere gratitude to many people who strongly encouraged his work over the last ten years.

CHAPTER 1

INTRODUCTION

1.1. Operator-valued functions, Laplace transforms and closed operators

In what follows, we assume that E is a complex Banach space and that I is a (bounded or unbounded) segment in $\mathbb{R}^n$, where $n \in \{1,2\}$.

DEFINITION 1.1.1. (i) It is said that a function $f : I \to E$ is *simple* if there exist $k \in \mathbb{N}$, elements $z_i \in E$, $1 \leqslant i \leqslant k$ and Lebesgue measurable subsets Ω_k, $1 \leqslant i \leqslant k$ of I, such that $m(\Omega_i) < \infty$, $1 \leqslant i \leqslant k$ and that

$$f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t), \quad t \in I. \tag{1}$$

(ii) It is said that a function $f : I \to E$ is *measurable* if there exists a sequence (f_n) in E^I such that, for every $n \in \mathbb{N}$, f_n is a simple function and that $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$.

(iii) A function $f : I \to E$ is said to be *weakly measurable* iff for every $x^* \in E^*$, the function $t \mapsto x^*(f(t))$, $t \in I$ is measurable.

(iv) Let $-\infty < a < b < \infty$ and $a < \tau \leqslant \infty$. A function $f : [a,b] \to E$ is said to be *absolutely continuous* iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of open subintervals (a_i, b_i), $1 \leqslant i \leqslant k$ of $[a,b]$ with $\sum_{i=1}^{k}(b_i - a_i) < \delta$, the following holds $\sum_{i=1}^{k} \|f(b_i) - f(a_i)\| < \varepsilon$; a function $f : [a,\tau) \to E$ is said to be absolutely continuous iff for every $\tau_0 \in (a,\tau)$, the function $f_{|[a,\tau_0]} : [a,\tau_0] \to E$ is absolutely continuous.

If E is a separable Banach space, then a function $f(\cdot)$ is measurable iff $f(\cdot)$ is weakly measurable. Suppose, further, that $f : I \to E$ and that (f_n) is a sequence of measurable functions satisfying $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$. Then $f(\cdot)$ is also measurable. Next, the Bochner integral of a simple function $f : I \to E$, $f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t)$, $t \in I$, is defined by $\int_I f(t)\,dt := \sum_{i=1}^{k} z_i m(\Omega_i)$. One can simply prove that the definition of Bochner integral does not depend on the representation (1).

A measurable function $f : I \to E$ is said to be *Bochner integrable* if there exists a sequence of simple functions (f_n) in E^I such that $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e.

$t \in I$ and

$$\lim_{n\to\infty} \int\limits_I \|f_n(t) - f(t)\|\, dt = 0; \tag{2}$$

in this case, the Bochner integral of $f(\cdot)$ is defined by $\int_I f(t)dt := \lim_{n\to\infty} \int_I f_n(t)dt$. The definition of Bochner integrability of a measurable function makes a sense and is independent of the choice of a sequence of simple functions (f_n) in E^I satisfying $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$ and (2). It can be verified that $f : I \to E$ is Bochner integrable iff $f(\cdot)$ is measurable and the function $t \mapsto \|f(t)\|$, $t \in I$ is integrable as well as that, for every Bochner integrable function $f : [0, \infty) \to E$, we have $\int_0^\infty f(t)\, dt = \lim_{\tau\to+\infty} \int_0^\tau f_{|[0,\tau]}(t)\, dt$. The space of all Bochner integrable functions from I into E is denoted by $L^1(I : E)$; equipped with the norm $\|f\|_1 := \int_I \|f(t)\|\, dt$, $L^1(I : E)$ becomes a Banach space. A function $f : [0, \infty) \to E$ is said to be locally (Bochner) integrable iff $f(\cdot)_{|[0,\tau]}$ is Bochner integrable for every $\tau > 0$. The space of all locally integrable functions from $[0, \infty)$ into E is denoted by $L^1_{\mathrm{loc}}([0, \infty) : E)$. If $f \in L^1_{\mathrm{loc}}([0, \infty) : E)$ and $\lim_{\tau\to+\infty} \int_0^\tau f_{|[0,\tau]}(t)\, dt$ exists, then we say that $\int_0^\infty f(t)\, dt$ converges as an improper integral and define $\int_0^\infty f(t)\, dt := \lim_{\tau\to+\infty} \int\limits_0^\tau f_{|[0,\tau]}(t)\, dt$. If there is no risk for confusion, we will not distinguish a function and its restriction to any subinterval of its domain.

The following proposition will be used frequently throughout the book.

PROPOSITION 1.1.2. *Let A be a closed linear operator in E* (cf. the final part of this subsection) *and let $f : I \to E$ be Bochner integrable. If $f(t) \in D(A)$, $t \in I$ and $A \circ f : I \to E$ is Bochner integrable, then $\int_I f(t)\, dt \in D(A)$ and $A \int_I f(t)\, dt = \int_I A(f(t))\, dt$.*

Now we state the operator valued version of the dominated convergence theorem and the Fubini theorem (cf. also [**300**, p. 325]).

THEOREM 1.1.3. (i) *Suppose that (f_n) is a sequence of Bochner integrable functions from E^I and that there exists an integrable function $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \leqslant g(t)$ for a.e. $t \in I$ and $n \in \mathbb{N}$. If $f : I \to E$ and $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$, then $f(\cdot)$ is Bochner integrable, $\int_I f(t)\, dt = \lim_{n\to\infty} \int_I f_n(t)\, dt$ and $\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\, dt = 0$.*

(ii) *Let I_1 and I_2 be segments in $\mathbb{R}$ and let $I = I_1 \times I_2$. Suppose that $F : I \to E$ is measurable and that $\int_{I_1} \int_{I_2} \|f(s, t)\|\, dt\, ds < \infty$. Then $f(\cdot, \cdot)$ is Bochner integrable, the repeated integrals $\int_{I_1} \int_{I_2} f(s, t)\, dt\, ds$ and $\int_{I_2} \int_{I_1} f(s, t)\, ds\, dt$ exist and equal to the integral $\int_I f(s, t)\, ds\, dt$.*

Let $1 \leqslant p < \infty$ and let $(\Omega, \mathcal{R}, \mu)$ be a measure space. Then the space $L^p(\Omega : E)$ consists of all strongly μ-measurable functions $f : \Omega \to E$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p d\mu)^{1/p}$ is finite. The space $L^\infty(\Omega : E)$ consists of all strongly μ-measurable, essentially bounded functions and is equipped with the norm $\|f\|_\infty := ess \sup_{t\in\Omega} \|f(t)\|$, $f \in L^\infty(\Omega : E)$. Herein we identify functions that are equal μ-almost everywhere on Ω. By Riesz–Fischer theorem, $(L^p(\Omega : E), \|\cdot\|_p)$ is a Banach space for all $p \in [1, \infty]$, and furthermore, we know that $(L^2(\Omega : E), \| \cdot \|_2)$ is a Hilbert space. If $\lim_{n\to\infty} f_n = f$ in $L^p(\Omega : E)$, then there exists a subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} f_{n_k}(t) = f(t)$ μ-almost everywhere. If the Banach space E is reflexive, then $L^p(\Omega : E)$ is reflexive for all $p \in (1, \infty)$ and its dual is isometrically isomorphic to $L^{\frac{p}{p-1}}(\Omega : E)$. The next proposition clarifies the basic properties of operator valued absolutely continuous functions.

PROPOSITION 1.1.4. (i) *Suppose* $-\infty < a < b < \infty$, $f \in L^1([a, b] : E)$ *and* $F(t) := \int_0^t f(s)\, ds$, $t \in [a, b]$. *Then* $F(\cdot)$ *is absolutely continuous,* $F'(t) = f(t)$ *for a.e.* $t \in [a, b]$ *and* $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, ds = 0$ *for a.e.* $t \in [a, b]$, *i.e., almost every point of* $[a, b]$ *is a Lebesgue point of the function* $f(\cdot)$. *Furthermore, if* $f \in C([a, b] : E)$, *then the preceding equality holds for all* $t \in [a, b]$.

(ii) *Suppose* $-\infty < a < b < \infty$, $F : [a, b] \to E$ *is absolutely continuous and* $F'(t)$ *exists for a.e.* $t \in [a, b]$. *Then* $F'(\cdot)$ *is Bochner integrable on* $[a, b]$ *and* $F(t) = F(a) + \int_a^t F'(s)\, ds$, $t \in [a, b]$.

A Banach space E is said to possess the *Radon–Nikodym property* iff every absolutely continuous function $F : [0, 1] \to E$ is differentiable a.e. It is well known that every reflexive Banach space possesses the Radon–Nikodym property and that the space $L^1[0, 1]$ does not possess the Radon–Nikodym property.

PROPOSITION 1.1.5. *Suppose* X *is a Banach space,* $f \in L^1_{\mathrm{loc}}([0, \infty) : E)$ *and* $T : [0, \infty) \to L(E, X)$ *is strongly continuous, i.e., the mapping* $t \mapsto T(t)x$, $t \geqslant 0$ *is continuous for every fixed* $x \in E$. *Define the mapping* $T *_0 f : [0, \infty) \to X$ *by* $(T *_0 f)(t) := \int_0^t T(t - s) f(s)\, ds$, $t \geqslant 0$. *Then* $T *_0 f \in C([0, \infty) : X)$.

DEFINITION 1.1.6. Let $f \in L^1_{\mathrm{loc}}([0, \infty) : E)$. Then we say that $f(\cdot)$ is *Laplace transformable* iff there exists $\omega \in \mathbb{R}$ such that

$$\mathcal{L}\big(f(t)\big)(\lambda) := \tilde{f}(\lambda) := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda s} f(s)\, ds := \int_0^\infty e^{-\lambda s} f(s)\, ds$$

exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. The *abscissa of the convergence* of $\tilde{f}(\cdot)$ is defined by $\operatorname{abs}(f) := \inf\{\omega \in \mathbb{R} : \tilde{f}(\lambda)$ exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega\}$. Given a measurable function $f : [0, \infty) \to E$, we define the *exponential growth bound* $\omega(f)$ by setting

$$\omega(f) := \inf\big\{\omega \in \mathbb{R} : \text{exists } M \geqslant 0 \text{ such that } \|f(t)\| \leqslant M e^{\omega t},\ t \geqslant 0\big\}.$$

Obviously, $\operatorname{abs}(f) \leqslant \operatorname{abs}(\|f\|) \leqslant \omega(f)$, but in general, there exist examples where one has the strict inequalities. We refer the reader to [**14**, Appendix A] for the basic properties of operator valued analytic functions. If $f(\cdot)$ is Laplace transformable, then $\tilde{f}(\lambda)$ exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \operatorname{abs}(f)$, the mapping $\lambda \mapsto \tilde{f}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \operatorname{abs}(f)$ is analytic, $\frac{d^n}{d\lambda^n} \tilde{f}(\lambda) = (-1)^n \int_0^\infty e^{-\lambda t} t^n f(t)\, dt$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \operatorname{abs}(f)$ (understood in the sense of improper integral) and $\tilde{f}(\lambda)$ does not exist if $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda < \operatorname{abs}(f)$.

THEOREM 1.1.7. *Suppose* $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$ *is Laplace transformable,* $z \in \mathbb{C}$ *and* $s \geqslant 0$.

(i) *Put* $g(t) := e^{-zt} f(t)$, $t \geqslant 0$. *Then* $g(\cdot)$ *is Laplace transformable,* $\mathrm{abs}(g) = \mathrm{abs}(f) - \mathrm{Re}\, z$ *and* $\tilde{g}(\lambda) = \tilde{f}(\lambda + z)$, $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > \mathrm{abs}(f) - \mathrm{Re}\, z$.

(ii) *Put* $f_s(t) := f(t+s)$, $t \geqslant 0$. *Then* $f_s(\cdot)$ *is Laplace transformable,* $\mathrm{abs}(f_s) = \mathrm{abs}(f)$ *and* $\tilde{f}_s(\lambda) = e^{\lambda s}(\tilde{f}(\lambda) - \int_0^s e^{-\lambda t} f(t)\, dt)$, $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > \mathrm{abs}(f)$.

(iii) *Suppose* $h \in L^1_{\mathrm{loc}}([0,\infty))$ *is Laplace transformable and* $(h *_0 f)(t) := \int_0^t h(t-s) f(s)\, ds$, $t \geqslant 0$. *Then* $h *_0 f$ *is Laplace transformable,* $\mathrm{abs}(h *_0 f) \leqslant \max(\mathrm{abs}(|h|), \mathrm{abs}(f))$ *and*

$$\widetilde{h *_0 f}(\lambda) = \tilde{h}(\lambda)\tilde{f}(\lambda),\ \ \lambda\ \in \mathbb{C},\ \mathrm{Re}\,\lambda > \max(\mathrm{abs}(|h|), \mathrm{abs}(f)).$$

(iv) *Let* $F(t) := \int_0^t f(s)ds$, $t \geqslant 0$. *Then* $F(\cdot)$ *is Laplace transformable,* $\mathrm{abs}(F) \leqslant \max(0, \mathrm{abs}(f))$ *and* $\tilde{F}(\lambda) = \frac{\tilde{f}(\lambda)}{\lambda}$, $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > \max(0, \mathrm{abs}(f))$.

(v) *Suppose, in addition,* $\omega(f) < \infty$ *and put*

$$j(t) := \int\limits_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} f(s)\, ds \text{ and } k(t) := \int\limits_0^\infty \frac{s e^{-s^2/4t}}{2\sqrt{\pi} t^{\frac{3}{2}}} f(s)\, ds,\ t > 0.$$

Then $j(\cdot)$ *and* $k(\cdot)$ *are Laplace transformable,*

$$\max(\mathrm{abs}(j), \mathrm{abs}(k)) \leqslant (\max(\omega(f), 0))^2,\ \tilde{j}(\lambda) = \frac{\tilde{f}(\sqrt{\lambda})}{\sqrt{\lambda}} \text{ and } \tilde{k}(\lambda) = \tilde{f}(\sqrt{\lambda})$$

for all $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\,\lambda > (\max(\omega(f), 0))^2$.

(vi) (The uniqueness theorem for the Laplace transform) *Suppose* $\lambda_0 > \mathrm{abs}(f)$ *and* $\tilde{f}(\lambda) = 0$ *for all* $\lambda \in (\lambda_0, \infty)$. *Then* $f(t) = 0$ *for a.e.* $t \geqslant 0$.

(vii) *Let* $l(\cdot)$ *be Laplace transformable and let* $\omega > \max(\mathrm{abs}(f), \mathrm{abs}(l))$. *For a closed linear operator* A*, the following assertions are equivalent:*

(vii.1) $f(t) \in D(A)$ *and* $Af(t) = l(t)$ *for a.e.* $t \geqslant 0$.
(vii.2) $\tilde{f}(\lambda) \in D(A)$ *and* $A\tilde{f}(\lambda) = \tilde{l}(\lambda)$, $\lambda \in (\omega, \infty)$.

(viii) *Suppose* $\varepsilon > 0$. *Then the following assertions are equivalent:*

(viii.1) $\limsup_{\lambda\to\infty} \frac{\ln \|\tilde{f}(\lambda)\|}{\lambda} \leqslant -\varepsilon$.
(viii.2) $f(t) = 0$ *for a.e.* $t \in [0, \varepsilon]$.

(ix) (Post–Widder inversion) *Suppose* $t > 0$ *is a Lebesgue point of* $f(\cdot)$. *Then the following holds:*

$$f(t) = \lim_{n\to\infty} (-1)^n \frac{1}{n!} \Big(\frac{n}{t}\Big)^{n+1} \tilde{f}^{(n)}\Big(\frac{n}{t}\Big).$$

DEFINITION 1.1.8. A sequence (λ_n) in $\mathbb{C}$ is called a *uniqueness sequence* if for every Laplace transformable function $f(\cdot)$ which satisfies that $\tilde{f}(\lambda_n)$ is defined for every $n \in \mathbb{N}$ and that $\tilde{f}(\lambda_n) = 0$, $n \in \mathbb{N}$, one has $f(t) = 0$ for a.e. $t \geqslant 0$.

THEOREM 1.1.9. *Suppose* $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$ *is Laplace transformable and* $a \in (\max(0, \mathrm{abs}(f)), \infty)$. *If* (λ_n) *is a sequence in* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \mathrm{abs}(f)\}$ *without accumulation points such that* $|\{\lambda_n : n \in \mathbb{N}\}| = \infty$ *and that*

$$\sum_{n=1}^{\infty}\left(1 - \frac{|1-\lambda_n|}{|1+\lambda_n|}\right) < \infty, \tag{3}$$

then (λ_n) *is a uniqueness sequence. Suppose, conversely, that* (λ_n) *is a sequence in* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ *and that* (λ_n) *does not possess any accumulation point in the open right half plane as well as that the sum appearing in* (3) *is finite. Then there exists a Laplace transformable function* $f(\cdot)$ *such that* $0 \neq f \in L^1_{\mathrm{loc}}([0,\infty) : E)$ *and that* $\tilde{f}(\lambda_n) = 0$, $n \in \mathbb{N}$.

Let $a > 0$, $b > 0$, $\gamma \in \mathbb{N}$ and $\delta \in \mathbb{R}$. Then $(a + bn^{\gamma})_n$, resp. $(1 + in^{\delta})_n$, is a uniqueness sequence whenever $\gamma \in (-\infty, 0) \cup (0, 1]$, resp. $\delta \in (0, \frac{1}{2}]$.

Denote $\Sigma_\alpha := \{re^{i\theta} : r > 0,\, \theta \in (-\alpha, \alpha)\}$, $\alpha \in (0, \pi]$. The following important characterization of analytic properties of operator valued Laplace transform is due to Sova [**391**].

THEOREM 1.1.10. *Let* $\alpha \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$ *and* $q : (\omega, \infty) \to E$. *Then the following assertions are equivalent:*

(i) *There exists an analytic function* $f : \Sigma_\alpha \to E$ *such that* $\sup_{z \in \Sigma_\beta} \|e^{-\omega z} f(z)\| < \infty$ *for all* $\beta \in (0, \alpha)$ *and* $q(\lambda) = \tilde{f}(\lambda)$ *for all* $\lambda \in (\omega, \infty)$.

(ii) *The function* $q(\cdot)$ *has an analytic extension* $\tilde{q} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to E$ *such that* $\sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}} \|(\lambda - \omega)\tilde{q}(\lambda)\| < \infty$ *for all* $\gamma \in (0, \alpha)$.

We need the assertions of [**14**, Proposition 2.6.3] and [**14**, Proposition 2.6.4].

THEOREM 1.1.11. (i) *Suppose* $\alpha \in (0, \pi]$, $f : \Sigma_\alpha \to E$ *is analytic and* $\sup_{z \in \Sigma_\beta} \|f(z)\| < \infty$ *for all* $\beta \in (0, \alpha)$. *Let* $x \in E$. *Then the following holds:*

(i.1) *If* $\lim_{t \to \infty} f(t) = x$, *then* $\lim_{z \in \Sigma_\beta,\, |z| \to \infty} f(z) = x$ *for all* $\beta \in (0, \alpha)$.

(i.2) *If* $\lim_{t \downarrow 0} f(t) = x$, *then* $\lim_{z \in \Sigma_\beta,\, z \to 0} f(z) = x$ *for all* $\beta \in (0, \alpha)$.

(ii) *Let* α, ω *and* $q(\cdot)$ *have the same meaning as in the formulation of Theorem* 1.1.10 *and let* $x \in E$. *Then the following holds:*

(ii.1) $\lim_{t \downarrow 0} f(t) = x$ *iff* $\lim_{\lambda \to +\infty} \lambda q(\lambda) = x$.

(ii.2) *Let* $\omega = 0$. *Then* $\lim_{t \to \infty} f(t) = x$ *iff* $\lim_{\lambda \downarrow 0} \lambda q(\lambda) = x$.

The complex inversion theorem for the operator valued Laplace transform reads as follows.

THEOREM 1.1.12. *Assume* $a \geqslant 0$, $q : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a\} \to E$ *is analytic and there exist* $M > 0$ *and* $r > 1$ *such that* $\|q(\lambda)\| \leqslant \frac{M}{|\lambda|^r}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > a$. *Then there exist a continuous function* $f : [0, \infty) \to E$ *and* $M' > 0$ *such that* $\|f(t)\| \leqslant M' t^{r-1} e^{at}$, $t \geqslant 0$ *and that* $q(\lambda) = \tilde{f}(\lambda)$ *for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > a$.

Notice that the continuous function $f(\cdot)$ given in the formulation of Theorem 1.1.12 is given by:

$$f(t)=\frac{1}{2\pi i}\int\limits_{\overline{a}-i\infty}^{\overline{a}+i\infty} e^{\lambda t}q(\lambda)\,d\lambda,\ \ t\geqslant 0,$$

and that the previous improper integral does not depend on the choice of a number $\overline{a}>a$. The Arendt–Widder theorem has been reconsidered in a series of papers (see e.g. [**41**], [**68**], [**202**] and [**431**]); the following version is sufficient for our purposes.

THEOREM 1.1.13 (Hieber [**149**], Xiao-Liang [**434**]). *Let $a\geqslant 0$, $\alpha\in(0,1]$, $\omega\in(-\infty,a]$, $M>0$ and let $q:(a,\infty)\to E$ be an infinitely differentiable function. Then (i) $\Leftrightarrow$ (ii), where:*

(i) *The following holds* $\left\|(\lambda-\omega)^{k+1}\frac{q^{(k)}(\lambda)}{k!}\right\|\leqslant M,\ \lambda>a,\ k\in\mathbb{N}_0$.

(ii) *There exists a function $F\in C([0,\infty):E)$ satisfying $F(0)=0$,*

$$q(\lambda)=\lambda^{\alpha}\int\limits_0^{\infty}e^{-\lambda t}F(t)\,dt,\ \ \lambda>a,$$

$$\left\|\int\limits_0^{t+h}\frac{(t+h-s)^{-\alpha}}{\Gamma(1-\alpha)}F(s)\,ds-\int\limits_0^{t}\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}F(s)\,ds\right\|\leqslant Mhe^{\omega t}\max(e^{\omega h},1),$$

for any $t\geqslant 0$ and $h\geqslant 0$, if $\alpha\in(0,1)$, and

$$\|F(t+h)-F(t)\|\leqslant Mhe^{\omega t}\max(e^{\omega h},1),\ \ t\geqslant 0,\ h\geqslant 0$$

if $\alpha=1$. Moreover, in this case,

$$\|F(t+h)-F(t)\|\leqslant\frac{2M}{\alpha\Gamma(\alpha)}h^{r}\max(e^{\omega(t+h)},1),\ \ t\geqslant 0,\ h\geqslant 0.$$

A linear operator $A:D(A)\to E$ is *closed* iff the graph of the operator A, defined by $G_A:=\{(x,Ax):x\in D(A)\}$, is a closed subset of $E\times E$. A necessary and sufficient condition for a linear operator $A:D(A)\to E$ to be closed is that for every sequence (x_n) in $D(A)$ such that $\lim_{n\to\infty}x_n=x$ and that $\lim_{n\to\infty}Ax_n=y$, the following holds: $x\in D(A)$ and $Ax=y$. For a linear operator A, we introduce the *graph norm* on $D(A)$ by $\|x\|_{[D(A)]}:=\|x\|+\|Ax\|$, $x\in D(A)$. Then $(D(A),\|\cdot\|_{[D(A)]})$ is a Banach space iff A is closed. A subspace $Y\subseteq D(A)$ is called a *core* for A iff Y is dense in $D(A)$ with respect to the graph norm. The closed graph theorem states that every closed linear operator defined on the whole space E is a bounded, linear operator; henceforth we denote by $L(E)$ the space of all bounded, linear operators on E and by $\mathrm{Kern}(A)$ and $\mathrm{R}(A)$, the kernel and the range of the operator A, respectively.

Further on, a linear operator A is *closable* iff there exists a closed linear operator B such that $A\subseteq B$. It can be simply shown that a linear operator A is closable iff for every sequence (x_n) in $D(A)$ such that $\lim_{n\to\infty}x_n=0$ and that $\lim_{n\to\infty}Ax_n=y$, we have $y=0$. Suppose that A is a closable linear operator. The *closure* of A, denoted by $\overline{A}$, is defined as the set of all elements $(x, y) \in E \times E$ such that there exists a sequence (x_n) in $D(A)$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$; then $\overline{A}$ is a closed linear operator and, for every other closed linear operator B which contains A, one has $\overline{A} \subseteq B$. Suppose $A : D(A) \to E$ is a linear operator. We define the powers of A recursively by setting: $A^0 =: I$, $D(A^n) := \{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\}$ and $A^n x := A(A^{n-1}x)$, $x \in D(A^n)$, $n \in \mathbb{N}$. Then $D(A^n) = D(A-\lambda)^n$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$. Put $D_\infty(A) := \bigcap_{n\geqslant 1} D(A^n)$. The *resolvent set* of a linear operator A, denoted by $\rho(A)$, is the set of all complex numbers λ such that the operator $\lambda - A$ is bijective; we write $R(\lambda : A) := (\lambda - A)^{-1}$. Recall that the assumption $\rho(A) \neq \emptyset$ implies that, for every $n \in \mathbb{N}$, we have that the operator A^n is closed; furthermore, if A is densely defined, i.e., $\overline{D(A)} = E$, and $\rho(A) \neq \emptyset$, then $D(A^n)$ is a core for A, $n \in \mathbb{N}$. The *spectrum* of the operator A, denoted by $\sigma(A)$, is defined to be the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$. We know that $\rho(A)$ is an open subset of $\mathbb{C}$ and that, in the case $\rho(A) \neq \emptyset$, the mapping $\lambda \to R(\lambda : A)$, $\lambda \in \rho(A)$ is an analytic mapping from $\rho(A)$ into $L(E)$. Furthermore, the resolvent equation states that $R(\lambda : A) - R(\xi : A) = (\xi - \lambda)R(\lambda : A)R(\xi : A)$, and as a consequence, one obtains inductively that $\frac{d^n}{d\lambda^n} R(\lambda : A) = (-1)^n n! R(\lambda : A)^{n+1}$, $\lambda \in \rho(A)$, $n \in \mathbb{N}$. A closed, linear operator A is said to be the *Hille–Yosida operator* ([**82**]) if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and that $\|R(\lambda : A)^n\| \leqslant \frac{M}{(\lambda-\omega)^n}$, $\lambda > \omega$, $n \in \mathbb{N}$. For a closed linear operator A, we introduce the following subset of $E^* \times E^*$:

$$A^* =: \big\{(x^*, y^*) \in E^* \times E^* : x^*(Ax) = y^*(x) \text{ for all } x \in D(A)\big\}.$$

If A is densely defined, then the adjoint operator A^* of A is a closed linear operator in E^*. Suppose F is a closed subspace of E. Then the *part of A in F,* denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and let A and B be linear operators. We define αA, $A + B$ and AB in the following way: $D(\alpha A) =: D(A)$, $D(A+B) = D(A) \cap D(B)$ and $D(AB) := \{x \in D(B) : Bx \in D(A)\}$, $(\alpha A)x := \alpha Ax$, $x \in D(\alpha A)$, $(A+B)x := Ax+Bx$, $x \in D(A+B)$ and $(AB)x := A(Bx)$, $x \in D(AB)$.

Proposition 1.1.14. (i) *Suppose A is closed, $B \in L(E)$, $\alpha \in \mathbb{C} \setminus \{0\}$ and F is a closed subspace of E. Then $\alpha(A + B)$, $\alpha(AB)$ and $A_{|F}$ are also closed.*

(ii) *Let A be densely defined and $\rho(A) \neq \emptyset$. Then $D_\infty(A)$ is dense in E.*

(iii) *Suppose A is a closed operator, $U \subseteq \mathbb{C}$ is open and connected and $U \cap \rho(A) \neq \emptyset$. If there exists an analytic mapping $F : U \to L(E)$ such that the set $\{\lambda \in U \cap \rho(A) : F(\lambda) = R(\lambda : A)\}$ has a limit point in U, then $U \subseteq \rho(A)$ and $F(\lambda) = R(\lambda : A)$, $\lambda \in \rho(A)$.*

(iv) *Suppose A is a closed, densely defined operator. Then $D(A^*)$ is dense in E^* with respect to the weak* topology, and in the case when E is reflexive, we have that A^* is dense in E^* with respect to the strong topology. Furthermore, $\sigma(A^*) = \sigma(A)$ and $R(\lambda : A^*) = R(\lambda : A)^*$, $\lambda \in \rho(A)$.*

(v) *If A is closable and densely defined, then $\overline{A}^* = A^*$.*

It is noteworthy that $D(A^*)$ is weak* dense in E^* even in the case when A is not densely defined in E. This follows from [**235**, Lemma 2.4] and the proof of [**14**, Proposition B.10].

EXAMPLE 1.1.15 (Multiplication Operators). Let $(\Omega, \mathcal{R}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{C}$ be a measurable function. Put

$$\mathrm{Essran} f := \big\{\lambda \in \mathbb{C} : \mu(\{x \in \Omega : |f(x) - \lambda| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\big\}.$$

Suppose that $\Omega \subseteq \mathbb{R}^n$ and that open subsets of Ω are measurable with non-zero measure; then it can be simply verified that, for every continuous function $f : \Omega \to \mathbb{C}$, we have $\mathrm{Essran} f = \overline{\mathrm{R}(f)}$. Define, for $p \in [1, \infty]$, the multiplication operator A_f in $L^p(\Omega, \mu)$ by setting: $D(A_f) =: \{g \in L^p(\Omega, \mu) : fg \in L^p(\Omega, \mu)\}$ and $A_f g := fg$, $g \in D(A_f)$. Then the following holds:

(i) A_f is a closed operator, and $A_f \in L(L^p(\Omega, \mu))$ iff $f \in L^\infty(\Omega, \mu)$.
(ii) Suppose $f \notin L^\infty(\Omega, \mu)$. Then A_f is densely defined iff $p < \infty$.
(iii) $\sigma(A_f) = \mathrm{Essran} f$.

DEFINITION 1.1.16 (Kunstmann [**249**]). A closed linear operator A is said to be *stationary dense* iff

$$n(A) := \inf\big\{k \in \mathbb{N}_0 : D(A^m) \subseteq \overline{D(A^{m+1})} \text{ for all } m \geqslant k\big\} < \infty.$$

Generally, a densely defined operator A need not be stationary dense, but in the case $\rho(A) \neq \emptyset$, A must be stationary dense with $n(A) = 0$. Furthermore, if A is not necessarily densely defined and $\rho(A) \neq \emptyset$, then one can simply prove that $n(A) = \inf\{k \in \mathbb{N}_0 : D(A^k) \subseteq \overline{D(A^{k+1})}\}$.

1.2. C-regularized semigroups and cosine functions

Throughout this section we assume that $L(E) \ni C$ is injective. Recall, the *C-resolvent set* of a closed linear operator A, denoted by $\rho_C(A)$, is defined as the set of all complex numbers λ such that the operator $\lambda - A$ is injective and that $\mathrm{R}(C) \subseteq \mathrm{R}(\lambda - A)$.

DEFINITION 1.2.1. Let $\tau \in (0, \infty]$. A strongly continuous family $(T(t))_{t \in [0,\tau)}$, resp. $(C(t))_{t \in [0,\tau)}$, in $L(E)$ is said to be a (*local,* if $\tau < \infty$) *C-regularized semigroup*, resp. *C-regularized cosine function*, if:

(i.1) $T(t + s)C = T(t)T(s)$, for all t, $s \in [0, \tau)$ with $t + s < \tau$,
(i.2) $T(0) = C$,

resp.,

(ii.1) $C(t + s)C + C(|t - s|)C = 2C(t)C(s)$, for all t, $s \in [0, \tau)$ with $t + s < \tau$,
(ii.2) $C(0) = C$.

A closed linear operator A which satisfies:

(i.3) $T(t)A \subseteq AT(t)$, $t \in [0, \tau)$,
(i.4) $\int_0^t T(s)x\, ds \in D(A)$, $t \in [0, \tau)$, $x \in E$ and $A \int_0^t T(s)x\, ds = T(t)x - Cx$, $t \in [0, \tau)$, $x \in E$,

resp.,

(ii.3) $C(t)A \subseteq AC(t)$, $t \in [0, \tau)$,

(ii.4) $\int_0^t (t-s)C(s)x\,ds \in D(A)$, $t \in [0, \tau)$, $x \in E$ and $A\int_0^t (t-s)C(s)x\,ds = C(t)x - Cx$, $t \in [0, \tau)$, $x \in E$,

is called a *subgenerator* of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$. It is said that a global C-regularized semigroup $(T(t))_{t\geqslant 0}$, resp. a global C-regularized cosine function $(C(t))_{t\geqslant 0}$, is *exponentially bounded* if there exist $M > 0$ and $\omega \geqslant 0$ such that $\|T(t)\| \leqslant Me^{\omega t}$, resp. $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

The *(integral) generator* of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, is defined by

$$\hat{A} := \Big\{(x,y) \in E \times E : T(t)x - Cx = \int_0^t T(s)y\,ds,\ t \in [0,\tau)\Big\}, \text{ resp.}$$

$$\hat{A} := \Big\{(x,y) \in E \times E : C(t)x - Cx = \int_0^t (t-s)C(s)y\,ds,\ t \in [0,\tau)\Big\},$$

and it is the maximal subgenerator of $(T(t))_{t\geqslant 0}$, resp. $(C(t))_{t\geqslant 0}$. In both cases, $C^{-1}\hat{A}C = \hat{A}$. Moreover, the integral generator of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, coincides with the (infinitesimal) generator $\hat{A}$ of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, defined by:

$$\Big\{(x,y) \in E \times E : \lim_{t\to 0+} \frac{T(t)x - Cx}{t} = Cy\Big\}, \text{ resp.}$$

$$\Big\{(x,y) \in E \times E : \lim_{t\to 0+} 2\frac{C(t)x - Cx}{t^2} = Cy\Big\}.$$

In the case $C = I$ or $\rho(\hat{A}) \neq \emptyset$, the set of all subgenerators of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, denoted by $\wp(T)$, resp. $\wp(C)$, is monomial. In general, the set $\wp(T)$, resp. $\wp(C)$, need not be finite and, endowed with corresponding algebraic operations, forms a complete lattice whose partial ordering coincides with the usual set inclusion. For further information concerning such lattices, we refer the reader to [**422, 451**] and Subsection 2.1.1. The well known result of van Casteren [**53**] says that, for every local semigroup $(T(t))_{t\in[0,\tau)}$, i.e., local I-regularized semigroup, there exists a strongly continuous semigroup $(\tilde{T}(t))_{t\geqslant 0}$ such that $\tilde{T}(t) = T(t)$, $t \in [0, \tau)$. The same assertion holds for local cosine functions, but in general, a local C-regularized semigroup ($C \neq I$), resp. C-regularized cosine function, need not be extendible to a larger interval. Every strongly continuous semigroup (global cosine function) must be exponentially bounded and this is no longer true for global C-regularized semigroups and cosine functions.

The following Hille–Yosida characterization of global exponentially bounded C-regularized semigroups and cosine functions will be proved in Subsection 2.1.2 in a more general context.

Theorem 1.2.2 (Hille–Yosida). *Let A be densely defined and let $CA \subseteq AC$.*

(i) *The operator A is a subgenerator of an exponentially bounded C-regularized semigroup $(T(t))_{t\geqslant 0}$ satisfying $\|T(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for appropriate constants $M > 0$*

and $\omega \in \mathbb{R}$ *iff* $(\omega, \infty) \subseteq \rho_C(A)$, *the mapping* $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda > \omega$ *is infinitely differentiable and*

$$\left\| \frac{d^k}{d\lambda^k}[(\lambda - A)^{-1}C] \right\| \leqslant \frac{Mk!}{(\lambda - \omega)^{k+1}}, \quad k \in \mathbb{N}_0, \ \lambda > \omega.$$

(ii) *The operator* A *is a subgenerator of an exponentially bounded* C*-regularized cosine function* $(C(t))_{t\geqslant 0}$ *satisfying* $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ *for appropriate constants* $M > 0$ *and* $\omega \geqslant 0$ *iff* $(\omega^2, \infty) \subseteq \rho_C(A)$, *the mapping* $\lambda \mapsto \lambda(\lambda^2 - A)^{-1}C$, $\lambda > \omega$ *is infinitely differentiable and*

$$\left\| \frac{d^k}{d\lambda^k}[\lambda(\lambda^2 - A)^{-1}C] \right\| \leqslant \frac{Mk!}{(\lambda - \omega)^{k+1}}, \quad k \in \mathbb{N}_0, \ \lambda > \omega.$$

The definition of an analytic C-regularized semigroup was introduced independently by deLaubenfels [**92**] and Tanaka [**398**].

Definition 1.2.3. (i) Let $\alpha \in (0, \frac{\pi}{2}]$. A C-regularized semigroup $(T(t))_{t\geqslant 0}$ is said to be an *analytic* C*-regularized semigroup of angle* α if there exists a function $\mathbf{T} : \Sigma_\alpha \cup \{0\} \to L(E)$ such that $\mathbf{T}(t) = T(t)$, $t \geqslant 0$ and:

(i.1) the mapping $z \mapsto \mathbf{T}(z)$, $z \in \Sigma_\alpha$ is analytic,
(i.2) $\mathbf{T}(z)\mathbf{T}(\omega) = \mathbf{T}(z + \omega)C$, $z, \ \omega \in \Sigma_\alpha$ and
(i.3) the mapping $z \mapsto \mathbf{T}(z)x$, $z \in \overline{\Sigma_\beta}$ is continuous for every fixed $x \in E$ and $\beta \in (0, \alpha)$.

(ii) [**89**] An *entire* C*-regularized group* is an entire family of bounded linear operators $(T(z))_{z\in\mathbb{C}}$ such that $T(0) = C$ and $T(z+\omega)C = T(z)T(\omega)$, $z, \ \omega \in \mathbb{C}$. The generator of an entire C-regularized group is said to be the generator of $(T(t))_{t\geqslant 0}$.

Theorem 1.2.4. *Assume* $M > 0$, $\omega \geqslant 0$ *and* A *is a subgenerator of an exponentially bounded* C*-regularized cosine function* $(C(t))_{t\geqslant 0}$. *Then* A *is a subgenerator of an exponentially bounded, analytic* C*-regularized semigroup* $(T(t))_{t\geqslant 0}$ *of angle* $\frac{\pi}{2}$, *where* $T(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C(s)x \, ds$, $t > 0$, $x \in E$. *Furthermore,* $\|T(t)\| \leqslant 2Me^{\omega^2 t}$, $t \geqslant 0$ *provided* $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

Proposition 1.2.5. *Suppose* A *is a subgenerator of a* (*local*) C*-regularized semigroup* $(S(t))_{t\in[0,\tau)}$, *resp.* C*-regularized cosine function* $(C(t))_{t\in[0,\tau)}$. *Then* $T(t)T(s) = T(s)T(t)$, *resp.* $C(t)C(s) = C(s)C(t)$ *for all* $t, \ s \in [0, \tau)$ *and* $\mathrm{R}(T(t)) \subseteq \overline{D(A)}$, $t \in [0, \tau)$, *resp.* $\mathrm{R}(C(t)) \subseteq \overline{D(A)}$, $t \in [0, \tau)$.

Proposition 1.2.6. *Suppose* $(T(t))_{t\in[0,\tau)}$, *resp.* $(C(t))_{t\in[0,\tau)}$, *is a strongly continuous family in* $L(E)$ *and* A *is a closed linear operator. If*

$$\mathrm{R}\left(\int_0^t S(s)\, ds\right) \subseteq D(A) \ \textit{and} \ \int_0^t S(s)A\, ds \subseteq A\int_0^t S(s)\, ds = S(t) - C \ \textit{for all} \ t \in [0, \tau),$$

resp.,

$$\mathrm{R}\left(\int_0^t (t-s)C(s)\, ds\right) \subseteq D(A) \ \textit{and} \ \int_0^t (t-s)C(s)A\, ds \subseteq A\int_0^t (t-s)C(s)\, ds = C(t) - C,$$

for all $t \in [0,\tau)$, *then* $(T(t))_{t\in[0,\tau)}$ *is a (local) C-regularized semigroup, resp.* $(C(t))_{t\in[0,\tau)}$ *is a (local) C-regularized cosine function, generated by* $C^{-1}AC$.

The following is an extension type theorem for local C-regularized semigroups and cosine functions; in Subsection 2.1.1, we will consider automatic extension type theorems for local convoluted C-semigroups and cosine functions. It seems that the assertions of Theorem 2.1.9 and Theorem 2.1.14 (proved in the case $n = 2$) can be additionally refined following the approach of Wang and Gao [**424**]:

THEOREM 1.2.7. *Suppose that A is a subgenerator (the integral generator) of a local C-regularized semigroup $(T(t))_{t\in[0,\tau)}$, resp. C-regularized cosine function $(C(t))_{t\in[0,\tau)}$. Then, for every $n \in \mathbb{N}$, A is a subgenerator (the integral generator) of a local C^n-regularized semigroup $(T_n(t))_{t\in[0,n\tau)}$, resp. C^n-regularized cosine function $(C_n(t))_{t\in[0,n\tau)}$.*

The most important additive perturbation results for (local) C-regularized semigroups and cosine functions have been proved by Shaw and his collaborators:

THEOREM 1.2.8. [**381**] (i) *Assume $(T(t))_{t\in[0,\tau)}$ is a (local, global exponentially bounded) C-regularized semigroup having A as a subgenerator, resp. the integral generator, $B \in L(E)$, $\mathrm{R}(B) \subseteq \mathrm{R}(C)$ and $BCx = CBx$, $x \in D(A)$. Then $A+B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) C-regularized semigroup $(T_B(t))_{t\in[0,\tau)}$ which satisfies the integral equation:*

$$T_B(t)x = T(t)x + \int_0^t T(t-s)C^{-1}BT_B(s)x\,ds, \;\; t \in [0,\tau), \; x \in E.$$

(ii) *Assume $(C(t))_{t\in[0,\tau)}$ is a (local, global exponentially bounded) C-regularized cosine function having A as a subgenerator, resp. the integral generator, $B \in L(E)$, $\mathrm{R}(B) \subseteq \mathrm{R}(C)$ and $BCx = CBx$, $x \in D(A)$. Then $A + B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) C-regularized cosine function $(C_B(t))_{t\in[0,\tau)}$ which satisfies the integral equation:*

$$C_B(t)x = C(t)x + \int_0^t \int_0^{t-s} C(t-r)C^{-1}BC_B(s)x\,dr\,ds, \;\; t \in [0,\tau), \; x \in E.$$

EXAMPLE 1.2.9. (i) [**89**, Example 8.6] Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $E := L^p(\Omega)$, $1 \leqslant p < \infty$. Put $D(A) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $A := \Delta$. Then $-A$ generates an entire C-regularized group for some injective operator $C \in L(L^p(\Omega))$.

(ii) [**100**, Example 2.11], [**422**, Example 2.14] Let $E := L^\infty(\mathbb{R})$ and let $G := d/dx$ with maximal domain. Put $D(A) := D(G^2)$, $Ax := Gx$, $x \in D(A)$ and $(T(t)f)x := (R(1:G)^2 f)(x+t)$, $t \geqslant 0$, $x \in \mathbb{R}$, $f \in E$. Then A is closable, $(T(t))_{t\geqslant 0}$ is a global $R(1:G)^2$-regularized semigroup generated by G, $\overline{A} \in \wp(T)$, $|\wp(T)| = \infty$ and every subgenerator of $(T(t))_{t\geqslant 0}$ contains $\overline{A}$.

(iii) [**89**, Example 16.3] Let $E := \{f : \mathbb{R} \to \mathbb{C}$ is continuous $: \lim_{|x|\to\infty} e^{x^2} f(x) = 0\}$, $\|f\| := \sup_{x\in\mathbb{R}} |e^{x^2} f(x)|$, $f \in E$ and $A := \frac{d}{dx}$ with maximal domain. Then there does not exist an injective operator $C \in L(E)$ such that A generates a global C-regularized semigroup.

(iv) [**381**] Let $E := L^2([1,\infty))$ and let $(C(t)f)(s) := \frac{1}{2}(e^{st} + e^{-st})e^{-s}f(s)$, $t \in [0,1)$, $s \geqslant 1$, $f \in E$. Then $(C(t))_{t\in[0,1)}$ is a local $C(0)$-regularized cosine function, $2\|C(t)\| = e^{t-1} + e^{-t-1}$, $t \in [0,1)$ and $(C(t))_{t\in[0,1)}$ cannot be extended beyond the interval $[0,1]$.

(v) [**420**, Example 3.2] Let $E := L^1(\mathbb{R})$ and let $A := -\frac{d^2}{ds^2}$ with maximal domain. Put

$$G_t(s) := \int\limits_{-\infty}^{\infty} e^{-isu}\cosh(tu)\, e^{-u^2}\, du, \quad t \geqslant 0, \ s \in \mathbb{R},$$

$$(C(t)f)(s) := \int\limits_{-\infty}^{\infty} G_t(s-u)f(u)\, du, \quad t \geqslant 0, \ s \in \mathbb{R}, \ f \in E.$$

Then $(C(t))_{t\geqslant 0}$ is a global not exponentially bounded $C(0)$-regularized cosine function and $\|C(t)\| \geqslant \sqrt{\pi}e^{t^2/4}$, $t \geqslant 0$.

(vi) [**89, 454, 221**] The one-dimensional equation describing sound propagation in a viscous gas [**138**] has the form $u_{tt} = 2u_{txx} + u_{xx}$ and after standard matrix reduction to a first order system becomes

$$\frac{d}{dt}\vec{u}(t) = P(D)\vec{u}(t), \ t \geqslant 0, \ \text{ where } D \equiv -i\frac{d}{dx} \text{ and } P(x) \equiv \begin{bmatrix} 0 & 1 \\ -x^2 & -2x^2 \end{bmatrix}.$$

We assume that E is a function space on which translations are uniformly bounded and strongly continuous; herein it is worth noting that E can be consisted of functions defined on some bounded domain (cf. [**454**, p. 189] for further information). Further on, we assume that iD, resp. $\Delta \equiv -D^2$, generates a bounded strongly continuous group, resp. a strongly continuous semigroup in E, and that the operator $P(D)$ is taken with its maximal domain in $E \times E$ [**89, 454**]. Then the following holds (for the definition of fractional powers and the functional calculus for commuting generators of bounded C_0-groups, see [**89**, Section XII]):

(vi.1) Let $r > \frac{1}{2}$. Then $\overline{P(D)}$ generates an exponentially bounded, analytic $(1-\Delta)^{-r}$-regularized semigroup $(T_r(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$. Furthermore, the semigroup $(T_r(t))_{t\in\Sigma_{\frac{\pi}{2}}}$ can be extended to $\overline{\Sigma_{\frac{\pi}{2}}}$ and there exists $K > 0$ such that $\|T_r(t)\| \leqslant K(1+|t|)^{\frac{3}{2}} e^{\frac{1}{2}t\sin(\arg(t))}$, $t \in \overline{\Sigma_{\frac{\pi}{2}}} \smallsetminus \{0\}$.

(vi.2) $(T_r(it))_{t\in\mathbb{R}}$ is an exponentially bounded $(1-\Delta)^{-r}$-regularized group generated by $i\overline{P(D)}$.

(vi.3) The mapping $t \mapsto T_r(t)\binom{f}{g}$, $t \in \overline{\Sigma_{\frac{\pi}{2}}}$ is continuous for every fixed pair $\binom{f}{g} \in E \times E$.

(vi.4) The mapping $t \mapsto T_r(t)$, $t \in \overline{\Sigma_{\frac{\pi}{2}}}$ is norm continuous provided $r > \frac{3}{4}$.

(vii) [**43, 125, 219, 454, 461**] The isothermal motion of a one-dimensional body with small viscosity and capillarity is described, in the simplest situation, by the system:

$$\begin{cases} u_t = 2au_{xxx} + bv_x - cv_{xxx}, \\ v_t = u_x, \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where a, b and c are positive constants. The associated polynomial matrix is $P(x) \equiv \left[\begin{smallmatrix} -2ax^2 & ibx+icx^3 \\ ix & 0 \end{smallmatrix}\right]$. It is well known [**125**] that $P(D)$ does not generate a strongly continuous semigroup in $L^1(\mathbb{R}) \times L^1(\mathbb{R})$.

(vii.1) Let $a^2 - c < 0$ and $r' \geqslant \frac{1}{2}$. Then $\overline{P(D)}$ generates an analytic $(1-\Delta)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geqslant 0}$ of angle $\arctan \frac{a}{\sqrt{c-a^2}}$ and there exists a function $p : (-\arctan \frac{a}{\sqrt{c-a^2}}, \arctan \frac{a}{\sqrt{c-a^2}}) \to (0,\infty)$ such that $\|T_{r'}(z)\| \leqslant K(1+|z|)^{\frac{3}{2}} e^{p(\arg(z))\sin(\arg(z))|z|}$, $z \in \Sigma_{\arctan \frac{a}{\sqrt{c-a^2}}}$.

(vii.2) Let $a^2 - c > 0$ and $r' \geqslant \frac{1}{2}$. Then $\overline{P(D)}$ generates an analytic $(1-\Delta)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$ which satisfies $\|T_{r'}(z)\| \leqslant K(1+|z|)^{\frac{3}{2}} e^{\sqrt{b}\sin(\arg(z))|z|}$, $z \in \Sigma_{\frac{\pi}{2}}$.

(vii.3) Let $a^2 - c > 0$ and $r' \geqslant 1$. Then $\overline{P(D)}$ generates an exponentially bounded $(1-\Delta)^{-r'}$-regularized cosine function $(C_{r'}(t))_{t\geqslant 0}$.

It would take too long to go into a further analysis of C–regularized semigroups and cosine functions. We strongly recommend for the reader [**14**], [**27**], [**87**], [**125**], [**128**], [**155**], [**181**], [**201**], [**298**], [**322**], [**355**] and [**390**] for the theory of strongly continuous semigroups and cosine functions as well as [**89**]–[**104**], [**152**], [**260**]–[**261**], [**272**], [**324**], [**381**]–[**383**], [**398**], [**403**]–[**404**], [**421**], [**424**], [**431**], [**434**], [**436**]–[**440**], [**451**]–[**454**], [**460**] and [**464**] for the theory of C-regularized semigroups and cosine functions.

In the remaining part of this section, we study regularization of different types of operator semigroups that are strongly continuous for $t > 0$. Let us recall that a one-parameter family $(T(t))_{t\geqslant 0}$ in $L(E)$ is called a *semigroup* if $T(t+s) = T(t)T(s)$, $t, s \geqslant 0$, $T(0) = I$ and the mapping $t \mapsto T(t)x$, $t > 0$ is continuous for every fixed $x \in E$. The infinitesimal generator A_0 of the semigroup $(T(t))_{t\geqslant 0}$ is defined by $A_0x := \lim_{t\to 0+} \frac{T(t)x-x}{t}$ whenever the above limit exists. If A_0 is closable, then the operator $\overline{A_0}$ is called the *complete infinitesimal generator,* in short the c.i.g., of $(T(t))_{t\geqslant 0}$. Following Kunstmann [**253**], we introduce the generator of $(T(t))_{t\geqslant 0}$ by $A := \{(x, y) \in E \times E : (T(s)x, T(s)y) \in A_0$ for every $s \geqslant 0\}$. The generator A of $(T(t))_{t\geqslant 0}$ is a closed, linear operator in E. The set $\Sigma := \{x \in E : \lim_{t\to 0+} T(t)x = x\}$ is called the *continuity set* of $(T(t))_{t\geqslant 0}$. Note, if $(T(t))_{t\geqslant 0}$ is a semigroup, then the limit $\omega_0 := \lim_{t\to+\infty} \frac{\ln\|T(t)\|}{t}$ exists and $\omega_0 \in [-\infty, \infty)$; such a number ω_0 is called the *type of* $(T(t))_{t\geqslant 0}$. Now we recall the basic facts about semigroups of growth order $r > 0$ which were introduced by G. Da Prato [**84**] in 1966. A fairly complete information on the general theory of this class of operator semigroups can be obtained by consulting the papers [**324**]–[**326**], [**358**], [**389**], [**394**], [**398**] and [**449**]–[**450**].

The following notion will be generalized in Section 1.4.

DEFINITION 1.2.10. [**84**] An operator family $(T(t))_{t>0}$ in $L(E)$ is said to be a *semigroup of growth order* $r>0$ iff the following holds:

(i) $T(t+s)=T(t)T(s)$, $t,s>0$,
(ii) for every $x\in E$, the mapping $t\mapsto T(t)x$, $t>0$ is continuous,
(iii) $\|t^rT(t)\|=O(1)$, $t\to 0+$,
(iv) $T(t)x=0$ for all $t>0$ implies $x=0$, and
(v) $E_0:=\bigcup_{t>0}T(t)E$ is dense in E.

The infinitesimal generator A_0 of $(T(t))_{t>0}$ always exists and A_0 is a closable linear operator. The closure $A:=\overline{A_0}$ is called the complete infinitesimal generator, in short, c.i.g., of $(T(t))_{t>0}$.

DEFINITION 1.2.11. [**398**] Let $(T(t))_{t>0}$ be a semigroup of growth order $r>0$. If $(T(t))_{t>0}$ has an analytic extension to Σ_γ, for some $\gamma\in(0,\frac{\pi}{2}]$, denoted by the same symbol, and if additionally there exists $\omega\in\mathbb{R}$ such that, for every $\delta\in(0,\gamma)$, there exists a suitable constant $M_\delta>0$ with $\|z^rT(z)\|\leqslant M_\delta e^{\omega\operatorname{Re}z}$, $z\in\Sigma_\delta$, then the family $(T(t))_{t\in\Sigma_\gamma}$ is called an *analytic semigroup of growth order* r.

Let $t\in\mathbb{R}$. Henceforward $\lfloor t\rfloor$ and $\lceil t\rceil$ denote the largest integer $\leqslant t$ and the smallest integer $\geqslant t$, respectively, and $\Gamma(\cdot)$ denotes the *Gamma function.* The following generation results for (analytic) semigroups of growth order $r>0$ were established by Okazawa [**348**], Zabreiko–Zafievskii [**449**] and Tanaka [**398**].

THEOREM 1.2.12. (i) [**348**] *Let $r>0$ and $n=\lfloor r\rfloor$. A closed linear operator A is the c.i.g. of a semigroup of growth order $r>0$ iff the following holds:*

(i.1) *There exists $\omega\in\mathbb{R}$ such that $D(A^{n+1})\subseteq\mathrm{R}(\lambda-A)$ and that $\lambda-A$ is injective for all $\lambda>\omega$.*
(i.2) *There exists $M>0$ such that, for every $x\in D(A^{n+1})$:*

$$\|(\lambda-A)^{-m}x\|\leqslant\frac{M}{(m-1)!}\frac{\Gamma(m-r)}{(\lambda-\omega)^{m-r}}\|x\|,\ \ \lambda>\omega,\ m=k(n+1),\ k\in\mathbb{N},$$

(i.3) *$D(A)$ is dense in E, $D(A^{n+2})$ is a core for A and*
(i.4) *There exists $b\in(\omega,\infty)$ such that $(b-A)^{n+1}$ is closable.*

(ii) [**449**] *Let $r\in(0,1)$. Then A is the c.i.g. of a semigroup of growth order $r>0$ iff the following holds:*

(ii.1) *There exists $\omega\in\mathbb{R}$ such that $(\omega,\infty)\subseteq\rho(A)$.*
(ii.2) *There exists $M>0$ such that, for every $\lambda\in(\omega,\infty)$ and $m\in\mathbb{N}$,*

$$\|(\lambda-A)^{-m}\|\leqslant\frac{M}{(m-1)!}\frac{\Gamma(m-r)}{(\lambda-\omega)^{m-r}}\ \ \textit{and}$$

(ii.3) *$D(A)$ is dense in E.*

(iii) [**398**] *Let $r>0$, $\alpha\in(0,\frac{\pi}{2}]$ and $n=\lfloor r\rfloor$. A closed linear operator A is the c.i.g. of an analytic semigroup $(T(t))_{t\in\Sigma_\alpha}$ of growth order $r>0$ iff the following holds:*

(iii.1) *There exists $\omega\in\mathbb{R}$ such that $D(A^{n+1})\subseteq\mathrm{R}(\lambda-A)$ and that $\lambda-A$ is injective for all $\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\alpha}$.*

(iii.2) *For every* $\varepsilon \in (0, \alpha)$, *there exists* $M_\varepsilon > 0$ *such that, for every* $x \in D(A^{n+1})$ *and* $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha-\varepsilon}$:

$$\|(\lambda - A)^{-(n+1)}x\| \leqslant \frac{M_\varepsilon}{n!} \frac{\Gamma(n+1-r)}{|\lambda - \omega|^{n+1-r}}\|x\|$$

and the mapping $\lambda \mapsto \overline{(\lambda - A)^{-(n+1)}_{|D(A^{n+1})}}$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}$ *is analytic.*

(iii.3) $D(A)$ *is dense in* E, $D(A^{n+2})$ *is a core for* A *and*

(iii.4) *There exists* $b \in (\omega, \infty)$ *such that* $(b - A)^{n+1}$ *is closable.*

(iv) [**398**] *Let* $r \in (0, 1)$ *and* $\alpha \in (0, \frac{\pi}{2}]$. *Then* A *is the c.i.g. of an analytic semigroup* $(T(t))_{t\in\Sigma_\alpha}$ *of growth order* r *iff the following holds:*

(ii.1) *There exists* $\omega \in \mathbb{R}$ *such that* $\omega + \Sigma_{\frac{\pi}{2}+\alpha} \subseteq \rho(A)$.

(ii.2) *For every* $\varepsilon \in (0, \alpha)$, *there is* $M_\varepsilon > 0$ *such that, for every* $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha-\varepsilon}$:

$$\|(\lambda - A)^{-1}\| \leqslant \frac{M_\varepsilon \Gamma(1-r)}{|\lambda - \omega|^{1-r}} \text{ and}$$

(ii.3) $D(A)$ *is dense in* E.

Let A be the c.i.g. of an (analytic) semigroup of growth order $r > 0$. Then there exists a Banach space that is densely and continuously embedded in E on which A generates an (analytic) strongly continuous semigroup (of the same angle); for a proof, see [**358**].

DEFINITION 1.2.13. ([**324**], cf. also [**326**] and [**349**]) Suppose $\mathrm{R}(C)$ is dense in E and $(T(t))_{t\geqslant 0}$ is an exponentially bounded C-regularized semigroup. The complete infinitesimal generator, in short c.i.g, of $(T(t))_{t\geqslant 0}$ is defined as the closure $\overline{G}$ of the operator G, where

$$G = \Big\{(x, y) \in E \times E : x \in \mathrm{R}(C),\ \lim_{t\to 0+} \frac{C^{-1}T(t)x - x}{t} = y\Big\}.$$

It is well known that the operator G is closable and that the operator $\overline{G}$ satisfies $\overline{G} \subseteq \hat{A}$ and $C^{-1}\overline{G}C = \hat{A}$, where $\hat{A}$ is the integral generator of $(T(t))_{t\geqslant 0}$. Furthermore, it can be easily seen by the use of [**324**, Lemma 1.2, p. 361] and elementary operational properties of the Laplace transform that $\overline{G}$ is a subgenerator of $(T(t))_{t\geqslant 0}$. In general, it is not known whether the c.i.g. of $(T(t))_{t\geqslant 0}$ coincides with $\hat{A}$.

DEFINITION 1.2.14. [**343**] Let $k \in \mathbb{N}_0$. A semigroup $(T(t))_{t\geqslant 0}$ is said to be of *class* $(C_{(k)})$ iff the following conditions hold:

(i) $E_0 := \bigcup_{t>0} \mathrm{R}(T(t))$ is dense in E,

(ii) there exists $\omega \in (\omega_0, \infty)$ such that for every $\lambda \in (\omega, \infty)$ there exists an injective bounded linear operator R_λ so that $R_\lambda x = \int_0^\infty e^{-\lambda t}T(t)x\,dt$, $x \in E_0$ and

(iii) $D(A^k) \subseteq \Sigma$, where A is the c.i.g. of $(T(t))_{t\geqslant 0}$.

Notice that the conditions (i)–(ii) imply the existence of the c.i.g. A of $(T(t))_{t\geqslant 0}$ and $(\omega, \infty) \subseteq \rho(A)$. It is checked at once that every semigroup $(T(t))_{t\geqslant 0}$ of class

$(C_{(k)})$ is also of class $(C_{(k+1)})$ and that the class $(C_{(0)})$ coincides with the usual class of strongly continuous semigroups. The generation results for semigroups of class $(C_{(k)})$ can be found in [**324**] and [**343**].

THEOREM 1.2.15. [**324, 398**] (i) *Suppose that* $(T(t))_{t\geqslant 0}$ *is a semigroup of class* $(C_{(k)})$ *and that* A *is the c.i.g. of* $(T(t))_{t\geqslant 0}$. *Put* $S(t) := R(\omega+1 : A)^k T(t)$, $t \geqslant 0$. *Then* A *is the c.i.g. of the exponentially bounded* C*-regularized semigroup* $(S(t))_{t\geqslant 0}$.

(ii) *Suppose that* $(T(t))_{t\geqslant 0}$ *is a semigroup of growth order* $r > 0$ *and that* A *is the c.i.g. of* $(T(t))_{t\geqslant 0}$. *Put*

$$Cx := \frac{1}{n!}\int\limits_0^\infty e^{-(\omega_0+1)t} t^n T(t)x\,dt \text{ and } S(t) := CT(t),\ x \in E,\ t \geqslant 0,$$

where ω_0 *is the type of* $(T(t))_{t\geqslant 0}$ *and* $n = \lfloor r \rfloor$. *Then* $L(E) \ni C$ *is injective and* A *is the c.i.g. of the exponentially bounded* C*-regularized semigroup* $(S(t))_{t\geqslant 0}$.

(iii) *Let* $\alpha \in (0, \frac{\pi}{2}]$. *Suppose that* $(T(t))_{t\in\Sigma_\alpha}$ *is an analytic semigroup of growth order* $r > 0$ *and that* A *is the c.i.g. of* $(T(t))_{t\geqslant 0}$. *Define* C *as above and put* $S(t) := CT(t)$, $t \in \Sigma_\alpha$. *Then* A *is the c.i.g. of the exponentially bounded, analytic* C*-regularized semigroup* $(S(t))_{t\in\Sigma_\alpha}$.

The following theorem presents a most valuable result with regard to regularization of semigroups that are strongly continuous in $t > 0$.

THEOREM 1.2.16. [**253**] *Suppose* A *is the integral generator of a semigroup* $(T(t))_{t\geqslant 0}$ *which satisfies* $\bigcap_{t>0} \operatorname{Kern}(T(t)) = \{0\}$. *Then there exists an injective operator* $C \in L(E)$ *such that* A *is the integral generator of a global* C*-regularized semigroup* $(S(t))_{t\geqslant 0}$.

Let $P(x) = [p_{ij}(x)]_{m\times m}$, $x \in \mathbb{R}^n$ be an $m \times m$ polynomial matrix and let $\lambda_j(x)$, $1 \leqslant j \leqslant m$ be the eigenvalues of $P(x)$, $x \in \mathbb{R}^n$; see [**454**] for the definition of the closable operator $P(A)$. Set $k := 1 + \lfloor \frac{n}{2} \rfloor$, $\Lambda(P(x)) := \sup_{1\leqslant j\leqslant m} \operatorname{Re}\lambda_j(x)$, $x \in \mathbb{R}^n$, $N := \max(\operatorname{dg}(p_{ij}(x)))$ and assume that $r \in (0, N]$. Then it is said that $P(x)$ is *Shilov* r*-parabolic* [**138**] iff there exist $\omega > 0$ and $\omega' \in \mathbb{R}$ such that $\Lambda(P(x)) \leqslant -\omega|x|^r + \omega'$, $x \in \mathbb{R}^n$; in the case $r = N$, it is also said that $P(x)$ is *Petrovskii parabolic.* In what follows, we discuss the properties of various types of abstract Shilov parabolic systems. First of all, define

$$\pi_1(r) := \min_{1\leqslant j\leqslant m,\ |x|=r} |\lambda_j(x)|,\ r > 0, \quad \pi_2(r) := \max_{1\leqslant j\leqslant m,\ |x|=r} |\lambda_j(x)|,\ r > 0,$$

$$S(P) := \bigcup_{x\in\mathbb{R}^n,\ 1\leqslant j\leqslant m} \lambda_j(x)$$

and notice that a corollary of Seidenberg–Tarski theorem (cf. [**142**] and [**410**, Lemma 10.2]) implies that there exist real numbers a_1, a_2, α_1 and α_2 such that $\pi_1(r) = a_1 r^{\alpha_1}(1 + o(1))$ as $r \to \infty$ and $\pi_2(r) = a_2 r^{\alpha_2}(1 + o(1))$ as $r \to \infty$. Obviously, $r \leqslant \alpha_1 \leqslant \alpha_2 \leqslant N$ and, by the proof of [**410**, Proposition 10.4], Shilov r-parabolicity of $P(x)$, for some $r \in (0, N]$, implies that there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -\alpha|\operatorname{Im}\lambda|^{r/\alpha_2} + \beta\right\} \cap S(P) = \emptyset$.

THEOREM 1.2.17. [**219**] *Let $P(x)$ be Shilov r-parabolic for some $r \in (0, N)$.*

(i) *Put $\kappa := \frac{1}{r}(N-r)(m-1+n/2)$. Then the operator $\overline{P(A)}$ is the c.i.g. of a semigroup $(T_0(t))_{t>0}$ of growth order κ which additionally satisfies that the mapping $t \mapsto T_0(t)$, $t > 0$ is infinitely differentiable in the uniform operator topology and that there exists $K > 0$ such that:*

$$\Big\|\frac{d^l}{dt^l}T_0(t)\Big\| \leqslant K^l(1+t)^{m-1+\frac{n}{2}}e^{\omega t}l!^{N/r}(1+t^{-\kappa-\frac{Nl}{r}}),\ t>0,$$

where $\omega \equiv \sup_{x\in\mathbb{R}^n} \Lambda(P(x))$.

(ii) *Suppose, additionally, that there exist $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$ such that $\sigma(P(x)) \subseteq \omega + (\mathbb{C} \smallsetminus \Sigma_{\pi/2+\alpha})$, $x \in \mathbb{R}^n$. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order κ.*

REMARK 1.2.18. Set $T_{r'}(t) := \big(e^{tP(x)}(1+|x|^2)^{-r'}\big)(A)$, $t \geqslant 0$, $r' \geqslant 0$. Then the supposition $(N-r)(m-1)+Nl-2r' \leqslant -(N-r)k$ implies

$$\Big\|\frac{d^l}{dt^l}T_{r'}(t)\Big\| \leqslant K^l(1+t)^{m-1+n/2}e^{\omega t},\ \ t>0,$$

and the supposition $(N-r)(m-1)+Nl-2r' \in (-(N-r)k, 0)$ implies

$$\Big\|\frac{d^l}{dt^l}T_{r'}(t)\Big\| \leqslant \begin{cases} K^l l!^{N/r}t^{-\frac{1}{r}((N-r)(m-1+n/2)+Nl-2r')}, & t\in(0,1], \\ K^l l!^{N/r}(1+t)^{m-1+n/2}e^{\omega t}, & t>1. \end{cases}$$

Now we focus our attention to the *numerical range* of $P(x)$, defined by

$$\text{n.r.}(P(x)) := \{(P(x)y, y) : y \in \mathbb{R}^n,\ \|y\| = 1\},\ \ x\in\mathbb{R}^n,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{C}^n$ and $\|y\| \equiv (y,y)^{1/2}$. Set $\tilde{\Lambda}(P(x)) := \sup\{\operatorname{Re} z : z \in \text{n.r.}(P(x))\}$, $x \in \mathbb{R}^n$.

THEOREM 1.2.19. [**219**] *Let $r \in (0, N)$, $\omega' > 0$ and $\omega'' > 0$.*

(i) *Assume $\tilde{\Lambda}(P(x)) \leqslant -\omega'|x|^r + \omega''$, $x \in \mathbb{R}^n$ and put $\kappa_{n.r.} := \frac{n(N-r)}{2r}$. Then the operator $\overline{P(A)}$ is the c.i.g. of a semigroup $(T_0(t))_{t>0}$ of growth order $\kappa_{n.r.}$ which additionally satisfies that the mapping $t \mapsto T_0(t)$, $t > 0$ is infinitely differentiable in the uniform operator topology and that there exists $K > 0$ such that:*

$$\Big\|\frac{d^l}{dt^l}T_0(t)\Big\| \leqslant K^l e^{\omega t} l!^{N/r}(1+t)^{n/2}(1+t^{-\frac{1}{r}((N-r)\frac{n}{2}+Nl)}),\ \ t>0,$$

where $\omega \equiv \sup_{x\in\mathbb{R}^n} \tilde{\Lambda}(P(x))$.

(ii) *Let $\alpha \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$, n.r.$(P(x)) \subseteq \omega + (\mathbb{C} \smallsetminus \Sigma_{\frac{\pi}{2}+\alpha})$, $x \in \mathbb{R}^n$ and let $P(x)$ be Shilov r-parabolic. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order $\frac{nN}{2r}$.*

(iii) *Let $\alpha \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$, n.r.$(P(x)) \subseteq \omega + (\mathbb{C} \smallsetminus \Sigma_{\frac{\pi}{2}+\alpha})$, $x \in \mathbb{R}^n$ and $\tilde{\Lambda}(P(x)) \leqslant -\omega'|x|^r + \omega''$, $x \in \mathbb{R}^n$. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order $\kappa_{n.r.}$.*

REMARK 1.2.20. (i) The decay rate of derivatives of $(T_{r'}(t))_{t\geqslant 0}$ in a neighborhood of zero (cf. Theorem 1.2.19) improves the corresponding one given in the formulation of [**454**, Theorem 3.2] provided $2Nl > (N-r)n$.

(ii) Suppose $p \in (1,\infty)$, $E = L^p(\mathbb{R}^n)$ and set $n_E := n|\frac{1}{2} - \frac{1}{p}|$. Then the growth order of $(T_0(t))_{t>0}$ in Theorem 1.2.17 and Theorem 1.2.19 can be slightly refined by interchanging the term $\frac{n}{2}$ with n_E.

(iii) With some obvious modifications, the assertions of Theorem 1.2.17 and Theorem 1.2.19 remain true in the case $E = C_b(\mathbb{R}^n)$ or $E = L^\infty(\mathbb{R}^n)$.

(iv) Suppose that $P(x)$ is Shilov r-parabolic for some $r \in (0,N)$, and denote by $\Sigma(T_0)$ the continuity set of the semigroup $(T_0(t))_{t>0}$ given in Theorem 1.2.17, resp. Theorem 1.2.19. Then $\Sigma(T_0)$ contains $\mathrm{R}\big((1+|A|^2)^{-r'}\big)$ for all $r' > \frac{1}{2}(N-r) \times (m-1+\frac{n}{2})$, resp. $r' > \frac{1}{4}n(N-r)$ [**454**], and the abstract Cauchy problem

$$(ACP): \begin{cases} \vec{u} \in C([0,\infty):E^m) \cap C^\infty((0,\infty):E^m), \\ \vec{u}'(t) = \overline{p_{ij}(A)}\vec{u}(t),\ t>0, \\ \vec{u}(0) = \vec{x}, \end{cases}$$

has a unique solution for all $\vec{x} \in \Sigma(T_0)$, improving the corresponding result of Zheng and Li (cf. [**454**, Lemma 1.2(b)]). In general, $\mathrm{R}\big((1+|A|^2)^{-r'}\big)$ can be strictly contained in $\Sigma(T_0)$.

(v) Semigroups of growth order $r > 0$ can be also applied in the analysis of time-dependent Shilov parabolic systems ([**454**], [**219**]).

Recall that Webb considered in [**427**] a class of abstract semilinear Volterra equations appearing in thermodynamics of materials with memory [**74, 75**]. An insignificant modification of the proofs of [**427**, Theorems 2.1–2.2, Corollary 2.1] implies the following theorem.

THEOREM 1.2.21. (i) *Assume A is a subgenerator of a (local) C-regularized semigroup $(T(t))_{t\in[0,\tau)}$ and there exists $t_0 \in (0,\tau)$ such that:*

(i.1) $C^{-1}f \in C^1([0,t_0]:E)$,

(i.2) $C^{-1}g \in C([0,t_0]\times D:E)$, *where D is an open subset of $[D(A)]$, $C^{-1}g(t,x)$ is continuously differentiable with respect to t, and for each $x \in D$ there is a neighborhood D_x about x and continuous functions $b:[0,t_0]\to[0,\infty)$ and $c:[0,t_0]\to[0,\infty)$ such that, for every $t\in[0,t_0]$ and $x_1,x_2\in D_x$:*

$$\|C^{-1}g(t,x_1) - C^{-1}g(t,x_2)\| \leqslant b(t)\|x_1-x_2\|_{[D(A)]},$$

$$\Big\|\frac{\partial}{\partial t}C^{-1}g(t,x_1) - \frac{\partial}{\partial t}C^{-1}g(t,x_2)\Big\| \leqslant c(t)\|x_1-x_2\|_{[D(A)]}.$$

Then, for each $x \in C(D)$, there exist a number $t_1 \in (0,t_0)$ and a unique function $u:[0,t_1]\to E$ such that $u \in C^1([0,t_1]:E)\cap C([0,t_1]:[D(A)])$,

$$u'(t) = Au(t) + \int_0^t g(t-s,u(s))ds + f(t),\ t\in[0,t_1] \text{ and } u(0)=x. \tag{4}$$

Assume further $n \in \mathbb{N}$, $x \in C(D(A^n))$, $\tau = \infty$ as well as (i.1) *and* (i.2) *hold with $C^{-1}f$, $C^{-1}g$, $D = D_y$ ($y \in D(A)$), $[0,t_0]$, $b:[0,t_0]\to[0,\infty)$ and $c:[0,t_0]\to$*

$[0,\infty)$, replaced by $C^{-n}f$, $C^{-n}g$, $[D(A)]$, $[0,nt_0]$, $b_n : [0,nt_0] \to [0,\infty)$ and $c_n : [0,nt_0] \to [0,\infty)$, respectively. Then there exists a unique function $u_n : [0,nt_1] \to E$ such that $u_n \in C^1([0,nt_1] : E) \cap C([0,nt_1] : [D(A)])$ and that (4) *holds with $u(t)$ and $[0,t_1]$ replaced by $u_n(t)$ and $[0,nt_1]$, respectively.*

(ii) *Assume $x \in D$,* (i.1)–(i.2) *hold, $M \geqslant 1$, $\omega \in \mathbb{R}$, $\|T(t)\| \leqslant Me^{\omega t}$, $t \in [0,\tau)$ and x_1, $x_2 \in C(D)$. Denote by $u_1(t)$ and $u_2(t)$ the solutions of* (4) *with initial values x_1 and x_2, respectively, and set $\alpha(t) = \int_0^t e^{-\omega s}(b(s)+c(s))\,ds$, $t \in [0,t_1]$ and $\beta(t) = \max_{s\in[0,t]} e^{-\omega s} b(s)$, $t \in [0,t_1]$. Then the assumption $\{u_1(t), u_2(t)\} \subseteq D_x$, $t \in [0,t_1]$ implies:*

$$\|u_1(t) - u_2(t)\| \leqslant M\|C^{-1}x_1 - C^{-1}x_2\|_{[D(A)]}e^{(M\alpha(t)+\beta(t)+Mb(0)+\omega)t},\ t \in [0,t_1].$$

Furthermore, if $D = D_x = [D(A)]$, $x \in D(A)$ and $M\alpha(t) + \beta(t) + Mb(0) + \omega \leqslant \gamma$, for some $\gamma \in \mathbb{R}$ and every $t \in [0,t_1]$, then

$$\|u_1(t) - u_2(t)\| \leqslant M\|C^{-1}x_1 - C^{-1}x_2\|_{[D(A)]}e^{\gamma t},\ \ t \in [0,t_1].$$

1.3. Function spaces

In this section, we shall analyze various types of generalized function spaces used throughout the book. We begin with the recollection of the most important properties of operator valued distribution spaces.

The *Schwartz spaces of test functions* $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ [**2, 397**] carry the usual inductive limit topologies while the topology of the *space of rapidly decreasing functions* $\mathcal{S}$ defines the following system of seminorms

$$p_{m,n}(\psi) := \sup_{x\in\mathbb{R}} |x^m \psi^{(n)}(x)|,\ \psi \in \mathcal{S},\ m,\ n \in \mathbb{N}_0.$$

By $\mathcal{D}_0$ we denote the subspace of $\mathcal{D}$ which consists of the elements supported by $[0,\infty)$. Further on, $\mathcal{D}'(E) := L(\mathcal{D} : E)$, $\mathcal{E}'(E) := L(\mathcal{E} : E)$ and $\mathcal{S}'(E) := L(\mathcal{S} : E)$ are the spaces of continuous linear functions $\mathcal{D} \to E$, $\mathcal{E} \to E$ and $\mathcal{S} \to E$, respectively; $\mathcal{D}_0'(E)$, $\mathcal{E}_0'(E)$ and $\mathcal{S}_0'(E)$ are the subspaces of $\mathcal{D}'(E)$, $\mathcal{E}'(E)$ and $\mathcal{S}'(E)$, respectively, containing the elements supported by $[0,\infty)$. Denote by $\mathcal{B}$ the family of all bounded subsets of $\mathcal{D}$. Put $p_B(f) := \sup_{\varphi\in B}\|f(\varphi)\|$, $f \in \mathcal{D}'(E)$, $B \in \mathcal{B}$. Then p_B, $B \in \mathcal{B}$ is a seminorm on $\mathcal{D}'(E)$ and the system $(p_B)_{B\in\mathcal{B}}$ defines the topology on $\mathcal{D}'(E)$. The topology on $\mathcal{E}'(E)$, resp., $\mathcal{S}'(E)$, is defined similarly. Notice that the spaces $\mathcal{D}(\Omega : E)$, $\mathcal{E}(\Omega : E)$, $\mathcal{D}'(\Omega : E)$ and $\mathcal{E}'(\Omega : E)$, where Ω is an open non-empty subset of $\mathbb{R}^n$, can be defined along the same lines. Let $\rho \in \mathcal{D}$ satisfy $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$ and $\operatorname{supp}\rho \subseteq [0,1]$. By a *regularizing sequence* in $\mathcal{D}$ we mean a sequence (ρ_n) in $\mathcal{D}_0$ obtained by $\rho_n(t) := n\rho(nt)$, $t \in \mathbb{R}$. If φ, $\psi : \mathbb{R} \to \mathbb{C}$ are measurable functions, we use the *convolutions* $\varphi * \psi$ and $\varphi *_0 \psi$ defined by

$$\varphi * \psi(t) := \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,ds \text{ and } \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\,ds,\ t\ \in \mathbb{R}.$$

Notice that $\varphi * \psi = \varphi *_0 \psi$, φ, $\psi \in \mathcal{D}_0$. Given $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$, or $\varphi \in \mathcal{E}$ and $f \in \mathcal{E}'$, we define the convolution $f * \varphi$ by $(f * \varphi)(t) := f(\varphi(t - \cdot))$, $t \in \mathbb{R}$. For $f \in \mathcal{D}'$, or for $f \in \mathcal{E}'$, define $\check{f}$ by $\check{f}(\varphi) := f(\varphi(-\cdot))$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{E}$). Generally, the convolution of two distribution $f,\ g \in \mathcal{D}'$, denoted by $f * g \in \mathcal{D}'$, is defined by $(f * g)(\varphi) := g(\check{f} * \varphi)$, $\varphi \in \mathcal{D}$. It is well known that $\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g$. We transfer the preceding notion to operator valued distributions by means of [**252**, Proposition 1.1].

PROPOSITION 1.3.1. *Suppose X, Y and Z are Banach spaces and $b : X \times Y \to Z$ is bilinear and continuous. Then there is a unique bilinear, separately continuous mapping $*_b : \mathcal{D}_0'(X) \times \mathcal{D}_0'(Y) \to \mathcal{D}_0'(Z)$ such that $(S \otimes x) *_b (T \otimes y) = S * T \otimes b(x, y)$, for all $S,\ T \in \mathcal{D}_0'$ and $x \in X,\ y \in Y$. Moreover, this mapping is continuous.*

We need the following structural theorems for the spaces $\mathcal{D}'(E)$ and $\mathcal{S}'(E)$ (cf. for instance [**307**, Theorem 2.1.1, Theorem 2.1.2]):

THEOREM 1.3.2. (i) *Let $G \in \mathcal{D}'(E)$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}$ be open and bounded. Then there exist a number $n \in \mathbb{N}$ and a continuous function $f : \mathbb{R} \to E$ such that*

$$G(\varphi) = (-1)^n \int_{\mathbb{R}} \varphi^{(n)}(t) f(t)\, dt, \tag{5}$$

for all $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subseteq \Omega$. Furthermore, if $\Omega \subseteq (-\infty,\ a)$ and $G = 0$ on $(-\infty, a)$, then $f(t) = 0$ for $t < a$.

(ii) *Let $G \in \mathcal{S}'(E)$. Then there exist $n \in \mathbb{N}$, $r > 0$ and a continuous function $f : \mathbb{R} \to E$ such that* (5) *holds for all $\varphi \in \mathcal{S}$, and $|f(t)| =_{|t| \to \infty} O(|t|^r)$. Furthermore, if $G = 0$ on $(-\infty,\ a)$, then $f(t) = 0$ for $t < a$.*

Let $k \in \mathbb{N}$, $p \in [1, \infty]$ and let Ω be an open non-empty subset of $\mathbb{R}^n$. Then the *Sobolev space* $W^{k,p}(\Omega : E)$ consists of those operator valued distributions $u \in \mathcal{D}'(\Omega : E)$ such that, for every $i \in \{0, \ldots, k\}$ and for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant k$, one has $D^\alpha u \in L^p(\Omega : E)$. In this place, the derivative D^α is taken in the sense of distributions. Notice that the space $W^{k,p}((0, \tau) : E)$, where $\tau \in (0, \infty)$, can be characterized by means of corresponding spaces of absolutely continuous functions (cf. for example [**27**, Chapter I, Section 2.2]).

In the sequel, we assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and that the following condition is fulfilled:

$$M_p^2 \leqslant M_{p+1} M_{p-1}, \ \ p \in \mathbb{N}. \tag{M.1}$$

Every employment of the conditions:

$$M_p \leqslant AH^p \sup_{0 \leqslant i \leqslant p} M_i M_{p-i}, \ \ p \in \mathbb{N}, \text{ for some } A,\ H > 1, \tag{M.2}$$

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty, \tag{M.3$'$}$$

and the condition

$$\sup_{p \in \mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1} M_{p+1}}{p M_p M_q} < \infty, \tag{M.3}$$

which is slightly stronger than (M.3′), will be explicitly emphasized.

Let $s > 1$. The *Gevrey sequences* $(p!^s)$, (p^{ps}) or $(\Gamma(1+ps))$ satisfy the above conditions. The *associated function* of (M_p) is defined by $M(\rho) := \sup_{p\in\mathbb{N}} \ln \frac{\rho^p}{M_p}$, $\rho > 0$; $M(0) := 0$. If $\lambda \in \mathbb{C}$, then we define $M(\lambda) := M(|\lambda|)$; put $m_p := \frac{M_p}{M_{p-1}}$, $p \in \mathbb{N}$ and notice that, thanks to (M.1), the sequence (m_p) is increasing. We know that the function $t \mapsto M(t)$, $t \geqslant 0$ is increasing as well as that $\lim_{\lambda\to\infty} M(\lambda) = \infty$ and that the function $M(\cdot)$ vanishes in some open neighborhood of zero. Denote by $m(\lambda)$ the number of $m_p \leqslant \lambda$. Since (M_p) satisfies (M.1), it follows that (cf. [**207**, p. 50]) $M(t) = \int_0^t \frac{m(\lambda)}{\lambda}\, d\lambda$, $t \geqslant 0$. This implies that the mapping $t \mapsto M(t)$, $t \geqslant 0$ is absolutely continuous and that the mapping $t \mapsto M(t)$, $t \in [0,\infty) \smallsetminus \{m_p : p \in \mathbb{N}\}$ is continuously differentiable with $M'(t) = \frac{m(t)}{t}$, $t \in [0,\infty) \smallsetminus \{m_p : p \in \mathbb{N}\}$. The following inequalities have been proved by Petzsche [**361**] and Komatsu [**207**]:

(i) If (M_p) satisfies (M.1), then $M(a+b) \leqslant M(2a) + M(2b)$, $a,\ b \geqslant 0$.

(ii) If (M_p) satisfies (M.1) and (M.2), then there exist $K > 0$ and $B > 0$ such that $2M(a) \leqslant M(Ha) + \ln(AM_0)$, $a \geqslant 0$, $M(La) \leqslant \frac{3}{2}LM(a) + K$, $a \geqslant 0$, $L \geqslant 1$ and $LM(a) \leqslant M(B^{L-1}a) + E_L$, $a \geqslant 0$, $L \geqslant 1$ and E_L is a constant depending only on L and (M_p). Herein A and H denotes the constants in (M.2).

In the remnant of this section, we assume that (M_p) satisfies (M.1), (M.2) and (M.3$'$) (cf. [**47, 64, 207**] and [**209**] for different approaches to the theory of ultradistributions). Recall that the spaces of *Beurling,* respectively, *Roumieu ultradifferentiable functions* are defined by $\mathcal{D}^{(M_p)} := \mathcal{D}^{(M_p)}(\mathbb{R}) := \operatorname{ind}\lim_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{(M_p)}$, respectively, $\mathcal{D}^{\{M_p\}} := \mathcal{D}^{\{M_p\}}(\mathbb{R}) := \operatorname{ind}\lim_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)} := \operatorname{proj}\lim_{h\to\infty} \mathcal{D}_K^{M_p,h}$, respectively, $\mathcal{D}_K^{\{M_p\}} := \operatorname{ind}\lim_{h\to 0} \mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \big\{\phi \in C^\infty(\mathbb{R}) : \operatorname{supp}\phi \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty\big\} \text{ and}$$

$$\|\phi\|_{M_p,h,K} := \sup\Big\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K,\ p \in \mathbb{N}_0\Big\}.$$

Henceforth the asterisk $*$ stands for the Beurling case (M_p) or for the Roumieu case $\{M_p\}$. Denote by $\mathcal{D}'^*(E) := L(\mathcal{D}^*(\mathbb{R}) : E)$ the space consisted of all continuous linear functions from $\mathcal{D}^*(\mathbb{R})$ into E; $\mathcal{D}_0^*$ denotes the space of elements in $\mathcal{D}^*$ which are supported by $[0,\infty)$ whereas $\mathcal{E}_0'^*$ denotes the space of ultradistributions whose supports are compact subsets of $[0,\infty)$. Recall [**207**], an entire function of the form $P(\lambda) = \sum_{p=0}^\infty a_p\lambda^p$, $\lambda \in \mathbb{C}$, is *of class* (M_p), resp., *of class* $\{M_p\}$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leqslant Cl^p/M_p$, $p \in \mathbb{N}$. The corresponding ultradifferential operator $P(D) = \sum_{p=0}^\infty a_pD^p$ is of class (M_p), resp., of class $\{M_p\}$. We introduce the topology of above spaces as well as the convolution of scalar valued ultradistributions (ultradifferentiable functions) in the same way as in the case of corresponding distribution spaces. It is well known that there exists $\rho \in \mathcal{D}^*$ satisfying $\operatorname{supp}\rho \subseteq [0,1]$, $\rho \geqslant 0$ and $\int_{-\infty}^\infty \rho(t)\,dt = 1$. Put $\rho_n(t) := n\rho(nt)$, $t \in \mathbb{R}$; then (ρ_n) is said to be a regularizing sequence in $\mathcal{D}^*$. In the next analogue of Proposition 1.3.1, the convolution of Banach space valued ultradistributions is taken in the sense of [**255**, Corollary 3.6].

PROPOSITION 1.3.3. *Suppose X, Y and Z are Banach spaces and $b : X \times Y \to Z$ is bilinear and continuous. Then there is a unique bilinear, separately continuous mapping $*_b : \mathcal{D}_0'^*(X) \times \mathcal{D}_0'^*(Y) \to \mathcal{D}_0'^*(Z)$ such that $(S \otimes x) *_b (T \otimes y) = S * T \otimes b(x, y)$, for all S, $T \in \mathcal{D}_0'^*$ and $x \in X$, $y \in Y$. Moreover, this mapping is hypo-continuous with respect to bounded sets.*

The following structural theorems for operator valued ultradistributions are located in [**130**] and [**209**].

THEOREM 1.3.4. (i) *Let $G \in \mathcal{D}'^*(E)$. Then, for each relatively compact non-empty open set $\Omega \subseteq \mathbb{R}$, there exists a sequence of continuous function (f_n) in $E^{\overline{\Omega}}$ such that*

$$G_{|\Omega} = \sum_{n=0}^{\infty} D^n f_n$$

and that there exist $K > 0$ and $L > 0$ in the Beurling case, resp., for every $L > 0$ there exists $K > 0$ in the Roumieu case, such that $\sup_{t \in \overline{\Omega}} \|f_n(t)\| \leqslant K \frac{L^n}{M_n}$, $n \in \mathbb{N}$.

(ii) *Suppose, additionally, that (M_p) satisfies* (M.3). *Then for each relatively compact non-empty open set $\Omega \subseteq \mathbb{R}$ there exist an ultradifferential operator of $*$-class and a continuous function $f : \overline{\Omega} \to E$ such that $G_{|\Omega} = P(D)f$.*

The following is a characterization of operator valued (ultra-)distributions supported by a point.

THEOREM 1.3.5. (i) *Suppose $G \in \mathcal{D}'(E)$ and* $\operatorname{supp} G \subseteq \{0\}$. *Then there exist $n \in \mathbb{N}$ and $x_i \in E$, $0 \leqslant i \leqslant n$ such that $G(\varphi) = \sum_{i=0}^{n} \delta^{(i)}(\varphi) x_i$, $\varphi \in \mathcal{D}$.*

(ii) *Suppose that (M_p) additionally satisfies* (M.3) *as well as that $G \in \mathcal{D}'^*(E)$ and* $\operatorname{supp} G \subseteq \{0\}$. *Then there exists a sequence (x_n) in E such that $G(\varphi) = \sum_{n=0}^{\infty} \delta^{(n)}(\varphi) x_n$, $\varphi \in \mathcal{D}^*(E)$ and that there exist $K > 0$ and $L > 0$ in the Beurling case, resp., for every $L > 0$ there exists $K > 0$ in the Roumieu case, such that $\|x_n\| \leqslant K \frac{L^n}{M_n}$, $n \in \mathbb{N}$.*

The spaces of *tempered ultradistributions* of the Beurling, resp. the Roumieu type, are defined in [**364**] as duals of the corresponding test spaces

$$\mathcal{S}^{(M_p)}(\mathbb{R}) := \operatorname*{proj\,lim}_{h \to \infty} \mathcal{S}^{M_p,h}(\mathbb{R}), \text{ resp. } \mathcal{S}^{\{M_p\}}(\mathbb{R}) := \operatorname*{ind\,lim}_{h \to 0} \mathcal{S}^{M_p,h}(\mathbb{R}),$$

where

$$\mathcal{S}^{M_p,h}(\mathbb{R}) := \bigl\{\phi \in C^\infty(\mathbb{R}) : \|\phi\|_{M_p,h} < \infty\bigr\},\ h > 0,$$

$$\|\phi\|_{M_p,h} := \sup\Bigl\{\frac{h^{\alpha+\beta}}{M_\alpha M_\beta}(1+t^2)^{\beta/2}|\phi^{(\alpha)}(t)| : t \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{N}_0\Bigr\}.$$

We also refer to [**50**], [**69**], [**143**], [**168**], [**175**]–[**176**] and [**244**]–[**245**] for the analysis of these spaces.

It could be of importance to stress that

$$\mathcal{D}^{(M_p)}(\mathbb{R}) \hookrightarrow \mathcal{S}^{(M_p)}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{S}^{'(M_p)}(\mathbb{R}) \hookrightarrow \mathcal{D}^{'(M_p)}(\mathbb{R})$$

and

$$\mathcal{D}^{\{M_p\}}(\mathbb{R}) \hookrightarrow \mathcal{S}^{\{M_p\}}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{S}^{'\{M_p\}}(\mathbb{R}) \hookrightarrow \mathcal{D}^{'\{M_p\}}(\mathbb{R})$$

where $\hookrightarrow$ means the continuous and dense embedding. The space $\mathcal{S}'^*(E)$ consists of all linear continuous mappings from $\mathcal{S}^*$ into E and its topology is defined as before. Arguing as in [**364**, Theorem 2], one can prove the following structural theorem for the space $\mathcal{S}'^*(E)$.

THEOREM 1.3.6. *Let $G \in \mathcal{S}_0'^*(E)$ and let (M_p) additionally satisfy* (M.3). *Then there exist an ultradifferential operator $P(D)$ of $*$-class and a continuous function $f : \mathbb{R} \to E$ such that* $\operatorname{supp} f \subseteq (-\infty, 0]$, *$G = P(D)f$ and $\|f(t)\| \leqslant Ke^{M(h|t|)}$, $t \in \mathbb{R}$, for some $h > 0$ and $K > 0$ in the Beurling case, resp., for every $h > 0$ and a corresponding $K > 0$ in the Roumieu case.*

In the sequel, we employ the Paley-Wiener type theorems for ultradifferentiable functions and infinitely differentiable functions with compact support. For further information, we refer the reader to [**207**, Section 9] and [**365**, Section 11.6].

1.4. Complex powers of operators

Chronologically, the theory of fractional powers of operators dates from the papers [**154**] of Hille, who studied the semigroup formed from the fractional powers of a bounded linear operator in 1939, and Bochner [**42**], who constructed the fractional powers of $-\Delta$ in 1949. From then on, many different techniques have been established in the framework of this theory (see e.g. [**20**], [**24**], [**98**]–[**99**], [**158**], [**211**], [**300**]–[**302**], [**335**], [**358**]–[**359**], [**394**] and [**428**]). The monograph [**300**] is of fundamental importance and contains the essential part of the theory of fractional powers of non-negative operators including topics related to extensions of Hirsch functional calculus, fractional powers of operators in locally convex spaces, interpolation spaces and the famous Dore–Venni theorem.

1.4.1. Complex powers of densely defined operators. In this subsection, we follow the approach of Straub [**394**] who defined the complex powers of a closed, densely defined operator A satisfying

$$\Sigma(\gamma) := \big\{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leqslant \gamma\big\} \cup \{0\} \subseteq \rho(A),\ \text{ for some } \gamma \in (0, \tfrac{\pi}{2}), \tag{1.4.1}$$

$$\|R(\lambda\,{:}\,A)\| \leqslant M(1 + |\lambda|)^n,\ \lambda \in \Sigma(\gamma),\ \text{ for some } M > 0 \text{ and } n \in \mathbb{N}_0. \tag{1.4.2}$$

For such an operator A, Straub defined in [**394**] the fractional powers $(-A)^b$, for all $b \in \mathbb{C}$. If A fulfills (1.4.1) and (1.4.2), then one can employ the construction given in [**98**] to obtain the definition of the fractional power of $-A$, but only for $b \geqslant 0$. In general, the definitions given in [**98**] and [**394**] do not coincide; see [**98**] and [**358**] for further information. The ideas developed in [**394**] can be applied to an essentially larger class of closed, densely defined operators.

Throughout this subsection, E denotes a complex Banach space and A denotes a closed, densely defined operator in E. Let $a \in (0, 1)$, $C \in (0, 1]$ and $d \in (0, 1]$; $B_d := \{z \in \mathbb{C} : |z| \leqslant d\}$ and $P_{a,C} := \{\xi + i\eta : \xi \in (0, \infty),\ \eta \in \mathbb{R},\ \ |\eta| \leqslant C\xi^a\}$. We assume that A satisfies the following conditions:

$$P_{a,C} \cup B_d \subseteq \rho(A), \tag{§}$$

$$\|R(\lambda\,{:}\,A)\| \leqslant M(1 + |\lambda|)^{\alpha},\ \lambda \in P_{a,C} \cup B_d,\ \text{ for some } M > 0 \text{ and } \alpha \geqslant 0. \tag{§§}$$

Note, if $P_{a,C} \cup \{0\} \subseteq \rho(A)$ and $\|R(\lambda : A)\| = O((1 + |\lambda|)^{\alpha})$, $\lambda \in P_{a,C} \cup \{0\}$, then there exist $d \in (0, 1]$ and $M > 0$ such that (§) and (§§) are fulfilled. It is worth noting that if A satisfies the assumptions (1.4.1) and (1.4.2), then for every $a \in (0, 1)$, there exist $C \in (0, 1]$ and $d \in (0, 1]$ such that (§) and (§§) are valid (with $\alpha = n$). It is clear that there exist a great number of multiplication, differential and pseudo-differential operators acting on L^p type spaces which fulfill (§) and (§§), but not (1.4.1). Suppose, for example, that $E := L^2(\mathbb{R})$ and that $A := \Delta^2 - i\Delta - I$ with maximal distributional domain. Then the spectrum of A is $\{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \eta^2 = \xi + 1\}$ and, for every $b \in \mathbb{C}$, a slight modification of the construction given in [**394**] gives the definition of $(I + i\Delta - \Delta^2)^b$. For further information related to operators which satisfy (1.4.1) and (1.4.2), the reader may consult [**335**, p. 158] and Subsection 2.1.1.

In Subsection 1.4.1, resp., Subsection 1.4.2 we assume that the number $\alpha \geqslant 0$, resp. $\alpha \geqslant -1$, is minimal with respect to the property (§§) and employ the following notations. Given $a \in (0, 1)$, $C \in (0, 1]$ and $d \in (0, 1]$, put $\Gamma_1(a, C, d) := \{\xi + i\eta : \xi > 0,\ \eta \in \mathbb{R},\ \eta = -C\xi^a,\ \xi^2 + \eta^2 \geqslant d^2\}$. It is clear that there exists a unique number $\varepsilon(a, C, d) \in (0, d)$ such that $(\varepsilon(a, C, d), -C\varepsilon(a, C, d)^a) \in \partial B_d$. We define $\Gamma_2(a, C, d) := \{\xi + i\eta : \xi > 0,\ \eta \in \mathbb{R},\ \xi^2 + \eta^2 = d^2,\ \xi \leqslant \varepsilon(a, C, d)\}$ and $\Gamma_3(a, C, d) := \{\xi + i\eta : \xi > 0,\ \eta \in \mathbb{R},\ \eta = C\xi^a,\ \xi^2 + \eta^2 \geqslant d^2\}$. The upwards oriented curve $\Gamma(a, C, d)$ is defined by $\Gamma(a, C, d) := \Gamma_1(a, C, d) \cup \Gamma_2(a, C, d) \cup \Gamma_3(a, C, d)$; put now $H(a, C, d) := \{\xi + i\eta : \xi > 0,\ \eta \in \mathbb{R},\ |\eta| \leqslant C\xi^a\} \cup B_d$. Given $\tilde{d} \in (0, d]$ and $\tilde{a} \in (0, a]$, one can find a suitable constant $\tilde{C}$ so that $\Gamma(\tilde{a}, \tilde{C}, \tilde{d}) \subseteq H(a, C, d)$, where we define $\Gamma(\tilde{a}, \tilde{C}, \tilde{d})$ in the same way as $\Gamma(a, C, d)$. In order to construct the complex power $(-A)^b$, for every $b \in \mathbb{C}$, we first define a closable linear operator J^b. As in [**394**], the construction is based on improper integrals of the form $\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^b R(\lambda : A)\, d\lambda$.

PROPOSITION 1.4.1. *Let $b \in \mathbb{C}$ satisfy $\operatorname{Re} b < -(\alpha+1)$ and let $x \in E$. Then the integral $I(b)x := \frac{1}{2\pi i}\int_{\Gamma(a,C,d)}(-\lambda)^b R(\lambda : A)x\, d\lambda$ exists and defines a bounded linear operator $I(b) \in L(E)$. Moreover, if for some $\tilde{a} \in (0, a]$, $\tilde{C} \in (0, C]$ and $\tilde{d} \in (0, d]$: $\Gamma(\tilde{a}, \tilde{C}, \tilde{d}) \subseteq H(a, C, d)$, then $I(b)x = \frac{1}{2\pi i}\int_{\Gamma(\tilde{a},\tilde{C},\tilde{d})}(-\lambda)^b R(\lambda : A)x\, d\lambda$.*

PROOF. The proof is essentially contained in that of [**394**, Lemma 1.1]. Note that the function $\lambda \mapsto (-\lambda)^b$ $(1^b = 1)$ is analytic in $\mathbb{C} \smallsetminus [0, \infty)$ and that

$$|(-\lambda)^b| \leqslant |\lambda|^{\operatorname{Re} b} e^{\pi |\operatorname{Im} b|},\ \lambda \in \mathbb{C} \smallsetminus \{0\}.$$

The integral over $\Gamma_2(a, C, d)$ exists since $\Gamma_2(a, C, d)$ is a finite path. By (§§), we obtain that there exists a constant $M(a, C, d, b) > 0$ such that

$$\left\| \frac{1}{2\pi i} \int\limits_{\Gamma_3(a,C,d)} (-\lambda)^b R(\lambda : A)x\, d\lambda \right\| \leqslant M(a, C, d, b)\|x\| \int\limits_{\varepsilon(a,C,d)}^{\infty} \sqrt{t^2 + C^2 t^{2a}}^{\operatorname{Re} b}\, t^{\alpha}\, dt.$$

Since $(t^2 + C^2 t^{2a})^{\operatorname{Re} b/2} t^{\alpha} \sim t^{\alpha + Reb}$, $t \to +\infty$, the integral over $\Gamma_3(a, C, d)$ exists. Similarly, the integral over $\Gamma_1(a, C, d)$ exists. It remains to be shown that the integral $I(b)$ is independent of the choice of a curve $\Gamma(a, C, d)$. Let R be sufficiently large and let the curve $\Gamma_R = \{Re^{it} : t \in [\arctan(\tilde{C}R^{\tilde{a}-1}), \arctan(CR^{a-1})]\}$ be upwards oriented. Then

$$\left\| \int_{\Gamma_R} (-\lambda)^b R(\lambda : A)x \, d\lambda \right\| \leqslant 2\pi e^{\pi|\operatorname{Im} b|} R^{\operatorname{Re} b}(1+R)^\alpha R \to 0, \ R \to +\infty.$$

The proof completes an elementary application of Cauchy's theorem. □

If no confusion seems likely, we shall simply denote $\Gamma(a, C, d)$, $H(a, C, d)$ and $\varepsilon(a, C, d)$ by Γ, H and ε, respectively.

Notice, if $b \in \mathbb{C}$, then $\operatorname{Re}(b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2) \in [-(\alpha+2), -(\alpha+1))$. Hence, the following definition of the operator J^b makes a sense.

Definition 1.4.2. Let $b \in \mathbb{C}$. The *operator* J^b is defined by

$$D(J^b) := D(A^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2})$$

$$J^b x := \begin{cases} I(b)x, (\alpha+2) \leqslant \operatorname{Re} b < -(\alpha+1), \\ I(b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2)(-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} x, \text{ otherwise.} \end{cases}$$

Remark 1.4.3. If a densely defined operator A satisfies (1.4.1) and (1.4.2), then we have already seen that, for every $a \in (0,1)$, there exist $C \in (0,1]$ and $d \in (0,1]$ such that (§) and (§§) are fulfilled. In this case, the definition of J^b is equivalent to the corresponding one given in [**394**, Definition 1.2].

In what follows, we will use the generalized resolvent equation

$$(6) \qquad (-\lambda)^{-n-1} R(\lambda : A)(-A)^{n+1} x = R(\lambda : A)x + \sum_{i=0}^{n} (-\lambda)^{-i-1}(-A)^i x,$$

if $\lambda \in \rho(A)$, $\lambda \neq 0$, $n \in \mathbb{N}_0$, $x \in D(A^{n+1})$, and the simple equality $\int_\Gamma (-\lambda)^b d\lambda = 0$, if $\operatorname{Re} b < -1$.

Proposition 1.4.4. *Let* $x \in D(A^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2})$. *Then:*

$$(7) \qquad J^b x = \begin{cases} \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b R(\lambda : A)x \, d\lambda, \ \operatorname{Re} b < 0, \\ \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b \rfloor - 1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor + 1} x \, d\lambda, \ \operatorname{Re} b \geqslant 0. \end{cases}$$

Proof. Suppose $\operatorname{Re} b < 0$. If $\operatorname{Re} b \in [-(\alpha+2), -(\alpha+1))$, the conclusion follows directly from the definition of J^b. If $\operatorname{Re} b \notin [-(\alpha+2), -(\alpha+1))$, then by Definition 1.4.2 and (6):

$$\begin{aligned} J^b x &= J^{b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2}(-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} x \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} x \, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b \left(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b + \alpha \rfloor + 1} (-\lambda)^{-i-1}(-A)^i x \right) d\lambda. \end{aligned}$$

If $\operatorname{Re} b < 0$ and $i = 0, 1, \dots, \lfloor \operatorname{Re} b + \alpha \rfloor + 1$, then $\operatorname{Re} b - i - 1 < -1$ and the last term equals $\frac{1}{2\pi i} \int_\Gamma (-\lambda)^b R(\lambda : A)x \, d\lambda$ as claimed. Suppose now $\operatorname{Re} b \geqslant 0$. Then (6) implies

$$(-\lambda)^{b-\lfloor \operatorname{Re} b \rfloor -1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor +1} x = (-\lambda)^b \left(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b \rfloor} (-\lambda)^{-i-1} (-A)^i x \right).$$

Then one gets

$$\begin{aligned} J^b x &= J^{b-\lfloor \operatorname{Re} b + \alpha \rfloor - 2} (-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} x \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} x \, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b \left(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b + \alpha + 1 \rfloor} (-\lambda)^{-i-1} (-A)^i x \right) d\lambda, \end{aligned}$$

and since for $j = \lfloor \operatorname{Re} b + 1 \rfloor, \dots, \lfloor \operatorname{Re} b + \alpha + 1 \rfloor$, $\operatorname{Re} b - i - 1 < -1$, we obtain

$$\begin{aligned} &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b \left(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b \rfloor} (-\lambda)^{-i-1} (-A)^i x \right) d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b \rfloor - 1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b + 1 \rfloor} x \, d\lambda. \end{aligned}$$

The proof is completed. □

Put $C^b := (-A)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} J^{b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2}$. Then, for every $b \in \mathbb{C}$, C^b is a closed linear operator. Arguing similarly as in the proof of [**394**, Proposition 1.3], one obtains that, for every $b \in \mathbb{C}$ with $\operatorname{Re} b \geqslant -(\alpha + 1)$, $J^b \subseteq C^b$ and, consequently, J^b is a closable operator. Clearly, $J^b \in L(E)$ for every $b \in \mathbb{C}$ with $\operatorname{Re} b < -(\alpha + 1)$.

LEMMA 1.4.5. *Let $b \in \mathbb{C}$. Then the following holds:*

(i) *$J^b x = J^{b+k} (-A)^{-k} x$, $k \in \mathbb{N}_0$, $x \in D(J^b)$, and*
(ii) *$J^b x = J^{b+k} (-A)^{-k} x$, if $-k \in \mathbb{N}$ and $x \in D(A^{\max(-k, \lfloor \operatorname{Re} b + \alpha + 2 \rfloor)})$.*

PROOF. (i) If $k = 0$, the proof is trivial. Suppose now $k = 1$. If $-(\alpha + 2) \leqslant \operatorname{Re} b < -(\alpha + 1)$, then $\operatorname{Re} b + 1 < 0$ and by Proposition 1.4.4, we obtain

$$\begin{aligned} J^{b+1} (-A)^{-1} x &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b+1} R(\lambda : A)(-A)^{-1} x \, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b+1} \frac{R(\lambda : A)x - (-A)^{-1} x}{(-\lambda)} \, d\lambda = \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b R(\lambda : A)x \, d\lambda. \end{aligned}$$

If $\operatorname{Re} b \notin [-(\alpha + 2), -(\alpha + 1))$, the assertion follows from Definition 1.4.2. Now (i) follows by induction; the assertion (ii) follows immediately from (i). □

For $b \in \mathbb{C}$, denote $\langle b \rangle := \max(0, \lfloor \operatorname{Re} b + \alpha \rfloor + 2)$. Note that $\langle b + c \rangle \leqslant \langle b \rangle + \langle c \rangle$. The expected semigroup property of the family $(J^b)_{b \in \mathbb{C}}$ can be proved similarly as in [**394**, Lemma 1.4]. More precisely, we have:

PROPOSITION 1.4.6. *Let b, $c \in \mathbb{C}$. Then $J^b J^c x = J^{b+c} x$, $x \in D(A^{\langle b \rangle + \langle c \rangle})$.*

If $k \in \mathbb{N}$ and $x \in D(A^k)$, put $\|x\|_k := \|x\| + \|Ax\| + \cdots + \|A^k x\|$. Now we will prove the following lemma which naturally corresponds to [**394**, Lemma 1.5].

LEMMA 1.4.7. *Let $b \in \mathbb{Z}$ and $x \in D(A^{\lfloor b+\alpha \rfloor + 2})$. Then $J^b x = (-A)^b x$.*

PROOF. By Lemma 1.4.5(ii), it suffices to consider the case $b = 1$. We have to prove that

$$\frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{-1} R(\lambda : A)(-A)^2 x \, d\lambda = -Ax.$$

By the resolvent equation, it follows that $x \in D(A^{\lfloor \alpha+1 \rfloor})$ implies that there exists a suitable constant $M > 0$ such that

$$\|R(\lambda : A)x\| \leqslant M|\lambda|^{\alpha - \lfloor \alpha \rfloor - 1} \|x\|_{\lfloor \alpha+1 \rfloor}, \quad \lambda \in H \cup \Gamma, \ |\lambda| \geqslant d.$$

Let $R > d$. Then there exists a unique number $\kappa(R) \in (0, R)$ such that $\kappa(R)^2 + C^2 \kappa(R)^{2a} = R^2$. Denote $\Gamma_R = \{Re^{i\theta} : |\theta| \leqslant \arctan(C\kappa(R)^{a-1})\}$; we assume that Γ_R is upwards oriented. If $x \in D(A^{\lfloor \alpha \rfloor + 3})$, then $A^2 x \in D(A^{\lfloor \alpha \rfloor + 1})$ and the previous inequality implies

$$\left\| \int_{\Gamma_R} (-\lambda)^{-1} R(\lambda : A)(-A)^2 x \, d\lambda \right\| \leqslant 2\pi \frac{M}{R} R^{\alpha - \lfloor \alpha \rfloor - 1} \|x\|_{\lfloor \alpha+3 \rfloor}, \quad R \to 0, \ R \to +\infty.$$

The remaining part of proof follows by an application of Cauchy's formula. □

Proceeding as in [**394**], one can prove that:

(i) If $b \in \mathbb{Z}$, then $\overline{J^b} = (-A)^b$.
(ii) If $\operatorname{Re} b > \alpha + 1$, then $\overline{J^b} = C^b$.

Now we are in a position to introduce complex powers.

DEFINITION 1.4.8. Let $b \in \mathbb{C}$. Then *the complex power* $(-A)^b$ of the operator $-A$ is defined by $(-A)^b := \overline{J^b}$.

The next theorem clarifies the basic structural properties of powers. See [**394**] for a proof.

THEOREM 1.4.9. *Let b, $c \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Then we have the following.*

(i) *$D(A^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2 + k})$ is a core for $(-A)^b$.*
(ii) *$(-A)^{b+c} \subseteq \overline{(-A)^b (-A)^c}$.*
(iii) *$(-A)^{b+c} = \overline{(-A)^b (-A)^c}$, if $(-A)^{b+c} = C^{b+c}$.*
(iv) *$\overline{(-A)^{-b} (-A)^b} = I$; $(-A)^{-b} (-A)^b x = x$, $x \in D((-A)^b)$.*
(v) *$(-A)^b$ is injective.*

The following facts should be stated. If a closed, densely defined operator A satisfies (§) and (§§) with $\alpha \in [-1,0)$ therein, resp. (1.4.1) and (1.4.2) with $\alpha \in [-1,0)$ therein, then one can define the complex powers since, in this case, the resolvent is bounded on the region $P_{a,C} \cup B_d$, resp. $\Sigma(a) \cup B_d$. Owing to [**394**, Lemma 1.8], Proposition 1.4.4 and Theorem 1.4.9(i), it can be easily seen that the construction given in [**335**] and [**394**] (cf. for example [**335**, pp. 157–158]) coincide with the construction established in this section. The former conclusion remains true if (1.4.1) holds for some $\alpha \geqslant 0$; anyway, we have that $(-A)^b \in L(E)$ for all $b \in \mathbb{C}$ with $\operatorname{Re} b < -(\alpha+1)$.

1.4.2. Complex powers of non-densely defined operators. Unless stated otherwise, in this subsection we assume that A is a closed linear operator and that the following conditions hold:

(§) $P_{a,C} \cup B_d \subseteq \rho(A)$ and

(§§) $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\alpha}$, $\lambda \in P_{a,C} \cup B_d$, for some $M > 0$ and $\alpha \geqslant -1$.

Suppose, for the time being, that a closed, densely defined operator A satisfies (§) and (§§) with $\alpha \in [-1,0)$, or

($§_1$) $\Sigma(\gamma) := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leqslant \gamma\} \cup \{0\} \subset \rho(A)$, for some $\gamma \in (0, \frac{\pi}{2})$ and

($§§_1$) $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\alpha}$, $\lambda \in \Sigma(\gamma)$, for some $M > 0$ and $\alpha \in [-1,0)$.

Then there exists $d \in (0,1]$ such that $\|R(\cdot : A)\|$ is bounded on the region $P_{a,C} \cup B_d$, resp. $\Sigma(\gamma) \cup B_d$. We define the complex powers of $-A$ as in the preceding paragraph with $\alpha = 0$. Then the formula (9) holds for every $b \in \mathbb{C} \setminus \mathbb{Z}$ and $x \in D(A^{\lfloor \operatorname{Re} b \rfloor + 2})$ and it can be easily seen that the construction given on pages 157 and 158 of [**335**] coincides with the construction established in the preceding subsection for real values of exponents. The former conclusion remains true if ($§§_1$) holds for some $\alpha \geqslant 0$; in any case, $(-A)^b$ is a closed, densely defined linear operator, and furthermore, $(-A)^b \in L(E)$ if $\operatorname{Re} b < -(\alpha+1)$. Let $\operatorname{Re} b \in (-1,0)$ and $x \in D(A^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2})$. Then there exists $y \in E$ such that $x = (-A)^{-\lfloor \operatorname{Re} b + \alpha \rfloor - 2} y$ and Proposition 1.4.4 implies $J^b x = \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^b R(\lambda : A)(-A)^{-\lfloor \operatorname{Re} b + \alpha \rfloor - 2} y \, d\lambda$. By the resolvent equation, one easily infers that

$$\begin{aligned} R(\lambda : A)(-A)^{-\lfloor \operatorname{Re} b + \alpha \rfloor - 2} y = & \sum_{j=1}^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2} (-1)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2 - j} \lambda^{j - \lfloor \operatorname{Re} b + \alpha \rfloor - 3} (-A)^{-j} y \\ & + \frac{(-1)^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2}}{\lambda^{\lfloor \operatorname{Re} b + \alpha \rfloor + 2}} R(\lambda : A) y, \quad \lambda \in \rho(A) \setminus \{0\}. \end{aligned} \tag{8}$$

Combined with the inequality $|(-\lambda)^b| \leqslant |\lambda|^{\operatorname{Re} b} e^{\pi |\operatorname{Im} b|}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and the residue theorem, (8) indicates that, for all sufficiently small positive real numbers ε, one can deform the path of integration Γ, appearing in the definition of J^b, into the upwards oriented boundary of the region $B_{\varepsilon} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant 0,\ |\operatorname{Im} \lambda| \leqslant \frac{\varepsilon}{2}\}$. In such a way, we obtain that $J^b x = -\frac{\sin b\pi}{\pi} \int_0^{\infty} t^b R(t : A) x \, dt$. Using Lemma 1.4.5, one gets that, for every $b \in \mathbb{C}$ such that $\operatorname{Re} b \notin \mathbb{Z}$ and $x \in D(A^{\lfloor \operatorname{Re} b+\alpha\rfloor+2})$:

$$(9) \qquad (-A)^b x = \frac{\sin(\lfloor \operatorname{Re} b\rfloor + 1 - b)\pi}{\pi} \int\limits_0^\infty t^{b-\lfloor \operatorname{Re} b\rfloor -1} R(t:A)(-A)^{\lfloor \operatorname{Re} b\rfloor+1} x\, dt.$$

Notice that the equality (9) generalizes the assertion (P2) given on page 158 of [**335**]. In the case $\alpha < 0$, the formula (9) holds for every $b \in \mathbb{C}$ with $\operatorname{Re} b \notin \mathbb{Z}$ and $x \in D(A^{\lfloor \operatorname{Re} b\rfloor+2})$. The proof of following extension of [**249**, Lemma 1.5] is omitted.

Lemma 1.4.10. *Suppose $\alpha \geqslant -1$ and A is a closed linear operator. If there exist a constant $M > 0$ and a sequence (λ_n) in $\rho(A)$ such that* $\lim_{n\to\infty} |\lambda_n| = \infty$ *and that $\|R(\lambda_n : A)\| \leqslant M(1 + |\lambda_n|)^\alpha$, $n \in \mathbb{N}$, then A is stationary dense and $n(A) \leqslant \lfloor\alpha\rfloor + 2$.*

Suppose now that A is a closed, non-densely defined linear operator such that (§) and (§§) hold. By Lemma 1.4.10, we have that $n(A) \leqslant \lfloor\alpha\rfloor + 2$, and thanks to [**249**, Remark 1.2(iii)], the equality $\overline{D(A^{\lfloor\alpha\rfloor+m})} = \overline{D(A^{\lfloor\alpha\rfloor+n})}$ holds for all m, $n \in \mathbb{N} \smallsetminus \{1\}$. Put $F := \overline{D(A^{n(A)})}$. By [**249**, Proposition 2.1], one gets that $A_{|F}$ is densely defined in F as well as that $\rho(A : E) = \rho(A_{|F} : F)$ and that $\|R(\lambda : A_{|F})\|_F \leqslant \|R(\lambda\!:\!A)\|_E$. This implies:

$$(10) \quad P_{a,C} \cup B_d \subseteq \rho(A_{|F} : F) \text{ and } \|R(\lambda : A_{|F})\|_F \leqslant M(1 + |\lambda|)^\alpha,\ \lambda \in P_{a,C} \cup B_d.$$

By the foregoing, one can construct the complex powers of the operator $(-A_{|F})^b = \overline{J_F^b}$ in the Banach space F. Following the approach of Martinez and Sanz [**301**] for non-negative operators, we introduce the complex powers of the operator $-A$ as follows.

Definition 1.4.11. Let $b \in \mathbb{C}$. The *complex power* $(-A)^b$ is defined by

$$(-A)^b := (-A)^{n(A)}(-A_{|F})^b(-A)^{-n(A)}.$$

It is straightforwardly checked that, for every $\lambda \in P_{a,C} \cup B_d$ and $b \in \mathbb{C}$, we have $(-A)^b = (\lambda - A)^{n(A)}(-A_{|F})^b(\lambda - A)^{-n(A)}$. The definition of power $(-A)^b$ coincides with the above given definition when A is densely defined, and does not depend on the choice of a number $\alpha \geqslant -1$ satisfying (§§). Furthermore, by Lemma 1.4.10, $(-A_{|F})^b \subseteq (-A)^b \subseteq (-A)^{\lfloor\alpha\rfloor+2}(-A_{|F})^b(-A)^{-\lfloor\alpha\rfloor-2}$ and it is not clear whether, in general, $(-A)^{\lfloor\alpha\rfloor+2}(-A_{|F})^b(-A)^{-\lfloor\alpha\rfloor-2} \subseteq (-A)^b$.

Theorem 1.4.12. *Suppose b, $c \in \mathbb{C}$ and $n \in \mathbb{N}$. Then the complex powers of the operator $-A$ satisfy the following properties:*

(i) *$(-A)^b$ is a closed linear operator.*
(ii) *$(-A)^b$ is injective.*
(iii) *$(-A)^b \in L(E)$ if $\operatorname{Re} b < -(\alpha + 1)$, $D(A^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}) \subseteq D((-A)^b)$, $b \in \mathbb{C}$, $\alpha \geqslant 0$ and $D(A^{\lfloor \operatorname{Re} b\rfloor+2}) \subseteq D((-A)^b)$, $b \in \mathbb{C}$, $\alpha \in [-1, 0)$.*
(iv) *$(-A)^{-b}(-A)^b x = x$, $x \in D((-A)^b)$, $(-A)^{-b} = ((-A)^b)^{-1}$ and $I_{|F} \subseteq \overline{(-A)^{-b}(-A)^b} \subseteq I$.*
(v) *$(-A)^n = (-1)^n A \cdots A$ n-times, $(-A)^{-n} = R(0 : A)^n$ and $(-A)^0 = I$.*

(vi) *Let* $x \in D((-A)^{b+c})$. *Then there is a sequence* (x_k) *in* $D((-A)^b(-A)^c)$ *such that*

$$\lim_{k\to\infty} x_k = (-A)^{-n(A)}x \text{ and } \lim_{k\to\infty}(-A)^b(-A)^c x_k = (-A)^{-n(A)}(-A)^{b+c}x.$$

(vii) $\overline{(-A)^b(-A)^c} \subseteq (-A)^{b+c}$ *if* $(-A_F)^{b+c} = C_F^{b+c}$. *In particular, the preceding inclusion holds whenever* $|\operatorname{Re}(b+c)| > \alpha+1$ *or* $b+c \in \mathbb{Z}$.

(viii) $I_{|F} \subseteq \overline{(-A)^{-b}(-A)^b} \subseteq I$.

(ix) *Suppose* $\operatorname{Re} b \notin \mathbb{Z}$ *and* $\alpha \geqslant 0$, *resp.* $\alpha \in [-1,0)$. *Then the equality* (9) *holds for every* $x \in D(A^{\lfloor \operatorname{Re} b+\alpha\rfloor+2})$, *resp. for every* $x \in D(A^{\lfloor \operatorname{Re} b\rfloor+2})$.

(x) *Let* $(-A)^{c-b} \in L(E)$. *Then*

$$D((-A)^b) \subseteq D((-A)^c) \text{ and } (-A)^c x = (-A)^{c-b}(-A)^b x,\ x \in D((-A)^b).$$

(xi) *Let* $c \in \mathbb{C}$ *and* $x \in D(A^{\lfloor \operatorname{Re} c+\alpha\rfloor+2})$. *Then*

$$\lim_{b\to c}(-A)^b x = (-A)^c x. \tag{11}$$

Proof. By Theorem 1.4.9, we know that the properties (i)-(iv) hold for the complex powers $(-A_{|F})^b$ in F as well as that the powers $(-A_{|F})^b$, $b \in \mathbb{Z}$ coincide with the usual powers of the operator $A_{|F}$. Furthermore, $(-A_{|F})^{b+c} \subseteq \overline{(-A_{|F})^b(-A_{|F})^c}$, with the equality if $(-A_{|F})^{b+c} = C_F^{b+c}$, and $\overline{(-A_{|F})^{-b}(-A_{|F})^b} = I_{|F}$. The proofs of (i), (ii), (iv), (v) and (vi) follow from the corresponding properties of powers $(-A_{|F})^b$ and elementary definitions. We will prove the first assertion in (iii) only in the case $\alpha \in [-1,0)$ since the consideration is similar if $\alpha \geqslant 0$. Suppose $x \in E$ and $\operatorname{Re} b < -(\alpha+1)$. Then $n(A) = 1$ and one sees directly that $(-A_{|F})^b y = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A_{|F})y\,d\lambda$, $y \in F$. Arguing as in the proof of Proposition 1.4.1, one gets that the integral $\frac{1}{2\pi i}\int_\Gamma(-\lambda)^b R(\lambda{:}A)x\,d\lambda$ converges. Hence,

$$\begin{aligned}(-A_{|F})^b(-A)^{-1}x &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A_{|F})(-A)^{-1}x\,d\lambda\\ &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda{:}A)(-A)^{-1}x\,d\lambda\\ &= \frac{1}{2\pi i}(-A)^{-1}\int_\Gamma (-\lambda)^b R(\lambda{:}A)x\,d\lambda \in D(A).\end{aligned}$$

Hence, $x \in D((-A)^b)$, $(-A)^b x = \frac{1}{2\pi i}\int_\Gamma(-\lambda)^b R(\lambda : A)x\,d\lambda$, $x \in E$ and the closed graph theorem implies $(-A)^b \in L(E)$. We will prove the second assertion in (iii) in the case $\alpha \geqslant 0$. Notice that the first part of (iii) implies $D(A^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}) \subseteq D((-A)^b)$ if $b \in \mathbb{C}$ and $\lfloor \operatorname{Re} b+\alpha\rfloor+2 \leqslant 0$. Suppose $\lfloor \operatorname{Re} b+\alpha\rfloor+2 \geqslant 1$. Then one obtains inductively $D(A^{k+n(A)}) \subseteq D(A_F^k)$, $k \in \mathbb{N}_0$, and consequently, $(-A)^{-n(A)}x \in D(A^{\lfloor \operatorname{Re} b+\alpha+n(A)\rfloor}) \subseteq D(A_{|F}^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}) = D(J_F^b)$. Taking into account the proof of Proposition 1.4.1, the equality $R(\lambda : A_{|F})(-A)^{-n(A)}x = R(\lambda : A)(-A)^{-n(A)}x$, $\lambda \in \rho(A)$ and Proposition 1.4.4, one yields $J_F^b(-A)^{-n(A)}x \in D(A^{n(A)})$ and $x \in$

$D((-A)^b)$. Put $n := n(A)$. Then (vii) follows from the following

$$\begin{aligned}\overline{(-A)^b(-A)^c} &= \overline{[(-A)^n(-A_{|F})^b(-A)^{-n}][(-A)^n(-A_{|F})^c(-A)^{-n}]}\\ &\subseteq \overline{(-A)^n(-A_{|F})^b(-A_{|F})^c(-A)^{-n}} \subseteq \overline{(-A)^n\overline{(-A_{|F})^b(-A_{|F})^c}(-A)^{-n}}\\ &= \overline{(-A)^n(-A_{|F})^{b+c}(-A)^{-n}} = (-A)^n(-A_{|F})^{b+c}(-A)^{-n} = (-A)^{b+c}.\end{aligned}$$

In order to prove (ix), notice that the improper integral appearing in the formulation of (9) still converges. Without loss of generality, we may assume that $\alpha \geqslant 0$. Suppose $x \in D(A^{\lfloor \operatorname{Re} b+\alpha \rfloor +2})$. Owing to the assertion (iii) and its proof, one gets $x \in D((-A)^b)$ and

$$\begin{aligned}&(-A)^{-n(A)}(-A)^b x = (-A_{|F})^b(-A)^{-n(A)}x = J_F^b(-A)^{-n(A)}x\\ &= \frac{\sin(\lfloor \operatorname{Re} b\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \operatorname{Re} b\rfloor -1}R(t:A_F)(-A_{|F})^{\lfloor \operatorname{Re} b\rfloor +1}(-A)^{-n(A)}x\,dt\\ &= \frac{\sin(\lfloor \operatorname{Re} b\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \operatorname{Re} b\rfloor -1}R(t:A)(-A)^{\lfloor \operatorname{Re} b\rfloor +1}(-A)^{-n(A)}x\,dt.\end{aligned}$$

Due to the closedness of $(-A)^{n(A)}$,

$$\begin{aligned}(-A)^b x &= \frac{\sin(\lfloor \operatorname{Re} b\rfloor + 1 - b)\pi}{\pi}\\ &\quad\times (-A)^{n(A)}\int_0^\infty t^{b-\lfloor \operatorname{Re} b\rfloor -1}R(t:A)(-A)^{\lfloor \operatorname{Re} b\rfloor +1}(-A)^{-n(A)}x\,dt\\ &= \frac{\sin(\lfloor \operatorname{Re} b\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \operatorname{Re} b\rfloor -1}R(t:A)(-A)^{\lfloor \operatorname{Re} b\rfloor +1}x\,dt,\end{aligned}$$

as required. If A is densely defined, then the property (ix) follows directly from Proposition 1.4.6, Theorem 1.4.9(i) and the boundedness of $(-A)^{c-b}$. Assume now $x \in D((-A)^b)$ and A is not densely defined. Using Theorem 1.4.9(i) and the equality (12) given below, one can simply prove that $(-A)^{-1}(-A_F)^{c-b} \subseteq (-A_F)^{c-b}(-A)^{-1}$. This implies $(-A)^{c-b}(-A)^{-k} = (-A)^{-k}(-A)^{c-b}$, $k \in \mathbb{N}_0$ and $(-A)^{n(A)}(-A)^{c-b}y = (-A)^{c-b}(-A)^{n(A)}y$, $y \in D(A^{n(A)})$. One can simply prove that $(-A_F)^{c-b} \in L(F)$. Hence,

$$\begin{aligned}(-A_F)^c(-A)^{-n(A)}x &= (-A_F)^{c-b}(-A_F)^b(-A)^{-n(A)}x\\ &\in (-A)^{c-b}(D(A^{n(A)})) \subseteq D(A^{n(A)}),\ \ x \in D((-A)^c)\\ (-A)^c x &= (-A)^{n(A)}(-A_F)^{c-b}(-A_F)^b(-A)^{-n(A)}x\\ &= (-A)^{n(A)}(-A)^{c-b}(-A_F)^b(-A)^{-n(A)}x\\ &= (-A)^{c-b}(-A)^{n(A)}(-A_F)^b(-A)^{-n(A)}x = (-A)^{c-b}(-A)^b x\end{aligned}$$

finishing the proof of (x). We will prove (xi) only in the case $\alpha \geqslant 0$. Suppose first that $\operatorname{Re} c \notin \mathbb{Z}$. It is clear that there exists $\sigma > 0$ such that $\lfloor \operatorname{Re} b+\alpha \rfloor \leqslant \lfloor \operatorname{Re} c+\alpha \rfloor$ for any $b \in \mathbb{C}$ such that $|b-c| < \sigma$. This implies that $(-A)^b x$ is given by the formula (12) (cf. Remark 1.4.13 given below) in a neighborhood of the point c. Now the required continuity property follows from the formula (9) and the dominated convergence theorem. Assume $\operatorname{Re} c \in \mathbb{Z}$. In the case $\alpha \notin \mathbb{N}_0$, (11) can be proved by means of (13) and the dominated convergence theorem. The case $\alpha \in \mathbb{N}_0$ can be considered analogically. As a matter of fact, (13) and the dominated convergence theorem imply that $\lim_{b \to c,\ \operatorname{Re} b \geqslant \operatorname{Re} c} (-A)^b x = (-A)^c x$. Since

$$
\begin{aligned}
(-A)^b x &= \frac{1}{2\pi i} \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} b+\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor +\alpha+2} x \, d\lambda \\
&= \frac{1}{2\pi i} \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} b+\alpha \rfloor -2} R(\lambda : A)(-A)^{-1}(-A)^{\lfloor \operatorname{Re} c \rfloor +\alpha+2} x \, d\lambda \\
&= \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} b+\alpha \rfloor -3} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} c \rfloor +\alpha+2} x \, \frac{d\lambda}{2\pi i} \\
&= \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} c+\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} c \rfloor +\alpha+2} x \, \frac{d\lambda}{2\pi i},
\end{aligned}
$$

$\operatorname{Re} b \in (\operatorname{Re} c - 1, \operatorname{Re} c)$, it follows that $\lim_{b \to c,\ \operatorname{Re} b < \operatorname{Re} c} (-A)^b x = (-A)^c x$. The proof is completed. □

REMARK 1.4.13. (i) It is clear that the inclusion $(-A)^{b+c} \subseteq \overline{(-A)^b(-A)^c}$, $b,\ c \in \mathbb{C}$ clarified in the previous subsection cannot be expected if the domain of the operator A is not dense in E. The assertion (vi) quoted in the formulation of Theorem 1.4.12 presents an interpretation of this property in the case of non-densely defined operators.

(ii) Put $(-A)^b_\alpha := (-A)^{\lfloor \alpha \rfloor +2}(-A_{|F})^b(-A)^{-\lfloor \alpha \rfloor -2}$, $b \in \mathbb{C}$. Then the properties (i)–(xi) of Theorem 1.4.12 still hold with $n(A)$ and $(-A)^b$, replaced by $\lfloor \alpha \rfloor + 2$ and $(-A)^b_\alpha$, respectively, therein.

(iii) The proof of assertion (iii) of Theorem 1.4.12 implies that, for every $b \in \mathbb{C}$ and, $x \in D(A^{\lfloor \operatorname{Re} b+\alpha \rfloor +2})$ if $\alpha \geqslant 0$, resp. $x \in D(A^{\lfloor \operatorname{Re} b \rfloor +2})$, if $\alpha \in [-1, 0)$, we have:

$$
(12) \qquad (-A)^b x = \begin{cases} \frac{1}{2\pi i} \int\limits_{\Gamma} (-\lambda)^b R(\lambda : A) x \, d\lambda, \ \operatorname{Re} b < 0, \\ \frac{1}{2\pi i} \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} b \rfloor -1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor +1} x \, d\lambda, \ \operatorname{Re} b \geqslant 0 \end{cases}
$$

$$
(13) \qquad (-A)^b x = \frac{1}{2\pi i} \int\limits_{\Gamma} (-\lambda)^{b-\lfloor \operatorname{Re} b-\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b+\alpha \rfloor +2} x \, d\lambda.
$$

(iv) Let $(-A)^{-b} \in L(E)$ and $D((-A)^b) \subseteq D((-A)^c)$. Then $(-A)^{c-b} \in L(E)$ [**233**].

(v) Suppose that a closed linear operator A satisfies (§$_1$) and (§§$_1$). Then $-A$ falls under the scope of operators considered by Periago and Straub in [**359**] and one can construct the complex powers of $-A$ by the use of an extension of McIntosh functional calculus given in [**359**, Section 2]. It can be proved that the complex powers constructed in this subsection coincide with those of [**359**].

The following definition of an (analytic) semigroup of growth order $r > 0$ is an extension of the previous one given in Section 1.2.

DEFINITION 1.4.14. An operator family $(T(t))_{t>0}$ in $L(E)$ is a *semigroup of growth order* $r > 0$ if the following conditions hold:

(i) $T(t+s) = T(t)T(s)$, $t, s > 0$,
(ii) the mapping $t \mapsto T(t)x$, $t > 0$ is continuous for every fixed $x \in E$,
(iii) $\|t^r T(t)\| = O(1)$, $t \to 0+$ and
(iv) $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

If a semigroup $(T(t))_{t>0}$ of growth order $r > 0$ has an analytic extension to Σ_γ, for some $\gamma \in (0, \frac{\pi}{2}]$, denoted by the same symbol, and if additionally there exists $\omega \in \mathbb{R}$ such that, for every $\delta \in (0, \gamma)$, there exists a suitable constant $M_\delta > 0$ with $\|z^r T(z)\| \leqslant M_\delta e^{\omega \operatorname{Re} z}$, $z \in \Sigma_\delta$, then the family $(T(z))_{z\in\Sigma_\gamma}$ is called an *analytic semigroup of growth order* r.

Notice only that we have removed the density of the set $E_0 := \bigcup_{t>0} T(t)E$ in E from Definition 1.2.10 and Definition 1.2.11. The *infinitesimal generator* of $(T(t))_{t>0}$ is defined as before

$$G := \Big\{(x, y) \in E \times E : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y\Big\}.$$

By [**348**, Lemma 3.1], G is a closable linear operator. The closure of G, denoted by $\overline{G}$, is said to be the *complete infinitesimal generator,* in short, c.i.g., of $(T(t))_{t>0}$. The *continuity set* of $(T(t))_{t>0}$, resp. $(T(z))_{z\in\Sigma_\gamma}$, is defined to be the set $\{x \in E : \lim_{t\to 0+} T(t)x = x\}$, resp. $\{x \in E : \lim_{z\to 0,\ z\in\Sigma_{\gamma'}} T_b(z)x = x \text{ for all } \gamma' \in (0, \gamma)\}$.

Suppose that $\overline{G}$ is the c.i.g. of a semigroup $(T(t))_{t>0}$, resp. an analytic semigroup $(T(z))_{z\in\Sigma_\gamma}$, of growth order $r > 0$. Repeating literally the arguments given in [**348**] and [**398**] (cf. also [**324**, Section 5]), one gets that the conditions (I), (II) and (IV) quoted in the formulation [**348**, Theorem 1.2], resp. (b2), (b3) and (b4) quoted in the formulation [**398**, Theorem 3], remain true if the denseness of E_0 in E is disregarded. It is an open problem whether such conditions are sufficient for the generation of non-dense (analytic) semigroups of growth order $r > 0$. Further on, suppose that $(T(z))_{z\in\Sigma_\gamma}$ is an analytic semigroup of growth order r. It is clear that, for every $\theta \in (0, \gamma)$, $(T(te^{i\theta}))_{t>0}$ is a semigroup of growth order $r > 0$. With the help of C-regularized semigroups, one can prove that the integral generator of $(T(te^{i\theta}))_{t>0}$ is always $e^{i\theta}\hat{G}$ and that the c.i.g. of $(T(te^{i\theta}))_{t>0}$ is $e^{i\theta}\overline{G}$ whenever E_0 is dense in E or $r \in (0, 1)$ (cf. also [**466**, Theorem 1]). Unfortunately, it is quite questionable whether the last assertion remains true if $\overline{E_0} \neq E$ and $r \geqslant 1$.

THEOREM 1.4.15. *Let* $b \in (0, \frac{1}{2})$ *and let* A *be a closed linear operator with not necessarily densely defined domain. Then the operator* $-(-A_{|F})^b$ *is the c.i.g. of an analytic semigroup* $(T_b(z))_{z\in\Sigma_{\arctan(\cos \pi b)}}$ *of growth order* $\frac{\alpha+1}{b}$, *where*

$$T_b(z) := \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda,\ z \in \Sigma_{\arctan(\cos \pi b)}. \tag{14}$$

Furthermore, the integral generator of $(T_b(z))_{z\in\Sigma_{\arctan(\cos\pi b)}}$ *is just the operator* $-(-A)^{\lfloor\alpha\rfloor+2}(-A_{|F})^b(-A)^{-\lfloor\alpha\rfloor-2}$*; in particular,* $-(-A)^b \subseteq \hat{G}$ *and* $-(-A)^b = \hat{G}$ *if* $D(A)$ *is not dense in* E *and* $\alpha \in (-1, 0)$.

Proof. The choice of b implies $b\pi < \frac{\pi}{2}$. Put $\gamma := \arctan(\cos \pi b)$. Then $\gamma \in (0, \frac{\pi}{2})$ and, for every $z = \xi + i\eta \in \Sigma_\gamma$, we have $\xi\cos(b\pi) - |\eta| > 0$. Furthermore, $|e^{-z(-\lambda)^b}| = e^{-\xi|\lambda|^b\cos(b\arg(-\lambda))+\eta|\lambda|^b\sin(b\arg(-\lambda))} \leqslant e^{-(\xi\cos(b\pi)-|\eta|)|\lambda|^b}$. Without loss of generality, one can assume that $\varepsilon \in (0, 1)$. The convergence of the curve integral over Γ_1 and Γ_3 follows from the computation

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int\limits_{\Gamma_1} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda\right\| \\
&\leqslant \frac{M}{2\pi}\int\limits_{\varepsilon}^{\infty} e^{-(\xi\cos(b\pi)-|\eta|)\sqrt{t^2+t^{2a}}^{\,b}}\Big(1+\sqrt{t^2+t^{2a}}\Big)^{\alpha}(1+a\varepsilon^{a-1})\, dt \\
&\leqslant \frac{M(1+a\varepsilon^{a-1})}{2\pi}\Bigg[\int\limits_{\varepsilon}^{1} e^{-(\xi\cos(b\pi)-|\eta|)t^b}\big(1+\sqrt{2}\big)^{\alpha} dt \\
&\qquad\qquad + \int\limits_{1}^{\infty} e^{-(\xi\cos(b\pi)-|\eta|)t^b}\big(1+\sqrt{2}\big)^{\alpha} t^{\alpha} dt\Bigg] \\
&\leqslant \frac{M\big(1+\sqrt{2}\big)^{\alpha}(1+a\varepsilon^{a-1})}{2\pi}\Bigg[(1-\varepsilon)e^{-(\xi\cos(b\pi)-|\eta|)\varepsilon^b} + \int\limits_{0}^{\infty} e^{-(\xi\cos(b\pi)-|\eta|)t^b} t^{\alpha} dt\Bigg] \\
&= \frac{M\big(1+\sqrt{2}\big)^{\alpha}(1+a\varepsilon^{a-1})}{2\pi}\Bigg[(1-\varepsilon)e^{-(\xi\cos(b\pi)-|\eta|)\varepsilon^b} \\
&\qquad\qquad + \frac{1}{b}\Gamma\Big(\frac{\alpha+1}{b}\Big)(\xi\cos(b\pi)-|\eta|)^{-\frac{\alpha+1}{b}}\Bigg].
\end{aligned}$$

The convergence of the integral over Γ_2 is obvious and one obtains

$$\left\|\frac{1}{2\pi i}\int\limits_{\Gamma_2} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda\right\| \leqslant M e^{-(\xi\cos(b\pi)-|\eta|)d^b}(1+d)^{\alpha+1}.$$

Hence, for every $\delta \in (0, \gamma)$, we have $\|z^{\frac{\alpha+1}{b}} T_b(z)\| = O(1)$, $z \in \Sigma_\delta$. By an elementary application of Cauchy's formula, it follows that the integral in (14) does not depend on the choice of a curve $\Gamma(a, C, d)$. Denote by $\Omega_b(A)$, resp. $\Omega_b(A)$, the continuity set of $(T_b(z))_{z\in\Sigma_\gamma}$, resp. $(T_b(t))_{t>0}$. Fix a number $\lambda_0 \in \rho(A) \smallsetminus H$. Here we would like to point out that $\rho(A) \smallsetminus H$ is a nonempty set since $\rho(A)$ is an open subset of $\mathbb{C}$. Using the same arguments as in [**394**, Proposition 2.3, Proposition 2.5, Proposition 2.6, Proposition 2.7 and Proposition 2.8], one obtains:

1. Let $m \in \{0,1\}$ and $\gamma' \in (0,\gamma)$. Then the improper integral

$$\int_{\Gamma} \frac{-(-\lambda)^{mb} e^{-z(-\lambda)^b}}{(\lambda-\lambda_0)^{\lfloor b+\alpha \rfloor+2}} R(\lambda : A)\, d\lambda$$

converges uniformly for $z \in \overline{\Sigma_{\gamma'}}$.

2. The mapping $z \mapsto T_b(z)$, $z \in \Sigma_\gamma$ is analytic and

$$\frac{d^n}{dz^n} T_b(z) = \frac{(-1)^n}{2\pi i} \int_{\Gamma} (-\lambda)^{nb} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda, \quad n \in \mathbb{N}.$$

3. $T_b(z_1+z_2) = T_b(z_1)T_b(z_2)$, $z_1, z_2 \in \Sigma_\gamma$.
4. $D(A^{\lfloor b+\alpha \rfloor+1}) \subseteq \Omega_b(A)$, if $\lfloor b+\alpha \rfloor \geqslant 0$.
5. If $\lfloor b+\alpha \rfloor \leqslant 0$, $x \in D(A^{\lfloor b+\alpha \rfloor+2}) = D(J^b)$ and $\gamma' \in (0,\gamma)$, then

$$\lim_{z \to 0,\ z \in \Sigma_{\gamma'}} \frac{T_b(z)x - x}{z} = \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{b-1} R(\lambda : A) Ax\, d\lambda.$$

6. For every $z \in \Sigma_\gamma$, $T_b(z)$ is an injective operator.

By the foregoing, we obtain that $(T_b(z))_{z \in \Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{\alpha+1}{b}$. Suppose, for the time being, that A is densely defined and denote by A_b the generator of $(T_b(t))_{t>0}$. By 4, we obtain that $-J^b \subseteq A_b$. Consequently, $-(-A)^b \subseteq \overline{A_b}$. Since $\int_\Gamma e^{-z(-\lambda)^b} \lambda^n d\lambda = 0$, $n \in \mathbb{N}_0$, $z \in \Sigma_\gamma$, one can repeat literally the arguments given in [**394**, Lemma 2.10] in order to obtain that, for every $x \in E$ and $z \in \Sigma_\gamma$, $T_b(z)x \in D(A^n)$ and

$$A^n T_b(z)x = \frac{1}{2\pi i} \int_{\Gamma} e^{-z(-\lambda)^b} \lambda^n R(\lambda : A)x\, d\lambda. \tag{15}$$

To prove that $\overline{A_b} \subseteq -(-A)^b$, one can argue in the same manner as in [**394**]. Actually, it is enough to replace the natural number n in the proofs of [**394**, Proposition 2.11 and Proposition 2.12] with $\lfloor b+\alpha \rfloor$. Suppose now $\overline{D(A)} \neq E$. Denote by G the infinitesimal generator of $(T_b(z))_{z \in \Sigma_\gamma}$ and put $S_b(z)x := T_b(z)x$, $z \in \Sigma_\gamma$, $x \in F$. Since $T_b(z)x \in D_\infty(A)$, $z \in \Sigma_\gamma$, $x \in E$, we obtain that $S_b(z) \in L(F)$, $t \in \Sigma_\gamma$. Furthermore, for every $\lambda \in \rho(A)$, we have $R(\lambda : A)D(A^{n(A)}) \subseteq D(A^{n(A)+1})$ and $R(\lambda : A)F \subseteq F$. Hence, $R(\lambda : A)x = R(\lambda : A_{|F})x$, $x \in F$, $\lambda \in \rho(A : E) = \rho(A_{|F} : F)$ and

$$\begin{aligned} S_b(z)x = T_b(z)x &= \frac{1}{2\pi i} \int_{\Gamma} e^{-z(-\lambda)^b} R(\lambda : A)x\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} e^{-z(-\lambda)^b} R(\lambda : A_{|F})x\, d\lambda,\ x \in F,\ z \in \Sigma_\gamma. \end{aligned} \tag{16}$$

Since $A_{|F}$ is densely defined in F and satisfies (10), one can apply the first part of the proof in order to see that $-(-A_{|F})^b$ is the c.i.g. of an analytic semigroup

$(S_b(z))_{z\in\Sigma_\gamma}$ of growth order $\frac{\alpha+1}{b}$ in F. If $x \in D(G)$, then $\lim_{t\to 0+} T_b(t)x = x \in F$, and consequently, $Gx \in F$. With this in view, one immediately gets:

$$G = \Big\{(x, y) \in F \times F : \lim_{t\to 0+} \frac{S_b(t)x - x}{t} = y\Big\}.$$

We will prove that $-(-A)^b$ is the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$ only in the case of non-densely defined operators. Since, by (ii), $\bigcup_{z\in\Sigma_\gamma} \mathrm{R}(T_b(z)) \subseteq D_\infty(A)$, the following equivalence relation is obvious:

$$(x, y) \in \hat{G} \text{ iff } T_b(s)y = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-1} R(\lambda : A)AT_b(s)x\, d\lambda \text{ for all } s > 0. \tag{17}$$

Let $(x, y) \in D((-A)^b)$ and $n = \lfloor\alpha\rfloor + 2$. Then $(-A)^{-n}y = (-A_{|F})^b(-A)^{-n}x$ and this implies the existence of a sequence $(x_n, y_n) \in J_F^b$ such that $\lim_{n\to\infty} x_n = (-A)^n x$ and $\lim_{n\to\infty} J_F^b x_n = (-A_{|F}^b)(-A)^n x$. Keeping in mind (15), we infer that, for every $s > 0$:

$$\begin{aligned}
(-A)^n T_b(s)y &= \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} T_b(s)R(\lambda : A_{|F})(-A_{|F})^{\lfloor b+\alpha\rfloor+2}x_n d\lambda\\
&= \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)x_n d\lambda\\
&= \frac{(-1)^{\lfloor b+\alpha\rfloor+2}}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)\\
&\quad\times\left[\frac{1}{2\pi i}\int_\Gamma e^{-s(-\xi)^b}\xi^{\lfloor b+\alpha\rfloor+2}R(\xi : A)T_b(s)x_n d\xi\, d\lambda\right].
\end{aligned}$$

Using the dominated convergence theorem, one can continue the computation as follows:

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)(-A)^{-n}x\, d\lambda\\
&= \frac{1}{2\pi i}(-A)^{-n}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)x\, d\lambda, \quad s > 0.
\end{aligned}$$

The injectiveness of $(-A)^{-n}$ yields (17) and $-(-A)^b \subseteq \hat{G}$. Next, we will show that $D(A^{\lfloor\alpha\rfloor+2}) \subseteq \Omega_b(A)$. If $b + \alpha \geqslant 0$, the proof is obvious; suppose $b + \alpha < 0$, $\gamma' \in (0, \gamma)$ and $\lambda_0 \in \rho(A) \smallsetminus H(a, C, d)$. Then $\lfloor\alpha\rfloor + 2 = 1$ and

$$\begin{aligned}
T_b(z)(-A)^{-1}x - (-A)^{-1}x &= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)(-A)^{-1}x\, d\lambda - (-A)^{-1}x\\
= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)&(-A)^{-1}x\, d\lambda - e^{-z(-\lambda_0)^b}(-A)^{-1}x\\
&+ [e^{-z(-\lambda_0)^b}(-A)^{-1}x - (-A)^{-1}x]
\end{aligned}$$

$$= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}\Big[R(\lambda : A)(-A)^{-1}x - \frac{(-A)^{-1}x}{\lambda-\lambda_0}\Big]d\lambda + \big[e^{-z(-\lambda_0)^b}(-A)^{-1}x - (-A)^{-1}x\big]$$

$$= \frac{(-1)}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}\left[\frac{R(\lambda : A)x}{\lambda} + \frac{\lambda_0(-A)^{-1}x}{\lambda(\lambda-\lambda_0)}\right]d\lambda + \big[e^{-z(-\lambda_0)^b}(-A)^{-1}x - (-A)^{-1}x\big],$$

for all $z \in \Sigma_{\gamma'}$ and $x \in E$. The preceding equality combined with the residue theorem, the inequality $|e^{-z(-\lambda)^b}| \leqslant e^{-(\operatorname{Re} z\cos(b\pi)-|\operatorname{Im} z|)|\lambda|^b}$ and the dominated convergence theorem, indicates that $\lim_{z\to 0,\, z\in\Sigma_{\gamma'}} T_b(z)(-A)^{-1}x = (-A)^{-1}x$, $x \in E$ and that $D(A) \subseteq \Omega_b(A)$. Then it is checked at once that $\hat{G}$ is the the integral generator of an exponentially bounded, analytic $(-A)^{-n}$-regularized semigroup $\big(S_b(z) := T_b(z)(-A)^{-n}\big)_{z\in\Sigma_\gamma}$. The assumption $(x, y) \in \hat{G}$ implies

$$\lim_{t\to 0+}\frac{T_b(t)(-A)^{-n}x - (-A)^{-n}x}{t} = (-A)^{-n}y,\ \ (-A)^{-n}x \in (-A_{|F})^b,$$
$$(-A_{|F})^b(-A)^{-n}x = (-A)^{-n}y \in D(A^n).$$

Thereby, $x \in D((-A)^b)$, $(-A)^b x = y = \hat{G}x$ and the proof is completed. □

THEOREM 1.4.16. *Let $n \in \mathbb{N}$, $n \geqslant 3$ and let A be a closed operator which satisfies (§) and (§§). Suppose $|\theta| < \arctan(\cos\frac{\pi}{n})$. Then, for every $x \in \Omega_{\frac{1}{n}}(A)$, the abstract Cauchy problem*

$$(P_n) : \begin{cases} u \in C((0,\infty) : [D(A)]) \cap C^n((0,\infty) : E), \\ \frac{d^n}{dt^n}u(t) = (-1)^{n+1}e^{in\theta}Au(t),\ t > 0, \\ \lim_{t\to 0+} u(t) = x,\ \sup_{t>0}\|u(t)\| < \infty, \end{cases}$$

has a solution $u(t) = T_{\frac{1}{n}}(te^{i\theta})x$, $t > 0$. Furthermore, $u(\cdot)$ can be analytically extended to$\Sigma_{\arctan(\cos\frac{\pi}{n})-|\theta|}$ and, for every $\delta \in (0, \arctan(\cos\frac{\pi}{n}) - |\theta|)$ and $i \in \mathbb{N}_0$,

$$\sup_{z\in\Sigma_\delta}\Big\|z^{i+n\alpha+n}\frac{d^i}{dz^i}u(z)\Big\| < \infty.$$

PROOF. We will prove the theorem only in the case $\theta = 0$. One can use the assertion 2 used in the proof of preceding theorem and (15) to obtain that $\frac{d^n}{dt^n}u(t) = (-1)^{n+1}Au(t)$, $t > 0$. By Theorem 1.4.15, $u(\cdot)$ can be analytically extended to $\Sigma_{\arctan(\cos\frac{\pi}{n})}$. Due to the proof of Theorem 1.4.15 (see the assertion 3), we easily infer that $\lim_{t\to 0+} u(t) = x$. Let us fix a number $\delta \in (0, \arctan(\cos\frac{\pi}{n}))$ and a number $z \in \Sigma_\delta$. Since

$$\frac{d^i}{dz^i}u(z) = \frac{(-1)^n}{2\pi i}\int_\Gamma (-\lambda)^{i/n}e^{-z(-\lambda)^{1/n}}R(\lambda : A)\,d\lambda,$$

and $\|(-\lambda)^{\frac{i}{n}}R(\lambda : A)x\| \leqslant M(1+|\lambda|)^{\alpha+\frac{i}{n}}$, it follows (see the proofs of Theorem 1.4.15 and [**394**, Proposition 2.2]) that

$$\Big\|\frac{d^i}{dz^i}u(z)\Big\| = O\Big(\big(\xi\cos(\pi/n)-|\eta|\big)^{-\frac{\alpha+i/n+1}{i/n}}\Big),\ \ z\in\Sigma_\delta.$$

Hence, $\sup_{z\in\Sigma_\delta}\|z^{i+n\alpha+n}\frac{d^i}{dz^i}u(z)\|<\infty$, as required. □

REMARK 1.4.17. (i) [**394**], [**358**] Suppose A is a closed operator as well as (1.4.1) holds and (1.4.2) holds with n replaced by α therein. Let $b\in\big(0,\frac{\pi}{2(b-a)}\big)$ (in particular, this holds good for $b\in(0,\frac{1}{2}]$) and let

$$T_b(t):=\frac{1}{2\pi i}\int\limits_{\Gamma} e^{-t(-\lambda)^b}R(\lambda\colon A)\,d\lambda,\ \ t\in\Sigma_{\arctan(\pi(b-a))},$$

where Γ is upwards oriented boundary of $\Omega_{a,d}=\{z\in\mathbb{C}:z\neq 0,\ |\arg(z)|\in(-a,a)\}\cup B_d$. Then, for every $t\in\Sigma_{\arctan(\pi(b-a))}$, $T_b(t)$ is an injective bounded operator and $(T_b(t))_{t\in\Sigma_{\arctan(\pi(b-a))}}$ is an analytic semigroup of growth order $\frac{\alpha+1}{b}$. Using the above conclusions, one can simply reformulate Theorem 1.4.15 in the case $n=2$; in this case, the uniqueness of solutions of the abstract Cauchy problem (P_n) holds provided $n(A)\leqslant 1$ ([**233**]).

(ii) Suppose that A is a closed linear operator and that there exist $M>0$ and $\alpha\in(-1,0)$ such that $[0,\infty)\subseteq\rho(A)$ and $\|R(\lambda\colon A)\|\leqslant M(1+|\lambda|)^{\alpha}$, $\lambda\geqslant 0$. Then the usual series argument implies the existence of numbers $C>0$, $d\in(0,1]$ and $M'>0$ satisfying $P_{-\alpha,C}\cup B_d\subseteq\rho(A)$ and $\|R(\lambda:A)\|\leqslant M'(1+|\lambda|)^{\alpha}$, $\lambda\in P_{-\alpha,C}\cup B_d$. By Theorem 1.4.15, one obtains that, for every $b\in(0,\frac{1}{2})$, the operator $-\big(A_{|\overline{D(A)}}\big)^b$ generates an analytic semigroup $(T_b(z))_{z\in\Sigma_{\arctan(\cos\pi b)}}$ of growth order $\frac{\alpha+1}{b}$. Hence, (P_n) has a solution for every $n\in\mathbb{N}\setminus\{1,2\}$ and $x\in\Omega_{\frac{1}{n},\theta}(A)$.

(iii) In general, $D\big(A^{\lfloor\frac{1}{n}+\alpha\rfloor+1}\big)$ is strictly contained in $\Omega_{\frac{1}{n},\theta}(A)$ ([**358**]).

(iv) The c.i.g. of $(T_b(z))_{z\in\Sigma_\gamma}$ can be strictly contained in the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$ for all $b\in(0,\frac{1}{2})$. Indeed, suppose that $-A$ is a non-densely defined positive operator and denote by $\widehat{(-A)^b}$ the complex power of $-A$ in the sense of [**300**, Section 5]. By [**300**, Theorem 5.2.1, Corollary 5.1.12(i)], we have that $\widehat{(-A)^b}=-A\big(-A_{|\overline{D(A)}}\big)^b(-A)^{-1}$. Obviously, $\big(-\widehat{A_{|\overline{D(A)}}}\big)^b=\big(-A_{|\overline{D(A)}}\big)^b$, $b\in(0,\frac{1}{2})$ and this implies that $\widehat{(-A)^b}=(-A)^b$, $b\in(0,\frac{1}{2})$. On the other hand, it is clear that A satisfies $(\S_1)$ and $(\S\S_1)$ with some $\alpha\in(-1,0)$ and the claimed assertion follows by making use of [**300**, Corollary 5.1.12(ii)], which asserts that $\big(-A_{|\overline{D(A)}}\big)^b\neq(-A)^b$, $b\in(0,\frac{1}{2})$.

Assume now that a closed, possibly non-densely defined operator A satisfies:

$(\Diamond)$ $(0,\infty)\subseteq\rho(A)$ and $(\Diamond\Diamond)$ $\sup_{\lambda>0}(1+|\lambda|)^{-\alpha}\|R(\lambda\colon A)\|<\infty$,

for an appropriate constant $\alpha\geqslant -1$. The complex power $(-A)^b$, $b\in\mathbb{C}$ has been recently constructed in [**233**] following the above described method. First of all, notice that the usual series argument implies that, under the hypotheses $(\Diamond)$ and $(\Diamond\Diamond)$, there exist $d\in(0,1]$, $C\in(0,1)$, $\varepsilon\in(0,1]$ and $M>0$ such that:

$$(\odot)\qquad P_{\alpha,\varepsilon,C}\cup B_d\subseteq\rho(A),\ (\varepsilon,C(1+\varepsilon)^{-\alpha})\in\partial B_d,$$

$$(\odot\odot) \qquad \|R(\lambda\colon A)\| \leqslant M(1+|\lambda|)^{\alpha},\ \lambda \in P_{\alpha,\varepsilon,C} \cup B_d.$$

Put

$$\Gamma_1(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta = -C(1+\xi)^{-\alpha}\},$$
$$\Gamma_2(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi^2 + \eta^2 = d^2,\ \xi \leqslant \varepsilon\},$$
$$\Gamma_3(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta = C(1+\xi)^{-\alpha}\}.$$

The curve $\Gamma(\alpha,\varepsilon,C) := \Gamma_1(\alpha,\varepsilon,C) \cup \Gamma_2(\alpha,\varepsilon,C) \cup \Gamma_3(\alpha,\varepsilon,C)$ is oriented so that $\operatorname{Im}\lambda$ increases along $\Gamma_2(\alpha,\varepsilon,C)$ and that $\operatorname{Im}\lambda$ decreases along $\Gamma_1(\alpha,\varepsilon,C)$ and $\Gamma_3(\alpha,\varepsilon,C)$. Suppose for a moment that A is densely defined and that $\alpha \geqslant 0$. Using the arguments given in the proof of Proposition 1.4.1, we have that, for every $b \in \mathbb{C}$ with $\operatorname{Re} b < -(\alpha+1)$, the integral $I(b) := \frac{1}{2\pi i}\int_{\Gamma(\alpha,\varepsilon,C)} (-\lambda)^b R(\lambda\colon A)\, d\lambda$ exists and defines a bounded linear operator. Let $b \in \mathbb{C}$. Then the operator J^b is defined as in Definition 1.4.2. Arguing as in the proof of Proposition 1.4.4, we have that (7) holds for all $x \in D(A^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}) = D(J^b)$. Put $C^b := (-A)^{\lfloor \operatorname{Re} b+\alpha\rfloor+2} J^{b-\lfloor \operatorname{Re} b+\alpha\rfloor-2}$. Then, for every $b \in \mathbb{C}$, C^b is a closed linear operator which contains J^b and one can prove that $C^b = \overline{J_b}$ if $|\operatorname{Re} b| > \alpha+1$ or $b \in \mathbb{Z}$. The complex power $(-A)^b$, $b \in \mathbb{C}$ is defined by $(-A)^b := \overline{J^b}$ and coincides with the usual power of the operator A if $b \in \mathbb{Z}$. It is worthwhile to note that the assertions of Lemma 1.4.5, Lemma 1.4.7, Proposition 1.4.6 and Theorem 1.4.9 as well as the equality (9) still hold in the case of operators satisfying ($\lozenge$) and ($\lozenge\lozenge$). Suppose now that a closed, densely defined operator A satisfies ($\lozenge$) and ($\lozenge\lozenge$) with $\alpha \in [-1,0)$. Then it is clear that $\|R(\cdot : A)\|$ is bounded on the region $P_{\alpha,\varepsilon,C} \cup B_d$. We define the complex powers of $-A$ by assuming that $\alpha = 0$. Then, as before, $(-A)^b$ is a closed, densely defined linear operator and $(-A)^b \in L(E)$ provided $\operatorname{Re} b < -(\alpha+1)$. Fix a number $\alpha \geqslant -1$ satisfying ($\lozenge\lozenge$). Then the construction of powers of densely defined operators does not depend on the choice of numbers $d \in (0,1]$, $C \in (0,1)$, $\varepsilon \in (0,1]$ and $M > 0$ satisfying ($\odot$) and ($\odot\odot$). Furthermore, $\sup_{\lambda>0}(1+|\lambda|)^{-\beta}\|R(\lambda\colon A)\| < \infty$ for all $\beta \in [\alpha,\infty)$, and the construction of powers of densely defined operators does not depend on the choice of such a number β. If A is not densely defined and satisfies ($\lozenge$)–($\lozenge\lozenge$), then we define the power $(-A)^b$ and the operator $(-A)^b_\alpha$ ($b \in \mathbb{C}$) as in Definition 1.4.11 and Remark 1.4.13(ii), respectively. Then the assertions of Theorems 1.4.12, 1.4.15, 1.4.16 and Remark 1.4.13 continue to hold in the case of operators satisfying ($\lozenge$) and ($\lozenge\lozenge$). Finally, suppose that ($\lozenge$) and ($\lozenge\lozenge$) hold with $\alpha > 0$. Set $T(t) := (-A)^{it}(-A)^{-\lfloor\alpha\rfloor-2}$, $t \in \mathbb{R}$. Then the closed graph theorem implies $T(t) \in L(E)$, $t \in \mathbb{R}$, and by Theorem 1.4.12(xi), we obtain that the mapping $t \mapsto T(t)x$, $t \in \mathbb{R}$ is continuous for every fixed $x \in E$. One can simply prove that $(T(t))_{t\in\mathbb{R}}$ is a global $(-A)^{-\lfloor\alpha\rfloor-2}$-regularized group. Denote by B the integral generator of $(T(t))_{t\in\mathbb{R}}$. The *logarithm of* $-A$, denoted by $\log(-A)$, is defined by $\log(-A) := -iB$. Clearly, the definition of $\log(-A)$ is independent of the choice of a number $\alpha > 0$ satisfying ($\lozenge$) and ($\lozenge\lozenge$), and $\pm i\log(-A)$ are the integral generators of local $(-A)^{-\lfloor\alpha\rfloor-2}$-regularized semigroups. For further information concerning operator logarithms, we refer to [**46**], [**73**], [**89**], [**113**], [**146**], [**264**], [**341**], [**350**] and [**443**].

CHAPTER 2

CONVOLUTED C-SEMIGROUPS AND COSINE FUNCTIONS

Throughout this chapter, E and $L(E)$ denote a non-trivial complex Banach space and the Banach algebra of bounded linear operators on E. For a closed linear operator A acting on E, $D(A)$, $\mathrm{Kern}(A)$, $\mathrm{R}(A)$ and $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively. We assume henceforth $C \in L(E)$ and C is injective; recall, the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \mathrm{R}(C) \subseteq \mathrm{R}(\lambda - A)$ and $\lambda - A$ is injective$\}$. From now on, $D(A)$ is equipped with the graph norm $\|x\|_{[D(A)]} := \|x\| + \|Ax\|$, $x \in D(A)$; $\tau \in (0,\infty]$, K is a complex-valued locally integrable function in $[0,\tau)$ and $K(\cdot)$ is not identical to zero. Put $\Theta(t) := \int_0^t K(s)ds$ and $\Theta^{-1}(t) := \int_0^t \Theta(s)\,ds$, $t \in [0,\tau)$; then $\Theta(\cdot)$ is an absolutely continuous function in $[0,\tau)$ and $\Theta'(t) = K(t)$ for a.e. $t \in [0,\tau)$. Let us recall that a function $K \in L^1_{\mathrm{loc}}([0,\tau))$ is called a *kernel* if, for every $\phi \in C([0,\tau))$, the assumption $\int_0^t K(t-s)\phi(s)\,ds = 0$, $t \in [0,\tau)$, implies $\phi \equiv 0$; due to the famous Titchmarsh's theorem [**14**], the condition $0 \in \operatorname{supp} K$ implies that K is a kernel.

We use occasionally the following conditions:

(P1) K is Laplace transformable, i.e., $K \in L^1_{\mathrm{loc}}([0,\infty))$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{K}(\lambda) := \mathcal{L}(K(t))(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}K(t)\,dt := \int_0^\infty e^{-\lambda t}K(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$.
Put $\mathrm{abs}(K) := \inf\{\operatorname{Re}\lambda : \tilde{K}(\lambda)$ exists$\}$.

(P2) K satisfies (P1) and $\tilde{K}(\lambda) \neq 0$, $\operatorname{Re}\lambda > \beta$, where $\beta \geqslant \mathrm{abs}(K)$.

2.1. Definitions and main structural properties

Definition 2.1.1. [**61**], [**228**]–[**230**] Suppose A is a closed operator, $K \in L^1_{\mathrm{loc}}([0,\tau))$ and $0 < \tau \leqslant \infty$. If there exists a strongly continuous operator family $(S_K(t))_{t\in[0,\tau)}$ ($S_K(t) \in L(E)$, $t \in [0,\tau)$) such that:

(i) $S_K(t)A \subseteq AS_K(t)$, $t \in [0,\tau)$,

(ii) $S_K(t)C = CS_K(t)$, $t \in [0,\tau)$ and

(iii) for all $x \in E$ and $t \in [0,\tau)$: $\int_0^t S_K(s)x\,ds \in D(A)$ and

$$A\int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx, \tag{18}$$

then it is said that A is a subgenerator of a *(local) K-convoluted C-semigroup* $(S_K(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then it is said that $(S_K(t))_{t\geqslant 0}$ is an *exponentially bounded,*

K-convoluted C-semigroup with a subgenerator A if, in addition, there are constants $M>0$ and $\omega\in\mathbb{R}$ such that $\|S_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$.

DEFINITION 2.1.2. [**228**]–[**230**] Suppose A is a closed operator, $K\in L^1_{\mathrm{loc}}([0,\tau))$ and $0<\tau\leqslant\infty$. If there exists a strongly continuous operator family $(C_K(t))_{t\in[0,\tau)}$ such that:

(i) $C_K(t)A\subseteq AC_K(t)$, $t\in[0,\tau)$,
(ii) $C_K(t)C=CC_K(t)$, $t\in[0,\tau)$ and
(iii) for all $x\in E$ and $t\in[0,\tau)$: $\int_0^t(t-s)C_K(s)x\,ds\in D(A)$ and

$$A\int_0^t(t-s)C_K(s)x\,ds=C_K(t)x-\Theta(t)Cx, \tag{19}$$

then it is said that A is a subgenerator of a *(local) K-convoluted C-cosine function* $(C_K(t))_{t\in[0,\tau)}$. If $\tau=\infty$, then it is said that $(C_K(t))_{t\geqslant 0}$ is an *exponentially bounded, K-convoluted C-cosine function* with a subgenerator A if, in addition, there are constants $M>0$ and $\omega\in\mathbb{R}$ such that $\|C_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$.

Plugging $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in Definition 2.1.1 and Definition 2.1.2, where $\alpha>0$, we obtain the well-known classes of *fractionally integrated C-semigroups* and *cosine functions*; in the case $C=I$, we obtain the classes of *K-convoluted semigroups* and *cosine functions.*

The integral generator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, is defined by

$$\left\{(x,y)\in E\times E: S_K(t)x-\Theta(t)Cx=\int_0^t S_K(s)y\,ds,\ t\in[0,\tau)\right\},\ \text{resp.},$$

$$\left\{(x,y)\in E\times E: C_K(t)x-\Theta(t)Cx=\int_0^t (t-s)C_K(s)y\,ds,\ t\in[0,\tau)\right\}.$$

The integral generator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, is a closed linear operator which is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. In what follows, we designate by $\wp(S_K)$, resp. $\wp(C_K)$, the set which consists of all subgenerators of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. It is well known that such sets can be consisted of infinitely many elements [**228, 422**]; before illustrate these facts, we clarify the following proposition which can be simply justified with the help of Proposition 1.1.2 and Proposition 1.1.5.

PROPOSITION 2.1.3. *Let A be a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, resp. K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, and let $H\in L^1_{\mathrm{loc}}([0,\tau))$ satisfy $H*_0K\neq 0$ in $L^1_{\mathrm{loc}}([0,\tau))$. Then A is a subgenerator of an $(H*_0K)$-convoluted C-semigroup $\big((H*_0S_K)(t)\big)_{t\in[0,\tau)}$, resp. $(H*_0K)$-convoluted C-cosine function $\big((H*_0C_K)(t)\big)_{t\in[0,\tau)}$.*

For example, if $(S(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, is a (local) C-semigroup, resp. C-cosine function, with a subgenerator A, define $S_K(t)x:=\int_0^t K(t-s)S(s)x\,ds$

and $C_K(t)x := \int_0^t K(t-s)C(s)x\,ds$, $x \in E$, $t \in [0,\tau)$. Then $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, is a (local) K-convoluted C-semigroup, resp. K-convoluted C-cosine function with a subgenerator A.

EXAMPLE 2.1.4. (a) [**422**] Let $E := l^2$, $n \in \mathbb{N}$,

$$C\langle x_k\rangle := \langle\overbrace{0,\dots,0}^{n}, x_1, x_2, \dots\rangle \text{ and } S(t) := e^tC,\ t \geqslant 0,\ \langle x_k\rangle \in E.$$

Then $(S(t))_{t\geqslant 0}$ is a global exponentially bounded C-regularized semigroup with the integral generator I and $|\wp(S)| = 2^n$.

(b) [**228**] Choose an arbitrary $K \in L^1_{\mathrm{loc}}([0,\infty))$. Put $E := l^\infty$, $C\langle x_n\rangle := \langle 0, x_1, 0, x_2, 0, x_3, \dots\rangle$ and $C_K(t)\langle x_n\rangle := \Theta(t)C\langle x_n\rangle$, $t \geqslant 0$, $\langle x_n\rangle \in E$. If $I \subseteq 2\mathbb{N}+1$, define $E_I := \{\langle x_n\rangle \in E : x_i = 0 \text{ for all } i \in (2\mathbb{N}+1) \smallsetminus I\}$. It is clear that E_I is a closed subspace of E which contains $\mathrm{R}(C)$ and that $E_{I_1} \neq E_{I_2}$, if $I_1 \neq I_2$. Define a closed linear operator A_I on E by $D(A_I) := E_I$ and $A_I\langle x_n\rangle := 0$, $\langle x_n\rangle \in D(A_I)$. It is straightforward to see that, for every $I \subseteq 2\mathbb{N}+1$, A_I is a subgenerator of the global K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$ and that $\wp(C_K) = \{A_I : I \subseteq 2\mathbb{N}+1\}$. This implies that there exist the continuum many subgenerators of $(C_K(t))_{t\geqslant 0}$.

Suppose A is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$. Then $CA \subseteq AC$; in order to verify this, suppose $x \in D(A)$, $t \in [0,\tau)$ and $\Theta(t) \neq 0$. Combining the closedness of A with the conditions (i) and (iii) quoted in the formulation of Definition 2.1.2, it follows that

$$\begin{aligned}C_K(t)Ax - \Theta(t)CAx = A\int_0^t (t-s)C_K(s)Ax\,ds &= A^2\int_0^t (t-s)C_K(s)x\,ds\\ &= A(C_K(t)x - \Theta(t)Cx).\end{aligned}$$

Since $C_K(t)x \in D(A)$ and $\Theta(t) \neq 0$, we immediately obtain $Cx \in D(A)$ and $CAx = ACx$. The same conclusion holds if A is a subgenerator of a K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$; in this chapter we always assume that $CA \subseteq AC$.

The following composition property of local convoluted C-semigroups follows from the argumentation given in the proof of [**275**, Proposition 2.4] (cf. also [**61**] and [**230**, Proposition 5.4]); notice only that the equality

$$\Theta(s)\Theta(t-s) - \int_{t-s}^t K(t-r)\Theta(r)\,dr + \int_0^s K(t-r)\Theta(r)\,dr = 0,\quad 0 \leqslant t < \tau,\ 0 \leqslant s < t$$

implies that the coefficient of C^2x appearing in the proof of [**275**, Proposition 2.4] equals zero.

PROPOSITION 2.1.5. *Assume A is a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Then the following holds:*

$$S_K(t)S_K(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)S_K(r)Cx\,dr, \tag{20}$$

for every $x \in E$ and $t,\ s \in [0,\tau)$ with $t+s<\tau$. Hence, $S_K(t)S_K(s)=S_K(s)S_K(t)$ for all $t,\ s \in [0,\tau)$ with $t+s \leqslant \tau$.

The following proposition is a generalization of [**230**, Proposition 5.5(3)–(4)], [**422**, Corollary 2.9, Proposition 3.3] and some results given in [**228**, Section 2].

PROPOSITION 2.1.6. *Assume $\hat{A}$ is the integral generator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$ and $\{A,B\} \subseteq \wp(S_K)$. Then:*

(i) $\hat{A}=C^{-1}AC \in \wp(S_K)$.
(ii) $C^{-1}AC=C^{-1}BC$, $C(D(A)) \subseteq D(B)$ *and* $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$.
(iii) *If* $A \neq \hat{A}$, *then* $\rho(A)=\emptyset$.
(iv) *For every* $\lambda \in \rho_C(A)$:

$$(\lambda-A)^{-1}CS_K(t)=S_K(t)(\lambda-A)^{-1}C,\ t \in [0,\tau). \tag{21}$$

(v) *A and B have the same eigenvalues.*
(vi) *If* $A \subseteq B$, *then* $\rho_C(A) \subseteq \rho_C(B)$.
(vii) $|\wp(S_K)|=1$, *if* $C(D(\hat{A}))$ *is a core for* $\hat{A}$.
(viii) $Ax=Bx,\ x \in D(A)\cap D(B)$.
(ix) *Define the operators $A \wedge B$ and $A \vee B$ as follows:* $A\wedge B=\frac{1}{2}(A+B)$, $D(A\vee_0 B):=\operatorname{span}[D(A)\cup D(B)]$ *and*

$$A\vee_0 B(ax+by):=aAx+bBy,\ x\in D(A),\ y\in D(B),\ a,\ b\in\mathbb{C}.$$

Then $A\vee_0 B$ is closable and $\{A\wedge B,\ A\vee B\}\subseteq \wp(S_K)$, where $A\vee B:=\overline{A\vee_0 B}$.

PROOF. Obviously, $CA \subseteq AC$, $A \subseteq C^{-1}AC$ and $C^{-1}AC$ is closed. Assume $(x,y) \in \hat{A}$, i.e., $S_K(t)x-\Theta(t)Cx=\int_0^t S_K(s)y\,ds,\ t\in[0,\tau)$. Thereby, $A\int_0^t S_K(s)x\,ds=\int_0^t S_K(s)y\,ds,\ t\in[0,\tau)$, which simply implies $S_K(t)x\in D(A)$, $AS_K(t)x=S_K(t)y$ and $A[\Theta(t)Cx+\int_0^t S_K(s)y\,ds]=S_K(t)y,\ t\in[0,\tau)$. Since $\int_0^t S_K(s)y\,ds\in D(A),\ t\in[0,\tau)$ and $\Theta\neq 0$ in $C([0,\tau))$, one gets $Cx\in D(A)$ and $\Theta(t)ACx+S_K(t)y-\Theta(t)Cy=S_K(t)y,\ t\in[0,\tau)$. This implies $ACx=Cy$, $(x,y)\in C^{-1}AC$ and $\hat{A}\subseteq C^{-1}AC$. Clearly, $\int_0^t S_K(s)x\,ds\in D(A)\subseteq D(C^{-1}AC)$ and $C^{-1}AC\int_0^t S_K(s)x\,ds=A\int_0^t S_K(s)x\,ds=S_K(t)x-\Theta(t)Cx,\ t\in[0,\tau),\ x\in E$. Suppose now $x\in D(C^{-1}AC)$ and $t\in[0,\tau)$. Since $Cx\in D(A)$ and $S_K(t)A\subseteq AS_K(t)$, one obtains $CS_K(t)x=S_K(t)Cx\in D(A)$ and $ACS_K(t)x=AS_K(t)Cx=S_K(t)ACx=S_K(t)C[C^{-1}AC]x=CS_K(t)[C^{-1}AC]x\in \mathrm{R}(C)$ and $[C^{-1}AC]S_K(t)x=S_K(t)[C^{-1}AC]x$. So, $S_K(t)[C^{-1}AC]\subseteq[C^{-1}AC]S_K(t)$, $C^{-1}AC$ is a subgenerator of $(S_K(t))_{t\in[0,\tau)}$ and $C^{-1}AC\subseteq\hat{A}$. Therefore, $\hat{A}=C^{-1}AC$ and the proof of (i) is completed; (ii) and (iii) follow automatically from (i). To prove (iv), assume $\lambda\in\rho_C(A)$, $t\in[0,\tau)$ and $x\in E$. Then $(\lambda-A)^{-1}Cx\in D(A)$, $S_K(t)(\lambda-A)^{-1}Cx\in D(A)$ and $(\lambda-A)S_K(t)(\lambda-A)^{-1}Cx=S_K(t)(\lambda-A)(\lambda-A)^{-1}Cx=S_K(t)Cx=CS_K(t)x$. This gives (21). To prove (v) and (vi), observe only that $\operatorname{Kern}(\lambda-A)\subseteq\operatorname{Kern}(\lambda-\hat{A})$ and that $C^{-1}BC=\hat{A}$ implies $C\big(\operatorname{Kern}(\lambda-\hat{A})\big)\subseteq\operatorname{Kern}(\lambda-B)$, $\lambda\in\mathbb{C}$. Suppose now $A\in\wp(S_K)$, $x\in D(\hat{A})$ and $C(D(\hat{A}))$ is a core for $\hat{A}$. Let (x_n) be a sequence in $D(\hat{A})$ such that $\lim_{n\to\infty}Cx_n=x$ and $\lim_{n\to\infty}\hat{A}Cx_n=\hat{A}x$. Since $C(D(\hat{A}))\subseteq D(A)$, we obtain that $\lim_{n\to\infty} Cx_n = x$ and $\lim_{n\to\infty} ACx_n = \hat{A}x$. The closedness of A implies $x \in D(A)$, $D(\hat{A}) \subseteq D(A)$ and $\hat{A} = A$. The proofs of (viii) and (ix) are left to the reader. $\square$

REMARK 2.1.7. There exist examples of local C-regularized semigroups and local C-regularized cosine functions whose integral generators possess the empty C-resolvent sets [**275**]. Moreover, $|\wp(S_K)| = 1$ provided $C = I$ [**227**].

Assume $(S_K(t))_{t\in[0,\tau)}$ is a (local) K-convoluted C-semigroup and K is a kernel. By [**230**, Proposition 5.5], $S_K(t)S_K(s) = S_K(s)S_K(t)$, $0 \leqslant t,\ s < \tau$ and $(S_K(t))_{t\in[0,\tau)}$ is uniquely determined by one of its subgenerators.

REMARK 2.1.8. (i) Define the operator A_1 by

$$D(A_1) := \left\{ \sum_{k=1}^{m} \int_0^{t_k} S_K(s)x_k\,ds : x_k \in E,\ t_k \in [0,\tau),\ k = 1,\ldots,m \right\},$$

$$A_1\left(\sum_{k=1}^{m} \int_0^{t_k} S_K(s)x_k\,ds \right) := \sum_{k=1}^{m} \Big(S(t_k)x_k - \Theta(t_k)Cx_k \Big).$$

It is straightforward to verify that A_1 is well-defined and closable. Suppose, additionally, $\tau = \infty$ or K is a kernel. Then $S_K(t)S_K(s) = S_K(s)S_K(t)$, $t,\ s \in [0,\tau)$ and this enables one to see that: $S_K(t)(D(A_1)) \subseteq D(A_1)$, $S_K(t)A_1 \subseteq A_1S_K(t)$, $S_K(t)\overline{A_1} \subseteq \overline{A_1}S_K(t)$, $t \in [0,\tau)$ and $\overline{A_1} \in \wp(S_K)$. Certainly, $\overline{A_1} \subseteq A$, if $A \in \wp(S_K)$.

(ii) It can be proved that $(\wp(S_K), \wedge, \vee)$ is a complete lattice whose partial ordering coincides with the usual set inclusion and that $\wp(S_K)$ is totally ordered iff $|\wp(S_K)| \leqslant 2$ [**422**].

(iii) Suppose $|\wp(S_K)| < \infty$. Arguing as in [**422**, Section 2], one can prove that $(\wp(S_K), \wedge, \vee)$ is a Boolean lattice; this implies the existence of a non-negative integer n satisfying $|\wp(S_K)| = 2^n$.

The following extension type theorem for local convoluted C-semigroups essentially follows from the analysis obtained by Ciorănescu and Lumer in [**61**] (cf. also [**5**], [**216**] and [**275**] for some special cases).

THEOREM 2.1.9. *Let A be a subgenerator of a local K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, $\tau_0 \in (\frac{\tau}{2},\tau)$ and let $K = K_{1|[0,\tau)}$ for an appropriate complex-valued function $K_1 \in L^1_{\mathrm{loc}}([0,2\tau))$. $\big($Put $\Theta_1(t) = \int_0^t K_1(s)\,ds$ and $\Theta_1^{-1}(t) = \int_0^t \Theta_1(s)\,ds$, $t \in [0,2\tau)$; since it makes no misunderstanding, we will also write K, Θ and Θ^{-1} for K_1, Θ_1 and Θ_1^{-1}, respectively, and denote by $K *_0 K$ the restriction of this function to any subinterval of $[0,2\tau)$.$\big)$ Then A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-semigroup $(S_{K*_0K}(t))_{t\in[0,2\tau_0)}$, where: $S_{K*_0K}(t)x = \int_0^t K(t-s) \times S_K(s)Cx\,ds$, $t \in [0,\tau_0]$ and*

$$S_{K*_0K}(t)x = S_K(\tau_0)S_K(t-\tau_0)x + \left(\int_0^{t-\tau_0} + \int_0^{\tau_0} \right) K(t-r)S_K(r)Cx\,dr,$$

for any $t \in (\tau_0, 2\tau_0)$ *and* $x \in E$. *Furthermore, the condition* $0 \in \operatorname{supp} K$ *implies that* A *is a subgenerator of a local* $(K *_0 K)$*-convoluted* C^2*-semigroup on* $[0, 2\tau)$.

COROLLARY 2.1.10. *Suppose* $\alpha > 0$ *and* A *is a subgenerator of a local* α*-times integrated* C*-semigroup* $(S_\alpha(t))_{t\in[0,\tau)}$. *Then* A *is a subgenerator of a local* (2α)*-times integrated* C^2*-semigroup* $(S_{2\alpha}(t))_{t\in[0,2\tau)}$.

We need the following useful theorem which enables one to clarify several important characterizations of (local) convoluted C-cosine functions by a trustworthy passing to the corresponding theory of convoluted C-semigroups; notice that one can relate (local) C-regularized cosine functions and (local) once integrated C-regularized semigroups analogically.

THEOREM 2.1.11. *Suppose* A *is a closed operator,* $0 < \tau \leqslant \infty$ *and* $K \in L^1_{\mathrm{loc}}([0, \tau))$. *Then the following assertions are equivalent:*

(i) *A is a subgenerator of a K-convoluted C-cosine function* $(C_K(t))_{t\in[0,\tau)}$ *in* E.

(ii) *The operator* $\mathcal{A} := \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ *is a subgenerator of a* Θ*-convoluted* $\mathcal{C}$*-semigroup* $(S_\Theta(t))_{t\in[0,\tau)}$ *in* $E \times E$, *where* $\mathcal{C} := \left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right)$.

In this case:

$$S_\Theta(t) = \begin{pmatrix} \int_0^t C_K(s)\,ds & \int_0^t (t-s)C_K(s)\,ds \\ C_K(t) - \Theta(t)C & \int_0^t C_K(s)\,ds \end{pmatrix}, \quad 0 \leqslant t < \tau,$$

and the integral generators of $(C_K(t))_{t\in[0,\tau)}$ *and* $(S_\Theta(t))_{t\in[0,\tau)}$, *denoted respectively by* B *and* $\mathcal{B}$, *satisfy* $\mathcal{B} = \left(\begin{smallmatrix} 0 & I \\ B & 0 \end{smallmatrix}\right)$. *Furthermore, the integral generator of* $(C_K(t))_{t\in[0,\tau)}$, *resp.* $(S_\Theta(t))_{t\in[0,\tau)}$, *is* $C^{-1}AC$, *resp.* $\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \equiv \left(\begin{smallmatrix} 0 & I \\ C^{-1}AC & 0 \end{smallmatrix}\right)$.

PROOF. (i) $\Rightarrow$ (ii) It is checked at once that $(S_\Theta(t))_{t\in[0,\tau)}$ is a strongly continuous operator family in $E \times E$ satisfying $S_\Theta(t)\mathcal{A} \subseteq \mathcal{A}S_\Theta(t)$ and $S_\Theta(t)\mathcal{C} = \mathcal{C}S_\Theta(t)$, $0 \leqslant t < \tau$. The proof of (ii) follows from the next simple computation:

$$\begin{aligned}
\mathcal{A}\int_0^t S_\Theta(s)\begin{pmatrix} x \\ y \end{pmatrix} ds &= \mathcal{A}\int_0^t \begin{pmatrix} \int_0^s C_K(r)x\,dr + \int_0^s (s-r)C_K(r)y\,dr \\ C_K(s)x - \Theta(s)Cx + \int_0^s C_K(r)y\,dr \end{pmatrix} ds \\
&= \mathcal{A}\begin{pmatrix} \int_0^t (t-s)C_K(s)x\,ds + \int_0^t \frac{(t-s)^2}{2} C_K(s)y\,ds \\ \int_0^t C_K(s)x\,ds - \int_0^t \Theta(s)Cx\,ds + \int_0^t (t-s)C_K(s)y\,ds \end{pmatrix} \\
&= \begin{pmatrix} \int_0^t C_K(s)x\,ds - \int_0^t \Theta(s)Cx\,ds + \int_0^t (t-s)C_K(s)y\,ds \\ C_K(t)x - \Theta(t)Cx + \int_0^t C_K(s)y\,ds - \int_0^t \Theta(s)Cy\,ds \end{pmatrix} \\
&= S_\Theta(t)\begin{pmatrix} x \\ y \end{pmatrix} - \int_0^t \Theta(s)\begin{pmatrix} Cx \\ Cy \end{pmatrix} ds, \quad 0 \leqslant t < \tau, \ x, \ y \in E.
\end{aligned}$$

(ii) $\Rightarrow$ (i) Put

$$S_\Theta(t) = \begin{pmatrix} S^1_\Theta(t) & S^2_\Theta(t) \\ S^3_\Theta(t) & S^4_\Theta(t) \end{pmatrix}_{t\in[0,\tau)},$$

where $S^i_\Theta(t) \in L(E)$, $i \in \{1,2,3,4\}$ and $0 \leqslant t < \tau$. A simple consequence of $S_\theta(t)\mathcal{C} = \mathcal{C}S_\Theta(t)$, $t \in [0,\tau)$ is $S^i_\Theta(t)C = CS^i_\Theta(t)$, $t \in [0,\tau)$, $i \in \{1,2,3,4\}$. Since $S_\Theta(t)\mathcal{A} \subseteq \mathcal{A}S_\Theta(t)$, $t \in [0,\tau)$, one gets:

$$\begin{aligned}
&S^1_\Theta(t)x + S^2_\Theta(t)y \in D(A),\\
&S^1_\Theta(t)y + S^2_\Theta(t)Ax = S^3_\Theta(t)x + S^4_\Theta(t)y,\\
&S^3_\Theta(t)y + S^4_\Theta(t)Ax = A(S^1_\Theta(t)x + S^2_\Theta(t)y),\ \ 0 \leqslant t < \tau,\ x \in D(A),\ y \in E.
\end{aligned}$$

Hence, $S^3_\Theta(t)x = S^2_\Theta(t)Ax$, $x \in D(A)$ and $S^3_\Theta(t)y = AS^2_\Theta(t)y$, $y \in E$, $0 \leqslant t < \tau$. This implies that, for every $x \in D(A)$, $S^3_\Theta(t)Ax = AS^2_\Theta(t)Ax = AS^3_\Theta(t)x$, $t \in [0,\tau)$. Thereby, $S^3_\Theta(t)A \subseteq AS^3_\Theta(t)$, $t \in [0,\tau)$ and $(S^3_\Theta(t) + \Theta(t)C)_{t\in[0,\tau)}$ is a strongly continuous operator family in E. By making use of the following equality

$$\mathcal{A}\int_0^t S_\Theta(s)\binom{x}{y}\,ds = S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds,$$

one yields

$$\begin{aligned}
&\int_0^t S^3_\Theta(s)x\,ds + \int_0^t S^4_\Theta(s)y\,ds = S^1_\Theta(t)x + S^2_\Theta(t)y - \int_0^t \Theta(s)Cx\,ds,\\
&A\left(\int_0^t S^1_\Theta(s)x\,ds + \int_0^t S^2_\Theta(s)y\,ds\right) = S^3_\Theta(t)x + S^4_\Theta(t)y - \int_0^t \Theta(s)Cy\,ds,
\end{aligned}$$

provided $0 \leqslant t < \tau, x,\ y \in E$. Hence,

$$\begin{aligned}
&\int_0^t S^3_\Theta(s)x\,ds = S^1_\Theta(t)x - \int_0^t \Theta(s)Cx\,ds, \qquad A\int_0^t S^1_\Theta(s)x\,ds = S^3_\Theta(t)x,\\
&A\left(\int_0^t (t-s)(S^3_\Theta(s)x + \Theta(s)Cx)\,ds\right) = A\left(\int_0^t (t-s)\Big(\frac{d}{dv}S^1_\Theta(v)x\Big)_{v=s}ds\right)\\
&= A\int_0^t S^1_\Theta(s)x\,ds = \big(S^3_\Theta(t)x + \Theta(t)Cx\big) - \Theta(t)Cx,\ \ 0 \leqslant t < \tau,\ x \in E,
\end{aligned}$$

and we have proved that A is a subgenerator of the K-convoluted C-cosine function $(S^3_\Theta(t) + \Theta(t)C)_{t\in[0,\tau)}$. Clearly, $S^1_\Theta(t) = S^4_\Theta(t)$ and $S^2_\Theta(t) = \int_0^t S^1_\Theta(s)\,ds$, $0 \leqslant t < \tau$. To prove that $\mathcal{B} = \left(\begin{smallmatrix} 0 & I \\ B & 0 \end{smallmatrix}\right)$, let us fix elements $x, y, x_1, y_1 \in E$. Then

$$S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds = \int_0^t S_\Theta(s)\binom{x_1}{y_1}\,ds \ \text{ for all } t \in [0,\tau)$$

iff $C_K(t)x-\Theta(t)Cx=\int_0^t(t-s)C_K(s)y_1ds$ for all $t\in[0,\tau)$ and $y=x_1$. Namely, if $S_\Theta(t)\binom{x}{y}-\int_0^t\Theta(s)\binom{Cx}{Cy}\,ds=\int_0^t S_\Theta(s)\binom{x_1}{y_1}\,ds$ for all $t\in[0,\tau)$, then

$$\int_0^t C_K(s)x\,ds+\int_0^t(t-s)C_K(s)y\,ds-\int_0^t\Theta(s)Cx\,ds \\ =\int_0^t(t-s)C_K(s)x_1ds+\int_0^t\frac{(t-s)^2}{2}C_K(s)y_1ds, \tag{22}$$

$$C_K(t)x-\Theta(t)Cx+\int_0^t C_K(s)y\,ds-\int_0^t\Theta(s)Cy\,ds \\ =\int_0^t C_K(s)x_1ds-\int_0^t\Theta(s)Cx_1ds+\int_0^t(t-s)C_K(s)y_1ds,\ t\in[0,\tau). \tag{23}$$

Differentiating (22) with respect to t, one obtains

$$C_K(t)x+\int_0^t C_K(s)y\,ds-\Theta(t)Cx=\int_0^t C_K(s)x_1ds+\int_0^t(t-s)C_K(s)y_1ds.$$

The preceding equality and (23) together imply $\int_0^t\Theta(s)Cy\,ds=\int_0^t\Theta(s)Cx_1ds$, $t\in[0,\tau)$ and $y=x_1$. Thanks to (23), one yields $C_K(t)x-\Theta(t)Cx=\int_0^t(t-s)C_K(s)y_1ds$, $t\in[0,\tau)$ and $(x,y_1)\in B$. Suppose conversely $y=x_1$ and $(x,y_1)\in B$. Then $C_K(t)x-\Theta(t)Cx=\int_0^t(t-s)C_K(s)y_1ds$, $t\in[0,\tau)$ and (23) holds. Integrating (23) with respect to t one obtains (22) and $\left(\binom{x}{y},\binom{x_1}{y_1}\right)\in\mathcal{B}$. Therefore, $\left(\binom{x}{y},\binom{x_1}{y_1}\right)\in\mathcal{B}$ iff $y=x_1$ and $(x,\,y_1)\in B$. Further on, $CA\subseteq AC$ implies $\mathcal{CA}\subseteq\mathcal{AC}$ and one can employ Proposition 2.1.6(i) in order to see that the integral generator of $(S_\Theta(t))_{t\in[0,\tau)}$ is $\mathcal{C}^{-1}\mathcal{AC}$. Clearly, $\mathcal{C}^{-1}\mathcal{AC}=\begin{pmatrix}0 & I\\ C^{-1}AC & 0\end{pmatrix}$, which implies that the integral generator of $(C_K(t))_{t\in[0,\tau)}$ is $C^{-1}AC$. □

By Theorem 2.1.11 and Remark 2.1.7, it follows that $|\wp(C_K)|=1$ provided $C=I$. In order to prove the composition property of convoluted C-cosine functions, we will make use of the following auxiliary lemma whose proof is left to the reader as an easy exercise.

Lemma 2.1.12. *Let $0<\tau\leqslant\infty$ and $K\in C([0,\tau))$. Then:*

$$\left[\int_0^{t+s}-\int_0^t-\int_0^s\right]K(t+s-r)K(r)\,dr=0,\quad 0\leqslant t,\ s,\ t+s<\tau.$$

Theorem 2.1.13. *Let A be a subgenerator of a (local) K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, $x\in E$, $t,\ s\in[0,\tau)$ and $t+s<\tau$. Then the following* *formulae hold:*

$$
\begin{aligned}
&2C_K(t)C_K(s)x = \left(\int_t^{t+s} - \int_0^s\right) K(t+s-r)C_K(r)Cx\,dr \\
&\quad + \int_{t-s}^{t} K(r-t+s)C_K(r)Cx\,dr + \int_0^s K(r+t-s)C_K(r)Cx\,dr, \quad t \geqslant s; \\
&2C_K(t)C_K(s)x = \left(\int_s^{t+s} - \int_0^t\right) K(t+s-r)C_K(r)Cx\,dr \\
&\quad + \int_{s-t}^{s} K(r+t-s)C_K(r)Cx\,dr + \int_0^t K(r-t+s)C_K(r)Cx\,dr, \quad t < s.
\end{aligned} \tag{24}
$$

Proof. First of all, we will prove the composition property in case K is an absolutely continuous function in $[0, \tau)$. In order to do that, suppose $\tau_0 \in (0, \tau)$, $x \in E$ and put $D_{\tau_0} := \{(t, s) \in \mathbb{R}^2 : 0 \leqslant t,\ s,\ t + s \leqslant \tau_0,\ s \leqslant t\}$. Define

$$
u(t, s) := \int_0^t C_K(r)\big(C_K(s)x - \Theta(s)Cx\big)\,dr, \ (t, s) \in D_{\tau_0} \text{ and} \tag{25}
$$

$$
F(t, s) := \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)C_K(r)Cx\,dr - \Theta(s)C_K(t)Cx,
$$

for any $(t, s) \in D_{\tau_0}$. Designate by $C^1(D_{\tau_0} : E)$ the vector space which consists of those functions from D_{τ_0} into E that are continuously differentiable in $\mathrm{int}D_{\tau_0}$ and whose partial derivatives can be extended continuously throughout D_{τ_0}. Consider the problem

$$
(P) : \begin{cases} u \in C^1(D_{\tau_0} : E), \\ u_t(t, s) + u_s(t, s) = F(t, s), \ (t, s) \in D_{\tau_0}, \\ u(t, 0) = 0. \end{cases}
$$

The uniqueness of solutions of the problem (P) can be proved by means of the elementary theory of quasi-linear partial differential equations of first order. On the other hand, an application of Theorem 2.1.11 yields that $\mathcal{A}$ is a subgenerator of the Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in $E \times E$. Thanks to Proposition 2.1.5, one obtains:

$$
\begin{aligned}
&S_\Theta(t)S_\Theta(s)(0\ \ x)^T \\
&= \Bigg(\int_0^t C_K(v)\int_0^s (s-r)C_K(r)x\,dr\,dv + \int_0^t (t-v)C_K(v)\int_0^s C_K(r)x\,dr\,dv \\
&C_K(t)\int_0^s (s-r)C_K(r)x\,dr - \Theta(t)\int_0^s (s-r)C_K(r)Cx\,dr + \int_0^t\int_0^s C_K(v)C_K(r)x\,dr\,dv\Bigg)^T
\end{aligned}
$$

$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-r) \left(\int_0^r (r-v) C_K(v) Cx \, dv \ \int_0^r C_K(v) Cx dv \right)^T dr,$$

for any $(t,s) \in D_{\tau_0}$. Hence,

$$A \left[\int_0^t C_K(v) \int_0^s (s-r) C_K(r) x \, dr \, dv + \int_0^t (t-v) C_K(v) \int_0^s C_K(r) x \, dr \, dv \right]$$
$$= A \left\{ \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-r) \int_0^r (r-v) C_K(v) Cx \, dv \, dr \right\}, \quad (t,s) \in D_{\tau_0}.$$

The last equality and Lemma 2.1.12 imply:

$$\int_0^t C_K(v) \Big(C_K(s)x - \Theta(s)Cx \Big) \, dv + C_K(t) \int_0^s C_K(r) x \, dr - \Theta(t) \int_0^s C_K(r) Cx \, dr$$
$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-r) C_K(r) Cx \, dr, \quad (t,s) \in D_{\tau_0}. \tag{26}$$

Fix, for the time being, a number $t \in [0,\tau)$. The standard arguments gives:

$$\frac{d}{ds} \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-r) C_K(r) Cx \, dr$$
$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-r) C_K(r) Cx \, dr - \Theta(t) C_K(s) Cx, \quad s \in [0, \tau - t).$$

Differentiate (26) with respect to s in order to see that the function $u(t,s)$, given by (25), is a solution of (P). Further on, put

$$v_1(t,s) := \frac{1}{2} \int_0^s \left(\int_s^{s+v} - \int_0^v \right) K(v+s-r) C_K(r) Cx \, dr \, dv,$$

$$v_2(t,s) := \frac{1}{2} \int_0^s \int_{s-v}^s K(r-s+v) C_K(r) Cx \, dr \, dv,$$

$$v_3(t,s) := \frac{1}{2} \int_0^s \int_0^v K(r+s-v) C_K(r) Cx \, dr \, dv,$$

$$v_4(t,s) := \frac{1}{2} \int_s^t \left(\int_v^{s+v} - \int_0^s \right) K(v+s-r) C_K(r) Cx \, dr \, dv,$$

$$
\begin{aligned}
v_5(t,s) &:= \frac{1}{2}\int\limits_s^t \int\limits_{v-s}^{v} K(r-v+s)C_K(r)Cx\,dr\,dv,\\
v_6(t,s) &:= \frac{1}{2}\int\limits_s^t \int\limits_0^{s} K(r+v-s)C_K(r)Cx\,dr\,dv - \Theta(s)\int\limits_0^t C_K(r)Cx\,dr\\
v(t,s) &:= \sum_{i=1}^{6} v_i(t,s), \quad (t,s)\in D_{\tau_0}.
\end{aligned}
$$

To prove that $v(t,s)$ is also a solution of (P), notice that the usual limit procedure implies:

$$
\begin{aligned}
2\frac{\partial v_1}{\partial s}(t,s) &= \Bigg(\int\limits_s^{2s} - \int\limits_0^{s}\Bigg)K(2s-r)C_K(r)Cx\,dr - \int\limits_0^s\int\limits_0^v K'(v+s-r)C_K(r)Cx\,dr\,dv\\
&+ \int\limits_0^s\int\limits_s^{s+v} K'(v+s-r)C_K(r)Cx\,dr\,dv - \Theta(s)C_K(s)Cx + K(0)\int\limits_0^s C_K(s+v)Cx\,dv,\\
2\frac{\partial v_2}{\partial s}(t,s) &= \int\limits_0^s K(r)C_K(r)Cx\,dr - \int\limits_0^s\int\limits_{s-v}^{s} K'(r-s+v)C_K(r)Cx\,dr\,dv\\
&\quad - K(0)\int\limits_0^s C_K(s-v)Cx\,dv + \Theta(s)C_K(s)Cx,\\
2\frac{\partial v_3}{\partial s}(t,s) &= \int\limits_0^s K(r)C_K(r)Cx\,dr + \int\limits_0^s\int\limits_0^v K'(r+s-v)C_K(r)Cx\,dr\,dv,\\
2\frac{\partial v_4}{\partial s}(t,s) &= \int\limits_s^t\Bigg(\int\limits_v^{s+v} - \int\limits_0^{s}\Bigg)K'(v+s-r)C_K(r)Cx\,dr\,dv + K(0)\int\limits_s^t C_K(s+v)Cx\,dv\\
&\quad - \int\limits_s^t K(v)\,dv C_K(s)Cx - \Bigg(\int\limits_s^{2s} - \int\limits_0^{s}\Bigg)K(2s-r)C_K(r)Cx\,dr,\\
2\frac{\partial v_5}{\partial s}(t,s) &= \int\limits_s^t\int\limits_{v-s}^{v} K'(r-v+s)C_K(r)Cx\,dr + K(0)\int\limits_s^t C_K(v-s)Cx\,dv\\
&\quad - \int\limits_0^s K(r)C_K(r)Cx\,dr,
\end{aligned}
$$

$$2\frac{\partial v_6}{\partial s}(t,s) = -\int\limits_s^t \int\limits_0^s K'(r+v-s)C_K(r)Cx\,dr\,dv + \int\limits_s^t K(r)\,drC_K(s)Cx \\ - \int\limits_0^s K(r)C_K(r)Cx\,dr - 2K(s)\int\limits_0^t C_K(r)Cx\,dr,$$

for any $(t,s) \in D_{\tau_0}$. Adding these six summands, one gets:

$$2\frac{\partial v}{\partial s}(t,s) = K(0)\Biggl(\int\limits_0^{t-s} + \int\limits_s^{2s} - \int\limits_0^{s} + \int\limits_{2s}^{t+s}\Biggr)C_K(r)Cx\,dr \\ - 2K(s)\int\limits_0^t C_K(r)Cx\,dr + I_1 + I_2 + I_3, \quad (t,s) \in D_{\tau_0}, \tag{27}$$

where

$$I_1 := \Biggl(\int\limits_0^s \int\limits_s^{s+v} - \int\limits_0^s \int\limits_0^v + \int\limits_s^t \int\limits_v^{s+v} - \int\limits_s^t \int\limits_0^s\Biggr)K'(v+s-r)C_K(r)Cx\,dr\,dv, \tag{28}$$

$$I_2 := \Biggl(\int\limits_0^s \int\limits_0^v + \int\limits_s^t \int\limits_{v-s}^v\Biggr)K'(r+s-v)C_K(r)Cx\,dr\,dv, \tag{29}$$

$$I_3 := \Biggl(-\int\limits_0^s \int\limits_{s-v}^s - \int\limits_s^t \int\limits_0^s\Biggr)K'(r-s+v)C_K(r)Cx\,dr\,dv, \quad (t,s) \in D_{\tau_0}. \tag{30}$$

An elementary calculus shows that:

$$I_3 = -\int\limits_0^s \int\limits_{s-r}^s K'(r-s+v)C_K(r)Cx\,dv\,dr - \int\limits_0^s \int\limits_s^t K'(r-s+v)C_K(r)Cx\,dv\,dr$$

$$= -\int\limits_0^s \bigl(K(r) - K(0)\bigr)C_K(r)Cx\,dr - \int\limits_0^s \bigl(K(t-s+r) - K(r)\bigr)C_K(r)Cx\,dr. \tag{31}$$

Applying the same arguments, one yields:

$$I_1 = K(s)\int\limits_0^t C_K(r)Cx\,dr - K(0)\int\limits_s^{t+s} C_K(r)Cx\,dr \\ + \Biggl(\int\limits_t^{t+s} - \int\limits_0^s\Biggr)K(t+s-r)C_K(r)Cx\,dr \tag{32}$$

$$I_2 = -\int_{t-s}^{t} K(r+s-t)C_K(r)Cx\,dr + K(s)\int_0^t C_K(r)Cx\,dr - K(0)\int_0^{t-s} C_K(r)Cx\,dr,\ (t,s)\in D_{\tau_0}. \tag{33}$$

Furthermore, $v(t,0)=0$, $t\in[0,\tau_0]$,

$$2\frac{\partial v}{\partial t}(t,s) = \left(\int_t^{t+s} - \int_0^s\right) K(t+s-r)C_K(r)Cx\,dr + \int_{t-s}^{t} K(r-t+s)C_K(r)Cx\,dr + \int_0^s K(r+t-s)C_K(r)Cx\,dr - 2\Theta(s)C_K(t)Cx, \tag{34}$$

$v\in C^1(D_{\tau_0}:E)$ and a simple computation involving (27)–(34) implies that the function v solves (P). By the uniqueness of solutions of (P), we obtain:

$$C_K(t)C_K(s)x = v_t(t,s) + \Theta(s)C_K(t)Cx,\ (t,s)\in D_{\tau_0}. \tag{35}$$

By (35) and arbitrariness of τ_0, one yields that the composition property holds whenever K is an absolutely continuous function in $[0,\tau)$, $x\in E$, $0\leqslant t,\ s,\ t+s<\tau$ and $s\leqslant t$. Put $C_\Theta(t)x := \int_0^t C_K(r)x\,dr$, $t\in[0,\tau)$, $x\in E$; then $(C_\Theta(t))_{t\in[0,\tau)}$ is a Θ-convoluted C-cosine function with a subgenerator A and the first part of the proof implies that, for every $x\in E$ and $(t,s)\in[0,\tau)\times[0,\tau)$ with $t+s<\tau$ and $s\leqslant t$:

$$2C_\Theta(t)C_\Theta(s)x = \left(\int_t^{t+s} - \int_0^s\right)\Theta(t+s-r)C_\Theta(r)Cx\,dr + \int_{t-s}^{t}\Theta(r-t+s)C_\Theta(r)Cx\,dr + \int_0^s \Theta(r+t-s)C_\Theta(r)Cx\,dr. \tag{36}$$

Notice also that the partial integration implies that, for every $x\in E$ and $(t,s)\in[0,\tau)\times[0,\tau)$ with $t+s<\tau$ and $s\leqslant t$:

$$\left(\int_t^{t+s} - \int_0^s\right)\Theta(t+s-r)C_\Theta(r)Cx\,dr = \Theta^{-1}(s)C_\Theta(t)Cx + \Theta^{-1}(t)C_\Theta(s)Cx + \int_t^{t+s}\Theta^{-1}(t+s-r)C_K(r)Cx\,dr, \tag{37}$$

$$\int_{t-s}^{t}\Theta(r-t+s)C_\Theta(r)Cx\,dr \tag{38}$$

$$= \Theta^{-1}(s)C_\Theta(t)Cx - \int_{t-s}^{t} \Theta^{-1}(r-t+s)C_K(r)Cx\,dr,$$

$$(39)\quad \int_0^s \Theta(r+t-s)C_\Theta(r)Cx\,dr = \Theta^{-1}(t)C_\Theta(s)Cx - \int_0^s \Theta^{-1}(r+t-s)C_K(r)Cx\,dr.$$

Now one can rewrite (36) by means of (37)–(39):

$$(40)\quad 2C_\Theta(t)C_\Theta(s)x = 2\Theta^{-1}(s)C_\Theta(t)Cx + 2\Theta^{-1}(t)C_\Theta(s)Cx + \left(\int_t^{t+s} - \int_0^s\right)\Theta^{-1}(t+s-r)C_K(r)Cx\,dr - \int_{t-s}^{t} \Theta^{-1}(r-t+s)C_K(r)Cx\,dr - \int_0^s \Theta^{-1}(r+t-s)C_K(r)Cx\,dr.$$

Taking into account (40), it can be straightforwardly proved that, for every $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s < \tau$ and $s \leqslant t$:

$$(41)\quad 2C_K(t)C_\Theta(s)x = 2\frac{d}{dt}C_\Theta(t)C_\Theta(s)x = 2\Theta(t)C_\Theta(s)Cx + \left(\int_t^{t+s} - \int_0^s\right)\Theta(t+s-r)C_K(r)Cx\,dr + \int_{t-s}^{t} \Theta(r-t+s)C_K(r)Cx\,dr - \int_0^s \Theta(r+t-s)C_K(r)Cx\,dr.$$

Differentiation of (41) with respect to s immediately implies the validity of composition property for all $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s < \tau$ and $s \leqslant t$. The proof of composition property in the case $s > t$ can be obtained along the same lines. $\square$

Now we are in a position to prove the following extension type theorem for local convoluted C-cosine functions.

THEOREM 2.1.14. *Let A be a subgenerator of a local K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ and let $\tau_0 \in (\frac{\tau}{2},\tau)$. Assume that there exists a complex valued function $K_1 \in L^1_{\mathrm{loc}}([0,2\tau))$ such that $K = K_{1_{|[0,\tau)}}$* (cf. also the formulation of Theorem 2.1.9). *Then A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-cosine function $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$, which is given by:*

$$C_{K*_0K}(t)x = \begin{cases} \int_0^t K(t-s)C_K(s)Cxds, \ t \in [0,\tau_0], \\ 2C_K(\tau_0)C_K(t-\tau_0)x + \bigl(\int_0^{t-\tau_0} + \int_0^{\tau_0}\bigr)K(t-r)C_K(r)Cx\,dr \\ \quad - \int_{2\tau_0-t}^{\tau_0} K(r+t-2\tau_0)C_K(r)Cx\,dr \\ \quad - \int_0^{t-\tau_0} K(r+2\tau_0-t)C_K(r)Cx\,dr, \ t \in (\tau_0, 2\tau_0), \ x \in E. \end{cases}$$

Furthermore, the condition $0 \in \operatorname{supp} K$ implies that A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-cosine function on $[0, 2\tau)$.

Proof. Notice that $K *_0 K \in L^1_{\mathrm{loc}}([0,2\tau))$ and that $K *_0 K$ is not identical to zero. Clearly, $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$ is a strongly continuous operator family which commutes with A and C. By Proposition 2.1.3, one gets that $((K*_0C_KC)(t))_{t\in[0,\tau)}$ is a local $(K *_0 K)$-convoluted C^2-cosine function having A as a subgenerator, and consequently, the condition (iii) quoted in the formulation of Definition 2.1.2 holds for every $t \in [0,\tau_0]$ and $x \in E$. It remains to be shown that this condition holds for every $t \in (\tau_0, 2\tau_0)$ and $x \in E$; to this end, denote $\Phi = \int_0^t (t-s)C_{K*_0K}(s)x\,ds$ and notice that:

$$\Phi = \int_0^{\tau_0}(\tau_0 - s)\int_0^s K(s-r)C_K(r)Cx\,dr\,ds + \int_0^{\tau_0}(t-\tau_0)\int_0^s K(s-r)C_K(r)Cx\,dr\,ds \\ + 2C_K(\tau_0)\int_0^{t-\tau_0}(t-\tau_0-s)C_K(s)x\,ds + I_1 + I_2 - I_3 - I_4, \tag{42}$$

where:

$$I_1 := \int_{\tau_0}^t (t-s)\int_0^{s-\tau_0} K(s-r)C_K(r)Cx\,dr\,ds, \tag{43}$$

$$I_2 := \int_{\tau_0}^t (t-s)\int_0^{\tau_0} K(s-r)C_K(r)Cx\,dr\,ds, \tag{44}$$

$$I_3 := \int_{\tau_0}^t (t-s)\int_{2\tau_0-s}^{\tau_0} K(r+s-2\tau_0)C_K(r)Cx\,dr\,ds \tag{45}$$

$$I_4 := \int_{\tau_0}^t (t-s)\int_0^{s-\tau_0} K(r+2\tau_0-s)C_K(r)Cx\,dr\,ds. \tag{46}$$

We compute I_1 as follows:

$$I_1=\int\limits_{\tau_0}^{t}(t-s)\int\limits_0^{s-\tau_0}K(s-r)C_K(r)Cx\,dr\,ds=\int\limits_0^{t-\tau_0}\int\limits_{r+\tau_0}^{t}(t-s)K(s-r)C_K(r)Cx\,ds\,dr$$

$$=\int\limits_0^{t-\tau_0}[-\Theta(\tau_0)(t-\tau_0-r)+\int\limits_{r+\tau_0}^{t}\Theta(s-r)ds]C_K(r)Cx\,dr$$

$$=-\Theta(\tau_0)\int\limits_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cx\,dr+\int\limits_0^{t-\tau_0}[\Theta^{-1}(t-r)-\Theta^{-1}(\tau_0)]C_K(r)Cx\,dr$$

$$=-\Theta(\tau_0)\int\limits_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cx\,dr+\int\limits_0^{t-\tau_0}\Theta(t-r)\int\limits_0^{r}C_K(v)Cx\,dv\,dr$$

$$=-\Theta(\tau_0)\int\limits_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cx\,dr+\Theta(\tau_0)\int\limits_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cx\,dr$$

$$+\int\limits_0^{t-\tau_0}K(t-r)\int\limits_0^{r}(r-v)C_K(v)Cx\,dv\,dr=\int\limits_0^{t-\tau_0}K(t-r)\int\limits_0^{r}(r-v)C_K(v)Cx\,dv\,dr. \tag{47}$$

Applying the same argumentation, we easily infer that:

$$I_2=-\int\limits_0^{\tau_0}(t-\tau_0)\Theta(\tau_0-r)C_K(r)Cx\,dr+\Theta(t-\tau_0)\int\limits_0^{\tau_0}(\tau_0-r)C_K(r)Cx\,dr \tag{48}$$

$$+\Theta^{-1}(t-\tau_0)\int\limits_0^{\tau_0}C_K(r)Cx\,dr+\int\limits_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\int\limits_0^{r}(r-v)C_K(v)Cx\,dv\,dr,$$

$$I_3=-\Theta(t-\tau_0)\int\limits_0^{\tau_0}(\tau_0-r)C_K(r)Cx\,dr+\Theta^{-1}(t-\tau_0)\int\limits_0^{\tau_0}C_K(r)Cx\,dr \tag{49}$$

$$+\int\limits_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\int\limits_0^{r}(r-v)C_K(v)Cx\,dv\,dr$$

$$I_4=\int\limits_0^{t-\tau_0}K(r+2\tau_0-t)\int\limits_0^{r}(r-v)C_K(v)Cx\,dv\,dr. \tag{50}$$

Exploiting (42)–(50) and the following simple equality:

$$\int_0^{\tau_0}(t-\tau_0)\int_0^s K(s-r)C_K(r)Cx\,dr\,ds=\int_0^{\tau_0}(t-\tau_0)\Theta(\tau_0-r)C_K(r)Cx\,dr,$$

one obtains:

(51)

$$\begin{aligned}\Phi=\int_0^{\tau_0}(\tau_0-s)\int_0^s K(s-r)C_K(r)Cx\,dr\,ds+2C_K(\tau_0)\int_0^{t-\tau_0}(t-\tau_0-s)C_K(s)x\,ds\\
+\int_0^{t-\tau_0}K(t-r)\int_0^r(r-v)C_K(v)Cx\,dv\,dr\\
+\int_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\int_0^r(r-v)C_K(v)Cx\,dv\,dr\\
-\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\int_0^r(r-v)C_K(v)Cx\,dv\,dr\\
-\int_0^{t-\tau_0}K(r+2\tau_0-t)\int_0^r(r-v)C_K(v)Cx\,dv\,dr\\
+2\Theta(t-\tau_0)\int_0^{\tau_0}(\tau_0-r)C_K(r)Cx\,dr.\end{aligned}$$

The last equality implies $\Phi\in D(A)$ and

$$A(\Phi)=C_{K*_0K}(t)-f(t)C^2x, \tag{52}$$

where

$$\begin{aligned}f(t)=\int_0^{\tau_0}K(\tau_0-r)\Theta(r)\,dr+\int_0^{t-\tau_0}K(t-r)\Theta(r)\,dr\\
+\int_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\Theta(r)\,dr-\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\Theta(r)\,dr\\
-\int_0^{t-\tau_0}K(r+2\tau_0-t)\Theta(r)\,dr+2\Theta(\tau_0)\Theta(t-\tau_0).\end{aligned}$$

Notice also that $(\Theta(t)I)_{t\in[0,2\tau)}$ is a local K-convoluted cosine function generated by $\mathbf{0}$ and that the following identity follows immediately from an application of Theorem 2.1.13:

$$(53)\quad 2\Theta(\tau_0)\Theta(t-\tau_0)=\Biggl(\int\limits_{\tau_0}^{t}-\int\limits_{0}^{t-\tau_0}\Biggr)K(t-r)\Theta(r)\,dr$$
$$+\int\limits_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\Theta(r)\,dr+\int\limits_{0}^{t-\tau_0}K(r+2\tau_0-t)\Theta(r)\,dr.$$

In view of (52)–(53), it follows that $f(t)=(K*_0\Theta)(t)$ and that A is a subgenerator of a local $(K*_0K)$-convoluted C^2-cosine function $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$. The preassumption $0\in \operatorname{supp}K$ implies that the function $(K*_0K)_{|[0,\tau')}$ is a kernel for all $\tau'\in(0,2\tau]$; in this case, $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$ is a unique local $(K*_0K)$-convoluted C^2-cosine function with a subgenerator A [**228**] and the proof of Theorem 2.1.14 ends a routine argument. □

Corollary 2.1.15. *Suppose $\alpha>0$ and A is a subgenerator of a local α-times integrated C-cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. Then A is a subgenerator of a local (2α)-times integrated C^2-cosine function $(C_{2\alpha}(t))_{t\in[0,2\tau)}$.*

Keeping in mind the proofs of Proposition 2.1.6, [**218**, Corollary 3.3] and Theorem 2.1.9, one immediately gets the following assertion.

Proposition 2.1.16. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ and $\{A,B\}\subseteq\wp(C_K)$. Then the assertions* (ii), (iii), (v) *and* (vi)*, given in the formulation of Proposition* 2.1.6 *still hold while the assertions* (i), (iv) *and* (vii) *hold with S_K replaced by C_K. Furthermore, if $0\in\operatorname{supp}K$, then $C_K(t)C_K(s)=C_K(s)C_K(t)$, $0\leqslant t,\ s<\tau$.*

Question. Suppose K is not a kernel and A is a subgenerator of a local K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, resp. K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$. Does the equality $S_K(t)S_K(s)=S_K(s)S_K(t)$, $0\leqslant t,\ s<\tau$, resp. $C_K(t)C_K(s)=C_K(s)C_K(t)$, $0\leqslant t,\ s<\tau$, hold?

Proposition 2.1.17. *Suppose $\pm A$ are subgenerators of (local, global exponentially bounded) K-convoluted C-semigroups $(S_K^{\pm}(t))_{t\in[0,\tau)}$ and A^2 is closed. Then A^2 is a subgenerator of a (local, global exponentially bounded) K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, which is given by $C_K(t)x:=\frac{1}{2}\bigl(S_K^+(t)x+S_K^-(t)x\bigr)$, $x\in E$, $t\in[0,\tau)$.*

Proof. It is straightforward to verify that $(C_K(t))_{t\in[0,\tau)}$ is a strongly continuous operator family which commutes with A^2 and C as well as that

$$A^2\int\limits_0^t(t-s)C_K(s)x\,ds=A^2\int\limits_0^t\int\limits_0^s C_K(r)x\,dr\,ds=A\int\limits_0^t A\int\limits_0^s C_K(r)x\,dr\,ds$$
$$=\frac{1}{2}A\int\limits_0^t\bigl[S_K^+(s)x-\Theta(s)Cx-S_K^-(s)x+\Theta(s)Cx\bigr]\,ds$$

$$= \frac{1}{2}\big[S_K^+(t)x - \Theta(t)Cx + S_K^-(t)x - \Theta(t)Cx\big]$$
$$= C_K(t)x - \Theta(t)Cx, \ \ x \in E, \ t \in [0,\tau).$$

This completes the proof. □

Notice only that, under assumptions of Proposition 2.1.17, one can simply prove that the operator A^2 is closed when $C = I$; it is not clear whether the operator A^2 is closed in the case $C \neq I$ and $\rho(A) = \emptyset$. Next, we point out that there exists a somewhat different definition of a (local) K-convoluted C-semigroup, resp. K-convoluted C-cosine function. For the sake of consistency, we will give these definitions only in global case.

Definition 2.1.18. Let $0 \neq K \in L^1_{\mathrm{loc}}([0,\infty))$. A strongly continuous operator family $(S_K(t))_{t\geqslant 0}$ is called a (global) K-convoluted C-semigroup iff the following holds:

(i) $S_K(0) = 0$,
(ii) $S_K(t)C = CS_K(t)$, $t \geqslant 0$ and
(iii) $S_K(t)S_K(s)x = \big[\int_0^{t+s} - \int_0^t - \int_0^s\big] K(t+s-r)S_K(r)Cx\,dr$, $x \in E$, t, $s \geqslant 0$;

$(S_K(t))_{t\geqslant 0}$ is said to be *non-degenerate* if the assumption $S_K(t)x = 0$ for all $t \geqslant 0$ implies $x = 0$. For a non-degenerate K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ we define its (integral) generator A by

$$A := \left\{(x,y) \in E \times E : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds, \ t \geqslant 0\right\}.$$

A closed linear operator A is said to be a subgenerator of $(S_K(t))_{t\geqslant 0}$ if the conditions (i) and (iii) of Definition 2.1.1 hold.

Definition 2.1.19. Let $0 \neq K \in L^1_{\mathrm{loc}}([0,\infty))$. A strongly continuous operator family $(C_K(t))_{t\geqslant 0}$ is called a (global) K-convoluted C-cosine function iff the following holds:

(i) $C_K(0) = 0$,
(ii) $C_K(t)C = CC_K(t)$, $t \geqslant 0$ and
(iii) (24) holds for $x \in E$, $t \geqslant 0$ and $s \geqslant 0$;

$(C_K(t))_{t\geqslant 0}$ is said to be non-degenerate if the assumption $C_K(t)x = 0$ for all $t \geqslant 0$ implies $x = 0$. For a non-degenerate K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$ we define its (integral) generator A by

$$A := \left\{(x,y) \in E \times E : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y\,ds, \ t \geqslant 0\right\}.$$

A closed linear operator A is said to be a subgenerator of $(C_K(t))_{t\geqslant 0}$ if the conditions (i) and (iii) of Definition 2.1.2 hold.

Remark 2.1.20. Let $0 \neq K \in L^1_{\mathrm{loc}}([0,\infty))$ and let $(S_K(t))_{t\geqslant 0}$, resp. $(C_K(t))_{t\geqslant 0}$, be a non-degenerate strongly continuous operator family. By the argumentation given in the proofs of [**227**, Proposition 2.2], [**258**, Proposition 1.5], Proposition 2.1.5 and Theorem 2.1.13, we have that $(S_K(t))_{t\geqslant 0}$, resp. $(C_K(t))_{t\geqslant 0}$, is a global K-convoluted C-semigroup, resp. K-convoluted C-cosine function, having A as a subgenerator (the integral generator) in the sense of Definition 2.1.1, resp. Definition 2.1.2 iff $(S_K(t))_{t\geqslant 0}$, resp. $(C_K(t))_{t\geqslant 0}$, is a global K-convoluted C-semigroup, resp. K-convoluted C-cosine function having A as a subgenerator (the integral generator) in the sense of Definition 2.1.18, resp. Definition 2.1.19.

In Proposition 2.1.21 and Proposition 2.1.23, we give upper bounds for the stationarity of generators of fractionally integrated semigroups and cosine functions.

Proposition 2.1.21. [**249**] *Suppose $\alpha > 0$ and A generates a (local) α-times integrated semigroup. Then A is stationary dense and $n(A) \leqslant \lceil \alpha \rceil$.*

Lemma 2.1.22. *Let A be a closed operator. Then A is stationary dense iff $\mathcal{A}$ is stationary dense. In this case, $n(\mathcal{A}) = 2n(A)$.*

Proof. Assume that A is stationary dense and $n(A) = n \in \mathbb{N}_0$. Let us prove that $D(\mathcal{A}^m) \subseteq \overline{D(\mathcal{A}^{m+1})}$ for all $m \in \mathbb{N}_0$ with $m \geqslant 2n$. Let $m = 2i$ for some $i \geqslant n$. We have to prove that $D(A^i) \times D(A^i) \subseteq \overline{D(A^{i+1}) \times D(A^i)}$. This is a consequence of $D(A^i) \subseteq \overline{D(A^{i+1})}$. If $m = 2i + 1$ for some $i \geqslant n$, then $D(\mathcal{A}^m) \subseteq \overline{D(\mathcal{A}^{m+1})}$ is equivalent with $D(A^{i+1}) \times D(A^i) \subseteq \overline{D(A^{i+1}) \times D(A^{i+1})}$, which holds since $i \geqslant n$. Thus, $\mathcal{A}$ is stationary dense and $n(\mathcal{A}) \leqslant 2n(A)$. Furthermore, $n(\mathcal{A}) = 0$ if $n(A) = 0$. Suppose $n(\mathcal{A}) < 2n(A)$. If $n(\mathcal{A}) = 2i$ for some $i \in \{0, 1, \ldots, n-1\}$, then $D(A^i) \times D(A^i) \subseteq \overline{D(A^{i+1}) \times D(A^i)}$. Hence, $D(A^i) \subseteq \overline{D(A^{i+1})}$ and the contradiction is obvious. Similarly, if $n(\mathcal{A}) = 2i + 1$ for some $i \in \{0, 1, \ldots, n-1\}$, then $D(A^{i+1}) \times D(A^i) \subseteq \overline{D(A^{i+1}) \times D(A^{i+1})}$. Again, $D(A^i) \subseteq \overline{D(A^{i+1})}$ and this is in contradiction with $n(A) = n$. Hence, we have proved that $\mathcal{A}$ is stationary dense and that $n(\mathcal{A}) = 2n(A)$. Assume now that $\mathcal{A}$ is stationary dense. Similarly as in the first part of the proof, one obtains that A is stationary dense and that $n(\mathcal{A}) = 2n(A)$. $\square$

Proposition 2.1.23. *Let A be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$ for some $0 < \tau \leqslant \infty$ and $\alpha > 0$. Then $n(A) \leqslant \left\lfloor \frac{\lceil \alpha \rceil + 1}{2} \right\rfloor$.*

Proof. By Theorem 2.1.11, the operator $\mathcal{A}$ is the generator of an $(\lceil \alpha \rceil + 1)$-times integrated semigroup $(S_{\lceil \alpha \rceil + 1}(t))_{t\in[0,\tau)}$ in $E \times E$. Now one can apply Proposition 2.1.21 to see that $n(\mathcal{A}) \leqslant \lceil \alpha \rceil + 1$. The claimed assertion follows by Lemma 2.1.22. $\square$

Comment and Problem. By [**14**, Example 3.15.5, p. 224], the generator B of the standard translation group on $L^1(\mathbb{R})$ satisfies the following: The operator $A := (B^*)^2$ is the non-densely defined generator of a sine function in $L^\infty(\mathbb{R})$. Then Proposition 2.1.21 implies $n(A) = 1$, and, in particular, we have that, in the general situation of previous proposition, the estimate $n(A) \leqslant \left\lfloor \frac{\lceil \alpha \rceil + \beta}{2} \right\rfloor$, where $[0, 1) \ni \beta$ is an arbitrary number, cannot be proved since here $n(A) = 1$ and $\alpha = 1$. Finally, the following problem can be proposed: Given $\alpha > 0$, is it possible to construct a Banach space E_α, a closed linear operator A_α on E_α which generates a (local) α-times integrated cosine function and satisfies $n(A_\alpha) = \lfloor \frac{\lceil \alpha \rceil + 1}{2} \rfloor$?

We use later on the following generalization of [**222**, Lemma 1.10].

PROPOSITION 2.1.24. *Suppose A is a closed linear operator, $CA \subseteq AC$ and $\lambda \in \mathbb{C}$. Then $\lambda \in \rho_{\mathcal{C}}(\mathcal{A}) \Leftrightarrow \lambda^2 \in \rho_C(A)$. If this is the case, then:*

$$(\lambda - \mathcal{A})^{-1}\mathcal{C} = \begin{pmatrix} \lambda(\lambda^2 - A)^{-1}C & (\lambda^2 - A)^{-1}C \\ A(\lambda^2 - A)^{-1}C & \lambda(\lambda^2 - A)^{-1}C \end{pmatrix},$$

$$\|(\lambda - \mathcal{A})^{-1}\mathcal{C}\| \leqslant (1 + |\lambda|)\sqrt{1 + |\lambda|^2}\,\|(\lambda^2 - A)^{-1}C\|$$

$$\|(\lambda^2 - A)^{-1}C\| \leqslant \|(\lambda - \mathcal{A})^{-1}\mathcal{C}\|.$$

REMARK 2.1.25. Let $k \in C([0, \infty))$ be a scalar kernel and let a satisfy (P1). Assume that A is a closed linear operator. Following Lizama [**286**] and Kim [**197**]-[**198**], a strongly continuous operator family $(R(t))_{t\geqslant 0}$ is called an (a, k)-*regularized resolvent* iff the following holds:

(i) $R(t)A \subseteq AR(t)$, $t \geqslant 0$, $R(0) = k(0)I$ and

(ii) $R(t)x = k(t)x + \int_0^t a(t - s)AR(s)x\, ds$, $t \geqslant 0$, $x \in D(A)$.

By [**286**, Lemma 2.2], $\rho(A) \neq \emptyset$ implies that, for every $x \in E$ and $t \geqslant 0$, $\int_0^t a(t - s)R(s)x\, ds \in D(A)$ and $A\int_0^t a(t - s)R(s)x\, ds = R(t)x - k(t)x$, $t \geqslant 0$, $x \in E$; in this case, the notion of (a, k)-regularized resolvents unify the notions of global convoluted semigroups ($a(t) \equiv 1$) and global convoluted cosine functions ($a(t) \equiv t$). It is also worth noticing that the condition $k(0) = 0$ is not necessary in the analysis given in [**286**] (cf. also [**330**]), and that global convoluted C-semigroups and cosine functions cannot be linked to (a, k)-regularized resolvents in the case $C \neq I$. In order to overcome the above described difficulties, the class of (a, k)-*regularized C-resolvent families* extending the classes of (a, k)-regularized resolvent families [**286**], regularized resolvent families [**276**] as well as (local) convoluted C-semigroups and cosine functions has been recently introduced in [**235**]:

DEFINITION 2.1.26. Let $0 < \tau \leqslant \infty$, $k \in C([0, \tau))$, $k \neq 0$ and let $a \in L^1_{\mathrm{loc}}([0, \tau))$, $a \neq 0$. Assume that A is a closed linear operator and that $L(E) \ni C$ is an injective operator. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a, k)-*regularized C-resolvent family* having A as a subgenerator iff the following holds:

(i) $R(t)A \subseteq AR(t)$, $t \in [0, \tau)$, $CA \subseteq AC$ and $R(0) = k(0)C$,

(ii) $R(t)C = CR(t)$, $t \in [0, \tau)$ and

(iii) $R(t)x = k(t)Cx + \int_0^t a(t - s)AR(s)x\, ds$, $t \in [0, \tau)$, $x \in D(A)$.

In the case $\tau = \infty$, $(R(t))_{t\geqslant 0}$ is said to be exponentially bounded (a, k)-regularized C-resolvent family with a subgenerator A if, additionally, there exist $M > 0$ and $\omega \geqslant 0$ such that $\|R(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

In the case $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, $\alpha > 0$, it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-*times integrated C-resolvent family*; in such a way, we unify the notion of (local) α-times integrated C-semigroups ($a(t) \equiv 1$) and cosine functions ($a(t) \equiv t$); see [**286**].

Furthermore, in case $k(t) = \int_0^t K(s)\,ds$, $t \in [0,\tau)$, where $K \in L^1_{\mathrm{loc}}([0,\tau))$ and $K \neq 0$, we obtain the unification concept for (local) K-convoluted C-semigroups and cosine functions. In the case $k(t) \equiv 1$, $(R(t))_{t\in[0,\tau)}$ is said to be a *(local) (a,C)-regularized resolvent family* with a subgenerator A. Designate by $\wp(R)$ the set which consists of all subgenerators of $(R(t))_{t\in[0,\tau)}$. Then the following holds:

(i) $A \in \wp(R)$ implies $C^{-1}AC \in \wp(R)$.

(ii) If $A \in \wp(R)$ and $\lambda \in \rho_C(A)$, then
$$R(t)(\lambda - A)^{-1}C = (\lambda - A)^{-1}CR(t),\ t \in [0,\tau).$$

(iii) Assume, additionally, $a(t)$ is a kernel. Then one can define the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting
$$\hat{A} := \left\{(x,y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds,\ t \in [0,\tau)\right\}.$$
The integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ is a closed linear operator satisfying $C^{-1}\hat{A}C = \hat{A}$. Furthermore, $\hat{A}$ extends an arbitrary subgenerator of $(R(t))_{t\in[0,\tau)}$, and $\hat{A}$ itself is a subgenerator if $R(t)R(s) = R(s)R(t)$, $0 \leqslant t,\ s < \tau$.

In what follows, we employ the following conditions:

(H1): A is densely defined.

(H2): $\rho(A) \neq \emptyset$.

(H3): $\rho_C(A) \neq \emptyset$ and $\overline{\mathrm{R}(C)} = E$.

(H4): A is densely defined or $\rho_C(A) \neq \emptyset$.

(H5): (H1) $\vee$ (H2) $\vee$ (H3).

Assume temporarily $\lambda \in \rho_C(A)$, $x \in \mathrm{R}(C)$, $t \in [0,\tau)$ and put$z = (a * R)(t)x$, where $*$ denotes the finite convolution product.Following the proof of [**286**, Lemma 2.2], we have
$$\begin{aligned} z &= \lambda(a * R)(t)(\lambda - A)^{-1}x - (a * R)(t)A(\lambda - A)^{-1}x \\ &= \lambda(a * R)(t)(\lambda - A)^{-1}x - \big(R(t)(\lambda - A)^{-1}x - k(t)C(\lambda - A)^{-1}x\big) \\ &= \lambda(\lambda - A)^{-1}C(a * R)(t)C^{-1}x - \big((\lambda - A)^{-1}R(t)x - k(t)(\lambda - A)^{-1}Cx\big), \end{aligned}$$
where the last two equalities follow on account of $CA \subseteq AC$, $R(s)A \subseteq AR(s)$ and $R(s)(\lambda - A)^{-1}C = (\lambda - A)^{-1}CR(s)$, $s \in [0,\tau)$. Hence, $(\lambda - A)z = \lambda z - (R(t)x - Cx)$,
$$\int_0^t a(t-s)R(s)x\,ds \in D(A) \text{ and } A\int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx. \tag{54}$$
The closedness of A implies that (54) holds for every $t \in [0,\tau)$ and $x \in \overline{\mathrm{R}(C)}$.

Let $\alpha > 0$, $\beta > 0$ and $\gamma \in (0,1)$. Denote by $\mathbf{D}_t^{\alpha}$ the *Caputo fractional derivative of order* α and by $E_\beta(z)$ the *Mittag–Leffler function* $E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+1)}$, $z \in \mathbb{C}$; then the *Wright function* $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geqslant 0$ (for further information, see e.g. [**36**], [**141**], [**393**] and references therein). In the subsequent theorem, we assume that the scalar-valued kernels $k,\ k_1,\ k_2,...$ are continuous on $[0,\tau)$ and that $a \neq 0$ in $L^1_{\mathrm{loc}}([0,\tau))$; we use the notion and notation given in [**286**].

THEOREM 2.1.27. [**235**]–[**236**], [**286**] (i) *Let A be a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ and let* (H5) *hold. Then* (54) *holds for every $t \in [0,\tau)$ and $x \in E$; if $\rho_C(A) \neq \emptyset$, then* (54) *holds for every $t \in [0,\tau)$ and $x \in \overline{\mathrm{R}(C)}$.*

(ii) *Suppose A is a subgenerator of an (a,k_i)-regularized C-resolvent family $(R_i(t))_{t\in[0,\tau)}$, $i = 1,2$. Then $(k_2 * R_1)(t) = (k_1 * R_2)(t)$, $t \in [0,\tau)$, whenever* (H4) *holds.*

(iii) *Let $(R_1(t))_{t\in[0,\tau)}$ and $(R_2(t))_{t\in[0,\tau)}$ be two (a,k)-regularized C-resolvent families having A as a subgenerator. Then $R_1(t)x = R_2(t)x$, $t \in [0,\tau)$, $x \in \overline{D(A)}$, and $R_1(t) = R_2(t)$, $t \in [0,\tau)$, provided that* (H4) *holds.*

(iv) *Suppose A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. If $k(t)$ is absolutely continuous and $k(0) \neq 0$, then A is a subgenerator of a* (*local*) *(a,C)-regularized resolvent family.*

(v) *Let $(R(t))_{t\in[0,\tau)}$ be an (a,k)-regularized C-resolvent family with a subgenerator A and let $b \in L^1_{\mathrm{loc}}([0,\tau))$ satisfy that $k*b$ is a kernel. Then A is a subgenerator of an $(a,k*b)$-regularized C-resolvent family $((b*R)(t))_{t\in[0,\tau)}$.*

(vi) *Suppose $(R(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family with a subgenerator A,* (H1) *or* (H3) *holds, and $a(t)$ is a kernel. Then the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ satisfies $\hat{A} = C^{-1}AC$. If* (H2) *holds, then $\hat{A} = C^{-1}AC = A$.*

(vii) *Let $B \in \wp(R)$ and let* (H5) *hold for B and C. Then we have the following:*

(vii.1) *$C^{-1}AC = C^{-1}BC$ and $C(D(A)) \subseteq D(B)$.*
(vii.2) *A and B have the same eigenvalues.*
(vii.3) *The assumption $A \subseteq B$ implies $\rho_C(A) \subseteq \rho_C(B)$.*
(vii.4) *The set $\wp(R)$ is monomial if $C(D(\hat{A}))$ is a core for $\hat{A}$.*
(vii.5) *$A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$ and $Ax = Bx$, $x \in D(A) \cap D(B)$; furthermore, the property* (vii.5) *holds whenever $\{A,B\} \subseteq \wp(R)$ and $a(t)$ is a kernel.*

(viii) *Define the mapping $K_C : C([0,\tau) : E) \to C([0,\tau) : E)$ by $K_C u := k * Cu$, $u \in C([0,\tau) : E)$. Suppose $f \in C([0,\tau) : E)$, A is a subgenerator of a* (*local*) *(a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ and* (H5) *holds. Then the problem*

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \in [0,\tau), \tag{55}$$

has a unique solution iff $R * f \in \mathrm{R}(K_C)$.

(ix) *Assume $n \in \mathbb{N}$, $f \in C([0,\tau) : E)$, A is a subgenerator of a* (*local*) *n-times integrated C-resolvent family $(R(t))_{t\in[0,\tau)}$, and* (H5) *holds. Then* (55) *has a unique solution iff $C^{-1}(R * f) \in C_0^{n+1}([0,\tau) : E)$.*

(x) *Let* (H5) *hold. Assume $n \in \mathbb{N}$, A is a subgenerator of an n-times integrated C-regularized resolvent and $a \in BV_{\mathrm{loc}}([0,\tau))$, resp. A is a subgenerator of an (a,C)-regularized resolvent family. Assume, further, that $C^{-1}f \in C^{(n+1)}([0,\tau) : E)$,*

$f^{(k-1)}(0) \in D(A^{n+1-k})$ *and* $A^{n+1-k}f^{(k-1)}(0) \in \mathrm{R}(C)$, $1 \leqslant k \leqslant n+1$, *resp.* $C^{-1}f \in C([0,\tau):E) \cap W^{1,1}_{\mathrm{loc}}([0,\tau):E)$. *Then* (55) *has a unique solution.*

(xi) *Assume* (H5) *holds, A is a subgenerator of an (a,k)-regularized C-resolvent family, $k \in AC([0,\tau))$ and $k(0) \neq 0$. If $C^{-1}f \in C^1([0,\tau):E)$, then there exists a unique solution of* (55).

(xii) *Let k and a satisfy* (P1) *and let $(R(t))_{t\geqslant 0}$ be a strongly continuous operator family which satisfies $\|R(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for some $M > 0$ and $\omega \geqslant 0$. Put* $\omega_0 := \max\big(\omega, \mathrm{abs}(a), \mathrm{abs}(k)\big)$.

(xii.1) *Assume A is a subgenerator of the exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ and* (H5) *holds. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)A$ is injective,* $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)A)$,

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in E, \ \operatorname{Re}\lambda > \omega_0, \ \tilde{k}(\lambda) \neq 0. \tag{56}$$

$$\Big\{\frac{1}{\tilde{a}(\lambda)} : \lambda \in \mathbb{C}, \ \operatorname{Re}\lambda > \omega_0, \ \tilde{k}(\lambda) \neq 0, \ \tilde{a}(\lambda) \neq 0\Big\} \subseteq \rho_C(A). \tag{57}$$

(xii.2) *Assume that* (56)–(57) *hold. Then A is a subgenerator of the exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$.*

(xiii) *Suppose $\pm A$ are subgenerators of* (*local, global exponentially bounded*) *(a,k)-regularized C-resolvent families $(R^{\pm}(t))_{t\in[0,\tau)}$ and A^2 is closed. Then A^2 is a subgenerator of a* (*local, global exponentially bounded*) *$(a*a,k)$-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, which is given by $R(t)x := \frac{1}{2}\big(R^+(t)x + R^-(t)x\big)$, $x \in E$, $t \in [0,\tau)$.*

(xiv) *Assume $\tau \in (0,\infty]$, $L^1_{\mathrm{loc}}([0,\tau)) \ni a_1$ is a kernel, $L^1_{\mathrm{loc}}([0,\tau)) \ni k$ is a kernel, $a(t) = (a_1 * a_1)(t)$, $t \in [0,\tau)$ and $k_1(t) = (k * a_1)(t)$, $t \in [0,\tau)$. Assume that* (H5) *holds. Then A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ iff $\mathcal{A}$ is a subgenerator of an (a_1,k_1)-regularized $\mathcal{C}$-resolvent family $(S(t))_{t\in[0,\tau)}$. If this is the case, then we have*

$$S(t) = \begin{pmatrix} (a_1 * R)(t) & (a * R)(t) \\ R(t) - k(t)C & (a_1 * R)(t) \end{pmatrix}, \quad 0 \leqslant t < \tau,$$

and the integral generators of $(R(t))_{t\in[0,\tau)}$ and $(S(t))_{t\in[0,\tau)}$, denoted respectively by B and $\mathcal{B}$, satisfy $\mathcal{B} = \left(\begin{smallmatrix} 0 & I \\ B & 0 \end{smallmatrix}\right)$.

(xv) *Assume $a(t)$ and $k(t)$ satisfy* (P1), *$\lim_{\lambda\to\infty,\,\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda) = k(0)$, there exists $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|\,dt < \infty$ and A is a subgenerator of an exponentially bounded (a,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$. Then*

$$\lim_{\lambda\to\infty,\,\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = k(0)Cx, \ x \in \overline{D(A)}.$$

(xvi) *Assume $a(t)$ and $k(t)$ satisfy* (P1), *$M \geqslant 1$, $\omega \geqslant 0$, $(S(t))_{t\geqslant 0}$ is an (a,k)-regularized C-resolvent family satisfying $\|S(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ and $AC \notin L(E)$.*

Then, for every $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \max\bigl(\omega, \operatorname{abs}(a), \operatorname{abs}(k)\bigr)$ *and* $\tilde{k}(\lambda) \neq 0$*, we have that* $\tilde{a}(\lambda) \neq 0$ *and that* $1/\tilde{a}(\lambda) \in \rho_C(A)$*.*

(xvii) *Assume* $\alpha \in (0,1)$*,* A *is a subgenerator of a global* $\bigl(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k\bigr)$*-regularized* C*-resolvent family* $(S_\alpha(t))_{t\geqslant 0}$*,* $D(A) \neq \{0\}$*, and* $\lim_{t\to+\infty} |k(t)|$ *does not exist in* $[0,\infty]$ *or* $\lim_{t\to+\infty} |k(t)| \neq 0$*. Then there do not exist* $M \geqslant 1$ *and* $\omega > 0$ *such that* $\|S_\alpha(t)\| \leqslant Me^{-\omega t}$*,* $t \geqslant 0$*.*

(xviii) *Assume that* $\alpha \in (1,2)$ *and* A *is a subgenerator of a* $\bigl(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C\bigr)$*-regularized resolvent family* $(S_\alpha(t))_{t\geqslant 0}$ *which satisfies* $\|S_\alpha(t)\| \leqslant Me^{\omega t}$ *for appropriate constants* $M \geqslant 1$ *and* $\omega \geqslant 0$*. Let* $(B(t))_{t\geqslant 0} \subseteq L(E)$*,* $\mathrm{R}(B(t)) \subseteq \mathrm{R}(C)$*,* $t \geqslant 0$ *and* $C^{-1}B(\cdot) \in C([0,\infty) : L(E))$*. Then, for every* $x \in D(A)$*, there exists a unique solution* $u(t)$ *of the problem*

$$\begin{cases} \mathbf{D}_t^\alpha u(t,x) = (A + B(t))u(t,x), \ t > 0, \\ u(0,x) = Cx, \ u'(0,x) = 0. \end{cases}$$

The solution $u(t,x)$ *is given by* $u(t,x) = \sum_{n=0}^{\infty} S_{\alpha,n}(t)x$*,* $t \geqslant 0$*, where we define* $S_{\alpha,n}(t)$ $(t \geqslant 0)$ *recursively by* $S_{\alpha,0}(t) := S_\alpha(t)$ *and*

$$S_{\alpha,n}(t) := \int\limits_0^t \int\limits_0^{t-\sigma} \frac{(t-\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} S_\alpha(s) C^{-1} B(\sigma) S_{\alpha,n-1}(\sigma)\, ds\, d\sigma.$$

Denote $K(T) = \max_{t\in[0,T]} \|C^{-1}B(t)\|$*,* $T \geqslant 0$*. Then*

$$\|u(t,x)\| \leqslant Me^{\omega t} E_\alpha(MK_T t^\alpha)\|x\|, \ t \in [0,T]$$
$$\|u(t,x) - S_\alpha(t)x\| \leqslant Me^{\omega t}\bigl(E_\alpha(MK_T t^\alpha) - 1\bigr)\|x\|, \ t \in [0,T].$$

(xix) *Assume* $k_\beta(t)$ *satisfies* (P1)*,* $0 < \alpha < \beta$*,* $\gamma = \frac{\alpha}{\beta}$ *and* A *is a subgenerator of a* $\bigl(\frac{t^{\beta-1}}{\Gamma(\beta)}, k_\beta\bigr)$*-regularized* C*-resolvent family* $(S_\beta(t))_{t\geqslant 0}$ *which satisfies* $\|S_\beta(t)\| = O(e^{\omega t})$*,* $t \geqslant 0$ *for some* $\omega \geqslant \max(0, \operatorname{abs}(k_\beta))$*. Assume additionally that* (H5) *holds and that there exists a function* $k_\alpha(t)$ *satisfying* (P1)*,* $k_\alpha(0) = k_\beta(0)$ *and* $\tilde{k_\alpha}(\lambda) = \lambda^{\frac{\alpha}{\beta}-1}\tilde{k_\beta}(\lambda^{\frac{\alpha}{\beta}})$ *for all sufficiently large positive real numbers* λ*. Then* A *is a subgenerator of a* $\bigl(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_\alpha\bigr)$*-regularized* C*-resolvent family* $(S_\alpha(t))_{t\geqslant 0}$ *which satisfies* $\|S_\alpha(t)\| = O(e^{\omega^{\beta/\alpha}t})$*,* $t \geqslant 0$ *and*

$$S_\alpha(t)x = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})S_\beta(s)x\, ds, \quad x \in E, \ t > 0.$$

Furthermore:

- (xix.1) *The mapping* $t \mapsto S_\alpha(t)$*,* $t > 0$ *has an analytic extension to the sector* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$*.*
- (xix.2) *If* $\omega = 0$ *and* $\varepsilon \in \bigl(0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi)\bigr)$*, then there exists* $M_{\gamma,\varepsilon} > 0$ *such that* $\|S_\alpha(z)\| \leqslant M_{\gamma,\varepsilon}$*,* $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}$*.*
- (xix.3) *If* $\omega > 0$ *and* $\varepsilon \in \bigl(0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2})\bigr)$*, then there exist* $\delta_{\gamma,\varepsilon} > 0$ *and* $M_{\gamma,\varepsilon} > 0$ *such that* $\|S_\alpha(z)\| \leqslant M_{\gamma,\varepsilon}e^{\delta_{\gamma,\varepsilon}\operatorname{Re} z}$*,* $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}$*.*

(xix.4) *Let* $\zeta \geqslant 0$. *Then the assumption* $\|S_\beta(t)\| = O(1+t^\zeta)$, $t \geqslant 0$, *resp.* $\|S_\beta(t)\| = O(t^\zeta)$, $t \geqslant 0$, *implies* $\|S_\alpha(t)\| = O(1+t^{\gamma\zeta})$, $t \geqslant 0$, *resp.* $\|S_\alpha(t)\| = O(t^{\gamma\zeta})$, $t \geqslant 0$.

(xx) *Suppose* $\beta \in (0,2]$, $\Sigma_{\frac{\beta\pi}{2}} \subseteq \rho_C(A)$, $\sup_{\lambda\in\Sigma_{\frac{\beta\pi}{2}}} \|\lambda(\lambda-A)^{-1}C\| < \infty$ *and, for every* $x \in E$, *the mapping* $\lambda \mapsto (\lambda-A)^{-1}Cx$, $\lambda \in \Sigma_{\frac{\beta\pi}{2}}$ *is continuous. Then, for every* $r \in (0,1]$, A *is the integral generator of a global* $(\frac{t^{\beta-1}}{\Gamma(\beta)}, \frac{t^r}{\Gamma(r+1)})$*-regularized* C^2*-resolvent family* $(S_r(t))_{t\geqslant 0}$ *satisfying that the mapping* $t \mapsto S_r(t)$, $t \geqslant 0$ *is locally Hölder continuous with exponent* r*; if* A *is densely defined, then* A *is the integral generator of a global bounded* $(\frac{t^{\beta-1}}{\Gamma(\beta)}, C^2)$*-resolvent family* $(S(t))_{t\geqslant 0}$.

Denote by a^{*n} the n-th convolution power of the kernel $a(t)$, $n \in \mathbb{N}$, and see [**369**] for the definition of completely positive functions and the notion used in the subsequent theorem. An insignificant modification of the proofs of [**286**, Theorem 3.7] and [**369**, Theorems 4.1, 4.3, 4.5] implies the following subordination principles.

THEOREM 2.1.28. [**235**] (i) *Let* $a(t)$, $b(t)$ *and* $c(t)$ *satisfy* (P1) *and let* $\beta \geqslant 0$ *be such that* $\int_0^\infty e^{-\beta t}|b(t)|dt < \infty$. *Let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int\limits_0^\infty c(t)\,dt > \frac{1}{\beta},\ \alpha = 0 \text{ otherwise,}$$

and let $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda \geqslant \alpha$. *Let* A *be a subgenerator of a* (b,k)*-regularized* C*-resolvent family* $(R_b(t))_{t\geqslant 0}$ *satisfying that* $\|R_b(t)\| = O(e^{\omega_b t})$, $t \geqslant 0$ *for some* $\omega_b \geqslant 0$, *and let* (H2) *or* (H3) *hold. Assume, further, that* $c(t)$ *is completely positive and that there exists a function* $k_1(t)$ *satisfying* (P1) *and*

$$\tilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big),\ \lambda > \omega_0,\ \tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0,\ \text{for some } \omega_0 > 0.$$

Put

$$\omega_a := \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int\limits_0^\infty c(t)dt > \frac{1}{\omega_b},\ \omega_a := 0 \text{ otherwise.}$$

Then, for every $r \in (0,1]$, A *is a subgenerator of a global* $(a, k_1 * \frac{t^{r-1}}{\Gamma(r)})$*-regularized* C*-resolvent family* $(R_r(t))_{t\geqslant 0}$ *such that* $\|R_r(t)\| = O(e^{\omega_a t})$, $t \geqslant 0$ *and that the mapping* $t \mapsto R_r(t)$, $t \geqslant 0$ *is locally Hölder continuous with exponent* r, *if* $\omega_b = 0$ *or* $\omega_b\tilde{c}(0) \neq 1$, *resp., for every* $\varepsilon > 0$, *there exists* $M_\varepsilon \geqslant 1$ *such that* $\|R_r(t)\| = O(e^{\varepsilon t})$, $t \geqslant 0$ *and that the mapping* $t \mapsto R_r(t)$, $t \geqslant 0$ *is locally Hölder continuous with exponent* r, *if* $\omega_b > 0$ *and* $\omega_b\tilde{c}(0) = 1$. *Furthermore, if* A *is densely defined, then* A *is a subgenerator of a global* (a, k_1)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *such that* $\|R(t)\| = O(e^{\omega_a t})$, $t \geqslant 0$, *resp., for every* $\varepsilon > 0$, $\|R(t)\| = O(e^{\varepsilon t})$, $t \geqslant 0$.

(ii) *Suppose* $\alpha \geqslant 0$, A *is a subgenerator of a global exponentially bounded* α*-times integrated* C*-semigroup,* $a(t)$ *is completely positive and satisfies* (P1), $k(t)$ *satisfies* (P1) *and* $\tilde{k}(\lambda) = \tilde{a}(\lambda)^\alpha$, λ *sufficiently large. Then, for every* $r \in (0,1]$, A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially* *bounded* $(a, k * \frac{t^{r-1}}{\Gamma(r)})$*-regularized* C*-resolvent family* $((a, a^{*n} * \frac{t^{r-1}}{\Gamma(r)})$*-regularized* C*-resolvent family if* $\alpha = n \in \mathbb{N}$*, resp.* $(a, \frac{t^r}{\Gamma(r+1)})$*-regularized* C*-resolvent family if* $\alpha = 0)$*. If, additionally,* A *is densely defined, then* A *is a subgenerator of an exponentially bounded* $(a, 1*k)$*-regularized* C*-resolvent family* $((a, 1*a^{*n})$*-regularized* C*-resolvent family if* $\alpha = n \in \mathbb{N}$*, resp.* (a, C)*-regularized resolvent family if* $\alpha = 0)$*.*

(iii) *Suppose* $\alpha \geqslant 0$ *and* A *is a subgenerator of an exponentially bounded* α*-times integrated* C*-cosine function. Let* $L^1_{\mathrm{loc}}([0,\infty)) \ni c$ *be completely positive and let* $a(t) = (c * c)(t)$*,* $t \geqslant 0$*. (Given* $L^1_{\mathrm{loc}}([0,\infty)) \ni a$ *in advance, such a function* $c(t)$ *always exists provided* $a(t)$ *is completely positive or* $a(t) \neq 0$ *is a creep function and* $a_1(t)$ *is log-convex.) Assume* $k(t)$ *satisfies* (P1) *and* $\tilde{k}(\lambda) = \tilde{c}(\lambda)^{\alpha}/\lambda$*,* λ *sufficiently large. Then, for every* $r \in (0, 1]$*,* A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially bounded* $(a, k * \frac{t^{r-1}}{\Gamma(r)})$*-regularized* C*-resolvent family* $((a, c^{*n} * \frac{t^{r-1}}{\Gamma(r)})$*-regularized* C*-resolvent family if* $\alpha = n \in \mathbb{N}$*, resp.* $(a, \frac{t^r}{\Gamma(r+1)})$*-regularized* C*-resolvent family if* $\alpha = 0)$*. If, additionally,* A *is densely defined, then* A *is a subgenerator of an exponentially bounded* $(a, 1 * k)$*-regularized* C*-resolvent family* $((a, 1 * c^{*n})$*-regularized* C*-resolvent family if* $\alpha = n \in \mathbb{N}$*, resp.* (a, C)*-regularized resolvent family if* $\alpha = 0)$*.*

Denote by A_p the realization of the Laplacian with Dirichlet or Neumann boundary conditions on $L^p([0,\pi]^n)$, $1 \leqslant p < \infty$. By [**195**, Theorem 4.2], A_p generates an exponentially bounded α-times integrated cosine function for every $\alpha \geqslant (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$. In what follows, we employ the notation given in [**369**]. Assume $c \in BV_{\mathrm{loc}}([0,\infty))$ and $m(t)$ is a bounded creep function with $m_0 = m(0+) > 0$. Thanks to [**369**, Proposition 4.4, p. 94], we have that there exists a completely positive function $b(t)$ such that $dm * b = 1$. After the usual procedure, the problem [**369**, (5.34)] describing heat conduction in materials with memory is equivalent to

$$u(t) = (a * A_p)(t) + f(t),\ t \geqslant 0, \tag{58}$$

where $a(t) = (b * dc)(t)$, $t \geqslant 0$ and $f(t)$ contains $r * b$ as well as the temperature history. Assume that:

(i) $p \neq 2$, (ii) $\Gamma_b = \emptyset$ or $\Gamma_f = \emptyset$, and
(iii) there exists a completely positive function $c_1(t)$ such that $a(t) = (c_1 * c_1)(t)$, $t \geqslant 0$.

We refer the reader to [**369**, pp. 140–141] for the analysis of the problem (58) in the case: $p = 2$ and m, $c \in \mathcal{BF}$. Applying Theorem 2.1.28(iii), one gets that A_p is the integral generator of an exponentially bounded $\left(a, 1 * \mathcal{L}^{-1}\left(\frac{1}{\lambda}\tilde{c_1}(\lambda)^{(n-1)|\frac{1}{2}-\frac{1}{p}|}\right)(t)\right)$-regularized resolvent family, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Notice also that [**369**, Lemma 4.3, p. 105] implies that, for every $\beta \in [0, 1]$, the function $\lambda \mapsto \tilde{c_1}(\lambda)^{\beta}/\lambda$ is the Laplace transform of a Bernstein function, and that the function $k(t)$ appearing in the formulations of Theorem 2.1.28(ii)–(iii) always exists (provided $\alpha > 0$ in (ii)). On the other hand, an application of Proposition 2.3.12 given below (it seems that this result can be slightly improved in the case of fractionally integrated cosine functions) gives that there exists $\omega > 0$ such that A_p is the integral generator of an exponentially bounded $(\omega - A_p)^{-\lceil \frac{1}{2}(n-1)|\frac{1}{2}-\frac{1}{p}|\rceil}$-regularized cosine function. Using Theorem 2.1.28(iii) again, we have that A_p is the integral generator of an exponentially bounded $\big(a, (\omega - A_p)^{-\lceil \frac{1}{2}(n-1)|\frac{1}{2}-\frac{1}{p}|\rceil}\big)$-regularized resolvent family, and Theorem 2.1.27(x) can be applied. In both approaches, regrettably, we must restrict ourselves to the study of pure Dirichlet or Neumann problem. It is also worthwhile to note that Theorem 2.1.28(iii) can be applied in the analysis of the Rayleigh problem of viscoelasticity in L^∞ type spaces; as a matter of fact, the operator A defined on [**369**, p. 136] generates an exponentially bounded α-times integrated cosine function in $L^\infty((0,\infty))$ for all $\alpha > 0$. Noticing that, for every $\alpha > 0$, the operator $Au(x) := u''(x)$, $x \in [0,1]$,

$$u \in D(A) := \{u \in L^\infty(0,1) : u', u'' \in L^\infty(0,1),\ u(0) = u'(1) = 0\},$$

generates a polynomially bounded α-times integrated cosine function $(C_\alpha(t))_{t\geqslant 0}$ in $L^\infty(0,1)$, we are in a position to apply Theorem 2.1.28(iii) in the analysis of motion for the axial extension of a viscoelastic rod [**369**, (5.49), p. 138]. It could be of interest to know in which classes of non-Hilbert spaces the problem of torsion of a rod [**369**, (5.46), p. 137] can be considered.

Let $\beta > 0$. Concerning fractional powers of sectorial operators generating $\frac{t^{\beta-1}}{\Gamma(\beta)}$-resolvent families (i.e. $(\frac{t^{\beta-1}}{\Gamma(\beta)}, I)$-regularized resolvent families), and Landau-Kolmogorov type inequalities for subgenerators of (a,k)-regularized C-resolvent families, the reader may consult [**280**] and [**293**]. Further on, it is worth noting that Karczewska and Lizama [**180**] have recently analyzed the following stochastic fractional oscillation equation

$$u(t) + \int_0^t (t-s)\big[A\mathbf{D}_s^\alpha u(s) + u(s)\big]\,ds = W(t),\ t > 0, \tag{59}$$

where $1 < \alpha < 2$, A is the generator of a bounded analytic C_0-semigroup on a Hilbert space H and $W(t)$ denotes an H-valued Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, P)$. The theory of (a,k)-regularized C-resolvent families is essentially applied in the study of deterministic counterpart of the equation (59) in integrated form

$$u(t) + \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} Au(s)\,ds + \int_0^t (t-s)u(s)\,ds = \int_0^t (t-s)f(s)\,ds,\ t > 0,$$

where $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$.

Using the argumentation given in [**288**] and [**292**], one can prove the following.

THEOREM 2.1.29. [**235**] (i) *Suppose that the next conditions hold:*

(i.1) *The mapping $t \mapsto |k(t)|$, $t \in [0,\tau)$ is nondecreasing.*

(i.2) *There exist $\varepsilon_{a,k} > 0$ and $t_{a,k} \in [0,\tau)$ such that*

$$\Big|\int_0^t a(t-s)k(s)\,ds\Big| \geqslant \varepsilon_{a,k} \int_0^t |a(t-s)k(s)|\,ds,\quad t \in [0, t_{a,k}).$$

(i.3) *A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ and* (H5) *holds.*

(i.4) $\limsup_{t\to 0+}\frac{\|R(t)\|}{|k(t)|}<\infty$.

Then, for every $x\in D(A_{\overline{D(A)}})$,

$$ACx=\lim_{t\to 0+}\frac{R(t)x-k(t)Cx}{(a*k)(t)}. \tag{60}$$

Assume, further, $x\in\overline{D(A)}$ and $\lim_{t\to 0+}\frac{R(t)x-k(t)Cx}{(a*k)(t)}$ exists. Then $Cx\in D(A_{\overline{D(A)}})$ and (60) *holds.*

(ii) *Suppose A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying $\|R(t)\|=O(k(t))$, $t\to 0+$ and $\min(a(t),k(t))>0$, $t\in(0,\tau)$. Then the following holds:*

(ii.1) $\lim_{t\to 0+}\frac{(a*R)(t)x}{(a*k)(t)}=Cx$, $x\in\overline{D(A)}$.

(ii.2) *Suppose* (H5). *If $x\in\overline{D(A)}$, $y\in E$ and $\lim_{t\to 0+}\frac{R(t)x-k(t)Cx}{(a*k)(t)}=y$, then $Cx\in D(A)$ and $y=ACx$.*

(ii.3) *Let E be reflexive, let* (H5) *hold and let $R(s)R(t)=R(t)R(s)$, $0\leqslant t$, $s<\tau$. If $x\in\overline{D(A)}$ and $\lim_{t\to 0+}\|\frac{R(t)x-k(t)Cx}{(a*k)(t)}\|<\infty$, then $Cx\in D(A)$.*

(iii) *Suppose A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying that $\|R(t)\|=O(k(t))$, $t\to 0+$, $\min(a(t),k(t))>0$, $t\in(0,\tau)$ and that* (H5) *holds. Then, for every $x\in D(A_{\overline{D(A)}})$,* (60) *holds. Furthermore, if $x\in\overline{D(A)}$ and $\lim_{t\to 0+}\frac{R(t)x-k(t)Cx}{(a*k)(t)}$ exists, then $Cx\in D(A_{\overline{D(A)}})$ and* (60) *holds.*

(iv) *Suppose $\alpha>0$ and A is a subgenerator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$, resp. α-times integrated C-cosine function $(C_\alpha(t))_{t\in[0,\tau)}$, such that $\limsup_{t\to 0+}\frac{\|S_\alpha(t)\|}{t^\alpha}<\infty$, resp. $\limsup_{t\to 0+}\frac{\|C_\alpha(t)\|}{t^\alpha}<\infty$. Then, for every $x\in D(A)$ such that $Ax\in\overline{D(A)}$:*

$$CAx=\lim_{t\to 0+}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)S_\alpha(t)x-t^\alpha Cx}{t^{\alpha+1}},\quad \textit{resp.}$$
$$CAx=\lim_{t\to 0+}\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)C_\alpha(t)x-t^\alpha Cx}{t^{\alpha+2}}.$$

Assume (M_p) satisfies (M.1), (M.2) and (M.3′). Put $L_p:=M_p^{1/p}$ and $\omega_L(t):=\sum_{p=0}^{\infty}\frac{t^p}{L_p^p}$, $t\geqslant 0$ (cf. also [**28**], Section 3.2 and Subsection 3.5.3).

Definition 2.1.30. Let $(R(t))_{t\in[0,\tau)}$ be a (local) (a,k)-regularized C-resolvent family having A as a subgenerator and let the mapping $t\mapsto R(t)$, $t\in(0,\tau)$ be infinitely differentiable (in the uniform operator topology). Then it is said that $(R(t))_{t\in(0,\tau)}$ is of *class C^L*, resp. of *class C_L*, iff for every compact set $K\subseteq(0,\tau)$ there exists $h_K>0$, resp. for every compact set $K\subseteq(0,\tau)$ and for every $h>0$:

$$\sup_{t\in K,\,p\in\mathbb{N}_0}\left\|\frac{h_K^p\frac{d^p}{dt^p}R(t)}{L_p^p}\right\|<\infty,\ \text{resp.}\ \sup_{t\in K,\,p\in\mathbb{N}_0}\left\|\frac{h^p\frac{d^p}{dt^p}R(t)}{L_p^p}\right\|<\infty;$$

$(R(t))_{t\in[0,\tau)}$ is said to be *ρ-hypoanalytic,* $1 \leqslant \rho < \infty$, if $(R(t))_{t\in[0,\tau)}$ is of class C^L with $L_p = p!^{\rho/p}$ (notice only that (M.3′) does not hold provided $M_p = p!$ and $L_p = p!^{1/p}$).

By the proof of the scalar-valued version of the Pringsheim theorem (cf. for example [**206**, Theorem 2.1, p. 34]), it follows that the mapping $t \mapsto R(t)$, $t \in (0,\tau)$ is real analytic iff $(R(t))_{t\in[0,\tau)}$ is ρ-hypoanalytic with $\rho = 1$.

THEOREM 2.1.31. [**28**], [**235**] (i) *Suppose A is a closed linear operator, $k(t)$ and $a(t)$ satisfy* (P1), *$r \geqslant -1$ and there exists $\omega \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$ such that, for every $z \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0\}$, we have that the operator $I - \tilde{a}(z)A$ is injective and that* $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(z)A)$. *If, additionally, for every $\sigma > 0$, there exist $C_\sigma > 0$, $M_\sigma > 0$ and an open neighborhood $\Omega_{\sigma,\omega}$ of the region*

$$\Lambda_{\sigma,\omega} := \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma\bigr\} \cup \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\bigr\},$$

and an analytic mapping $h_\sigma : \Omega_{\sigma,\omega} \to L(E)$ such that $h_\sigma(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$, and that $\|h_\sigma(\lambda)\| \leqslant M_\sigma|\lambda|^r$, $\lambda \in \Lambda_{\sigma,\omega}$, then, for every $\zeta > 1$, A is a subgenerator of a norm continuous, exponentially bounded $\bigl(a, k * \frac{t^{\zeta+r-1}}{\Gamma(\zeta+r)}\bigr)$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto R(t)$, $t > 0$ is infinitely differentiable.

(ii) *Suppose $k(t)$ and $a(t)$ satisfy* (P1), (H5) *holds and A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying $\|R(t)\| \leqslant Me^{\omega' t}$, $t \geqslant 0$ for appropriate constants $\omega' \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$ and $M > 0$. If there exists $\omega > \omega'$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ so that:*

- (ii.1) *there exist an open neighborhood $\Omega_{\sigma,\omega}$ of the region $\Lambda_{\sigma,\omega}$, and the analytic mappings $f_\sigma : \Omega_{\sigma,\omega} \to \mathbb{C}$, $g_\sigma : \Omega_{\sigma,\omega} \to \mathbb{C}$ and $h_\sigma : \Omega_{\sigma,\omega} \to L(E)$ such that $f_\sigma(\lambda) = \tilde{k}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \omega$ and $g_\sigma(\lambda) = \tilde{a}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \omega$,*
- (ii.2) *for every $\lambda \in \Lambda_{\sigma,\omega}$ with $\operatorname{Re}\lambda \leqslant \omega$, the operator $I - \tilde{a}(\lambda)A$ is injective and* $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)A)$,
- (ii.3) *$h_\sigma(\lambda) = f_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}C$, $\lambda \in \Lambda_{\sigma,\omega}$ and*
- (ii.4) *$\|h_\sigma(\lambda)\| \leqslant M_\sigma|\operatorname{Im}\lambda|$, $\lambda \in \Lambda_{\sigma,\omega}$, $\operatorname{Re}\lambda \leqslant \omega$ and $\max(|f_\sigma(\lambda)|, |g_\sigma(\lambda)|) \leqslant M_\sigma$, $\lambda \in \Lambda_{\sigma,\omega}$,*

then the mapping $t \mapsto R(t)x$, $t > 0$ is infinitely differentiable for every fixed $x \in D(A^2)$. Furthermore, if $D(A^2)$ is dense in E, then the mapping $t \mapsto R(t)$, $t > 0$ is infinitely differentiable.

(iii) *Suppose $k(t)$ and $a(t)$ satisfy* (P1), *A is a subgenerator of a* (*local*) *(a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $\omega \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$ and $m \in \mathbb{N}$. Denote, for every $\varepsilon \in (0,1)$ and a corresponding $K_\varepsilon > 0$,*

$$F_{\varepsilon,\omega} =: \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -\ln \omega_L(K_\varepsilon|\operatorname{Im}\lambda|) + \omega\bigr\}.$$

Assume that, for every $\varepsilon \in (0,1)$, there exist $C_\varepsilon > 0$, $M_\varepsilon > 0$, an open neighborhood $O_{\varepsilon,\omega}$ of the region

$$G_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega,\ \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : \operatorname{Re}\lambda \leqslant \omega\},$$

and analytic mappings $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ *such that:*

(iii.1) $f_\varepsilon(\lambda) = \tilde{k}(\lambda)$, $\operatorname{Re}\lambda > \omega$; $g_\varepsilon(\lambda) = \tilde{a}(\lambda)$, $\operatorname{Re}\lambda > \omega$,

(iii.2) *for every* $\lambda \in F_{\varepsilon,\omega}$, *the operator* $I - g_\varepsilon(\lambda)A$ *is injective and* $\mathrm{R}(C) \subseteq \mathrm{R}(I - g_\varepsilon(\lambda)A)$,

(iii.3) $h_\varepsilon(\lambda) = f_\varepsilon(\lambda)(I - g_\varepsilon(\lambda)A)^{-1}C$, $\lambda \in G_{\varepsilon,\omega}$,

(iii.4) $\|h_\varepsilon(\lambda)\| \leqslant M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|\operatorname{Re}\lambda|}$, $\lambda \in F_{\varepsilon,\omega}$, $\operatorname{Re}\lambda \leqslant \omega$ *and* $\|h_\varepsilon(\lambda)\| \leqslant M_\varepsilon(1+|\lambda|)^m$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \omega$.

Then $(R(t))_{t\in[0,\tau)}$ *is of class* C^L.

(iv) *Suppose* $k(t)$ *and* $a(t)$ *satisfy* (P1), A *is a subgenerator of a* (*local*) (a,k)-*regularized* C-*resolvent family* $(R(t))_{t\in[0,\tau)}$, $\omega \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$ *and* $m \in \mathbb{N}$. *Denote, for every* $\varepsilon \in (0,1)$, $\rho \in [1,\infty)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega,\rho} =: \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -K_\varepsilon|\operatorname{Im}\lambda|^{1/\rho} + \omega\bigr\}.$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $C_\varepsilon > 0$, $M_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega,\ \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega,\rho} : \operatorname{Re}\lambda \leqslant \omega\}$, *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ *such that the conditions* (iii.1)–(iii.4) *of this theorem hold with* $F_{\varepsilon,\omega}$, *resp.* $G_{\varepsilon,\omega}$, *replaced by* $F_{\varepsilon,\omega,\rho}$, *resp.* $G_{\varepsilon,\omega,\rho}$. *Then* $(R(t))_{t\in[0,\tau)}$ *is* ρ-*hypoanalytic.*

(v) *Suppose* $\alpha > 0$, $j \in \mathbb{N}$ *and* $(R(t))_{t\in[0,\tau)}$ *is a* (*local*) (a,k)-*regularized* C-*resolvent family with a subgenerator* A. *Set* $R_\alpha(t)x := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} R(s)x\,ds$, $t \in [0,\tau)$, $x \in E$. *Then* $(R_\alpha(t))_{t\in[0,\tau)}$ *is an* $\bigl(a, k * \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigr)$-*regularized* C-*resolvent family with a subgenerator* A. *Furthermore, if the mapping* $t \mapsto R(t)$, $t \in (0,\tau)$ *is* j-*times differentiable, then the mapping* $t \mapsto R_\alpha(t)$, $t \in (0,\tau)$ *is likewise* j-*times differentiable. If this is the case, then we have, for every* $t \in [0,\tau)$, $b \in (0,t)$ *and* $x \in E$:

$$\begin{aligned}\frac{d^j}{dt^j}R_\alpha(t)x = \int\limits_0^b \frac{(t-s)^{\alpha-1-j}}{\Gamma(\alpha)} \prod_{i=1}^{j}(\alpha-i)R(s)x\,ds + \sum_{i=0}^{j}\frac{(t-b)^{\alpha+i-j}}{\Gamma(\alpha+i+1)}\\ \times \prod_{k=0}^{j-1}(\alpha+i-k)R^{(i)}(b)x + \int\limits_b^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\frac{d^j}{ds^j}R(s)x\,ds,\end{aligned}$$

and:

(v.1) *If* $(R(t))_{t\in[0,\tau)}$ *is of class* C^L, *resp. of class* C_L, *then* $(R_\alpha(t))_{t\in[0,\tau)}$ *is likewise of class* C^L, *resp. of class* C_L.

(v.2) *If* $(R(t))_{t\in[0,\tau)}$ *is* ρ-*hypoanalytic,* $1 \leqslant \rho < \infty$, *then* $(R_\alpha(t))_{t\in[0,\tau)}$ *is likewise* ρ-*hypoanalytic.*

THEOREM 2.1.32. [**234**]–[**235**] (i) *Suppose* $j \in \mathbb{N}$, $\tau > 0$, $K \in L^1_{\mathrm{loc}}([0,\tau))$, $0 \in \operatorname{supp} K$, $K \in C^j((0,\tau))$ ($K \in C^\infty((0,\tau))$), A *is a subgenerator of a local* K-*convoluted* C-*semigroup* $(S_K(t))_{t\in[0,\tau)}$ *satisfying that the mapping* $t \mapsto S_K(t)$,

$t \in (0,\tau)$ is j-times (infinitely) differentiable and $K = K_{1_{|[0,\tau)}}$ for an appropriate complex-valued function $K_1 \in L^1_{\text{loc}}([0,2\tau))$ (cf. the formulation of Theorem 2.1.9*). Then A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-semigroup $(S_{K*_0K}(t))_{t\in[0,2\tau)}$ satisfying that the mapping $t \mapsto S_{K*_0K}(t)$, $t \in (0,2\tau)$ is j-times (infinitely) differentiable.*

(ii) *Suppose $\alpha \geqslant 0$, $j \in \mathbb{N}$ and A is a subgenerator of a local α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Then A is a subgenerator of a local (2α)-times integrated C^2-semigroup $(S_{2\alpha}(t))_{t\in[0,2\tau)}$ and the following holds: If the mapping $t \mapsto S_\alpha(t)$, $t \in (0,\tau)$ is j-times (infinitely) differentiable, then the mapping $t \mapsto S_{2\alpha}(t)$, $t \in (0,2\tau)$ is j-times (infinitely) differentiable.*

(iii) *Assume $\alpha \geqslant 0$ and A generates a local α-times integrated semigroup $(S_\alpha(t))_{t\in[0,\tau)}$ satisfying that the mapping $t \mapsto S_\alpha(t)$, $t \in (0,\tau)$ is $(\lceil\alpha\rceil + 1)$-times differentiable. Then A generates a global α-times integrated semigroup $(\tilde{S}_\alpha(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto \tilde{S}_\alpha(t)$, $t > 0$ is infinitely differentiable.*

(iv) *Suppose A is a subgenerator of a local K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, $0 \in \operatorname{supp} K$, $K \in C^\infty((0,\tau))$ ($K \in C^j((0,\tau))$, $j \in \mathbb{N}$) resp. K is of class C^L (C_L), and let $K = K_{1_{|[0,\tau)}}$ for an appropriate complex-valued function $K_1 \in L^1_{\text{loc}}([0,2\tau))$. Let the mapping $t \mapsto C_K(t)$, $t \in (0,\tau)$ be infinitely differentiable (j-times differentiable, $j \in \mathbb{N}$), resp. let $(C_K(t))_{t\in[0,\tau)}$ be of class C^L (C_L). Then A is a subgenerator of a local $(K*K)$-convoluted C^2-cosine function $(C_{K*K}(t))_{t\in[0,2\tau)}$ satisfying that the mapping $t \mapsto C_{K*K}(t)$, $t \in (0,2\tau)$ is infinitely differentiable ($(j-1)$-times differentiable), resp. $(C_{K*K}(t))_{t\in[0,2\tau)}$ is of class C^L (C_L). Furthermore, the preassumptions $j \in \mathbb{N}$, and $K \in C^j((0,\tau)) \cap C^{j-1}([0,\tau))$, imply the following: If the mapping $t \mapsto C_K(t)$, $t \in (0,\tau)$ is j-times differentiable, then the mapping $t \mapsto C_{K*K}(t)$, $t \in (0,2\tau)$ is likewise j-times differentiable.*

(v) *Suppose $\alpha \geqslant 0$, $j \in \mathbb{N}$ and A is a subgenerator of a local α-times integrated C-cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. Then A is a subgenerator of a local (2α)-times integrated C^2-cosine function $(C_{2\alpha}(t))_{t\in[0,2\tau)}$ and the following holds:*

- (v.1) *If the mapping $t \mapsto C_\alpha(t)$, $t \in (0,\tau)$ is infinitely differentiable (j-times differentiable, $j \in \mathbb{N}$), then the mapping $t \mapsto C_{2\alpha}(t)$, $t \in (0,2\tau)$ is infinitely differentiable ($(j-1)$-times differentiable; j-times differentiable, provided $\alpha \geqslant j$).*
- (v.2) *If $(C_\alpha(t))_{t\in[0,\tau)}$ is of class C^L, resp. C_L, then $(C_{2\alpha}(t))_{t\in[0,2\tau)}$ is likewise of class C^L, resp. C_L.*
- (v.3) *Assume $\alpha \in \mathbb{N}_0$, $j \in \mathbb{N}$ and the mapping $t \mapsto C_\alpha(t)$, $t \in (0,\tau)$ is infinitely differentiable (j-times differentiable). Then the mapping $t \mapsto C_{2\alpha}(t)$, $t \in (0,2\tau)$ is j-times differentiable.*

(vi) *Suppose that $\alpha \geqslant 0$ and that A generates a (local) α-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$ satisfying that the mapping $t \mapsto C_\alpha(t)$, $t > 0$ is $(\lceil\alpha\rceil + 2)$-times differentiable. Then A must be bounded.*

Assume that $\min(a(t), k(t)) > 0$, $t \in (0,\tau)$ and that A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. The *Favard class* $F_{a,k}$ is defined by setting

$$F_{a,k} := \Big\{ x \in E : \sup_{t\in(0,\tau)} \frac{\|R(t)x - k(t)Cx\|}{(a*k)(t)} < \infty \Big\}.$$

Equipped with the norm $|\cdot|_{a,k} := \|\cdot\| + \sup_{t\in(0,\tau)} \frac{\|R(t)\cdot - k(t)C\cdot\|}{(a*k)(t)}$, $F_{a,k}$ becomes a Banach space, and in the case when $\|R(t)\| = O(k(t))$, $t \in [0,\tau)$, we have $D(A) \subseteq F_{a,k}$. The proof of [**292**, Theorem 3.4] immediately implies the following assertion.

THEOREM 2.1.33. *Assume* $\min(a(t), k(t)) > 0$*,* $t \in (0,\tau)$*,* $\operatorname{abs}(k) = \operatorname{abs}(a) = 0$*, A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying* $\|R(t)\| = O(1)$*,* $t \geqslant 0$ *and* (H5) *holds.*

(i) *Let $x \in F_{a,k}$. Then*

$$\sup_{\lambda>0,\ \tilde{k}(\lambda)\neq 0} \big\|A(I - \tilde{a}(\lambda)A)^{-1}Cx\big\| < \infty. \tag{61}$$

(ii) *Assume, in addition, that the mapping $\tilde{a} : (0,\infty) \to (0,\infty)$ is surjective and that* $\sup_{t>0} \frac{(1*a)(t)}{(a*k)(t)} < \infty$*. Then* (61) *implies* $Cx \in F_{a,k}$*.*

For further information concerning Volterra integro-differential equations, we recommend [**79**], [**289**], [**369**], [**373**] and references cited there.

2.2. Exponentially bounded convoluted C-semigroups and cosine functions

The most important interplay between exponentially bounded convoluted C-cosine functions and operator valued Laplace transform is described in the following theorem (cf. also Theorem 2.1.27(xii)).

THEOREM 2.2.1. *Let K satisfy* (P1) *and let A be a closed linear operator.*

(i) *Assume A is a subgenerator of an exponentially bounded, K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$ satisfying $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M > 0$ and $\omega \geqslant 0$. If* $\omega_1 = \max(\omega, \operatorname{abs}(K))$*, then:*

$$\big\{\lambda^2 :\ \operatorname{Re}\lambda > \omega_1,\ \tilde{K}(\lambda) \neq 0\big\} \subseteq \rho_C(A) \ \textit{and} \tag{62}$$

$$\lambda(\lambda^2 - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)x\,dt,\ \ x \in E,\ \operatorname{Re}\lambda > \omega_1,\ \tilde{K}(\lambda) \neq 0. \tag{63}$$

(ii) *Assume $M > 0$, $\omega \geqslant 0$, $(C_K(t))_{t\geqslant 0}$ is a strongly continuous operator family satisfying $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega_1 = \max(\omega, \operatorname{abs}(K))$ and* (62)–(63) *hold. Then $(C_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-cosine function with a subgenerator A.*

PROOF. Fix temporarily a complex number λ such that $\tilde{K}(\lambda) \neq 0$ and $\operatorname{Re}\lambda > \omega_1$. Since (19) is assumed and A is closed, one obtains

$$\mathcal{L}\big(C_K(t)x\big)(\lambda) = \frac{\tilde{K}(\lambda)}{\lambda}Cx + \frac{1}{\lambda^2}A\mathcal{L}(C_K(t)x)(\lambda),\ \text{i.e.,}$$

$$(\lambda^2 - A)\mathcal{L}\big(C_K(t)x\big)(\lambda) = \lambda\tilde{K}(\lambda)Cx,\ \ x \in E. \tag{64}$$

Hence, $\mathrm{R}(C) \subseteq \mathrm{R}(\lambda^2 - A)$. Assuming $(\lambda^2 - A)x = 0$, one has

$$C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)Ax\,ds = \lambda^2 \int_0^t (t-s)C_K(s)x\,ds,\ t \geqslant 0,$$

and consequently, $\mathcal{L}\big(C_K(t)x\big)(\lambda) = \frac{\tilde{K}(\lambda)}{\lambda}Cx + \mathcal{L}\big(C_K(t)x\big)(\lambda)$, $Cx = 0$ and $x = 0$. This implies the injectiveness of $\lambda^2 - A$; thanks to (64) one gets $(w_1^2, \infty) \subseteq \rho_C(A)$ and (63), which completes the proof of (i). Now we will prove (ii). Using (63) and $CA \subseteq AC$, we infer that $(\lambda^2 - A)^{-1}C^2x = C(\lambda^2 - A)^{-1}Cx$, $x \in E$. Hence,

$$\frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)Cx\,dt = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} CC_K(t)x\,dt,\ x \in E.$$

Since $K \neq 0$ in $L^1_{\mathrm{loc}}([0,\infty))$, it follows that

$$\{z \in \mathbb{C} : \operatorname{Re} z \geqslant \omega_1\} = \overline{\{z \in \mathbb{C} : \operatorname{Re} z > \omega_1,\ \tilde{K}(z) \neq 0\}}$$

and

$$\int_0^\infty e^{-\lambda t} C_K(t)Cx\,dt = \int_0^\infty e^{-\lambda t} CC_K(t)x\,dt,\ \ \operatorname{Re}\lambda > \omega_1,\ x \in E.$$

The uniqueness theorem for the Laplace transform implies $CC_K(t) = C_K(t)C$, $t \geqslant 0$, $x \in E$. Then we obtain

$$\lambda(\lambda^2 - A)^{-1}CAx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)Axdt,\ \ \operatorname{Re}\lambda > \omega_1,\ \tilde{K}(\lambda) \neq 0,\ x \in D(A)$$

$$\lambda A(\lambda^2 - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)Axdt,\ \ \operatorname{Re}\lambda > \omega_1,\ \tilde{K}(\lambda) \neq 0,\ x \in D(A).$$

An immediate consequence is $A\int_0^\infty e^{-\lambda t}C_K(t)x\,dt = \int_0^\infty e^{-\lambda t}C_K(t)Ax\,dt$, $\operatorname{Re}\lambda > \omega_1$, $\tilde{K}(\lambda) \neq 0$, $x \in D(A)$. Using the closedness of A and the above arguments, we obtain that the last equality holds for every $x \in D(A)$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$. Now one can apply Theorem 1.1.7(vii) in order to see that $C_K(t)A \subseteq AC_K(t)$, $t \geqslant 0$. The following equalities hold for every $x \in E$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$ and $\tilde{K}(\lambda) \neq 0$,

$$\begin{aligned}\mathcal{L}\Bigg(\int_0^t (t-s)C_K(s)x\,ds\Bigg)(\lambda) &= \mathcal{L}(t)(\lambda)\mathcal{L}(C_K(t)x)(\lambda)\\ &= \frac{1}{\lambda^2}\tilde{K}(\lambda)\lambda(\lambda^2 - A)^{-1}Cx = \frac{\tilde{K}(\lambda)}{\lambda}(\lambda^2 - A)^{-1}Cx\end{aligned}$$

and

$$A\Bigg(\mathcal{L}\Bigg(\int_0^t (t-s)C_K(s)x\,ds\Bigg)(\lambda)\Bigg) = \tilde{K}(\lambda)\lambda(\lambda^2 - A)^{-1}Cx - \frac{\tilde{K}(\lambda)}{\lambda}Cx$$

$$= \mathcal{L}\big(C_K(t)x - \Theta(t)Cx\big)(\lambda), \ \operatorname{Re}\lambda > \omega_1, \ \tilde{K}(\lambda) \neq 0.$$

Using the closedness of A and Theorem 1.1.7(vii), one immediately gets (19). $\square$

The following characterization of exponentially bounded convoluted C-semigroups can be proved along the same lines.

Theorem 2.2.2. *Let K satisfy* (P1) *and let A be a closed linear operator.*

(i) *Assume $M > 0$, $\omega \geqslant 0$, A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ and $\omega_1 = \max(\omega, \operatorname{abs}(K))$. Then the following holds:*

$$\big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_1, \ \tilde{K}(\lambda) \neq 0\big\} \subseteq \rho_C(A) \ \textit{and} \tag{65}$$

$$(\lambda - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} S_K(t)x\, dt, \ \ x \in E, \ \operatorname{Re}\lambda > \omega_1, \ \tilde{K}(\lambda) \neq 0. \tag{66}$$

(ii) *Assume $M > 0$, $\omega \geqslant 0$, $(S_K(t))_{t\geqslant 0}$ is a strongly continuous operator family, $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega_1 = \max(\omega, \operatorname{abs}(K))$ and* (65)–(66) *hold. Then $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A.*

Remark 2.2.3. Assume that (62)–(63), resp. (65)–(66), hold only for real values of λ's. Then $(C_K(t))_{t\geqslant 0}$, resp. $(S_K(t))_{t\geqslant 0}$, is still an exponentially bounded, K-convoluted C-cosine function, resp. K-convoluted C-semigroup, with a subgenerator A.

Using Theorem 2.2.1, Theorem 2.2.2 and Theorem 1.1.12, one can simply prove the following assertion.

Theorem 2.2.4. (i) *Suppose K satisfies* (P1), *$\omega > \max(0, \operatorname{abs}(K))$, A is a closed linear operator with $\{\lambda^2 : \operatorname{Re}\lambda > \omega, \ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A)$ and the function $\lambda \mapsto \lambda\tilde{K}(\lambda)(\lambda^2 - A)^{-1}C$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$, can be extended to an analytic function $\Upsilon : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \to L(E)$ satisfying $\|\Upsilon(\lambda)\| \leqslant M_0|\lambda|^r$, $\operatorname{Re}\lambda > \omega$, where $r \geqslant -1$. Then, for every $\alpha > 1$, there exist a continuous function $C : [0, \infty) \to L(E)$ with $C(0) = 0$ and $M_1 > 0$ such that $\|C(t)\| \leqslant M_1e^{\omega t}$, $t \geqslant 0$ and*

$$\lambda\tilde{K}(\lambda)(\lambda^2 - A)^{-1}C = \lambda^{\alpha+r} \int_0^\infty e^{-\lambda t} C(t)\, dt, \ \ \operatorname{Re}\lambda > \omega, \ \tilde{K}(\lambda) \neq 0.$$

Furthermore, $(C(t))_{t\geqslant 0}$ is a norm continuous, exponentially bounded $\big(K *_0 \frac{t^{\alpha+r-1}}{\Gamma(\alpha+r)}\big)$-convoluted C-cosine function with a subgenerator A.

(ii) *Suppose K satisfy* (P1), *$\omega > \max(0, \operatorname{abs}(K))$, A is a closed linear operator with $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega, \ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A)$ and there exists an analytic function $\Upsilon : \{z \in \mathbb{C} : \operatorname{Re} z > \omega\} \to L(E)$ so that $\Upsilon(\lambda) = \tilde{K}(\lambda)(\lambda - A)^{-1}C$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$, and $\|\Upsilon(\lambda)\| \leqslant M_0|\lambda|^r$, $\operatorname{Re}\lambda > \omega$, for some $r \geqslant -1$. Then, for every $\alpha > 1$, there exist a continuous function $S : [0, \infty) \to L(E)$ with $S(0) = 0$ and*

$M_1 > 0$ such that $\|S(t)\| \leqslant M_1 e^{\omega t}$, $t \geqslant 0$ and

$$\tilde{K}(\lambda)(\lambda - A)^{-1}C = \lambda^{\alpha+r} \int_0^\infty e^{-\lambda t} S(t)\,dt, \quad \operatorname{Re}\lambda > \omega, \ \tilde{K}(\lambda) \neq 0.$$

Furthermore, $(S(t))_{t\geqslant 0}$ is a norm continuous, exponentially bounded $\big(K *_0 \frac{t^{\alpha+r-1}}{\Gamma(\alpha+r)}\big)$-convoluted C-semigroup with a subgenerator A.

Theorem 2.2.5. *Suppose K satisfies* (P1) *and A is a closed linear operator. Then the following holds:*

(i.1) *Let $M > 0$ and $\omega \geqslant 0$. Then the operator A is a subgenerator of an exponentially bounded, Θ-convoluted C-semigroup $(S_\Theta(t))_{t\geqslant 0}$ satisfying the condition:*

$$\|S_\Theta(t+h) - S_\Theta(t)\| \leqslant Mhe^{\omega(t+h)}, \quad t \geqslant 0, \ h \geqslant 0 \tag{67}$$

iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that:

$$\{\lambda \in (a,\infty) : \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A), \tag{68}$$

$$\lambda \mapsto \tilde{K}(\lambda)(\lambda - A)^{-1}C, \quad \lambda > a, \ \tilde{K}(\lambda) \neq 0 \text{ is infinitely differentiable and} \tag{69}$$

$$\Big\|\frac{d^k}{d\lambda^k}[\tilde{K}(\lambda)(\lambda - A)^{-1}C]\Big\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}, \quad k \in \mathbb{N}_0, \ \lambda > a, \ \tilde{K}(\lambda) \neq 0. \tag{70}$$

(i.2) *Assume $M > 0$, $\omega \geqslant 0$ and A is densely defined. Then A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that* (68)–(70) *hold.*

(ii.1) *Let $M > 0$ and $\omega \geqslant 0$. Then the operator A is a subgenerator of an exponentially bounded, Θ-convoluted C-cosine function $(C_\Theta(t))_{t\geqslant 0}$ satisfying the condition:*

$$\|C_\Theta(t+h) - C_\Theta(t)\| \leqslant Mhe^{\omega(t+h)}, \quad t \geqslant 0, \ h \geqslant 0 \tag{71}$$

iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that:

$$\{\lambda^2 : \lambda \in (a,\infty), \ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A), \tag{72}$$

$$\lambda \mapsto \lambda\tilde{K}(\lambda)(\lambda^2 - A)^{-1}C, \ \lambda > a, \ \tilde{K}(\lambda) \neq 0 \text{ is infinitely differentiable and} \tag{73}$$

$$\Big\|\frac{d^k}{d\lambda^k}[\lambda\tilde{K}(\lambda)(\lambda^2 - A)^{-1}C]\Big\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}, \ k \in \mathbb{N}_0, \ \ \lambda > a, \ \tilde{K}(\lambda) \neq 0. \tag{74}$$

(ii.2) *Assume $M > 0$, $\omega \geqslant 0$ and A is densely defined. Then A is a subgenerator of an exponentially bounded, K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$ satisfying $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that* (72)–(74) *hold.*

Proof. (i.1) Let (68)–(70) hold and let $a := \max(\omega, \operatorname{abs}(K))$. Assuming $\lambda > a$ and $\tilde{K}(\lambda) \neq 0$, (70) implies that the power series

$$\sum_{k\geqslant 0} \frac{[\tilde{K}(\lambda)(\lambda - A)^{-1}C]^{(k)}(\lambda)}{k!}(z-\lambda)^k,$$

converges for every $z \in \mathbb{C}$ satisfying $|z-\lambda| < \lambda - \omega$. This, in turn, implies that there exists a C^∞-function $\Upsilon : (a,\infty) \to L(E)$ satisfying $\Upsilon(\lambda) = \tilde{K}(\lambda)(\lambda - A)^{-1}C$, $\lambda > a$, $\tilde{K}(\lambda) \neq 0$ and $\left\| \frac{d^k}{d\lambda^k}\Upsilon(\lambda)\right\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}$, $k \in \mathbb{N}_0$, $\lambda > a$. An application of Theorem 1.1.13 gives that there exists a function $S_\Theta : [0,\infty) \to L(E)$ such that (67) holds and that $\Upsilon(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S_\Theta(t)\,dt$, $\lambda > a$. Then it is straightforward to see that $\tilde{\Theta}(\lambda)(\lambda - A)^{-1}C = \int_0^\infty e^{-\lambda t} S_\Theta(t)\,dt$, $\lambda > a$, $\tilde{\Theta}(\lambda) \neq 0$. Now one can proceed as in the proof of Theorem 2.2.1 in order to see that $(S_\Theta(t))_{t\geqslant 0}$ is an exponentially bounded, Θ-convoluted C-semigroup with a subgenerator A (cf. also Remark 2.2.3). Assume conversely that A is a subgenerator of an exponentially bounded, Θ-convoluted C-semigroup $(S_\Theta(t))_{t\geqslant 0}$ which satisfies (67). Arguing as before, one obtains (68) and

$$\lambda(\lambda - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int\limits_0^\infty e^{-\lambda t} S_\Theta(t)x\,dt, \ \ x \in E, \ \operatorname{Re}\lambda > a, \ \tilde{K}(\lambda) \neq 0.$$

This implies (69). To prove (70), fix $x \in E$, $x^* \in E^*$ and put afterwards $f(t) := x^*(S_\Theta(t)x)$, $t \geqslant 0$. Then (67) implies that $f(\cdot)$ is differentiable almost everywhere in $[0,\infty)$ with $|f'(t)| \leqslant C\|x\|\,\|x^*\|e^{\omega t}$ for a.e. $t \geqslant 0$. Moreover, $x^*(\tilde{K}(\lambda)(\lambda - A)^{-1}Cx) = \int_0^\infty e^{-\lambda t} f'(t)\,dt$, $\lambda > a$, $\tilde{K}(\lambda) \neq 0$. Therefore, (70) holds. Using the same arguments as in the proof of [**434**, Theorem 3.4, p. 14], one obtains (i.2). The proofs of (ii.1) and (ii.2) are similar to those of (i.1) and (i.2). $\square$

The next profiling of C-pseudoresolvents follows from the proofs of [**259**, Proposition 2.2] and [**384**, Theorem 1.3]:

PROPOSITION 2.2.6. *Let $M > 0$, let K satisfy* (P1) *and let $\omega \geqslant \max(0, \operatorname{abs}(K))$.*

(i) *Suppose $(S_K(t))_{t\geqslant 0}$ is a strongly continuous operator family and $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$. Put $R_\lambda x := \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} S_K(t)x\,dt$, $x \in E$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$. Then $(\lambda - \mu)R_\lambda R_\mu x = R_\mu Cx - R_\lambda Cx$, $\lambda,\ \mu > \omega$, $\tilde{K}(\lambda)\tilde{K}(\mu) \neq 0$, $x \in E$ iff* (20) *holds for $x \in E$, $t \geqslant 0$ and $s \geqslant 0$.*

(ii) *Suppose $(C_K(t))_{t\geqslant 0}$ is a strongly continuous operator family and $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$. Put $R_{\lambda^2} x := \frac{1}{\lambda\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)x\,dt$, $x \in E$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$. Then $(\lambda^2 - \mu^2)R_{\lambda^2} R_{\mu^2} x = R_{\mu^2} Cx - R_{\lambda^2} Cx$, $\lambda,\ \mu > \omega$, $\tilde{K}(\lambda)\tilde{K}(\mu) \neq 0$, $x \in E$ iff* (24) *holds for $x \in E$, $t \geqslant 0$ and $s \geqslant 0$.*

The following adjoint type theorem is motivated by [**325**, Theorem 4.2]; the proof is only sketched here without giving full details.

THEOREM 2.2.7. (i) *Suppose A is a subgenerator of a* (*local, global exponentially bounded*) *K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, resp. K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, $D(A)$ and $\mathrm{R}(C)$ are dense in E and $\alpha > 0$. Then A^* is a subgenerator of a* (*local, global exponentially bounded*) *$\bigl(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigr)$-convoluted C^*-semigroup $(S^*_{K,\alpha}(t))_{t\in[0,\tau)}$, resp. $\bigl(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigr)$-convoluted C^*-cosine* *function* $(C^*_{K,\alpha}(t))_{t\in[0,\tau)}$ *in* E^*, *where*

$$S_{K,\alpha}(t)x^* := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_K(s)^* x^* ds, \;\; x^* \in E^*, \; t \in [0,\tau), \;\; \textit{resp.}$$

$$C_{K,\alpha}(t)x^* := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} C_K(s)^* x^* ds, \;\; x^* \in E^*, \; t \in [0,\tau).$$

(ii) *Suppose* $D(A)$ *and* $\mathrm{R}(C)$ *are dense in* E, *and* A *is a subgenerator of a* K*-convoluted* C*-semigroup* $(S_K(t))_{t\in[0,\tau)}$, *resp.* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}$. *Then the part of* A^* *in* $\overline{D(A^*)}$ *is a subgenerator of a* K*-convoluted* $C^*_{|\overline{D(A^*)}}$*-semigroup* $\bigl(S^*_K(t)_{|\overline{D(A^*)}}\bigr)_{t\in[0,\tau)}$, *resp.* K*-convoluted* C^**-cosine function* $\bigl(C^*_K(t)_{|\overline{D(A^*)}}\bigr)_{t\in[0,\tau)}$.

(iii) *Suppose* E *is reflexive,* $D(A)$ *and* $\mathrm{R}(C)$ *are dense in* E, *and* A *is a subgenerator of a* K*-convoluted* C*-semigroup* $(S_K(t))_{t\in[0,\tau)}$, *resp.* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}$. *Then* A^* *is a subgenerator of a* (*local, global exponentially bounded*) K*-convoluted* C^**-semigroup* $(S^*_K(t))_{t\in[0,\tau)}$, *resp.* K*-convoluted* C^**-cosine function* $(C^*_K(t))_{t\in[0,\tau)}$ *in* E^*.

(iv) *Suppose* A *is a subgenerator of a* (*local, global exponentially bounded*) (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$, $D(A)$ *and* $\mathrm{R}(C)$ *are dense in* E *and* $\alpha > 0$. *Then* A^* *is a subgenerator of a* (*local, global exponentially bounded*) $\bigl(a, k *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigr)$*-regularized* C^**-resolvent family* $(R^*_\alpha(t))_{t\in[0,\tau)}$, *which is given by*

$$R_\alpha(t)x^* := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} R(s)^* x^* ds, \;\; x^* \in E^*, \; t \in [0,\tau).$$

(v) *Suppose* $D(A)$ *and* $\mathrm{R}(C)$ *are dense in* E, *and* A *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$. *Then the part of* A^* *in* $\overline{D(A^*)}$ *is a subgenerator of an* (a,k)*-regularized* $C^*_{|\overline{D(A^*)}}$*-resolvent family in* E^*.

(vi) *Suppose* E *is reflexive,* $D(A)$ *and* $\mathrm{R}(C)$ *are dense in* E, *and* A *is a subgenerator of a* (*local, global exponentially bounded*) (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$. *Then* A^* *is a subgenerator of a* (*local, global exponentially bounded*) (a,k)*-regularized* C^**-resolvent family* $(R^*(t))_{t\in[0,\tau)}$.

Proof. We will only prove (i)-(iii) provided that A is a subgenerator of a K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Since $\mathrm{R}(C)$ is dense in E, the operator C^* is injective. By the proof of [**325**, Theorem 4.2], we have that $(S^*_{K,\alpha}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family in E^* and

$$A^* \int_0^t S^*_{K,\alpha}(s)x^* ds = S^*_{K,\alpha}(t)x^* - \Bigl(\Theta *_0 \frac{\cdot^{\alpha-1}}{\Gamma(\alpha)}\Bigr)(t)C^*x^*, \;\; x^* \in E^*, \; t \in [0,\tau).$$

The simple computation shows that $S^*_{K,\alpha}(t)A^* \subseteq A^*S^*_{K,\alpha}(t)$ and $S^*_{K,\alpha}(t)C^* = C^*S^*_{K,\alpha}(t)$, $t \in [0,\tau)$, which completes the proof of (i). The proof of (ii) follows exactly in the same way as in the proof of [**325**, Theorem 4.2] while the proof of (iii) follows immediately from (ii) and Proposition 1.1.14(iv). □

It is clear that the notions of K-convoluted C-semigroups and cosine functions, or more generally (a, k)-regularized C-resolvent families, can be considered if E is a sequentially complete locally convex space; with minor exceptions, the results established here continue to hold in this setting ([**241**]). We continue by observing that Wu and Zhang [**429**] have recently introduced a new topological concept for the purpose of researches of semigroups on L^{∞}-type spaces and the L^1-uniqueness of the Fokker-Planck equation (cf. also [**268**, Theorem 2.1, Theorem 2.2]). Let us explain in more detail the importance of such an approach. Let E be a sequentially complete locally convex space. Then one can define on E^* the *topology of uniform convergence on compacts of* E, denoted by $\mathcal{C}(E^*, E)$; more precisely, given a functional $x_0^* \in E^*$, the basis of open neighborhoods of x_0^* w.r.t. $\mathcal{C}(E^*, E)$ is given by $N(x_0^* : \mathbf{K}, \varepsilon) := \{x^* \in E^* : \sup_{x \in \mathbf{K}} |\langle x^* - x_0^*, x\rangle| < \varepsilon\}$, where $\mathbf{K}$ runs over all compacts of E and $\varepsilon > 0$. Then $(E^*, \mathcal{C}(E^*, E))$ is locally convex and complete. On the other hand, E^* can be equipped with the Hausdorff locally convex topology defined by the system $(|\cdot|_B)_{B \in \mathcal{B}}$ of seminorms on E^*, where $\mathcal{B}$ denotes the family of all bounded subsets of E and $|x^*|_B := \sup_{x \in B} |\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. In this case, E^* is sequentially complete provided that E is barreled. Furthermore, one can simply prove that the topology $\mathcal{C}(E^*, E)$ is finer than the topology induced by the calibration $(|\cdot|_B)_{B \in \mathcal{B}}$. With the notion explained in [**241**], we have the following theorem which is not so easily comparable to Theorem 2.2.7.

THEOREM 2.2.8. *Suppose $D(A)$ and $\mathrm{R}(C)$ are dense in E, and A is a subgenerator of a locally equicontinuous (a, k)-regularized C-resolvent family $(R(t))_{t \in [0,\tau)}$. Then A^* is a subgenerator of a locally equicontinuous (a, k)-regularized C^*-resolvent family $(R(t)^*)_{t \in [0,\tau)}$ in $(E^*, \mathcal{C}(E^*, E))$. Furthermore, if $\tau = \infty$ and $(R(t))_{t \geqslant 0}$ is exponentially equicontinuous, then $(R(t)^*)_{t \geqslant 0}$ is also exponentially equicontinuous.*

2.3. Abstract Cauchy problems

We are turning back to our standing hypothesis in which E is a Banach space.

Convoluted C-semigroups and functions are important tools in the study of the following abstract Cauchy problems:

$$(\Theta C) : \begin{cases} u \in C([0, \tau) : [D(A)]) \cap C^1([0, \tau) : E), \\ u'(t) = Au(t) + \Theta(t)Cx, \ t \in [0, \tau), \\ u(0) = 0, \end{cases}$$

and

$$(ACP_2)_{\Theta} : \begin{cases} u \in C([0, \tau) : [D(A)]) \cap C^2([0, \tau) : E), \\ u''(t) = Au(t) + \Theta(t)Cx + \int\limits_0^t \Theta(s)Cyds, \ t \in [0, \tau), \\ u(0) = 0, \ u'(0) = 0. \end{cases}$$

It is said that (ΘC), resp. $(ACP_2)_{\Theta}$, is *well-posed* if for every x, resp. x, $y \in E$, there exists a unique solution of (ΘC), resp. $(ACP_2)_{\Theta}$. The existence of a unique solution of (ΘC), resp. $(ACP_2)_\Theta$, is closely connected with the existence of a unique K-*convoluted mild solution* of the problem (ACP_1), resp. (ACP_2), where

$$(ACP_1): \begin{cases} u \in C([0,\tau):[D(A)]) \cap C^1([0,\tau):E), \\ u'(t) = Au(t),\ t \in [0,\tau), \\ u(0) = x, \end{cases}$$

and

$$(ACP_2): \begin{cases} u \in C([0,\tau):[D(A)]) \cap C^2([0,\tau):E), \\ u''(t) = Au(t),\ t \in [0,\tau), \\ u(0) = x,\ u'(0) = y. \end{cases}$$

The notion of mild solutions of (ACP_2) was introduced by Wang and Huang [**420**] in case $\tau = \infty$ and $K(t) = \frac{t^{n-1}}{(n-1)!}$, $t \geqslant 0,\ n \in \mathbb{N}$.

The subsequent assertions follow from the use of arguments given in the proofs of [**5**, Proposition 2.3], [**275**, Proposition 2.4, Theorem 2.5] and [**418**, Theorem 2.4] (cf. also [**230**, Propositions 5.3, 5.4 and 5.5]).

PROPOSITION 2.3.1. *Suppose* $0 < \tau \leqslant \infty$ *and* (ΘC) *is well-posed. Then there exists a unique strongly continuous operator family* $(S_K(t))_{t\in[0,\tau)}$ *such that, for every* $x \in E$, $\int_0^t S(s)x\,ds \in D(A)$ *and* $A\int_0^t S(s)x\,ds = S(t)x - \Theta(t)Cx$, $t \in [0,\tau)$. *Furthermore,* $S_K(t)S_K(s) = S_K(s)S_K(t)$, $0 \leqslant t,\ s < \tau$, $(S_K(t))_{t\in[0,\tau)}$ *is a local* K-*convoluted* C-*semigroup with a subgenerator* A *and the integral generator* $C^{-1}AC$.

PROPOSITION 2.3.2. *Suppose* $0 < \tau \leqslant \infty$, K *is a kernel and* A *is a subgenerator of a* K-*convoluted* C-*semigroup* $(S_K(t))_{t\in[0,\tau)}$. *Then* (ΘC) *is well-posed.*

The next proposition can be proved by using the closedness of A and the functional equality of K-convoluted C-semigroups (cosine functions).

PROPOSITION 2.3.3. *Let* A *be a subgenerator of a* (*local*) K-*convoluted* C-*semigroup* $(S_K(t))_{t\in[0,\tau)}$, *resp.* K-*convoluted* C-*cosine function* $(C_K(t))_{t\in[0,\tau)}$.

(i) *Suppose* $k \in \mathbb{N}$, $x \in D(A^k)$ *and* $K \in C^{k-1}([0,\tau))$. *Then*

$$\frac{d^k}{dt^k}S_K(t)x = S_K(t)A^kx + \sum_{i=0}^{k-1} K^{(i)}(t)CA^{k-1-i}x,\ \ t \in [0,\tau).$$

(ii) *Suppose* $k \in \mathbb{N}$, $x \in D(A^k)$ *and* $K \in C^{2k-1}([0,\tau))$. *Then*

$$\frac{d^{2k}}{dt^{2k}}C_K(t)x = C_K(t)A^kx + \sum_{i=0}^{k-1} K^{(2i+1)}(t)CA^{k-1-i}x,\ \ t \in [0,\tau).$$

(iii) *Suppose* $k \in \mathbb{N}$, $x \in D(A^k)$ *and* $K \in C^{2k-2}([0,\tau))$. *Then*

$$\frac{d^{2k-1}}{dt^{2k-1}}C_K(t)x = \int_0^t C_K(s)A^kx\,ds + \sum_{i=0}^{k-1} K^{(2i)}(t)CA^{k-1-i}x,\ \ t \in [0,\tau).$$

PROPOSITION 2.3.4. *Suppose $k \in \mathbb{N}$, $K \in C^k([0,\tau))$ and (ΘC) is well-posed. Then, for every $x \in D(A^{k+1})$, there exists a unique solution of the problem (ΘC_k), where:*

$$(\Theta C_k) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \frac{d^k}{dt^k} K(t)Cx, \ t \in [0,\tau), \\ u(0) = \sum_{i=0}^{k-1} K^{(i)}(0)A^{k-1-i}Cx. \end{cases}$$

PROOF. The prescribed assumptions imply that A is a subgenerator of a K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Now one can simply verify that

$$u(t) := \int_0^t S_K(s)A^{k+1}x\,ds + \sum_{i=0}^{k} \Theta^{(i)}(t)A^{k-i}Cx, \quad t \in [0,\tau), \ x \in D(A^{k+1}),$$

is a solution of (ΘC_k). The uniqueness of solutions of (ΘC_k) follows from the well-posedness of the problem (ΘC) at $x = 0$. □

Suppose $H \in L^1_{\text{loc}}([0,\tau))$, $H *_0 K \neq 0$ in $L^1_{\text{loc}}([0,\tau))$ and (ΘC) is well-posed. Then it can be simply checked that the problem $(H *_0 \Theta, C)$ is also well-posed.

PROPOSITION 2.3.5. *Suppose $k \in \mathbb{N}$, $k \geqslant 2$, $K \in C^k([0,\tau))$, $K^{(i)}(0) = 0$, $0 \leqslant i \leqslant k-2$, A is a closed linear operator, $\lambda_0 \in \rho(A)$ and the problem (ΘC_k) has a unique solution for every $x \in D(A^{k+1})$. Then (ΘC) is well-posed.*

PROOF. Let $y \in D(A^{k+1})$ and $z = Cy$. Define $u_1(t) := (\lambda_0 - A)\int_0^t u_y(s)\,ds$, $t \in [0,\tau)$, where $u_y(\cdot)$ is a solution of (ΘC_k) with x replaced by y there. Direct computation shows that $u_1(\cdot)$ is a solution of the problem

$$(\Theta C_{k-1}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \frac{d^{k-1}}{dt^{k-1}} K(t)x, \ t \in [0,\tau), \\ u(0) = 0, \end{cases}$$

where $x = (\lambda_0 - A)z$. Therefore, the problem (ΘC_{k-1}) has a solution for all $x \in (\lambda_0 - A)CD(A^{k+1})$. Similarly, the problem

$$(\Theta C_{k-2}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \frac{d^{k-2}}{dt^{k-2}} K(t)x, \ t \in [0,\tau), \\ u(0) = 0, \end{cases}$$

has a solution for all $x \in (\lambda_0 - A)^2 CD(A^{k+1})$ and we obtain inductively the existence of a solution $u_{k+1}(\cdot)$ of the problem

$$(\Theta C_{-1}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \Theta(t)x, \ t \in [0,\tau), \\ u(0) = 0, \end{cases}$$

for all $x \in (\lambda_0 - A)^{k+1}CD(A^{k+1})$. Since $CA \subseteq AC$ and $\lambda_0 \in \rho(A)$, we have $\mathrm{R}(C) \subseteq (\lambda_0 - A)^{k+1}CD(A^{k+1})$ and this implies that (ΘC) has a solution for all $x \in E$; the uniqueness of solutions of (ΘC) follows from the well-posedness of the problem (ΘC_k) at $x = 0$. □

The next statements can be shown following the lines of the proofs of Proposition 2.3.4 and Proposition 2.3.5.

THEOREM 2.3.6. *Suppose $k \in \mathbb{N}$, $K \in C^{k-1}([0,\tau))$, A is a closed linear operator, $\lambda_0 \in \rho(A)$ and, in the case $k \geqslant 2$, $K^{(i)}(0) = 0$, $0 \leqslant i \leqslant k-2$. Then the following assertions are equivalent:*

(i) (ΘC) *is well-posed.* (ii) $(\Theta^{(k)}, R(\lambda_0 : A)^k C)$ *is well-posed.*

COROLLARY 2.3.7. *Suppose $k \in \mathbb{N}$, A is a closed linear operator and $\lambda_0 \in \rho(A)$. Then the following assertions are equivalent:*

(i) *A is a subgenerator of a local k-times integrated C-semigroup on $[0,\tau)$.*

(ii) *A is a subgenerator of a local $(R(\lambda_0 : A)^k C)$-regularized semigroup on $[0,\tau)$.*

Our objective in the sequel of this subsection is to prove the analogues of Proposition 2.3.4-Corollary 2.3.7 for K-convoluted C-cosine functions.

PROPOSITION 2.3.8. *Suppose $(ACP_2)_\Theta$ is well-posed and $k \in \mathbb{N}$. Then A $(C^{-1}AC)$ is a subgenerator (the integral generator) of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ and the following holds:*

(i) *If $x \in D(A^k)$, $y \in D(A^k)$ and $K \in C^{2k-1}([0,\tau))$, then the abstract Cauchy problem:*

$$(ACP_2)_{\Theta,2k-1} : \begin{cases} u \in C^2([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + \frac{d^{2k-1}}{dt^{2k-1}} K(t)Cx + \frac{d^{2k-2}}{dt^{2k-2}} K(t)Cy,\ t \in [0,\tau), \\ u(0) = \sum_{i=0}^{k-2} K^{(2i+1)}(0)CA^{k-2-i}x + \sum_{i=0}^{k-2} K^{(2i)}(0)CA^{k-2-i}y, \\ u'(0) = \sum_{i=0}^{k-1} K^{(2i)}(0)CA^{k-1-i}x + \sum_{i=0}^{k-2} K^{(2i+1)}(0)CA^{k-2-i}y, \end{cases}$$

has a unique solution given by:

$$u(t) = C_K(t)A^{k-1}x + \sum_{i=0}^{k-2} K^{(2i+1)}(t)CA^{k-2-i}x + \int_0^t C_K(s)A^{k-1}y\,ds + \sum_{i=0}^{k-2} K^{(2i)}(t)CA^{k-2-i}y,\ \ t \in [0,\tau).$$

(ii) *If $x \in D(A^{k+1})$, $y \in D(A^k)$ and $K \in C^{2k}([0,\tau))$, then the abstract Cauchy problem:*

$$(ACP_2)_{\Theta,2k} : \begin{cases} u \in C^2([0,\tau) : E) \cap C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + \frac{d^{2k}}{dt^{2k}} K(t)Cx + \frac{d^{2k-1}}{dt^{2k-1}} K(t)Cy,\ t \in [0,\tau), \\ u(0) = \sum_{i=0}^{k-1} K^{(2i)}(0)CA^{k-1-i}x + \sum_{i=0}^{k-1} K^{(2i+1)}(0)CA^{k-1-i}y, \\ u'(0) = \sum_{i=0}^{k-1} K^{(2i+1)}(0)CA^{k-1-i}x + \sum_{i=0}^{k-1} K^{(2i)}(0)CA^{k-1-i}y, \end{cases}$$

has a unique solution given by:

$$u(t) = \int_0^t C_K(s)A^k x\,ds + \sum_{i=0}^{k-1} K^{(2i)}(t)CA^{k-1-i}x$$

$$+ C_K(t)A^{k-1}y + \sum_{i=0}^{k-2} K^{(2i+1)}(t)CA^{k-2-i}y,\ t \in [0,\tau).$$

Proof. We will only outline the proof of (i). By Proposition 2.1.16 and Proposition 2.3.15 given below, A ($C^{-1}AC$) is a subgenerator (the integral generator) of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$. Clearly, $u \in C^2([0,\tau) : E) \cap C([0,\tau) : [D(A)])$, and by Proposition 2.3.3, one gets:

$$u'(t) = \int_0^t C_K(s)A^k x\, ds + K(t)CA^{k-1}x + \sum_{i=0}^{k-2} K^{(2i+2)}(t)CA^{k-2-i}x$$
$$+ C_K(t)A^{k-1}y + \sum_{i=0}^{k-2} K^{(2i+1)}(t)CA^{k-2-i}y,$$
$$u''(t) = C_K(t)A^k x + K'(t)CA^{k-1}x + \sum_{i=0}^{k-1} K^{(2i+1)}(t)CA^{k-1-i}x$$
$$+ \int_0^t C_K(s)A^k y\, ds + K(t)CA^{k-1}y + \sum_{i=0}^{k-2} K^{(2i+2)}(t)CA^{k-2-i}y,$$
$$u''(t) - Au(t) = \sum_{i=0}^{k-1} K^{(2i+1)}(t)CA^{k-1-i}x - \sum_{i=0}^{k-2} K^{(2i+1)}(t)CA^{k-1-i}x$$
$$+ \sum_{i=0}^{k-2} K^{(2i+2)}(t)CA^{k-2-i}y - \sum_{i=0}^{k-2} K^{(2i)}(t)CA^{k-1-i}y$$
$$= K^{(2k+1)}(t)Cx + K^{(2i-2)}(t)Cy,\ t \in [0,\tau).$$

The uniqueness of solutions of $(ACP_2)_\Theta$ at $x = y = 0$ completes the proof. □

In the context of integrated C-cosine functions considered in Banach spaces, the previous proposition extends [**434**, Theorem 6.10, p. 40].

Theorem 2.3.9. (i) *Suppose $k \in \mathbb{N}$, $K \in C^{2k-1}([0,\tau))$, $\lambda_0 \in \rho(A)$, the abstract Cauchy problem $(ACP_2)_{\Theta,2k-1}$ has a unique solution for all x, $y \in D(A^k)$ and, in the case $k \geqslant 2$, $K^{(i)}(0) = 0$, $0 \leqslant i \leqslant 2k-4$. Then $(ACP_2)_\Theta$ is well-posed.*

(ii) *Suppose $k \in \mathbb{N}$, $K \in C^{2k}([0,\tau))$, $\lambda_0 \in \rho(A)$, the abstract Cauchy problem $(ACP_2)_{\Theta,2k}$ has a unique solution for all $x \in D(A^{k+1})$ and $y \in D(A^k)$, and, in the case $k \geqslant 2$, $K^{(i)}(0) = 0$, $0 \leqslant i \leqslant 2k-3$. Then $(ACP_2)_\Theta$ is well-posed.*

Proof. We will only prove (i). Suppose $x \in D(A^k)$, $y \in D(A^k)$ and designate by $u_{x,y}(\cdot)$ the unique solution of the problem $(ACP_2)_{\Theta,2k-1}$. Then the injectiveness of the operator $\lambda_0 - A$ and the supposition $K^{(i)}(0) = 0$, $0 \leqslant i \leqslant 2k-4$ (provided that $k \geqslant 2$) easily imply that the function $u_1(t) := (\lambda_0 - A)\int_0^t (t-s)u_{x,y}(s)ds$,

$t \in [0, \tau)$ is a solution of the problem

$$\begin{cases} u \in C^2([0, \tau) : E) \cap C([0, \tau) : [D(A)]), \\ u''(t) = Au(t) + \frac{d^{2k-3}}{dt^{2k-3}} K(t)(\lambda_0 - A)Cx + \frac{d^{2k-4}}{dt^{2k-4}} K(t)(\lambda_0 - A)Cy, \ t \in [0, \tau), \\ u(0) = u'(0) = 0. \end{cases}$$

Arguing as in the proof Theorem 2.3.5, we obtain that the problem

$$\begin{cases} u \in C^2([0, \tau) : E) \cap C([0, \tau) : [D(A)]), \\ u''(t) = Au(t) + \Theta(t)x + \int_0^t \Theta(s)y\, ds, \ t \in [0, \tau), \\ u(0) = u'(0) = 0, \end{cases}$$

has a unique solution for all $x,\ y \in (\lambda_0 - A)^k CD(A^k)$. Since $\lambda_0 \in \rho(A)$ and $CA \subseteq AC$, we immediately obtain that $(ACP_2)_\Theta$ is well-posed. □

REMARK 2.3.10. (i) It can be simply justified that in the assertion (i) of Theorem 2.3.9 one can assume the well-posedness of the problem $(ACP_2)_{\Theta,2k-1}$ only for $x \in D(A^k)$ and $y = 0$; analogically, in the assertion (ii) one can only assume that $x = 0$. In such a way, we obtain a generalization of [**434**, Theorem 6.9, p. 40].

(ii) In the formulations of Theorem 2.3.5 and Theorem 2.3.9, we need not to restrict ourselves to the case $\rho(A) \neq \emptyset$. The following changes must be done to cover the newly arisen situation:

(ii.1) Theorem 2.3.5: $\lambda_0 \in \rho_C(A)$,

$$\mathrm{R}(C) \subseteq \big\{(\lambda_0 - A)^{k+1}x : x \in D(A^{k+1})\big\} = D\big((\lambda_0 - A)^{-(k+1)}\big)$$

and, for every $x \in (\lambda_0 - A)^{-(k+1)}\, \mathrm{R}(C)$, the problem (ΘC_k) has a unique solution.

(ii.2) Theorem 2.3.9(i): $\lambda_0 \in \rho_C(A)$, $\mathrm{R}(C) \subseteq D((\lambda_0 - A)^{-k})$ and the problem $(ACP_2)_{\Theta,2k-1}$ has a unique solution for every $x \in (\lambda_0 - A)^{-k}\, \mathrm{R}(C)$ and $y = 0$.

(ii.3) Theorem 2.3.9(ii): $\lambda_0 \in \rho_C(A)$, $\mathrm{R}(C) \subseteq D((\lambda_0 - A)^{-k})$ and the problem $(ACP_2)_{\Theta,2k}$ has a unique solution for every $y \in (\lambda_0 - A)^{-k}\, \mathrm{R}(C)$ and $x = 0$. At the end of this remark, let us point out that the closed graph theorem and an induction argument imply $(\lambda_0 - A)^{-k}C \in L(E)$.

Taking into account preceding remark and the method described in the proofs of Theorem 2.3.5, Theorem 2.3.9 and [**261**, Theorem 2.1] (cf. also Proposition 2.3.12 given below), the next corollary follows instantly.

COROLLARY 2.3.11. (i) *Suppose* $k \in \mathbb{N}$, $\lambda_0 \in \rho_C(A)$, $\mathrm{R}(C) \subseteq D((\lambda_0 - A)^{-(k+1)})$ *and the abstract Cauchy problem* (ACP_1) *has a unique solution for every* $x \in (\lambda_0 - A)^{-(k+1)}\, \mathrm{R}(C)$. *Then* A *is a subgenerator of a (local)* k*-times integrated* C*-semigroup* $(S_k(t))_{t \in [0,\tau)}$, *and moreover,* A *is a subgenerator of a (local)* $\big((\lambda_0 - A)^{-k}C\big)$*-regularized semigroup* $(S(t))_{t \in [0,\tau)}$ *which satisfies*

$$S_k(t)x = (\lambda_0 - A)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} S(s)x\, ds, \quad x \in E, \ t \in [0, \tau). \tag{75}$$

The formula (75) *is also applicable to semigroups appearing in the formulation of Corollary* 2.3.7.

(ii) *Suppose* $k \in \mathbb{N}$, $\lambda_0 \in \rho_C(A)$, $\mathrm{R}(C) \subseteq D\big((\lambda_0 - A)^{-(k+1)}\big)$ *and the abstract Cauchy problem* (ACP_2) *has a unique solution for every* $x \in (\lambda_0 - A)^{-(k+1)}\,\mathrm{R}(C)$ *and* $y = 0$. *Then* A *is a subgenerator of a* (*local*) $(2k)$*-times integrated* C*-cosine function* $(S_k(t))_{t\in[0,\tau)}$.

(iii) *Let* $k \in \mathbb{N}$, $\lambda_0 \in \rho_C(A)$, $\mathrm{R}(C) \subseteq D((\lambda_0 - A)^{-(k+1)})$ *and the abstract Cauchy problem* (ACP_2) *have a unique solution for every* $y \in (\lambda_0 - A)^{-(k+1)}\,\mathrm{R}(C)$ *and* $x = 0$. *Then* A *is a subgenerator of a* (*local*) $(2k+1)$*-times integrated* C*-cosine function* $(C_k(t))_{t\in[0,\tau)}$.

A careful inspection of the proof of [**420**, Theorem 2.1] implies the following.

PROPOSITION 2.3.12. *Suppose* A *is a closed operator,* $k \in \mathbb{N}_0$, $\lambda_0^2 \in \rho_C(A)$ *and* $\mathrm{R}(C) \subseteq D\big((\lambda_0^2 - A)^{-(k+1)}\big)$. *Then* A *is a subgenerator of a* (*local, global exponentially bounded*) $(2k)$*-times* (*resp.* $(2k+1)$*-times*) *integrated* C*-cosine function* $(C_{2k}(t))_{t\in[0,\tau)}$ (*resp.* $(C_{2k+1}(t))_{t\in[0,\tau)}$) *iff* A *is a subgenerator of a* (*local, global exponentially bounded*) $\big((\lambda_0^2 - A)^{-k}C\big)$*-regularized cosine* (*resp.* $\big((\lambda_0^2 - A)^{-(k+1)}C\big)$*-regularized cosine function*) $(C_0(t))_{t\in[0,\tau)}$, *and moreover, the following formulae hold:*

$$C_{2k}(t)x = (\lambda_0^2 - A)^k \int_0^t \frac{(t-s)^{2k-1}}{(2k-1)!} C_0(s)x\,ds,\ \ t \in [0,\tau),\ x \in E,$$

$$\begin{aligned} C_0(t)x &= \bigg\{[(-1)^k \sum_{i=1}^{k} \binom{k}{i} \lambda_0^{2i} (P_{i-1}h_{\lambda_0}) *_0 (P_{i-1}h_{-\lambda_0})] *_0 C_{2k}\bigg\}(t)x \\ &\quad + (-1)^k C_{2k}(t)x + \sum_{i=1}^{k} (-1)^{k-i} \frac{d}{dt}\big[(P_{k-i}h_{\lambda_0}) *_0 (P_{k-i}h_{-\lambda_0})\big](t)(\lambda_0^2 - A)^{-k}Cx, \end{aligned}$$

for any $t \in [0,\tau)$ *and* $x \in E$, *where* $P_i(t) = \frac{t^i}{i!}$, $t \in [0,\tau)$, $0 \leqslant i \leqslant k$ *and* $h_{\pm\lambda_0}(t) = e^{\pm\lambda_0 t}$, $t \in [0,\tau)$.

The following proposition is an analogue of Proposition 2.3.12 for integrated C-semigroups.

PROPOSITION 2.3.13. *Suppose* $k \in \mathbb{N}$, $\lambda_0 \in \rho_C(A)$ *and* $\mathrm{R}(C) \subseteq D\big((\lambda_0 - A)^{-k}\big)$. *Then* A *is a subgenerator of a* (*local, global exponentially bounded*) k*-times integrated* C*-semigroup* $(S_k(t))_{t\in[0,\tau)}$ *iff* A *is a subgenerator of a* (*local, global exponentially bounded*) $\big((\lambda_0 - A)^{-k}C\big)$*-regularized semigroup* $(S_0(t))_{t\in[0,\tau)}$. *Furthermore, the following holds:*

$$\begin{aligned} (76)\quad S_0(t)x = (-1)^k \bigg[S_k(t)x &+ \sum_{i=1}^{k} \binom{k}{i} \lambda_0^i \int_0^t e^{\lambda_0(t-s)} \frac{(t-s)^{i-1}}{(i-1)!} S_k(s)x\,ds \\ &+ \sum_{i=0}^{k-1} \frac{e^{\lambda_0 t} t^{k-1-i}}{(k-1-i)!} (A - \lambda_0)^{-(i+1)} Cx \bigg],\ t \in [0,\tau),\ x \in E \end{aligned}$$

$$S_k(t)x = (\lambda_0 - A)^k \int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!} S_0(s)x\,ds, \quad t \in [0,\tau),\ x \in E.$$

Proof. Suppose that A is a subgenerator of a (local) k-times integrated C-semigroup $(S_k(t))_{t\in[0,\tau)}$. Set $A_{\lambda_0} := A - \lambda_0 I$. Then a rescaling result for (local) integrated C-semigroups (cf. [**5**, Lemma 3.2], the proof of [**216**, Theorem 4.9] and Subsection 2.1.5) implies that A_{λ_0} is a subgenerator of a (local) k-times integrated C-semigroup $(S_{k,\lambda_0}(t))_{t\in[0,\tau)}$, where, for every $t \in [0,\tau)$ and $x \in E$:

$$S_{k,\lambda_0}(t)x = e^{-\lambda_0 t} S_k(t)x + \int\limits_0^t e^{-\lambda_0 s} \sum_{i=1}^{k} \binom{k}{i} \lambda_0^i \frac{(t-s)^{i-1}}{(i-1)!} S_k(s)x\,ds.$$

Define
(77)

$$S_{0,\lambda_0}(t)x := (-1)^k \Bigg[S_{k,\lambda_0}(t)x + \sum_{i=0}^{k-1} \frac{t^{k-i-1}}{(k-i-1)!} A_{\lambda_0}^{-(i+1)} Cx \Bigg], \quad t \in [0,\tau),\ x \in E.$$

It can be simply verified that A_{λ_0} is a subgenerator of a (local) $\big((\lambda_0 - A)^{-k}C\big)$-regularized semigroup $(S_{0,\lambda_0}(t))_{t\in[0,\tau)}$. This clearly implies that A is a subgenerator of a (local) $\big((\lambda_0 - A)^{-k}C\big)$-regularized semigroup $(S_0(t) := e^{\lambda_0 t} S_{0,\lambda_0}(t))_{t\in[0,\tau)}$ and that (76) holds. The converse statement follows from the formula (77) and an easy computation. $\square$

Definition 2.3.14. It is said that a function $v \in C([0,\tau) : E)$ is a *K-convoluted mild solution* of (ACP_1), resp. (ACP_2), at $x \in E$, resp. $(x,y) \in E \times E$, if, for every $t \in [0,\tau)$, $\int_0^t v(s)\,ds \in D(A)$, resp. $\int_0^t (t-s)v(s)\,ds \in D(A)$, and $A\int_0^t v(s)\,ds = v(t) - \Theta(t)x$, $t \in [0,\tau)$, resp. $A\int_0^t (t-s)v(s)\,ds = v(t) - \Theta(t)x - \int_0^t \Theta(s)y\,ds$, $t \in [0,\tau)$.

Let $C = I$. Then it is clear that $u \in C^1([0,\tau) : E) \cap C([0,\tau) : [D(A)])$, resp. $u \in C^2([0,\tau) : E) \cap C([0,\tau) : [D(A)])$, is a (unique) solution of (ΘC), resp. $(ACP_2)_\Theta$, on $[0,\tau)$ iff $v = u' \in C([0,\tau) : E)$, resp. $v = u'' \in C([0,\tau) : E)$, is a (unique) K-convoluted mild solution of (ACP_1), resp. (ACP_2) on $[0,\tau)$. Let A be a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, $0 < \tau \leqslant \infty$ and $x,\ y \in E$. Then $v(t) = C_K(t)x + \int_0^t C_K(s)y\,ds$, $t \in [0,\tau)$ is a K-convoluted mild solution of (ACP_2) at (Cx, Cy) and $u(t) = \int_0^t (t-s)v(s)\,ds$, $t \in [0,\tau)$ is a solution of $(ACP_2)_\Theta$. If K is a kernel, then the function $v(\cdot)$ is a unique K-convoluted mild solution of (ACP_2) and $u(\cdot)$ is a unique solution of $(ACP_2)_\Theta$; see [**230**, Proposition 4.2] and the proof of [**420**, Theorem 1.5]. Without any substantial changes, one obtains the corresponding statements for K-convoluted C-semigroups. Now we state:

Proposition 2.3.15. *Assume that for each $x \in E$ there exists a unique K-convoluted mild solution of (ACP_2) at $(Cx, 0)$, $0 < \tau \leqslant \infty$. Then A is a subgenerator of a K-convoluted C-cosine function on $[0,\tau)$.*

PROOF. Let $t \in [0, \tau)$ and $x \in E$. Define $C_K(t)x := v(t)$, where $v(\cdot)$ is a unique K-convoluted mild solution of (ACP_2) at $(Cx, 0)$. The uniqueness of mild solutions implies that $(C_K(t))_{t\in[0,\tau)}$ is a strongly continuous family of linear operators satisfying (iii) of Definition 2.1.2. The proof of (i) and (ii) of Definition 2.1.2 follows analogously as in the proof of [**420**, Theorem 1.5]. For the sake of completeness, we will prove (i). Fix an $x \in D(A)$ and define

$$\overline{C}_K(t)x := \int\limits_0^t (t-s)C_K(s)Ax\,ds + \Theta(t)Cx,\ t \in [0, \tau).$$

Clearly, the mapping $t \mapsto \overline{C}_K(t)x$ belongs to $C([0, \tau) : E)$ and, for every $t \in [0, \tau)$,

$$A\int\limits_0^t (t-s)\overline{C}_K(s)x\,ds = A\int\limits_0^t (t-s)\left[\int\limits_0^s (s-r)C_K(r)Ax\,dr + \Theta(s)Cx\right]ds$$

$$= \int\limits_0^t (t-s)A\int\limits_0^s (s-r)C_K(r)Ax\,dr\,ds + \int\limits_0^t (t-s)\Theta(s)ACx\,ds$$

$$= \int\limits_0^t (t-s)[C_K(s)Ax - \Theta(s)CAx]\,ds + \int\limits_0^t (t-s)\Theta(s)ACx\,ds = \overline{C}_K(t)x - \Theta(t)Cx.$$

Using again the uniqueness of K-convoluted mild solutions, one yields $\overline{C}_K(t)x = C_K(t)x$, $t \in [0, \tau)$, i.e., $\int_0^t (t-s)C_K(s)Ax\,ds = A\int_0^t (t-s)C_K(s)x\,ds$ for all $t \in [0, \tau)$. Differentiate the last equality twice with respect to t to obtain $C_K(t)x \in D(A)$ and $AC_K(t)x = C_K(t)Ax$, $t \in [0, \tau)$. It remains to be shown that $C_K(t)$ is a bounded operator for all $t \in [0, \tau)$. To this end, we will slightly modify the proof of [**5**, Proposition 2.3]. Consider the mapping $\Phi : E \to C([0, \tau) : [D(A)])$ given by

$$\Phi(x)(t) := \int\limits_0^t (t-s)C_K(s)x\,ds,\ t \in [0, \tau),\ x \in E,$$

where $C([0, \tau) : [D(A)])$ is a Fréchet space equipped with the sequence of seminorms $(p_n)_n$, where

$$p_n(v) := \sup_{t\in[0,\tau-\frac{1}{n}]} \|v(t)\|_{[D(A)]},\ v \in C([0, \tau) : [D(A)]),\ \text{if } \tau \in (0, \infty),\ \text{resp.}$$

$$p_n(v) := \sup_{t\in[0,n]} \|v(t)\|_{[D(A)]},\ v \in C([0, \tau) : [D(A)]),\ \text{if } \tau = \infty.$$

It can be easily seen that Φ is well defined and that Φ is linear. Let us show that Φ has a closed graph. Without loss of generality, one can assume that $\tau \in \mathbb{R}$. Suppose $x_n \to x$, and $\Phi(x_n) \to f$, $n \to \infty$. Choose an integer $k \in \mathbb{N}$ with $k > \frac{1}{\tau}$. Then

$$\sup_{t\in[0,\tau-\frac{1}{k}]} \left\| \int\limits_0^t (t-s)C_K(s)x_n ds - f(t) \right\|_{[D(A)]} \to 0,\ \ n \to \infty.$$

Hence,

$$Af(t) = \lim_{n\to\infty} A\int_0^t (t-s)C_K(s)x_n ds = \lim_{n\to\infty}[C_K(t)x_n - \Theta(t)Cx_n], \quad t \in [0,\tau),$$

and $\lim_{n\to\infty} C_K(t)x_n = Af(t) + \Theta(t)Cx$, $t \in [0,\tau)$. Using the dominated convergence theorem, we have

$$f(t) = \lim_{n\to\infty}\int_0^t (t-s)C_K(s)x_n ds = \int_0^t (t-s)[Af(s) + \Theta(s)Cx]\, ds, \ t \in [0,\tau).$$

Therefore, $f(0) = f'(0) = 0$, $f \in C^2([0,\tau) : E)$, $Af(t) = f''(t) - \Theta(t)Cx$, $t \in [0,\tau)$ and $A\int_0^t (t-s)v(s)\, ds = v(t) - \Theta(t)Cx$, $t \in [0,\tau)$, where $v = f''$. Hence, $v(t) = C_K(t)x$, $t \in [0,\tau)$, $f = \Phi(x)$ and, for all sufficiently large $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$\left\| A\int_0^t (t-s)C_K(s)x\, ds \right\| \leqslant c_n\|x\|, \quad x \in E, \ t \in [0,\tau - 1/n).$$

Since $A\int_0^t (t-s)C_K(s)x\, ds = C_K(t)x - \Theta(t)Cx$, $x \in E$, $t \in [0,\tau)$, one gets $C_K(t) \in L(E)$, $t \in [0,\tau)$. □

Corollary 2.3.16. *Suppose K is a kernel and $0 < \tau \leqslant \infty$. Then the following statements are equivalent.*

(i) $(ACP_2)_\Theta$ *is well-posed.*
(ii) *For every $x \in E$, there exists a unique K-convoluted mild solution of (ACP_2) at $(Cx, 0)$.*
(iii) *For every x, $y \in E$, there exists a unique K-convoluted mild solution of (ACP_2) at (Cx, Cy).*
(iv) *A is a subgenerator of a K-convoluted C-cosine function on $[0,\tau)$.*

Corollary 2.3.17. *Suppose K is a kernel and $0 < \tau \leqslant \infty$. Then the following statements are equivalent.*

(i) (ΘC) *is well-posed.*
(ii) *For every $x \in E$, there exists a unique K-convoluted mild solution of (ACP_1) at Cx.*
(iii) *A is a subgenerator of a K-convoluted C-semigroup on $[0,\tau)$.*

Suppose $0 < \tau \leqslant \infty$, (ΘC) is well-posed and define

$$L_\gamma(\lambda) := \int_0^\gamma e^{-\lambda s}S(s)\, ds, \ \gamma \in [0,\tau), \ \lambda \in [0,\infty),$$

where $S_K(\cdot)$ is given by Proposition 2.3.1. We summarize the basic properties of the operators $L_\gamma(\lambda)$ in the following proposition whose proof is analogous to that of [**275**, Proposition 5.1].

PROPOSITION 2.3.18. *Let $x \in E$ and $\gamma \in [0, \tau)$. Then the following holds:*

(i) *The function $\lambda \to L_\gamma(\lambda)x$ belongs to $C^\infty([0,\infty) : E)$ and there exists $M_\gamma > 0$ such that*

$$\left\| \frac{\lambda^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} L_\gamma(\lambda) \right\| \leqslant M_\gamma, \quad \lambda \geqslant 0, \ n \in \mathbb{N}.$$

(ii) *$L_\gamma(\lambda)$ commutes with C and A for all $\lambda \geqslant 0$.*

(iii) *$(\lambda - A)L_\gamma(\lambda)x = -e^{-\lambda\gamma}S(\gamma)x + \int_0^\gamma e^{-\lambda s}K(s)Cx\,ds$, $\lambda \geqslant 0$.*

(iv) *$L_\gamma(\lambda)L_\gamma(\eta) = L_\gamma(\eta)L_\gamma(\lambda)$, $\lambda \geqslant 0$, $\eta \geqslant 0$.*

An operator family $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \geqslant 0\}$ is called an *asymptotic ΘC-resolvent* for A if there exists a strongly continuous operator family $(V(t))_{t\in[0,\tau)}$ such that the conditions (i), (ii) and (iv) of Proposition 2.3.18 hold and that the condition (iii) of Proposition 2.3.18 holds with $S(\gamma)$ replaced by $V(\gamma)$. Using Theorem 1.1.13 and the arguments given in the proofs of [**275**, Theorem 5.2, Corollary 5.3], one can prove the following assertions.

THEOREM 2.3.19. *Let A be a closed operator and let K be a kernel. Assume that A has an asymptotic ΘC-resolvent $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \geqslant 0\}$. Then $\bigl(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta(s)\,ds, C\bigr)$ is well-posed for all $\alpha > 0$.*

THEOREM 2.3.20. *Suppose $D(A)$ is dense in E and K is a kernel. Then (ΘC) is well-posed for A on $[0,\tau)$ iff A has an asymptotic ΘC-resolvent $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \geqslant 0\}$.*

For further information concerning asymptotic C-resolvents, we refer the reader to [**381**]–[**382**], [**404**] and [**421**].

DEFINITION 2.3.21. The abstract Cauchy problem $(ACP_2)_\Theta$ is *exponentially well-posed* if for every x, $y \in E$ there exists a unique solution $u(\cdot)$ of $(ACP_2)_\Theta$ and if, additionally, for every x, $y \in E$, the solution $u(\cdot)$ satisfies the estimate $\|u(t)\| \leqslant Me^{\omega t}$, $0 \leqslant t < \tau$, for appropriate constants $M > 0$ and $\omega \in \mathbb{R}$. The exponential well-posedness of the problem (ΘC) is defined similarly.

We need an auxiliary lemma whose proof follows exactly in the same way as in the proof of [**355**, Lemma 4.1, p. 100].

LEMMA 2.3.22. *Suppose $T > 0$, $u \in C([0,T] : E)$ and there exist $\lambda > 0$ and $M > 0$ such that $\bigl\|\int_0^T e^{n\lambda s}u(s)\,ds\bigr\| \leqslant M$, $n \in \mathbb{N}$. Then $u(t) = 0$, $t \in [0,T]$.*

The following is a generalization of the Ljubich uniqueness theorem.

THEOREM 2.3.23. *Suppose $\lambda > 0$, $\{n\lambda : n \in \mathbb{N}\} \subseteq \rho_C(A)$ and, for every $\sigma > 0$ and $x \in E$, $\lim_{n\to\infty} \frac{\|(n\lambda - A)^{-1}Cx\|}{e^{n\lambda\sigma}} = 0$. Then, for every $x \in E$, there exists at most one solution of the initial value problem*

$$(CP)_1 : \begin{cases} u \in C^1((0,\infty) : E) \cap C([0,\infty) : E), \\ u'(t) = Au(t),\ t > 0, \\ u(0) = x. \end{cases}$$

PROOF. Put $\lambda_n := n\lambda$, $n \in \mathbb{N}$ and define, for every $n \in \mathbb{N}$, the function $z_n(\cdot)$ by setting $z_n(t) := (\lambda_n - A)^{-1}Cu(t)$, $t \geqslant 0$, where $u(t)$ is a solution of the problem $(CP)_1$ at $x = 0$. Then we have $z_n'(t) = (\lambda_n - A)^{-1}Cu'(t) = (\lambda_n - A)^{-1}CAu(t) = \lambda_n z_n(t) - Cu(t)$, $t > 0$ and $z_n(0) = 0$. This implies $z_n(t) = -\int_0^t e^{\lambda_n(t-s)}Cu(s)\,ds$, $t \geqslant 0$. Since $\lim_{n\to\infty}\int_{t-\sigma}^t e^{\lambda_n(t-\sigma-s)}Cu(s)\,ds = 0$, the prescribed assumptions imply $\lim_{n\to\infty}\int_0^{t-\sigma} e^{\lambda_n(t-\sigma-s)}Cu(s)\,ds = 0$ for every $t > 0$ and $\sigma \in (0, t]$. Hence, $\lim_{n\to\infty}\int_0^T e^{\lambda_n(T-s)}Cu(s)\,ds = 0$, $T \geqslant 0$ and the proof follows by Lemma 2.3.22. □

THEOREM 2.3.24. (i) *Suppose K satisfies* (P1), $\lambda_0 > \max(0, \operatorname{abs}(K))$ *is such that $\tilde{K}(n\lambda_0) \neq 0$, $n \in \mathbb{N}$, A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ and, for every $\varepsilon > 0$,*

$$\frac{1}{|\tilde{K}(\lambda)|} = O(e^{\varepsilon|\lambda|}),\ \lambda \to +\infty,\ \tilde{K}(\lambda) \neq 0. \tag{78}$$

Then the abstract Cauchy problem (ΘC) is exponentially well-posed.

(ii) *Suppose (ΘC) is exponentially well-posed. Then A is a subgenerator of an exponentially bounded Θ-convoluted C-semigroup $(S_\Theta(t))_{t\geqslant 0}$.*

(iii) *Suppose $(ACP_2)_\Theta$ is exponentially well-posed. Then A is a subgenerator of an exponentially bounded Θ^{-1}-convoluted C-cosine function $(C_{\Theta^{-1}}(t))_{t\geqslant 0}$.*

PROOF. It is straightforward to verify that $u(t) := \int_0^t S_K(s)x\,ds$, $t \geqslant 0$, $x \in E$ is an exponentially bounded solution of (ΘC). Then (78) and Theorem 2.2.2 imply that, for every $\sigma > 0$,

$$\lim_{\lambda\to+\infty,\ \tilde{K}(\lambda)\neq 0} \frac{\|(\lambda - A)^{-1}C\|}{e^{\sigma\lambda}} = 0.$$

Now the uniqueness of solutions of (ΘC) at $x = 0$ follows by the use of Theorem 2.3.23, finishing the proof of (i). Suppose (ΘC) is exponentially well-posed and define $S_\Theta(t)x := u(t, x)$, $t \geqslant 0$, $x \in E$, where $u(\cdot, x)$ is a unique solution of (ΘC). Arguing as before, one yields that $(S_\Theta(t))_{t\geqslant 0}$ is a global Θ-convoluted C-semigroup with a subgenerator A and that, for every $x \in E$, there exist $M_x > 0$ and $\omega_x > 0$ such that $\|S_\Theta(t)x\| \leqslant M_x e^{\omega_x t}$, $t \geqslant 0$. Using the uniform exponential boundedness principle (cf. [**5**, Proposition 5.4]), it follows that there exist $M > 0$ and $\omega > 0$ such that $\|S_\Theta(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, which completes the proof of (ii). The proof of (iii) is analogous to that of (ii). □

Recall that the function $u(\cdot)$ is a *mild solution* of the abstract Cauchy problem (ACP_1), resp. (ACP_2), iff the mapping $t \mapsto u(t)$, $t \geqslant 0$ is continuous, $\int_0^t u(s)\,ds \in D(A)$ and $A\int_0^t u(s)\,ds = u(t) - x$, $t \geqslant 0$, resp. $\int_0^t (t-s)u(s)\,ds \in D(A)$ and $A\int_0^t (t-s)u(s)\,ds = u(t) - x - ty$, $t \geqslant 0$. The following remarkable result can be attributed to van Neerven and Straub [**335**].

THEOREM 2.3.25. (i) *Let $\alpha > 0$ and let A be densely defined and the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Then, for every $\varepsilon > 0$, $\sigma > 0$ and*

$x_0 \in D((\omega+\sigma-A)^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP_1) has a unique mild solution. Moreover, this solution is exponentially bounded and its exponential type is at most ω.

(ii) *Let $\alpha > 0$ and let A be densely defined and the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant M(1+t^\gamma)$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\gamma \geqslant 0$. Then, for every $\varepsilon > 0$, $\sigma > 0$ and $x_0 \in D((\sigma-A)^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP_1) has a unique mild solution. Moreover, this solution is polynomially bounded and its polynomial type is at most* $\max(\alpha-1+\varepsilon, \gamma+\varepsilon, 2\gamma-\alpha+\varepsilon)$.

The preceding theorem has been essentially utilized by Li and Zheng in [**277**]:

THEOREM 2.3.26. (i) *Let $\alpha > 0$ and let A be densely defined and the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Then, for every $\varepsilon > 0$ and $\sigma > 0$, A is the integral generator of an exponentially bounded $(\omega+\sigma-A)^{-(\alpha+\varepsilon)}$-regularized semigroup $(T(t))_{t\geqslant 0}$ which satisfies that, for every $\sigma' > \sigma$, there exists $M' \geqslant 1$ such that $\|T(t)\| \leqslant M'e^{(\omega+\sigma')t}$, $t \geqslant 0$.*

(ii) *Suppose that A is a densely defined closed operator and that there exist constants $M \geqslant 1$, $\omega \geqslant 0$, $\beta > 0$ and $\gamma \in (0, \frac{\pi}{2})$ such that $\omega + \Sigma_\gamma \subseteq \rho(A)$ and that $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\beta-1}$, $\lambda \in \omega+\Sigma_\gamma$. If A generates an exponentially bounded $(\omega+\sigma-A)^{-\alpha}$-regularized semigroup for some $\alpha > \beta$ and $\sigma > 0$, then, for every $\varepsilon > 0$, A generates an exponentially bounded $(\alpha+\varepsilon)$-times integrated semigroup.*

It is worth noting that Theorem 2.3.25 and Theorem 2.3.26 still hold in the case of non-densely defined generators of fractionally integrated semigroups [**233**], which can be applied to non-densely defined convolution operators considered by Hieber in [**149**, Section 4]. In such a way, one can prove an extension of [**277**, Theorem 3.7] for the operators acting in $L^\infty(\mathbb{R}^n)$ and $C_b(\mathbb{R}^n)$.

THEOREM 2.3.27. [**233**] (i) *Let $\alpha > 0$ and let A be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\geqslant 0}$ satisfying $\|C_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Then, for every $\varepsilon > 0$, $\sigma > 0$ and $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+1})$, the abstract Cauchy problem (ACP_2) has a unique mild solution, where $\mathcal{A}_{\omega+\sigma} \equiv \left(\begin{smallmatrix} 0 & I \\ A-(\omega+\sigma) & 0 \end{smallmatrix}\right)$. Moreover, this solution is exponentially bounded and its exponential type is at most ω. If $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+2})$, the solution is classical.*

(ii) *Let $\alpha > 0$ and let A be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\geqslant 0}$ satisfying $\|C_\alpha(t)\| \leqslant M(1+t^\gamma)$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\gamma \geqslant 0$. Then, for every $\varepsilon > 0$, $\sigma > 0$ and $(x_0, y_0) \in D((-\mathcal{A}_\sigma)^{\alpha+\varepsilon+1})$, the abstract Cauchy problem (ACP_2) has a unique mild solution. Moreover, this solution is polynomially bounded and its polynomial type is at most*

$$\max\big(\alpha+\varepsilon, \max(\alpha, \gamma+2)+\varepsilon, 2\max(\alpha, \gamma+2)-(\alpha+1)+\varepsilon\big).$$

If $(x_0, y_0) \in D((-\mathcal{A}_\sigma)^{\alpha+\varepsilon+2})$, the solution is classical.

The following remark can be reformulated in the case of fractionally integrated semigroups ([**233**]).

REMARK 2.3.28. Let $\alpha \in (2n, 2n+1)$ for some $n \in \mathbb{N}_0$, resp. $\alpha \in (2n-1, 2n)$ for some $n \in \mathbb{N}$. By Proposition 2.3.8, we know that the classical solution of (ACP_2) exists for all $(x_0, y_0) \in D(A^{n+2}) \times D(A^{n+1}) = D(\mathcal{A}^{2n+3})$, resp. for all $(x_0, y_0) \in D(A^{n+1}) \times D(A^{n+1}) = D(\mathcal{A}^{2n+2})$. It can be proved that the set $\bigcup_{\varepsilon \in (0, \lfloor \alpha \rfloor + 1 - \alpha]} D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+2})$ strictly contains $D(\mathcal{A}^{2n+3})$, respectively the set $\bigcup_{\varepsilon \in (0, \lfloor \alpha \rfloor + 1 - \alpha]} D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+2})$ strictly contains $D(\mathcal{A}^{2n+2})$. The same conclusion holds in the case of mild solutions.

For further information concerning inhomogeneous Cauchy problems and generalized variation of parameters formula, the reader may consult [**14**], [**128**], [**186**], [**241**], [**259**]-[**261**], [**280**], [**286**], [**298**] and [**381**].

2.4. Analytical properties

We start by recalling that $\Sigma_\gamma = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ \arg(\lambda) \in (-\gamma, \gamma)\}$ $(\gamma \in (0, \pi])$.

DEFINITION 2.4.1. Let $0 < \alpha \leqslant \frac{\pi}{2}$ and let $(S_K(t))_{t \geqslant 0}$ be a K-convoluted C-semigroup. Then we say that $(S_K(t))_{t \geqslant 0}$ is an *analytic K-convoluted C-semigroup of angle α,* if there exists an analytic function $\mathbf{S}_K : \Sigma_\alpha \to L(E)$ which satisfies

(i) $\mathbf{S}_K(t) = S_K(t)$, $t > 0$ and
(ii) $\lim_{z \to 0,\ z \in \Sigma_\gamma} \mathbf{S}_K(z)x = 0$ for all $\gamma \in (0, \alpha)$ and $x \in E$.

It is said that $(S_K(t))_{t \geqslant 0}$ is an *exponentially bounded, analytic K-convoluted C-semigroup,* resp. *bounded analytic K-convoluted C-semigroup, of angle* α, if for every $\gamma \in (0, \alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $\|\mathbf{S}_K(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$.

Since no confusion seems likely, we also write $\mathbf{S}_K$ for S_K. Plugging $K(t) = \frac{t^{r-1}}{\Gamma(r)}$, $t > 0$ in Definition 2.4.1, where $r > 0$, we obtain the well-known classes of analytic r-times integrated C-semigroups; an analytic 0-times integrated C-semigroup is defined to be an analytic C-regularized semigroup. The notion of (exponential) boundedness of an analytic r-times integrated C-semigroup, $r \geqslant 0$, is understood in the sense of Definition 2.4.1. The author proved in [**224**] that, in the case $r \in \mathbb{N}$, the definition of an analytic r-times integrated semigroup is equivalent to the corresponding one given by R. deLaubenfels in [**92**].

The following assertion is an extension of [**92**, Proposition 3.7(a)].

PROPOSITION 2.4.2. *Suppose K satisfies* (P1), *$\alpha \in (0, \frac{\pi}{2}]$ and A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup $(S_K(t))_{t \geqslant 0}$ of angle α. Suppose, further, that the condition* (H) *holds, where:*

(H): *There exist functions $c : (-\alpha, \alpha) \to \mathbb{C} \setminus \{0\}$, $\omega_0 : (-\alpha, \alpha) \to [0, \infty)$ and a family of functions $(K_\theta)_{\theta \in (-\alpha, \alpha)}$ satisfying* (P1) *and* $\operatorname{abs}(K_\theta) \leqslant \omega_0(\theta)$, $\frac{\operatorname{abs}(K)}{\cos\theta} \leqslant \omega_0(\theta)$,

$$\Phi_\theta := \{\lambda \in (\omega_0(\theta), \infty) : \tilde{K}(\lambda e^{-i\theta}) = 0\} = \{\lambda \in (\omega_0(\theta), \infty) : \widetilde{K_\theta}(\lambda) = 0\}, \tag{79}$$

$$\frac{\widetilde{K_\theta}(\lambda)}{\tilde{K}(\lambda e^{-i\theta})} = c(\theta),\ \lambda > \omega_0(\theta),\ \lambda \notin \Phi_\theta,\ \theta \in (-\alpha, \alpha). \tag{80}$$

Then, for every $\theta \in (-\alpha, \alpha)$, the operator $e^{i\theta}A$ is a subgenerator of an exponentially bounded, analytic K_θ-convoluted C-semigroup $\big(c(\theta)S_K(te^{i\theta})\big)_{t\geqslant 0}$ of angle $\alpha - |\theta|$. Furthermore, $S_K(te^{i\theta})A \subseteq AS_K(te^{i\theta})$, $t \geqslant 0$ and

$$A\int\limits_0^{te^{i\theta}} S_K(s)x\,ds = S_K(te^{i\theta})x - \frac{1}{c(\theta)}\int\limits_0^t K_\theta(s)\,dsCx,\ t\geqslant 0,\ x\in E,\ \theta\in(-\alpha,\alpha).$$

PROOF. Let $\theta \in (-\alpha, \alpha)$ and let $\lambda \in \mathbb{R}$ be sufficiently large with $\widetilde{K_\theta}(\lambda) \neq 0$. Denote $\Gamma_\theta := \{te^{-i\theta} : t \geqslant 0\}$ and notice that $\big(c(\theta)S_K(te^{i\theta})\big)_{t\geqslant 0}$ is a strongly continuous, exponentially bounded operator family. Clearly, $\tilde{K}(\lambda e^{-i\theta}) \neq 0$ and:

$$\begin{aligned}(81)\quad \widetilde{K_\theta}(\lambda)(\lambda - e^{i\theta}A)^{-1}Cx &= \widetilde{K_\theta}(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx\\ &= e^{-i\theta}\frac{\widetilde{K_\theta}(\lambda)}{\tilde{K}(\lambda e^{-i\theta})}\int\limits_0^\infty e^{-\lambda e^{-i\theta}t}S_K(t)x\,dt = e^{-i\theta}c(\theta)\int\limits_{\Gamma_\theta} e^{-\lambda t}e^{i\theta}S_K(te^{i\theta})x\,dt\\ &= \int\limits_0^\infty e^{-\lambda t}\Big(c(\theta)S_K(te^{i\theta})x\Big)\,dt,\ x\in E,\end{aligned}$$

where the last equality in (81) follows from an elementary application of Cauchy theorem. Invoking Remark 2.2.3 and Definition 2.4.1, we conclude that $e^{i\theta}A$ is a subgenerator of an exponentially bounded, analytic K_θ-convoluted C-semigroup $\big(c(\theta)S_K(te^{i\theta})\big)_{t\geqslant 0}$, as required. □

COROLLARY 2.4.3. *Suppose $r > 0$, $\alpha \in (0, \frac{\pi}{2}]$, $\theta \in (-\alpha, \alpha)$ and A is a subgenerator of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant 0}$ of angle α. Then $e^{i\theta}A$ is a subgenertor of an exponentially bounded, analytic r-times integrated C-semigroup $\big(e^{-i\theta r}S_r(te^{i\theta})\big)_{t\geqslant 0}$ of angle $\alpha - |\theta|$. Furthermore, $S_r(z)A \subseteq AS_r(z)$, $z \in \Sigma_\alpha$ and $A\int_0^z S_r(s)x\,ds = S_r(z)x - \frac{z^r}{\Gamma(r+1)}Cx$, $z \in \Sigma_\alpha$, $x \in E$.*

The subsequent theorems clarify the main structural properties of exponentially bounded, analytic K-convoluted C-semigroups.

THEOREM 2.4.4. *Suppose $\alpha \in (0, \frac{\pi}{2}]$, K satisfies* (P1) *and $\tilde{K}(\cdot)$ can be analytically continued to a function $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$, where $\omega \in [\max(0, \mathrm{abs}(K)), \infty)$. Suppose, further, A is a subgenerator of an analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ of angle α and*

$$(82)\qquad \sup_{z\in\Sigma_\gamma}\|e^{-\omega z}S_K(z)\| < \infty \text{ for all } \gamma \in (0,\alpha).$$

Denote by $\hat{A}$ the integral generator of $(S_K(t))_{t\geqslant 0}$ and put

$$(83)\qquad N := \big\{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha} : g(\lambda) \neq 0\big\}.$$

Then:

$$(84)\qquad N \subseteq \rho_C(\hat{A}),$$

$$\sup_{\lambda\in\, N\cap\,(\omega+\Sigma_{\frac{\pi}{2}+\gamma_1})}\big\|(\lambda-\omega)g(\lambda)(\lambda-\hat{A})^{-1}C\big\|<\infty \textit{ for all } \gamma_1\in(0,\alpha) \tag{85}$$

$$\lim_{\lambda\to+\infty,\ \tilde{K}(\lambda)\neq 0}\lambda\tilde{K}(\lambda)(\lambda-A)^{-1}Cx=0,\ x\in E \textit{ and} \tag{86}$$

$$\textit{the mapping } \lambda\mapsto(\lambda-\hat{A})^{-1}C,\ \lambda\in N \textit{ is analytic.} \tag{87}$$

Assuming

(H_1): (H) *holds with* $c(\cdot)$, $\omega_0(\cdot)$, $(K_\theta)_{\theta\in(-\alpha,\alpha)}$ *and* $\operatorname{abs}(K_\theta)\leqslant\omega\cos\theta$, $\theta\in(-\alpha,\alpha)$, *one has* (84)–(85) *and* (87) *with* $\hat{A}$ *replaced by* A *therein.*

Proof. By the foregoing, $\{\lambda\in\mathbb{C}\colon \operatorname{Re}\lambda>\omega,\ \tilde{K}(\lambda)\neq 0\}\subseteq\rho_C(A)$ and

$$\tilde{K}(\lambda)(\lambda-A)^{-1}Cx=\int\limits_0^\infty e^{-\lambda t}S_K(t)x\,dt,\ \operatorname{Re}\lambda>\omega,\ \tilde{K}(\lambda)\neq 0,\ x\in E.$$

Put $q(\lambda):=\int_0^\infty e^{-\lambda t}S_K(t)\,dt$, $\operatorname{Re}\lambda>\omega$. An application of Theorem 1.1.10 gives that the function $q(\cdot)$ can be extended to an analytic function $\tilde{q}:\omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(E)$ satisfying $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\|(\lambda-\omega)\tilde{q}(\lambda)\|<\infty$ for all $\gamma\in(0,\alpha)$. Further on, N is an open subset of $\mathbb{C}$ and it can be easily seen that every two point in N can be connected with a C^∞ curve lying in N; in particular, N is an open connected subset of $\mathbb{C}$. The function $F:N\to L(E)$ given by $F(\lambda):=\frac{\tilde{q}(\lambda)}{g(\lambda)}$, $\lambda\in N$ is analytic and

$$\big\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\omega,\ \tilde{K}(\lambda)\neq 0\big\}\subseteq\big\{\lambda\in N\cap\rho_C(A):F(\lambda)=(\lambda-A)^{-1}C\big\}.$$

Denote $V=\{\lambda\in N\cap\rho_C(A):F(\lambda)=(\lambda-A)^{-1}C\}$ and suppose $\mu\in\rho_C(A)$, $x\in D(A)$ and $y\in E$. Since

$$F(\lambda)(\lambda-A)x=(\lambda-A)^{-1}C(\lambda-A)x=Cx,\ \ \lambda\in V, \tag{88}$$

$$F(\lambda)Cy=CF(\lambda)y,\ \lambda\in V \text{ and} \tag{89}$$

$$F(\lambda)Cy=(\lambda-A)^{-1}C^2y=(\mu-A)^{-1}C^2y-(\lambda-\mu)(\mu-A)^{-1}CF(\lambda)y,\ \lambda\in V, \tag{90}$$

the uniqueness theorem for analytic functions (cf. [**14**, Proposition A2, Proposition B.5] and Proposition 1.1.14(iii)) implies that (88)–(90) remain true for all $\lambda\in N$. Suppose now that $(\lambda-A)x=0$ for some $\lambda\in N$ and $x\in D(A)$. Owing to (88), one gets $Cx=0$, $x=0$ and the injectiveness of $\lambda-A$. By Proposition 2.1.6, we obtain that the operator $\lambda-\hat{A}$ is also injective. Furthermore,

$$\begin{aligned}(\lambda-A)CF(\lambda)y&=(\lambda-A)F(\lambda)Cy\\&=(\lambda-A)\big[(\mu-A)^{-1}C^2y-(\lambda-\mu)(\mu-A)^{-1}CF(\lambda)y\big]\\&=C^2y+(\lambda-\mu)\big[(\mu-A)^{-1}C^2y-CF(\lambda)y-(\lambda-\mu)(\mu-A)^{-1}CF(\lambda)y\big],\end{aligned}$$

and thanks to the validity of (90) for all $\lambda\in N$, one obtains that

$$(\lambda-A)CF(\lambda)y=C^2y,\ \lambda\in N.$$

The last equality, injectiveness of C and Proposition 2.1.6(ii) together imply:

$$\lambda F(\lambda)y = C^{-1}AC[F(\lambda)y] + Cy = \hat{A}F(\lambda)y + Cy,\ \lambda \in N,$$

i.e., $(\lambda-\hat{A})F(\lambda)y = Cy,\ \lambda \in N$. This implies $\mathrm{R}(C) \subseteq \mathrm{R}(\lambda-\hat{A})$, $\lambda \in N$, $N \subseteq \rho_C(\hat{A})$, $F(\lambda) = (\lambda-\hat{A})^{-1}C,\ \lambda \in N$, (84) and (87). The estimate (85) is an immediate consequence of Theorem 1.1.10. Let $x \in E$ be fixed. Then $z \mapsto S_K(z)x,\ z \in \Sigma_\alpha$ is an analytic function which satisfies the condition (i) quoted in the formulation of Theorem 1.1.10. Since $\lim_{t\downarrow 0} S_K(t)x = 0$, an application of Theorem 1.1.11(i) implies that $\lim_{\lambda\to+\infty}\lambda q(\lambda) = 0$. This gives $\lim_{\lambda\to+\infty,\ \tilde{K}(\lambda)\neq 0}\lambda\tilde{K}(\lambda)(\lambda-A)^{-1}Cx = 0$, i.e., (86) and the first part of the proof is completed. Suppose now that (H_1) holds. Then $\mathrm{abs}(K_\theta) \leqslant \omega\cos\theta,\ \theta \in (-\alpha,\alpha)$ and one easily infers that, for every $\theta \in (-\alpha,\alpha)$, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\cos\theta,\ \widetilde{K_\theta}(\lambda) \neq 0\} \subseteq \rho_C(e^{i\theta}A)$ and that:

$$\widetilde{K_\theta}(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx = \int\limits_0^\infty e^{-\lambda t}\big(c(\theta)S_K(te^{i\theta})\big)x\,dt, \tag{91}$$

for all $x \in E$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega\cos\theta$ and $\widetilde{K_\theta}(\lambda) \neq 0$. Fix a number $\theta \in (-\alpha,\alpha)$ and define $G_\theta : \{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-\frac{\pi}{2}-\theta,\ \frac{\pi}{2}-\theta)\} \cap N \to \mathbb{C}$ by $G_\theta(\lambda) := \frac{\widetilde{K_\theta}(\lambda e^{i\theta})}{g(\lambda)}$, $\lambda \in D(G_\theta(\cdot))$. Then it is clear that $D(G_\theta(\cdot))$ is an open, connected subset of $\mathbb{C}$ and that, by (79)–(80), there exists $a > 0$ such that $\Phi_{\theta,a} := \{te^{-i\theta} \cap N : t \geqslant a\} \subseteq D(G_\theta(\cdot))$ and that $G_\theta(\lambda) = c(\theta)$, $\lambda \in \Phi_{\theta,a}$. By the uniqueness theorem for analytic functions, one obtains that $G_\theta(\lambda) = c(\theta),\ \lambda \in D(G_\theta(\cdot))$. Hence, (91) implies $\{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-\frac{\pi}{2}-\theta,\ \frac{\pi}{2}-\theta)\} \cap N \subseteq \rho_C(A)$,

$$(z-A)^{-1}Cx = \frac{e^{i\theta}}{g(z)}\int\limits_0^\infty e^{-ze^{i\theta}t}S_K(te^{i\theta})x\,dt, \tag{92}$$

for all $z \in \{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-(\frac{\pi}{2}+\theta),\ \frac{\pi}{2}-\theta)\} \cap N$ and $x \in E$, and the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $\arg(z-\omega) \in (-(\frac{\pi}{2}+\theta),\ \frac{\pi}{2}-\theta)$ is analytic. One can apply the same argument to $e^{-i\theta}A$ in order to see that $\{z \in N : \arg(z-\omega) \in (\theta-\frac{\pi}{2}, \frac{\pi}{2}+\theta)\} \subseteq \rho_C(A)$ and that the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $\arg(z-\omega) \in (\theta-\frac{\pi}{2}, \theta+\frac{\pi}{2})$ is analytic. Thereby, $\{z \in N : |\arg(z-\omega)| < \theta+\frac{\pi}{2}\} \subseteq \rho_C(A)$ and the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $|\arg(z-\omega)| < \theta+\frac{\pi}{2}$ is analytic. This completes the proof of theorem. □

THEOREM 2.4.5. *Assume* $\alpha \in (0,\frac{\pi}{2}]$, K *satisfies* (P1) *and* $\omega \geqslant \max(0,\mathrm{abs}(K))$. *Suppose* A *is a closed linear operator with* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A)$ *and the function* $\lambda \mapsto \tilde{K}(\lambda)(\lambda-A)^{-1}C$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$, *can be analytically extended to a function* $\tilde{q} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(E)$ *satisfying*

$$\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\big\|(\lambda-\omega)\tilde{q}(\lambda)\big\| < \infty \text{ for all } \gamma \in (0,\alpha) \text{ and}$$

$$\lim_{\lambda\to+\infty}\lambda\tilde{q}(\lambda)x = 0,\ \ x \in E,\ \text{if } \overline{D(A)} \neq E. \tag{93}$$

Then A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup of angle α.

PROOF. The use of Theorem 1.1.10 implies that there exists an analytic function $S_K : \Sigma_\alpha \to L(E)$ such that $\sup\limits_{z\in\Sigma_\gamma} \|e^{-\omega z} S_K(z)\| < \infty$ for all $\gamma \in (0,\alpha)$ and

$$\tilde{q}(\lambda) = \int\limits_0^\infty e^{-\lambda t} S_K(t)\,dt,\ \operatorname{Re}\lambda > \omega.$$

Let us define $S_K(0) := 0$. Let $x \in E$ and $\gamma \in (0,\alpha)$. We will prove that $\lim_{z\to 0,\, z\in\Sigma_\gamma} S_K(z)x = 0$. Note that the mapping $f(z) := e^{-\omega z} S_K(z)x$, $z \in \Sigma_\alpha$ is analytic and that $\sup_{z\in\Sigma_\gamma} \|f(z)\| < \infty$ for all $\gamma \in (0,\alpha)$. By Theorem 1.1.11, it is enough to show $\lim_{t\downarrow 0} S_K(t)x = 0$. This is a consequence of the assumption $\lim_{\lambda\to+\infty} \lambda\tilde{q}(\lambda)x = 0$. It follows that $(S_K(t))_{t\geqslant 0}$ is a strongly continuous, exponentially bounded operator family which satisfies, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{K}(\lambda) \neq 0$, $\tilde{K}(\lambda)(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t} S_K(t)x\,dt$. Similarly as in the proof of Theorem 2.2.1, we have that A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$. Since $(S_K(t))_{t\geqslant 0}$ verifies the conditions (i) and (ii), given in Definition 2.4.1, $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, analytic K-convoluted C-semigroup of angle α having A as a subgenerator. Suppose now that A is densely defined. We will prove that (93) holds. By Theorem 1.1.11 and the first part of the proof, it suffices to show that $\lim_{t\downarrow 0} S_K(t)x = 0$. Suppose, for the time being, $x \in D(A)$. Since $\tilde{q}(\lambda)x = \tilde{K}(\lambda)(\lambda - A)^{-1}Cx$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$ we get

$$\begin{aligned}\mathcal{L}\Bigg(\int\limits_0^t S_K(s)Ax\,ds\Bigg)(\lambda) &= \frac{\tilde{q}(\lambda)}{\lambda}Ax = \tilde{q}(\lambda)x - \frac{\tilde{K}(\lambda)}{\lambda}Cx\\ &= \mathcal{L}\big(S_K(t)x - \Theta(t)Cx\big)(\lambda),\ \ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \omega,\ \tilde{K}(\lambda) \neq 0.\end{aligned}$$

The uniqueness theorem for Laplace transform implies $\int_0^t S_K(s)Ax\,ds = S_K(t)x - \Theta(t)Cx$, $t \geqslant 0$. Therefore, $\|S_K(t)x\| \leqslant |\Theta(t)|Cx + te^{\omega t}\|Ax\|$, $t \geqslant 0$ and $\lim_{t\downarrow 0} S_K(t)x = 0$. Combined with the exponential boundedness of $S_K(\cdot)$, this indicates that $\lim_{t\downarrow 0} S_K(t)x = 0$ for every $x \in E$. $\square$

We need the following useful profiling of C-pseudoresolvents.

PROPOSITION 2.4.6. [**98**, Proposition 2.6, Remark 2.7, Corollary 2.8] *Let $\Omega \subseteq \rho_C(A)$ be open.*

(i) *The local boundedness of the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ implies the analyticity of the mapping $\lambda \mapsto (\lambda - A)^{-1}C^3$, $\lambda \in \Omega$.*

(ii) *Suppose that* $\mathrm{R}(C)$ *is dense in E. Then the local boundedness of the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ implies its analyticity as well as* $\mathrm{R}(C) \subseteq \mathrm{R}\big((\lambda - A)^n\big)$, *$n \in \mathbb{N}$ and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda - A)^{-1}C = (-1)^{n-1}(n-1)!(\lambda - A)^{-n}C,\ n \in \mathbb{N}. \tag{94}$$

(iii) *The continuity of mapping* $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ *implies its analyticity as well as* $\mathrm{R}(C) \subseteq \mathrm{R}((\lambda - A)^n)$, $n \in \mathbb{N}$ *and* (94).

It is checked at once that the function $K(t) = \frac{t^{r-1}}{\Gamma(r)}$, $t > 0$, $r > 0$ satisfies the condition (H_1) with $c(\theta) = e^{-ir\theta}$, $\omega_0(\theta) = 0$ and $K_\theta(t) = K(t)$, $\theta \in (-\alpha, \alpha)$, $t > 0$. Keeping in mind Theorem 2.4.5, Proposition 2.4.6 and the above remark, one immediately obtains the proof of subsequent theorem; notice only that, in the case $r = 0$, the equality (95) follows from Theorem 1.1.11 and elementary definitions.

THEOREM 2.4.7. *Suppose* $r \geqslant 0$ *and* $\alpha \in (0, \frac{\pi}{2}]$. *Then* A *is a subgenerator of an exponentially bounded, analytic* r*-times integrated* C*-semigroup* $(S_r(t))_{t \geqslant 0}$ *of angle* α *iff for every* $\gamma \in (0, \alpha)$, *there exist* $M_\gamma > 0$ *and* $\omega_\gamma \geqslant 0$ *such that:*

$$\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subseteq \rho_C(A),$$
$$\|(\lambda - A)^{-1}C\| \leqslant M_\gamma(1 + |\lambda|)^{r-1}, \ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma},$$

the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}$ *is analytic* (*continuous*) *and*

$$\lim_{\lambda \to +\infty} \frac{(\lambda - A)^{-1}Cx}{\lambda^{r-1}} = \chi_{\{0\}}(r)Cx, \ \ x \in E, \ \text{if } \overline{D(A)} \neq E. \tag{95}$$

THEOREM 2.4.8 (The abstract Weierstrass formula). *Assume* $M > 0$, $\beta \geqslant 0$, $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$ *and* A *is a subgenerator of an exponentially bounded,* K*-convoluted* C*-cosine function* $(C_K(t))_{t \geqslant 0}$. *Then* A *is a subgenerator of an exponentially bounded, analytic* K_1*-convoluted* C*-semigroup* $(S(t))_{t \geqslant 0}$ *of angle* $\frac{\pi}{2}$, *where:*

$$K_1(t) := \int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}} K(s)\, ds \ \text{and} \ S(t) := \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C_K(s)\, ds, \ t > 0.$$

PROOF. We follow the proof of the abstract Weierstrass formula (cf. [**14**, p. 220]). Due to Theorem 1.1.7(v), the function $K_1(\cdot)$ fulfills (P1), $\mathrm{abs}(K_1) \geqslant \beta^2$ and $\widetilde{K_1}(\lambda) = \tilde{K}(\sqrt{\lambda})$, $\operatorname{Re}\lambda > \beta^2$. Let $x \in E$ be fixed. Putting $r = \frac{s}{\sqrt{t}}$, and using the dominated convergence theorem after that, one obtains

$$S(t)x = \int_0^\infty \frac{e^{-r^2/4}}{\sqrt{\pi}} C_K(r\sqrt{t})x\, dr \to 0, \ \ t \to 0+. \tag{96}$$

Define $S(0) := 0$. By (96), $(S(t))_{t \geqslant 0}$ is a strongly continuous, exponentially bounded operator family. Furthermore, one can employ Theorem 2.2.1 and Theorem 1.1.7(v) to obtain that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta^2$ and $\widetilde{K_1}(\lambda) \neq 0$, the following holds:

$$\int_0^\infty e^{-\lambda t} S(t)x\, dt = \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C_K(s)x\, ds\, dt = \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}s} C_K(s)x\, ds$$
$$= \frac{1}{\sqrt{\lambda}} \sqrt{\lambda}\tilde{K}(\sqrt{\lambda})(\lambda - A)^{-1}Cx = \widetilde{K_1}(\lambda)(\lambda - A)^{-1}Cx.$$

As above, one concludes that $(S(t))_{t\geqslant 0}$ is an exponentially bounded K_1-convoluted C-semigroup with a subgenerator A. If $\operatorname{Re} z > 0$, we define $S(z)$ in a natural way: $S(z)x =: \frac{1}{\sqrt{\pi z}} \int_0^\infty e^{-s^2/4z} C_K(s)x\,ds$. Then $S : \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \to L(E)$ is analytic, and using the same arguments as in the proof of the classical Weierstrass formula, one obtains that, for all $\beta \in (0, \frac{\pi}{2})$, there exist $M_\beta > 0$ and $\omega_\beta > 0$ such that $\|S(z)\| \leqslant M_\beta e^{\omega_\beta |z|}$, $z \in \Sigma_\beta$. It remains to be shown that, for every fixed $\beta \in (0, \frac{\pi}{2})$, $\lim_{z\in\Sigma_\beta,\, z\to 0} S(z)x = 0$. For this, choose an $\omega_2 > \frac{\omega_\beta}{\cos\beta}$. Then the function $z \mapsto e^{-\omega_2 z}S(z)x$, $z \in \Sigma_\beta$ is analytic and satisfies $\sup_{z\in\Sigma_\beta} \|e^{-\omega_2 z}S(z)\| < \infty$. Since $\lim_{t\to 0+} e^{-\omega_2 t}S(t)x = 0$, an employment of Theorem 1.1.11 implies $\lim_{z\in\Sigma_\beta,\, z\to 0} e^{-\omega_2 z}S(z)x = 0$. The proof is now completed. $\square$

The assumption of previous theorem is satisfied for the function $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha \geqslant 1$; then $K_1(t) = \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}$. Furthermore, the proof of Theorem 2.4.8 still work in the singular case $\alpha \in (0,1)$, since in this case, $K_1(\cdot)$ again fulfills (P1) as well as $\operatorname{abs}(K_1) \geqslant 0$ and $\widetilde{K_1}(\lambda) = \tilde{K}(\sqrt{\lambda})$, $\operatorname{Re}\lambda > 0$. Therefore, as an immediate consequence of the proof of Theorem 2.4.8, we obtain the following corollary:

COROLLARY 2.4.9. *Suppose $\alpha > 0$ and A is a subgenerator of an exponentially bounded α-times integrated C-cosine function $(C_\alpha(t))_{t\geqslant 0}$. Then A is a subgenerator of an exponentially bounded, analytic $(\frac{\alpha}{2})$-times integrated C-semigroup $(S_{\alpha/2}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where $S_{\alpha/2}(t)x := \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C_\alpha(s)x\,ds$, $t > 0$, $x \in E$.*

The following is Kato's analyticity criterion for K-convoluted C-semigroups.

THEOREM 2.4.10. *Suppose $\alpha \in (0, \frac{\pi}{2}]$, K satisfies* (P1)*, $\omega \geqslant \max(0, \operatorname{abs}(K))$, there exists an analytic function $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$ such that $g(\lambda) = \tilde{K}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$ and* (H$_1$) *holds. Then A is a subgenerator of an analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying* (82) *iff:*

(i) *For every $\theta \in (-\alpha, \alpha)$, $e^{i\theta}A$ is a subgenerator of a K_θ-convoluted C-semigroup $(S_\theta(t))_{t\geqslant 0}$, and*

(ii) *for every $\beta \in (0, \alpha)$, there exists $M_\beta > 0$ such that*

$$\Big\| \frac{1}{c(\theta)} S_\theta(t) \Big\| \leqslant M_\beta e^{\omega t\cos\theta},\ t \geqslant 0,\ \theta \in (-\beta, \beta).$$

PROOF. Suppose A is a subgenerator of an analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying (82). By Proposition 2.4.2, we have that (i) and (ii) hold with $S_\theta(t) = c(\theta)S_K(te^{i\theta})$, $t \geqslant 0$, $\theta \in (-\alpha, \alpha)$. To prove the converse statement, notice that the argumentation given in the final part of the proof of Theorem 2.4.5 implies that $(\omega + \Sigma_{\frac{\pi}{2}+\alpha}) \cap N \subseteq \rho_C(A)$ and that there exists an analytic mapping $G : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(E)$ such that $G(\lambda) = g(\lambda)(\lambda - A)^{-1}C$, $\lambda \in (\omega + \Sigma_{\frac{\pi}{2}+\alpha}) \cap N$, where N is defined by (83). Furthermore, for every $\theta \in (-\alpha, \alpha)$:

$$G(\lambda) = e^{i\theta} \int_0^\infty e^{-\lambda t e^{i\theta}} \Big(\frac{1}{c(\theta)} S_\theta(t) \Big)\, dt \text{ if } \arg(\lambda - \omega) \in \Big(-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta \Big), \tag{97}$$

$$(98)\quad G(\lambda)=e^{-i\theta}\int\limits_0^\infty e^{-\lambda te^{-i\theta}}\Big(\frac{1}{c(-\theta)}S_{-\theta}(t)\Big)\,dt \text{ if } \arg(\lambda-\omega)\in\Big(\theta-\frac{\pi}{2},\theta+\frac{\pi}{2}\Big).$$

Keeping in mind (ii) as well as (97)–(98), we have that, for every $\beta\in(0,\alpha)$, $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\beta}}\|(\lambda-\omega)G(\lambda)\|<\infty$. By Theorem 1.1.10, one gets the existence of an analytic mapping $S_K:\Sigma_\alpha\to L(E)$ such that $\sup_{z\in\Sigma_\beta}\|e^{-\omega z}S_K(z)\|<\infty$ for all $\beta\in(0,\alpha)$ and that $G(\lambda)=\widetilde{S_K}(\lambda)$ for all $\lambda\in(\omega,\infty)$. Furthermore, the uniqueness theorem for Laplace transform implies $S_K(z)=\frac{1}{c(\arg(z))}S_{\arg(z)}(|z|)$, $z\in\Sigma_\alpha$, and since $c(0)=1$ and $K_0=K$, it is enough to show that, for every fixed $x\in E$ and $\beta\in(0,\alpha)$, $\lim_{z\in\Sigma_\beta,\,z\to0}S_K(z)x=0$. To this end, notice that $\lim_{t\downarrow0}S_K(t)x=\lim_{t\downarrow0}S_0(t)x=0$ and that Theorem 1.1.11 implies

$$\lim_{z\in\Sigma_\beta,\,z\to0}e^{-\omega z}S_K(z)x=\lim_{z\in\Sigma_\beta,\,z\to0}S_K(z)x=0.\qquad\square$$

In the following corollary, we remove any density assumption from [**455**, Theorem]:

Corollary 2.4.11. *Suppose $r\geqslant0$, $\alpha\in(0,\frac{\pi}{2}]$ and $\omega\in[0,\infty)$ if $r>0$, resp. $\omega\in\mathbb{R}$, if $r=0$. Then A is a subgenerator of an analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant0}$ of angle α satisfying* $\sup_{\lambda\in\Sigma_\beta}\|e^{-\omega z}S_r(z)\|<\infty$ *for all $\beta\in(0,\alpha)$ iff the following conditions hold:*

(i) *For every $\theta\in(-\alpha,\alpha)$, $e^{i\theta}A$ is a subgenerator of an r-times integrated C-semigroup $(S_\theta(t))_{t\geqslant0}$, and*
(ii) *for every $\beta\in(0,\alpha)$, there exists $M_\beta>0$ such that $\|S_\theta(t)\|\leqslant M_\beta e^{\omega t\cos\theta}$, $t\geqslant0$, $\theta\in(-\beta,\beta)$.*

The proof of the next generalization of [**14**, Theorem 3.9.7] and [**14**, Corollary 3.9.9] follows from Theorem 2.4.5, Theorem 2.4.7 and the proof of [**14**, Corollary 2.6.1].

Theorem 2.4.12. *Suppose $\alpha\in(0,\frac{\pi}{2})$, $r\geqslant0$, $\omega\geqslant0$ and $e^{\pm i\alpha}A$ are subgenerators of exponentially bounded r-times integrated C-regularized semigroups $(S_r^{\pm\alpha}(t))_{t\geqslant0}$. Then, for every $\zeta>0$, A is a subgenerator of an exponentially bounded, analytic $(r+\zeta)$-times integrated C-regularized semigroup $(T_{r+\zeta}(t))_{t\geqslant0}$ of angle α*; *if A is densely defined, then A is a subgenerator of an exponentially bounded, analytic r-times integrated C-regularized semigroup $(T_r(t))_{t\geqslant0}$ of angle α.*

The following theorem is an extension of [**89**, Theorem 8.2] and can be applied to differential operators considered in [**82**], [**89**, Section XXIV], [**359**, Example 2.3] and [**416**].

Theorem 2.4.13. *Suppose $r\geqslant0$, $\theta\in(0,\frac{\pi}{2})$ and $-A$ is a subgenerator of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant0}$ of angle θ. Then there exists an injective operator $C_1\in L(E)$ such that A is a subgenerator of an entire C_1-regularized group in E. Furthermore, if A is densely defined, then C_1 can be chosen such that* $\mathrm{R}(C_1)$ *is dense in E.*

Proof. Let $\frac{\pi}{2} > \phi > \frac{\pi}{2} - \theta$ and $\phi < \alpha\phi < \frac{\pi}{2}$. Keeping in mind Theorem 2.4.7, one can assume that there exist a number $d \in (0,1]$ and an open neighborhood $\Omega_{\phi,d}$ of the region $\Sigma_\phi \ \cup \ \{z \in \mathbb{C} : |z| \leqslant d\}$ such that the set $\{(1+|\lambda|)^{1-r}(\lambda - A)^{-1}C : \lambda \in \Omega_{\phi,d}\}$ is bounded and that the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\lambda \in \Omega_{\phi,d}$ is analytic for every fixed $x \in E$. Denote by Γ_ϕ the boundary of $\Sigma_\phi \ \cup \ \{z \in \mathbb{C} : |z| \leqslant d\}$ and assume that Γ_ϕ is oriented in such a way that $\operatorname{Im}\lambda$ decreases along Γ_ϕ. Define $T_\alpha(z)x := \frac{1}{2\pi i}\int_{\Gamma_\phi} e^{-z\lambda^\alpha}(\lambda - A)^{-1}Cx\,d\lambda$, $x \in E$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$. Then the argumentation given in the proofs of [**394**, Proposition 2.3-Proposition 2.8] enables one to see that $T_\alpha(z)$ is injective for all $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ and that there exists $n_{r,\alpha} \in \mathbb{N}$ such that $\lim_{t\to 0+}\frac{T_\alpha(t)x - Cx}{t} = -\frac{1}{2\pi i}\int_{\Gamma_\phi}\lambda^{\alpha-1}(\lambda - A)^{-1}CAx\,d\lambda$, $x \in D(A^{n_{r,\alpha}})$. Define, for every $z_0 \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$,

$$S_{\alpha,z_0}(z)x := \frac{1}{2\pi i}\int\limits_{\Gamma_\phi} e^{\lambda z}e^{-z_0\lambda^\alpha}(\lambda - A)^{-1}Cx\,d\lambda,\ x \in E,\ z \in \mathbb{C}.$$

Then $S_{\alpha,z_0}(z) \in L(E)$ ($z_0 \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$, $z \in \mathbb{C}$) and the dominated convergence theorem implies that, for every $z_0 \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ and $x \in E$, $S_{\alpha,z_0}(z_1+z_2)T_\alpha(z_0) = S_{\alpha,z_0}(z_1)S_{\alpha,z_0}(z_2)$, $z_1,\ z_2 \in \mathbb{C}$ and that the mapping $z \mapsto S_{\alpha,z_0}(z)x$, $z \in \mathbb{C}$ is entire. Now it can be easily seen that, for every $z_0 \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$, $(S_{\alpha,z_0}(z))_{z\in\mathbb{C}}$ is an entire $T_\alpha(z_0)$-regularized group with a subgenerator A and the integral generator $T_\alpha(z_0)^{-1}AT_\alpha(z_0)$ (the last operator equals A provided $\rho(A) \neq \emptyset$). Assume now A is densely defined. Let $z_0 \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ be fixed. We will prove that $\mathrm{R}(T_\alpha(z_0))$ is dense in E by using a slight modification of the proof of [**89**, Lemma 8.8]. Assume $x^* \in E^*$, $\langle x^*, T_\alpha(z_0)x\rangle = 0$, $x \in E$ and denote, for every $x \in E$ and $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$,

$$F_{\alpha,z,x}(\zeta) = \Big\langle x^*, \frac{1}{2\pi i}\int\limits_{\Gamma_\phi}\lambda^\zeta e^{-z\lambda^\alpha}(\lambda - A)^{-1}Cx\,d\lambda\Big\rangle,\ \operatorname{Re}\zeta > 0.$$

By the dominated convergence theorem,

$$\lim_{h\to 0}\Big|\Big\langle x^*, \frac{T_\alpha(z+h)x - T_\alpha(z)x}{h} + F_{\alpha,z,x}(\alpha)\Big\rangle\Big| = 0,\ x \in E$$

and the convergence is uniform on bounded subsets of E. This implies that the mapping $z \mapsto T_\alpha(z)^*x^*$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ is differentiable and that $\langle \frac{d}{dz}T_\alpha(z)^*x^*, x\rangle = -F_{\alpha,z,x}(\alpha)$, $x \in E$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$. By induction, the mapping $z \mapsto T_\alpha(z)^*x^*$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ is infinitely differentiable and $\langle \frac{d^k}{dz^k}T_\alpha(z)^*x^*, x\rangle = (-1)^kF_{\alpha,z,x}(k\alpha)$, $x \in E$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$. Since $\rho_C(A) \neq \emptyset$, A is densely defined and $\mathrm{R}(C)$ is dense in E, it is obvious that the set $((\lambda - A)^{-1}C)^k(D(A^n))$ is dense in E for every $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Denote by B a linear operator $\{(x,y) \in E \times E : \lim_{t\to 0+}\frac{T_\alpha(t)x - Cx}{t} = y\}$ and notice that, for every $k \in \mathbb{N}$, there exists $n_{\alpha,r,k} \in \mathbb{N}$ such that $((\lambda - A)^{-1}C)^{n_{\alpha,r,k}}(D(A^{n_{r,\alpha}})) \subseteq D(B^k)$. One obtains inductively $\langle \frac{d^k}{dz^k}T_\alpha(z)^*x^*, x\rangle = \langle x^*, T_\alpha(z)B^kx\rangle$, $x \in ((\lambda - A)^{-1}C)^{n_{\alpha,r,k}}(D(A^{n_{r,\alpha}}))$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$, which implies $(\frac{d^k}{dz^k}T_\alpha(z)^*x^*)_{z=z_0} = 0$, $k \in \mathbb{N}_0$. Choose an arbitrary $x^{**} \in E^{**}$ and notice that the preceding equality and the infinite differentiability of the mapping $z \mapsto T_\alpha(z)^*x^*$,

$z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ together imply that the mapping $z \mapsto \langle x^{**}, T_\alpha(z)^* x^* \rangle$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ is analytic and that $(\frac{d^k}{dz^k}\langle x^{**}, T_\alpha(z)^* x^* \rangle)_{z=z_0} = \langle x^{**}, (\frac{d^k}{dz^k} T_\alpha(z)^* x^*)_{z=z_0} \rangle = 0$, $k \in \mathbb{N}_0$. Therefore, $T_\alpha(z)^* x^* = 0$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$, $\langle x^*, T_\alpha(z)x \rangle = 0$, $x \in E$, $z \in \Sigma_{\frac{\pi}{2}-\alpha\phi}$ and $x^* = 0$. The proof of theorem is completed. □

It is noteworthy that Definition 2.4.1 is a special case of the following definition which has been recently introduced in [**235**]:

DEFINITION 2.4.14. Let $0 < \alpha \leqslant \frac{\pi}{2}$ and let $(R(t))_{t \geqslant 0}$ be an (a,k)-regularized C-resolvent family. Then it is said that $(R(t))_{t \geqslant 0}$ is an *analytic (a,k)-regularized C-resolvent family of angle α,* if there exists an analytic function $\mathbf{R} : \Sigma_\alpha \to L(E)$ which satisfies:

(i) $\mathbf{R}(t) = R(t)$, $t > 0$ and

(ii) $\lim_{z \to 0,\, z \in \Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0, \alpha)$ and $x \in E$.

It is said that $(R(t))_{t \geqslant 0}$ is an *exponentially bounded, analytic (a,k)-regularized C-resolvent family,* resp. *bounded analytic (a,k)-regularized C-resolvent family, of angle* α, if for every $\gamma \in (0, \alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $\|\mathbf{R}(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$.

Since no confusion seems likely, we also write $R(\cdot)$ for $\mathbf{R}(\cdot)$. The next theorem can be proved by means of the arguments given in [**276**, Section 3] and [**369**, Chapter 2].

PROPOSITION 2.4.15. *Let $k(t)$ and $a(t)$ satisfy* (P1), $\lim_{\lambda \to +\infty} \lambda \tilde{k}(\lambda) = k(0) \neq 0$, *$A$ is densely bounded, $A \notin L(E)$ and there exists $\omega_0 \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$ such that $\int_0^\infty e^{-\omega t}|a(t)|\,dt < \infty$. Assume that A is a subgenerator of an exponentially bounded, analytic (a,k)-regularized C-resolvent family $(R(t))_{t \geqslant 0}$ of angle $\alpha \in (0, \frac{\pi}{2}]$ and that there exists $\omega \geqslant \omega_0$ such that*

$$\sup_{z \in \Sigma_\gamma} \|e^{-\omega z} R(z)\| < \infty \text{ for all } \gamma \in (0, \alpha). \tag{99}$$

Then the function $\tilde{a}(\lambda)$ can be extended to a meromorphic function defined on the sector $\omega + \Sigma_{\frac{\pi}{2}+\alpha}$.

It is worthwhile to mention that it is not clear, all assumptions of Proposition 2.4.16 being satisfied, whether A must be a subgenerator of an (a,C)-regularized resolvent family on $[0,\tau)$. Further on, let us notice that the assertions (i) and (ii) of [**369**, Theorem 2.2, p. 57] still hold in the case of exponentially bounded, analytic (a,C)-regularized resolvent families.

The subsequent theorem clarifies the basic analytical properties of (a,k)-regularized C-resolvent families. Notice only that the assertion which naturally corresponds to [**276**, Lemma 3.7] (cf. also [**369**, Corollary 2.2, p. 53]) does not seem attainable in the case of a general (a,k)-regularized C-resolvent family.

THEOREM 2.4.16. [**235**] (i) *Suppose $\alpha \in (0, \frac{\pi}{2}]$, $k(t)$ and $a(t)$ satisfy* (P1), (H5) *holds and $\tilde{k}(\lambda)$ can be analytically continued to a function $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$, where $\omega \geqslant \max\bigl(0, \operatorname{abs}(k), \operatorname{abs}(a)\bigr)$. Suppose, further, that A is a subgenerator of* *an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle α and that* (99) *holds. Set $N := \{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha} : g(\lambda) \neq 0\}$. Then N is an open connected subset of $\mathbb{C}$. Assume that there exists an analytic function $\hat{a} : N \to \mathbb{C}$ such that $\hat{a}(\lambda) = \tilde{a}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$. Then the operator $I - \hat{a}(\lambda)A$ is injective for every $\lambda \in N$, $\mathrm{R}(C) \subseteq \mathrm{R}\big(I - \hat{a}(\lambda)C^{-1}AC\big)$ for every $\lambda \in N_1 := \{\lambda \in N : \hat{a}(\lambda) \neq 0\}$,*

$$\sup_{\lambda \in N_1 \cap (\omega + \Sigma_{\pi/2+\gamma_1})} \big\|(\lambda-\omega)g(\lambda)(I - \hat{a}(\lambda)C^{-1}AC)^{-1}C\big\| < \infty, \ \ \gamma_1 \in (0,\alpha),$$

$$\textit{the mapping } \lambda \mapsto (I - \hat{a}(\lambda)C^{-1}AC)^{-1}C, \ \lambda \in N_1 \textit{ is analytic}$$

$$\lim_{\lambda \to +\infty,\ \tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = k(0)Cx, \ \ x \in E.$$

(ii) *Assume $k(t)$ and $a(t)$ satisfy* (P1), *$\omega \geqslant \max(0, \operatorname{abs}(k), \operatorname{abs}(a))$ and $\alpha \in (0, \frac{\pi}{2}]$. Assume, further, that A is a closed linear operator and that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have that the operator $I - \tilde{a}(\lambda)A$ is injective and that $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)A)$. If there exists an analytic function $q : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(E)$ such that*

$$q(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C, \ \lambda \in \mathbb{C}, \ \operatorname{Re}\lambda > \omega, \ \tilde{k}(\lambda) \neq 0, \tag{100}$$

$$\sup_{\lambda \in \omega + \Sigma_{\pi/2+\gamma}} \|(\lambda - \omega)q(\lambda)\| < \infty \textit{ for all } \gamma \in (0,\alpha), \tag{101}$$

$$\lim_{\lambda \to +\infty} \lambda q(\lambda)x = k(0)Cx, \ \ x \in E, \ \textit{if } \overline{D(A)} \neq E, \tag{102}$$

then A is a subgenerator of an exponentially bounded, analytic (a,k)-regularized C-resolvent family of angle α.

Example 2.4.17. Let $\beta \in (0,2)$, $\alpha > 0$, $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ and $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. Let A be densely defined. Then A is a subgenerator of an exponentially bounded, analytic (a,k)-regularized C-resolvent family of angle $\gamma \in (0, \frac{\pi}{2}]$ iff for every $\delta \in (0,\gamma)$, there exist $M_\delta > 0$ and $\omega_\delta \geqslant 0$ such that:

$$\big(\omega_\delta + \Sigma_{\frac{\pi}{2}+\delta}\big)^{1/\beta} \subseteq \rho_C(A),$$
$$\big\|(\lambda^\beta - A)^{-1}C\big\| \leqslant M_\delta(1+|\lambda|)^{\alpha-\beta}, \ \lambda \in (\omega_\delta + \Sigma_{\frac{\pi}{2}+\delta})^{1/\beta} \text{ and}$$
$$\text{the mapping } \lambda \mapsto (\lambda^\beta - A)^{-1}C, \ \lambda \in (\omega_\delta + \Sigma_{\frac{\pi}{2}+\delta})^{1/\beta} \text{ is analytic (continuous).}$$

The next theorem is an extension of Theorem 1.2.4 and Theorem 2.4.8:

Theorem 2.4.18. [**235**] (i) *Assume $k(t)$ and $a(t)$ satisfy* (P1), *and there exist $M > 0$ and $\omega > 0$ such that $|k(t)| \leqslant Me^{\omega t}$, $t \geqslant 0$. Assume, further, that there exist a number $\omega' \geqslant \omega$ and a function $a_1(t)$ satisfying* (P1) *and $\tilde{a_1}(\lambda) = \tilde{a}\big(\sqrt{\lambda}\big)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega'$. Let A be a subgenerator of an exponentially bounded (a,k)-regularized C-resolvent family $(C(t))_{t\geqslant 0}$ and let* (H5) *hold. Then A is a subgenerator of an exponentially bounded, analytic (a_1,k_1)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where:*

$$k_1(t) := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} k(s)\, ds, \ \ t > 0, \ k_1(0) := k(0), \tag{103}$$

$$R(t)x := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} C(s)x\,ds,\ \ t>0,\ x\in E,\ R(0):=k(0)C. \tag{104}$$

(ii) *Assume $k(t)$ satisfy* (P1), *$\beta>0$ and there exist $M>0$ and $\omega>0$ such that $|k(t)|\leqslant Me^{\omega t}$, $t\geqslant 0$. Let A be a subgenerator of an exponentially bounded $\big(\frac{t^{2\beta-1}}{\Gamma(2\beta)},k\big)$-regularized C-resolvent family $(C(t))_{t\geqslant 0}$ and let* (H5) *hold. Then A is a subgenerator of an exponentially bounded, analytic $\big(\frac{t^{\beta-1}}{\Gamma(\beta)},k_1\big)$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where $k_1(t)$ and $R(t)$ are defined through* (103)–(104).

Notice that $a_1(t)=\int_0^\infty s\frac{e^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}a(s)\,ds$, $t>0$, whenever the function $a(t)$ is exponentially bounded. Further on, Kato's analyticity criterion for exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},C)$-regularized resolvent families ($0<\alpha<2$) has been recently proved by Chen and Li in [**56**]; it seems plausible that the assertion of [**56**, Theorem 4.8] (cf. also Theorem 2.4.10, Corollary 2.4.11 and Theorem 2.4.12) can be reformulated and proved in the case of a general $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized C-resolvent family. For further information on the interplay between exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},\frac{t^{\beta}}{\Gamma(\beta+1)})$-regularized C-resolvent families ($0<\alpha<2$, $\beta>0$) with corresponding growth order at zero and exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},C)$-regularized resolvent families, we refer the reader to [**56**, Theorem 4.9] and [**224**, Theorem 1.1].

Concerning subordination principle established in Theorem 2.1.27(ixx), we have the following theorem whose purpose is to improve the angle of analyticity of the resolvent $(S_\alpha(t))_{t\geqslant 0}$ provided that $(S_\beta(t))_{t\geqslant 0}$ is an exponentially bounded, analytic $(\frac{t^{\beta-1}}{\Gamma(\beta)},k_\beta)$-regularized C-resolvent family. We employ the same notation as in the formulation of Theorem 2.1.27(ixx).

THEOREM 2.4.19. *Assume $(S_\beta(t))_{t\geqslant 0}$ is an exponentially bounded, analytic $(\frac{t^{\beta-1}}{\Gamma(\beta)},k_\beta)$-regularized C-resolvent family of angle $\delta\in(0,\frac{\pi}{2}]$ and there exist functions $c:(-\frac{\pi}{2}\alpha,\frac{\pi}{2}\alpha)\rightarrow\mathbb{C}\smallsetminus\{0\}$ and $\omega_0:(-\frac{\pi}{2}\alpha,\frac{\pi}{2}\alpha)\rightarrow[0,\infty)$ such that*

$$\Phi_\theta=\big\{\lambda\in(\omega_0(\theta),\infty):\widetilde{k_\beta}(\lambda e^{-i\theta/\beta})=0\big\}=\big\{\lambda\in(\omega_0(\theta),\infty):\widetilde{k_\beta}(\lambda)=0\big\} \tag{105}$$

and

$$\widetilde{k_\beta}(\lambda)/\widetilde{k_\beta}(\lambda e^{-i\theta/\beta})=c(\theta),\ \ \lambda>\omega_0(\theta),\ \ \lambda\notin\Phi_\theta,\ \ \theta\in\Big(-\frac{\pi}{2}\alpha,\frac{\pi}{2}\alpha\Big). \tag{106}$$

Assume, further, that there exist a number $\omega'>\omega$ and an analytic function $\hat{k_\beta}:\omega'+\Sigma_{\frac{\pi}{2}+\delta}\rightarrow\mathbb{C}$ such that $\hat{k_\beta}(\lambda)=\widetilde{k_\beta}(\lambda)$, $\operatorname{Re}\lambda>\omega'$. Set $\nu:=\min(\frac{1}{\gamma}(\frac{\pi}{2}+\delta)-\frac{\pi}{2},\frac{\pi}{2})$, $\vartheta:=\min(\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)+\frac{\delta}{\gamma},\pi)$, $\vartheta_s:=\vartheta$ if $(S_\beta(t))_{t\geqslant 0}$ is a bounded, analytic $(\frac{t^{\beta-1}}{\Gamma(\beta)},k_\beta)$-regularized C-resolvent family and $\vartheta_s:=\min(\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})+\frac{\delta}{\gamma},\pi)$, otherwise. Then the mapping $t\mapsto S_\alpha(t)$, $t>0$ is analytically extendible to the sector Σ_ϑ and the mapping $z\mapsto S_\alpha(z)x$, $z\in\overline{\Sigma_{\vartheta_s-\varepsilon}}$ is continuous for every $\varepsilon\in(0,\vartheta_s)$ and $x\in E$.

PROOF. Designate $N := \{\lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\delta} : \hat{k_\beta}(\lambda) \neq 0\}$. Then it is obvious that there exists an analytic function $q_\beta : \omega' + \Sigma_{\frac{\pi}{2}+\delta} \to L(E)$ satisfying that $q_\beta(\lambda)x = \widetilde{k_\beta}(\lambda)(I - \lambda^{-\beta}A)^{-1}Cx$, $\operatorname{Re}\lambda > \omega'$, $\widetilde{k_\beta}(\lambda) \neq 0$, $x \in E$. By the proof of [**235**, Theorem 2.16], the following holds:

(a) The operator $I - \lambda^{-\beta}A$ is injective for all $\lambda \in N$ and $R(C) \subseteq R(I - \lambda^{-\beta}C^{-1}AC)$, $\lambda \in N$.
(b) For every $\delta_1 \in (0, \delta)$, the set $\{(\lambda - \omega')q_\beta(\lambda) : \lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\delta_1}\}$ is bounded.
(c) For every $x \in E$, $q_\beta(\lambda)x = \hat{k_\beta}(\lambda)(I - \lambda^{-\beta}C^{-1}AC)^{-1}Cx$, $\lambda \in N$ and $\lim_{\lambda\to+\infty} \lambda q_\beta(\lambda)x = k_\beta(0)Cx = k_\alpha(0)Cx$.

Let $\varepsilon \in (0, \nu)$. Put now, for every $x \in E$ and for every $\lambda \in \mathbb{C}$ such that $\lambda^\gamma \in \omega' + \Sigma_{\frac{\pi}{2}+\delta}$, $q_\alpha(\lambda)x := \lambda^{\gamma-1}q_\beta(\lambda^\gamma)x$. Taking into account (a)-(c) and the equality $\widetilde{k_\alpha}(\lambda) = \lambda^{\frac{\alpha}{\beta}-1}\widetilde{k_\beta}(\lambda^{\frac{\alpha}{\beta}})$, $\lambda > a$, we easily infer that there exists a sufficiently large number $\omega'_\varepsilon > 0$ such that $\lim_{\lambda\to+\infty} \lambda q_\alpha(\lambda)x = k_\alpha(0)Cx$ as well as that the set $\{(\lambda - \omega'_\varepsilon)q_\alpha(\lambda) : \lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2}+\nu-\varepsilon}\}$ is bounded and that $q_\alpha(\lambda)x = \widetilde{k_\alpha}(\lambda)(I - \lambda^{-\alpha}A)^{-1}Cx$, $\operatorname{Re}\lambda > \omega'_\varepsilon$, $\widetilde{k_\alpha}(\lambda) \neq 0$, $x \in E$. Using Theorem 2.4.16(ii), we get that $(S_\alpha(t))_{t\geqslant 0}$ is an exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_\alpha)$-regularized C-resolvent family of angle $\nu - \varepsilon$ having A as a subgenerator. By the arbitrariness of ε, A is a subgenerator of the exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_\alpha)$-regularized C-resolvent family $(S_\alpha(t))_{t\geqslant 0}$ of angle ν. Suppose, for the time being, $\theta \in (-\beta\delta, \beta\delta) \cap (-\nu\alpha, \nu\alpha)$ and put $\Gamma_{\theta,\zeta} := \{te^{-i\frac{\theta}{\zeta}} : t \geqslant 0\}$, $\zeta > 0$. By (105)-(106) we have that, for every $x \in E$ and $\lambda > \omega_0(\theta)$ with $\lambda \notin \Phi_\theta$:

$$
\begin{aligned}
\widetilde{k_\beta}(\lambda)(I - \lambda^{-\beta}e^{i\theta}A)^{-1}Cx &= \widetilde{k_\beta}(\lambda)\lambda^\beta e^{-i\theta}((\lambda e^{-i\frac{\theta}{\beta}})^\beta - A)^{-1}Cx \\
&= \widetilde{k_\beta}(\lambda)\lambda^\beta e^{-i\theta}\frac{1}{\widetilde{k_\beta}(\lambda e^{-i\frac{\theta}{\beta}})\lambda^\beta e^{-i\theta}}\int_0^\infty e^{-\lambda e^{-i\frac{\theta}{\beta}}t}S_\beta(t)x\,dt \\
&= c(\theta)\int_0^\infty e^{-\lambda e^{-i\frac{\theta}{\beta}}t}S_\beta(t)x\,dt = c(\theta)e^{i\frac{\theta}{\beta}}\int_{\Gamma_{\theta,\beta}} e^{-\lambda z}S_\beta(ze^{i\frac{\theta}{\beta}})x\,dz
\end{aligned}
$$

and by Cauchy formula

$$
= \int_0^\infty e^{-\lambda t}\big(c(\theta)e^{i\frac{\theta}{\beta}}S_\beta(te^{i\frac{\theta}{\beta}})x\big)\,dt.
$$

This implies that $e^{i\theta}A$ is a subgenerator of an exponentially bounded $(\frac{t^{\beta-1}}{\Gamma(\beta)}, k_\beta)$-regularized C-resolvent family $(S_{\beta,\theta}(t) \equiv c(\theta)e^{i\frac{\theta}{\beta}}S_\beta(te^{i\frac{\theta}{\beta}}))_{t\geqslant 0}$. By making use of Theorem 2.1.27(ixx), we get that $e^{i\theta}A$ is a subgenerator of an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_\alpha)$-regularized C-resolvent family $(S_{\alpha,\theta}(t))_{t\geqslant 0}$ such that the mapping $t \mapsto S_{\alpha,\theta}(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$. By (105)-(106), we obtain that $\widetilde{k_\alpha}(\lambda) = c(\theta)e^{i\frac{\theta}{\alpha}(\frac{\alpha}{\beta}-1)}\widetilde{k_\alpha}(\lambda e^{-i\frac{\theta}{\alpha}}) \neq 0$, provided $\lambda > \omega_0(\theta)^{1/\gamma}$ and $\lambda^\gamma \notin \Phi_\theta$. Using this equality and Cauchy formula, we obtain that there exists

$\omega''_\theta > \omega_0(\theta)^{1/\gamma}$ such that, for every $x \in E$ and $\lambda > \omega''_\theta$ with $\lambda^\gamma \notin \Phi_\theta$:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\big(c(\theta)e^{i\frac{\theta}{\beta}}S_\alpha(te^{i\frac{\theta}{\alpha}})x\big)\,dt &= c(\theta)e^{i\frac{\theta}{\beta}}e^{-i\frac{\theta}{\alpha}}\int_{\Gamma_{\theta,\alpha}} e^{-\lambda z e^{-i\frac{\theta}{\alpha}}}S_\alpha(z)x\,dz\\ &= c(\theta)e^{i\frac{\theta}{\beta}}e^{-i\frac{\theta}{\alpha}}\int_0^\infty e^{-\lambda t e^{-i\frac{\theta}{\alpha}}}S_\alpha(t)x\,dt\\ &= c(\theta)e^{i\frac{\theta}{\beta}}e^{-i\frac{\theta}{\alpha}}\widetilde{k_\alpha}(\lambda e^{-i\frac{\theta}{\alpha}})\lambda^\alpha e^{-i\theta}(\lambda^\alpha e^{-i\theta}-A)^{-1}Cx\\ &= c(\theta)e^{i\frac{\theta}{\beta}}e^{-i\frac{\theta}{\alpha}}c(\theta)^{-1}\widetilde{k_\alpha}(\lambda)e^{-i\frac{\theta}{\alpha}(\frac{\alpha}{\beta}-1)}\lambda^\alpha e^{-i\theta}(\lambda^\alpha e^{-i\theta}-A)^{-1}Cx\\ &= \widetilde{k_\alpha}(\lambda)\lambda^\alpha(\lambda^\alpha - e^{i\theta}A)^{-1}Cx.\end{aligned}$$

This, in combination with the obvious equality

$$\int_0^\infty e^{-\lambda t}S_{\alpha,\theta}(t)x\,dt = \widetilde{k_\alpha}(\lambda)\lambda^\alpha(\lambda^\alpha - e^{i\theta}A)^{-1}Cx,\ \ \lambda > \omega''_\theta,\ \widetilde{k_\alpha}(\lambda) \neq 0,\ x \in E$$

and the uniqueness theorem for Laplace transform, yields:

$$S_{\alpha,\theta}(t)x = c(\theta)e^{i\frac{\theta}{\beta}}S_\alpha(te^{i\frac{\theta}{\alpha}})x,\ t \geqslant 0,\ \theta \in (-\beta\delta,\beta\delta)\cap(-\nu\alpha,\nu\alpha),\ x \in E. \tag{107}$$

Set $\Omega := \{\lambda^{1/\gamma} : \lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\delta}\}$ and $\hat{k_\alpha}(\lambda) := \lambda^{\gamma-1}\hat{k_\beta}(\lambda^\gamma)$, $\lambda \in \Omega$. By the uniqueness theorem for analytic functions we have that, for every $\theta \in (-\beta\delta,\beta\delta)$ and for every $\lambda \in \Omega$ with $\lambda e^{-i\frac{\theta}{\alpha}} \in \Omega$:

$$\hat{k_\alpha}(\lambda) = c(\theta)e^{i\frac{\theta}{\alpha}(\frac{\alpha}{\beta}-1)}\hat{k_\alpha}(\lambda e^{-i\frac{\theta}{\alpha}}). \tag{108}$$

The next step is to prove that the assumptions $\theta_1,\ \theta_2 \in (-\beta\delta,\beta\delta)$, $z \in \mathbb{C}$ and $ze^{-i\frac{\theta_1}{\alpha}},\ ze^{-i\frac{\theta_2}{\alpha}} \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})}$ imply:

$$e^{-i\frac{\theta_1}{\beta}}c(\theta_1)^{-1}S_{\alpha,\theta_1}(ze^{-i\frac{\theta_1}{\alpha}})x = e^{-i\frac{\theta_2}{\beta}}c(\theta_2)^{-1}S_{\alpha,\theta_2}(ze^{-i\frac{\theta_2}{\alpha}})x,\ x \in E. \tag{109}$$

By making use of the argument that has been appeared twice in the proof so far, it follows that there exists $\omega''' > 0$ such that:

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}S_{\alpha,\theta_j}\big(te^{i\arg(ze^{-i\frac{\theta_j}{\alpha}})}\big)x\,dt \\ &= \lambda^\alpha e^{-i\arg(ze^{-i\frac{\theta_j}{\alpha}})(1+\alpha)}\widetilde{k_\alpha}\big(\lambda e^{-i\arg(ze^{-i\frac{\theta_j}{\alpha}})}\big)\big(\lambda^\alpha e^{-i\arg(ze^{-i\frac{\theta_j}{\alpha}})\alpha} - e^{i\theta_j}A\big)^{-1}Cx,\end{aligned} \tag{110}$$

provided $x \in E,\ \lambda > \omega''',\ \widetilde{k_\alpha}(\lambda e^{-i\arg(ze^{-i\frac{\theta_1}{\alpha}})})\widetilde{k_\alpha}(\lambda e^{-i\arg(ze^{-i\frac{\theta_2}{\alpha}})}) \neq 0$ and $j = 1, 2$. By (108), (110) and an elementary computation, we obtain that

$$\begin{aligned}&e^{-i\frac{\theta_1}{\beta}}c(\theta_1)^{-1}\int_0^\infty e^{-\lambda t}S_{\alpha,\theta_1}(te^{i\arg(ze^{-i\frac{\theta_1}{\alpha}})})x\,dt\\ &\qquad = e^{-i\frac{\theta_2}{\beta}}c(\theta_2)^{-1}\int_0 e^{-\lambda t}S_{\alpha,\theta_2}(te^{i\arg(ze^{-i\frac{\theta_2}{\alpha}})})x\,dt,\ \lambda > \omega''',\ x \in E,\end{aligned}$$

which implies (109) by the uniqueness theorem for Laplace transform. In the remnant of the proof, we consider three possible cases.

Case 1. $(\frac{1}{\gamma}-1)\frac{\pi}{2}\geqslant\pi$. The assertion of theorem in this case trivially follows from an application of Theorem 2.1.27(ixx).

Case 2. $(\frac{1}{\gamma}-1)\frac{\pi}{2}\in[\frac{\pi}{2},\pi)$. In this case, $\vartheta=\min((\frac{1}{\gamma}-1)\frac{\pi}{2}+\min(\frac{\delta}{\gamma},\nu),\pi)$. Let $\varepsilon'\in(0,\vartheta)$ be sufficiently small and let $x\in E$. Then there exists $\theta\in(0,\beta\delta)\cap(0,\nu\alpha)$ such that $\frac{\theta}{\alpha}>\vartheta-\frac{\varepsilon'}{2}-(\frac{1}{\gamma}-1)\frac{\pi}{2}$. Define, for every $z\in\Sigma_{\vartheta-\varepsilon'}$,

$$S_\alpha(z)x:=\begin{cases} S_\alpha(z)x, \ z\in\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}},\\ e^{-i\frac{\theta}{\beta}}c(\theta)^{-1}S_{\alpha,\theta}\big(ze^{-i\frac{\theta}{\alpha}}\big)x, \ z\in\Sigma_{\vartheta-\varepsilon'}\smallsetminus\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}.\end{cases}$$

By (107), one can simply prove that the mapping $z\mapsto S_\alpha(z)x$, $z\in\Sigma_{\vartheta-\varepsilon'}$ is analytic, which implies by arbitrariness of ε' that the mapping $t\mapsto S_\alpha(t)x$, $t>0$ can be analytically extended to the sector Σ_ϑ. If $(S_\beta(t))_{t\geqslant0}$ is a bounded analytic $(\frac{t^{\beta-1}}{\Gamma(\beta)},k_\beta)$-regularized C-resolvent family, then $(S_{\beta,\theta}(t))_{t\geqslant0}$ is also bounded. By Theorem 2.1.27(ixx), the mappings $z\mapsto S_{\alpha,\theta}(z)x$ and $z\mapsto S_\alpha(z)x$ are continuous on the closure of any proper subsector of $\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}$, which implies that the mapping $z\mapsto S_\alpha(z)x$, $z\in\overline{\Sigma_{\vartheta-2\varepsilon'}}$ is continuous. Suppose $(S_\beta(t))_{t\geqslant0}$ is an exponentially bounded, analytic $(\frac{t^{\beta-1}}{\Gamma(\beta)},k_\beta)$-regularized C-resolvent family. Then one can simply prove that there exists $\theta'\in(0,\beta\delta)\ \cap\ (0,\nu\alpha)$ such that $\frac{\theta'}{\alpha}>\frac{\delta}{\gamma}-\frac{\varepsilon'}{2}$ and $S_\alpha(z)x=e^{-i\frac{\theta'}{\beta}}c(\theta')^{-1}S_{\alpha,\theta'}(ze^{-i\frac{\theta'}{\alpha}})x$, $z\in\Sigma_{\vartheta_s-\varepsilon'}\smallsetminus\Sigma_{\frac{\pi}{2}-\frac{\varepsilon'}{2}}$, which implies by Theorem 2.1.27(ixx) the continuity of the mapping $z\mapsto S_\alpha(z)x$, $z\in\overline{\Sigma_{\vartheta_s-2\varepsilon'}}$.

Case 3. $(\frac{1}{\gamma}-1)\frac{\pi}{2}\in(0,\frac{\pi}{2})$. Then $\vartheta=\vartheta_s=\min((\frac{1}{\gamma}-1)\frac{\pi}{2}+\frac{\delta}{\gamma},\pi)$. Let $\varepsilon''\in(0,(\frac{1}{\gamma}-1)\frac{\pi}{2})$ be sufficiently small and let $x\in E$. Define, for every $\zeta\in[(\frac{1}{\gamma}-1)\frac{\pi}{2},\vartheta-\varepsilon'')\cup(-(\vartheta-\varepsilon''),-(\frac{1}{\gamma}-1)\frac{\pi}{2}]$:

$$\begin{aligned}
&\varepsilon_\zeta=:\tfrac{1}{2}\min((\tfrac{1}{\gamma}-1)\tfrac{\pi}{2}-\varepsilon''+\tfrac{\delta}{\gamma}-|\zeta|,(\tfrac{1}{\gamma}-1)\tfrac{\pi}{2}-\varepsilon'',\vartheta-\varepsilon''-|\zeta|),\\
&\Omega_\zeta:=\{z\in\mathbb{C}:z\neq0,\ \arg(z)\in(\zeta-\varepsilon_\zeta,\zeta+\varepsilon_\zeta)\},\\
&\theta_\zeta:=\alpha[|\zeta|-((\tfrac{1}{\gamma}-1)t\tfrac{\pi}{2}-\varepsilon'')+\varepsilon_\zeta],
\end{aligned}$$

$$S_\alpha^\zeta(z)x:=e^{-i\frac{\theta_\zeta}{\beta}}c(\theta_\zeta)^{-1}S_{\alpha,\theta_\zeta}(ze^{-i\frac{\theta_\zeta}{\alpha}})x,\ z\in\Omega_\zeta \text{ and } S_\alpha^\zeta(0)x:=k_\alpha(0)Cx.$$

Notice that, for every $\zeta\in[(\frac{1}{\gamma}-1)\frac{\pi}{2},\vartheta-\varepsilon'')\cup(-(\vartheta-\varepsilon''),-(\frac{1}{\gamma}-1)\frac{\pi}{2}]$, the mapping $z\mapsto S_\alpha^\zeta(z)x$, $z\in\Omega_\zeta$ is analytic as well as $\Omega_\zeta\subseteq\Sigma_{\vartheta-\varepsilon''}$, $ze^{-i\frac{\theta_\zeta}{\alpha}}\in\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}-\varepsilon''}$, $z\in\Omega_\zeta$ and $\theta_\zeta\in(0,\beta\delta)$. By Theorem 2.1.27(ixx), the mapping $z\mapsto S_\alpha^\zeta(z)x$, $z\in\Omega_\zeta\cup\{0\}$ is continuous provided $|\zeta|\in[(\frac{1}{\gamma}-1)\frac{\pi}{2},\vartheta-\varepsilon'')$. Furthermore, there exist $k\in\mathbb{N}$ and $\zeta_1,\cdots,\zeta_k\in[(\frac{1}{\gamma}-1)\frac{\pi}{2},\vartheta-\varepsilon'')\cup(-(\vartheta-\varepsilon''),-(\frac{1}{\gamma}-1)\frac{\pi}{2}]$ such that:

$$\overline{\Sigma_{\vartheta_s-2\varepsilon''}}\subseteq\overline{\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}-\varepsilon''}}\cup\Omega_{\zeta_1}\cup\cdots\cup\Omega_{\zeta_k}. \tag{111}$$

By (109), one has $S_\alpha^{\zeta_1}(z)x=S_\alpha^{\zeta_2}(z)x$ for all $z\in\Omega_{\zeta_1}\cap\Omega_{\zeta_2}$ and ζ_1,ζ_2 satisfying the properties stated above. Assume $z\in\Sigma_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}\cap\Omega_{(\frac{1}{\gamma}-1)\frac{\pi}{2}}$. Using the Laplace transform, Cauchy formula, (108) and (110), we get that $S_\alpha(z)x=S_\alpha^{(\frac{1}{\gamma}-1)\frac{\pi}{2}}(z)x$.

Now it is clear that the mapping $t \mapsto S_\alpha(t)x$, $t > 0$ admits an analytic extension to the sector Σ_ϑ. The continuity of mapping $z \mapsto S_\alpha(z)x$, $z \in \overline{\Sigma_{\vartheta_s - 2\varepsilon''}}$ follows from (111) and Theorem 2.1.27(ixx), which completes the proof of theorem. $\square$

2.5. Perturbation theorems

We start this section by stating the following rescaling result for subgenerators of K-convoluted C-semigroups.

THEOREM 2.5.1. *Suppose $z \in \mathbb{C}$, K and F satisfy* (P1), *there exists $a > 0$ such that*

$$\frac{\tilde{K}(\lambda) - \tilde{K}(\lambda + z)}{\tilde{K}(\lambda + z)} = \int_0^\infty e^{-\lambda t} F(t)\, dt, \quad \operatorname{Re}\lambda > a, \ \tilde{K}(\lambda + z) \neq 0, \tag{112}$$

and A is a subgenerator, resp. the integral generator, of a (*local*) *K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Then $A - z$ is a subgenerator, resp. the integral generator, of a* (*local*) *K-convoluted C-semigroup $(S_{K,z}(t))_{t\in[0,\tau)}$, where:*

$$S_{K,z}(t) := e^{-tz} S_K(t) + \int_0^t F(t-s) e^{-zs} S_K(s)\, ds, \quad t \in [0, \tau). \tag{113}$$

Furthermore, in the case $\tau = \infty$, $(S_{K,z}(t))_{t\geqslant 0}$ is exponentially bounded provided that F and $(S_K(t))_{t\geqslant 0}$ are exponentially bounded.

PROOF. It is clear that $(S_{K,z}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family which commutes with C and $A - z$. Furthermore,

$$\begin{aligned}
(A-z)\int_0^t S_{K,z}(s)x\, ds &= (A-z)\int_0^t \Bigg[e^{-zs}S_K(s)x + \int_0^s F(s-r)e^{-zr}S_K(r)x\, dr\Bigg] ds \\
&= (A-z)\Bigg[e^{-zt}\int_0^t S_K(s)x\, ds + z\int_0^t e^{-sz}\int_0^s S_K(r)x\, dr\, ds\Bigg] \\
&\quad + (A-z)\int_0^t\int_0^s F(s-r)e^{-zr}S_K(r)x\, dr\, ds \\
&= e^{-zt}\big[S_K(t)x - \Theta(t)Cx\big] - ze^{-zt}\int_0^t S_K(s)x\, ds + z\int_0^t e^{-sz}\big[S_K(s)x - \Theta(s)Cx\big]\, ds \\
&\quad - z^2\int_0^t e^{-sz}\int_0^s S_K(r)x\, dr\, ds + (A-z)\int_0^t F(t-s)\int_0^s e^{-zr}S_K(r)x\, dr\, ds \\
&= e^{-zt}\big[S_K(t)x - \Theta(t)Cx\big] - ze^{-zt}\int_0^t S_K(s)x\, ds
\end{aligned}$$

$$+ z\int_0^t e^{-sz}\big[S_K(s)x - \Theta(s)Cx\big]\,ds - z^2\int_0^t e^{-sz}\int_0^s S_K(r)x\,dr\,ds$$
$$+ \int_0^t F(t-s)(A-z)\left[e^{-zs}\int_0^s S_K(r)x\,dr + z\int_0^s e^{-zr}\int_0^r S_K(v)x\,dv\,dr\right]ds$$
$$= S_{K,z}(t)x - f_1(t) - f_2(t)Cx,$$

where:

$$f_1(t) := ze^{-zt}\int_0^t S_K(s)x\,ds - z\int_0^t e^{-sz}S_K(s)x\,ds$$
$$+ z^2\int_0^t e^{-sz}\int_0^s S_K(r)x\,dr\,ds + z\int_0^t e^{-zs}F(t-s)\int_0^s S_K(r)x\,dr\,ds$$
$$- z\int_0^t F(t-s)\int_0^s e^{-zr}\big[S_K(r)x - \Theta(r)Cx\big]\,dr\,ds \tag{114}$$
$$+ z^2\int_0^t F(t-s)\int_0^s e^{-zr}\int_0^r S_K(v)x\,dv\,dr\,ds,\ t\in[0,\tau)$$
$$f_2(t) := \Theta(t)e^{-zt} + z\int_0^t e^{-zs}\Theta(s)\,ds$$
$$+ \int_0^t F(t-s)e^{-zs}\Theta(s)\,ds - \int_0^t F(t-s)\int_0^s e^{-zr}\Theta(r)\,dr\,ds,\quad t\in[0,\tau).$$

Fix a number $t\in(0,\tau)$ and define afterwards a function $\widetilde{S}_K : [0,\infty)\to L(E)$ by setting

$$\widetilde{S}_K(s) := \begin{cases} S_K(s), & s\in[0,t],\\ S_K(t), & s>t.\end{cases}$$

Clearly, $(\widetilde{S}_K(t))_{t\geqslant 0}$ is a strongly continuous operator family and there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|\widetilde{S}_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$. Define $\tilde{f}_1 : [0,\infty)\to L(E)$ by replacing $S_K(\cdot)$ in (114) with $\widetilde{S}_K(\cdot)$. Then $\tilde{f}_1(\cdot)$ extends continuously the function $f_1(\cdot)$ to the whole non-negative real axis, and moreover, $\tilde{f}_1(\cdot)$ is Laplace transformable. Using the elementary operational properties of Laplace transforms, one obtains $\mathcal{L}\big(f_1(t)\big)(\lambda) = \mathcal{L}\big(f_2(t)\big)(\lambda) = 0$ for all sufficiently large real numbers λ. An application of the uniqueness theorem for the Laplace transform gives that $A-z$ is a subgenerator of a (local) K-convoluted C-semigroup $(S_{K,z}(t))_{t\in[0,\tau)}$. Suppose now that A is the integral generator of $(S_K(t))_{t\in[0,\tau)}$. Then one has $C^{-1}AC = A$ and this implies that $C^{-1}(A-z)C = A-z$ is the integral generator of $(S_{K,z}(t))_{t\in[0,\tau)}$.

Finally, the exponential boundedness of $(S_{K,z}(t))_{t\geqslant 0}$ simply follows from (113) and the exponential boundedness of F and $(S_K(t))_{t\geqslant 0}$. $\square$

Suppose $K = \mathcal{L}^{-1}\big(\frac{p_m(\lambda)}{p_k(\lambda)}\big)$, where p_k (p_m) is a polynomial of degree k (m) and $k > m$. Then the condition (112) holds for a suitable exponentially bounded function F. Suppose now $\alpha > 0$ and $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$. Then there exists a sufficiently large positive real number a such that, for every $\lambda > a$,

$$\frac{\tilde{K}(\lambda) - \tilde{K}(\lambda+z)}{\tilde{K}(\lambda+z)} = \Big(1 + \frac{z}{\lambda}\Big)^{\alpha} - 1 = \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n}{\lambda^n} = \mathcal{L}\Big(\sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!}\Big)(\lambda),$$

where $1^{\alpha} = 1$. Since $\sup_{n\in\mathbb{N}} |\binom{\alpha}{n}| =: L_0 < \infty$, we obtain

$$\Big|\sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!}\Big| \leqslant L_0 |z| e^{|z|t}, \ t \geqslant 0.$$

Hence, we have the following.

Theorem 2.5.2. *Suppose $z \in \mathbb{C}$, $\alpha > 0$ and A is a subgenerator, resp. the integral generator, of a* (*local, global exponentially bounded*) *α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Then $A - z$ is a subgenerator, resp. the integral generator, of a* (*local, global exponentially bounded*) *α-times integrated C-semigroup $(S_{\alpha,z}(t))_{t\in[0,\tau)}$, which is given by:*

$$S_{\alpha,z}(t)x = e^{-zt} S_\alpha(t)x + \int_0^t \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!} e^{-zs} S_\alpha(s)x\, ds, \ \ t \in [0,\tau), \ x \in E.$$

Theorem 2.5.3. *Suppose $B \in L(E)$, K is a kernel and satisfies* (P1), *A is a subgenerator* (*the integral generator*) *of a* (*local*) *K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, $BA \subseteq AB$, $BC = CB$ and there exists $a > 0$ such that the following conditions hold:*

(i) *For every $n \in \mathbb{N}$, there is a function $K_n(\cdot)$ satisfying* (P1) *and*

$$\widetilde{K_n}(\lambda) = \tilde{K}(\lambda)\Big(\frac{1}{\tilde{K}(\cdot)}\Big)^{(n)}(\lambda), \ \ \lambda > a, \ \tilde{K}(\lambda) \neq 0.$$

Put $\Theta_n(t) := \int_0^t |K_n(s)|\, ds$, $t \geqslant 0$, $n \in \mathbb{N}$.

(ii) *$\sum_{n=1}^{\infty} \Theta_n(t) < \infty$, $t \geqslant 0$.*

(iii) *The function $t \mapsto \max_{s\in[0,t]} |\Theta(s)| e^{-at} \sum_{n=1}^{\infty} \Theta_n(t)$, $t \geqslant 0$ is an element of the space $L^1([0,\infty) : \mathbb{R})$.*

Then $A+B$ is a subgenerator (*the integral generator*) *of a* (*local*) *K-convoluted C-semigroup $(S_K^B(t))_{t\in[0,\tau)}$, given by*

$$S_K^B(t) := e^{tB} S_K(t) + \sum_{i=1}^{\infty} \sum_{n=1}^{i} \frac{B^i}{i!} (-1)^n \binom{i}{n} \int_0^t K_n(t-s) s^{i-n} S_K(s)\, ds. \tag{115}$$

Furthermore, the following holds:

(a) $\left\|S_K^B(t)-e^{tB}S_K(t)\right\| \leqslant e^{\|B\|}\max_{s\in[0,t]}\|S_K(s)\|\sum_{n=1}^{\infty}\Theta_n(t)e^{t\|B\|}$ *for every* $t\in[0,\tau)$.

(b) *Suppose* $\tau=\infty$, $(S_K(t))_{t\geqslant 0}$ *is exponentially bounded and there exist constants* $M>0$ *and* $\omega\geqslant 0$ *such that*

$$\sum_{n=1}^{\infty}\Theta_n(t)\leqslant Me^{\omega t},\ t\geqslant 0. \tag{116}$$

Then $(S_K^B(t))_{t\in[0,\tau)}$ *is also exponentially bounded.*

PROOF. Notice that the commutation of B with C and A implies that the function $u_1(\cdot)$, resp. $u_2(\cdot)$, given by $u_1(t):=\int_0^t S_K(s)Bx\,ds$, $t\in[0,\tau)$, resp. $u_2(t):=\int_0^t BS_K(s)x\,ds$, $t\in[0,\tau)$, solves the initial value problem

$$\begin{cases} u\in C([0,\tau):[D(A)])\cap C^1([0,\tau):E),\\ u'(t)=Au(t)+\Theta(t)CBx,\ t\in[0,\tau),\\ u(0)=0.\end{cases}$$

Since K is a kernel, the preceding problem has at most one solution and one easily infers that $BS_K(t)x=S_K(t)Bx$, $t\in[0,\tau)$, $x\in E$. Let $x\in E$ be fixed. Clearly,

$$\begin{aligned}\left\|S_K^B(t)-e^{tB}S_K(t)\right\| &\leqslant \max_{s\in[0,t]}\|S_K(s)\|\sum_{n=1}^{\infty}\Theta_n(t)\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{\|B\|^i}{i!}\binom{i}{n}t^{i-n}\\ &=\max_{s\in[0,t]}\|S_K(s)\|\sum_{n=1}^{\infty}\Theta_n(t)\sum_{i=1}^{\infty}\frac{\|B\|^i}{i!}t^i\sum_{n=1}^{i}\binom{i}{n}t^{-n}\\ &\leqslant \max_{s\in[0,t]}\|S_K(s)\|\sum_{n=1}^{\infty}\Theta_n(t)\sum_{i=1}^{\infty}\frac{\|B\|^i}{i!}t^i\frac{(t+1)^i}{t^i}\\ &=e^{\|B\|}\max_{s\in[0,t]}\|S_K(s)\|\sum_{n=1}^{\infty}\Theta_n(t)e^{t\|B\|},\ t\in(0,\tau),\end{aligned}$$

and this implies (a). The previous computation also shows that $(S_K^B(t))_{t\in[0,\tau)}$ is a strongly continuous operator family which commutes with $A+B$ and C. Then the dominated convergence theorem, the closedness of A and integration by parts, as well as the argumentation used in the estimation of term $\|S_K^B(t)-e^{tB}S_K(t)\|$, $t\in[0,\tau)$, imply:

$$\begin{aligned}&(A+B)\int_0^t S_K^B(s)x\,ds=(A+B)\int_0^t e^{sB}S_K(s)x\,ds\\ &\quad+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\int_0^t\int_0^s K_n(s-r)r^{i-n}S_K(r)x\,dr\,ds\\ &=e^{tB}\big[S_K(t)x-\Theta(t)Cx\big]+B\int_0^t e^{sB}\Theta(s)Cx\,ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\int_0^t K_n(t-s)\int_0^s r^{i-n}S_K(r)x\,dr\,ds\\
&= e^{tB}\big[S_K(t)x-\Theta(t)Cx\big]+B\int_0^t e^{sB}\Theta(s)Cx\,ds\\
&\quad+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\\
&\quad\times\int_0^t K_n(t-s)\Bigg[s^{i-n}\int_0^s S_K(r)xdr-(i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr\Bigg]\,ds\\
&= e^{tB}\big[S_K(t)x-\Theta(t)Cx\big]+B\int_0^t e^{sB}\Theta(s)Cx\,ds\\
&\quad+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\\
&\quad\times\Bigg[s^{i-n}\int_0^s S_K(r)x\,dr-(i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr\Bigg]\,ds\\
&\quad+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}[S_K(s)x-\Theta(s)Cx]\,ds\\
&\quad+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}[S_K(r)x-\Theta(r)Cx]\,dr\,ds\\
&= S_K^B(t)x-f_1(t)-f_2(t)Cx,\quad t\in[0,\tau),
\end{aligned}
$$

where:

$$
\begin{aligned}
f_1(t) := &\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\\
&\times\Bigg[s^{i-n}\int_0^s S_K(r)x\,dr-(i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr\Bigg]\,ds\\
&+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}S_K(r)x\,dr\,ds,\quad t\in[0,\tau)
\end{aligned}
$$

$$
\begin{aligned}
f_2(t) &:= e^{tB}\Theta(t) - B\int_0^t e^{sB}\Theta(s)\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}\Theta(s)\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\Theta(r)\,dr\,ds, \quad t\in[0,\tau).
\end{aligned}
$$

Then the partial integration implies:

$$
\begin{aligned}
f_1(t) &= \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\int_0^s r^{i-n}\int_0^r S_K(r)x\,dr\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\int_0^r S_K(r)x\,dr\,ds, \ t\in[0,\tau).
\end{aligned}
$$

The coefficient of B^i, $i\geqslant 2$ in the expression of $f_1(t)$ equals

$$
\sum_{n=1}^{i-1}(-1)^n\Big(\frac{n-i}{i!}\binom{i}{n}+\frac{1}{(i-1)!}\binom{i-1}{n}\Big)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\int_0^r S_K(r)x\,dr\,ds = 0,
$$

because $\frac{n-i}{i!}\binom{i}{n}+\frac{1}{(i-1)!}\binom{i-1}{n}=0$. Thereby, $f_1(t)=0$, $t\in[0,\tau)$. On the other hand, the usual series arguments imply that the coefficient of B^i in the expression of $f_2(\cdot)$ equals $\Theta(t)$, $t\geqslant 0$ if $i=0$, and

$$
\begin{aligned}
f_{2,i}(t) &:= \frac{t^i}{i!}\Theta(t) - \int_0^t \frac{s^{i-1}}{(i-1)!}\Theta(s)\,ds + \sum_{n=1}^{i}\frac{1}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}\Theta(s)\,ds \\
&+ \sum_{n=1}^{i}\frac{1}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\Theta(r)\,dr\,ds, \quad t\geqslant 0,
\end{aligned}
$$

if $i\geqslant 1$. Proceeding as before, one obtains, as a consequence of the condition (iii), that the function $t\mapsto f_{2,i}(t)$, $t\geqslant 0$ satisfies (P1) and that there exists $a''>0$ such that

$$
\begin{aligned}
\mathcal{L}\big(f_{2,i}(t)\big)(\lambda) &= \frac{1}{i!}(-1)^i\Big(\frac{\tilde K(\cdot)}{\cdot}\Big)^{(i)}(\lambda) - \frac{1}{\lambda}\frac{1}{(i-1)!}(-1)^{i-1}\Big(\frac{\tilde K(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda) \\
&+ \sum_{n=1}^{i}\frac{1}{i!}(-1)^i\binom{i}{n}\tilde K(\lambda)\Big(\frac{1}{\tilde K(\cdot)}\Big)^{(n)}(\lambda)\frac{1}{\lambda}\tilde K^{(i-n)}(\lambda) \\
&= \frac{1}{i!}(-1)^i\Big(\frac{\tilde K(\cdot)}{\cdot}\Big)^{(i)}(\lambda) - \frac{1}{\lambda}\frac{1}{(i-1)!}(-1)^{i-1}\Big(\frac{\tilde K(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda)
\end{aligned}
$$

$$+\frac{\tilde{K}(\lambda)}{\lambda}\frac{(-1)^i}{i!}\Big(-\frac{1}{\tilde{K}(\lambda)}\Big)\tilde{K}^{(i)}(\lambda)$$
$$=\frac{1}{i!}(-1)^i\Big(\frac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i)}(\lambda)-\frac{1}{\lambda}\frac{1}{(i-1)!}(-1)^{i-1}\Big(\frac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda)$$
$$+\frac{(-1)^{i+1}}{i!}\frac{\tilde{K}^{(i)}(\lambda)}{\lambda}=0,$$

for all $\lambda > a''$ with $\tilde{K}(\lambda)\neq 0$. This implies $f_2(t)=\Theta(t)$, $t\in[0,\tau)$, and consequently, $(S_K^B(t))_{t\in[0,\tau)}$ is a (local) K-convoluted C-semigroup with a subgenerator $A+B$. The proof of (b) follows from a simple computation; furthermore, if A is the integral generator of $(S_K(t))_{t\in[0,\tau)}$, then $C^{-1}AC=A$ and $C^{-1}(A+B)C=A+B$ is the integral generator of $(S_K^B(t))_{t\in[0,\tau)}$. This completes the proof of theorem. □

Remark 2.5.4. (i) The assumption (i) of Theorem 2.5.3 is satisfied for the function $K(\cdot)=\mathcal{L}^{-1}\big(\frac{a}{p_k(\lambda)}\big)$, where $p_k(\cdot)$ is a polynomial of degree $k\in\mathbb{N}$ and $a\in\mathbb{C}\smallsetminus\{0\}$. Then $n_0=k$ and $K_n\equiv 0$, $n\geqslant k+1$. Furthermore, in this case we have the existence of positive real numbers M and ω such that (116) holds.

(ii) Let $n>1$ and let $P(\cdot)$ be an analytic function in the right half plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\lambda_0\}$ for some $\lambda_0\geqslant 1$. Suppose that $P(\lambda)\neq 0$, $\operatorname{Re}\lambda>\lambda_0$, and that there exist $C>0$ and $r\in(1/2,1]$ with:

$$|P(\lambda)|\geqslant C|\lambda|^n,\ \operatorname{Re}\lambda>\lambda_0, \tag{117}$$

$$\Big|\frac{d^i}{d\lambda^i}P(\lambda)\Big|\leqslant C|\lambda|^{-ir}|P(\lambda)|,\ \operatorname{Re}\lambda>\lambda_0,\ i\in\mathbb{N}, \tag{118}$$

$$\frac{P'}{P}\in LT(\mathbb{C}), \tag{119}$$

where we denote by $LT(\mathbb{C})$ the set of all Laplace transforms of exponentially bounded functions. We will prove that the condition (i) of Theorem 2.5.3 holds for the function $K=\mathcal{L}^{-1}(1/P)$ as well as that there exist $M>0$ and $\omega\geqslant 0$ such that (116) holds. First, note that the assumption (117) and Theorem 1.1.12 imply that there exists $K\in C([0,\infty):E)$ such that $K(0)=0$, $|K(t)|\leqslant Me^{\lambda_0 t}$, $t\geqslant 0$, for a suitable $M>0$, and $\mathcal{L}(K(t))(\lambda)=1/P(\lambda)$, $\operatorname{Re}\lambda>\lambda_0$. Let us show that $P^{(j)}/P$ is an element of $LT(\mathbb{C})$ for all $j\in\mathbb{N}$. This is clear for $j=1$ since we have assumed (119). Suppose $j\geqslant 2$. Then the assumption (118) implies $|P^{(j)}(\lambda)/P(\lambda)|\leqslant C|\lambda|^{-jr}$, $\operatorname{Re}\lambda>\lambda_0$. Since $r\in(1/2,1]$ and $j\geqslant 2$, one can apply Theorem 1.1.12 in order to see that $P^{(j)}/P\in LT(\mathbb{C})$. Put $K_j=\mathcal{L}^{-1}(P^{(j)}/P)$, $j\in\mathbb{N}$. In case $j\geqslant 2$, the proof of Theorem 1.1.12 implies

$$K_j(t)=\frac{1}{2\pi i}\int\limits_{\bar{\lambda}_0-i\infty}^{\bar{\lambda}_0+i\infty}e^{\mu t}\frac{P^{(j)}(\mu)}{P(\mu)}\,d\mu,\ t\geqslant 0,$$

where the last integral is independent of $\bar{\lambda}_0>\lambda_0$. Now it is enough to prove that there exists $C_1>0$ such that

$$|K_j(t)|\leqslant C_1e^{\lambda_0 t},\ t\geqslant 0,\ j=4,5,\ldots. \tag{120}$$

Suppose $j \geqslant 4$. Then

$$|K_j(t)| \leqslant \frac{1}{2\pi} e^{\bar{\lambda}_0 t} \int\limits_{-\infty}^{\infty} \frac{C}{(\bar{\lambda}_0^2 + s^2)^{j/4}}\, ds \leqslant \frac{C}{2\pi} e^{\bar{\lambda}_0 t} \int\limits_{-\infty}^{\infty} \frac{1}{(\bar{\lambda}_0^2 + s^2)}\, ds,\ t \geqslant 0.$$

Letting $\bar{\lambda}_0 \to \lambda_0$, we have (120) and the required properties automatically follow. Notice also that it is possible to assume that $r \in (0,1]$. If this is the case, one has to replace (119) by $\frac{P^{(j)}}{P} \in LT(\mathbb{C})$, $j \leqslant 1/r$, $j \in \mathbb{N}$. Finally, let us recall that, in the theory of pseudodifferential operators, a smooth symbol P is called hypoelliptic if the conditions (117), (118) hold as well as $|P(\lambda)| \leqslant C|\lambda|^q$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant a$, for some $q \in \mathbb{R}$, $C > 0$ and $a > 0$.

(iii) The conditions (ii) and (iii) quoted in the formulation of Theorem 2.5.3 can be replaced with:

(ii)$'$ $\sum_{i=1}^{\infty} \|2B\|^i (1+t)^i \sum_{n=1}^{i} \frac{\Theta_n(t)}{i!} < \infty$ for all $t \geqslant 0$ and

(iii)$'$ to every $i \in \mathbb{N}$, there exists $a_i > 0$ such that the function

$$t \mapsto \max_{s \in [0,t]} |\Theta(s)| e^{-a_i t} \sum_{n=1}^{i} \frac{(2t+2)^i}{i!} \Theta_n(t),\ t \geqslant 0$$

belongs to the space $L^1([0,\infty) : \mathbb{R})$.

Then the estimate (a) reduces to

$$\left\| S_K^B(t) - e^{tB} S_K(t) \right\| \leqslant \max_{s \in [0,t]} \|S_K(s)\| \sum_{i=1}^{\infty} \|2B\|^i (1+t)^i \frac{1}{i!} \sum_{n=1}^{i} \Theta_i(t)$$

and a corresponding analogue of the assertion (b) can be simply stated. Notice only that one can prove that $f_1 \equiv 0$ by direct computation of coefficient of B^i, $i \in \mathbb{N}$ and that the condition (iii)$'$ is necessary in our striving to show that, for every $i \in \mathbb{N}$, the function $t \mapsto f_{2,i}(t)$, $t \geqslant 0$ satisfies (P1); it is also clear that (iii)$'$ holds provided that Θ is exponentially bounded and that, for every $n \in \mathbb{N}$, Θ_n is exponentially bounded, too. Let us prove now that (ii)$'$ and (iii)$'$ hold for the function $K = \mathcal{L}^{-1}(e^{-\lambda^\sigma})$, where $\sigma \in (0,1)$. First of all, we know that K is an exponentially bounded, continuous kernel. Let $f(\lambda) = e^{\lambda^\sigma}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Then the mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ is analytic, $f'(\lambda) = \sigma \lambda^{\sigma-1} f(\lambda)$ and

$$f^{(n)}(\lambda) = \sum_{i=0}^{n-1} \binom{n-1}{i} (\cdot^{\sigma-1})^{(n-i-1)}(\lambda) f^{(i)}(\lambda),\ \lambda \in \mathbb{C} \setminus (-\infty, 0]. \tag{121}$$

Using (121), one concludes inductively that, for every $n \in \mathbb{N}$, there exist real numbers $p_{i,n}(\sigma)$, $1 \leqslant i \leqslant n$ such that, for every $t \geqslant 0$:

$$\widetilde{K_n}(\lambda) = \sum_{i=1}^{n} p_{i,n}(\sigma) \lambda^{i\sigma - n},\ \operatorname{Re}\lambda > 0 \text{ and } \Theta_n(t) \leqslant \sum_{i=1}^{n} |p_{i,n}(\sigma)| \frac{t^{n-i\sigma}}{\Gamma(n+1-i\sigma)}.$$

Put $p_{0,n}(\sigma) := 0$, $n \in \mathbb{N}$. By the foregoing,

$$\Big(e^{\cdot^\sigma}\Big)^{(n)}(\lambda) = e^{\lambda^\sigma} \sum_{i=1}^{n} p_{i,n}(\sigma)\lambda^{i\sigma-n}$$

$$\Big(e^{\cdot^\sigma}\Big)^{(n+1)}(\lambda) = e^{\lambda^\sigma} \sum_{i=1}^{n+1} \big(p_{i,n}(\sigma)(i\sigma-n)+\sigma p_{i-1,n}(\sigma)\big)\lambda^{i\sigma-(n+1)},$$

for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Hence, $p_{1,n}(\sigma) = \sigma(\sigma-1)\cdots(\sigma-(n-1))$, $n \in \mathbb{N} \smallsetminus \{1\}$, $p_{n,n}(\sigma) = \sigma^n$, $n \in \mathbb{N}$ and

$$p_{i,n+1}(\sigma) = p_{i,n}(\sigma)(i\sigma-n)+\sigma p_{i-1,n}(\sigma),\ n \in \mathbb{N},\ 2 \leqslant i \leqslant n. \tag{122}$$

Clearly, $L_\sigma := \sup_{n\in\mathbb{N}_0} |\binom{\sigma}{n}| < \infty$. Applying (122) we infer that for every $n \geqslant 2$:

$$\begin{aligned}
&\sum_{i=1}^{n+1} i!|p_{i,n+1}(\sigma)| \\
&\leqslant |\sigma(\sigma-1)\cdots(\sigma-n)| + \sum_{i=2}^{n}\big[\sigma i!|p_{i-1,n}(\sigma)| + n(\sigma+1)i!|p_{i,n}(\sigma)|\big] + (n+1)! \\
&\leqslant L_\sigma(\sigma+n)n! + n\sigma\sum_{i=1}^{n-1} i!|p_{i,n}(\sigma)| + n(\sigma+1)\sum_{i=2}^{n} i!|p_{i,n}(\sigma)| + (n+1)!.
\end{aligned}$$

The preceding inequality implies that, for every $\zeta \geqslant 2+4\sigma+2L_\sigma$, the following holds:

$$\sum_{i=1}^{n} i!|p_{i,n}(\sigma)| \leqslant \zeta^n n! \text{ for all } n \in \mathbb{N}. \tag{123}$$

Denote by ζ_σ the minimum of all numbers satisfying (123). Then a simple computation shows that, for every $x \in E$:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{n=1}^{i} \frac{\|B\|^i}{i!}\binom{i}{n}\int_0^t \|K_n(t-s)s^{i-n}S_K(s)x\|\,ds \\
&\leqslant \max_{s\in[0,t]}\|S_K(s)x\| \sum_{i=1}^{\infty}\frac{\|B\|^i\zeta_\sigma^i}{i!}\sum_{n=1}^{i}\sum_{l=1}^{n}\frac{t^{i+1-l\sigma}i!}{\Gamma(i+2-l\sigma)l!},\ t \geqslant 0.
\end{aligned} \tag{124}$$

On the other hand, it is easily verified that:

$$\sum_{n=1}^{i}\sum_{l=1}^{n}\frac{i!}{\Gamma(i+2-l\sigma)l!} \leqslant i2^{(2-\sigma)i},\ i \in \mathbb{N}. \tag{125}$$

Combining (124) and (125), it follows that

$$\Big\|S_K^B(t) - e^{tB}S_K(t)\Big\| \leqslant t\|B\|\zeta_\sigma 2^{2-\sigma}e^{\|B\|\zeta_\sigma 2^{2-\sigma}} \max_{s\in[0,t]}\|S_K(s)\|,\ t \in [0,\min(1,\tau))$$

$$\Big\|S_K^B(t) - e^{tB}S_K(t)\Big\| \leqslant t^2\|B\|\zeta_\sigma 2^{2-\sigma}e^{\|B\|\zeta_\sigma 2^{2-\sigma}t} \max_{s\in[0,t]}\|S_K(s)\|,\ t \in [1,\tau),\ \text{if } \tau > 1,$$

proving the condition (ii)'; furthermore, if $\tau = \infty$ and $(S_K(t))_{t\geqslant 0}$ is exponentially bounded, then $(S_K^B(t))_{t\geqslant 0}$ is also exponentially bounded. These conclusions still hold for the function $K = \mathcal{L}^{-1}(e^{-a\lambda^{\sigma}})$, where $\sigma \in (0,1)$ and $a > 0$, which will be of importance in Section 3.5.

Suppose $\alpha > 0$, $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, $L_0 := \sup_{n\in\mathbb{N}} |\binom{\alpha}{n}|$ and A is a subgenerator of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Then $L_0 < \infty$, $K_n(t) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}t^{n-1}$, $\Theta_n(t) = |\binom{\alpha}{n}|t^n$, $t \geqslant 0$, $n \in \mathbb{N}$ and this implies that the condition (iii) of Theorem 2.5.3 does not hold if $\alpha \notin \mathbb{N}$. Fortunately, the series appearing in (115) still converges and the estimate $\|S_\alpha^B(t) - e^{tB}S_\alpha(t)\| \leqslant L_0 \max_{s\in[0,t]} \|S_\alpha(s)\|e^{2t\|B\|}$, $t \in [0,\tau)$ follows similarly; furthermore, the proof of Theorem 2.5.3 can be repeated verbatim. Having in mind these observations, we are in a position to clarify the following important generalization of [**227**, Corollary 4.5] and [**423**, Theorem 2.3]:

THEOREM 2.5.5. *Suppose $\alpha > 0$, A is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$, $B \in L(E)$, $BA \subseteq AB$ and $BC = CB$. Then $A + B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha^B(t))_{t\in[0,\tau)}$, which is given by*

(126)
$$S_\alpha^B(t) := e^{tB}S_\alpha(t) + \sum_{i=1}^{\infty}\sum_{n=1}^{i} \frac{B^i}{i!}(-1)^n n \binom{i}{n}\binom{\alpha}{n} \int_0^t (t-s)^{n-1}s^{i-n}S_\alpha(s)\,ds,\ t \in [0,\tau).$$

Notice that the previous formula can be rewritten in the following form:

$$S_\alpha^B(t) = e^{tB}S_\alpha(t) + \sum_{i\geqslant 1}\binom{\alpha}{i}(-B)^i \int_0^t \frac{(t-s)^{i-1}}{(i-1)!}e^{Bs}S(s)\,ds,\ t \in [0,\tau).$$

The main objective in the following theorem is to clarify a perturbation result for subgenerators of exponentially bounded, analytic integrated C-semigroups.

THEOREM 2.5.6. *Suppose $r > 0$, $\alpha \in (0, \frac{\pi}{2}]$, A is a subgenerator, resp. the integral generator, of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r(t))_{t\geqslant 0}$ of angle α; $B \in L(E)$, $BA \subseteq AB$ and $BC = CB$. Then $A + B$ is a subgenerator, resp. the integral generator, of an exponentially bounded, analytic r-times integrated C-semigroup $(S_\alpha^B(t))_{t\geqslant 0}$ of angle α, where*

$$S_r^B(z) := e^{zB}S_r(z) + \sum_{i=1}^{\infty}\binom{\alpha}{i}(-B)^i \int_0^z \frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_r(s)\,ds,\ z \in \Sigma_\alpha. \tag{127}$$

PROOF. Put $R_0 := \sup_{n\in\mathbb{N}} |\binom{r}{n}|$. Notice that, for every $z \in \Sigma_\alpha$, the series appearing in (127) is absolutely convergent and that, for every $\gamma \in (-\alpha,\alpha)$ such that $|\gamma| > \arg(z)$, we have the following:

$$\begin{aligned}\big\|S_r^B(z) - e^{zB}S_r(z)\big\| &\leqslant \sum_{i\geqslant 1} R_0\|B\|^i \int\limits_0^{\operatorname{Re} z} \frac{|z|^{i-1}}{(i-1)!} e^{\|B\||z|} M_\gamma e^{\omega_\gamma \operatorname{Re} z} ds \\ &\leqslant \operatorname{Re} z\, M_\gamma R_0 \|B\| e^{(2\|B\|+\omega_\gamma)\operatorname{Re} z}.\end{aligned}$$

This implies that $(S_r(z))_{z\in\Sigma_\alpha}$ is a strongly continuous operator family and that the conditions (i) and (ii) given in the formulation of Definition 2.4.1 hold. It remains to be shown that the mapping

$$z \mapsto \sum_{i=1}^{\infty} \binom{\alpha}{i} (-B)^i \int\limits_0^z \frac{(z-s)^{i-1}}{(i-1)!} e^{Bs} S_r(s)\, ds,\ z \in \Sigma_\alpha$$

is analytic. By standard arguments, the mapping $f_0(z) = \int_0^z e^{-Bs}S_r(s)\,ds$, $z \in \Sigma_\alpha$ is analytic and $f_0'(z) = e^{-Bz}S_r(z)$, $z \in \Sigma_\alpha$. This yields that, for every $i \in \mathbb{N}$, the mapping $f_i(t) = \int_0^z \frac{(z-s)^{i-1}}{(i-1)!} e^{Bs} S_r(s)\,ds$, $z \in \Sigma_\alpha$ is analytic and that $f_i'(z) = f_{i-1}(z)$, $z \in \Sigma_\alpha$. Furthermore, the series in (127) is locally uniformly convergent since

$$\begin{aligned}&\left\| \binom{\alpha}{i} (-B)^i \int\limits_0^z \frac{(z-s)^{i-1}}{(i-1)!} e^{Bs} S_r(s)\, ds \right\| \\ &\qquad\leqslant M_\gamma \sup_{z\in K} |z|\, \|B\| \frac{1}{(i-1)!} \Big((\|B\| \sup_{z\in K} |z|)^{i-1} \Big) e^{(\|B\|+\omega)\sup_{z\in K}|z|},\end{aligned}$$

where K is an arbitrary compact subset of Σ_α and γ is chosen so that $K \subseteq \Sigma_\gamma$. An application of the Weierstrass theorem completes the proof of theorem. □

The following theorem extends the assertion of [**423**, Theorem 2.4, Theorem 2.5, Corollary 2.6] (cf. also [**457**, Theorem 2.3]). The proof is omitted since it follows by the use of argumentation given in [**423**], [**381**, Section 10] and [**457**].

THEOREM 2.5.7. *Suppose $n \in \mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is a (local, global exponentially bounded) n-times integrated C-semigroup having A as a subgenerator, resp. the integral generator, $B \in L(\overline{D(A)}, E)$, $\mathrm{R}(B) \subseteq C(D(A^n))$ and $BCx = CBx$, $x \in D(A)$. Then $A + B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) n-times integrated C-semigroup $(S_B(t))_{t\in[0,\tau)}$, which satisfies the integral equation:*

$$S_B(t)x = S(t)x + \int\limits_0^t \frac{d^n}{dt^n} S(t-s)C^{-1}BS_B(s)x\, ds,\ t \in [0,\tau),\ x \in E.$$

THEOREM 2.5.8. [**263**], [**242**] *Suppose $\alpha \geqslant 0$, $(C(t))_{t\in[0,\tau)}$ is a (local, global exponentially bounded) α-times integrated C-cosine function having A as a subgenerator, resp. the integral generator, $B \in L(\overline{D(A)}, E)$, $\mathrm{R}(B) \subseteq C\big(D(A^{\lceil \frac{\alpha-1}{2} \rceil})\big)$ and*

$BCx = CBx$, $x \in D(A)$. *Then* $A + B$ *is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded)* α*-times integrated* C*-cosine function* $(C_B(t))_{t\in[0,\tau)}$.

The following theorem mimics an interesting perturbation result of Kaiser and Weis [**171**] which can be additionally refined if the Fourier type of the space E (cf. [**14**], [**171**] and [**242**]) is also taken into consideration.

THEOREM 2.5.9. *Assume* K *satisfies* (P1), (P2) *and there is* $\beta \in (\mathrm{abs}(K), \infty)$ *such that, for every* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *satisfying:*

$$\frac{1}{|\tilde{K}(\lambda)|} \leqslant C_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \beta. \tag{128}$$

(i) *Assume* A *generates an exponentially bounded* K*-convoluted semigroup* $(S_K(t))_{t\geqslant 0}$ *such that* $\|S_K(t)\| \leqslant M_1 e^{\omega t}$, $t \geqslant 0$ *for some* $M_1 > 0$ *and* $\omega \geqslant 0$. *Let* B *be a linear operator such that* $D(A) \subseteq D(B)$ *and that there exist* $M \in (0,1)$ *and* $\lambda_0 \in (\max(\beta,\omega),\infty)$ *satisfying* $\|BR(\lambda : A)\| \leqslant M$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda = \lambda_0$. *Then, for every* $\alpha > 1$, *the operator* $A + B$ *generates an exponentially bounded,* $\big(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\big)$*-convoluted semigroup.*

(ii) *Assume* A *generates an exponentially bounded* K*-convoluted semigroup* $(S_K(t))_{t\geqslant 0}$ *such that* $\|S_K(t)\| \leqslant M_1 e^{\omega t}$, $t \geqslant 0$ *for some* $M_1 > 0$ *and* $\omega \geqslant 0$. *Let* B *be a densely defined linear operator such that there exist* $M \in (0,1)$ *and* $\lambda_0 \in (\max(\beta,\omega),\infty)$ *satisfying* $\|R(\lambda : A)Bx\| \leqslant M\|x\|$, $x \in D(B)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda = \lambda_0$. *Then there exists a closed extension* D *of the operator* $A + B$ *such that, for every* $\alpha > 1$, *the operator* D *generates an exponentially bounded,* $\big(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\big)$*-convoluted semigroup. Furthermore, if* A *and* A^* *are densely defined, then* D *is the part of the operator* $(A^* + B^*)^*$ *in* E.

(iii) *Assume* A *generates an exponentially bounded* K*-convoluted cosine function* $(C_K(t))_{t\geqslant 0}$ *such that* $\|C_K(t)\| \leqslant M_1 e^{\omega t}$, $t \geqslant 0$ *for some* $M_1 > 0$ *and* $\omega \geqslant 0$. *Let* B *be a linear operator such that* $D(A) \subseteq D(B)$ *and that there exist* $M > 0$ *and* $\lambda_0 \in (\max(\beta,\omega),\infty)$ *satisfying* $\|BR(\lambda^2 : A)\| \leqslant \frac{M}{|\lambda|}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda = \lambda_0$. *Then, for every* $\alpha > 1$, *the operator* $A + B$ *generates an exponentially bounded,* $\big(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\big)$*-convoluted cosine function.*

(iv) *Assume* A *generates an exponentially bounded* K*-convoluted cosine function* $(C_K(t))_{t\geqslant 0}$ *such that* $\|C_K(t)\| \leqslant M_1 e^{\omega t}$, $t \geqslant 0$ *for some* $M_1 > 0$ *and* $\omega \geqslant 0$. *Let* B *be a densely defined linear operator such that there exist* $M \in (0,1)$ *and* $\lambda_0 \in (\max(\beta,\omega),\infty)$ *satisfying* $\|R(\lambda^2 : A)Bx\| \leqslant \frac{M}{|\lambda|}\|x\|$, $x \in D(B)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda = \lambda_0$. *Then there exists a closed extension* D *of the operator* $A + B$ *such that, for every* $\alpha > 1$, *the operator* D *generates an exponentially bounded,* $\big(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\big)$*-convoluted cosine function. Furthermore, if* A *and* A^* *are densely defined, then* D *is the part of the operator* $(A^* + B^*)^*$ *in* E.

PROOF. We will prove (iii) and (iv). By Theorem 2.2.1(i), $\{\lambda^2 : \lambda \in \mathbb{C}, \operatorname{Re}\lambda > \max(\beta,\omega)\} \subseteq \rho(A)$ and $\|R(\lambda^2 : A)\| \leqslant \frac{M_1}{|\lambda||\tilde{K}(\lambda)|(\operatorname{Re}\lambda - \omega)}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \max(\beta,\omega)$.

Suppose $z \in \mathbb{C}$ and $\operatorname{Re} z > \lambda_0$. Put $\lambda = \lambda_0 + i \operatorname{Im} z$ and notice that

$$
\begin{aligned}
(129)\quad \big\|BR(z^2 : A)\big\| &= \big\|BR(\lambda^2 : A)(I + (\lambda^2 - z^2)R(z^2 : A))\big\| \\
&\leqslant \big\|BR(\lambda^2 : A)\big\|\big(1 + |\lambda - z||\lambda + z|\,\|R(z^2 : A)\|\big) \\
&\leqslant \frac{M}{|\lambda|}\Big(1 + |\lambda - z||\lambda + z|\frac{M_1}{|z||\tilde{K}(z)|(\operatorname{Re} z - \omega)}\Big) \\
&\leqslant \frac{M}{|\lambda|}\Big(1 + |\lambda + z|\frac{M_1}{|z||\tilde{K}(z)|}\Big) \leqslant M\Big(\frac{1}{|\lambda|} + (1 + \frac{|z|}{|\lambda|})\frac{M_1}{|z||\tilde{K}(z)|}\Big) \\
&\leqslant M\Big(\frac{1}{\lambda_0} + \frac{M_1}{|z||\tilde{K}(z)|} + \frac{M_1}{\lambda_0|\tilde{K}(z)|}\Big).
\end{aligned}
$$

Consider now the function $h : \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\} \to \mathbb{C}$ defined by $h(z) := zBR((z + \lambda_0)^2 : A)$, $\operatorname{Re} z \geqslant 0$. Then $\|h(it)\| \leqslant M$, $t \in \mathbb{R}$ and, by (128)–(129), we have that, for every $\varepsilon > 0$, there exists $\overline{C_\varepsilon} > 0$ such that $\|h(z)\| \leqslant \overline{C_\varepsilon} e^{\varepsilon|z|}$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geqslant 0$. An application of the Phragmén-Lindelöf type theorems (cf. for instance [**14**, Theorem 3.9.8, p. 179]) gives that $\|h(z)\| \leqslant M$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geqslant 0$. This, in turn, implies that there exists $a > \lambda_0$ such that $\|BR(\lambda^2 : A)\| < \frac{1}{2}$, $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > a$, so that $\lambda^2 \in \rho(A + B)$ and

$$
\big\|\lambda R(\lambda^2 : A + B)\big\| = \big\|\lambda R(\lambda^2 : A)(I - BR(\lambda^2 : A))^{-1}\big\| \leqslant \frac{1}{|\tilde{K}(\lambda)|},\ \lambda \in \mathbb{C},\ \operatorname{Re} \lambda > a.
$$

The proof of (iii) completes an application of Theorem 2.2.4(i) while the proof of (iv) follows from [**171**, Lemma 3.2] and a similar argumentation. □

The proof of Theorem 2.5.9 immediately implies the following corollary.

COROLLARY 2.5.10. (i) *Assume A generates a cosine function $(C(t))_{t \geqslant 0}$ satisfying $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for appropriate $M > 0$ and $\omega \geqslant 0$. If B is a linear operator such that $D(A) \subseteq D(B)$ and that there exist $M' > 0$ and $\lambda_0 \in (\omega, \infty)$ satisfying $\|BR(\lambda^2 : A)\| \leqslant \frac{M}{|\lambda|}$, $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda = \lambda_0$, then, for every $\alpha > 1$, the operator $A + B$ generates an exponentially bounded, α-times integrated cosine function.*

(ii) *Assume A generates a cosine function $(C(t))_{t \geqslant 0}$ satisfying $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for appropriate $M > 0$ and $\omega \geqslant 0$. Let B be a densely defined linear operator such that there exist $M' > 0$ and $\lambda_0 \in (\omega, \infty)$ satisfying $\|R(\lambda^2 : A)Bx\| \leqslant \frac{M}{|\lambda|}\|x\|$, $x \in D(B)$, $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda = \lambda_0$. Then there exists a closed extension D of the operator $A+B$ such that, for every $\alpha > 1$, the operator D generates an exponentially bounded, α-times integrated cosine function. Furthermore, if A and A^* are densely defined, then D is the part of the operator $(A^* + B^*)^*$ in E.*

In the remnant of this section, we consider (multiplicative) perturbations of subgenerators of (a, k)-regularized C-resolvent families.

THEOREM 2.5.11. (i) [**288**], [**235**] *Suppose $M > 0$, $\omega_1 \geqslant \omega \geqslant 0$, A is a subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t \geqslant 0}$ satisfying $||R(t)|| \leqslant Me^{\omega t}$, $t \geqslant 0$ and $z \in \mathbb{C}$. Let $B : \overline{D(A)} \to E$ be a linear operator such that $BCx = CBx$, $x \in D(A)$ and that $||C^{-1}Bx|| \leqslant c||x||$, $x \in \overline{D(A)}$ for some $c > 0$. Let* (P1) *hold for*

$a(t)$, $k(t)$, $b(t)$ *and let* $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z$, $\operatorname{Re}\lambda > \omega_1$, $\tilde{k}(\lambda) \neq 0$. *Suppose* $\mu > \omega_1$, $\gamma \in [0,1)$ *and:*

$$\overline{D(A)} = E \text{ and } \int_0^\infty e^{-\mu t} \Big\| C^{-1}B \int_0^t b(t-s)R(s)x\,ds + zC^{-1}BR(t)x \Big\| dt \leqslant \gamma \|x\|, \ x \in D(A) \tag{130}$$

or

$$(R(t))_{t\geqslant 0} \text{ satisfies } (54), \ \overline{D(A)} \neq E \text{ and } (130) \text{ holds for any } x \in E. \tag{131}$$

Then the operator $A+B$ *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(R_B(t))_{t\geqslant 0}$ *satisfying* (54) *with* A *replaced by* $A+B$ *therein. Furthermore,*

$$\|R_B(t)\| \leqslant \frac{M}{1-\gamma} e^{\mu t}, \ t \geqslant 0, \tag{132}$$

$$R_B(t)x = R(t)x + \int_0^t R_B(t-r)\Big(C^{-1}B \int_0^r b(r-s)R(s)x\,ds + zC^{-1}BR(t)x\Big)dr, \ t \geqslant 0, \ x \in D(A), \tag{133}$$

and (133) *holds for any* $t \geqslant 0$ *and* $x \in E$ *provided* (131).

(ii) [**289**], [**235**] *Assume* $C([0,\infty)) \ni a$ *satisfies* (P1), (H5) *holds,* $B \in L(E)$, $\mathrm{R}(B) \subseteq \mathrm{R}(C)$ *and* A *is a subgenerator of an exponentially bounded* (a,a)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$. *Assume, further, that there exists* $\omega \geqslant 0$ *such that, for every* $h \geqslant 0$ *and for every function* $f \in C([0,\infty) : E)$,

- (Ma) $\int_0^h R(h-s)C^{-1}Bf(s)\,ds \in D(A)$,
- (Mb) $\big\|A\int_0^h R(h-s)C^{-1}Bf(s)\,ds\big\| \leqslant e^{\omega t}\mu_B(h)\|f\|_{[0,h]}$, $t \geqslant 0$, *where* $\|f\|_{[0,h]} := \sup_{t\in[0,h]} \|f(t)\|$, $\mu_B(t) : [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing and satisfies* $\mu_B(0) = 0$, *and*
- (Mc) *there exists an injective operator* $C_1 \in L(E)$ *such that* $\mathrm{R}(C_1) \subseteq \mathrm{R}(C)$ *and* $C_1A(I+B) \subseteq A(I+B)C_1$.

Then $A(I+B)$ *is a subgenerator of an exponentially bounded* (a,a)*-regularized* C_1*-resolvent family* $(S(t))_{t\geqslant 0}$ *which satisfies the following integral equation*

$$S(t)x = R(t)C^{-1}C_1x + A\int_0^t R(t-s)C^{-1}BS(s)x\,ds, \ \ t \geqslant 0, \ x \in E.$$

(iii) [**289**], [**235**] *Let* A *be a subgenerator of an exponentially bounded, once integrated* C*-cosine function and let* ω, B *and* C_1 *be as in* (ii). *Then* $A(I+B)$ *is a subgenerator of an exponentially bounded, once integrated* C_1*-cosine function.*

(iv) [**289**], [**235**] *Assume that* A *is a subgenerator of an exponentially bounded* (a,a)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *and that a Banach space* $(Z, |\cdot|_Z)$

satisfies the conditions (Za), (Zb) *and* (Zc) *given in the formulation of* [**289**, Definition 4.1]. (*In particular, these conditions hold for* $[D(A)]$.) *Then* (Ma) *and* (Mb) *are fulfilled provided* $C^{-1}B \in L(X,Z)$.

(v) [**374**], [**441**] *Let* $B \in L(E)$ *and* $BC = CB$.

(v.1) *Assume* BA *is a subgenerator of a* (*local*) (a,k)*-regularized* C*-resolvent family and* (H5) *holds for* BA *and* C. *Then* AB *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family.*

(v.2) *Assume* AB *is a subgenerator of a* (*local*) (a,k)*-regularized* C*-resolvent family and* (H5) *holds for* AB *and* C. *Then* BA *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family, provided* $\rho(BA) \neq \emptyset$.

Recall that V. Keyantuo and M. Warma analyzed in [**195**] the generation of fractionally integrated cosine functions in L^p-spaces by elliptic differential operators with variable coefficients. Notice that Theorem 2.5.11(v) can be applied to these operators (cf. [**195**, Theorem 2.2 and pp. 78-79] and [**374**, Example 3.1]).

Assume $\alpha > 0$, $l \in \mathbb{N}$ and $f(t)$ is an E-valued function satisfying (P1). Set $F_\alpha(z) := \int_0^\infty e^{-z^{1/\alpha}t} f(t)\,dt$, $z > \max(\operatorname{abs}(f),0)^\alpha$. Using induction and elementary operational properties of vector-valued Laplace transform, one can simply prove that there exist real numbers $(c_{l_0,l,\alpha})_{1\leqslant l_0\leqslant l}$, independent of E and $f(t)$, such that:

$$\frac{d^l}{dz^l}F_\alpha(z) = \sum_{l_0=1}^{l} c_{l_0,l,\alpha} z^{\frac{l_0}{\alpha}-l} \int\limits_0^\infty e^{-z^{1/\alpha}t} t^{l_0} f(t)\,dt,\ z > \max(\operatorname{abs}(f),0)^\alpha.$$

Furthermore, $c_{l,l,\alpha} = \frac{(-1)^l}{\alpha^l}$, $l \geqslant 1$, $c_{1,l,\alpha} = \frac{(-1)}{\alpha}(\frac{1}{\alpha}-1)\cdots(\frac{1}{\alpha}-(l-1))$, $l \geqslant 2$ and the following non-linear recursive formula holds:

$$c_{l_0,l+1,\alpha} = \frac{(-1)}{\alpha} c_{l_0-1,l,\alpha} + \Big(\frac{l_0}{\alpha} - l\Big) c_{l_0,l,\alpha},\ l_0 = 2,\cdots,l.$$

Then there exists $\zeta \geqslant 1$ such that $\sum_{l_0=1}^{l} l_0!|c_{l_0,l,\alpha}| \leqslant \zeta^l l!$ for all $l \in \mathbb{N}$.

Now we are able to state the following perturbation theorem for abstract time-fractional equations ([**242**]).

THEOREM 2.5.12. *Suppose* $\alpha > 0$, *scalar-valued continuous kernels* $k(t)$ *and* $k_1(t)$ *satisfy* (P1), A *is a subgenerator of an exponentially bounded* $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying*

$$A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} R(s)x\,ds = R(t)x - k(t)Cx,\ t \geqslant 0,\ x \in E \tag{134}$$

$$\sup_{t\geqslant 0} e^{-\omega t}\|R(t)\| < \infty \text{ for some } \omega \geqslant \max(\operatorname{abs}(k),0). \tag{135}$$

Let the following conditions hold:

(i) $B \in L(E)$, $BA \subseteq AB$ *and* $BC = CB$. *There exist* $M \geqslant 1$, $\omega' \geqslant 0$, $\omega'' \geqslant 0$ *and* $\omega''' \geqslant \max(\omega+\omega', \omega+\omega'', \operatorname{abs}(k_1))$ *such that*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega''',\ \widetilde{k_1}(\lambda) \neq 0\} \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega''',\ \tilde{k}(\lambda) \neq 0\}$$

as well as:

(ii) *For every* i, l_0, $l \in \mathbb{N}$ *with* $1 \leqslant l \leqslant i$ *and* $1 \leqslant l_0 \leqslant l$, *there exists a function* $k_{i,l_0,l}(t)$ *satisfying* (P1) *and*

$$\mathcal{L}\big(k_{i,l_0,l}(t)\big)(\lambda) = c_{l_0,l,\alpha}\lambda^{l_0-\alpha(l-1)}\widetilde{k_1}(\lambda)\Big(\frac{1}{z\tilde{k}\big(z^{1/\alpha}\big)}\Big)^{(i-l)}_{z=\lambda^\alpha},\ \operatorname{Re}\lambda > \omega''',\ \widetilde{k_1}(\lambda) \neq 0.$$

(iii) *For every* $i \in \mathbb{N}_0$, *there exist a constant* $c_i \in \mathbb{C}$ *and a function* ${}_ik(t)$ *satisfying* (P1) *such that*

$$c_i + \widetilde{{}_ik}(\lambda) = \lambda^\alpha \widetilde{k_1}(\lambda)\Big(\frac{1}{z\tilde{k}\big(z^{1/\alpha}\big)}\Big)^{(i)}_{z=\lambda^\alpha},\ \operatorname{Re}\lambda > \omega''',\ \widetilde{k_1}(\lambda) \neq 0.$$

(iv) $\sum_{i=0}^\infty |c_i|\frac{\|B\|^i}{i!} < \infty$ *and* $\sum_{i=0}^\infty \frac{\|B\|^i}{i!}\int_0^t |{}_ik(s)|ds \leqslant Me^{\omega' t}$, $t \geqslant 0$,

(v) $\sum_{i=1}^\infty \sum_{l=1}^i \sum_{l_0=1}^l \frac{\|B\|^i}{i!}\binom{i}{l}\int_0^t (t-s)^{l_0}|k_{i,l_0,l}(s)|ds \leqslant Me^{\omega' t},\ t \geqslant 0,$

(vi) $\sum_{i=2}^\infty \sum_{l=2}^i \sum_{l_0=1}^{l-1} \frac{\|B\|^i}{i!}l\binom{i}{l}\int_0^t (t-s)^{l_0}|k_{i-1,l_0,l-1}(s)|ds \leqslant Me^{\omega'' t},\ t \geqslant 0.$

Then the operator $A+B$ *is a subgenerator of an exponentially bounded* $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_1)$-*regularized* C-*resolvent family* $(R_B(t))_{t\geqslant 0}$, *which is given by the following formula:*

$$\begin{aligned}R_B(t)x := &\sum_{i=0}^\infty \frac{(-B)^i}{i!}\Big[c_iR(t)x + \big({}_ik * R(\cdot)x\big)(t)\Big]\\ &+\sum_{i=1}^\infty\sum_{l=1}^i\sum_{l_0=1}^l \frac{(-B)^i}{i!}\binom{i}{l}\Big(k_{i,l_0,l} *^{\cdot l_0} R(\cdot)x\Big)(t),\ t \geqslant 0,\ x \in E.\end{aligned}$$

Furthermore,

$$\big(A+B\big)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}R_B(s)x\,ds = R_B(t)x - k_1(t)Cx,\ t \geqslant 0,\ x \in E, \tag{136}$$

$\sup_{t\geqslant 0} e^{-(\omega+\omega')t}\|R_B(t)\| < \infty$ *and*

$$R_B(t)R_B(s) = R_B(s)R_B(t),\ t,\ s \geqslant 0. \tag{137}$$

It is noteworthy that $(R_B(t))_{t\geqslant 0}$ is a unique $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_1)$-regularized C-resolvent family with the properties stated in the formulation of Theorem 2.5.12 and that it is not clear whether there exist functions $k(t)$ and $k_1(t)$ such that Theorem 2.5.12 is applicable in the case $\alpha \in (0,1)$; cf. also [**36**, Example 2.24]. In the following theorem, we analyze stability of analytical properties under bounded perturbations described in Theorem 2.5.12 (cf. also Theorem 2.5.3 and Theorem 2.5.6).

THEOREM 2.5.13. [**242**] *Consider the situation of* Theorem 2.5.12. *Assume additionally that* $(R(t))_{t\geqslant 0}$ *is an exponentially bounded, analytic* $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k)$-*regularized* C-*resolvent family of angle* $\beta \in (0, \frac{\pi}{2}]$ *and that, for every* $\gamma \in (0,\beta)$, *the set* $\{e^{-\omega_\gamma \operatorname{Re} z}R(z) : z \in \Sigma_\gamma\}$ *is bounded for some* $\omega_\gamma \geqslant 0$. *Let* $\varepsilon > 0$ *be such that, for every* $\gamma \in (0,\beta)$, *there exist* $\omega_{\gamma,1} \geqslant \max(\sup\{\operatorname{abs}({}_ik) : i \geqslant 1\}, \omega_\gamma)$ *and* $\omega_{\gamma,2} \geqslant \max(\sup\{\operatorname{abs}(k_{i,l_0,l}) : 1 \leqslant l \leqslant i,\ 1 \leqslant l_0 \leqslant l\}, \omega_\gamma + \varepsilon)$ *with the following properties:*

(i) *For every $i \in \mathbb{N}_0$, the function $\lambda \mapsto {}_i\widetilde{k}(\lambda)$, $\lambda > \omega_{\gamma,1}$ can be analytically extended to the sector $\omega_{\gamma,1} + \Sigma_{\frac{\pi}{2}+\gamma}$ and the following holds:*

$$\sum_{i=0}^{\infty} \frac{\|B\|^i}{i!} \sup_{\lambda \in \omega_{\gamma,1}+\Sigma_{\frac{\pi}{2}+\gamma}} \bigl|{}_i\widetilde{k}(\lambda)\bigr| < \infty.$$

(ii) *For every i, l_0, $l \in \mathbb{N}$ with $1 \leqslant l \leqslant i$ and $1 \leqslant l_0 \leqslant l$, the function $\lambda \mapsto \mathcal{L}(k_{i,l_0,l}(t))(\lambda)$, $\lambda > \omega_{\gamma,2}$ can be analytically extended to the sector $\omega_{\gamma,2} + \Sigma_{\frac{\pi}{2}+\gamma}$ and the following holds:*

$$\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{\|B\|^i}{i!}\binom{i}{l}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \sup_{\lambda \in \omega_{\gamma,2}+\Sigma_{\frac{\pi}{2}+\gamma}} \bigl|\mathcal{L}\bigl(k_{i,l_0,l}(t)\bigr)(\lambda)\bigr| < \infty.$$

Then $(R_B(t))_{t\geqslant 0}$ is an exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_1)$-regularized C-resolvent family of angle β.

The assumptions of Theorem 2.5.12 and Theorem 2.5.13 are satisfied provided $\alpha > 1$ and $k(t) = k_1(t) = \frac{t^r}{\Gamma(r+1)}$, where $r \geqslant 0$. In this case, $\zeta_\alpha = 1$,

$$\sum_{l_0=1}^{l} l_0!|c_{l_0,l,\alpha}| = \frac{1}{\alpha}\Bigl(\frac{1}{\alpha}+1\Bigr)\cdots\Bigl(\frac{1}{\alpha}+(l-1)\Bigr) \text{ for all } l \in \mathbb{N},$$

$c_0 = 1$, $k_0(t) = 0$, $c_i = 0$, $i \geqslant 1$,

$${}_ik(t) = \Bigl(\frac{r+1}{\alpha}-1\Bigr)\cdots\Bigl(\frac{r+1}{\alpha}-i\Bigr)\frac{t^{\alpha i-1}}{\Gamma(\alpha i)},\ t \geqslant 0,\ i \geqslant 1$$

and, for every i, l_0, $l \in \mathbb{N}$ with $1 \leqslant l \leqslant i$ and $1 \leqslant l_0 \leqslant l$,

$$k_{i,l_0,l}(t) = c_{l_0,l,\alpha}\Bigl(\frac{r+1}{\alpha}-1\Bigr)\cdots\Bigl(\frac{r+1}{\alpha}-(i-l)\Bigr)\frac{t^{\alpha i-l_0-1}}{\Gamma(\alpha i-l_0)},\ t > 0,$$

where $(\frac{r+1}{\alpha}-1)\cdots(\frac{r+1}{\alpha}-(i-i)) := 1$.

Corollary 2.5.14. *Suppose $\alpha > 1$, $\omega \geqslant 0$, $r \geqslant 0$, A is a subgenerator of an r-times integrated $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family $(R(t))_{t\geqslant 0}$ satisfying* (134)-(135) *for some $\omega \geqslant 0$. Let $B \in L(E)$ satisfy the condition* (i) *quoted in the formulation of* Theorem 2.5.12. *Then $A + B$ is a subgenerator of an exponentially bounded r-times integrated $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family $(R_B(t))_{t\geqslant 0}$ satisfying* (136)-(137) *and $\sup_{t\geqslant 0} \frac{1}{(t+1)} e^{-(\omega+\|B\|^{1/\alpha})t}\|R_B(t)\| < \infty$. Furthermore, $(R_B(t))_{t\geqslant 0}$ is an exponentially bounded, analytic r-times integrated $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family of angle $\beta \in (0, \frac{\pi}{2}]$ provided that $(R(t))_{t\geqslant 0}$ is.*

Assume now $\alpha > 1$, $\varrho > 0$, $\sigma \in (0, 1)$ and $k(t) = k_1(t) = \mathcal{L}^{-1}\bigl(\lambda^{-\alpha}e^{-\varrho\lambda^\sigma}\bigr)(t)$, $t \geqslant 0$. Then, for every $l \in \mathbb{N}$, there exist real numbers $(p_{m,l,\alpha,\varrho,\sigma})_{1\leqslant m\leqslant l}$ such that, for every $l \in \mathbb{N}$, $p_{1,l,\alpha,\varrho,\sigma} = \varrho\frac{\sigma}{\alpha}(\frac{\sigma}{\alpha}-1)\cdots(\frac{\sigma}{\alpha}-(l-1))$, $p_{l,l,\alpha,\varrho,\sigma} = (\varrho\frac{\sigma}{\alpha})^l$, and that the following holds:

$$\frac{d^l}{dz^l}\Big(\frac{1}{z\tilde{k}(z^{1/\alpha})}\Big)=\frac{d^l}{dz^l}e^{\varrho z^{\sigma/\alpha}}=e^{\varrho z^{\sigma/\alpha}}\sum_{m=1}^{l}p_{m,l,\alpha,\varrho,\sigma}z^{m\frac{\sigma}{\alpha}-l},\ z>0$$

and

$$p_{m,l+1,\alpha,\varrho,\sigma}=\varrho\frac{\sigma}{\alpha}p_{m-1,l,\alpha,\varrho,\sigma}+\Big(m\frac{\sigma}{\alpha}-l\Big)p_{m,l,\alpha,\varrho,\sigma},\ 2\leqslant m\leqslant l. \tag{138}$$

This implies $c_0=1$, $k_0(t)=0$, $c_i=0$, $i\geqslant 1$,

$$_ik(t)=\sum_{m=1}^{i}p_{m,i,\alpha,\varrho,\sigma}\frac{t^{\alpha i-m\sigma-1}}{\Gamma(\alpha i-m\sigma)},\ t>0,\ i\geqslant 1,$$

$$k_{i,l_0,l}(t)=c_{l_0,l,\alpha}\sum_{m=1}^{i-l}p_{m,i-l,\alpha,\varrho,\sigma}\frac{t^{\alpha i-l_0-m\sigma-1}}{\Gamma(\alpha i-l_0-m\sigma)},\ t>0,\ 1\leqslant l<i,\ 1\leqslant l_0\leqslant l$$

$$k_{i,l_0,i}(t)=c_{l_0,i,\alpha}\frac{t^{\alpha i-l_0-1}}{\Gamma(\alpha i-l_0)},\ t>0,\ 1\leqslant l_0\leqslant i.$$

In view of (138), we easily obtain the existence of a constant $\zeta_{\alpha,\varrho,\sigma}\geqslant 1$ such that

$$\sum_{m=1}^{l}m!|p_{m,l,\alpha,\varrho,\sigma}|\leqslant\zeta^l_{\alpha,\varrho,\sigma}l!\text{ for all }l\in\mathbb{N}. \tag{139}$$

In what follows, we assume that $\zeta_{\alpha,\varrho,\sigma}\geqslant 1$ is minimal with respect to (139).

Corollary 2.5.15. *Let us suppose that $\alpha>1$, $\omega\geqslant 0$, $\varrho>0$, $\sigma\in(0,1)$, $k(t)=\mathcal{L}^{-1}\big(\lambda^{-\alpha}e^{-\varrho\lambda^\sigma}\big)(t)$, $t\geqslant 0$ and let A be a subgenerator of an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ satisfying* (134)–(135). *Let $B\in L(E)$ satisfy the condition* (i) *quoted in the formulation of* Theorem 2.5.12. *Then $A+B$ is a subgenerator of an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized C-resolvent family $(R_B(t))_{t\geqslant 0}$ satisfying* (136)-(137) *and* $\sup_{t\geqslant 0}e^{-(\omega+(\|B\|\zeta_{\alpha,\varrho,\sigma})^{1/\alpha}+\varepsilon)t}\|R_B(t)\|<\infty$ *for every $\varepsilon>0$. Furthermore, $(R_B(t))_{t\geqslant 0}$ is an exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized C-resolvent family of angle $\beta\in(0,\frac{\pi}{2}]$ provided that $(R(t))_{t\geqslant 0}$ is.*

One can simply prove that A is the integral generator of an exponential distribution cosine function iff A is the integral generator of an exponentially bounded α-times integrated cosine function for some $\alpha\geqslant 0$, and that there exists a tempered ultradistribution fundamental sine solution for a closed linear operator A iff A is the integral generator of a global exponentially bounded $\mathcal{L}^{-1}\big(\lambda^{-1}e^{-\varrho\lambda^{1/s}}\big)$-convoluted cosine function for some (for every) $\varrho>0$; cf. Section 3.4 and Section 3.5 for more details. By Corollary 2.5.14–Corollary 2.5.15, we obtain that the classes of exponential distribution cosine functions and tempered ultradistribution fundamental sine solutions of Beurling (Roumieu) class persist under bounded commuting perturbations. Finally, it seems to be really difficult to prove an analogue of Theorem 2.5.12 in the context of local $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized C-resolvent families. By reason of that, it is not clear whether the classes of distribution cosine functions and ultradistribution fundamental sine solutions of Beurling (Roumieu) class retain the property stated above.

2.6. Convoluted C-groups

We introduce the class of K-convoluted C-groups as follows.

DEFINITION 2.6.1. Let A and B be closed operators. A strongly continuous operator family $(S_K(t))_{t\in(-\tau,\tau)}$ is called a (*local,* if $\tau<\infty$) *K-convoluted C-group* with a subgenerator A if:

(i) $\big(S_{K,+}(t):=S_K(t)\big)_{t\in[0,\tau)}$, resp. $\big(S_{K,-}(t):=S_K(-t)\big)_{t\in[0,\tau)}$, is a (local) K-convoluted C-semigroup with a subgenerator A, resp. B, and

(ii) for every t, $s\in(-\tau,\tau)$ with $t<0<s$ and $x\in E$:

$$S_K(t)S_K(s)x=S_K(s)S_K(t)x$$
$$=\begin{cases}\int\limits_{t+s}^{s}K(r-t-s)S_K(r)Cx\,dr+\int\limits_{t}^{0}K(t+s-r)S_K(r)Cx\,dr, & t+s\geqslant 0,\\ \int\limits_{t}^{t+s}K(t+s-r)S_K(r)Cx\,dr+\int\limits_{0}^{s}K(r-t-s)S_K(r)Cx\,dr, & t+s<0.\end{cases}$$

It is said that $(S_K(t))_{t\in\mathbb{R}}$ is exponentially bounded if there exist $M>0$ and $\omega\geqslant 0$ such that $\|S_K(t)\|\leqslant Me^{\omega|t|}$, $t\in\mathbb{R}$. A closed linear operator $\hat{A}$ is the integral generator of $(S_K(t))_{t\in(-\tau,\tau)}$ if $\hat{A}$ is the integral generator of $(S_K(t))_{t\in[0,\tau)}$.

Plugging $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t\in[0,\tau)$ in Definition 2.6.1, where $\alpha>0$, we obtain the class of *α-times integrated C-groups* (cf. also [**137**, Definition 3.6], [**220**, Definition 4.1] and [**315**, Definition 5]).

Suppose $(S_K(t))_{t\in(-\tau,\tau)}$ is a (local) K-convoluted C-group. As before, $\wp(S_K)$ designates the set of all subgenerators of $(S_K(t))_{t\in(-\tau,\tau)}$, i.e., $\wp(S_K)=\wp(S_{K,+})$; then one can simply construct a global exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$ with the continuum many subgenerators.

The proof of the next proposition is omitted.

PROPOSITION 2.6.2. *Suppose $(S_K(t))_{t\in(-\tau,\tau)}$ is a (local) K-convoluted C-group and $A\in\wp(S_K)$. Put $\check{S}_K(t):=S_K(-t)$, $t\in(-\tau,\tau)$. Then $(\check{S}_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group, $B\in\wp(\check{S}_K)$ and the integral generator of $(\check{S}_K(t))_{t\in(-\tau,\tau)}$ coincides with that of $(S_{K,-}(t))_{t\in[0,\tau)}$.*

PROPOSITION 2.6.3. *Suppose $\tau\in(0,\infty]$, $K_1\in L^1_{\mathrm{loc}}([0,\tau))$, $\hat{A}$ is the integral generator of a K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $A\in\wp(S_K)$ and $K*_0K_1\neq 0$ in $L^1_{\mathrm{loc}}([0,\tau))$. Put $S_{K*_0K_1}(t)x:=\int_0^t K_1(t-s)S_K(s)x\,ds$, $t\in[0,\tau)$, $x\in E$ and $S_{K*_0K_1}(t)x:=\int_0^{-t}K_1(-t-s)S_K(-s)x\,ds$, $t\in(-\tau,0)$, $x\in E$. Then $(S_{K*_0K_1}(t))_{t\in(-\tau,\tau)}$ is a $(K*_0K_1)$-convoluted C-group, $A\in\wp(S_{K*_0K_1})$ and the integral generator of $(S_{K*_0K_1}(t))_{t\in(-\tau,\tau)}$ is $\hat{A}$.*

Proof. It is clear that

$$\big(S_{K*_0K_1,+}(t) := S_{K*_0K_1}(t)\big)_{t\in[0,\tau)} \text{ and } \big(S_{K*_0K_1,-}(t) := S_{K*_0K_1}(-t)\big)_{t\in[0,\tau)}$$

are $(K *_0 K_1)$-convoluted C-semigroups whose integral generators are $\hat{A}$ and $\hat{B}$, respectively. Furthermore, $A \in \wp(S_{K*_0K_1,+})$, $B \in \wp(S_{K*_0K_1,-})$ and

$$S_{K*_0K_1}(t)S_{K*_0K_1}(s) = S_{K*_0K_1}(s)S_{K*_0K_1}(t),\ -\tau < t < 0 < s < \tau.$$

So, it suffices to prove the composition property for $S_{K*_0K_1}(t)S_{K*_0K_1}(s)$, $-\tau < t < 0 < s < \tau$. This will be done only in the case $t + s \geqslant 0$. Fix an $x \in E$ and observe that:

$$\begin{aligned}
S_{K*_0K_1}(t)S_{K*_0K_1}(s)x &= \int_0^{-t} K_1(-t-v)S_K(-v)S_{K*_0K_1}(s)x\,dv \\
&= \int_0^{-t}\int_0^{s} K_1(-t-v)K_1(s-u)S_K(-v)S_K(u)x\,du\,dv \\
&= \int_0^{-t} K_1(-t-v)\Bigg[\int_0^{v} K_1(s-u)S_K(-v)S_K(u)x\,du\Bigg]\,dv \\
&\quad + \int_0^{-t} K_1(-t-v)\Bigg[\int_v^{s} K_1(s-u)S_K(-v)S_K(u)x\,du\Bigg]\,dv \\
&= \int_0^{-t} K_1(-t-v)\int_0^{v} K_1(s-u) \\
&\Bigg[\int_{-v}^{u-v} K(u-v-r)S_K(r)Cx\,dr + \int_0^{u} K(r-u+v)S_K(r)Cx\,dr\Bigg]\,du\,dv \\
&+ \int_0^{-t} K_1(-t-v)\int_v^{s} K_1(s-u) \\
&\Bigg[\int_{u-v}^{u} K(r-u+v)S_K(r)Cx\,dr + \int_{-v}^{0} K(u-v-r)S_K(r)Cx\,dr\Bigg]\,du\,dv \\
&= S_1 + S_2,
\end{aligned}$$

where

$$S_1 := \int_0^{-t} K_1(-t-v)\int_0^{v} K_1(s-u)\int_0^{u} K(r-u+v)S_K(r)Cx\,dr\,du\,dv$$

$$+\int_0^{-t} K_1(-t-v)\int_v^s K_1(s-u)\int_{u-v}^u K(r-u+v)S_K(r)Cx\,dr\,du\,dv$$

$$S_2 := \int_0^{-t} K_1(-t-v)\int_0^v K_1(s-u)\int_{-v}^{u-v} K(u-v-r)S_K(r)Cx\,dr\,du\,dv$$

$$+\int_0^{-t} K_1(-t-v)\int_v^s K_1(s-u)\int_{-v}^0 K(u-v-r)S_K(r)Cx\,dr\,du\,dv.$$

The proof is completed if one shows:

$$S_1 = \int_{t+s}^s (K *_0 K_1)(\xi-t-s)\int_0^\xi K_1(\xi-z)S_K(z)Cx\,dz\,d\xi, \tag{140}$$

$$S_2 = \int_t^0 (K *_0 K_1)(t+s-\xi)\int_0^{-\xi} K_1(-\xi-z)S_K(-z)Cx\,dz\,d\xi. \tag{141}$$

To prove (140), one can use the equality

$$\int_{t+s}^s (K *_0 K_1)(\xi-t-s)\int_0^\xi K_1(\xi-z)S_K(z)Cx\,dz\,d\xi$$

$$= \int_{t+s}^s \left[\int_0^{\xi-t-s} K_1(\xi-t-s-\sigma)K(\sigma)d\sigma\right]\int_0^\xi K_1(\xi-z)S_K(z)Cx\,dz\,d\xi$$

and the substitution of variables $v = s+\sigma-\xi$, $u = s+z-\xi$ and $r = z$; the proof of (141) can be obtained along the same lines. □

Proposition 2.6.4. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $A \in \wp(S_K)$, $B \in \wp(S_{K,-})$ and $\hat{B}$ is the integral generator of $(S_{K,-}(t))_{t\in[0,\tau)}$. Then:*

(i) $\hat{A}S_K(t)x = S_K(t)Ax$, $x \in D(A)$, $t \in (-\tau,0]$ *and* $\hat{B}S_K(s)x = S_K(s)Bx$, $x \in D(B)$, $s \in [0,\tau)$.

(ii) $S_K(t)\hat{A} \subseteq \hat{A}S_K(t)$, $t \in (-\tau,0]$ *and* $S_K(s)\hat{B} \subseteq \hat{B}S_K(s)$, $s \in [0,\tau)$.

Proof. Put $\Theta_1(t) = \int_0^t \Theta(s)\,ds$, $t \in [0,\tau)$, $S_{\Theta_1}(t)x = \int_0^t (t-s)S_K(s)x\,ds$, $t \in [0,\tau)$, $x \in E$ and $S_{\Theta_1}(t)x = \int_0^{-t}(-t-s)S_K(-s)x\,ds$, $t \in (-\tau,0)$, $x \in E$. By Proposition 2.6.3, $(S_{\Theta_1}(t))_{t\in(-\tau,\tau)}$ is a Θ_1-convoluted C-group, $A \in \wp(S_{\Theta_1})$, the integral generator of $(S_{\Theta_1}(t))_{t\in(-\tau,\tau)}$ is $\hat{A}$ and the integral generator of $(S_{\Theta_1,-}(t))_{t\in[0,\tau)}$ is

$\hat{B}$. Clearly,

$$
\begin{aligned}
&S_{\Theta_1}(t)A\int_0^s S_{\Theta_1}(r)x\,dr = S_{\Theta_1}(t)\bigg(S_{\Theta_1}(s)x-\int_0^s \Theta_1(r)\,dr Cx\bigg)\\
&= S_{\Theta_1}(s)S_{\Theta_1}(t)x-\int_0^s \Theta_1(r)\,dr CS_{\Theta_1}(t)x\\
&= A\int_0^s S_{\Theta_1}(r)S_{\Theta_1}(t)x\,dr+\int_0^s \Theta_1(r)\,dr CS_{\Theta_1}(t)x-\int_0^s \Theta_1(r)\,dr CS_{\Theta_1}(t)x\\
&= A\int_0^s S_{\Theta_1}(r)S_{\Theta_1}(t)x\,dr = AS_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr,\ \ t\in(-\tau,0),\ s\in[0,\tau),\ x\in E.
\end{aligned}
$$

Suppose now $x\in D(A)$. Then we obtain

$$
S_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)Ax\,dr = AS_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr,\ \ t\in(-\tau,0),\ s\in[0,\tau).
$$

The previous equality and closedness of A imply $S_{\Theta_1}(t)S_{\Theta_1}(s)x\in D(A)$, $t\in(-\tau,0)$, $s\in[0,\tau)$ and $AS_{\Theta_1}(t)S_{\Theta_1}(s)x=S_{\Theta_1}(t)S_{\Theta_1}(s)Ax$, $t\in(-\tau,0)$, $s\in[0,\tau)$. Suppose, for a moment, $t\in(-\tau,0)$, $s\in[0,\tau)$ and $t+s\geqslant 0$. The composition property of $S_{\Theta_1}(\cdot)$ allows one to establish the following equality:

$$
\begin{aligned}
&\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)CAx\,dr+\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)CAx\,dr\\
&\qquad = A\bigg[\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr+\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr\bigg].
\end{aligned}
$$

Since $S_{\Theta_1}(r)A\subseteq AS_{\Theta_1}(r)$, $r\in[0,\tau)$ and $CA\subseteq AC$, one gets

$$
\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr\in D(A)
$$

$$
A\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr=\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)CAx\,dr.
$$

Hence, $\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr\in D(A)$ and

$$
A\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr=\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)CAx\,dr. \tag{142}
$$

Put now $\Omega=\{(t,s)\in(-\tau,0)\times(0,\tau):t+s>0\}$ and

$$f_y(t,s)=\int\limits_t^0\Theta_1(t+s-r)S_{\Theta_1}(r)y\,dr,\ \ (t,s)\in\Omega,\ y\in E.$$

Then the dominated convergence theorem implies:

$$\frac{\partial}{\partial t}f_y(t,s)=\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)y\,dr-\Theta_1(s)S_{\Theta_1}(t)y,$$

$$\frac{\partial}{\partial s}f_y(t,s)=\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)y\,dr,\ \ (t,s)\in\Omega,\ y\in E.$$

By the closedness of A and (142), one gets $A\frac{\partial}{\partial s}f_{Cx}(t,s)=\frac{\partial}{\partial s}f_{CAx}(t,s)$, $(t,s)\in\Omega$. In other words,

$$(143)\quad A\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr=\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)CAx\,dr,\ (t,s)\in\Omega.$$

Analogously, $A\frac{\partial}{\partial t}f_{Cx}(t,s)=\frac{\partial}{\partial t}f_{CAx}(t,s)$, $(t,s)\in\Omega$, i.e., for every $(t,s)\in\Omega$,

$$(144)\quad A\Bigg[\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\Theta_1(s)S_{\Theta_1}(t)Cx\Bigg]$$
$$=\int\limits_t^0\Theta(t+s-r)S_{\Theta_1}(r)CAx\,dr-\Theta_1(s)S_{\Theta_1}(t)CAx\,dr.$$

An employment of (143)–(144) gives $\Theta_1(s)S_{\Theta_1}(t)Cx\in D(A)$, $(t,s)\in\Omega$ and $A(\Theta_1(s)S_{\Theta_1}(t)Cx)=\Theta_1(s)S_{\Theta_1}(t)CAx$, $(t,s)\in\Omega$. Similarly, $A(\Theta_1(s)S_{\Theta_1}(t)Cx)=\Theta_1(s)S_{\Theta_1}(t)CAx$, if $(t,s)\in(-\tau,0)\times(0,\tau)$ and $t+s\leqslant 0$. Thus,

$$(145)\qquad A\big(\Theta_1(s)S_{\Theta_1}(t)Cx\big)=\Theta_1(s)S_{\Theta_1}(t)CAx,\ \ t\in(-\tau,0),\ s\in[0,\tau).$$

It is evident that there exists $s\in[0,\tau)$ with $\Theta_1(s)\neq 0$ and one can apply (145) in order to conclude that $A(S_{\Theta_1}(t)Cx)=S_{\Theta_1}(t)CAx$, $t\in(-\tau,0)$. Differentiate the last equality twice with respect to t to obtain that $S_K(t)Cx\in D(A)$ and that $AS_K(t)Cx=S_K(t)CAx$, $t\in(-\tau,0)$. The last equality gives $ACS_K(t)x=CS_K(t)Ax$, $S_K(t)x\in D(C^{-1}AC)$ and $[C^{-1}AC]S_K(t)x=S_K(t)Ax$, $t\in(-\tau,0]$. On the other hand, Proposition 2.1.6 implies $\hat{A}=C^{-1}AC$, and consequently, $S_K(t)x\in D(\hat{A})$, $x\in D(A)$, $t\in(-\tau,0]$. Since $\hat{A}\in\wp(S_K)$ and $C^{-1}\hat{A}C=\hat{A}$, one obtains that $S_K(t)\hat{A}x=[C^{-1}\hat{A}C]S_K(t)x=\hat{A}S_K(t)x$, $t\in(-\tau,0]$, $x\in D(\hat{A})$. The remnant of proof follows by Proposition 2.6.2. □

Assume $\alpha>0$, $(S_\alpha(t))_{t\in\mathbb{R}}$ is an exponentially bounded, α-times integrated group generated by A and B is the generator of $(S_\alpha(-t))_{t\geqslant 0}$. Let us recall that El-Mennaoui proved in [**117**] that $B=-A$; this result can be generalized as follows:

THEOREM 2.6.5. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $\hat{B}$ is the integral generator of $(S_{K,-}(t))_{t\in[0,\tau)}$, $A \in \wp(S_K)$ and $B \in \wp(S_{K,-})$. Then the following holds:*

(i) $S_K(t)x \in D(B)$ *and* $BS_K(t)x = -S_K(t)\hat{A}x$, $x \in D(\hat{A})$, $t \in (-\tau, 0]$; $S_K(s)x \in D(A)$ *and* $AS_K(s)x = -S_K(s)\hat{B}x$, $x \in D(\hat{B})$, $s \in [0,\tau)$,

(ii) $\hat{B} = -\hat{A}$,

(iii) $BCx = -C\hat{A}x$, $x \in D(\hat{A})$; $ACx = -C\hat{B}x$, $x \in D(\hat{B})$ *and*

(iv) $\int_0^t S_K(r)Cx\,dr \in D(A)$, $t \in (-\tau, 0]$; $\int_0^s S_K(r)Cx\,dr \in D(B)$, $s \in [0,\tau)$.

PROOF. Let

$$\Theta_i(t) = \int_0^t (t-s)^{i-1}\Theta(s)\,ds, \ \ i = 1, 2, \ t \in [0,\tau),$$

$$S_{\Theta_1}(t)x = \int_0^t (t-s)S_K(s)x\,ds, \ \ t \in [0,\tau), \ x \in E,$$

$$S_{\Theta_1}(t)x = \int_0^{-t} (-t-s)S_K(-s)x\,ds, \ \ t \in (-\tau, 0), \ x \in E.$$

Suppose now $t < 0 < s$, $t + s \leqslant 0$ and $x \in E$. Then the preceding proposition and the composition property of $S_{\Theta_1}(\cdot)$ imply:

$$S_{\Theta_1}(t)\Bigg(S_{\Theta_1}(s)x - \int_0^s \Theta_1(r)\,drCx\Bigg) = S_{\Theta_1}(t)\hat{A}\int_0^s S_{\Theta_1}(r)x\,dr \tag{146}$$

$$= \hat{A}S_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr = \hat{A}\int_0^s S_{\Theta_1}(t)S_{\Theta_1}(r)x\,dr$$

$$= \hat{A}\int_0^s \Bigg[\int_t^{t+r} \Theta_1(t+r-v)S_{\Theta_1}(v)Cx\,dv + \int_0^r \Theta_1(v-t-r)S_{\Theta_1}(v)Cx\,dv\Bigg]dr$$

$$= \int_t^{t+s} \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr + \int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr - \int_0^s \Theta_1(r)\,drS_{\Theta_1}(t)Cx.$$

Differentiate (146) with respect to s in order to conclude that:

$$\hat{A}\Bigg[\int_t^{t+s} \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr + \int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr\Bigg] \tag{147}$$

$$= \int_t^{t+s} \Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr - \int_0^s \Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr$$

$$+\Theta_1(-t)S_{\Theta_1}(s)Cx-\Theta_1(s)S_{\Theta_1}(t)Cx.$$

Further on, it is clear that $\int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr \in D(\hat{A})$ and that

$$\hat{A}\int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr=\int_0^s \Theta_1(r-t-s)\hat{A}\int_0^r S_\Theta(v)Cx\,dv\,dr$$
$$=\int_0^s \Theta_1(r-t-s)(S_\Theta(r)Cx-\Theta_1(r)C^2x)\,dr$$
$$=\int_0^s \Theta_1(r-t-s)S_\Theta(r)Cx\,dr-\int_0^s \Theta_1(r-t-s)\Theta_1(r)C^2x\,dr.$$

This equality and (147) imply $\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \in D(\hat{A})$ and:

$$(148)\quad \hat{A}\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr$$
$$=\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\int_0^s\Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr-\int_0^s\Theta_1(r-t-s)S_\Theta(r)Cx\,dr$$
$$+\Theta_1(-t)S_{\Theta_1}(s)Cx-\Theta_1(s)S_{\Theta_1}(t)Cx+\int_0^s\Theta_1(r-t-s)\Theta_1(r)\,drC^2x.$$

The partial integration yields

$$-\int_0^s\Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr-\int_0^s\Theta_1(r-t-s)S_\Theta(r)Cx\,dr=-\Theta_1(-t)S_{\Theta_1}(s)Cx$$

and, due to (148), one gets:

$$(149)\quad \hat{A}\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr$$
$$=\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr+\int_0^s\Theta_1(r-t-s)\Theta_1(r)\,drC^2x-\Theta_1(s)S_{\Theta_1}(t)Cx.$$

Further on,

$$B\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr$$

$$= B\int\limits_t^{t+s}\Theta_1(t+s-r)\int\limits_0^{-r}S_\Theta(-v)Cx\,dv\,dr$$

$$= \int\limits_t^{t+s}\Theta_1(t+s-r)[S_\Theta(r)Cx-\Theta_1(-r)C^2x]\,dr$$

$$= \int\limits_t^{t+s}\Theta_1(t+s-r)S_\Theta(r)Cx\,dr-\int\limits_t^{t+s}\Theta_1(t+s-r)\Theta_1(-r)C^2x\,dr$$

$$= \Theta_1(s)S_{\Theta_1}(t)Cx-\int\limits_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\int\limits_t^{t+s}\Theta_1(t+s-r)\Theta_1(-r)C^2x\,dr,$$

where the last equality follows from integration by parts. Hence,

$$B\int\limits_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \tag{150}$$

$$= \Theta_1(s)S_{\Theta_1}(t)Cx-\int\limits_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\int\limits_0^{s}\Theta_1(r-t-s)\Theta_1(r)C^2x\,dr.$$

By (149)–(150), we obtain:

$$\hat{A}\int\limits_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr=-B\int\limits_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{151}$$

Suppose $x\in D(\hat{A})$; then $Cx\in D(\hat{A})$ and, thanks to Proposition 2.6.3 and (151), we easily infer that:

$$\int\limits_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr=-B\int\limits_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{152}$$

Differentiate the previous equality with respect to s to conclude that $\int\limits_t^{t+s}\Theta(t+s-r)$ $\times\, S_{\Theta_1}(r)Cx\,dr\in D(B)$ and that:

$$\int\limits_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr=-B\int\limits_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{153}$$

On the other hand, differentiation of (152) with respect to t leads us to the following: $\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr+\Theta_1(s)S_{\Theta_1}(t)Cx\in D(B)$ and

$$\int\limits_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr+\Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx \tag{154}$$

$$= -B\left[\int_t^{t+s} \Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr + \Theta_1(s)S_{\Theta_1}(t)Cx\right].$$

Making use of (153)–(154), it readily follows that $\Theta_1(s)S_{\Theta_1}(t)Cx \in D(B)$ and $-B(\Theta_1(s)S_{\Theta_1}(t)Cx) = \Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx$. Using the similar arguments, one obtains that the last equality remains true if $t+s \geqslant 0$ and $x \in D(\hat{A})$. So, $\Theta_1(s)S_{\Theta_1}(t)Cx \in D(B)$ and

(155)
$$-B\big(\Theta_1(s)S_{\Theta_1}(t)Cx\big) = \Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx,\ t \in (-\tau, 0],\ s \in [0,\tau),\ x \in D(\hat{A}).$$

Choose a number $s \in [0,\tau)$ with $\Theta_1(s) \neq 0$ and notice that (155) implies $S_{\Theta_1}(t)Cx \in D(B)$ and

(156)
$$-B(S_{\Theta_1}(t)Cx) = S_{\Theta_1}(t)\hat{A}Cx,\ t \in (-\tau, 0],\ x \in D(\hat{A}).$$

A consequence of (156) is

$$\begin{aligned} S_{\Theta_1}(t)Cx - \Theta_2(-t)C^2x &= B\int_0^{-t} S_{\Theta_1}(-v)Cx\,dv = -\int_0^{-t} S_{\Theta_1}(-v)\hat{A}Cx\,dv \\ &= -C\int_0^{-t} S_{\Theta_1}(-v)\hat{A}x\,dv,\quad t \in (-\tau, 0],\ x \in D(\hat{A}). \end{aligned}$$

Therefore,

(157)
$$S_{\Theta_1}(t)x - \Theta_2(-t)Cx = -\int_0^{-t} S_{\Theta_1}(-v)\hat{A}x\,dv,\ t \in (-\tau, 0],\ x \in D(\hat{A}),$$

which clearly implies

$$B\int_0^{-t} S_{\Theta_1}(-v)x\,dv = -\int_0^{-t} S_{\Theta_1}(-v)\hat{A}x\,dv,\ t \in (-\tau, 0],\ x \in D(\hat{A}).$$

The closedness of B enables one to see that $S_{\Theta_1}(t)x \in D(B)$ and that $BS_{\Theta_1}(t)x = -S_{\Theta_1}(t)\hat{A}x,\ t \in (-\tau, 0],\ x \in D(\hat{A})$. Differentiate the last equality twice with respect to t so as to conclude that $S_K(t)x \in D(B)$ and that $BS_K(t)x = -S_K(t)\hat{A}x$, $t \in (-\tau, 0],\ x \in D(\hat{A})$. This equality and Proposition 2.6.2 imply: $\check{S}_K(-s)x \in D(A)$ and $A\check{S}_K(-s)x = -\check{S}_K(-s)\hat{B}x,\ s \in [0,\tau),\ x \in D(\hat{B})$, i.e., $S_K(s)x \in D(A)$ and $AS_K(s)x = -S_K(s)\hat{B}x,\ x \in D(\hat{B}),\ s \in [0,\tau)$. The proof of (i) is completed. Further on, (157) implies $-\hat{A} \subseteq \hat{B}$. Now one can apply Proposition 2.6.2 and the first part of proof to obtain that $-\hat{B} \subseteq \hat{A}$; hence, $\hat{B} = -\hat{A}$ and this ends the proof of (ii). Finally, (iii) and (iv) are simple consequences of the assertion (ii) of this theorem and Proposition 2.1.6(i)-(ii). $\square$

COROLLARY 2.6.6. *Assume K satisfies* (P1) *and $\hat{A}$ is the integral generator of an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$. If there exist $M > 0$*

and $\beta > 0$ such that $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, then $C^{-1}\hat{A}^2C$ is the integral generator of an exponentially bounded, analytic K_1-convoluted C-semigroup $(S_{K_1}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where

$$K_1(t) := \int\limits_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}} K(s)\,ds, \quad t > 0,$$

$$S_{K_1}(t)x := \frac{1}{2\sqrt{\pi t}} \int\limits_0^\infty e^{-s^2/4t}\big(S_K(s)x + S_K(-s)x\big)\,ds, \quad t > 0,\ x \in E.$$

Before proceeding further, let us point out that the previous corollary remains true in the case $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha \in (0,1)$.

Theorem 2.6.7. *Assume $\tau \in (0,\infty]$ and $\pm\hat{A}$ are the integral generators of K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\in[0,\tau)}$. Put $S_K(t) := S_{K,+}(t)$, $t \in [0,\tau)$ and $S_K(t) := S_{K,-}(-t)$, $t \in (-\tau, 0)$. Then $(S_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group whose integral generator is $\hat{A}$.*

Proof. Assume $-\tau < t < 0 < s < \tau$ and $t + s \geqslant 0$. We will prove the composition property for $S_K(t)S_K(s)$. Fix an $x \in E$ and define

$$f(r) := S_K(t + s - r) \int\limits_0^r S_K(\sigma)x\,d\sigma,\ r \in [t+s, s].$$

Clearly, $\hat{A}S_K(\sigma) \subseteq S_K(\sigma)\hat{A}$, $\sigma \in (-\tau,\tau)$ and the semigroup property of a K-convoluted C-semigroup implies:

$$\frac{d}{dr} f(r) = S_K(t+s-r)S_K(r)x - \hat{A}S_K(t+s-r)\int\limits_0^r S_K(\sigma)x\,d\sigma$$

$$+K(r-s-t)C\int\limits_0^r S_K(\sigma)x\,d\sigma = \Theta(r)S_K(t+s-r)Cx + K(r-s-t)C\int\limits_0^r S_K(\sigma)x\,d\sigma,$$

for a.e. $r \in (t+s, s)$. Integrate the last equality with respect to r from $t+s$ to s to obtain:

$$S_K(t)\int\limits_0^s S_K(\sigma)x\,d\sigma = \int\limits_{t+s}^s \Theta(r)S_K(t+s-r)Cx\,dr + \int\limits_{t+s}^s K(r-s-t)C\int\limits_0^r S_K(\sigma)x\,d\sigma\,dr.$$

Since $\hat{A} \in \wp(S_{K,+})$, the last equality yields:

$$S_K(t)S_K(s)x = S_K(t)\left[\hat{A}\int\limits_0^s S_K(\sigma)x\,d\sigma + \Theta(s)Cx\right] \tag{158}$$

$$= \hat{A}\Biggl[\int\limits_{t+s}^{s}\Theta(r)S_K(t+s-r)Cx\,dr+\int\limits_{t+s}^{s}K(r-s-t)C\int\limits_{0}^{r}S_K(\sigma)x\,d\sigma\,dr\Biggr]+\Theta(s)S_K(t)Cx$$

$$= \hat{A}\int\limits_{t+s}^{s}\Theta(r)S_{K,-}(r-t-s)Cx\,dr$$

$$+\int\limits_{t+s}^{s}K(r-s-t)\Bigl[S_K(r)Cx-\Theta(r)C^2x\Bigr]\,dr+\Theta(s)S_K(t)Cx.$$

Furthermore,

$$\hat{A}\int\limits_{t+s}^{s}\Theta(r)S_{K,-}(r-t-s)Cx\,dr=\hat{A}\int\limits_{0}^{-t}\Theta(v+t+s)S_{K,-}(v)Cx\,dv \tag{159}$$

$$= \hat{A}\Biggl[\Theta(s)\int\limits_{0}^{-t}S_{K,-}(r)Cx\,dr-\int\limits_{0}^{-t}K(t+s+r)\int\limits_{0}^{r}S_{K,-}(v)Cx\,dv\,dr\Biggr]$$

$$= -\Theta(s)\Bigl[S_K(t)Cx-\Theta(-t)C^2x\Bigr]+\int\limits_{0}^{-t}K(t+s+r)\Bigl[S_K(-r)Cx-\Theta(r)C^2x\Bigr]\,dr.$$

With (158)–(159) in view, one gets:

$$S_K(t)S_K(s)x$$

$$= -\Theta(s)\Bigl[S_K(t)Cx-\Theta(-t)C^2x\Bigr]+\int\limits_{0}^{-t}K(t+s+r)\Bigl[S_K(-r)Cx-\Theta(r)C^2x\Bigr]\,dr$$

$$+\int\limits_{t+s}^{s}K(r-s-t)\Bigl[S_K(r)Cx-\Theta(r)C^2x\Bigr]\,dr+\Theta(s)S_K(t)Cx$$

$$= \int\limits_{t+s}^{s}K(r-t-s)S_K(r)Cx\,dr+\int\limits_{t}^{0}K(t+s-r)S_K(r)Cx\,dr$$

$$+\Theta(s)\Theta(-t)C^2x+\int\limits_{-t}^{0}K(t+s+r)\Theta(r)C^2x\,dr-\int\limits_{t+s}^{s}K(r-s-t)\Theta(r)C^2x\,dr,$$

and the composition property for $S_K(t)S_K(s)$ follows from the next computation:

$$\Theta(s)\Theta(-t)C^2x+\int\limits_{-t}^{0}K(t+s+r)\Theta(r)C^2x\,dr-\int\limits_{t+s}^{s}K(r-s-t)\Theta(r)C^2x\,dr$$

$$= \Theta(s)\Theta(-t)C^2x - \int_{t+s}^{s} K(r)\Theta(r-t-s)C^2x\,dr - \int_{t+s}^{s} K(r-s-t)\Theta(r)C^2x\,dr$$

$$= \Theta(s)\Theta(-t)C^2x - \Big[\Theta(s)\Theta(-t)C^2x - \int_{t+s}^{s} K(r-s-t)\Theta(r)C^2x\,dr\Big]$$
$$- \int_{t+s}^{s} K(r-s-t)\Theta(r)C^2x\,dr = 0.$$

The proof of composition property in the case $t+s<0$ can be obtained as follows. Since $\hat{A}\int_0^r S_K(\sigma)x\,d\sigma = S_K(r)x - \Theta(-r)Cx$, $r \in (-\tau, 0]$, we get

$$\frac{d}{dr}f(r) = S_K(t+s-r)S_K(r)x$$
$$- \hat{A}S_K(t+s-r)\int_0^r S_K(\sigma)x\,d\sigma + K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma$$
$$= \Theta(|r|)S_K(t+s-r)Cx + K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma,$$

for a.e. $r \in (t+s, s)$. Integrate the last equality with respect to r from $t+s$ to s to obtain

$$S_K(t)\int_0^s S_K(\sigma)x\,d\sigma = \int_{t+s}^{0} \Theta(-r)S_K(t+s-r)Cx\,dr + \int_{t+s}^{0} K(r-t-s)\int_0^r S_K(\sigma)Cx\,d\sigma\,dr$$

$$+ \int_0^s \Theta(r)S_K(t+s-r)Cx\,dr + \int_0^s K(r-s-t)\int_0^r S_K(\sigma)Cx\,d\sigma\,dr. \tag{160}$$

Clearly,

$$S_K(t)S_K(s)x = S_K(t)\Big[\hat{A}\int_0^s S_K(\sigma)x\,d\sigma + \Theta(s)Cx\Big]$$
$$= \hat{A}S_K(t)\int_0^s S_K(\sigma)x\,d\sigma + \Theta(s)S_K(t)Cx,$$

and a tedious computation involving (160) leads us to the next equality:

$$S_K(t)S_K(s)x = \int_t^{t+s} K(t+s-r)S(r)Cx\,dr + \int_0^s K(r-t-s)S(r)Cx\,dr \tag{161}$$

$$+\left[\Theta(s)\Theta(-t)-\int_t^{t+s}K(t+s-r)\Theta(-r)\,dr-\int_0^s K(r-s-t)\Theta(r)\,dr\right]C^2x.$$

Since

$$\Theta(s)\Theta(-t)-\int_t^{t+s}K(t+s-r)\Theta(-r)\,dr-\int_0^s K(r-s-t)\Theta(r)\,dr$$

$$=\Theta(s)\Theta(-t)+\int_s^0 K(r)\Theta(r-t-s)\,dr-\Theta(-t)\Theta(s)+\int_0^s \Theta(r-t-s)K(r)\,dr=0,$$

(161) implies the composition property for $S_K(t)S_K(s)$. By the foregoing,

$$S_K(s)S_K(t)x=\check{S}_K(-s)\check{S}_K(-t)x$$

$$=\begin{cases}\int_{-t-s}^{-t}K(r+t+s)\check{S}_K(r)Cx\,dr+\int_{-s}^{0}K(-t-s-r)\check{S}_K(r)Cx\,dr, & t+s<0,\\ \int_{-s}^{-t-s}K(-t-s-r)\check{S}_K(r)Cx\,dr+\int_{0}^{-t}K(r+t+s)\check{S}_K(r)Cx\,dr, & t+s\geqslant 0,\end{cases}$$

$$=\begin{cases}\int_{t}^{t+s}K(t+s-r)S_K(r)Cx\,dr+\int_{0}^{s}K(r-t-s)S_K(r)Cx\,dr, & t+s<0,\\ \int_{t+s}^{s}K(r-t-s)S_K(r)Cx\,dr+\int_{t}^{0}K(t+s-r)S_K(r)Cx\,dr, & t+s\geqslant 0,\end{cases}$$

for every $x\in E$. The composition property for $S_K(t)S_K(s)$ and previous equality imply $S_K(t)S_K(s)=S_K(s)S_K(t)$, $t<0<s$, which ends the proof of theorem. $\square$

QUESTIONS. (i) Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $A\in\wp(S_K)$ and $A\neq\hat{A}$. Is it true that $-A\in\wp(S_{K,-})$?

(ii) Suppose A is the integral generator of a (local) K-convoluted (semi-)group $(S_K(t))_{t\in(-\tau,\tau)}$. Does there exist an injective operator $C\in L(E)$ such that A generates a *global* C-(semi-)group?

COROLLARY 2.6.8. *Suppose $\tau\in(0,\infty]$, $\hat{A}$ is a closed linear operator and $(S_K(t))_{t\in(-\tau,\tau)}$ is a strongly continuous operator family. Then $\hat{A}$ is the integral generator of a K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$ iff $\pm\hat{A}$ are the integral generators of K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\in[0,\tau)}$.*

The following theorem is a consequence of Corollary 2.6.8 and the corresponding assertions for exponentially bounded convoluted C-semigroups.

THEOREM 2.6.9. *Let K satisfy* (P1) *and $\hat{A}$ be a closed linear operator. Then the following holds.*

(i) *Let $M>0$ and $\omega\geqslant 0$. Then $\hat{A}$ is the integral generator of an exponentially bounded, Θ-convoluted C-group $(S_\Theta(t))_{t\in\mathbb{R}}$ such that*

$$\big\|S_\Theta(\pm t\pm h)-S_\Theta(\pm t)\big\|\leqslant Mhe^{\omega(t+h)},\ \ t\geqslant 0,\ h\geqslant 0$$

iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ *such that:*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(\pm\hat{A}), \tag{162}$$

$$\lambda \mapsto \tilde{K}(\lambda)(\lambda \pm \hat{A})^{-1}C,\ \ \lambda > a,\ \tilde{K}(\lambda) \neq 0 \text{ is infinitely differentiable,} \tag{163}$$

$$\Big\|\frac{d^k}{d\lambda^k}[\tilde{K}(\lambda)(\lambda \pm \hat{A})^{-1}C]\Big\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}},\ \ k \in \mathbb{N}_0,\ \lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{164}$$

(ii) *Suppose* $M > 0$, $\omega \geqslant 0$ *and* $\hat{A}$ *is densely defined. Then* $\hat{A}$ *is the integral generator of an exponentially bounded,* K*-convoluted* C*-group* $(S_K(t))_{t\in\mathbb{R}}$ *satisfying* $\|S_K(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, $\omega \geqslant 0$ *iff there exists* $a \geqslant \max(\omega, \operatorname{abs}(K))$ *such that* (162)–(164) *is fulfilled.*

(iii) *Suppose that* $\hat{A}$ *is the integral generator of an exponentially bounded,* K*-convoluted* C*-group* $(S_K(t))_{t\in\mathbb{R}}$ *satisfying* $\|S_K(t)\| = O(e^{\omega|t|})$, $t \in \mathbb{R}$, $\omega \geqslant 0$. *Put* $a := \max(\omega, \operatorname{abs}(K))$. *Then:*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(\pm\hat{A}) \tag{165}$$

$$(\lambda \pm \hat{A})^{-1}Cx = \frac{1}{\tilde{K}(\lambda)}\int_0^\infty e^{-\lambda t}S_K(\mp t)\,dt,\ \ \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{166}$$

(iv) *Suppose* $(S_K(t))_{t\in\mathbb{R}}$ *is a strongly continuous operator family and* $\|S_K(t)\| = O(e^{\omega|t|})$, $t \in \mathbb{R}$, $\omega \geqslant 0$. *Put* $a := \max(\omega, \operatorname{abs}(K))$. *If* (165)–(166) *hold, then* $\hat{A}$ *is the integral generator of an exponentially bounded,* K*-convoluted* C*-group* $(S_K(t))_{t\in\mathbb{R}}$.

Keeping in mind Corollary 2.6.8, one can simply formulate several other structural characterizations of convoluted C-groups. The remainder of this section is devoted to the study of relations between fractionally integrated cosine functions, analytic semigroups of growth order $r > 0$, some special subclasses of differentiable C-regularized groups and (local) convoluted groups whose derivatives possess some expected properties of operator valued ultradifferentiable functions of the Beurling type (cf. also the next chapter). We need some auxiliary notations.

1. Let $a > 0$ and $b > 0$. Then the exponential region $E(a, b)$ was primarily defined by Arendt, El–Mennaoui and Keyantuo in [**5**]:

$$E(a, b) := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\}.$$

Put $E^2(a, b) := \{\lambda^2 : \lambda \in E(a, b)\}$.

2. Suppose $s > 1$. Following Chazarain [**54**] (cf. also [**210**] and [**307**, Section 2.3]), we define the ultra-logarithmic region of type l:

$$\Omega_{\alpha,\beta,l} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha M(l|\operatorname{Im}\lambda|) + \beta\},\ \ l > 0,\ \alpha > 0,\ \beta \in \mathbb{R},$$

where $M(t) := \sup_{p\in\mathbb{N}_0} \ln t^p/p!^s$, $t > 0$ and $M(0) := 0$.

3. If $\theta \in (0, \pi]$ and $d \in (0, 1]$, put $B_d := \{\lambda \in \mathbb{C} : |\lambda| \leqslant d\}$ and $\Omega_{\theta,d} := \Sigma_\theta \cup B_d$.

The following family of continuous exponentially bounded kernels (cf. [**14**, p. 107]) plays an important role in our analysis:

$$K_\delta(t) := \frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} e^{\lambda t-\lambda^\delta}\,d\lambda,\ t \geqslant 0,\ \delta \in (0,1),\ r > 0,\ \text{where } 1^\delta = 1.$$

Put, for $c > 0$ and $\delta \in (0,1)$, $K_{\delta,c}(t) := K_\delta(ct)$, $t \geqslant 0$. It is well known that, for every $\delta \in (0,1)$, $c > 0$ and $s = 1/\delta$:

$$\begin{aligned}\big|\widetilde{K_{\delta,c}}(\lambda)\big| &= \Big|\frac{1}{c}\widetilde{K_\delta}\Big(\frac{\lambda}{c}\Big)\Big| = \frac{1}{c}\big|e^{-(\lambda/c)^\delta}\big| = \frac{1}{c}\,e^{-\cos(\delta\arg(\lambda/c))|\lambda/c|^\delta}\\ &\leqslant \frac{1}{c}\,e^{-\cos(\pi/2s)c^{-1/s}|\lambda|^{1/s}},\ \ \operatorname{Re}\lambda > 0.\end{aligned}$$

For the sake of simplicity, in the following theorem, we consider only Gevrey type sequences $p!^s$, $s \in (1,2)$ and the functions $K_{1/s,\,c}$, $c > 0$. Actually, the argumentation given in [**225**] and [**307**, Section 1.3] can serve one to prove a more general result.

Theorem 2.6.10. [**231**] *Suppose $\alpha > 0$ and A generates a (local) α-times integrated cosine function. Then the following holds:*

(i) *For every $b \in (\frac{1}{2},1)$ and $\gamma \in \big(0,\arctan(\cos(\frac{b\pi}{2}))\big)$, there exist two analytic operator families $\big(T_{b,+}(t)\big)_{t\in\Sigma_\gamma} \subseteq L(E)$ and $\big(T_{b,-}(t)\big)_{t\in\Sigma_\gamma} \subseteq L(E)$ which satisfy:*

- (i.1) *For every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators.*
- (i.2) $\|t^{\frac{\alpha}{2b}}T_{b,\pm}(t)\| = O(1)$, $t \to 0+$.
- (i.3) *For every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, the operator iA is the generator of a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized group $(S_{b,t_1,t_2}(r))_{r\in\mathbb{R}}$.*
- (i.4) *For every $x \in E$, $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, the mapping $r \mapsto S_{b,t_1,t_2}(r)x$, $r \in \mathbb{R}$ is infinitely differentiable in $(-\infty,0)\cup(0,\infty)$.*
- (i.5) *Suppose K is a compact subset of $\mathbb{R}$ and $0 \notin K$. Then, for every $h > 0$ and $s \in (\frac{1}{b},2)$:*

$$\sup_{p\in\mathbb{N}_0,\ r\in K}\frac{1}{p!^s}h^p\Big\|\frac{d^p}{dr^p}S_{b,t_1,t_2}(r)x\Big\| < \infty.$$

(ii) *For every $s \in (1,2)$ and $\tau \in (0,\infty)$, there exists $c_\tau > 0$ such that iA generates a local $K_{1/s,\,c_\tau}$-convoluted group $\big(S_{K_{1/s,\,c_\tau}}(t)\big)_{t\in(-\tau,\tau)}$ which satisfies:*

- (ii.1) *The mappings $t \mapsto S_{K_{1/s,c_\tau}}(\pm t)$, $t \in [0,\tau)$ are infinitely differentiable.*
- (ii.2) *There exists $h > 0$ such that*

$$\sup_{t\in(-\tau,\tau)\smallsetminus\{0\},\ p\in\mathbb{N}_0}\frac{1}{p!^s}h^p\Big\|\frac{d^p}{dt^p}S_{K_{\frac{1}{s},c_\tau}}(t)\Big\| < \infty.$$

Proof. According to Theorem 2.7.3(ii) given below we have the existence of positive real numbers a and b such that $E^2(a,b) \subseteq \rho(A)$ and that $\|R(\lambda^2 : A)\| \leqslant M|\lambda|^\alpha$, $\lambda \in E(a,b)$. Suppose now $s \in (\frac{1}{b},2)$. Proceeding as in the proof of [**223**, Theorem 4.3], we get the existence of numbers $\delta > 0$, $\varepsilon \in \mathbb{R}$ and $l \geqslant 1$ (cf. also [**54**]) so that $\Omega_{\delta,\varepsilon,l} \subseteq \rho(\pm iA)$ and $\|R(\lambda : \pm iA)\| \leqslant M|\lambda|^{\frac{\alpha}{2}}$, $\lambda \in \Omega_{\delta,\varepsilon,l}$. Further on, it is clear that there exist numbers $a \in (0, \frac{\pi}{2})$, $d \in (0,1]$ and $\omega \in \mathbb{R}$ so that: $b \in \big(0, \frac{\pi}{2(\pi-a)}\big)$, $\gamma \in \big(0, \arctan(\cos(b(\pi-a)))\big)$ and $\Omega_{a,d} \subseteq \Omega_{\delta,\varepsilon-\omega,l} \subseteq \rho(\pm iA-\omega)$. Let the curve $\Gamma_{a,d} := \partial\Omega_{a,d}$ be upwards oriented. Define $T_{b,\pm}(t)$, $t \in \Sigma_\gamma$ by:

$$T_{b,\pm}(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : \pm iA - \omega)x\,d\lambda,\ x \in E.$$

The arguments given in Section 1.1.4 show that $(T_{b,\pm}(t))_{t\in\Sigma_\gamma}$ are analytic operator families which fulfill the claimed properties (i.1) and (i.2). Assume K is a compact subset of $(0,\infty)$, $t \in \Sigma_\gamma$ and $x \in E$. Then $\pm iA$ generate global $T_{b,\pm}(t)$-semigroups $(S_{b,t,\pm}(r))_{r\geqslant 0}$ [**225**]. Furthermore, the mappings $r \mapsto S_{b,t,\pm}(r)x$, $r > 0$ are infinitely differentiable and, for every $h > 0$:

$$\sup_{p\in\mathbb{N}_0,\ r\in K} \frac{1}{p!^s} h^p \Big\| \frac{d^p}{dr^p} S_{b,t,\pm}(r)x \Big\| < \infty. \tag{167}$$

Suppose $t_1 \in \Sigma_\gamma$, $t_2 \in \Sigma_\gamma$ and $x \in E$. Evidently, $T_{b,+}(t_1)(\pm iA) \subseteq (\pm iA)T_{b,+}(t_1)$, $T_{b,-}(t_2)(\pm iA) \subseteq (\pm iA)T_{b,-}(t_2)$ and $T_{b,+}(t_1)T_{b,-}(t_2) = T_{b,-}(t_2)T_{b,+}(t_1)$. Then one obtains

$$\begin{aligned} &T_{b,-}(t_2)\big(S_{b,t_1,+}(r)x - T_{b,+}(t_1)x\big) \\ &\qquad = T_{b,-}(t_2)iA\int\limits_0^r S_{b,t_1,+}(v)x\,dv = iAT_{b,-}(t_2)\int\limits_0^r S_{b,t_1,+}(v)xdv, \end{aligned}$$

$$iA\int\limits_0^r \big(T_{b,-}(t_2)S_{b,t_1,+}(v)\big)x\,dv = T_{b,-}(t_2)S_{b,t_1,+}(r)x - T_{b,+}(t_1)T_{b,-}(t_2)x,\ r \geqslant 0.$$

Clearly, we have that $[T_{b,-}(t_2)S_{b,t_1,+}(r)]T_{b,+}(t_1) = T_{b,+}(t_1)[T_{b,-}(t_2)S_{b,t_1,+}(r)]$, $r \geqslant 0$, and $[T_{b,-}(t_2)S_{b,t_1,+}(r)]iA \subseteq iA[T_{b,-}(t_2)S_{b,t_1,+}(r)]$, $r \geqslant 0$. The above given arguments simply imply that $\big(T_{b,-}(t_2)S_{b,t_1,+}(r)\big)_{r\geqslant 0}$ is a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized semigroup generated by iA. Analogously, $\big(T_{b,+}(t_1)S_{b,t_2,-}(r)\big)_{r\geqslant 0}$ is a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized semigroup generated by $-iA$. Hence, iA generates a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-regularized group $(S_{b,t_1,t_2}(r))_{r\in\mathbb{R}}$ given by: $S_{b,t_1,t_2}(r) = T_{b,-}(t_2)S_{b,t_1,+}(r)$, $r \geqslant 0$ and $S_{b,t_1,t_2}(r) = T_{b,+}(t_1)S_{b,t_2,-}(-r)$, $r < 0$. This yields (i.3) and (i.4) while the proof of (i.5) follows immediately from (i4) and (167). To prove (ii), choose arbitrarily numbers $\tau \in (0,\infty)$ and $s \in (1,2)$. Denote by Γ_l the upwards oriented boundary of $\Omega_{\delta,\varepsilon,l}$ and notice that [**207**] there exists an appropriate constant $d_1 > 0$ such that $M(\lambda) \leqslant d_1\lambda^{\frac{1}{s}}$, $\lambda \geqslant 0$. Put $c_\tau := \frac{1}{2}\big[\frac{1}{\cos(\pi/2s)}\tau\delta d_1 l^{1/s}\big]^{-s}$,

$$S_{K_{\frac{1}{s},c_\tau},\pm}(t) := \frac{1}{2\pi i}\int\limits_{\Gamma_l} e^{\lambda t}\widetilde{K_{\frac{1}{s},c_\tau}}(\lambda)R(\lambda : \pm iA)d\lambda,\ t \in [0,\tau),$$

$S_{K_{1/s,c_\tau}}(t) = S_{K_{1/s,c_\tau},+}(t)$, $t \in [0,\tau)$ and $S_{K_{1/s,c_\tau}}(t) = S_{K_{1/s,c_\tau},-}(-t)$, $t \in (-\tau,0)$. Arguing as in the proof of [**307**, Theorem 1.3.2, p. 58], one obtains that $S_{K_{1/s,c_\tau},\pm}(t) \in L(E)$, $t \in [0,\tau)$ and that $\pm iA$ generate local $K_{1/s,c_\tau}$-convoluted semigroups

$\big(S_{K_{1/s,\,c_\tau},\pm}(t)\big)_{t\in[0,\tau)}$. An employment of Corollary 2.6.8 shows that iA generates the local $K_{1/s,\,c_\tau}$-convoluted group $(S_{K_{1/s,\,c_\tau}}(t))_{t\in(-\tau,\tau)}$. The elementary inequality $|e^{\lambda h}-1|\leqslant h|\lambda|e^{\operatorname{Re}\lambda h}$, $\lambda\in\mathbb{C}$, $h>0$ and the dominated convergence theorem imply that the mappings $t\mapsto S_{K_{1/s,\,c_\tau}}(\pm t)$, $t\in[0,\tau)$ are infinitely differentiable and that

$$\frac{d^p}{dt^p}S_{K_{\frac{1}{s},\,c_\tau}}(\pm t)=\frac{1}{2\pi i}\int\limits_{\Gamma_l}\lambda^p e^{\lambda t}\widetilde{K_{\frac{1}{s},\,c_\tau}}(\lambda)R(\lambda:\pm iA)\,d\lambda,\ t\in[0,\tau),\ p\in\mathbb{N}_0. \tag{168}$$

Due to the choice of c_τ, there exists $h>0$ such that:

$$d_1h^{\frac{1}{s}}+\tau\delta d_1l^{\frac{1}{s}}<\cos\Big(\frac{\pi}{2s}\Big)c_\tau^{-\frac{1}{s}}. \tag{169}$$

Taking into account (168)–(169), one gets

$$\begin{aligned}
&\sup_{t\in(-\tau,\tau)\smallsetminus\{0\},\,p\in\mathbb{N}_0}\frac{1}{p!^s}h^p\Big\|\frac{d^p}{dt^p}S_{K_{1/s,\,c_\tau}}(t)\Big\|\\
&\leqslant \text{Const}\sup_{t\in(-\tau,\tau)\smallsetminus\{0\},\,p\in\mathbb{N}_0}\int\limits_{\Gamma_l}\frac{(h|\lambda|)^p}{p!^s}e^{\operatorname{Re}\lambda|t|}|\widetilde{K_{\frac{1}{s},\,c_\tau}}(\lambda)|\|R(\lambda:\pm iA)\|\,|d\lambda|\\
&\leqslant \text{Const}\sup_{t\in(-\tau,\tau)\smallsetminus\{0\}}\int\limits_{\Gamma_l}e^{M(h|\lambda|)}e^{|t|(\delta M(l|\operatorname{Im}\lambda|)+\varepsilon)}e^{-\cos(\frac{\pi}{2s})c_\tau^{-\frac{1}{s}}|\lambda|^{\frac{1}{s}}}|\lambda|^{\frac{\alpha}{2}}|d\lambda|\\
&\leqslant \text{Const}\sup_{t\in(-\tau,\tau)\smallsetminus\{0\}}\int\limits_{\Gamma_l}e^{d_1h^{\frac{1}{s}}|\lambda|^{\frac{1}{s}}}e^{|t|(\delta d_1l^{\frac{1}{s}}|\lambda|^{\frac{1}{s}}+\varepsilon)}e^{-\cos(\frac{\pi}{2s})c_\tau^{-\frac{1}{s}}|\lambda|^{\frac{1}{s}}}|\lambda|^{\frac{\alpha}{2}}|d\lambda|\\
&\leqslant \text{Const}\,e^{|\varepsilon|\tau}\int\limits_{\Gamma_l}e^{(d_1h^{\frac{1}{s}}+\tau\delta d_1l^{\frac{1}{s}}-\cos(\frac{\pi}{2s})c_\tau^{-\frac{1}{s}})|\lambda|^{\frac{1}{s}}}|\lambda|^{\frac{\alpha}{2}}|d\lambda|<\infty.
\end{aligned}$$

The proof is thereby completed. □

We close this section with the analysis of certain classes of abstract Volterra equations on the line. Of concern are the following equations:

$$u(t)=\int_0^\infty a(s)Au(t-s)\,ds+\int_{-\infty}^t k(t-s)g'(s)\,ds, \tag{170}$$

where $g:\mathbb{R}\to E$, $a\in L^1_{\text{loc}}([0,\infty))$, $a\neq 0$, $k\in C([0,\infty))$, $k\neq 0$, and

$$u(t)=f(t)+\int_0^t a(t-s)Au(s)\,ds,\ t\in(-\tau,\tau), \tag{171}$$

where $\tau\in(0,\infty]$ and $f\in C((-\tau,\tau):E)$. Notice that the equation (170) appears in the study of the problem of heat flow with memory [**342**].

PROPOSITION 2.6.11. *Assume A is a subgenerator of a global (a,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$, $g:\mathbb{R}\to \mathrm{R}(C)$, $C^{-1}g(\cdot)$ is differentiable for a.e. $t\in\mathbb{R}$, $C^{-1}g(t)\in D(A)$ for a.e. $t\in\mathbb{R}$,*

(i) *the mapping $s\mapsto S(t-s)(C^{-1}g)'(s)$, $s\in(-\infty,t]$ is an element of the space $L^1((-\infty,t]:[D(A)])$ for a.e. $t\in\mathbb{R}$, and*

(ii) *the mapping* $s \mapsto k(t-s)g'(s)$, $s \in (-\infty, t]$ *is an element of the space* $L^1((-\infty, t] : E)$ *for a.e.* $t \in \mathbb{R}$.

Put $u(t) := \int_{-\infty}^{t} S(t-s)(C^{-1}g)'(s)\,ds$, $t \in \mathbb{R}$. *Then* $C(\mathbb{R} : E) \ni u$ *satisfies* (170).

Proof. The continuity of $u(t)$ can be proved by using the dominated convergence theorem and the strong continuity of $(S(t))_{t\geqslant 0}$. The proof of (170) follows from the following computation:

$$\begin{aligned}
&\int_0^\infty a(s)Au(t-s)\,ds + \int_{-\infty}^{t} k(t-s)g'(s)\,ds\\
&= \int_0^\infty a(s)A\int_{-\infty}^{t-s} S(t-s-r)(C^{-1}g)'(r)\,dr\,ds + \int_{-\infty}^{t} k(t-s)g'(s)\,ds\\
&= \int_0^\infty\int_0^{s'} a(s'-r')AS(r')(C^{-1}g)'(t-s')\,dr'\,ds' + \int_{-\infty}^{t} k(t-s)g'(s)\,ds\\
&= \int_0^\infty (S(s')-k(s')C)(C^{-1}g)'(t-s')\,ds + \int_{-\infty}^{t} k(t-s)g'(s)\,ds\\
&= u(t) - \int_0^\infty k(s)g'(t-s')\,ds' + \int_{-\infty}^{t} k(t-s)g'(s)\,ds = u(t),\ t \in \mathbb{R}. \qquad \square
\end{aligned}$$

Denote by $AP(E)$, $AA(E)$, $AA_c(E)$ and $AAA(E)$ the spaces which consist of all almost periodic functions, almost automorphic functions, compact almost automorphic functions and asymptotically almost automorphic functions defined on $\mathbb{R}$, respectively, and assume that the function $(C^{-1}g)'(t)$ belongs to one of these spaces [**339**]. By [**49**, Theorem 4.6], the uniform integrability of $(S(t))_{t\geqslant 0}$ implies that the solution $u(t)$ of (170) belongs to the same space as $(C^{-1}g)'(t)$. The above assertion remains true in nonscalar case (cf. Appendix).

Proposition 2.6.12. (i) *Assume* $a \in L^1_{\text{loc}}((-\tau,\tau))$, $k \in C((-\tau,\tau))$, $a \neq 0$ *and* $k \neq 0$. *Let* $k_+(t) = k(t)$, $a_+(t) = a(t)$, $t \in [0,\tau)$, $k_-(t) = k(-t)$ *and* $a_-(t) = a(-t)$, $t \in (-\tau, 0]$. *If* $\pm A$ *are subgenerators of* $(a_\pm, k_\pm)$*-regularized* C*-resolvent families* $(S_\pm(t))_{t\in[0,\tau)}$, *then, for every* $x \in D(A)$, *the function* $u : (-\tau,\tau) \to E$ *given by* $u(t) = S_+(t)x$, $t \in [0,\tau)$ *and* $u(t) = S_-(-t)x$, $t \in (-\tau, 0]$ *is a solution of* (171) *with* $f(t) = k(t)Cx$, $t \in (-\tau,\tau)$. *Furthermore, the solutions of* (171) *are unique provided that* $k_\pm(t)$ *are kernels.*

(ii) *Assume* $n_\pm \in \mathbb{N}$, $f \in C((-\tau,\tau) : E)$, $a \in L^1_{\text{loc}}((-\tau,\tau))$, $a \neq 0$, $f_+(t) = f(t)$, $a_+(t) = a(t)$, $t \in [0,\tau)$, $f_-(t) = f(-t)$, $a_-(t) = a(-t)$, $t \in (-\tau, 0]$, *and* $\pm A$ *are subgenerators of* $(n_\pm - 1)$*-times integrated* $(a_\pm, C_\pm)$*-regularized resolvent families. Assume, additionally,* $a_\pm \in BV_{\text{loc}}([0,\tau))$ *if* $n_\pm > 1$ *(that is:* $a_+ \in BV_{\text{loc}}([0,\tau))$ *if* $n_+ > 1$, *and* $a_- \in BV_{\text{loc}}([0,\tau))$ *if* $n_- > 1$*) as well as:*

(ii.1) $C_{\pm}^{-1}f_{\pm} \in C^{(n_{\pm})}([0,\tau) : E)$, $f_{\pm}^{(k-1)}(0) \in D(A^{n_{\pm}-k})$ *and* $A^{n_{\pm}-k}f^{(k-1)}(0) \in R(C_{\pm})$, $1 \leqslant k \leqslant n_{\pm}$, *if* $n_{\pm} > 1$, *resp.*

(ii.2) $C_{\pm}^{-1}f_{\pm} \in C([0,\tau) : E) \cap W^{1,1}_{\mathrm{loc}}([0,\tau) : E)$ *if* $n_+ = n_- = 1$.

Then there exists a unique solution of (171).

EXAMPLE 2.6.13. (cf. also Subsection 2.1.8) (i) Assume $-\infty < \alpha \leqslant \beta < \infty$, $1 \leqslant p \leqslant \infty$, $0 < \tau \leqslant \infty$, $n \in \mathbb{N}$, $E = L^p(\mathbb{R}^n)$ or $E = C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m \in \mathbb{N}$, $\alpha \leqslant \mathrm{Re}(P(x)) \leqslant \beta$, $x \in \mathbb{R}^n$ and $A = P(D)$. Then there exists $\omega > 0$ such that, for every $r > n|\frac{1}{2} - \frac{1}{p}|$, $\pm A$ are the integral generators of exponentially bounded $(\omega \mp A)^{-r}$-regularized semigroups in E. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R})$, $a \neq 0$, be such that the mappings $t \mapsto a_+(t) = a(t)$, $t \geqslant 0$ and $t \mapsto a_-(t) = a(-t)$, $t \geqslant 0$ are completely positive. By Theorem 2.1.28(ii), $\pm A$ are the integral generators of exponentially bounded $(a_{\pm}, (\omega \mp A)^{-r})$-regularized resolvent families provided $E = L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$, resp. $(a_{\pm}, t)$-regularized $(\omega \mp A)^{-r}$-resolvent families provided $E = L^{\infty}(\mathbb{R}^n)$ $(C_b(\mathbb{R}^n))$. Let $f \in C((-\tau,\tau) : E)$ and let $f_{\pm}(t)$ satisfy the assumption of Proposition 2.6.12(ii.2), resp. Proposition 2.6.12(ii.1), with $n_{\pm} = 1$, resp. $n_{\pm} = 2$. Then there exists a unique solution of (171); it is noteworthy that the above example can be reformulated in the case when A is the integral generator of an exponentially bounded integrated group or C-regularized group, and that obtained conclusions continue to hold in many other function spaces.

(ii) Assume $E = L^2[0,\pi]$, $A = -\Delta$ with the Dirichlet or Neumann boundary conditions, $\tau = \infty$, $\beta \in [\frac{1}{2}, 1)$, $\alpha > 1 + \beta$, $a(t) = \frac{|t|^{\beta-1}}{\Gamma(\beta)}$, $t \in (-\tau,\tau)$ and $f(t) = \mathcal{L}^{-1}((\mathbf{h}_{\alpha,\beta}(\lambda))(|t|))$, $t \in (-\tau,\tau)$, where $\mathbf{h}_{\alpha,\beta}(\lambda)$ is defined through [**235**, (2.64)]. Then Proposition 2.6.12(i) implies that there exists a unique solution $u(t)$ of (171) and that $u_{|\mathbb{R}\smallsetminus\{0\}}$ is analytically extendible to the sector $\Sigma_{\frac{\pi}{2}(\frac{1}{\beta}-1)}$. By Proposition 2.6.12(i) and [**235**, Example 2.31(iii)], it follows that, for every $n \in \mathbb{N}$, there exists an exponentially bounded kernel $k_n(t)$ such that (171) has a unique solution $u_n(t)$ with A replaced by the polyharmonic operator Δ^{2^n} and $f(t)$ replaced by $k_n(t)$; moreover, $u_{n|\mathbb{R}\smallsetminus\{0\}}$ is analytically extendible to the sector $\Sigma_{\frac{\pi}{2}}$. The analysis of preceding example in the case $\beta \in [1,2)$ is given in [**235**].

2.7. Spectral characterizations

We start this subsection with following result which is necessary in our striving to reveal the satisfactory relationship between local K-convoluted semigroups and hyperfunction semigroups of Ōuchi [**353**].

THEOREM 2.7.1. *Suppose $M > 0$, $\beta \geqslant 0$, $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, $(S_K(t))_{t\in[0,\tau)}$ is a* (*local*) *K-convoluted semigroup generated by A and, for every $\varepsilon > 0$, there exist $\varepsilon_0 \in (0, \tau\varepsilon)$ and $T_\varepsilon > 0$ such that $1/|\tilde{K}(\lambda)| \leqslant T_\varepsilon e^{\varepsilon_0|\lambda|}$, $\mathrm{Re}\,\lambda > \beta$, $\tilde{K}(\lambda) \neq 0$. Then, for every $\varepsilon > 0$, there exist $\overline{C}_\varepsilon > 0$ and $\overline{K}_\varepsilon > 0$ such that*

$$\Omega^1_\varepsilon := \left\{\lambda \in \mathbb{C} : \tilde{K}(\lambda) \neq 0,\ \mathrm{Re}\,\lambda > \beta,\ \mathrm{Re}\,\lambda \geqslant \varepsilon|\lambda| + \overline{C}_\varepsilon\right\} \subseteq \rho(A),$$
$$\|R(\lambda\colon A)\| \leqslant \overline{K}_\varepsilon e^{\varepsilon_0|\lambda|},\ \lambda \in \Omega^1_\varepsilon,\ \tilde{K}(\lambda) \neq 0.$$

PROOF. Let $\varepsilon \in (0,1)$ be fixed. Define

$$R(\lambda,t) := \frac{1}{\tilde{K}(\lambda)} \int_0^t e^{-\lambda s} S_K(s)\,ds, \quad \operatorname{Re}\lambda > \beta,\ \tilde{K}(\lambda) \neq 0,\ t \in [0,\tau),$$

and fix an element $x \in E$. Proceeding as in the proof of [**307**, Theorem 1.3.1], one gets that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$ and $\tilde{K}(\lambda) \neq 0$:

$$\begin{aligned}(\lambda I - A)R(\lambda,t)x &= \frac{1}{\tilde{K}(\lambda)}\left(\lambda \int_0^t e^{-\lambda s} S_K(s)x\,ds - A\int_0^t e^{-\lambda s} S_K(s)x\,ds\right)\\ &= \frac{1}{\tilde{K}(\lambda)}\left(\lambda \int_0^t e^{-\lambda s}\Theta(s)x\,ds - e^{-\lambda t}A\int_0^t S_K(s)x\,ds\right)\\ &= \frac{1}{\tilde{K}(\lambda)}\left(\int_0^t e^{-\lambda s}K(s)x\,ds - e^{-\lambda t}S_K(t)x\right)\\ &= x - \frac{1}{\tilde{K}(\lambda)}\left(e^{-\lambda t}S_K(t)x + \int_t^\infty e^{-\lambda s}K(s)x\,ds\right) := x - B_t(\lambda)x.\end{aligned}$$

Our goal is to find the domain Ω^1_ε such that, for all $\lambda \in \Omega^1_\varepsilon$, we can estimate $B_t(\lambda)$ as follows:

$$\begin{aligned}\|B_t(\lambda)\| &\leqslant \frac{1}{|\tilde{K}(\lambda)|}\left(e^{-\operatorname{Re}\lambda t}\|S_K(t)\| + M\int_t^\infty e^{(\beta-\operatorname{Re}\lambda)s}ds\right)\\ &\leqslant \frac{1}{|\tilde{K}(\lambda)|}\left(e^{-\operatorname{Re}\lambda t}\|S_K(t)\| + M\frac{e^{(\beta-\operatorname{Re}\lambda)t}}{\operatorname{Re}\lambda-\beta}\right)\\ &\leqslant T_\varepsilon e^{\varepsilon_0|\lambda|}e^{(\beta-\operatorname{Re}\lambda)t}\left(e^{-\beta t}\|S_K(t)\| + \frac{M}{\operatorname{Re}\lambda-\beta}\right).\end{aligned}$$

Let $t = \frac{\varepsilon_0}{\varepsilon} \in (0,\tau)$, $\|S_K(t)\| = C_0$ and let $\beta_1 \in (\beta,\infty)$. Assume $\operatorname{Re}\lambda > \beta_1$, $\tilde{K}(\lambda) \neq 0$ and let us find an additional condition such that:

$$\begin{aligned}\|B_t(\lambda)\| &\leqslant T_\varepsilon\left(e^{-\beta t}\|S_K(t)\| + \frac{M}{\operatorname{Re}\lambda-\beta}\right)e^{\varepsilon_0|\lambda|+(\beta-\operatorname{Re}\lambda)t}\\ &\leqslant T_\varepsilon\left(e^{-\beta t}C_0 + \frac{M}{\beta_1-\beta}\right)e^{\varepsilon_0|\lambda|+(\beta_1-\operatorname{Re}\lambda)t} \leqslant \delta < 1.\end{aligned}$$

One can simply verify that, with

$$\overline{C}_\varepsilon := \max\left(\beta(1-\varepsilon), \beta_1 + \frac{\varepsilon_0}{\varepsilon}\ln\frac{\delta}{T_\varepsilon(e^{-\beta t}C_0 + \frac{M}{\beta_1-\beta})}\right),$$

$$\overline{K}_\varepsilon := \int\limits_0^{\varepsilon_0/\varepsilon} \|S_K(s)\|\,ds \frac{T_\varepsilon}{1-\delta},$$

and Ω^1_ε as in the formulation of theorem, $\|B_t(\lambda)\| \leqslant \delta$, $\lambda \in \Omega^1_\varepsilon$. Since $R(\lambda, t)$ and $B_t(\lambda)$ commute with A, one yields $\Omega^1_\varepsilon \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| = \big\|R(\lambda, t)(I - B_t(\lambda))^{-1}\big\| \leqslant \frac{1}{|\tilde{K}(\lambda)|}\Big|\int\limits_0^t e^{-\lambda s} S_K(s)\,ds\Big|\frac{1}{1-\delta}$$
$$\leqslant \overline{K}_\varepsilon e^{\varepsilon_0|\lambda|} \leqslant \overline{K}_\varepsilon e^{\varepsilon\tau|\lambda|},\ \lambda \in \Omega^1_\varepsilon. \qquad \square$$

Assume that, for every $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $M_\varepsilon > 0$ satisfying

$$\Omega_\varepsilon := \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \varepsilon|\lambda| + C_\varepsilon\big\} \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant M_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \Omega_\varepsilon,$$

i.e., that there exists a hyperfunction fundamental solution for A (cf. Definition 3.5.32, Theorem 3.5.33, Example 3.5.35 and questions preceding Corollary 2.6.8). Then it is not clear whether there exist $\tau > 0$ and $K \in L^1_{\mathrm{loc}}([0, \tau))$, $K \neq 0$ such that A is the integral generator of a K-convoluted semigroup on $[0, \tau)$.

THEOREM 2.7.2. (i) *Suppose* $\alpha > 0$, $M > 0$, $\beta \geqslant 0$, $\Phi : \mathbb{C} \to [0, \infty)$, $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, $(S_K(t))_{t\in[0,\tau)}$ *is a local K-convoluted semigroup generated by A and* $1/|\tilde{K}(\lambda)| \leqslant e^{\Phi(\alpha\lambda)}$, $\operatorname{Re}\lambda > \beta$, $\tilde{K}(\lambda) \neq 0$. *Then, for every* $t \in (0, \tau)$, *there exist* $\beta(t) > 0$ *and* $M(t) > 0$ *such that*

$$\Lambda_{t,\alpha,\beta(t)} := \Big\{\lambda \in \mathbb{C} : \tilde{K}(\lambda) \neq 0,\ \operatorname{Re}\lambda \geqslant \frac{\Phi(\alpha\lambda)}{t} + \beta(t)\Big\} \subseteq \rho(A),$$
$$\|R(\lambda : A)\| \leqslant M(t)e^{\Phi(\alpha\lambda)},\ \lambda \in \Lambda_{t,\alpha,\beta(t)},\ \tilde{K}(\lambda) \neq 0.$$

Furthermore, the existence of a sequence (t_n) *in* $[0, \tau)$ *satisfying* $\lim_{n\to\infty} t_n = \tau$ *and* $\sup_{n\in\mathbb{N}} \ln\|S_K(t_n)\| < \infty$ *implies that there exist* $\beta' > 0$ *and* $M' > 0$ *such that* $\Lambda_{\tau,\alpha,\beta'} \subseteq \rho(A)$ *and* $\|R(\lambda : A)\| \leqslant M'e^{\Phi(\alpha\lambda)}$, $\lambda \in \Lambda_{\tau,\alpha,\beta'}$.

(ii) *Suppose K satisfies* (P1), $r_0 \geqslant \max(0, \operatorname{abs}(K))$, $\Phi : [r_0, \infty) \to [0, \infty)$ *is a continuously differentiable, strictly increasing mapping,* $\lim_{t\to\infty}\Phi(t) = +\infty$, $\Phi'(\cdot)$ *is bounded on* $[r_0, \infty)$ *and there exist* $\alpha > 0$, $\gamma > 0$ *and* $\beta > r_0$ *such that*

$$\Psi_{\alpha,\beta,\gamma} := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \frac{\Phi(\alpha|\operatorname{Im}\lambda|)}{\gamma} + \beta\Big\} \subseteq \rho_C(A).$$

Denote by $\Gamma_{\alpha,\beta,\gamma}$ *the upwards oriented boundary of* $\Psi_{\alpha,\beta,\gamma}$ *and by* $\Omega_{\alpha,\beta,\gamma}$ *the open region which lies to the right of* $\Gamma_{\alpha,\beta,\gamma}$. *Let the following conditions hold.*

(ii.1) *The mapping* $\lambda \mapsto \tilde{K}(\lambda)(\lambda - A)^{-1}C$ *is analytic on* $\Omega_{\alpha,\beta,\gamma}$ *and continuous on* $\Gamma_{\alpha,\beta,\gamma}$.

(ii.2) *There exist* $M > 0$ *and* $\sigma > 0$ *such that:*

$$\|\tilde{K}(\lambda)(\lambda - A)^{-1}C\| \leqslant Me^{-\Phi(\sigma|\lambda|)},\ \lambda \in \overline{\Omega_{\alpha,\beta,\gamma}}.$$

(ii.3) *There exists a function* $m : [0, \infty) \to (0, \infty)$ *such that* $m(s) = 1$, $s \in [0, 1]$ *and that, for every* $s > 1$, *there exists an* $r_s > r_0$ *so that* $\frac{\Phi(t)}{\Phi(st)} \geqslant m(s)$, $t \geqslant r_s$.

(ii.4) $\lim_{t\to\infty} te^{-\Phi(\sigma t)} = 0$.

(ii.5) $(\exists a \geqslant 0)(\exists r'_a > r_0)(\forall t > r'_a)\frac{\ln t}{\Phi(t)} \geqslant a$.

Then the operator A is a subgenerator of a local K-convoluted C-semigroup on $[0, a + m(\frac{\alpha}{\sigma\gamma}))$.

(iii) [**277**] *Suppose $\alpha > 0$ and A generates a (local) α-times integrated semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Then, for every $a \in (0, \frac{\tau}{\alpha})$, there exist $b > 0$ and $M > 0$ such that:*

$$E(a,b) \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^\alpha,\ \lambda \in E(a,b). \tag{172}$$

(iv) *Suppose $\alpha > 0$, $a > 0$, $b > 0$, $M > 0$,*

$$E(a,b) \subseteq \rho_C(A),\ \|(\lambda - A)^{-1}C\| \leqslant M(1 + |\lambda|)^\alpha,\ \lambda \in E(a,b),$$

and the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in E(a,b)$ is continuous. Then, for every $\beta \in (\alpha + 1, \infty)$, A is a subgenerator of a local β-times integrated C-semigroup $(S_\beta(t))_{t\in[0,a(\beta-\alpha-1))}$.

Proof. The proof of (i) follows from the argumentation given in the proofs of Theorem 2.7.1 and [**307**, Theorem 1.3.2]. To prove (ii), set

$$S_K(t) := \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta,\gamma}} e^{\lambda t}\tilde{K}(\lambda)(\lambda - A)^{-1}C\, d\lambda,\ t \in [0, a + m(\alpha/\sigma\gamma)). \tag{173}$$

Let us show that the improper integral in (173) converges for all $t \in [0, a+m(\frac{\alpha}{\sigma\gamma}))$. Denote by $\Gamma^1_{\alpha,\beta,\gamma} := \{\lambda \in \Gamma_{\alpha,\beta,\gamma} : \operatorname{Im}\lambda \geqslant 0\}$ and $\Gamma^2_{\alpha,\beta,\gamma} := \{\lambda \in \Gamma_{\alpha,\beta,\gamma} : \operatorname{Im}\lambda \leqslant 0\}$. Taking into account the equality $\lim_{t\to\infty}\Phi(t) = +\infty$ as well as (ii.3) and (ii.5), we easily infer that there exist a sufficiently large real number r' and a number $\zeta > 1$ so that $t\Phi(\frac{\alpha s}{\gamma}) - \Phi(\sigma s) \leqslant \ln M - \zeta \ln s$, $s \geqslant r'$. Hence, there exists $M' > 0$ such that:

$$e^{t\Phi(\frac{\alpha s}{\gamma})-\Phi(\sigma s)} \leqslant M's^{-\zeta},\ s \geqslant r'. \tag{174}$$

Then the estimate (174) implies:

$$\begin{aligned}
&\Bigg\| \int_{\Gamma^1_{\alpha,\beta,\gamma}\cap\{\lambda\in\mathbb{C} \mid \operatorname{Im}\lambda \geqslant r'\}} e^{\lambda t}\tilde{K}(\lambda)(\lambda - A)^{-1}C\, d\lambda \Bigg\| \\
&\qquad \leqslant \int_{r'}^{\infty} e^{(\frac{\Phi(\alpha s)}{\gamma}+\beta)t} e^{-\Phi(\sigma s)} \Big(1 + \frac{\alpha\Phi'(\alpha s)}{\gamma}\Big)\, ds \\
&\qquad\qquad \leqslant \text{Const}\, e^{\beta t} \int_{r'}^{\infty} e^{t\frac{\Phi(\alpha s)}{\gamma} - \Phi(\sigma s)} ds \leqslant \text{Const}\, e^{\beta t} \int_1^\infty \frac{ds}{s^\zeta} < \infty.
\end{aligned}$$

This implies the convergence of the curve integral over $\Gamma^1_{\alpha,\beta,\gamma}$; the convergence of the curve integral over $\Gamma^2_{\alpha,\beta,\gamma}$ can be proved similarly. This implies $S_K(t) \in L(E)$, $S_K(t)A \subseteq AS_K(t)$ and $S_K(t)C = CS_K(t)$, $t \in [0, a+m(\frac{\alpha}{\sigma\gamma}))$. Using Cauchy formula and the estimates (ii.2) and (ii.4), one can simply prove that $\int_{\Gamma_{\alpha,\beta,\gamma}} \tilde{K}(\lambda)(\lambda -$

$A)^{-1}Cd\lambda = 0$. Proceeding as in the final part of the proof of [**307**, Theorem 1.3.2], one gets that $A\int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx$, $x \in E$, $t \in [0, a+m(\frac{\sigma\gamma}{\alpha}))$, which completes the proof of (ii).

The assertion (iv) is a simple consequence of the assertion (ii). $\square$

THEOREM 2.7.3. (i) *Suppose K is a kernel, $M > 0$, $\beta \geqslant 0$, $\alpha > 0$, $\Phi : \mathbb{C} \to [0,\infty)$, $|\Theta(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, $(C_K(t))_{t\in[0,\tau)}$ is a local K-convoluted cosine function generated by A and $1/|\tilde{\Theta}(\lambda)| \leqslant e^{\Phi(\alpha\lambda)}$, $\operatorname{Re}\lambda > \beta$, $\tilde{K}(\lambda) \neq 0$. Then, for every $t \in (0,\tau)$, there exist $\beta(t) > 0$ and $M(t) > 0$ such that*

$$\Lambda^2_{t,\alpha,\beta(t)} := \Big\{\lambda^2 \in \mathbb{C} : \tilde{K}(\lambda) \neq 0,\ \operatorname{Re}\lambda \geqslant \frac{\Phi(\alpha\lambda)}{t} + \beta(t)\Big\} \subseteq \rho(A), \tag{175}$$

$$\|R(\lambda^2 : A)\| \leqslant M(t)\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|},\ \lambda \in \Lambda_{t,\alpha,\beta(t)},\ \tilde{K}(\lambda) \neq 0.$$

Furthermore, the existence of a sequence (t_n) in $[0,\tau)$ satisfying $\lim_{n\to\infty} t_n = \tau$ and $\sup_{n\in\mathbb{N}} \ln\|C_K(t_n)\| < \infty$ implies that there exist $\beta' > 0$ and $M' > 0$ such that $\Lambda^2_{\tau,\alpha,\beta'} \subseteq \rho(A)$ and $\|R(\lambda^2 : A)\| \leqslant M'\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}$, $\lambda \in \Lambda_{\tau,\alpha,\beta'}$.

(ii) *Suppose $\alpha > 0$ and A generates a (local) α-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. Then, for every $a \in (0, \frac{\tau}{\alpha+1})$, there exist $b > 0$ and $M > 0$ such that:*

$$E^2(a,b) \subseteq \rho(A) \text{ and } \|R(\lambda^2 : A)\| \leqslant M(1+|\lambda|)^\alpha,\ \lambda \in E(a,b). \tag{176}$$

(iii) *Suppose K satisfies (P1), $r_0 \geqslant \max(0, \operatorname{abs}(K))$, $\Phi : [r_0,\infty) \to [0,\infty)$ is a continuously differentiable, strictly increasing mapping, $\lim_{t\to\infty}\Phi(t) = +\infty$, $\Phi'(\cdot)$ is bounded on $[r_0,\infty)$ and there exist $\alpha > 0$, $\gamma > 0$ and $\beta > r_0$ such that*

$$\Psi^2_{\alpha,\beta,\gamma} := \Big\{\lambda^2 : \lambda \in \mathbb{C},\ \operatorname{Re}\lambda \geqslant \frac{\Phi(\alpha|\operatorname{Im}\lambda|)}{\gamma} + \beta\Big\} \subseteq \rho_C(A). \tag{177}$$

Denote by $\Gamma_{\alpha,\beta,\gamma}$ the upwards oriented boundary of $\Psi_{\alpha,\beta,\gamma}$ (cf. also the formulation of preceding theorem) and by $\Omega_{\alpha,\beta,\gamma}$ the open region which lies to the right of $\Gamma_{\alpha,\beta,\gamma}$. Let the following conditions hold.

- (iii.1) *The mapping $\lambda \mapsto \tilde{K}(\lambda)(\lambda^2 - A)^{-1}C$ is analytic on $\Omega_{\alpha,\beta,\gamma}$ and continuous on $\Gamma_{\alpha,\beta,\gamma}$.*
- (iii.2) *There exist $M > 0$ and $\sigma > 0$ such that:*

$$\big\|\tilde{K}(\lambda)[(\lambda^2 - A)^{-1}C + C/\lambda]\big\| \leqslant Me^{-\Phi(\sigma|\lambda|)},\ \lambda \in \overline{\Omega_{\alpha,\beta,\gamma}}.$$

- (iii.3) *The conditions (ii.3), (ii.4) and (ii.5) given in the formulation of Theorem 2.7.2 hold.*

Then A is a subgenerator of a local K-convoluted C-cosine function on $[0, a + m(\frac{\alpha}{\sigma\gamma}))$.

(iv) *Suppose $\alpha > 0$, $a > 0$, $b > 0$, $M > 0$, $E^2(a,b) \subseteq \rho_C(A)$, $\|(\lambda^2 - A)^{-1}C\| \leqslant M(1+|\lambda|)^\alpha$, $\lambda \in E(a,b)$, and the mapping $\lambda \mapsto (\lambda^2 - A)^{-1}C$, $\lambda \in E(a,b)$ is continuous. Then, for every $\beta \in (\alpha+2,\infty)$, A is a subgenerator of a local β-times integrated C-cosine function $(C_\beta(t))_{t\in[0,a(\beta-\alpha-1))}$.*

PROOF. Suppose $t \in (0,\tau)$, $\sigma \in (0,1)$ and $\mathrm{proj}_1 : E \times E \to E$ is defined by $\mathrm{proj}_1\binom{x}{y} := x$, $x, y \in E$. Then it is clear from Theorem 2.1.11 that $\mathcal{A}$ generates a (local) Θ-convoluted semigroup $(S_\Theta(s))_{s\in[0,\tau)}$ in $E \times E$ and that, thanks to Theorem 2.7.1, there exist $\beta(t) > 0$ and $M(t) > 0$ such that (175) holds and that, for every $x \in E$,

$$R(\lambda^2 : A)x = \mathrm{proj}_1[R(\lambda : \mathcal{A})\tbinom{0}{x}]$$
$$= \mathrm{proj}_1\left[\frac{1}{\tilde{\Theta}(\lambda)}\int_0^t e^{-\lambda s}\begin{pmatrix}\int_0^s C_K(v)\,dv & \int_0^s (s-v)C_K(v)\,dv\\ C_K(s) - \Theta(s)C & \int_0^s C_K(v)\,dv\end{pmatrix}(I - B_t(\lambda))^{-1}\begin{pmatrix}0\\x\end{pmatrix}ds\right],$$

for all $\lambda \in \Lambda_{t,\alpha,\beta(t)}$, where

$$B_t(\lambda) = \frac{1}{\tilde{\Theta}(\lambda)}\left(e^{-\lambda t}S_\Theta(t)I + \int_t^\infty e^{-\lambda s}\Theta(s)I\,ds\right),\ \|B_t(\lambda)\| \leqslant \sigma;$$
$$\left\|(I - B_t(\lambda))^{-1}\right\| \leqslant \frac{1}{1-\sigma}, \quad \lambda \in \Lambda_{t,\alpha,\beta(t)}.$$

Since K is a kernel, we have $C_K(t)C_K(s) = C_K(s)C_K(t)$, $0 \leqslant t, s < \tau$ and the last equality implies $(I - B_t(\lambda))^{-1}S_\Theta(s) = S_\Theta(s)(I - B_t(\lambda))^{-1}$, $0 \leqslant t, s < \tau$. Then the partial integration yields:

$$R(\lambda^2 : A)x$$
$$= \mathrm{proj}_1\left[\frac{1}{\tilde{\Theta}(\lambda)}\int_0^t e^{-\lambda s}(I - B_t(\lambda))^{-1}\begin{pmatrix}\int_0^s C_K(v)\,dv & \int_0^s (s-v)C_K(v)\,dv\\ C_K(s) - \Theta(s)C & \int_0^s C_K(v)\,dv\end{pmatrix}\begin{pmatrix}0\\x\end{pmatrix}ds\right]$$
$$= \mathrm{proj}_1\left[-\frac{(I - B_t(\lambda))^{-1}}{\tilde{K}(\lambda)}e^{-\lambda t}\begin{pmatrix}\int_0^t (t-s)C_K(s)x\,ds\\ \int_0^t C_K(s)x\,ds\end{pmatrix}\right]$$
$$+ \mathrm{proj}_1\left[\frac{(I - B_t(\lambda))^{-1}}{\tilde{K}(\lambda)}\int_0^t e^{-\lambda s}\begin{pmatrix}\int_0^s C_K(r)x dr\\ C_K(s)x\end{pmatrix}ds\right]$$

and

$$\|R(\lambda^2 : A)x\| \leqslant \frac{1}{1-\sigma}\left(\left\|\int_0^t (t-s)C_K(s)x\,ds\right\| + \left\|\int_0^t C_K(s)x\,ds\right\|\right)\frac{e^{-\operatorname{Re}\lambda t}}{|\lambda||\tilde{\Theta}(\lambda)|}$$
$$+ \frac{1}{1-\sigma}\frac{1}{|\lambda||\tilde{\Theta}(\lambda)|}\int_0^t e^{-\operatorname{Re}\lambda s}\left(\left\|\int_0^s C_K(r)x\,dr\right\| + \left\|C_K(s)x\right\|\right)ds$$
$$\leqslant \frac{1}{1-\sigma}\left(\left\|\int_0^t (t-s)C_K(s)x\,ds\right\| + \left\|\int_0^t C_K(s)x\,ds\right\|\right)\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}$$

$$+\frac{1}{1-\sigma}\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}\int\limits_0^t\Biggl(\Biggl\|\int\limits_0^s C_K(r)x\,dr\Biggr\|+\Bigl\|C_K(s)x\Bigr\|\Biggr)\,ds$$

and this, in turn, implies that (i) holds good. The proof of (ii) follows from (i) and Remark 3.4.14. The proof of (iii) can be obtained by passing to the theory of semigroups. Indeed, the assumption (177) and Lemma 2.1.24 imply that $\Omega_{\alpha,\beta,\gamma}\subseteq\rho_{\mathcal{C}}(\mathcal{A})$ and (iii.1) gives that the mapping $\lambda\mapsto\tilde{\Theta}(\lambda)(\lambda-\mathcal{A})^{-1}\mathcal{C}$, $\lambda\in\Omega_{\alpha,\beta,\gamma}$ is analytic on $\Omega_{\alpha,\beta,\gamma}$ and continuous on $\Gamma_{\alpha,\beta,\gamma}$. By the estimate (iii.2), we easily infer that there exists a number $M'>0$ such that $\|\tilde{\Theta}(\lambda)(\lambda-\mathcal{A})^{-1}\mathcal{C}\|\leqslant M'e^{-\Phi(\sigma|\lambda|)}$, $\lambda\in\Omega_{\alpha,\beta,\gamma}$. Since (iii.3) holds, we obtain that the operator $\mathcal{A}$ is a subgenerator of a local Θ-convoluted $\mathcal{C}$-cosine function on $[0,a+m(\frac{\alpha}{\sigma\gamma}))$. The proof of (iii) completes an employment of Theorem 2.1.11. Notice only that we have the following structural equality

$$C_K(t)=\frac{1}{2\pi i}\int\limits_{\Gamma_{\alpha,\beta,\gamma}}e^{\lambda t}\lambda\tilde{K}(\lambda)(\lambda^2-A)^{-1}C\,d\lambda,\ \lambda\in[0,a+m(\alpha/\sigma\gamma)).$$

In order to prove (iv), let us set, for every $t\in[0,a(\beta-\alpha-1))$,

$$C_\beta(t):=\frac{1}{2\pi i}\int\limits_\Gamma e^{\lambda t}\frac{(\lambda^2-A)^{-1}C}{\lambda^{\beta-1}}\,d\lambda,$$

where Γ is the upwards oriented boundary of $E(a,b)$. Having Lemma 2.1.24 and Theorem 2.7.2(iv) in mind, the proof of (iv) follows from that of (iii). $\square$

REMARK 2.7.4. (i) Suppose $\alpha>0$, $0<\tau<\infty$ and A generates an α-times integrated semigroup $(S_\alpha(t))_{t\in[0,\tau)}$, resp. an α-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. If there exists a sequence (t_n) in $[0,\tau)$ satisfying $\lim_{n\to\infty}t_n=\tau$ and $\sup_{n\in\mathbb{N}}\ln\|S_\alpha(t_n)\|<\infty$, resp. $\sup_{n\in\mathbb{N}}\ln\|C_\alpha(t_n)\|<\infty$, then, for every $a\in(0,\frac{\tau}{\alpha}]$, resp. $a\in(0,\frac{\tau}{\alpha+1}]$, there exist $b>0$ and $M>0$ such that (172), resp. (176), holds.

(ii) The assumptions of Theorem 2.7.2 and Theorem 2.7.3 are satisfied for the function $\Phi(t)=At^{\frac{1}{s}}+B$, where $s>1$, $A>0$ and $B\in\mathbb{R}$. For example, the item (ii.3) holds for the function $m(\varsigma)=\varepsilon/\varsigma^{\frac{1}{s}}$, where $\varepsilon>0$ can be chosen arbitrarily, and the item (ii.5) holds with $a=0$. If $K(t)=\mathcal{L}^{-1}\bigl(e^{-\lambda^{\frac{1}{s}}}\bigr)(t)$, $t\geqslant0$ and $\|R(\lambda\colon A)\|=O(e^{(\cos(\frac{\pi}{2s})-c\sigma^{\frac{1}{s}})|\lambda|^{\frac{1}{s}}})$, $\lambda\in\overline{\Omega_{\alpha,\beta,\gamma}}$, then we may apply Theorem 2.7.2 to deduce that A generates a local K-convoluted semigroup on $[0,\sigma^{\frac{1}{s}}\gamma/\alpha^{\frac{1}{s}})$. Further on, the assumption on continuous differentiability of the function $\Phi(\cdot)$, given in the formulation of Theorem 2.7.2(ii), can be slightly weakened. In fact, one can assume that there exists an increasing sequence (n_p) in $[r_0,\infty)$ such that the function $\Phi(\cdot)$ is of class C^1 in $[r_0,\infty)\smallsetminus\{n_p:p\in\mathbb{N}\}$. Suppose now that (M_p) satisfies (M.1), (M.2) and (M.3′) and that there exist numbers $\alpha>0$, $\beta\in\mathbb{R}$ and $l\geqslant1$ such that the (M_p)-ultralogarithmic region of type l, $\Lambda_{\alpha,\,\beta,\,l}=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant\alpha M(l|\operatorname{Im}\lambda|)+\beta\}$, belongs to $\rho(A)$ and that $\|R(\lambda\colon A)\|=O(e^{M(l|\lambda|)})$, $\lambda\in\ \Lambda_{\alpha,\,\beta,l}$. Since, for every $L\geqslant1$, there exist constants $K>1$ and $B>0$, and a number $E_L>0$, such that $M(Lt)\leqslant\frac{3}{2}LM(t)+K$, $t\geqslant0$ and $LM(t)\leqslant M(B^{L-1}t)+E_L$ (cf. [**51**, Lemma

2.1.3] and Section 1.3), it can be proved by means of Theorem 2.7.2(ii) (with $a = 0$ and $m(s) = 1/(\frac{3}{2}s + \varepsilon)$, $s > 1$, $0 < \varepsilon$ given in advance) that, for every $\varsigma > 0$, A generates a local $\mathcal{L}^{-1}\big(1/\prod_{p=1}^{\infty}\big(1 + \frac{B(l+\varsigma)\lambda}{m_p}\big)\big)$-convoluted semigroup on $[0, \frac{2}{3}\frac{\sigma}{l\alpha})$. By Theorem 2.7.3, the previous example can be simply reformulated in the case of local K-convoluted cosine functions.

2.8. Examples and applications

EXAMPLE 2.8.1. Suppose $E := L^2[0, \pi]$ and $A := -\Delta$ with the Dirichlet or Neumann boundary conditions (cf. [**14**, Section 7.2] and [**307**]) and

$$h(\lambda) := \frac{1}{\lambda^2}\prod_{n=0}^{\infty}\frac{n^2 - \lambda}{n^2 + \lambda}, \ \ \operatorname{Re}\lambda > 0, \ \lambda \neq n^2, \ n \in \mathbb{N}.$$

Define $\mathbf{h} : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \to \mathbb{C}$ by setting: $\mathbf{h}(\lambda) = h(\lambda)$, $\operatorname{Re}\lambda > 0$, $\lambda \neq n^2$, $n \in \mathbb{N}$ and $\mathbf{h}(n^2) = 0$, $n \in \mathbb{N}$. Then the function $\mathbf{h}(\cdot)$ is analytic and there exists an exponentially bounded, continuous function K such that $\tilde{K}(\lambda) = \mathbf{h}(\lambda)$, $\operatorname{Re}\lambda > 0$. Bäumer [**33**] proved that

$$\|\tilde{K}(\lambda)R(\lambda : A)\| \leqslant \frac{\text{Const} + |1/\lambda|}{|\lambda|^2}, \text{ if } \operatorname{Re}\lambda > 0, \ \lambda \neq n^2, \ n \in \mathbb{N},$$

and that $0 \in \operatorname{supp} K$. Moreover, the function $\lambda \mapsto \tilde{K}(\lambda)R(\lambda : A)$, $\operatorname{Re}\lambda > 0$ and $\tilde{K}(\lambda) \neq 0$, can be extended to an analytic function $\Upsilon : \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \to L(E)$ which satisfies $\|\Upsilon(\lambda)\| \leqslant \frac{\text{Const} + |1/\lambda|}{|\lambda|^2}$, $\operatorname{Re}\lambda > 0$. Then the use of Theorem 1.1.12 implies that there exists a continuous function $S_K : [0, \infty) \to L(E)$ such that, for every $\varepsilon > 0$, $\|S_K(t)\| = O(e^{\varepsilon t})$, $t \geqslant 0$ and that $\Upsilon(\lambda) = \int_0^{\infty} e^{-\lambda t}S_K(t)\,dt$, $\operatorname{Re}\lambda > 0$. By the proof of Theorem 1.1.12 (see also [**434**, Theorem 1.12]), it follows that, for every $r > 0$, $S_K(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\lambda t}\Upsilon(\lambda)\,d\lambda$, $t \geqslant 0$. Let $t > 0$ be fixed. With $r = 1/t$ one easily obtains $\|S_K(t)\| = O(t + t^2)$, $t \geqslant 0$. As an outcome, one gets that $(S_K(t))_{t\geqslant 0}$ is a polynomially bounded K-convoluted semigroup generated by A; let us point out that the dividing of the term $\prod_{n=0}^{\infty}\frac{n^2-\lambda}{n^2+\lambda}$ by λ^2 has been done only for the sake of brevity and that, for every $l > 0$, $-\Delta$ generates an exponentially bounded, analytic $(K_l *_0 K)$-convoluted semigroup of angle $\frac{\pi}{2}$ [**234**], where $K_l(t) = \mathcal{L}^{-1}\big((\prod_{i=0}^{\infty}(1 + \frac{l\lambda}{p^s}))^{-1}\big)(t)$, $t \geqslant 0$, $l > 0$ and $s \in (1, 2)$. Assume now $\beta \in [\frac{1}{2}, 1)$, $\alpha > 1 + \beta$, $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $\mathbf{h}_{\alpha,\beta}(\lambda) := \frac{1}{\lambda^{\alpha}}\prod_{n=0}^{\infty}\frac{n^2-\lambda^{\beta}}{n^2+\lambda^{\beta}}$, $\operatorname{Re}\lambda > 0$, $\lambda \neq n^{2/\beta}$, $n \in \mathbb{N}$ and $\mathbf{h}_{\alpha,\beta}(n^{2/\beta}) = 0$, $n \in \mathbb{N}$. Let $k(t) = \mathcal{L}^{-1}\big(\mathbf{h}_{\alpha,\beta}(\lambda)\big)(t)$, $t \geqslant 0$. Then A generates an exponentially bounded, analytic (a, k)-regularized resolvent $(R(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}(\frac{1}{\beta} - 1)$, and $\|R(t)\| = O(t^{\alpha-1} + t^{\alpha+\beta-1})$, $t \geqslant 0$ [**235**].

EXAMPLE 2.8.2. Let $A := -\Delta$, $E := L^2[0, \pi]$ and K be as in the previous example, and suppose that $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$ for appropriate real numbers $M > 0$ and $\beta > 0$. Clearly, $|K_1(t)| \leqslant Mte^{\beta t}$, $t \geqslant 0$ and A generates an exponentially bounded K_1-convoluted semigroup $(S_{K_1}(t))_{t\geqslant 0}$, where $S_{K_1}(t)x = \int_0^t S_K(s)x\,ds$,

$x \in E$, $t \geqslant 0$ and $K_1(t) = \int_0^t K(s)\,ds$, $t \geqslant 0$. Moreover, $-A$ also generates an exponentially bounded K-convoluted semigroup $(V_K(t))_{t\geqslant 0}$ in E since it is the generator of an analytic C_0-semigroup of angle $\frac{\pi}{2}$. An employment of Proposition 2.1.11 implies that the biharmonic operator Δ^2, endowed with the corresponding boundary conditions, generates an exponentially bounded K-convoluted cosine function $(C_K(t))_{t\geqslant 0}$, where $C_K(t) := 1/2(S_K(t) + V_K(t))$, $t \geqslant 0$. This implies that Δ^2 generates an exponentially bounded K_1-cosine function $(C_{K_1}(t))_{t\geqslant 0}$, and owing to Theorem 2.4.8, Δ^2 generates an exponentially bounded, analytic K_2-convoluted semigroup of angle $\frac{\pi}{2}$, where the function $K_2(\cdot)$ is taken in the sense of Theorem 2.4.8. Herein it is worth noting that we have integrated once the function K so as to prove that the function K_2 is exponentially bounded. Actually, one gets that, for every $t > 0$:

$$
\begin{aligned}
|K_2(t)| &\leqslant M \int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}} se^{\beta s} ds = \frac{M}{2\sqrt{\pi}} \int_0^\infty r^2 e^{\beta r\sqrt{t} - \frac{r^2}{4}}\,dr \\
&= \frac{M}{2\sqrt{\pi}} \int_0^\infty r^2 e^{\beta^2 t - \left(\frac{r}{2} - \beta\sqrt{t}\right)^2} dr \\
&= \frac{M}{2\sqrt{\pi}} e^{\beta^2 t} \int_0^\infty r^2 e^{-\left(\frac{r}{2} - \beta\sqrt{t}\right)^2} dr = \frac{M}{2\sqrt{\pi}} e^{\beta^2 t} \int_{-\beta\sqrt{t}}^\infty 8\big(v^2 + 2v\beta\sqrt{t} + \beta^2 t\big) e^{-v^2}\,dv \\
&\leqslant \frac{4M}{\sqrt{\pi}} e^{\beta^2 t} \left[\int_{-\infty}^\infty v^2 e^{-v^2} dv + 2\beta\sqrt{t} \int_0^\infty v e^{-v^2} dv + \beta^2 t \int_{-\infty}^\infty e^{-v^2} dv \right] \leqslant \overline{M} e^{(\beta^2+1)t},
\end{aligned}
$$

for an appropriate constant $\overline{M} > 0$. Furthermore, K_2 is a kernel since

$$
\limsup_{\lambda\to\infty} \frac{\ln|\widetilde{K_2}(\lambda)|}{\lambda} = \limsup_{\lambda\to\infty} \frac{\ln|\tilde{K}_1(\sqrt{\lambda})|}{\lambda} = 0.
$$

On the other side, Δ^2 cannot be the generator of a (local) α-times integrated semigroup, $\alpha > 0$, since the resolvent set of Δ^2 does not contain any ray (ω, ∞). Hence, in the analysis of Δ^2 and $-\Delta$, we do not need any C, but the use of regularized operator families enables several advantages which hardly can be considered by the use of asymptotic Laplace transform techniques. More generally, suppose $n \in \mathbb{N}$. Since $\Delta = -A$ generates a cosine function (cf. for instance [**14**, Example 7.2.1, p. 418]), one can employ an old result of Goldstein (cf. [**89**, p. 215]), in order to see that $-\Delta^{2n}$ generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$. Hence, an application of [**89**, Theorem 8.2] shows that there exists an injective operator $C_n \in L(L^2[0,\pi])$ so that Δ^{2n} generates an entire C_n-regularized group. Further on, one can apply Proposition 2.1.11 in order to see that the polyharmonic operator Δ^4 generates an exponentially bounded K_2-convoluted cosine function. Put $\overline{K_3}(t) := \int_0^t K_2(s)\,ds$, $t \geqslant 0$. Then $\overline{K_3}(\cdot)$ is a kernel and we have $|\overline{K_3}(t)| \leqslant \overline{M} t e^{(\beta^2+1)t}$, $t \geqslant 0$. Clearly, Δ^4

generates an exponentially bounded $\overline{K_3}$-convoluted cosine function. Then Theorem 2.4.8 can be applied again and, as a conclusion, one obtains that Δ^4 generates an exponentially bounded, analytic K_3-convoluted semigroup of angle $\frac{\pi}{2}$, where $K_3(t) := \int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}\overline{K_3}(s)\,ds,\ t > 0$. Arguing as before, we have that $K_3(\cdot)$ is an exponentially bounded kernel. Continuing this procedure leads us to the fact that, for every $n \in \mathbb{N}$, there exist exponentially bounded kernels K_n and K_{n+1} such that Δ^{2^n} generates an exponentially bounded, K_n-convoluted cosine function, and simultaneously, an exponentially bounded, analytic K_{n+1}-convoluted semigroup of angle $\frac{\pi}{2}$. Note that this procedure can be done only with loss of regularity, since we need to apply Theorem 2.4.8, and that it is not clear whether there exists a kernel $\overline{K_n}$ such that Δ^{2n} generates an exponentially bounded, $\overline{K_n}$-convoluted cosine function. The preceding analysis also enables one to prove that Δ^{2^n} generates an exponentially bounded K_{n+1}-convoluted group. Observe that the operator $-\Delta$, considered in the first part of this example, generates an exponentially bounded K-convoluted group.

Further on, assume that A is a self-adjoint operator in a Hilbert space H and that A has a discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$, where we write the eigenvalues in increasing order and repeat them according to multiplicity. Suppose $\operatorname{Re}\lambda_n > 0$, $n \geqslant n_0$ and m is a natural number greater than any multiplicity of λ_n, $n \geqslant n_0$. If

$$\sum_{n\geqslant n_0}\Big(1 - \frac{|\sqrt{\lambda_n} - 1|}{\sqrt{\lambda_n} + 1}\Big) < \infty,$$

then, according to Theorem 1.1.9, there exists an exponentially bounded function K such that $\tilde{K}\big(\sqrt{\lambda_n}\,\big) = 0$, $n \geqslant n_0$. This implies that the function $\lambda \mapsto \widetilde{K^{*m}}(\lambda)R(\lambda^2 : A)$ can be analytically extended to a right half plane, where K^{*m} denotes the mth convolution power of K. If, additionally,

$$\big\|\widetilde{K^{*m}}(\lambda)R(\lambda^2 : A)\big\| \leqslant M|\lambda|^{-3},\ \operatorname{Re}\lambda > \omega\ (\geqslant 0),\ \lambda \neq \sqrt{\lambda_n},\ n \geqslant n_0,$$

for some $M > 0$, then A generates an exponentially bounded K^{*m}-convoluted cosine function. It is evident that this procedure cannot be done if $(\sqrt{\lambda_n})_{n\geqslant n_0}$ is a uniqueness sequence, see for instance [**14**] and [**32**]. Therefore, the theory of convoluted cosine functions cannot be applied if $\lambda_n \sim n^{2s}$, $n \to +\infty$, for some $s \in (0,1]$, and this, in turn, implies that the operator $-\Delta$, considered in the first part of this example, cannot be the generator of any exponentially bounded, convoluted cosine function. It is also worth noting that, for every $n \in \mathbb{N}$,there exists an exponentially bounded kernel $k_n(t)$ such that the polyharmonic operator Δ^{2^n} generates an exponentially bounded, analytic (a, k_n)-regularized resolvent family of angle $\frac{\pi}{2}$ ([**235**]), where $a(\cdot)$ has the same meaning as in Example 2.8.1. The case $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, where $\beta \in (0, 1/2)$, is more delicate [**235**].

Before going any further, we would like to note that the method described in Example 2.8.1 and Example 2.8.2 can be applied with minor modifications to the Legendre differential operator $(Af)(x) := -((1 - x^2)f')'$ and to the operator $(A_m f)(x) := -((1 - x^2)f')' + \frac{m^2}{1-x^2}f(x)$, where $m \in \mathbb{N}$. Strictly speaking, the operator A with domain $D(A) = C^\infty[-1, 1]$ is essentially self-adjoint in

$E = L^2((-1,1))$ and $\overline{A}$ has the point spectrum which consists of simple poles $\lambda_n = n(n+1)$, with characteristic functions being the Legendre polynomials $L_n(x) = \sqrt{\frac{2n+1}{2}}2^{-n}\frac{1}{n!}\frac{d^n}{dx^n}[(1-x^2)^n]$, $n = 0,1,2,\ldots$. The operator A_m considered with domain $D(A_m) = \{f : f(x) = (1-x^2)^{m/2}P(x),\ P(x) \text{ polynomial}\}$ is essentially self-adjoint in $L^2((-1,1))$ and the point spectrum consists of simple poles $\lambda_n = n(n+1)$, $n = m, m+1, \ldots$, with $L_n^m = \sqrt{\frac{(n-m)!}{(n+m)!}}(1-x^2)^{m/2}\frac{d^m}{dx^m}L_n(x)$ being the characteristic functions. Concerning time-fractional equations, a possible application can be made to the Laguerre's differential operator, $(Af)(x) = -4(xf'(x))' + (x + \frac{\alpha^2}{x})f(x)$ $(\alpha > -1)$ in $L^2((0,\infty))$, to the Hermite's differential operator $(Af)(x) = -f''(x) + x^2 f(x)$ in $L^2(\Omega)$, where Ω is an open subset of $\mathbb{R}$, and to the harmonic oscillator H considered on [**406**, p. 178]. For further information, we refer the reader to [**409**, pp. 283–285] and [**406**, Sections 8.2, 8.3, 10.3].

The following example is motivated by [**249**, Example 1.6].

EXAMPLE 2.8.3. Let (M_p) satisfy (M.1), (M.2) and (M.3′). Define

$$E_{M_p} := \Big\{f \in C^\infty[0,1] : \|f\|_{M_p} =: \sup_{p \geqslant 0}\frac{\|f^{(p)}\|_\infty}{M_p} < \infty\Big\},$$
$$A_{M_p} := -d/ds,\ \ D(A_{M_p}) := \big\{f \in E_{M_p} : f' \in E_{M_p},\ f(0) = 0\big\}.$$

Arguing as in [**249**, Example 1.6], one can verify that A_{M_p} is not stationary dense and that: $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant 0\} \subseteq \rho(A_{M_p})$ and $\|R(\lambda : A_{M_p})\| \leqslant Ce^{M(\tilde{r}|\lambda|)}$, $\operatorname{Re}\lambda \geqslant 0$, for some $C > 0$ and $\tilde{r} > 0$. Moreover, A cannot be the generator of a local integrated semigroup and Theorem 2.2.4 implies that A_{M_p} is the generator of a global exponentially bounded K-convoluted semigroup, where K is any function satisfying (P1) and $|\tilde{K}(\lambda)| \leqslant e^{-M(r|\lambda|)}$, $\operatorname{Re}\lambda > 0$, for some $r > \tilde{r}$.

1. Let $M_p = p!^2$, $p \in \mathbb{N}_0$. Then, by the well-known estimates for associated functions ([**207**], [**210**]), we obtain $\|R(\lambda : A_{M_p})\| \leqslant Ce^{m_1|\lambda|^{1/2}}$, $\operatorname{Re}\lambda > 0$, for some $m_1 > 0$ and $C > 0$. Let $K(t) = \frac{1}{2\sqrt{\pi t^3}}e^{-\frac{a^2}{4t}}$, $t > 0$, for some $a > m_1\sqrt{2}$. Then $\widetilde{K}(\lambda) = e^{-a\sqrt{\lambda}}$, $\operatorname{Re}\lambda > 0$, where $\sqrt{1} = 1$, and one can straightforwardly prove that $\|\tilde{K}(\lambda)R(\lambda : A)\| \leqslant Ce^{\left(m_1 - a\frac{\sqrt{2}}{2}\right)|\lambda|^{1/2}}$, $\operatorname{Re}\lambda > 0$. Thus, Theorem 2.2.4 implies that A_{M_p} generates an exponentially bounded K-convoluted semigroup $(S_K(t))_{t\geqslant 0}$. By the proof of [**434**, Theorem 1.12], we have

$$S_K(t) = \frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} e^{\lambda t}\tilde{K}(\lambda)R(\lambda : A)\,d\lambda, \quad t \geqslant 0,\ \text{for any } r > 0.$$

Let $t > 0$ and $k \in \mathbb{N}_0$ be fixed. With $r = t^{-1}$, $b = -m_1 + a\frac{\sqrt{2}}{2}$ and a suitable $C > 0$, one easily obtains

$$\|S_K(t)\| \leqslant C\int\limits_{-\infty}^{+\infty}\frac{dx}{e^{b(t^{-2}+x^2)^{1/4}}},$$

$$\Big\|\frac{S_K(t)}{t^k}\Big\| \leqslant C\int\limits_{-\infty}^{+\infty}\frac{dx}{t^k e^{bt^{-1/2}/2}e^{bx^{1/2}/2}} \leqslant C\int\limits_{-\infty}^{+\infty}\frac{(2k)!dx}{t^k(bt^{-1/2}/2)^{2k}e^{bx^{1/2}/2}}.$$

Thus, for every $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that

$$\Big\|\frac{S_K(t)}{t^k}\Big\| \leqslant C_k\int\limits_{-\infty}^{+\infty}\frac{dx}{e^{(\frac{a\sqrt{2}}{4}-\frac{m_1}{2})\sqrt{x}}}.$$

This implies that, for every $k \in \mathbb{N}_0$, $\|S_K(t)\| = O(t^k)$, $t \geqslant 0$. Similarly, if $m_1 = \frac{a\sqrt{2}}{2}$, then A_{M_p} generates a polynomially bounded $\big(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\big)$-convoluted semigroup for all $\alpha > 1$. In this example, one can also use the well known complementary error function.

2. Assume that $M_p = p!$. Then (M.3$'$) does not hold but we continue to consider $E = E_{p!}$ and $A = A_{p!}$. Let C be an injective operator in $L(E)$. Note that E is a subspace of the space of functions analytic in some neighborhood of $[0,1]$ and that we do not require that $\mathrm{R}(C)$ is dense in E. It is easy to see that $\rho(A)$ contains the right half-plane and that $\|R(\lambda\colon A)\| \leqslant Me^{|\lambda|}$, $\operatorname{Re}\lambda > 0$, for some positive constant M. Arguing as in [**227**, Example 6.2], we reveal that A cannot be a subgenerator of any local C-regularized semigroup.

3. Let $M_p = p!^s$ $(s > 1)$, $\beta \in (0,1)$ and let, for every $l > 0$, $k_l(t) = \mathcal{L}^{-1}\big(1/\prod_{p=1}^{\infty}\big(1+\frac{l\lambda}{p^{s/\beta}}\big)\big)(t)$, $t \geqslant 0$ and $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. Then it is obvious that there exist $l' > 0$ and $M > 0$ such that $\|\lambda\tilde{k_{l'}}(\lambda)(I-\tilde{a}(\lambda)A_{p!^s})^{-1}\| \leqslant M$, $\lambda \in \Sigma_{\frac{\pi}{2\beta}}$. This implies that, for every $l > l'$, the operator $A \equiv A_{p!^s}$ generates an analytic (a,k_l)-regularized resolvent of angle $\frac{\pi}{2}\big(\frac{1}{\beta}-1\big)$. In the meantime, A cannot be the generator of an exponentially bounded $\big(a, \frac{t^\alpha}{\Gamma(\alpha+1)}\big)$-regularized resolvent $(\alpha \geqslant 0)$ since A is not stationary dense. Furthermore, $\rho(A) = \mathbb{C}$. If $f \in E \equiv E_{p!^s}$, $t \in [0,1]$ and $\lambda \in \mathbb{C}$, set $f^1_\lambda(t) := \int_0^t e^{-\lambda(t-s)}f(s)ds$ and $f^2_\lambda(t) := \int_0^t e^{\lambda(t-s)}f(s)ds$. Then $f^1_\lambda(\cdot)$, $f^2_\lambda(\cdot) \in E$, $\lambda \in \mathbb{C}$ and there exist $b > 0$ and $M \geqslant 1$, independent of $f(\cdot)$, such that

$$\big\|f^1_\lambda(\cdot)\big\| \leqslant M\|f\|e^{b|\lambda|^{1/s}},\ \operatorname{Re}\lambda \geqslant 0,\ f \in E. \tag{178}$$

t is clear that $\|f^2_\lambda(\cdot)\|_{L^\infty[0,1]} \leqslant e^{|\lambda|}\|f\|$, $\operatorname{Re}\lambda \geqslant 0$ and $\|\frac{d}{dt}f^2_\lambda(\cdot)\|_{L^\infty[0,1]} \leqslant (|\lambda|e^{|\lambda|}+1)\|f\|$, $\operatorname{Re}\lambda \geqslant 0$. Proceeding by induction, we obtain

$$\frac{d^n}{dt^n}f^2_\lambda(t) = \frac{d^{n-1}}{dt^{n-1}}f(t) + \sum_{k=1}^{n-1}\lambda^k\frac{d^{n-1-k}}{dt^{n-1-k}}f(t) + \lambda^n f^2_\lambda(t),\ n \geqslant 2,\ t \in [0,1],\ \operatorname{Re}\lambda \geqslant 0. \tag{179}$$

On the other hand, [**207**, Proposition 4.5] implies that there exists $c > 0$ such that $\sum_{p=0}^{\infty} t^p/p!^s = O(e^{ct^{1/s}})$, $t \geqslant 0$. Combined with (282) and the logarithmic convexity, the last estimate yields:

$$\begin{aligned}\frac{1}{n!^s}\Big\|\frac{d^n}{dt^n}f^2_\lambda(\cdot)\Big\|_{L^\infty[0,1]} &\leqslant \|f\| + \|f\|e^{c|\lambda|^{1/s}} + \frac{|\lambda|^n}{n!^s}e^{|\lambda|}\|f\| \\ &\leqslant \big(1+e^{c|\lambda|^{1/s}}+e^{c|\lambda|^{1/s}}e^{|\lambda|}\big)\|f\|,\ \operatorname{Re}\lambda \geqslant 0,\ \lambda \neq 0.\end{aligned} \tag{180}$$

In view of (180) we get that, for every $\eta > 1$, there exists $M_\eta \geqslant 1$, independent of $f(\cdot)$, such that

$$\big\|f_\lambda^2(\cdot)\big\| \leqslant M_\eta \|f\| e^{\eta|\lambda|}, \ \operatorname{Re}\lambda \geqslant 0, \ f \in E. \tag{181}$$

Consider now the complex polynomial $P(z) = \sum_{j=0}^n a_j z^j$, $z \in \mathbb{C}$, $a_n \neq 0$, $n \geqslant 2$. Set, for every $\lambda \in \mathbb{C}$, $P_\lambda(\cdot) := P(\cdot) - \lambda$ and consider the operator $P(A)$ defined by

$$D(P(A)) := D(A^n) \text{ and } P(A)f := \sum_{j=0}^n a_j A^j f, \ f \in D(P(A)).$$

Clearly, $P(A)$ is not stationary dense. Let $r > 0$ and $d > 0$ be such that $P(z) \neq 0$, $|z| \geqslant r$ and $P'(z) \neq 0$, $|z| \geqslant d$. Let $z_{1,\lambda}, \cdots, z_{n,\lambda}$ denote the zeros of the polynomial $z \mapsto P_\lambda(z)$, $z \in \mathbb{C}$ and let $0 < m := \min_{|z| \geqslant d+1} |P'(z)|$. Then an old result of J. L. Walsh [**417**] says that $|z_{j,\lambda}| \leqslant r + |a_n|^{-1/n} |\lambda|^{1/n}$, $1 \leqslant j \leqslant n$, $\lambda \in \mathbb{C}$. Furthermore, it is checked at once that there exists a sufficiently large $\lambda_0 > 0$ such that $z_{j,\lambda}$ is a simple zero of $P_\lambda(z)$ and that $|z_{j,\lambda}| \geqslant d+1$, provided $|\lambda| \geqslant \lambda_0$ and $1 \leqslant j \leqslant n$. Therefore, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant \lambda_0$ and for every $i, \ j \in \{1, \cdots, n\}$ with $i \neq j$, the following holds:

$$d + 1 \leqslant |z_{j,\lambda}| \leqslant r + |a_n|^{-1/n} |\lambda|^{1/n} \text{ and } |P'(z_{j,\lambda})| \geqslant m, \ z_{i,\lambda} \neq z_{j,\lambda}. \tag{182}$$

One can simply prove that

$$\rho(p(A)) = \mathbb{C} \text{ and } R(\lambda : p(A)) = (-1)^{n+1} a_n^{-1} R(z_{1,\lambda} : A) \cdots R(z_{n,\lambda} : A), \ \lambda \in \mathbb{C}. \tag{183}$$

Assume now $|\lambda| \geqslant \lambda_0$. Then de L'Hospital's rule implies:

$$a_n \prod_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (z_{i,\lambda} - z_{j,\lambda}) = (-1)^{n+1} P'(z_{j,\lambda}), \ 1 \leqslant j \leqslant n. \tag{184}$$

Using the resolvent equation, (178), (181)-(182) and (184), one can rewrite and evaluate the right hand side of equality appearing in (183) as follows:

$$\begin{aligned}
&\big\|(-1)^{n+1} a_n^{-1} R(z_{1,\lambda} : A) \cdots R(z_{n,\lambda} : A)\big\| \\
&\qquad = \Big\|(-1)^{n+1} a_n^{-1} \sum_{j=1}^n \frac{R(z_{j,\lambda} : A)}{\prod_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (z_{i,\lambda} - z_{j,\lambda})}\Big\| \\
&\qquad = \Big\|\sum_{j=1}^n \frac{R(z_{j,\lambda} : A)}{P'(z_{j,\lambda})}\Big\| \leqslant \frac{1}{m} \sum_{j=1}^n \big\|R(z_{j,\lambda} : A)\big\|.
\end{aligned} \tag{185}$$

By (183) and (185) we finally get that, for every $\eta > 1$,

$$\big\|R(\lambda : p(A))\big\| = O\big(e^{b|a_n|^{-1/n} |\lambda|^{1/ns}} + e^{\eta |a_n|^{-1/n} |\lambda|^{1/n}}\big), \ \lambda \in \mathbb{C}. \tag{186}$$

Since the preceding estimate holds for any $\lambda \in \mathbb{C}$, it is quite complicated to inscribe here all of its consequences; for example, $P(A)$ generates a tempered ultradistribution sine of $(p!^s)$-class provided $n \geqslant 2s$, and $P(A)$ generates an exponentially bounded, $\mathcal{L}^{-1}(e^{-\varrho\lambda^{1/n}})$-convoluted group provided $\varrho > |a_n|^{-1/n} / \cos(\pi/2n)$. In what follows, we will present an illustrative application of Corollary 2.5.15. Suppose $n > \alpha \geqslant 1$, $\delta \in (0, \frac{\pi}{2}]$, $\frac{\alpha}{n}(\frac{\pi}{2} + \delta) < \frac{\pi}{2}$, $\varrho \geqslant 1/\cos(\frac{\alpha}{n}(\frac{\pi}{2} + \delta))$ and

$k(t)=\mathcal{L}^{-1}(\lambda^{-\alpha}e^{-\varrho\lambda^{\alpha/n}})(t)$, $t\geqslant 0$. By (186), $P(A)$ is the integral generator of an exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized resolvent family of angle δ. Let $\varphi\in E$ and $Bf(t):=(\varphi*f)(t)$, $t\in[0,1]$, $f\in E$. Then $B\in L(E)$, $BP(A)\subseteq P(A)B$ and, therefore, $P(A)+B$ is the integral generator of an exponentially bounded, analytic $(\frac{t^{\alpha-1}}{\Gamma(\alpha)},k)$-regularized resolvent family of angle δ.

EXAMPLE 2.8.4. Let $E=L^p(\mathbb{R})$, $1\leqslant p\leqslant\infty$. Consider the next multiplication operator with maximal domain in E:

$$Af(x):=(x+ix^2)^2f(x),\ x\in\mathbb{R},\ f\in E.$$

It is clear that A is dense and stationary dense if $1\leqslant p<\infty$ and that A cannot the generator of any (local) integrated cosine function, $1\leqslant p\leqslant\infty$. Moreover, if $p=\infty$, then A is not stationary dense since, for example, the function $x\mapsto\frac{1}{x^{2n}+1}$ belongs to $D(A^n)\smallsetminus\overline{D(A^{n+1})}$, $n\in\mathbb{N}$. Further on, one can easily verify that A generates an ultradistribution sine of $*$-class, if $M_p=p!^s$, $s\in(1,2)$. If $M_p=p!^2$, then the analysis given in [**223**, Example 4.4] shows that A does not generate an ultradistribution sine of the Roumieu class and that A generates an ultradistribution sine of the Beurling class. Suppose now $M_p=p!^s$, for some $s\in(1,2)$, and put $\delta=\frac{1}{s}$. Then A generates a global (not exponentially bounded) K_δ-cosine function since, for every $\tau\in(0,\infty)$, A generates a K_δ-cosine function on $[0,\tau)$. Indeed, suppose $M(\lambda)\leqslant C_s|\lambda|^{1/s}$, $\lambda\in\mathbb{C}$, $\tau\in(0,\infty)$ and choose an $\alpha>0$ with $\tau\leqslant\frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}$. It is evident that for such an $\alpha>0$ there exists a sufficiently large $\beta>0$ such that $\Lambda^2_{\alpha,\beta,1}\subseteq\rho(A)$ and that the resolvent of A is bounded on $\Lambda^2_{\alpha,\beta,1}$, where $\Lambda_{\alpha,\beta,l}=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant\alpha M(l|\lambda|)+\beta\}$. Put $\Gamma:=\partial(\Lambda_{\alpha,\beta,1})$. We assume that Γ is upwards oriented. Define

$$(C_\delta(t)f)(x):=\frac{1}{2\pi i}\int\limits_\Gamma\frac{\lambda e^{\lambda t-\lambda^\delta}}{\lambda^2-(x+ix^2)^2}\,d\lambda f(x),\ f\in E,\ x\in\mathbb{R},\ t\in\Big[0,\frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}\Big).$$

Note that the above integral is convergent since $\big|e^{-\lambda^\delta}\big|\leqslant e^{-\cos(\delta\pi/2)|\lambda|^\delta}$, $\operatorname{Re}\lambda>0$ and

$$\big|e^{\lambda t-\lambda^\delta}\big|\leqslant e^{\beta t}e^{M(\alpha\lambda)t-\cos(\delta\pi/2)|\lambda|^\delta}\leqslant e^{\beta t}e^{C_s\alpha^{1/s}|\lambda|^\delta t-\cos(\delta\pi/2)|\lambda|^\delta},\ \lambda\in\Gamma.$$

It is checked at oncethat $(C_\delta(t))_{t\in[0,\tau)}$ is a local K_δ-convoluted cosine function generated by A. At the end of this example, let us point out that there exists $\tau_0\in(0,\infty)$ such that A generates a local $K_{1/2}$-convoluted cosine function on $[0,\tau_0)$ and that the preceding example can be set in the context of (a,k)-regularized resolvent families ([**235**]); in such a way, one can simply construct examples of global not exponentially bounded (a,k)-regularized resolvent families.

EXAMPLE 2.8.5. (i) ([**271**]) Let $E=C_0(\mathbb{R})\oplus C_0(\mathbb{R})\oplus C_0(\mathbb{R})$, $C(f,g,h):=(f,g,\sin(\cdot)h(\cdot))$, $f,g,h\in C_0(\mathbb{R})$ and $A(f,g,h):=(f'+g',g',(\chi_{[0,\infty)}-\chi_{(-\infty,0]})h)$, $(f,g,h)\in D(A):=\{(f,g,h)\in E:f'\in C_0(\mathbb{R}),\ g'\in C_0(\mathbb{R}),\ h(0)=0\}$. Arguing as in [**271**, Example 8.1, Example 8.2], one gets that A is the integral generator of an exponentially bounded once integrated C-semigroup and that A is not a subgenerator of any local C-regularized semigroup. Suppose now $m_i\in C^1(\mathbb{R})$,

$i = 1, 2$, the mappings $t \mapsto |t|m_i(t)$, $t \in \mathbb{R}$ and $t \mapsto |t|m_i'(t)$, $t \in \mathbb{R}$ are bounded for $i = 1, 2$; $C(\mathbb{R}) \ni m_3$ is bounded and satisfies $m_3(0) = 0$. Put, for every f, $g, h \in C_0(\mathbb{R})$,

$$B(f, g, h) := \Biggl(m_1(\cdot) \int\limits_0^{\cdot} f(s)\, ds,\ m_2(\cdot) \int\limits_0^{\cdot} g(s)\, ds,\ \sin(\cdot) m_3(\cdot) h(\cdot)\Biggr).$$

Then $B \in L(E)$, $\mathrm{R}(B) \subseteq C(D(A))$ and $BC(f, g, h) = CB(f, g, h)$, $(f, g, h) \in E$. By Theorem 2.5.7, one obtains that $A+B$ is the integral generator of an exponentially bounded once integrated C-semigroup.

(ii) Let $E := L^1(\mathbb{R})$ and let $D := d/dx$ with maximal distributional domain. Then it is well known (cf. also [**171**, Corollary 3.4, Example 7.1]) that E has the Fourier type 1, and in particular, that E is not a B-convex Banach space. Furthermore, $A := D^2 = d^2/dx^2$ generates a bounded cosine function $(C(t))_{t \geqslant 0}$ given by

$$(C(t)f)(x) := \frac{1}{2}\bigl(f(x+t) + f(x-t)\bigr),\ t \geqslant 0,\ x \in \mathbb{R},\ f \in L^1(\mathbb{R}),$$

and Sobolev imbedding theorem implies $D(A) = W^{1,2}(\mathbb{R}) \subseteq C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Suppose $g \in L^1(\mathbb{R}) \smallsetminus L^\infty(\mathbb{R})$ and define a linear operator $B : L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \to L^1(\mathbb{R})$ by $Bf(x) := f(x)g(x)$, $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In general, B cannot be extended to a bounded linear operator from $L^1(\mathbb{R})$ into $L^1(\mathbb{R})$ and $\mathrm{R}(B) \nsubseteq D(A)$. It is clear that, for every $f \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$:

$$\begin{aligned}\bigl\|B(2\lambda R(\lambda^2 : A)f)\bigr\| &= \int\limits_{-\infty}^{\infty} |g(x)| \Biggl| \int\limits_0^{\infty} e^{-\lambda t}\bigl(f(x+t) + f(x-t)\bigr)\, dt \Biggr|\, dx \\ &\leqslant \int\limits_{-\infty}^{\infty} |g(x)| \int\limits_0^{\infty} \bigl(|f(x+t)| + |f(x-t)|\bigr)\, dt\, dx \\ &\leqslant 2\|g\|\|f\|.\end{aligned}$$

This implies that the assumptions quoted in the formulation of Corollary 2.5.10(i) hold with $\lambda_0 = 1$. Hence, $A + B$ generates an exponentially bounded α-times integrated cosine function for every $\alpha > 1$; let us also note that it is not clear whether there exists $\beta \in [0, 1)$ such that $A+B$ generates a (local) β-times integrated cosine function although one can simply prove that there exist $a > 0$ and $M > 0$ such that $\|\lambda R(\lambda^2 : A + B)\| \leqslant \frac{M}{\operatorname{Re}\lambda}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > a$.

(iii) Suppose A generates a (local) α-times integrated cosine function for some $\alpha > 0$, $B \in L(E)$ and $BA \subseteq AB$. Then the proof of [**223**, Theorem 4.3] and the analysis given in [**228**, Example 7.3] (cf. also Theorem 2.6.10) imply that, for every $s \in (1, 2)$, $\pm iA$ generate global $K_{1/s}$-convoluted semigroups and that $\pm iA$ generate local $K_{1/2}$-semigroups, where $K_\sigma(t) = \mathcal{L}^{-1}\bigl(e^{-\lambda^\sigma}\bigr)(t)$, $t \geqslant 0$, $\sigma \in (0, 1)$. By Theorem 2.5.3 and Remark 2.5.4(iii), we have that $\pm i(A+B)$ generate global $K_{1/s}$-semigroups for every $s \in (1, 2)$ and that $\pm i(A + B)$ generate local $K_{1/2}$-convoluted semigroups. Therefore, a large class of differential operators generating integrated cosine functions can be used to provide applications of Theorem 2.5.3.

(iv) ([**36**, Example 2.24]) Let $E := l^1$, $0 < \alpha < 1$ and $l := \lceil \frac{1-\alpha}{\alpha} \rceil$. Define a closed densely defined linear operator A_α on E by $D(A_\alpha) := \{\langle x_n \rangle \in l^1 : \sum_{n=1}^{\infty} n|x_n| < \infty\}$ and $A_\alpha \langle x_n \rangle := \langle e^{i\alpha\frac{\pi}{2}} n x_n \rangle$, $\langle x_n \rangle \in D(A_\alpha)$. Then A_α is the integral generator of a bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized resolvent family, $A_\alpha + I$ is not the integral generator of an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized resolvent family and $\sigma(A_\alpha) = \{e^{i\alpha\frac{\pi}{2}} n : n \in \mathbb{N}\}$. Suppose

$$B \in L(E) \text{ and } R(B) \subseteq D(A^l) = \left\{ \langle x_n \rangle \in l^1 : \sum_{n=1}^{\infty} n^l |x_n| < \infty \right\}.$$

Then $A + B$ generates an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized resolvent family [**242**].

(v) [**114**] Consider the Laplace-Beltrami operator $-\Delta_T$ on the torus $T := \mathbb{R}^2/\Gamma$, where $\Gamma := \mathbb{Z}(a,0) + \mathbb{Z}(0,b)$ and $a,\ b > 0$. Then $i\Delta_T$ generates on $L^p(T)$ $(1 < p < \infty)$ an exponentially bounded n-times integrated group for any $n > |\frac{1}{p} - \frac{1}{2}|$, and $\sigma(i\Delta_T) = \sigma_p(i\Delta_T) = \{(\frac{2\pi}{a})^2 m^2 + (\frac{2\pi}{b})^2 n^2 : m,\ n \in \mathbb{Z}\}$. Let $\alpha := a^2/b^2$ be the algebraic number of degree $d \geqslant 2$ and let $\{\lambda_n : n \in \mathbb{N}\}$ be the set of eigenvalues of $i\Delta_T$ on $L^p(T)$. Then there exist projectors T_k on $L^p(T)$ such that

$$\sum_{k=1}^{\infty} T_k x = x,\ x \in D\big((-\Delta)^{n(d-1)+2}\big),\ \text{where } n > \Big|\frac{1}{p} - \frac{1}{2}\Big|.$$

(vi) Let us recall that a Banach space E has *Fourier type* $p \in [1,2]$ iff the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R} : E)$ to $L^q(\mathbb{R} : E)$, where $1/p + 1/q = 1$. Each Banach space E has Fourier type 1, and E^* has the same Fourier type as E. A space of the form $L^p(\Omega, \mu)$ has Fourier type $\min(p, \frac{p}{p-1})$ and there exist examples of non-reflexive Banach spaces which do have non-trivial Fourier type. As mentioned in Section 2.5, the assertions of Theorem 2.5.9 and Corollary 2.5.10 can be refined if E has non-trivial Fourier type, which will be indicated in the following fractional analogue of [**171**, Proposition 8.1]. Let $1 < p < \infty$, $1/p + 1/q = 1$, $k \in \mathbb{N}_0$, $0 < \beta \leqslant 2$ and $E := L^p(\mathbb{R})$. Define a closed linear operator $A_{\beta,k}$ on E by $D(A_{\beta,k}) := W^{4k+2,p}(\mathbb{R})$ and $A_{\beta,k} f := e^{i(2-\beta)\frac{\pi}{2}} f^{(4k+2)}$, $f \in D(A_{\beta,k})$. Put $Bf(x) := V(x) f^{(l)}(x)$, $x \in \mathbb{R}$ with maximal domain $D(B) := \{f \in E : V \cdot f^{(l)} \in E\}$; here $V(x)$ is a potential and $l \in \mathbb{N}_0$. Assume first that

$$V \in L^p(\mathbb{R}) \text{ and } l \leqslant \frac{1}{p}\Big((4k+2)p - 1 - \frac{(4k+2)(p-1)}{\beta}\Big). \tag{187}$$

Given $\operatorname{Re}\lambda > 0$, denote by $\mu_{j,\lambda}$ $(1 \leqslant j \leqslant 2k+1)$ $(2k+1)$ solutions of the equation $\mu_{j,\lambda}^{4k+2} = \lambda^\beta e^{i(\beta\frac{\pi}{2} - \pi)}$ with $\operatorname{Re}\mu_{j,\lambda} > 0$. Then $D(A) \subseteq D(B)$,

$$\big(R\big(\lambda^\beta : A_{\beta,k}\big) f\big)(x) = \frac{e^{i\beta\frac{\pi}{2}}}{4k+2} \int_{-\infty}^{\infty} \sum_{j=1}^{2k+1} \frac{e^{-\mu_{j,\lambda}|x-s|}}{(-\mu_{j,\lambda})^{4k+1}} f(s)\,ds,\ f \in E,\ x \in \mathbb{R},\ \operatorname{Re}\lambda > 0,$$

$$\big(BR\big(\lambda^{\beta} : A_{\beta,k}\big)f\big)(x) = \frac{e^{i\beta\frac{\pi}{2}}}{4k+2}V(x)\sum_{j=1}^{2k+1}\Bigg(\int_{-\infty}^{x}\frac{e^{-\mu_{j,\lambda}(x-s)}}{(-\mu_{j,\lambda})^{4k-l+1}}f(s)ds$$
$$-\int_{x}^{\infty}\frac{e^{\mu_{j,\lambda}(x-s)}}{\mu_{j,\lambda}^{4k-l+1}}f(s)ds\Bigg),\ f \in E,\ x \in \mathbb{R},\ \operatorname{Re}\lambda > 0,$$

(188) $$\big\|R\big(\lambda^{\beta} : A_{\beta,k}\big)\big\| \leqslant \Big(|\lambda|^{\beta(1-\frac{1}{4k+2})}\min\big(\operatorname{Re}\mu_{1,\lambda},\cdots,\operatorname{Re}\mu_{2k+1,\lambda}\big)\Big)^{-1},\ \operatorname{Re}\lambda > 0,$$

and

(189) $$\Big\|BR\big(\lambda^{\beta} : A_{\beta,k}\big)\big\| \leqslant ||V||_p\Big(|\lambda|^{\beta(1-\frac{l+1}{4k+2})}\min\big(\big(\operatorname{Re}\mu_{1,\lambda}\big)^{1/q},\cdots,\big(\operatorname{Re}\mu_{2k+1,\lambda}\big)^{1/q}\big)\Big)^{-1},$$

provided $\operatorname{Re}\lambda > 0$. Furthermore, $\operatorname{Re}\mu_{j,\lambda} = |\lambda|^{\frac{\beta}{4k+2}}\cos(\arg(\mu_{j,\lambda}))$, $\operatorname{Re}\lambda > 0$, $1 \leqslant j \leqslant 2k+1$ and

$$\min(\operatorname{Re}\mu_{1,\lambda},\cdots,\operatorname{Re}\mu_{2k+1,\lambda}) = |\lambda|^{\frac{\beta}{4k+2}}\min\Big(\cos\big(\frac{\arg(\lambda)\beta + (\beta\pi)/(2)}{4k+2} + \frac{(2k-1)\pi}{4k+2}\big),$$
$$-\cos\big(\frac{\arg(\lambda)\beta + (\beta\pi)/(2)}{4k+2} + \frac{\pi}{2}\big)\Big),\ \operatorname{Re}\lambda > 0,$$

which implies that there exists a constant $c_{\beta,k} > 0$ such that

(190) $$|\lambda|^{\frac{\beta}{4k+2}}\cos(\arg(\lambda))\Big/\min(\operatorname{Re}\mu_{1,\lambda},\cdots,\operatorname{Re}\mu_{2k+1,\lambda}) \leqslant c_{\beta,k},\ \operatorname{Re}\lambda > 0.$$

Keeping in mind (187)-(190), we obtain that

(191) $$\big\|R\big(\lambda^{\beta} : A_{\beta,k}\big)\big\| = O\big(|\lambda|^{1-\beta}(\operatorname{Re}\lambda)^{-1}\big),\ \operatorname{Re}\lambda > 0$$

and

(192) $$\big\|BR\big(\lambda^{\beta} : A_{\beta,k}\big)\big\| = O\big(||V||_p(\operatorname{Re}\lambda)^{-\beta(1-\frac{l+1}{4k+2}+\frac{1}{(4k+2)q})}\big) = O\big(||V||_p(\operatorname{Re}\lambda)^{(-1)/q}\big),$$

provided $\operatorname{Re}\lambda > 0$. Denote by β_k the infimum of all non-negative real numbers $r \geqslant 0$ such that the operator $A_{\beta,k}$ generates an exponentially bounded $(\frac{t^{\beta-1}}{\Gamma(\beta)}, \frac{t^r}{\Gamma(r+1)})$-regularized resolvent family. The precise computation of integration rate β_k is non-trivial problem (cf. also the representation formula [**36**, Example 3.7, (3.15)]). Clearly, (191) yields the imprecise estimate $\beta_k \leqslant 1$, and $\beta_k \leqslant |\frac{1}{2} - \frac{1}{p}|$ provided $\beta \in \{1, 2\}$ ([**147**], [**456**]). Set $\kappa_p := \min(\frac{1}{p}, \frac{p-1}{p})$. By [**242**, Theorem 3.2], $A_{\beta,k} + B$ generates an exponentially bounded $(\frac{t^{\beta-1}}{\Gamma(\beta)}, \frac{t^{\sigma_{\beta,k,p}}}{\Gamma(\sigma_{\beta,k,p}+1)})$-regularized resolvent family for any $\sigma_{\beta,k,p} > \beta_k + \kappa_p$. By (191)-(192) and the proof of [**171**, Proposition 8.1], the above remains true provided $(4k+2)p - 1 - \frac{(4k+2)(p-1)}{\beta} \geqslant 0$, $l = 0$ and $V \in L^p(\mathbb{R}) + L^{\infty}(\mathbb{R})$; similarly, one can consider the operators $A^1_{\beta,k}$ ($k \in \mathbb{N}$, $0 < \beta \leqslant 2$) and $A^2_{\beta,k}$ ($k \in \mathbb{N}$, $0 < \beta \leqslant 1$) given by $A^1_{\beta,k}f := e^{-i\beta\frac{\pi}{2}}f^{(4k)}$, $f \in W^{4k,p}(\mathbb{R}) := D(A^1_{\beta,k})$ and $A^2_{\beta,k}f := e^{\pm i\frac{\pi}{2}(1-\beta)}f^{(2k+1)}$, $f \in W^{2k+1,p}(\mathbb{R}) := D(A^2_{\beta,k})$.

In the following example, we use the standard multi-index notation.

EXAMPLE 2.8.6. Let $k \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \leqslant |\alpha| \leqslant k$, $a_\alpha \neq 0$ for some α with $|\alpha| = k$, $P(x) = \sum_{|\alpha| \leqslant k} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, $P(\cdot)$ is an elliptic polynomial, i.e., there exist $C > 0$ and $L > 0$ such that $|P(x)| \geqslant C|x|^k$, $|x| \geqslant L$, $\omega := \sup_{x \in \mathbb{R}^n} \operatorname{Re}(P(x)) < \infty$, E is one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$,

$$P(D) := \sum_{|\alpha| \leqslant k} a_\alpha f^{(\alpha)} \text{ and } D(P(D)) := \big\{ f \in E : P(D)f \in E \text{ distributionally} \big\}.$$

Put $n_E = n|\frac{1}{2} - \frac{1}{p}|$, if $E = L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$ and $n_E > \frac{n}{2}$, otherwise. Then the following holds:

(i) [**14**] The operator $P(D)$ generates an exponentially bounded r-times integrated semigroup in E for any $r > n_E$.
(ii) [**434**] The operator $P(D)$ generates an exponentially bounded n_E-times integrated semigroup in $L^p(\mathbb{R}^n)$ provided $p \in (1, \infty)$.
(iii) [**277**], [**233**] For every $\omega' \in (\omega, \infty)$ and $r > n|\frac{1}{2} - \frac{1}{p}|$, the operator $P(D)$ generates an exponentially bounded $(\omega' - P(D))^{-r}$-regularized semigroup in E.

It is noteworthy that the theory of C-regularized semigroups can be applied to non-elliptic differential operators. More precisely, one can prove that, in the case of a general polynomial $P(x)$ satisfying changebar $\omega := \sup_{x \in \mathbb{R}^n} \operatorname{Re}(P(x)) < \infty$, $P(D)$ is the integral generator of a global exponentially bounded $(1 - \Delta)^{-n_E k/2}$-regularized semigroup. If $\omega = \infty$, then the operator $P(D)$ generates an entire C-regularized group. The above estimates can be slightly improved if $P(x)$ is r-coercive for some $r \in (0, k]$ (that is, $|P(x)|^{-1} = O(|x|^{-r})$ as $|x| \to \infty$). For time-dependent PDE's, we recommend for the reader [**463**]. Finally, we would like to draw attention to the recent paper of Nagaoka [**332**] for the generation of fractionally integrated semigroups by superelliptic differential operators.

EXAMPLE 2.8.7. Let $p \in [1, \infty]$ and $n \in \mathbb{N}$. Then the following holds:

(i) [**156**] The Schrödinger operator $i\Delta_p$, considered with its maximal distributional domain, generates a C_0-semigroup (group) in $L^p(\mathbb{R}^n)$ iff $p = 2$.

(ii) [**147**]–[**148**] Let $p > 1$. Then the Schrödinger operator $i\Delta_p$ generates an exponentially bounded r-times integrated semigroup in $L^p(\mathbb{R}^n)$ iff $r \geqslant n|\frac{1}{2} - \frac{1}{p}|$. The Schrödinger operator $i\Delta_p$, where $p \in \{1, \infty\}$, generates an exponentially bounded r-times integrated semigroup in $L^p(\mathbb{R}^n)$ iff $r > \frac{n}{2}$.

(iii) [**119**] Let $1 \leqslant p < \infty$. Then the Schrödinger operator $i\Delta_p$, considered with the Dirichlet or Neumann boundary conditions, generates an exponentially bounded r-times integrated semigroup (group) in $L^p((-\pi, \pi)^n)$ for any $r > \frac{n}{2}|\frac{1}{2} - \frac{1}{p}|$, and moreover, $i\Delta_p$ does not generate an exponentially bounded r-times integrated semigroup (group) in $L^p((-\pi, \pi)^n)$ if $r < \frac{n}{2}|\frac{1}{2} - \frac{1}{p}|$; the previous assertion remains true in the case of Banach space $L^p(T^n)$, where T^n is the n-dimensional torus.

For further information concerning Schrödinger type evolution equations in L^p type spaces, we refer the reader to [**14**]–[**23**], [**89**], [**147**]–[**148**], [**187**]–[**189**], [**257**], [**335**], [**355**] and [**434**]–[**435**].

EXAMPLE 2.8.8. Let $p \in [1, \infty)$ and $n \in \mathbb{N}$. Then we have the following:

(i) [**294**] The Laplacian Δ_p, considered with its maximal distributional domain, generates a cosine function in $L^p(\mathbb{R}^n)$ iff $p = 2$ or $n = 1$.

(ii) [**147**], [**120**] The Laplacian Δ_p generates an exponentially bounded r-times integrated cosine function in $L^p(\mathbb{R}^n)$ iff $r \geqslant (n-1)|\frac{1}{2} - \frac{1}{p}|$.

(iii) [**460**] The Laplacian Δ_p generates a polynomially bounded $(1–\Delta_p)^{-\frac{n}{2}|\frac{1}{2}-\frac{1}{p}|}$-regularized cosine function in $L^p(\mathbb{R}^n)$ if $p \in (1, \infty)$, resp. $(1 - \Delta_p)^{-s}$-regularized cosine function $(s > \frac{n}{4})$ if $p = 1$.

(iv) [**195**] The Laplacian Δ_p with the Dirichlet or Neumann boundary conditions generates an exponentially bounded r-times integrated cosine function in $L^p((0, \pi)^n)$ for any $r \geqslant (n-1)|\frac{1}{2} - \frac{1}{p}|$.

Assume further that $P(x)$ is not necessarily elliptic polynomial of order k and that E is one of the spaces listed in Example 2.8.6. Set $h_{t,\beta}(x) := (1 + |x|^2)^{-\beta/2} \times \sum_{j=0}^{\infty} \frac{t^{2j} P(x)^j}{(2j)!}$, $x \in \mathbb{R}^n$, $t \geqslant 0$, $\beta \geqslant 0$, $\Omega(\omega) := \{\lambda^2 : \operatorname{Re} \lambda > \omega\}$, if $\omega > 0$ and $\Omega(\omega) := \mathbb{C} \smallsetminus (-\infty, \omega^2]$, if $\omega \leqslant 0$. Assume $r \in [0, k]$ and (H') holds with some $\omega \in \mathbb{R}$, where:

(H'): $P(x) \notin \Omega(\omega)$, $x \in \mathbb{R}^n$ and, in the case $r \in (0, k]$, there exist $\sigma > 0$ and $\sigma' > 0$ such that $\operatorname{Re}(P(x)) \leqslant -\sigma|x|^r + \sigma'$, $x \in \mathbb{R}^n$.

Then the proof of [**460**, Theorem 2.2] implies that there exists $M \geqslant 1$ such that, for every $\beta > (m - \frac{r}{2})\frac{n}{4}$, $P(D)$ generates an exponentially bounded $(1 - \Delta)^{-\beta}$-regularized cosine function $(C_\beta(t))_{t \geqslant 0}$ in E which satisfies $C_\beta(t)f = \mathcal{F}^{-1} h_{t,\beta} * f$, $t \geqslant 0$, $f \in E$ and $\|C_\beta(t)\| \leqslant M g_{n/2}(t)$, $t \geqslant 0$, where the function $g_{n/2}(t)$ is defined on [**460**, p. 40] and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The previous estimate can be additionally refined in the case that $E = L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ by allowing that β takes the value $\frac{1}{2}(m - \frac{r}{2})n|\frac{1}{p} - \frac{1}{2}|$.

EXAMPLE 2.8.9. [**37**], [**62**], [**225**] Suppose that $\omega : [0, \infty) \to [0, \infty)$ is a continuous, concave, increasing function satisfying $\lim_{t\to\infty} \omega(t) = \infty$, $\lim_{t\to\infty} \frac{\omega(t)}{t} = 0$ and $\int_1^\infty \frac{\omega(t)}{t^2}\, dt < \infty$. Given $x_0 \in (0, \infty)$, define

$$\Omega(\omega) := \big\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant \max\big(x_0, \omega(|\operatorname{Im} \lambda|)\big)\big\},$$

and assume further that A is a closed, linear operator which satisfies $\Omega(\omega) \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n e^{\omega(\sigma|\lambda|)}$, $\lambda \in \Omega(\omega)$, for some $M > 0$, $\sigma \geqslant 0$ and $n \in \mathbb{N}$. Then there exist $\tau > 0$ and an exponentially bounded, continuous kernel K such that A generates a local $K_{|[0,\tau)}$-convoluted semigroup $(S_K(t))_{t\in[0,\tau)}$; in the case $\sigma = 0$, one can prove that there exists a family of bounded injective operators $(C(k, \varepsilon))_{\varepsilon>0}$ such that, for every $\varepsilon > 0$, A is a subgenerator of a global $C(k, \varepsilon)$-regularized semigroup that is infinitely differentiable in $t > 0$.

CHAPTER 3

ABSTRACT CAUCHY PROBLEMS IN THE SPACES OF OPERATOR VALUED (ULTRA-)DISTRIBUTIONS AND HYPERFUNCTIONS

3.1. C-Distribution semigroups

3.1.1. Elementary properties of C-distribution semigroups. Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. If

(C.D.S.1) $$\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi),\ \varphi,\ \psi \in \mathcal{D},$$

then $\mathcal{G}$ is called a *pre*-(C-DS) and if additionally

(C.D.S.2) $$\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi \in \mathcal{D}_0} \operatorname{Kern}(\mathcal{G}(\varphi)) = \{0\},$$

then $\mathcal{G}$ is called a *C-distribution semigroup,* (C-DS) in short. If, moreover

(C.D.S.3) $$\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi \in \mathcal{D}_0} \mathrm{R}(\mathcal{G}(\varphi)) \text{ is dense in } E,$$

then $\mathcal{G}$ is called a *dense* (C-DS).

This definition, with $C = I$, was introduced in [**252**], where Kunstmann defined a distribution semigroup, (DS) in short. It is clear, if $\mathcal{G}$ is a pre-(C-DS), then $\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi)$, φ, $\psi \in \mathcal{D}$. Also, in this case, $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

Recall, the polars of nonempty sets $M \subseteq E$ and $N \subseteq E^*$ are defined as follows:

$$M^\circ = \{y \in E^* : |y(x)| \leqslant 1 \text{ for all } x \in M\},$$
$$N^\circ = \{x \in E : \ |y(x)| \leqslant 1 \text{ for all } y \in N\}.$$

Repeating literally the arguments given in [**252**], one can prove the following assertion describing the structural properties of a pre-(C-DS) on its kernel space.

PROPOSITION 3.1.1. *Let $\mathcal{G}$ be a pre-(C-DS). Then, with $N = \mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N, ($G_1 = \mathcal{G}_{|N}$) we have: There exists a unique operators $T_0,\ T_1, \ldots, T_m \in L(E)$ such that $G_1 = \sum_{j=1}^{m} \delta^{(j)} \otimes T_j$, $T_iC^i = (-1)^i T_0^{i+1}$, $i = 0, 1, \ldots, m-1$ and $T_0T_m = T_0^{m+2} = 0$.*

In the next proposition we present some analogues of results known for distribution semigroups (cf. [**252**]).

PROPOSITION 3.1.2. *Let $\mathcal{G}$ be a pre-(C-DS), $F := E/\mathcal{N}(\mathcal{G})$ and let q be the corresponding canonical mapping $q : E \to F$.*

(i) *Let $H \in L(\mathcal{D} : L(F))$ be defined by $q\mathcal{G}(\varphi) := H(\varphi)q$ for all $\varphi \in \mathcal{D}$ and let $\tilde{C}$ be a linear operator in F defined by $\tilde{C}q := qC$. Then $\tilde{C} \in L(F)$ and $\tilde{C}$ is injective. Moreover, H is a ($\tilde{C}$-DS) in F.*

(ii) *$C(\overline{\langle\mathcal{R}(\mathcal{G})\rangle}) \subseteq \overline{\mathcal{R}(\mathcal{G})}$, where $\langle\mathcal{R}(\mathcal{G})\rangle$ denotes the linear span of $\mathcal{R}(\mathcal{G})$.*

(iii) *Assume $\mathcal{G}$ is not dense and $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. Put $R := \overline{\mathcal{R}(\mathcal{G})}$ and $H := \mathcal{G}_{|R}$. Then H is a dense pre-(C_1-DS) on R with $C_1 = C_{|R}$.*

(iv) *Assume $\overline{\mathrm{R}(C)} = E$. Then the dual $\mathcal{G}(\cdot)^*$ is a pre-(C^*-DS) on E^* and $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.*

(v) *If E is reflexive and $\overline{\mathrm{R}(C)} = E$, then $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.*

(vi) *Assume $\overline{\mathrm{R}(C)} = E$. Then $\mathcal{G}^*$ is a (C^*-DS) in E^* iff $\mathcal{G}$ is a dense pre-(C-DS). If E is reflexive, then $\mathcal{G}^*$ is a dense pre-(C^*-DS) in E^* iff $\mathcal{G}$ is a (C-DS).*

PROOF. The proof will be given only for (i). First of all, notice that the definition of $\tilde{C}(q(x))$ does not depend on the representative of a class $q(x)$. As a matter of fact, the assumption $q(x) = q(y)$, i.e., $\mathcal{G}(\varphi)(x - y) = 0$, $\varphi \in \mathcal{D}_0$, and $C\mathcal{G} = \mathcal{G}C$, imply $\mathcal{G}(\varphi)(Cx - Cy) = 0$, $\varphi \in \mathcal{D}_0$, and $\tilde{C}(q(x)) = \tilde{C}(q(y))$. Now it is clear that $\tilde{C}$ is a linear operator in F. To prove that $\tilde{C}$ is continuous, suppose $x \in E$. Then $\|\tilde{C}(q(x))\| = \inf_{y\in\mathcal{N}(\mathcal{G})} \|Cx + y\|$. Let $y \in \mathcal{N}(\mathcal{G})$ be fixed. Applying again $C\mathcal{G} = \mathcal{G}C$, we have that $Cy \in \mathcal{N}(\mathcal{G})$. Thus, $\|\tilde{C}(q(x))\| \leqslant \|Cx+Cy\| \leqslant \|C\|\,\|x+y\|$; this implies $\|\tilde{C}(q(x))\| \leqslant \|C\|\,\|q(x)\|$, $\tilde{C} \in L(F)$ and $\|\tilde{C}\| \leqslant \|C\|$. Let $\tilde{C}(q(x)) = 0$. Then $Cx \in \mathcal{N}(\mathcal{G})$ and $C\mathcal{G}(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. Since C is an injective operator, one has $x \in \mathcal{N}(\mathcal{G})$ and $q(x) = 0$. Therefore, $\tilde{C} \in L(F)$ and $\tilde{C}$ is injective. One sees directly that H satisfies ($\tilde{C}$.D.S.1) and $\tilde{C}H = H\tilde{C}$. Suppose $H(\varphi)q(x) = 0$, $\varphi \in \mathcal{D}_0$, i.e., $\mathcal{G}(\varphi)x \in \mathcal{N}(\mathcal{G})$, $\varphi \in \mathcal{D}_0$. This implies $\mathcal{G}(\psi)\mathcal{G}(\varphi)x = 0$, $C\mathcal{G}(\varphi * \psi)x = 0$ and $\mathcal{G}(\varphi * \psi)x = 0$, φ, $\psi \in \mathcal{D}_0$. Choose a regularizing sequence (ρ_n) to obtain $\mathcal{G}(\varphi)x = \lim_{n\to\infty} \mathcal{G}(\varphi * \rho_n)x = 0$, $\varphi \in \mathcal{D}_0$ and $q(x) = 0$. $\square$

Let $\mathcal{G}$ be a (C-DS) and let $T \in \mathcal{E}_0'(\mathbb{C})$, i.e., T is a scalar-valued distribution with compact support in $[0, \infty)$. Define $G(T)$ on a subset of E by

$$y = G(T)x \text{ iff } \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \quad \text{for all} \quad \varphi \in \mathcal{D}_0.$$

Denote its domain by $D(G(T))$. By (C.D.S.2), $G(T)$ is a function. Moreover, $G(T)$ is a closed linear operator and $G(\delta) = I$. The *(infinitesimal) generator* of a (C-DS) $\mathcal{G}$ is defined by $A := G(-\delta')$. Since for $\psi \in \mathcal{D}$, $\psi_+ := \psi\mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'(\mathbb{C})$, ($\mathbf{1}_{[0,\infty)}$ stands for the characteristic function of $[0, \infty)$) the definition of $G(\psi_+)$ is clear. Further on, it is visible that C does not appear in the definition of $G(T)$. Someone may think that the notion of $G(T)$ is misleading without C. This is not the case; this just simplifies the definition of A. Namely, let us define the operator $G_C(T)$ ($T \in \mathcal{E}_0'(\mathbb{C})$) by $G_C(T) = \{(x, y) \in E \times E : \mathcal{G}(T * \varphi)Cx = \mathcal{G}(\varphi)y, \varphi \in \mathcal{D}_0\}$. It can be easily seen that $G_C(T)$ is a closed linear operator as well as that $G_C(\delta) = C$ and $G(T)C = G_C(T)$, $T \in \mathcal{E}_0'(\mathbb{C})$. Further on, if $\mathcal{G}$ is a (C-DS), $T \in \mathcal{E}_0'(\mathbb{C})$ and

$\varphi \in \mathcal{D}$, then $G(\varphi)G(T) \subseteq G(T)\mathcal{G}(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$. If $f : \mathbb{R} \to \mathbb{C}$, put $(\tau_t f)(s) := f(s-t)$, $s \in \mathbb{R}$, $t \in \mathbb{R}$. Note, if $\mathcal{G}$ is a pre-(C-DS) and φ, $\psi \in \mathcal{D}$, then the assumption $\varphi(t) = \psi(t)$, $t \geqslant 0$, implies $\mathcal{G}(\varphi) = \mathcal{G}(\psi)$. Indeed, put $\eta = \varphi - \psi$. Then $\eta \in \mathcal{D}_{(-\infty,0]}$ and the continuity of $\mathcal{G}$ implies $\lim_{h\to 0-} \mathcal{G}(\tau_h \eta)x = \mathcal{G}(\eta)x = 0$, $x \in E$. Now we state:

PROPOSITION 3.1.3. *If $\mathcal{G}$ is a (C-DS), then $G(\psi_+)C = \mathcal{G}(\psi)$, $\psi \in \mathcal{D}$.*

PROOF. Let $x \in E$ and $\psi \in \mathcal{D}$. Then $G(\psi_+)Cx = \mathcal{G}(\psi)x$ iff $\mathcal{G}(\psi_+ * \varphi)Cx = \mathcal{G}(\varphi)\mathcal{G}(\psi)x$ for all $\varphi \in \mathcal{D}_0$ iff $\mathcal{G}(\psi_+ * \varphi)Cx = \mathcal{G}(\varphi *_0 \psi)Cx$ for all $\varphi \in \mathcal{D}_0$. The last statement is true since, for every fixed $\varphi \in \mathcal{D}_0$, one has $\varphi *_0 \psi = \psi_+ * \varphi$. □

Using the same arguments as in [**252**, Lemma 3.6], one can prove the following.

PROPOSITION 3.1.4. *Let S, $T \in \mathcal{E}_0'$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x \in E$. Then the following holds:*

(i) $(\mathcal{G}(\varphi)x, \mathcal{G}(\overbrace{T * \cdots * T}^{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.
(ii) $G(S)G(T) \subseteq G(S*T)$ *with* $D(G(S)G(T)) = D(G(S*T)) \cap D(G(T))$, *and* $G(S) + G(T) \subseteq G(S+T)$.
(iii) $(\mathcal{G}(\psi)x, \mathcal{G}(-\psi')x - \psi(0)Cx) \in G(-\delta')$.
(iv) *If $\mathcal{G}$ is dense, then its generator is densely defined.*

EXAMPLE 3.1.5. (i) Let A be the infinitesimal generator of a C-regularized semigroup $(T(t))_{t\geqslant 0}$ and $\mathcal{G}(\varphi) := \int_0^\infty \varphi(t)T(t)\,dt$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a (C-DS) with the generator A.

PROOF. We will only prove that A is the generator of $\mathcal{G}$. The following is well known: $C^{-1}AC = A$, $T(t)C = CT(t)$, $T(t)A \subseteq AT(t)$, $t \geqslant 0$. Suppose now $(x, y) \in C^{-1}AC = A$. Then $A\int_0^t T(s)Cx\,ds = T(t)Cx - C^2x$ and $\int_0^t T(s)ACx\,ds = T(t)Cx - C^2x$, $t \geqslant 0$. Hence, $\int_0^t T(s)Cy\,ds = CT(t)x - C^2x$ and $\int_0^t T(s)y\,ds = T(t)x - Cx$, $t \geqslant 0$. We have to prove $-\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi(t)T(t)y\,dt$ for all $\varphi \in \mathcal{D}_0$. This follows from

$$\int_0^\infty \varphi(t)T(t)y\,dt = -\int_0^\infty \varphi'(t)\int_0^t T(s)y\,ds\,dt = -\int_0^\infty \varphi'(t)(T(t)x - Cx)\,dt$$
$$= -\int_0^\infty \varphi'(t)T(t)x\,dt.$$

Consequently, $(x, y) \in B$, where B is the generator of $\mathcal{G}$. Suppose $(x, y) \in B$. Then $-\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi(t)T(t)y\,dt$ and $\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi'(t)\int_0^t T(s)y\,ds\,dt$, $\varphi \in \mathcal{D}_0$. Thus, $T(t)x - \int_0^t T(s)yds = \text{Const.}$ and $\int_0^t T(s)y\,ds = T(t)x - Cx$, $t \geqslant 0$. Hence, $A\int_0^t T(s)x\,ds = \int_0^t T(s)y\,ds$ for all $t \geqslant 0$. Since A is closed, we obtain $T(t)x \in D(A)$ and $AT(t)x = T(t)y$ for all $t \geqslant 0$. Accordingly, $(x, y) \in C^{-1}AC = A$. □

(ii) If $\mathcal{G}$ is a (DS) with the generator A and $\mathcal{G}C = C\mathcal{G}$, then $\mathcal{G}C$ is a (C-DS) with the generator A.

(iii) [**185**] Let P be a bounded projector on E with $PC = CP$. Define $\mathcal{G}(\varphi) := \int_0^\infty \varphi(t)\,dtPC$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-(C-DS) and $\mathcal{N}(\mathcal{G}) = \mathrm{Kern}(P)$.

(iv) [**252**, Example 2.8] Let $m \in \mathbb{N}$ and let H^m denote the completion of $C^m[0,1]$ with respect to the norm $f \mapsto \|f\|_{H^m} := \sum_{\alpha \leqslant m} \|f^\alpha\|_{L^2}$. Then H^m is a separable Hilbert space and the next expression $G(\varphi)f := (\varphi *_0 f)\mathbf{1}_{[0,1]}$, $\varphi \in \mathcal{D}$, $f \in H^m$, defines a non-dense (DS) in H^m.

3.1.2. Connections with integrated C-semigroups. Exponential C-distribution semigroups. Let remind us that the abstract Cauchy problem

$$(C_{n+1}(\tau)) : \begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^1([0,\tau) : E), \\ u'(t) = Au(t) + \frac{t^n}{n!}Cx,\ t \in [0,\tau), \\ u(0) = 0, \end{cases}$$

is C-well posed if for any $x \in E$ there exists a unique solution of $C_{n+1}(\tau)$. In this subsection, we investigate relations between C-distribution semigroups and the corresponding $C_{n+1}(\tau)$ problems with (local) integrated C-semigroups.

LEMMA 3.1.6. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then $C^{-1}AC = A$.*

PROOF. Let $(x,y) \in A$. Then $\mathcal{G}(-\varphi')x = \mathcal{G}(\varphi)y$, $C\mathcal{G}(-\varphi')x = C\mathcal{G}(\varphi)y$ and $\mathcal{G}(-\varphi')Cx = \mathcal{G}(\varphi)Cy$, $\varphi \in \mathcal{D}_0$. So, $(Cx, Cy) \in A$ and $A \subseteq C^{-1}AC$. Assume $(x,y) \in C^{-1}AC$. Then $ACx = Cy$ and $\mathcal{G}(-\varphi')Cx = \mathcal{G}(\varphi)Cy$, $\varphi \in \mathcal{D}_0$. Since $C\mathcal{G} = \mathcal{G}C$ and C is injective, one has $\mathcal{G}(-\varphi')x = \mathcal{G}(\varphi)y$, $\varphi \in \mathcal{D}_0$, $(x,y) \in A$ and $C^{-1}AC = A$. ☐

THEOREM 3.1.7. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then, for every $\tau > 0$, there exist $n_\tau \in \mathbb{N}$ and a non-degenerate operator family $(W(t))_{t\in[0,\tau)}$ such that:*

(i) $A\int_0^t W(s)x\,ds = W(t)x - \frac{t^{n_\tau}}{n_\tau!}Cx$, $t \in [0,\tau)$, $x \in E$,
(ii) $CA \subseteq AC$, $W(t)A \subseteq AW(t)$, $CW(t) = W(t)C$, $t \in [0,\tau)$ *and*
(iii) $(W(t))_{t\in[0,\tau)}$ *is a local n_τ-times integrated C-semigroup generated by A.*

PROOF. It is clear that $A\mathcal{G}(\varphi)x = -\mathcal{G}(\varphi')x - \varphi(0)Cx$, $\varphi \in \mathcal{D}$, $x \in E$. This implies that $\mathcal{G}$ is a continuous linear mapping from $\mathcal{D}$ into $L(E, [D(A)])$. By Theorem 1.3.2, one obtains that, for every $\tau > 0$, there exist $n_\tau \in \mathbb{N}$ and $W \in C([-\tau,\tau] : L(E,[D(A)]))$ such that $\mathcal{G}(\varphi)x = (-1)^{n_\tau}\int_{-\tau}^{\tau} \varphi^{(n_\tau)}(t)W(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$. Moreover, $\operatorname{supp} W \subseteq [0,\tau]$,

$$\begin{aligned}(-1)^{n_\tau}\int_0^\tau \varphi^{(n_\tau)}(t)AW(t)x\,dt &= A\mathcal{G}(\varphi)x = \mathcal{G}(-\varphi')x - \varphi(0)Cx \\ &= (-1)^{n_\tau+1}\int_0^\tau \varphi^{(n_\tau+1)}(t)W(t)x\,dt - \varphi(0)Cx\end{aligned}$$

and, for every $\varphi \in \mathcal{D}_{[0,\tau)}$ and $x \in E$, $\int_0^\tau \varphi^{(n_\tau+1)}(t)\big[\int_0^t AW(s)x\,ds - W(t)x\big]dt = 0$. This implies $\int_0^t AW(s)x\,ds - W(t)x = \sum_{j=0}^{n_\tau} t^j B_j x$, $t \in [0,\tau)$, for some operators

$B_j \in L(E)$, $j = 0, 1, \ldots, n_\tau$. By the proof of [**418**, Theorem 3.8], we get $B_j = 0$ for $j = 0, 1, \ldots, n_\tau - 1$ and $B_{n\tau} = -\frac{t^{n_\tau}}{n_\tau!}C$, which implies

$$\int_0^t AW(s)x\,ds = W(t)x - \frac{t^{n_\tau}}{n_\tau!}Cx, \ \ t \in [0,\tau), \ x \in E.$$

Since $C\mathcal{G} = \mathcal{G}C$, $CA \subseteq AC$ and $\mathcal{G}(\varphi)A \subseteq A\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$, the remaining part of (ii) can be obtained along the same lines. Then the assertion (iii) immediately follows. $\square$

Remark 3.1.8. Notice that Theorem 3.1.7 generalizes [**216**, Theorem 4.2] and implies that every (C-DS) is uniquely determined by its generator.

Theorem 3.1.9. *Suppose that there exists a sequence $\langle(p_k, \tau_k)\rangle$ ($p_k \in \mathbb{N}_0$, $\tau_k \in (0,\infty)$; $k \in \mathbb{N}_0$) such that $\lim_{k\to\infty} \tau_k = \infty$ and that $C_{p_k+1}(\tau_k)$ is C-well posed for A. If $CA \subseteq AC$, then $C^{-1}AC$ generates a (C-DS).*

Proof. Clearly, we may assume $\tau_k < \tau_{k+1}$, and $p_k \geqslant 2$, $k \in \mathbb{N}_0$. Let $(W_{p_k}(t))_{t\in[0,\tau_k)}$ be the local p_k-times integrated C-regularized semigroup generated by $C^{-1}AC$; here $W_{p_k}(\cdot)$ is given by [**275**, Theorem 2.5]. Because every local integrated C-semigroup is uniquely determined by its generator (cf. also [**259**, Proposition 1.3]), the following definition is independent of $k \in \mathbb{N}_0$. Let $\varphi \in \mathcal{D}_{(-\infty,\tau_k)}$ and $\mathcal{G}(\varphi)x := (-1)^{p_k}\int_0^\infty \varphi^{(p_k)}(t)W_{p_k}(t)x\,dt$, $x \in E$. Then $\mathcal{G} \in \mathcal{D}'_0(L(E))$ and $\mathcal{G}C = C\mathcal{G}$. Furthermore, for every $x \in E$ and φ, $\psi \in \mathcal{D}_{(-\infty,\tau_k)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\infty, \tau_k)$,

$$\begin{aligned}
\mathcal{G}(\varphi)\mathcal{G}(\psi)x &= \int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi^{(p_k)}(s)W_{p_k}(t)W_{p_k}(s)x\,ds\,dt \\
&= \int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi^{(p_k)}(s)\left[\left(\int_t^{t+s} - \int_0^s\right)\frac{(t+s-r)^{p_k-1}}{(p_k-1)!}W_{p_k}(r)Cx\,dr\right]ds\,dt \\
&= -\int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi^{(p_k-1)}(s)\frac{d}{ds}\left[\left(\int_t^{t+s} - \int_0^s\right)\frac{(t+s-r)^{p_k-1}}{(p_k-1)!}W_{p_k}(r)Cx\,dr\right]ds\,dt \\
&= -\int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi^{(p_k-1)}(s) \\
&\quad\times\left[\left(\int_t^{t+s} - \int_0^s\right)\frac{(t+s-r)^{p_k-2}}{(p_k-2)!}W_{p_k}(r)Cx\,dr - \frac{t^{p_k-1}}{(p_k-1)!}W_{p_k}(s)Cx\right]ds\,dt \\
&= -\int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi^{(p_k-1)}(s)\left[\left(\int_t^{t+s} - \int_0^s\right)\frac{(t+s-r)^{p_k-2}}{(p_k-2)!}W_{p_k}(r)Cx\,dr\right]ds\,dt
\end{aligned}$$

$$+(-1)^{p_k}\varphi(0)\int_0^\infty \psi^{(p_k-1)}(s)W_{p_k}(s)Cx\,ds.$$

Applying the same argument sufficiently many times, we obtain:

$$\begin{aligned}\mathcal{G}(\varphi)\mathcal{G}(\psi)x &= (-1)^{p_k}\Biggl(\int_0^\infty \varphi^{(p_k)}(t)\int_0^\infty \psi(s)W_{p_k}(t+s)Cx\,ds\,dt\\ &\qquad + \sum_{j=0}^{p_k-1}\varphi^{(j)}(0)\int_0^\infty \psi^{(p_k-1-j)}(s)W_{p_k}(s)Cxds\Biggr)\\ &= (-1)^{p_k}\Biggl(\int_0^\infty \varphi^{(p_k)}(t)\int_t^\infty \psi(t-s)W_{p_k}(s)Cx\,ds\,dt\\ &\qquad + \sum_{j=0}^{p_k-1}\varphi^{(j)}(0)\int_0^\infty \psi^{(p_k-1-j)}(s)W_{p_k}(s)Cx\,ds\Biggr)\\ &= (-1)^{p_k}\int_0^\infty\Biggl[(\varphi^{(p_k)}*_0\psi)(s)+\sum_{j=0}^{p_k-1}\varphi^{(j)}(0)\psi^{(p_k-1-j)}(s)\Biggr]W_{p_k}(s)Cx\,ds\\ &= (-1)^{p_k}\int_0^\infty (\varphi*_0\psi)^{(p_k)}(s)W_{p_k}(s)Cx\,ds=\mathcal{G}(\varphi*_0\psi)Cx,\ x\in E.\end{aligned}$$

So (C.D.S.1) holds. Suppose $x\in E$ satisfies $\mathcal{G}(\varphi)x=0$, $\varphi\in\mathcal{D}_{[0,\tau_k]}$, for some $k\in\mathbb{N}$. Then we obtain $W_{p_k}(t)x=\sum_{j=0}^{p_k-1}t^jz_j$, $t\in[0,\tau_k)$, for some $z_j\in E$, $j=0,1,\ldots,p_k-1$. Using the closedness of A and the functional equality $A\int_0^t W_{p_k}(s)x\,ds=W_{p_k}(t)x-\frac{t^{p_k}}{p_k!}Cx$, $t\in[0,\tau_k)$, we easily get $z_j=0$, $j=0,1,\ldots,p_k-1$. Hence, $x=0$ and (C.D.S.2) holds.

Let us prove that $C^{-1}AC$ is the generator of $\mathcal{G}$. Suppose $(x,y)\in C^{-1}AC$ and $\varphi\in\mathcal{D}_{[0,\tau_k)}$ for some $k\in\mathbb{N}$. Then

$$\begin{aligned}\mathcal{G}(-\varphi')x &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)W_{p_k}(t)x\,dt\\ &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\Biggl(\frac{t^{p_k}}{p_k!}Cx+\int_0^t W_{p_k}(s)y\,ds\Biggr)dt\\ &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\int_0^t W_{p_k}(s)y\,ds\,dt=\mathcal{G}(\varphi)y,\end{aligned}$$

and $C^{-1}AC\subseteq B$, where B is the generator of $\mathcal{G}$. Assume now $(x,y)\in B$. Then

$$(-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)W_{p_k}(t)x\,dt = (-1)^{p_k}\int_0^\infty \varphi^{(p_k)}(t)W_{p_k}(t)y\,dt$$
$$= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\int_0^t W_{p_k}(s)y\,ds\,dt, \quad \varphi \in \mathcal{D}_{[0,\tau_k)}.$$

Thereby,

$$W_{p_k}(t)x - \int_0^t W_{p_k}(s)y\,ds = \sum_{j=0}^{p_k} t^j z_j, \quad t \in [0,\tau_k), \tag{193}$$

for some $z_j \in E$, $j = 0, 1, \ldots, p_k$. We can take $t = 0$ to obtain $z_0 = 0$. Using (193), we have $\frac{d}{dt}W_{p_k}(t)x - W_{p_k}(t)y = \sum_{j=1}^{p_k} jt^{j-1}z_j$, and

$$AW_{p_k}(t)x + \frac{t^{p_k-1}}{(p_k-1)!}Cx - A\int_0^t W_{p_k}(s)y\,ds - \frac{t^{p_k}}{p_k!}Cy = \sum_{j=1}^{p_k} jt^{j-1}z_j, \ t \in [0,\tau_k).$$

Hence,

$$A\sum_{j=1}^{p_k} t^j z_j = \sum_{j=1}^{p_k} jt^{j-1}z_j - \frac{t^{p_k-1}}{(p_k-1)!}Cx + \frac{t^{p_k}}{p_k!}Cy, \quad t \in [0,\tau_k). \tag{194}$$

Since A is closed, one can differentiate both sides of (194) sufficiently many times to get $z_j = 0$, $j = 1, 2, \ldots, p_k - 1$ and $z_{p_k} = \frac{Cx}{p_k!}$. This implies

$$W_{p_k}(t)x - \int_0^t W_{p_k}(s)y\,ds = \frac{t^{p_k}}{p_k!}Cx, \quad t \in [0,\tau_k),$$

and $(x, y) \in C^{-1}AC$. The proof is completed. □

REMARK 3.1.10. If $C = I$, then the well-posedness of $C_{k+1}(\tau)$ for some $k \in \mathbb{N}$ and $\tau > 0$ implies that A generates a (DS) (see [**252**]). This fact follows directly from Theorem 3.1.9 and an additional observation that the well-posedness of $C_{k+1}(\tau)$ implies the well-posedness of $C_{2k+1}(2\tau)$ (cf. [**5**, Theorem 4.1] and Subsection 2.1.1). Due to [**275**, Theorem 4.1], the C-wellposedness of $C_{k+1}(\tau)$ implies the C^2-wellposedness of $C_{2k+1}(2\tau)$. Finally, combining Theorem 3.1.7 and Theorem 3.1.9, we obtain that a closed linear operator A generates a (C-DS) iff for every $\tau > 0$ there exists $n_\tau \in \mathbb{N}$ such that A is the integral generator of a local n_τ-times integrated C-semigroup on $[0, \tau)$.

The following proposition is a consequence of the above considerations.

PROPOSITION 3.1.11. (i) *Let $\mathcal{G}$ be a (C-DS) generated by A. Then, for every $\tau > 0$, there exist $n_\tau \in \mathbb{N}$ and a local n_τ-times integrated C-semigroup* $(W_{n_\tau}(t))_{t\in[0,\tau)}$

generated by A such that

$$\mathcal{G}(\varphi)x=(-1)^{n_\tau}\int\limits_0^\infty \varphi^{(n_\tau)}(t)W_{n_\tau}(t)x\,dt,\ \ \varphi\in\mathcal{D}_{(-\infty,\tau)},\ x\in E.$$

(ii) *Let $n\in\mathbb{N}_0$ and let $(W_n(t))_{t\geqslant 0}$ be an n-times integrated C-semigroup generated by A. Put $\mathcal{G}(\varphi)x:=(-1)^n\int_0^\infty\varphi^{(n)}(t)W_n(t)x\,dt$, $\varphi\in\mathcal{D}$, $x\in E$. Then $\mathcal{G}$ is a (C-DS) generated by A.*

LEMMA 3.1.12. *Let $(S(t))_{t\in[0,\tau)}$ be an n-times integrated C-semigroup generated by A, $0<\tau\leqslant\infty$, $n\in\mathbb{N}$. If $x\in D(A^k)$ for some $k\in\mathbb{N}$ with $k\leqslant n$, then $\frac{d^k}{dt^k}S(t)x=S(t)A^kx+\sum_{i=0}^{k-1}\frac{t^{n-i-1}}{(n-i-1)!}CA^{k-i-1}x$, $t\in[0,\tau)$.*

If $\mathcal{G}\in\mathcal{D}'(L(E))$ and $\omega\in\mathbb{R}$, define $e^{-\omega t}\mathcal{G}$ by $e^{-\omega t}\mathcal{G}(\varphi):=\mathcal{G}(e^{-\omega\cdot}\varphi)$, $\varphi\in\mathcal{D}$. Clearly, $e^{-\omega t}\mathcal{G}\in\mathcal{D}'(L(E))$.

DEFINITION 3.1.13. A (C-DS) $\mathcal{G}$ is said to be an *exponential C-distribution semigroup* if there exists $\omega\in\mathbb{R}$ such that $e^{-\omega t}\mathcal{G}\in\mathcal{S}'(L(E))$.

In the sequel, if $\mathcal{G}\in\mathcal{D}'(E)$ and $\varphi\in\mathcal{D}$, then we also write $\langle\mathcal{G},\varphi\rangle$ for $\mathcal{G}(\varphi)$. Now we state the following important relationship between exponential C-distribution semigroups and exponentially bounded integrated C-semigroups.

THEOREM 3.1.14. *Let A be a closed linear operator. Then:*

(i) *A is the generator of an exponential C-distribution semigroup $\mathcal{G}$ iff*
(ii) *there exists $n\in\mathbb{N}$ such that A is the (integral) generator of an exponentially bounded n-times integrated C-semigroup $(W_n(t))_{t\geqslant 0}$.*

PROOF. (ii) $\Rightarrow$ (i). Let A be the generator of $(W_n(t))_{t\geqslant 0}$ and let $\|W_n(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$ for some $M>0$. Put $\mathcal{G}(\varphi)x:=(-1)^n\int_0^\infty\varphi^{(n)}(t)W_n(t)x\,dt$, $\varphi\in\mathcal{D}$, $x\in E$. By Proposition 3.1.11, $\mathcal{G}$ is a (C-DS) generated by A. For any $\varepsilon>0$ and $\varphi\in\mathcal{D}$, we have

$$\begin{aligned}\big\|\big\langle e^{-(\omega+\varepsilon)t}\mathcal{G},\ \varphi\big\rangle\big\| &\leqslant M\int\limits_0^\infty e^{\omega t}\big|\big(e^{-(\omega+\varepsilon)\cdot}\varphi\big)^{(n)}(t)\big|\,dt\\ &\leqslant M2^n\int\limits_0^\infty e^{\omega t}\sum_{i=0}^n|\omega+\varepsilon|^{n-i}e^{-(\omega+\varepsilon)t}|\varphi^{(i)}(t)|\,dt\\ &\leqslant M_1\int\limits_0^\infty e^{-\varepsilon t}\sum_{i=0}^n\big|\varphi^{(i)}(t)\big|\,dt\leqslant\frac{M_1}{\varepsilon}\sum_{i=0}^n p_{0,i}(\varphi),\end{aligned}$$

for a suitable constant M_1 independent of φ, where $p_{0,i}(\psi):=\sup_{x\in\mathbb{R}}|\psi^{(i)}(x)|$, $\psi\in\mathcal{S}$, is a continuous seminorm on $\mathcal{S}$. This implies $e^{-(\omega+\varepsilon)t}\mathcal{G}\in\mathcal{S}'(L(E))$ if $\varepsilon>0$.

(i) $\Rightarrow$ (ii). Suppose that $\mathcal{G}$ is C-distribution semigroup generated by A and $e^{-\omega t}$ $\mathcal{G}\in\mathcal{S}'(L(E))$. Clearly, $e^{-\omega t}\mathcal{G}$ is a (C-DS) generated by $A-\omega I$ and Lemma 3.1.6 implies $C^{-1}(A-\omega I)C=A-\omega I$. Then, for every $\varphi\in\mathcal{D}$, $A\langle e^{-\omega t}\mathcal{G},\varphi\rangle x=$

$\langle e^{-\omega t}\mathcal{G}, -\varphi'\rangle x + \omega\langle e^{-\omega t}\mathcal{G}, \varphi\rangle x - \varphi(0)Cx$, which gives $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E, [D(A)]))$. Now we may apply Theorem 1.3.2 to obtain that there exist $n \in \mathbb{N}$, $r > 0$ and a continuous function $V : \mathbb{R} \to L(E, [D(A)])$ supported by $[0, \infty)$ such that

$$\langle e^{-\omega t}\mathcal{G},\ \varphi\rangle x = (-1)^n \int_0^\infty V(t)\varphi^{(n)}(t)x\,dt,$$

for all $\varphi \in \mathcal{D}$, $x \in E$, and $|V(t)| \leqslant Mt^r$, $t \geqslant 0$. Since $e^{-\omega t}\mathcal{G}$ is a C-distribution semigroup generated by $A - \omega I$, arguing as in the proofs of the statements (i) and (ii) of Theorem 3.1.7, one can conclude that: $(A-\omega I)\int_0^t V(s)x\,ds = V(t)x - \frac{t^n}{n!}Cx$, $t \geqslant 0$, $x \in E$; $V(t)(A - \omega I) \subseteq (A - \omega I)V(t)$ and $CV(t) = V(t)C$, $t \geqslant 0$. Therefore, $(V(t))_{t\geqslant 0}$ is an exponentially bounded, n-times integrated C-semigroup generated by $C^{-1}(A - \omega I)C = A - \omega I$. Define $W_n(t) := e^{\omega t}V(t) + \int_0^t e^{\omega s}p_n(t - s)V(s)\,ds$, $t \geqslant 0$, where p_n is the polynomial of degree $(n - 1)$ such that

$$\sum_{i=1}^{n}\binom{n}{i}(-\omega)^i\lambda^{-i} = \int_0^\infty e^{-\lambda t}p_n(t)\,dt,\ \lambda > 0.$$

A standard perturbation argument shows that A is the generator of an exponentially bounded, n-times integrated C-semigroup $(W_n(t))_{t\geqslant 0}$. □

Remark 3.1.15. Recall, if A is the (integral) generator a (local) n-times integrated C-semigroup $(T_n(t))_{t\in[0,\tau)}$, $n \in \mathbb{N}_0$, then $C^{-1}AC = A$. Note also that we do not require $\overline{D(A)} = E$ in the previous theorem.

One can simply prove the following proposition.

Proposition 3.1.16. *Let A be a subgenerator of an n-times integrated C-cosine function $(C_n(t))_{t\geqslant 0}$, $n \in \mathbb{N}_0$. Then the operator $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$ generates a $\mathcal{C}$-distribution semigroup in $E \times E$.*

The verification of the following proposition is left to the reader.

Proposition 3.1.17. *Let A be a closed linear operator and let $\lambda \in \rho(A)$. Then the following assertions are equivalent.*

(i) *A is the generator of a (DS).*
(ii) *A is the generator of an $R(\lambda : A)^n$-distribution semigroup for all $n \in \mathbb{N}$.*
(iii) *There exists $n \in \mathbb{N}$ such that A is the generator of an $R(\lambda : A)^n$-distribution semigroup.*

Finally, we raise the issue:

Problem. Does any generator A of a *local* integrated C-semigroup generate a $(\tilde{C}-\mathrm{DS})$ for some $\tilde{C}$ which may be different from C?

3.1.3. Dense C-distribution semigroups. We will consider in this section some new conditions for $\mathcal{G} \in \mathcal{D}_0'(L(E))$:

(d_1): $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}_0$,

(d_2): the same as (C.D.S.2),

(d_3): $\mathcal{R}(\mathcal{G})$ is dense in E,

(d_4): for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0,\infty) : E)$ so that $u_x(0) = Cx$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt$, $\varphi \in \mathcal{D}$,

(d_5): if (d_2) holds then (d_5) means $G(\varphi_+)C = \mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$.

Proposition 3.1.18. *Suppose $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}C = C\mathcal{G}$. Then $\mathcal{G}$ is a (C-DS) iff (d_1), (d_2) and (d_5) hold.*

Proof. Keeping in mind Proposition 3.1.3, we only have to prove that the suppositions (d_1), (d_2) and (d_5) imply (C.D.S.1). In order to do that, let us notice that (d_1) and (d_2) imply that $G(T)$ commutes with $\mathcal{G}(\eta)$ and C for all $T \in \mathcal{E}_0'$ and $\eta \in \mathcal{D}_0$. By (d_1) and (d_2), we get $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$, $T \in \mathcal{E}_0'$ and $G(S)G(T) \subseteq G(S * T)$, $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, $T, S \in \mathcal{E}_0'$. Let φ, $\psi \in \mathcal{D}$ and $x \in E$ be fixed; then the property (C.D.S.1) follows from the next computation involving (d_5):

$$\mathcal{G}(\varphi *_0 \psi)Cx = \mathcal{G}(\varphi)\mathcal{G}(\psi)x$$
$$\Updownarrow$$
$$\mathcal{G}(\varphi *_0 \psi)Cx = G(\varphi_+)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$\forall \eta \in \mathcal{D}_0 : \mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi *_0 \psi)Cx$$
$$\Updownarrow$$
$$\forall \eta \in \mathcal{D}_0 : \mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx = \mathcal{G}(\eta)G((\varphi *_0 \psi)_+)C^2x$$
$$\Updownarrow$$
$$\forall \beta, \eta \in \mathcal{D}_0 : \mathcal{G}((\varphi *_0 \psi)_+ * \beta)\mathcal{G}(\eta)C^2x = \mathcal{G}(\beta)\mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$\forall \beta, \eta \in \mathcal{D}_0 : \mathcal{G}(\varphi_+ * (\beta * \psi_+))\mathcal{G}(\eta)C^2x = \mathcal{G}(\beta)\mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$\forall \beta, \eta \in \mathcal{D}_0 : \mathcal{G}(\beta * \psi_+)G(\varphi_+)\mathcal{G}(\eta)C^2x = \mathcal{G}(\beta)\mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$\forall \beta, \eta \in \mathcal{D}_0 : \mathcal{G}(\beta)G(\psi_+)G(\varphi_+)\mathcal{G}(\eta)C^2x = \mathcal{G}(\beta)\mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$\forall \eta \in \mathcal{D}_0 : G(\psi_+)G(\varphi_+)\mathcal{G}(\eta)C^2x = \mathcal{G}(\varphi_+ * \eta)CG(\psi_+)Cx$$
$$\Updownarrow$$
$$G(\varphi_+)CG(\psi_+)Cx = G(\psi_+)G(\varphi_+)C^2x$$
$$\Updownarrow$$

$$G(\varphi_+)G(\psi_+)C^2x = G(\psi_+)G(\varphi_+)C^2x$$
$$\Updownarrow$$
$$G((\varphi * \psi)_+)C^2x = G((\psi * \varphi)_+)C^2x. \qquad \square$$

PROPOSITION 3.1.19. *Suppose $\mathcal{G} \in \mathcal{D}'_0(L(E))$ satisfies (d_1), (d_2), (d_3), (d_4) and $\mathcal{G}C = C\mathcal{G}$. Then $\mathcal{G}$ is a (C-DS).*

PROOF. Owing to the previous proposition, the proof automatically follows if one shows that (d_5) holds. We will prove that $(Cx, \mathcal{G}(\varphi)x) \in G(\varphi_+)$, $x \in \mathcal{R}(\mathcal{G})$, $\varphi \in \mathcal{D}$. Suppose (ρ_n) is a regularizing sequence and u_x is a function appearing in the formulation of the property (d_4). Clearly, for every $\eta \in \mathcal{D}$,

$$\begin{aligned}\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)Cx &= \mathcal{G}((\varphi_+ * \rho_n) * \eta)C^2x = \mathcal{G}(\eta)C\mathcal{G}(\varphi_+ * \rho_n)x \\ &= \mathcal{G}(\eta)C\int_0^\infty (\varphi_+ * \rho_n)(t)u_x(t)\,dt \to \mathcal{G}(\eta)C\mathcal{G}(\varphi)x,\ n \to \infty, \\ \mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)Cx &= \mathcal{G}(\varphi_+ * \eta * \rho_n)C^2x \to \mathcal{G}(\varphi_+ * \eta)C^2x,\ n \to \infty.\end{aligned}$$

Hence, $\mathcal{G}(\varphi_+ * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi)x$, $\eta \in \mathcal{D}_0$, and the closedness of $G(\varphi_+)$ gives $(Cx, \mathcal{G}(\varphi)x) \in G(\varphi_+)$, $x \in E$, $\varphi \in \mathcal{D}$. The last equality implies (d_5) and completes the proof. $\square$

REMARK 3.1.20. [**252**] Suppose $\mathcal{G}$ is a (C-DS) generated by A. Following Lions [**282**], we introduce the operator A_0 as the set of all $(x, y) \in E \times E$ such that there exists a regularizing sequence (ρ_n) in $\mathcal{D}_0$ such that $\lim_{n\to\infty} \mathcal{G}(\rho_n)x = Cx$ and $\lim_{n\to\infty} \mathcal{G}(-\rho'_n)x = Cy$. Then it can be easily proved $(\mathcal{G}(\varphi)x, \mathcal{G}(-\varphi')x) \in A_0$, $\varphi \in \mathcal{D}_0$, $x \in E$ and that A_0 is a closable linear operator whose closure is contained in A. Furthermore, the denseness of $\mathcal{G}$ implies $\overline{A_0} = A$ and here it is worth noting that the last equality does not remain true in the general case of a non-dense (C-DS). Even in the case $C = I$, (d_1), (d_2) and (d_4) taken together do not imply (C.D.S.1); in this case, we also know that $\overline{A_0}$ coincides with the closure of the restriction of A to $D_\infty(A)$.

PROPOSITION 3.1.21. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then, for every $x \in D_\infty(A)$, there exists a unique function u_x satisfying:*

$$\begin{cases} u_x \in C^\infty([0,\infty) : E), \\ \mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt,\ \varphi \in \mathcal{D}_0, \\ u_x(0) = Cx. \end{cases}$$

PROOF. It suffices to show that, for every $\tau > 0$, there exists a unique function $u_{x,\tau} \in C^\infty([0,\tau] : E)$ so that $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_{x,\tau}(t)\,dt$, $\varphi \in \mathcal{D}_{[0,\tau)}$ and $u_{x,\tau}(0) = Cx$. To this end, assume $\mathcal{G}(\varphi)x = (-1)^{n_\tau}\int_0^\infty \varphi^{(n_\tau)}(t)W_{n_\tau}(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\infty,2\tau)}$, $x \in E$, for some $n_\tau \in \mathbb{N}$ and an n_τ-times integrated C-semigroup $(W_{n_\tau}(t))_{t\in[0,2\tau)}$

generated by A. Define $u_{x,\tau}(t) := \frac{d^{n_\tau}}{dt^{n_\tau}} W_{n_\tau}(t)x$, $0 \leqslant t \leqslant \tau$. By Lemma 3.1.12,

$$u_{x,\tau}(t) = W_{n_\tau}(t)A^{n_\tau}x + \sum_{i=0}^{n_\tau - 1} \frac{t^{n_\tau - i - 1}}{(n_\tau - i - 1)!} CA^{n_\tau - i - 1}x, \quad 0 \leqslant t \leqslant \tau.$$

Since $x \in D_\infty(A)$, one obtains $u_{x,\tau} \in C^\infty([0,\tau] : E)$. Moreover, $u_{x,\tau}(0) = Cx$ and the uniqueness of such a function $u_{x,\tau}$ follows from the next simple observation: let (ρ_n) be a regularization sequence in $\mathcal{D}$. Then $u_{x,\tau}(t) = \lim_{n\to\infty} \mathcal{G}(\rho_n(\cdot - t))x$, $t \in (0, \tau)$. ☐

REMARK 3.1.22. Let $\mathcal{G}$ and A be as above. Then $C(D_\infty(A)) \subseteq \overline{\mathcal{R}(\mathcal{G})}$. Assume now $D(A)$ and $\mathrm{R}(C)$ are dense in E. Combining Proposition 2.2.7 and Remark 3.1.10, we have that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that the operator A, resp. $(C^*)^{-1}A^*C^*$, is the integral generator of a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$, resp. a local $(n_\tau + 1)$-times integrated C^*-semigroup $\bigl(\int_0^t S_{n_\tau}(s)^* ds\bigr)_{t\in[0,\tau)}$. Therefore, $(C^*)^{-1}A^*C^*$ is the generator of a (C^*-DS) $\mathcal{G}^*$ in E^*.

PROPOSITION 3.1.23. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then $C(D_\infty(A)) \subseteq \overline{\mathcal{R}(\mathcal{G})}$. Assume additionally $\overline{\mathrm{R}(C)} = E$. Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is dense.* (ii) *A is densely defined.* (iii) *$\mathcal{G}^*$ is a (C^*-DS) in E^*.*

PROPOSITION 3.1.24. *Let $\mathcal{G}$ be a (C-DS). Then $\mathcal{G}$ satisfies* (d_4).

PROOF. Let $x = \mathcal{G}(\psi)y$, $\psi \in \mathcal{D}_0$, $y \in E$. Then the continuity of $\mathcal{G}$ on $\mathcal{D}$ implies $\mathcal{G}(\varphi)x = \mathcal{G}(\varphi)\mathcal{G}(\psi)y = \mathcal{G}(\varphi *_0 \psi)Cy = \mathcal{G}\bigl(\int_0^\infty \varphi(t)\tau_t\psi\, dt\bigr)Cy = \int_0^\infty \varphi(t)\mathcal{G}(\tau_t\psi)Cy\, dt$, $\varphi \in \mathcal{D}$. The function $u_x : t \to \mathcal{G}(\tau_t\psi)Cy$, $t \geqslant 0$ has the desired properties. ☐

In the remainder of this subsection, we also consider non-dense C-distribution semigroups. First of all, we state the following important characterization of distribution semigroups.

THEOREM 3.1.25. [**252**], [**418**] *A closed linear operator A is the generator of a distribution semigroup iff there exist $a > 0$, $b > 0$, $M > 0$ and $n \in \mathbb{N}$ such that*

$$E(a,b) \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n$$

iff there exist $\tau \in (0, \infty]$ and $n \in \mathbb{N}$ such that A generates a (local) n-times integrated semigroup on $[0, \tau)$.

Suppose that A is a closed linear operator. Then, for each $n \in \mathbb{N}$, the space $D(A^n)$ equipped with the norm $\|x\|_n := \sum_{i=0}^n \|A^i x\|$, $x \in D(A^n)$ is complete and the projective limit of the Banach spaces $(D(A^n), \|\cdot\|_n)$, i.e., the space $D_\infty(A)$, equipped with the family of norms $(\|\cdot\|_n)$, is a Fréchet space. The restriction of the operator A to $D_\infty(A)$ is clearly a continuous linear mapping in $D_\infty(A)$. The reader may consult [**18**], [**58**], [**107**], [**212**]–[**213**], [**250**], [**327**], [**410**]–[**411**] and [**434**] for the basic theory of semigroups of operators in locally convex spaces.

THEOREM 3.1.26. [**249**] (i) *A closed linear operator A is the generator of a distribution semigroup iff A is stationary dense, $\rho(A) \neq \emptyset$ and the restriction of the operator A to $D_\infty(A)$ generates a strongly continuous semigroup in $D_\infty(A)$.*

(ii) *A closed linear operator A is the generator of an exponential distribution semigroup iff A is stationary dense, $\rho(A) \neq \emptyset$ and the restriction of the operator A to $D_\infty(A)$ generates a quasi-equicontinuous semigroup in $D_\infty(A)$.*

THEOREM 3.1.27. [**234**] *Let $a > 0$, $b > 0$, $\alpha > 0$, $M > 0$, $E(a,b) \subseteq \rho_C(A)$, the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in E(a,b)$ is continuous, $CA \subseteq AC$ and $\|(\lambda - A)^{-1}C\| \leqslant M(1+|\lambda|)^\alpha$, $\lambda \in E(a,b)$. Set*

$$\tilde{\varphi}(\lambda) := \int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)\,dt, \quad \varphi \in \mathcal{D},$$

$$\mathcal{G}(\varphi)x := \frac{1}{2\pi i}\int_\Gamma \tilde{\varphi}(\lambda)(\lambda - A)^{-1}Cx\,d\lambda, \quad x \in E, \ \varphi \in \mathcal{D},$$

where Γ is the upwards oriented boundary of $E(a,b)$. Then $\mathcal{G}$ is a (C-DS) generated by $C^{-1}AC$.

PROOF. By Proposition 2.4.6, we may assume that the mapping $\lambda \mapsto (\lambda - A)^{-1}C$ is analytic on some open neighborhood of the region $E(a,b)$. Using the Paley–Wiener theorem, the Cauchy formula, the inverse Fourier transform as well as the simple equalities $A(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}Cx - Cx$, $\lambda \in \rho_C(A)$, $x \in E$ and $\widetilde{\varphi'}(\lambda) = -\lambda\tilde{\varphi}(\lambda)$, $\lambda \in \mathbb{C}$, it follows that: $\operatorname{supp}\mathcal{G} \subseteq [0,\infty)$, $\frac{1}{2\pi i}\int_\Gamma \tilde{\varphi}(\lambda)\,d\lambda = \varphi(0)$, $\varphi \in \mathcal{D}$ and

$$A\mathcal{G}(\varphi)x = \mathcal{G}(-\varphi')x - \varphi(0)Cx, \ \varphi \in \mathcal{D}, \ x \in E. \tag{195}$$

Let $\psi \in \mathcal{D}$ and $x \in E$ be fixed [**252**]. Put $P := \delta' \otimes I - \delta \otimes A$, $U := \mathcal{G}(\cdot)\mathcal{G}(\psi)x$, $V := \mathcal{G}(\cdot *_0 \psi)Cx$ and consider $\mathcal{G}$ as an element of the space $\mathcal{D}_0'(L(E,[D(A)]))$; clearly, $P \in \mathcal{D}_0'(L([D(A)],E))$, $U \in \mathcal{D}_0'([D(A)])$ and $V \in \mathcal{D}_0'([D(A)])$. Since $\mathcal{G}(\varphi)A \subseteq A\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ and (195) holds, we have $G * P = \delta \otimes C_{[D(A)]} \in \mathcal{D}_0'(L([D(A)]))$, $P * G = \delta \otimes C \in \mathcal{D}_0'(L(E))$ and $P * U = P * V = \delta \otimes G(\psi)Cx \in \mathcal{D}_0'(E)$. The associativity of convolution implies $(G * P) * U = (G * P) * V$, i.e., $(\delta \otimes C_{[D(A)]}) * U = (\delta \otimes C_{[D(A)]}) * V \in \mathcal{D}_0'([D(A)])$ and $C\mathcal{G}(\varphi)\mathcal{G}(\psi)x = C\mathcal{G}(\varphi *_0 \psi)Cx$, $\varphi \in \mathcal{D}$. The injectiveness of C implies (C.D.S.1). The proof of (C.D.S.2) follows as in [**282**] and [**252**]. In fact, the preassumption $\mathcal{G}(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$ implies that $\operatorname{supp}\mathcal{G}(\cdot)x \subseteq \{0\}$ and that there exist $k \in \mathbb{N}$ and $y_0, \cdots, y_k \in D(A)$ such that $\mathcal{G}(\cdot)x = \sum_{j=0}^k \delta^{(j)} \otimes y_j$ and that $\sum_{j=0}^k \delta^{(j+1)} \otimes y_j - \sum_{j=0}^k \delta^{(j)} \otimes Ay_j = \delta \otimes Cx$. This implies $y_k = \cdots = y_0 = Cx = x = 0$ and (C.D.S.2). Hence, $\mathcal{G}$ is a (C-DS) whose generator, denoted by $\tilde{A}$, satisfies $C^{-1}AC \subseteq \tilde{A}$. By Theorem 3.1.7, there exist $n_1 \in \mathbb{N}$ and $\tau_1 \in (0,\infty)$ such that $\tilde{A}$ is the integral generator of a local n_1-times integrated C-semigroup $(S_{n_1}(t))_{t\in[0,\tau_1)}$. Furthermore, Theorem 2.7.2 implies that there exist $n_2 \in \mathbb{N}$ and $\tau_2 \in (0,\infty)$ such that $C^{-1}AC$ is the integral generator of a local n_2-times integrated C-semigroup $(S_{n_2}(t))_{t\in[0,\tau_2)}$. Without loss of generality, we may assume $n = n_1 = n_2$ and $\tau = \tau_1 = \tau_2$. Set $S(t)x := S_n(t)x - S_n(t)x$, $t \in [0,\tau)$, $x \in E$. Then $\tilde{A}\int_0^t S(s)x\,ds = S(t)x$, $t \in [0,\tau)$, $x \in E$ and arguing as in the proof of [**328**, Proposition 2.6], one gets $S(t)=0$, $t\in[0,\tau)$. This implies $\tilde{A}=C^{-1}AC$ and completes the proof of theorem. $\square$

The *solution space* for a closed, linear operator A, denoted by $Z(A)$, is defined to be the set of all $x\in E$ for which there exists a continuous mapping $u(\cdot,x)\in C([0,\infty):E)$ satisfying $\int_0^t u(s,x)\,ds\in D(A)$ and $A\int_0^t u(s,x)\,ds=u(t,x)-x$, $t\geqslant 0$.

PROPOSITION 3.1.28. [**238**] (i) *Assume A generates a (C-DS) $\mathcal{G}$. Denote by $D(\mathcal{G})$ the set of all $x\in\bigcap_{t\geqslant 0}D(G(\delta_t))$ satisfying that the mapping $t\mapsto G(\delta_t)x$, $t\geqslant 0$ is continuous; here $\delta_t(\varphi)=\varphi(t)$, $t\in\mathbb{R}$, $\varphi\in\mathcal{D}$. Then $Z(A)=D(\mathcal{G})$. If $x\in Z(A)$, then $u(t,x)=G(\delta_t)x$, $t\geqslant 0$ and*

$$\mathcal{G}(\psi)x=\int_0^\infty \psi(t)Cu(t,x)\,dt=\int_0^\infty \psi(t)G_C(\delta_t)x\,dt,\quad \psi\in\mathcal{D}_0.$$

(ii) *Assume that, for every $\tau>0$, there exists $n_\tau\in\mathbb{N}$ such that A is a subgenerator of a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Then the solution space $Z(A)$ is the space which consists of all elements $x\in E$ such that, for every $\tau>0$, $S_{n_\tau}(t)x\in \mathrm{R}(C)$ and that the mapping $t\mapsto C^{-1}S_{n_\tau}(t)x$, $t\in[0,\tau)$ is n_τ-times continuously differentiable.*

3.1.4. Chaotic C-distribution semigroups. Chronologically, the first examples of hypercyclic operators were given on the space $\mathrm{H}(\mathbb{C})$ of entire functions equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. More precisely, Birkhoff proved in 1929 that the translation operator $f\mapsto f(\cdot+a)$, $f\in\mathrm{H}(\mathbb{C})$, $a\in\mathbb{C}\smallsetminus\{0\}$ is hypercyclic in $\mathrm{H}(\mathbb{C})$, and MacLane proved in 1952 the hypercyclicity of the derivative operator $f\mapsto f'$, $f\in\mathrm{H}(\mathbb{C})$. The first example of a hypercyclic operator on a Banach space was given by Rolewicz in 1969. The underlying Banach space in his analysis is chosen to be $l^2(\mathbb{N})$. The first systematic investigation into the hypercyclicity and chaos of strongly continuous semigroups was obtained by Desch, Schappacher and Webb [**109**] in 1997. The basic references concerning hypercyclic and chaotic behavior of distribution semigroups and strongly continuous semigroups are [**77**], [**106**], [**109**], [**173**] and [**303**]–[**304**]. The notion of hypercyclicity and chaos of distribution semigroups as well as unbounded semigroups of linear operators was introduced by deLaubenfels, Emamirad and Grosse-Erdmann in [**106**]. The main objective in this subsection is to enquire into the chaotic and hypercyclic properties of C-distribution semigroups and integrated C-semigroups.

We assume that E is a separable infinite-dimensional complex Banach space. Let S be a non-empty closed subset of $\mathbb{C}$ satisfying $\mathrm{S}\smallsetminus\{0\}\neq\emptyset$. A linear operator T on E is said to be *hypercyclic* if there exists an element $x\in D_\infty(T)$ whose orbit $\{T^n x : n\in\mathbb{N}_0\}$ is dense in E; T is said to be *topologically transitive,* resp. *topologically mixing*, if for every pair of open non-empty subsets U, V of E, there exist $x\in D_\infty(T)$ and $n\in\mathbb{N}$ such that $T^n(U)\cap V\neq\emptyset$, resp. if for every pair of open non-empty subsets U, V of E, there exists $n_0\in\mathbb{N}$ such that, for every $n\in\mathbb{N}$ with $n\geqslant n_0$, $T^n(U)\cap V\neq\emptyset$. A *periodic point* for T is an element $x\in$

$D_\infty(T)$ satisfying that there exists $n \in \mathbb{N}$ with $T^n x = x$. Finally, T is said to be *chaotic* if T is hypercyclic and the set of periodic points of T is dense in E. By Proposition 3.1.4(ii), we have

$$(196)\quad D(G(\delta_s)G(\delta_t)) = D(G(\delta_s * \delta_t)) \cap D(G(\delta_t)) = D(G(\delta_{t+s})) \cap D(G(\delta_t)),\ t,\ s \geqslant 0.$$

Then (196) implies $G(\delta_t)(D(\mathcal{G})) \subseteq D(\mathcal{G})$, $t \geqslant 0$. A closed linear subspace $\tilde{E}$ of E is said to be *$\mathcal{G}$-admissible* iff $G(\delta_t)(D(\mathcal{G}) \cap \tilde{E}) \subseteq D(\mathcal{G}) \cap \tilde{E}$, $t \geqslant 0$. Define $\mathbf{G}(\varphi)\binom{x}{y} := \binom{\mathcal{G}(\varphi)x}{\mathcal{G}(\varphi)y}$ and $\mathcal{C}\binom{x}{y} := \binom{Cx}{Cy}$, $x,\ y \in E$, $\varphi \in \mathcal{D}$. Then $\mathbf{G}$ is a ($\mathcal{C}$-DS) in $E \oplus E$, and $\tilde{E} \oplus \tilde{E}$ is $\mathbf{G}$-admissible provided that $\tilde{E}$ is $\mathcal{G}$-admissible.

Definition 3.1.29. Let $\mathcal{G}$ be a (C-DS) and let $\tilde{E}$ be $\mathcal{G}$-admissible. Then it is said that $\mathcal{G}$ is:

(i) *$\tilde{E}$-hypercyclic*, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that the set $\{G(\delta_t)x : t \geqslant 0\}$ is dense in $\tilde{E}$,
(ii) *$\tilde{E}$-chaotic*, if $\mathcal{G}$ is $\tilde{E}$-hypercyclic and the set of *$\tilde{E}$-periodic points* of $\mathcal{G}$, $\mathcal{G}_{\tilde{E},per}$, defined by $\{x \in D(\mathcal{G}) \cap \tilde{E} : G(\delta_{t_0})x = x \text{ for some } t_0 > 0\}$, is dense in $\tilde{E}$,
(iii) *$\tilde{E}$-topologically transitive*, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in D(\mathcal{G}) \cap \tilde{E}$ and $t \geqslant 0$ such that $\|y - v\| < \varepsilon$ and $\|z - G(\delta_t)v\| < \varepsilon$,
(iv) *$\tilde{E}$-topologically mixing*, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exists $v_t \in D(\mathcal{G}) \cap \tilde{E}$ such that $\|y - v_t\| < \varepsilon$ and $\|z - G(\delta_t)v_t\| < \varepsilon$, $t \geqslant t_0$,
(v) *$\tilde{E}$-weakly mixing*, if $\mathbf{G}$ is $(\tilde{E} \oplus \tilde{E})$-hypercylic in $E \oplus E$,
(vi) *$\tilde{E}$-supercyclic*, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its *projective orbit* $\{cG(\delta_t)x : c \in \mathbb{C},\ t \geqslant 0\}$ is dense in $\tilde{E}$,
(vii) *$\tilde{E}$-positively supercyclic*, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its *positive projective orbit* $\{cG(\delta_t)x : c \geqslant 0,\ t \geqslant 0\}$ is dense in $\tilde{E}$,
(viii) *$\tilde{E}_{\mathrm{S}}$-hypercyclic*, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its S-*projective orbit* $\{cG(\delta_t)x : c \in \mathrm{S},\ t \geqslant 0\}$ is dense in $\tilde{E}$,
(ix) *$\tilde{E}_{\mathrm{S}}$-topologically transitive*, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in D(\mathcal{G}) \cap \tilde{E}$, $t \geqslant 0$ and $c \in \mathrm{S}$ such that $\|y - v\| < \varepsilon$ and that $\|z - cG(\delta_t)v\| < \varepsilon$,
(x) *sub-chaotic*, if there exists a $\mathcal{G}$-admissible subset $\hat{E}$ such that $\mathcal{G}$ is $\hat{E}$-chaotic.

Let $\alpha \in (0, \infty)$, $\alpha \notin \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil \alpha \rceil$. Recall [**315**], the *Weyl fractional derivatives* W_+^α and W_-^α of order α (cf. also Subsections 3.2.1 and 3.3.1) are defined by

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\, ds,\ t \in \mathbb{R},$$

$$W_-^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^t (t-s)^{n-\alpha-1} f(s)\, ds,\ t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}$, put $W_+^n := (-1)^n \frac{d^n}{dt^n}$ and $W_-^n := \frac{d^n}{dt^n}$. Then $W_\pm^{\alpha+\beta} = W_\pm^\alpha W_\pm^\beta$, $\alpha,\ \beta > 0$. Assume that A is the integral generator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geqslant 0}$ for some $\alpha \geqslant 0$. Set $\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}_\alpha$ is a (C-DS) generated by A [**315**].

Definition 3.1.30. Let $\tilde{E}$ be a closed linear subspace of E. Then it is said that $\tilde{E}$ is $(S_\alpha(t))_{t\geqslant 0}$-*admissible* iff $\tilde{E}$ is $\mathcal{G}_\alpha$-admissible, and that $(S_\alpha(t))_{t\geqslant 0}$ is $\tilde{E}$-hypercyclic iff $\mathcal{G}_\alpha$ is; all other dynamical properties of $(S_\alpha(t))_{t\geqslant 0}$ are understood in the same sense. A point $x \in \tilde{E}$ is said to be a $\tilde{E}$-*periodic point of* $(S_\alpha(t))_{t\geqslant 0}$ iff x is a $\tilde{E}$-periodic point of $\mathcal{G}_\alpha$.

It is clear that the notion of $\tilde{E}_{\mathrm{S}}$-hypercyclicity generalizes the notions of (positive) $\tilde{E}$-supercyclicity and $\tilde{E}$-hypercyclicity. In the case $\tilde{E} = E$, it is also said that $\mathcal{G}$ ($(S_\alpha(t))_{t\geqslant 0}$) is hypercyclic, chaotic,..., S-hypercyclic, S-topologically transitive, and we write $\mathcal{G}_{per}$ instead of $\mathcal{G}_{\tilde{E},per}$. Let $\beta > \alpha$ and $S_\beta(t)x = \int_0^t \frac{(t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} S_\alpha(s)x\,ds$, $t \geqslant 0$, $x \in E$. Then $\mathcal{G}_\alpha(\varphi)x = \int_0^\infty W_+^\beta \varphi(t) S_\beta(t)x\,dt = \mathcal{G}_\beta(\varphi)x$, $x \in E$, $\varphi \in \mathcal{D}$, and this implies that a closed linear subspace $\tilde{E}$ is $(S_\alpha(t))_{t\geqslant 0}$-admissible iff $\tilde{E}$ is $(S_\beta(t))_{t\geqslant 0}$-admissible, and that $(S_\alpha(t))_{t\geqslant 0}$ is $\tilde{E}$-hypercyclic ($\tilde{E}$-chaotic,..., subchaotic) iff $(S_\beta(t))_{t\geqslant 0}$ is; because of this, we assume in the sequel that $\alpha = n \in \mathbb{N}_0$. Assume $\mathcal{G}$ is a (C-DS) and $\tilde{E}$ is $\mathcal{G}$-admissible. If $\mathcal{G}$ is $\tilde{E}$-weakly mixing, then one can simply prove that $\mathcal{G}$ is both $\tilde{E}$-topologically transitive and $\tilde{E}$-hypercyclic. Assume that the semigroup $(e^{tA})_{t\geqslant 0}$ is hypercyclic (chaotic) in the sense of [**106**, Definition 3.2] and let $L(E) \ni C$ be an injective operator such that $(W(t) := e^{tA}C)_{t\geqslant 0}$ is a C-regularized semigroup generated by A. Put $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)W(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then it can be simply proved that $(W(t))_{t\geqslant 0}$, resp. $\mathcal{G}$, is hypercyclic (chaotic) in the sense of Definition 3.1.29, resp. Definition 3.1.30. Hence, examples given in [**106**, Section 5] can be used for the construction of chaotic C-regularized semigroups.

Example 3.1.31. [**106**] Let Ω be an open bounded subset of $\mathbb{R}^2$ and let Δ act on $L^2(\Omega)$ with the Dirichlet boundary conditions; the complex power $(-\Delta)^b$ ($b \in \mathbb{C}$) is understood in the sense of [**300**]. Then there exists an injective operator $C \in L(L^2(\Omega))$ such that $((-\Delta)^t C)_{t\geqslant 0}$ is a chaotic C-regularized semigroup.

Assume A generates a (C-DS) $\mathcal{G}$ and $x \in Z(A)$. Then $C(Z(A)) \subseteq \overline{\mathcal{R}(\mathcal{G})}$ and $\mathcal{G}(\psi)x \in \mathrm{R}(C)$, $\psi \in \mathcal{D}$. Further on, $\mathcal{R}(\mathcal{G}) \subseteq Z(A)$, $G(\delta_t)(Z(A)) \subseteq Z(A) \subseteq \overline{D(A)}$, $t \geqslant 0$ and $\tilde{E}_{\mathrm{S}}$-hypercyclicity ($\tilde{E}_{\mathrm{S}}$-topological transitivity) of $\mathcal{G}$ implies $\overline{\tilde{E} \cap Z(A)} = \tilde{E}$ and $\tilde{E} \subseteq \overline{D(A)}$. Given $t > 0$ and $\sigma > 0$, set

$$\Phi_{t,\sigma} := \Big\{\varphi \in \mathcal{D}_0 : \operatorname{supp}\varphi \subseteq (t-\sigma, t+\sigma),\ \varphi \geqslant 0,\ \int \varphi(s)\,ds = 1\Big\}.$$

Keeping in mind Proposition 3.1.28 and the proofs of [**106**, Proposition 3.3, Theorem 4.6], we have the following theorem.

THEOREM 3.1.32. (i) *Assume $n \in \mathbb{N}_0$, A is the integral generator of an n-times integrated C-semigroup $(S_n(t))_{t\geqslant 0}$, $\overline{C(\tilde{E})} = \tilde{E}$ and $\tilde{E}$ is $\mathcal{G}_n$-admissible. Then the following holds.*

(i.1) $(S_n(t))_{t\geqslant 0}$ *is $\tilde{E}_{\mathrm{S}}$-hypercyclic iff there exists $x \in \tilde{E}$ such that the mapping $t \mapsto S_n(t)x$, $t \geqslant 0$ is n-times continuously differentiable and that the set $\{c\frac{d^n}{dt^n}S_n(t)x : c \in S,\ t \geqslant 0\}$ is dense in $\tilde{E}$.*

(i.2) $(S_n(t))_{t\geqslant 0}$ *is $\tilde{E}_{\mathrm{S}}$-topologically transitive iff for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in \tilde{E}$, $t_0 > 0$ and $c \in \mathrm{S}$ such that the mapping $t \mapsto S_n(t)v$, $t \geqslant 0$ is n-times continuously differentiable and that $\|y - v\| < \varepsilon$ as well as $\|z - c(\frac{d^n}{dt^n}S_n(t)v)_{t=t_0}\| < \varepsilon$.*

(i.3) $(S_n(t))_{t\geqslant 0}$ *is $\tilde{E}$-chaotic iff $(S_n(t))_{t\geqslant 0}$ is $\tilde{E}$-hypercyclic and there exists a dense subset of $\tilde{E}$ consisting of those vectors $x \in \tilde{E}$ for which there exists $t_0 > 0$ such that the mapping $t \mapsto S_n(t)x$, $t \geqslant 0$ is n-times continuously differentiable and that $(\frac{d^n}{dt^n}S_n(t)x)_{t=t_0} = Cx$.*

(ii) *Let A be the generator of a (C-DS) $\mathcal{G}$ and let $\tilde{E}$ be $\mathcal{G}$-admissible. Then:*

(ii.1) $\mathcal{G}$ *is $\tilde{E}_{\mathrm{S}}$-hypercyclic iff there exists $x_0 \in D(\mathcal{G}) \cap \tilde{E}$ such that, for every $x \in \tilde{E}$ and $\varepsilon > 0$, there exist $t_0 > 0$, $c \in \mathrm{S}$ and $\sigma > 0$ such that*

$$\|cC^{-1}\mathcal{G}(\varphi)x_0 - x\| < \varepsilon,\ \varphi \in \Phi_{t_0,\sigma}.$$

(ii.2) $\mathcal{G}$ *is $\tilde{E}_{\mathrm{S}}$-topologically transitive iff for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $t_0 > 0$, $c \in \mathrm{S}$, $\sigma > 0$ and $v \in D(\mathcal{G}) \cap \tilde{E}$ such that, for every $\varphi \in \Phi_{t_0,\sigma}$,*

$$\|y - v\| < \varepsilon \text{ and } \|z - cC^{-1}\mathcal{G}(\varphi)v\| < \varepsilon.$$

(ii.3) $\mathcal{G}$ *is $\tilde{E}$-chaotic iff $\mathcal{G}$ is $\tilde{E}$-hypercyclic and there exists a dense set in $\tilde{E}$ of vectors $x \in D(\mathcal{G}) \cap \tilde{E}$ for which there exists $\tau > 0$ such that, for every $\varepsilon > 0$, there exists $\sigma > 0$ satisfying*

$$\|C^{-1}\mathcal{G}(\varphi)x - x\| < \varepsilon,\ \varphi \in \Phi_{\tau,\sigma}.$$

COROLLARY 3.1.33. *Let A be the generator of a (C-DS) $\mathcal{G}$. Assume $\tilde{E}$ is $\mathcal{G}$-admissible and $\mathcal{G}$ is $\tilde{E}_{\mathrm{S}}$-hypercyclic ($\tilde{E}_{\mathrm{S}}$-topologically transitive). Then $\overline{C(\tilde{E})} \subseteq \overline{\mathcal{R}(\mathcal{G})}$.*

The Hypercyclicity Criterion for C-distribution semigroups reads as follows.

THEOREM 3.1.34. *Let A be the generator of a (C-DS) $\mathcal{G}$ and let $\tilde{E}$ be $\mathcal{G}$-admissible. Assume that there exist subsets $\overline{Y_1}$, $\overline{Y_2} \subseteq Z(A) \cap \tilde{E}$, both dense in $\tilde{E}$, a mapping $\overline{S} : \overline{Y_1} \to \overline{Y_1}$ and a bounded linear operator D in $\tilde{E}$ such that:*

(i) $G(\delta_1)\overline{S}y = y$, $y \in \overline{Y_1}$,
(ii) $\lim_{n\to\infty} \overline{S}^n y = 0$, $y \in \overline{Y_1}$,
(iii) $\lim_{n\to\infty} G(\delta_n)\omega = 0$, $\omega \in \overline{Y_2}$,
(iv) $\mathrm{R}(D)$ *is dense in* $\tilde{E}$,
(v) $\mathrm{R}(D) \subseteq Z(A) \cap \tilde{E}$, $G(\delta_n)D \in L(\tilde{E})$, $n \in \mathbb{N}$ *and*
(vi) $DG(\delta_1)x = G(\delta_1)Dx$, $x \in Z(A) \cap \tilde{E}$.

Then $\mathbf{G}$ *is both* $(\tilde{E} \oplus \tilde{E})$*-hypercyclic and* $(\tilde{E} \oplus \tilde{E})$*-topologically transitive; in particular,* $\mathcal{G}$ *is* $\tilde{E}$*-weakly mixing.*

PROOF. Let T_1 be the restriction of the operator $G(\delta_1)$ to $Z(A) \cap \tilde{E}$, $T_1 = G(\delta_1)_{|Z(A) \cap \tilde{E}}$. Put $T := T_1 \oplus T_1$, $Y_1 := \overline{Y_1} \oplus \overline{Y_1}$, $Y_2 := \overline{Y_2} \oplus \overline{Y_2}$, $\tilde{D} := D \oplus D$ and define $S : Y_1 \to Y_1$ by $S(x,y) := (\overline{S}x, \overline{S}y)$, $x,\ y \in \overline{Y_1}$. Since $G(\delta_1)(Z(A) \cap \tilde{E}) \subseteq Z(A) \cap \tilde{E}$, $D_\infty(T) = Z(A) \cap \tilde{E}$ and $G(\delta_1)^n x = G(\delta_n)x$, $x \in Z(A) \cap \tilde{E}$, one can apply [**106**, Theorem 2.3] in an effort to see that the operator T is hypercyclic in $\tilde{E} \oplus \tilde{E}$. Under the aegis of the proof of [**106**, Theorem 2.3], it follows that T is also topologically transitive. The proof of theorem completes a routine argument. $\square$

Let $\mathrm{R}(C)$ be dense in E. Assume $\tilde{E} = E$, A is the integral generator of a C-regularized semigroup $(T(t))_{t \geqslant 0}$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)T(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then the conditions (iv)–(vi) quoted in the formulation of Theorem 3.1.34 hold with $D = C$ and, in this case, Theorem 3.1.34 reduces to the Hypercyclicity Criterion for C-regularized semigroups (cf. [**106**, Theorem 3.4]).

EXAMPLE 3.1.35. (i) Let $n \in \mathbb{N}$, $\rho(t) := \frac{1}{t^{2n}+1}$, $t \in \mathbb{R}$, $Af := f'$, $D(A) := \{f \in C_{0,\rho}(\mathbb{R}) : f' \in C_{0,\rho}(\mathbb{R})\}$, $E_n := (C_{0,\rho}(\mathbb{R}))^{n+1}$, $D(A_n) := D(A)^{n+1}$ and $A_n(f_1, \cdot\cdot\cdot, f_{n+1}) := (Af_1 + Af_2, Af_2 + Af_3, \cdot\cdot\cdot, Af_n + Af_{n+1}, Af_{n+1})$, $(f_1, \cdot\cdot\cdot, f_{n+1}) \in D(A_n)$. Then it is well known that $\pm A_n$ generate global polynomially bounded n-times integrated semigroups $(S_{n,\pm}(t))_{t \geqslant 0}$ and that neither A_n nor $-A_n$ generates a local $(n-1)$-times integrated semigroup. Denote by $G_{\pm,n}$ distribution semigroups generated by $\pm A_n$. Then it can be easily proved that for every $\varphi_1, ..., \varphi_{n+1} \in \mathcal{D}$:

$$G_{\pm,n}(\delta_t)\big(\varphi_1, ..., \varphi_{n+1}\big)^T = \big(\psi_1, ..., \psi_{n+1}\big)^T,$$

where $\psi_i(\cdot) = \sum_{j=0}^{n+1-i} \frac{(\pm t)^j}{j!} \varphi^{(j)}_{i+j}(\cdot \pm t)$, $1 \leqslant i \leqslant n+1$. This immediately implies the concrete representation formula for $(S_{n,\pm}(t))_{t \geqslant 0}$. It can be proved (cf. [**239**] for further information) that, for every $t > 0$, the operators $G_{\pm,n}(\delta_t) \oplus G_{\pm,n}(\delta_t)$ are hypercyclic in $E_n \oplus E_n$. This implies that $(S_{n,\pm}(t))_{t \geqslant 0}$ are weakly mixing. Arguing in a similar way, one can construct a closed linear operator ${}_nA$, a Banach space ${}_nE$ and an injective operator ${}_nC \in L({}_nE)$ such that ${}_nA$ is a subgenerator of a global weakly mixing n-times integrated ${}_nC$-semigroup $({}_nS(t))_{t \geqslant 0}$ on ${}_nE$ and that ${}_nA$ is not a subgenerator of any local $(n-1)$-times integrated ${}_nC$-semigroup on ${}_nE$.

(ii) Let $n \in \mathbb{N}$, $\Omega := (0,\infty)^n$, $\alpha_i > 0$, $1 \leqslant i \leqslant n$ and $\alpha := \min\{\alpha_i : 1 \leqslant i \leqslant n\}$. Set $\rho(x) := e^{-(x_1^\alpha + \cdots + x_n^\alpha)}$ and

$$\varphi(t,x) := \big((t + x_1^{\alpha_1})^{1/\alpha_1}, \ldots, (t + x_n^{\alpha_n})^{1/\alpha_n}\big), \ \ t \geqslant 0,\ x = (x_1, \ldots, x_n) \in \Omega.$$

Let remind us that the space $C_{0,\rho}(\Omega, \mathbb{C})$ consists of all continuous functions $f : \Omega \to \mathbb{C}$ satisfying that, for every $\varepsilon > 0$, $\{x \in \Omega : |f(x)|\rho(x) \geqslant \varepsilon\}$ is a compact subset of Ω; equipped with the norm $\|f\| := \sup_{x \in \Omega} |f(x)|\rho(x)$, $C_{0,\rho}(\Omega, \mathbb{C})$ becomes a Banach space. The space of all continuous functions $f : \Omega \to \mathbb{C}$ whose support is a compact subset of Ω, denoted by $C_c(\Omega, \mathbb{C})$, is dense in $C_{0,\rho}(\Omega, \mathbb{C})$. Define $(T_\varphi(t)f)(x) := f(\varphi(t,x))$, $t \geqslant 0$, $x \in \Omega$ and $Cf(x) := e^{-(x_1 + \cdots + x_n)}f(x)$, $x \in \Omega$,

$f \in C_{0,\rho}(\Omega, \mathbb{C})$. Then one can simply prove that $T_\varphi(t) \notin L(C_{0,\rho}(\Omega, \mathbb{C}))$, $t > 0$ and that $(T_\varphi(t)C)_{t \geqslant 0}$ is a bounded C-regularized semigroup. Given $f \in C_c(\Omega, \mathbb{C})$, define $\tilde{f} : [0, \infty)^n \to \mathbb{C}$ by $\tilde{f}(x) := f(x)$, $x \in \Omega$ and $\tilde{f}(x) := 0$, $x \in [0, \infty)^n \setminus \Omega$. Applying [**106**, Theorem 3.4] with $Y_1 = Y_2 = C_c(\Omega, \mathbb{C})$ and

$$\begin{aligned} Sf(x_1, \ldots, x_n) = \tilde{f}((x_1^{\alpha_1} - 1)^{1/\alpha_1} &\chi_{[(a_1^{\alpha_1}+1)^{1/\alpha_1}, (b_1^{\alpha_1}+1)^{1/\alpha_1}]}(x_1), \ldots, \\ &(x_n^{\alpha_n} - 1)^{1/\alpha_n} \chi_{[(a_n^{\alpha_n}+1)^{1/\alpha_n}, (b_n^{\alpha_n}+1)^{1/\alpha_n}]}(x_n)), \end{aligned}$$

$x \in \Omega$, $f \in Y_1$, $\operatorname{supp} f \subseteq \prod_{i=1}^n [a_i, b_i] \subseteq \Omega$, we get that $(T_\varphi(t)C)_{t \geqslant 0}$ is weakly mixing. Furthermore, $(T_\varphi(t)C)_{t \geqslant 0}$ is topologically mixing and, thanks to the proof of [**174**, Theorem 5.7], $(T_\varphi(t)C)_{t \geqslant 0}$ is chaotic.

(iii) [**106**], [**239**] Assume that ω_1, ω_2, V_{ω_2,ω_1}, Q, $Q(B)$, N, h_μ and E possess the same meaning as in [**106**, Section 5] and that $Q(\operatorname{int}(V_{\omega_2,\omega_1})) \cap i\mathbb{R} \neq \emptyset$. Then $\pm Q(B)h_\mu = \pm Q(\mu)h_\mu$, $e^{-(-B^2)^N} h_\mu = e^{-(-\mu^2)^N} h_\mu$, $\mu \in \operatorname{int}(V_{\omega_2,\omega_1})$ and $h_\mu \in (\operatorname{Kern}(Q(B)) \setminus \{0\})$, provided $\operatorname{Re}\mu \in (\omega_2, \omega_1)$. Define $\hat{E}$ as the closure of the set $\operatorname{span}\{(h_\mu, Q(\mu)h_\mu)^T : \mu \in \operatorname{int}(V_{\omega_2,\omega_1})\}$. Then $Q^2(B)$ is the integral generator of a global $(e^{-(-z^2)^N})(B)$-regularized cosine function $((\cosh(tQ(z))e^{-(-z^2)^N})(B))_{t \geqslant 0}$ and the operator $\left(\begin{smallmatrix} 0 & I \\ Q^2(B) & 0 \end{smallmatrix}\right)$ generates of an entire $\left(\begin{smallmatrix} (e^{-(-z^2)^N})(B) & 0 \\ 0 & (e^{-(-z^2)^N})(B) \end{smallmatrix}\right)$-regularized semigroup $(S_0(t))_{t \geqslant 0}$ satisfying that $(S_0(t))_{t \geqslant 0}$ is $\hat{E}$-topologically mixing and that the set of all $\hat{E}$-periodic points of $(S_0(t))_{t \geqslant 0}$ is dense in $\hat{E}$. Furthermore, the analysis given in [**106**, Theorem 5.8] can serve one to construct important examples of regular ultradistribution semigroups of Beurling class.

Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$. Then $(T(t))_{t \geqslant 0}$ is S-topologically transitive in the sense of Definition 3.1.30 iff $(T(t))_{t \geqslant 0}$ is S-topologically transitive in the sense of the definition introduced on pages 50–51 of [**237**]. It is well known that S-topological transitivity of $(T(t))_{t \geqslant 0}$ is equivalent to its S-hypercyclicity and that $(T(t))_{t \geqslant 0}$ is weakly mixing provided that $(T(t))_{t \geqslant 0}$ is chaotic [**237**]; it is not clear whether the above assertions continue to hold in the case of C-distribution semigroups. In the sequel of this subsection, we will use the fact that the notions of $\tilde{E}$-topological transitivity and $\tilde{E}$-periodic points of a (C-DS) $\mathcal{G}$ (or an n-times integrated C-semigroup $(S_n(t))_{t \geqslant 0}$) can be understood in the sense of Definition 3.1.29 even if the set $\tilde{E}$ is not $\mathcal{G}$-admissible.

The next theorem is a strengthening of [**109**, Theorem 3.1] and [**26**, Criterion 2.3].

THEOREM 3.1.36. [**238**]-[**239**] *Let $t_0 > 0$.*

(i) *Let A be the generator of a (C-DS) $\mathcal{G}$. Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in \operatorname{Kern}(A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. Assume, further, that $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$, implies $x^* = 0$. Then $\mathcal{G}$ is topologically mixing, every single operator $G(\delta_{t_0})$ is topologically mixing and has a dense set of periodic points in E.*

(ii) *Let A be the generator of a (C-DS) $\mathcal{G}$. Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in \mathrm{Kern}(A-\lambda) \smallsetminus \{0\}$, $\lambda \in \Omega$. Put $E_0 := \mathrm{span}\{f(\lambda) : \lambda \in \Omega\}$ and $\tilde{E} := \overline{E_0}$. Then $\mathcal{G}$ is $\tilde{E}$-topologically mixing, the part of the operator $G(\delta_{t_0})$ in the Banach space $\tilde{E}$ is topologically mixing in $\tilde{E}$ and the set of $\tilde{E}$-periodic points of such an operator is dense in $\tilde{E}$.*

REMARK 3.1.37. (i) It is not clear whether the set $\tilde{E}$, appearing in the formulation of the assertion (ii) of the previous theorem, is $\mathcal{G}$-admissible.

(ii) Assume A is the integral generator of a C-regularized semigroup $(T(t))_{t\geqslant 0}$ and $\mathrm{R}(C)$ is dense in E. Let Ω and $f(\cdot)$ satisfy the assumptions quoted in the formulation of Theorem 3.1.36(i). Then $(T(t))_{t\geqslant 0}$ is chaotic, weakly mixing and, for every $t > 0$, the operator $C^{-1}T(t)$ is chaotic.

THEOREM 3.1.38. *Let $\theta \in (0, \frac{\pi}{2})$ and let $-A$ generate an analytic strongly continuous semigroup of angle θ. Assume $n \in \mathbb{N}$, $a_n > 0$, $a_{n-i} \in \mathbb{C}$, $1 \leqslant i \leqslant n$, $D(p(A)) = D(A^n)$, $p(A) = \sum_{i=0}^{n} a_i A^i$ and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$.*

(i) *Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in \mathrm{Kern}(-A-\lambda)\smallsetminus\{0\}$, $\lambda \in \Omega$. Assume, also, that the supposition $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$, implies $x^* = 0$. Then, for every $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, there exists $\omega \in \mathbb{R}$ such that $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha}$-regularized group $(T(t))_{t\in\mathbb{C}}$. Furthermore, $(T(t))_{t\geqslant 0}$ is chaotic, topologically mixing and, for every $t > 0$, the operator $C^{-1}T(t)$ is chaotic.*

(ii) *Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in \mathrm{Kern}(-A-\lambda) \smallsetminus \{0\}$, $\lambda \in \Omega$. Let E_0 and $\tilde{E}$ be as in the formulation of Theorem 3.1.36(ii). Then there exists $\omega \in \mathbb{R}$ such that, for every $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha}$-regularized group $(T(t))_{t\in\mathbb{C}}$ such that $(T(t))_{t\geqslant 0}$ is $\tilde{E}$-topologically mixing and that the set of $\tilde{E}$-periodic points of $(T(t))_{t\geqslant 0}$ is dense in $\tilde{E}$.*

PROOF. The proof of (i) can be obtained as follows. By the arguments given in [**89**, Section XXIV], we have that the operator $-p(A)$ generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. Let $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$. By [**89**, Theorem 8.2], one gets that there exists a convenable chosen number $\omega \in \mathbb{R}$ such that $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha} \equiv C$-regularized group $(T(t))_{t\in\mathbb{C}}$. Thanks to the proof of [**106**, Lemma 5.6], $\sigma_p(-p(A)) = -p(-\sigma_p(-A))$ and $f(\lambda) \in \mathrm{Kern}(-p(A) + p(-\lambda))$, $\lambda \in \Omega$. Without loss of generality, one can assume that $p'(z) \neq 0$, $z \in -\Omega$; otherwise, one can replace Ω by $\Omega \smallsetminus \{\gamma_1, \ldots, \gamma_{n-1}\}$, where $\gamma_1, \ldots, \gamma_{n-1}$ are not necessarily distinct zeros of the polynomial $z \mapsto p'(z)$, $z \in \mathbb{C}$. Hence, the mapping $\lambda \mapsto p(-\lambda)$, $\lambda \in \Omega$ and its inverse mapping $z \mapsto -p^{-1}(z)$, $z \in p(-\Omega)$, are analytic and open. The set $-p(-\Omega)$ is open, connected and intersects the imaginary axis. Moreover, the mapping $z \mapsto f(-p^{-1}(-z))$, $z \in -p(-\Omega)$ is analytic, $f(-p^{-1}(-z)) \in \mathrm{Kern}(-p(A) - z)$, $z \in -p(-\Omega)$ and $x^*(f(-p^{-1}(-z))) = 0$, $z \in$

$-p(-\Omega)$, for some $x^* \in E^*$, implies $x^* = 0$. Therefore, it suffices to prove (i) in the case $p(z) = z$. In order to do that, notice that $-\Omega \subseteq \sigma_p(A)$, $f(-\lambda) \in D_\infty(A)$ and $A^k f(-\lambda) = \lambda^k f(-\lambda)$, $\lambda \in -\Omega$, $k \in \mathbb{N}$. It can be simply proved that $f(-\lambda) \in Z(A)$, $\lambda \in -\Omega$, $f(-\lambda) \in \mathrm{Kern}(A-\lambda)$, $\lambda \in -\Omega$ and $C^{-1}T(t)f(\lambda) = e^{-\lambda t} f(\lambda)$, $t \geqslant 0$, $\lambda \in \Omega$. By Theorem 3.1.36(i), one has that $(T(t))_{t\geqslant 0}$ is topologically mixing and that the set of periodic points of $(T(t))_{t\geqslant 0}$ is dense in E. Since $\mathrm{R}(C)$ is dense in E [**89**], one can apply [**106**, Theorem 3.4] with $Y_1 = X_0 \oplus X_0$, $Y_2 = X_\infty \oplus X_\infty$ and $S : Y_1 \to Y_1$, defined by $S\big(\sum_{i=1}^k \alpha_i f(\lambda_i), \sum_{i=1}^l \beta_i f(z_i)\big) := \big(\sum_{i=1}^k \alpha_i e^{\lambda_i} f(\lambda_i), \sum_{i=1}^l \beta_i e^{z_i} f(z_i)\big)$, k, $l \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $\mathrm{Re}\,\lambda_i < 0$, $1 \leqslant i \leqslant k$, $\beta_i \in \mathbb{C}$, $\mathrm{Re}\, z_i < 0$, $1 \leqslant i \leqslant l$, in order to see that, for every $t > 0$, the operator $C^{-1}T(t) \oplus C^{-1}T(t)$ is hypercyclic. This implies that $(T(t))_{t\geqslant 0}$ is weakly mixing and chaotic. The chaoticity of the operator $C^{-1}T(t)$ $(t > 0)$ can be shown as in the proof of [**173**, Theorem 4.9] and this completes the proof of (i).

The proof of (ii) can be obtained similarly. $\square$

Remark 3.1.39. (i) Assume that $\mathcal{G}$ is a (C-DS) and that the set $\tilde{E}$ is not $\mathcal{G}$-admissible. Then one can define the notion of $\tilde{E}$-hypercyclicity ($\tilde{E}_{\mathrm{S}}$-hypercyclicity) of $\mathcal{G}$ in several different ways. In the second part of this remark, it will be said that $\mathcal{G}$ is $\tilde{E}$-hypercyclic iff there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that the set $\{G(\delta_t)x : t \geqslant 0\}$ is a dense subset of $\tilde{E}$, and that $\mathcal{G}$ is $\tilde{E}$-chaotic iff $\mathcal{G}$ is $\tilde{E}$-hypercyclic and the set $\mathcal{G}_{\tilde{E},per}$ is dense in $\tilde{E}$.

(ii) Under the assumptions of Theorem 3.1.38(ii), $(T(t))_{t\geqslant 0}$ is $\tilde{E}$-chaotic. We will prove this statement only in the case $p(z) = z$. Clearly, for every $\lambda \in \Omega$, $R(\xi : A)f(\lambda) = \frac{f(\lambda)}{\xi-\lambda}$, $\xi \in \rho(A) \smallsetminus \{\lambda\}$. By the representation formula [**89**, p. 70, l. 2], one can show that there exists a mapping $g : \Omega \to \mathbb{C} \smallsetminus \{0\}$ such that $Cf(\lambda) = g(\lambda)f(\lambda)$, $\lambda \in \Omega$. This implies that $C(E_0) = E_0$ and that $\mathrm{R}(C_{\tilde{E}})$ is dense in $\tilde{E}$. Let $D(T_1) = \{x \in Z(A) \cap \tilde{E} : G(\delta_1)x \in Z(A) \cap \tilde{E}\}$ and $T_1 x = G(\delta_1)x$, $x \in D(T_1)$. Using [**106**, Theorem 2.3] with $T = T_1 \oplus T_1$, $Y_1 = X_0 \oplus X_0$, $Y_2 = X_\infty \oplus X_\infty$, $S(x,y) = (e^\lambda x, e^\lambda y)$, x, $y \in X_0$, and $C_{\tilde{E}}$, one yields that the operator T is hypercyclic in $\tilde{E}$. As an outcome, we get that $(T(t))_{t\geqslant 0}$ is $\tilde{E}$-chaotic.

Example 3.1.40. [**109**, Example 4.12] In what follows, we analyze chaotic properties of a convection-diffusion type equation of the form

$$\begin{cases} u_t = au_{xx} + bu_x + cu := -Au, \\ u(0,t) = 0, \ t \geqslant 0, \\ u(x,0) = u_0(x), \ x \geqslant 0. \end{cases}$$

It is well known that the operator $-A$, considered with the domain $D(-A) = \{f \in W^{2,2}([0,\infty)) : f(0) = 0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $E = L^2([0,\infty))$, provided a, b, $c > 0$ and $c < \frac{b^2}{2a} < 1$. The same conclusion holds if we consider the operator $-A$ with the domain $D(-A) = \{f \in W^{2,1}([0,\infty)) : f(0) = 0\}$ in $E = L^1([0,\infty))$. Let

$$\Omega = \Big\{\lambda \in \mathbb{C} : \Big|\lambda - \Big(c - \frac{b^2}{4a}\Big)\Big| \leqslant \frac{b^2}{4a}, \ \mathrm{Im}\,\lambda \neq 0 \text{ if } \mathrm{Re}\,\lambda \leqslant c - \frac{b^2}{4a}\Big\}$$

and let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a nonconstant polynomial such that $a_n > 0$ and that $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$. Notice that the last condition holds provided $a_0 \in i\mathbb{R}$. By Theorem 3.1.38, one gets that there exists an injective operator $C \in L(E)$ such that $p(A)$ generates an entire C-regularized group $(T(t))_{t\in\mathbb{C}}$ satisfying that $(T(t))_{t\geqslant 0}$ is chaotic and topologically mixing. Set $\hat{E} := \overline{\{(f_\lambda(\cdot), p(-\lambda)f_\lambda(\cdot))^T : \lambda \in \Omega\}}$, where the function f_λ is defined in [**109**, Example 4.12]. Then [**239**] the operator $\bigl(\begin{smallmatrix} 0 & I \\ p^2(A) & 0 \end{smallmatrix}\bigr)$ is the integral generator of an entire $\mathcal{C}$-regularized semigroup $(S_0(t))_{t\geqslant 0}$ satisfying that $(S_0(t))_{t\geqslant 0}$ is $\hat{E}$-topologically mixing and that the set of all $\hat{E}$-periodic points of $(S_0(t))_{t\geqslant 0}$ is dense in $\hat{E}$. Using the composition property of regularized semigroups, it simply follows that there exist x, $y \in \hat{E}$ such that the set $\{\mathcal{C}^{-1}S_0(nt)\binom{x}{y} : n \in \mathbb{N}_0\}$ is a dense subset of $\hat{E}$. Since $\mathrm{R}(\mathcal{C}_{\hat{E}})$ is dense in $\hat{E}$, one gets that $\{S_0(nt)\binom{x}{y} : n \in \mathbb{N}_0\}$ is also a dense subset of $\hat{E}$. This implies that $(S_0(t))_{t\geqslant 0}$ is $\hat{E}$-hypercyclic.

Ji and Weber [**163**] have recently investigated the dynamics of L^p heat semigroups ($p > 2$) on symmetric spaces of non-compact type. It is noteworthy that Theorem 3.1.38 and Remark 3.1.39 can be applied to the operators considered in [**163**, Theorem 3.1(a), Theorem 3.2, Corollary 3.3] and that convenable chosen shifts (polynomials) of the backwards heat operator, acting on such spaces, has a certain (sub-)chaotical behavior. More precisely, we have the following.

EXAMPLE 3.1.41. Let X be a symmetric space of non-compact type (of rank one) and $p > 2$. Then there exist a closed linear subspace $\tilde{X}$ of X (X, if the rank of X is one), a number $c_p > 0$ and an injective operator $C \in L(L^p_\natural(X))$ such that for any $c > c_p$ the operator $(-\Delta^\natural_{X,p} + c)^2$ generates a global C-regularized cosine function $(C(t))_{t\geqslant 0}$ in $L^p_\natural(X)$. Furthermore, there exists a closed linear subspace $\hat{X}$ of $X \oplus X$ such that the operator $\bigl(\begin{smallmatrix} 0 & I \\ (-\Delta^\natural_{X,p}+c)^2 & 0 \end{smallmatrix}\bigr)$ generates an entire $\mathcal{C}$-regularized semigroup $(S_0(t))_{t\geqslant 0}$ satisfying that $(S_0(t))_{t\geqslant 0}$ is $\hat{X}$-topologically mixing and that the set of all $\hat{X}$-periodic points of $(S_0(t))_{t\geqslant 0}$ is dense in $\hat{X}$.

The following theorem is an extension of [**329**, Theorem 2.1], [**76**, Proposition 2.1], [**78**, Theorem 1.1] and Theorem 3.1.36(i).

THEOREM 3.1.42. (i) *Assume $\mathcal{G}$ is a (C-DS) generated by A, $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$ and $t_0 > 0$. If $\sigma_p(A) \cap i\mathbb{R} \supseteq (i\omega_1, i\omega_2) \cap \frac{2\pi i\mathbb{Q}}{t_0}$, $k \in \mathbb{N}$ and $g_j : (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0} \to E$ is a function satisfying that, for every $j = 1, \ldots, k$, $Ag_j(s) = isg_j(s)$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$, then every point in* span$\{g_j(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$, $1 \leqslant j \leqslant k\}$ *is a periodic point of $G(\delta_{t_0})$. Assume now that $f_j : (\omega_1, \omega_2) \to E$ is a Bochner integrable function such that, for every $j = 1, \ldots, k$, $Af_j(s) = isf_j(s)$ for a.e. $s \in (\omega_1, \omega_2)$. Put $\psi_{r,j} := \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\, ds$, $r \in \mathbb{R}$, $1 \leqslant j \leqslant k$.*

(i.1) *Assume* span$\{f_j(s) : s \in (\omega_1, \omega_2) \smallsetminus \Omega$, $1 \leqslant j \leqslant k\}$ *is dense in E for every subset Ω of (ω_1, ω_2) with zero measure. Then $\mathcal{G}$ is topologically mixing and $G(\delta_{t_0})$ is topologically mixing.*

(i.2) *Put* $\tilde{E} := \overline{\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}}$. *Then* $\mathcal{G}$ *is* $\tilde{E}$*-topologically mixing and the part of* $G(\delta_{t_0})$ *in* $\tilde{E}$ *is topologically mixing in the Banach space* $\tilde{E}$.

(ii) *Assume* $\mathcal{G}$ *is a* (*C-DS*) *generated by* A, $t_0 > 0$, $\tilde{E}$ *is a closed linear subspace of* E, $E_0 := \operatorname{span}\{x \in Z(A) : \exists\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0,\ G(\delta_t)x = e^{\lambda t}x,\ t \geqslant 0\}$, $E_\infty := \operatorname{span}\{x \in Z(A) : \exists\lambda \in \mathbb{C},\ \operatorname{Re}\lambda > 0,\ G(\delta_t)x = e^{\lambda t}x,\ t \geqslant 0\}$ *and* $E_{per} := \operatorname{span}\{x \in Z(A) : \exists\lambda \in \mathbb{Q},\ G(\delta_t)x = e^{\pi\lambda it}x,\ t \geqslant 0\}$. *Then the following holds:*

(ii.1) *If* $E_0 \cap \tilde{E}$ *is dense in* $\tilde{E}$ *and if* E_∞ *is a dense subspace of* $\tilde{E}$, *then* $\mathcal{G}$ *is* $\tilde{E}$*-topologically mixing; if* $G(\delta_t)(E_0 \cap \tilde{E}) \subseteq \tilde{E}$, $t \geqslant 0$, *then the part of* $G(\delta_{t_0})$ *in* $\tilde{E}$ *is topologically mixing in the Banach space* $\tilde{E}$.

(ii.2) *If* $E_{per} \cap \tilde{E}$ *is dense in* $\tilde{E}$, *then the set of* $\tilde{E}$*-periodic points of* $\mathcal{G}$ *is dense in* $\tilde{E}$; *if, additionally,* E_{per} *is a dense subspace of* $\tilde{E}$, *then the set of all periodic points of the part of the operator* $G(\delta_{t_0})$ *in* $\tilde{E}$ *is dense in* $\tilde{E}$.

PROOF. We will prove the assertion (i.1). By Riemann–Lebesgue lemma and the dominated convergence theorem, we have that $\lim_{|r|\to\infty}\psi_{r,j} = 0$ and that the mapping $r \mapsto \psi_{r,j}$, $r \in \mathbb{R}$ is continuous ($1 \leqslant j \leqslant k$). Then $G(\delta_t)f_j(s) = e^{its}f_j(s)$ for a.e. $s \in (\omega_1, \omega_2)$, $G(\delta_t)\psi_{r,j} = \psi_{r+t,j}$, $t \geqslant 0$, $r \in \mathbb{R}$, $1 \leqslant j \leqslant k$ and $\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R}, 1 \leqslant j \leqslant k\} \subseteq D(\mathcal{G})$. Using the proof of [**329**, Theorem 2.1], it can be easily seen that $\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}$ is dense in E. So, it suffices to show that, given y, $z \in \operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}$ and $\varepsilon > 0$ in advance, there exists $t_0 \geqslant 0$ such that, for every $t \geqslant t_0$, there exists $x_t \in Z(A) = D(\mathcal{G})$ such that:

$$\|y - x_t\| < \varepsilon \text{ and } \|z - G_1(\delta_t)x_t\| < \varepsilon. \tag{197}$$

Let $y = \sum_{l=1}^{m}\alpha_l\psi_{r_l,i_l}$ and $z = \sum_{l=1}^{n}\beta_l\psi_{\tilde{r}_l,\tilde{i}_l}$ for some α_l, $\beta_l \in \mathbb{C}$, r_l, $\tilde{r}_l \in \mathbb{R}$ and $1 \leqslant i_l, \tilde{i}_l \leqslant k$. Then there exists $t_0(\varepsilon) > 0$ such that $\|\sum_{l=1}^{n}\beta_l\psi_{\tilde{r}_l-t,\tilde{i}_l}\| < \varepsilon$ and $G(\delta_t)\sum_{l=1}^{n}\beta_l\psi_{\tilde{r}_l-t,\tilde{i}_l} = z$, $t \geqslant t_0(\varepsilon)$. Furthermore, there exists $t_1(\varepsilon) > 0$ such that $\|G(\delta_t)y\| = \|\sum_{l=1}^{m}\alpha_l\psi_{r_l+t,i_l}\| < \varepsilon$, $t \geqslant t_1(\varepsilon)$. Then (197) holds with $t_0 = \max(t_0(\varepsilon), t_1(\varepsilon))$ and $x_t = \sum_{l=1}^{n}\beta_l\psi_{\tilde{r}_l-t,\tilde{i}_l} + y$, $t \geqslant t_0$. The operator $G(\delta_{t_0})$ is obviously topologically mixing, which completes the proof. $\square$

REMARK 3.1.43. (i) Assume the function $f_j : (\omega_1, \omega_2) \to E$ is weakly continuous for every $j = 1, \ldots, k$, $t_0 > 0$ and Ω is a subset of (ω_1, ω_2) with zero measure. Then

$$\begin{aligned}&\overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \cap 2\pi\mathbb{Q}/t_0,\ 1 \leqslant j \leqslant k\}}\\ &= \overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2),\ 1 \leqslant j \leqslant k\}} = \overline{\operatorname{span}\bigcup_{j=1}^{k}\{f_j(s) : s \in (\omega_1, \omega_2) \smallsetminus \Omega\}}.\end{aligned}$$

(ii) Let Ω be a subset of (ω_1, ω_2) with zero measure, let $r \in \mathbb{R}$ and let $1 \leqslant j \leqslant k$. Then $\psi_{r,j} = \int_{\omega_1}^{\omega_2} e^{irs}f_j(s)\,ds \in \overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \smallsetminus \Omega\}}$.

(iii) Assume that the mapping $r \mapsto \psi_{r,j}$, $r \in \mathbb{R}$ is an element of the space $L^1(\mathbb{R} : E)$ for every $j = 1, \ldots, k$. Then the inversion theorem for the Fourier transform implies that there exists a subset Ω of (ω_1, ω_2) with zero measure such that $\overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \smallsetminus \Omega,\ 1 \leqslant j \leqslant k\}} = \overline{\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leqslant j \leqslant k\}}$.

(iv) By multiplying with an appropriate scalar-valued function, we may assume that, for every $j = 1, \ldots, k$, the function $f_j(\cdot)$ is strongly measurable (cf. also [**329**, Remark 2.4]).

The following example illustrates an application of Theorem 3.1.42(i) and can be formulated in a more general setting.

Example 3.1.44. Assume $\alpha > 0$, $\tau \in i\mathbb{R} \smallsetminus \{0\}$ and $E := BUC(\mathbb{R})$. After the usual matrix reduction to a first order system, the equation $\tau u_{tt} + u_t = \alpha u_{xx}$ becomes

$$\frac{d}{dt}\vec{u}(t) = P(D)\vec{u}(t),\ t \geqslant 0,\ \text{where } D \equiv -i\frac{d}{dx},\ P(x) \equiv \begin{bmatrix} 0 & 1 \\ -\frac{\alpha}{\tau}x^2 & -\frac{1}{\tau} \end{bmatrix}$$

and $P(D)$ acts on $E \oplus E$ with its maximal distributional domain. The polynomial matrix $P(x)$ is not Petrovskii correct and [**89**, Theorem 14.1] implies that there exists an injective operator $C \in L(E \oplus E)$ such that $P(D)$ generates an entire C-regularized group $(T(z))_{z\in\mathbb{C}}$, with $\mathrm{R}(C)$ dense. Put $\omega_1 = -\infty$ and $\omega_2 = 0$, resp. $\omega_1 = 0$ and $\omega_2 = +\infty$, if $\operatorname{Im}\tau > 0$, resp. $\operatorname{Im}\tau < 0$. Then $\frac{-\tau s^2+is}{\alpha} \in (-\infty, 0)$, $s \in (\omega_1, \omega_2)$. Let $h_1(s) := \cos\big(\cdot(\frac{\tau s^2-is}{\alpha})^{1/2}\big)$, $h_2(s) := \sin\big(\cdot(\frac{\tau s^2-is}{\alpha})^{1/2}\big)$, $s \in (\omega_1, \omega_2)$ and let $f \in C^\infty((0,\infty))$ be such that the mapping $s \mapsto f_j(s) := \big(f(s)h_j(s), isf(s)h_j(s)\big)^T$, $s > 0$ is Bochner integrable and that the mapping

$$s \mapsto \begin{cases} f_j(s), & s \in (\omega_1, \omega_2) \\ 0, & s \notin (\omega_1, \omega_2) \end{cases}$$

belongs to the space $H^1(\mathbb{R})$ for $j = 1, 2$. Put $\psi_{r,j} = \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\,ds$, $r \in \mathbb{R}$, $j = 1, 2$ and $\tilde{E} = \overline{\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ j = 1, 2\}}$. By Bernstein lemma [**14**, Lemma 8.2.1, p. 429], Theorem 3.1.42(i.2) and Remark 3.1.43(i)–(iii), one gets that $(T(t))_{t\geqslant 0}$ is $\tilde{E}$-topologically mixing as well as that for each $t_0 > 0$ the part of the operator $C^{-1}T(t_0)$ in $\tilde{E}$ is topologically mixing in $\tilde{E}$ and that the set of $\tilde{E}$-periodic points of such an operator is dense in $\tilde{E}$.

The hypercyclic and topologically mixing properties of abstract time-fractional equations have been recently considered in [**243**]. The results obtained there can be applied in the study of time-fractional equations on symmetric spaces of non-compact type and time-fractional equations involving bounded perturbations of Ornstein-Uhlenbeck operators ([**76**]).

3.2. Various classes of distribution semigroups

In the remaining part of the book, we primarily consider the case $C = I$ although one can reformulate a great part of our results in the case of general C. We start with the recollection of fundamental properties of *smooth distribution semigroups* introduced by Balabane and Emamirad [**21**]–[**23**] and further studied by Arendt and Kellermann [**7**], El–Mennaoui [**117**], Hieber [**148**], Kunstmann, Mijatović and Pilipović [**257**] and Miana [**311**]–[**315**]. Denote by $\mathcal{S}_+$ the space of all infinitely differentiable functions $f : [0,\infty) \to \mathbb{C}$ such that $q_{m,n}(f) :=$

$\sup_{t\geqslant 0}|t^m f^{(n)}(t)| < \infty$ for all m, $n \in \mathbb{N}_0$. It is well known that $q_{m,n}(\cdot)$ is a seminorm on $\mathcal{S}_+$ for all m, $n \in \mathbb{N}_0$ and that the system $(q_{m,n})$ defines the Frechét topology on $\mathcal{S}_+$. The dual space $\mathcal{S}'_+$ is said to be the space of tempered distributions on $[0,\infty)$. In what follows, we assume that $\mathcal{S}'_+$ is equipped with the strong topology. Further on, let $\mathcal{D}_+ := \{f \in C^\infty([0,\infty)) : f \text{ is compactly supported}\}$. Define $\mathcal{K} : \mathcal{D} \to \mathcal{D}_+$ by $\mathcal{K}(\varphi)(t) := \varphi(t)$, $t \geqslant 0$, $\varphi \in \mathcal{D}$. We know that $\mathcal{D}_+$ is an (LF) space, and due to the theorem of Seeley [**380**], there exists a linear continuous operator $\Lambda : \mathcal{D}_+ \to \mathcal{D}$ satisfying $\mathcal{K}\Lambda = I_{\mathcal{D}_+}$. Let $q_k(f) := \int_0^\infty t^k|f^{(k)}(t)|\,dt$, $f \in \mathbb{N}_0$, $f \in \mathcal{S}_+$. Following Miana [**311**], we denote by $AC^{(k)}(t^k)$ the completion of $\mathcal{S}_+$ in the norm $q_k(\cdot)$. It is checked at once that $AC^{(k)}(t^k) \hookrightarrow AC^{(j)}(t^j)$ if $0 \leqslant j \leqslant k$. Given $f \in \mathcal{D}_+$, the Weyl fractional integral of order $\alpha > 0$ is defined by $(W^{-\alpha}f)(t) := \int_t^\infty \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds$, $f \in \mathcal{D}_+$, $t \geqslant 0$. It is well known that, for every $\alpha > 0$, the mapping $W^{-\alpha} : \mathcal{D}_+ \to \mathcal{D}_+$ is bijective. The inverse mapping of $W^{-\alpha}(\cdot)$, denoted by $W^\alpha(\cdot)$, is called the Weyl fractional derivative of order $\alpha > 0$. If $\alpha \in \mathbb{N}$, then $W^\alpha f = (-1)^n f^{(n)}$, $f \in \mathcal{D}_+$. Furthermore, $W^\alpha W^\beta = W^{\alpha+\beta}$ for all α, $\beta \in \mathbb{R}$, where we put $W^0 := I$. In Subsection 3.3.1 we will employ a somewhat different notion. By $AC^\alpha(t^\alpha)$ we denote the completion of the normed space $(\mathcal{D}_+, \omega_\alpha)$, where $\omega_\alpha(f) := \int_0^\infty t^\alpha|(W^\alpha f)(t)|\,dt$, $f \in \mathcal{D}_+$.

Definition 3.2.1. A *smooth distribution semigroup* is a continuous linear mapping $G : \mathcal{S}_+ \to L(E)$ which satisfies:

(i) $G(\varphi * \psi) = G(\varphi)G(\psi)$, φ, $\psi \in \mathcal{S}_+$ and
(ii) there exists a dense subset D of E such that for all $x \in D$ there exists a continuous function $u_x : [0,\infty) \to E$ such that $G(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt$, $\varphi \in \mathcal{S}_+$, $x \in D$.

If G is a smooth distribution semigroup, then for every regularizing sequence (ρ_n) in $\mathcal{D}_0$, we have $\lim_{n\to\infty} G(\rho_n) = I$. Put $\overline{G}(\varphi) := G(\mathcal{K}(\varphi))$, $\varphi \in \mathcal{D}$. By [**311**, Proposition 4.3], it follows that $\mathcal{D}'_0(L(E)) \ni \overline{G}$ is a dense distribution semigroup. The infinitesimal generator of G is said to be the infinitesimal generator of $\overline{G}$.

Proposition 3.2.2. [**311**, Proposition 4.7] *Suppose $\alpha \geqslant 0$ and A is the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\|S_\alpha(t)\| \leqslant Mt^\alpha(1+t^\beta)$, $t \geqslant 0$ for some $M > 0$ and $\beta \geqslant 0$. If A is densely defined, then A is the generator of a smooth distribution semigroup.*

Suppose $\alpha > 0$ and G is a smooth distribution semigroup. If G can be continuously extended to a mapping from $AC^{(\alpha)}(t^\alpha)$ into $L(E)$, then we say that G is a *smooth distribution semigroup of order α*.

Theorem 3.2.3. *Suppose A is densely defined and $\alpha > 0$. Then the following assertions are equivalent:*

(i) *A is the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\|S_\alpha(t)\| \leqslant Mt^\alpha$, $t \geqslant 0$ for some $M > 0$.*
(ii) *A is the generator of a smooth distribution semigroup of order α.*

The class of strong distribution semigroups has been recently introduced and analyzed in [**257**].

Let $\varphi \in \mathcal{D}$ and $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$. Put, for every $\varepsilon > 0$, $\varphi_\varepsilon(t) := \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$ and $\theta_\varepsilon(t) := \int_{-\infty}^{t/\varepsilon} \varphi(s)\,ds$, $t \in \mathbb{R}$.

Definition 3.2.4. An element $G \in \mathcal{D}_0'(L(E))$ is said to be a *strong distribution semigroup* if (d_1) and (d_2) hold with $C = I$ (cf. Subsection 3.1.3) and

(d_5^s): There exists a dense subset D of E such that, for every $x \in D$ and $\varphi \in \mathcal{D}$ with $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$, we have $\lim_{\varepsilon \to 0+} G(\varphi\theta_\varepsilon)x = G(\varphi)x$.

By [**257**, Theorem 3], we know that every strong distribution semigroup satisfies (d_5) with $C = I$ and, owing to Proposition 3.1.18, G is a distribution semigroup. It can be simply verified that every generator A of a strong distribution semigroup is stationary dense with $n(A) \leqslant 1$ as well as that every dense distribution semigroup is a strong distribution semigroup with $D = \mathcal{R}(G)$. Furthermore, the condition (d_5^s) can be characterized by the value of the operator-valued distribution $\mathcal{G}^{-1}$ in the sense of Łojasiewicz (cf. [**257**, Corollary 1]).

The following theorem follows from the proof of [**257**, Corollary 1] and a simple reasoning.

Theorem 3.2.5. *Suppose $\alpha > 0$, $\beta \in (0,1)$, $\tau \in (0,\infty)$, $M > 0$ and A generates an α-times integrated semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Denote by E_0 the set which consists of all elements $x \in E$ such that $\lim_{t\to 0+} \frac{\|S_\alpha(t)x\|}{t^{\alpha-1}} = 0$. If E_0 is dense in E, then A generates a strong distribution semigroup. In particular, this holds provided $\|S_\alpha(t)\| \leqslant Mt^{\alpha-\beta}$, $t \in [0, \frac{\tau}{2})$.*

It is predictable that there exists a distribution semigroup which does not satisfy (d_5^s).

Example 3.2.6. Let B be the generator of the standard translation group on $L^1(\mathbb{R})$. We have already seen that the operator $A = (B^*)^2$ is the non-densely defined generator of a sine function in $L^\infty(\mathbb{R})$ and that $n(A) = 1$. Taking into account Theorem 2.1.11 and Lemma 2.1.22, one gets that the operator $\mathcal{A}$ generates a twice integrated semigroup in $E \times E$. This implies that the operator $\mathcal{A}$ generates a distribution semigroup $\mathcal{G}$ in $E \times E$. Since $n(\mathcal{A}) = 2$, $\mathcal{G}$ does not satisfy (d_5^s).

Questions. (i) Suppose A generates a distribution semigroup G and $n(A) = 1$. Does it imply that G is a strong distribution semigroup?

(ii) Suppose G is a strong distribution semigroup and

(d_5^{glob}): For every $x \in E$ and $\varphi \in \mathcal{D}$ with $\int_{-\infty}^{\infty} \varphi(t)dt = 1$, $\lim_{\varepsilon\to 0+} G(\varphi\theta_\varepsilon)x = G(\varphi)x$.

Does (d_5^{glob}) automatically hold?

Now we would like to inscribe the basic structural properties of $[r]$-semigroups [**217**]. Suppose $r \geqslant 0$, $k \in \mathbb{N}_0$ and set

$$p_{rk}(\varphi) := \sum_{i=0}^{k} \big\| e^{rt} t^i \varphi^{(i)} \big\|_{L^1([0,\infty))}, \quad q_{rk}(\varphi) := \sum_{i=0}^{k} \big\| t^i (e^{rt}\varphi)^{(i)} \big\|_{L^1([0,\infty))}, \ \varphi \in \mathcal{D}_+.$$

It is well known that the inclusion mapping $id : (\mathcal{D}_+, p_{rk}) \to (\mathcal{D}_+, q_{rk})$ is a continuous mapping between normed spaces. Denote by T_{rk}, resp. D_{rk}, the completion of the space $(\mathcal{D}_+, p_{rk})$, resp. $(\mathcal{D}_+, q_{rk})$. Further on, put $h_\lambda(t) := e^{-\lambda t}H(t)$, $t \in \mathbb{R}$, where $H(\cdot)$ denotes the Heaviside function. Then $h_\lambda(t) \in T_{rk} \cap D_{rk}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > r$, and moreover, T_{rk} and D_{rk} are algebras for the convolution product $*_0$.

Definition 3.2.7. Suppose $r \geqslant 0$, $k \in \mathbb{N}_0$ and G is a distribution semigroup. Then it is said that G is an $[r,k]$-*semigroup,* resp. $\{r,k\}$-*semigroup,* if $G\Lambda$ can be continuously extended to a continuous linear mapping from T_{rk}, resp. D_{rk}, into $L(E)$. We say that G is an $[r]$-*semigroup*, resp. $\{r\}$-*semigroup*, if there exists $k \in \mathbb{N}_0$ such that G is an $[r,k]$-semigroup, resp. $\{r,k\}$-semigroup.

It is obvious that every $\{r,k\}$-semigroup is also an $[r,k]$-semigroup, $r \geqslant 0$, $k \in \mathbb{N}_0$ and that, for every $r > 0$, there exists a densely defined operator A such that A is the generator of an $[r,1]$-semigroup and that there is no $k \in \mathbb{N}_0$ such that A is the generator of an $\{r,k\}$-semigroup. Further on, the class of $[r,0]$-semigroups, $\{r,0\}$-semigroups and $(r,0)$-semigroups, introduced by Wang in [**418**], coincide for every $r \geqslant 0$. Therefore, the necessary and sufficient condition for a closed linear operator A to be the generator of an $[r,0]$-*semigroup* [**418**] is that there exists $M > 0$ such that $(r,\infty) \subseteq \rho(A)$ and that

$$\Big\|\frac{d^n}{d\lambda^n}R(\lambda : A)\Big\| \leqslant \frac{Mn!}{(\lambda - r)^{n+1}},\ \lambda > r,\ n \in \mathbb{N}_0.$$

As a consequence, we have that every Hille–Yosida operator is the generator of an $[r,0]$-semigroup for a convenable chosen $r \geqslant 0$.

Theorem 3.2.8. [**91**], [**217**] *Suppose $r \geqslant 0$, $k \in \mathbb{N}_0$ and $D(A)$ is dense in E. Then the following assertions are equivalent:*

(i) *The operator A is the generator of an $\{r,k\}$-semigroup.*

(ii) *The operator $A - r$ is the generator of an exponentially bounded k-times integrated semigroup $(S(t))_{t \geqslant 0}$ such that $\|S(t)\| = O(t^k)$, $t \geqslant 0$.*

(iii) *There exists $M > 0$ such that $(r,\infty) \subseteq \rho(A)$ and that*

$$\Bigg\|\frac{d^j}{d\lambda^j}\Big[\frac{R(\lambda + r : A)}{\lambda^k}\Big]\Bigg\| \leqslant M\frac{(k+j)!}{\lambda^{k+j+1}}, \quad \lambda > 0,\ j \in \mathbb{N}_0.$$

(iv) *The operator $r - A$ admits a smooth semispectral distribution of degree k.*

We summarize the properties of $[r]$-semigroups in the following theorem.

Theorem 3.2.9. [**217**] (i) *Suppose A is the generator of an $[r,k]$-semigroup for some $r \geqslant 0$ and $k \in \mathbb{N}$. Then $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > r\} \subseteq \rho(A)$ and there exists $M > 0$ such that, for every $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > r$, the following holds:*

$$\big\|R(\lambda : A)^n\big\| \leqslant \frac{Mn(n+1)\cdots(n+k-1)|\lambda|^k}{(\operatorname{Re}\lambda - r)^{n+k}}.$$

(ii) *Suppose A is the generator of an $[r,k]$-semigroup G for some $r \geqslant 0$. Then G is a smooth distribution semigroup and $n(A) \leqslant 1$. Furthermore, if E is reflexive, then A must be densely defined.*

(iii) *Suppose m, $m - k \in \mathbb{N}_0$, $r > 0$ and A is the generator of a k-times integrated semigroup $(S_k(t))_{t \geqslant 0}$ such that $\|S_k(t)\| = O\big(e^{rt}(t^k + t^m)\big)$, $t \geqslant 0$. Then A is the generator of an $[r,m]$-semigroup.*

(iv) *Suppose that* $A - r$ *is the generator of a* k*-times integrated semigroup* $(S_k(t))_{t\geqslant 0}$ *such that* $\|S_k(t)\| = O(t^k + t^m)$, $t \geqslant 0$. *Then* A *is the generator of an* $[r, m]$*-semigroup.*

(v) *Suppose that* A *is the generator of an* $[r, k]$*-semigroup for some* $r \geqslant 0$ *and* $k \in \mathbb{N}_0$. *Then the part of* $A - r$ *in* $\overline{D(A)}$ *generates a* k*-times integrated semigroup* $(S_k(t))_{t\geqslant 0}$ *in* $\overline{D(A)}$ *satisfying* $\|S_k(t)\| = O(t^k + t^{2k})$, $t \geqslant 0$.

(vi) *Suppose* A *is a closed linear operator,* $r > 0$ *and* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > r\} \subseteq \rho(A)$. *If there exist* $k \in \mathbb{N}_0$ *and* $M > 0$ *such that*

$$\|R(\lambda : A)\| \leqslant M \frac{|\lambda|^k}{(\operatorname{Re}\lambda - r)^{k+1}}, \ \operatorname{Re}\lambda > r, \tag{198}$$

then A *is the generator of a* $(k + 2)$*-times integrated semigroup* $(S_k(t))_{t\geqslant 0}$ *which satisfies* $\|S_k(t)\| = O\big(\min(e^{rt}t^{k+1}, e^{rt}t^{k+2})\big)$, $t \geqslant 0$ *and* A *is the generator of an* $[r, k + 2]$*-semigroup.*

(vii) *Suppose that* A *is the generator of an* $[r, k]$*-semigroup for some* $r > 0$ *and* $k \in \mathbb{N}_0$. *Then* A *is the generator of a* $(k + 2)$*-times integrated semigroup* $(S_{k+2}(t))_{t\geqslant 0}$ *such that* $\|S_{k+2}(t)\| = O\big(\min(e^{rt}t^{k+1}, e^{rt}t^{k+2})\big)$, $t \geqslant 0$.

(viii) *Suppose* A *is a closed linear operator and* $r \geqslant 0$. *Then* A *is the generator of an* $[r]$*-semigroup iff there exist* $k \in \mathbb{N}$ *and* $M > 0$ *such that* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > r\} \subseteq \rho(A)$ *and that* (198) *holds iff there exists* $k \in \mathbb{N}$ *such that* A *is the generator of a* k*-times integrated semigroup* $(S_k(t))_{t\geqslant 0}$ *such that* $\|S_k(t)\| = O(e^{rt}t^k)$, $t \geqslant 0$.

(ix) *Suppose* A *is a closed linear operator and* $r \geqslant 0$. *Then* A *is the generator of an* $\{r\}$*-semigroup iff there exist* $k \in \mathbb{N}$ *and* $M>0$ *such that* $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > r\} \subseteq \rho(A)$ *and*

$$\|R(\lambda : A)\| \leqslant M \frac{|\lambda - r|^k}{(\operatorname{Re}\lambda - r)^{k+1}}, \ \operatorname{Re}\lambda > r$$

iff there exists $k \in \mathbb{N}$ *such that* $A - r$ *is the generator of a* k*-times integrated semigroup* $(S_k(t))_{t\geqslant 0}$ *such that* $\|S_k(t)\| = O(t^k)$, $t \geqslant 0$.

DEFINITION 3.2.10. [**91**] Denote by $\mathcal{A}$ the space consisted of all Laplace transforms of functions from $\mathcal{S}_+$, equipped with the following system of seminorms:

$$\|g\|_{j,k} := \|t^j \varphi^{(k)}(t)\|_{L^1([0,\infty))}, \ j, \ k \in \mathbb{N}_0, \ g = \mathcal{L}(\varphi) \in \mathcal{A}.$$

A *smooth semispectral distribution for* A is a continuous algebraic homomorphism $f : \mathcal{A} \to L(E)$ which satisfies:

(i) $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \subseteq \rho(A)$ and $f\big(\frac{1}{\lambda - \cdot}\big) = R(\lambda : A)$ whenever $\operatorname{Re}\lambda < 0$,

(ii) $f\big(g(\frac{\cdot}{n})\big)x \to x$, $n \to \infty$ for all $x \in E$ and $g \in \mathcal{A}$ such that $g(0) = 1$.

If $-A$ admits a smooth semispectral distribution, then A must be densely defined. We refer the interested reader to [**104**] for the notion of a *regularized quasi-spectral distribution* which removes any density assumption from Definition 3.2.10. Suppose that A is the densely defined generator of a global k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ which satisfies $\|S_k(t)\| = O(t^k(1 + t^n))$, $t \geqslant 0$ for some n, $k \in \mathbb{N}_0$. Then it is well known that $-A$ admits a smooth semispectral distribution.

THEOREM 3.2.11. (i) *Suppose that A is a closed, densely defined linear operator, $m, k \in \mathbb{N}$ and $m \geqslant k$. Then the following assertions are equivalent:*

(i.1) *A is the generator of a distribution semigroup G and there exists $C > 0$ such that $\|G(\varphi)\| \leqslant C \int_0^\infty (t^k + t^m)|\varphi^{(k)}(t)|\,dt$, $\varphi \in \mathcal{D}$.*

(i.2) *A is the generator of a k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ such that $\|S(t)\| = O(t^k + t^m)$, $t \geqslant 0$.*

(i.3) *$-A$ admits a smooth semispectral distribution $f(\cdot)$ such that, for an appropriate $C > 0$, $\|f(\hat{\varphi})\| \leqslant C\|(t^k + t^m)\varphi^{(k)}\|_1$, $\varphi \in \mathcal{D}$.*

(ii) *Suppose A is a closed, densely defined linear operator and $r > 0$. Then the following assertions are equivalent:*

(ii.1) *A is the generator of an $[r]$-semigroup.*

(ii.2) *$r - A$ admits a smooth semispectral distribution $f(\cdot)$ such that there exists an appropriate $C > 0$ with $\|f(\hat{\varphi})\| \leqslant C \int_0^\infty (t^k + t^m)|\varphi^{(k)}(t)|\,dt$, $\varphi \in \mathcal{D}$, for some k, $m \in \mathbb{N}$ with $m \geqslant k$.*

(ii.3) *A is the generator of a k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ such that $\|S_k(t)\| = O(e^{rt}(t^k + t^m))$, $t \geqslant 0$ for some k, $m \in \mathbb{N}$ with $m \geqslant k$.*

(ii.4) *$A - r$ is the generator of a k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ such that $\|S_k(t)\| = O(t^k + t^m)$, $t \geqslant 0$ for some k, $m \in \mathbb{N}$ with $m \geqslant k$.*

It is also worth noting that, for every generator A of an $[r, k]$-semigroup, where $r \geqslant 0$ and $k \in \mathbb{N}$, the operator $r - A$ admits $\mathcal{A}_{k+2,n}$ functional calculus for all $n \in \mathbb{N}$ with $n \geqslant k + 1$ (cf. [**91**] for the notion).

EXAMPLE 3.2.12. (i) [**8**] Let $p \in (1, \infty)$. Denote by J_p the Riemann–Liouville semigroup on $L^p((0, 1))$, that is

$$(J_p(z)f)(x) := \frac{1}{\Gamma(z)} \int_0^x (x - y)^{z-1} f(y)\,dy,\ f \in L^p((0, 1)),\ x \in (0, 1),\ \operatorname{Re} z > 0.$$

Designate by A_p the generator of J_p. Then the operator iA_p generates a C_0-group $(T_p(t))_{t\in\mathbb{R}}$ on $L^p((0, 1))$ and the following holds $\|T_p(t)\| = O((1 + t^2)e^{|t|\frac{\pi}{2}})$, $t \in \mathbb{R}$. Set $G_p(\varphi) := \int_0^\infty \varphi(t)T_p(t)\,dt$, $\varphi \in \mathcal{D}$. Then G_p is a dense $[\frac{\pi}{2}, 2]$-semigroup in $L^p((0, 1))$ with the generator iA_p. Evidently, $-iA_p$ is also the generator of a $[\frac{\pi}{2}, 2]$-semigroup in $L^p((0, 1))$.

(ii) [**101**] Suppose $p \in [1, \infty)$ and $m : \mathbb{R} \to (0, \infty)$ is a measurable function which satisfies:

$$\Big(\sup_{s\in\mathbb{R}} \frac{m(s - t)}{m(s)}\Big)^{\frac{1}{p}} \leqslant M(1 + t^k),\ t \geqslant 0, \tag{199}$$

for some $k \in \mathbb{N}$ and $M > 0$. Let $r > 0$ be fixed. Put $(T_p(t)f)(x) := e^{rt}f(x + t)$, $x \in \mathbb{R}$, $t \geqslant 0$, $f \in L^p(\mathbb{R}, m(t)dt)$. Then $(T_p(t))_{t\geqslant 0}$ is a C_0-semigroup in $L^p(\mathbb{R}, m(t)dt)$ and

$$\|T_p(t)\| = e^{rt}\Big(\sup_{s\in\mathbb{R}} \frac{m(s - t)}{m(s)}\Big)^{\frac{1}{p}} = O\big(e^{rt}(1 + t^k)\big).$$

Put now $G_p(\varphi) := \int_0^\infty \varphi(t)T_p(t)\,dt$, $\varphi \in \mathcal{D}$. Then G_p is a dense $[r, k]$-semigroup in $L^p(\mathbb{R}, m(t)dt)$. Notice, if $m(\cdot)$ is a positive polynomial, then (199) holds for appropriate numbers $k \in \mathbb{N}$ and $M > 0$.

(iii) [**14**] Suppose $r > 0$,

$$E := \Big\{f \in C([0,\infty)) : \lim_{x\to\infty} \frac{f(x)}{x+1} = 0\Big\},$$

$$\|f\| := \sup_{x\geqslant 0} \frac{|f(x)|}{x+1},\ f \in E,\ \ (T(t)f)(x) := f(x+t),\ f \in E,\ t \geqslant 0,\ x \geqslant 0.$$

Then $(T(t))_{t\geqslant 0}$ is a C_0-semigroup and $\|T(t)\| = 1+t$, $t \geqslant 0$. The generator A of $(T(t))_{t\geqslant 0}$ is just the operator $\frac{d}{dx}$ with maximal domain in E. Set

$$G(\varphi) := \int_0^\infty \varphi(t)e^{rt}T(t)\,dt,\ \varphi \in \mathcal{D}.$$

Then G is a dense $[r,1]$-semigroup with the generator $A+r$. Suppose that G is an $\{r,k\}$-semigroup for some $k \in \mathbb{N}$. Then A generates a k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ such that $\|S_k(t)\| = O(t^k)$, $t \geqslant 0$. Since $S_k(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}T(s)\,ds$, $t \geqslant 0$, it follows that

$$\sup_{x\geqslant 0} \frac{1}{x+1}\left|\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!} f(x+s)\,ds\right| \leqslant Mt^k \sup_{x\geqslant 0} \frac{|f(x)|}{x+1},\ \ f \in E,\ t \geqslant 0.$$

Let $f(t) = \sqrt{t}$, $t \geqslant 0$. Then one gets

$$\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!}\sqrt{s}\,ds \leqslant \sup_{x\geqslant 0} \frac{1}{x+1}\left|\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!}\sqrt{x+s}\,ds\right| \leqslant \frac{Mt^k}{2},\ \ t \geqslant 0.$$

This is a contradiction. Furthermore, for every $k \in \mathbb{N}_0$, the operator A generates a k-times integrated semigroup $(S_k(t))_{t\geqslant 0}$ such that $\|S_k(t)\| = O(t^k + t^{k+1})$, $t \geqslant 0$ and there does not exist a number $\alpha \in [0,k+1)$ such that $\|S_k(t)\| = O(t^k + t^\alpha + 1)$, $t \geqslant 0$.

(iv) [**217**] For every $r > 0$ and $k \in \mathbb{N}$ there exists a dense $[r,k]$-semigroup which is not an $[r,k-1]$-semigroup. Indeed, suppose that $T \in L(E)$ is nilpotent and that $T^{k+1} = 0$. Define

$$T(t) := e^{rt}\sum_{i=0}^k \frac{T^i t^i}{i!},\ t \geqslant 0.$$

Then $\|T(t)\| = O\big(e^{rt}(1+t^k)\big)$, $t \geqslant 0$, $(T(t))_{t\geqslant 0}$ is a C_0-semigroup generated by $T+r$ and $T+r$ generates a dense $[r,k]$-semigroup. Put now $E := \mathbb{R}^{k+1}$ with the sup-norm and

$$T(x_1,x_2,\ldots,x_{k+1}) := (x_2,\ldots,x_{k+1},0),\ x_i \in \mathbb{R},\ i = 1,2,\ldots,k+1.$$

Then $T^{k+1} = 0$ and $T+r$ generates an $[r,k]$-semigroup G. Suppose that G is an $[r,k-1]$-semigroup. By Theorem 3.2.9, it follows that the operator T generates a $(k-1)$-times integrated semigroup $(S_{k-1}(t))_{t\geqslant 0}$ such that $\|S_{k-1}(t)\| = O(t^{k-1} + t^{2k-2})$. If $k = 1$, this means that T generates a bounded C_0-semigroup. Then the contradiction is obvious since $\|e^{-rt}T(t)\| = 1 + t + \cdots + \frac{t^k}{k!}$, $t \geqslant 0$. If $k > 1$ and $t \geqslant 0$, then

$$S_{k-1}(t)(x_1, x_2, \dots, x_{k+1}) = \int_0^t \frac{(t-s)^{k-2}}{(k-2)!} e^{-rs} T(s)(x_1, x_2, \dots, x_{k+1})\, ds.$$

On the other hand, direct computation shows that

$$\|S_{k-1}(t)\| = \frac{t^{k-1}}{(k-1)!} + \cdots + \frac{t^{2k-1}}{(2k-1)!},\ t \geqslant 0.$$

This is in contradiction with $\|S_{k-1}(t)\| = O(t^{k-1} + t^{2k-2})$.

Now we clarify the basic properties of *differentable* and *analytic* distribution semigroups. Recall, a scalar valued distribution δ_t is defined by $\delta_t(\varphi) = \varphi(t)$, $\varphi \in \mathcal{D}$ $(t \in \mathbb{R})$.

We introduce differentiable and analytic semigroups following the approach of Barbu [**28**] (cf. also Da Prato, Mosco [**85**]–[**86**] and Fujiwara [**135**]).

Definition 3.2.13. Suppose that G is a distribution semigroup and that $\alpha \in (0, \frac{\pi}{2}]$. Then it is said that G is an *(infinitely) differentiable distribution semigroup*, resp. an *analytic distribution semigroup of angle* α, iff the mapping $t \mapsto G(\delta_t) \in L(E)$, $t > 0$ is (infinitely) differentiable, resp. iff the mapping $t \mapsto G(\delta_t) \in L(E)$, $t > 0$ can be analytically extended to the sector Σ_α, where we assume that $L(E)$ is equipped with the strong topology.

The next characterization of differentiable distribution semigroups with densely defined generators was proved by Barbu.

Theorem 3.2.14. [**28**] *Suppose that A is a closed, densely defined linear operator. Then A generates a differentiable distribution semigroup iff there exist $n \in \mathbb{N}$ and $\omega \geqslant 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ such that*

(200)
$$\Upsilon_{\sigma,\omega} := \big\{\lambda \in \mathbb{C} : -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma \leqslant \operatorname{Re}\lambda \leqslant \omega\big\} \cup \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\big\} \subseteq \rho(A)$$

(201)
$$\|R(\lambda : A)\| \leqslant M_\sigma(1 + |\lambda|)^n,\ \ \lambda \in \Upsilon_{\sigma,\omega}.$$

Further on, every dense differentiable (DS) G must be infinitely differentiable and exponential [**28**]. Now we state the following important extension of Theorem 3.2.14.

Theorem 3.2.15. *Suppose A is a closed, linear operator. Then the assertions* (i), (ii), (iii), (iv), (v) *and* (vi) *are equivalent, where:*

(i) *There exists $n \in \mathbb{N}$ such that A generates an exponentially bounded n-times integrated semigroup $(S_n(t))_{t \geqslant 0}$ such that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable.*
(ii) *There exists $n \in \mathbb{N}$ such that A generates an exponentially bounded n-times integrated semigroup $(S_n(t))_{t \geqslant 0}$ such that the mapping $t \mapsto S_n(t)$, $t > 0$ is $(n+1)$-times differentiable.*
(iii) *A generates a differentiable distribution semigroup.*

(iv) *There exist* $n \in \mathbb{N}$ *and* $\omega \geqslant 0$ *such that, for every* $\sigma > 0$, *there exist* $C_\sigma > 0$ *and* $M_\sigma > 0$ *such that* (200)-(201) *hold.*

(v) *There exists* $n \in \mathbb{N}$ *such that* A *generates a global* n*-times integrated semigroup* $(S_n(t))_{t\geqslant 0}$ *such that the mapping* $t \mapsto S_n(t)$, $t > 0$ *is infinitely differentiable.*

(vi) *There exists* $n \in \mathbb{N}$ *such that* A *generates a global* n*-times integrated semigroup* $(S_n(t))_{t\geqslant 0}$ *such that the mapping* $t \mapsto S_n(t)$, $t > 0$ *is* $(n+1)$*-times differentiable.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. Suppose that (ii) holds and put $G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)S_n(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then G is an (EDSG) and, by the closed graph theorem, the differentiability of G follows immediately if one shows that $\frac{d^n}{dt^n}S_n(t) \subseteq G(\delta_t)$, $t > 0$. Since

$$G(\delta_t) = \big\{(x,y) \in E \times E : G(\varphi(\cdot - t))x = G(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\big\},\ t > 0,$$

we have to prove that:

$$\int_0^\infty \varphi^{(n)}(s-t)S_n(s)x\,ds = \int_0^\infty \varphi^{(n)}(s)S_n(s)S_n^{(n)}(t)x\,ds,\ x \in E,\ t > 0,\ \varphi \in \mathcal{D}_0. \tag{202}$$

Towards this end, notice that, for every $\varphi \in \mathcal{D}_0$, $t > 0$ and $x \in E$,

$$\int_0^\infty \varphi^{(n)}(s)S_n(s)S_n^{(n)}(t)x\,ds = \int_0^\infty \varphi^{(n)}(s)\frac{d^n}{dt^n}S_n(s)S_n(t)x\,ds$$

$$= \int_0^\infty \varphi^{(n)}(s)\frac{d^n}{dt^n}\left[\left(\int_s^{t+s} - \int_0^t\right)\frac{(t+s-r)^{n-1}}{(n-1)!}S_n(r)x\,dr\right]ds$$

$$= \int_0^\infty \varphi^{(n)}(s)\frac{d^{n-1}}{dt^{n-1}}\left[\left(\int_s^{t+s} - \int_0^t\right)\frac{(t+s-r)^{n-2}}{(n-2)!}S_n(r)x\,dr - \frac{s^{n-1}}{(n-1)!}S_n(t)x\right]ds$$

$$= \int_0^\infty \varphi^{(n)}(s)\frac{d^{n-1}}{dt^{n-1}}\left[\left(\int_s^{t+s} - \int_0^t\right)\frac{(t+s-r)^{n-2}}{(n-2)!}S_n(r)x\,dr\right]ds.$$

Repeating this procedure sufficiently many times leads us to the following:

$$\int_0^\infty \varphi^{(n)}(s)S_n(s)S_n^{(n)}(t)x\,ds = \int_0^\infty \varphi^{(n)}(s)S_n(t+s)x\,ds = \int_0^\infty \varphi^{(n)}(s-t)S_n(s)x\,ds.$$

Thus, (202) holds and G is differentiable, as claimed. Suppose now that A generates a differentiable distribution semigroup G and that $t > 0$. By [**249**, Theorem 3.5], the operator A is stationary dense. Put $F := \overline{D(A^{n(A)})}$. Then one can simply verify that $G_{|F}$ is a dense (DS) in the Banach space F with the generator $A_{|F}$. This implies that there exists $n \in \mathbb{N}$ such that (cf. [**216**], [**252**] and [**418**]), for every $k \in \mathbb{N}$, $A_{|F}$ generates a local (kn)-times integrated semigroup $(S^F_{kn}(s))_{s\in[0,kt)}$ in F which additionally satisfies:

$$G_{|F}(\varphi)x = (-1)^{kn} \int_0^\infty \varphi^{(kn)}(s) S^F_{kn}(s)x\,ds,\ x \in F,\ \varphi \in \mathcal{D}_{(-\infty,kt)}.$$

Let $x \in D_\infty(A)$. We will prove that $G(\delta_t)x \in F$ and that $G_{|F}(\delta_t)x = \frac{d^{2n}}{dt^{2n}} S^F_{2n}(t)x$. Suppose $\varphi \in \mathcal{D}_{[0,(k-1)t)}$ for some $k \in \mathbb{N}$ with $k \geqslant 3$. Since

$$S_{kn}(s) = \int_0^s \frac{(s-r)^{(k-2)n-1}}{((k-2)n-1)!} S_{2n}(r)x\,dr,\ s \in [0,2t),$$

we have

$$S^F_{kn}(s)\Big(\frac{d^{2n}}{dr^{2n}} S^F_{2n}(r)x\Big)_{r=t} = S^F_{kn}(s)\Big(\frac{d^{kn}}{dr^{kn}} S^F_{kn}(r)x\Big)_{r=t} = \Big(\frac{d^{kn}}{dr^{kn}} S^F_{kn}(s)S^F_{kn}(r)x\Big)_{r=t}$$

for all $s \in [0,(k-1)t)$. The proof of the implication (ii) $\Rightarrow$ (iii) implies

$$\int_0^\infty \varphi^{(kn)}(s-t)S^F_{kn}(s)x\,ds = \int_0^\infty \varphi^{(kn)}(s)\Big(\frac{d^{kn}}{dr^{kn}} S^F_{kn}(s)S^F_{kn}(r)x\Big)_{r=t} ds.$$

Hence, $G(\delta_t)x = G_{|F}(\delta_t)x = \frac{d^{2n}}{dt^{2n}} S^F_{2n}(t)x$, $x \in D_\infty(A)$. On the other hand, exploitation of [**249**, Proposition 2.1(iv)] gives that $\overline{D_\infty(A)} = F$, and the continuity of mapping $t \mapsto G(\delta_t)$, $t > 0$ implies $G(\delta_t)x = G_{|F}(\delta_t)x \in F$. By the foregoing, one has that $G_{|F}$ is a dense differentiable (DS) in F generated by $A_{|F}$. The assertion (iv) is a consequence of Theorem 3.2.14 and [**249**, Corollary 2.2]. The implication (iv) $\Rightarrow$ (i) can be proved by the next insignificant modification of the proof of [**355**, Theorem 4.7, p. 54]. Suppose $\omega_0 > \omega$ and put

$$\Gamma_1 := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 2C_\sigma - \sigma \ln(-\operatorname{Im}\lambda),\ -\infty < \operatorname{Im}\lambda \leqslant -e^{\frac{2C_\sigma}{\sigma}}\},$$
$$\Gamma_2 := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \omega_0,\ e^{\frac{2C_\sigma}{\sigma}} \leqslant \operatorname{Im}\lambda \leqslant e^{\frac{2C_\sigma}{\sigma}}\}$$
$$\Gamma_3 := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 2C_\sigma - \sigma \ln(\operatorname{Im}\lambda),\ e^{\frac{2C_\sigma}{\sigma}} \leqslant \operatorname{Im}\lambda < +\infty\},$$

$\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_k := \{\lambda \in \Gamma : |\lambda| \leqslant k\}$. The curves Γ and Γ_i are oriented so that $\operatorname{Im}\lambda$ increases along Γ and Γ_i, $i = 1,2,3$. Set, for a sufficiently large $k_0 \in \mathbb{N}$,

$$S^k(t) := \frac{1}{2\pi i} \int_{\Gamma_k} e^{\lambda t} \frac{R(\lambda : A)}{\lambda^{n+2}}\,d\lambda,\ \ t \geqslant 0,\ k \geqslant k_0.$$

It is simple to prove that $\frac{d^j}{dt^j} S^k(t) = \frac{1}{2\pi i} \int_{\Gamma_k} e^{\lambda t} \lambda^{j-n-2} R(\lambda : A)\,d\lambda$, $t \geqslant 0$, $k \geqslant k_0$, $j \in \mathbb{N}$. Let $k_0 < k < l$ and $\varsigma > 0$. Then we obtain

$$\Big\|\frac{d^j}{dt^j} S^k(t) - \frac{d^j}{dt^j} S^l(t)\Big\| = \frac{1}{2\pi i} \Big\| \int_{\Gamma_l \cap \{\lambda \in \mathbb{C}\ k \leqslant |\lambda| \leqslant l\}} e^{\lambda t} \lambda^{j-n-2} R(\lambda : A)\,d\lambda \Big\|$$

$$\leqslant \text{Const}\,\frac{M_\sigma}{2\pi}e^{2C_\sigma t}\int\limits_{\Gamma_l\,\cap\,\{\lambda\in\mathbb{C}\ k\leqslant|\lambda|\leqslant l\}}|\operatorname{Im}\lambda|^{-\sigma t}\big(|2C_\sigma-\ln|\operatorname{Im}\lambda|+i\operatorname{Im}\lambda|\big)^{j-2}|d\lambda|,$$

for all $j\in\mathbb{N}_0$. Since $|\operatorname{Im}\lambda|^{-\sigma t}\big(1+|2C_\sigma-\ln|\operatorname{Im}\lambda|+i\operatorname{Im}\lambda|\big)^{j-2}\sim|\operatorname{Im}\lambda|^{j-2-\sigma t}$, $|\lambda|\to\infty$, $\lambda\in\Gamma$, one gets that, for every $t>\max\big(\frac{j-1}{\sigma},0\big)$ and $j\in\mathbb{N}_0$, the sequence $\big(\frac{d^j}{dt^j}S^k(t)\big)_k$ is convergent in $L(E)$ and that the convergence is uniform on every compact subset of $\big[\max\big(\frac{j-1}{\sigma},0\big)+\varsigma,\infty\big)$. Put $S_j(t):=\lim_{k\to\infty}\frac{d^j}{dt^j}S^k(t)$, $j\in\mathbb{N}_0$, $t>\frac{j}{\sigma}$. It is obvious that $\frac{d}{dt}S_j(t)=S_{j+1}(t)$, $j\in\mathbb{N}_0$, $t>\frac{j}{\sigma}+\varsigma$. This implies that the mapping $t\mapsto S_0(t)$, $t>\frac{j+2}{\sigma}+\varsigma$ is j-times differentiable and that $\frac{d^j}{dt^j}S_0(t)=S_j(t)$, $t>\frac{j}{\sigma}+\varsigma$. Set also

$$S(t):=\frac{1}{2\pi i}\int\limits_{\omega_0-i\infty}^{\omega_0+i\infty}e^{\lambda t}\frac{R(\lambda:A)}{\lambda^{n+2}}\,d\lambda,\ t\geqslant 0.$$

Then the proof of [**14**, Theorem 2.5.1] implies that $(S(t))_{t\geqslant 0}$ is an exponentially bounded $(n+2)$-times integrated semigroup, and thanks to the residue theorem, we have that $S_0(t)=S(t)$, $t>\frac{1}{\sigma}$. Consequently, the mapping $t\mapsto S(t)$, $t>\frac{j}{\sigma}+\varsigma$ is j-times differentiable. The arbitrariness of σ shows that the mapping $t\mapsto S(t)$, $t>0$ is infinitely differentiable, as required. Therefore, we have proved (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). The proof of implication (v) $\Rightarrow$ (vi) is trivial and the proof of implication (vi) $\Rightarrow$ (iii) can be obtained following the lines of the proof of implication (ii) $\Rightarrow$ (iii). Certainly, (iii) $\Rightarrow$ (i) $\Rightarrow$ (v) and the proof of theorem is completed. $\square$

Multiplication operators in L^∞-type spaces can serve as examples of non-densely defined generators of differentiable (DS)'s.

REMARK 3.2.16. (i) Suppose that A generates a differentiable (DS) G. Then the proof of Theorem 3.2.15 yields that the mapping $t\mapsto G(\delta_t)$, $t>0$ is infinitely differentiable and that $G(\varphi)x=\int_0^\infty\varphi(t)G(\delta_t)x\,dt$ for all $x\in E$ and $\varphi\in\mathcal{D}$ with $\operatorname{supp}\varphi\subseteq(0,\infty)$. Furthermore, G is an (EDS), $G_{|F}$ is a dense (DS) in the Banach space F generated by $A_{|F}$ and $G_{|F}$ is differentiable whenever G is; in this case, $G(\delta_t)x=G_{|F}(\delta_t)x$, $t>0$, $x\in F$. By Theorem 3.2.15, there exists $n\in\mathbb{N}$ such that A generates an exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ such that the mapping $t\mapsto S_n(t)$, $t>0$ is infinitely differentiable. Hence, $G(\delta_t*\rho_k)x=(-1)^n\int_0^\infty\rho_k^{(n)}(s-t)S_n(s)x\,ds=\int_0^\infty\rho_k(s-t)\frac{d^n}{ds^n}S_n(s)x\,ds$ and $\lim_{k\to\infty}G(\delta_t*\rho_k)x=G(\delta_t)x$, $x\in E$. Furthermore, the Lebesgue dominated convergence theorem implies

$$\frac{d}{dt}G(\delta_t*\rho_k)x=(-1)^{n+1}\int\limits_0^\infty\rho_k^{(n+1)}(s-t)S_n(s)x\,ds=\int\limits_0^\infty\rho_k(s-t)\frac{d^{n+1}}{ds^{n+1}}S_n(s)x\,ds,$$

$$\lim_{k\to\infty}\frac{d}{dt}G(\delta_t*\rho_k)x=\frac{d}{dt}G(\delta_t)x=\frac{d^{n+1}}{dt^{n+1}}S_n(t)x,\ \ x\in E.$$

Inductively,

$$\frac{d^l}{dt^l}G(\delta_t * \rho_k)x = (-1)^{n+l}\int\limits_0^\infty \rho_k^{(n+l)}(s-t)S_n(s)x\,ds = \int\limits_0^\infty \rho_k(s-t)\frac{d^{n+l}}{ds^{n+l}}S_n(s)x\,ds,$$

$$\lim_{k\to\infty}\frac{d^l}{dt^l}G(\delta_t * \rho_k)x = \frac{d^l}{dt^l}G(\delta_t)x = \frac{d^{n+l}}{dt^{n+l}}S_n(t)x,\ l \in \mathbb{N}_0,\ x \in E. \tag{203}$$

Having in mind [**249**, Proposition 2.1, Corollary 2.2], we get that the spectral characterizations clarified in [**28**, Theorem 4, Corollaries p. 423 and 427, Theorem 5] present necessary conditions for the generation of non-dense *distribution semigroups of class* C^L*,* A^ϱ*,* $\varrho \geqslant 1$ *and* A^ϱ_γ*,* $\varrho \geqslant 1$*,* $\gamma \geqslant 0$*.* The sufficiency of such spectral characterizations follows from (203) and the proofs of cited results. In particular, [**28**, Corollary, p. 423] completely describes the spectral properties of generators of non-dense real analytic (DS)'s. Important examples of distribution semigroups of class C^L with $L_j = j!^{s/j}$, $s > 1$, $j \in \mathbb{N}_0$ follows from the researches of Belinskiy, Lasiecka [**39**], Chen, Triggiani [**57**], Favini, Triggiani [**131**], Markin [**299**] and Shubov [**386**].

(ii) Suppose that A generates a global n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto S_n(t)$, $t > 0$ is n-times continuously differentiable. By [**224**, Corollary 3.3], we have that $(\frac{d^n}{dt^n}S_n(t))_{t\geqslant 0}$ is a semigroup [**155**], and therefore, there exist $M > 0$ and $\omega \geqslant 0$ such that $\|\frac{d^n}{dt^n}S_n(t)\| \leqslant Me^{\omega t}$, $t \geqslant 1$. This implies that $(S_n(t))_{t\geqslant 0}$ must be exponentially bounded, and the equivalence (ii) $\Leftrightarrow$ (vi) of Theorem 3.2.15 is not surprising. If, additionally, the mapping $t \mapsto S_n(t)$, $t > 0$ is $(n+1)$-times differentiable, then the proof of Theorem 3.2.15 implies that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable.

(iii) Let us note that Renardy [**372**] proved that there exists a differentiable C_0-semigroup and its bounded perturbation that is not differentiable. Some other references on differentiability of perturbed semigroups are [**29**]–[**31**], [**112**], [**165**] and [**234**].

PROPOSITION 3.2.17. [**234**] (cf. also Theorem 2.1.31) *Suppose A is a closed linear operator, K satisfies* (P1)*, $r \geqslant 1$ and there exists $\omega \geqslant 0$ such that, for every $\sigma > 0$, there exists $C_\sigma > 0$ such that the function $\tilde{K}$ can be analytically extended to an open neighborhood $\Omega_{\sigma,\omega}$ of the region $\Upsilon_{\sigma,\omega}$. Denote, for every $\sigma > 0$, by $g_\sigma(\cdot)$ the analytic extension of the function $\tilde{K}$ to the region $\Omega_{\sigma,\omega}$ and suppose that, for every $\sigma > 0$:*

(i) $\Omega_{\sigma,\omega} =: \{\lambda \in \Upsilon_{\sigma,\omega} : g_\sigma(\lambda) \neq 0\} \subseteq \rho_C(A)$,
(ii) *there exists an analytic function $h_\sigma : \Omega_{\sigma,\omega} \to L(E)$ such that $h_\sigma(\lambda) = g_\sigma(\lambda)(\lambda - A)^{-1}C$, $\lambda \in \Omega_{\sigma,\omega}$ and*
(iii) *there exists $M_\sigma > 0$ such that $\|h_\sigma(\lambda)\| \leqslant M_\sigma|\lambda|^r$, $\lambda \in \Upsilon_{\sigma,\omega}$.*

Then, for every $\zeta > 1$, A is a subgenerator of a norm continuous, exponentially bounded $\big(K*_0\frac{t^{\zeta+r-1}}{\Gamma(\zeta+r)}\big)$-convoluted C-semigroup $(S(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto S(t)$, $t > 0$ is infinitely differentiable.

THEOREM 3.2.18 (Differentiability of integrated semigroups, [**234**]). (i) *Suppose $n \in \mathbb{N}$, $\omega' \geqslant 0$, $M > 0$ and A is the densely defined generator of an n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ which satisfies $\|S_n(t)\| \leqslant Me^{\omega' t}$, $t \geqslant 0$. Then the following assertions are equivalent:*

(i.1) *The mapping $t \mapsto S_n(t)$, $t > 0$ is ($(n+1)$-times) infinitely differentiable.*

(i.2) *There exists $\omega \geqslant \omega'$ such that, for every $\sigma > 0$, there exists $C_\sigma > 0$ and $M_\sigma > 0$, such that $\Upsilon_{\sigma,\omega} \subseteq \rho(A)$ and that*

$$\|R(\lambda\colon A)\| \leqslant M_\sigma |\lambda|^n |\operatorname{Im}\lambda|, \ \lambda \in \Upsilon_{\sigma,\omega}, \ \operatorname{Re}\lambda \leqslant \omega.$$

(ii) *Suppose $\alpha \geqslant 0$, $\omega' \geqslant 0$, $M > 0$ and A is the densely defined generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\|S_\alpha(t)\| \leqslant Me^{\omega' t}$, $t \geqslant 0$. Then the following assertions are equivalent:*

(ii.1) *The mapping $t \mapsto S_\alpha(t)$, $t > 0$ is ($(\lceil\alpha\rceil+1)$-times) infinitely differentiable.*

(ii.2) *There exists $\omega \geqslant \omega'$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ such that $\Upsilon_{\sigma,\omega} \subseteq \rho(A)$ and that*

$$\|R(\lambda\colon A)\| \leqslant M_\sigma |\lambda|^{\lceil\alpha\rceil+1} |\operatorname{Im}\lambda|, \ \lambda \in \Upsilon_{\sigma,\omega}, \ \operatorname{Re}\lambda \leqslant \omega.$$

(iii) *Suppose $\alpha \geqslant 0$, $\omega' \geqslant 0$, $M > 0$ and A generates a global α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$. Then the following assertions are equivalent:*

(iii.1) *The mapping $t \mapsto S_\alpha(t)$, $t > 0$ is ($(\lceil\alpha\rceil+1)$-times) infinitely differentiable.*

(iii.2) *There exist $m \in \mathbb{N}$ and $\omega \geqslant 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ satisfying $\Upsilon_{\sigma,\omega} \subseteq \rho(A)$ and*

$$\|R(\lambda\colon A)\| \leqslant M_\sigma |\lambda|^m, \ \lambda \in \Upsilon_{\sigma,\omega}, \ \operatorname{Re}\lambda \leqslant \omega.$$

The following result is closely related to [**28**, Theorem 6], [**86**, Theorem 1.1], [**135**, Theorem 4] and clarifies the basic structural properties of non-dense analytic distribution semigroups.

THEOREM 3.2.19. *Let $\alpha \in (0, \frac{\pi}{2}]$ and let A be a closed linear operator.*

(i) *The following assertions are equivalent.*

(i.1) *A generates an analytic (DS) G of angle α.*

(i.2) *There exist $n \in \mathbb{N}$, $M > 0$, $\omega \geqslant 0$ and an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ so that A generates an n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ which satisfies $\mathbf{S}_n(t) = S_n(t)$, $t > 0$ and $\|S_n(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.*

(i.3) *There are an $n \in \mathbb{N}$ and an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ so that A generates an n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ which satisfies $\mathbf{S}_n(t) = S_n(t)$, $t > 0$.*

(ii) *A generates an analytic n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ of angle α for some $n \in \mathbb{N}$ iff A generates an analytic (DS) of angle α which additionally satisfies the next condition:*

$$\exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0\ \forall \gamma \in (0,\alpha)\ \forall x \in E\ \exists \lim_{z\to 0,\ z\in\Sigma_\gamma} \int_1^z \frac{(z-s)^{n-1}}{(n-1)!} G(\delta_s)x\,ds. \tag{204}$$

(iii) *There is an $n \in \mathbb{N}$ such that operator A generates an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ of angle α iff A generates an analytic (DS) of angle α which additionally satisfies* (204) *and the next condition:*

$$\exists n_1 \in \mathbb{N}\ \forall n \geqslant n_1\ \forall \gamma \in (0,\alpha)\ \exists M_\gamma,\ \omega_\gamma > 0\ \forall z \in \Sigma_\gamma :$$

$$\Bigg\| \int\limits_1^z \frac{(z-s)^{n-1}}{(n-1)!} G(\delta_s)\, ds \Bigg\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}.$$

(iv) *Suppose that, for every $\gamma \in (0,\alpha)$, there exist $M_\gamma > 0$, $\omega_\gamma \geqslant 0$ and $n_\gamma \in \mathbb{N}$ such that:*

$$\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subseteq \rho(A) \text{ and } \|R(\lambda\colon A)\| \leqslant M_\gamma (1+|\lambda|)^{n_\gamma - 1},\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}. \tag{205}$$

Then A generates an analytic (DS) of angle α.

(v) *Suppose $r > 0$ and $\alpha \in (0, \frac{\pi}{2}]$. If there exists $n \in \mathbb{N}$ such that A generates an analytic n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ of angle α such that, for every $\gamma \in (0,\alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that $\|z^{r-n} S_n(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$, then A generates an analytic (DS) G of angle α such that $(G(\delta_z))_{z\in\Sigma_\alpha}$ is an analytic semigroup of growth order r.*

(vi) *Suppose $r \in (0,1)$ and $\alpha \in (0, \frac{\pi}{2}]$. Then A generates an analytic (DS) G of angle α satisfying that $(G(\delta_z))_{z\in\Sigma_\alpha}$ is an analytic semigroup of growth order r iff A generates an exponentially bounded, analytic once integrated semigroup $(S_1(t))_{t\geqslant 0}$ of angle α satisfying that, for every $\gamma \in (0,\alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that $\|z^{r-1} S_1(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$ iff A generates an exponentially bounded, analytic r-times integrated semigroup of angle α.*

Proof. In order to prove (i), notice that the implication (i.2) $\Rightarrow$ (i.3) is trivial and that the proof of implication (ii) $\Rightarrow$ (iii) of Theorem 3.2.15 shows that the analyticity of G follows automatically from (i.2). It remains to be proved the implication (i.1) $\Rightarrow$ (i.2). Let A generate an analytic (DS) G of angle α. By Theorem 3.2.15, we know that there exists $n \in \mathbb{N}$ such that A generates an exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable. One can use again the proof of implication (ii) $\Rightarrow$ (iii) of Theorem 3.2.15 to deduce that $G(\delta_t) = \frac{d^n}{dt^n} S_n(t)$, $t > 0$. The analyticity of G shows that the mapping $t \mapsto \frac{d^n}{dt^n} S_n(t)$, $t > 0$ can be analytically extended to the sector Σ_α. Now it is checked at once that the mapping $t \mapsto S_n(t)$, $t > 0$ can be analytically extended to the sector Σ_α, as required. This completes the proof of (i). To prove (ii), suppose first that A generates an analytic n_0-times integrated semigroup $(S_{n_0}(t))_{t\geqslant 0}$ of angle α. Then A generates an analytic n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ of angle α for all $n \geqslant n_0$. By (i), one immediately obtains that A generates an analytic (DS) G of angle α satisfying $G(\delta_t) = \frac{d^n}{dt^n} S_n(t)$, $t > 0$, $n \geqslant n_0$. Using integration by parts, we have

$$\int\limits_1^t \frac{(t-s)^{n-1}}{(n-1)!} G(\delta_s) x\, ds = S_n(t)x - \sum_{i=0}^{n} \frac{(t-1)^{n-i}}{(n-i)!} \Big(\frac{d^{n-i}}{dt^{n-i}} S_n(t)x \Big)_{t=1},\ t > 0,\ n \geqslant n_0,$$

and the uniqueness theorem for analytic functions implies that, for every $z \in \Sigma_\alpha$ and $n \geqslant n_0$:

$$\int\limits_1^z \frac{(z-s)^{n-1}}{(n-1)!} G(\delta_s)x\,ds = S_n(z)x - \sum_{i=0}^{n} \frac{(z-1)^{n-i}}{(n-i)!} \Big(\frac{d^{n-i}}{dt^{n-i}} S_n(t)x\Big)_{t=1}.$$

Having in mind the definition of an analytic convoluted semigroup, one directly sees that (204) holds. The converse statement can be proven similarly. The proof of (iii) can be deduced along the same lines. As an outcome of the hypothesis in (iv), one yields that, for every $\gamma \in (0, \alpha)$, A generates an exponentially bounded, analytic $(n_\gamma + 1)$-times integrated semigroup of angle γ. By (iii), the operator A generates an analytic (DS) of angle α. In order to prove (v), suppose that A generates an analytic n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ of angle α with prescribed property. By the assertion (i) of this theorem, we have that A generates an analytic (DS) G of angle α satisfying $G(\delta_t) = \frac{d^n}{dt^n} S_n(t)$, $t > 0$. By [**224**, Proposition 3.2], $\big(\frac{d^n}{dz^n} S_n(z)\big)_{z\in\Sigma_\alpha}$ is an analytic operator family, $T(z_1+z_2) = T(z_1)T(z_2)$, z_1, $z_2 \in \Sigma_\alpha$ and $T(t)x = 0$ for all $t > 0$ implies $x = 0$. By the uniqueness theorem for analytic functions, it follows that, for every $z \in \Sigma_\alpha$, $G(\delta_z) = \frac{d^n}{dz^n} S_n(z)$. Let $\gamma \in (0, \alpha)$ and let $\varepsilon \in (0, \alpha - \gamma)$. Then the Cauchy integral formula gives that, for every $z \in \Sigma_\gamma$ with $\operatorname{Im} z \geqslant 0$:

$$(206)\quad \big\|z^r G(\delta_z)\big\| = \left\| z^r \frac{n!}{2\pi i} \oint\limits_{|\lambda - z| = |z|\sin(\gamma+\varepsilon-\arg(z))} \frac{S_n(\lambda)}{(\lambda-z)^{n+1}}\,d\lambda \right\|$$

$$\leqslant \frac{n!}{2\pi}|z|^r \int\limits_0^{2\pi} \frac{\big\|S_n\big(z + |z|\sin(\gamma+\varepsilon-\arg(z))e^{i\theta}\big)\big\|}{|z|^{n+1}\sin^{n+1}(\gamma+\varepsilon-\arg(z))} |z|\sin(\gamma+\varepsilon-\arg(z))\,d\theta$$

$$\leqslant n!|z|^r M_{\gamma+\varepsilon} e^{\omega_{\gamma+\varepsilon}(\operatorname{Re} z + |z|)} \frac{|2z|^{n-r}}{|z|^n \sin^n \varepsilon} \text{ if } n \geqslant r.$$

In the case $r > n$, we have that

$$|z + |z|\sin(\gamma+\varepsilon-\arg(z))e^{i\theta}| \leqslant |z| - |z|\sin(\gamma+\varepsilon-\arg(z)) \leqslant |z|(1-\sin\varepsilon),$$

and one gets from (206):

$$\|z^r G(\delta_z)\| \leqslant n!|z|^r M_{\gamma+\varepsilon} e^{\omega_{\gamma+\varepsilon}(\operatorname{Re} z + |z|)} \frac{|z|^{n-r}(1-\sin\varepsilon)^{n-r}}{|z|^n \sin^n\varepsilon} \text{ if } n < r.$$

The preceding estimates also hold for every $z \in \Sigma_\gamma$ with $\operatorname{Im} z \leqslant 0$ and this completes the proof of (v).

To prove (vi), let $r \in (0, 1)$ and let A generate an analytic (DS) G of angle α satisfying that $(G(\delta_z))_{z\in\Sigma_\alpha}$ is an analytic semigroup of growth order r. Put $S_1(t)x := \int_0^t G(\delta_s)x\,ds$, $t \geqslant 0$, $x \in E$ and assume that $\hat{A}$ is the integral generator of $(G(\delta_t))_{t\geqslant 0}$. Then it can be simply verified that $A \subseteq \hat{A}$ and that $(S_1(t))_{t\geqslant 0}$ is a once integrated semigroup generated by $\hat{A}$. By the foregoing, there exist $n \in \mathbb{N} \smallsetminus \{1\}$ and an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ so that A generates an n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ which satisfies $\mathbf{S}_n(t)=S_n(t)$, $t>0$ and $\frac{d^n}{dt^n}S_n(t)=G(\delta_t)$, $t>0$. Put $G(\varphi)x := (-1)^n \int_0^\infty \varphi^{(n)}(t)S_n(t)x\,dt$ and $H(\varphi)x := -\int_0^\infty \varphi'(t)\int_0^t G(\delta_s)x\,ds\,dt$, $\varphi \in \mathcal{D}$, $x \in E$. Then G, resp. H, is a (DS) generated by A, resp. $\hat{A}$, and the partial integration implies that $G(\varphi)x = H(\varphi)x = \int_0^\infty \varphi(t)G(\delta_t)x\,dt$ for all $\varphi \in \mathcal{D}_{(0,\infty)}$ and $x \in E$. So, $G(\varphi) = H(\varphi)$, $\varphi \in \mathcal{D}_0$ and this forces $\hat{A} = A$. Since every (local) n-times integrated semigroup is uniquely determined by its generator, we have that $S_n(t)x = \int_0^t \frac{(t-s)^{n-2}}{(n-2)!}S_1(s)x\,ds$, $t \geqslant 0$, $x \in E$. Hence, the mapping $t \mapsto \frac{d}{dt}S_1(t)$, $t > 0$ can be analytically extended to the sector Σ_α and $G(\delta_z) = \frac{d}{dz}S_1(z)$, $z \in \Sigma_\alpha$. Let $\gamma \in (0, \alpha)$. Then the Lagrange mean value theorem implies that there exists $M'_\gamma > 0$ such that

$$\|S_1(z) - S_1(\operatorname{Re} z)\| \leqslant |z \tan\gamma| \sup_{\xi\in[\operatorname{Re} z,z]} \Big\| \frac{d}{d\xi}S_1(\xi) \Big\| \leqslant M'_\gamma e^{\omega_\gamma \operatorname{Re} z}|z|^{1-r},\ z \in \Sigma_\gamma.$$

Hence, $\|S_1(z)\| \leqslant \|S_1(\operatorname{Re} z)\| + M'_\gamma e^{\omega_\gamma \operatorname{Re} z}|z|^{1-r} \leqslant \frac{2M'_\gamma}{1-r}e^{\omega_\gamma \operatorname{Re} z}|z|^{1-r}$, $z \in \Sigma_\gamma$, and consequently, $(S_1(t))_{t\geqslant 0}$ is an exponentially bounded, analytic once integrated semigroup of angle α which clearly fulfills the required property.

The converse statement in (vi) follows from an application of (v). Let A be the generator of an exponentially bounded, analytic r-times integrated semigroup $(S_r(t))_{t\geqslant 0}$ of angle α. Then it is clear that A generates an exponentially bounded, analytic once integrated semigroup $(S_1(t))_{t\geqslant 0}$ of angle α, where $S_1(t)x = \int_0^t \frac{(t-s)^{-r}}{\Gamma(1-r)}S_r(s)x\,ds$, $t \geqslant 0$, $x \in E$. Let $\gamma \in (-\alpha, \alpha)$ be fixed. Then Theorem 2.4.10 yields that, for every $r' \geqslant r$, the operator $e^{i\gamma}A$ generates an exponentially bounded r'-times integrated semigroup $(S_{r',\gamma}(t))_{t\geqslant 0}$ and that, for every $t \geqslant 0$ and $x \in E$:

$$e^{-i\gamma}S_1(te^{i\gamma})x = S_{1,\gamma}(t)x = \int_0^t \frac{(t-s)^{-r}}{\Gamma(1-r)}S_{r,\gamma}(s)x\,ds = e^{-i\gamma r}\int_0^t \frac{(t-s)^{-r}}{\Gamma(1-r)}S_r(se^{i\gamma})x\,ds.$$

Hence,

$$S_1(z)x = e^{i\arg(z)(1-r)}\int_0^{|z|} \frac{(|z|-s)^{-r}}{\Gamma(1-r)}S_r(se^{i\arg(z)})x\,ds,\ z \in \Sigma_\alpha,\ x \in E,$$

and there exist $N_\gamma > 0$ and $\omega'_\gamma \geqslant 0$ such that $\|S_1(z)\| \leqslant N_\gamma|z|^{1-r}e^{\omega'_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$, as required. Assume again that A generates an analytic (DS) G of angle α satisfying that $(G(\delta_z))_{z\in\Sigma_\alpha}$ is an analytic semigroup of growth order r. Put, for every $x \in E$, $t \geqslant 0$ and $\gamma \in (-\alpha, \alpha) : S_{r,\gamma}(t)x =: \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}G(\delta_{se^{i\gamma}})x\,ds$. The first part of proof shows that $(G(\delta_t))_{t\geqslant 0}$ is a semigroup of growth order $r > 0$ whose integral generator is A and this implies that, for every $\gamma \in (-\alpha, \alpha)$, $(G(\delta_{te^{i\gamma}}))_{t\geqslant 0}$ is a semigroup of growth order $r > 0$ whose integral generator is $e^{i\gamma}A$ and that $(S_{r,\gamma}(t))_{t\geqslant 0}$ is an exponentially bounded, r-times integrated semigroup generated by $e^{i\gamma}A$. Furthermore, for every $\gamma \in (-\alpha, \alpha)$, there exist $M_\gamma > 0$, $M'_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that $\|S_{r,\gamma}(t)\| \leqslant M_\gamma e^{\omega_\gamma t\cos\gamma}\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}s^{-r}dr \leqslant M'_\gamma e^{\omega_\gamma t\cos\gamma}$, $t \geqslant 0$. By

Theorem 2.4.10, it follows that A generates an exponentially bounded, analytic r-times integrated semigroup $(S_r(t))_{t\geqslant 0}$ of angle α and the proof of (vi) is completed. $\square$

REMARK 3.2.20. (i) Suppose that $r \in (0,1)$ and that A generates a differentiable (DS) G such that $(G(\delta_t))_{t>0}$ is a semigroup of growth order r. Using the same arguments as in the proof of preceding theorem, one can conclude that A generates an exponentially bounded r-times integrated semigroup $(S_r(t))_{t\geqslant 0}$ that is infinitely differentiable in $t > 0$. It is not clear whether the converse statement holds.

(ii) Suppose Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary and $\alpha \in (0,1)$. A large class of differential operators acting in the space $C^\alpha(\overline{\Omega})$ of Hölder continuous functions analyzed by Von Wahl [**416**] and Periago, Straub [**357**]–[**358**] can be used for the construction of analytic (DS)'s satisfying the property (vi) stated in the formulation of Theorem 3.2.19. By [**357**, Example 3.3], fractional powers of $L^p(\mathbb{R})$-realization of the Kourteweg-de Vries operator (see [**359**] for the notion) $A = \frac{\partial^3}{\partial x^3} + \frac{\partial}{\partial x}$ also generate (DS)'s with above property.

COROLLARY 3.2.21. *Suppose that $\alpha \in (0, \frac{\pi}{2}]$ and that, for every $\gamma \in (-\alpha, \alpha)$, the operator $e^{i\gamma}A$ generates an (EDS). Then A generates an analytic (DS) G of angle α.*

COROLLARY 3.2.22. *Suppose that $\alpha \in (0, \frac{\pi}{2}]$ and that A generates an analytic (DS) of angle α. Then the abstract Cauchy problem*

$$\begin{cases} u \in C^\infty((0,\infty) : E), \\ u'(t) = Au(t),\ t > 0, \end{cases}$$

has a non-trivial solution u which can be analytically extended to the sector Σ_α.

PROOF. By Theorem 3.2.19, we infer that there exist an $n \in \mathbb{N}$ and an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ such that A generates a global n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ which satisfies $\mathbf{S}_n(t) = S_n(t)$, $t > 0$. Using the functional relation $A\int_0^t S_n(s)x\,ds = S_n(t)x - \frac{t^n}{n!}x$, $t \geqslant 0$, $x \in E$ and the closedness of A, one can simply verify that, for every $x \in E$, the function $u(t) = \frac{d^n}{dt^n}S_n(t)x$, $t > 0$, $x \in E$ is a solution of the above problem. This solution is analytically extendible to the sector Σ_α and, by [**224**, Corollary 3.3], non-trivial provided $x \neq 0$. $\square$

The next proposition shows that the analyticity of distribution semigroups is preserved under bounded commuting perturbations.

PROPOSITION 3.2.23. *Let $\alpha \in (0, \frac{\pi}{2}]$ and let A be the generator of an analytic (DS) of angle α. If $B \in L(E)$, then the operator $A + B$ generates an analytic (DS) of the same angle.*

PROOF. By Theorem 3.2.19, there exist $n \in \mathbb{N}$, $M > 0$, $\omega \geqslant 0$, an exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ generated by A and an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ such that $\mathbf{S}_n(t) = S_n(t)$, $t > 0$ and that $\|S_n(t)\| \leqslant$

$Me^{\omega t}$, $t \geqslant 0$. By Theorem 2.5.5, we infer that the operator $A+B$ generates an exponentially bounded n-times integrated semigroup $(S_n^B(t))_{t\geqslant 0}$, which is given by

$$S_n^B(t)x := e^{tB}S_n(t)x + \sum_{i=1}^{n}\binom{n}{i}(-B)^i \int_0^t \frac{(t-s)^{i-1}}{(i-1)!}e^{Bs}S_n(s)x\,ds,\ z \in \Sigma_\alpha,\ x \in E.$$

It remains to be proven that $(S_n^B(t))_{t\geqslant 0}$ can be analytically extended to the sector Σ_α. To this end, define, for every $z \in \Sigma_\alpha$ and $x \in E$,

$$\begin{aligned} S_n^B(z)x &:= e^{zB}S_n(z)x \\ &+ \sum_{i=1}^{n}\binom{n}{i}(-B)^i\Bigg(\int_0^1 \frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_n(s)x\,ds + \int_1^z \frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_n(s)x\,ds\Bigg). \end{aligned}$$

It is clear that the mapping $z \mapsto S_n^B(z)$, $z \in \Sigma_\alpha$ is analytic and the proof completes an application of Theorem 3.2.19. $\square$

EXAMPLE 3.2.24. For every $n \geqslant 1$ there exists a closed densely defined operator B acting on a Banach (Hilbert) space such that B generates a global exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ satisfying that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable, and that B does not generate a local $(n-1)$-times semigroup. To this end, suppose that A generates a contractive, immediately differentiable C_0-semigroup $(T(t))_{t\geqslant 0}$ in a Banach (Hilbert) space E which additionally satisfies that $(T(t))_{t\geqslant 0}$ cannot be analytically extended in a sector around the nonnegative real axis. For concrete examples, we refer to [**125**, p. 24–33, p. 409]. Let E^{n+1} be equipped with the sup-norm and let $D(B) := D(A)^{n+1}$ and $B(x_1,\ldots,x_{n+1}) := (Ax_1+Ax_2, Ax_2+Ax_3,\ldots,Ax_n+Ax_{n+1}, Ax_{n+1})$, $(x_1,\ldots,x_{n+1}) \in D(B)$. Arguing as in [**337**, Proposition 2.4] (cf. also [**14**, Theorem 3.2.13, p. 133]), we have that B generates a global exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ and that B does not generate a local $(n-1)$-times semigroup. It remains to be proved that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable. Let $\sigma > 0$. Then there exist $\omega \geqslant 0$, $C_\sigma > 0$ and $M_\sigma > 0$ such that $\Upsilon_{\sigma,\omega} \subseteq \rho(A)$ and $\|R(\lambda\colon A)\| \leqslant M_\sigma|\operatorname{Im}\lambda|$. The claimed assertion follows from Theorem 3.2.18(iii) and the computation given in the proof of [**337**, Proposition 2.4]:

$$\begin{aligned} \|R(\lambda : B)\| &\leqslant \sum_{k=0}^{n}\|A^kR(\lambda\colon A)^{k+1}\| \leqslant M_\sigma|\operatorname{Im}\lambda|\sum_{k=0}^{n}\|A^kR(\lambda\colon A)^k\| \\ &\leqslant M_\sigma|\operatorname{Im}\lambda|\sum_{k=0}^{n}\|\lambda R(\lambda\colon A)-I\|^k \leqslant M_\sigma|\operatorname{Im}\lambda|\sum_{k=0}^{n}(1+M_\sigma|\lambda||\operatorname{Im}\lambda|)^k \\ &\leqslant (n+1)M_\sigma|\operatorname{Im}\lambda|(1+M_\sigma|\lambda|^2)^{n+1} \leqslant (n+1)M'_\sigma|\lambda|^{2n+2}|\operatorname{Im}\lambda|, \end{aligned}$$

$\lambda \in \Upsilon_{\sigma,\omega}$, for some $M'_\sigma > 0$.

For further information related to differential and analytical properties of convoluted C-semigroups, C-distribution semigroups and (a,k)-regularized C-resolvent families, we refer the reader to [**234**]-[**235**].

3.3. Distribution groups

3.3.1. Introduction and basic properties of distribution groups. We start with the following notion. If $\varphi \in \mathcal{D}$ and $G \in \mathcal{D}'(L(E))$, then we define $\check{\varphi}(\cdot) := \varphi(-\cdot)$ and $\check{G}(\cdot) := G(\check{\cdot})$. Clearly, $\varphi \check{*} \psi = \check{\varphi} * \check{\psi}$ and $\check{\varphi}^{(n)} = (-1)^n \check{\varphi^{(n)}}$, φ, $\psi \in \mathcal{D}$, $n \in \mathbb{N}$. We focus our attention to the following system of convolution type equations:

$$G * (\delta' \otimes I - \delta \otimes A) = 0 \otimes I_{[D(A)]} \text{ and } (\delta' \otimes I - \delta \otimes A) * G = 0 \otimes I_E, \tag{207}$$

where A is a closed operator acting on a Banach space E, $G \in \mathcal{D}'(L(E, [D(A)]))$, $\delta' \otimes I - \delta \otimes A \in \mathcal{D}'(L([D(A)], E))$ and I denotes the inclusion $D(A) \to E$. Here we stress that every operator-valued distribution G satisfying, for every $\varphi \in \mathcal{D}$ and $x \in E$,

$$G \in \mathcal{D}'(L(E)), \ \ G(\varphi)x \in D(A), \ \ AG(\varphi)x = G(-\varphi')x, \ \ G(\varphi)A \subseteq AG(\varphi), \tag{208}$$

can be viewed as an element of the space $\mathcal{D}'(L(E, [D(A)]))$ which solves (207) (cf. also [**315**]). It turns out that the introduced class of $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-groups presents a natural framework for investigation of equations involving operators satisfying (208).

Definition 3.3.1. An element $G \in \mathcal{D}'(L(E))$ is called a *pre-distribution group, pre*-(DG) in short, if the next condition holds:
$(\mathrm{DG})_1$: $G(\varphi * \psi) = G(\varphi)G(\psi)$ for all φ, $\psi \in \mathcal{D}$.
If G additionally satisfies:
$(\mathrm{DG})_2$: $\mathcal{N}(G) =: \bigcap_{\varphi \in \mathcal{D}} \mathrm{Kern}(G(\varphi)) = \{0\}$,
then it is said that G is a *distribution group,* (DG) shortly. A pre-(DG) G is called *dense* iff
$(\mathrm{DG})_3$: The set $\mathcal{R}(G) := \bigcap_{\varphi \in \mathcal{D}} \mathrm{R}(G(\varphi))$ is dense in E.
Suppose $G \in \mathcal{D}'(L(E))$, G satisfies $(DG)_2$ and $T \in \mathcal{E}'$. We define $G(T)$ by

$$G(T) := \big\{(x, y) \in E \times E : G(T * \varphi)x = G(\varphi)y \text{ for all } \varphi \in \mathcal{D}\big\}.$$

Due to $(\mathrm{DG})_2$, $G(T)$ is a function and it is straightforward to see that $G(T)$ is a closed linear operator in E. The *generator* A of a (DG) G is defined by $A := G(-\delta')$. Notice, if G is a (DG) generated by A, then (208) holds.

An element $G \in \mathcal{D}'(L(E))$ is called *regular (representable)* if the following holds:
$(\mathrm{DG})_4$: For every $x \in \mathcal{R}(G)$, there is a function $t \mapsto u(t; x)$, $t \in \mathbb{R}$ satisfying:

$$u(\cdot; x) \in C(\mathbb{R} : E), \ u(0; x) = x \text{ and } G(\psi)x = \int_{-\infty}^{\infty} \psi(t)u(t; x)\, dt, \ \psi \in \mathcal{D}.$$

Let us observe that the function $u(\cdot; x)$ is unique. Indeed, let $x \in \mathcal{R}(G)$ and $t \in \mathbb{R}$ be fixed and let (ζ_n) be a sequence in $\mathcal{D}$ satisfying $\lim_{n\to\infty} \zeta_n = \delta_t$, in the sense of distributions. Then $u(t; x) = \lim_{n\to\infty} G(\zeta_n)x$.

Example 3.3.2. (i) Let A and $-A$ generate C-distribution semigroups G_+ and G_-, respectively. Put $G(\varphi) := G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. Then A and G fulfill (208).

Indeed, $G \in \mathcal{D}'(L(E))$, $G(\varphi)A \subseteq AG(\varphi)$, $\varphi \in \mathcal{D}$, $AG_+(\varphi)x = G_+(-\varphi')x - \varphi(0)Cx$ and $-AG_-(\varphi)x = G_-(-\varphi')x - \varphi(0)Cx$, $\varphi \in \mathcal{D}$, $x \in E$. Thereby, $AG(\varphi)x = G_+(-\varphi')x - \varphi(0)Cx + G_-(\check{\varphi}')x + \check{\varphi}(0)Cx = G_+(-\varphi')x + G_-(-\check{\varphi}')x = G(-\varphi')x$, $x \in E$, $\varphi \in \mathcal{D}$. Furthermore, it can be proved the following: $G(\varphi*\psi)C = G(\varphi)G(\psi)$, $\varphi,\ \psi \in \mathcal{D}$ [**216**], [**315**], $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\varphi)) = \{0\}$ and $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\check{\varphi})) = \{0\}$.

(ii) Assume G is a (DG), $P \in L(E)$, $P^2 = P$ and $GP = PG$. Set $G_P(\varphi)x := G(\varphi)Px$, $\varphi \in \mathcal{D}$, $x \in E$. Then G_P is a pre-(DG) and $\mathcal{N}(G_P) = \operatorname{Kern}(P)$.

(iii) Assume A and G fulfill (208). Define G_T ($T \in \mathcal{E}'$) by $G_T(\varphi)x := G(T*\varphi)x$, $\varphi \in \mathcal{D}$, $x \in E$. Then (208) holds for A and G_T.

(iv) [**89**, Example 16.3] Let $E := \{f : \mathbb{R} \to \mathbb{C}$ is continuous: $\lim_{|x|\to\infty} e^{x^2} f(x) = 0\}$, $\|f\| := \sup_{x\in\mathbb{R}} |e^{x^2} f(x)|$, $f \in E$ and $A := \frac{d}{dx}$ with maximal domain. Put $(S(t)f)(x) := e^{-(x+t)^2} f(x+t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, $f \in E$. Then $S(t)f \in E$, $\|S(t)\| \leqslant e^{2t^2}$, $\int_0^t S(s)f\,ds \in D(A)$ and $A\int_0^t S(s)f\,ds = S(t)f - S(0)f$, $t \in \mathbb{R}$, $f \in E$. Put $G(\varphi)f := \int_{-\infty}^{\infty} \varphi(t)S(t)f\,dt$, $f \in E$, $\varphi \in \mathcal{D}$. Clearly, $G \in \mathcal{D}'(L(E))$ and the partial integration yields $G(\varphi)f \in D(A)$, $AG(\varphi)f = G(-\varphi')f$ and

$$\big(G(\varphi)Af - AG(\varphi)f\big)(x) = 2\int_{-\infty}^{\infty} \varphi(t)(x+t)e^{-(x+t)^2} f(x+t)\,dt,\ x \in \mathbb{R},\ \varphi \in \mathcal{D}.$$

Therefore, A does not commute with $G(\cdot)$ and (208) does not hold. Furthermore, it can be verified that G fulfills $(DG)_2$ and that G is not regular.

(v) Let $\mathcal{F}$ denote the Fourier transform on the real line,

$$\mathcal{F}(f)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t} f(t)\,dt,\ \xi \in \mathbb{R}.$$

Suppose that $\mathcal{E}$ is a quasi-spectral distribution in the sense of [**104**, Definition 2.2] and that $\mathcal{E}$ can be continuously extended to $\mathcal{S}$. Put $\mathcal{F}(\mathcal{D}) := \{\mathcal{F}(\varphi) : \varphi \in \mathcal{D}\}$ and $G(\varphi) := \mathcal{E}(\mathcal{F}^{-1}(\varphi))$, $\varphi \in \mathcal{S}$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then $G \in \mathcal{S}'(L(E))$, $G(\varphi * \psi) = G(\varphi)G(\psi)$, $\varphi,\ \psi \in \mathcal{S}$ and $\bigcap_{\varphi\in\mathcal{F}(\mathcal{D})} \operatorname{Kern}(G(\varphi)) = \{0\}$. Suppose, additionally, that for every $x \in E$ and $\phi \in \mathcal{S}$ with $\phi(0) = 1$:

$$\lim_{n\to\infty} \mathcal{E}(\phi_n)x = x,\ \text{where } \phi_n(t) = \phi(t/n),\ t \in \mathbb{R},\ n \in \mathbb{N}. \tag{209}$$

Notice that (209) implies that $\mathcal{E}$ is a spectral distribution in the sense of [**104**, Definition 2.4] (cf. also [**25**, Definition 1.1]). We will show that $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\varphi)) = \{0\}$. Indeed, suppose $\rho \in \mathcal{D}$, $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, $\operatorname{supp}\rho \subseteq [0,1]$ and $G(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$, i.e., $\mathcal{E}(\mathcal{F}^{-1}(\varphi))x = 0$, $\varphi \in \mathcal{D}_0$. Put $\phi(t) = \mathcal{F}^{-1}(\rho)(t) = \int_{-\infty}^{\infty} e^{i\xi t}\rho(\xi)\,d\xi$, $t \in \mathbb{R}$. Then $\phi \in \mathcal{S}$ and $\phi(0) = 1$. Put $\rho_n(t) = n\rho(nt)$ and $\phi_n(t) = \mathcal{F}^{-1}(\rho_n)(t)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Clearly, $\phi_n(t) = \phi(\frac{t}{n})$, $t \in \mathbb{R}$, $n \in \mathbb{N}$ and (209) implies $x = \lim_{n\to\infty} \mathcal{E}(\phi_n)x = \lim_{n\to\infty} \mathcal{E}(\mathcal{F}^{-1}(\rho_n))x = 0$. Analogously, $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\check{\varphi})) = \{0\}$ and this implies that $(DG)_2$ holds for G.

A closed linear operator A satisfying (208) need not be the generator of a (DG) and this implies that relations between distribution groups and convolution type equations are, at least, quite unclear.

The proofs of the subsequent assertions are omitted.

LEMMA 3.3.3. *Let G be a pre-(DG). Then $\check{G}$ is a pre-(DG). If, in addition, G is a (DG) generated by A, then $\check{G}$ is a (DG) generated by $-A$.*

PROPOSITION 3.3.4. *Let G be a pre-(DG), $F := E/\mathcal{N}(G)$ and q be the corresponding canonical mapping $q : E \to F$.*

(i) *Let $H \in L(\mathcal{D} : L(F))$ be defined by $qG(\varphi) := H(\varphi)q$ for all $\varphi \in \mathcal{D}$. Then H is a (DG) in F.*
(ii) *$\overline{\langle \mathcal{R}(G) \rangle} = \overline{\mathcal{R}(G)}$, where $\langle \mathcal{R}(G) \rangle$ denotes the linear span of $\mathcal{R}(G)$.*
(iii) *Assume that G is not dense. Put $R := \overline{\mathcal{R}(G)}$ and $H := G_{|R}$. Then H is a dense pre-(DG) in R. Moreover, if G is a (DG) generated by A, then H is a (DG) in R generated by $A_{|R}$.*
(iv) *The adjoint G^* of G is a pre-(DG) in E^* with $\mathcal{N}(G^*) = \overline{\mathcal{R}(G)}^{\circ}$.*
(v) *If E is reflexive, then $\mathcal{N}(G) = \overline{\mathcal{R}(G^*)}^{\circ}$.*
(vi) *G^* is a (DG) in E^* iff G is a dense pre-(DG). If E is reflexive, then G^* is a dense pre-(DG) in E^* iff G is a (DG).*
(vii) *$\mathcal{N}(G) \cap \langle \mathcal{R}(G) \rangle = \{0\}$.*
(viii) *Suppose $x = G(\varphi)y$, for some $\varphi \in \mathcal{D}$ and $y \in E$. Put $u(t;x) := G(\tau_t\varphi)y$, $t \in \mathbb{R}$. Then $u(0;x) = x$, $u(\cdot;x) \in C^\infty(\mathbb{R} : E)$, $\frac{d^n}{dt^n}u(t;x) = A^n u(t;x)$, $t \in \mathbb{R}$, $n \in \mathbb{N}_0$, $G(\psi)x = \int_{-\infty}^{\infty} \psi(t)u(t;x)\,dt$, $\psi \in \mathcal{D}$ and G is regular.*

PROPOSITION 3.3.5. *Let G be a (DG) and let S, $T \in \mathcal{E}'$, $\varphi \in \mathcal{D}$ and $x \in E$. Then:*

(i) *$(G(\varphi)x,\ G(\overbrace{T * \cdots * T}^{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.*
(ii) *$G(S)G(T) \subseteq G(S * T)$, $D(G(S)\,G(T)) = D(G(S * T)) \cap D(G(T))$ and $G(S) + G(T) \subseteq G(S + T)$. In general, $G(S)\,G(T) \neq G(S * T)$.*
(iii) *$G(\varphi)\,G(T) \subseteq G(T)\,G(\varphi)$.*
(iv) *If G is dense, its generator is densely defined.*

REMARK 3.3.6. Suppose $G \in \mathcal{D}'(L(E))$ and G fulfills $(DG)_3 - (DG)_4$. Then G is a pre-(DG) iff:
(210)
$$\bigcup_{t\in\mathbb{R},\ x\in\mathcal{R}(G)} u(t;x) \subseteq \mathcal{R}(G) \text{ and } u(t+s;x) = u(t;u(s;x)),\ t,\ s \in \mathbb{R},\ x \in \mathcal{R}(G).$$

The necessity of (210) follows directly from Proposition3.3.4(j). To prove the sufficiency, notice that

$$G(\varphi * \psi)x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\varphi(t-s)\psi(s)\,ds]u(t;x)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t)\psi(s)u(t+s;x)\,ds\,dt$$

$$= \int_{-\infty}^{\infty} \varphi(t) \int_{-\infty}^{\infty} \psi(s)u(s;u(t;x))\,ds\,dt = \int_{-\infty}^{\infty} \varphi(t)G(\psi)u(t;x)\,dt$$

$$= G(\psi) \int_{-\infty}^{\infty} \varphi(t)u(t;x)\,dt = G(\psi)G(\varphi)x,$$

for every $x \in \mathcal{R}(G)$. The denseness of $\mathcal{R}(G)$ in E implies $(DG)_1$.

Theorem 3.3.7. (i) *Suppose* $0 < \tau \leqslant \infty$, $\alpha > 0$ *and* $\pm A$ *generate* α*-times integrated semigroups* $(S_{\pm}(t))_{t\in[0,\tau)}$. *Then* A *generates a (DG).*

(ii) *Suppose* $\pm A$ *generate distribution semigroups* $G_{\pm}$. *Put* $G(\varphi) := G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. *Then* G *is a (DG) generated by* A.

Proof. (i) Let us prove that A generates a (DG). Put $n := \lceil \alpha \rceil$. It is obvious that, for every $k \in \mathbb{N}$, $\pm A$ generate (kn)-times integrated semigroups $(S^k_{\pm}(t))_{t\in[0,k\tau)}$. Let $\varphi \in \mathcal{D}_{(-\infty,k\tau)}$ and $x \in E$. Set

$$G_+(\varphi)x := (-1)^{kn} \int_0^{\infty} \varphi^{(kn)}(t)S^k_+(t)x\,dt,\ G_-(\varphi)x := (-1)^{kn} \int_0^{\infty} \varphi^{(kn)}(t)S^k_-(t)x\,dt$$

and $G(\varphi) := G_+(\varphi) + G_-(\check{\varphi})$. Certainly, G_+ and G_- are distribution semigroups generated by A and $-A$, respectively. In order to prove that G is a (DG) generated by A, assume $x \in \mathcal{N}(G)$. Then, for every $\varphi \in \mathcal{D}_0$, $G(\varphi)x = 0$, and this implies $G_+(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. Since G_+ is a (DS) generated by A, we get $x = 0$ and $(DG)_2$ holds for G. Further on, A generates a local (kn)-times integrated group $(S^{kn}(t))_{t\in(-k\tau,k\tau)}$. Now one can repeat literally the arguments given in the proof of [**315**, Theorem 6] so as to conclude that $(DG)_1$ holds for all φ, $\psi \in \mathcal{D}_{(-k\tau/2,k\tau/2)}$. Hence, G fulfills $(DG)_1$. It remains to be proved that $B = A$, where B is the generator of G. Suppose $(x, y) \in B$. Then $G(-\varphi')x = G(\varphi)y$, $\varphi \in \mathcal{D}$, i.e., $G_+(-\varphi')x + G_-(\check{-\varphi'})x = G_+(\varphi)x + G_-(\check{\varphi})x$, $\varphi \in \mathcal{D}$. This, in particular, holds for every $\varphi \in \mathcal{D}_0$ and one obtains $G_+(-\varphi')x = G_+(\varphi)x$, $\varphi \in \mathcal{D}_0$. In other words, $B \subseteq A$. Assume now $(x, y) \in A$. Then the definition of G and Proposition 3.1.4(iii) imply:

$$G(\varphi)y = G(\varphi)Ax = G_+(\varphi)Ax + G_-(\check{\varphi})Ax = G_+(-\varphi')x - \varphi(0)x - G_-(-\check{\varphi}')x + \varphi(0)x$$

$$= G_+(-\varphi')x + G_-(\check{-\varphi'})x = G(-\varphi')x,\ \varphi \in \mathcal{D}.$$

This gives $A \subseteq B$ and ends the proof of (i). To prove (ii), notice that an application of Corollary 2.1.10 and Theorem 3.1.25 yields that there exist $\tau \in (0, \infty)$ and $n \in \mathbb{N}$ so that, for every $k \in \mathbb{N}$, $\pm A$ generate (kn)-times integrated semigroups $(S^{kn}_{\pm}(t))_{t\in[0,k\tau)}$. Assume $x \in E$ and $\varphi \in \mathcal{D}_{(-\infty,k\tau)}$, for some $k \in \mathbb{N}$. Put $G_1(\varphi)x := (-1)^{kn} \int_0^{\infty} \varphi^{(kn)}(t)S^{kn}_+(t)x\,dt$ and $G_2(\varphi)x := (-1)^{kn} \int_0^{\infty} \varphi^{(kn)}(t)S^{kn}_-(t)x\,dt$. Then [**418**] G_1 and G_2 are distribution semigroups generated by A and $-A$, respectively. Hence, $G_+ = G_1$, $G_- = G_2$, $G = G_1 + \check{G}_2$ and the remnant of the proof of (ii) follows by the use of arguments already given in the proof of (i). $\square$

The previous theorem implies that a wide class of multiplication operators acting on $L^p(\mathbb{R}^n)$-type spaces can be used for the construction of (exponential) distribution groups. In particular, several examples presented in [**5**] offers one to construct local once integrated groups which can be explicitly calculated.

Let $\alpha \in (0,\infty)$, $\alpha \notin \mathbb{N}$ and $f \in \mathcal{S}$. Put $n := \lceil \alpha \rceil$. Recall [**317**], the Weyl fractional derivatives W_+^α and W_-^α of order α are defined by:

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_t^\infty (s-t)^{n-\alpha-1} f(s)\,ds,\ t \in \mathbb{R},$$

$$W_-^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_{-\infty}^t (t-s)^{n-\alpha-1} f(s)\,ds,\ t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}$, put $W_+^n := (-1)^n \frac{d^n}{dt^n}$ and $W_-^n := \frac{d^n}{dt^n}$. Then we know [**315**] that $W_\pm^{\alpha+\beta} = W_\pm^\alpha W_\pm^\beta$, $\alpha > 0$, $\beta > 0$. The next result can be attributed to Miana [**315**].

THEOREM 3.3.8. *Suppose $\alpha > 0$ and $(S(t))_{t\in\mathbb{R}}$ is an α-times integrated group generated by A. Put $G(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t)S(t)x\,dt + \int_0^\infty W_+^\alpha \check{\varphi}(t)S(-t)x\,dt$, $\varphi \in \mathcal{D}$, $x \in E$. Then G is a (DG) generated by A.*

PROOF. In order to prove that $(DG)_1$ holds for G, one can argue as in the proof of [**315**, Theorem 6] (cf. also [**137**, Lemma 1.6]). Define now $G_+(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t)S_+(t)x\,dt$, $\varphi \in \mathcal{D}$, $x \in E$. Then G_+ is a (DSG) generated by A (cf. [**315**]–[**316**] and [**252**, Theorem 3.10]). The assumption $G(\varphi)x = 0$, $\varphi \in \mathcal{D}$ implies $G_+(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. Therefore, $(DG)_2$ holds for G and G is a (DG). Let us prove that $A = A_1$, where A_1 is the generator of G (cf. also the proof of [**315**, Theorem 7]). Notice that $-A$ is the generator of $(S_-(t))_{t\geqslant 0}$. Assume $x \in D(A_1)$. Then $G(-\varphi')x = G(\varphi)A_1x$, $\varphi \in \mathcal{D}$, i.e., $G_+(-\varphi')x + G_-(-\check{\varphi'})x = G_+(\varphi)A_1x + G_-(\varphi)A_1x$, $\varphi \in \mathcal{D}$. Especially, $G_+(-\varphi')x = G_+(\varphi)A_1x$, $\varphi \in \mathcal{D}_0$ and $G_-(-\check{\varphi}')x = -G_-(\check{\varphi})A_1x$, $\check{\varphi} \in \mathcal{D}_0$. So, $x \in D(A)$ and $Ax = A_1x$. Assume now $x \in D(A)$. Then $G(\varphi)Ax = G_+(\varphi)Ax + G_-(\check{\varphi})Ax$, $\varphi \in \mathcal{D}$ and the use of Proposition 3.1.4(iii) implies: $G(\varphi)Ax = G_+(-\varphi')x - \varphi(0)x - G_-(-\check{\varphi}')x + \varphi(0)x = G_+(-\varphi')x + G_-(-\check{\varphi'})x = G(-\varphi')x$, $\varphi \in \mathcal{D}$. Hence, $A_1x = Ax$ as claimed. $\square$

REMARK 3.3.9. Due to the definition of $W_\pm^\alpha$, we have the following: If $\alpha = n \in \mathbb{N}$, $\varphi \in \mathcal{D}$ and $x \in E$, then

$$G(\varphi)x = (-1)^n \int\limits_0^\infty \varphi^{(n)}(t)S(t)x\,dt + (-1)^n \int\limits_0^\infty \check{\varphi}^{(n)}(t)S(-t)x\,dt,$$

$$G(\varphi)x = (-1)^n \int\limits_0^\infty \varphi^{(n)}(t)S(t)x\,dt + \int\limits_{-\infty}^0 \varphi^{(n)}(t)S(t)x\,dt.$$

The next theorem clarifies an interesting relation between integrated groups and *global differentiable* C-regularized groups.

THEOREM 3.3.10. *Assume $\alpha > 0$, $\tau \in (0,\infty]$ and A is the generator of an α-times integrated group $(S_\alpha(t))_{t\in(-\tau,\tau)}$. Then, for every $b \in (0,1)$ and $\gamma \in \big(0, \arctan(\cos(b\frac{\pi}{2}))\big)$, there exist two analytic operator families $(T_{b,+}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ and $(T_{b,-}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ so that:*

(i) *For every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators.*
(ii) *For every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, A generates a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-regularized group $(V_{b,t_1,t_2}(s))_{s\in\mathbb{R}}$.*
(iii) *For every $x \in E$, $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, the mapping $s \mapsto V_{b,t_1,t_2}(s)x$, $s \in \mathbb{R}$ is infinitely differentiable in $(-\infty,0)\cup(0,\infty)$.*

PROOF. The existence of numbers $c > 0$, $d > 0$ and $M > 0$ so that $E(c,d) \subseteq \rho(\pm A)$ and that $\|R(\lambda : \pm A)\| \leqslant M|\lambda|^\alpha$, $\lambda \in E(c,d)$ is obvious. Choose a number $a \in (0, \frac{\pi}{2})$ such that $b \in (0, \frac{\pi}{2(\pi-a)})$ and that $\gamma \in \big(0, \arctan(\cos(b(\pi-a)))\big)$. It is clear that there are numbers $d \in (0,1]$ and $\omega \in (d+1,\infty)$ so that

$$\Omega_{a,d} := \{z \in \mathbb{C} : |z| \leqslant d\} \cup \{re^{i\theta} : r > 0,\ \theta \in [-a,a]\} \subseteq \rho(A-\omega)\cap\rho(-A-\omega).$$

Let the curve $\Gamma_{a,d} = \partial\Omega_{a,d}$ be upwards oriented. Define $T_{b,\pm}(t)$, $t \in \Sigma_\gamma$ by:

$$T_{b,\pm}(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : \pm A - \omega)x\,d\lambda,\ x \in E.$$

Certainly, $(T_{b,\pm}(t))_{t\in\Sigma_\gamma}$ are analytic operator families and, for every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators. Clearly, $T_{b,+}(t_1)(-A-\omega) \subseteq (-A-\omega)T_{b,+}(t_1)$ and $T_{b,-}(t_2)(A-\omega) \subseteq (A-\omega)T_{b,-}(t_2)$, t_1, $t_2 \in \Sigma_\gamma$. It is straightforward to prove that $T_{b,+}(t_1)T_{b,-}(t_2) = T_{b,-}(t_2)T_{b,+}(t_1)$, t_1, $t_2 \in \Sigma_\gamma$ and the argumentation given in Subsection 2.1.6 shows that $\pm A - \omega$ are generators of global $T_{b,\pm}(t)$-regularized semigroups $(U_{b,t,\pm}(s))_{s\geqslant 0}$. Suppose t_1, $t_2 \in \Sigma_\gamma$ and $x \in E$. Then one obtains

$$\begin{aligned} &T_{b,-}(t_2)(U_{b,t_1,+}(s)x - T_{b,+}(t_1)x) \\ &\quad = T_{b,-}(t_2)(A-\omega)\int\limits_0^s U_{b,t_1,+}(v)x\,dv = (A-\omega)T_{b,-}(t_2)\int\limits_0^s U_{b,t_1,+}(v)x\,dv. \end{aligned}$$

Hence,

$$(A-\omega)\int\limits_0^s \Big(T_{b,-}(t_2)U_{b,t_1,+}(v)\Big)x\,dv = T_{b,-}(t_2)U_{b,t_1,+}(s)x - T_{b,+}(t_1)T_{b,-}(t_2)x,$$

for all $s \geqslant 0$. Furthermore,

$$\begin{aligned} &[T_{b,-}(t_2)U_{b,t_1,+}(s)]T_{b,+}(t_1) = T_{b,+}(t_1)[T_{b,-}(t_2)U_{b,t_1,+}(s)],\ s \geqslant 0, \\ &[T_{b,-}(t_2)U_{b,t_1,+}(s)](A-\omega) \subseteq (A-\omega)[T_{b,-}(t_2)U_{b,t_1,+}(s)],\ s \geqslant 0. \end{aligned}$$

This implies that $\big(T_{b,-}(t_2)U_{b,t_1,+}(s)\big)_{s\geqslant 0}$ is a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized semigroup generated by $(T_{b,+}(t_1)T_{b,-}(t_2))^{-1}(A-\omega)(T_{b,+}(t_1)T_{b,-}(t_2)) = A-\omega$. Therefore, $\big(e^{\omega s}T_{b,-}(t_2)U_{b,t_1,+}(s)\big)_{s\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-regularized semigroup generated by A, and, for every t_1, $t_2 \in \Sigma_\gamma$, $\big(e^{\omega s}T_{b,+}(t_1)U_{b,t_2,-}(s)\big)_{s\geqslant 0}$ is a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized semigroup generated by $-A$. Hence, for every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, A generates a global $\big(T_{b,+}(t_1)T_{b,-}(t_2)\big)$-regularized group $(V_{b,t_1,t_2}(s))_{s\in\mathbb{R}}$ which is given by: $V_{b,t_1,t_2}(s) =: e^{\omega s}T_{b,-}(t_2)U_{b,t_1,+}(s)$, $s \geqslant 0$ and $V_{b,t_1,t_2}(s) := e^{-\omega s}T_{b,+}(t_1)U_{b,t_2,-}(-s)$, $s < 0$. The mapping $s \mapsto V_{b,t_1,t_2}(s)x$, $s \in \mathbb{R}$ is infinitely differentiable in $(-\infty, 0) \cup (0,\infty)$ since the corresponding mappings $s \mapsto T_{b,-}(t_2)U_{b,t_1,+}(s)x$ and $s \mapsto T_{b,+}(t_1)U_{b,t_2,-}(s)x$ are infinitely differentiable in $s > 0$ (cf. Subsection 2.1.6). The proof is completed. $\square$

3.3.2. $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$**-groups.** We introduce the class of $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-groups as follows.

Definition 3.3.11. Let A be a closed linear operator. Suppose, further, $0 < \tau \leqslant \infty$, $n \in \mathbb{N}$ and $B_0,\dots,B_n,C_0,\dots,C_{n-1} \in L(E)$. A strongly continuous operator family $(S(t))_{t\in(-\tau,\tau)}$ is said to be a $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-*group* with a subgenerator A iff:

(i) $A\int_0^t S(s)x\,ds = S(t)x + \sum_{j=0}^n t^jB_jx$, $t \in (-\tau,\tau)$, $x \in E$ and

(ii) $AS(t)x - S(t)Ax = \sum_{j=0}^{n-1} t^jC_jx$, $t \in (-\tau,\tau)$, $x \in D(A)$.

It is said that $(S(t))_{t\in(-\tau,\tau)}$ is *non-degenerate* if the assumption $S(t)x = 0$, for all $t \in (-\tau,\tau)$, implies $x = 0$. The (*integral*) *generator* of a non-degenerate $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$ is defined by

$$\hat{A} = \Bigg\{(x,y) \in E\times E : S(t)x+\sum_{j=0}^n t^jB_jx-\sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx = \int_0^t S(s)y\,ds,\ t \in (-\tau,\tau)\Bigg\}.$$

The integral generator $\hat{A}$ of a non-degenerate $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$ is a function and it ischecked at once that$\hat{A}$ is a closed linear operator which is an extension of any subgenerator of $(S(t))_{t\in(-\tau,\tau)}$. Further on, the injectiveness of B_i for some $i \in \{0,\dots,n\}$ implies that $(S(t))_{t\in(-\tau,\tau)}$ is non-degenerate. In general, a subgenerator A of $(S(t))_{t\in(-\tau,\tau)}$ does not commute with $S(\cdot)$ and the set of all subgenerators of $(S(t))_{t\in(-\tau,\tau)}$ need not be monomial.

Example 3.3.12. (i) Let $E := \mathbb{R}^2$. Put $A(x_1,x_2) := (x_1-x_2,0)$, $B_0(x_1,x_2) := (0,-x_2)$, $B_1(x_1,x_2) := (-x_1-x_2,-x_1)$, $B_2(x_1,x_2) := (0,0)$, $C_0(x_1,x_2) = (-x_2,0)$, $C_1(x_1,x_2) := (-x_1+x_2,-x_1+x_2)$ and $S(t)(x_1,x_2) := (tx_1,tx_1+x_2)$, $t \in \mathbb{R}$, $(x_1,x_2) \in E$. Then $(S(t))_{t\in\mathbb{R}}$ is a $[B_0,B_1,B_2,C_0,C_1]$-group with a subgenerator A and: $S(t)S(s) = S(s)S(t)$ iff $t = s$, $S(t)D \neq DS(t)$, $t \in \mathbb{R}$, $D \in \{B_1,C_0,C_1\}$ and $D_0D_1 \neq D_1D_0$, $D_i \in \{B_i,C_i\}$, $i = 1,2$.

(ii) Assume $C_j = 0$, $j = 0,\dots,n-1$ and the bounded linear operators B_j, $j = 0,\dots,n$ fulfill $E \neq \overline{\sum_{i=0}^n \mathrm{R}(B_i)}$. Put $S(t)x := -\sum_{j=0}^n t^jB_jx$, $t \in (-\tau,\tau)$, $x \in E$ and denote by $\mathfrak{D}$ the family of all closed subspaces of E containing $\overline{\sum_{i=0}^n \mathrm{R}(B_i)}$. If $F \in \mathfrak{D}$, define a closed linear operator A_F by $D(A_F) := F$ and $A_Fx := 0$, $x \in D(A_F)$. Then A_F is a subgenerator of a $[B_0,\dots,B_n,\underbrace{0,\dots,0}_{n}]$-group $(S(t))_{t\in(-\tau,\tau)}$.

QUESTION. Suppose $n \in \mathbb{N}$, $\tau \in (0,\infty]$, $B_0,\ldots,B_n,C_0,\ldots,C_{n-1} \in L(E)$ as well as $(S_1(t))_{t\in(-\tau,\tau)}$ and $(S_2(t))_{t\in(-\tau,\tau)}$ are $[B_0,\ldots,B_n,C_0,\ldots,C_{n-1}]$-groups having A as a subgenerator. Is it true that $S_1(t)=S_2(t)$, $t\in(-\tau,\tau)$?

REMARK 3.3.13. (i) Assume $n\in\mathbb{N}$, $\tau\in(0,\infty]$ and A generates an n-times integrated group $(S(t))_{t\in(-\tau,\tau)}$. Put $\overline{S}(t):=S(t)$, $t\in[0,\tau)$ and $\overline{S}(t):=(-1)^nS(t)$, $t\in(-\tau,0)$. Then $(\overline{S}(t))_{t\in(-\tau,\tau)}$ is a $[\overbrace{0,\ldots,0}^{n},\frac{(-1)}{n!}I,\overbrace{0,\ldots,0}^{n}]$-group having A as a subgenerator. A similar assertion holds for n-times integrated C-groups.

(ii) Let $(S(t))_{t\in(-\tau,\tau)}$ be a $[B_0,\ldots,B_n,C_0,\ldots,C_{n-1}]$-group having A as a subgenerator. Put $\check{S}(t):=S(-t)$, $t\in(-\tau,\tau)$, $\check{B}_j:=(-1)^jB_j$ and $\check{C}_j:=(-1)^{j+1}C_j$. Then $(\check{S}(t))_{t\in(-\tau,\tau)}$ is a $[\check{B}_0,\ldots,\check{B}_n,\check{C}_0,\ldots,\check{C}_{n-1}]$-group with a subgenerator $-A$.

(iii) Let $k\in\mathbb{N}$ and let $D_1,\ldots,D_k\in L(E)$. Given $i\in\{1,\ldots,k\}$, put $\overline{D_i}:=\prod_{j=1}^{i}D_j$. Define $S_i(\cdot)$, $i\in\{0,\ldots,k\}$ recursively by:

$$S_0(t)x:=S(t)x,\ \ldots,\ S_i(t)x:=\int_0^t S_{i-1}(s)D_ix\,ds,\ x\in E,\ t\in(-\tau,\tau),$$

and suppose, additionally, that $D_iA\subseteq AD_i$, $i\in\{1,\ldots,k\}$. By a simple induction argument, one can deduce that, for every $i\in\{1,\ldots,k\}$, $(S_i(t))_{t\in(-\tau,\tau)}$ is a

$$\Biggl[\overbrace{0,\ldots,0}^{i},\frac{0!B_0\overline{D_i}}{i!},\ldots,\frac{n!B_n\overline{D_i}}{(i+n)!},\overbrace{0,\ldots,0}^{i},\frac{0!C_0\overline{D_i}}{i!},\ldots,\frac{(n-1)!C_{n-1}\overline{D_i}}{(i+n-1)!}\Biggr]\text{-group}$$

with a subgenerator A.

(iv) Suppose A generates a (local) C-regularized group $(T(t))_{t\in(-\tau,\tau)}$. Then $(T(t))_{t\in(-\tau,\tau)}$ is a $[-C,0,0]$-group with the integral generator A. Put

$$T_n(t)x:=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}T(s)x\,ds,\quad t\in\mathbb{R},\ x\in E,\ n\in\mathbb{N}.$$

Then $(T_n(t))_{t\in(-\tau,\tau)}$ is a $\bigl[\underbrace{0,\ldots,0}_{n},-\frac{1}{n!}C,\underbrace{0,\ldots,0}_{n}\bigr]$-group with the integral generator A.

Let A be closed and $B_0,\ldots,B_n\in L(E)$. Put

$$\rho_{B_0,\ldots,B_n}(A):=\Biggl\{\lambda\in\mathbb{C}:\mathrm{R}\Biggl(\sum_{j=0}^{n}\frac{j!}{\lambda^j}B_j\Biggr)\subseteq\mathrm{R}(\lambda-A)\Biggr\}.$$

The next characterization of exponentially bounded $[B_0,\ldots,B_n,C_0,\ldots,C_{n-1}]$-groups can be simply proved (cf. [**227**]).

PROPOSITION 3.3.14. (i) *Assume A is a subgenerator of a $[B_0^+,\ldots,B_n^+,C_0^+,\ldots,C_{n-1}^+]$-group $(S(t))_{t\in\mathbb{R}}$ satisfying $\|S(t)\|\leqslant Me^{\omega|t|}$, $t\in\mathbb{R}$, for some $M>0$ and $\omega\geqslant 0$. Set $B_j^-:=(-1)^jB_j^+$ and $C_j^-:=(-1)^{j+1}C_j^+$. Then:*

(i.1) $\rho_{B_0^+,\ldots,B_n^+}(A)\cap\rho_{B_0^-,\ldots,B_n^-}(-A)\supseteq\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\omega\}$, (i.2) $\int_0^\infty e^{-\lambda t} S(\pm t)x\,dt = -(\lambda \mp A)^{-1} \sum_{j=0}^n \frac{j!}{\lambda^j} B_j^{\pm} x$, $\operatorname{Re}\lambda > \omega$, $x \in E$ *and*

(i.3) $\pm A \int_0^\infty e^{-\lambda t} S(\pm t)x\,dt - \int_0^\infty e^{-\lambda t} S(\pm t) \pm Ax\,dt = \sum_{j=0}^{n-1} \frac{j!}{\lambda^{j+1}} C_j^{\pm} x$, $\operatorname{Re}\lambda > \omega$, $x \in D(A)$.

(ii) *Assume A is a closed operator and $(S(t))_{t\in\mathbb{R}}$ is a strongly continuous operator family satisfying $\|S(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, for some $M > 0$ and $\omega \geqslant 0$. If* (i.1), (i.2) *and* (i.3) *hold, then $(S(t))_{t\in\mathbb{R}}$ is a $[B_0^+, \ldots, B_n^+, C_0^+, \ldots, C_{n-1}^+]$-group with a subgenerator A.*

Let $n \in \mathbb{N}$. If A is a closed operator and $B_0, \ldots, B_n \in L(E)$, then we define linear operators Y_i, $i \in \{0, \ldots, n\}$ recursively by:

$$Y_0 := B_0,\ Y_{i+1} := (i+1)!B_{i+1} + AY_i,\ i \in \{0, \ldots, n-1\}.$$

Note that Y_1 is closed and that the assumption $0 \in \rho(A)$ simply implies the closedness of Y_i, $i \in \{0, \ldots, n\}$.

Proposition 3.3.15. *Suppose $\tau \in (0, \infty]$, $n \in \mathbb{N} \setminus \{1\}$ and A is a subgenerator of a $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$. Then:*

(i) *$iB_ix - C_{i-1}x \in D(A)$, $x \in D(A)$, $i \in \{1, \ldots, n\}$, $A(iB_ix - C_{i-1}x) = iB_iAx - iC_ix$, $i \in \{1, \ldots, n-1\}$ and $A(nB_nx - C_{n-1}x) = nB_nAx$, $x \in D(A)$.*

(ii) *$D(A^k) \subseteq \bigcap_{j=0}^k D(Y_j)$, $k \in \{0, \ldots, n\}$ and $Y_kx = -\bigl(\frac{d^k}{dt^k} S(t)x\bigr)_{t=0}$, $x \in D(A^k)$, $k \in \{0, \ldots, n\}$.*

(iii) *For every $k \in \{0, \ldots, n-1\}$ and $x \in D(A^{k+1})$:*

$$C_kx + \frac{1}{k!} AY_k(x) = \frac{1}{k!} Y_k(Ax). \tag{211}$$

(iv) *If $\mathrm{R}(B_0) \subseteq D(A)$, then Y_2 is closed, $D(A^k) \subseteq \bigcap_{j=0}^{k+1} D(Y_j)$, $k \in \{0, \ldots, n-1\}$,* (211) *holds for every $k \in \{1, \ldots, n-1\}$ and $x \in D(A^k)$, and there exists an appropriate constant $M > 0$ so that $\|Y_{k+1}x\| \leqslant M\|x\|_k$, $k \in \{0, \ldots, n-1\}$, $x \in D(A^k)$.*

(v) *$A(-Y_nx + Y_{n-1}Ax) = -n!B_nAx$, $x \in D(A^n)$; if $\mathrm{R}(B_0) \subseteq D(A)$, then $AY_nx = Y_nAx$, $x \in D(A^n)$.*

Proof. Suppose $x \in D(A)$. Clearly, $\frac{d}{dt} S(t)x = AS(t)x - \sum_{i=1}^n it^{i-1}B_ix$, $t \in (-\tau, \tau)$ and

$$\sum_{i=0}^{n-1} t^iC_ix = AS(t)x - S(t)Ax = AS(t)x - \left[A \int_0^t S(s)Ax\,ds - \sum_{i=0}^n t^iB_iAx\right]$$

for every $t \in (-\tau, \tau)$. Hence,

$$\sum_{i=0}^{n-1} t^iC_ix - \sum_{i=0}^n t^iB_iAx = A\left[S(t)x - \int_0^t S(s)Ax\,ds\right],\ t \in (-\tau, \tau).$$

Since

$$\frac{d}{dt}\left[S(t)x - \int_0^t S(s)Ax\,ds\right] = AS(t)x - \sum_{i=1}^n it^{i-1}B_ix - S(t)Ax = \sum_{i=0}^{n-1} t^iC_ix - \sum_{i=1}^n it^{i-1}B_ix,$$

$t \in (-\tau, \tau)$, the closedness of A implies

$$\sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} i t^{i-1} B_i x = \frac{d}{dt}\Biggl[S(t)x - \int\limits_0^t S(s)Ax\,ds\Biggr] \in D(A),\ t \in (-\tau, \tau),$$

$$(212)\quad A\Biggl[\sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} i t^{i-1} B_i x\Biggr] = \sum_{i=1}^{n-1} i t^{i-1} C_i x - \sum_{i=1}^{n} i t^{i-1} B_i Ax,\ t \in (-\tau, \tau).$$

Differentiation both sides of (212) implies $iB_i x - C_{i-1}x \in D(A)$, $i \in \{1, \ldots, n\}$ and (i). To prove (ii), notice that the closedness of A and the argumentation used in the proof of (i) enable one to conclude that the mapping $t \mapsto \frac{d^k}{dt^k} S(t)x$, $t \in (-\tau, \tau)$ is k-times continuously differentiable for every $k \in \{0, \ldots, n\}$ and $x \in D(A^k)$. Let $k \in \{0, \ldots, n\}$ be fixed. Then we obtain:

$$(213)\qquad \frac{d^{l+1}}{dt^{l+1}} S(t)x = A\frac{d^l}{dt^l} S(t)x - \sum_{j=l+1}^{n} j \cdots (j-l) t^{j-l-1} B_j x,\ t \in (-\tau, \tau),$$

for every $l \in \{0, \ldots, k-1\}$. Since $Y_0 = B_0$, the proof of (ii) follows by induction. Suppose now $x \in D(A^{k+1})$. Then the mapping $t \mapsto S(t)Ax$, $t \in (-\tau, \tau)$ is k-times continuously differentiable. Since $C_k x = \frac{1}{k!}\frac{d^k}{dt^k}[AS(t)x - S(t)Ax]$, $t \in (-\tau, \tau)$, the mapping $t \mapsto AS(t)x$, $t \in (-\tau, \tau)$ is k-times continuously differentiable. The closedness of A implies $\frac{d^k}{dt^k} AS(t)x = A\frac{d^k}{dt^k} S(t)x$, $t \in (-\tau, \tau)$ and $C_k x = \frac{1}{k!}[A\frac{d^k}{dt^k} S(t)x - \frac{d^k}{dt^k} S(t)Ax]$, $t \in (-\tau, \tau)$. Put $t = 0$ in the last equality to finish the proof of (iii). To prove (iv), notice that $\mathrm{R}(B_0) \subseteq D(A)$ and that the closed graph theorem implies $Y_1 = AB_0 + B_1 \in L(E)$; the closedness of Y_2 simply follows from this fact. Let $x \in D(A^k)$. Since $\frac{d}{dt} S(t)x - S(t)Ax = \sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} i t^{i-1} B_i x$, $t \in (-\tau, \tau)$, one obtains $\frac{d^k}{dt^k} S(t)x - \frac{d^{k-1}}{dt^{k-1}} S(t)Ax = \frac{d^{k-1}}{dt^{k-1}}\bigl[\sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} i t^{i-1} B_i x\bigr]$, $t \in (-\tau, \tau)$. This implies $-Y_k x + Y_{k-1}Ax = (k-1)!C_{k-1}x - k!B_k x$ and, by (i), $-Y_k x + Y_{k-1}Ax \in D(A)$ and:

$$(214)\quad A[-Y_k x + Y_{k-1}Ax] = k!C_k x - k!B_k Ax,\ k \in \{1, \ldots, n-1\},\ x \in D(A^k).$$

Because $\mathrm{R}(B_0) \subseteq D(A)$, one concludes inductively from (214) that $Y_k x \in D(A)$, $x \in D(A^k)$, $k \in \{0, \ldots, n-1\}$, i.e., $D(A^k) \subseteq \bigcap_{i=0}^{k+1} D(Y_i)$, $k \in \{0, \ldots, n-1\}$, and $k!C_k x + AY_k x = k!B_k Ax + AY_{k-1}Ax = Y_k Ax$, $k \in \{1, \ldots, n-1\}$, $x \in D(A^k)$. The existence of a constant $M > 0$ satisfying $\|Y_{k+1}x\| \leqslant M\|x\|_k$, $k \in \{0, \ldots, n-1\}$, $x \in D(A^k)$ essentially follows from an application of (211) and an induction argument. This completes the proof of (iv). The proof of (v) follows from that of (iv). $\square$

Remark 3.3.16. (i) Let $(S(t))_{t\in[0,\tau)}$ be a $[B_0, B_1, C_0]$-group having A as a subgenerator. Arguing as in the proof of previous proposition, one obtains $A(B_1 x - C_0 x) = B_1 Ax$, $x \in D(A)$ and $AY_1 x = Y_1 Ax$, $x \in D(A^2)$. Furthermore, $AY_1 x = Y_1 Ax$, $x \in D(A)$ if $\mathrm{R}(B_0) \subseteq D(A)$.

(ii) The next question is motivated by the analysis of Arendt, El-Mennaoui and Keyantuo [**5**]: Let A be a subgenerator of a $[\underbrace{0, \ldots, 0}_{n}, \frac{-1}{n!}I, C_0, \ldots, C_{n-1}]$-group

$(S(t))_{t\in(-\tau,\tau)}$, $n \in \mathbb{N}$, $0 < \tau \leqslant \infty$. Does the inclusion $S(t)A \subseteq AS(t)$ hold for every $t \in (-\tau,\tau)$? The answer is affirmative and we will show this only in the non-trivial case $n > 1$. Indeed, $S(0) = 0$ and this implies $C_0x = 0$, $x \in D(A)$. By Proposition 3.3.15(i), $AC_{i-1}x = iC_ix$, $i \in \{1,\dots,n-1\}$, $x \in D(A)$. Inductively, $C_ix = 0$, $i \in \{1,\dots,n-1\}$, $x \in D(A)$ and an immediate consequence is $S(t)A \subseteq AS(t)$, $t \in (-\tau,\tau)$.

(iii) Suppose A, resp. $\hat{A}$, is a subgenerator, resp. the integral generator, of a non-degenerate $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$, $n \geqslant 2$. Then $iB_ix - C_{i-1}x \in D(A)$, $x \in D(\hat{A})$, $i \in \{1,\dots,n\}$, $A(iB_ix - C_{i-1}x) = iB_i\hat{A}x - iC_ix$, $i \in \{1,\dots,n-1\}$ and $A(nB_nx - C_{n-1}x) = nB_n\hat{A}x$, $x \in D(\hat{A})$. To prove this, suppose $(x,y) \in \hat{A}$. Clearly, $A\int_0^t S(s)x\,ds = \int_0^t (S(s)y + \sum_{j=0}^{n-1} s^jC_jx)\,ds$, $t \in (-\tau,\tau)$. Differentiate this equality to obtain $S(t)x \in D(A)$ and $AS(t)x = S(t)y + \sum_{j=0}^{n-1} t^jC_jx$, $t \in (-\tau,\tau)$. Hence,

$$A\left[\int_0^t S(s)y\,ds + \sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx - \sum_{j=0}^{n} t^jB_jx\right] = S(t)y + \sum_{j=0}^{n-1} t^jC_jx,\ t \in (-\tau,\tau),$$

$$A\left[\sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx - \sum_{j=0}^{n} t^jB_jx\right] = \sum_{j=0}^{n-1} t^jC_jx - \sum_{j=0}^{n} t^jB_jy,\ t \in (-\tau,\tau).$$

Differentiation of the previous equality leads us to the desired assertion. Notice that we have proved an extension of Proposition 3.3.15(i) and that, in the case $n = 1$, $A(B_1x - C_0x) = B_1Ax$, $x \in D(\hat{A})$.

The main objective in the following proposition is to clarify a composition property of $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-groups satisfying the condition (215) quoted below.

PROPOSITION 3.3.17. *Suppose* $0 < \tau \leqslant \infty$ *and* A *is a subgenerator of a* $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$*-group* $(S(t))_{t\in(-\tau,\tau)}$. *If*

$$C_jA \subseteq AC_j,\ j = 0,\dots,n-1 \text{ and } B_jA \subseteq AB_j,\ j = 1,\dots,n, \tag{215}$$

then, for every $x \in E$,

$$\begin{aligned} S(t)S(s)x = {} & \sum_{j=1}^{n}\left[\int_0^s j(t+s-r)^{j-1}B_jS(r)x\,dr - \int_t^{t+s} j(t+s-r)^{j-1}S(r)B_jx\,dr\right] \\ & - \sum_{j=0}^{n-1}\left[\int_0^s (t+s-r)^jC_jS(r)x\,dr + t^jC_j\int_0^s S(r)x\,dr\right] \\ & - S(t+s)B_0x - \sum_{j=0}^{n-1}\sum_{i=0}^{n}\int_0^s (t+s-r)^jr^i\,dr\,C_jB_ix,\ t,\ s \in (-\tau,\tau),\ |t+s| < \tau. \end{aligned} \tag{216}$$

PROOF. Let $y \in D(A)$ and $t \in (-\tau, \tau)$. Then we obtain $\int_0^t AS(s)y\,ds = S(t)y + \sum_{j=0}^{n} t^j B_j y$, i.e., $\int_0^t [S(s)Ay + \sum_{j=0}^{n-1} s^j C_j y]\,ds = S(t)y + \sum_{j=0}^{n} t^j B_j y$. Hence,

$$\frac{d}{dt}S(t)y = S(t)Ay + \sum_{j=0}^{n-1} t^j C_j y - \sum_{j=1}^{n} j t^{j-1} B_j y,\ t \in (-\tau, \tau). \tag{217}$$

Let $x \in E$ be fixed and let $t,\ s \in (-\tau, \tau)$ satisfy $|t+s| < \tau$. Define the function $f : (t+s-\tau, t+s+\tau) \cap (-\tau, \tau) \to E$ by $f(r) := S(t+s-r)\int_0^r S(s)x\,ds$. Then:

$$\begin{aligned}
\frac{d}{dr}f(r) &= \frac{d}{dr}\Bigg[S(t+s-r)\int_0^r S(v)x\,dv\Bigg] = S(t+s-r)S(r)x \\
&\quad - \Bigg[S(t+s-r)A\int_0^r S(v)x\,dv + \sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv \\
&\qquad\qquad - \sum_{j=1}^{n} j(t+s-r)^{j-1} B_j \int_0^r S(v)x\,dv\Bigg] \\
&= S(t+s-r)S(r)x - S(t+s-r)\Bigg[S(r)x + \sum_{j=0}^{n} r^j B_j x\Bigg] \\
&\quad - \sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv + \sum_{j=1}^{n} j(t+s-r)^{j-1} B_j \int_0^r S(v)x\,dv \\
&= -\sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv + \sum_{j=1}^{n} j(t+s-r)^{j-1} B_j \int_0^r S(v)x\,dv \\
&\qquad - \sum_{j=0}^{n} r^j S(t+s-r) B_j x,
\end{aligned}$$

for all $r \in (t+s-\tau, t+s+\tau) \cap (-\tau, \tau)$. Integrate the last equality with respect to r from 0 to s to conclude that:

$$\begin{aligned}
S(t)\int_0^s S(v)x\,dv &= -\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \int_0^r S(v)x\,dv\,dr \\
&\quad + \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1}\int_0^r S(v)x\,dv\,dr - \sum_{j=0}^{n}\int_0^s r^j S(t+s-r)B_j x\,dr.
\end{aligned} \tag{218}$$

This implies

$$S(t)S(s)x = S(t)\Bigg[A\int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j B_j x\Bigg] \tag{219}$$

$$= \left[AS(t) \int_0^s S(v)x\,dv - \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv \right] - \sum_{j=0}^{n} s^j S(t)B_j x$$

$$= A\Bigg[-\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \int_0^r S(v)x\,dv\,dr$$

$$+ \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1} \int_0^r S(v)x\,dv\,dr - \sum_{j=0}^{n} \int_0^s r^j S(t+s-r)B_j x\,dr \Bigg]$$

$$- \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j S(t)B_j x.$$

Taking into consideration (215), we get:

$$= -\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \left[S(r)x + \sum_{i=0}^{n} r^i B_i x \right] dr \tag{220}$$

$$+ \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1} \left[S(r)x + \sum_{i=0}^{n} r^i B_i x \right] dr$$

$$- A\sum_{j=0}^{n} \int_0^s r^j S(t+s-r)B_j x\,dr - \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j S(t)B_j x.$$

Observe that:

$$A\int_0^s S(t+s-r)B_0 x\,dr = A\int_t^{t+s} S(v)B_0 x\,dv = A\left[\int_0^{t+s} S(v)B_0 x\,dv - \int_0^t S(v)B_0 x\,dv \right]$$

$$= S(t+s)B_0 x + \sum_{i=0}^{n} (t+s)^i B_i B_0 x - S(t)B_0 x - \sum_{i=0}^{n} t^i B_i B_0 x,$$

$$A\int_0^s r^j S(t+s-r)B_j x\,dr = A\int_t^{t+s} (t+s-v)^j S(v)B_j x\,dv$$

$$= A\left[-s^j \int_0^t S(\sigma)B_j x\,d\sigma + \int_t^{t+s} j(t+s-v)^{j-1} \int_0^v S(\sigma)B_j x\,d\sigma\,dv \right]$$

$$= -s^j \left[S(t)B_j x + \sum_{i=0}^{n} t^i B_i B_j x \right] + \int_t^{t+s} j(t+s-v)^{j-1} \left[S(v)B_j x + \sum_{i=0}^{n} v^i B_i B_j x \right] dv,$$

for all $j = 1, \ldots, n$. By (220),

$$S(t)S(s)x = -\sum_{j=0}^{n-1} \int_0^s (t+s-r)^j C_j S(r)x\,dr - \sum_{j=0}^{n-1} \int_0^s (t+s-r)^j \sum_{i=0}^{n} r^i C_j B_i x\,dr$$

$$+\sum_{j=1}^{n} j \int_0^s (t+s-r)^{j-1} B_j S(r)x\,dr + \sum_{j=1}^{n} j \int_0^s (t+s-r)^{j-1} \sum_{i=0}^{n} r^i B_j B_i x\,dr$$

$$-S(t+s)B_0 x - \sum_{i=0}^{n} (t+s)^i B_i B_0 x + S(t)B_0 x + \sum_{i=0}^{n} t^i B_i B_0 x$$

$$+\sum_{j=1}^{n} s^j \Biggl[S(t)B_j x + \sum_{i=0}^{n} t^i B_i B_j x\Biggr] - \sum_{j=1}^{n} j \int_t^{t+s} (t+s-v)^{j-1} \Biggl[S(v)B_j x + \sum_{i=0}^{n} v^i B_i B_j x\Biggr] dv$$

$$-\sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j S(t)B_j x. \tag{221}$$

Clearly, $S(t)B_0 x + \sum_{j=1}^{n} s^j S(t)B_j x - \sum_{j=0}^{n} s^j S(t)B_j x = 0$ and:

$$-\sum_{j=1}^{n}\sum_{i=0}^{n} \int_t^{t+s} j(t+s-v)^{j-1} v^i dv B_i B_j x = -\sum_{j=1}^{n}\sum_{i=0}^{n} \int_0^s (t+s-r)^i j r^{j-1} dr B_i B_j x$$

$$= -\sum_{j=1}^{n}\sum_{i=0}^{n} t^i s^j B_i B_j x - \sum_{j=1}^{n}\sum_{i=1}^{n} \int_0^s i(t+s-r)^{i-1} r^j dr B_i B_j x$$

$$= -\sum_{j=1}^{n}\sum_{i=0}^{n} t^i s^j B_i B_j x - \sum_{j=1}^{n}\sum_{i=1}^{n} \int_0^s j(t+s-r)^{j-1} r^i dr B_j B_i x.$$

Therefore,

$$-\sum_{j=1}^{n}\sum_{i=0}^{n} \int_t^{t+s} j(t+s-v)^{j-1} v^i dv B_i B_j x + \sum_{j=1}^{n}\sum_{i=0}^{n} s^j t^i B_i B_j x \tag{222}$$

$$+\sum_{j=1}^{n}\sum_{i=0}^{n} \int_0^s j(t+s-r)^{j-1} r^i dr B_j B_i x$$

$$= \sum_{j=1}^{n} \int_0^s j(t+s-r)^{j-1} dr B_j B_0 x = \sum_{j=1}^{n} [(t+s)^j - t^j] B_j B_0 x.$$

Finally, (216) follows from an application of (221)–(222). □

REMARK 3.3.18. The composition property does not remain true if the condition (215) is neglected. Namely, let A, B_0, B_1, B_2, C_0, C_1 and $(S(t))_{t\in\mathbb{R}}$ have the same meaning as in Example 3.3.12(i). Then $(S(t))_{t\in\mathbb{R}}$ is a $[B_0, B_1, B_2, C_0, C_1]$-group with a subgenerator A and a tedious matrix computation shows that (215) and (216) do not hold. Moreover, $\rho_{B_0,B_1,B_2}(A) \supseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $\mathrm{R}(B_0) + \mathrm{R}(B_1) + \mathrm{R}(B_2) \nsubseteq \mathrm{R}(1 - A)$ (cf. also Proposition 3.3.14).

3.3.3. Further relations between distribution groups, local integrated groups and $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-groups. In order to establish a satisfactory relationship of distribution groups with local integrated groups, we need the following definition introduced by Tanaka and Okazawa in [**404**] (cf. [**404**, Definition 4]):

($\triangle$) Suppose $n \in \mathbb{N}$ and $\tau \in (0, \infty]$. A strongly continuous operator family $(S(t))_{t\in[0,\tau)}$ is called a (local) n-times integrated semigroup if:

(i) for every $x \in E$ and $(t, s) \in [0, \tau) \times [0, \tau)$ with $t + s < \tau$:

$$S(t)S(s)x = \left[\int_0^{t+s} - \int_0^{t} - \int_0^{s}\right] \frac{(t + s - r)^{n-1}}{(n-1)!} S(r)x\,dr,$$

and $S(0) = 0$,

(ii) $S(t)x = 0$ for every $t \in [0, \tau)$ implies $x = 0$.

Suppose $(S(t))_{t\in[0,\tau)}$ is an n-times integrated semigroup in the sense of ($\triangle$). The infinitesimal generator A_0 of $(S(t))_{t\in[0,\tau)}$ is defined in [**404**] via: $D(A_0) =: \bigl\{x \in \bigcup_{\sigma\in(0,\tau]} C^n(\sigma) : \lim_{h\to 0+} \frac{S^{(n)}(h)x-x}{h} \text{ exists}\bigr\}$ and $A_0x := \lim_{h\to 0+} \frac{S^{(n)}(h)x-x}{h}$, $x \in D(A_0)$, where

$$C^n(\sigma) := \bigl\{x \in E : S(\cdot)x : [0, \sigma) \to E \text{ is } n\text{-times continuously differentiable}\bigr\}.$$

The infinitesimal generator A_0 of $(S(t))_{t\in[0,\tau)}$ is a closable linear operator and the closure of A_0, $\overline{A_0}$, is said to be the complete infinitesimal generator, c.i.g in short, of $(S(t))_{t\in[0,\tau)}$. Suppose $(S(t))_{t\in[0,\tau)}$ is a (local) n-times integrated semigroup in the sense of Definition 2.1.1. Then $(S(t))_{t\in[0,\tau)}$ is an n-times integrated semigroup in the sense of ($\triangle$); in general, the converse statement does not hold (cf. [**5**], [**227**, Proposition 2.1], [**252**] and [**404**, Proposition 4.5]).

THEOREM 3.3.19. (i) *Suppose $G \in \mathcal{D}'(L(E))$ and A is a closed linear operator so that* (208) *holds. Then, for every $\tau \in (0, \infty)$, there exist $n_0 = n_0(\tau) \in \mathbb{N}$ and $B_0, \ldots, B_{n_0}, C_0, \ldots, C_{n_0-1} \in L(E)$ such that A is a subgenerator of a $[B_0, \ldots, B_{n_0}, C_0, \ldots, C_{n_0-1}]$-group $(S_\tau(t))_{t\in(-\tau,\tau)}$ satisfying $S_\tau(t)x \in D(A)$ for all $x \in E$ and $t \in (-\tau, \tau)$.*

(ii) *Let G and A be as in the formulation of* (i) *and let $A_1 := A_{|\overline{\mathcal{R}(G)}}$. Suppose, in addition, that G is regular and put $S(t) := S_\tau(t)$, $t \in (-\tau, \tau)$, where $(S_\tau(t))_{t\in(-\tau,\tau)}$ is the $[B_0, \ldots, B_{n_0}, C_0, \ldots, C_{n_0-1}]$-group constructed in* (i)*. Then:*

(ii.1) *$\mathcal{R}(G) \subseteq \bigcap_{i=0}^{n_0} D(Y_i)$, $Y_{n_0}x = -x$, $x \in \mathcal{R}(G)$, the function $t \mapsto u(t; x)$, $t \in \mathbb{R}$ is infinitely differentiable, $u(t; x) \in D_\infty(A)$ and $\frac{d^n}{dt^n}u(t; x) = u(t; A^n x)$, $t \in \mathbb{R}$, $x \in \mathcal{R}(G)$, $n \in \mathbb{N}$.*

(ii.2) *If* $n_0=1$, *then* $\pm A_1$ *generate once integrated semigroups* $(S^1_\pm(t))_{t\in[0,\tau)}\subseteq L(\overline{\mathcal{R}(G)})$ *given by* $S^1_\pm(t)x:=S(\pm t)(\pm x)\pm B_0x$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$. *Furthermore,* A_1 *generates a* C_0*-group in* $\overline{\mathcal{R}(G)}$.

(ii.3) *Let* $n_0=2$ *and put* $S^2_\pm(t)x:=S(\pm t)x+B_0x+t(\pm AB_0\pm B_1)x$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. *Then* $S^2_\pm(t)\in L(\overline{\mathcal{R}(G)})$, $\bigl(\int_0^t S^2_\pm(s)x ds, S^2_\pm(t)x-\frac{t^2}{2}x\bigr)\in\pm A_1$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$, $S^2_\pm(t)A_1x=A_1S^2_\pm(t)x$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$, $S^2_\pm(t)x\in D(A)$, $x\in\overline{\mathcal{R}(G)}$, *the mapping* $t\mapsto\frac{d}{dt}S^2_\pm(t)x$, $t\in[0,\tau)$ *is continuously differentiable for every* $x\in\overline{\mathcal{R}(G)}$, $\overline{\mathcal{R}(G)}\subseteq\bigcap_{i=0}^2 D(Y_i)$ *and* $Y_2x=-x$, $x\in\overline{\mathcal{R}(G)}$. *Furthermore,* $\pm A_1$ *are generators of local once integrated semigroups* $(\frac{d}{dt}S^2_\pm(t))_{t\in[0,\tau)}$.

(ii.4) *Assume* $n_0\geqslant 3$,

$$\overline{\mathcal{R}(G)}\subseteq\bigcap_{i=2}^{n_0-1}D(Y_i) \tag{223}$$

and there exists $M>0$ *with*

$$\|Y_ix\|\leqslant M\|x\|,\ x\in\overline{\mathcal{R}(G)},\ i=2,\dots,n_0-1. \tag{224}$$

The following holds: $\overline{\mathcal{R}(G)}\subseteq D(Y_{n_0})$ *and* $Y_{n_0}x=-x$, $x\in\overline{\mathcal{R}(G)}$. *Set*

$$S^{n_0}_+(t)x:=S(t)x+\sum_{i=0}^{n_0-1}\frac{t^i}{i!}Y_ix$$

$$S^{n_0}_-(t)x:=(-1)^{n_0}S(-t)x+\sum_{i=0}^{n_0-1}\frac{(-1)^{n_0+i}t^i}{i!}Y_ix,\ x\in\overline{\mathcal{R}(G)},\ t\in[0,\tau).$$

Then: $S^{n_0}_\pm(t)\in L(\overline{\mathcal{R}(G)})$, $\bigl(\int_0^t S^{n_0}_\pm(s)x\,ds, S^{n_0}_\pm(t)x-\frac{t^{n_0}}{n_0!}x\bigr)\in\pm A_1$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$, $S^{n_0}_\pm(t)A_1x=A_1S^{n_0}_\pm(t)x$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$ *and* $S^{n_0}_\pm(t)x\in D(A)$, $x\in\overline{\mathcal{R}(G)}$. *Put*

$$A^{'}_{n_0-1,\pm}=\Bigl\{(x,y)\in\pm A_1: C_ix+\frac{1}{i!}AY_ix=\pm\frac{1}{i!}Y_iy,\ i=2,\dots,n_0-1\Bigr\}.$$

Then $A^{'}_{n_0-1,\pm}$ *are generators of local* (n_0-1)*-times integrated semigroups* $(\frac{d}{dt}S^{n_0}_\pm(t))_{t\in[0,\tau)}$.

(ii.5) *Let* $n_0\geqslant 3$ *and* $\rho(A)\neq\emptyset$. *Then, for every* $\tau_0\in(0,\infty)$, *there exists* $n(\tau_0)\in\mathbb{N}$ *such that* A_1 *generates a local* $n(\tau_0)$*-times integrated group on* $(-\tau_0,\tau_0)$.

Proof. (i) Let $\tau\in(0,\infty)$ be chosen arbitrarily. Since $AG(\varphi)x=G(-\varphi')x$, $\varphi\in\mathcal{D}$, $x\in E$ we have $G\in\mathcal{D}'(L(E,[D(A)]))$, and Theorem 1.3.2 implies that there exist a natural number $n_0=n_0(\tau)$ and a continuous function $S_\tau:[-\tau,\tau]\to L(E,[D(A)])$ such that $G(\varphi)x=(-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)S_\tau(t)x\,dt$, $\varphi\in\mathcal{D}_{(-\tau,\tau)}$, $x\in E$.

Then we obtain:

$$AG(\varphi)x=(-1)^{n_0}\int\limits_{-\tau}^{\tau}\varphi^{(n_0)}(t)AS_\tau(t)x\,dt=(-1)^{n_0+1}\int\limits_{-\tau}^{\tau}\varphi^{(n_0+1)}(t)\int\limits_0^t AS_\tau(s)x\,ds\,dt$$
$$=G(-\varphi')x=(-1)^{n_0+1}\int\limits_{-\tau}^{\tau}\varphi^{(n_0+1)}(t)S_\tau(t)x\,dt,\ \varphi\in\mathcal{D}_{(-\tau,\tau)},\ x\in E.$$

An immediate consequence is:

$$\int\limits_{-\tau}^{\tau}\varphi^{(n_0+1)}(t)\Bigg[\int\limits_0^t AS_\tau(s)xds-S_\tau(t)x\Bigg]\,dt=0,\ \varphi\in\mathcal{D}_{(-\tau,\tau)},\ x\in E.$$

The well-known arguments of distribution theory (cf. for instance [**128**, Lemma 8.1.1]) imply that there exist $B_0,\dots,B_{n_0}\in L(E)$ which satisfy (i) of Definition 3.3.11. Similarly, if $x\in D(A)$, then $G(\varphi)Ax=AG(\varphi)x$, $\varphi\in\mathcal{D}$ and we get:

$$\int\limits_{-\tau}^{\tau}\varphi^{(n_0)}(t)\big[AS_\tau(t)x-S_\tau(t)Ax\big]\,dt=0,\quad\varphi\in\mathcal{D}_{(-\tau,\tau)},\ x\in E.$$

So, there exist $C_0,\dots,C_{n_0-1}\in L(E)$ satisfying $AS_\tau(t)x-S_\tau(t)Ax=\sum_{j=0}^{n-1}t^jC_jx$ for all $t\in(-\tau,\tau)$ and $x\in D(A)$. To prove (ii.1), we need the following notion. Suppose $\zeta\in\mathcal{D}$ and $\int_{-\infty}^{\infty}\zeta(t)\,dt=1$. Given $\varphi\in\mathcal{D}$, we define $I_\zeta(\varphi)$ by

$$I_\zeta(\varphi)(t):=\int\limits_{-\infty}^{t}\Bigg[\varphi(u)-\zeta(u)\int\limits_{-\infty}^{\infty}\varphi(v)\,dv\Bigg]\,du,\ t\in\mathbb{R}.$$

Then we have: $I_\zeta(\varphi)\in\mathcal{D}$, $I_\zeta(\varphi')=\varphi$ and $\frac{d}{dt}I_\zeta(\varphi)(t)=\varphi(t)-\zeta(t)\int_{-\infty}^{\infty}\varphi(v)\,dv$, $t\in\mathbb{R}$. Suppose $x\in\mathcal{R}(G)$. Since $AG(\varphi)x=G(-\varphi')x$, $\varphi\in\mathcal{D}$ one obtains $-\int_{-\infty}^{\infty}\varphi'(t)u(t;x)dt=A\int_{-\infty}^{\infty}\varphi(t)u(t;x)\,dt$, $\varphi\in\mathcal{D}$. Then the partial integration yields:

$$A\int\limits_{-\infty}^{\infty}\varphi'(t)\int\limits_0^t u(s;x)\,ds\,dt=\int\limits_{-\infty}^{\infty}\varphi'(t)u(t;x)\,dt,\ \varphi\in\mathcal{D}.\tag{225}$$

Suppose (ρ_n) is a regularizing sequence and put $\varphi_n=I_\zeta(\rho_n)$ in (225) in an effort to see that:

$$A\int\limits_{-\infty}^{\infty}\big[\rho_n(t)-\zeta(t)\big]\int\limits_0^t u(s;x)\,ds\,dt=\int\limits_{-\infty}^{\infty}\big[\rho_n(t)-\zeta(t)\big]u(t;x)\,dt.$$

The closedness of A and $u(0;x)=x$ imply, for every $\zeta\in\mathcal{D}$ with $\int_{-\infty}^{\infty}\zeta(t)\,dt=1$:

$$A\int\limits_{-\infty}^{\infty}\zeta(t)\int\limits_0^t u(s;x)\,ds\,dt=\int\limits_{-\infty}^{\infty}\zeta(t)u(t;x)\,dt-x.\tag{226}$$

It is evident that, for every $t \in \mathbb{R}$, there exists a sequence (ζ_n) in $\mathcal{D}$ such that $\int_{-\infty}^{\infty} \zeta_n(t)\,dt = 1$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} \zeta_n = \delta_t$, in the sense of distributions. Put (ζ_n) in (226). As above, the closedness of A implies $\int_0^t u(s;x)\,ds \in D(A)$ and $A\int_0^t u(s;x)\,ds = u(t;x) - x$, $t \in \mathbb{R}$. Inductively,

$$A\int_0^t \frac{(t-s)^k}{k!}u(s;x)\,ds = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}u(s;x)\,ds - \frac{t^k}{k!}x,\ t \in \mathbb{R},\ k \in \mathbb{N}. \tag{227}$$

Clearly, $Ax \in \mathcal{R}(G)$ and A commutes with $G(\cdot)$. Hence,

$$A\int_{-\infty}^{\infty} \varphi(t)u(t;x)\,dt = \int_{-\infty}^{\infty} \varphi(t)u(t;Ax)\,dt,\ \varphi \in \mathcal{D}. \tag{228}$$

An application of (228) gives $u(t;x) \in D(A)$, $Au(t;x) = u(t;Ax)$, $t \in \mathbb{R}$, which implies $u(t;x) \in D_\infty(A)$, $t \in \mathbb{R}$. Since $A\int_0^t u(s;x)\,ds = u(t;x) - x$, $t \in \mathbb{R}$, one obtains by induction that the function $t \mapsto u(t;x)$, $t \in \mathbb{R}$ is infinitely differentiable and that $\frac{d^n}{dt^n}u(t;x) = u(t;A^n x)$, $t \in \mathbb{R}$, $x \in \mathcal{R}(G)$, $n \in \mathbb{N}$. Furthermore,

$$A\int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}u(s;x)\,ds = \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}u(s;Ax)\,ds. \tag{229}$$

Since

$$\begin{aligned} G(\varphi)x &= (-1)^{n_0}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)S(t)x\,dt = \int_{-\infty}^{\infty} \varphi(t)u(t;x)\,dt \\ &= (-1)^{n_0}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)\int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}u(s;x)\,ds\,dt,\quad \varphi \in \mathcal{D}_{(-\tau,\tau)}, \end{aligned}$$

there is a subset $\{y_0(x), \ldots, y_{n_0-1}(x)\}$ of E such that:

$$S(t)x - \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}u(s;x)\,ds = -\sum_{i=0}^{n_0-1} t^i y_i(x),\ t \in (-\tau,\tau). \tag{230}$$

Put $t = 0$ to obtain $y_0(x) = B_0 x$. By (230), it follows that

$$\int_0^t S(s)x\,ds - \int_0^t \frac{(t-s)^{n_0}}{n_0!}u(s;x)\,ds = -\sum_{i=0}^{n_0-1} \frac{t^{i+1}}{i+1}y_i(x),\ t \in (-\tau,\tau). \tag{231}$$

By (227), one can apply A on both sides of (231) to deduce that, for every $t \in (-\tau,\tau)$:

$$\left[S(t)x + \sum_{i=0}^{n_0} t^i B_i x\right] - \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}u(s;x)\,ds + \frac{t^{n_0}}{n_0!}x = -A\sum_{i=0}^{n_0-1} \frac{t^{i+1}}{i+1}y_i(x),$$

which implies:

$$-\sum_{i=0}^{n_0-1} t^i y_i(x) + \sum_{i=0}^{n_0} t^i B_i x + \frac{t^{n_0}}{n_0!}x = -A\sum_{i=0}^{n_0-1} \frac{t^{i+1}}{i+1} y_i(x),\ t \in (-\tau, \tau). \tag{232}$$

Since A is closed, one can differentiate (232) sufficiently many times to obtain that: $x \in \bigcap_{i=0}^{n_0} D(Y_i)$, $Y_{n_0}x = -x$ and $y_i(x) = \frac{1}{i!}Y_i x$, $i \in \{0, \ldots, n_0 - 1\}$. This completes the proof of (ii.1). To prove (ii.2), fix an $x \in \mathcal{R}(G)$. Let $S_+^{n_0}(t)x = S(t)x + \sum_{i=0}^{n_0-1} \frac{t^i}{i!} Y_i(x)$, $t \in [0, \tau)$. By (230), $S_+^{n_0}(t)x = \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!} u(s; x)\,ds$, $t \in [0, \tau)$ and (229) implies $AS_+^{n_0}(t)x = S_+^{n_0}(t)Ax$, $t \in [0, \tau)$. Let $n_0 = 1$. Then $S_+^1(t)x = S(t)x + B_0x$, $t \in [0, \tau)$, $x \in \overline{\mathcal{R}(G)}$. By the proof of (ii.1), $S_+^1(t)x = \int\limits_0^t u(s; x)\, ds$, $t \in [0, \tau)$, $x \in \mathcal{R}(G)$. Accordingly, $S_+^1(t)(\overline{\mathcal{R}(G)}) \subseteq \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$. By (227), $\bigl(\int_0^t S_+^1(s)x\, ds, S_+^1(t)x - tx\bigr) \in A_1$, $t \in [0, \tau)$, $x \in \mathcal{R}(G)$ and the closedness of A implies $\bigl(\int_0^t S_+^1(s)x ds, S_+^1(t)x - tx\bigr) \in A_1$, $t \in [0, \tau)$, $x \in \overline{\mathcal{R}(G)}$. Clearly, $S_+^1(t)A_1 \subseteq A_1 S_+^1(t)$, $t \in [0, \tau)$ and this proves that $(S_+^1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A_1. The similar arguments (see also the proof of (ii.3)) work for $-A_1$ and $(S_-^1(t))_{t\in[0,\tau)}$. To prove that A_1 generates a C_0-group in $\overline{\mathcal{R}(G)}$, we argue as follows. Since $\bigl(\int_0^t S_+^1(s)x\, ds, S_+^1(t)x - tx\bigr) \in A_1$, $t \in [0, \tau)$, $x \in \overline{\mathcal{R}(G)}$ and $\mathrm{R}(S_+^1(t)) \subseteq D(A)$, $t \in [0, \tau)$ one gets that the mapping $t \mapsto \frac{d}{dt}S_+^1(t)x$, $t \in [0, \tau)$ is continuously differentiable for every $x \in \overline{\mathcal{R}(G)}$ and that $\frac{d}{dt}S_+^1(t)x = AS_1(t)x + x$, $t \in [0, \tau)$, $x \in \overline{\mathcal{R}(G)}$. Moreover, it can be easily checked that, for every fixed $x \in \overline{\mathcal{R}(G)}$, the function $u(t) = S_+^1(t)x$, $t \in [0, \tau)$ is a unique solution of the problem:

$$C_1(\tau): \begin{cases} u \in C([0, \tau) : [D(A_1)]) \cap C^1([0, \tau) : \overline{\mathcal{R}(G)}), \\ u'(t) = A_1 u(t) + x,\ t \in [0, \tau), \\ u(0) = 0. \end{cases}$$

An application of [**5**, Theorem 1.2] gives that A_1 generates a C_0-semigroup in $\overline{\mathcal{R}(G)}$. Similarly, $-A_1$ generates a C_0-semigroup in $\overline{\mathcal{R}(G)}$ and this clearly implies that A_1 generates a C_0-group in $\overline{\mathcal{R}(G)}$. Let us prove (ii.3). The proof of (b1) implies that $S_+^2(t)x = \int\limits_0^t (t - s)u(s; x)\, ds$, $t \in [0, \tau)$, $x \in \mathcal{R}(G)$. So, $S_+^2(t)(\overline{\mathcal{R}(G)}) \subseteq \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$. Note also that $S(0) = -B_0$ and that the closed graph theorem gives $S_+^2(t) \in L(\overline{\mathcal{R}(G)})$, $t \in [0, \tau)$. Next, the closedness of A and (227) imply $\bigl(\int_0^t S_+^2(s)x ds, S_+^2(t) - \frac{t^2}{2}x\bigr) \in A_1$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$. Since $\int_0^t S_+^2(s)x\, ds \in D(A)$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$ and $\mathrm{R}(B_0) \subseteq D(A)$, we immediately obtain $AB_0x + B_1x \in D(A)$, $x \in \overline{\mathcal{R}(G)}$. Further on, $A\int_0^t S_+^2(s)x\, ds = S(t)x + B_0x + tB_1x + t^2B_2x + tAB_0x + \frac{t^2}{2}A(AB_0x + B_1x) = S_+^2(t)x - \frac{t^2}{2}x$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$. Therefore, $A(AB_0x + B_1x) = -x - 2B_2x$, $x \in \overline{\mathcal{R}(G)}$ and $S_+^2(t)x \in D(A)$, $AS_+^2(t)x = A(S(t)x + B_0x) + t(-x - 2B_2x)$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$, $\overline{\mathcal{R}(G)} \subseteq \bigcap_{i=0}^2 D(Y_i)$ and $Y_2x = -x$, $x \in \overline{\mathcal{R}(G)}$. Suppose $x \in D(A_1)$. Since

$\mathrm{R}(B_0) \subseteq D(A)$ and $AS_+^2(t)x - S_+^2(t)Ax = [AS(t)x + AB_0x + tA(AB_0 + B_1)x] - [S(t)Ax + B_0Ax + t(AB_0 + B_1)Ax] = tC_1x + t[A(AB_0 + B_1)x - (AB_0 + B_1)Ax]$, $t \in [0, \tau)$, Proposition 3.6.8(iv) immediately implies $\big(S_+^2(t)x, S_+^2(t)A_1x\big) \in A_1$, $t \in [0, \tau)$. Thus, $(S_+^2(t))_{t\in[0,\tau)}$ is a twice integrated semigroup generated by A_1. Because $\mathrm{R}(S_+^2(t)) \subseteq D(A)$, $t \in [0, \tau)$, the mapping $t \mapsto S_+^2(t)x$ is continuously differentiable for every fixed $x \in \overline{\mathcal{R}(G)}$ and the following holds: $\frac{d}{dt}S_+^2(t)x = AS_+^2(t)x + tx = A(S(t)x + B_0x) - 2tB_2x$, $t \in [0, \tau)$, $x \in \overline{\mathcal{R}(G)}$. Then it is straightforward to see that $\frac{d}{dt}S_+^2(t) \in L(\overline{\mathcal{R}(G)})$, $t \in [0, \tau)$ and that $\big(\int_0^t \frac{d}{ds}S_+^2(s)x\,ds, (\frac{d}{dt}S_+^2(t)x) - tx\big) \in A_1$, $t \in [0, \tau)$. Suppose now $x \in D(A_1)$. Then $\frac{d}{dt}S_+^2(t)x = AS_+^2(t)x + tx = S_+^2(t)Ax + tx \in D(A)$ and $A\frac{d}{dt}S_+^2(t)x = AS_+^2(t)Ax + tAx = \frac{d}{dt}S_+^2(t)Ax$, $t \in [0, \tau)$, $\big(\frac{d}{dt}S_+^2(t)x, \frac{d}{dt}S_+^2(t)A_1x\big) \in A_1$, $t \in [0, \tau)$, and consequently, $(\frac{d}{dt}S_+^2(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A_1. In order to obtain the corresponding statement for the operator $-A_1$ and $(S_-^2(t))_{t\in[0,\tau)}$, notice the following facts: (208) holds for $-A$ and $\check{G}$, $\check{G}$ fulfills $(DG)_4$ with $u(\check{\cdot}; x)$, $\check{G}(\varphi)x = (-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)[(-1)^{n_0}\check{S}(t)]x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$ and $((-1)^{n_0}\check{S}(t))_{t\in(-\tau,\tau)}$ is a $[(-1)^{n_0}B_0, \ldots, (-1)^{n_0+n_0}B_{n_0}, (-1)^{n_0+1}C_0, \ldots, (-1)^{n_0+n_0}C_{n_0-1}]$-group with a subgenerator $-A$. To prove (ii.4), assume $x \in \overline{\mathcal{R}(G)}$. Let (x_n) be a sequence in $\mathcal{R}(G)$ with $\lim_{n\to\infty} x_n = x$. Due to (224) and (ii.1), $\lim_{n\to\infty} Y_{n_0-1}(x_n) = Y_{n_0-1}x$ and $\lim_{n\to\infty} AY_{n_0-1}(x_n) = -x - n_0!B_{n_0}x$. Hence, $Y_{n_0-1}x \in D(A)$, $x \in D(Y_{n_0})$ and $Y_{n_0}x = -x$ as claimed. This yields $S_\pm^{n_0}(t)x \in D(A)$, $x \in \overline{\mathcal{R}(G)}$. As in the proofs of (ii.1), (ii.2) and (ii.3), one obtains $S_\pm^{n_0}(t) \in L(\overline{\mathcal{R}(G)})$, $\big(\int_0^t S_\pm^{n_0}(s)x ds, S_\pm^{n_0}(t)x - \frac{t^{n_0}}{n_0!}x\big) \in \pm A_1$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$ and $S_\pm^{n_0}(t)A_1x = A_1S_\pm^{n_0}(t)x$, $t \in [0, \tau)$, $x \in \mathcal{R}(G)$. We will sketch the rest of the proof of (ii.4) only for A and $S_+^{n_0}(\cdot)$. Suppose t, $s \in [0, \tau)$ and $t + s < \tau$. Since $AS_+^{n_0}(\cdot)x = S_+^{n_0}(\cdot)Ax$, $x \in \mathcal{R}(G)$, one can repeat literally the arguments given in the proof of [**275**, Propostion 2.4] so as to conclude that:

$$S_+^{n_0}(t)S_+^{n_0}(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] \frac{(t+s-r)^{n_0-1}}{(n_0-1)!} S_+^2(r)x\,dr,\ x \in \mathcal{R}(G). \tag{233}$$

The standard limit procedure implies that (233) remains true for every $x \in \overline{\mathcal{R}(G)}$ and t, $s \in [0, \tau)$ with $t + s < \tau$. Then it is straightforward to verify that $(S_+^{n_0}(t))_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ is a local n_0-times integrated semigroup in the sense of $(\triangle)$. To prove that $A'_{n_0-1,+}$ is the generator of a local n_0-times integrated semigroup $(S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 2.1.1, we argue as follows. First of all, let us observe that $A'_{n_0-1,+}$ is a closed operator and that the arguments employed in the proof of Proposition 3.3.15 also show that $D(A^{n_0}) \cap D(A_1) \subseteq C^{n_0}(\tau)$. Suppose now $x \in D(A_0)$, where A_0 is the infinitesimal generator of $(S_+^{n_0}(t))_{t\in[0,\tau)}$. This implies the existence of a number $\sigma \in (0, \tau)$ so that the mapping $t \mapsto S_+^{n_0}(t)x$, $t \in [0, \sigma)$ is n_0-times continuously differentiable and that $A_0x = \lim_{t\to0+} \frac{1}{t}\big(\frac{d^{n_0}}{dt^{n_0}}(S_+^{n_0}(t)x) - x\big)$. On the other hand, the closedness of A implies $\frac{d^k}{dt^k}S(t)x \in D(A)$ and $\frac{d^{k+1}}{dt^{k+1}}S(t)x = A\frac{d^k}{dt^k}S(t)x - \sum_{j=k+1}^{n} j\cdots(j-k)t^{j-k-1}B_jx$, for every $t \in [0,\sigma)$ and $k \in \{0,\dots,n_0-1\}$. Therefore, $x \in \bigcap_{i=0}^{n_0} D(Y_i)$ and $Y_k x = -(\frac{d^k}{dt^k}S(t)x)_{t=0}$, $k \in \{0,\dots,n_0\}$. Moreover,

$$\begin{aligned}\frac{d^{n_0}}{dt^{n_0}}(S_+^{n_0}(t)x) - x &= A\Big[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x\Big] - n_0!B_{n_0}x - x\\ &= A\Big[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x\Big] + AY_{n_0-1}x = A\Big[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x - \Big(\frac{d^{n_0-1}}{dt^{n_0-1}}S(t)x\Big)_{t=0}\Big].\end{aligned}$$

It is also evident that

$$x = \Big(\frac{d^{n_0}}{dt^{n_0}}S(t)x\Big)_{t=0} = \lim_{t\to 0+}\frac{1}{t}\Big(\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x - \Big(\frac{d^{n_0-1}}{dt^{n_0-1}}S(t)x\Big)_{t=0}\Big).$$

The closedness of A implies $x \in D(A_1)$, $A_0x = A_1x$ and, because of that, $\overline{A_0} \subseteq A_1$. Further on, $\mathcal{R}(G) \subseteq D(A^{n_0}) \cap D(A_1) \subseteq C^{n_0}(\tau)$ and an application of [**404**, Proposition 4.5] gives $\big(\int_0^t S_+^{n_0}(s)x\,ds, S_+^{n_0}(t)x - \frac{t^{n_0}}{n_0!}x\big) \in \overline{A_0}$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$ and $S_+^{n_0}(t)\overline{A_0}x = \overline{A_0}S_+^{n_0}(t)x$, $t \in [0,\tau)$, $x \in D(\overline{A_0})$. Suppose $(x,y) \in \overline{A_0}$. Then $0 = AS_+^{n_0}(t)x - S_+^{n_0}(t)y = \sum_{i=0}^{n_0-1} t^iC_ix + \sum_{i=0}^{n_0-1}\frac{t^i}{i!}AY_ix - \sum_{i=0}^{n_0-1}\frac{t^i}{i!}Y_iy$, $t \in [0,\tau)$, which implies $\overline{A_0} \subseteq A^{'}_{n_0-1,+}$. Further on, fix an $x \in D(A'_{n_0-1,+})$ and notice that $AS_+^{n_0}(t)x = S_+^{n_0}(t)Ax$, $t \in [0,\tau)$ and

$$\begin{aligned}{A_{n_0-1,+}}'\ni\Bigg(\int_0^t S_+^{n_0}(s)x\,ds, S_+^{n_0}(t)x - \frac{t^{n_0}}{n_0!}x\Bigg) &= \Bigg(\int_0^t S_+^{n_0}(s)x\,ds, A_1\int_0^t S_+^{n_0}(s)x\,ds\Bigg)\\ &= \Bigg(\int_0^t S_+^{n_0}(s)x\,ds, \int_0^t S_+^{n_0}(s)Ax\,ds\Bigg),\ t \in [0,\tau).\end{aligned}$$

This implies $C_i\int_0^t S_+^{n_0}(s)x\,ds + \frac{1}{i!}AY_i\int_0^t S_+^{n_0}(s)x\,ds = \frac{1}{i!}Y_i\int_0^t S_+^{n_0}(s)Ax\,ds$, $t \in [0,\tau)$, $i \in \{2,\dots,n_0-1\}$. Differentiate this equality to obtain $C_iS_+^{n_0}(t)x + \frac{1}{i!}AY_iS_+^{n_0}(t)x = \frac{1}{i!}Y_iS_+^{n_0}(t)Ax$, $t \in [0,\tau)$, $i \in \{2,\dots,n_0-1\}$. Thus, $S_+^{n_0}(t)A'_{n_0-1,+} \subseteq A^{'}_{n_0-1,+}S_+^{n_0}(t)$, $t \in [0,\tau)$ and $A^{'}_{n_0-1,+}$ is the generator of a local n_0-times integrated semigroup $(S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 2.1.1. An application of the arguments given in the proof of [**227**, Proposition 2.1] gives $\overline{A_0} = A'_{n_0-1,+}$. Since $\mathrm{R}(S_+^{n_0}(t)) \subseteq D(A)$, the mapping $t \mapsto S_+^{n_0}(t)x$, $t \in [0,\tau)$ is continuously differentiable for every fixed $x \in \overline{\mathcal{R}(G)}$ and $\frac{d}{dt}S_+^{n_0}(t)x = AS_+^{n_0}(t)x + \frac{t^{n_0-1}}{(n_0-1)!}x = AS(t)x + \sum_{i=0}^{n_0-1}\frac{t^i}{i!}AY_ix + \frac{t^{n_0-1}}{(n_0-1)!}x$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. Then it can be easily verified that $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ is a local (n_0-1)-times integrated semigroup in the sense of $(\triangle)$. The c.i.g of $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)}$ is $\overline{A_0} = A^{'}_{n_0-1,+}$ and an application of [**404**, Proposition 4.5] enables one to see that $A^{'}_{n_0-1,+}$ is the generator of a local (n_0-1)-times integrated semigroup $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 2.1.1. To prove (ii.5), we argue as follows. Suppose $\lambda \in \rho(A)$ and set $A_\lambda := A - \lambda$, $G_\lambda := e^{-\lambda\cdot}G$ and $u_\lambda(\cdot;x) := e^{-\lambda\cdot}u(\cdot;x)$, $x \in \mathcal{R}(G) = \mathcal{R}(G_\lambda)$. It is straightforward to check that A_λ and G_λ fulfill (208) and that G_λ is regular with $G_\lambda(\varphi)x = \int_{-\infty}^{\infty}\varphi(t)u_\lambda(t;x)\,dt$,

$\varphi \in \mathcal{D}$, $x \in \mathcal{R}(G_\lambda)$. Clearly,

$$G_\lambda(\varphi)x = G(e^{-\lambda\cdot}\varphi)x = (-1)^{n_0}\int_{-\tau}^{\tau}\big(e^{-\lambda\cdot}\varphi\big)^{(n_0)}(t)\,S(t)x\,dt$$

$$= (-1)^{n_0}\int_{-\tau}^{\tau}\sum_{i=0}^{n_0}(-1)^{n_0-i}\binom{n_0}{i}\lambda^{n_0-i}e^{-\lambda t}\varphi^{(i)}(t)S(t)x\,dt$$

$$= \sum_{i=0}^{n_0}(-1)^i\binom{n_0}{i}\lambda^{n_0-i}\int_{-\tau}^{\tau}\varphi^{(i)}(t)\big(e^{-\lambda t}S(t)x\big)\,dt = (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)e^{-\lambda t}S(t)x\,dt$$

$$+\sum_{i=1}^{n_0}(-1)^i\binom{n_0}{i}\lambda^{n_0-i}(-1)^{n_0-i}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds\,dt$$

$$= (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)\Bigg[e^{-\lambda t}S(t)x+\sum_{i=1}^{n_0}\binom{n_0}{i}\lambda^{n_0-i}\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds\Bigg]\,dt,$$

for every $\varphi \in \mathcal{D}_{(-\tau,\tau)}$ and $x \in E$. Put, for every $t \in (-\tau,\tau)$ and $x \in E$,

$$S_\lambda(t)x = e^{-\lambda t}S(t)x + \sum_{i=1}^{n_0}\binom{n_0}{i}\lambda^{n_0-i}\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds.$$

Then the mapping $S_\lambda : (-\tau,\tau) \to L(E,[D(A_\lambda)])$ is continuous and $G_\lambda(\varphi)x = (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)S_\lambda(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$. The proof of (i) implies that there exist bounded linear operators $B_0^\lambda,\dots,B_{n_0}^\lambda,C_0^\lambda,\dots,C_{n_0-1}^\lambda$ such that A_λ is a subgenerator of a $[B_0^\lambda,\dots,B_{n_0}^\lambda,C_0^\lambda,\dots,C_{n_0-1}^\lambda]$-group $(S_\lambda(t))_{t\in(-\tau,\tau)}$. Define Y_i^λ recursively by $Y_0^\lambda := B_0^\lambda$ and $Y_{i+1}^\lambda := (i+1)!B_{i+1}^\lambda + A_\lambda Y_i^\lambda$, $i \in \{0,\dots,n_0-1\}$. Since $0 \in \rho(A_\lambda)$, we have that Y_i^λ is closed, $i = 1,\dots,n_0$. Suppose, for the time being, $x \in \overline{\mathcal{R}(G)}$ and (x_n) is a sequence in $\mathcal{R}(G)$ such that $\lim_{n\to\infty}x_n = x$. A consequence of $Y_{n_0}^\lambda x_n = -x_n$, $n \in \mathbb{N}$ is $\lim_{n\to\infty}A_\lambda Y_{n_0-1}^\lambda x_n = -x-n_0!B_{n_0}^\lambda x$ and the boundedness of A_λ^{-1} implies $\lim_{n\to\infty}Y_{n_0-1}^\lambda x_n = A_\lambda^{-1}(-x-n_0!B_{n_0}^\lambda x)$. Continuing this procedure enables one to establish that, for every $i = 1,\dots,n_0-1$, $\lim_{n\to\infty}Y_i^\lambda x_n$ exists. The closedness of Y_i^λ yields $x \in \bigcap_{i=0}^{n_0}D(Y_i^\lambda)$ and $Y_{n_0}^\lambda x = -x$. Put $A_{1,\lambda} := (A_\lambda)_{\overline{\mathcal{R}(G_\lambda)}}$ and $Cx := A_\lambda^{-(n_0-1)}x$, $x \in \overline{\mathcal{R}(G)}$. Because $G_\lambda A_\lambda \subseteq A_\lambda G_\lambda$, we have $A_\lambda^{-k}G_\lambda = G_\lambda A_\lambda^{-k}$, $k \in \mathbb{N}$, $A_\lambda^{-k}(\mathcal{R}(G)) \subseteq \mathcal{R}(G)$, $k \in \mathbb{N}$ and $A_\lambda^{-k}(\overline{\mathcal{R}(G)}) \subseteq \overline{\mathcal{R}(G)}$, $k \in \mathbb{N}$. This implies $0 \in \rho(A_{1,\lambda})$ and the injectiveness of $C \in L(\overline{\mathcal{R}(G)})$. Assume now $x \in D(A_{1,\lambda}^{n_0-1})$. Then $A_{1,\lambda}^{n_0-1}x \in \overline{\mathcal{R}(G)}$, $x = A_\lambda^{-(n_0-1)}(A_\lambda^{n_0-1}x) = C(A_\lambda^{n_0-1}x) \in \mathrm{R}(C)$. and $D(A_{1,\lambda}^{n_0-1}) \subseteq \mathrm{R}(C)$. Proceeding as in the proof of Proposition 3.3.15, one obtains that the mapping $t \mapsto S_\lambda(t)x$, $t \in (-\tau,\tau)$ is n_0-times continuously differentiable and that there exists a function $M : (-\tau,\tau) \to (0,\infty)$, independent of x, so that $\|\frac{d^{n_0}}{dt^{n_0}}S_\lambda(t)x\| \leqslant M(|t|)\sum_{i=0}^{n_0}\|A_\lambda^i x\|$, $t \in (-\tau,\tau)$. Put $u^\lambda(t;x) =$

$\frac{d^{n_0}}{dt^{n_0}}S_\lambda(t)x$, $t \in [0,\tau)$, $x \in D(A_\lambda^{n_0-1})$ and $T(t)x = u^\lambda(t;Cx)$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. Due to Proposition 3.3.15, $D(A_\lambda^{n_0-1}) \subseteq D(Y_\lambda^{n_0})$ and $u^\lambda(0;x) = -Y_\lambda^{n_0}x$, $x \in D(A_\lambda^{n_0-1})$. Moreover, $\mathrm{R}(C) \subseteq \overline{\mathcal{R}(G)} \cap D(A_\lambda^{n_0-1})$ and this implies $u^\lambda(0;Cx) = -Y_\lambda^{n_0}Cx = Cx$. The mapping $t \mapsto T(t)x$, $t \in [0,\tau)$ is continuous for every fixed $x \in \overline{\mathcal{R}(G)}$ and $\|T(t)x\| = \|u^\lambda(t;A_\lambda^{-(n_0-1)}x)\| \leqslant M(t)\sum_{i=0}^{n_0-1}\|A_\lambda^{-1}\|^i\|x\|$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. The partial integration shows $G_\lambda(\varphi)x = \int_{-\tau}^{\tau}\varphi(t)u^\lambda(t;x)\,dt$, $\varphi \in \mathcal{D}_{[0,\tau)}$, $x \in D(A_\lambda^{n_0-1})$ and this yields $u^\lambda(t;x) \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$, $x \in D(A_\lambda^{n_0-1})$. Therefore, $T(t)x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$ and $T(t) \in L(\overline{\mathcal{R}(G)})$, $t \in [0,\tau)$. As in the proof of (ii.1), we infer $A_\lambda\int_0^t u^\lambda(s;Cx)\,ds = A_\lambda\int_0^t T(s)x\,ds = u^\lambda(t;Cx) - u^\lambda(0;Cx) = T(t)x - Cx$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$ and $u^\lambda(t;A_\lambda x) = A_\lambda u^\lambda(t;x)$, $t \in [0,\tau)$, $x \in D(A_\lambda^{n_0})$. By the previous equality, $T(t)A_{1,\lambda} \subseteq A_{1,\lambda}T(t)$ and $T(t)C = CT(t)$, $t \in [0,\tau)$. Now it can be easily seen that the abstract Cauchy problem:

$$\begin{cases} v \in C([0,\tau):[D(A_{1,\lambda})]) \cap C^1([0,\tau):\overline{\mathcal{R}(G)}), \\ v'(t) = A_{1,\lambda}v(t) + Cx,\ t \in [0,\tau), \\ v(0) = 0, \end{cases}$$

has a unique solution for every $x \in \overline{\mathcal{R}(G)}$, given by $v(t) = \int_0^t T(s)xds$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. This implies that the abstract Cauchy problem:

$$(ACP,\tau): \begin{cases} f \in C([0,\tau):[D(A_{1,\lambda})]) \cap C^1([0,\tau):\overline{\mathcal{R}(G)}), \\ f'(t) = A_{1,\lambda}f(t),\ t \in [0,\tau), \\ f(0) = x, \end{cases}$$

has a unique solution for every $x \in C(D(A_{1,\lambda}))$ and that $A_{1,\lambda}$ is the integral generator of a local C-regularized semigroup $(T(t))_{t\in[0,\tau)}$. As before, $D(A_\lambda^{n_0}) \subseteq C(D(A_\lambda))$ and this implies that that $A_{1,\lambda}$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. A rescaling result for local integrated semigroups implies that A_1 generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. Similarly, $-A_1$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. ☐

THEOREM 3.3.20. *Suppose G is a* (DG) *generated by A. Then the group $(S(t))_{t\in(-\tau,\tau)}$, constructed in Theorem* (3.3.19)(i), *is non-degenerate. If $n_0 = 1$, then A generates a C_0-group. If $n_0 = 2$, then:*

(a) $\big(S_\pm^1(t) := \pm A(S(\pm t)x + B_0x) - 2tB_2x\big)_{t\in[0,\tau)}$ *are local once integrated semigroups in the sense of $(\triangle)$.*
(b) *The c.i.g of $(S_+^1(t))_{t\in[0,\tau)}$ $((S_-^1(t))_{t\in[0,\tau)})$ is $A_{|\overline{\mathcal{R}(G)}}$ $((-A)_{|\overline{\mathcal{R}(G)}})$.*
(c) *Suppose A is densely defined or $\lambda - A$ is surjective for some $\lambda \in \mathbb{C}$. Then $\pm A$ are generators of local once integrated semigroups $(S_\pm^1(t))_{t\in[0,\tau)}$.*

Furthermore:

(i) *For every $x \in E$ and φ, $\psi \in \mathcal{D}_{(-\tau,\tau)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\tau,\tau)$:*

$$G(\varphi)G(\psi)x = \sum_{i=0}^{n_0}(-1)^{i+1}i!\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\int_{-\infty}^{\infty}\psi^{(n_0-i)}(s)S(t+s)B_ix\,ds\,dt. \tag{234}$$

(ii) $Y_{n_0}x = -x$, $x \in \bigcap_{i=0}^{n_0} D(Y_i)$.
(iii) *Suppose* $x \in D(A^{n_0-1})$. *Then* $\{x, Ax\} \subseteq \bigcap_{i=2}^{n_0} D(Y_i)$, $Y_{n_0}x = -x$, $Y_{n_0}Ax = -Ax$ *and* $D(A^{n_0-1}) \subseteq \overline{\mathcal{R}(G)}$.
(iv) A *is stationary dense with* $n(A) \leqslant n_0 - 1$.
(v) *If* $\rho(A) \neq \emptyset$, *then for every* $\tau_0 \in (0, \infty)$, *there is an* $n(\tau_0) \in \mathbb{N}$ *so that* A *generates a local* $n(\tau_0)$*-times integrated group on* $(-\tau_0, \tau_0)$.
(vi) G *is dense iff* $D_\infty(A)$ *is dense in* E. *In the case* $\rho(A) \neq \emptyset$, G *is dense iff* A *is densely defined.*
(vii) $\bigcap_{\varphi \in \mathcal{D}_0} \operatorname{Kern}(G(\varphi)) = \{0\}$ *and* $\bigcap_{\varphi \in \mathcal{D}_0} \operatorname{Kern}(G(\check{\varphi})) = \{0\}$.

PROOF. Assume $S(t)x = 0$, $t \in (-\tau, \tau)$. This implies $G(\psi)x = 0$, $\psi \in \mathcal{D}_{(-\tau,\tau)}$ and $G(\psi)x = \lim_{n\to\infty} G(\psi * \rho_n)x = \lim_{n\to\infty} G(\psi)G(\rho_n)x = 0$, $\psi \in \mathcal{D}$ where (ρ_n) is a regularizing sequence. Owing to $(DG)_2$, one can deduce that $x \in \mathcal{N}(G)$ and that $(S(t))_{t\in(-\tau,\tau)}$ is non-degenerate. Put now $S_1(t)x = S(t)x + B_0x$, $t \in (-\tau, \tau)$, $x \in E$. We will prove that $(S_1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A. Observe that $S_1(t)A \subseteq AS_1(t)$, $t \in (-\tau, \tau)$ and that $S_1 : (-\tau, \tau) \to L(E, [D(A)])$ is continuous. This clearly implies $\frac{d}{dt}S_1(t)x = AS_1(t)x + Bx$, $t \in (-\tau, \tau)$, $x \in E$ where $B = -B_1 - AB_0 \in L(E)$. Further on, $\int_0^t S_1(s)x\,ds \in D(A)$, $t \in (-\tau, \tau)$, $x \in E$,

$$\begin{aligned} A\int_0^t S_1(s)x\,ds = A\int_0^t (S(s)x + B_0x)\,ds &= S(t)x + B_0x + tB_1x + tAB_0x \\ &= S_1(t)x - tBx,\ \ t \in (-\tau, \tau),\ x \in E \end{aligned}$$

and $(S_1(t))_{t\in(-\tau,\tau)}$ is a $[0, -B, 0]$-group with a subgenerator A. We will prove that $B = I$. Suppose ζ, $\eta \in \mathcal{D}_{(-\tau/4,\tau/4)}$ and (ρ_n) is a regularizing sequence. We know that $\operatorname{supp} I_\zeta(\varphi) \subseteq [\min(-\tau/4, \inf(\operatorname{supp}\varphi)), \max(\tau/4,\ \sup(\operatorname{supp}\varphi))]$ and that there exists $k \in \mathbb{N}$ such that $\operatorname{supp} I_\zeta(\rho_n) \cup \operatorname{supp} I_\eta(\rho_n) \subseteq [-\tau/4, \tau/4]$, $n \geqslant k$. Fix an $x \in E$. By $(\mathrm{DG})_1$, it follows that, for every φ, $\psi \in \mathcal{D}_{(-\tau/4,\tau/4)}$:

$$\int_{-\infty}^{\infty} \varphi'(t) \int_{-\infty}^{\infty} \psi'(s)S_1(t)S_1(s)x\,ds\,dt = -\int_{-\infty}^{\infty} \varphi'(t) \int_{-\infty}^{\infty} \psi(s)S_1(t+s)x\,ds\,dt. \tag{235}$$

Put $\varphi = I_\zeta(\rho_n)$, $n \geqslant k$ in (235). Then one obtains, for every φ, $\psi \in \mathcal{D}_{(-\tau/4,\tau/4)}$:

$$\int_{-\infty}^{\infty} [\rho_n(t) - \zeta(t)] \int_{-\infty}^{\infty} \psi'(s)S_1(t)S_1(s)x\,ds\,dt = -\int_{-\infty}^{\infty} [\rho_n(t) - \zeta(t)] \int_{-\infty}^{\infty} \psi(s)S_1(t+s)x\,ds\,dt.$$

Letting $n \to \infty$ and applying the partial integration, one gets that, for every $\psi \in \mathcal{D}_{(-\tau/4,\tau/4)}$:

$$-\int_{-\infty}^{\infty} \zeta(t) \int_{-\infty}^{\infty} \psi'(s)S_1(t)S_1(s)x\,ds\,dt \tag{236}$$

$$= \int_{-\infty}^{\infty} \psi'(s) \int_0^s S_1(v)x\,dv\,ds - \int_{-\infty}^{\infty} \zeta(t) \int_{-\infty}^{\infty} \psi'(s) \int_0^s S_1(t+v)x\,dv\,ds\,dt.$$

Plug $\psi = I_\eta(\rho_n)$, $n \geqslant k$ into (236). Then we obtain, for every $\psi \in \mathcal{D}_{(-\tau/4,\tau/4)}$:

$$- \int_{-\infty}^{\infty} \zeta(t) \int_{-\infty}^{\infty} \big[\rho_n(s) - \eta(s)\big] S_1(t)S_1(s)x\,ds\,dt$$

$$= \int_{-\infty}^{\infty} \big[\rho_n(s) - \eta(s)\big] \int_0^s S_1(v)x\,dv\,ds - \int_{-\infty}^{\infty} \zeta(t) \int_{-\infty}^{\infty} \big[\rho_n(s) - \eta(s)\big] \int_0^s S_1(t+v)x\,dv\,dt\,ds.$$

The standard limit procedure leads us to the following

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta(t)\eta(s)S_1(t)S_1(s)x\,ds\,dt \tag{237}$$
$$= - \int_{-\infty}^{\infty} \eta(s) \int_0^s S_1(v)x\,dv\,ds + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta(t)\eta(s) \int_0^s S_1(t+v)x\,dv\,dt\,ds.$$

Let $t,\ s \in (-\tau/4, \tau/4)$ be fixed and let $(\zeta_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{D}_{(-\tau/4,\tau/4)}$ satisfying $\int_{-\infty}^{\infty} \zeta_n(t)\,dt = 1$, $\int_{-\infty}^{\infty} \eta_n(t)\,dt = 1$, $n \in \mathbb{N}$, $\lim_{n\to\infty} \zeta_n = \delta_t$ and $\lim_{n\to\infty} \eta_n = \delta_s$, in the sense of distributions. By virtue of (237),

$$S_1(t)S_1(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] S_1(r)x\,dr. \tag{238}$$

Notice that (238) implies $S_1(t)(\frac{d}{dr}S_1(r)x)_{r=s} = S_1(t+s)x - S_1(s)x$ and $S_1(t)[AS_1(s)x + Bx] = S_1(t+s)x - S_1(s)x$. Since $S_1(t)A \subseteq AS_1(t)$, $t \in (-\tau,\tau)$, one yields:

$$A\left[\int_0^{t+s} - \int_0^t - \int_0^s\right] S_1(r)x\,dr + S_1(t)Bx = S_1(t+s)x - S_1(s)x, \text{ i.e.,}$$

$S_1(t+s)x-(t+s)Bx-S_1(t)x+tBx-S_1(s)x+sBx+S_1(t)Bx = S_1(t+s)x-S_1(s)x$. So, $S_1(v)[Bx - x] = 0$, $v \in (-\tau/4, \tau/4)$. Since $G(\varphi)x = -\int_{-\infty}^{\infty} \varphi'(v)S_1(v)x\,dv$, $\varphi \in \mathcal{D}_{(-\tau/4,\tau/4)}$, we obtain that $(S_1(t))_{t\in(-\tau/4,\tau/4)}$ is a non-degenerate operator family. Hence, $B = I$ and $(S_1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A. Analogously, $(-S(-t) - B_0)_{t\in[0,\tau)}$ is a once integrated semigroup generated by $-A$ and one can repeat literally the arguments given in the proof of Theorem 3.3.19(ii.2) in order to see that A generates a C_0-group. Suppose now $n_0 = 2$ and denote $A_1 = A_{|\overline{\mathcal{R}(G)}}$. We will only prove that A_1 is the c.i.g of $(S^1_+(t))_{t\in[0,\tau)}$. Evidently, $AB_0+B_1 \in L(E)$, $G(\varphi)x = \int_{-\infty}^{\infty} \varphi''(t)[S(t)x+B_0x+t(AB_0+B_1)x]\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$ and the mapping $t \mapsto S(t)x + B_0x + t(AB_0 + B_1)x$, $t \in [0,\tau)$ is continuously differentiable with $\frac{d}{dt}[S(t)x + B_0x + t(AB_0 + B_1)x] = AS(t)x - B_1x - 2tB_2x + (AB_0 + B_1)x$, $t \in [0, \tau)$, $x \in E$. Therefore,

$$G(\varphi)x = -\int_{-\infty}^{\infty} \varphi'(t) S_+^1(t)x\, dt,\ \varphi \in \mathcal{D}_{[0,\tau)},\ x \in E. \tag{239}$$

Suppose $x \in E$, φ, $\psi \in \mathcal{D}_{[0,\tau)}$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq [0, \tau)$. Since G satisfies $(DG)_1$, we obtain

$$\int_{-\infty}^{\infty} \varphi'(t) \int_{-\infty}^{\infty} \psi'(s) S_+^1(t) S_+^1(s)x\, dt\, ds = -\int_{-\infty}^{\infty} \varphi'(t) \int_{-\infty}^{\infty} \psi(s) S_+^1(t+s)x\, dt\, ds.$$

Arguing as in the case $n_0 = 1$, one gets, for every t, $s \in [0, \tau)$ with $t + s < \tau$:

$$S_+^1(t) S_+^1(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] S_+^1(r)x\, dr.$$

Further on, $S_+^1(0) = 0$ and the mapping $t \mapsto S_+^1(t)x$, $t \in [0, \tau)$ is continuous. It can be simply verified that $(S_+^1(t))_{t\in[0,\tau)}$ is a non-degenerate operator family, and consequently, $(S_+^1(t))_{t\in[0,\tau)}$ is a local once integrated semigroup in the sense of $(\triangle)$. Suppose $x \in D(A_0)$. Then there exists $\sigma \in (0, \tau]$ such that the mapping $t \mapsto S_+^1(t)x$, $t \in [0, \sigma)$ is continuously differentiable and that $A_0x = \lim_{t\to 0+} \frac{1}{t}\big(\frac{d}{dt}(S_+^1(t)x) - x\big)$. The partial integration and (239) yield:

$$G(\varphi)x = \int_{-\infty}^{\infty} \varphi(t) \frac{d}{dt} S_+^1(t)x\, dt,\ \varphi \in \mathcal{D}_{[0,\sigma)}. \tag{240}$$

Owing to (240) and Theorem 3.3.20(ii.3), we get $\lim_{n\to\infty} G(\rho_n)x = x \in \overline{\mathcal{R}(G)}$, $S_+^2(t)x = S(t)x + B_0x + t(AB_0 + B_1)x \in \overline{\mathcal{R}(G)}$, $t \in [0, \tau)$ and $\overline{\mathcal{R}(G)} \ni \frac{d}{dt} S_+^2(t)x = AS(t)x - B_1x - 2tB_2x + (AB_0 + B_1)x = S_+^1(t)x$, $t \in [0, \tau)$. Consequently, $\frac{d}{dt} S_+^1(t)x \in \overline{\mathcal{R}(G)}$, $t \in [0, \sigma)$, $A_0x = \lim_{t\to 0+} \frac{1}{t}\big(\frac{d}{dt}(S_+^1(t)x) - x\big) \in \overline{\mathcal{R}(G)}$ and:

$$\{x, A_0x\} \subseteq \overline{\mathcal{R}(G)}. \tag{241}$$

Furthermore, $\frac{d}{dt} S(t)x = AS(t)x - B_1x - 2tB_2x$, $t \in [0, \tau)$, $\frac{d}{dt} S_+^1(t)x + 2B_2x = \lim_{h\to 0} \frac{A[S(t+h)x - S(0)x] - A[S(t)x - S(0)x]}{h} = \lim_{h\to 0} A\frac{S(t+h)x - S(t)x}{h}$, $t \in [0, \sigma)$ and $\lim_{h\to 0} \frac{S(t+h)x - S(t)x}{h} = AS(t)x - B_1x - 2tB_2x$, $t \in [0, \tau)$. The closedness of A gives $AS(t)x - B_1x - 2tB_2x \in D(A)$, $t \in [0, \sigma)$ and $A[AS(t)x - B_1x - 2tB_2x] = \frac{d}{dt} S_+^1(t)x + 2B_2x$, $t \in [0, \sigma)$. Put $t = 0$ in the previous equality to obtain $A(AB_0 + B_1)x = -x - 2B_2x$. Notice also that

$$\begin{aligned} A_0x = \lim_{t\to 0+} \frac{\frac{d}{dt}(S_+^1(t)x) - x}{t} &= \lim_{t\to 0+} \frac{A[AS(t)x - B_1x - 2tB_2x] - 2B_2x - x}{t} \\ &= \lim_{t\to 0+} \frac{A[AS(t)x - B_1x - 2tB_2x] + A(AB_0 + B_1)x}{t} \end{aligned}$$

$$= \lim_{t\to 0+} A\frac{A[S(t)x - S(0)x] - 2tB_2x}{t}.$$

On the other hand,

$$\lim_{t\to 0+} \frac{A[S(t)x - S(0)x] - 2tB_2x}{t} = \lim_{t\to 0+} \frac{S^1_+(t)x - S^1_+(0)x}{t} = \Big(\frac{d}{dt}S^1_+(t)x\Big)_{t=0} = x.$$

Therefore, $x \in D(A)$, $A_0x = Ax$, $A_0 \subseteq A$ and (241) enables one to see that $A_0 \subseteq A_1$ and $\overline{A_0} \subseteq A_1$. Furthermore, Theorem 3.3.19(ii.3) shows that A_1 is the generator of a once integrated semigroup $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ in the sense of Definition 2.1.1. Accordingly, $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ is a local once integrated semigroup in the sense of $(\triangle)$ and it can be easily proved that the c.i.g of $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)}$ is A_1. But, the c.i.g of $(S^1_+(t))_{t\in[0,\tau)}$ is an extension of the c.i.g of $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)}$. Hence, $A_1 \subseteq \overline{A_0}$ and $A_1 = \overline{A_0}$. Further on, it is straightforward to see that $\frac{d}{dt}S^1_+(t)x = AS(t)Ax - B_1Ax - 2tB_2Ax - (2B_2 - C_1)x$, $t \in [0,\tau)$, $x \in D(A)$. Due to [**404**, Lemma 4.3(b)], we obtain that $x = \big(\frac{d}{dt}S^1_+(t)x\big)_{t=0}$, $x \in D(A)$ and an immediate consequence of this equality and (240) is $\lim_{n\to\infty} G(\rho_n)x = x$, $x \in D(A)$. By Theorem 3.3.19(ii.3), we have $D(A) \subseteq \overline{\mathcal{R}(G)} \subseteq \bigcap_{i=0}^{2} D(Y_i)$ and $Y_2x = -x$, $x \in D(A)$. Suppose $x \in D(A)$. By Proposition 3.3.15(iv), $Ax \in D(Y_1)$, $C_1x + AY_1x = Y_1Ax$ and, because of that, $2B_2x + Y_1Ax = C_1x + Y_2x = C_1x - x$. Now an application of Proposition 3.3.15(i) shows that $Y_1Ax = -(2B_2x - C_1x) - x \in D(A)$ and that $AY_1Ax = -2B_2Ax - -Ax$. In other words, $Ax \in \bigcap_{i=0}^{2} D(Y_i)$ and $Y_2Ax = -Ax$. Let us prove (c). First of all, assume $\lambda \in \mathbb{C}$, $\lambda - A$ is surjective and $x = (\lambda - A)y$ for some $y \in D(A)$. Then we obtain $E = \bigcap_{i=0}^{2} D(Y_i) \ni x$ and $Y_2x = Y_2(\lambda y - Ay) = -\lambda y + Ay = -x$. Proceeding as in the proof of (ii.3) of Theorem 3.3.19, one gets that $A\int_0^t S^2_+(s)x\,ds = S(t)x + B_0x + tB_1x + t^2B_2x + tAB_0x + \frac{t^2}{2}A(AB_0x + B_1x) = S^2_+(t)x - \frac{t^2}{2}x$, $x \in E$, $t \in [0,\tau)$. This implies $\int_0^t S^1_+(s)x\,ds \in D(A)$, $t \in [0,\tau)$ and $A\int_0^t S^1_+(s)xds = S^1_+(t)x - tx$, $x \in E$, $t \in [0,\tau)$. Assume $x \in D(A)$. Due to Proposition 3.3.15(i), we get $S^1_+(t)x = (S(t) - S(0))Ax - t(2B_2x - C_1x) \in D(A)$, $t \in [0,\tau)$ and $AS^1_+(t)x = A(S(t) - S(0))Ax - 2tB_2Ax = S^1_+(t)Ax$, $t \in [0,\tau)$. Suppose now that A is densely defined. Since $D(A) \subseteq \overline{\mathcal{R}(G)}$, we obtain that $\overline{\mathcal{R}(G)} = E$ and that A is the c.i.g of $(S^1_+(t))_{t\in[0,\tau)}$. Due to [**404**, Proposition 4.5], $(S^1_+(t))_{t\in[0,\tau)}$ is a local once integrated semigroup in the sense of Definition 2.1.1. To prove (i), suppose $x \in E$, φ, $\psi \in \mathcal{D}_{(-\tau,\tau)}$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\tau,\tau)$. Note that:

$$\begin{aligned} G(\varphi)G(\psi)x &= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0)}(s)S(t)S(s)x\,ds\,dt \\ &= -\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)S(t)\int_0^s S(v)x\,dv\,ds\,dt. \end{aligned}$$

Repeating literally the arguments given in the proof of Proposition 3.3.17, one obtains (218). Then the last equality implies:

$$G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\Bigg[-\sum_{j=0}^{n_0-1}\int_0^s (t+s-r)^j \int_0^r C_j S(v)x\,dv\,dr$$

$$+\sum_{j=1}^{n_0} j\int_0^s (t+s-r)^{j-1}\int_0^r B_j S(v)x\,dv\,dr - \sum_{j=0}^{n_0}\int_0^s r^j S(t+s-r)B_j x\,dr\Bigg]\,ds\,dt.$$

Noticing that $\int_{-\infty}^{\infty} \varphi^{(n)}(t)t^j\,dt = 0$, $n \in \mathbb{N}$, $j \in \mathbb{N}_0$, $n > j$, we get:

$$I_1 := \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \sum_{j=0}^{n_0-1}\int_0^s (t+s-r)^j \int_0^r C_j S(v)x\,dv\,dr\,ds\,dt = 0.$$

Indeed,

$$I_1 = \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\times$$

$$\times \sum_{j=0}^{n_0-1}\int_0^s \sum_{\substack{(k_1,k_2,k_3)\in\mathbb{N}_0^3\\ k_1+k_2+k_3=j}} \frac{j!}{k_1!k_2!k_3!} t^{k_1}s^{k_2}(-r)^{k_3}\int_0^r C_j S(v)x\,dv\,dr\,ds\,dt.$$

Suppose $j \in \{0,\ldots,n_0-1\}$, $(k_1,k_2,k_3)\in\mathbb{N}_0^3$ and $k_1+k_2+k_3=j$. Then one gets:

$$\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\int_0^s \frac{j!}{k_1!k_2!k_3!} t^{k_1}s^{k_2}(-r)^{k_3}\int_0^r C_j S(v)x\,dv\,dr\,ds\,dt$$

$$= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t)t^{k_1}\,dt \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\int_0^s \frac{j!}{k_1!k_2!k_3!} s^{k_2}(-r)^{k_3}\int_0^r C_j S(v)x\,dv\,dr\,ds\,dt = 0.$$

Hence, $I_1 = 0$. Proceeding in a similar way, we infer that

$$\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\sum_{j=1}^{n_0} j\int_0^s (t+s-r)^{j-1}\int_0^r B_j S(v)x\,dv\,dr\,ds\,dt = 0$$

and

$$G(\varphi)G(\psi)x = \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\sum_{j=0}^{n_0}\int_0^s r^j S(t+s-r)B_j x\,dr\,ds\,dt$$

$$= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\sum_{j=0}^{n_0}\int_t^{t+s} (t+s-r)^j S(r)B_j x\,dr\,ds\,dt. \tag{242}$$

Put, for every $t \in (-\tau,\tau)$ and $j \in \{1,\ldots,n_0+1\}$, $g_{j,t}(s) := \int_t^{t+s}(t+s-r)^{j-1} S(r)B_jx\,dr$, $s \in (-\tau-t,\tau-t)$. It is straightforward to check that $\frac{d}{ds}g_{j,t}(s) = (j-1)\int_t^{t+s}(t+s-r)^{j-2}S(r)B_jx\,dr$, $j>1$, $s \in (-\tau-t,\tau-t)$ and that $\frac{d}{ds}g_{1,t}(s) = S(t+s)B_1x$, $s \in (-\tau-t,\tau-t)$. The partial integration and (242) imply:

$$G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0)}(s)\sum_{j=1}^{n_0}\int_t^{t+s} j(t+s-r)^{j-1}S(r)B_jx\,dr\,ds\,dt$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0)}(s)S(t+s)B_0x\,ds\,dt.$$

Applying again the partial integration, we get

$$G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0)}(s)S(t+s)B_0x\,ds\,dt$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0-1)}(s)S(t+s)B_1x\,ds\,dt$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0-1)}(s)\sum_{j=2}^{n_0}j(j-1)\int_t^{t+s}(t+s-r)^{j-2}S(r)B_jx\,dr\,ds\,dt.$$

Continuing this procedure, we finally obtain (234).

To prove (ii), suppose $\varphi,\psi \in \mathcal{D}_{(-\tau,\tau)}$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\tau,\tau)$. Certainly,

$$(243)\quad G(\varphi*\psi)x = (-1)^{n_0}\int_{-\infty}^{\infty}(\varphi*\psi)^{(n_0)}(t)S(t)x\,dt$$
$$= (-1)^{n_0}\int_{-\infty}^{\infty}(\varphi^{(n_0)}*\psi)(t)S(t)x\,dt = (-1)^{n_0}\int_{-\infty}^{\infty}\Bigg[\int_{-\infty}^{\infty}\varphi^{(n_0)}(t-s)\psi(s)\,ds\Bigg]S(t)x\,dt$$
$$= (-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\int_{-\infty}^{\infty}\psi(s)S(t+s)x\,ds\,dt,\ x \in E.$$

Owing to (243),

$$(244)\quad G(\psi)x = \lim_{n\to\infty}G(\psi*\rho_n)x$$
$$= \lim_{n\to\infty}(-1)^{n_0}\int_{-\infty}^{\infty}\rho_n^{(n_0)}(t)\int_{-\infty}^{\infty}\psi(s)S(t+s)x\,ds\,dt,\ x \in E.$$

Combining $(DG)_1$, (234) and (244), we obtain:

$$(245)\qquad (-1)^{n_0}G(\psi)x = \sum_{i=0}^{n_0}(-1)^{i+1}i!G(\psi^{(n_0-i)})B_ix,\ x \in E.$$

Suppose now $x \in \bigcap_{i=0}^{n_0} D(Y_i)$. A consequence of the definition of Y_{n_0} and (208) is $n_0!G(\psi)B_{n_0}x+AG(\psi)Y_{n_0-1}x = n_0!G(\psi)B_{n_0}x-G(\psi')Y_{n_0-1}x = G(\psi)Y_{n_0}x$. If $n_0 \geqslant 2$, then we obtain $n_0!G(\psi)B_{n_0}x - G(\psi')(AY_{n_0-2}x + (n_0-1)!B_{n_0-1}x) = G(\psi)Y_{n_0}x$ and $n_0!G(\psi)B_{n_0}x - (n_0-1)!G(\psi')B_{n_0-1}x + G(\psi'')Y_{n_0-2}x = G(\psi)Y_{n_0}x$. By the definition of Y_i and (208), one concludes inductively:

$$\sum_{i=0}^{n_0}(-1)^{n_0+i}i!G(\psi^{(n_0-i)})B_i x = G(\psi)Y_{n_0}x. \tag{246}$$

This equality and (245) imply $G(\psi)(Y_{n_0}x + x) = 0$; a simple consequence is $G(\eta)(Y_{n_0}x + x) = 0$, $\eta \in \mathcal{D}$ and the proof of (ii) follows from an application of $(DG)_2$. To prove (iii), one can argue as in the proof of (ii.5) of Theorem 3.3.19. We sketch the proof for the sake of completeness. Fix an $x \in D(A^{n_0-1})$. Since $S : (-\tau, \tau) \to L(E, [D(A)])$ is continuous, the arguments given in the proof of Proposition 3.3.15 imply that the mapping $t \mapsto S(t)x$, $t \in (-\tau, \tau)$ is n_0-times continuously differentiable and that there exists a function $M : (-\tau, \tau) \to (0, \infty)$ satisfying $\left\|\frac{d^{n_0}}{dt^{n_0}}S(t)x\right\| \leqslant M(t)\|x\|_{n_0-1}$, $t \in (-\tau, \tau)$. Furthermore, (213) holds for every $l \in \{0, \ldots, n_0-1\}$ and one obtains inductively $Y_k x = -\left(\frac{d^k}{dt^k}S(t)x\right)_{t=0}$, $k \in \{0, \ldots, n\}$. Denote $u(t;x) = \frac{d^{n_0}}{dt^{n_0}}S(t)x$, $t \in (-\tau, \tau)$; then the partial integration shows $G(\varphi)x = \int_{-\infty}^{\infty}\varphi(t)u(t;x)\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$. The previous equality and (ii) imply $\lim_{n\to\infty} G(\rho_n)x = u(0;x) = -Y_{n_0}x = x \in \overline{\mathcal{R}(G)}$. Therefore, $D(A^{n_0-1}) \subseteq \overline{\mathcal{R}(G)}$. Further on, Proposition 3.3.15(iv) implies $C_{n_0-1}x + \frac{1}{(n_0-1)!}AY_{n_0-1}x = \frac{1}{(n_0-1)!}Y_{n_0-1}(Ax)$, i.e., $C_{n_0-1}x+\frac{1}{(n_0-1)!}[-x-n_0!B_{n_0}x] = \frac{1}{(n_0-1)!}Y_{n_0-1}(Ax)$. Due to Proposition 3.3.15(i), $Y_{n_0-1}(Ax) \in D(A)$ and a simple computation gives $Y_{n_0}Ax = -Ax$, which completes the proof of (iii). Further on, let us observe that (iii) implies $D(A^n) \subseteq D(A^{n_0-1}) \subseteq \overline{\mathcal{R}(G)} \subseteq \overline{D_\infty(A)} \subseteq \overline{D(A^{n+1})}$ for every $n \in \mathbb{N}$ such that $n \geqslant n_0-1$. Hence, A is stationary dense and $n(A) \leqslant n_0-1$. Assuming $\lambda \in \rho(A)$, we will prove that A generates a local (n_0-1)-times integrated group on $(-\tau, \tau)$. Repeating literally the arguments given in the proof of Theorem 3.3.19, one gets $A\int_0^t u(s;x)\,ds = u(t;x) - x$, $t \in (-\tau, \tau)$, $x \in D(A^{n_0-1})$ and $Au(t;x) = u(t;Ax)$, $t \in (-\tau, \tau)$, $x \in D(A^{n_0})$. Set $S^{n_0-1}(t)x =: u(t; R(\lambda : A)^{n_0-1}x)$, $t \in [0, \tau)$, $x \in E$. Clearly, the mapping $t \mapsto S^{n_0-1}(t)x$, $t \in [0, \tau)$ is continuous for every $x \in E$ and an induction argument shows that, for every $k \in \mathbb{N}_0$, there exists an appropriate constant $M(k, \lambda) \in (0, \infty)$ which fulfills $\|A^k R(\lambda : A)^k x\| \leqslant M(k, \lambda)\|x\|$, $x \in E$. This implies $\|S^{n_0-1}(t)x\| = \|u(t; R(\lambda : A)^{n_0-1}x)\| \leqslant M(t)\|R(\lambda : A)^{n_0-1}x\|_{n_0-1} \leqslant M(t)\sum_{i=0}^{n_0-1} M(i, \lambda)\|R(\lambda : A)\|^{n_0-1-i}\|x\|$, $x \in E$ and $S^{n_0-1}(t) \in L(E)$, $t \in [0, \tau)$. Let $C = R(\lambda : A)^{n_0-1}$. Then $A\int_0^t S^{n_0-1}(s)x\,ds = A\int_0^t u(s;Cx)\,ds = u(t;Cx) - Cx = S^{n_0-1}(t)x - Cx$, $t \in [0, \tau)$, $x \in E$. Since $Au(t;x) = u(t;Ax)$, $t \in (-\tau, \tau)$, $x \in D(A^{n_0})$, one easily obtains $S^{n_0-1}(t)A \subseteq AS^{n_0-1}(t)$, $S^{n_0-1}(t)R(\lambda : A) = R(\lambda : A)S^{n_0-1}(t)$ and, by induction, $S^{n_0-1}(t)C = CS^{n_0-1}(t)$, $t \in [0, \tau)$. Now it is straightforward to prove that the abstract Cauchy problem:

$$\begin{cases} v \in C([0,\tau) : [D(A)]) \cap C^1([0,\tau) : E), \\ v'(t) = Av(t) + Cx,\ t \in [0,\tau), \\ v(0) = 0, \end{cases}$$

has a unique solution for every $x \in E$, given by $v(t) = \int_0^t S^{n_0-1}(s)x\,ds$, $t \in [0,\tau)$, $x \in E$. Consequently, A generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. Since $-A$ generates a (DG) $\check{G}$, we also obtain that $-A$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. Therefore, A generates a local (n_0-1)-times integrated group on $(-\tau,\tau)$. This completes the proof of (v). To prove (vi), notice that the assumption $\overline{\mathcal{R}(G)} = E$ and $\mathcal{R}(G) \subseteq D_\infty(A)$ imply that $D_\infty(A)$ is dense in E. The converse statement is obvious since $\overline{D_\infty(A)} \subseteq \overline{D(A^{n_0-1})} \subseteq \overline{\mathcal{R}(G)}$ (cf. the proofs of (iii) and (iv)). In the case $\rho(A) \neq \emptyset$, the denseness of $\overline{D_\infty(A)}$ in E is equivalent to the denseness of $D(A)$ in E and the proof of (vi) completes a routine argument. It remains to be proved (vii). Suppose $G(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. This implies $(-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{n_0}(t)S(t)x\,dt = 0$, $\varphi \in \mathcal{D}_{[0,\tau)}$ and the existence of bounded linear operators $D_0,\ldots,D_{n_0-1} \in L(E)$ satisfying $S(t)x = \sum_{j=0}^{n_0-1} t^j D_j x$, $t \in [0,\tau)$. Hence,

$$A\sum_{j=0}^{n_0-1}\frac{t^{j+1}}{j+1}D_jx = \sum_{j=0}^{n_0-1}t^jD_jx + \sum_{j=0}^{n_0}t^jB_jx,\ t \in [0,\tau). \tag{247}$$

Substitute $t = 0$ in (247) to obtain $D_0 = -B_0$. Differentiating (247), it is straightforward to see that: $x \in \bigcap_{i=0}^{n_0} D(Y_i)$, $\bigcup_{i=0}^{n_0-1}\{D_ix\} \subseteq D(A)$, $D_ix = \frac{(-1)}{i!}Y_ix$, $i = 1,\ldots,n_0-1$ and $A(D_{n_0-1}x) = n_0B_{n_0}x$. This implies $(n_0B_{n_0} + \frac{1}{(n_0-1)!}AY_{n_0-1})x = 0$, i.e., $x \in N(Y_{n_0})$. Due to (ii), $x = 0$ and $\bigcap_{\varphi\in\mathcal{D}_0}\mathrm{Kern}(G(\varphi)) = \{0\}$. The second equality in (vii) follows by passing to $-A$ and $\check{G}$. $\square$

Example 3.3.21. Put $E := L^\infty(\mathbb{R})$ and $A := d/dt$ with maximal domain. Then A is not densely defined and generates a once integrated group $(S_1(t))_{t\in\mathbb{R}}$ given by $(S_1(t)f)(s) := \int_0^t f(r+s)\,dr$, $s \in \mathbb{R}$, $t \in \mathbb{R}$ (cf. also [**104**, Example 4.1]). Put $S_2(t)f := \int_0^t S_1(s)f\,ds$, $t \geqslant 0$, $f \in E$, $S_2(t)f := \int_0^{-t} S_1(-s)f\,ds$, $t < 0$, $f \in E$ and $G(\varphi)f := \int_{-\infty}^{\infty}\varphi''(t)S_2(t)f\,dt$, $\varphi \in \mathcal{D}$, $f \in E$. Then $(S_2(t))_{t\in\mathbb{R}}$ is a twice integrated group generated by A, the mapping $S_2 : \mathbb{R} \to L(E,[D(A)])$ is continuous and G is a non-dense (DG) generated by A (cf. Theorem 3.3.20 with $n_0 = 2$). We would like to point out that there exists $f \in D(A)$ such that $Af \notin \overline{\mathcal{R}(G)}$. Suppose to the contrary that $\mathrm{R}(A) \subseteq \overline{\mathcal{R}(G)}$. Due to Theorem 3.3.20, $D(A) \subseteq \overline{\mathcal{R}(G)}$ and we obtain $(\lambda - A)f \in \overline{\mathcal{R}(G)}$, $\lambda \in \mathbb{C}$, $f \in D(A)$. Since $\mathbb{C}\smallsetminus i\mathbb{R} \subseteq \rho(A)$, one yields $E = \overline{\mathcal{R}(G)}$ and the contradiction is obvious. Hence, Theorem 3.3.20 implies that $(S_1(t))_{t\geqslant 0}$ is a once integrated semigroup generated by A in the sense of Definition 2.1.1 and that the c.i.g of $(S_1(t))_{t\in\mathbb{R}}$ is $A_{|\overline{\mathcal{R}(G)}}$ $(\neq A)$. Furthermore, $\overline{\mathcal{R}(G)} \subsetneq \bigcap_{i=0}^{2} D(Y_i) = E$.

Proposition 3.3.22. *Suppose G_1 and G_2 are distribution groups generated by A and $\rho(A) \neq \emptyset$. Then $G_1 = G_2$.*

Proof. Suppose $x \in E$, $\lambda \in \rho(A)$ and $\varphi \in \mathcal{D}_{(-\tau,\tau)}$ for some $\tau \in (0,\infty)$. We will prove that $G_1(\varphi)x = G_2(\varphi)x$. Clearly, $G_i \in \mathcal{D}'(L(E,[D(A)]))$, $i = 1,2$ and an application of Theorem 1.3.2 gives that there exist $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ and continuous mappings $S_i : (-\tau,\tau) \to L(E,[D(A)])$, $i = 1,2$ so that $G_i(\psi)x =$

$(-1)^{n_i}\int_{-\infty}^{\infty}\psi^{(n_i)}(t)S_i(t)x\,dt$, $\psi\in\mathcal{D}_{(-\tau,\tau)}$, $x\in E$, $i=1,2$. The proof of Theorem 3.3.20 shows that there are bounded linear operators $B_0,\ldots,B_{n_1},\overline{B_0},\ldots,\overline{B_{n_2}}$, $C_0,\ldots,C_{n_1-1},\overline{C_0},\ldots,\overline{C_{n_2-1}}$ such that $(S_1(t))_{t\in(-\tau,\tau)}$, resp., $(S_2(t))_{t\in(-\tau,\tau)}$ is a $[B_0,\ldots,B_{n_1},C_0,\ldots,C_{n_1-1}]$-group, resp., $\left[\overline{B_0},\ldots,\overline{B_{n_2}},\overline{C_0},\ldots,\overline{C_{n_2-1}}\right]$-group with a subgenerator A. Without loss of generality, we may assume $n_1=n_2$. The proof of Theorem 3.3.20 implies that $\left(\frac{d^{n_1}}{dt^{n_1}}S_i(t)R(\lambda:A)^{n_1-1}\right)_{t\in[0,\tau)}$, $i=1,2$ are local $\left(R(\lambda:A)^{n_1-1}\right)$-regularized semigroups generated by A. Hence, there exist $x_0,\cdots,x_{n_1-1}\in E$ such that $S_1(t)R(\lambda{:}A)^{n_1-1}x-S_2(t)R(\lambda{:}A)^{n_1-1}x=\sum_{i=0}^{n_1-1}t^ix_i$, $t\in[0,\tau)$. An immediate consequence is:

$$\begin{aligned}&R(\lambda{:}A)^{n_1-1}G_1(\varphi)x-R(\lambda{:}A)^{n_1-1}G_2(\varphi)x\\&=G_1(\varphi)R(\lambda{:}A)^{n_1-1}x-G_2(\varphi)R(\lambda{:}A)^{n_1-1}x=(-1)^{n_1}\int\limits_{-\infty}^{\infty}\varphi^{(n_1)}(t)\sum_{i=0}^{n_1-1}t^ix_i\,dt=0,\end{aligned}$$

which clearly implies $G_1(\varphi)x=G_2(\varphi)x$. $\square$

Remark 3.3.23. (i) Suppose A generates a (DG) G and $\rho(A)\neq\emptyset$. Then there exist $a>0$ and $b>0$ such that $E(a,b)\subseteq\rho(\pm A)$ and that the following representation formula holds for G:

$$G(\varphi)x=\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{-\infty}^{\infty}\varphi(t)\Big[e^{\lambda t}R(\lambda{:}A)x+e^{-\lambda t}R(\lambda:-A)x\Big]\,dt\,d\lambda,\ x\in E,\ \varphi\in\mathcal{D},$$

where we assume that the curve $\Gamma=\partial E(a,b)$ is upwards oriented.

(ii) Suppose $G\in\mathcal{D}'(L(E))$ is regular, A is a closed linear operator so that (208) holds and there are no non-trivial solutions of the abstract Cauchy problem:

$$(ACP_1):\left\{\begin{array}{l}u\in C(\mathbb{R}:[D(A)])\cap C^1(\mathbb{R}:E),\\u'(t)=Au(t),\ t\in\mathbb{R},\\u(0)=x,\end{array}\right.$$

when $x=0$. Then $G(\varphi*\psi)x=G(\varphi)G(\psi)x$, $x\in\overline{\mathcal{R}(G)}$, $\varphi,\,\psi\in\mathcal{D}$. Towards this end, observe that $G(\varphi*\psi)x=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)\psi(s)u(t+s;x)\,ds\,dt$ and that $G(\psi)G(\varphi)x=\int_{-\infty}^{\infty}\varphi(t)G(\psi)u(t;x)\,dt$, $x\in\mathcal{R}(G)$, $\varphi,\,\psi\in\mathcal{D}$. The consideration is over if we prove that $G(\psi)u(t;x)=\int_{-\infty}^{\infty}\psi(s)u(t+s;x)\,ds$, $\psi\in\mathcal{D}$, $x\in\mathcal{R}(G)$, $t\in\mathbb{R}$. Set, for fixed $\psi\in\mathcal{D}$ and $x\in\mathcal{R}(G)$, $f(t):=G(\psi)u(t;x)-\int_{-\infty}^{\infty}\psi(s)u(t+s;x)\,ds$, $t\in\mathbb{R}$. Then $A\int_0^tf(s)\,ds=G(\psi)[u(t;x)-x]-\int_{-\infty}^{\infty}\psi(s)A\int_s^{t+s}u(r;x)\,dr\,ds=G(\psi)[u(t;x)-x]-\int_{-\infty}^{\infty}\psi(s)[u(t+s;x)-u(s;x)]\,ds=f(t)$, $t\in\mathbb{R}$. So, the function $u(t)=\int_0^tf(s)ds$, $t\in\mathbb{R}$ solves (ACP_1) and $u(0)=0$. This proves $f\equiv0$.

(iii) Suppose $G\in\mathcal{D}'(L(E))$ is regular, (208) holds for A and G, $\tau\in(0,\infty)$ and $\rho(A)\neq\emptyset$. Set $G_1:=G_{|\overline{\mathcal{R}(G)}}$. Then G_1 is a dense (DG) in $\overline{\mathcal{R}(G)}$ generated by A_1. To prove this, we employ the same terminology as in the proof of Theorem 3.3.19; without loss of generality, one can assume $0\in\rho(A)$ so that $A_\lambda=A$, $u_\lambda=u$ and

$G_\lambda = G$. Suppose (ρ_n) is a regularizing sequence. Choose an arbitrary $\tau \in (0,\infty)$ and notice that

$$\begin{aligned} C^2G(\varphi * \psi)x &= CG(\varphi*\psi)Cx = C\int_{-\infty}^{\infty} (\varphi*\psi)(t)u(t;Cx)\,dt \\ &= C\int_{-\infty}^{\infty} (\varphi*\psi)(t)T(t)x\,dt = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t)\psi(s)T(t+s)x\,ds\,dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t)\psi(s)T(t)T(s)x\,ds\,dt = G(\varphi)CG(\psi)Cx = C^2G(\varphi)G(\psi)x, \end{aligned}$$

for every $x \in \overline{\mathcal{R}(G)}$ and $\varphi,\ \psi \in \mathcal{D}_{[0,\tau)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq [0,\tau)$. The injectiveness of C combining with the argumentation used in the proof of Theorem 3.3.19 enables one to deduce that $G(\varphi*\psi)x = G(\varphi)G(\psi)x$, $\varphi,\ \psi \in \mathcal{D}$, $x \in \overline{\mathcal{R}(G)}$ and that $G_1 \in \mathcal{D}'(L(\overline{\mathcal{R}(G)}))$ satisfies $(DG)_1$. The assumption $G_1(\varphi)x = 0$, $\varphi \in \mathcal{D}$ implies $G_1(\varphi)Cx = \int_{-\infty}^{\infty}\varphi(t)u(t;Cx)\,dt = \int_{-\infty}^{\infty}\varphi(t)T(t)x\,dt = 0$, for every $\varphi \in \mathcal{D}_{[0,\tau)}$ and $Cx = T(0)x = \lim_{n\to\infty} G_1(\rho_n)Cx = 0$. Hence, $x = 0$ and G_1 is a (DG) in $\overline{\mathcal{R}(G)}$. It can be easily seen that G_1 is generated by A_1.

(iv) Suppose G and A possess the same meaning as in (ii), and $\lambda \in \rho(A)$. Then the $[B_0,\dots,B_{n_0},C_0,\dots,C_{n_0-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$, constructed in Theorem 3.3.19(i), satisfies (223), (224) as well as $Y_{n_0}x = -x$, $x \in \overline{\mathcal{R}(G)}$. Indeed, due to (ii), $G_1 = G_{|\overline{\mathcal{R}(G)}}$ is a (DG) in $\overline{\mathcal{R}(G)}$ generated by A_1. The proof of Theorem 3.3.19 implies that A_1 generates a local n_0-times integrated group $(S_{n_0}(t))_{t\in(-\tau,\tau)}$ in $L(\overline{\mathcal{R}(G)})$. It is not hard to prove that $G_1(\varphi)x = (-1)^{n_0}\int_0^\infty \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt + \int_{-\infty}^0 \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in \overline{\mathcal{R}(G)}$. Hence, $\int_0^\infty \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt = \int_0^\infty \varphi^{(n_0)}(t)S(t)x\,dt$, $\varphi \in \mathcal{D}_{[0,\tau)}$, $x \in \overline{\mathcal{R}(G)}$ and an application of [**128**, Theorem 8.1.1] gives the existence of operators $D_i \in L(\overline{\mathcal{R}(G)},E)$, $i = 0,\dots,n_0-1$ satisfying $S(t)x = S_{n_0}(t)x + \sum_{i=0}^{n_0-1} t^iD_ix$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. Since $A\int_0^t S(s)x\,ds = S(t)x + \sum_{i=0}^{n_0} t^iB_ix$, $t \in [0,\tau)$, $x \in E$, one obtains $A\sum_{i=0}^{n_0-1}\frac{t^{i+1}}{i+1}D_ix = \sum_{i=0}^{n_0-1} t^i(D_ix + B_ix) + t^{n_0}B_{n_0}x + \frac{t^{n_0}}{n_0!}x$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. This implies $\overline{\mathcal{R}(G)} \subseteq D(Y_{n_0})$, $D_ix = \frac{(-1)}{i!}Y_ix$, $i = 1,\dots,n_0-1$, (223)-(224) and $Y_{n_0}x = -x$, $x \in \overline{\mathcal{R}(G)}$.

REMARK 3.3.24. Suppose G is a (DG) and $\varphi \in \mathcal{D}$. Then $G(\varphi) = G(\varphi_+) + G(\varphi_-)$ iff $\{G(\varphi_+),\ G(\varphi_-)\} \subseteq L(E)$ iff $G(\varphi_+) \in L(E)$. Namely, the assumption $G(\varphi) = G(\varphi_+) + G(\varphi_-)$ immediately implies $D(G(\varphi_+)) = D(G(\varphi_-)) = E$ and, by the closed graph theorem, $G(\varphi_+) \in L(E)$ and $G(\varphi_-) \in L(E)$. Clearly, $\{G(\varphi_+),\ G(\varphi_-)\} \subseteq L(E)$ implies $G(\varphi_+) \in L(E)$. Suppose now $G(\varphi_+) \in L(E)$. We will prove that $G(\varphi_-) \in L(E)$ and $G(\varphi_-) = G(\varphi) - G(\varphi_+)$. Fix an $x \in E$ and notice that $G(\varphi*\psi)x = G(\psi)G(\varphi)x$, $\psi \in \mathcal{D}$ implies $G(\varphi_+*\psi)x + G(\varphi_-*\psi)x = G(\psi)G(\varphi)x$, $\psi \in \mathcal{D}$. Since $x \in D(G(\varphi_+))$, we obtain $G(\varphi_- * \psi)x = G(\psi)[G(\varphi)x - G(\varphi_+)x]$, $x \in D(G(\varphi_-))$ and $G(\varphi_-)x = G(\varphi)x - G(\varphi_+)x$.

PROPOSITION 3.3.25. *Let G be a (DG) and $G(\varphi_+) \in L(E)$, $\varphi \in \mathcal{D}$. Put $G_+(\varphi) := G(\varphi_+)$ and $G_-(\varphi) := G((\check{\varphi})_-)$, $\varphi \in \mathcal{D}$. Then $\pm A$ are generators of distribution semigroups $G_\pm$.*

PROOF. By the previous remark, $G_\pm(\varphi) \in L(E)$, $\varphi \in \mathcal{D}$ and $G(\varphi) = G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. Clearly, $\operatorname{supp} G_+ \cup \operatorname{supp} G_- \subseteq [0,\infty)$, $G_\pm \in \mathcal{D}_0'(L(E))$ and Theorem 3.3.20(vii) implies $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G_\pm(\varphi)) = \{0\}$. Since $(\varphi *_0 \psi)_+ = \varphi_+ * \psi_+$, φ, $\psi \in \mathcal{D}$, Proposition 3.3.5 yields that G_+ is a pre-(DS). Analogously, G_- is a pre-(DS) and one obtains that G_+, resp., G_- is a (DS). Designate by A_+, resp., A_-, the generator of G_+, resp., G_-. Then it is straightforward to verify that $A_\pm$ are extensions of $\pm A$. We continue by proving that $A_+ = -A_-$. Let $x \in E$ and φ, $\psi \in \mathcal{D}$. Then one obtains: $G(\varphi_+ * \psi)x = G(\psi)G(\varphi_+)x$, $G_+(\varphi_+ * \psi)x + G_-(\varphi_+ \check{*} \psi)x = (G_+(\psi) + G_-(\check{\psi}))G_+(\varphi)x$ and:

$$G_+\big(\varphi_+ * \psi\big)x + G_-\big(\varphi_+ \check{*} \psi\big)x = G_+(\varphi *_0 \psi)x + G_-(\check{\psi})G_+(\varphi)x. \tag{248}$$

Notice that $(\psi * \varphi_+ - \varphi * \psi_- - \varphi *_0 \psi)(t) = 0$, $t \geqslant 0$, which implies in combination with (248) that: $G_+(\psi * \varphi_+)x = G_+(\varphi * \psi_-)x + G_+(\varphi *_0 \psi)x$ and

$$G_+\big(\varphi * (\check{\psi})_-\big)x + G_-\big(\varphi_+ \check{*} \check{\psi}\big)x = G_-(\psi)G_+(\varphi)x. \tag{249}$$

Suppose now $(x,y) \in D(A_+)$, $a > 0$, $\psi \in \mathcal{D}_{(a,\infty)}$ and (ρ_n) is a regularizing sequence satisfying $\operatorname{supp}\rho_n \subseteq [0,\frac{1}{n}]$, $n \in \mathbb{N}$. Since $G_+(-\varphi')x = G_+(\varphi)y$, $\varphi \in \mathcal{D}_0$, (249) implies:

$$\begin{aligned} G_+\big(\rho_n * (\check{\psi})_-\big)y + G_-\big(\rho_n \check{*} \check{\psi}\big)y &= G_-(\psi)G_+(\rho_n)y = G_-(\psi)G_+(-\rho_n')x \\ &= G_+\big(-\rho_n' * (\check{\psi})_-\big)x + G_-\big(-\rho_n' \check{*} \check{\psi}\big)x. \end{aligned} \tag{250}$$

Clearly, $\operatorname{supp}(\rho_n * (\check{\psi})_-) \cup \operatorname{supp}(-\rho_n' * (\check{\psi})_-) \subseteq [0,\frac{1}{n}] + (-\infty,-a) \subseteq (-\infty,0]$, $n \geqslant \frac{1}{a}$ and an application of (250) yields:

$$G_-\big(\rho_n \check{*} \check{\psi}\big)y = G_-\big((-\rho_n * (\check{\psi})')^\vee\big)x. \tag{251}$$

Letting $n \to \infty$ in (251), one concludes that $G_-(\psi)y = -G_-\big(((\check{\psi})')^\vee\big)x = G_-(\psi')x$. It readily follows that the previous equalities remain true for every $\psi \in \mathcal{D}_0$, so that $(x,-y) \in A_-$ and $A_+ \subseteq -A_-$; one can similarly prove that $A_+ \supseteq -A_-$. Therefore, $A_+ = -A_-$ as claimed. Taking into account Proposition 3.1.4(iii), one gets: $G(\varphi)A_+x = G_+(\varphi)A_+x + G_-(\check{\varphi})A_+x = G_+(-\varphi')x - \varphi(0)x - A_-G_-(\check{\varphi})x = G_+(-\varphi')x - \varphi(0)x - \big(G_-(-\check{\varphi}')x - \check{\varphi}(0)x\big) = G_+(-\varphi')x + G_-(-\check{\varphi}')x = G(-\varphi')x$, $\varphi \in \mathcal{D}$. Hence, $(x, A_+x) \in A$, $A_+ \subseteq A$ and $A_+ = A$. A similar argumentation implies $A_- \subseteq -A$ and $A_- = -A$, finishing the proof. $\square$

REMARK 3.3.26. Suppose G is a (DG) generated by A and $\rho(A) \neq \emptyset$. By Theorem 3.3.7 and Theorem 3.3.20, we have that A, resp., $-A$, is the generator of a (DS) G_1, resp., G_2. The proofs of Theorem 3.3.7 and Proposition 3.3.22 imply $G(\varphi) = G_1(\varphi) + G_2(\check{\varphi})$, $\varphi \in \mathcal{D}$ and $G_1(\varphi)G_2(\psi) = G_2(\psi)G_1(\varphi)$, φ, $\psi \in \mathcal{D}$. Let $x \in E$ and $\varphi \in \mathcal{D}$ be fixed. We will prove that $G(\psi * \varphi_+)x = G(\psi)G_1(\varphi)x$, $\psi \in \mathcal{D}$, i.e., that:

$$G_1\big(\psi * \varphi_+\big)x + G_2\big(\psi \check{*} \varphi_+\big)x = G_1(\varphi)G_2(\check{\psi})x + G_1(\varphi)G_1(\psi),\ \psi \in \mathcal{D}. \tag{252}$$

Notice that the proof of [**315**, Theorem 6] (see [**315**, (9), p. 61]) enables one to see that $G_1\big(\varphi * \psi_-\big)x + G_2\big(\varphi_+ \check{*} \psi\big)x = G_1(\varphi)G_2(\check{\psi})x$, $\psi \in \mathcal{D}$. As in the proof of Proposition 3.3.25, one has $(\psi * \varphi_+ - \varphi * \psi_- - \varphi *_0 \psi)(t) = 0$, $t \geqslant 0, \psi \in \mathcal{D}$, which gives $G_1(\psi * \varphi_+)x = G_1(\varphi * \psi_-)x + G_1(\varphi *_0 \psi)x = G_1(\varphi * \psi_-)x + G_1(\varphi)G_1(\psi)x$, $\psi \in \mathcal{D}$. Hence, $G_1(\psi * \varphi_+)x + G_2\big(\psi \check{*} \varphi_+\big)x = G_1(\varphi * \psi_-)x + G_1(\varphi)G_1(\psi)x + G_2\big(\psi \check{*} \varphi_+\big)x = G_1(\varphi * \psi_-)x + G_1(\varphi)G_1(\psi)x + G_1(\varphi)G_2(\check{\psi})x - G_1(\varphi * \psi_-)x = G_1(\varphi)G_2(\check{\psi})x + G_1(\varphi)G_1(\psi)x$, $\psi \in \mathcal{D}$ and this proves (252). As a consequence, one obtains that $G_+(\varphi) = G(\varphi_+) = G_1(\varphi)$. Accordingly, A is the generator of $G_+ = G_1$ and Remark 3.3.24 implies that $G(\varphi_-) = G_2(\check{\varphi}) \in L(E)$, $\varphi \in \mathcal{D}$ and that $G\big((\check{\cdot})_-\big)$ is a (DS) generated by $-A$.

THEOREM 3.3.27. *Suppose $B_0, \ldots, B_n, C_0, \ldots, C_{n-1} \in L(E)$ and A is a subgenerator of a $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-group $(S(t))_{t \in \mathbb{R}}$. Put*

$$G(\varphi)x := (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t)S(t)x\,dt,\ \varphi \in \mathcal{D},\ x \in E.$$

Then:

(i) (208) *holds and* (234) *holds for every $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}$.*
(ii) $\mathcal{N}(G) \subseteq \operatorname{Kern}(Y_n)$ *and, in particular, the injectiveness of Y_n implies $(DG)_2$ for G.*
(iii) *For every $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}$,* $\operatorname{Kern}(Y_n) \subseteq \operatorname{Kern}(G(\varphi)G(\psi))$*; especially, if G is regular, then* $\operatorname{Kern}(Y_n) = \mathcal{N}(G)$.
(iv) *Assume $B_0 = \cdots = B_{n-1} = 0$ and $B_n = -\frac{1}{n!}I$. Then G is a (DG) generated by A.*

PROOF. (i) Clearly, $G \in \mathcal{D}'(L(E))$. Suppose $x \in E$ and $\varphi \in \mathcal{D}$. To prove $AG(\varphi)x = G(-\varphi')x$, notice that

$$G(\varphi)x = (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t)S(t)x\,dt = (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t S(s)x\,ds\,dt \in D(A)$$

and that

$$\begin{aligned} AG(\varphi)x &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t)\Big[S(t)x + \sum_{j=0}^{n} t^j B_j x\Big]\,dt \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t)S(t)x\,dt = G(-\varphi')x. \end{aligned}$$

Further on,

$$G(\varphi)Ax = (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t S(s)Ax\,ds\,dt$$

$$= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t \Big[AS(s)x - \sum_{j=0}^{n-1} s^j C_j x \Big] \, ds \, dt$$

$$= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) A \int_0^t S(s)x \, ds \, dt = AG(\varphi)x, \ x \in D(A).$$

Hence, $G(\varphi)A \subseteq AG(\varphi)$ and (208) holds. The assertions of (234) and (ii) follow from the arguments given in the proof of Theorem 3.3.20. Let $\varphi \in \mathcal{D}$, $\psi \in \mathcal{D}$ and $x \in \mathrm{Kern}(Y_n)$ be fixed. Arguing as in the proof of Theorem 3.3.20, one gets the validity of (246). Hence,

$$0 = \sum_{i=0}^{n} (-1)^{i+1} i! G\big((\varphi^{(n)} * \psi)^{(n-i)}\big) B_i x = \sum_{i=0}^{n} (-1)^{i+1} i! G(\varphi^{(n)} * \psi^{(n-i)}) B_i x$$

$$= \sum_{i=0}^{n} (-1)^{i+1} i! (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t) \int_{-\infty}^{\infty} \psi^{(n-i)}(s) S(t+s) B_i x \, ds \, dt.$$

Owing to (234), $G(\varphi)G(\psi)x = 0$ and $\mathrm{Kern}(Y_n) \subseteq \mathrm{Kern}(G(\varphi)G(\psi))$. Let (ρ_k) be a regularizing sequence and let G be regular. Then $G(\psi)x = \lim_{k\to\infty} G(\rho_k)G(\psi)x = 0$ and $x \in \mathcal{N}(G)$. Therefore. $\mathrm{Kern}(Y_n) \subseteq \mathcal{N}(G)$, and due to (ii), $\mathcal{N}(G) \subseteq \mathrm{Kern}(Y_n)$. The proof of (iii) is completed; to prove (iv), notice that the proof of Theorem 3.3.20 implies (243) for G. Since $B_0 = \cdots = B_{n-1} = 0$ and $B_n = -\frac{1}{n!} I$, we immediately obtain $(DG)_1$ from (234). Clearly, $Y_n = n! B_n = -I$ and $(DG)_2$ follows from an application of (ii). Hence, G is a (DG). Put now $\overline{S}(t) := S(t)$, $t \geqslant 0$ and $\overline{S}(t) := (-1)^n S(t)$, $t < 0$. It is obvious that $(\overline{S}(t))_{t\in\mathbb{R}}$ is an n-times integrated group generated by A. Furthermore, it is clear that $G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)\overline{S}(t)x \, dt + \int_{-\infty}^0 \varphi^{(n)}(t)\overline{S}(t)x \, dt$, $\varphi \in \mathcal{D}$, $x \in E$. Arguing as in the proof of Theorem 3.3.8, one yields that G is generated by A. □

3.4. Distribution cosine functions

3.4.1. Definition and elementary properties. Throughout this section we assume that the space $E \times E$ is topologized by the norm $\|(x,y)\|_{E\times E} := \|x\| + \|y\|$. Let $\alpha \in \mathcal{D}_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty} \alpha(t)\, dt = 1$. Then, with α chosen in this way, for every fixed $\varphi \in \mathcal{D}$, we define $I(\varphi)$ as follows

$$I(\varphi)(x) := \int_{-\infty}^{x} \Big[\varphi(t) - \alpha(t) \int_{-\infty}^{\infty} \varphi(u)\, du \Big] \, dt, \ \ x \in \mathbb{R}.$$

It is clear that $I(\varphi) \in \mathcal{D}$, $I(\varphi') = \varphi$ and $\frac{d}{dx} I(\varphi)(x) = \varphi(x) - \alpha(x) \int_{-\infty}^{\infty} \varphi(u)\, du$, $x \in \mathbb{R}$. Then, for $G \in \mathcal{D}'(L(E))$, we define G^{-1} by $G^{-1}(\varphi) := -G(I(\varphi))$, $\varphi \in \mathcal{D}$. Then $G^{-1} \in \mathcal{D}'(L(E))$ and $(G^{-1})' = G$; more precisely, $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi)$, $\varphi \in \mathcal{D}$. Let $\varphi \in \mathcal{D}$ and $\operatorname{supp} \varphi \subseteq (-\infty, 0)$. The choice of the function $\alpha(\cdot)$ implies $\frac{d}{dx} I(\varphi)(x) = 0$, $x > a$, for a suitable $a \in (-\infty, 0)$. Accordingly,

$\operatorname{supp} I(\varphi) \subseteq (-\infty, 0)$. This implies the following: $\operatorname{supp} G \subseteq [0,\infty) \Rightarrow \operatorname{supp} G^{-1} \subseteq [0,\infty)$. Moreover, it can be easily proved that, for every $\varphi \in \mathcal{D}$,

$$\operatorname{supp} I(\varphi) \subseteq [\min(-2,\ \inf(\operatorname{supp}\varphi)), \max(-1, \sup(\operatorname{supp}\varphi))].$$

We recall the assertion of Proposition 2.1.24 with $C = I$.

LEMMA 3.4.1. (i) *Let A be a closed linear operator and let $\lambda \in \mathbb{C}$. Then $\lambda \in \rho(\mathcal{A}) \Leftrightarrow \lambda^2 \in \rho(A)$. In this case,* $\|R(\lambda : \mathcal{A})\| \leqslant (1+|\lambda|)\sqrt{1+|\lambda|^2}\|R(\lambda^2 : A)\|+1$, $\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\|$ *and*

$$R(\lambda : \mathcal{A})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R(\lambda^2 : A)(\lambda x + y) \\ AR(\lambda^2 : A)x + \lambda R(\lambda^2 : A)y \end{pmatrix},\ x,\ y \in E,\ \lambda \in \rho(\mathcal{A}).$$

(ii) *Let $\emptyset \neq \Omega \subseteq \mathbb{C}$. Then $\Omega \subseteq \rho(\mathcal{A})$ iff $\Omega^2 \subseteq \rho(A)$; if this is the case, then $\|R(\cdot : A)\|$ is polynomially bounded on Ω^2 iff $\|R(\cdot : \mathcal{A})\|$ is polynomially bounded on Ω.*

We introduce distribution cosine functions as follows.

DEFINITION 3.4.2. An element $G \in \mathcal{D}_0'(L(E))$ is called a *pre-*(DCF) if it satisfies

$$(\mathrm{DCF}_1) \qquad G^{-1}(\varphi *_0 \psi) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi), \varphi,\ \psi \in \mathcal{D}.$$

A pre-(DCF) G is called a *distribution cosine function,* in short (DCF), if, additionally,

$$(\mathrm{DCF}_2) \qquad x = y = 0 \text{ iff } G(\varphi)x + G^{-1}(\varphi)y = 0 \text{ for all } \varphi \in \mathcal{D}_0.$$

A pre-(DCF) G is *dense* if the set $\mathcal{R}(G) := \bigcup_{\varphi\in\mathcal{D}_0} \mathrm{R}(G(\varphi))$ is dense in E.

Notice that (DCF_2) implies $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\varphi)) = \{0\} = \bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G^{-1}(\varphi))$. From Definition 3.4.2, it is also clear that $G(\varphi) = 0$ provided G is a pre-(DCF) and $\varphi \in \mathcal{D}_{(-\infty,0]}$. It is not clear whether the condition $\big(\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G(\varphi)) \supseteq\big)$ $\bigcap_{\varphi\in\mathcal{D}_0} \operatorname{Kern}(G^{-1}(\varphi)) = \{0\}$ implies (DCF_2).

Someone may think that (DCF_2) is a crude assumption. But, this is a right "non-degenerate" condition as the next proposition shows.

PROPOSITION 3.4.3. *Let $G \in \mathcal{D}_0'(L(E))$. Then G is a pre-*(DCF) *in E iff $\mathcal{G} \equiv \begin{pmatrix} G & G^{-1} \\ G' - \delta & G \end{pmatrix}$ is a pre-*(DSG) *in $E \times E$. Moreover, $\mathcal{G}$ is a* (DS) *iff G is a pre-*(DCF) *which satisfies (DCF_2).*

PROOF. Since $\alpha \in \mathcal{D}_{[-2,-1]}$, one gets $\mathcal{G} \in \mathcal{D}_0'(L(E \times E))$ and the simple calculation shows that $\mathcal{G}$ satisfies $\mathcal{G}(\varphi *_0 \psi) = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}$ iff the following holds:

(i) $G^{-1}(\varphi *_0 \psi) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi)$,
(ii) $G(\varphi *_0 \psi) = G(\varphi)G(\psi) + G^{-1}(\varphi)(G' - \delta)(\psi)$ and
(iii) $G'(\varphi *_0 \psi) = (G' - \delta)(\varphi)G(\psi) + G(\varphi)(G' - \delta)(\psi)$, φ, $\psi \in \mathcal{D}$.

We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose (i) holds. Since $(\varphi *_0 \psi)' = \varphi' *_0 \psi + \varphi(0)\psi = \varphi *_0 \psi' + \psi(0)\varphi$, φ, $\psi \in \mathcal{D}$, we infer that

$$G(\varphi *_0 \psi) = -G^{-1}((\varphi *_0 \psi)') = -G^{-1}(\varphi *_0 \psi' + \psi(0)\varphi)$$

$$= -\big(G^{-1}(\varphi)G(\psi') + G(\varphi)G^{-1}(\psi') + \delta(\psi)G^{-1}(\varphi)\big)$$
$$= G^{-1}(\varphi)G'(\psi) + G(\varphi)G(\psi) - \delta(\psi)G^{-1}(\varphi),\ \varphi,\ \psi \in \mathcal{D}.$$

This implies (ii). With assumed (ii), one obtains (iii) from the computation

$$\begin{aligned} G'(\varphi *_0 \psi) &= -G((\varphi *_0 \psi)') = -G(\varphi' *_0 \psi + \varphi(0)\psi) \\ &= -\big(G(\varphi')G(\psi) + G^{-1}(\varphi')(G' - \delta)(\psi) + \delta(\varphi)G(\psi)\big) \\ &= (G' - \delta)(\varphi)G(\psi) + G(\varphi)(G' - \delta)(\psi),\ \varphi,\ \psi \in \mathcal{D}. \end{aligned}$$

It is also clear that $\mathcal{G}$ satisfies (d_2) if G satisfies (DCF_2). Suppose that $\mathcal{G}$ satisfies (d_2). Let us prove that G satisfies (DCF_2). In order to do that, assume $x,\ y \in E$ and $G(\varphi)x + G^{-1}(\varphi)y = 0$, $\varphi \in \mathcal{D}_0$. Then one gets

$$(G' - \delta)(\varphi)x + G(\varphi)y = -G(\varphi')x - \varphi(0)x - G^{-1}(\varphi')y = 0,\ \varphi \in \mathcal{D}_0.$$

Since $\mathcal{G}$ satisfies (d_2), it follows that $x = y = 0$. □

Properties (DCF_1) and (DCF_2) can be interpreted respectively as $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and the linear independence of $\cos(\cdot)$ and $\sin(\cdot)$. Next, we characterize distribution cosine functions as follows.

PROPOSITION 3.4.4. *Let $G \in \mathcal{D}_0'(L(E))$. Then G is a* (DCF) *iff* (DCF_2) *holds and*

$$G^{-1}(\varphi * \psi_+) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi),\ \varphi \in \mathcal{D}_0,\ \psi \in \mathcal{D}. \tag{253}$$

PROOF. Assume G is a (DCF). Then $\mathcal{G}$ is a (DS) in $E \times E$ and the use of Proposition 3.1.18 gives $\mathcal{G}(\psi_+) = \mathcal{G}(\psi)$, $\psi \in \mathcal{D}$. Hence,

$$\begin{pmatrix} G(\varphi * \psi_+) & G^{-1}(\varphi * \psi_+) \\ (G' - \delta)(\varphi * \psi_+) & G(\varphi * \psi_+) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} G(\varphi) & G^{-1}(\varphi) \\ (G' - \delta)(\varphi) & G(\varphi) \end{pmatrix} \begin{pmatrix} G(\psi) & G^{-1}(\psi) \\ (G' - \delta)(\psi) & G(\psi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for every $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x,\ y \in E$. Choose $x = 0$ to obtain (253). Let us suppose now that (DCF_2) and (253) are fulfilled. Then $\mathcal{G}$ satisfies (d_2). The assumption (253) implies $G^{-1}(\varphi * \psi) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi)$, $\varphi,\ \psi \in \mathcal{D}_0$, and consequently,

$$\begin{aligned} G(\varphi * \psi) &= G(\varphi)G(\psi) + G^{-1}(\varphi)(G' - \delta)(\psi), \\ (G' - \delta)(\varphi * \psi) &= (G' - \delta)(\varphi)G(\psi) + G(\varphi)(G' - \delta)(\psi),\ \varphi,\ \psi \in \mathcal{D}_0. \end{aligned}$$

As an outcome, we get that (d_1) holds for $\mathcal{G}$. Let $\varphi \in \mathcal{D}_0$ and $\psi \in \mathcal{D}$. Then we obtain, for every $\varphi \in \mathcal{D}_0$ and $\psi \in \mathcal{D}$:

$$\begin{aligned} G(\varphi * \psi_+) &= -G^{-1}((\varphi *_0 \psi_+)') = -G^{-1}(\varphi' *_0 \psi_+ + \varphi(0)\psi_+) \\ &= -\big(G^{-1}(\varphi')G(\psi) + G(\varphi')G^{-1}(\psi)\big) = G(\varphi)G(\psi) + G'(\varphi)G^{-1}(\psi). \end{aligned}$$

Since $(\varphi *_0 \psi_+)' = (\varphi *_0 (\psi')_+) + \psi(0)\varphi$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$, we get

$$\begin{aligned} G(\varphi *_0 \psi_+) &= -G^{-1}(\varphi *_0 \psi_+)') = -G^{-1}(\varphi *_0 (\psi')_+ + \psi(0)\varphi) \\ &= -\big(G^{-1}(\varphi)G(\psi') + G(\varphi)G^{-1}(\psi')\big) - \psi(0)G^{-1}(\varphi) \end{aligned}$$

$$\begin{aligned}
&= G(\varphi)G(\psi) + G^{-1}(\varphi)(G' - \delta)(\psi),\\
(G' - \delta)(\varphi * \psi_+) &= G'(\varphi * \psi_+) = -G((\varphi *_0 \psi_+)') = -G(\varphi' *_0 \psi_+)\\
&= -\big[G(\varphi')G(\psi) + G^{-1}(\varphi')(G' - \delta)(\psi)\big]\\
&= (G' - \delta)(\varphi)G(\psi) + G(\varphi)(G' - \delta)(\psi).
\end{aligned}$$

Thus, (d_5) holds for $\mathcal{G}$ and $\mathcal{G}$ is a (DS) in $E \times E$. The remainder of proof follows by the use of preceding proposition. □

Proposition 3.4.5. *Let $G \in \mathcal{D}_0'(L(E))$. Then G is a pre-*(DCF) *iff*

$$G^{-1}(\varphi)(G' - \delta)(\psi) = (G' - \delta)(\varphi)G^{-1}(\psi),\ \varphi,\ \psi \in \mathcal{D}. \tag{254}$$

Proof. We have proved in Proposition 3.4.3 that G is a pre-(DCF) in E iff $\mathcal{G}$ is a pre-(DS) in $E \times E$. Then the use of [**199**, Proposition 2] gives that any of these conditions is also equivalent to:

$$\mathcal{G}(\varphi')\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}(\psi') = \psi(0)\mathcal{G}(\varphi) - \varphi(0)\mathcal{G}(\psi),\ \varphi,\ \psi \in \mathcal{D}. \tag{255}$$

As in the proofs of Proposition 3.4.3 and Proposition 3.4.4, it follows that (255) holds iff (254) holds. □

The infinitesimal generator of a (DCF) can be defined in several different ways; here we follow an idea of Shiraishi and Hirata [**385**] which has been utilized by Kunstmann in [**252**].

Definition 3.4.6. The *generator* A of a (DCF) G is given by

$$\big\{(x, y) \in E \times E : G^{-1}(\varphi'')x = G^{-1}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\big\}.$$

Because of (DCF_2), A is a function and it is easy to see that A is a closed linear operator in E.

Lemma 3.4.7. *Let A be the generator of a* (DCF) *G. Then $\mathcal{A} \subseteq \mathcal{B}$, where $\mathcal{A} \equiv \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ and $\mathcal{B}$ is the generator of $\mathcal{G}$. Furthermore,*

$$(x, y) \in A \Leftrightarrow \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) \in \mathcal{B}.$$

Proof. Let $\left(\binom{x}{y}, \binom{u}{v}\right) \in \mathcal{A}$. Then $x \in D(A)$, $y = u$ and $Ax = v$. We have to prove that, for every $\varphi \in \mathcal{D}_0$, $\mathcal{G}(-\varphi')\binom{x}{y} = \mathcal{G}(\varphi)\binom{u}{v}$. Towards this end, fix a $\varphi \in \mathcal{D}_0$. Then the definition of A implies

$$\begin{aligned}
G(-\varphi')x = G^{-1}(\varphi'')x = G^{-1}(\varphi)Ax = G^{-1}(\varphi)v,\\
G'(-\varphi')x = -G^{-1}(\varphi''')x = -G^{-1}(\varphi')Ax = -G^{-1}(\varphi')v = G(\varphi)v.
\end{aligned}$$

Since $\varphi(0) = \varphi'(0) = 0$ and $y = u$, we obtain

$$\begin{aligned}
G(-\varphi')x + G^{-1}(-\varphi')y = G(\varphi)u + G^{-1}(\varphi)v,\\
(G' - \delta)(-\varphi')x + G(-\varphi')y = (G' - \delta)(\varphi)u + G(\varphi)v.
\end{aligned}$$

This, in turn, implies $-\mathcal{G}(\varphi')\binom{x}{y} = \mathcal{G}(\varphi)\binom{u}{v}$, $\varphi \in \mathcal{D}_0$ and $\left(\binom{x}{y}, \binom{u}{v}\right) \in \mathcal{B}$. Assume that $(x, y) \in A$. Then $\left(\binom{x}{0}, \binom{0}{y}\right) \in \mathcal{A}$, and consequently, $\left(\binom{x}{0}, \binom{0}{y}\right) \in \mathcal{B}$. Suppose now $\left(\binom{x}{0},\binom{0}{y}\right)\in\mathcal{B}$ and fix again a $\varphi\in\mathcal{D}_0$. Then $\mathcal{G}(-\varphi')\binom{x}{0}=\mathcal{G}(\varphi)\binom{0}{y}$, and by the definition of $\mathcal{G}$,

$$\begin{pmatrix} G(-\varphi') & G^{-1}(-\varphi') \\ G'(-\varphi')-\delta(-\varphi') & G(-\varphi')\end{pmatrix}\begin{pmatrix} x\\ 0\end{pmatrix}=\begin{pmatrix} G(\varphi) & G^{-1}(\varphi) \\ G'(\varphi)-\delta(\varphi) & G(\varphi)\end{pmatrix}\begin{pmatrix} 0\\ y\end{pmatrix}.$$

Thereby, $G(-\varphi')x=G^{-1}(\varphi)y$, i.e., $G^{-1}(\varphi'')x=G^{-1}(\varphi)y$. This implies $(x,y)\in A$ and completes the proof. □

The following proposition will help to get relations between distribution cosine functions and local integrated cosine functions; notice that the property (DCF_2) has an important role again.

Proposition 3.4.8. *Let G be a* (DCF) *generated by A. Then the following holds:*

(i) $(G(\psi)x,\,G(\psi'')x+\psi'(0)x)\in A$, $\psi\in\mathcal{D}$, $x\in E$.
(ii) $(G^{-1}(\psi)x,-G(\psi')x-\psi(0)x)\in A$, $\psi\in\mathcal{D}$, $x\in E$.
(iii) $G(\psi)A\subseteq AG(\psi)$, $\psi\in\mathcal{D}$.
(iv) $G^{-1}(\psi)A\subseteq AG^{-1}(\psi)$, $\psi\in\mathcal{D}$.

Proof. Let $x,\,y\in E$.

(i) Clearly, $(G(\psi)x,y)\in A$ iff $G'(\varphi)G(\psi)x=G^{-1}(\varphi)y$, $\varphi\in\mathcal{D}_0$. If $\varphi\in\mathcal{D}_0$, then $\varphi(0)=0$ and by (iii) in the proof of Proposition 3.4.3, this is equivalent to:

$$G'(\varphi *_0 \psi)x-G(\varphi)G'(\psi)x+\psi(0)G(\varphi)x=G^{-1}(\varphi)y,\ \ \varphi\in\mathcal{D}_0$$
$$\Leftrightarrow$$
$$-G(\varphi *_0 \psi'+\psi(0)\varphi)x-G(\varphi)G'(\psi)x+\psi(0)G(\varphi)x=G^{-1}(\varphi)y,\ \ \varphi\in\mathcal{D}_0$$
$$\Leftrightarrow$$
$$-G(\varphi *_0 \psi')x-G(\varphi)G'(\psi)x=G^{-1}(\varphi)y,\ \ \varphi\in\mathcal{D}_0.$$

By (ii) in the proof of Proposition 3.4.3, this is equivalent to

$$-\big[G(\varphi)G(\psi')x+G^{-1}(\varphi)(G'(\psi')x-\psi'(0)x)\big]-G(\varphi)G'(\psi)x=G^{-1}(\varphi)y,\ \varphi\in\mathcal{D}_0$$
$$\Leftrightarrow$$
$$G(\varphi)\big[-G(\psi')x-G'(\psi)x\big]+G^{-1}(\varphi)\big[-G'(\psi')x+\psi'(0)x-y\big]=0,\ \varphi\in\mathcal{D}_0$$
$$\Leftrightarrow$$
$$y=G(\psi'')x+\psi'(0)x.$$

(ii) Let us recall that $G^{-1}(\psi)=-G(I(\psi))$ and that $\frac{d}{dt}I(\psi)(t)=\psi(t)-\alpha(t)\times\int_{-\infty}^{\infty}\psi(u)\,du$, $t\in\mathbb{R}$. Hence, $\frac{d^2}{dt^2}I(\psi)(t)=\psi'(t)-\alpha'(t)\int_{-\infty}^{\infty}\psi(u)\,du$, $t\in\mathbb{R}$. Since $\alpha\in\mathcal{D}_{[-2,-1]}$ and $G\in\mathcal{D}_0'(L(E))$, we obtain $(I(\psi))'(0)=\psi(0)$ and $G\big((I(\psi))''\big)=G\big(\psi'-\alpha'\int_{-\infty}^{\infty}\psi(u)\,du\big)=G(\psi')$. The use of (i) gives

$$AG^{-1}(\psi)x=-AG(I(\psi))x=-\big[G((I(\psi))'')x+(I(\psi))'(0)x\big]=-G(\psi')x-\psi(0)x.$$

(iii) Let $x\in D(A)$. Then $\left(\binom{x}{0},\binom{0}{Ax}\right)\in\mathcal{B}$, and by Proposition 3.1.4(iii),

$$\mathcal{G}(\psi)\begin{pmatrix}0\\ Ax\end{pmatrix}=-\mathcal{G}(\psi')\begin{pmatrix}x\\ 0\end{pmatrix}-\psi(0)\begin{pmatrix}x\\ 0\end{pmatrix},$$

which implies $G(\psi)Ax = G(\psi'')x + \psi'(0)x$. Thus, (iii) is a consequence of (i). To prove (iv), fix a test function $\psi \in \mathcal{D}$ and apply (iii) to see that $G^{-1}(\psi)A = -G(I(\psi))A \subseteq -AG(I(\psi)) = AG^{-1}(\psi)$. This completes the proof. $\square$

3.4.2. Relationship to integrated cosine functions, convolution equations and local C-regularized cosine functions.

THEOREM 3.4.9. *Let A be the generator of a* (DCF) *G. Then there exist $\tau > 0$, $n \in \mathbb{N}$ and a local n-times integrated cosine function $(C_n(t))_{t\in[0,\tau)}$ generated by A.*

PROOF. By Proposition 3.4.8(i), we have that, for every $\varphi \in \mathcal{D}$ and $x \in E$, $AG(\varphi)x = G(\varphi'')x + \varphi'(0)x$. This implies that G is a continuous linear mapping from $\mathcal{D}$ into $L(E, [D(A)])$, and as before, we get that there are $\tau > 0$, $n \in \mathbb{N}$ and a strongly continuous function $C_n : [-\tau, \tau] \to L(E, [D(A)])$ such that

$$G(\varphi)x = (-1)^n \int_{-\tau}^{\tau} \varphi^{(n)}(t)C_n(t)x\,dt,$$

for all $x \in E$ and $\varphi \in \mathcal{D}_{(-\tau,\tau)}$. Moreover, $\operatorname{supp} G \subseteq [0, \infty)$ implies $C_n(t) = 0$, $t \in [-\tau, 0]$ and

$$\begin{aligned}(-1)^n \int_0^{\tau} \varphi^{(n)}(t)AC_n(t)x\,dt &= AG(\varphi)x = G(\varphi'')x + \varphi'(0)x \\ &= (-1)^{n+2} \int_0^{\tau} \varphi^{(n+2)}(t)C_n(t)x\,dt + \varphi'(0)x,\end{aligned}$$

for all $x \in E$ and $\varphi \in \mathcal{D}_{(-\tau,\tau)}$. Thus, there exist $B_0, \ldots, B_{n+1} \in L(E)$ such that

$$\int_0^t (t-s)AC_n(s)x\,ds - C_n(t)x = \sum_{j=0}^{n+1} t^j B_j x,\ x \in E,\ t \in [0, \tau).$$

Hence,

$$(-1)^{n+2} \int_0^{\tau} \varphi^{(n+2)}(t) \sum_{j=0}^{n+1} t^j B_j x\,dt = \varphi'(0)x,\ \ \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E,\ \text{i.e.,}$$

$$(-1)^{n+2} \sum_{j=0}^{n+1} (-1)^{j+1} j! \varphi^{(n+1-j)}(0)B_j x = \varphi'(0)x,\ \ \varphi \in \mathcal{D}_{(-\tau,\tau)},\ \ x \in E.$$

One can choose a sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ in $\mathcal{D}_{(-\tau,\tau)}$ with $\varphi_k^{(j)}(0) = \delta_{jk}$, $j, k \in \mathbb{N}_0$, to conclude that $B_j = 0$, $j \in \{0, 1, \ldots, n+1\} \setminus \{n\}$, $B_n = \frac{(-1)}{n!}I$, and that

$$A \int_0^t (t-s)C_n(s)x\,ds = C_n(t)x - \frac{t^n}{n!}x,\ \ x \in E,\ t \in [0, \tau).$$

Since $G(\varphi)$ commutes with A, $\varphi \in \mathcal{D}$, it follows that there exist $F_0, \ldots, F_{n-1} \in L(E)$ such that

$$AC_n(t)x - C_n(t)Ax = \sum_{j=0}^{n-1} t^j F_j x, \quad x \in D(A),\ t \in [0, \tau).$$

Arguing similarly as in the first part of the proof, one has $F_j = 0$, $0 \leqslant j \leqslant n-1$. Thereby, $(C_n(t))_{t\in[0,\tau)}$ is a local n-times integrated cosine function generated by A. $\square$

THEOREM 3.4.10. *Let A be the generator of a (local) n-times integrated cosine function $(C_n(t))_{t\in[0,\tau)}$. Then A is the generator of a* (DCF).

PROOF. It is clear that $\mathcal{A}$ is the generator of an $(n+1)$-times integrated semigroup $(S_{n+1}(t))_{t\in[0,\tau)}$, where S_{n+1} is given in Theorem 2.1.11. By Corollary 2.1.9 and induction, one can prove that, for every $k \in \mathbb{N}$, $\mathcal{A}$ is the generator of a $(k(n+1))$-times integrated semigroup $(S_{k(n+1)}(t))_{t\in[0,k\tau)}$. Denote

$$S_{k(n+1)}(t) = \begin{pmatrix} S^1_{k(n+1)}(t) & S^2_{k(n+1)}(t) \\ S^3_{k(n+1)}(t) & S^4_{k(n+1)}(t) \end{pmatrix}, \quad 0 \leqslant t < k\tau.$$

The proof of the implication (ii) $\Rightarrow$ (i) of Theorem 2.1.11 yields:

$$S^1_{k(n+1)}(t) = S^4_{k(n+1)}(t), S^2_{k(n+1)}(t) = \int_0^t S^1_{k(n+1)}(s)\, ds,$$
$$S^3_{k(n+1)}(t) = \frac{d}{dt} S^1_{k(n+1)}(t) - \frac{t^{k(n+1)-1}}{(k(n+1)-1)!} I, \quad 0 \leqslant t < k\tau.$$

Furthermore, by the proof of Theorem 2.1.11, we have that the operator A is the generator of a $(k(n+1)-1)$-times integrated cosine function $\big(C_{k(n+1)-1}(t)\big)_{t\in[0,k\tau)}$ which is given by $C_{k(n+1)-1}(t) = S^3_{k(n+1)}(t) + \frac{t^{k(n+1)-1}}{(k(n+1)-1)!} I$, $t \in [0, k\tau)$. This implies that $S^1_{k(n+1)}(t) = \int_0^t C_{k(n+1)-1}(s)\, ds$, $t \in [0, k\tau)$ and that $(S^1_{k(n+1)}(t))_{t\in[0,k\tau)}$ is a $(k(n+1))$-times integrated cosine function generated by A. Given $\varphi \in \mathcal{D}$, choose $k \in \mathbb{N}$ such that $\varphi \in \mathcal{D}_{(-\infty,k\tau)}$. Define

$$\mathcal{G}(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} := (-1)^{k(n+1)} \int_0^\infty \varphi^{(k(n+1))}(t) S_{k(n+1)}(t) \begin{pmatrix} x \\ y \end{pmatrix} dt, \quad x,\, y \in E,$$
$$G(\varphi)x := (-1)^{k(n+1)} \int_0^\infty \varphi^{(k(n+1))}(t) S^1_{k(n+1)}(t) x\, dt, \quad x \in E.$$

One can easily prove that these definitions are independent of $k \in \mathbb{N}$. Moreover, $\mathcal{G}$ is a (DS) in $E \times E$ generated by $\mathcal{A}$; see the proof of [**418**, Theorem 3.8]. Let

$\varphi \in \mathcal{D}_{(-\infty,k\tau)}$ and $x \in E$. Then we obtain:

$$
\begin{aligned}
G^{-1}(\varphi)x &= -G(I(\varphi))x = -\int_0^\infty (I(\varphi))^{(k(n+1))}(t)S^1_{k(n+1)}(t)x\,dt \\
&= -\int_0^\infty \Bigl(\varphi^{(k(n+1)-1)}(t) - \alpha^{(k(n+1)-1)}(t)\int_{-\infty}^\infty \varphi(u)\,du\Bigr)S^1_{k(n+1)}(t)x\,dt \\
&= -\int_0^\infty \varphi^{(k(n+1)-1)}(t)S^1_{k(n+1)}(t)x\,dt = \int_0^\infty \varphi^{(k(n+1))}(t)\int_0^t S^1_{k(n+1)}(s)x\,ds\,dt \\
&= \int_0^\infty \varphi^{(k(n+1))}(t)S^2_{k(n+1)}(t)x\,dt,
\end{aligned}
$$

$$
\begin{aligned}
(G'-\delta)(\varphi)x &= -\int_0^\infty \varphi^{(k(n+1)+1)}(t)S^1_{k(n+1)}(t)x\,dt - \varphi(0)x \\
&= \int_0^\infty \varphi^{(k(n+1))}(t)\frac{d}{dt}S^1_{k(n+1)}(t)x\,dt - \int_0^\infty \varphi^{(k(n+1))}(t)\frac{t^{k(n+1)-1}}{(k(n+1)-1)!}x\,dt \\
&= \int_0^\infty \varphi^{(k(n+1))}(t)S^3_{k(n+1)}(t)x\,dt.
\end{aligned}
$$

Hence, $\mathcal{G}(\varphi) = \begin{pmatrix} G(\varphi) & G^{-1}(\varphi) \\ (G'-\delta)(\varphi) & G(\varphi) \end{pmatrix}$, $\varphi \in \mathcal{D}$. From Proposition 3.4.3, it follows that G is a (DCF). Suppose that the generator of G is B. Owing to Lemma 3.4.7, one gets $(x,y) \in B \Leftrightarrow \bigl(\binom{x}{0},\binom{0}{y}\bigr) \in \mathcal{A} \Leftrightarrow (x,y) \in A$. This completes the proof of theorem. $\square$

Corollary 3.4.11. *Let A be the generator of an n-times integrated cosine function $(C_n(t))_{t\geqslant 0}$. Put $G(\varphi)x := (-1)^n \int_0^\infty \varphi^{(n)}(t)C_n(t)x\,dt$, $\varphi \in \mathcal{D}$, $x \in E$. Then G is a* (DCF) *generated by A.*

Corollary 3.4.12. *Let G be a* (DCF) *generated by A. Then there exist $\tau > 0$, $n \in \mathbb{N}$ and a local n-times integrated cosine function $(C_n(t))_{t\in[0,\tau)}$ generated by A such that $G(\varphi)x = (-1)^n \int_0^\tau \varphi^{(n)}(t)C_n(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\infty,\tau)}$, $x \in E$.*

Given $\alpha > 0$ and $\beta > 0$, define the logarithmic region $\tilde{\Lambda}_{\alpha,\beta}$ by

$$\tilde{\Lambda}_{\alpha,\beta} := \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha + \beta\ln(1+|\lambda|)\bigr\}.$$

The following theorem is the main result of this subsection.

Theorem 3.4.13. *Let A be a closed operator. Then the following statements are equivalent:*

(i) *A is the generator of a* (DCF).

(ii) *There exist $\tau > 0$ and $n \in \mathbb{N}$ such that A is the generator of an n-times integrated cosine function on $[0, \tau)$.*

(iii) *For every $\tau > 0$ there is an $n \in \mathbb{N}$ such that A is the generator of an n-times integrated cosine function on $[0, \tau)$.*

(iv) *The operator $\mathcal{A}$ is the generator of a* (DS) *in $E \times E$.*

(v) *For every $\tau > 0$ there is an $n \in \mathbb{N}$ such that for all $(x, y) \in E \times E$ there exists a unique n-times integrated mild solution of (ACP_2).*

(vi) *There are constants $\alpha, \beta, M > 0$ and $n \in \mathbb{N}_0$ such that*

$\tilde{\Lambda}^2_{\alpha,\beta} := \{\lambda^2 : \lambda \in \tilde{\Lambda}_{\alpha,\beta}\} \subseteq \rho(A)$ *and* $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n, \lambda \in \tilde{\Lambda}^2_{\alpha,\beta}$.

Proof. The implication (i) $\Rightarrow$ (ii) is Theorem 3.4.9 and the implication (ii) $\Rightarrow$ (i) is Theorem 3.4.10. Assume that (ii) is true. Then the operator $\mathcal{A}$ is the generator of an $(n+1)$-times integrated semigroup $(S_{n+1}(t))_{t\in[0,\tau)}$. By Theorem 3.1.25, the operator $\mathcal{A}$ generates a (DS) in $E\times E$ and (iv) holds. If (iv) holds, then for all $\tau > 0$ there is an $n \in \mathbb{N}_0$ such that the operator $\mathcal{A}$ generates an $(n + 1)$-times integrated semigroup $(S_{n+1}(t))_{t\in[0,\tau)}$. Fix a number $\tau > 0$ and choose a non-negative integer n such that $\mathcal{A}$ generates an $(n+1)$-times integrated semigroup $(S_{n+1}(t))_{t\in[0,\tau)}$. By Theorem 2.1.11, the operator A must be the generator of a local n-times integrated cosine function on $[0, \tau)$ and (iii) is proved. The implication (iii) $\Rightarrow$ (ii) is trivial. The equivalence of (iii) and (v) has been already proved. Hence, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). The equivalence of (iv) and (vi) is an easy application of Theorem 3.1.25. $\square$

Remark 3.4.14. The difference between the logarithmic region $\tilde{\Lambda}_{\alpha,\beta}$ and the exponential region $E(\alpha, \beta)$ is inessential here; more precisely, one may replace (vi) with:

(vi′) There are constants $\alpha > 0$, $\beta > 0$, $M > 0$ and $n \in \mathbb{N}_0$ such that $E^2(\alpha, \beta) \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n, \lambda \in E^2(\alpha, \beta)$.

Namely, let $\alpha > 0$ and $\beta > 0$ be fixed. Arguing as in [**5**, Lemma 2.6], one can prove that $\tilde{\Lambda}_{\alpha,\beta} \subseteq E(\frac{1}{\beta}, \alpha)$ and that, for every $\alpha' > \alpha$, there exists $\beta' \geqslant \beta$ such that $E(\alpha, \beta) \subseteq \tilde{\Lambda}_{\beta', \frac{1}{\alpha'}}$. Furthermore, the logarithmic region in the formulation of Theorem 3.4.13 can be replaced by the region

$$\Lambda_{\alpha,\beta} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha + \beta \ln(1 + |\operatorname{Im}\lambda|)\}.$$

This follows from the following estimate (cf. [**199**, p. 199] for a proof):

$$\tilde{\Lambda}_{\alpha,\beta} \subseteq \Lambda_{\alpha,\beta} \subseteq \tilde{\Lambda}_{(1+\frac{\ln(1+\alpha)}{\alpha}\beta)^{-1}\alpha,(1+\frac{\ln(1+\alpha)}{\alpha}\beta)^{-1}\beta}.$$

Proposition 3.4.15. *Let A be a closed operator such that $\pm A$ generate distribution semigroups $G_\pm$. Then A^2 generates a* (DCF) *G, which is given by $G(\varphi) = \frac{1}{2}(G_+(\varphi) + G_-(\varphi))$, $\varphi \in \mathcal{D}$.*

Next, we would like to point out an interesting interplay between distribution cosine functions and convolution type equations; as a matter of fact, we use Proposition 3.4.3 to reduce our investigations to the corresponding theory of distribution semigroups.

THEOREM 3.4.16. (i) *Let A be a closed operator and let $G \in \mathcal{D}_0'(L(E))$. Then G is a* (DCF) *generated by A iff $G \in \mathcal{D}_0'(L(E,[D(A)]))$, $G * P = \delta' \otimes Id_{D(A)}$ and $P*G = \delta' \otimes Id_E$, where $D(A)$ is supplied with the graph norm, $P := \delta'' \otimes I - \delta \otimes A \in \mathcal{D}_0'(L([D(A)],E))$ and I denotes the inclusion $D(A) \to E$.*

(ii) *Let $G \in \mathcal{D}_0'(L(E))$. Then G is* (DCF) *in E generated by A iff $\mathcal{G} \equiv \begin{pmatrix} G & G^{-1} \\ G'-\delta & G \end{pmatrix}$ is a* (DS) *in $E \times E$ generated by $\mathcal{A}$.*

PROOF. (i) Let $X = L(E,[D(A)])$, $Y = L([D(A)],E)$, $Z = L([D(A)])$ and let $b : X \times Y \to Z$ be defined by $b(B,C) := BC$, $B \in X$, $C \in Y$. The definition of $G * P$ is given by Proposition 1.3.1; the convolution $P * G$ can be understood similarly. Let $x \in D(A)$, $k \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}$. Then it can be proved that:

$$\big(G*(\delta^{(k)} \otimes I)\big)(\varphi)x = (-1)^k G(\varphi^{(k)})x \text{ and } \big(G*(\delta^{(k)} \otimes A)\big)(\varphi)x = (-1)^k G(\varphi^{(k)})Ax.$$

Analogically,

$$\big((\delta^{(k)} \otimes I) * G\big)(\varphi)x = (-1)^k G(\varphi^{(k)})x,$$
$$\big((\delta^{(k)} \otimes A) * G\big)(\varphi)x = (-1)^k AG(\varphi^{(k)})x, \ \ \varphi \in \mathcal{D},\ x \in E,\ k \in \mathbb{N}_0.$$

Suppose that G is a (DCF) generated by A and $x \in E$. Then Proposition 3.4.8(i) implies $AG(\varphi)x = G(\varphi'')x + \varphi'(0)x$. Therefore, $G \in \mathcal{D}_0'(L(E,[D(A)]))$,

$$(P * G)(\varphi)x = G(\varphi'')x - AG(\varphi)x = -\varphi'(0)x \text{ and } P * G = \delta' \otimes Id_E.$$

We obtain $G * P = \delta' \otimes Id_{[D(A)]}$ in the same manner. Let $G \in \mathcal{D}_0'(L(E,[D(A)]))$ satisfy $G*P = \delta' \otimes Id_{[D(A)]}$ and $P*G = \delta' \otimes Id_E$. Since $\operatorname{supp} G \subseteq [0,\infty)$, it follows that $\operatorname{supp} G^{-1} \subseteq [0,\infty)$ and $\operatorname{supp} \mathcal{G} \subseteq [0,\infty)$. If $x \in E$, then the assumptions $G * P = \delta' \otimes Id_{[D(A)]}$ and $P * G = \delta' \otimes Id_E$ imply (i) of Proposition 3.4.8 and $G(\varphi)Ax = G(\psi'')x + \psi'(0)x$, $\varphi \in \mathcal{D}$, $x \in D(A)$. By the proof of Proposition 3.4.8, one obtains

$$AG^{-1}(\varphi)x = -G(\varphi')x - \varphi(0)x, \ \ \varphi \in \mathcal{D},\ x \in E,$$
$$G^{-1}(\varphi)Ax = -G(\varphi')x - \varphi(0)x, \ \ \varphi \in \mathcal{D},\ x \in E.$$

It follows that $\mathcal{G} \in \mathcal{D}_0'(L(E \times E,[D(\mathcal{A})]))$, where $D(\mathcal{A})$ is endowed with the graph norm. Let $x \in D(A)$. Then, for every $\varphi \in \mathcal{D}$:

$$-\mathcal{G}(\varphi')\begin{pmatrix} x \\ y \end{pmatrix} - \mathcal{G}(\varphi)\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = -\mathcal{G}(\varphi')\begin{pmatrix} x \\ y \end{pmatrix} - \mathcal{G}(\varphi)\begin{pmatrix} y \\ Ax \end{pmatrix}$$
$$= \begin{pmatrix} -G(\varphi')x - G^{-1}(\varphi')y - G(\varphi)y - G^{-1}(\varphi)Ax \\ -G'(\varphi')x + \varphi'(0)x - G(\varphi')y - G'(\varphi)y + \varphi(0)y - G(\varphi)Ax \end{pmatrix} = \varphi(0)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Similarly, if $x,\, y \in E$, then $-\mathcal{G}(\varphi')\binom{x}{y} - \mathcal{A}\mathcal{G}(\varphi)\binom{x}{y} = \varphi(0)\binom{x}{y}$, $\varphi \in \mathcal{D}$. This implies $\mathcal{G} * P_1 = \delta \otimes Id_{[D(\mathcal{A})]}$ and $P_1 * \mathcal{G} = \delta \otimes Id_{E \times E}$, where $P_1 := \delta' \otimes Id - \delta \otimes \mathcal{A} \in \mathcal{D}_0'(L([D(\mathcal{A})], E \times E))$ and Id denotes the inclusion $D(\mathcal{A}) \to E \times E$. This fact combined with the proof of [**252**, Theorem 3.10] enables one to see that $\mathcal{G}$ is a (DS) in $E \times E$ generated by $\mathcal{A}$. Thus, G is a (DCF) in E. If B is the generator of G, then $(x,y) \in B \Leftrightarrow \big(\binom{x}{0}, \binom{0}{y}\big) \in \mathcal{A} \Leftrightarrow (x,y) \in A$.

(ii) Suppose that G is a (DCF) generated by A. Then $\mathcal{G}$ is a (DS) in $E \times E$. Let P be as in (i). Then we obtain $G * P = \delta' \otimes Id_{[D(A)]}$ and $P * G = \delta' \otimes Id_E$. Then the proof of (i) implies that the generator of $\mathcal{G}$ is $\mathcal{A}$. Conversely, if $\mathcal{G}$ is a (DS) generated by $\mathcal{A}$, then G is a (DCF). It can be easily seen that the generator of G is A. $\square$

Let us recall that G is a (DS) generated by A if and only if G is a distribution fundamental solution for A (cf. [**252**, p. 844–845]). Since there is at most one distribution fundamental solution for a closed linear operator A, it follows that every (DS) is uniquely determined by its generator. Herein it is worthwhile to notice that Kisyński introduced in [**199**] the generator of a pre-(DS) G; in his approach, this is a closed linear operator from E into $E/\mathcal{N}(G)$. He proved that every pre-(DS) is uniquely determined by its generator, see [**199**, Corollary 2]. Now we state:

Corollary 3.4.17. *Every distribution cosine function is uniquely determined by its generator.*

Proof. Suppose that G_1 and G_2 are distribution cosine functions generated by a closed linear operator A. Then

$$\mathcal{G}_1 \equiv \begin{pmatrix} G_1 & G_1^{-1} \\ G_1' - \delta & G_1 \end{pmatrix} \quad \text{and} \quad \mathcal{G}_2 \equiv \begin{pmatrix} G_2 & G_2^{-1} \\ G_2' - \delta & G_2 \end{pmatrix}$$

are distribution semigroups generated by $\mathcal{A}$. Thereby, for every $x \in E$ and $\varphi \in \mathcal{D}$, $\mathcal{G}_1(\varphi)\binom{x}{0} = \mathcal{G}_2(\varphi)\binom{x}{0}$. This implies $G_1(\varphi)x = G_2(\varphi)x$ and completes the proof. $\square$

Now we clarify the interplay between distribution cosine functions and local C-regularized cosine functions; we refer to [**381**] for the introduction to the theory of local C-regularized cosine functions.

Proposition 3.4.18. *Let A be a closed operator. Then the following statements are equivalent:*

(i) *A generates a* (DCF),
(ii) *$\rho(A) \neq \emptyset$ and there exist $n \in \mathbb{N}$ and $\tau \in (0, \infty]$ such that A is the generator of an $R(\lambda : A)^n$-regularized cosine function on $[0, \tau)$ for all $\lambda \in \rho(A)$,*
(iii) *$\rho(A) \neq \emptyset$ and there exist $\lambda \in \rho(A)$, $n \in \mathbb{N}$ and $\tau \in (0, \infty]$ such that A is the generator of an $R(\lambda : A)^n$-regularized cosine function on $[0, \tau)$.*

We close this subsection by stating the following proposition.

Proposition 3.4.19. *Let G be a* (DCF). *Then $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, φ, $\psi \in \mathcal{D}$.*

3.4.3. Exponential distribution cosine functions.

Definition 3.4.20. A distribution cosine function G is said to be an *exponential distribution cosine function,* (EDCF) in short, if $\mathcal{G} \equiv \begin{pmatrix} G & G^{-1} \\ G' - \delta & G \end{pmatrix}$ is an (EDS) in $E \times E$.

The following might be surprising:

PROPOSITION 3.4.21. *Let G be a* (DCF). *Then G is an* (EDCF) *iff there exists $\varepsilon \in \mathbb{R}$ such that $e^{-\varepsilon t}G^{-1} \in \mathcal{S}'_0(L(E))$.*

PROOF. Let remind us that $\mathcal{S}$ is topologized by the seminorms

$$\|\psi\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}} |x^\alpha \psi^{(\beta)}(x)|, \ \ \alpha,\, \beta \in \mathbb{N}_0,\ \psi \in \mathcal{S}.$$

Assume that G is an (EDCF). By Definition 3.4.20, this implies that there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\left(\begin{smallmatrix} G & G^{-1} \\ G'-\delta & G\end{smallmatrix}\right) \in \mathcal{S}'(L(E\times E))$, i.e., there exist $M > 0$ and α, $\beta \in \mathbb{N}_0$ such that, for every $\varphi \in \mathcal{D}$,

$$\left\| \left\langle e^{-\omega t} \begin{pmatrix} G & G^{-1} \\ G' - \delta & G \end{pmatrix}, \varphi \right\rangle \right\|_{L(E\times E)} \leqslant M\|\varphi\|_{\alpha,\beta}.$$

Therefore, for all $\varphi \in \mathcal{D}$ and x, $y \in E$, the following holds:

$$\begin{aligned} &\big\|\big\langle e^{-\omega t}G,\varphi\big\rangle\, x + \big\langle e^{-\omega t}G^{-1},\varphi\big\rangle\, y\big\| \\ &\qquad + \big\|\big\langle e^{-\omega t}(G'-\delta),\varphi\big\rangle\, x + \big\langle e^{-\omega t}G,\varphi\big\rangle\, y\big\| \leqslant M\|\varphi\|_{\alpha,\beta}(\|x\|+\|y\|).\end{aligned}$$

Choose $x = 0$ to obtain $e^{-\omega t}G^{-1} \in \mathcal{S}'_0(L(E))$. Suppose now $e^{-\omega t}G^{-1} \in \mathcal{S}'_0(L(E))$. Then there exist $M > 0$ and α, $\beta \in \mathbb{N}_0$ so that $\|G^{-1}(e^{-\omega t}\varphi)\| \leqslant M\|\varphi\|_{\alpha,\beta}$, $\varphi \in \mathcal{D}$. Then one gets:

$$\begin{aligned} &\big\|(e^{-\omega t}G)(\varphi)\big\| = \big\|G(e^{-\omega t}\varphi)\big\| = \big\|G^{-1}(-\omega e^{-\omega t}\varphi + e^{-\omega t}\varphi')\big\| \\ &\qquad \leqslant M|\omega|\|\varphi\|_{\alpha,\beta} + M\|\varphi'\|_{\alpha,\beta} \leqslant M|\omega|\|\varphi\|_{\alpha,\beta} + M\|\varphi\|_{\alpha,\beta+1},\ \varphi \in \mathcal{D}.\end{aligned}$$

Hence, $e^{-\omega t}G \in \mathcal{S}'_0(L(E))$; similarly, $e^{-\omega t}(G'-\delta) \in \mathcal{S}'_0(L(E))$ and we finally obtain

$$e^{-\omega t}\begin{pmatrix} G & G^{-1} \\ G'-\delta & G\end{pmatrix} \in \mathcal{S}'_0(L(E\times E)). \qquad \square$$

THEOREM 3.4.22. *Let A be a closed operator. Then the following assertions are equivalent:*

(i) *A is the generator of an* (EDCF) *in E.*
(ii) *$\mathcal{A}$ is the generator of an* (EDS) *in $E \times E$.*
(iii) *A is the generator of a global exponentially bounded n-times integrated cosine function for some $n \in \mathbb{N}$.*
(iv) *There are constants $\omega > 0$, $M > 0$ and $k \in \mathbb{N}$ such that*

$$\Pi_\omega := \left\{\eta + i\xi : \eta > \omega^2 - \frac{\xi^2}{4\omega^2}\right\} \subseteq \rho(A) \ \text{and}\ \|R(\lambda\colon A)\| \leqslant M|\lambda|^k,\ \lambda \in \Pi_\omega.$$

PROOF. The implication (i) $\Rightarrow$ (ii) follows from Theorem 3.4.17(ii). In order to prove the converse, suppose that $\mathcal{G}$ is a (DS) in $E\times E$ generated by $\mathcal{A}$ and that $e^{-\omega t}\mathcal{G} \in \mathcal{S}'_0(L(E\times E))$. Clearly, $e^{-\omega t}\mathcal{G}$ is a (DS) in $E\times E$ generated by $\mathcal{A} - \left(\begin{smallmatrix}\omega I & 0 \\ 0 & \omega I\end{smallmatrix}\right)$. By Proposition 3.1.4(iii), we have that, for every $\varphi \in \mathcal{D}$ and x, $y \in E$:

$$\mathcal{A}\left\langle e^{-\omega t}\mathcal{G},\varphi\right\rangle \begin{pmatrix} x \\ y\end{pmatrix} = \left\langle e^{-\omega t}\mathcal{G},-\varphi'\right\rangle \begin{pmatrix} x \\ y\end{pmatrix} + \omega\left\langle e^{-\omega t}\mathcal{G},\varphi\right\rangle \begin{pmatrix} x \\ y\end{pmatrix} - \varphi(0)\begin{pmatrix} x \\ y\end{pmatrix},$$

which gives $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E \times E, [D(\mathcal{A})]))$. The use of Theorem 1.3.2(ii) implies that there exist $n \in \mathbb{N}$, $r > 0$ and a continuous function $\bar{S}_{n+1} : \mathbb{R} \to L(E \times E, [D(\mathcal{A})])$ supported by $[0, \infty)$ such that

$$\langle e^{-\omega t}\mathcal{G}, \varphi \rangle \binom{x}{y} = (-1)^{n+1} \int\limits_0^\infty \varphi^{(n+1)}(t)\bar{S}_{n+1}(t)\binom{x}{y}\, dt,$$

for all $\varphi \in \mathcal{D}$, x, $y \in E$, and $|\bar{S}_{n+1}(t)| \leqslant Mt^r$, $t \geqslant 0$. By the proof of [**5**, Theorem 7.2], one gets that $(\bar{S}_{n+1}(t))_{t\geqslant 0}$ is an $(n+1)$-times integrated semigroup generated by $\mathcal{A} - \left(\begin{smallmatrix} \omega I & 0 \\ 0 & \omega I \end{smallmatrix}\right)$. A standard rescaling argument shows that $\mathcal{A}$ is the generator of a global exponentially bounded $(n+1)$-times integrated semigroup $(S_{n+1}(t))_{t\geqslant 0}$. Hence, the operator A generates a global exponentially bounded n-times integrated cosine function $(C_n(t))_{t\geqslant 0}$. We have proved (ii) $\Rightarrow$ (iii). We continue the proof of (ii) $\Rightarrow$ (i) by applying the argumentation given in the final part of the proof of Theorem 3.4.10. It is easy to see that $\mathcal{G}(\varphi) = (-1)^{n+1} \int_0^\infty \varphi^{(n+1)}(t)S_{n+1}(t)\, dt$, $\varphi \in \mathcal{D}$. Define G by $G(\varphi)x := (-1)^{(n+1)} \int_0^\infty \varphi^{(n+1)}(t)S^1_{(n+1)}(t)x\, dt$, $x \in E$, $\varphi \in \mathcal{D}$, with the same terminology as in the proof of Theorem 3.4.10. Using the same arguments as in the proof of cited theorem, one obtains $\mathcal{G} = \left(\begin{smallmatrix} G & G^{-1} \\ G'-\delta & G \end{smallmatrix}\right)$. Therefore, G is an (EDCF) generated by A and (i) follows. Suppose that (iii) holds. Then $\mathcal{A}$ generates an exponentially bounded $(n+1)$-times integrated semigroup $(S_{n+1}(t))_{t\geqslant 0}$. Set $\mathcal{G}(\varphi) := (-1)^{n+1} \int_0^\infty \varphi^{(n+1)}(t)S_{n+1}(t)\, dt$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is an (EDS) in $E \times E$ generated by $\mathcal{A}$, and (ii) follows. If (ii) holds, then there exists $\omega > 0$ such that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subseteq \rho(\mathcal{A})$ and that $\|R(\cdot : \mathcal{A})\|$ is polynomially bounded on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$. Therefore, $\Pi_\omega = \{\lambda^2 : \lambda \in \mathbb{C}, \operatorname{Re}\lambda > \omega\}$ is contained in $\rho(A)$ and $\|R(\cdot : A)\|$ is polynomially bounded on Π_ω. So (iv) holds. Assume that (iv) is true. Then $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subseteq \rho(\mathcal{A})$ and $\|R(\cdot : \mathcal{A})\|$ is polynomially bounded on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$. This implies that $\mathcal{A}$ generates an exponentially bounded $(n+1)$-times integrated semigroup for some $n \in \mathbb{N}$. This implies (iii) and ends the proof. □

Let A be a densely defined operator and let A be the generator of an exponentially bounded α-times integrated cosine function for some $\alpha \geqslant 0$. By Proposition 2.2.7, the adjoint A^* of A is the generator of an exponentially bounded $(\alpha + 1)$-times integrated cosine function. Then the previous theorem immediately implies the following theorem which remains true in the case of a general distribution cosine function.

Proposition 3.4.23. *Let A be a densely defined operator. If A is the generator of an* (EDCF) *in E, then A^* is the generator of an* (EDCF) *in E^*.*

The proofs of subsequent assertions are standard and therefore omitted.

Proposition 3.4.24. *If A is the generator of an* (EDCF)*, then for every $z \in \mathbb{C}$, the operator $A + z$ is also the generator of an* (EDCF).

Proposition 3.4.25. *Suppose that A and $-A$ generate exponential distribution semigroups. Then A^2 is the generator of an* (EDCF).

3.4.4. Dense distribution cosine functions. In this subsection we focus our attention to dense distribution cosine functions and their generators. First of all, we will prove the next proposition.

PROPOSITION 3.4.26. *Let G be a* (DCF)*. Then for all $\binom{x}{y} \in \mathcal{R}(\mathcal{G})$ there exists a function $u \in C^1([0,\infty) : E)$ satisfying $u(0) = x$, $u'(0) = y$ and*

$$G(\psi)x + G^{-1}(\psi)y = \int_0^\infty \psi(t)u(t)\,dt, \quad \psi \in \mathcal{D}.$$

PROOF. It is clear that $\mathcal{G}$ is a (DS) in $E \times E$. Since $\mathcal{G}$ satisfies (d_4) we have that for all $\binom{x}{y} \in \mathcal{R}(\mathcal{G})$ there exist two functions u, $v \in C([0,\infty) : E)$ such that $u(0) = x$, $v(0) = y$ and that

$$G(\psi)x + G^{-1}(\psi)y = \int_0^\infty \psi(t)u(t)\,dt,$$

$$(G' - \delta)(\psi)x + G(\psi)y = \int_0^\infty \psi(t)v(t)\,dt, \quad \psi \in \mathcal{D}.$$

With $y = 0$ and $x \in E$, integration by parts implies

$$\int_0^\infty \varphi'(t)\left(u(t) - \int_0^t v(s)\,ds\right)dt = G(\varphi')x + G'(\varphi)x = 0, \ \varphi \in \mathcal{D}_0.$$

Then one obtains $u(t) = x + \int_0^t v(s)\,ds$, $t \geqslant 0$, and the function u has the desired properties. □

PROPOSITION 3.4.27. *Let G be a* (DCF) *generated by A. Then for all x, $y \in D_\infty(A)$ there exists a function $u \in C^1([0,\infty) : E)$ satisfying $u(0) = x$, $u'(0) = y$ and*

$$G(\varphi)x + G^{-1}(\varphi)y = \int_0^\infty \varphi(t)u(t)\,dt, \quad \varphi \in \mathcal{D}_0.$$

PROOF. Applying [**418**, Corollary 3.9], we obtain that for all $\binom{x}{y} \in D_\infty(\mathcal{A}) = D_\infty(A) \times D_\infty(A)$ there exist two functions u, $v \in C([0,\infty) : E)$ such that $u(0) = x$, $v(0) = y$ and that $\mathcal{G}(\varphi)\binom{x}{y} = \int_0^\infty \varphi(t)\binom{u(t)}{v(t)}\,dt$ for all $\varphi \in \mathcal{D}_0$. This, in turn, implies

$$G(\varphi)x + G^{-1}(\varphi)y = \int_0^\infty \varphi(t)u(t)\,dt,$$

$$(G' - \delta)(\varphi)x + G(\varphi)y = \int_0^\infty \varphi(t)v(t)\,dt, \quad \varphi \in \mathcal{D}_0.$$

Now one can repeat verbatim the final part of the proof of previous proposition. □

THEOREM 3.4.28. *Let G be a* (DCF) *generated by A. Then $\overline{D(A)} = E$ iff G is dense.*

PROOF. Assume G is dense. Then Proposition 3.4.8(i) implies $\overline{D(A)} = E$. Conversely, suppose that $\overline{D(A)} = E$. Since $\rho(A) \neq \emptyset$, we have $\overline{D_\infty(A)} = E$ and it suffices to show $D_\infty(A) \subseteq \overline{\mathcal{R}(G)}$. Let $x \in D_\infty(A)$. By Proposition 3.4.27, we obtain that there exists a function $u \in C^1([0,\infty) : E)$ satisfying $u(0) = x$, $u'(0) = 0$ and

$$G(\varphi)x = \int_0^\infty \varphi(t)u(t)\,dt,\ \varphi \in \mathcal{D}_0.$$

Let (ρ_n) be a regularizing sequence. Then $x = \lim_{n\to\infty} G(\rho_n)x \in \overline{\mathcal{R}(G)}$ and this completes the proof. $\square$

REMARK 3.4.29. Let A be the generator of a (DCF) G. Then it can be also proved that G is dense iff $\mathcal{G}$ is a dense distribution semigroup in $E \times E$.

3.4.5. Almost-distribution cosine functions, cosine convolution products and their relations with distribution cosine functions. Assume that $\tau_0 : [0,\infty) \to [0,\infty)$ is a measurable function such that $\inf_{t\geqslant 0} \tau_0(t) > 0$ and that there exists $C_0 > 0$ satisfying:

$$\tau_0(t+s) \leqslant C_0\tau_0(t)\tau_0(s),\ t,\ s \geqslant 0 \text{ and } \tau_0(t-s) \leqslant C_0\tau_0(t)\tau_0(s),\ 0 < s < t.$$

Then $(L^1([0,\infty) : \tau_0), \|\cdot\|_{\tau_0})$ denotes the Banach space consisting of those measurable functions $f : [0,\infty) \to \mathbb{C}$ for which $\|f\|_{\tau_0} := \int_0^\infty |f(t)|\tau_0(t)\,dt < \infty$. If $f,\ g \in L^1([0,\infty) : \tau_0)$, put $f \circ g(t) := \int_t^\infty f(s-t)g(s)\,ds,\ t \geqslant 0$. Clearly, $f *_0 g \in L^1([0,\infty) : \tau_0)$ and $f \circ g \in L^1([0,\infty) : \tau_0)$. The *cosine convolution product* $f *_c g$ is defined by $f *_c g := \frac{1}{2}(f *_0 g + f \circ g + g \circ f)$; the *sine convolution product* by $f *_s g := \frac{1}{2}(f *_0 g - f \circ g - g \circ f)$ and the *sine-cosine convolution product* by $f *_{sc} g := \frac{1}{2}(f *_0 g - f \circ g + g \circ f)$. Notice, $f *_c g,\ f *_s g, f *_{sc} g \in L^1([0,\infty) : \tau_0)$, resp. $\mathcal{D}_+$, if $f,\ g \in L^1([0,\infty) : \tau_0)$, resp. $f,\ g \in \mathcal{D}_+$; see for example [**388**].

The following proposition can be viewed as an analogue of the well-known formula $\cos(t+s) = \cos t \cos s - \sin t \sin s$ for distribution cosine functions.

PROPOSITION 3.4.30. *Let G be a* (DCF) *generated by A.Then $G(\varphi *_0 \psi)x = G(\varphi)G(\psi)x + AG^{-1}(\varphi)G^{-1}(\psi)x,\ \varphi,\ \psi \in \mathcal{D},\ x \in E$.*

PROOF. Notice, if $\varphi,\ \psi \in \mathcal{D}$, then $(\varphi *_0 \psi)'(t) = \varphi' *_0 \psi(t) + \varphi(0)\psi(t),\ t \in \mathbb{R}$. Since A generates G and $G(\varphi) = -G^{-1}(\varphi'),\ \varphi \in \mathcal{D}$, we infer that

$$\begin{aligned} G(\varphi *_0 \psi)x &= -\varphi(0)G^{-1}(\psi)x - G^{-1}(\varphi' *_0 \psi)x \\ &= G(\varphi)G(\psi)x + (-\varphi(0) - G(\varphi'))G^{-1}(\psi)x \\ &= G(\varphi)G(\psi)x + AG^{-1}(\varphi)G^{-1}(\psi)x, \end{aligned}$$

for any $x \in E$. $\square$

In the next theorem, we characterize pre-distribution cosine functions by convolution products.

THEOREM 3.4.31. *Let $G \in \mathcal{D}'_0(L(E))$ satisfy $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, φ, $\psi \in \mathcal{D}$. Then the following assertions are equivalent:*

(i) *G is a pre-*(DCF) *and $G^{-1}(\Lambda(f \circ g - g \circ f)) = G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g))$ for all f, $g \in \mathcal{D}_+$.*
(ii) *$G^{-1}(\Lambda(f *_{sc} g)) = G^{-1}(\Lambda(f))G(\Lambda(g))$ for all f, $g \in \mathcal{D}_+$.*

PROOF. (i) $\Rightarrow$ (ii). Note, $f *_0 g(t) = (g *_{sc} f + f *_{sc} g)(t)$, $(f \circ g - g \circ f)(t) = (g *_{sc} f - f *_{sc} g)(t)$ and $\Lambda(f *_0 g)(t) = \Lambda(f) *_0 \Lambda(g)(t)$, for $t \geqslant 0$ and f, $g \in \mathcal{D}_+$. Moreover, $G^{-1}(\varphi) = 0$ if $\varphi \in \mathcal{D}_{(-\infty,0]}$ and we obtain

$$G^{-1}\big(\Lambda(g *_{sc} f + f *_{sc} g)\big) = G^{-1}(\Lambda(f))G(\Lambda(g)) + G(\Lambda(f))G^{-1}(\Lambda(g)),$$
$$G^{-1}\big(\Lambda(g *_{sc} f - f *_{sc} g)\big) = G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g)).$$

This, in turn, implies $G^{-1}(\Lambda(f *_{sc} g)) = G^{-1}(\Lambda(f))G(\Lambda(g))$ for all f, $g \in \mathcal{D}_+$. (In this direction, we do not use the assumption $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, φ, $\psi \in \mathcal{D}$).

(ii) $\Rightarrow$ (i). Fix φ, $\psi \in \mathcal{D}$. Since $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, we have $G^{-1}(\varphi)G(\psi) = G(\psi)G^{-1}(\varphi)$. Certainly, $\mathcal{K}(\varphi) *_0 \mathcal{K}(\psi)(t) = \big(\mathcal{K}(\psi) *_{sc} \mathcal{K}(\varphi) + \mathcal{K}(\varphi) *_{sc} \mathcal{K}(\psi)\big)(t)$ for all $t \geqslant 0$ and this enables one to see that:

$$\begin{aligned} G^{-1}(\varphi *_0 \psi) &= G^{-1}\big(\Lambda(\mathcal{K}(\varphi) *_0 \mathcal{K}(\psi))\big) = G^{-1}\big(\Lambda(\mathcal{K}(\psi) *_{sc} \mathcal{K}(\varphi) + \mathcal{K}(\varphi) *_{sc} \mathcal{K}(\psi))\big) \\ &= G^{-1}\big(\Lambda\mathcal{K}(\varphi)\big)G\big(\Lambda\mathcal{K}(\psi)\big) + G^{-1}\big(\Lambda\mathcal{K}(\psi)\big)G\big(\Lambda\mathcal{K}(\varphi)\big) \\ &= G^{-1}(\varphi)G(\psi) + G^{-1}(\psi)G(\varphi) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi). \end{aligned}$$

Hence, G is a pre-(DCF). Since $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, φ, $\psi \in \mathcal{D}$, the second equality follows from (ii):

$$\begin{aligned} G^{-1}\big(\Lambda(f \circ g - g \circ f)\big) &= G^{-1}\big(\Lambda(g *_{sc} f - f *_{sc} g)\big) \\ &= G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g)), \end{aligned}$$

for all f, $g \in \mathcal{D}_+$. $\square$

DEFINITION 3.4.32. [**312**] An element $G \in L(\mathcal{D}_+ : L(E))$ is called an *almost-distribution cosine function* if:

(i) $G(f *_c g) = G(f)G(g)$, f, $g \in \mathcal{D}_+$ and
(ii) $\bigcap_{f \in \mathcal{D}_+} \mathrm{Kern}(G(f)) = \{0\}$.

The (infinitesimal) generator A of G is defined by

$$A := \big\{(x, y) \in E \times E : G(f)y = G(f'')x + f'(0)x \ \text{ for all } f \in \mathcal{D}_+\big\}.$$

It is known that A is a closed linear operator. Further on, $G(f)A \subseteq AG(f)$, $G(f)x \in D(A)$ and $AG(f)x = G(f'')x + f'(0)x$, $f \in \mathcal{D}_+$. Recall, a global n-times integrated cosine function $(C_n(t))_{t \geq 0}$ defines an almost-distribution cosine functions G (cf. [**312**]) by

$$G(f)x = (-1)^n \int_0^\infty f^{(n)}(t)\, C_n(t)x\, dt, \ x \in E, \ f \in \mathcal{D}_+.$$

THEOREM 3.4.33. *Let G be a* (DCF) *generated by A. Then $G\Lambda$ is an almost-distribution cosine function generated by A.*

PROOF. Notice that $G\Lambda \in L(\mathcal{D}_+ : L(E))$. Since G is a (DCF) generated by A, it follows that $\bigcap_{\varphi\in\mathcal{D}_0} \mathrm{Kern}(G(\varphi)) = \{0\}$. Hence, the condition (ii) in the definition of an almost-distribution cosine function is fulfilled. In order to prove (i), let us fix $f,\ g \in \mathcal{D}_+$. Let

$$\operatorname{supp} f \cup \operatorname{supp} g \cup (\operatorname{supp} f + \operatorname{supp} g) \cup \operatorname{supp}(f \circ g) \cup \operatorname{supp}(g \circ f) \subseteq [0, a], \tag{256}$$

for some $a \in (0,\infty)$. This implies $\operatorname{supp}(f *_c g) \subseteq [0,a]$ and $\operatorname{supp}\big(\Lambda(f *_c g)\big) \subseteq (-\infty, a]$. Due to Theorem 3.4.13, there exists $n \in \mathbb{N}$ such that A is the generator of an n-times integrated cosine function $(C_n(t))_{t\in[0,2a)}$. Then the proofs of Theorem 3.4.24 and Corollary 3.4.18 imply

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)\, C_n(t)x\, dt,\quad x \in E,\ \varphi \in \mathcal{D}_{(-\infty,2a)}.$$

Therefore,

$$G\Lambda(f *_c g)x = (-1)^n \int_0^\infty \big(\Lambda(f *_c g)\big)^{(n)}(t)\, C_n(t)x\, dt = (-1)^n \int_0^\infty (f *_c g)^{(n)}(t)\, C_n(t)x\, dt.$$

Clearly, $G\Lambda(f)x = (-1)^n \int_0^\infty f^{(n)}(t)\, C_n(t)x\, dt$. Hence, we have to prove

$$(-1)^n \int_0^\infty (f *_c g)^{(n)}(t)\, C_n(t)x\, dt = (-1)^n \int_0^\infty f^{(n)}(t)\, C_n(t) \int_0^\infty g^{(n)}(s)\, C_n(s)x\, ds\, dt. \tag{257}$$

This can be obtained as in the proof of [**312**, Theorem 4] with $\alpha = n \in \mathbb{N}$. Note only that (256) implies that Fubini theorem can be applied in the proofs of [**311**, Proposition 1.1] and [**312**, Theorem 4]. Let B be the generator of $G\Lambda$. We will prove $A = B$. Suppose $(x, y) \in A$. Then $G^{-1}(\varphi'')x = G^{-1}(\varphi)y$ for all $\varphi \in \mathcal{D}_0$. Our goal is to prove that

$$G\Lambda(f)y = G\Lambda(f'')x + f'(0)x \text{ for all } f \in \mathcal{D}_+, \tag{258}$$

which implies $(x, y) \in B$ and $A \subseteq B$. Fix an $f \in \mathcal{D}_+$. Taking into account Proposition 3.4.8, we obtain

$$G\Lambda(f)y = G\Lambda(f)Ax = AG(\Lambda(f))x = G\big((\Lambda(f))''\big)x + (\Lambda(f))'(0)x.$$

Since $(\Lambda(f))''(t) = \Lambda(f'')(t)$, $t \geqslant 0$, one can continue as follows

$$= G(\Lambda(f''))x + f'(0)x,$$

and (258) holds. Suppose now $(x, y) \in B$. Then:

$$G\Lambda(f)y = G\Lambda(f'')x + f'(0)x,\ f \in \mathcal{D}_+. \tag{259}$$

One must prove that $G^{-1}(\varphi'')x = G^{-1}(\varphi)y$, $\varphi \in \mathcal{D}_0$. Let $\operatorname{supp}\varphi \subseteq [0, b]$, for some $b > 0$. Obviously, $\operatorname{supp} I(\varphi) \subseteq [-2, b]$. Note, $\frac{d^2}{dt^2} I(\varphi)(t) = \varphi'(t) - \alpha'(t) \int_{-\infty}^\infty \varphi(u)\, du$, $t \in \mathbb{R}$, and consequently, $\frac{d^2}{dt^2} I(\varphi)(t) = \varphi'(t)$, $t \geqslant 0$. Then $I(\varphi)(t) = \Lambda\big(\mathcal{K}(I(\varphi))\big)(t)$,

$t \geqslant 0$, $\big(\mathcal{K}(I(\varphi))\big)'(0) = \varphi(0) - \alpha(0)\int_{-\infty}^{\infty}\varphi(u)\,du = \varphi(0) = 0$ and $\Lambda\big((\mathcal{K}(I(\varphi)))''\big)(t) = (I(\varphi))''(t) = \varphi'(t)$, $t \geqslant 0$. Now one obtains from (259):

$$\begin{aligned} G^{-1}(\varphi)y &= -G(I(\varphi))y = -G\big(\Lambda(\mathcal{K}(I(\varphi)))\big)y \\ &= -\big(G\Lambda\big((\mathcal{K}(I(\varphi)))''\big)x + \big(\mathcal{K}(I(\varphi))\big)'(0)x\big) \\ &= -G\Lambda\big((\mathcal{K}(I(\varphi)))''\big)x = -G(\varphi')x = G^{-1}(\varphi'')x, \end{aligned}$$

which gives $(x, y) \in A$ and ends the proof. $\square$

Corollary 3.4.34. *Let G be a* (DCF) *generated by A. Then*

$$G\big(\Lambda(f *_s g)\big) = AG^{-1}(\Lambda(f))G^{-1}(\Lambda(g)),\ f,\, g \in \mathcal{D}_+.$$

Proof. Take $f,\ g \in \mathcal{D}_+$. Since $f *_0 g = f *_c g + f *_s g$, one can apply Proposition 3.4.30 and Theorem 3.4.31 to obtain the equality. $\square$

The use of Theorem 3.4.16 enables one to briefly prove the following fundamental relationship between distribution cosine functions and almost-distribution cosine functions as well as to establish directly some other results (cf. for example Proposition 3.4.20):

Theorem 3.4.35. *Let G_1 be an almost-distribution cosine function generated by A. Then A is the generator of a* (DCF) *G given by $G(\varphi) := G_1(\mathcal{K}(\varphi))$, $\varphi \in \mathcal{D}$.*

Proof. Note, if $\operatorname{supp}\varphi \subseteq (-\infty, 0)$, then $\mathcal{K}(\varphi) = 0$ in $\mathcal{D}_+$, which clearly implies $\operatorname{supp} G \subseteq [0, \infty)$ and $G \in \mathcal{D}_0'(L(E))$. Recall, $G(f)A \subseteq AG(f)$, $G(f)x \in D(A)$ and $AG(f)x = G(f'')x + f'(0)x$, $f \in \mathcal{D}_+$; see [**312**, p. 178]. We will prove that:

$$\begin{aligned} &AG(\varphi)x = G(\varphi'')x + \varphi'(0)x,\ \ x \in E,\ \varphi \in \mathcal{D}, \\ &G(\varphi)Ax = G(\varphi'')x + \varphi'(0)x,\ \ x \in D(A),\ \varphi \in \mathcal{D}. \end{aligned} \tag{260}$$

Let $x \in E$ and $\varphi \in \mathcal{D}$. Then

$$\begin{aligned} AG(\varphi)x &= AG_1(\mathcal{K}(\varphi))x = G_1\big((\mathcal{K}(\varphi))''\big)x + \varphi'(0)x \\ &= G_1\big(\mathcal{K}(\varphi'')\big)x + \varphi'(0)x = G(\varphi'')x + \varphi'(0)x. \end{aligned}$$

Since $G_1A \subseteq AG_1$, the second equality in (260) can be proved similarly. It is evident that (260) implies $G \in \mathcal{D}_0'(L(E, [D(A)]))$. Moreover, $G * P = \delta' \otimes Id_{[D(A)]}$ and $P * G = \delta' \otimes Id_E$, where we use the terminology given in the formulation of Theorem 3.4.16: $P = \delta'' \otimes I - \delta \otimes A \in \mathcal{D}_0'(L([D(A)], E))$, $Id_{[D(A)]}$ denotes the inclusion $D(A) \to E$ and $\big(\delta^{(k)} \otimes Id_{[D(A)]}\big)(\varphi)x = (-1)^k\varphi^{(k)}(0)x$, $\big(\delta^{(k)} \otimes I\big)(\varphi)x = (-1)^k\varphi^{(k)}(0)x$, $\big(\delta \otimes A\big)(\varphi)x = \varphi(0)Ax$, $\varphi \in \mathcal{D}$, $x \in D(A)$, $k \in \mathbb{N}_0$ and $\big(\delta' \otimes Id_E\big)(\varphi)x = -\varphi'(0)x$, $\varphi \in \mathcal{D}$, $x \in E$. By Theorem 3.4.16, one yields that G is a (DCF) generated by A. $\square$

Corollary 3.4.36. *Every almost-distribution cosine function is uniquely determined by its generator.*

PROOF. Suppose G_1 and G_2 are almost-distribution cosine functions generated by a closed linear operator A. Put $\mathcal{G}_i(\varphi) := G_i(\mathcal{K}(\varphi))$, $\varphi \in \mathcal{D}$, $i = 1, 2$. By Theorem 3.4.35, $\mathcal{G}_1$ and $\mathcal{G}_2$ are distribution cosine functions generated by A and one can use Corollary 3.4.18 to obtain that $\mathcal{G}_1 = \mathcal{G}_2$, i.e., $G_1(\mathcal{K}(\varphi)) = G_2(\mathcal{K}(\varphi))$, $\varphi \in \mathcal{D}$. Since $\mathcal{K} : \mathcal{D} \to \mathcal{D}_+$ is a surjective mapping, we have that $G_1 = G_2$. This ends the proof. $\square$

THEOREM 3.4.37. *Let A be a closed linear operator. Then, the following are equivalent.*

(i) *A is the generator of a* (DCF).
(ii) *A is the generator of an almost-distribution cosine function.*

Now we will recollect some results proved by Miana in [**312**]. We consider almost-distribution cosine functionsand their relationship with global fractionally integrated cosine functions with corresponding growth order. Let remind us [**312**] that the family of *Bochner–Riesz functions* (R_t^θ), $\theta > -1$, $t > 0$, is defined by $R_t^\theta(s) = \frac{(t-s)^\theta}{\Gamma(\theta+1)}\chi_{(0,t)}$. The Weyl functional calculus can be applied to the functions which do not belong to the space $\mathcal{D}_+$; for example, in the case of Bochner–Riesz functions we have that $W_+^\alpha R_t^\theta = R_t^{\theta-\alpha}$, $\theta+1 > \alpha \geqslant 0$. Denote by Ω_α, $\alpha > 0$ the set of all nondecreasing continuous functions $\tau_\alpha(\cdot)$ on $(0,\infty)$ so that $\inf_{t>0} t^{-\alpha}u(t) > 0$ and that there exists a constant $C_\alpha > 0$ satisfying

$$\int\limits_{[0,t]\,\cap\,[s,s+t]} u^{\alpha-1}\tau_\alpha(t+s-u)\,du \leqslant C_\alpha \tau_\alpha(t)\tau_\alpha(s),\ 0 < t \leqslant s.$$

The typical functions $\tau_\alpha(t) = t^\alpha$; $t^\beta(1+t)^\gamma$ ($\beta \in [0,\alpha]$, $\beta+\gamma \geqslant \alpha$); $t^\beta e^{\tau t}$ ($\beta \in [0,\alpha]$, $\tau > 0$) belong to Ω_α. Suppose $\tau_\alpha \in \Omega_\alpha$ and $\nu > \alpha$; then the function $\tau_\nu = t^{\nu-\alpha}\tau_\alpha$, $t > 0$ belongs to Ω_ν. Designate by Ω_α^h the subset of Ω_α, $\alpha > 0$ which consists of all functions of the form $\tau_\alpha = t^\alpha\omega_0(t)$, $t > 0$, where the continuous nondecreasing function $\omega_0 : [0,\infty) \to [0,\infty)$ satisfies $\inf_{t>0}\omega_0(t) > 0$ and $\omega_0(t+s) \leqslant \omega_0(t)\omega_0(s)$, t, $s > 0$. Suppose $\alpha > 0$, $\tau_\alpha \in \Omega_\alpha$ and define

$$q_{\tau_\alpha}(\varphi) := \int\limits_0^\infty \frac{\tau_\alpha(t)}{\Gamma(\alpha+1)}|W_+^\alpha\varphi(t)|\,dt,\ \varphi \in \mathcal{D}_+.$$

Then $q_{\tau_\alpha}(\cdot)$ is a norm on $\mathcal{D}_+$ and there exists a constant $C_\alpha > 0$ such that $q_{\tau_\alpha}(\varphi *_c \phi) \leqslant C_\alpha q_{\tau_\alpha}(\varphi)q_{\tau_\alpha}(\phi)$, $\varphi, \phi \in \mathcal{D}_+$ [**312**]. Let $\mathfrak{T}_+^\alpha(\tau_\alpha, *_c)$ denote the completion of the normed space $(\mathcal{D}_+, q_{\tau_\alpha})$; then $\mathfrak{T}_+^\alpha(\tau_\alpha, *_c)$ is invariant under the cosine convolution cosine product $*_c$ and the following holds (cf. [**312**, Theorem 3]):

(i) $\mathfrak{T}_+^\alpha(\tau_\alpha, *_c) \hookrightarrow \mathfrak{T}_+^\alpha(t^\alpha, *_c) \hookrightarrow L^1([0,\infty), *_c)$, where $\hookrightarrow$ denotes the dense and continuous embedding,
(ii) $\mathfrak{T}_+^\beta(t^\beta, *_c) \hookrightarrow \mathfrak{T}_+^\alpha(t^\alpha, *_c)$, $\beta > \alpha > 0$,
(iii) $R_t^{\nu-1} \in \mathfrak{T}_+^\alpha(\tau_\alpha, *_c)$, $\nu > \alpha$, $t > 0$ and there exists a constant $C_{\nu,\alpha} > 0$ such that $q_{\tau_\alpha}(R_t^{\nu-1}) \leqslant C_{\nu,\alpha}t^{\nu-\alpha}\tau_\alpha(t)$, $t > 0$.

An almost-distribution cosine function G_+ is said to be of *order* $\alpha > 0$ and *growth* $\tau_\alpha \in \Omega_\alpha$ if G_+ can be extended to a continuous linear mapping from $\mathfrak{T}^\alpha_+(\tau_\alpha,\, *_c)$ into $L(E)$. Now we are in a position to clarify the following important result:

THEOREM 3.4.38. [**312**] (i) *Let A be the generator of an α-times integrated cosine functions $(C_\alpha(t))_{t\geqslant 0}$ such that $\|C_\alpha(t)\| \leqslant C\tau_\alpha(t)$, $t > 0$. Then the mapping $G_+ : \mathfrak{T}^\alpha_+(\tau_\alpha,\, *_c) \to L(E)$, given by*

$$G_+(f)x := \int\limits_0^\infty W^\alpha_+ f(t) C_\alpha(t)x\, dt,\ f \in \mathfrak{T}^\alpha_+(\tau_\alpha,\, *_c),\ x \in E,$$

is a continuous algebra homomorphism which satisfies:

$$\int\limits_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)} C_\alpha(s)x\, ds = G_+(R^{\nu-1}_t)x,\ \ \nu > \alpha,\ x \in E$$

$$\int\limits_0^\infty W^\alpha_+ f(t) C_\alpha(t)x\, dt = \int\limits_0^\infty W^\nu_+ f(t) \int\limits_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)} C_\alpha(s)x\, ds\, dt,$$

for all $f \in \mathfrak{T}^\nu_+(t^{\nu-\alpha}\tau_\alpha,\, *_c)$, $x \in E$. Furthermore, the restriction of G_+ to $\mathcal{D}_+$ is an almost-distribution cosine function of order $\alpha > 0$ and growth τ_α with the generator A.

(ii) *Suppose A is the generator of an almost-distribution cosine function G_+ of order $\alpha > 0$ and growth $\tau_\alpha \in \Omega_\alpha$. Then, for every $\nu > \alpha$, A generates a ν-times integrated cosine function $(C_\nu(t))_{t\geqslant 0}$ such that $\|C_\nu(t)\| \leqslant C_\nu t^{\nu-\alpha}\tau_\alpha(t)$, $t > 0$ and that*

$$G_+(f)x = \int\limits_0^\infty W^\nu_+ f(t) \int\limits_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)} C_\alpha(s)x\, ds\, dt,\ f \in \mathcal{D}_+,\ x \in E.$$

(iii) *Let $\alpha > 0$, $\tau_\alpha \in \Omega^h_\alpha$ and let $D(A)$ be dense in E. Then the following assertions are equivalent:*

- (iii.1) *A generates an α-times integrated cosine function $(C_\alpha(t))_{t\geqslant 0}$ such that $\|C_\alpha(t)\| \leqslant C\tau_\alpha(t)$, $t > 0$.*
- (iii.2) *A generates an almost-distribution cosine function G_+ of order $\alpha > 0$ and growth τ_α such that $G_+(\mathcal{D}_+)$ is dense in E.*

Suppose $a > 0$, $b > 0$, $\alpha > 0$, $M > 0$, $E^2(a,b) \subseteq \rho(A)$ and $\|R(\lambda^2 : A)\| \leqslant M(1+|\lambda|)^\alpha$, $\lambda \in E(a,b)$. Put $\tilde{\varphi}(\lambda) := \int_{-\infty}^\infty e^{\lambda t}\varphi(t)\, dt$, $\varphi \in \mathcal{D}$ and

$$G(\varphi)x := \frac{1}{2\pi i} \int\limits_\Gamma \lambda\tilde{\varphi}(\lambda) R(\lambda^2 \colon A)x\, d\lambda,\ x \in E,\ \varphi \in \mathcal{D},$$

where Γ is the upwards oriented boundary of $E(a,b)$. Then one can simply prove that G is a (DCF) generated by A. This assertion can be reformulated, with minor changes, in the case of ultradistribution sines considered in Subsections 3.5.4 and 3.6.3. Now we pay our attention to the study of mild solutions of second order abstract Cauchy problems. A function $u(t)$ is said to be a mild solution of the abstract Cauchy problem

$$(ACP_2) : u''(t) = Au(t),\ t \geqslant 0,\ u(0) = x,\ u'(0) = y,$$

iff the mapping $t \mapsto u(t)$, $t \geqslant 0$ is continuous, $\int_0^t (t-s)u(s)\,ds \in D(A)$ and $A\int_0^t (t-s)u(s)\,ds = u(t) - x - ty$, $t \geqslant 0$. Recall that there exists at most one mild solution of (ACP_2), provided that there exists $\alpha \geqslant 0$ such that A is a subgenerator of a local α-times integrated C-cosine function. Denote by $Z_2(A)$ the set which consists of those elements $x \in E$ for which there exists a solution of (ACP_2) with $y = 0$. Assume now that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that A is a subgenerator of a local n_τ-times integrated C-cosine function $(C_{n_\tau}(t))_{t\in[0,\tau)}$. Then the solution space $Z(\mathcal{A})$ consists of those pairs $(x\,,\,y)^T$ in $E \times E$ satisfying that, for every $\tau > 0$, $C_{n_\tau}(t)x + \int_0^t C_{n_\tau}(s)y\,ds \in \mathrm{R}(C)$, $t \in [0,\tau)$ and that the mapping $t \mapsto C^{-1}\big(C_{n_\tau}(t)x+\int_0^t C_{n_\tau}(s)y\,ds\big)$, $t \in [0,\tau)$ is $(n_\tau+1)$-times continuously differentiable. Moreover, the solution space $Z_2(A)$ consists exactly of those vectors $x \in E$ such that, for every $\tau > 0$, $C_{n_\tau}(t)x \in \mathrm{R}(C)$ and that the mapping $t \mapsto C^{-1}C_{n_\tau}(t)x$, $t \in [0,\tau)$ is n_τ-times continuously differentiable. If $x \in Z_2(A)$ and $t \in [0,\tau)$, then the mild solution $u(\cdot,x)$ is given by the formula $u(\cdot,x) = \frac{d^{n_\tau}}{dt^{n_\tau}}C^{-1}C_{n_\tau}(t)x$, $t \geqslant 0$. If A generates a (DCF) G and $x \in Z_2(A)$, then we also denote by $G(\delta_t)$ the above solution. Then it is easily seen that: $G(\delta_t)(Z_2(A)) \subseteq Z_2(A)$, $t \geqslant 0$, $2G(\delta_s)G(\delta_t)x = G(\delta_{t+s})x + G(\delta_{|t-s|})x$, $t,\ s \geqslant 0$ and $G(\varphi)x = \int_0^\infty \varphi(t)G(\delta_t)x\,dt$, $\varphi \in \mathcal{D}_0$. Furthermore, $\mathcal{R}(G) \subseteq Z_2(A)$. In order to see that, assume $x \in \mathcal{R}(G)$ and $x = G(\varphi)y$ for some $\varphi \in \mathcal{D}_0$ and $y \in E$. Put

$$u(t;x) := \frac{1}{2}\big[G\big(\varphi(\cdot - t)\big)y + G\big(\varphi(\cdot + t)\big)y + G\big(\varphi(t - \cdot)\big)y\big],\ \ t \geqslant 0. \tag{261}$$

Using the continuity of G, one gets that $u(\cdot;x) \in C([0,\infty) : E)$. Denote $f(t) := G\big(\varphi(\cdot - t)\big)y$, $g(t) := G\big(\varphi(\cdot + t)\big)y$ and $h(t) := G\big(\varphi(t - \cdot)\big)y$, $t \geqslant 0$. Then $f, g, h \in C^2([0,\infty) : E)$, $f'(t) = -G\big(\varphi'(\cdot-t)\big)y$, $f''(t) = G\big(\varphi''(\cdot-t)\big)y$, $g'(t) = G\big(\varphi'(\cdot+t)\big)y$, $g''(t) = G\big(\varphi''(\cdot + t)\big)y$, $h'(t) = -G\big(\varphi'(t - \cdot)\big)y$ and $h''(t) = G\big(\varphi''(t - \cdot)\big)y$, $t \geqslant 0$. The above equalities, the partial integration and the representation formula (261) taken together imply:

$$\begin{aligned}
&A\int_0^t (t-s)u(s;x)\,ds\\
&= \frac{1}{2}\int_0^t (t-s)\Big[G\big(\varphi''(\cdot-s)\big)y+G\big(\varphi''(\cdot+s)\big)y+\varphi'(s)Cy+G\big(\varphi''(s-\cdot)\big)y-\varphi'(s)Cy\Big]\,ds\\
&= \frac{1}{2}\Bigg[-\int_0^t G\big(\varphi'(\cdot-s)\big)y\,ds+\int_0^t G\big(\varphi'(\cdot+s)\big)y\,ds-\int_0^t G\big(\varphi'(s-\cdot)\big)y\,ds\Bigg] = u(t;x)-x,
\end{aligned}$$

$t \geqslant 0$, as required.

Let (N_p) and (R_p) be sequences of positive numbers which satisfy (M.1). Following Chou (cf. for example [**207**, Definition 3.9, p. 53]), we write $N_p \prec R_p$ iff, for every $\delta \in (0, \infty)$, $\sup_{p \in \mathbb{N}_0} \frac{N_p \delta^p}{R_p} < \infty$.

Now we state the following relationship between distribution cosine functions and ultradistribution semigroups. It is an extension of [**189**, Theorem 3.1] where the corresponding result is proved for the class of dense exponential distribution cosine functions.

THEOREM 3.4.39. *Suppose that a closed linear operator A generates a* (DCF)*. If (M_p) additionally satisfies $M_p \prec p!^s$, for some $s \in (1,2)$, then $\pm iA$ generate (M_p)-ultradistribution semigroups of $*$-class.*

PROOF. We will prove the assertion only for iA since the same arguments work for $-iA$. The existence of numbers α, β, $M > 0$ and $n \in \mathbb{N}$ such that $E^2(\alpha, \beta) \subseteq \rho(A)$ and that $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n$, $\lambda \in E^2(\alpha, \beta)$ is obvious. Put $\Gamma' := \partial E^2(\alpha, \beta)$ and $\Gamma := i\Gamma'$. Then it can be easily seen that $\Gamma' = \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$, where:

1. Γ'_1 is a part of the parabola $\{\eta + i\xi : \eta = \beta^2 - \frac{\xi^2}{4\beta^2}\}$; further on, Γ'_1 is contained in some compact subset of $\mathbb{C}$, 2. $\Gamma'_2 = \{t^2 - e^{2\alpha t} + 2te^{\alpha t} i : t \geqslant \beta\}$ and $\Gamma'_3 = \{t^2 - e^{2\alpha t} - 2te^{\alpha t} i : t \geqslant \beta\}$. This implies that, for every $c \in (\frac{1}{2}, 1)$,

$$\lim_{\lambda \in \Gamma,\, |\lambda| \to \infty} \frac{|\operatorname{Im} \lambda|^c}{|\operatorname{Re} \lambda|} = \infty. \tag{262}$$

It is clear that the curve Γ divides the complex plane into two disjunct open sets. Denote by Ω one of such two sets which contains a ray (ω, ∞), for some $\omega > 0$. Let $k > 0$ be fixed. Since $\Omega \subseteq \rho(iA)$ and $\|R(\cdot : iA)\|$ is polynomially bounded on Ω, the proof will be completed if one shows that there exists a suitable $C_k > 0$ with

$$\big\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant M(k|\lambda|) + C_k\big\} \subseteq \Omega, \tag{263}$$

where $M(\cdot)$ denotes the associated function of the sequence $(p!^s)$. Note, (262) implies that, for every $c \in (\frac{1}{2}, 1)$, there exists a sufficiently large $K_c > 0$ satisfying

$$\big\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant |\operatorname{Im} \lambda|^c + K_c\big\} \subseteq \Omega. \tag{264}$$

Choose an $s \in (1,2)$ with $M_p \prec p!^s$. Then an application of [**207**, Lemma 3.10] gives that there exists a constant $C_{k,s} > 0$ with $\rho^{\frac{1}{s}} \leqslant M(k\rho) + \ln C_{k,s}$, $\rho \geqslant 0$. Moreover, there exists a suitable $K_{\frac{1}{s}} > 0$ such that (264) holds with $c = \frac{1}{s}$. Now it is straightforward to see that (263) is valid with $C_k = \ln C_{k,s} + K_{\frac{1}{s}}$. Indeed, if $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant M(k|\lambda|) + \ln C_{k,s} + K_{\frac{1}{s}}$, then $\operatorname{Re} \lambda \geqslant |\lambda|^{\frac{1}{s}} + K_{\frac{1}{s}}$, and due to (264), $\lambda \in \Omega$. $\square$

Since

$$\lim_{\xi \to +\infty} \frac{\Gamma(\xi)}{\xi^{\xi - \frac{1}{2}} e^{-\xi}} = \sqrt{2\pi},$$

Gevrey's type sequence (M_p) fulfills the assumption of Theorem 3.4.39 iff $s \in (1,2)$.

Before going any further, we would like to recommend for the reader [**239**] for the basic properties of hypercyclic C-distribution cosine functions. Given a number

$\tau > 0$ and a function $K \in L^1_{\mathrm{loc}}([0,\infty))$, define the mappings $T_K : L^1[0,\tau] \to L^1[0,\tau]$ and $T'_K : L^1[0,\tau] \to L^1[0,\tau]$ by $T_K f := K *_0 f$ and $T'_K f := K \circ f$, $f \in L^1[0,\tau]$. The next generalization of Titchmarsh–Foiaş theorem has been recently proved in [**196**]:

THEOREM 3.4.40. *Let $\tau > 0$ and $K \in L^1_{\mathrm{loc}}([0,\infty))$. Then the following assertions are equivalent:*

(i) $0 \in \operatorname{supp} K$.
(ii) *The mapping T_K is injective.*
(iii) *The mapping T_K has dense range.*
(iv) *The mapping T'_K is injective.*
(v) *The mapping T'_K has dense range.*

In the remaining part of this subsection, which is of some independent interest, we will assume that $K \in L^1_{\mathrm{loc}}([0,\infty))$ and that $0 \in \operatorname{supp} K$. Set $D_K := T'_K(\mathcal{D}_+)$ and $W_K := (T'_K)^{-1}$. Notice that, in the case $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$, we have that $D_K = \mathcal{D}_+$ as well as that the operator T'_K, resp. W_K, is just the Weyl fractional integral of growth α, resp. the Weyl fractional derivative of growth α. It is known that the space D_K is invariant under differentiation and the convolution products $*_0$, $\circ$ and $*_c$.

THEOREM 3.4.41. [**196**] (i) *Let $K \in L^1_{\mathrm{loc}}([0,\infty))$, let $0 \in \operatorname{supp} K$ and let $(S_K(t))_{t\geqslant 0}$ be a global K-convoluted C-semigroup having A as a subgenerator. Define $G_K : D_K \to L(E)$ by $G_K(f)x := \int_0^\infty W_K f(t) S_K(t)x\,dt$, $f \in D_K$, $x \in E$. Then:*

(i.1) $G_K(f *_0 g)C = G_K(f)G_K(g)$, $f, g \in D_K$.
(i.2) $G_K(f)A \subseteq AG_K(f)$ *and* $AG_K(f)x = G_K(-f')x - f(0)Cx$, $f \in D_K$, $x \in E$.
(i.3) *Let $x \in D(A)$. Then* $G_K(f')x = -\int_0^\infty W_K f(t) \frac{d}{dt} S_K(t)x\,dt$, $f \in D_K$.
(i.4) *Let $L \in L^1_{\mathrm{loc}}([0,\infty))$, $0 \in \operatorname{supp} L$ and $S_{K*_0 L}(t)x = \int_0^t L(t-s)S_K(t)x\,dt$, $t \geqslant 0$, $x \in E$. Then $G_{K*_0 L}(f) = G_K(f)$, $f \in D_{K*_0 L}$.*

(ii) *Let $K \in L^1_{\mathrm{loc}}([0,\infty))$, $0 \in \operatorname{supp} K$ and let $(C_K(t))_{t\geqslant 0}$ be a global K-convoluted C-cosine function having A as a subgenerator. Define $G^c_K : D_K \to L(E)$ by $G^c_K(f)x := \int_0^\infty W_K f(t) C_K(t)x\,dt$, $f \in D_K$, $x \in E$. Then:*

(ii.1) $G^c_K(f *_c g)C = G^c_K(f)G^c_K(g)$, $f, g \in D_K$.
(ii.2) $G^c_K(f)A \subseteq AG^c_K(f)$ *and* $AG_K(f)x = G_K(f'')x + f'(0)Cx$, $f \in D_K$, $x \in E$.
(ii.3) *Let $L \in L^1_{\mathrm{loc}}([0,\infty))$, $0 \in \operatorname{supp} L$ and $C_{K*_0 L}(t)x = \int_0^t L(t-s)C_K(t)x\,dt$, $t \geqslant 0$, $x \in E$. Then $G^c_{K*_0 L}(f) = G^c_K(f)$, $f \in D_{K*_0 L}$.*

Let $\tau : [0,\infty) \to [0,\infty)$ be a locally integrable function. Then it is said that $\tau \in A_K$ iff there exists $M > 0$ such that

$$\int_0^s \tau(u)|K(r+s-u)|\,du + \int_r^{r+s} \tau(u)|K(r+s-u)|\,du \leqslant C\tau(s)\tau(r),\ 0 \leqslant s \leqslant r,$$

and that $\tau \in B_K$ iff $A_K \ni \tau$ is non-decreasing, continuous and there exist $M > 0$ such that $\int_0^t |K(s)|\,ds \leqslant M\tau(t)$, $t \geqslant 0$. Define $\|f\|_{K,\tau} := \int_0^\infty |W_K f(t)|\tau(t)\,dt$, $f \in D_K$. Then $\|\cdot\|_{K,\tau}$ is a norm on D_K and the completion of D_K in the norm $\|\cdot\|_{K,\tau}$ is denoted by $\mathcal{T}_K(\tau)$. Recall [**196**], $\mathcal{T}_K(\tau)$ is densely and continuously embedded in $L^1([0,\infty))$ if $\tau \in B_K$. The next theorem can be simply reformulated in the case $C \neq I$.

THEOREM 3.4.42. [**196**] (i) *Let $K \in L^1_{\text{loc}}([0,\infty))$, let $0 \in \operatorname{supp} K$ and let $(S_K(t))_{t\geqslant 0}$ be a global K-convoluted semigroup generated by A. Assume $\tau \in A_K$ and $\|S_K(t)\| \leqslant M\tau(t)$, $t \geqslant 0$, for some $M > 0$. Define $\Psi_K : (\mathcal{T}_K(\tau), *_0) \to L(E)$ by $\Psi_K(f)x := \int_0^\infty W_K f(t) S_K(t)x\,dt$, $f \in \mathcal{T}_K(\tau)$, $x \in E$. Then Ψ_K is a bounded algebra homomorphism.*

(ii) *Let $K \in L^1_{\text{loc}}([0,\infty))$, $0 \in \operatorname{supp} K$ and let $(C_K(t))_{t\geqslant 0}$ be a global K-convoluted cosine function generated by A. Assume $\tau \in B_K$ and $\|C_K(t)\| \leqslant M\tau(t)$, $t \geqslant 0$, for some $M > 0$. Define $\Phi_K : (\mathcal{T}_K(\tau), *_c) \to L(E)$ by $\Phi_K(f)x := \int_0^\infty W_K f(t) C_K(t)x\,dt$, $f \in \mathcal{T}_K(\tau)$, $x \in E$. Then Φ_K is a bounded algebra homomorphism.*

3.4.6. Examples. Recall, if A generates an (EDCF) then the spectrum $\sigma(A)$ of A must be contained in the parabolic domain $\{\eta + i\xi : \eta \leqslant \omega^2 - \frac{\xi^2}{4\omega^2}\}$ for some $\omega > 0$.

EXAMPLE 3.4.43. Let

$$E := \Big\{ f \in \bigcap_{k\in\mathbb{N}_0} C^k([k,\infty)) : f(0) = 0,\ \|f\|_E := \sup_{k\in\mathbb{N}_0} \sup_{t\geqslant k} |f^{(k)}(t)| < \infty \Big\}.$$

Consider now the operator

$$Af := f'', \quad D(A) := \{f \in E : f',\ f'' \in E\},$$

and suppose that A generates a (DCF). Then there are constants $\omega > 0$, $M > 0$ and $k \in \mathbb{N}$ such that $\{\lambda \in \mathbb{R} : \lambda > \omega^2\} \subseteq \rho(A)$ and $\|R(\lambda^2 : A)\| \leqslant M\lambda^k$, $\lambda > \omega$. Choose

$$g(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2} \\ -2x + 2, & \frac{1}{2} < x < 1 \\ 0, & x \geqslant 1. \end{cases}$$

Then $\|g\|_E = 1$ and

$$\begin{aligned} M\lambda^{k+1} &\geqslant \|\lambda R(\lambda^2 : A) g\|_E = \|\sinh(\lambda t) *_0 g\|_E \\ &\geqslant \frac{1}{2}\sup_{t\geqslant 0}\Big|\int_0^t \big(e^{\lambda(t-s)} - e^{-\lambda(t-s)}\big) g(s)\,ds\Big| \geqslant \sup_{t\in[0,\frac{1}{2}]}\Big|\int_0^t \big(e^{\lambda(t-s)} - e^{-\lambda(t-s)}\big) s\,ds\Big| \\ &\geqslant \sup_{t\in[0,\frac{1}{2}]}\Big|\frac{-2t}{\lambda} + \frac{e^{\lambda t} - e^{-\lambda t}}{\lambda^2}\Big| \geqslant \Big|\frac{-1}{\lambda} + \frac{e^{\frac{\lambda}{2}} - e^{\frac{-\lambda}{2}}}{\lambda^2}\Big| \geqslant \frac{e^{\frac{\lambda}{2}}}{\lambda^2} - \frac{1}{\lambda} - \frac{e^{\frac{-\lambda}{2}}}{\lambda^2},\ \ \lambda > \omega. \end{aligned}$$

Hence, A does not generate a (DCF). Note that operator $-d/dx$ with maximal domain in E is the generator of a (DS) in E (cf. [**252**, Example 3.5]).

EXAMPLE 3.4.44. Let $E := L^p((0,\infty))$, $1 \leqslant p \leqslant \infty$, $m(x) := (x+ie^x)^2$, $x > 0$, $(Af)(x) := m(x)f(x)$, $D(A) := \{f \in E : mf \in E\}$. Clearly, $\{x + ie^x : x > 0\} \cap \tilde{\Lambda}_{1,1} = \emptyset$. Denote $d := \operatorname{dist}(\{\pm(x+ie^x) : x > 0\}, \partial\tilde{\Lambda}_{1,1})$. Then $d > 0$, $\tilde{\Lambda}^2_{1,1} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant \frac{1}{d^2}, \lambda \in \tilde{\Lambda}^2_{1,1}$. Therefore, A generates a (DCF). Since $\sigma(A) = \{(x + ie^x)^2 : x > 0\}$, we have $\sigma(A) \cap \Pi_\omega \neq \emptyset$ for all $\omega > 0$. Thus, A does not generate an (EDCF). Moreover, one can easily see that A generates a local once integrated cosine function $(C_1(t))_{t\in[0,1)}$ on E which is given by

$$(C_1(t)f)(x) := \frac{\sinh((x+ie^x)t)f(x)}{x+ie^x}, \quad 0 \leqslant t < 1,\ x > 0,\ f \in E.$$

It is clear that $(C_1(t))_{t\in[0,1)}$ can be extended on $[0,1]$ and that $\sup_{t\in[0,1]} \|C_1(t)\| \leqslant 1$. However, A does not generate a local sine function on $[0,\tau)$ for any $\tau > 1$.

EXAMPLE 3.4.45. Let us consider now Hardy spaces of holomorphic functions in the upper half-plane. Denote $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Then the Hardy space $H^p(\mathbb{R}^2_+)$, $1 \leqslant p < \infty$, is defined as the space of all holomorphic functions defined on $\mathbb{R}^2_+$ such that

$$\|F\|_{H^p(\mathbb{R}^2_+)} := \Bigg(\sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^p dx\Bigg)^{1/p} < \infty,$$

for all $F \in H^p(\mathbb{R}^2_+)$. Let $B(\cdot)$ be a holomorphic function on $\mathbb{R}^2_+$ with $B(\mathbb{R}^2_+) \subseteq \{\eta + i\xi : \eta \leqslant \omega^2 - \frac{\xi^2}{4\omega^2}\}$, for some $\omega > 0$, and

$$(AF)(z) := B(z)F(z),\ \operatorname{Im} z > 0, D(A) := \{F \in H^p(\mathbb{R}^2_+) : AF \in H^p(\mathbb{R}^2_+)\}.$$

One can simply verify that $\Pi_{\omega+1} \subseteq \rho(A)$ and $\|\lambda R(\lambda^2 : A)\| \leqslant \frac{|\lambda|}{2\omega+1}$, $\operatorname{Re}\lambda > \omega + 1$. The last estimate implies that, for every $\alpha > 2$, the operator A is the generator of an exponentially bounded α-times integrated cosine function. Thus, A generates an (EDCF) in $H^p(\mathbb{R}^2_+)$. Particularly, one can take

$$B(z) = \Big(\frac{1}{\pi i}\ln\frac{z-1}{z+1} + a\Big)^2,\ \operatorname{Im} z > 0\ (a \in \mathbb{C}),\ \text{or}\ B(z) = -\ln^2 z,\ \operatorname{Im} z > 0,$$

where $\ln z = \ln|z| + i\arg(z)$, $z \in \mathbb{C} \setminus \{0\}$.

EXAMPLE 3.4.46. Let E be an arbitrary Banach space, $P \in L(E)$ and $P^2 = P$. Define $G(\varphi)x := \int_0^\infty \varphi(t)\,dt\,Px$, $x \in E, \varphi \in \mathcal{D}$. Then $G^{-1}(\varphi)x = \int_0^\infty t\varphi(t)\,dt\,Px$, $x \in E$, $\varphi \in \mathcal{D}$, and

$$\int_0^\infty t\varphi(t)\,dt \int_0^\infty \psi(s)s\,ds + \int_0^\infty \varphi(t)\,dt \int_0^\infty s\psi(s)\,ds = \int_0^\infty\int_0^\infty (t+s)\varphi(t)\psi(s)\,ds\,dt$$
$$= \int_0^\infty\int_0^u u\varphi(u-v)\psi(v)\,dv\,du = \int_0^\infty u\int_0^u \varphi(u-v)\psi(v)\,dv\,du = \int_0^\infty u(\varphi *_0 \psi)(u)\,du,$$

for all φ, $\psi \in \mathcal{D}$. Hence, G is a pre-(DCF) in E, and

$$\{x,y\} \subseteq \operatorname{Kern}(P) \Leftrightarrow G(\varphi)x + G^{-1}(\varphi)y = 0 \text{ for all } \varphi \in \mathcal{D}_0.$$

Note also that G is a pre-(DSG) in E satisfying $\mathcal{N}(G) = \mathrm{Kern}(P)$.

The next illustrative example shows that Theorem 3.4.39 does not hold in the case of a general sequence (M_p).

EXAMPLE 3.4.47. Let $E := L^p(\mathbb{R})$, $1 \leqslant p < \infty$ and $m(x) := (1-\frac{x^2}{4})+ix$, $x \in \mathbb{R}$. Define a closed linear operator A on E by $Af(x) =: m(x)f(x)$, $x \in \mathbb{R}$, $D(A) := \{f \in E : mf \in E\}$. Then it is obvious that A generates a dense exponential (DCF) and that $\sigma(iA) = \{x + (1 - \frac{x^2}{4})i : x \in \mathbb{R}\}$. Suppose now $M_p = p!^2$. We will show that iA generates an ultradistribution semigroup of the Beurling class and that iA is not the generator of an ultradistribution semigroup of the Roumieu class. First of all, we know that there exist constants $\omega > 0$, $a > 0$ and $b > 0$ with $a\rho^{1/2} \leqslant M(\rho)$, $\rho \geqslant \omega$ and $M(\rho) \leqslant b\rho^{1/2}$, $\rho \geqslant 0$. The validity of above statements immediately follows if we prove that (cf. the next section for further information):

$$\partial\Omega_{k,C} \cap \sigma(iA) = \emptyset, \text{ for every } k \in (4/a^2, \infty) \text{ and a sufficiently large } C > 0, \tag{265}$$

$$\partial\Omega_{k,C} \cap \sigma(iA) \neq \emptyset, \text{ for every } k \in (0, 4/b^2) \text{ and } C > 0. \tag{266}$$

Let $k \in (\frac{4}{a^2}, \infty)$. Let $C \geqslant \frac{\omega}{k}$. In order to prove that (265) holds, note that: $\eta + i\xi \in \partial\Omega_{k,C} \Rightarrow \eta \geqslant C$, $k\sqrt{\eta^2 + \xi^2} \geqslant k\eta \geqslant kC \geqslant \omega$. Hence, $\eta = M\big(k\sqrt{\eta^2+\xi^2}\,\big) + C \geqslant a\sqrt{k}\sqrt[4]{\eta^2+\xi^2} + C$. This estimate ensures that, for a sufficiently large $C > 0$, the curve $\partial\Omega_{k,C}$ lies above the graph of the function $f(\eta) = -\sqrt{\frac{(\eta-C)^4}{a^4k^2} - \eta^2}$; moreover, $f(\eta) \sim -\frac{\eta^2}{a^2k}$, $\eta \to +\infty$. Therefore, the choice of k implies that there exists a suitable $\beta > 0$ such that the part of the parabola $\xi = -\frac{\eta^2}{a^2k}$, $\eta \geqslant \beta$ has the empty intersection with $\sigma(iA)$. This immediately implies (265), while (266) follows similarly from the fact that, for every $k \in (0, \frac{4}{b^2})$ and $C > 0$, the interior of the parabola $\eta = -\frac{\xi^2}{b^2k}$ is strictly contained in that of $\xi = -\frac{\eta^2}{4}$ and that, for $\eta + i\xi \in \partial\Omega_{k,C}$, we have $\eta = M\big(k\sqrt{\eta^2+\xi^2}\,\big) + C \leqslant b\sqrt{k}\sqrt[4]{\eta^2+\xi^2} + C$. At the end of this analysis, we point out that the implication: G is an ultradistribution fundamental solution of $*$-class $\Rightarrow \bigcap_{\varphi\in\mathcal{D}_0^*} \mathrm{Kern}(G(\varphi)) = \{0\}$, is not true in general (see [**59**]). In the case of densely defined operators, the concept of regular ultradistribution semigroups of Beurling class was introduced by Ciorănescu in [**59**] for this purpose. An application of [**59**, Proposition 2.6] gives that the operator iA, considered above, generates a regular ultradistribution semigroup of (p^{2p})-class G (cf. [**59**] and the next section for the notion). Similarly, if $M_p = p!^s$, $s > 2$, then it can be proved that iA does not generate an ultradistribution semigroup of Beurling, resp., Roumieu class. The same assertions hold for $-iA$.

3.5. Ultradistribution and hyperfunction semigroups

Unless stated otherwise, we assume in this section that (M_p) satisfies (M.1), (M.2) and (M.3′). The use of condition (M.3) will be explicitly emphasized.

3.5.1. The structural properties of ultradistribution semigroups. We define L-ultradistribution semigroups following [**282**] and ultradistribution semigroups following [**252**] and [**418**].

Definition 3.5.1. Let $G \in \mathcal{D}_0'^*(L(E))$. Then it is said that G is an *L-ultradistribution semigroup of $*$-class* iff:

(U.1) $G(\phi * \psi) = G(\phi)G(\psi)$, ϕ, $\psi \in \mathcal{D}_0^*$,
(U.2) $\mathcal{N}(G) := \bigcap_{\phi \in \mathcal{D}_0^*} \mathrm{Kern}(G(\phi)) = \{0\}$,
(U.3) $\mathcal{R}(G) := \bigcup_{\phi \in \mathcal{D}_0^*} \mathrm{R}(G(\phi))$ is dense in E, and
(U.4) For every $x \in \mathcal{R}(G)$ there exists a function $u \in C([0,\infty) : E)$ satisfying $u(0) = x$ and $G(\phi)x = \int_0^\infty \phi(t)u(t)\,dt$, $\phi \in \mathcal{D}^*$.

The continuity of mapping $*_0 : \mathcal{D}^* \times \mathcal{D}^* \to \mathcal{D}^*$ is obvious and the continuity properties of $*_0$ remain similar to those of $*$. This justifies the next definition of a (non-dense) ultradistribution semigroup.

Definition 3.5.2. Let $G \in \mathcal{D}_0'^*(L(E))$. If G satisfies

$$\text{(U.5)} \qquad G(\phi *_0 \psi) = G(\phi)G(\psi), \quad \phi,\, \psi \in \mathcal{D}^*,$$

then it is said that G is a *pre*-(UDS) of $*$-class. If (U.5) and (U.2) are fulfilled for G, then G is said to be an *ultradistribution semigroup of $*$-class,* in short, (UDS). We say that a pre-(UDS) is *dense* if G additionally satisfies (U.3).

If $G \in \mathcal{D}_0'^*(L(E))$, then the condition:

$$\text{(U.2}'\text{)} \qquad \operatorname{supp} G(\cdot)x \nsubseteq \{0\} \text{ for every } x \in E \smallsetminus \{0\},$$

is equivalent to (U.2). This follows from the fact that

$$0 \neq x_0 \in \bigcap_{\phi \in \mathcal{D}_0^*} \{x \in E : \ G(\phi)x = 0\} \Longleftrightarrow \operatorname{supp} G(\cdot)x_0 = \{0\}.$$

As in the case of distribution semigroups, if (U.3) holds, then:

$$[(\text{U.1}) \wedge (\text{U.2}) \wedge (\text{U.4})] \Longleftrightarrow [(\text{U.5}) \wedge (\text{U.2})].$$

It is clear that if G is a pre-(UDS) of $*$-class, then $\mathcal{N}(G)$ is a closed subspace of E. The next example is an insignificant modification of [**252**, Example 2.3].

Example 3.5.3. Let E be a Banach space and let T be a bounded linear operator on E such that there exist $C > 0$ and $L > 0$, in the Beurling case, resp., for every $L > 0$ there exists $C > 0$, in the Roumieu case, with $\|T^{p+1}\| \leqslant C\frac{L^p}{M_p}$, $p \in \mathbb{N}_0$. Define $G(\phi) := \sum_{p=0}^\infty \phi^{(p)}(0)T^{p+1}$, $\phi \in \mathcal{D}^*$. Then G is a pre-(UDS) of $*$-class satisfying $\mathcal{N}(G) = E$. The verification of this fact is left to the reader. Note that we do not require that T is a nilpotent operator as in [**252**, Example 2.3]. The concrete construction in the Beurling case goes as follows. Let $E := l_\infty$ and $T(\langle x_p \rangle) := \langle x_{p+1}/m_p \rangle$, $\langle x_p \rangle \in l_\infty$. Since (m_p) is increasing, it is straightforward to see that $\|T^p\| = 1/M_p$, $p \in \mathbb{N}_0$. Define G as above. Then G is a pre-(UDS) of the Beurling class and T is not a nilpotent operator. The corresponding example for the Roumieu case can be constructed similarly.

Borel's type theorem for ultradifferentiable functions (cf. [**360**]) implies that, for every complex sequence (a_n) such that $a_n = 0$, $n \geqslant n_0$, there exists an $f \in \mathcal{D}^*$ satisfying $f^{(n)}(0) = a_n$, $n \in \mathbb{N}_0$. This allows us to obtain the structural characterization of a pre-(UDS) of $*$-class on its kernel space $\mathcal{N}(G)$.

PROPOSITION 3.5.4. *Assume additionally that* (M.3) *holds. Let* $\mathcal{G}$ *be a pre-*(UDS) *of* $*$*-class and* $G := \mathcal{G}(\cdot)|_{\mathcal{N}(\mathcal{G})}$. *Then* G *is a pre-*(UDS) *of* $*$*-class on* $\mathcal{N}(\mathcal{G})$ *and there exists an operator* $T \in L(\mathcal{N}(\mathcal{G}))$ *such that there exist* $C > 0$ *and* $L > 0$ *in the Beurling case, resp., for every* $L > 0$ *there exists a suitable* $C > 0$*, in the Roumieu case, such that*

$$\|T^{j+1}\| \leqslant C\frac{L^j}{M_j},\ j \in \mathbb{N}_0 \ \text{ and } G = \sum_{j=0}^{\infty} \delta^{(j)} \otimes (-1)^j T^{j+1}.$$

PROOF. It is clear that (U.5) implies that $\mathcal{N}(\mathcal{G})$ is invariant under G and that G is a pre-(UDS) of $*$-class on $\mathcal{N}(\mathcal{G})$ with $\mathcal{N}(\mathcal{G}) = \mathcal{N}(G)$. Moreover, $\operatorname{supp} G \subseteq \{0\}$ and an application of Theorem 1.3.5(ii) yields that there exists a sequence $(T_j)_{j\in\mathbb{N}_0}$ in $L(E)$ such that there exist $C > 0$ and $L > 0$ in the Beurling case, resp., for every $L > 0$ there exists $C > 0$ in the Roumieu case, satisfying

$$\|T_j\| \leqslant C\frac{L^j}{M_j},\ j \in \mathbb{N}_0 \ \text{ and } G = \sum_{j=0}^{\infty} \delta^{(j)} \otimes T_j.$$

Because of (U.5) and $(\phi *_0 \varphi)^{(j)}(0) = \sum_{k=0}^{j-1} \phi^{(k)}(0)\varphi^{(j-1-k)}(0)$, $j \in \mathbb{N}$, one obtains

$$\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}(-1)^j\phi^{(k)}(0)\varphi^{(j-1-k)}(0)T_jx = \sum_{j,\,k=0}^{\infty}(-1)^{j+k}\phi^{(j)}(0)\varphi^{(k)}(0)T_jT_kx,$$

for all $x \in \mathcal{N}(G)$ and $\phi,\ \varphi \in \mathcal{D}^*$. Choose $\phi \in \mathcal{D}^*$ with $\phi(0) = 1$ and $\phi^{(j)}(0) = 0$, $j \in \mathbb{N}$, to obtain

$$\sum_{j=1}^{\infty}(-1)^j\varphi^{(j-1)}(0)T_jx = \sum_{j=0}^{\infty}(-1)^j\varphi^{(j)}(0)T_0T_jx,\ \ x \in \mathcal{N}(G),\ \varphi \in \mathcal{D}^*.$$

One can choose a sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ in $\mathcal{D}^*$ with $\varphi_k^{(j)}(0) = \delta_{jk}$, $j,\ k \in \mathbb{N}_0$ to conclude that $T_k = (-1)^k T_0^{k+1}$, $k \in \mathbb{N}$. This proves the proposition. □

Let G be a (UDS) of $*$-class and let $T \in \mathcal{E}_0'^*$. Then we define $G(T)$ as in the case of distribution semigroups:

$$G(T) = \{(x, y) \in E \times E : G(T * \phi)x = G(\phi)y \text{ for all } \phi \in \mathcal{D}_0^*\}.$$

Clearly, $G(\delta) = I$ and $G(T)$, $T \in \mathcal{E}_0'^*$ is a closed linear operator. The generator of G is defined by $A := G(-\delta')$.

Since for $\varphi \in \mathcal{D}_0^*$, $\varphi_+ := \varphi\mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'^*$, the definition of $G(\varphi_+)$ is clear.

Notice, the adjoint of a pre-(UDS) of $*$-class G, denoted by G^*, is also a pre-(UDS) of $*$-class.

THEOREM 3.5.5. *Let* G *be a pre-*(UDS) *of* $*$*-class,* $F := E/\mathcal{N}(G)$ *and* q *be the corresponding canonical mapping* $q : E \to F$.

(i) *Let* $H \in L(\mathcal{D}^*, L(F))$ *be defined by* $qG(\varphi) := H(\varphi)q$ *for all* $\varphi \in \mathcal{D}^*$. *Then* H *is a* (UDS) *of* $*$*-class in* F.

(ii) $\overline{\langle\mathcal{R}(G)\rangle} = \overline{\mathcal{R}(G)}$, *where* $\langle\mathcal{R}(G)\rangle$ *denotes the linear span of* $\mathcal{R}(G)$.

(iii) *Assume that* G *is not dense. Put* $R := \overline{\mathcal{R}(G)}$ *and* $H := G_{|R}$. *Then* H *is a dense pre-*(UDS) *of* $*$*-class in* R.
(iv) *The adjoint* G^* *of* G *is a pre-*(UDS) *of* $*$*-class on* E^* *with* $\mathcal{N}(G^*) = \overline{\mathcal{R}(G)}^{\circ}$.
(v) *If* E *is reflexive, then* $\mathcal{N}(G) = \overline{\mathcal{R}(G^*)}^{\circ}$.
(vi) G^* *is a* (UDS) *of* $*$*-class on* E^* *iff* G *is a dense pre-*(UDS) *of* $*$*-class. If* E *is reflexive, then* G^* *is a dense pre-*(UDS) *of* $*$*-class on* E^* *iff* G *is a* (UDS) *of* $*$*-class.*
(vii) G *is a* (UDS) *of* $*$*-class iff* (U.1), (U.2) *and* (U.6) *hold, where*
(U.6) $\quad G(\varphi_+) = G(\varphi)$, $\varphi \in \mathcal{D}^*$.
(viii) $\mathcal{N}(G) \cap \langle \mathcal{R}(G) \rangle = \{0\}$.

THEOREM 3.5.6. *Let* G *be a* (UDS) *of* $*$*-class and let* S, $T \in \mathcal{E}_0'^*$, $\varphi \in \mathcal{D}_0^*$, $\psi \in \mathcal{D}^*$, $x \in E$. *Then the following holds:*

(i) $(G(\varphi)x, G(\overbrace{T * \cdots * T}^{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.
(ii) $G(S)G(T) \subseteq G(S * T)$ *with* $G(S)G(T) = D(G(S * T)) \cap D(G(T))$, *and* $G(S) + G(T) \subseteq G(S + T)$.
(iii) $(G(\psi)x, G(-\psi')x - \psi(0)x) \in G(-\delta')$.
(iv) *If* G *is dense, its generator is densely defined.*

Before proceeding further, we would like to emphasize that [**59**, Lemma 2.7] implies that every (M_p)-ultradistribution semigroup G of [**54**] is also a pre-(UDS) of the Beurling class. Using Theorem 3.5.5, we are in a position to introduce the generator of a pre-(UDS) of $*$-class.

DEFINITION 3.5.7. Let G be a pre-(UDS) of $*$-class in E. Then A is the *generator of* G if A is the generator of a (UDS) H in $F = E/\mathcal{N}(G)$, which is given in Theorem 3.5.5(i).

The case in which G is a (UDS) of $*$-class is not excluded: one can simply identify E with F. The generator of a pre-(UDS) of $*$-class is a closed linear operator from F into F. Our definition is slightly different from the definition of the generator of a pre-(DS) given in [**199**]. In fact, following Definition 2 and Corollary 1 of [**199**], the generator of a pre-(DS) in E is a closed linear operator from E into F.

DEFINITION 3.5.8. Let D be another Banach space and let $P \in \mathcal{D}_0'^*(L(D, E))$. Then $G \in \mathcal{D}_0'^*(L(E, D))$ is an *ultradistribution fundamental solution for* P if

$$P * G = \delta \otimes I_E \text{ and } G * P = \delta \otimes I_D.$$

As in the case of distributions, an ultradistribution fundamental solution for $P \in \mathcal{D}_0'^*(L(D, E))$ is uniquely determined. This, combined with the next theorem, implies that every (UDS) is uniquely determined by its generator.

THEOREM 3.5.9. *Let* A *be a closed operator in* E. *If* A *generates a* (UDS) G *of* $*$*-class, then* G *is an ultradistribution fundamental solution for*

$$P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{D}_0'^*\big(L([D(A)], E)\big).$$

In particular, if $T \in \mathcal{D}_0'^*(E)$, then $u = G * T$ is a unique solution of the problem

$$-Au + \frac{\partial}{\partial t}u = T,\ u \in \mathcal{D}_0'^*([D(A)]). \tag{267}$$

Furthermore, $\operatorname{supp} T \subseteq [\alpha, \infty)$ *implies* $\operatorname{supp} u \subseteq [\alpha, \infty)$*, where $\alpha \geqslant 0$. Conversely, if $G \in \mathcal{D}_0'^*(L(E, [D(A)]))$ is an ultradistribution fundamental solution for P, then G is a pre-*(UDS) *of $*$-class in E and the generator of G is the closure of the operator* $\mathrm{A} := \{(q(x), q(y)) : (x, y) \in A\}$.

Proof. If A generates a (UDS) G of $*$-class, then Theorem 3.5.6(iii) implies that G is an ultradistribution fundamental solution for P, and consequently, $u = G*T$ is a unique solution of (267). Clearly, if $\operatorname{supp} T \subseteq [\alpha, \infty)$, then $\operatorname{supp} u \subseteq [\alpha, \infty)$. Let $G \in \mathcal{D}_0'^*(L(E, [D(A)]))$ be an ultradistribution fundamental solution for P. Using the same arguments as in the proof of [**252**, Theorem 3.10], we have that G is a pre-(UDS) of $*$-class in E. We will only prove that the generator of G is the closure of A. First of all, let us prove that A is a closable operator. Let (x_n) be a sequence in $D(A)$ such that $q(x_n) \to 0$ and $\mathrm{A}(q(x_n)) \to q(y)$, $n \to \infty$, for some $y \in E$. These assumptions imply the existence of a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that

$$\inf_{z \in \mathcal{N}(G)} \|x_{n_k} + z\| < 1/k \text{ and } \inf_{z \in \mathcal{N}(G)} \|Ax_{n_k} - y + z\| < 1/k, \quad k \in \mathbb{N}.$$

Hence, there exist two sequences $(z_k)_k$ and $(z_k^1)_k$ in $\mathcal{N}(G)$ satisfying

$$\|x_{n_k} + z_k\| < 1/k \text{ and } \|Ax_{n_k} - y + z_k^1\| < 1/k,\ k \in \mathbb{N}.$$

Let $\phi \in \mathcal{D}_0^*$ be fixed. Since G is an ultradistribution fundamental solution for P, one has $G(\phi)Ax_n = -G(\phi')x_n$, $n \in \mathbb{N}$. Then we obtain:

$$\begin{aligned} \|-G(\phi)y\| &= \|G(\phi)(Ax_{n_k} - y + z_k^1) - G(\phi)(Ax_{n_k} + z_k^1)\| \\ &\leqslant \|G(\phi)\|/k + \|G(\phi)Ax_{n_k}\| = \|G(\phi)\|/k + \|G(-\phi')(x_{n_k} + z_k)\| \\ &\leqslant \|G(\phi)\|/k + \|G(\phi')\|/k, \quad k \in \mathbb{N}. \end{aligned}$$

Letting $k \to \infty$, we obtain that $q(y) = 0$ and that A is a closable linear operator in F. Suppose that A_1 generates G. If $(q(x), q(y))$ belongs to the closure of A for some x, $y \in E$, then there exists a sequence $((x_n, y_n))_n$ in A such that $(q(x_n), q(y_n)) \to (q(x), q(y))$, $n \to \infty$, in $F \times F$. Applying the same arguments as above, we get that there exist a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and two sequences $(z_k)_k$ and $(z_k^1)_k$ in $\mathcal{N}(G)$ such that

$$\|x_{n_k} - x + z_k\| < 1/k \text{ and } \|y_{n_k} - y + z_k^1\| < 1/k,\ k \in \mathbb{N}.$$

Let $\phi \in \mathcal{D}_0^*$ be fixed. If $\varphi \in \mathcal{D}_0^*$, then

$$\begin{aligned} &\big\|G(\varphi)\big(G(-\phi')x - G(\phi)y\big)\big\| \\ &= \big\|G(\varphi)[G(\phi')(x_{n_k} - x + z_k) - G(\phi')(x_{n_k} + z_k) + G(\phi)(y_{n_k} - y + z_k^1) - G(\phi)(y_{n_k} + z_k^1)]\big\| \\ &= \big\|G(\varphi)[G(\phi')(x_{n_k} - x + z_k) + G(\phi)(y_{n_k} - y + z_k^1)]\big\| \\ &\leqslant \big(\|G(\varphi *_0 \phi)\| + \|G(\varphi *_0 \phi')\|\big)/k, \quad k \in \mathbb{N}. \end{aligned}$$

Thus, $G(-\phi')x - G(\phi)y \in \mathcal{N}(G)$, i.e., $H(-\phi')q(x) = H(\phi)q(y)$, $\phi \in \mathcal{D}_0^*$, and A_1 contains the closure of A. This implies that $\mathcal{D}_0'^*([D(\overline{\mathrm{A}})])$ is isomorphic to a subspace of $\mathcal{D}_0'^*([D(\mathrm{A}_1)])$. The first part of the proof gives that H is an ultradistribution fundamental solution for $P := \delta' \otimes I_{D(\mathrm{A}_1)} - \delta \otimes \mathrm{A}_1$. Applying again the arguments given in the final part of the proof of [**252**, Theorem 3.10], we obtain $D(\mathrm{A}_1) = D(\overline{\mathrm{A}})$. The proof is completed. □

COROLLARY 3.5.10. *Let $G \in \mathcal{D}_0'^*(L(E, [D(A)]))$. Then G is an ultradistribution fundamental solution for $P = \delta' \otimes I_{[D(A)]} - \delta \otimes A \in \mathcal{D}_0'^*(L([D(A)], E))$ and $\mathcal{N}(G) = \{0\}$ iff G is an* (UDS) *of $*$-class generated by A.*

REMARK 3.5.11. In the case of distribution semigroups, if G is a fundamental solution for P in the sense of [**252**, Definition 3.9], then G is a (DS), i.e., the following also holds: $\mathcal{N}(G) = \{0\}$. Generally, it is not true in the case of ultradistribution semigroups. There exist a Banach space E, a closed linear operator A in E and an ultradistribution fundamental solution G for P such that $\mathcal{N}(G) \neq \{0\}$; see, for example, [**64**, p. 156].

QUESTION. If G is an ultradistribution fundamental solution for P, and simultaneously, a (UDS) of $*$-class, then it can be proved that the operator A defined above is closed. Is it true if G is just an ultradistribution fundamental solution for P?

THEOREM 3.5.12. *Assume that* (M.3) *holds. Let $T \in \mathcal{D}_0'^*(E)$ and let A be a closed, densely defined operator. Assume, further, that the equation*

$$-Au + \frac{\partial}{\partial t}u = T,\ u \in \mathcal{D}_0'^*([D(A)])$$

has a unique solution depending continuously on T so that if $\operatorname{supp} T \subseteq [\alpha, \infty)$*, then* $\operatorname{supp} u \subseteq [\alpha, \infty)$*. Moreover, assume that for $T = \delta$ the corresponding ultradistribution fundamental solution u satisfies* $\operatorname{supp} u(\cdot, x) \nsubseteq \{0\}$*, $x \in E \setminus \{0\}$* (cf. (U.2′))*. Then A is the generator of an L-ultradistribution semigroup of $*$-class.*

OUTLINE OF THE PROOF. The proof is similar to that of [**282**, Theorem 5.1]. Since the mapping $H : u \mapsto -Au + \frac{\partial}{\partial t}u$ is an isomorphism of $\mathcal{D}_0'^*([D(A)])$ onto $\mathcal{D}_0'^*(E)$ which commutes with translations, one can prove that H is a convolution operator, i.e., that there exists $G \in \mathcal{D}_0'^*(L(E, [D(A)]))$ such that $H(T) = G * T$. Using this fact and Theorem 1.3.5(ii), one can repeat literally the proof of Lions given in [**282**]. The essential change concerning the proof of [**282**, Theorem 5.1] is related to the proof of (U.2) for the solution u of the equation $-Au + \frac{\partial}{\partial t}u = \delta$ (cf. [**282**, Part 5, p. 152]). Clearly, (U.2) is a consequence of the assumption $\operatorname{supp} u(\cdot, x) \nsubseteq \{0\}$, $x \in E \setminus \{0\}$. □

3.5.2. Exponential ultradistribution semigroups.

DEFINITION 3.5.13. (i) Let D be a Banach space and let $P \in \mathcal{D}_0'^*(L(D, E))$. Then we say that $G \in \mathcal{D}_0'^*(L(E, D))$ is an *exponential ultradistribution fundamental solution for P* if G is an ultradistribution fundamental solution for P and if there exists $\omega \geqslant 0$ such that $e^{-\omega t}G \in \mathcal{S}'^*(L(E))$.

(ii) Suppose G is a (UDS) of $*$-class. Then it is said that G is an *exponential ultradistribution semigroup* of $*$-class, in short (EUDS), iff there exists $\omega \geqslant 0$ such that $e^{-\omega t}G \in \mathcal{S}'^{*}(L(E))$.

THEOREM 3.5.14. *Suppose A is a closed linear operator. Then there exists an exponential ultradistribution fundamental solution of $*$-class for A iff there exist $a \geqslant 0$, $k > 0$ and $L > 0$, in the Beurling case, resp., there exists $a \geqslant 0$ such that, for every $k > 0$ there exists $L_k > 0$, in the Roumieu case, such that:*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a\} \subseteq \rho(A), \tag{268}$$

$$\|R(\lambda\colon A)\| \leqslant Le^{M(k|\lambda|)},\ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > a,\ \textit{resp.,} \tag{269}$$

$$\|R(\lambda\colon A)\| \leqslant L_k e^{M(k|\lambda|)}\ \textit{ for all } k > 0 \textit{ and } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re}\lambda > a. \tag{270}$$

PROOF. We will prove the assertion only in the Beurling case since the proof is quite similar in the Roumieu case. Suppose first that G is an exponential ultradistribution fundamental solution of (M_p)-class for A and that $e^{-\omega t}G \in \mathcal{S}'^{(M_p)}(L(E))$ for some $\omega \geqslant 0$. Let $h \in \mathcal{E}^{(M_p)}(\mathbb{R})$, $r > 0$ and let $h(t) = 0$, $t < -r$ and $h(t) = 1$, $t \geqslant 0$. Then the function $t \mapsto h(t)e^{(\omega-\lambda)t}$, $t \in \mathbb{R}$ belongs to the space $\mathcal{S}^{(M_p)}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. This implies that the definition of $\tilde{G}(\lambda) := G(h(t)e^{-\lambda t}) := G\big(e^{-\omega t}(h(t)e^{(\omega-\lambda)t})\big)$ is meaningful for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. Since $e^{-\omega t}G$ is an ultradistribution fundamental solution for $A - \omega$, we get that $(A-\omega)G(e^{-\omega t}\varphi)x = G(-e^{-\omega t}\varphi')x - \varphi(0)x$, $x \in E$, $\varphi \in \mathcal{D}^{(M_p)}$. The continuity of $e^{-\omega t}G$ on $\mathcal{S}^{(M_p)}$, and the denseness of $\mathcal{D}^{(M_p)}$ in $\mathcal{S}^{(M_p)}$, imply that the previous equality holds for all $\varphi \in \mathcal{S}^{(M_p)}$. Then a simple computation with $\varphi(t) = h(t)e^{(\omega-\lambda)t}$, $t \in \mathbb{R}$, and the obvious inclusion $\operatorname{supp} G \subseteq [0,\infty)$, imply:

$$A\tilde{G}(\lambda)x = \lambda\tilde{G}(\lambda)x - x,\ x \in E,\ \operatorname{Re}\lambda > \omega.$$

Hence, $(\lambda - A)\tilde{G}(\lambda)x = x$, $x \in E$, $\operatorname{Re}\lambda > \omega$ and, since $\tilde{G}(\lambda)A \subseteq A\tilde{G}(\lambda)$, $\operatorname{Re}\lambda > \omega$, $\tilde{G}(\lambda)(\lambda - A)x = x$, $x \in D(A)$, $\operatorname{Re}\lambda > \omega$. This implies (268) with $a = \omega$ and $R(\lambda : A)x = \tilde{G}(\lambda)x$, $x \in E$, $\operatorname{Re}\lambda > a$. It is obvious that there exists $h > 0$ such that $\|\tilde{G}(\lambda)\| = \|(e^{-\omega t}G)\big(h(t)e^{(\omega-\lambda)t}\big)\| \leqslant \|h(t)e^{(\omega-\lambda)t}\|_{M_p,h}$ provided $\operatorname{Re}\lambda > \omega$. Then the estimate (269) follows easily from the preceding inequality, which ends the proof of necessity.

To prove sufficiency, suppose $\beta > a$, $l \geqslant \max(k,1)$, $\alpha > 0$ and put $\tilde{\varphi}(\lambda) := \int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)\,dt$, $\lambda \in \mathbb{C}$, $\varphi \in \mathcal{D}^{(M_p)}$ and $G(\varphi) := \frac{1}{2\pi i}\int_{\Gamma_l} \tilde{\varphi}(\lambda)R(\lambda\colon A)\,d\lambda$, $\varphi \in \mathcal{D}^{(M_p)}$, where Γ_l denotes the upwards oriented boundary of the ultra-logarithmic region $\Lambda_{\alpha,\beta,l} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha M(l|\operatorname{Im}\lambda|) + \beta\}$. Proceeding as in [**59**], [**210**] and [**307**, Section 2.3], one gets that G is an ultradistribution fundamental solution for A. By the Paley–Wiener type theorem (cf. [**59**, Theorem 1.1] and [**207**]) and Cauchy theorem, we can deform the path of integration Γ_l to the straight line connecting $\bar{a} - i\infty$ and $\bar{a} + i\infty$, where $\bar{a} \in (a,\beta)$. Hence,

$$G(\varphi) = \frac{1}{2\pi i}\int\limits_{\bar{a}-i\infty}^{\bar{a}+i\infty} \tilde{\varphi}(\lambda)R(\lambda\colon A)\,d\lambda,\ \varphi \in \mathcal{D}^{(M_p)}.$$

We will prove that $e^{-\omega t}G \in \mathcal{S}'^{(M_p)}(L(E))$ for all $\omega \in (\bar{a}, \infty)$. Suppose $\varphi \in \mathcal{D}^{(M_p)}$, $\operatorname{supp}\varphi \subseteq [a,b]$, $a < 0$ and $b > 0$. Let $\zeta \in \mathcal{D}^{(M_p)}$, $\zeta(t) = 1$, $t \in [-1,1]$ and $\zeta(t) = 0$, $t \geqslant 2$ and let a function $\xi \in \mathcal{D}^{(M_p)}$ satisfy $\xi(t) = 1, 1 \leqslant t \leqslant b+1$, $\xi(t) = 0$, $t \leqslant -2$ and $\xi(t) = \zeta(t-b-1)$, $t \geqslant b+1$. Then $\varphi\xi \in \mathcal{D}^{(M_p)}$, $\varphi(t)\xi(t) = \varphi(t)$, $t \geqslant 0$ and $G(\varphi) = G(\varphi\xi)$. The partial integration and the proof of implication (III) $\Rightarrow$ (II) of [**307**, Theorem 2.3.1] imply that, for every $h > 0$, $n \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = \bar{a}$:

$$\left|\widetilde{e^{-\omega\cdot}\varphi\xi}(\lambda)\right| = \left|\frac{(-1)^n}{(\lambda-\omega)^n}\int\limits_{-2}^{b} e^{(\lambda-\omega)t}(\varphi\xi)^{(n)}(t)dt\right| \leqslant \frac{1}{|\lambda-\omega|^n}\int\limits_{-2}^{b} e^{-(\omega-\bar{a})t}|(\varphi\xi)^{(n)}(t)|dt$$

$$\leqslant \frac{1}{|\lambda-\omega|^n}\int\limits_{-2}^{b}\frac{e^{-(\omega-\bar{a})t}}{(1+t^2)^{\frac{\beta}{2}}}dt\frac{\|\varphi\xi\|_{M_p,h}}{h^{n+\beta}}M_nM_\beta \leqslant \frac{e^{2(\omega-\bar{a})}-e^{-(\omega-\bar{a})b}}{(\omega-\bar{a})|\lambda-\omega|^n}\frac{M_n}{h^n}\frac{M_\beta}{h^\beta}\|\varphi\xi\|_{M_p,h}.$$

Since the preceding equality holds for every $n \in \mathbb{N}$ and

$$\inf_{n\in\mathbb{N}_0}\frac{M_n}{h^n|\lambda-\omega|^n} = e^{-M(h|\lambda-\omega|)},$$

one gets that, for every $h > 0$, $\beta \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = \bar{a}$, there exists a constant M, independent of a and b, such that:

$$\left|\widetilde{e^{-\omega\cdot}\varphi\xi}(\lambda)\right| \leqslant M\frac{M_\beta}{h^\beta}e^{-M(h|\lambda-\omega|)}\|\varphi\xi\|_{M_p,h}. \tag{271}$$

By the definition of $\|\cdot\|_{M_p,h}$ and the logarithmic convexity $M_{p+q} \geqslant M_pM_q$, $p,q \in \mathbb{N}_0$, one gets that, for every $h > 0$:

$$\begin{aligned}
\|\varphi\xi\|_{M_p,h} &= \sup_{t\in\mathbb{R},\ \alpha,\ \beta\in\mathbb{N}_0}\left\{\frac{h^{\alpha+\beta}}{M_\alpha M_\beta}(1+t^2)^{\beta/2}|(\varphi\xi)^{(\alpha)}(t)|\right\}\\
&\leqslant \sup_{t\in\mathbb{R},\ \alpha,\ \beta\in\mathbb{N}_0}\left\{\frac{h^{\alpha+\beta}}{M_\alpha M_\beta}(1+t^2)^{\beta/2}\sum_{i=0}^{\alpha}\binom{\alpha}{i}|\varphi^{(\alpha-i)}(t)||\xi^{(i)}(t)|\right\}\\
&\leqslant \sup_{t\in\mathbb{R},\ \alpha,\ \beta\in\mathbb{N}_0}\left\{\frac{h^{\alpha+\beta}}{M_{\alpha-i}M_iM_\beta}(1+t^2)^{\beta/2}\sum_{i=0}^{\alpha}\binom{\alpha}{i}|\varphi^{(\alpha-i)}(t)||\xi^{(i)}(t)|\right\}\\
&\leqslant \sup_{t\in\mathbb{R},\ \alpha,\ \beta\in\mathbb{N}_0}\left\{\frac{h^{\alpha-i}h^\beta h^i}{M_i}\sum_{i=0}^{\alpha}\binom{\alpha}{i}|\xi^{(i)}(t)|\frac{1}{(4h)^{\alpha+\beta-i}}\right\}\|\varphi\|_{M_p,4h} \qquad (272)\\
&\leqslant \sup_{t\in\mathbb{R},\ \alpha,\ \beta\in\mathbb{N}_0}\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{1}{2^i}\frac{(2h)^i|\xi^{(i)}(t)|}{M_i}\frac{1}{4^{\alpha+\beta-i}}\|\varphi\|_{M_p,4h}\\
&\leqslant \sup_{\alpha\in\mathbb{N}_0}\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{1}{2^i}\frac{1}{4^{\alpha-i}}\|\varphi\|_{M_p,4h}\|\xi\|_{M_p,2h,[-2,b+3]}\\
&\leqslant \|\varphi\|_{M_p,4h}\|\xi\|_{M_p,2h,[-2,b+3]}.
\end{aligned}$$

In view of (271)–(272), one yields that there exists $M > 0$, independent of a and b, such that, for every $\beta \in \mathbb{N}_0$ and $h > 0$:

$$\left|\widetilde{e^{-\omega\cdot}\varphi}\xi(\lambda)\right| \leqslant M\frac{M_\beta}{h^\beta}e^{-M(h|\lambda-\omega|)}\|\varphi\|_{M_p,4h}. \tag{273}$$

The estimate (273) combined with [**51**, Lemma 2.1.3] indicates that there exists $h > 0$ such that $\|e^{-\omega t}G(\varphi)\| \leqslant \mathrm{Const}\,\|\varphi\|_{M_p,4h}$, $\varphi \in \mathcal{D}^{(M_p)}$. The proof of theorem is thereby complete. □

The relations between exponential ultradistribution semigroups of the Beurling class and exponentially bounded convoluted semigroups have been recently discussed in [**227**].Notice that the preceding theorem enables one to transfer the assertion of [**227**, Theorem 3.10] to Roumieu case as well as to remove any density assumption from the formulation of [**227**, Theorem 3.10(i)].

Example 3.5.15. ([**249**, Example 1.6], cf. also Example 2.8.3) Define

$$E_{M_p} := \Big\{f \in C^\infty[0,1] : \|f\|_{M_p} := \sup_{p\geqslant 0}\frac{\|f^{(p)}\|_\infty}{M_p} < \infty\Big\},$$

and $A_{M_p} := -d/ds$, $D(A_{M_p}) =: \{f \in E_{M_p} : f' \in E_{M_p},\ f(0) = 0\}$. Put now

$$(G(\varphi)f)(x) := \int_0^x \varphi(x-t)f(t)\,dt,\ \varphi \in \mathcal{D}^{(M_p)},\ f \in E_{M_p},\ x \in [0,1].$$

Clearly, $G(\varphi)f \in C^\infty[0,1]$ and

$$\frac{d^p}{dx^p}(G(\varphi)f)(x) = \int_0^x \varphi^{(p)}(x-t)f(t)\,dt + \sum_{k=0}^{p-1}\varphi^{(p-1-k)}(0)f^{(k)}(x),$$

for every $\varphi \in \mathcal{D}^{(M_p)}$, $f \in E_{M_p}$, $x \in [0,1]$ and $p \in \mathbb{N}_0$. Since $M_{p+q} \geqslant M_pM_q$, p, $q \in \mathbb{N}_0$, the preceding equality implies that, for every $p \in \mathbb{N}_0$, $x \in [0,1]$, $\varphi \in \mathcal{D}^{(M_p)}$ and $f \in E_{M_p}$:

$$\begin{aligned}\Big|\frac{1}{M_p}\frac{d^p}{dx^p}(G(\varphi)f)x)\Big| &\leqslant \|\varphi\|_{M_p,1,[0,1]}\|f\| + \sum_{k=0}^{p-1}\Big|\frac{\varphi^{(p-1-k)}(0)}{M_{p-k}}\Big|\|f\|\\ &\leqslant \|\varphi\|_{M_p,1,[0,1]}\Big(1+\sum_{k=0}^{p-1}\frac{1}{m_{p-k}}\Big)\|f\| \leqslant \|\varphi\|_{M_p,1,[0,1]}\Big(1+\sum_{p=0}^{\infty}\frac{1}{m_p}\Big)\|f\|.\end{aligned}$$

Hence, $\|G(\varphi)\| \leqslant \|\varphi\|_{M_p,1,[0,1]}\big(1+\sum_{p=0}^\infty \frac{1}{m_p}\big)$ and $G \in \mathcal{D}_0'^{(M_p)}(L(E))$. The conditions (U.1) and (U.2) can be proved trivially, and consequently, G is an (EUDS) of (M_p)-class whose generator is obviously the operator A_{M_p}. By Theorem 3.6.4 given in the next subsection, we have that there exists an injective operator $C \in L(E_{M_p})$ such that A_{M_p} generates a differentiable local C-regularized semigroup $(S(t))_{t\in[0,2)}$.

Put, for every fixed $f \in E_{M_p}$, $x \in [0,1]$ and $t \in [0,1]$, $u(t,x) := (S(t)f)(x)$. According to the differentiability of $(S(t))_{t\in[0,2)}$ and the proof of Theorem 3.6.4, one immediately obtains that u is a solution of the problem

$$(P): \begin{cases} u \in C^1([0,1]\times[0,1]) \\ u_x + u_t = 0 \\ u(0,x) = (Cf)(x),\ u(t,0) = 0. \end{cases}$$

Hence, for every $(t,x) \in [0,1]\times[0,1]$,

$$(S(t)f)(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t \\ [Cf](x-t), & 1 \geqslant x > t. \end{cases}$$

In particular, $S(t) = 0$, $t \in [1,2)$. Define now $\tilde{S}(t)$, $t \geqslant 0$ by $\tilde{S}(t) := S(t)$, $t \in [0,1]$ and $\tilde{S}(t) := 0$, $t > 1$. Then $(\tilde{S}(t))_{t\geqslant 0}$ is a global differentiable C-regularized semigroup generated by A_{M_p}. The previous analysis and Theorem 3.6.9 imply that there exists an injective operator $C_1 \in L(E)$ such that A_{M_p} generates a global differentiable C_1-regularized semigroup $(\tilde{S}_1(t))_{t\geqslant 0}$ such that $\tilde{S}_1(t) = 0$, $t \geqslant 1$ and that $\sup_{t\geqslant 0,\ p\in\mathbb{N}_0} \frac{h^p}{M_p}\big\|\frac{d^p}{dt^p}\tilde{S}_1(t)\big\| < \infty$ for every fixed number $h > 0$.

Example 3.5.16. [**252**] Let A_{M_p}, E_{M_p} and G be as in the preceding example. Choose an element $x \in E$ and a functional $x^* \in (D(A))^\circ$ with $\langle x^*, x\rangle = 1$. Define now $\tilde{G} := G + \delta \otimes \langle x^*, \cdot\rangle x$. Then $\tilde{G}$ satisfies (U.1), (U.2), (U.4), but not (U.5).

Ultradistribution semigroups are important in the analysis of some classes of pseudo-differential evolution systems with constant coefficients (cf. [**123**] and [**226**]). We also refer to [**54**], [**255**] and [**344**] for examples of differential operators generating ultradistribution semigroups.

The following result has been recently proved by using the theory of convoluted semigroups.

Theorem 3.5.17. [**226**] *Suppose $s > 1$, $M_p = p!^s$, there exists a (tempered) ultradistribution fundamental solution of $*$-class for A, $B \in L(E)$ and $BA \subseteq AB$. Then there exists a (tempered) ultradistribution fundamental solution of $*$-class for $A + B$.*

It is quite questionable whether Theorem 3.5.17 holds in the case of a general sequence (M_p) satisfying (M.1), (M.2) and (M.3$'$).

Problem. Suppose $s > 1$, $M_p = p!^s$, A generates a (UDS) of $*$-class, $B \in L(E)$ and $BA \subseteq AB$. Denote by G the ultradistribution fundamental solution of $*$-class for $A + B$ (cf. Theorem 3.5.17). Is it true that G satisfies (U.2)?

Example 3.5.18. [**226**] (i) Suppose $c > 1$, $\sigma > 0$ and $\varsigma \in \mathbb{R}$, $M > 0$, $k \in \mathbb{N}$,

$$\sigma(A) \subseteq \pm\Pi_{c,\sigma,\varsigma} \text{ and } \|R(\lambda\colon A)\| \leqslant M(1+|\lambda|)^k,\ \lambda \notin \pm\Pi_{c,\sigma,\varsigma}.$$

Let $p(x) := \sum_{i=0}^{n} a_i x^i$, $x \in \mathbb{C}$, where $n \in \mathbb{N}$, $a_n > 0$ and $a_{n-i} \in \mathbb{C}$, $i = 1, \ldots, n$. Then the operators $\pm ip(A)$ generate ultradistribution semigroups of $\Big(p!^{\overline{\frac{n}{n-1+\frac{1}{c}}}}\Big)$-class. Herein $\Pi_{c,\sigma,\varsigma} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \sigma|\operatorname{Im}\lambda|^c + \varsigma\}$.

(ii) Let p be as in (i) and let A (or $-A$) generate an exponentially bounded integrated cosine function. Then the operators $\pm ip(A)$ generate:

(ii.1) ultradistribution semigroups of $(p!^s)$-class provided $s \in \big(1, \frac{2n}{2n-1}\big]$,

(ii.2) ultradistribution semigroups of $\{p!^s\}$-class provided $s \in \big(1, \frac{2n}{2n-1}\big)$.

(iii) Let p be as in (i) and let A (or $-A$) generate a (local) integrated cosine function. Then the operators $\pm ip(A)$ generate ultradistribution semigroups of $*$-class provided $M_p = p!^s$ and $s \in \big(1, \frac{2n}{2n-1}\big)$.

(iv) Suppose $c \in (0,1)$, $\sigma > 0$, $\varsigma \in \mathbb{R}$, $\sigma(A) \subseteq \pm\big(\mathbb{C} \setminus \{\lambda^2 : \lambda \in \Pi_{c,\sigma,\varsigma}\}\big)$ and $\|R(\cdot : A)\|$ is polynomially bounded on the complement of $\{\lambda^2 : \lambda \in \Pi_{c,\sigma,\varsigma}\}$. Let p be as in (i). Then $\pm ip(A)$ generate ultradistribution semigroups of $*$-class provided $M_p = p!^s$ and $s \in \big(1, \frac{2n}{2n+c-1}\big)$.

The proof of the following structural result follows from the analysis of Komatsu [**210**], Melnikova, Filinkov [**307**, Section 2.3] and Melnikova, Anufrieva [**309**, Subsection 1.3.4].

Theorem 3.5.19. *There exists an ultradistribution fundamental solution G of $*$-class for a closed linear operator A iff there exist $l > 0$ and $\beta > 0$, in the Beurling case, resp., for every $l > 0$ there exists $\beta_l > 0$, in the Roumieu case, such that:*

$$\Omega^{(M_p)}_{l,\beta} := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant M(l|\lambda|) + \beta\Big\} \subseteq \rho(A), \quad \textit{resp.},$$
$$\Omega^{\{M_p\}}_{l,\beta_l} := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant M(l|\lambda|) + \beta_l\Big\} \subseteq \rho(A)$$

and

$$\|R(\lambda : A)\| \leqslant \beta e^{M(l|\lambda|)}, \ \ \lambda \in \Omega^{(M_p)}_{l,\beta}, \quad \textit{resp.},$$
$$\|R(\lambda : A)\| \leqslant \beta_l e^{M(l|\lambda|)}, \ \ \lambda \in \Omega^{\{M_p\}}_{l,\beta_l}.$$

It is obvious that the polynomial boundedness of $\|R(\cdot : A)\|$ on $\Omega^{(M_p)}_{l,\beta}$ ($\Omega^{\{M_p\}}_{l,\beta_l}$) implies that G is a (UDS) of (M_p)-class ($\{M_p\}$-class).

3.5.3. Differentiable ultradistribution semigroups.

Definition 3.5.20. Suppose G is a (UDS) of $*$-class and $\alpha \in (0, \frac{\pi}{2}]$. Then it is said that G is an *(infinitely) differentiable ultradistribution semigroup* (of $*$-class), resp. an *analytic ultradistribution semigroup of angle* α (and of $*$-class) iff the mapping $t \mapsto G(\delta_t) \in L(E)$, $t > 0$ is (infinitely) differentiable, resp. iff the mapping $t \mapsto G(\delta_t) \in L(E)$, $t > 0$ can be analytically extended to the sector Σ_α.

Let G be a (DS), resp., a (UDS) of $*$-class generated by A and let the mapping $t \mapsto G(\delta_t) \in L(E)$, $t > 0$ be continuous. Then $(G(\delta_t))_{t \geqslant 0}$ is a semigroup [**252, 226**] and this implies that there exists $\omega_0 \in [-\infty, \infty)$ such that $\omega_0 = \lim_{t\to\infty} \frac{\ln \|G(\delta_t)\|}{t}$. The asymptotic behavior of $(G(\delta_t))_{t \geqslant 0}$ in a neighborhood of zero is quite complicated. For instance, Da Prato and Mosco [**86**, p. 575] constructed an example of a densely defined operator A generating an analytic (DS) G of angle $\frac{\pi}{2}$ which satisfies $\frac{4}{e^2}\frac{1}{t} \leqslant \|G(\delta_t)\| \leqslant (2 + \frac{16}{e^4}\frac{1}{t^2})^{1/2}$, $t > 0$. Notice also that A generates an exponentially bounded, analytic once integrated semigroup of angle $\frac{\pi}{2}$ and that A cannot be the generator of a C_0-semigroup.

Let G be a dense (DS), resp., a dense (UDS) of $*$-class in E. Then G^* is a dense (DS), resp., a dense (UDS) of $*$-class [**252, 226**] in E^* and it can be simply proved that (infinite) differentiability, resp. analyticity, of G, implies (infinite) differentiability, resp. analyticity, of G^*. In order to characterize spectral properties of the infinitesimal generators of differentiable ultradistribution semigroups and their relationship to differentiable convoluted semigroups, we need the following family of kernels [**63**]. Since (M_p) satisfies (M.1), (M.2) and (M.3$'$), one can define, for every $l > 0$, the next entire function $\omega_l(\lambda) := \prod_{p=1}^{\infty}\big(1 + \frac{l\lambda}{m_p}\big)$, $\lambda \in \mathbb{C}$. Then it is clear that:

$$|\omega_l(\lambda)| \geqslant \sup_{k\in\mathbb{N}} \prod_{p=1}^{k}\Big|1 + \frac{l\lambda}{m_p}\Big| \geqslant \sup_{k\in\mathbb{N}} \prod_{p=1}^{k} \frac{l|\lambda|}{m_p} \geqslant \sup_{k\in\mathbb{N}} \frac{(l|\lambda|)^k}{M_p},\ \operatorname{Re}\lambda \geqslant 0.$$

Hence, $|\omega_l(\lambda)| \geqslant e^{M(l|\lambda|)}$, $\operatorname{Re}\lambda \geqslant 0$. It is also worth noting that, for every $\alpha \in (0, \frac{\pi}{2})$, $p \in \mathbb{N}_0$ and $\lambda \in \Sigma_{\frac{\pi}{2}+\alpha}$, we have $|1 + \frac{l\lambda}{m_p}| \geqslant \frac{l|\operatorname{Im}\lambda|}{m_p} \geqslant \frac{l(1+\tan\alpha)^{-1}|\lambda|}{m_p}$. Hence,

$$|\omega_l(\lambda)| \geqslant e^{M(l(1+\tan\alpha)^{-1}|\lambda|)},\ \ \alpha \in (0, \pi/2),\ l > 0,\ \lambda \in \Sigma_{\frac{\pi}{2}+\alpha}. \tag{274}$$

Put now $K_l(t) := \mathcal{L}^{-1}\big(\frac{1}{\omega_l(\lambda)}\big)(t)$, $t \geqslant 0$, $l > 0$. Then, for every $l > 0$, $0 \in \operatorname{supp} K_l$ and K_l is an infinitely differentiable function in $t \geqslant 0$.

Proposition 3.5.21. *Suppose A is a closed linear operator and there exists an exponential ultradistribution fundamental solution of $*$-class for A. Then there exists $k > 0$ such that, for every compact set $K \subseteq [0,\infty)$ and for every $l > k$, there exists $h_{l,K} > 0$, in the Beurling case, resp., for every compact set $K \subseteq [0,\infty)$ and for every $l > 0$, there exists $h_{l,K} > 0$, in the Roumieu case, such that A generates an exponentially bounded, K_l-convoluted semigroup $(S_{K_l}(t))_{t\geqslant 0}$ such that the mapping $t \mapsto S_{K_l}(t)$, $t \geqslant 0$ is infinitely differentiable and that* $\sup_{t\in K,\ p\in\mathbb{N}_0} \frac{h_{l,K}^p}{M_p}\big\|\frac{d^p}{dt^p}S_{K_l}(t)\big\| < \infty$.

Proof. Put, for some $\bar{a} > 0$ and $l > 0$, in the Beurling case, resp., for every $l > 0$, in the Roumieu case:

$$S_{K_l}(t)x =: \frac{1}{2\pi i}\int_{\bar{a}-i\infty}^{\bar{a}+i\infty} e^{\lambda t}\tilde{K}_l(\lambda)R(\lambda\colon A)x\,d\lambda,\ \ x \in E,\ t \geqslant 0$$

and notice that

$$\frac{d^p}{dt^p}S_{K_l}(t) = \frac{1}{2\pi i}\int_{\bar{a}-i\infty}^{\bar{a}+i\infty} e^{\lambda t}\lambda^p\tilde{K}_l(\lambda)R(\lambda\colon A)\,d\lambda,\ \ t \geqslant 0,\ p \in \mathbb{N}_0.$$

The proof immediately follows from the previous equality and a simple computation. □

Theorem 3.5.22. *Let A be a closed linear operator. Suppose that A generates a dense differentiable* (UDS) *G of $*$-class. Then there exists $h > 0$ such that, for* *every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$, in the Beurling case, resp. for every $h > 0$ and $\sigma > 0$, there exist $C_\sigma > 0$ and $M_{\sigma,h} > 0$ in the Roumieu case, such that* (200) *holds and that*

$$\|R(\lambda\colon A)\| \leqslant M_\sigma e^{M(h|\lambda|)}, \quad \lambda \in \Upsilon_\sigma, \quad \textit{resp.}, \tag{275}$$

$$\|R(\lambda\colon A)\| \leqslant M_{\sigma,h} e^{M(h|\lambda|)}, \ \lambda \in \Upsilon_\sigma. \tag{276}$$

Specifically, G is an (EUDS) *of $*$-class.*

Proof. We will prove the assertion in the Beurling case and remark the minor changes in the Roumieu case. Suppose $\varepsilon \in (0,1)$, $\psi \in \mathcal{D}^{(M_p)}$, $\psi(t) = 1$, $0 \leqslant t \leqslant \frac{1}{2}$, $\psi(t) = 0$, $t \geqslant 1$ and $\psi(t) = 0$, $t \leqslant -1$. Set $\psi_\varepsilon(t) := \psi(\frac{t}{\varepsilon})$, $t \in \mathbb{R}$. By the proof of implication (II) $\Rightarrow$ (III) of [**307**, Theorem 2.3.1], it follows that

$$(\lambda - A)G(e^{-\lambda t}\psi_\varepsilon(t)) = I + G(e^{-\lambda t}\psi'_\varepsilon(t)), \quad \lambda \in \mathbb{C}. \tag{277}$$

Since the set $\mathcal{R}(G)$ is dense in E, we have $G(e^{-\lambda t}\psi'_\varepsilon(t))x = \int_{\varepsilon/2}^{\varepsilon} e^{-\lambda t}\psi'_\varepsilon(t)\, G(\delta_t)x\, dt$ for all $x \in E$ and $\lambda \in \mathbb{C}$ (cf. [**27**, Remarks, p. 416] and [**226**]). Then the partial integration gives that $G(e^{-\lambda t}\psi'_\varepsilon(t))x = \frac{1}{\lambda}\int_{\varepsilon/2}^{\varepsilon} e^{-\lambda t}\frac{d}{dt}(\psi'_\varepsilon(t)\, G(\delta_t)x)\, dt$ for all $x \in E$ and $\lambda \in \mathbb{C} \smallsetminus \{0\}$. This implies

$$\big\|G(e^{-\lambda t}\psi'_\varepsilon(t))\big\| \leqslant \frac{1}{|\lambda|}\frac{1}{2}\,\varepsilon \sup_{\frac{1}{2}\varepsilon \leqslant t \leqslant \varepsilon} e^{-t\operatorname{Re}\lambda} M_\varepsilon, \tag{278}$$

with $M_\varepsilon = \sup_{\frac{1}{2}\varepsilon \leqslant t \leqslant \varepsilon} \|\frac{d}{dt}(\psi'_\varepsilon(t)G(\delta_t))\|$. Define Φ_ε by

$$\begin{aligned}\Phi_\varepsilon := &\big\{\lambda \in \mathbb{C} : \lambda \neq 0,\ \operatorname{Re}\lambda \leqslant 0,\ \ln|\lambda| \geqslant -\varepsilon\operatorname{Re}\lambda + \ln(\varepsilon M_\varepsilon)\big\}\\ &\cup \big\{\lambda \in \mathbb{C} : \lambda \neq 0,\ \operatorname{Re}\lambda \geqslant 0,\ \ln|\lambda| \geqslant -\tfrac{\varepsilon}{2}\operatorname{Re}\lambda + \ln(\varepsilon M_\varepsilon)\big\}.\end{aligned}$$

Then $\|G(e^{-\lambda t}\psi'_\varepsilon(t))\| \leqslant \frac{1}{2}$, $\lambda \in \Phi_\varepsilon$, and as in the proof of [**27**, Theorem 3], one gets $\Phi_\varepsilon \subseteq \rho(A)$ and

$$\|R(\lambda\colon A)\| \leqslant 2\|G(e^{-\lambda t}\psi_\varepsilon(t))\|, \quad \lambda \in \Phi_\varepsilon. \tag{279}$$

By the continuity of G, it follows that there exist $h' > 0$ and $C > 0$, independent of $\varepsilon > 0$, such that:

$$\|G(e^{-\lambda t}\psi_\varepsilon(t))\| \leqslant C\|e^{-\lambda t}\psi_\varepsilon(t)\|_{M_p,2h',[0,1]} = C\|e^{-\lambda t}\psi_\varepsilon(t)\|_{M_p,2h',[\frac{\varepsilon}{2},\varepsilon]}. \tag{280}$$

In the Roumieu case, the previous estimate holds for every $h' > 0$ and a corresponding $C_{h'} > 0$. By (279)–(280) and the inequality [**307**, (2.3.9), Theorem 2.3.1, p. 170], we reveal that, for every $\lambda \in \Phi_\varepsilon$:

$$\|R(\lambda\colon A)\| \leqslant 2C\|e^{-\lambda t}\psi_\varepsilon(t)\|_{M_p,2h',[\frac{\varepsilon}{2},\varepsilon]} \leqslant 2C\|\psi_\varepsilon\|_{M_p,h',[\frac{\varepsilon}{2},\varepsilon]} e^{M(\frac{\lambda}{h'}) - \frac{1}{2}\varepsilon\operatorname{Re}\lambda}. \tag{281}$$

With (281) in view, one obtains that there are an $h' > 0$ and a constant $M'_\varepsilon > 0$ (in the Roumieu case, such a constant M'_ε also depends on h') such that $\|R(\lambda\colon A)\| \leqslant M'_\varepsilon e^{M(\frac{\lambda}{h'})}(1 + |\lambda|)$, $\lambda \in \Phi_\varepsilon$. Especially, with $\varepsilon = \frac{1}{2}$, one yields that $\big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \max\big(1, 4|\ln(\frac{1}{2}M_{1/2})|\big)\big\} \subseteq \rho(A)$ and Theorem 3.5.14 yields that G is an (EUDS). Put $a_\varepsilon := \max\big(1, \frac{2|\ln(\varepsilon M_\varepsilon)|}{\varepsilon}\big)$. Let $N_\varepsilon > e^{\frac{\varepsilon}{2}a_{1/2} + |\ln(\varepsilon M_\varepsilon)|}$ and let $h > \frac{1}{h'}$. Then h

is independent of $\varepsilon \in (0,1)$ and there exists an appropriate constant $M''_\varepsilon > 0$ such that the set

$$\Delta_\varepsilon := \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant a_{1/2},\ \operatorname{Re}\lambda \geqslant -\tfrac{1}{\varepsilon}\ln|\operatorname{Im}\lambda| + \tfrac{1}{\varepsilon}\ln N_\varepsilon\bigr\} \cup \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant a_{1/2}\bigr\}$$

belongs to $\rho(A)$ and that $\|R(\lambda : A)\| \leqslant M''_\varepsilon e^{M(h|\lambda|)}$, $\lambda \in \Delta_\varepsilon$. In the case $\sigma \geqslant 1$, (275) follows with $\varepsilon = \frac{1}{\sigma}$, $\omega = a_{1/2}$, $C_\sigma = \frac{1}{\varepsilon}\ln N_\varepsilon$, $M_\sigma = M''_\varepsilon$, and the above chosen h; the case $\sigma \in (0,1)$ is completely regardless and this ends the proof of theorem. □

Let $a > 0$, $b \in (0,1)$ and set $K_{a,b}(t) := \mathcal{L}^{-1}\bigl(e^{-a\lambda^b}\bigr)(t)$, $t \geqslant 0$, where $1^b = 1$.

EXAMPLE 3.5.23. (i) Suppose $M_p = p!^s$, $s \in (1,2)$, $E := L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, $D(A) := \{f \in E : (x+ix^2)f(x) \in E\}$ and $Af(x) := (x+ix^2)f(x)$, $x \in \mathbb{R}$, $f \in D(A)$. Set $\delta := \frac{1}{s}$. Then A generates a global $K_{1,\delta}$-convoluted semigroup since, for every $\tau > 0$, A generates a $K_{1,\delta}$-convoluted semigroup on $[0,\tau)$. To see this, suppose $M(\lambda) \leqslant C_s|\lambda|^{1/s}$, $\lambda \in \mathbb{C}$, $\tau \in (0,\infty)$ and $\alpha > 0$ fulfills $\tau \leqslant \frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}$. It is checked at once that there exists $\beta > 0$ such that $\Lambda'_{\alpha,\beta,1} \subseteq \rho(A)$ and that the resolvent of A is bounded on $\Lambda'_{\alpha,\beta,1}$, where $\Lambda'_{\alpha,\beta,1} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant M(\alpha\lambda) + \beta\}$. Put $\Gamma := \partial\Lambda'_{\alpha,\beta,1}$ and define, for every $f \in E$, $x \in \mathbb{R}$ and $t \in \bigl[0, \frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}\bigr)$:

$$(S_{\delta,\tau}(t)f)(x) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t - \lambda^\delta}}{\lambda - (x+ix^2)}\,d\lambda f(x).$$

Since $\bigl|e^{-\lambda^\delta}\bigr| \leqslant e^{-\cos(\delta\pi/2)|\lambda|^\delta}$, $\operatorname{Re}\lambda > 0$ and

$$\bigl|e^{\lambda t - \lambda^\delta}\bigr| \leqslant e^{\beta t}e^{M(\alpha\lambda)t - \cos(\delta\pi/2)|\lambda|^\delta} \leqslant e^{\beta t}e^{C_s\alpha^{1/s}|\lambda|^{1/s}t - \cos(\delta\pi/2)|\lambda|^\delta},$$

for all $\lambda \in \Gamma$, one can straightforwardly verify that $(S_{\delta,\tau}(t))_{t\in[0,\tau)}$ is a local $K_{1,\delta}$-convoluted semigroup generated by A. Notice that the mapping $t \mapsto S_{\delta,\tau}(t)$, $t \in \bigl[0, \frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}\bigr)$ is infinitely differentiable and that

$$\frac{d^p}{dt^p}S_{\delta,\tau}(t)f(\cdot) = \frac{1}{2\pi i}\int_\Gamma \frac{\lambda^p e^{\lambda t - \lambda^\delta}}{\lambda - (\cdot + i\cdot^2)}\,d\lambda f(\cdot),\quad f \in E,\ t \in \Bigl[0, \frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}\Bigr).$$

This implies that A generates a global non-exponentially bounded $K_{1,\delta}$-convoluted semigroup $(S_\delta(t))_{t\geqslant 0}$ that is infinitely differentiable in the uniform operator topology for $t \geqslant 0$. Furthermore, for every $\tau > 0$, there exists $h_\tau > 0$ such that $\sup_{t\in[0,\tau),\, p\in\mathbb{N}_0} \frac{h_\tau^p}{M_p}\bigl\|\frac{d^p}{dt^p}S_\delta(t)\bigr\| < \infty$. On the other hand, it is obvious that A generates a (UDS) G of $*$-class which is given by $(G(\varphi)f)(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\varphi}(\lambda)}{\lambda-(x+ix^2)}\,d\lambda f(x)$, $\varphi \in \mathcal{D}^*$, $f \in E$, $x \in \mathbb{R}$. Let $t > 0$, $f \in E$ and $g \in E$. By the Paley–Wiener theorem and the Cauchy theorem, it follows that $(G(\varphi)f)(x) = \tilde{\varphi}(x+ix^2)f(x)$, $\varphi \in \mathcal{D}^*_0$, $f \in E$, $x \in \mathbb{R}$ and $G(\delta_t)f = g$ iff $e^{(x+ix^2)t}f(x) = g(x)$, $x \in \mathbb{R}$. Hence, $G(\delta_t) \notin L(E)$ and G is not differentiable.

(ii) Let E and A be as in examples 2.8.3 and 3.5.15. Then $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant 0\} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant C\sum_{p=0}^\infty \frac{|\lambda|^p}{M_p} \leqslant Ce^{M(\tilde{l}|\lambda|)}$, $\operatorname{Re}\lambda \geqslant 0$, for some $C > 0$ and

$\tilde{l} > 0$. Let $(G(\varphi)f)(t) = \int_0^t \varphi(t-s)f(s)\,ds$, $\varphi \in \mathcal{D}^{(M_p)}$, $f \in E$, $t \in [0,1]$. Then G is a non-dense (EUDS) generated by A and $G(\delta_t) = 0$, $t \geqslant 1$. Suppose now $t \in (0,1)$. Then it can be simply verified that, for every $s \in [0,1]$ and $f \in D(G(\delta_t))$,

$$(G(\delta_t)f)(s) = \begin{cases} 0, & 0 \leqslant s \leqslant t \\ f(s-t), & 1 \geqslant s > t. \end{cases}$$

In particular, $G(\delta_t) \notin L(E)$ and G is not differentiable. Furthermore, for every $h > 0$, there exists $l > \tilde{l}$ such that A generates a bounded K_l-convoluted semigroup $(S_{K_l}(t))_{t \geqslant 0}$ that is infinitely differentiable in $t \geqslant 0$ and satisfies

$$\sup_{p \in \mathbb{N}_0,\, t \in K} \frac{h^p}{M_p} \Big\| \frac{d^p}{dt^p} S_{K_l}(t) \Big\| < \infty$$

for every compact set $K \subseteq [0,\infty)$. This implies that relations between differentiable convoluted semigroups and differentiable (UDS)'s are more complicated than it looks at first sight. Further on, the previous example also shows that the existence of a number $h > 0$ satisfying that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ such that (200) and (276) hold, is not sufficient for the generation of differentiable (UDS)'s. Indeed, put $h_\lambda(t) = e^{-\lambda t} \int_0^t e^{\lambda s} f(s)\,ds$, $f \in E$, $\lambda \in \mathbb{C}$. Then we know that $R(\lambda : A)f = h_\lambda$ if $\operatorname{Re}\lambda \geqslant 0$. It is clear that, for every $t \in [0,1]$ and $n \in \mathbb{N}$ with $n \geqslant 2$:

$$\frac{d^n}{dt^n} h_\lambda(t) = \frac{d^{n-1}}{dt^{n-1}} f(t) + \sum_{k=1}^{n-1} (-\lambda)^k \frac{d^{n-1-k}}{dt^{n-1-k}} f(t) + (-\lambda)^n g(t). \tag{282}$$

Let $\sigma > 0$ and let $C_\sigma > 0$ be arbitrarily chosen. Then the supposition $\operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma$ implies

$$\begin{aligned} |h_\lambda(t)| &\leqslant \int_0^t e^{\operatorname{Re}\lambda(t-s)} ds \|f\|_{L^\infty[0,1]} \leqslant e^{-\operatorname{Re}\lambda} \|f\|_{L^\infty[0,1]} \\ &\leqslant e^{-C_\sigma} |\operatorname{Im}\lambda|^\sigma \|f\|_{L^\infty[0,1]}, \quad t \in [0,1]. \end{aligned}$$

By (282), the previous inequality and logarithmic convexity, one gets that

$$\frac{1}{M_n} \Big\| \frac{d^n}{dt^n} h_\lambda(\cdot) \Big\|_{L^\infty[0,1]} \leqslant C'_\sigma |\operatorname{Im}\lambda|^\sigma e^{M((\tilde{l}+\frac{1}{2})|\lambda|)} \|f\|, \ n \in \mathbb{N}, \ \ n \geqslant 2. \tag{283}$$

Since $\lambda h_\lambda(t) + h'_\lambda(t) = f(t)$, $t \in [0,1]$ and $h_\lambda(0) = 0$, one has that $\|h'_\lambda\|_{L^\infty[0,1]} \leqslant (1 + e^{-C_\sigma}|\lambda| |\operatorname{Im}\lambda|^\sigma) \|f\|_{L^\infty[0,1]}$. Keeping in mind (283), we obtain that there exists $M_\sigma > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \leqslant 0$ and $\operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma$, the following holds: $\|R(\lambda : A)\| \leqslant M_\sigma e^{M((\tilde{l}+1)|\lambda|)}$. Therefore, (200) and (276) hold but A does not generate a differentiable (UDS), as claimed. Notice also that, in this example, we have $\rho(A) = \mathbb{C}$. Suppose now that $E = \mathcal{S}^{M_p,1}$ and define A as above. Then $\|R(\cdot : A)\|$ can be estimated similarly and there exists $l > 0$ such that A generates a global bounded K_l-convoluted semigroups that is infinitely differentiable for $t \geqslant 0$.

THEOREM 3.5.24. *Suppose A generates a dense* (UDS) *G of $*$-class. Then the following assertions hold.*

(i) *Suppose $h > 0$, G is infinitely differentiable and the mapping $t \mapsto G(\delta_t)$, $t > 0$ satisfies that, for every compact set $K \subseteq (0,\infty)$,*

$$\|G(\delta_t)\|_{M_p,h,K} := \sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}G(\delta_t)\Big\| < \infty.$$

Then:

$$\exists\omega \geqslant 0\ \exists h' > 0\ \forall\sigma > 0\ \exists C_{\sigma,h} > 0\ \exists M_{\sigma,h} > 0, \quad \textit{in the Beurling case, resp.,}$$
$$\exists\omega \geqslant 0\ \forall h' > 0\ \forall\sigma > 0\ \exists C_{\sigma,h} > 0\ \exists M_{\sigma,h,h'} > 0, \ \textit{in the Roumieu case,}$$

such that

$$\Upsilon_{\sigma,h} := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant C_{\sigma,h} - \sigma M\Big(\frac{|\lambda|}{H+1/h}\Big)\Big\} \\ \cup \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\} \subseteq \rho(A)$$

and

$$\|R(\lambda : A)\| \leqslant M_{\sigma,h} e^{M(\frac{|\lambda|}{h'})+\frac{1}{2}M(\frac{|\lambda|}{H+1/h})},\ \lambda \in \Upsilon_{\sigma,h},\ \textit{in the Beurling case,} \tag{284}$$

resp.,

$$\|R(\lambda : A)\| \leqslant M_{\sigma,h,h'} e^{M(\frac{|\lambda|}{h'})+\frac{1}{2}M(\frac{|\lambda|}{H+1/h})},\ \lambda \in \Upsilon_{\sigma,h},\ \textit{in the Roumieu case.} \tag{285}$$

Herein H designates the constant appearing in the formulation of (M.2).

(ii) *G is infinitely differentiable and satisfies $\|G(\delta_t)\|_{M_p,h,K} < \infty$ for every compact set $K \subseteq (0,\infty)$ and for every $h > 0$ iff there exist $\omega \geqslant 0$, $k > 0$ and $h' > 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$, in the Beurling case, resp., there exist $\omega \geqslant 0$ and $k > 0$ such that, for every $h' > 0$ and $\sigma > 0$, there exist $C_\sigma > 0$ and $M_{\sigma,h'} > 0$, in the Roumieu case, such that*

$$\Xi_{\sigma,k} := \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant C_\sigma - \sigma M(k|\lambda|)\big\} \cup \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\big\} \subseteq \rho(A)$$

and

$$\|R(\lambda : A)\| \leqslant M_\sigma e^{M(h'|\lambda|)}, \quad \lambda \in \Xi_{\sigma,k},\ \textit{in the Beurling case, resp.,}$$
$$\|R(\lambda : A)\| \leqslant M_{\sigma,h'} e^{M(h'|\lambda|)},\ \lambda \in \Xi_{\sigma,k},\ \textit{in the Roumieu case.}$$

PROOF. We will employ the terminology given in the proof of Theorem 3.5.22. To prove (i), notice that the partial integration implies that, for every $n \in \mathbb{N}_0$,

$$\big\|G(e^{-\lambda t}\psi_\varepsilon'(t))\big\| \leqslant \frac{1}{|\lambda|^n}\int_{\varepsilon/2}^{\varepsilon}\Big\|e^{-\lambda t}\frac{d^n}{dt^n}\big(\psi_\varepsilon'(t)G(\delta_t)\big)\Big\|dt$$
$$\leqslant \frac{1}{|\lambda|^n}\int_{\varepsilon/2}^{\varepsilon} e^{-\operatorname{Re}\lambda t}\sum_{i=0}^{n}\binom{n}{i}\frac{1}{\varepsilon^{i+1}}\big|\psi^{(i+1)}(t/\varepsilon)\big|\Big\|\frac{d^{n-i}}{dt^{n-i}}G(\delta_t)\,dt\Big\|$$

$$\leqslant \frac{1}{|\lambda|^n} \sup_{\frac{\varepsilon}{2}\leqslant t\leqslant\varepsilon} e^{-\operatorname{Re}\lambda t} \sum_{i=0}^{n} \binom{n}{i} \|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]} \Big\| \frac{d^{n-i}}{dt^{n-i}} G(\delta_t) \Big\|_{L^\infty[\frac{\varepsilon}{2},\varepsilon]} M_{i+1},$$

and since (M.2) is assumed

$$\leqslant AHM_1 \|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]} \frac{1}{|\lambda|^n} \sup_{\frac{\varepsilon}{2}\leqslant t\leqslant\varepsilon} e^{-\operatorname{Re}\lambda t} \sum_{i=0}^{n} \binom{n}{i} \frac{M_{n-i}\|G(\delta_t)\|_{M_p,h,[\frac{\varepsilon}{2},\varepsilon]}}{h^{n-i}} H^i M_i.$$

By the logarithmic convexity

$$\leqslant AHM_1 \|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]} \frac{1}{|\lambda|^n} \sup_{\frac{\varepsilon}{2}\leqslant t\leqslant\varepsilon} e^{-\operatorname{Re}\lambda t} \sum_{i=0}^{n} \binom{n}{i} \frac{\|G(\delta_t)\|_{M_p,h,[\frac{\varepsilon}{2},\varepsilon]}}{h^{n-i}} H^i M_n$$

$$\leqslant AHM_1 \frac{M_n(H+1/h)^n}{|\lambda|^n} \|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]} \|G(\delta_t)\|_{M_p,h,[\frac{\varepsilon}{2},\varepsilon]} \sup_{\frac{\varepsilon}{2}\leqslant t\leqslant\varepsilon} e^{-\operatorname{Re}\lambda t}. \tag{286}$$

The validity of (286) for all $n \in \mathbb{N}_0$ enables one to conclude that

$$\big\|G(e^{-\lambda t}\psi'_\varepsilon(t))\big\| \leqslant AHM_1 e^{-M(\frac{|\lambda|}{H+1/h})} \|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]} \|G(\delta_t)\|_{M_p,h,[\frac{\varepsilon}{2}\varepsilon]} \sup_{\frac{\varepsilon}{2}\leqslant t\leqslant\varepsilon} e^{-\operatorname{Re}\lambda t}. \tag{287}$$

Suppose now that $\operatorname{Re}\lambda \leqslant 0$ and put $M_{\varepsilon,h} := AHM_1\|\psi\|_{M_p,\frac{1}{\varepsilon},[\frac{1}{2},1]}\|G(\delta_t)\|_{M_p,h,[\frac{\varepsilon}{2}\varepsilon]}$. Then (287) implies that, for every $\lambda \in \mathbb{C} \smallsetminus \{0\}$ with $\operatorname{Re}\lambda \leqslant 0$ and $\operatorname{Re}\lambda \geqslant -\frac{1}{\varepsilon}\ln\frac{1}{2M_{\varepsilon,h}} - \frac{1}{\varepsilon}M\big(\frac{|\lambda|}{H+1/h}\big)$, we have $\|G(e^{-\lambda t}\psi'_\varepsilon(t))\| \leqslant \frac{1}{2}$ and $\lambda \in \rho(A)$. Arguing as in the proof of Theorem 3.5.22, with $\sigma = \frac{1}{\varepsilon}$, one gets that there exist $\omega \geqslant 0$, $C_{\sigma,h} > 0$ and $M_{\sigma,h} > 0$, in the Beurling case, resp., $\omega \geqslant 0$, $C_{\sigma,h} > 0$ and $M_{\sigma,h,h'} > 0$, in the Roumieu case, such that $\Upsilon_{\sigma,h} \subseteq \rho(A)$ and that (284), resp., (285) holds, finishing the proof of (i).

The necessity in (ii) follows from (i) with $h = 1$. We will prove sufficiency by the following modification of the proof of [**28**, Theorem 3]. Suppose $h > 0$ and (ρ_n) is a regularization sequence in $\mathcal{D}^*$. By the Paley–Wiener theorem, one can simply prove that $G(\varphi)x = \frac{1}{2\pi i}\int_{\Gamma_\sigma}\tilde{\varphi}(\lambda)R(\lambda : A)x\,d\lambda$, $\varphi \in \mathcal{D}^*_0$, $x \in E$, where Γ_σ denotes the upwards oriented boundary of $\Xi_{\sigma,k}$. Define the curves Γ^1_σ, Γ^2_σ and Γ^3_σ similarly as in the proof of Theorem 3.2.19. Clearly,

$$\widetilde{\rho_n}(\lambda) = \int_0^{1/n} e^{\lambda t} n\rho(nt)\,dt = \frac{1}{\lambda^k}\int_0^{1/n} e^{\lambda t} n^{k+1}\rho^{(k)}(nt)\,dt, \quad \lambda \in \mathbb{C} \smallsetminus \{0\},\ k \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

This implies $|\widetilde{\rho_n}(\lambda)| \leqslant \big(\frac{n}{|\lambda|}\big)^k \frac{M_k}{h^k}\|\rho\|_{M_p,h,[0,1]}$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$, $k \in \mathbb{N}_0$, $h > 0$, $n \in \mathbb{N}$. Since this inequality holds for every $k \in \mathbb{N}_0$, one yields

$$|\widetilde{\rho_n}(\lambda)| \leqslant e^{-M(\frac{h|\lambda|}{n})}\|\rho\|_{M_p,h,[0,1]},\ \lambda \in \mathbb{C} \smallsetminus \{0\},\ h > 0,\ n \in \mathbb{N}. \tag{288}$$

On the other hand,

$$\frac{d^p}{dt^p}G(\delta_t * \rho_n)x = \frac{1}{2\pi i}\int_{\Gamma_\sigma} e^{\lambda t} R(\lambda : A)\lambda^p \widetilde{\rho_n}(\lambda)x\,d\lambda,\ t > 0,\ n \in \mathbb{N},\ p \in \mathbb{N}_0, \tag{289}$$

$$\lim_{n\to\infty}\frac{d^p}{dt^p}G(\delta_t*\rho_n)x=G(\delta_t)x,\ t>0,\ p\in\mathbb{N}_0,\ x\in\mathcal{R}(G), \tag{290}$$

uniformly on every compact of $(0,\infty)$. Then (288)–(290) and [**51**, Lemma 2.1.3] together imply the existence of constants $C_{\sigma,h}>0$ and $C'_{\sigma,h}>0$ so that:

$$\begin{aligned}&\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}G(\delta_t*\rho_n)\Big\|\\ &\leqslant\|\rho\|_{M_p,1,[0,1]}\frac{M_\sigma e^{C_\sigma}}{2\pi}\int\limits_{\Gamma^1_\sigma\cup\Gamma^3_\sigma}e^{-\sigma tM(k|\lambda|)}e^{M(h'|\lambda|)}e^{M(h|\lambda|)}|d\lambda|+e^{\omega t}\|\rho\|_{M_p,1,[0,1]}C'_{\sigma,h}\\ &\leqslant C_{\sigma,h}\Bigg(e^{\omega t}+\int\limits_{\Gamma^1_\sigma\cup\Gamma^3_\sigma}e^{-\sigma tM(k|\lambda|)+\frac{3}{2}(\frac{h}{k}+1)M(k|\lambda|)+\frac{3}{2}(\frac{h'}{k}+1)M(k|\lambda|)}|d\lambda|\Bigg)\\ &\leqslant C_{\sigma,h}\Bigg(e^{\omega t}+\int\limits_{\Gamma^1_\sigma\cup\Gamma^3_\sigma}e^{(3+\frac{3h+3h'}{2k}-\sigma t)M(k|\lambda|)}|d\lambda|\Bigg)<\infty,\end{aligned}$$

for every $t>\frac{1}{\sigma}\big(3+\frac{3h+3h'}{2k}\big)$. By the preceding estimate, (290), the arbitrariness of $\sigma>0$, and the denseness of $\mathcal{R}(G)$ in E, we infer that the mapping $t\mapsto G(\delta_t)$, $t>0$ is infinitely differentiable and that, for every compact subset K of $(0,\infty)$ and $h>0$, we have $\|G(\delta_t)\|_{M_p,h,K}<\infty$, as required. $\square$

REMARK 3.5.25. Suppose $s>1$, $M_p=p!^s$ and there exist $\sigma>0$, $k>0$, $M>0$, $\omega\geqslant 0$ and $n\in\mathbb{N}$ such that $\Xi_{\sigma,k}\subseteq\rho(A)$ and that $\|R(\lambda:A)\|\leqslant M(1+|\lambda|)^n$, $\lambda\in\Xi_{\sigma,k}$. Then, by [**28**, Theorem 5] and Remark 3.2.20(i), A generates a (DS) of class $A^{1/s}$. Scalar type operators generating Gevrey ultradifferentiable C_0-semigroups (cf. [**299**, Theorem 5.1]) as well as generators of Crandall–Pazy class of C_0-semigroups (cf. [**165**, Definition 5.4, Theorem 5.5] and [**117**]) and their adjoints can be used for the construction of (DS)'s of $A^{1/s}$ class.

THEOREM 3.5.26. *Let A generate a dense* (EUDS) *G of $*$-class and $\alpha\in(0,\frac{\pi}{2}]$.*

(i) (Real analyticity) *Suppose that there exist $\omega\geqslant 0$ and $h>0$, such that for every $\varepsilon\in(0,1)$, there exist $C_\varepsilon>0$ and $M_\varepsilon>0$, in the Beurling case, resp., there exists $\omega\geqslant 0$ such that, for every $h>0$ and $\varepsilon\in(0,1)$, there exist $C_\varepsilon>0$ and $M_{\varepsilon,h}>0$, in the Roumieu case, so that: $\mathrm{T}_\varepsilon:=\big\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda+C_\varepsilon|\operatorname{Im}\lambda|\geqslant\omega\big\}\subseteq\rho(A)$ and*

$$\|R(\lambda:A)\|\leqslant M_\varepsilon e^{M(h|\lambda|)}e^{\varepsilon|\operatorname{Re}\lambda|},\quad \lambda\in\mathrm{T}_\varepsilon,\ \textit{resp.,}$$

$$\|R(\lambda:A)\|\leqslant M_{\varepsilon,h}e^{M(h|\lambda|)}e^{\varepsilon|\operatorname{Re}\lambda|},\ \lambda\in\mathrm{T}_\varepsilon. \tag{291}$$

Then G is infinitely differentiable and the mapping $t\mapsto G(\delta_t)$, $t>0$ is real analytic.

(ii) (Analyticity) *Suppose that the following conditions hold.*

(ii.1) *For every $\gamma\in(0,\alpha)$, there exist $M_\gamma>0$ and $\omega_\gamma\geqslant 0$ such that $\omega_\gamma+\Sigma_{\frac{\pi}{2}+\gamma}\subseteq\rho(A)$.*

(ii.2) *For every $\gamma\in(0,\alpha)$, there exist $h_\gamma>0$ and $C_\gamma>0$, resp., for every $h>0$, there exists $C_{h,\gamma}>0$ such that:*

$$\|R(\lambda:A)\|\leqslant C_\gamma e^{M(h_\gamma|\lambda|)},\ \textit{resp.,}\ \|R(\lambda:A)\|\leqslant C_{h,\gamma}e^{M(h|\lambda|)},\ \ \lambda\in\omega_\gamma+\Sigma_{\frac{\pi}{2}+\gamma}.$$

Then G *is an analytic* (EUDS) *of angle* $\arctan(\sin\alpha)$.

PROOF. We will give the proof of theorem in the Beurling case. The differentiability in (i) has been already proven. Let us prove that the mapping $t \mapsto G(\delta_t)$, $t > 0$ is real analytic. By the proof of scalar valued version of the Pringsheim theorem, it is enough to show that, for every compact set $K \subseteq (0,\infty)$, there exists $h > 0$ such that $\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{p!}\big\|\frac{d^p}{dt^p}G(\delta_t)\big\| < \infty$. Denote by Γ_ε the upwards oriented boundary of T_ε. Thanks to the Paley–Wiener theorem and the proof of Theorem 3.5.24(ii), it follows that $\frac{d^p}{dt^p}G(\delta_t*\rho_n)x = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} e^{\lambda t}R(\lambda\colon A)\lambda^p\widetilde{\rho_n}(\lambda)x\,d\lambda$, $t > 0$, $n \in \mathbb{N}$, $p \in \mathbb{N}_0$, $x \in \mathcal{R}(G)$ and that (290) holds. Given a compact set $K \subseteq (0,\infty)$, one can find $\varepsilon > 0$ and $h > 0$ such that $\inf K > 2\big(\varepsilon + h\frac{1+C_\varepsilon}{C_\varepsilon}\big)$.

The next computation involves [**51**, Lemma 2.1.3], the denseness of $\mathcal{R}(G)$ in E and (288):

$$\begin{aligned}
\frac{h^p}{p!}\Big\|\frac{d^p}{dt^p}G(\delta_t)\Big\| &\leqslant \sup_{n\in\mathbb{N}_0}\frac{h^p}{p!}\Big\|\frac{d^p}{dt^p}G(\delta_t*\rho_n)\Big\| \\
&\leqslant \|\rho\|_{M_p,1,[0,1]}\frac{M_\varepsilon}{\pi p!}\int\limits_0^\infty e^{(\gamma-C_\varepsilon s)t}e^{M(h|\gamma-C_\varepsilon s+is|)}e^{\varepsilon|\gamma-C_\varepsilon s|}h^p|\gamma-C_\varepsilon s+is|^p ds \\
&\leqslant \|\rho\|_{M_p,1,[0,1]}\frac{1}{\pi}e^{M(2h\gamma)}M_\varepsilon e^{(\inf K+\sup K)\gamma} \\
&\qquad\times\int\limits_0^\infty e^{-C_\varepsilon s\inf K}e^{M(2h(1+C_\varepsilon)s)}e^{\varepsilon C_\varepsilon s}e^{h(1+C_\varepsilon)s}ds \\
&\leqslant \|\rho\|_{M_p,1,[0,1]}\frac{1}{\pi}e^{M(\inf K\gamma)}M_\varepsilon e^{(\inf K+\sup K)\gamma} \\
&\qquad\times\int\limits_0^\infty e^{-C_\varepsilon s\inf K}e^{M(\inf K(1+C_\varepsilon)s)}e^{\varepsilon C_\varepsilon s}e^{(\frac{\inf K}{2}-\varepsilon)C_\varepsilon s}ds < \infty,
\end{aligned}$$

for all $p \in \mathbb{N}_0$ and $t \in K$. The proof of (i) follows easily from the previous inequality.

Let us prove (ii). By Theorem 3.5.24(ii), we have that G is infinitely differentiable and that $\frac{d^p}{dt^p}G(\delta_t) = \lim_{n\to\infty}\frac{d^p}{dt^p}G(\delta_t*\rho_n)$, $t > 0$, where (ρ_n) is a regularizing sequence in $\mathcal{D}^{(M_p)}$. Suppose $\gamma \in (0,\alpha)$, $t_0 > 0$ and $t \in \mathbb{C}$ satisfies $|t-t_0| < \arctan(\sin\gamma)$. Thanks to the argumentation given in the proof of Theorem 3.5.24(ii), we infer that the Neumann series $\sum_{p=0}^\infty (t-t_0)^p\frac{1}{p!}\big(\frac{d^p}{dr^p}G(\delta_t)\big)_{r=t_0}$ is absolutely convergent. Indeed, let Γ_γ be the boundary of $1+\omega_\gamma+\Sigma_{\frac{\pi}{2}+\gamma}$, oriented in such a way that $\operatorname{Im}\lambda$ increases along Γ_γ, and let $\Gamma_\gamma^1 := \{\lambda \in \Gamma_\gamma : \operatorname{Im}\lambda \leqslant 0\}$ and $\Gamma_\gamma^2 := \{\lambda \in \Gamma_\gamma : \operatorname{Im}\lambda \geqslant 0\}$. Then $\Gamma_\gamma^1 = \{1+\omega_\gamma+e^{-i(\frac{\pi}{2}+\gamma)}s : s \geqslant 0\}$, $\Gamma_\gamma^2 = \{1+\omega_\gamma+e^{i(\frac{\pi}{2}+\gamma)}s : s \geqslant 0\}$, and the proof of Theorem 3.5.24(ii) combined with [**51**, Lemma 2.1.3] implies that:

$$\sum_{p=0}^\infty |t-t_0|^p\Big\|\frac{1}{p!}\Big(\frac{d^p}{dr^p}G(\delta_r)\Big)_{r=t_0}\Big\| \leqslant \sup_{n\in\mathbb{N}_0}\sum_{p=0}^\infty |t-t_0|^p\Big\|\frac{1}{p!}\Big(\frac{d^p}{dr^p}G(\delta_r*\rho_n)\Big)_{r=t_0}\Big\|$$

$$\leqslant \|\rho\|_{M_p,1,[0,1]} \sum_{p=0}^{\infty} |t-t_0|^p \frac{1}{2\pi p!} \int\limits_{\Gamma_\gamma} e^{\operatorname{Re}\lambda t_0} e^{M(h_\gamma|\lambda|)} |\lambda|^p |d\lambda|$$

$$\leqslant \frac{1}{\pi} \|\rho\|_{M_p,1,[0,1]} e^{M(2h_\gamma(1+\omega_\gamma))} e^{|t-t_0|(1+\omega_\gamma)}$$

$$\times \int\limits_0^{\infty} e^{(\omega_\gamma + s\cos(\frac{\pi}{2}+\gamma))t_0} e^{M(2h_\gamma s)} e^{|t-t_0|s} ds < \infty.$$

This completes the proof of theorem. □

The subsequent proposition clarifies some interesting properties of exponentially bounded, analytic $K_{a,b}$-convoluted semigroups. Of importance is to stress that (cf. the estimate (274)) the assertion (ii) cannot be reformulated in the case of exponentially bounded, analytic K_l-convoluted semigroups.

PROPOSITION 3.5.27. *Suppose $\alpha \in (0, \frac{\pi}{2}]$, $a > 0$ and $b > 0$.*

(i) *Then A generates an exponentially bounded, analytic $K_{a,b}$-convoluted semigroup of angle α iff for every $\gamma \in (0,\alpha)$, there exist $\omega_\gamma \geqslant 0$ and $M_\gamma > 0$ such that the following conditions are satisfied:*

$$\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subseteq \rho(A), \tag{292}$$

$$\sup_{\lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}} \left\| (\lambda - \omega_\gamma) e^{-a|\lambda|^b \cos(b \arg(\lambda))} R(\lambda : A) \right\| \leqslant M_\gamma,\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}, \tag{293}$$

$$\lim_{\lambda \to +\infty} \lambda e^{-a\lambda^b} R(\lambda : A)x = 0,\ \ x \in E,\ \textit{if}\ \overline{D(A)} \neq E.$$

(ii) *If, additionally,*

$$b(\pi/2 + \alpha) > \pi/2, \tag{294}$$

then there does not exist a closed linear operator A which generates an exponentially bounded, analytic $K_{a,b}$-convoluted semigroup of angle α.

PROOF. The proof of (i) follows immediately from the general characterization of exponentially bounded, analytic convoluted C-semigroups. To prove (ii), suppose to the contrary that (294) holds and that a closed, linear operator A generates an exponentially bounded, analytic $K_{a,b}$-convoluted semigroup of angle α. By (294), one gets that there exist $\varepsilon \in (0,1)$ and $\gamma \in (0,\alpha)$ such that $b(\frac{\pi}{2} + \gamma - \varepsilon) > \frac{\pi}{2} + \varepsilon$. The first part of proposition yields that there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that (292) and (293) hold. Set $A_\gamma := A - \omega_\gamma$. Thanks to the choice of γ and ε, we have that there exists a number $r > 0$ such that, for every $\lambda \in \mathbb{C}$ satisfying $|\lambda| \geqslant r$ and $\arg(\lambda) \in (\frac{\pi}{2} + \gamma - \frac{\varepsilon}{2}, \frac{\pi}{2} + \gamma)$:

$$\left\| R(\lambda : A_\gamma) \right\| \leqslant \frac{M_\gamma}{|\lambda|} e^{a|\lambda+\omega_\gamma|^b \cos(b(\frac{\pi}{2}+\gamma-\varepsilon))} \leqslant \frac{M_\gamma}{|\lambda|} e^{-a|\lambda+\omega_\gamma|^b \sin\varepsilon}. \tag{295}$$

An application of [**249**, Lemma 1.5] gives $n(A) \leqslant 1$. Denote by A_γ^F the part of A_γ in the Banach space $F = \overline{D(A)}$; then A_γ^F also satisfies (295). In particular, (295) implies that there exists sufficiently large M'_γ such that

$$\big\|R(\lambda : e^{-i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})}A^F_\gamma)\big\|_F = \big\|e^{i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})}R(\lambda e^{i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})} : A^F_\gamma)\big\|_F \leqslant \frac{1}{\lambda},$$

$\lambda \in (M'_\gamma, \infty)$. Hence, $\|R(\lambda : e^{-i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})}A^F_\gamma)^n\|_F \leqslant \frac{1}{\lambda^n}$, $\lambda > M'_\gamma$, and the Hille–Yosida theorem implies that the operator $e^{-i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})}A^F_\gamma$ generates a C_0-semigroup $(T^F(t))_{t\geqslant 0}$ in the Banach space F such that $\|T^F(t)\|_F \leqslant e^{M'_\gamma t}$, $t \geqslant 0$. By [**14**, Proposition 4.1.3, p. 248], one gets that $\lim_{\lambda\to\infty} \lambda R(\lambda : e^{-i(\frac{\pi}{2}+\gamma-\frac{\varepsilon}{4})}A^F_\gamma)x = x$ for all $x \in F$. On the other hand, (295) yields that the above limit equals zero for all $x \in F$ and the contradiction is obvious. □

Suppose (M_p) satisfies (M.1), (M.2) and (M.3′). Put $L_p := M_p^{1/p}$, $p \geqslant 0$. It is worth noting that the proof of Theorem 3.5.24 enables one to establish the following characterization of distribution semigroups of class C_L; herein, a differentiable (DS) G is said to be of class C_L iff for every compact set $K \subseteq (0,\infty)$ and for every $h > 0$: $\sup_{t\in K,\, p\in\mathbb{N}_0} \big\|\frac{h^p}{L_p^p}\frac{d^p}{dt^p}G(\delta_t)\big\| < \infty$.

THEOREM 3.5.28. *Let A be a closed linear operator and let $L_p = M_p^{1/p}$, $p \in \mathbb{N}_0$. Then A generates a* (DS) *of class C_L iff the following conditions hold:*

$$\exists\omega \geqslant 0\ \exists m \in \mathbb{N}\ \exists h', k > 0\ \forall\sigma > 0\ \exists C_\sigma,\ M_\sigma > 0 \tag{296}$$

such that

$$\Xi^k_{\sigma,\omega} \subseteq \rho(A), \tag{297}$$

$$\|R(\lambda : A)\| \leqslant M_\sigma(1+|\lambda|)^m,\ \operatorname{Re}\lambda \geqslant \omega, \tag{298}$$

$$\|R(\lambda : A)\| \leqslant M_\sigma e^{M(h'|\lambda|)},\ \lambda \in \Xi^k_{\sigma,\omega},\ \operatorname{Re}\lambda \leqslant \omega. \tag{299}$$

PROOF. Suppose first that A is densely defined. Let A be the generator of a (DS) G of class C_L. Then there exists $n \in \mathbb{N}$ such that A generates an exponentially bounded n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$ such that $G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)S_n(t)x\,dt$, $\varphi \in \mathcal{D}$, $x \in E$ and that the mapping $t \mapsto S_n(t)$, $t > 0$ is infinitely differentiable. Let $\varepsilon > 0$, $\psi \in \mathcal{D}^{(M_p)}$, $\psi(t) = 1$, $0 \leqslant t \leqslant \frac{1}{2}$, $\psi(t) = 0$, $t \geqslant 1$, $\psi(t) = 0$, $t \leqslant -1$ and $\psi_\varepsilon(t) = \psi(\frac{t}{\varepsilon})$, $t \in \mathbb{R}$. Then the proof of Theorem 3.5.24(i) implies that, for every $\lambda \in \mathbb{C} \smallsetminus \{0\}$ with $\operatorname{Re}\lambda \leqslant 0$ and $\operatorname{Re}\lambda \geqslant -\frac{1}{\varepsilon}\ln\frac{1}{2M_{\varepsilon,1}} - \frac{1}{\varepsilon}M(\frac{|\lambda|}{H+1})$, we have $\|G(e^{-\lambda t}\psi'_\varepsilon(t))\| \leqslant \frac{1}{2}$ and $\lambda \in \rho(A)$; analogically, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$ and $\operatorname{Re}\lambda \geqslant -\frac{2}{\varepsilon}\ln\frac{1}{2M_{\varepsilon,1}} - \frac{2}{\varepsilon}M\big(\frac{|\lambda|}{H+1}\big)$, we have $\|G(e^{-\lambda t}\psi'_\varepsilon(t))\| \leqslant \frac{1}{2}$ and $\lambda \in \rho(A)$. The necessity is a consequence of our previous analyses. To prove the converse, notice that A generates an (EDS) G and that $G(\varphi)x = \frac{1}{2\pi i}\int_{\omega_0-i\infty}^{\omega_0+i\infty} \tilde{\varphi}(\lambda)R(\lambda : A)x\,d\lambda$, $x \in E$, $\varphi \in \mathcal{D}$, where $\omega_0 > \omega$. Let (ρ_n) be a regularizing sequence in $\mathcal{D}_0^{(M_p)}$ and let Γ_σ designate the upwards oriented frontier of $\Xi^k_{\sigma,\omega}$. The Paley–Wiener theorem for ultradifferentiable functions implies that $G(\delta_t * \rho_n)x = \frac{1}{2\pi i}\int_{\Gamma_\sigma} e^{\lambda t}\widetilde{\rho_n}(\lambda)R(\lambda : A)x\,d\lambda$, $x \in E$, $t > 0$. Now one can repeat verbatim the arguments given in the proof of sufficiency in Theorem 3.5.24(ii) to end the proof in the case of densely defined operators. Suppose now that A is not densely defined. If A generates a (DS) G of class C_L, then $A_{|F}$ generates a dense (DS) $G_{|F}$ of class C_L in the Banach space $F = \overline{A^{n(A)}}$. The necessity follows from an application of [**249**, Proposition 2.1(iii)]. Let (296)–(299) hold. Then A generates an (EDS) G and, by [**249**, Proposition 2.1(iii)] and the first part of the proof, we have that $A_{|F}$ generates a dense (DS) $G_{|F}$ of class C_L in the Banach space F. In particular, $G_{|F}$ is differentiable and this, in turn, implies that G must be differentiable. The remnant of the proof follows as in the case of densely defined operators. □

COROLLARY 3.5.29. (i) *Let A be a closed linear operator and let $L_p = p!^{s/p}$, $s > 1$, $p \in \mathbb{N}_0$. Then A generates a* (DS) *of class C_L iff there exist $\omega \geqslant 0$, $m \in \mathbb{N}$, $h' > 0$ and $k > 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ such that:*

$$\Xi^s_{\sigma,\omega,k} \subseteq \rho(A), \quad \textit{where} \tag{300}$$

$$\Xi^s_{\sigma,\omega,k} := \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega, \operatorname{Re}\lambda \geqslant C_\sigma - \sigma k|\lambda|^{1/s}\big\} \,\cup\, \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\big\},$$

$$\|R(\lambda : A)\| \leqslant M_\sigma(1 + |\lambda|)^m, \ \operatorname{Re}\lambda \geqslant \omega, \tag{301}$$

$$\|R(\lambda : A)\| \leqslant M_\sigma e^{h'|\lambda|^{1/s}}, \ \lambda \in \Xi^s_{\sigma,\omega,k}, \ \operatorname{Re}\lambda \leqslant \omega. \tag{302}$$

(ii) *Let $\alpha \geqslant 0$, $L_p = M_p^{1/p}$, $p \in \mathbb{N}_0$ and let A be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$. Then $(S_\alpha(t))_{t\geqslant 0}$ is of class C_L iff* (296)–(299) *hold.*

(iii) *Let $\alpha \geqslant 0$, $s > 1$, $L_p = p!^{s/p}$, $p \in \mathbb{N}_0$ and let A be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$. Then $(S_\alpha(t))_{t\geqslant 0}$ is of class C_L iff there exist $\omega \geqslant 0$, $h' > 0$ and $k > 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ such that* (300)–(302) *hold.*

In the following example, we use the notion and notation given in [**14**, Chapter 8].

EXAMPLE 3.5.30. (i) Let $s > 1$, $k > 0$, $p \in [1, \infty)$, $m > 0$, $\rho \in [0, 1]$, $r > 0$, $a \in S^m_{\rho,0}$ satisfies (H_r),

$$n\Big|\frac{1}{2} - \frac{1}{p}\Big|\Big(\frac{m - r - \rho + 1}{r}\Big) < 1, \tag{303}$$

and assume that, for every $\sigma > 0$, there exists $C_\sigma > 0$ such that $a(\mathbb{R}^n) \cap \Xi^s_{\sigma,\omega,k} = \emptyset$. Let N_p be the smallest integer

$$\begin{cases} \geqslant n\big|\frac{1}{2} - \frac{1}{p}\big|\big(\frac{1+m-\rho}{r}\big), & p > 1, \\ > \frac{n}{2}\big(\frac{1+m-\rho}{r}\big), & p = 1. \end{cases}$$

By [**14**, Proposition 8.3.1, Theorem 8.3.6] and Corollary 3.5.29, the operator $\mathrm{Op}_p(a)$ generates an N_p-times integrated semigroup on $L^p(\mathbb{R}^n)$ which is of class C_L with $L_p = p!^{s/p}$. Herein we would like to note that the class C^L is more appropriate to deal with; for example, consider the elliptic polynomial $a(\xi) = -|\xi|^2 + i|\xi|^4$, $\xi \in \mathbb{R}^n$. Then the operator $\mathrm{Op}_p(a)$ generates an α-times integrated semigroup $(S(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^n)$ which is of class C^L with $L_p = p!^{2/p}$; by Corollary 3.5.29,

$(S(t))_{t\geqslant 0}$ is not of class C_L. It is of concern to accent that: $-\mathrm{Op}_p(a)$ generates a dense (UDS) of $(p!^s)$-class iff $s \in (1,2]$, $-\mathrm{Op}_p(a)$ generates a dense (UDS) of $\{p!^s\}$-class iff $s \in (1,2)$ and $-\mathrm{Op}_p(a)$ does not generate a (DS). Finally, let us mention that L^p-realizations of some special cases of pseudodifferential operators associated to the linearized Benjamin–Ono-Burgers equation [**110, 452**] likewise generate ρ-hypoanalytic integrated (C-)semigroups.

(ii) Let $s > 1$, $k > 0$, $p \in [1,\infty)$, $m > 0$, $\rho \in [0,1]$, $r > 0$, $a \in S^m_{\rho,0}$ satisfies (H_r), (303) holds, $E = L^p(\mathbb{R}^n)$ or $E = C_0(\mathbb{R}^n)$ (in the last case, we assume $p = \infty$), and $A = \mathrm{Op}_E(a)$. If $a(\cdot)$ is an elliptic polynomial of order m, then $m = r$, $\rho = 1$ and (303) holds. Assume first that there exist a sequence (M_p) satisfying (M.1), (M.2), (M.3$'$) and appropriate constants $l \geqslant 1$, $\zeta > 0$ and $\eta \in \mathbb{R}$ such that $a(\mathbb{R}^n) \cap \Lambda_{l,\zeta,\eta} = \emptyset$. By [**14**, Lemma 8.2.1, Proposition 8.2.6, the proof of Lemma 8.2.8], it follows that there exists $\eta' \geqslant \eta$ such that $\|R(\cdot : A)\|$ is polynomially bounded on $\Lambda_{l,\zeta,\eta'}$. This implies that A generates an ultradistribution semigroup of (M_p)-class. If $a(\mathbb{R}^n) \cap (P_{\nu,C'} \cup B_d) = \emptyset$ for some $\nu \in (0,1)$, $C' \in (0,1]$ and $d \in (0,1]$, then Theorem 1.4.15 can be applied with a convenable chosen $\alpha \geqslant -1$; the typical example is the operator $A = \xi\Delta^2 - i\varrho\Delta + \varsigma$ ($\xi > 0$, $\varrho \in \mathbb{R} \smallsetminus \{0\}$, $\varsigma < 0$).

REMARK 3.5.31. (i) The sufficient condition for the generation of dense (C-DS)'s of class C_L can be derived similarly [**234**]. Concerning the proof of Theorem 3.5.28, we also perceive the following interesting phenomenon. In order to deform the path of integration from the straight line connecting the points $\omega_0 - i\infty$ and $\omega_0 + i\infty$ into the upwards oriented frontier of the region $\Xi^k_{\sigma,\omega}$, we essentially utilize the regularizing sequence (ρ_n) in the space $\mathcal{D}_0^{(M_p)}$. Under the assumption (296)–(299), A generates an exponentially bounded $(m+2)$-times integrated semigroup $(S_{m+2}(t))_{t\geqslant 0}$ given by $S_{m+2}(t) = \frac{1}{2\pi i}\int_{\omega_0-i\infty}^{\omega_0+i\infty} e^{\lambda t}\frac{R(\lambda A)}{\lambda^{m+2}}\,d\lambda$, $t \geqslant 0$. On the other hand, the proof of Theorem 3.5.24(ii) implies that the improper integral $\frac{1}{2\pi i}\int_{\Gamma_\sigma} e^{\lambda t}\frac{R(\lambda A)}{\lambda^{m+2}}\,d\lambda =: S^0_{m+2}(t)$ converges for every $t > \frac{1}{\sigma}\big(3 + \frac{3h'}{2k}\big)$. Furthermore, for every compact set $K \subseteq \big(\frac{1}{\sigma}\big(3 + \frac{3h'}{2k}\big), \infty\big)$ and for every $h > 0$, one has $\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\big\|\frac{d^p}{dt^p}S^0_{m+2}(t)\big\| < \infty$. Unfortunately, it is not clear whether $S^0_{m+2}(t) = S_{m+2}(t)$ for $t > \frac{1}{\sigma}\big(3 + \frac{3h'}{2k}\big)$.

(ii) Let A generate a dense infinitely differentiable (UDS) G and let for every compact set $K \subseteq (0,\infty)$ there exists $h_K > 0$ such that $\|G(\delta_t)\|_{M_p,h_K,K} < \infty$. By the proof of Theorem 3.5.24, it follows that there exist $\omega \geqslant 0$ and $h' > 0$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$, $M_\sigma > 0$ and $k_\sigma > 0$ in the Beurling case, resp., there exists $\omega \geqslant 0$ such that, for every $h' > 0$ and $\sigma > 0$, there exist $C_\sigma > 0$, $k_\sigma > 0$ and $M_{\sigma,h'} > 0$, in the Roumieu case, such that $\Xi^{k_\sigma}_{\sigma,\omega} \subseteq \rho(A)$ and:

$$\|R(\lambda : A)\| \leqslant M_\sigma e^{M(\frac{|\lambda|}{h'}) + \frac{1}{2}M(k_\sigma|\lambda|)}, \quad \lambda \in \Xi_{\sigma,k_\sigma}, \text{ in the Beurling case, resp.,}$$

$$\|R(\lambda : A)\| \leqslant M_{\sigma,h'} e^{M(\frac{|\lambda|}{h'}) + \frac{1}{2}M(k_\sigma|\lambda|)}, \quad \lambda \in \Xi_{\sigma,k_\sigma}, \text{ in the Roumieu case.}$$

The proof of Theorem 3.5.24(ii) and the additional assumption $\limsup_{\sigma\to\infty} \sigma k_\sigma = \infty$ obviously imply that, for every compact set $K \subseteq (0,\infty)$, there exists $h_K > 0$

such that $\|G(\delta_t)\|_{M_p,h_K,K} < \infty$. It is quite questionable whether this holds in general.

The previous consideration clearly implies that the problem of finding a satisfactory Hille-Yosida's type theorem for differentiable ultradistribution semigroups is quite non-trivial. An additional difficulty is that the use of structural theorems for operator-valued ultradistributions does not take any effect.

Differential and analytical properties of hyperfunction fundamental solutions can be found in the papers of Kochubei [**205**] and Ōuchi [**353**]. Finally, we would like to propose the following questions.

QUESTIONS. 1. Suppose $\alpha \in (0, \frac{\pi}{2}]$, $\gamma \in (0, \alpha)$ and A generates an analytic (DS) of angle α. Are there $M_\gamma > 0$, $\omega_\gamma \geqslant 0$ and $n_\gamma \in \mathbb{N}$ such that (205) holds?

2. Suppose that G is a non-dense differentiable (UDS) of $*$-class. Must G be exponential (infinitely differentiable)?

3. Suppose A generates a dense (EUDS) G of $*$-class, $\alpha \in (0, \frac{\pi}{2}]$ and the assumptions (i) and (ii) quoted in the formulation of Theorem 3.5.26(ii) hold. Does there exist $\beta \in (\arctan(\sin\alpha), \alpha]$ such that G is an analytic (UDS) of angle β?

3.5.4. Hyperfunction spaces, semigroups and sines. The basic facts about Sato's hyperfunctions and Fourier hyperfunctions can be found in the monograph of Kaneko [**178**] (see also [**69**]–[**72**], [**157**]–[**170**], [**183**]–[**184**], [**207**], [**210**] and [**379**]). Let Ω be an open set in $\mathbb{C}$ containing an open set $I \subseteq \mathbb{R}$ as a closed subset and let $\mathcal{O}(\Omega)$ be the space of $E-$valued holomorphic functions on Ω endowed with the topology of uniform convergence on compact subsets of Ω. Then $\mathcal{O}(\Omega)$ is an (FS) space, and, as a closed subspace of $C^\infty(\Omega)$, the space $\mathcal{O}(\Omega)$ is nuclear. The space of $E-$valued hyperfunctions on I is defined as $\mathcal{B}(I, E) := \mathcal{O}(\Omega \setminus I, E)/\mathcal{O}(\Omega, E)$. A representative of $f = [f(z)] \in \mathcal{B}(I, E)$, $f \in \mathcal{O}(\Omega \setminus I, E)$ is called a defining function of f. The space of hyperfunctions supported by a compact set $K \subseteq I$ with values in E is denoted by $\Gamma_K(I, \mathcal{B}(E)) =: \mathcal{B}(K, E)$. It is the space of continuous linear mappings from $\mathcal{A}(K)$ into E, where $\mathcal{A}(K)$ is the space of analytic functions in neighborhoods of K endowed with the uniform convergence on compact neighborhoods of K (see [**206**, p. 107]). Since $\mathcal{A}(K)$ is a (DFS) space, we have that $\mathcal{B}(K, E)$ is an (FS) space; let us also mention that $\mathcal{A}(K)$ is an (LF) space. The space of all scalar valued hyperfunctions with the support in $[a, \infty)$, where $a \in \mathbb{R}$, is defined by $\mathcal{B}_{[a,\infty)} := \mathcal{O}(\mathbb{C} \setminus [a, \infty))/\mathcal{O}(\mathbb{C})$. The space of all $E-$valued hyperfunctions with the support contained in $[a, \infty)$ is defined similarly. By $\mathcal{B}_0(\mathbb{R}, E)$ is denoted the space which consists of all $E-$valued hyperfunctions supported by $[0, \infty)$. Recall, if $f \in \mathcal{B}(\mathbb{R}, E)$ and $\operatorname{supp} f \subseteq \{a\}$, then $f = \sum_{n=0}^{\infty} \delta^{(n)}(\cdot - a)x_n$, $x_n \in E$, where $\lim_{n\to\infty}(n!\|x_n\|)^{1/n} = 0$. For further information related to hyperfunction spaces, we refer to [**178**]; the convolution of operator valued hyperfunctions in the following definition is taken in the sense of [**205**].

DEFINITION 3.5.32. [**353**] Suppose A is a closed linear operator. By a *hyperfunction fundamental solution* for A we mean an element $G \in \mathcal{B}_0(\mathbb{R}, L(E, [D(A)]))$

satisfying $G * P = \delta \otimes I_{[D(A)]}$ and $P * G = \delta \otimes I_E$, where $P = \delta' \otimes A - \delta \otimes I \in \mathcal{B}_0(\mathbb{R}, L([D(A)], E))$.

Notice that there exists at most one hyperfunction fundamental solution for a closed linear operator A.

THEOREM 3.5.33. [**354**] *Suppose A is a closed linear operator. Then there exists a hyperfunction fundamental solution for A iff for every $\varepsilon \in (0,1)$, there exist $C_\varepsilon > 0$ and $M_\varepsilon > 0$ so that $\Omega_{\varepsilon,C_\varepsilon} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \varepsilon|\lambda| + C_\varepsilon\} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant M_\varepsilon e^{\varepsilon|\lambda|}$, $\lambda \in \Omega_{\varepsilon,C_\varepsilon}$.*

PROPOSITION 3.5.34. *Let $K \in L^1_{\mathrm{loc}}([0,\tau))$, for some $0 < \tau \leqslant 1$, and let A generate a K-convoluted semigroup $(S_K(t))_{t\in[0,\tau)}$. If K can be extended to a function K_1 in $L^1_{\mathrm{loc}}([0,\infty))$ which satisfies* (P1) *so that its Laplace transform has the same estimates as in Theorem* 2.7.1*, then there exists a hyperfunction fundamental solution for A.*

Relations between hyperfunction semigroups and C-regularized semigroups are complicated. The following instructive example shows that there exists a densely defined operator A on the Hardy space $H^2(\mathbb{C}_+)$ which has the following properties:

(i) There exists a hyperfunction fundamental solution for A.
(ii) A is not a subgenerator of a local integrated C-semigroup.

This example is essentially due to Beals [**37**].

EXAMPLE 3.5.35. Let $\psi(t) = \frac{t}{\ln(t+1)}$, $t > 0$, $\psi(0) = 1$. Then ψ is nonnegative, continuous, concave function on $[0,\infty)$ with $\psi(t) \to \infty$, $\frac{\psi(t)}{t} \to 0$, $t \to \infty$ and $\int_1^\infty \frac{\psi(t)}{t^2}\,dt = \infty$. It is clear that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\varepsilon t + C_\varepsilon \geqslant \psi(t)$, $t \geqslant 0$. Let A be a closed, densely defined linear operator acting on $E := H^2(\mathbb{C}_+)$ such that:

$$\Omega(\psi) := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \psi(|\operatorname{Im}\lambda|)\} \subseteq \rho(A), \quad \|R(\lambda : A)\| \leqslant \frac{M}{1+\operatorname{Re}\lambda},\ \lambda \in \Omega(\psi),$$

and that for all $\tau \in (0,\infty)$ there does not exist a solution of the following abstract Cauchy problem

$$\begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^1((0,\tau) : E), \\ u'(t) = Au(t),\ t \in (0,\tau), \\ u(0) = x, \end{cases}$$

unless $x = 0$. The existence of such an operator is proved in [**37**, Theorem 2']. Since $\rho(A) \neq \emptyset$, one gets that $D_\infty(A)$ is dense in E. Suppose that A is a subgenerator of a local k-times integrated C-semigroup on $[0,\tau)$, for some injective operator $C \in L(E)$, $k \in \mathbb{N}$ and $\tau \in (0,\infty)$. Then the problem

$$\begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^1([0,\tau) : E), \\ u'(t) = Au(t),\ t \in [0,\tau), \\ u(0) = x, \end{cases}$$

has a unique solution for all $x \in C(D(A^{k+1}))$. It follows $C(D(A^{k+1})) = \{0\}$ and this is a contradiction. On the other hand, it is easy to see that $\Omega_\varepsilon \subseteq \Omega_\psi \subseteq \rho(A)$.

The growth rate of resolvent shows that there exists a hyperfunction fundamental solution for A. It is clear that there exists an operator A which generates an entire C-regularized group but not a hyperfunction semigroup.

DEFINITION 3.5.36. It is said that a closed linear operator A admits a *hyperfunction fundamental sine solution* if the operator $\mathcal{A}$ admits a hyperfunction fundamental solution. The *ultradistribution fundamental sine solution of* $*$*-class* is defined in the same manner; a closed linear operator A generates an *ultradistribution sine of* $*$*-class* iff the operator $\mathcal{A}$ generates a (UDS) of $*$-class.

Let remind us that $K_\delta(t) = \mathcal{L}^{-1}\big(e^{-\lambda^\delta}\big)(t)$, $t \geqslant 0$.

THEOREM 3.5.37. [**228**] (i) *Suppose that A admits an ultradistribution fundamental sine solution of the Beurling, resp., the Roumieu class. Then A generates an exponentially bounded, analytic K_δ-semigroup of angle $\frac{\pi}{2}$, for all $\delta \in (\frac{1}{2s}, \frac{1}{2})$, resp., for all $\delta \in [\frac{1}{2s}, \frac{1}{2})$.*

(ii) *Suppose that A admits a hyperfunction fundamental sine solution. Then A generates an exponentially bounded, analytic $K_{1/2}$-semigroup of angle $\frac{\pi}{2}$.*

(iii) *Let A admit an ultradistribution fundamental sine solution of (M_p)-class, resp., $\{M_p\}$-class. Then, for every $\theta \in [0, \frac{\pi}{2})$, there exists an ultradistribution fundamental solution of (M_p^2)-class, resp., $\{M_p^2\}$-class for $e^{\pm i\theta}A$. Herein (M_p) satisfies* (M.1), (M.2) *and* (M.3′).

(iv) *Suppose that A admits a hyperfunction fundamental sine solution. Then, for every $\theta \in [0, \frac{\pi}{2})$, there exists an ultradistribution fundamental solution of $\{p!^2\}$-class for $e^{\pm i\theta}A$.*

PROOF. We will only give the proof of (i) in the Roumieu case. Let us fix $\gamma \in (0, \frac{\pi}{2})$ and $\delta \in [\frac{1}{2s}, \frac{1}{2})$. It is clear that there exists $C_s > 0$ with $M(|\lambda|) \leqslant C_s|\lambda|^{1/s}$, $\lambda \geqslant 0$. By the foregoing,

$$\big\{\lambda^2 : \lambda \in \mathbb{C},\ \operatorname{Re}\lambda \geqslant C_s(k|\lambda|)^{1/s} + C_k\big\} \subseteq \rho(A), \text{ i.e.,}$$

$$\big\{r^2e^{2i\theta} : r > 0,\ |\theta| < \pi/2,\ r\cos\theta \geqslant C_sk^{1/s}r^{1/s} + C_k\big\} \subseteq \rho(A).$$

Denote $\Gamma = \{re^{i\theta} : r\cos\theta = C_sk^{1/s}r^{1/s} + C_k\}$. Then $\lim_{|\lambda|\to\infty,\ \lambda\in\Gamma} |\arg(\lambda)| = \frac{\pi}{2}$. Therefore, there exist an $\omega_\gamma > 0$ and a suitable $\overline{C}_k > 0$ so that $\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subseteq \rho(A)$ and that

$$\|R(\lambda : A)\| \leqslant \overline{C}_k e^{M(k\sqrt{|\lambda|})} \leqslant \overline{C}_k e^{C_sk^{1/s}|\lambda|^{1/2s}},\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}.$$

The function $g : \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \to \mathbb{C}$, $g(\lambda) := e^{-\lambda^\delta}$, $\lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}$ is analytic, where $1^\delta = 1$. Furthermore,

$$\begin{aligned}\|g(\lambda)R(\lambda : A)\| &\leqslant \overline{C}_k \frac{1}{|e^{\lambda^\delta}|} e^{C_sk^{1/s}|\lambda|^{1/2s}} \\ &= \overline{C}_k \exp\big(-|\lambda|^\delta \cos(\delta\arg(\lambda)) + C_sk^{1/s}|\lambda|^{1/2s}\big) \\ &\leqslant \overline{C}_k e^{C_sk^{1/s}|\lambda|^{1/2s} - \cos(\pi\delta)|\lambda|^\delta},\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}.\end{aligned}$$

The choice of δ and arbitrariness of the number $k>0$ in the Roumieu case, combining with Theorem 2.4.5, imply that A generates an exponentially bounded, analytic K_δ-semigroup of angle γ. $\square$

For the properties of Laplace and Fourier hyperfunction fundamental solutions, we recommend for the reader [**166**]–[**167**], [**205**] and [**226**].

EXAMPLE 3.5.38. Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $1 \leqslant p < \infty$. Suppose that $E := H^p(\mathbb{C}_+)$. Let us recall that Beals constructed in the proof of [**37**, Theorem 2′] an analytic function $a_1 : \mathbb{C}_+ \to \{z \in \mathbb{C} : |z| \geqslant 1\}$ with the property that for every $\varepsilon > 0$, there exists a region of the form $\Omega_{\varepsilon,C_\varepsilon}$ satisfying $a_1(\mathbb{C}_+) \cap \Omega_{\varepsilon,C_\varepsilon} = \emptyset$. Let $B = a_1^2$. Then B is a holomorphic function on $\mathbb{C}_+$ and for all $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $K_\varepsilon > 0$ so that $B(\mathbb{C}_+) \subseteq (\Omega^2_{\varepsilon,C_\varepsilon})^c$. Define $(AF)(z) := B(z)F(z)$, $\operatorname{Im} z > 0$, $D(A) := \big\{F \in H^p(\mathbb{C}_+) : AF \in H^p(\mathbb{C}_+)\big\}$. Let $\varepsilon \in (0,1)$ be fixed and let $\varepsilon_1 \in (0,\varepsilon)$ satisfy $B(\mathbb{C}_+) \subseteq (\Omega^2_{\varepsilon_1,C_{\varepsilon_1}})^c$. Clearly, $\lim_{\lambda\to\infty,\ \lambda\in\partial\Omega_{\varepsilon_1,C_{\varepsilon_1}}} |\arg(\lambda)| = \arccos\varepsilon_1$ and there exists a sufficiently large $\overline{C}_\varepsilon > 0$ such that

$$\Omega_{\varepsilon,\overline{C}_\varepsilon} = \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \varepsilon|\lambda| + \overline{C}_\varepsilon\big\} \subseteq \Omega_{\varepsilon_1,C_{\varepsilon_1}} \text{ and } d := \operatorname{dist}(\partial\Omega_{\varepsilon_1,C_{\varepsilon_1}}, \partial\Omega_{\varepsilon,\overline{C}_\varepsilon}) > 0.$$

This implies $\Omega^2_{\varepsilon,\overline{C}_\varepsilon} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant d^{-2}$, $\lambda \in \Omega^2_{\varepsilon,\overline{C}_\varepsilon}$. Therefore, A admits a hyperfunction fundamental sine solution and it can be easily seen that A does not admit an ultradistribution fundamental sine solution.

EXAMPLE 3.5.39. [**226**] (i) Suppose $c > 1$, $\sigma > 0$, $\varsigma \in \mathbb{R}$, $M > 0$, $k \in \mathbb{N}$,

$$\sigma(A) \subseteq \pm\Pi_{c,\sigma,\varsigma} \text{ and } \|R(\lambda : A)\| \leqslant M(1+|\lambda|)^k,\ \lambda \notin \pm\Pi_{c,\sigma,\varsigma}.$$

Let p be as in the formulation of Example 3.5.18(i). Then the operator $(-1)^{n+1}p(A)$ generates an ultradistribution sine of $*$-class provided $M_p = p!^s$ and $s \in (1, \frac{n}{n-2+\frac{2}{c}})$.

(ii) Let p be as in (i) and let A generate a (local) integrated cosine function. Then the operator $(-1)^{n+1}p(A)$ generates an ultradistribution sine of $*$-class provided $M_p = p!^s$ and $s \in \big(1, \frac{n}{n-1}\big)$.

(iii) Suppose $c \in (0,1)$, $\sigma > 0$, $\varsigma \in \mathbb{R}$, $\sigma(A) \subseteq \pm\big(\mathbb{C} \setminus \{\lambda^2 : \lambda \in \Pi_{c,\sigma,\varsigma}\}\big)$ and $\|R(\cdot : A)\|$ is polynomially bounded on the complement of $\{\lambda^2 : \lambda \in \Pi_{c,\sigma,\varsigma}\}$. Let p be as in (i). Then the operator $(-1)^{n+1}p(A)$ generates an ultradistribution sine of $*$-class provided $M_p = p!^s$ and $s \in \big(1, \frac{n}{n+c-1}\big)$.

3.6. Regularization of ultradistribution semigroups and sines

3.6.1. Regularization of Gevrey type ultradistribution semigroups. In this subsection, we will always assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and that (M.1) holds. Every employment of the conditions (M.2), (M.3′) and (M.3) will be explicitly accented; the use of symbols A and M is clear from the context.

Let remind us of the following notations. Given $\theta \in (0,\pi]$ and $d \in (0,1]$, denote $\Sigma_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$, $B_d = \{\lambda \in \mathbb{C} : |\lambda| \leqslant d\}$ and $\Omega_{\theta,d} = \Sigma_\theta \cup B_d$. By $\Gamma_{\theta,d}$ is denoted the upwards oriented boundary of $\Omega_{\theta,d}$. Further, $\lfloor\beta\rfloor = \sup\{n \in \mathbb{Z} : n \leqslant \beta\}$, $\lceil\beta\rceil = \inf\{n \in \mathbb{Z} : \beta \leqslant n\}$ and (M_p)-ultralogarithmic region of type l,

$\Lambda_{\alpha,\beta,l}$, is defined by $\Lambda_{\alpha,\beta,l} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha M(l|\operatorname{Im}\lambda|) + \beta\}$, $\alpha > 0$, $\beta \in \mathbb{R}$, $l \geqslant 1$. From now on, we assume that the boundary of the ultra-logarithmic region $\Lambda_{\alpha,\beta,l}$, denoted by Γ_l, is upwards oriented.

Notice that the class of ultradistribution sines can be introduced following the approaches of Miana [**312**] and the author [**222**] for (almost-)distribution cosine functions, or by means of convolution type equations as it has been done by Komatsu [**210**]. The concepts presented in [**210, 312**] and [**222**] are not so easily comparable in ultradistribution case.

The assertions (i), (iv) and (v) of the subsequent theorem can be attributed to Straub [**394**]. Here we notice that the denseness of A is not used in the proofs of Propositions 2.2, 2.5, 2.6 and 2.8 as well as Lemmas 2.7 and 2.10 of [**394**] and that the assertion (v) extends [**38**, Lemma 1] and some estimates used in the proof of [**252**, Lemma 5.4] (cf. also [**20**, Lemma II-1, Theorem II-3]). The main problem in regularization of ultradistribution semigroups whose generators do not have polynomially bounded resolvent appears exactly in this place. Actually, if $\|R(\cdot : A)\|$ is not polynomially bounded on an appropriate ultra-logarithmic region, then it is not clear whether there exists an $n \in \mathbb{N}_0$ such that for every $x \in D(A^{n+2})$, the operator $T_b(t)$, defined in the formulation of the next theorem, fulfills $\lim_{t\to 0+} T_b(t)x = x$. Then it is not clear how one can prove that the operator $T_b(t)$ is injective; see also [**38**, Lemma 3], [**252**, Lemma 5.4] and the proof of [**394**, Proposition 2.8].

THEOREM 3.6.1. *Suppose that there exists a number $b \in (0,1)$ such that*

$$p^{\frac{p}{b}} \prec M_p \tag{304}$$

and that (M_p) satisfies (M.2). *If A is a closed linear operator such that there exist $\alpha > 0$, $l > 0$, $M > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$ satisfying*

$$\Lambda_{\alpha,\beta,l} \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant M(1+|\lambda|)^n, \ \lambda \in \Lambda_{\alpha,\beta,l},$$

then, for every $\gamma \in \big(0, \arctan(\cos(\frac{b\pi}{2}))\big)$, there are an $\omega \in \mathbb{R}$ and an analytic operator family $(T_b(t))_{t\in\Sigma_\gamma}$ of growth order $\frac{n+1}{b}$ satisfying:

(i) *For every $t \in \Sigma_\gamma$, the operator $T_b(t)$ is injective.*
(ii) *For every $t \in \Sigma_\gamma$, the operator A generates a global $T_b(t)$-regularized semigroup $(S_{b,t}(s))_{s\geqslant 0}$.*
(iii) *Let $K \subseteq [0,\infty)$ be a compact set, $t \in \Sigma_\gamma$ and $x \in E$. Then the mapping $s \mapsto S_{b,t}(s)x$ is infinitely differentiable in $s \geqslant 0$ and, for every $h > 0$,*

$$\sup_{p\in\mathbb{N}_0,\, s\in K} \frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S_{b,t}(s)x\Big\| < \infty.$$

(iv) *There is an $L > 0$ with $\|T_b(t)\| \leqslant L(\tan\gamma \operatorname{Re} t - |\operatorname{Im} t|)^{-\frac{n+1}{b}}$, $t \in \Sigma_\gamma$.*
(v) *If $x \in D(A^{n+2})$, then there exists $\lim_{t\to 0+} \frac{T_b(t)x - x}{t}$ and, in particular, $\lim_{t\to 0+} T_b(t)x = x$.*

Furthermore, $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{n+1}{b}$ whose c.i.g. is $-\big((\omega - A)_{|\overline{A^{n}(A)}}\big)^b$.

Proof. Let $\varphi : \mathbb{R} \to [0,\infty)$ be an infinitely differentiable function satisfying $\operatorname{supp}\varphi \subseteq [0,1]$ and $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. Put $M_1(t) := \int_0^t M(t-s)\varphi(s)\,ds,\ t \geqslant 0$ and notice that $M_1 \in C^{\infty}(\mathbb{R})$ and that there exist $m \in (0,\infty)$ and $M \in (0,\infty)$ such that $M(t) \leqslant mM_1(t) + M \leqslant mM(t) + M,\ t \geqslant 0$. Suppose $(0,1) \ni b$ satisfies $p^{\frac{p}{b}} \prec M_p$ and designate by $N(\cdot)$ the associated function of the sequence $(p^{\frac{p}{b}})$. Then $N(\lambda) \sim \frac{1}{be}|\lambda|^b,\ \lambda \to \infty$ and an application of [**207**, Lemma 3.10] gives that, for every $\mu > 0$, there exist positive real constants c_μ and C_μ such that $\lim_{\mu\to 0} c_\mu = 0$ and

$$M_1(l\lambda) \leqslant M(l\lambda) \leqslant N(\mu l\lambda) + C_\mu \leqslant c_\mu|\lambda|^b + C_\mu,\ \lambda \geqslant 0. \tag{305}$$

Denote, for $\sigma > 0$ and $\varsigma \in \mathbb{R}$, $\Lambda^1_{\sigma,\varsigma,l} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \sigma M_1(l|\operatorname{Im}\lambda|) + \varsigma\}$ and $\Pi_{b,\sigma,\varsigma} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \sigma|\operatorname{Im}\lambda|^b + \varsigma\}$. By the foregoing, we have $\Lambda^1_{\alpha m,\beta+\alpha M,l} \subseteq \Lambda_{\alpha,\beta,l} \subseteq \rho(A)$. Choose now a number $a \in (0,\frac{\pi}{2})$ such that $b \in \bigl(0,\frac{\pi}{2(\pi-a)}\bigr)$ and that $\gamma \in \bigl(0,\arctan(\cos(b(\pi-a)))\bigr)$. Thanks to (305), one obtains the existence of numbers $d \in (0,1]$, $\sigma \in (0,\infty)$ and $\omega \in \mathbb{R}$ such that $\Omega_{a,d} \subseteq \Lambda^1_{\alpha m,\beta+\alpha M-\omega,l} \subseteq \rho(A-\omega)$. Let $\Gamma_{a,d}$ and Γ denote the upwards oriented boundaries of $\Omega_{a,d}$ and $\Lambda^1_{\alpha m,\beta+\alpha M-\omega,l}$, respectively. Define $T_b(t),\ t \in \Sigma_\gamma$ by

$$T_b(t)x := \frac{1}{2\pi i}\int_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : A-\omega)x\,d\lambda,\ x \in E.$$

By Theorem 1.4.15, $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{n+1}{b}$ whose c.i.g. is $-\bigl((\omega-A)_{|\overline{A^{n}(A)}}\bigr)^b$. Define now, for every $t = t_1 + it_2 \in \Sigma_\gamma$, $s \geqslant 0$ and $x \in E$,

$$S^1_{b,t}(s)x := \frac{1}{2\pi i}\int_{\Gamma} e^{-t(-\lambda)^b} e^{\lambda s} R(\lambda : A-\omega)x\,d\lambda.$$

To prove that $S^1_{b,t}(s) \in L(E)$, notice that, for every $\lambda \notin \Omega_{a,d}$, we have $b\arg(-\lambda) \in (b(-\pi+a), b(\pi-a))$, $\cos(b\arg(-\lambda)) \in (\cos(b(\pi-a)),1]$, $\tan\gamma < \cos(b(\pi-a))$ and

$$\begin{aligned}\bigl|e^{-t(-\lambda)^b}\bigr| &= e^{-t_1|\lambda|^b\cos(b\arg(-\lambda))+t_2|\lambda|^b\sin(b\arg(-\lambda))}\\ &\leqslant e^{-(t_1\cos(b\arg(-\lambda))-|t_2|)|\lambda|^b} \leqslant e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \leqslant e^{-(t_1\tan\gamma-|t_2|)|\lambda|^b}.\end{aligned}$$

This inequality and (305) imply that, for all sufficiently small $\mu > 0$:

$$\begin{aligned}&\Bigl|e^{-t(-\lambda)^b}e^{\lambda s}\|R(\lambda : A-\omega)\|\Bigr|\\ &\quad\leqslant Me^{s(\alpha mM_1(l|\operatorname{Im}\lambda|)+\beta+\alpha M-\omega)}e^{-(t_1\tan\gamma-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n\\ &\leqslant M_\mu e^{s(\beta+\alpha M-\omega)}e^{s\alpha mc_\mu|\lambda|^b}e^{-(t_1\tan\gamma-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n,\ \lambda \in \Gamma,\ |\lambda| \geqslant r.\end{aligned} \tag{306}$$

The use of (306) with sufficiently small μ implies that $S^1_{b,t}(s) \in L(E)$, as required. Further on, the Cauchy formula and the previous argumentation enables one to see that

$$\int_{\Gamma} e^{\lambda s} e^{-t(-\lambda)^b} d\lambda = 0,\ s \geqslant 0,\ t \in \Sigma_\gamma \tag{307}$$

and that $T_b(t) = S^1_{b,t}(0)$, $t \in \Sigma_\gamma$. It is also clear that $S^1_{b,t}(s)T_b(t) = T_b(t)S^1_{b,t}(s)$ and that $S^1_{b,t}(s)(A-\omega) \subseteq (A-\omega)S^1_{b,t}(s)$, $s \geqslant 0$, $t \in \Sigma_\gamma$. Using the Fubini theorem, the resolvent equation and (307), one obtains

$$\begin{aligned}
(A-\omega)\int_0^s S_{b,t}(r)x\,dr &= \frac{1}{2\pi i}\int_0^s\int_{\Gamma} e^{\lambda r}e^{-t(-\lambda)^b}(A-\omega)R(\lambda : A-\omega)x\,d\lambda\,dr \\
&= \frac{1}{2\pi i}\int_0^s\int_{\Gamma} e^{\lambda r}e^{-t(-\lambda)^b}(\lambda R(\lambda : A-\omega)x - x)\,d\lambda\,dr \\
= \frac{1}{2\pi i}\int_0^s\int_{\Gamma} e^{\lambda r}e^{-t(-\lambda)^b}&\lambda R(\lambda : A-\omega)x\,d\lambda\,dr - \frac{1}{2\pi i}\int_0^s\int_{\Gamma} e^{\lambda r}e^{-t(-\lambda)^b}x\,d\lambda\,dr \\
&= \frac{1}{2\pi i}\int_0^s\int_{\Gamma} e^{\lambda r}e^{-t(-\lambda)^b}\lambda R(\lambda : A-\omega)x\,d\lambda\,dr \\
&= \frac{1}{2\pi i}\int_{\Gamma}\Bigg[\int_0^s e^{\lambda r}e^{-t(-\lambda)^b}\lambda R(\lambda : A-\omega)x\,dr\Bigg]d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma}(e^{\lambda s}-1)e^{-t(-\lambda)^b}R(\lambda : A-\omega)x\,d\lambda \\
= \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda s}e^{-t(-\lambda)^b}&R(\lambda : A-\omega)x\,d\lambda - \frac{1}{2\pi i}\int_{\Gamma} e^{-t(-\lambda)^b}R(\lambda : A-\omega)x\,d\lambda \\
&= S^1_{b,t}(s)x - T_b(t)x,\ s \geqslant 0,\ t \in \Sigma_\gamma,\ \ x \in E.
\end{aligned}$$

This implies that $(S^1_{b,t}(s))_{s\geqslant 0}$ is a global $T_b(t)$-regularized semigroup generated by $A-\omega$. In order to prove differentiability of $(S^1_{b,t}(s))_{s\geqslant 0}$, note that the arguments used in the proof of boundedness of the operator $S^1_{b,t}(s)$ also show that, for every $p \in \mathbb{N}$, the integral $\frac{1}{2\pi i}\int_{\Gamma}\lambda^p e^{\lambda s}e^{-t(-\lambda)^b}R(\lambda : A-\omega)\,d\lambda$ is convergent. Then the elementary inequality $|e^{\lambda h}-1| \leqslant h|\lambda|e^{\operatorname{Re}\lambda h}$, $\lambda \in \mathbb{C}$, $h > 0$ and the dominated convergence theorem yield $\frac{d}{ds}S^1_{b,t}(s) = \frac{1}{2\pi i}\int_{\Gamma}\lambda e^{\lambda s}e^{-t(-\lambda)^b}R(\lambda : A-\omega)\,d\lambda$, $s \geqslant 0$. Inductively,

$$\frac{d^p}{ds^p}S^1_{b,t}(s) = \frac{1}{2\pi i}\int_{\Gamma}\lambda^p e^{\lambda s}e^{-t(-\lambda)^b}R(\lambda : A-\omega)\,d\lambda,\ p \in \mathbb{N}_0,\ s \geqslant 0. \tag{308}$$

Taking into account (305) and (308), we easily infer that, for every compact set $K \subseteq [0,\infty)$, $t \in \Sigma_\gamma$ and $\mu > 0$:

$$\sup_{p\in\mathbb{N}_0,\ s\in K} \frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S^1_{b,t}(s)\Big\| \leqslant \frac{M}{2\pi}\int_\Gamma e^{M(h\lambda)}e^{\operatorname{Re}\lambda s}e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \times e^{-(t_1\tan\gamma-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{M}{2\pi}\int_\Gamma e^{M(h\lambda)}e^{s[\alpha M_1(l|\operatorname{Im}\lambda|)+\beta+\alpha M-\omega]}e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \times e^{-(t_1\tan\gamma-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{Me^{\sup K(\beta+\alpha M-\omega)}}{2\pi}\int_\Gamma e^{c_\mu\frac{h^b}{l^b}|\lambda|^b+C_\mu}e^{\alpha m\sup K[c_\mu\frac{h^b}{l^b}|\lambda|^b+C_\mu]} \times e^{-l(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{M}{2\pi}e^{\sup K[\beta+\alpha M-\omega+\alpha mC_\mu]+C_\mu}\int_\Gamma e^{c_\mu\frac{h^b}{l^b}|\lambda|^b(1+\sup K)}e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \times (1+|\lambda|+|\omega|)^n|d\lambda|.$$

Choosing sufficiently small μ, we obtain that $\sup_{p\in\mathbb{N}_0,\ s\in K}\frac{h^p}{M_p}\big\|\frac{d^p}{ds^p}S^1_{b,t}(s)\big\| < \infty$. Put now $S_{b,t}(s) := e^{\omega s}S^1_{b,t}(s)$, $s \geqslant 0$, $t \in \Sigma_\gamma$. Then it is clear that $(S_{b,t}(s))_{s\geqslant 0}$ is a global $T_b(t)$-regularized semigroup generated by A. Since (M_p) satisfies (M.1) and $M_0 = 1$, it can be easily seen that $M_{p+q} \geqslant M_pM_q$, p, $q \in \mathbb{N}_0$ (cf. for instance [**51**, Lemma 2.1.1]). Hence, for every $h_1 \in [h(2+2|\omega|),\infty)$:

$$\sup_{p\in\mathbb{N}_0,\ s\in K}\frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S_{b,t}(s)x\Big\|$$

$$\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\ s\in K}\frac{h^p2^p(1+|\omega|)^p}{M_p}\sum_{i=0}^p\Big\|\frac{d^{p-i}}{ds^{p-i}}S^1_{b,t}(s)x\Big\|$$

$$\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\ s\in K}h^p(2+2|\omega|)^p\sum_{i=0}^p\frac{CM_{p-i}}{h_1^{p-i}M_p}$$

$$\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\ s\in K}\big(h(2+2|\omega|)/h_1\big)^p\sum_{i=0}^p\frac{Ch_1^i}{M_i}$$

$$\leqslant Ce^{|\omega|\sup K}\sum_{i=0}^\infty\frac{h_1^i}{M_i} \leqslant Ce^{|\omega|\sup K}\sum_{i=0}^\infty\frac{h_1^i}{(2h_1)^i}\sup_{p\in\mathbb{N}_0}\frac{(2h_1)^p}{M_p}$$

$$\leqslant 2Ce^{|\omega|\sup K}e^{M(2h_1)} < \infty,\ \text{where } C = \sup_{p\in\mathbb{N}_0,\ s\in K}\frac{h_1^p}{M_p}\Big\|\frac{d^p}{ds^p}S^1_{b,t}(s)x\Big\|.$$

This implies (iii) and completes the proof. □

Before we go any further, let us notice that every Gevrey sequence satisfies (304) with $b \in (\frac{1}{s},1)$.

Corollary 3.6.2. *Suppose A is a closed operator and there exist $c \in (0,1)$, $\sigma > 0$, $M > 0$, $n \in \mathbb{N}$ and $\varsigma \in \mathbb{R}$ such that*

$$\Pi_{c,\sigma,\varsigma} = \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \sigma|\operatorname{Im}\lambda|^c + \varsigma\Big\} \subseteq \rho(A) \text{ and} \tag{309}$$

$$\|R(\lambda\colon A)\| \leqslant M(1+|\lambda|)^n,\ \lambda \in \Pi_{c,\sigma,\varsigma}. \tag{310}$$

Then, for every $b \in (c,1)$ *and* $\gamma \in \big(0, \arctan(\cos(\frac{b\pi}{2}))\big)$, *there is an analytic operator family* $(T_b(t))_{t\in\Sigma_\gamma}$ *in* $L(E)$ *satisfying the properties* (ii), (iv) *and* (v) *stated in the formulation of Theorem* 3.6.1. *Furthermore, the property* (iii) *holds for every compact set* $K \subseteq [0,\infty)$ *and* $M_p = p^{\frac{p}{c}}$, *and there exists* $\omega \in \mathbb{R}$ *such that* $(T_b(t))_{t\in\Sigma_\gamma}$ *is an analytic semigroup of growth order* $\frac{n+1}{b}$ *whose c.i.g. is* $-\big((\omega - A)_{|\overline{A^n(A)}}\big)^b$.

PROOF. Clearly, $p^{\frac{p}{b}} \prec M_p$ and $M(|\lambda|) \sim \frac{1}{ce}|\lambda|^c$, $\lambda \to \infty$. This implies that there exist $\alpha > 0$, $l > 0$ and $\beta \in \mathbb{R}$ with $\Lambda_{\alpha,\beta,l} \subseteq \Pi_{c,\sigma,\varsigma}$. An application of Theorem 3.6.1 ends the proof. $\square$

REMARK 3.6.3. Suppose A generates a (DS). Then, for every $c > 0$, there exist $\sigma > 0$, $M > 0$, $n \in \mathbb{N}$ and $\varsigma \in \mathbb{R}$ such that (309)-(310) hold. Hence, for every $b \in (0,1)$ and $\gamma \in \big(0, \arctan(\cos(b\frac{\pi}{2}))\big)$, A generates a global $T_b(t)$-regularized semigroup, where we define $T_b(t)$ as before; let remind us that Kunstmann [**252**] proved that this statement holds for every $b \in (0,1)$ and $\gamma \in \big(0, \frac{\pi(1-b)}{4}\big)$ (cf. also [**38**, p. 302]). Our estimate is better if $b \in (0, \frac{2}{\pi}]$. This follows from the following simple observation:

$$\Big|\arctan\Big(\cos\Big(b\frac{\pi}{2}\Big)\Big) - \frac{\pi}{4}\Big| \leqslant \Big|1 - \cos\Big(b\frac{\pi}{2}\Big)\Big| = 2\sin^2\Big(b\frac{\pi}{4}\Big) < \frac{b^2\pi^2}{8} \leqslant b\frac{\pi}{4}.$$

In conclusion, we obtain that there exists $\omega \in \mathbb{R}$ such that the solution of the incomplete Cauchy problem $u^{(k)}(t) = (-1)^{k+1}(A-\omega)u(t)$, $t > 0$ can be analytically extended to the larger sector $\Sigma_{\arctan(\cos(b\frac{\pi}{2}))}$ (cf. for instance [**89**, Section XXV] and [**394**]).

3.6.2. Regularization of ultradistribution semigroups whose generators possess ultra-polynomially bounded resolvent. In this subsection, we assume that (M_p) additionally satisfies (M.2) and (M.3). We define the *abstract Beurling space of* (M_p) *class* associated to a closed linear operator A as in [**59**]. Put $E^{(M_p)}(A) := \operatorname{proj\,lim}_{h\to+\infty} E_h^{\{M_p\}}(A)$, where

$$E_h^{\{M_p\}}(A) := \Big\{x \in D_\infty(A) : \|x\|_h^{\{M_p\}} = \sup_{p\in\mathbb{N}_0} \frac{h^p\|A^p x\|}{M_p} < \infty\Big\}.$$

Then $(E_h^{\{M_p\}}(A), \|\cdot\|_h^{\{M_p\}})$ is a Banach space, $E_{h'}^{\{M_p\}}(A) \subseteq E_h^{\{M_p\}}(A)$ if $0 < h < h' < \infty$ and $E^{(M_p)}(A)$ is a dense subspace of E whenever A is the generator of a regular (M_p)-ultradistribution semigroup [**59**]. In general, we do not know whether the space $E^{(M_p)}(A)$ is non-trivial (cf. [**38**, p. 301] and [**59**, p. 185]). Further on, we would like to point out that the equality

$$\sup_{p\in\mathbb{N}_0} \frac{h^p\|(A-z)^p x\|}{M_p} \leqslant 2e^{M(h(4+4|z|))}\|x\|_{h(2+2|z|)}^{\{M_p\}},\ h > 0,\ z \in \mathbb{C},$$

implies $E^{(M_p)}(A) = E^{(M_p)}(A-z)$, $z \in \mathbb{C}$ and that, thanks to (M.2), we have that the part of A in $E^{(M_p)}(A)$ is a continuous mapping from $E^{(M_p)}(A)$ into $E^{(M_p)}(A)$. (The previous assertions still hold if (M.3) is replaced by (M.3$'$).) It is noteworthy that the notions of quasi-analytic vectors and abstract Beurling spaces also appear in the papers of Chernoff [**67**], Lyubich [**296**], Spellmann [**392**] and that [**392**, Theorem 2] remains true in the case of non-densely defined Hille–Yosida operators.

The following entire function of exponential type zero [**207**] plays a crucial role in our investigation $\omega(z) := \prod_{i=1}^{\infty}\bigl(1+\frac{iz}{m_p}\bigr)$, $z \in \mathbb{C}$. We know the following (cf. for instance [**59**, pp. 169, 171, 182 and Lemma 3.2, p. 179]):

(P.1) there exist $l_0 \geqslant 1$ and $c_0 > 0$ such that $|\omega^n(z)| \leqslant c_0^n A^{n-1} e^{M(l_0 H^{n-1}|z|)}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$,

(P.2) there exist $L > 0$ and $\sigma \in (0,1]$ such that $|\omega(iz)| \geqslant L|\omega(|z|)|^{\sigma}$, $z \in \overline{(\Lambda_{\alpha,\beta,l})^c}$,

(P.3) due to [**207**, Proposition 4.6], the operator $\omega(lD) = \prod_{p=1}^{\infty}\bigl(1+\frac{ilD}{m_p}\bigr)$, $l \in \mathbb{C}$, is an ultradifferential operator of class (M_p). If we write $\omega^n(z) = \sum_{p=0}^{\infty} a_{n,p} z^p$, then $|a_{n,p}| \leqslant \text{Const}\,\frac{(l_0 H^{n-1})^p}{M_p}$, $p \in \mathbb{N}_0$, which implies that, for every $n \in \mathbb{N}$ and $l \in \mathbb{C}$, the operator $\omega^n(lD)$ is an ultradifferential operator of class (M_p) as long as (M_p) satisfies (M.3),

(P.4) for every $\alpha \geqslant 1$ and $z \in \mathbb{C}$: $|\omega(|z|)|^{\alpha} \geqslant \frac{1}{c_0}|\omega(\alpha l_0^{-1}|z|)|$, and

(P.5) $e^{(k+1)M(|z|)} \leqslant A^k e^{M(H^k|z|)}$, $z \in \mathbb{C}$.

Suppose that A is the generator of a (UDS) G of (M_p)-class. Then there exist constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ (cf. [**54**], [**59**, Theorem 1.5 and p. 181], [**210**] and [**307**]) which satisfy:

$$\Lambda_{\alpha,\beta,l} \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant \text{Const}\,\frac{e^{M(Hl|\lambda|)}}{|\lambda|^k},\ \lambda \in \Lambda_{\alpha,\beta,l},\ k \in \mathbb{N}. \tag{311}$$

Let $n \in \mathbb{N}$ and $n > Hl_0 l\sigma^{-1}$. Following the proof of [**59**, Proposition 3.1], we define a bounded linear operator D_n by setting $D_n := \frac{1}{2\pi i}\int_{\Gamma_l}\frac{R(\lambda A)}{\omega^n(i\lambda)}\,d\lambda$. Then it is obvious that $D_{nk} = D_n^k$, $k \in \mathbb{N}$. Arguing similarly as in the proofs of [**59**, Proposition 3.1] and [**59**, Theorem 3.8], it follows that $\mathcal{R}(G) \subseteq \mathrm{R}(D_n)$ and $E^{(M_p)}(A) = \bigcap_{k\in\mathbb{N}} \mathrm{R}(D_{nk})$; since we have assumed that G satisfies (U.2), D_n is injective. Unfortunately, it is not clear whether, for fixed $n \in \mathbb{N}$, $\mathrm{R}(D_n) \subseteq E^{(M_p)}(A)$. Now we clarify the following important interplay between ultradistribution semigroups and *local* C-regularized semigroups.

Theorem 3.6.4. *Suppose that A is the generator of a* (UDS) *G of (M_p)-class. Then, for every $\tau \in (0,\infty)$, there exists an injective operator $C_\tau \in L(E)$ such that A generates a local C_τ-regularized semigroup $(S(t))_{t\in[0,\tau)}$ satisfying that $(S(t))_{t\in[0,\tau)}$ is infinitely differentiable in $[0,\tau)$ and that there exists $h \in (0,\infty)$, independent of $\tau \in (0,\infty)$, such that*

$$\sup_{t\in[0,\tau),\ p\in\mathbb{N}_0}\frac{h^p}{M_p}\Bigl\|\frac{d^p}{dt^p}S(t)\Bigr\| < \infty. \tag{312}$$

Proof. The arguments given in the final part of the proof of Theorem 3.6.1 implies that one can translate A by a convenient multiple of the identity. Assume that constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ satisfy (311). Clearly,

$$|\omega(s)| = \prod_{k=1}^{\infty}\Big|1 + \frac{is}{m_k}\Big| \geqslant \sup_{p\in\mathbb{N}} \prod_{k=1}^{p} \frac{s}{m_k} = \sup_{p\in\mathbb{N}} \frac{s^p}{M_p} \geqslant e^{M(s)}, \quad s > 0. \tag{313}$$

Put $n_0 := \lfloor Hl_0 l\sigma^{-1}\rfloor + 1$, $k := \max(\lceil \tau\alpha\rceil, 2)$ and fix afterwards an element $x \in E$, an integer $n \in \mathbb{N}$ with $n \geqslant H^k + 2$ and a number $t \in [0, \tau)$. Then

$$(n-1)n_0\sigma \geqslant (n-1)Hll_0 > n-1 > 1, \tag{314}$$

$$(n-1)n_0\sigma l_0^{-1} \geqslant (H^k+1)Hl_0 l\sigma^{-1}\sigma l_0^{-1} > H^k l. \tag{315}$$

We define the bounded linear operator $S(t)$ (cf. also [**59**, pp. 188–189]) by

$$S(t) := \frac{1}{2\pi i}\int_{\Gamma_l} e^{\lambda t}\frac{R(\lambda : A)}{\omega^{nn_0}(i\lambda)}\,d\lambda. \tag{316}$$

In fact, $S(0) = D_{nn_0} := C_\tau \in L(E)$ is injective since G satisfies (U.2) (see the previous discussion). Notice that $n_0\sigma > 1$ and that (313)–(314), (P.2) and (P.4)–(P.5) together imply that, for every $p \in \mathbb{N}_0$:

$$\begin{aligned}
\Big\|\lambda^p \frac{e^{\lambda t}R(\lambda : A)}{\omega^{nn_0}(i\lambda)}\Big\| &\leqslant \text{Const}\,|\lambda|^p \frac{e^{t(\alpha M(l|\lambda|)+\beta)}e^{M(Hl|\lambda|)}}{|\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)|}\\
&\leqslant \text{Const}\,|\lambda|^p e^{t\beta}\frac{A^{\lceil t\alpha\rceil -1}e^{M(H^{\lceil t\alpha\rceil -1}l|\lambda|)}e^{M(Hl|\lambda|)}}{|\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)|}\\
&\leqslant \frac{\text{Const}\,|\lambda|^p e^{t\beta}A^{\lceil t\alpha\rceil -1}}{|\omega(|\lambda|)|^{n_0\sigma}}\frac{e^{2M(H^{k-1}l|\lambda|)}}{L^{nn_0}|\omega(|\lambda|)|^{(n-1)n_0\sigma}}\\
&\leqslant \frac{\text{Const}\,|\lambda|^p e^{t\beta}L^{-nn_0}A^{\lceil t\alpha\rceil}}{|\omega(|\lambda|)||\omega(|\lambda|)|^{n_0\sigma -1}}\frac{c_0|\omega(H^k l|\lambda|)|}{|\omega((n-1)n_0\sigma l_0^{-1}|\lambda|)|}\\
&\leqslant \frac{\text{Const}\,|\lambda|^p e^{t\beta}L^{-nn_0}A^{\lceil t\alpha\rceil}}{|\omega(|\lambda|)|e^{M(|\lambda|(n_0\sigma -1))}}\frac{c_0|\omega(H^k l|\lambda|)|}{|\omega((n-1)n_0\sigma l_0^{-1}|\lambda|)|}\\
&\leqslant \text{Const}\,|\lambda|^p \frac{1}{e^{M(|\lambda|(n_0\sigma -1))}|\lambda|^2},
\end{aligned} \tag{317}$$

where Const is independent of $p \in \mathbb{N}_0$. Then the Fubini theorem implies $S(s)C_\tau = C_\tau S(s)$, $s \in [0, \tau)$, and furthermore, it is checked at once that $S(s)A \subseteq AS(s)$, $s \in [0, \tau)$. Since $\rho(A) \neq \emptyset$, we have $C_\tau^{-1}AC_\tau = A$. In order to see that $(S(t))_{t\in[0,\tau)}$ is a local C_τ-regularized semigroup generated by A (cf. [**89**] and [**275**]), it is enough to prove that $A\int_0^t S(s)x\,ds = S(t)x - C_\tau x$, $t \in [0, \tau)$. To see this, one has to prove first the following equality:

$$\int_{\Gamma_l} \frac{e^{\lambda t}}{\omega^{nn_0}(i\lambda)}\,d\lambda = 0. \tag{318}$$

For a sufficiently large $R > 0$, put $\Gamma_R := \{z \in \mathbb{C} : |z| = R,\ z \notin \Lambda_{\alpha,\beta,l}\}$. As above, (314) and (P.4) imply

$$|\omega(|z|)|^{(n-1)n_0\sigma} \geqslant \frac{1}{c_0}\big|\omega((n-1)n_0\sigma l_0^{-1}|z|)\big|,\ \ z \in \mathbb{C}. \tag{319}$$

Taking into account (P.2) and (313), we obtain:

$$\begin{aligned}|\omega^{nn_0}(i\lambda)| &= |\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)| \geqslant L^{nn_0}|\omega^{(n-1)n_0\sigma}(R)|e^{M(R)n_0\sigma}\\ &\geqslant \text{Const}\,|\omega^{(n-1)n_0\sigma}(R)|R^2,\ \ \lambda \in \Gamma_R.\end{aligned}$$

An employment of (P.5) yields

$$\begin{aligned}\Big|\frac{e^{\lambda t}}{\omega^{nn_0}(i\lambda)}\Big| &\leqslant \frac{e^{t(\alpha M(l|\operatorname{Im}\lambda|)+\beta)}}{|\omega^{nn_0}(i\lambda)|} \leqslant \frac{\text{Const}}{R^2}e^{t\beta}\frac{e^{t\alpha M(lR)}}{|\omega^{(n-1)n_0\sigma}(R)|}\\ &\leqslant \frac{\text{Const}}{R^2}e^{t\beta}\frac{A^{\lceil t\alpha\rceil-1}e^{M(H^{\lceil t\alpha\rceil-1}lR)}}{|\omega^{(n-1)n_0\sigma}(R)|}.\end{aligned}$$

Owing to (319), one can continue the calculation as follows:

$$\leqslant \frac{\text{Const}}{R^2}c_0e^{t\beta}A^{\lceil t\alpha\rceil-1}\frac{|\omega(H^{\lceil t\alpha\rceil-1}lR)|}{|\omega((n-1)n_0\sigma l_0^{-1}R)|}.$$

The last inequality and (315) imply $\int_{\Gamma_R}\frac{e^{\lambda t}}{\omega^{nn_0}(i\lambda)}\,d\lambda \to 0,\ R \to +\infty$. Then the Cauchy theorem yields (318). Applying the Fubini theorem, the resolvent equation and (318), one obtains

$$\begin{aligned}A\int_0^t S(s)x\,ds &= \frac{1}{2\pi i}\int_0^t\int_{\Gamma_l} e^{\lambda s}\frac{AR(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda\,ds\\ &= \frac{1}{2\pi i}\int_0^t\int_{\Gamma_l} e^{\lambda s}\frac{\lambda R(\lambda : A)x - x}{\omega^{nn_0}(i\lambda)}\,d\lambda\,ds\\ &= \frac{1}{2\pi i}\int_0^t\int_{\Gamma_l} e^{\lambda s}\frac{\lambda R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda\,ds - \frac{1}{2\pi i}\int_0^t\int_{\Gamma_l} e^{\lambda s}\frac{x}{\omega^{nn_0}(i\lambda)}\,d\lambda\,ds\\ &= \frac{1}{2\pi i}\int_0^t\int_{\Gamma_l} e^{\lambda s}\frac{\lambda R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda\,ds = \frac{1}{2\pi i}\int_{\Gamma_l}\Big[\int_0^t e^{\lambda s}\frac{\lambda R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,ds\Big]\,d\lambda\\ &= \frac{1}{2\pi i}\int_{\Gamma_l}(e^{\lambda t}-1)\frac{R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda\\ &= \frac{1}{2\pi i}\int_{\Gamma_l} e^{\lambda t}\frac{R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda - \frac{1}{2\pi i}\int_{\Gamma_l}\frac{R(\lambda : A)x}{\omega^{nn_0}(i\lambda)}\,d\lambda = S(t)x - C_\tau x.\end{aligned}$$

As before, we have that, for every $p \in \mathbb{N}$, the integral $\frac{1}{2\pi i}\int_{\Gamma_l}\lambda^p e^{\lambda t}\frac{R(\lambda A)}{\omega^{nn_0}(i\lambda)}\,d\lambda$, $t \in [0,\tau)$ is convergent and that $\frac{d}{dt}S(t) = \frac{1}{2\pi i}\int_{\Gamma_l}\lambda e^{\lambda t}\frac{R(\lambda A)}{\omega^{nn_0}(i\lambda)}\,d\lambda$. Inductively,

$$\frac{d^p}{dt^p}S(t) = \frac{1}{2\pi i}\int\limits_{\Gamma_l}\lambda^p e^{\lambda t}\frac{R(\lambda : A)}{\omega^{nn_0}(i\lambda)}\,d\lambda,\ p \in \mathbb{N}_0,\ t \in [0,\tau). \tag{320}$$

It remains to be shown (312). Choose arbitrarily an $h \in (0, n_0\sigma - 1)$. An application of (317) and (320) gives:

$$\sup_{t\in[0,\tau),\, p\in\mathbb{N}_0}\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| \leqslant \frac{1}{2\pi}\sup_{t\in[0,\tau),\, p\in\mathbb{N}_0}\frac{h^p}{M_p}\int\limits_{\Gamma_l}\frac{|\lambda|^p|e^{\lambda t}|\,\|R(\lambda : A)\|}{|\omega^{nn_0}(i\lambda)|}\,|d\lambda|$$
$$\leqslant \text{Const}\int\limits_{\Gamma_l}\frac{e^{M(h|\lambda|)}}{e^{M(|\lambda|(n_0\sigma-1))}|\lambda|^2}\,|d\lambda| \leqslant \text{Const}\int\limits_{\Gamma_l}\frac{|d\lambda|}{|\lambda|^2} < \infty.$$

The proof is now completed. □

The proof of following lemma essentially follows from that of [**59**, Theorem 3.8].

Lemma 3.6.5. *Suppose G is a* (UDS) *of (M_p)-class generated by A, $l \geqslant 1$, $\alpha > 0$, $\beta > 0$, $n \in \mathbb{N}$, $n > Hl_0l\sigma^{-1}$* (cf. (P.1)–(P.5)) *and* (311) *holds. Then $E^{(M_p)}(A) = \bigcap_{k\in\mathbb{N}} D_{nk}(D_\infty(A))$ and*

$$D_{nk}^{-1}(E^{\{M_p\}}_{2l_0H^{nk+1}}) \subseteq \Big\{x \in D_\infty(A) : \sup_{p\in\mathbb{N}_0}\frac{\varsigma^p\|A^p x\|}{M_p} < \infty \text{ for all } \varsigma \in (0, 2l_0H^{nk})\Big\}. \tag{321}$$

Proof. Fix an integer $k \in \mathbb{N}$ and a number $\varsigma \in (0, 2l_0H^{nk})$. Put $h = 2l_0H^{nk+1}$. Let $y \in E^{\{M_p\}}_h$ and $\omega^{nk}(iz) = \sum_{p=0}^\infty a_{k,p}z^p$, $z \in \mathbb{C}$. Due to (P.3), we have $|a_{k,p}| \leqslant \text{Const}\,\frac{(l_0H^{nk})^p}{M_k}$, $p \in \mathbb{N}$ and the series $\sum_{p=0}^\infty a_{k,p}A^p y := x$ is convergent since

$$\|a_{k,p}A^p y\| \leqslant \text{Const}\,\frac{h^p\|A^p y\|}{M_p}\Big(\frac{l_0H^{nk}}{h}\Big)^p \leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\Big(\frac{1}{2H}\Big)^p.$$

Arguing as in the proof of [**59**, Theorem 3.8, p. 187], one gets that $y = D_{nk}x$ and the proof is completed if one shows that $x \in D_\infty(A)$ and that (321) holds with ς. First of all, let us observe that the series $\sum_{p=0}^\infty a_{k,p}A^{m+p}y$ is also convergent for all $m \in \mathbb{N}$. Indeed, (M.2) yields

$$\|a_{k,p}A^{m+p}y\| \leqslant \text{Const}\,\frac{h^{p+m}\|A^{p+m}y\|}{M_{p+m}}\Big(\frac{l_0H^{nk}}{h}\Big)^p\frac{M_{p+m}}{M_ph^m} \tag{322}$$
$$\leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\Big(\frac{1}{2H}\Big)^p\frac{M_{p+m}}{M_ph^m} \leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\Big(\frac{1}{2H}\Big)^p\frac{AH^{p+m}M_m}{h^m}$$
$$\leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\Big(\frac{1}{2}\Big)^p\Big(\frac{1}{2l_0H^{nk}}\Big)^m M_m.$$

By (322),

$$\sum_{p=0}^{\infty}|a_{k,p}|A^{p+m}y\| \leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\Big(\frac{1}{2l_0H^{nk}}\Big)^m M_m,\ \ x\in D_\infty(A)$$

and $A^m x=\sum_{p=0}^{\infty}a_{k,p}A^{m+p}y$. Finally, (322) implies

$$\sup_{m\in\mathbb{N}_0}\frac{\varsigma^m\|A^m x\|}{M_m}\leqslant \text{Const}\,\|y\|_h^{\{M_p\}}\sup_{m\in\mathbb{N}_0}\Big(\frac{\varsigma}{2l_0H^{nk}}\Big)^m\leqslant \text{Const}\,\|y\|_h^{\{M_p\}}. \qquad \square$$

Now we are in a position to clarify the following analogue of [**59**, Theorem 4.1, Corollary 4.2] for non-dense ultradistribution semigroups of (M_p)-class.

THEOREM 3.6.6. *Suppose that A generates a* (UDS) *of (M_p)-class. Then the abstract Cauchy problem*

$$(\text{ACP}):\begin{cases} u\in C^\infty([0,\infty):E)\cap C([0,\infty):[D(A)]),\\ u'(t)=Au(t),\ t\geqslant 0,\\ u(0)=x,\end{cases}$$

has a unique solution for all $x\in E^{(M_p)}(A)$. Furthermore, for every compact set $K\subseteq[0,\infty)$ and $h>0$, the solution u of (ACP) *satisfies*

$$\sup_{t\in K,\ p\in\mathbb{N}_0}\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}u(t)\Big\|<\infty.$$

PROOF. We basically follow the terminology given in the proof of Theorem 3.6.4 and Lemma 3.6.5 (cf. also (P.1)–(P.5)). The uniqueness of solutions of (ACP) is a consequence of the Ljubich uniqueness theorem. To prove the existence of solutions of (ACP), let us observe that the proof of Theorem 3.6.4 implies that there exist a number $n_0\in\mathbb{N}$ and a strictly increasing sequence (k_l) in $\mathbb{N}$ such that $n_0>Hl_0l\sigma^{-1}$ and that, for every $l\in\mathbb{N}$, the operator A is the generator of a differentiable $D_{n_0k_l}$-regularized semigroup $(S_l(t))_{t\in[0,l)}$. This implies that the abstract Cauchy problem

$$\begin{cases} u_l\in C^1([0,l):E)\cap C([0,l):[D(A)]),\\ u_l'(t)=Au_l(t),\ t\geqslant 0,\\ u_l(0)=x,\end{cases}$$

has a unique solution for every $x\in D_{n_0k_l}(D(A))$ given by $u_l(t)=D_{n_0k_l}^{-1}S_l(t)x$, $t\in[0,l)$. If $x\in E^{(M_p)}(A)$, then Lemma 3.6.5 implies that $u_l(t)=S_l(t)D_{n_0k_l}^{-1}x$, $t\in[0,l)$, and due to Theorem 3.6.4, we get $u_l\in C^\infty([0,l):E)$. Therefore, we automatically obtains the existence of a solution of (ACP) for $x\in E^{(M_p)}(A)$. Let $K\subseteq[0,\infty)$ be a compact set, $K\subseteq[0,l)$ for some $l\in\mathbb{N}$, $h>0$, $l'\in\mathbb{N}$, $l'>l$ and $2l_0H^{n_0k_{l'}}>h$. Then, for every $t\in K$:

$$\begin{aligned}\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}u(t)\Big\|&=\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}u_{l'}(t)\Big\|=\frac{h^p}{M_p}\Big\|A^pu_{l'}(t)\Big\|=\frac{h^p}{M_p}\Big\|A^pD_{n_0k_{l'}}^{-1}S_{l'}(t)x\Big\|\\&=\frac{h^p}{M_p}\Big\|S_{l'}(t)A^pD_{n_0k_{l'}}^{-1}x\Big\|\leqslant\sup_{s\in[0,l]}\Big\|S_{l'}(s)\Big\|\frac{h^p}{M_p}\Big\|A^pD_{n_0k_{l'}}^{-1}x\Big\|.\end{aligned}$$

This in combination with Lemma 3.6.5 completes the proof of theorem. □

The following lemma is closely linked to an old result of Roumieu (see e.g. [**207**, Lemma 4.3, p. 57]) in which it has been assumed that the corresponding sequences (M_p) and (N_p) satisfy (M.1) and (M.3').

LEMMA 3.6.7. *There exists a sequence* (N_p) *satisfying* $N_0 = 1$, (M.1), (M.2), (M.3) *and* $N_p \prec M_p$.

PROOF. Define a sequence (r_p) of positive real numbers recursively by:

$$r_1 := 1 \text{ and } r_{p+1} := r_p\Big[\frac{m_p}{m_{p+1}} + \min\Big(1 - \frac{m_p}{m_{p+1}}, \frac{1}{p}\frac{m_p}{m_{p+1}}\Big)\Big],\ p \in \mathbb{N}.$$

Then:

$$1 \geqslant \frac{r_{p+1}}{r_p} \geqslant \frac{m_p}{m_{p+1}} \text{ and } r_{p+1} \leqslant r_p\Big(1 + \frac{1}{p}\Big)\frac{m_p}{m_{p+1}},\ p \in \mathbb{N}. \tag{323}$$

Using (323), one obtains inductively:

$$r_p \leqslant p\frac{m_1}{m_p} \text{ and } \prod_{i=1}^{p} r_i \leqslant p!\frac{m_1^p}{M_p},\ p \in \mathbb{N}.$$

Since $p! \prec M_p$ (cf. [**207**, p. 74] and [**51**, Lemma 2.1.2]), one gets that, for every $\sigma > 0$,

$$\sup_{p\in\mathbb{N}_0} \sigma^p \prod_{i=1}^{p} r_i < \infty. \tag{324}$$

Put now $N_0 := 1$ and $N_p := M_p \prod_{i=1}^{p} r_i$, $p \in \mathbb{N}$. Keeping in mind (323), one can simply verify that (N_p) satisfies (M.1), (M.2) (with the same constants A and H) and (M.3). Furthermore, (324) implies that $N_p \prec M_p$. □

Now we are able to state the following important result.

THEOREM 3.6.8. *Suppose that* A *generates a* (UDS) G *of* (M_p)*-class. Then there exists an injective operator* C *such that* A *generates a global differentiable* C*-regularized semigroup* $(S(t))_{t\geqslant 0}$. *Furthermore, for every compact set* $K \subseteq [0,\infty)$ *and* $h > 0$, *one has* $\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\big\|\frac{d^p}{dt^p}S(t)\big\| < \infty$.

PROOF. By Lemma 3.6.7, there exist a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, (M.1), (M.2), (M.3) and $N_p \prec M_p$. As in the proof of Theorem 3.6.4, we may assume that numbers $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ satisfy (311). Denote by $N(\cdot)$ the associated function of (N_p) and notice that the previously given arguments combined with [**207**, Lemma 3.10] indicate that there exist $\alpha_1 > 0$ and $\beta_1 > 0$ such that $\Lambda^1_{\alpha_1,\beta_1,l} \subseteq \Lambda_{\alpha,\beta,l} \subseteq \rho(A)$. Furthermore, for every $\mu > 0$, there exists $C_\mu > 0$ such that $M(\lambda) \leqslant N(\mu\lambda) + C_\mu$, $\lambda \geqslant 0$, and thanks to [**361**, Lemma 1.7, p. 140] (cf. also [**51**, Lemma 2.1.3]), we know that, for every $L \geqslant 1$, there exist $B > 0$ and $E_L > 0$ such that

$$LN(\lambda) \leqslant N(B^{L-1}\lambda) + E_L,\ \lambda \geqslant 0. \tag{325}$$

Let Γ_1 and Γ_2 denote the upwards oriented boundaries of $\Lambda_{\alpha,\beta,l}$ and $\Lambda^1_{\alpha_1,\beta_1,l}$, respectively. Suppose that $\varrho \in \mathcal{D}^{(N_p)}_{[0,1]}$ satisfies $\rho(t) \geqslant 0$, $t \in \mathbb{R}$ and $\int_{\mathbb{R}} \varrho(t)\,dt = 1$. Put $\varrho_n(t) := n\varrho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Then it can be simply verified that, for every $\varphi \in \mathcal{D}^{(M_p)}$, $\varrho_n * \varphi \in \mathcal{D}^{(N_p)} \subseteq \mathcal{D}^{(M_p)}$, $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \varrho_n * \varphi = \varphi$ in $\mathcal{D}^{(M_p)}$. Define $G_1(\varphi) := G(\varphi)$, $\varphi \in \mathcal{D}^{(N_p)}$. Then $G_1 \in \mathcal{D}'^{(N_p)}_0(L(E))$ and satisfies (U.1). To prove (U.2), suppose $G_1(\varphi)x = 0$ for all $\varphi \in \mathcal{D}^{(N_p)}_0$. Then $G(\psi)x = \lim_{n\to\infty} G(\varrho_n * \psi)x = \lim_{n\to\infty} G_1(\varrho_n * \psi)x = 0$ for all $\psi \in \mathcal{D}^{(M_p)}_0$. So, $x = 0$, G_1 is a (UDS) of (N_p)-class and it can be simply checked that the generator of G_1 is A. Denote $\omega_{N_p}(z) := \prod_{i=1}^{\infty}\big(1 + \frac{izN_{p-1}}{N_p}\big)$, $z \in \mathbb{C}$ and notice that $|\omega_{N_p}(s)| \geqslant e^{N(s)}$, $s \geqslant 0$ and that, owing to (P.2), there exist $L_1 > 0$ and $\sigma_1 \in (0,1]$ such that $|\omega_{N_p}(iz)| \geqslant L_1|\omega_{N_p}(|z|)|^{\sigma_1}$, $z \in \overline{(\Lambda^1_{\alpha_1,\beta_1,l})^c}$. Since G_1 is a (UDS) generated by A, we have that there exists a sufficiently large integer $n \geqslant \lceil \frac{1}{\sigma_1} \rceil$ so that the bounded linear operator $C := \frac{1}{2\pi i}\int_{\Gamma_2} \frac{R(\lambda A)}{\omega^n_{N_p}(i\lambda)}\,d\lambda$ is injective. An elementary application of Cauchy formula implies that $C = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{R(\lambda A)}{\omega^n_{N_p}(i\lambda)}\,d\lambda$. Set now

$$S(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{e^{\lambda t}R(\lambda : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda,\ t \geqslant 0,\ x \in E.$$

Taking into account the simple equality $\int_{\Gamma_1} \frac{\lambda^p}{\omega^n_{N_p}(i\lambda)}\,d\lambda = 0$, one can repeat literally the proof of Theorem 3.6.4 in order to see that $(S(t))_{t\geqslant 0}$ is a global differentiable C-regularized semigroup generated by A and that, for every $p \in \mathbb{N}_0$, $\frac{d^p}{dt^p}S(t) = \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^p e^{\lambda t}\frac{R(\lambda A)}{\omega^n_{N_p}(i\lambda)}\,d\lambda$, $p \in \mathbb{N}_0$, $t \geqslant 0$. Suppose now that $K \subseteq [0,\infty)$ is a compact set and that $h > 0$. Then we get with the help of (325) that, for every $\mu > 0$,

$$\begin{aligned}
\sup_{t\in K,\ p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| &\leqslant \mathrm{Const}\, e^{\beta \sup K}\int\limits_{\Gamma_1} \frac{e^{M(h|\lambda|)+\alpha \sup K M(l|\lambda|)}e^{M(Hl|\lambda|)}}{|\omega^n_{N_p}(i\lambda)||\lambda|^2}\,|d\lambda| \\
&\leqslant \mathrm{Const}\, e^{\beta \sup K}\int\limits_{\Gamma_1} \frac{e^{M(h\lambda)+\alpha \sup K M(l|\lambda|)+M(Hl|\lambda|)}}{|\lambda|^2 L_1^n|\omega_{N_p}(|\lambda|)|^{n\sigma_1}}\,|d\lambda| \\
&\leqslant \mathrm{Const}\int\limits_{\Gamma_1} \frac{e^{M(h\lambda)+\alpha \sup K M(l|\lambda|)+M(Hl|\lambda|)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}\,|d\lambda| \\
&\leqslant M_\mu\int\limits_{\Gamma_1} \frac{e^{N(h|\lambda|\mu)+\alpha \sup K N(l|\lambda|\mu)+N(Hl|\lambda|\mu)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}\,|d\lambda| \qquad (326) \\
&\leqslant M_\mu e^{E_\alpha \sup K}\int\limits_{\Gamma_1} \frac{e^{N(h|\lambda|\mu)+N(B^{\alpha \sup K}l|\lambda|\mu)+E_{K,\alpha}+N(Hl|\lambda|\mu)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}\,|d\lambda|
\end{aligned}$$

$$\leqslant M_\mu e^{E_\alpha \sup K} \int\limits_{\Gamma_1} \frac{e^{3N(|\lambda|\mu[h+lB^{\alpha \sup K}+Hl])}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}\,|d\lambda|$$

$$\leqslant M_\mu e^{E_\alpha \sup K+E_3} \int\limits_{\Gamma_1} \frac{e^{N(B^2|\lambda|\mu[h+lB^{\alpha \sup K}+Hl])}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}\,|d\lambda|.$$

Suppose now that $\mu \in (0, \frac{1}{B^2(h+lB^{\alpha \sup K}+Hl)})$. Then we obtain from (326):

$$\leqslant C_\mu e^{E_\alpha \sup K+E_3} \int\limits_{\Gamma_1} \frac{|d\lambda|}{|\lambda|^2} < \infty. \qquad \square$$

The proof of previous theorem also implies:

THEOREM 3.6.9. *Suppose that A generates an exponential* (UDS) *G of* (M_p)-*class. Then there exists an injective operator $C \in L(E)$ such that A generates an exponentially bounded C-regularized semigroup $(S(t))_{t\geqslant 0}$ that is infinitely differentiable in $t \geqslant 0$. Furthermore, for every compact set $K \subseteq [0,\infty)$ and $h > 0$, one has* $\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p} \big\| \frac{d^p}{dt^p} S(t) \big\| < \infty$.

3.6.3. Higher order time-fractional equations. Regularization of ultradistribution sines. First of all, we recall the assertion of [**434**, Theorem 6.2, p. 132] with $\alpha = N \in \mathbb{N}$:

THEOREM 3.6.10. *Suppose $n \in \mathbb{N}$, $n \geqslant 2$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$, $z_0 \in \mathbb{C}$, $z_0 \neq 0$ and A is a closed linear operator. If*

$$e^{i\arg(z_0)}(|z_0| + \Sigma_\theta) \subseteq \rho(A), \tag{327}$$

$$\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^N,\ \lambda \in e^{i\arg(z_0)}(|z_0| + \Sigma_\theta), \tag{328}$$

then there exists a family $(C_\varepsilon)_{\varepsilon>0}$ of bounded injective operators on E such that:

(i) *For every $\varepsilon > 0$, there exists a unique solution u of the abstract Cauchy problem (ACP_n) with initial data $x_0, \ldots, x_{n-1} \in \mathrm{R}(C_\varepsilon)$ and*

$$\|u(t)\| \leqslant M(t) \sum_{i=0}^{n-1} \|C_\varepsilon^{-1} x_i\|,\ t \geqslant 0, \tag{329}$$

for some non-negative and locally bounded function $M(t)$, $t \geqslant 0$.

(ii) $\bigcup_{\varepsilon>0} C_\varepsilon(D(A^{N+2}))$ *is dense in* $D(A^{N+2})$.

Our intention in the first part of this section is to reconsider Theorem 3.6.10 and to prove its generalization in the framework of the theory of abstract time-fractional equations. Recall that J. Chazarain and H. O. Fattorini (cf. for instance [**434**]) proved that the abstract Cauchy problem (ACP_n) is not well posed in the classical sense if A is unbounded and $n \geqslant 3$. Concerning abstract time-fractional equations, it is worth noting that E. Bazhlekova proved (cf. [**36**, Theorem 2.6, p. 22]) that every generator of an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, I)$- regularized resolvent family must be bounded if $\alpha > 2$. The above statement is no longer true for the class of exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent families, as a class of very simple counterexamples show. We have established in [**240**] the sufficient conditions for generation of global not necessarily exponentially equicontinuous $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent families in sequentially complete locally convex space, SCLCS for short. The results obtained there can be used in the analysis of the following abstract time-fractional equation with $\alpha > 1$:

$$\mathbf{D}_t^{\alpha} u(t) = Au(t),\ t > 0;\ u^{(k)}(0) = x_k,\ k = 0, 1, \cdot\cdot\cdot, \lceil \alpha \rceil - 1, \tag{330}$$

where $x_k \in C(D(A))$, $k = 0, 1, \cdot\cdot\cdot, \lceil \alpha \rceil - 1$ and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α. In what follows, we will try to give the basic information on the C-wellposedness of (330). Assume that E is a SCLCS and that $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. Let $\alpha > 0$. A function $u \in C^{\lceil \alpha \rceil - 1}([0, \infty) : E)$ is said to be a (*strong*) *solution* of (330) if $Au \in C([0, \infty) : E)$, $\int_0^{\cdot} \frac{(\cdot - s)^{\lceil \alpha \rceil - \alpha - 1}}{\Gamma(\lceil \alpha \rceil - \alpha)} \bigl[u(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{s^k}{k!} x_k\bigr] ds \in C^{\lceil \alpha \rceil}([0, \infty) : E)$ and (330) holds. The abstract Cauchy problem (330) is said to be C-*wellposed* if:

(i) For every $x_0, \cdots, x_{\lceil \alpha \rceil - 1} \in C(D(A))$, there exists a unique solution $u(t; x_0, \cdot\cdot\cdot, x_{\lceil \alpha \rceil - 1})$ of (330).
(ii) For every $T > 0$ and $q \in \circledast$, there exist $c > 0$ and $r \in \circledast$ such that, for every $x_0, \cdot\cdot\cdot, x_{\lceil \alpha \rceil - 1} \in C(D(A))$, the following holds:

$$q\bigl(u(t; x_0, \cdot\cdot\cdot, x_{\lceil \alpha \rceil - 1})\bigr) \leqslant c \sum_{k=0}^{\lceil \alpha \rceil - 1} r\bigl(C^{-1} x_k\bigr),\ t \in [0, T]. \tag{331}$$

In case $C = I$ and E is a Banach space, the definition of C-wellposedness of (330) coincides with the one introduced on pages 19 & 20 of [**36**]. Assume that there exists a unique solution of (330) in case $x_0 \in C(D(A))$ and $x_j = 0$, $1 \leqslant j \leqslant \lceil \alpha \rceil - 1$. Applying [**36**, (1.21), p. 12], one gets that $u(t; x_0) \equiv u(t; x_0, 0, \cdot\cdot\cdot, 0)$, $t \geqslant 0$ is a unique function satisfying $u(\cdot; x_0),\ Au(\cdot; x_0) \in C([0, \infty) : E)$ and

$$u(t; x_0) = x_0 + \int_0^t \frac{(t-s)^{\alpha - 1}}{\Gamma(\alpha)} Au(s; x_0) ds,\ t \geqslant 0. \tag{332}$$

If, additionally, A is densely defined, E is complete and (331) holds provided $x_0 \in C(D(A))$ and $x_j = 0$, $1 \leqslant j \leqslant \lceil \alpha \rceil - 1$, then one can repeat literally the arguments given in the proof of [**369**, Proposition 1.1] in order to see that A is a subgenerator of a locally equicontinuous $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family $(S_\alpha(t))_{t \geqslant 0}$. Notice that we need the completeness of E here since one has to extend the operator $S_\alpha(t)x \equiv u(t; Cx, \cdot\cdot\cdot, 0)$ $(t \geqslant 0,\ x \in D(A))$ to the whole space E (cf. also [**306**, Lemma 22.19, p. 258]).

Assume now that, for every $x_0 \in C(D(A))$, there exists a unique function $u(t) \equiv u(t; x_0)$, $t \geqslant 0$ satisfying u, $Au \in C([0, \infty) : E)$ and (332). Then it is straightforward to see that $u(t)$ is a unique solution of (330) with $x_j = 0$, $1 \leqslant j \leqslant \lceil \alpha \rceil - 1$. If A is a subgenerator of a global $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family

$(S_\alpha(t))_{t\geqslant 0}$, then the unique solution of (330) is given by:

$$u(t) = S_\alpha(t)C^{-1}x_0 + \sum_{j=1}^{\lceil\alpha\rceil}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}S_\alpha(s)C^{-1}x_{j-1}ds,\ t\geqslant 0;$$

furthermore, the abstract Cauchy problem (330) is C-wellposed if, additionally, $(S_\alpha(t))_{t\geqslant 0}$ is locally equicontinuous.

In order to formulate the following theorem, we need to consider separately two possible cases: $\alpha > 2$ and $\alpha \in (1,2]$.

If $\alpha > 2$, then we assume that there exist $z_0 \in \mathbb{C} \setminus \{0\}$, $\beta \geqslant -1$, $d \in (0,1]$, $m \in (0,1)$, $\varepsilon \in (0,1]$ and $\gamma > -1$ such that:

($\flat$) $P_{z_0,\beta,\varepsilon,m} := e^{i\arg(z_0)}\big(|z_0| + (P_{\beta,\varepsilon,m} \cup B_d)\big) \subseteq \rho_C(A)$, $(\varepsilon, m(1+\varepsilon)^{-\beta}) \in \partial B_d$,

($\flat\flat$) the family $\{(1+|\lambda|)^{-\gamma}(\lambda - A)^{-1}C : \lambda \in P_{z_0,\beta,\varepsilon,m}\}$ is equicontinuous, and

($\flat\flat\flat$) the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\lambda \in P_{z_0,\beta,\varepsilon,m}$ is continuous for every fixed $x \in E$.

The case $\alpha \in (1,2]$ is more restrictive. We assume that there exist $z_0 \in \mathbb{C}\setminus\{0\}$, $\theta \in (\frac{\pi}{2}(2-\alpha), \frac{\pi}{2})$, $d \in (0,1]$ and $\gamma > -1$ such that:

($\flat_1$) $\Sigma(z_0,\theta,d) := e^{i\arg(z_0)}\big(|z_0| + (\Sigma(\theta) \cup B_d)\big) \subseteq \rho_C(A)$,

($\flat\flat_1$) the family $\{(1+|\lambda|)^{-\gamma}(\lambda - A)^{-1}C : \lambda \in \Sigma(z_0,\theta,d)\}$ is equicontinuous, and

($\flat\flat\flat_1$) the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\lambda \in \Sigma(z_0,\theta,d)$ is continuous for every fixed $x \in E$.

Given $b \in (0,1/2)$, set $\delta_b := \arctan(\cos \pi b)$.

THEOREM 3.6.11. *Let (M_p) be a sequence of positive real numbers satisfying $p! \prec M_p$.*

(i) *Let ($\flat$)-($\flat\flat\flat$) hold. Then, for every $b \in \left(\frac{1}{\alpha}, \frac{1}{2}\right)$, there exists an operator family $(T_b(z))_{z\in\Sigma_{\delta_b}}$ such that, for every $x \in E$, the mapping $z \mapsto T_b(z)x$, $z \in \Sigma_{\delta_b}$ is analytic and that the following holds:*

(i.1) *For every $z \in \Sigma_{\delta_b}$ and $p \in \circledast$, $T_b(z)$ is injective and there exist $c > 0$ and $q \in \circledast$ such that*

$$p\big(T_b(z)x\big) \leqslant c\big((\tan(\cos \pi b)\operatorname{Re} z - |\operatorname{Im} z|)^{-\frac{\gamma+1}{b}}\big)q(x),\ x \in E.$$

(i.2) *If $\lfloor b+\gamma\rfloor \geqslant 0$, $x \in D(A^{\lfloor b+\gamma\rfloor+2})$ and $\delta \in (0,\delta_b)$, then there exists $\lim_{z'\in\Sigma_\delta, z'\to 0}\frac{T_b(z')x-x}{z'}$, and particularly, $\lim_{z'\in\Sigma_\delta, z'\to 0}T_b(z')x = x$.*

(i.3) *For every $z \in \Sigma_{\delta_b}$, there exists a unique solution $u(\cdot;z)$ of the abstract Cauchy problem (330) with initial data $x_0, \cdots, x_{\lceil\alpha\rceil-1} \in \mathrm{R}(T_b(z))$ and $u(\cdot;z)$ can be extended to the whole complex plane. Furthermore, the mapping $\omega \mapsto u(\omega;z)$, $\omega \in \mathbb{C} \setminus (-\infty,0]$ is analytic and the abstract Cauchy problem (330) is $T_b(z)$-wellposed ($z \in \Sigma_{\delta_b}$). Let $K \subseteq \mathbb{C} \setminus (-\infty,0]$ be a compact set, let $h > 0$ and let $z \in \Sigma_{\delta_b}$. Then, for every seminorm $q \in \circledast$,* *there exist a constant $c_{K,h,z,q} > 0$ and a seminorm $r_q \in \circledast$ such that:*

$$\sum_{l=0}^{\lceil\alpha\rceil-1} \sup_{\omega\in K,\, p\in\mathbb{N}} \frac{h^p q\big(A^p \frac{d^l}{d\omega^l} u(\omega;z)\big)}{M_{\lfloor\alpha p\rfloor-1+l}} \leqslant c_{K,h,z,q} \sum_{i=0}^{\lceil\alpha\rceil-1} r_q\big(T_b(z)^{-1}x_i\big); \tag{333}$$

if $\alpha \in \mathbb{N}\setminus\{1,2\}$, then the mapping $\omega \mapsto u(\omega;z)$, $\omega \in \mathbb{C}$ is entire ($z \in \Sigma_{\delta_b}$) and (333) holds for any compact set $K \subseteq \mathbb{C}$, $h > 0$, $z \in \Sigma_{\delta_b}$ and $q \in \circledast$.

(ii) *Let ($\flat_1$)-($\flat\flat\flat_1$), $b \in \big(\frac{1}{\alpha}, \frac{\pi}{2(\pi-\theta)}\big)$ and $\vartheta \in (0, \arctan(\cos(b(\pi-\theta))))$. Then there exists an operator family $(T_b(z))_{z\in\Sigma_\vartheta}$ such that, for every $x \in E$, the mapping $z \mapsto T_b(z)x$, $z \in \Sigma_\vartheta$ is analytic and that the following holds:*

(ii.1) *For every $z \in \Sigma_\vartheta$ and $p \in \circledast$, $T_b(z)$ is injective and there exist $c > 0$ and $q \in \circledast$ such that*

$$p\big(T_b(z)x\big) \leqslant c\big((\tan(\vartheta)\operatorname{Re} z - |\operatorname{Im} z|)^{-\frac{\gamma+1}{b}}\big)q(x),\ x \in E.$$

(ii.2) *If $\lfloor b+\gamma\rfloor \geqslant 0$, $x \in D(A^{\lfloor b+\gamma\rfloor+2})$ and $\delta \in (0,\vartheta)$, then there exists $\lim_{z'\in\Sigma_\delta, z'\to 0} \frac{T_b(z')x-x}{z'}$, and particularly, $\lim_{z'\in\Sigma_\delta, z'\to 0} T_b(z')x = x$.*

(ii.3) *For every $z \in \Sigma_\vartheta$, there exists a unique solution $u(\cdot;z)$ of the abstract Cauchy problem (330) with initial data $x_0, x_1 \in \mathrm{R}(T_b(z))$ and $u(\cdot;z)$ can be extended to the whole complex plane. Furthermore, the mapping $\omega \mapsto u(\omega;z)$, $\omega \in \mathbb{C}\setminus(-\infty,0]$ is analytic and the abstract Cauchy problem (330) is $T_b(z)$-wellposed ($z \in \Sigma_\vartheta$). Let $K \subseteq \mathbb{C}\setminus(-\infty,0]$ be a compact set, let $h > 0$ and let $z \in \Sigma_\vartheta$. Then, for every seminorm $q \in \circledast$, there exist a constant $c_{K,h,z,q} > 0$ and a seminorm $r_q \in \circledast$ such that (333) holds with $\lceil\alpha\rceil = 2$; if $\alpha = 2$, then the mapping $\omega \mapsto u(\omega;z)$, $\omega \in \mathbb{C}$ is entire ($z \in \Sigma_\vartheta$) and (333) holds for any compact set $K \subseteq \mathbb{C}$, $h > 0$, $z \in \Sigma_\vartheta$ and $q \in \circledast$.*

PROOF. We will only prove the first part of theorem. Put $A_0 := e^{-i\arg(z_0)}A - |z_0|$. Then $P_{\beta,\varepsilon,m} \cup B_d \subseteq \rho_C(A_0)$ and, for every $q \in \circledast$, there exist $c_q > 0$ and $r_q \in \circledast$ such that

$$q\big((\lambda - A_0)^{-1}Cx\big) \leqslant c_q r_q(x)(1+|\lambda|)^\gamma,\ x \in E,\ \lambda \in P_{\beta,\varepsilon,m} \cup B_d. \tag{334}$$

Without loss of generality, we may assume that there exists an open neighborhood $\Omega_{\beta,\varepsilon,m,d}$ of the region $P_{\beta,\varepsilon,m} \cup B_d$ such that the mapping $\lambda \mapsto (\lambda - A_0)^{-1}Cx$, $\lambda \in \Omega_{\beta,\varepsilon,m,d}$ is continuous for every $x \in E$ and that (334) holds. By [**241**, Proposition 2.16(iii)], the mapping $\lambda \mapsto (\lambda - A_0)^{-1}Cx$, $\lambda \in \Omega_{\beta,\varepsilon,m,d}$ is analytic for every fixed $x \in E$. Then, for every $z \in \Sigma_{\delta_b}$ and $\lambda \in \Gamma$,

$$\begin{aligned}\big|e^{-z(-\lambda)^b}\big| &= e^{-\operatorname{Re} z|\lambda|^b\cos(b\arg(-\lambda))+\operatorname{Im} z|\lambda|^b\sin(b\arg(-\lambda))}\\ &\leqslant e^{-(\operatorname{Re} z\cos(b\arg(-\lambda))-|\operatorname{Im} z|)|\lambda|^b} \leqslant e^{-(\operatorname{Re} z\cos(b(\pi-a))-|\operatorname{Im} z|)|\lambda|^b}\\ &\leqslant e^{-(\operatorname{Re} z\tan\delta_b-|\operatorname{Im} z|)|\lambda|^b}.\end{aligned} \tag{335}$$

Define, for every $z \in \Sigma_{\delta_b}$,

$$T_b(z)x := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}(\lambda - A_0)^{-1}Cx\,d\lambda,\ x \in E. \tag{336}$$

By (334)-(335), $T_b(z) \in L(E)$ for all $z \in \Sigma_{\delta_b}$. One can simply prove that the mapping $z \mapsto T_b(z)x$, $z \in \Sigma_{\delta_b}$ is analytic for every fixed $x \in E$ ([**241**]). The injectiveness of each single operator $T_b(z)$ is a consequence of the proof of [**241**, Theorem 3.16]; for the sake of completeness, we will briefly sketch the proof of this fact. Let $\lambda_0 \in \Omega_{\beta,\varepsilon,m,d} \smallsetminus (P_{\beta,\varepsilon,m} \cup B_d)$. By induction, one gets that, for every $k \in \mathbb{N}_0$, $x \in D(A^{k+2})$ and $\lambda \in \rho_C(A_0) \smallsetminus \{\lambda_0\}$:

(337)

$$(\lambda - A_0)^{-1}Cx = \sum_{j=0}^{k+1} \frac{(-1)^j}{(\lambda-\lambda_0)^{j+1}}Cx + \frac{(-1)^k}{(\lambda-\lambda_0)^{k+2}}(\lambda - A_0)^{-1}C(\lambda_0 - A_0)^{k+2}x.$$

Using (337) and the proof of [**394**, Lemma 2.7], we obtain that the assumptions $\lfloor b+\gamma \rfloor \geqslant 0$, $x \in D(A^{\lfloor b+\gamma \rfloor+2})$ and $\delta \in (0, \delta_b)$ together imply

$$\begin{aligned} &\lim_{z' \in \Sigma_\delta, z' \to 0} \frac{T_b(z')x - Cx}{z'} \\ &= -(-\lambda_0)^b Cx - \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b \Big[(\lambda - A_0)^{-1}C - \frac{C}{\lambda - \lambda_0}\Big]x\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-1}(\lambda - A_0)^{-1}CAx\, d\lambda,\ x \in D(A^{\lfloor b+\gamma \rfloor+2}) \end{aligned} \tag{338}$$

and (i.2). The analyticity of the mapping $z' \mapsto T_b(z')x$, $z' \in \Sigma_{\delta_b}$ combined with the semigroup property $T_b(z'+z'')(0)Cx = T_b(z')T_b(z'')x$, $x \in E$, z', $z'' \in \Sigma_{\delta_b}$, and the injectiveness of C, implies $T_b(z')x = 0$, $z' \in \Sigma_{\delta_b}$. Since $\mathrm{R}(T_b(z')) \subseteq D_\infty(A_0)$ and $T_b(z')((\lambda_0 - A_0)^{-1}C)^{\lceil \gamma \rceil+2}x = ((\lambda_0 - A_0)^{-1}C)^{\lceil \gamma \rceil+2}T_b(z')x$, $z' \in \Sigma_{\delta_b}$, one can apply (338) in order to see that $\lim_{z' \to 0+} T_b(z')((\lambda_0 - A_0)^{-1}C)^{\lceil \gamma \rceil+2}x = ((\lambda_0 - A_0)^{-1}C)^{\lceil \gamma \rceil+2}x = 0$. Therefore, $x = 0$ and $T_b(z)$ is injective; the inequality stated in (i.1) readily follows. Define now, for every $t \geqslant 0$, $z \in \Sigma_{\delta_b}$ and $x \in E$,

$$S_{\alpha,b,z}(t)x := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^\infty \frac{e^{ij\arg(z_0)}t^{\alpha j}(\lambda + |z_0|)^j}{\Gamma(\alpha j+1)}(\lambda - A_0)^{-1}Cx\, d\lambda. \tag{339}$$

Notice that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that:

$$\begin{aligned} &\Bigg|\sum_{j=0}^\infty \frac{e^{ij\arg(z_0)}t^{\alpha j}(\lambda + |z_0|)^j}{\Gamma(\alpha j+1)}\Bigg| = \big|E_\alpha\big(e^{i\arg(z_0)}t^\alpha(\lambda + |z_0|)\big)\big| \\ &\leqslant E_\alpha\big(\big|e^{i\arg(z_0)}t^\alpha(\lambda + |z_0|)\big|\big) \leqslant c_\alpha e^{t(|\lambda|^{1/\alpha}+|z_0|^{1/\alpha})},\ t \geqslant 0,\ \lambda \in \Gamma,\ z \in \Sigma_{\delta_b}. \end{aligned}$$

The above implies $S_{\alpha,b,z}(t) \in L(E)$. Clearly, $S_{\alpha,b,z}(0) = T_b(z)$, $S_{\alpha,b,z}(t)T_b(z) = T_b(z)S_{\alpha,b,z}(t)$, $T_b(z)A \subseteq AT_b(z)$, $t \geqslant 0$ and $(S_{\alpha,b,z}(t))_{t\geqslant 0}$ is a strongly continuous operator family which commutes with A ($z \in \Sigma_{\delta_b}$). Furthermore, for every $z \in \Sigma_{\delta_b}$ and $T > 0$, the family $\{S_{\alpha,b,z}(t) : t \in [0,T]\}$ is equicontinuous. Using the Cauchy formula, we infer that, for every $z \in \Sigma_{\delta_b}$,

$$\frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^\infty \frac{e^{ij\arg(z_0)}t^{\alpha j}(\lambda + |z_0|)^j}{\Gamma(\alpha j+1)}Cx\, d\lambda = 0,\ t \geqslant 0,\ x \in E. \tag{340}$$

By making use of (340), the Fubini theorem, the resolvent equation and the closedness of A, we obtain that, for every $z \in \Sigma_{\delta_b}$,

$$
\begin{aligned}
&A \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_{\alpha,b,z}(s)x\,ds \\
&= A \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \left[\frac{1}{2\pi i} \int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^{\infty} \frac{e^{ij\arg(z_0)} s^{\alpha j} (\lambda + |z_0|)^j}{\Gamma(\alpha j+1)} (\lambda - A_0)^{-1} Cx\,d\lambda \right] ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^{\infty} \frac{e^{ij\arg(z_0)} s^{\alpha j} (\lambda + |z_0|)^j}{\Gamma(\alpha j+1)} \\
&\times e^{i\arg(z_0)} \left[e^{i\arg(z_0)} (\lambda + |z_0|) \left(e^{i\arg(z_0)} (\lambda + |z_0|) - A \right)^{-1} Cx - Cx \right] d\lambda\,ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^{\infty} \frac{e^{i(j+1)\arg(z_0)} s^{\alpha j} (\lambda + |z_0|)^{j+1}}{\Gamma(\alpha j+1)} \\
&\times e^{i\arg(z_0)} \left(e^{i\arg(z_0)} (\lambda + |z_0|) - A \right)^{-1} Cx\,d\lambda\,ds \\
&= \frac{1}{2\pi i} \int_\Gamma e^{-z(-\lambda)^b} \sum_{j=0}^{\infty} \frac{e^{i(j+1)\arg(z_0)} s^{\alpha(j+1)} (\lambda + |z_0|)^{j+1}}{\Gamma(\alpha(j+1)+1)} (\lambda - A_0)^{-1} Cx\,d\lambda \\
&= S_{\alpha,b,z}(t)x - T_b(z)x,\ t \geqslant 0,\ x \in E.
\end{aligned}
$$

Therefore, for every $z \in \Sigma_{\delta_b}$, $(S_{\alpha,b,z}(t))_{t\geqslant 0}$ is a locally equicontinuous $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, T_b(z))$-regularized resolvent family having A as a subgenerator, which immediately implies the $T_b(z)$-wellposedness of (330). Let $x_i \in R(T_b(z))$, $i = 0, 1, \cdot\cdot\cdot, \lceil\alpha\rceil - 1$. Then it is predictable that the solution of (330) is given by

(341)
$$
u(t;z) := S_{\alpha,b,z}(t)T_b(z)^{-1}x_0 + \sum_{i=1}^{\lceil\alpha\rceil} \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} S_{\alpha,b,z}(s)T_b(z)^{-1}x_{i-1}\,ds,\ t \geqslant 0.
$$

We will verify this without making no reference to our results stated in the beginning of this subsection. It is clear that the mapping $t \mapsto u(t;z)$, $t \geqslant 0$ $(z \in \Sigma_{\delta_b})$ can be extended to the whole complex plane by

$$
\begin{aligned}
u(\omega;z) = \frac{1}{2\pi i} \int_\Gamma e^{-z(-\lambda)^b} \sum_{k=0}^{\lceil\alpha\rceil-1} \sum_{j=0}^{\infty} \frac{e^{ij\arg(z_0)} \omega^{\alpha j+k} (\lambda + |z_0|)^j}{\Gamma(\alpha j+k+1)} \\
\times (\lambda - A_0)^{-1} CT_b(z)^{-1} x_k\,d\lambda,\ \omega \in \mathbb{C}.
\end{aligned}
\tag{342}
$$

Put, by common consent, $\frac{\omega^s}{\Gamma(s+1)} := 0$ if $-s \in \mathbb{N}$. Then the dominated convergence theorem and an elementary argumentation yield that the mapping $\omega \mapsto u(\omega; z)$, $\omega \in \mathbb{C} \smallsetminus (-\infty, 0]$ ($z \in \Sigma_{\delta_b}$) is analytic with

$$\frac{d}{d\omega} u(\omega; z) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-z(-\lambda)^b} \sum_{k=0}^{\lceil\alpha\rceil-1} \sum_{j=0}^{\infty} \frac{e^{ij \arg(z_0)} \omega^{\alpha j+k-1} (\lambda + |z_0|)^j}{\Gamma(\alpha j + k)} \times (\lambda - A_0)^{-1} C T_b(z)^{-1} x_k \, d\lambda, \ \omega \in \mathbb{C} \smallsetminus (-\infty, 0]. \tag{343}$$

Moreover, for every $p \in \mathbb{N}_0$, $l = 0, \cdots, \lceil\alpha\rceil - 1$, $\omega \in \mathbb{C} \smallsetminus (-\infty, 0]$ and $z \in \Sigma_{\delta_b}$,

$$A^p \frac{d^l}{d\omega^l} u(\omega; z) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-z(-\lambda)^b} \sum_{k=0}^{\lceil\alpha\rceil-1} \sum_{j=0}^{\infty} \frac{e^{i(j+p) \arg(z_0)} \omega^{\alpha j+k-l} (\lambda + |z_0|)^{j+p}}{\Gamma(\alpha j + k - l + 1)} \times (\lambda - A_0)^{-1} C T_b(z)^{-1} x_k \, d\lambda, \ \omega \in \mathbb{C} \smallsetminus (-\infty, 0]; \tag{344}$$

if $\alpha \in \mathbb{N}$, then the mapping $\omega \mapsto u(\omega; z)$, $\omega \in \mathbb{C}$ is entire and the formulae (343)-(344) hold for any $\omega \in \mathbb{C}$. The remaining part of the proof will be given in the case $\alpha \notin \mathbb{N}$. Owing to (344), $u(\cdot; z) \in C^{\lceil\alpha\rceil-1}([0, \infty) : E)$ and $Au(\cdot; z) \in C([0, \infty) : E)$. By the dominated convergence theorem, the definition of $T_b(z)$ as well as (342) and (344), it follows that $u(\cdot; z) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\cdot^k}{k!} x_k \in C^{\lceil\alpha\rceil}([0, \infty) : E)$ and that (330) holds, as claimed. The uniqueness of solutions of (330) follows from the uniqueness of solutions of (330) for $x_k = 0$, $0 \leqslant k \leqslant \lceil\alpha\rceil - 1$ and the fact that $(S_{\alpha,b,z}(t))_{t\geqslant 0}$ is a locally equicontinuous $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, T_b(z))$-regularized resolvent family having A as a subgenerator. Assume $K \subseteq \mathbb{C} \smallsetminus (-\infty, 0]$ is a compact set, $h > 0$, $z \in \Sigma_{\delta_b}$, $q \in \circledast$, and $|\omega| \leqslant L$, for every $\omega \in K$ and an appropriate $L \geqslant 1$. Put $N := \lceil\gamma\rceil + 2$, $M_1(z, q, b) := \max\{r_q(T_b(z)^{-1} x_j) : j = 0, \cdots, \lceil\alpha\rceil - 1\}$ and $\eta_{z,b} := \tan(\cos \pi b) \operatorname{Re} z - |\operatorname{Im} z|$. Since $|\lambda + |z_0||^{p+j} (1 + |\lambda + |z_0||)^N \leqslant (1 + |z_0|)^{p+j+N} (1 + |\lambda|)^{p+j+N}$, $j,\ p \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we get from (334) and (344) that:

$$\begin{aligned} q\big(A^p \frac{d^l}{d\omega^l} u(\omega; z)\big) &\leqslant c_q M_1(z, q, b) \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{1}{2\pi} \Bigg| \int\limits_{\Gamma} e^{-\eta_{z,b}|\lambda|^b} \\ &\times \sum_{j=0}^{\infty} \frac{L^{\alpha j+k-l}}{\Gamma(\alpha j + k - l + 1)} (1 + |z_0|)^{p+j+N} (1 + |\lambda|)^{p+j+N} \, d\lambda \Bigg| \\ &\leqslant c_q M_1(z, q, b) (1 + |z_0|)^{N+p} \\ &\times \sum_{k=0}^{\lceil\alpha\rceil-1} \sum_{j=0}^{\infty} \frac{L^{\alpha j+k-l} (1 + |z_0|)^j}{\Gamma(\alpha j + k - l + 1)} \Bigg| \frac{1}{2\pi} \int\limits_{\Gamma} e^{-\eta_{z,b}|\lambda|^b} (1 + |\lambda|)^{p+j+N} \, d\lambda \Bigg|. \end{aligned}$$

Using the proof of [**394**, Proposition 2.2] and Cauchy formula, we obtain that, for every $j,\ p \in \mathbb{N}_0$,

$$\Bigg| \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-\eta_{z,b}|\lambda|^b} (1 + |\lambda|)^{p+j+N} \, d\lambda \Bigg|$$

$$\leqslant 2^{p+j+N}\Big[e^{-\eta_{z,b}d^b}+\frac{1}{b}\Gamma\Big(\frac{p+j+N+1}{b}\Big)\eta^{-\frac{p+j+N+1}{b}}\Big]+2^{p+j+N}e^{-\eta_{z,b}d^b}$$
$$\leqslant 2^{p+j+N}\Big[2+\frac{1}{b}\Gamma\Big(\frac{p+j+N+1}{b}\Big)\eta_{z,b}^{-\frac{p+j+N+1}{b}}\Big],\ j\in\mathbb{N}_0.$$

Hence,

$$q\big(A^p\frac{d^l}{d\omega^l}u(\omega;z)\big)\leqslant c_qM_1(z,q,b)2^{p+N+1}(1+|z_0|)^{N+p}$$
$$\times\sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=0}^{\infty}\frac{L^{\alpha j+k-l}(2+2\omega)^j}{\Gamma(\alpha j+k-l+1)}$$
$$+\frac{c_qM_1(z,q,b)(2+2|z_0|)^{N+p}}{\eta_{z,b}^{\frac{N+p+1}{b}}b}$$
$$\times\sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=0}^{\infty}\frac{L^{\alpha j+k-l}(2+2\omega)^j}{\Gamma(\alpha j+k-l+1)}\frac{\Gamma(\frac{1}{b}(p+j+N+1))}{\eta_{z,b}^{\frac{j}{b}}}.$$

Put $B:=\sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=0}^{\infty}\frac{L^{\alpha j+k-l}(2+2|z_0|)^j}{\Gamma(\alpha j+k-l+1)}$. Then there exists $\nu_\alpha\geqslant 1$ such that $\alpha j(\alpha j+1)\cdots(\alpha j+\lceil\alpha\rceil)\leqslant\nu_\alpha^j$ for all $j\in\mathbb{N}$, which implies:

$$B\leqslant\sum_{k=0}^{\lceil\alpha\rceil-1}\Big(\frac{L^{k-l}}{(k-l)!}+\sum_{j=1}^{\infty}\frac{L^{\alpha j+k-l}(2+2|z_0|)^j}{\Gamma(\alpha j+k-l+1)}\Big)$$
$$\leqslant e^L+\sum_{k=l}^{\lceil\alpha\rceil-1}\sum_{j=1}^{\infty}\frac{L^{\alpha j+k-l}(2+2|z_0|)^j}{\Gamma(\alpha j+k-l+1)}+\sum_{k=0}^{l-1}\sum_{j=1}^{\infty}\frac{L^{\alpha j+k-l}(2+2|z_0|)^j}{\Gamma(\alpha j+k-l+1)}$$
$$\leqslant e^L+\lceil\alpha\rceil L^{\lceil\alpha\rceil-1}c_\alpha e^{L(2+2|z_0|)^{1/\alpha}}+c_\alpha(\lceil\alpha\rceil-1)e^{L\nu_\alpha(2+2|z_0|)^{1/\alpha}}.$$

As an outcome, we get

$$\sup_{p\in\mathbb{N}}\frac{h^p}{M_{\lfloor\alpha p\rfloor-1+l}}\Big[c_qM_1(z,q,b)2^{p+N+1}$$
$$\times(1+|z_0|)^{N+p}\sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=0}^{\infty}\frac{L^{\alpha j+k-l}(2+2|z_0|)^j}{\Gamma(\alpha j+k-l+1)}\Big]<\infty.$$

Let $\kappa\geqslant(2+2|z_0|)\eta^{-\frac{1}{b}}$ and let $j_0\in\mathbb{N}$ satisfy $j_0\geqslant\max(\frac{\lceil\alpha\rceil+1}{\alpha-\frac{1}{b}},2)$. We will prove that

$$S_{j_0}:=\sup_{p\in\mathbb{N}}\frac{\kappa^p}{M_{\lfloor\alpha p\rfloor-1+l}}\sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty}\frac{L^{\alpha j+k-l}\kappa^j}{\Gamma(\alpha j+k-l+1)}\Gamma\Big(\frac{p+j+N+1}{b}\Big)<\infty. \tag{345}$$

The choice of b implies $\frac{p+j+N+1}{b}\geqslant 2$ and since $\Gamma(\cdot)$ is increasing in (ξ,∞), where $\xi\sim 1.4616\ldots$, one has $\Gamma\big(\frac{p+j+N+1}{b}\big)\leqslant\big(\lceil\frac{p+j+N+1}{b}\rceil-1\big)!$ provided $p\in\mathbb{N}$ and $j\in\mathbb{N}_0$. Hereafter $c_{\alpha,K,b,N}$ is used as a generic symbol to denote a positive constant whose value depends only on α, K, b and N but may be different in different places. We get that

$$
\begin{aligned}
S_{j_0} &\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{\kappa^p(\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lfloor\alpha\rfloor-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j+k-l}\kappa^j}{\Gamma(\alpha j+k-l+1)(\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)! \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{2^{\lfloor\alpha p\rfloor}\kappa^p(\lfloor\alpha p\rfloor-1)!(\lfloor\alpha N\rfloor+1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lfloor\alpha\rfloor-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j+k-l}\kappa^j}{\Gamma(\alpha j+k-l+1)(\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)! \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(2^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j+k-l}\kappa^j 2^{\alpha j+\alpha p+k-l}}{(\lfloor\alpha j\rfloor+\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor+k-l)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)! \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j+k-l}(2^\alpha\kappa)^j}{(\lfloor\alpha j\rfloor+\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor+k-l)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)! \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(2^\alpha\kappa)^j\lceil\frac{1}{b}\rceil!2^{\alpha j+\alpha p}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!}{(\lfloor\alpha j\rfloor+\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor+k-l+\lceil\frac{1}{b}\rceil)!} \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(4^\alpha\kappa)^j}{(\lfloor\alpha j\rfloor+\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor+k-l+\lceil\frac{1}{b}\rceil-\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1)!} \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \\
&\times \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(4^\alpha\kappa)^j}{(\lfloor\alpha j\rfloor+\lfloor\alpha p\rfloor+\lfloor\alpha N\rfloor-l-\lceil\frac{p}{b}\rceil-\lceil\frac{j}{b}\rceil-\lceil\frac{N}{b}\rceil)!} \\
&\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^\alpha\kappa)^p(\lfloor\alpha p\rfloor-1)!}{M_{\lfloor\alpha p\rfloor-1+l}} \sum_{k=0}^{\lceil\alpha\rceil-1}\sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(4^\alpha\kappa)^j}{(\lfloor\alpha j\rfloor-\lceil\frac{j}{b}\rceil-l-1)!}
\end{aligned}
$$

$$\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(4^{\alpha}\kappa)^p(\lfloor \alpha p\rfloor - 1)!}{M_{\lfloor \alpha p\rfloor -1+l}} \sum_{k=0}^{\lceil \alpha\rceil -1} \sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(4^{\alpha}\kappa)^j}{((\alpha-\frac{1}{b})j-l-2)!}$$

$$\leqslant c_{\alpha,K,b,N} \sup_{p\in\mathbb{N}} \frac{(8^{\alpha}\kappa)^p(\lfloor \alpha p\rfloor - 1)!}{M_{\lfloor \alpha p\rfloor -1+l}} \sum_{k=0}^{\lceil \alpha\rceil -1} \sum_{j=j_0}^{\infty} \frac{L^{\alpha j}(8^{\alpha}\kappa)^j}{((\alpha-\frac{1}{b})j-2)!} < \infty.$$

On the other hand, it can easily be seen that, for every $j = 0, \cdots, j_0 - 1$,

$$\sup_{p\in\mathbb{N}} \frac{\kappa^p}{M_{\lfloor \alpha p\rfloor -1+l}} \sum_{k=0}^{\lceil \alpha\rceil -1} \frac{L^{\alpha j+k-l}}{\Gamma(\alpha j+k-l+1)} \kappa^j \Gamma\Big(\frac{p+j+N+1}{b}\Big) < \infty.$$

This implies (333) and completes the proof. $\square$

Remark 3.6.12. Let the conditions of Theorem 3.6.11(i) (resp. Theorem 3.6.11(ii)) hold. Then, for every fixed $z \in \Sigma_{\delta_b}$ (resp. $z \in \Sigma_{\vartheta}$), there exists a function $u_1 \in C([0,\infty) : E)$ so that the function $t \mapsto u(t;z)$, $t \geqslant 0$ satisfies $u(\cdot;z)_{|(0,\infty)} \in C^{\lceil \alpha\rceil}((0,\infty) : E)$ and $\frac{d^{\lceil \alpha\rceil}}{dt^{\lceil \alpha\rceil}} u(t;z) = \frac{t^{\alpha-\lceil \alpha\rceil}}{\Gamma(\alpha-\lceil \alpha\rceil+1)} u_1(t)$, $t > 0$. Using the same arguments as in the proof of Theorem 3.6.11, it follows that the estimate (333) holds for any compact set $K \subseteq [0,\infty)$. If, additionally, E is a Banach space and $C = I$, then $(T_b(z))_{z\in\Sigma_{\delta_b}}$ (resp. $(T_b(z))_{z\in\Sigma_{\vartheta}}$) is an analytic semigroup of growth order $\frac{\gamma+1}{b}$.

There is by now only a few references on (ultra)-distribution semigroups in SCLCSs (cf. [**411**]-[**412**] and [**415**]). In the sequel, we always assume that E is a Banach space.

Concerning regularization of ultradistribution sines whose generators possess ultra-polynomially bounded resolvent, we have the following interesting assertion which can be reformulated in the case of exponential ultradistribution sines.

Theorem 3.6.13. *Suppose (M_p) additionally satisfies* (M.2) *and* (M.3). *If A generates an ultradistribution sine of (M_p)-class, then there exists an injective operator C such that A generates a global C-regularized cosine function $(C(t))_{t\geqslant 0}$. Furthermore, the mapping $t \mapsto C(t)$, $t \geqslant 0$ is infinitely differentiable and, for every $h > 0$ and for every compact set $K \subseteq [0,\infty)$, the following holds:*

$$\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big(\Big\|\frac{d^{p+1}}{dt^{p+1}}C(t)\Big\| + \Big\|\frac{d^p}{dt^p}C(t)\Big\|\Big) < \infty. \tag{346}$$

Proof. We will use the same terminology as in the preceding subsection. The operator $\mathcal{A}$ generates a (UDS) of (M_p)-class and one can argue as in the proof of Theorem 3.6.4 to deduce that there exist constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ such that $\Lambda_{\alpha,\beta,l} \subseteq \rho(\mathcal{A})$ and that:

$$\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\| \leqslant \text{Const}\, \frac{e^{M(Hl|\lambda|)}}{|\lambda|^k},\ \lambda \in \Lambda_{\alpha,\beta,l},\ k \in \mathbb{N}.$$

By Lemma 3.6.7, we have the existence of a sequence (N_p) satisfying $N_0 = 1$, (M.1), (M.2), (M.3) and $N_p \prec M_p$. Furthermore, there exists a sufficiently large natural number n so that the operator $\mathcal{D}_n \in L(E \times E)$, defined by

$$\mathcal{D}_n(x\ y)^T := \frac{1}{2\pi i} \int\limits_{\Gamma_l} \frac{R(\lambda : \mathcal{A})}{\omega_{N_p}^n(i\lambda)} (x\ y)^T d\lambda, \quad x,\, y \in E$$

is injective and that the following expression defines a bounded linear operator for every $t \geqslant 0$:

$$C(t)x := \frac{1}{2\pi i} \int\limits_{\Gamma_l} \frac{\lambda \cosh(\lambda t) R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda,\ x \in E. \tag{347}$$

The standard argumentation implies $C(t)A \subseteq AC(t)$, $t \geqslant 0$. Moreover, $(C(t))_{t\geqslant 0}$ is strongly continuous and

$$\frac{1}{2\pi i} \int\limits_{\Gamma_l} \frac{e^{\pm\lambda s}\lambda^p}{\omega_{N_p}^n(i\lambda)}\, d\lambda = 0,\ p \in \mathbb{N}_0,\ s \geqslant 0. \tag{348}$$

Let us prove that $A\int_0^t (t-s)C(s)x\, ds = C(t)x - Cx$, $x \in E$, $t \geqslant 0$, where $C := C(0)$. Fix, for the time being, a number $t \geqslant 0$ and note that, for every $\lambda \in \Gamma_l$, we have $\lambda^3 \int_0^t (t-s)\cosh(\lambda s)\, ds = \lambda\cosh(\lambda t) - \lambda$. Then the Fubini theorem, the simple equality $AR(\lambda^2 : A)x = \lambda^2 R(\lambda^2 : A)x - x$, $\lambda \in \Gamma_l$, $x \in E$ and (348) imply:

$$\begin{aligned}
A\int\limits_0^t (t-s)C(s)x\, ds &= \int\limits_0^t (t-s)\frac{1}{2\pi i}\int\limits_{\Gamma_l}\left[\lambda\cosh(\lambda s)\frac{\lambda^2 R(\lambda^2 : A)x - x}{\omega_{N_p}^n(i\lambda)}\, d\lambda\right] ds \\
&= \int\limits_0^t (t-s)\frac{1}{2\pi i}\int\limits_{\Gamma_l}\left[\lambda^3\cosh(\lambda s)\frac{R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda\right] ds \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_l}\left[\lambda^3\int\limits_0^t (t-s)\cosh(\lambda s)\, ds\right]\frac{R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_l}\big(\lambda\cosh(\lambda t) - \lambda\big)\frac{R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda = C(t)x - Cx,
\end{aligned}$$

for every $x \in E$. Proceeding as in the proof of Theorem 3.6.4, one can differentiate (347) under the integral sign and, in such a way, one gets that, for every $t \in [0, \tau)$ and $x \in E$:

$$\frac{d^n}{dt^n}C(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_l}\frac{\lambda^{n+1}\cosh(\lambda t)R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda,\ 2|n,\ n \in \mathbb{N} \text{ and}$$

$$\frac{d^n}{dt^n}C(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_l}\frac{\lambda^{n+1}\sinh(\lambda t)R(\lambda^2 : A)x}{\omega_{N_p}^n(i\lambda)}\, d\lambda,\ 2|n-1,\ n \in \mathbb{N}. \tag{349}$$

One can prove that C is injective as follows. Suppose $Cx = 0$, for some $x \in E$. Put $C(-t) := C(t)$, $t \in (0, \tau)$ and notice that the previous argumentation simply implies that, for every $y \in E$ and $t,\ s \in \mathbb{R}$:

$$\int_0^s (s-r)C(r)\bigl(C(t)y - Cy\bigr)\,dr = \bigl(C(s) - C\bigr)\int_0^t (t-r)C(r)y\,dr.$$

Now an application of [**381**, Theorem 1.2] gives

$$C(t+s)y + C(|t-s|)y = 2C(t)C(s)y,\ y \in E,\ t \geqslant 0,\ s \geqslant 0.$$

Thereby, $C(t)x = 0$, $t \geqslant 0$ and the use of (349), with $n = 1$ and $t = 0$, yields:

$$\frac{1}{2\pi i}\int_{\Gamma_l} \frac{\lambda^2 R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = 0. \tag{350}$$

Using the equality $R(\lambda : \mathcal{A})\binom{x}{0} = \binom{\lambda R(\lambda^2 : A)x}{AR(\lambda^2 : A)x}$, $\lambda \in \Lambda_{\alpha,\beta,l}$, (348), (350) and the resolvent equation, we easily infer that

$$\frac{1}{2\pi i}\int_{\Gamma_l} \frac{AR(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = \frac{(-1)}{2\pi i}\int_{\Gamma_l} \frac{d\lambda}{\omega^n_{N_p}(i\lambda)}x + \frac{1}{2\pi i}\int_{\Gamma_l} \frac{\lambda^2 R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = 0.$$

Therefore, $\mathcal{D}_n(x\ 0)^T = 0$ and $x = 0$, as required. Hence, $(C(t))_{t\geqslant 0}$ is a global C-regularized cosine function with the integral generator A. The proof of (346) follows by means of the estimations already given in the proofs of Theorem 3.6.4 and Theorem 3.6.6. This completes the proof of theorem. □

Suppose that (M_p) additionally satisfies (M.2) and (M.3). Then one can simply prove that $E^{(M_p)}(\mathcal{A}) = E^{(M_p)}(A) \times E^{(M_p)}(A)$. Keeping in mind Theorem 3.6.6, the preceding equality immediately implies the following theorem.

THEOREM 3.6.14. *Suppose that* (M_p) *additionally satisfies* (M.2) *and* (M.3) *and that* G *is an ultradistribution sine of* (M_p)*-class generated by* A*. Then, for every* $x \in E^{(M_p)}(A)$ *and* $y \in E^{(M_p)}(A)$*, the abstract Cauchy problem*

$$(ACP_2): \begin{cases} u \in C^\infty([0,\infty) : E) \cap C([0,\infty) : [D(A)]), \\ u''(t) = Au(t),\ t \geqslant 0, \\ u(0) = x,\ u'(0) = y \end{cases}$$

has a unique solution. Furthermore, for every compact set $K \subseteq [0,\infty)$ *and* $h > 0$*, the solution* u *of* (ACP_2) *satisfies*

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \frac{h^p}{M_p}\Bigl(\Bigl\|\frac{d^p}{dt^p}u(t)\Bigr\| + \Bigl\|\frac{d^p}{dt^{p+1}}u(t)\Bigr\|\Bigr) < \infty.$$

APPENDIX

Abstract Volterra Equations of Nonscalar Type

Henceforth X and Y are Banach spaces, Y is continuously embedded in X, $L(X) \ni C$ is injective and $\tau \in (0, \infty]$. The norm in X, resp. Y, is denoted by $\| \cdot \|_X$, resp. $\| \cdot \|_Y$; $[\mathrm{R}(C)]$ denotes the Banach space $\mathrm{R}(C)$ equipped with the norm $\|x\|_{\mathrm{R}(C)} = \|C^{-1}x\|_X$, $x \in \mathrm{R}(C)$ and, for a given closed linear operator A in X, $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$, $x \in D(A)$. Let $A(t)$ be a locally integrable function from $[0, \tau)$ into $L(Y, X)$. Unless stated otherwise, we assume that $A(t)$ is not of scalar type, i.e., that there does not exist $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, and a closed linear operator A in X such that $Y = [D(A)]$ and that $A(t) = a(t)A$ for a.e. $t \in [0, \tau)$ (cf. also the short discussion preceding Proposition A.3 for full details). In what follows, the symbol $*$ denotes the finite convolution and the meaning of symbol A is clear from the context. We basically follow the terminology given in the monograph of Prüss [**369**].

Our intention is to enquire into the basic structural properties of a fairly general class of (local) (A, k)-regularized C-pseudoresolvent families. This class of pseudoresolvent families presents the main tool in the analysis of ill-posed hyperbolic Volterra equations of non-scalar type. It is worthwhile to mention that there are by now only a few references concerning non-scalar Volterra equations in their most general abstract form (cf. [**164, 179**] and [**369**]). We analyze Hille–Yosida type theorems, perturbations, regularity properties of solutions of non-scalar operator equations, and remove density assumptions from the previously known concepts.

Definition A.1. Let $k \in C([0, \tau))$ and $k \neq 0$. Consider the linear Volterra equation

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in [0, \tau), \tag{351}$$

where $\tau \in (0, \infty]$, $f \in C([0, \tau) : X)$ and $A \in L^1_{\mathrm{loc}}([0, \tau) : L(Y, X))$. A function $u \in C([0, \tau) : X)$ is said to be:

(i) a *strong solution* of (351) iff $u \in L^\infty_{\mathrm{loc}}([0, \tau) : Y)$ and (351) holds on $[0, \tau)$,

(ii) a *weak solution* of (351) iff there exist a sequence (f_n) in $C([0, \tau) : X)$ and a sequence (u_n) in $C([0, \tau) : X)$ such that $u_n(t)$ is a strong solution of (351) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ and $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0, \tau)$.

The abstract Cauchy problem (351) is said to be (kC)-*well posed* (C-*well posed*, if $k(t) \equiv 1$) iff for every $y \in Y$, there exists a unique strong solution of

$$u(t;y) = k(t)Cy + \int_0^t A(t-s)u(s;y)\,ds,\ t \in [0,\tau) \tag{352}$$

and if $u(t;y_n) \to 0$ in X, uniformly on compact subsets of $[0,\tau)$, whenever (y_n) is a zero sequence in Y; (351) is said to be a-*regularly* (kC)-*well posed* (a-*regularly* C-*well posed*, if $k(t) \equiv 1$), where $a \in L^1_{\text{loc}}([0,\tau))$, iff (351) is (kC)-well posed and if the equation

$$u(t) = (a*k)(t)Cx + \int_0^t A(t-s)u(s)\,ds,\ t \in [0,\tau)$$

admits a unique strong solution for every $x \in X$.

It is clear that every strong solution of (351) is also a mild solution of (351).

Definition A.2. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{\text{loc}}([0,\tau) : L(Y,X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X)$ is called an (A,k)-*regularized* C-*pseudoresolvent family* iff the following holds:

(S1) The mapping $t \mapsto S(t)x$, $t \in [0,\tau)$ is continuous in X for every fixed $x \in X$, $S(0) = k(0)C$ and $S(t)C = CS(t)$, $t \in [0,\tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. Then (S2) means $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ and $\bigl(U(t)_{|Y}\bigr)_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$S(t)y = k(t)Cy + \int_0^t A(t-s)dU(s)y\,ds,\ t \in [0,\tau),\ y \in Y, \tag{353}$$

$$S(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y, \tag{354}$$

hold; (353), resp. (354), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A,k)-*regularized* C-*resolvent family* if additionally:

(S4) For every $y \in Y$, $S(\cdot)y \in L^\infty_{\text{loc}}([0,\tau) : Y)$.

A family $(S(t))_{t\in[0,\tau)}$ in $L(X)$ is called a *weak* (A,k)-*regularized* C-*pseudoresolvent family* iff (S1) and (354) hold. A weak (A,k)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$ is said to be *exponentially bounded* iff there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S(t)\|_{L(X)} \leqslant Me^{\omega t}$, $t \geqslant 0$. Finally, a weak (A,k)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be a-*regular* ($a \in L^1_{\text{loc}}([0,\tau))$) iff $a * S(\cdot)x \in C([0,\tau) : Y)$, $x \in \overline{Y}^X$.

In this paragraph, we will ascertain a few lexicographical agreements. A (weak) (A, k)-regularized C-(pseudo)resolvent family with $k(t) \equiv \frac{t^\alpha}{\Gamma(\alpha+1)}$, where $\alpha \geqslant 0$, is also called a (weak) α-times integrated A-regularized C-(pseudo)resolvent family; a (weak) 0-times integrated A-regularized C-(pseudo)resolvent family is also said to be a (weak) A-regularized C-(pseudo)resolvent family. A (weak) (A, k)-regularized C-(pseudo)resolvent family is also said to be a (weak) (A, k)-regularized (pseudo)resolvent family ((weak) A-regularized (pseudo)resolvent family) if $C = I$ (if $C = I$ and $k(t) \equiv 1$).

It is worth noting that the integral appearing in the first resolvent equation (353) is understood in the sense of discussion following [**369**, Definition 6.2, p. 152] and that Jung considered in [**164**] a slightly different notion of A-regularized (pseudo)resolvent families. Moreover, (S3) can be rewritten in the following equivalent form:

$$\text{(S3}'\text{)} \qquad U(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds,\ t \in [0,\tau),\ y \in Y,$$

$$U(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y.$$

By the norm continuity we mean the continuity in $L(X)$ and, in many places, we do not distinguish $S(\cdot)$ $(U(\cdot))$ and its restriction to Y. The main reason why we assume that $A(t)$ is not of scalar type is the following: Let A be a subgenerator of a (local) (a, k)-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$ in the sense of Definition 2.1.26, let $Y = [D(A)]$ and let $A(t) = a(t)A$ for a.e. $t \in [0, \tau)$. Then $(S(t))_{t\in[0,\tau)}$ is an (A, k)-regularized C-resolvent family in the sense of Definition A.2, $S(t) \in L(Y)$, $t \in [0, \tau)$ and, for every $y \in Y$, $S(\cdot)y \in C([0, \tau) : Y)$ and the mapping $t \mapsto U(t)y$, $t \in [0, \tau)$ is continuously differentiable in Y with $\frac{d}{dt}U(t)y = S(t)y$, $t \in [0, \tau)$ (cf. also Remark A.10 as well as the proofs of Theorem A.7, Theorem A.9 and Theorem A.18). Assume conversely $A(t) = a(t)A$ for a.e. $t \in [0, \tau)$, $Y = [D(A)]$ and $(S(t))_{t\in[0,\tau)}$ is an (A, k)-regularized C-resolvent family in the sense of Definition A.2. If $CA \subseteq AC$ and $a(t)$ is kernel, then $(S(t))_{t\in[0,\tau)}$ is an (a, k)-regularized C-resolvent family in the sense of Definition 2.1.26. In order to verify this, notice that the second equality in (S3)' implies after differentiation $S(t)x = k(t)Cx + \int_0^t S(t-s)a(s)Axds = k(t)Cx + \int_0^t a(t-s)S(s)Axds$, $t \in [0, \tau)$, $x \in D(A)$, so that it suffices to show that $S(t)A \subseteq AS(t)$, $t \in [0, \tau)$. Combined with the first equality in (S3)', we get that, for every $t \in [0, \tau)$ and $x \in D(A)$:

$$\frac{d}{dt}\int_0^t a(t-s)AU(s)x\,ds = S(t)x - k(t)Cx = \int_0^t a(t-s)S(s)Ax\,ds,$$

$$\int_0^t a(t-s)AU(s)x\,ds = \int_0^t\int_0^s a(s-r)S(r)Ax\,dr\,ds = \int_0^t a(t-s)U(s)Ax\,ds.$$

Hence, $A\int_0^t S(s)x\,ds = \int_0^t S(s)Ax\,ds$, $t \in [0,\tau)$, $x \in D(A)$. Then the closedness of A gives $S(t)A \subseteq AS(t)$, $t \in [0,\tau)$, as required. In the formulations of Proposition A.6, Theorem A.12, Corollary A.13(i) as well as in the analyses given in Example A.14, Example A.20 and the paragraph preceding it, we also allow that $A(t)$ $((A+B)(t))$ is of scalar type; if this is the case, the notion of a corresponding (weak) (A,k)-regularized $((A+B,k)$-regularized) C-(pseudo)resolvent family is always understood in the sense of Definition A.2.

The subsequent propositions can be proved by means of the argumentation given in [**369**].

PROPOSITION A.3. (i) *Let $(S_i(t))_{t\in[0,\tau)}$ be an (A,k_i)-regularized C-pseudoresolvent family, $i=1,2$. Then $(k_2 * R_1)(t)x = (k_1 * R_2)(t)x$, $t\in[0,\tau)$, $x \in \overline{Y}^X$.*

(ii) *Let $(S_i(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family, $i=1,2$ and let $k(t)$ be a kernel. Then $S_1(t)x = S_2(t)x$, $t\in[0,\tau)$, $x\in\overline{Y}^X$.*

(iii) *Let $(S(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family. Assume any of the following conditions:*

(iii.1) *Y has the Radon–Nikodym property.*

(iii.2) *There exists a dense subset Z of Y such that $A(t)z\in Y$ for a.e. $t\in[0,\tau)$, $A(\cdot)z \in L^1_{\mathrm{loc}}([0,\tau):Y)$, $z\in Z$ and $C(Y)\subseteq Y$.*

(iii.3) *$(S(t))_{t\in[0,\tau)}$ is a-regular, $A(t)=(a*dB)(t)$, $t\in[0,\tau)$, where $C(Y)\subseteq Y$, $a\in L^1_{\mathrm{loc}}([0,\tau))$ and $B\in BV_{\mathrm{loc}}([0,\tau):L(Y,X))$ is such that $B(\cdot)y$ has a locally bounded Radon–Nikodym derivative w.r.t. $b(t)=\mathrm{Var}B|_0^t$, $t\in[0,\tau)$, $y\in Y$.*

Then $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family. Furthermore, if Y is reflexive, then $S(t)(Y)\subseteq Y$, $t\in[0,\tau)$ and the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is weakly continuous in Y for every $y\in Y$. In cases (ii) *and* (iii)*, the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is even strongly continuous in Y.*

PROPOSITION A.4. (i) *Assume $(S(t))_{t\in[0,\tau)}$ is a weak (A,k)-regularized C-pseudoresolvent family and $u(t)$ is a mild solution of* (351)*. Then*

$$(kC*u)(t) = (S*f)(t),\ t\in[0,\tau).$$

In particular, mild solutions of (351) *are unique provided that $k(t)$ is a kernel.*

(ii)*Assume $n\in\mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is an $(n-1)$-times integrated A-regularized C-pseudoresolvent family, $C^{-1}f\in C^{n-1}([0,\tau):X)$ and $f^{(i)}(0)=0$, $0\leqslant i\leqslant n-1$. Then the following assertions hold:*

(ii.1) *Let $(C^{-1}f)^{(n-1)}\in AC_{\mathrm{loc}}([0,\tau):Y)$ and $(C^{-1}f)^{(n)}\in L^1_{\mathrm{loc}}([0,\tau):Y)$. Then the function*

$$u(t)=\int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds=\int_0^t dU(s)(C^{-1}f)^{(n)}(t-s)\,ds,\ t\in[0,\tau)$$

is a unique strong solution of (351)*. Moreover, $u\in C([0,\tau):Y)$.*

(ii.2) *Let $(C^{-1}f)^{(n)}\in L^1_{\mathrm{loc}}([0,\tau):X)$ and $\overline{Y}^X=X$. Then the function $u(t)=\int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds$, $t\in[0,\tau)$ is a unique mild solution of* (351)*.*

(ii.3) *Let* $C^{-1}g \in W^{n,1}_{\mathrm{loc}}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $f(t) = \big(\frac{t^{n-1}}{(n-1)!} * a * g^{(n)}\big)(t)$, $t \in [0,\tau)$ *and* $(S(t))_{t\in[0,\tau)}$ *is* a*-regular. Then the function* $u(t) = \int_0^t S(t-s) \times (a * (C^{-1}g)^{(n)})(s)\, ds$, $t \in [0,\tau)$ *is a unique strong solution of* (351).

PROPOSITION A.5. (i) *Assume* $(S(t))_{t\in[0,\tau)}$ *is an* (A,k)*-regularized* C*-resolvent family. Put* $u(t;y) := S(t)y$, $t \in [0,\tau)$, $y \in Y$. *Then* $u(t;y)$ *is a strong solution of* (352), *and* (352) *is* (kC)*-well posed if* $k(t)$ *is a kernel.*

(ii) *Assume* $\overline{Y}^X = X$, (352) *is* (kC)*-well posed, all suppositions quoted in the formulation of Proposition* A.3(iii.2) *hold and* $A(t)Cz = CA(t)z$ *for all* $z \in Z$ *and a.e.* $t \in [0,\tau)$. *Then* (351) *admits an* (A,k)*-regularized* C*-resolvent family.*

(iii) *Assume* $\overline{Y}^X = X$, $L^1_{\mathrm{loc}}([0,\tau)) \ni a$ *is a kernel and* $A(t)Cy = CA(t)y$ *for all* $y \in Y$ *and a.e.* $t \in [0,\tau)$. *Then* (352) *is* a*-regularly* (kC)*-well posed iff* (351) *admits an* a*-regular* (A,k)*-regularized* C*-resolvent family.*

PROPOSITION A.6. *Assume* $A \in L^1_{\mathrm{loc}}([0,\tau) : L([D(A)],X))$ *is of the form*

$$A(t) = a(t)A + \int_0^t a(t-s)\, dB(s),\ t \in [0,\tau), \tag{355}$$

where $a \in L^1_{\mathrm{loc}}([0,\tau))$, $B \in BV_{\mathrm{loc}}([0,\tau) : L([D(A)],X))$ *is left continuous,* $B(0) = B(0+) = 0$, *and* A *is a closed linear operator with non-empty resolvent set. Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k)*-regularized* C*-pseudoresolvent family. Then* $(S(t))_{t\in[0,\tau)}$ *is* a*-regular.*

PROOF. Let $\mu \in \rho(A)$ and $K(t) := -B(t)(\mu - A)^{-1}$, $t \in [0,\tau)$. Then it is clear that $K \in BV_{loc}([0,\tau) : L(X))$. We define recursively $K_0(t) := K(t)$, $t \in [0,\tau)$ and $K_{n+1}(t) := \int_0^t dK(\tau)K_n(t-\tau)$, $t \in [0,\tau)$, $n \in \mathbb{N}$. By the proof of [**369**, Theorem 0.5, p. 13], the series $L(t) := \sum_{n=0}^{\infty}(-1)^n K_n(t)$, $t \in [0,\tau)$ converges absolutely in the norm of $BV^0([0,\tau) : L(X))$, $L \in BV^0([0,\tau) : L(X))$ and $L = K - dK * L = K - L * dK$. Repeating literally the proof of [**369**, Proposition 6.4, p. 137], we obtain that for every $y \in Y$:

$$A(a * S(\cdot)y) = S(\cdot)y - k(\cdot)Cy - dL * (S(\cdot)y - k(\cdot)Cy - \mu(a * S(\cdot))y).$$

Then the closedness of A immediately implies that, for every $x \in \overline{Y}^X$, one has $A(a * S(\cdot))x \in C([0,\tau) : X)$ and $a * S(\cdot)x \in C([0,\tau) : [D(A)])$. $\square$

The Hille–Yosida theorem for (A,k)-regularized C-pseudoresolvent families is given as follows.

THEOREM A.7. *Assume* $A \in L^1_{loc}([0,\tau) : L(Y,X))$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $a(t)$ *and* $k(t)$ *satisfy* (P1), $\varepsilon_0 \geqslant 0$ *and*

$$\int_0^\infty e^{-\varepsilon t}\big\|A(t)\big\|_{L(Y,X)} dt < \infty,\ \varepsilon > \varepsilon_0. \tag{356}$$

(i) *Let* $(S(t))_{t\geqslant 0}$ *be an* (A,k)*-regularized* C*-pseudoresolvent family such that there exists* $\omega \geqslant 0$ *with*

$$\sup_{t>0} e^{-\omega t}\Big(\big\|S(t)\big\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\big\|U(t)-U(s)\big\|_{L(Y)}\Big) < \infty. \tag{357}$$

Put $\omega_0 := \max(\omega, \operatorname{abs}(k), \varepsilon_0)$ *and* $H(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x \in X$, $\operatorname{Re}\lambda > \omega_0$. *Then the following holds:*

(N1) $C(Y) \subseteq Y$, $(\tilde{A}(\lambda))_{\operatorname{Re}\lambda>\omega_0}$ *is analytic in* $L(Y,X)$, $\mathrm{R}(C_{|Y}) \subseteq \mathrm{R}(I-\tilde{A}(\lambda))$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$, *and* $I - \tilde{A}(\lambda)$ *is injective,* $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$,

(N2) $H(\lambda)y = \lambda\tilde{U}(\lambda)y$, $y \in Y$, $\operatorname{Re}\lambda > \omega_0$, $(I-\tilde{A}(\lambda))^{-1}C_{|Y} \in L(Y)$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$, $(H(\lambda))_{\operatorname{Re}\lambda>\omega_0}$ *is analytic in both spaces,* $L(X)$ *and* $L(Y)$, $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$, *and for every* $y \in Y$ *and* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$*:*

$$H(\lambda)(I-\tilde{A}(\lambda))y = (I-\tilde{A}(\lambda))H(\lambda)y = \tilde{k}(\lambda)Cy. \tag{358}$$

(N3) $$\sup_{n\in\mathbb{N}_0}\ \sup_{\lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!}\Big(\Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(Y)}\Big) < \infty.$$

(ii) *Assume that* (N1)–(N3) *hold. Then there exists an exponentially bounded* (A,Θ)*-regularized* C*-resolvent family* $(S_1(t))_{t\geqslant 0}$.

(iii) *Assume that* (N1)–(N3) *hold and* $\overline{Y}^X = X$. *Then there exists an exponentially bounded* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geqslant 0}$ *such that* (357) *holds.*

(iv) *Assume* $(S(t))_{t\geqslant 0}$ *is an* (A,k)*-regularized* C*-pseudoresolvent family, there exists* $\omega \geqslant 0$ *such that* (357) *holds and* $\omega' \geqslant \omega$. *Then* $(S(t))_{t\geqslant 0}$ *is* a*-regular and* $\sup_{t\geqslant 0} e^{-\omega' t}\|a * S(t)\|_{L(\overline{Y}^X,Y)} < \infty$ *iff there exists* $\omega_1 \geqslant \max(\omega', \operatorname{abs}(a), \operatorname{abs}(k), \varepsilon_0)$ *such that*

$$\sup_{n\in\mathbb{N}_0}\ \sup_{\lambda>\omega_1,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega')^{n+1}}{n!}\Big\|\frac{d^n}{d\lambda^n}\tilde{a}(\lambda)H(\lambda)\Big\|_{L(\overline{Y}^X,Y)} < \infty. \tag{359}$$

Proof. In order to prove (i), notice that $\tilde{U}(\lambda) = H(\lambda)/\lambda$, $\operatorname{Re}\lambda > \omega_0$. Furthermore, $(\tilde{A}(\lambda))_{\operatorname{Re}\lambda>\omega_0}$ is analytic in $L(Y,X)$ and (357) in combination with (S1) yields that $(H(\lambda))_{\operatorname{Re}\lambda>\omega_0} \subseteq L(X) \cap L(Y)$ is analytic in both spaces, $L(X)$ and $L(Y)$, and that $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$. Fix, for the time being, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$. Using (S3)', one gets (358), $C(Y) \subseteq Y$, $\mathrm{R}(C_{|Y}) \subseteq \mathrm{R}(I-\tilde{A}(\lambda))$, $(I-\tilde{A}(\lambda))^{-1}C_{|Y} = \frac{\lambda\tilde{U}(\lambda)}{\tilde{k}(\lambda)} \in L(Y)$ and the injectiveness of the operator $I-\tilde{A}(\lambda)$. Therefore, we have proved (N1)-(N2). The assertion (N3) is an immediate consequence of Theorem 1.1.13, which completes the proof of (i). Assume now (N1)-(N3). By Theorem 1.1.13, we obtain that there exist $M \geqslant 1$ and continuous functions $S_1 : [0,\infty) \to L(X)$ and $S_1^Y : [0,\infty) \to L(Y)$ such that $S_1(0) = S_1^Y(0) = 0$,

$$\sup_{t>0} e^{-\omega t}\Big(\sup_{0<s<t}(t-s)^{-1}\|S_1(t)-S_1(s)\|_{L(X)}$$

$$+\sup_{0<s<t}(t-s)^{-1}\|S_1^Y(t)-S_1^Y(s)\|_{L(Y)}\Big)<\infty, \tag{360}$$

$$H(\lambda)x=\lambda\int_0^\infty e^{-\lambda t}S_1(t)x\,dt,\ x\in X,\ \operatorname{Re}\lambda>\omega_0, \tag{361}$$

$$H(\lambda)y=\lambda\int_0^\infty e^{-\lambda t}S_1^Y(t)y\,dt,\ y\in Y,\ \operatorname{Re}\lambda>\omega_0. \tag{362}$$

Using the inverse Laplace transform, (N2) and (361)-(362), we infer that $(S_1(t))_{t\geqslant 0}$ commutes with C and that $S_1(t)y=S_1^Y(t)y,\ t\geqslant 0,\ y\in Y$. It is evident that the mapping $t\mapsto S_1(t)y,\ t\geqslant 0$ is continuous in Y for every fixed $y\in Y$ and that $(U_1(t)\equiv\int_0^t S_1(s)ds)_{t\geqslant 0}$ is continuously differentiable in $L(Y)$ with $\frac{d}{dt}U_1(t)=S_1^Y(t),\ t\geqslant 0$. The above assures that (S1), (S2) and (S4) hold for $(S_1(t))_{t\geqslant 0}$. Combining the inverse Laplace transform and (358) one gets that $(S_1(t))_{t\geqslant 0}$ satisfies (S3)', which completes the proof of (ii). If $\overline{Y}^X=X$, then the proof of [**434**, Theorem 3.4, p. 14] implies that there exists a strongly continuous operator family $(S(t))_{t\geqslant 0}$ in $L(X)$ such that $S_1(t)x=\int_0^t S(s)xds,\ t\geqslant 0,\ x\in X$. The estimate (357) is a consequence of (360) and the remaining part of the proof of (iii) essentially follows from the corresponding part of the proof of [**369**, Theorem 6.2, p. 164]. Assuming $M'\geqslant 1,\ \omega'\geqslant 0$, a-regularity of $(S(t))_{t\geqslant 0}$ and $\|a*S(t)x\|_Y\leqslant M'e^{\omega' t}\|x\|_X,\ t\geqslant 0$, $x\in\overline{Y}^X$, the estimate (359) follows from a straightforward computation. The converse implication in (iv) follows from Theorem 1.1.13, the uniform boundedness principle and the final part of the proof of [**369**, Theorem 6.2, p. 165]. $\square$

REMARK A.8. Assume that $A(t)$ is of the form (355) and that $a(t)$ as well as $B(t)$, in addition to the assumptions prescribed in Proposition A.6, are of exponential growth. Then the condition (N3) can be replaced by a slightly weaker condition:

$$\Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(X)}\leqslant\frac{Mn!}{(\lambda-\omega)^{n+1}},\ n\in\mathbb{N}_0,\ \lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0. \tag{N3$'$}$$

Now we state the complex characterization theorem for (A,k)-regularized C-pseudoresolvent families.

THEOREM A.9. (i) *Assume $A(t)$ satisfies* (356) *with some $\varepsilon_0\geqslant 0$, $k(t)$ satisfies* (P1), *$\omega_1:=\max(\operatorname{abs}(k),\varepsilon_0)$ and there exists an analytic mapping $f:\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\omega_1\}\to L(X)$ such that $f(\lambda)(I-\tilde{A}(\lambda))y=\tilde{k}(\lambda)Cy$, $\operatorname{Re}\lambda>\omega_1$, $\tilde{k}(\lambda)\neq 0$, $y\in Y$, $f(\lambda)C=Cf(\lambda)$, $\operatorname{Re}\lambda>\omega_1$ and $\|f(\lambda)\|_{L(X)}\leqslant M|\lambda|^r$, $\operatorname{Re}\lambda>\omega_1$ for some $M\geqslant 1$ and $r>1$. Then, for every $\alpha>1$, there exists a norm continuous, exponentially bounded weak $(A,k*\frac{t^{r+\alpha-1}}{\Gamma(r+\alpha)})$-regularized C-pseudoresolvent family $(S_\alpha(t))_{t\geqslant 0}$.*

(ii) *Let $(S_\alpha(t))_{t\geqslant 0}$ be as in* (i) *and let $a(t)$ satisfy* (P1). *Then $(S_\alpha(t))_{t\geqslant 0}$ is a-regular provided that there exist $M_1\geqslant 1$, $r_1>1$, a set $P\subseteq\mathbb{C}$, which has a limit point in $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\max(\omega_1,\operatorname{abs}(a))\}$, and an analytic mapping $h:\{\lambda\in\mathbb{C}:$*

$\operatorname{Re}\lambda > \max(\omega_1, \operatorname{abs}(a))\} \to L(\overline{Y}^X, Y)$ *such that*

$$h(\lambda)(I - \tilde{A}(\lambda))y = k * \widetilde{\frac{t^{\alpha+r-1}}{\Gamma(\alpha+r)}}(\lambda)Cy,\ y \in Y,\ \operatorname{Re}\lambda > \max(\omega_1, \operatorname{abs}(a)),$$
$$\|h(\lambda)\|_{L(\overline{Y}^X, Y)} \leqslant M_1|\lambda|^{-r_1},\ \operatorname{Re}\lambda > \max(\omega_1, \operatorname{abs}(a)),$$

and that $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ *exists for all* $\lambda \in P$.

(iii) *Let, in addition to the assumptions given in* (i), *the mapping* $\lambda \mapsto f(\lambda) \in L(Y)$, $\operatorname{Re}\lambda > \omega_1$ *be analytic in* $L(Y)$. *Suppose*

$$(I - \tilde{A}(\lambda))f(\lambda)y = \tilde{k}(\lambda)Cy,\ \operatorname{Re}\lambda > \omega_1,\ \tilde{k}(\lambda) \neq 0,\ y \in Y, \tag{363}$$

$$\|f(\lambda)\|_{L(Y)} \leqslant M|\lambda|^r,\ \operatorname{Re}\lambda > \omega_1 \text{ for some } M \geqslant 1 \text{ and } r > 1. \tag{364}$$

Then, for every $\alpha > 1$, $(S_\alpha(t))_{t\geqslant 0}$ *is a norm continuous, exponentially bounded* $\big(A, k * \frac{t^{r+\alpha-1}}{\Gamma(r+\alpha)}\big)$*-regularized* C*-resolvent family, and* $(U_\alpha(t))_{t\geqslant 0}$ *is continuously differentiable in* $L(Y)$.

Proof. To prove (i), fix an $\alpha > 1$ and notice that $\frac{f(\lambda)}{\lambda^{r+\alpha}} - \tilde{A}(\lambda)\frac{f(\lambda)}{\lambda^{r+\alpha}} = \frac{\tilde{k}(\lambda)}{\lambda^{r+\alpha}}Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega_1$, $\tilde{k}(\lambda) \neq 0$. Hence, there exists an exponentially bounded, continuous function $S_\alpha : [0, \infty) \to L(X)$ such that $S_\alpha(0) = 0$ and that $\widetilde{S_\alpha}(\lambda) = \frac{f(\lambda)}{\lambda^{r+\alpha}}$, $\operatorname{Re}\lambda > \omega_1$. Using the inverse Laplace transform, one immediately yields that $(S_\alpha(t))_{t\geqslant 0}$ commutes with C and that the second resolvent equation holds, which completes the proof of (i). To prove (ii), notice that there exists an exponentially bounded function $S_\alpha^a : [0, \infty) \to L(\overline{Y}^X, Y)$ such that $S_\alpha^a(0) = 0$ and that $\widetilde{S_\alpha^a}(\lambda) = h(\lambda)$, $\operatorname{Re}\lambda > \omega_1$. Furthermore, it is checked at once that

$$\big(\tilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\tilde{S_\alpha}(\lambda)\big)(I - \tilde{A}(\lambda))y = 0,\ y \in Y,\ \operatorname{Re}\lambda > \omega_1. \tag{365}$$

Since the mapping $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ exists for all $\lambda \in P$, (365) implies that $(\tilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\tilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $\lambda \in P$. Hence, $(\tilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\tilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $\operatorname{Re}\lambda > \omega_1$ and this, in turn, implies $S_\alpha^a(t) = (a * S_\alpha)(t)$, $t \geqslant 0$, which shows that $(S_\alpha(t))_{t\geqslant 0}$ is a-regular. To prove (iii), it suffices to notice that (364) implies $S_\alpha \in C([0, \infty) : L(Y))$, $\frac{d}{dt}U_\alpha(t) = S_\alpha(t)$, $t \geqslant 0$ in $L(Y)$ and that the first resolvent equation follows instantly from (363). □

Remark A.10. Assume $a \in L^1_{loc}([0, \tau))$, $(S(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) (A, k)-regularized C-(pseudo)resolvent family and $L^1_{loc}([0, \tau)) \ni b$ satisfies $b * k \neq 0$. Set $S_b(t)x := (b * S)(t)x$, $t \in [0, \tau)$, $x \in X$. Then it readily follows that $(S_b(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) $(A, b * k)$-regularized C-(pseudo)resolvent family. Furthermore, $(U_b(t)_{|Y})_{t\in[0,\tau)}$ is continuously differentiable in $L(Y)$ (cf. the proofs of [**14**, Proposition 1.3.6, Proposition 1.3.7]), provided that (S2) holds for $(S(t))_{t\in[0,\tau)}$, and $a * S_b(\cdot)x \in AC_{loc}([0, \tau) : Y)$, $x \in \overline{Y}^X$, provided that $(S(t))_{t\in[0,\tau)}$ is a-regular.

By the proof of [**286**, Proposition 2.5] and the consideration given in Remark A.10, we have the following.

Proposition A.11. *Let $k \in AC_{\mathrm{loc}}([0,\tau))$, $k(0) \neq 0$ and let $(R(t))_{t\in[0,\tau)}$ be a (weak, weak a-regular) (A,k)-regularized C-(pseudo)resolvent family. Then there exists $b \in L^1_{\mathrm{loc}}([0,\tau))$ such that $\bigl(R(t) \equiv \frac{1}{k(0)}R(t) + (b*R)(t)\bigr)_{t\in[0,\tau)}$ is a (weak, weak a-regular) A-regularized C-(pseudo)resolvent family.*

The next theorem can be shown following the lines of the proof of [**369**, Theorem 6.1, p. 159] with $K_0 = S * C^{-1}B_0$ and $K_1 = S * C^{-1}B_1$.

Theorem A.12. *Assume $L^1_{\mathrm{loc}}([0,\tau)) \ni a$ is a kernel, $C(Y) \subseteq Y$, $\overline{Y}^X = X$, $B \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,[\mathrm{R}(C)]))$ is of the form $B(t)y = B_0(t)y + (a*B_1)(t)y$, $t \in [0,\tau)$, $y \in Y$, where $(B_0(t))_{t\in[0,\tau)} \subseteq L(Y) \cap L(X,[\mathrm{R}(C)])$, $(B_1(t))_{t\in[0,\tau)} \subseteq L(Y,[\mathrm{R}(C)])$,*

- (i) *$C^{-1}B_0(\cdot)y \in BV_{\mathrm{loc}}([0,\tau) : Y)$ for all $y \in Y$, $C^{-1}B_0(\cdot)x \in BV_{\mathrm{loc}}([0,\tau) : X)$ for all $x \in X$,*
- (ii) *$C^{-1}B_1(\cdot)y \in BV_{\mathrm{loc}}([0,\tau) : X)$ for all $y \in Y$, and*
- (iii) *$CB(t)y = B(t)Cy$, $y \in Y$, $t \in [0,\tau)$.*

Then there is an a-regular A-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$ iff there is an a-regular $(A+B)$-regularized C-(pseudo)resolvent family $(R(t))_{t\in[0,\tau)}$.

Before going any further, we would like to observe that it is not clear how one can prove an analogue of Theorem A.12 in the case of a general a-regular (A,k)-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$. From a practical point of view, the following corollary is crucially important; it is only worth noticing that one can remove density assumptions in any of cases set out below since the mapping $t \mapsto (a*S)(t)x$, $t \in [0,\tau)$ is continuous in Y for every fixed $x \in X$ (cf. [**369**, p. 160, l–9]):

Corollary A.13. (i) *Assume $L^1_{\mathrm{loc}}([0,\tau)) \ni a$ is a kernel, A is a subgenerator of an a-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$, $Y = [D(A)]$ and*

$$A(t) = a(t)A + (a*B_1)(t) + B_0(t),\ t \in [0,\tau),$$

where $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem A.3. *Assume that any of the following conditions holds:*

- (i.1) *A is densely defined.*
- (i.2) *$\rho(A) \neq \emptyset$.*
- (i.3) *$\rho_C(A) \neq \emptyset$ and $\overline{\mathrm{R}(C)}^X = X$.*

Then there exists an a-regular A-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$.

(ii) *Assume $(S(t))_{t\in[0,\tau)}$ is a (local) C-regularized semigroup having A as a subgenerator, and $B_0(\cdot)$ as well as $B_1(\cdot)$ satisfy the assumptions of* Theorem A.3 *with $Y = [D(A)]$. Then, for every $x \in D(A)$, there exists a unique solution of the problem*

$$\begin{cases} u \in C^1([0,\tau) : X)\ \cap\ C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + (dB_0 * u)(t)x + (B_1 * u)(t) + Cx,\ t \in [0,\tau), \\ u(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

(iii) *Assume A is a subgenerator of a (local) C-regularized cosine function $(C(t))_{t\in[0,\tau)}$, and $B_0(\cdot)$ as well as $B_1(\cdot)$ satisfy the assumptions of* Theorem A.3 *with $Y = [D(A)]$. Then, for every $x \in D(A)$, there exists a unique solution of the problem*

$$\begin{cases} u \in C^2([0,\tau) : X) \ \cap \ C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + (dB_0 * u')(t)x + (B_1 * u)(t) + Cx, \ t \in [0,\tau), \\ u(0) = u'(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is continuously differentiable in $[D(A)]$ and the mapping $t \mapsto u'(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

It is clear that Corollary A.13 can be applied to a wide class of integro-differential equations in Banach spaces and that all aspects of application cannot be easily perceived.

EXAMPLE A.14. Assume $1 \leqslant p \leqslant \infty$, $0 < \tau \leqslant \infty$, $n \in \mathbb{N}$, $X = L^p(\mathbb{R}^n)$ or $X = C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m \in \mathbb{N}$, $\omega = \sup_{x\in\mathbb{R}^n} \operatorname{Re} P(x) < \infty$ and $A = P(D)$. (Possible applications can be also made to non-elliptic abstract differential operators.) Then, for every $\omega' > \omega$ and $r > n|\frac{1}{2} - \frac{1}{p}|$, A generates an exponentially bounded $(\omega' - A)^{-r}$-regularized semigroup in X. Let a completely positive kernel $a(t)$ satisfy (P1) and let $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Corollary A.13(i). This implies that A is the integral generator of an exponentially bounded $(a, (\omega' - A)^{-r})$-regularized resolvent family provided $X = L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$; clearly, the same assertion holds if $a(t) \equiv 1$ and $X = L^\infty(\mathbb{R}^n)$ $(C_b(\mathbb{R}^n))$. An application of Corollary A.13 gives that, in any of these cases, there exists an a-regular A-regularized $(\omega' - A)^{-r}$-resolvent family $(R(t))_{t\in[0,\tau)}$, where $A(t) = a(t)P(D) + (a * B_1)(t) + B_0(t)$, $t \in [0,\tau)$. The preceding example can be set, with some obvious modifications, in the framework of the theory of C-regularized cosine functions.

The application of (A, k)-regularized C-pseudoresolvent families to problems in linear (thermo-)viscoelasticity and electrodynamics with memory (cf. [**369**, Chapter 9]) is almost completely confined to the case in which the underlying space X is Hilbert. In this context, we would like to propose the following problem (cf. also [**369**, p. 240] for the analysis of viscoelastic Timoshenko beam in case of non-synchronous materials).

PROBLEM. Suppose $\mu_0 > 0$, $\varepsilon_0 > 0$, $\Omega_1 \subseteq \mathbb{R}^3$ is an open set with smooth boundary Γ, $\Omega_2 = \mathbb{R}^3 \smallsetminus \Omega_1$ and $n(x)$ denotes the outer normal at $x \in \Gamma$ of Ω_1. Let $X := L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3) \times L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3)$, $p \in [1,\infty] \smallsetminus \{2\}$, and $\|(u_1,u_2,u_3,u_4)\| := (\mu_0\|u_1\|^2 + \varepsilon_0\|u_2\|^2 + \mu_0\|u_3\|^2 + \varepsilon_0\|u_4\|^2)^{1/2}$, u_1, $u_3 \in L^p(\Omega_1 : \mathbb{R}^3)$, u_2, $u_4 \in L^p(\Omega_2 : \mathbb{R}^3)$. Define the operator A_0 in X by setting

$$\begin{aligned} D(A_0) := \bigl\{u \in X : u_1,\ u_2 \in H^{1,p}(\Omega_1 : \mathbb{R}^3),\ u_3,\ u_4 \in H^{1,p}(\Omega_2 : \mathbb{R}^3), \\ n \times (u_1 - u_3) = n \times (u_2 - u_4) = 0\bigr\}, \end{aligned}$$

$$A_0 u := \bigl(-\mu_0^{-1}\operatorname{curl} u_2, \varepsilon_0^{-1}\operatorname{curl} u_1, -\mu_0^{-1}\operatorname{curl} u_4, \varepsilon_0^{-1}\operatorname{curl} u_2\bigr),\ u \in D(A_0).$$

Then one can simply prove that A_0 is closable. Does there exist an injective operator $C \in L(X)$ such that $\overline{A_0}$ generates a (local, global exponentially bounded) C-regularized semigroup in X?

Assuming the answer to the previous problem is in the affirmative and the functions $\varepsilon_i(\cdot)$, $\mu_i(\cdot)$, $\sigma_i(\cdot)$, $\nu_i(\cdot)$ and $\eta_i(\cdot)$ satisfy certain conditions (cf. [**369**, Subsection 9.6, p. 251–253] for further information), one can apply Corollary A.1(ii) in the study of C-wellposedness of transmission problem for media with memory.

Now we shall analyze differential and analytical properties of (A, k)-regularized C-pseudoresolvent families. Let (L_p) be a sequence of positive real numbers such that $L_0 = 1$ and that L_p^p satisfy (M.1), (M.2) and (M.3'). The associated function of (L_p) is defined by $M(\lambda) := \sup_{p\in\mathbb{N}_0} \ln \frac{|\lambda|^p}{L_p^p}$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$, $M(0) := 0$. Recall, the mapping $t \mapsto M(t)$, $t \geqslant 0$ is increasing, absolutely continuous, $\lim_{t\to\infty} M(t) = +\infty$ and $\lim_{t\to\infty} \frac{M(t)}{t} = 0$. Define $\omega_L(t) := \sum_{p=0}^{\infty} \frac{t^p}{L_p^p}$, $t \geqslant 0$, $M_p := L_p^p$ and $\Sigma_\alpha := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \alpha\}$ $(\alpha \in (0, \pi])$.

Definition A.15. (i) Assume that $(S(t))_{t\geqslant 0}$ be a (weak) (A, k)-regularized C-(pseudo)resolvent family. Then it is said that $(S(t))_{t\geqslant 0}$ is an *analytic (weak) (A, k)-regularized C-(pseudo)resolvent family of angle α*, if there exists an analytic function $\mathbf{S} : \mathbf{\Sigma}_\alpha \to \mathbf{L}(\mathbf{X})$ which satisfies $\mathbf{S}(t) = S(t)$, $t > 0$ and $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}(z)x = k(0)Cx$ for all $\gamma \in (0, \alpha)$ and $x \in X$. It is said that $(S(t))_{t\geqslant 0}$ is an exponentially bounded, analytic (weak) (A, k)-regularized C-(pseudo)resolvent family, resp. bounded analytic (weak) (A, k)-regularized C-(pseudo)resolvent family, of angle α, if for every $\gamma \in (0, \alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $\|\mathbf{S}(z)\|_{L(X)} \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$. (Since no confusion seems likely, we shall also write $S(\cdot)$ for $\mathbf{S}(\cdot)$.)

(ii) Assume $(S(t))_{t\in[0,\tau)}$ is a (weak) (A, k)-regularized C-(pseudo)resolvent family and the mapping $t \mapsto S(t)$, $t \in (0, \tau)$ is infinitely differentiable (in the strong topology of $L(X)$). Then it is said that $(S(t))_{t\in[0,\tau)}$ is of *class C^L*, resp. of *class C_L*, iff for every compact set $K \subseteq (0, \tau)$ there exists $h_K > 0$, resp. for every compact set $K \subseteq (0, \tau)$ and for every $h > 0$:

$$\sup_{t\in K,\, p\in\mathbb{N}_0} \Big\| \frac{h_K^p}{L_p^p} \frac{d^p}{dt^p} S(t) \Big\|_{L(X)} < \infty, \text{ resp. } \sup_{t\in K,\, p\in\mathbb{N}_0} \Big\| \frac{h^p}{L_p^p} \frac{d^p}{dt^p} S(t) \Big\|_{L(X)} < \infty;$$

$(S(t))_{t\in[0,\tau)}$ is said to be *ρ-hypoanalytic,* $1 \leqslant \rho < \infty$, if $(S(t))_{t\in[0,\tau)}$ is of class C^L with $L_p = p!^{\rho/p}$.

The careful inspection of the proofs of structural characterizations of analytic convoluted C-semigroups implies the following theorems.

Theorem A.16. (i) *Assume* $\alpha \in (0, \frac{\pi}{2}]$, $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1), (356) *holds with some* $\varepsilon_0 \geqslant 0$, $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$, $(S(t))_{t\geqslant 0}$ *is a (weak) analytic* (A, k)*-regularized C-(pseudo)resolvent family of angle* α *and*

$$\sup_{z\in\Sigma_\gamma} \big\| e^{-\omega z} S(z) \big\|_{L(X)} < \infty \text{ for all } \gamma \in (0, \alpha). \tag{366}$$

Then there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that* (i.1) $H(\lambda)(I-\tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$, $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega$,

(i.2) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}} \|(\lambda-\omega)H(\lambda)\|_{L(X)} < \infty$, $\gamma \in (0,\alpha)$ *and*

(i.3) $\lim_{\lambda\to+\infty,\, \tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = k(0)Cx$, $x \in X$.

(ii) *Let* $\alpha \in (0, \frac{\pi}{2}]$, $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfy* (P1), (356) *hold*, $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$, *there exists an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that* (i.1) *and* (i.2) *of* (i) *hold and that, in the case* $\overline{Y}^X \neq X$, (i.3) *also holds. Then there exists a (weak) analytic* (A, k)*-regularized* C*-(pseudo)resolvent family* $(S(t))_{t\geqslant 0}$ *of angle* α *such that* (366) *holds.*

THEOREM A.17. (i) *Assume* $\alpha \in (0, \frac{\pi}{2}]$, $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1), (356) *holds*, $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$, $(S(t))_{t\geqslant 0}$ *is an analytic* (A, k)*-regularized* C*-resolvent family of angle* α*, the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α *(the analytical extensions of* $U(\cdot)$ *and* $S(\cdot)$ *will be denoted by the same symbols), and*

$$\sup_{z\in\Sigma_\gamma} \left\|e^{-\omega z}S(z)\right\|_{L(X)} + \sup_{z\in\Sigma_\gamma} \left\|e^{-\omega z}S(z)\right\|_{L(Y)} < \infty \text{ for all } \gamma \in (0,\alpha). \tag{367}$$

Denote $H(\lambda)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x \in X$, $\operatorname{Re}\lambda > \omega$. *Then* (N1)–(N2) *hold*, $(H(\lambda))_{\operatorname{Re}\lambda>\omega}$ *is analytic in both spaces*, $L(X)$ *and* $L(Y)$,

(i.1) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}} \big(\|(\lambda-\omega)H(\lambda)\|_{L(X)} + \|(\lambda-\omega)H(\lambda)\|_{L(Y)}\big) < \infty$, $\gamma \in (0,\alpha)$, $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$ *and*

(i.2) $\lim_{\lambda\to+\infty,\, \tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = k(0)Cx = 0$, $x \in X$.

(ii) *Assume* $\alpha \in (0, \frac{\pi}{2}]$, $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1), (356) *and* (N1)–(N2) *hold*, $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$, $(H(\lambda))_{\operatorname{Re}\lambda>\omega}$ *is analytic in both spaces*, $L(X)$ *and* $L(Y)$, *and* $(I - \tilde{A}(\lambda))H(\lambda)y = H(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$. *Assume also that* (i.1) *of* (i) *of this theorem holds and that, in the case* $\overline{Y}^X \neq X$, (i.2) *also holds. Then there exists an analytic* (A, k)*-regularized* C*-resolvent family* $(S(t))_{t\geqslant 0}$ *of angle* α *such that* (367) *holds and that the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α.

The main objective in the subsequent theorems is to clarify the basic differential properties of (A, k)-regularized C-pseudoresolvent families.

THEOREM A.18. *Assume* $k(t)$ *satisfies* (P1), $r \geqslant -1$ *and* (356) *holds with some* $\varepsilon_0 \geqslant 0$. *Assume that there exists* $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$ *such that, for every* $\sigma > 0$, *there exist* $C_\sigma > 0$, $M_\sigma > 0$ *and an open neighborhood* $\Omega_{\sigma,\omega}$ *of the region*

$$\Lambda_{\sigma,\omega} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \omega,\ \operatorname{Re}\lambda \geqslant -\sigma \ln|\operatorname{Im}\lambda| + C_\sigma\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\},$$

and an analytic mapping $h_\sigma : \Omega_{\sigma,\omega} \to L(X)$ *such that* $h_\sigma(\lambda)C = Ch_\sigma(\lambda)$, $\operatorname{Re}\lambda > \omega$, $h_\sigma(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$, *and that* $\|h_\sigma(\lambda)\|_{L(X)} \leqslant M_\sigma|\lambda|^r$, $\lambda \in \Lambda_{\sigma,\omega}$. *Then, for every* $\zeta > 1$, *there exists a norm continuous, exponentially bounded weak* $\big(A, k * \frac{t^{\zeta+r-1}}{\Gamma(\zeta+r)}\big)$*-regularized* C*-pseudoresolvent family* $(S_\zeta(t))_{t\geqslant 0}$ *satisfying that the mapping* $t \mapsto S_\zeta(t)$, $t > 0$ *is infinitely differentiable in* $L(X)$. *If, additionally,* $h_\sigma(\lambda) \in L(Y)$ *for all* $\sigma > 0$, *and if the mapping* $\lambda \mapsto h_\sigma(\lambda)$, $\lambda \in \Omega_{\sigma,\omega}$

is analytic in $L(Y)$ *as well as* $(I-\tilde{A}(\lambda))h_\sigma(\lambda)y=\tilde{k}(\lambda)Cy$, $y\in Y$, $\operatorname{Re}\lambda>\omega$, $\tilde{k}(\lambda)\neq 0$, *and* $\|h_\sigma(\lambda)\|_{L(Y)}\leqslant M_\sigma|\lambda|^r$, $\lambda\in\Lambda_{\sigma,\omega}$, *then* $(S_\zeta(t))_{t\geqslant 0}$ *is a norm continuous, exponentially bounded* $\big(A,k*\frac{t^{\zeta+r-1}}{\Gamma(\zeta+r)}\big)$*-regularized* C*-resolvent family and the mapping* $t\mapsto S_\zeta(t)$, $t>0$ *is infinitely differentiable in* $L(Y)$.

THEOREM A.19. *Suppose* $k(t)$ *satisfies* (P1), (356) *holds with some* $\varepsilon_0\geqslant 0$, (M.1)–(M.3′) *hold for* (L_p), $(S(t))_{t\in[0,\tau)}$ *is a* (*local*) *weak* (A,k)*-regularized* C*-pseudoresolvent family,* $\omega\geqslant\max(\operatorname{abs}(k),\varepsilon_0)$ *and* $m\in\mathbb{N}$. *Denote, for every* $\varepsilon\in(0,1)$ *and a corresponding* $K_\varepsilon>0$, $F_{\varepsilon,\omega}=\big\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant-\ln\omega_L(K_\varepsilon|\operatorname{Im}\lambda|)+\omega\big\}$. *Assume that, for every* $\varepsilon\in(0,1)$, *there exist* $C_\varepsilon>0$, $M_\varepsilon>0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega}=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant\omega,\ \tilde{k}(\lambda)\neq 0\}\cup\{\lambda\in F_{\varepsilon,\omega}:\operatorname{Re}\lambda\leqslant\omega\}$, *and analytic mappings* $f_\varepsilon:O_{\varepsilon,\omega}\to\mathbb{C}$, $g_\varepsilon:O_{\varepsilon,\omega}\to L(Y,X)$ *and* $h_\varepsilon:O_{\varepsilon,\omega}\to L(X)$ *such that:*

(i) $f_\varepsilon(\lambda)=\tilde{k}(\lambda)$, $\operatorname{Re}\lambda>\omega$, $g_\varepsilon(\lambda)=\tilde{A}(\lambda)$, $\operatorname{Re}\lambda>\omega$, $h_\varepsilon(\lambda)C=Ch_\varepsilon(\lambda)$, $\operatorname{Re}\lambda>\omega$,

(ii) $h_\varepsilon(\lambda)(I-g_\varepsilon(\lambda))y=f_\varepsilon(\lambda)Cy$, $y\in Y$, $\lambda\in F_{\varepsilon,\omega}$,

(iii) $\|h_\varepsilon(\lambda)\|_{L(X)}\leqslant M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|\operatorname{Re}\lambda|}$, $\lambda\in F_{\varepsilon,\omega}$, $\operatorname{Re}\lambda\leqslant\omega$ *and* $\|h_\varepsilon(\lambda)\|_{L(X)}\leqslant M_\varepsilon(1+|\lambda|)^m$, $\operatorname{Re}\lambda\geqslant\omega$.

Then $(S(t))_{t\in[0,\tau)}$ *is of class* C^L. *Assume now that* $(S(t))_{t\in[0,\tau)}$ *is an* (A,k)*-regularized* C*-resolvent family, and that, in addition to the above assumptions,* $h_\varepsilon(\lambda)\in L(Y)$ *for all* $\varepsilon\in(0,1)$. *Let the mapping* $\lambda\mapsto h_\varepsilon(\lambda)$, $\lambda\in O_{\varepsilon,\omega}$ *be analytic in* $L(Y)$ *and let:*

(ii)′ $(I-g_\varepsilon(\lambda))h_\varepsilon(\lambda)y=f_\varepsilon(\lambda)Cy$, $y\in Y$, $\lambda\in F_{\varepsilon,\omega}$,

(iii)′ $\|h_\varepsilon(\lambda)\|_{L(Y)}\leqslant M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|\operatorname{Re}\lambda|}$, $\lambda\in F_{\varepsilon,\omega}$, $\operatorname{Re}\lambda\leqslant\omega$ *and* $\|h_\varepsilon(\lambda)\|_{L(Y)}\leqslant M_\varepsilon(1+|\lambda|)^m$, $\operatorname{Re}\lambda\geqslant\omega$ *for all* $\varepsilon\in(0,1)$.

Then the mapping $t\mapsto S(t)$, $t\in(0,\tau)$ *is of class* C^L *in* $L(Y)$.

Note that (M.3′) does not hold if $L_p=p!^{1/p}$ and that the preceding theorem remains true in this case; then, in fact, we obtain the sufficient conditions for the generation of real analytic C-(pseudo)resolvents. Further on, the set $F_{\varepsilon,\omega}$ appearing in the formulation of Theorem A.19 can be interchanged by the set $F_{\varepsilon,\omega,\rho}=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant-K_\varepsilon|\operatorname{Im}\lambda|^{1/\rho}+\omega\}$, provided $L_p=p!^{\rho/p}$ and $1\leqslant\rho<\infty$, and [**235**, Theorem 2.24] can be reformulated in nonscalar case.

By means of Corollary A.13(i) and the next observation, one can simply construct examples of (differentiable, in general, non-analytic) A-regularized C-resolvent families of class C^L (C_L). Let $(S(t))_{t\in[0,\tau)}$ be an (a,C)-regularized resolvent family of class C^L (C_L) and let the assumptions of Theorem A.12 hold with $Y=[D(A)]$ and $B_1=0$. Assume, in addition, $C^{-1}B_0\in C^\infty([0,\tau):L(X))$ is of class C^L (C_L) and $(C^{-1}B_0)^{(i)}(0)=0$, $i\in\mathbb{N}_0$. Denote by L the solution of the equation $L=K_0+dK_0*L$ in $BV_{\mathrm{loc}}([0,\tau):L(X))$, where $K_0(t)=(S*C^{-1}B_0)(t)$, $t\in[0,\tau)$. Let $A(t)=a(t)A+B_0(t)$, $t\in[0,\tau)$ and let $(R(t))_{t\in[0,\tau)}$ be an A-regularized C-resolvent family given by Corollary A.13(i). Then one can straightforwardly check that $L\in C^\infty([0,\tau):L(X))$ is of class C^L (C_L) and that $L^{(i)}(0)=0$, $i\in\mathbb{N}_0$. Taking into account the proof of [**369**, Theorem 6.1] (cf. [**369**, (6.20), p. 160] and [**369**, Corollary 0.3, p. 15]), it follows that $R^{(n)}(t) = S^{(n)}(t) + \int_0^t L^{(n+1)}(t-s)S(s)\,ds$, $t \in [0,\tau)$, $n \in \mathbb{N}_0$. This implies that $(R(t))_{t\in[0,\tau)}$ is of class C^L (C_L). Using the same method, we are in a position to construct examples of analytic A-regularized C-resolvent families:

Example A.20. The isothermal motion of a one-dimensional body with small viscosity and capillarity is described, in the simplest situation, by the system:

$$\begin{cases} u_t = 2au_{xx} + bv_x - cv_{xxx}, \\ v_t = u_x, \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where a, b and c are positive constants. The associated matrix of polynomials is $P(x) \equiv \begin{bmatrix} -2ax^2 & ibx + icx^3 \\ ix & 0 \end{bmatrix}$, and $P(x)$ is Shilov 2-parabolic. For the sake of brevity, we assume that $X = L^p(\mathbb{R}) \times L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$ and that X is equipped with the sup-norm. Then it is well known that the operator $P(D)$, considered with its maximal distributional domain, is closed and densely defined in X. Let us recall the following facts:

(i) Let $a^2 - c < 0$ and $r' \geqslant \frac{1}{2}$. Then $P(D)$ is the integral generator of an exponentially bounded, analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geqslant 0}$ of angle $\arctan \frac{a}{\sqrt{c-a^2}}$.

(ii) Let $a^2 - c = 0$ and $r' > \frac{3}{4}$. Then $P(D)$ is the integral generator of a bounded analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$.

(iii) Let $a^2 - c > 0$ and $r' \geqslant \frac{1}{2}$. Then $P(D)$ is the integral generator of an exponentially bounded, analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$.

Assume, in any of above cases, $\psi_1, \psi_2 \in \mathrm{S}^{2r',1}(\mathbb{R})$, where the fractional Sobolev space $\mathrm{S}^{2r',1}(\mathbb{R})$ is defined in the sense of [**300**, Definition 12.3.1, p. 297], $B_1 = 0$, $B_0(z)\binom{f}{g} = z\binom{\psi_1 * f}{\psi_2 * g}$ and $K(z)\binom{f}{g} = (S_{r'} * (1-\Delta)^{r'} B_0)(z)\binom{f}{g}$, $z \in \Sigma_\alpha$, $f, g \in L^p(\mathbb{R})$, where $\alpha = \arctan \frac{a}{\sqrt{c-a^2}}$, provided that (i) holds, resp. $\alpha = \frac{\pi}{2}$, provided that (ii) or (iii) holds. Let $K \subseteq \Sigma_\alpha$ be a compact set and let $\gamma \in (0,\alpha)$ satisfy $K \subseteq \Sigma_\gamma$. Then there exist

$$\delta \in \big(0, 1/\big((1+\sup K)(1 + \|(1-\Delta)^{r'}\psi_1\|_{L^1(\mathbb{R})} + \|(1-\Delta)^{r'}\psi_2\|_{L^1(\mathbb{R})})\big)\big),$$

$M_\gamma \geqslant 1$, $\omega_\gamma \geqslant 0$ and $\omega'_\gamma > \omega_\gamma$ such that

$$\left\| S_{r'}^{(-1)}(z) \equiv \int_0^z S_{r'}(s)\,ds \right\|_{L(X)} \leqslant M_\gamma |z| e^{\omega_\gamma \operatorname{Re} z} \leqslant \delta e^{\omega'_\gamma \operatorname{Re} z}, \quad z \in \Sigma\gamma.$$

This implies $\big\| \int_0^z S_{r'}^{(-1)}(z-s) S_{r'}^{(-1)}(s)\,ds \big\|_{L(X)} \leqslant \delta^2 |z| e^{\omega'_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$. Define recursively $(K_n(z))$ by $K_0(z) := K(z)$, $z \in \Sigma_\alpha$ and $K_{n+1}(z) := \int_0^z dK(s) K_n(z-s)$, $z \in \Sigma_\alpha$, $n \in \mathbb{N}_0$. As a matter of fact, $K_n(z) = (\underbrace{K' * \cdots * K'}_{n} * K)(z)$, $z \in \Sigma_\alpha$, $n \in \mathbb{N}$.

By Young's inequality,

$$\big\|K_1'(z)\big\|_{L(X)} \leqslant \delta^2|z|\big(\|(1-\Delta)^{r'}\psi_1\|_{L^1(\mathbb{R})}+\|(1-\Delta)^{r'}\psi_2\|_{L^1(\mathbb{R})}\big)^2 e^{\omega_\gamma' \operatorname{Re} z},\ \ z\in\Sigma_\gamma.$$

Inductively,

$$\big\|K_{n+1}'(z)\big\|_{L(X)} \leqslant \delta^{n+1}|z|^n\big(\|(1-\Delta)^{r'}\psi_1\|_{L^1(\mathbb{R})}+\|(1-\Delta)^{r'}\psi_2\|_{L^1(\mathbb{R})}\big)^{n+1} e^{\omega_\gamma' \operatorname{Re} z},$$

for any $z\in\Sigma_\gamma$ and $n\in\mathbb{N}_0$. Taken together, the preceding estimate and the Weierstrass theorem imply that the function $z\mapsto\int_0^z\sum_{n=0}^\infty K_n'(z-s)S_{r'}(s)ds$, $z\in\Sigma_\alpha$ is analytic and that there exist $M_\gamma'\geqslant 1$ and $\omega_\gamma''>\omega_\gamma'$ such that $\|\int_0^z\sum_{n=0}^\infty K_n'(z-s)S_{r'}(s)\,ds\|_{L(X)}\leqslant M_\gamma' e^{\omega_\gamma'' \operatorname{Re} z}$, $z\in\Sigma_\gamma$. Let $(R_{r'}(t))_{t\geqslant 0}$ be a $(P(D)+B_0(t))$-regularized C-resolvent family given by Corollary A.13(i). Since $R_{r'}(t)=S_{r'}(t)+\int_0^t\sum_{n=0}^\infty K_n'(t-s)S_{r'}(s)\,ds$, $t\geqslant 0$, we have that $(R_{r'}(t))_{t\geqslant 0}$ is an exponentially bounded, analytic 1-regular A-regularized C-resolvent family of angle α. On the other hand, $P(D)$ does not generate a strongly continuous semigroup in $L^1(\mathbb{R})\times L^1(\mathbb{R})$ and $\rho(P(D))\neq\emptyset$. Combining this with Theorem A.12 and Proposition A.6, we get that there does not exist a local $(P(D)+B_0(t))$-regularized pseudoresolvent family provided $p=1$.

EXAMPLE A.21. Let $X=L^p(\mathbb{R})$, $1\leqslant p\leqslant\infty$. Consider the next multiplication operators with maximal domain in X:

$$Af(x)=:2xf(x),\ Bf(x):=(-x^4+x^2-1)f(x),\ x\in\mathbb{R}.$$

Notice that $D(B)\subseteq D(A)$. Let $Y:=[D(B)]$ and let $A\in L^1_{\mathrm{loc}}([0,\infty):L(Y,X))$ be given by $A(t)f:=Af+tBf$, $t\geqslant 0$, $f\in D(B)$. Assume, further, $s\in(1,2)$, $\delta=\frac{1}{s}$, $L_p=p!^{s/p}$ and $K_\delta(t)=\mathcal{L}^{-1}\big(e^{-\lambda^\delta}\big)(t)$, $t\geqslant 0$. Then there exists a global (not exponentially bounded) (A,K_δ)-regularized resolvent family. Towards this end, it suffices to show that, for every $\tau>0$, there exists a local (A,K_δ)-regularized resolvent family on $[0,\tau)$. Denote by $M(t)$ the associated function of the sequence (L_p) and denote, with a little abuse of notation, $\Lambda_{\alpha,\beta,\gamma}=\big\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geqslant\frac{M(\alpha\lambda)}{\gamma}+\beta\big\}$, $\alpha>0$, $\beta>0$, $\gamma>0$. It is obvious that there exists $C_s>0$ such that $M(\lambda)\leqslant C_s|\lambda|^{1/s}$, $\lambda\in\mathbb{C}$. Given $\tau>0$ and $d>0$ in advance, one can find $\alpha>0$ and $\beta>0$ such that $\tau\leqslant\frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}$ and $|\lambda^2-2x\lambda+(x^4-x^2+1)|\geqslant d$, $\lambda\in\Lambda_{\alpha,\beta,1}$, $x\in\mathbb{R}$. Denote by Γ the upwards oriented frontier of the ultra-logarithmic region $\Lambda_{\alpha,\beta,1}$, and define, for every $f\in X$, $x\in\mathbb{R}$ and $t\in\big[0,\frac{\cos(\delta\pi/2)}{C_s\alpha^{1/s}}\big)$,

$$(S_\delta(t)f)(x):=\frac{1}{2\pi i}\int_\Gamma\frac{\lambda^2e^{\lambda t-\lambda^\delta}}{\lambda^2-2x\lambda+(x^4-x^2+1)}\,d\lambda f(x).$$

Then one can simply verify that $(S_\delta(t))_{t\in[0,\tau)}$ is a local (A,K_δ)-regularized resolvent family and that the mapping $t\mapsto S_\delta(t)$, $t\geqslant 0$ is infinitely differentiable in the strong topologies of $L(X)$ and $L(Y)$. Moreover, in both spaces, $L(X)$ and $L(Y)$,

$$\Big(\frac{d^p}{dt^p}S_\delta(t)f\Big)(x)=\frac{1}{2\pi i}\int_\Gamma\frac{\lambda^{p+2}e^{\lambda t-\lambda^\delta}}{\lambda^2-2x\lambda+(x^4-x^2+1)}\,d\lambda f(x),$$

for any $p \in \mathbb{N}_0$, $x \in \mathbb{R}$ and $f \in X$. This implies that, for every compact set $K \subseteq [0,\infty)$, there exists $h_K > 0$ such that

$$\sup_{t \in K,\ p \in \mathbb{N}_0} \Bigl(\Bigl\| \frac{h_K^p}{L_p^p} \frac{d^p}{dt^p} S_\delta(t) \Bigr\|_{L(X)} + \Bigl\| \frac{h_K^p}{L_p^p} \frac{d^p}{dt^p} S_\delta(t) \Bigr\|_{L(Y)} \Bigr) < \infty.$$

In particular, $(S_\delta(t))_{t \geqslant 0}$ is s-hypoanalytic; arguing in the same manner, we infer that there exists $\tau_0 > 0$ such that there exists a local 2-hypoanalytic $(A, K_{1/2})$-regularized resolvent family on $[0, \tau_0)$, where $K_{1/2}(t) = \mathcal{L}^{-1}\bigl(e^{-\lambda^{1/2}}\bigr)(t)$, $t \geqslant 0$. Note also that the use of Fourier multipliers enables one to reveal that the preceding conclusions remain true in the case of corresponding differential operators $\pm A(t)$, where $A(t)f = -tf'''' - tf'' - 2if' - tf$, $t \geqslant 0$, $1 < p < \infty$, $f \in Y = \mathrm{S}^{4,p}(\mathbb{R})$.

Let us consider the equations

$$u(t) = \int_0^\infty A(s)u(t-s)\,ds + \int_{-\infty}^t k(t-s)g'(s)\,ds, \tag{368}$$

where $g : \mathbb{R} \to X$, $A \in L^1_{\mathrm{loc}}([0,\infty) : L(Y,X))$, $A \neq 0$, $k \in C([0,\infty))$, $k \neq 0$, and

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in (-\tau,\tau), \tag{369}$$

where $\tau \in (0,\infty]$, $f \in C((-\tau,\tau) : X)$ and $A \in L^1_{\mathrm{loc}}((-\tau,\tau) : L(Y,X))$, $A \neq 0$. The following proposition can be applied to a class of nonscalar parabolic equations considered by Friedman and Shinbrot in [**133**].

Proposition A.22. *Assume that there exists an (A,k)-regularized C-resolvent family $(S(t))_{t \geqslant 0}$, $g : \mathbb{R} \to \mathrm{R}(C)$, $C^{-1}g(\cdot)$ is differentiable for a.e. $t \in \mathbb{R}$, $C^{-1}g(t) \in Y$ for a.e. $t \in \mathbb{R}$,*

(i) *the mapping $s \mapsto S(t-s)(C^{-1}g)'(s)$, $s \in (-\infty, t]$ is an element of the space $L^1((-\infty,t] : Y)$ for a.e. $t \in \mathbb{R}$, and*
(ii) *the mapping $s \mapsto k(t-s)g'(s)$, $s \in (-\infty,t]$ is an element of the space $L^1((-\infty,t] : X)$ for a.e. $t \in \mathbb{R}$.*

Put $u(t) := \int_{-\infty}^t S(t-s)(C^{-1}g)'(s)\,ds$, $t \in \mathbb{R}$. Then $C(\mathbb{R} : X) \ni u$ satisfies (368).

A function $u \in C((-\tau,\tau) : X)$ is said to be:

(i) a *strong solution* of (369) iff $u \in L^\infty_{\mathrm{loc}}((-\tau,\tau) : Y)$ and (369) holds on $(-\tau,\tau)$,
(ii) a *mild solution* of (369) iff there exist a sequence (f_n) in $C((-\tau,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : X)$ such that $u_n(t)$ is a strong solution of (369) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ and $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $(-\tau,\tau)$.

Proposition A.23. (i) *Assume $k \in C((-\tau,\tau))$, $k \neq 0$ and $A \in L^1_{\mathrm{loc}}((-\tau,\tau) : L(Y,X))$, $A \neq 0$. Let $k_+(t) = k(t)$, $A_+(t) = A(t)$, $t \in [0,\tau)$, $k_-(t) = k(-t)$ and $A_-(t) = -A(-t)$, $t \in (-\tau,0]$. If there exist $(A_\pm, k_\pm)$-regularized C-resolvent families $(S_\pm(t))_{t\in[0,\tau)}$, then for every $x \in Y$ the function $u : (-\tau,\tau) \to X$ given by $u(t) = S_+(t)x$, $t \in [0,\tau)$ and $u(t) = S_-(-t)x$, $t \in (-\tau,0]$ is a strong solution of*

(369) *with* $f(t)=k(t)Cx$, $t\in(-\tau,\tau)$. *Furthermore, strong solutions of* (369) *are unique provided that* $k_{\pm}(t)$ *are kernels.*

(ii) *Assume* $n_{\pm}\in\mathbb{N}$, $f\in C((-\tau,\tau):X)$, $A\in L^1_{\mathrm{loc}}((-\tau,\tau):L(Y,X))$, $A\neq 0$, $f_+(t)=f(t)$, $A_+(t)=A(t)$, $t\in[0,\tau)$, $f_-(t)=f(-t)$, $A_-(t)=-A(-t)$, $t\in(-\tau,0]$ *and there exist* $(n_{\pm}-1)$*-times integrated* $A_{\pm}$*-regularized* $C_{\pm}$*-resolvent families. Let* $f_{\pm}\in C^{(n_{\pm})}([0,\tau):X)$ *and* $f_{\pm}^{(i)}(0)=0$, $0\leqslant i\leqslant n_{\pm}-1$. *Then the following holds:*

- (ii.1) *Assume that* $(C_{\pm}^{-1}f_{\pm})^{(n_{\pm}-1)}\in AC_{\mathrm{loc}}([0,\tau):Y)$ *and* $(C_{\pm}^{-1}f_{\pm})^{(n_{\pm})}\in L^1_{\mathrm{loc}}([0,\tau):Y)$. *Then there exists a unique strong solution* $u(t)$ *of* (369), *and moreover* $u\in C((-\tau,\tau):Y)$.
- (ii.2) *Let* $(C_{\pm}^{-1}f_{\pm})^{(n_{\pm})}\in L^1_{\mathrm{loc}}([0,\tau):X)$ *and* $\overline{Y}^X=X$. *Then there exists a unique mild solution of* (369).

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