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**PROBABILITY QUANTIFIERS  
AND OPERATORS**

Miodrag Rašković and Radosav Đorđević

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## INTRODUCTION

The purpose of logics with probability quantifiers and operators is to give a model-theoretic approach to probability theory. The main subject of this monograph is to study the central problem of every logic, namely the completeness problem.

This subject since H.J. Keisler's initial work [32] has been studied by H.J. Keisler [34], D. Hoover [22] and others.

In Chapter 1 we give an introduction to nonstandard analysis, especially nonstandard measure theory which we use later (see [7]).

In Chapter 2 we study admissible sets and  $L_{\omega_1\omega}$  logic.

Chapter 3 contains results concerning the logic  $L_{AP}$  introduced by H.J. Keisler [34]. This logic is similar to the infinitary logic  $L_A$ , except that quantifiers  $(P\vec{x} \geq r)$  ( $\vec{x}$  is a finite sequence of variables,  $r$  is a real) are used instead of the usual  $\forall x$  and  $\exists x$  (see [34]).

In Chapter 4 we study logics with integrals and conditional expectations with stronger expressive power than the logic  $L_{AP}$ . The logic  $L_{AF}$ , introduced by Keisler in [34] as an equivalent of the logic  $L_{AP}$ , allows us to express many properties of random variables in an easier way. In this logic, the quantifiers  $\int \dots dx$  are incorporated instead of the quantifiers  $(P\vec{x} \geq r)$ . The logic  $L_{AF}$  is not rich enough to express probabilistic notions involving conditional expectations of random variables with respect to an  $\sigma$ -algebra, such as a martingale, a Markov process etc. These properties can be naturally expressed in a language with both integral quantifiers and conditional expectation operators. The logics  $L_{AE}$  and  $L_{ad}$  introduced by Keisler in [34] are appropriate for the study of stochastic processes.

Chapters from 5 to 10 contains the author's contributions to the subject.

The subject of Chapter 5 is to introduce the logic  $L_{AM}$ , which instead of probability measure has a  $\sigma$ -finite one. It also gives a method how to transfer results from  $L_{AP}$  to new logic (see [47]).

Chapter 6 contains a logic appropriate for a study of structures with two measures  $\mu_1$  and  $\mu_2$ . We are especially interested in absolutely continuous (i.e.,  $\mu_1 \ll \mu_2$ ) and singular (i.e.,  $\mu_1 \perp \mu_2$ ) cases. In both of them, we prove completeness, solv-

ing a problem proposed by Keisler in [34]. The proof uses the Barwise compactness theorem for  $L_A$  (see [48] for the first use of the method).

It is well known that the projection of a measurable set need not be measurable; so we cannot put together easily ordinary and probability quantifiers. In Chapter 7 we study a monadic logic for which this can be done.

In Chapter 8 we introduce the second-order probability logic  $L_{APV}^2$  which possesses the probability quantifiers  $(P\vec{x} \geq r)$  on individual variables and the ordinary quantifiers  $(\forall X)$  and  $(\exists X)$  on set variables.

In Chapter 9, unlike the others, we have to deal with intuitionism, the most important non-classical logic. The second characteristic of this chapter is a probability operator (like a modal operator) which we apply to either classical or intuitionistic propositional logic.

In Chapter 10 we introduce the notion of a cylindric probability algebra which can be considered as a common algebraic abstraction from a geometry associated with the basic set-theoretic notions on the one hand, and the theory of deductive systems of probability logic on the other hand.

Most of the results in the monography were presented at the Logic Seminar in Belgrade during the last decade. The Mathematical Institute of the Serbian Academy of Sciences has been a good working place.

We are happy to recognize that we have been strongly influenced by Žarko Mijajlović. We are indebted to him for his help and comments during the writing of this monography. We thank many mathematical colleagues and members of Seminar, particularly Zoran Marković and Rade Živaljević for their support and useful suggestions; Kosta Došen and Predrag Tanović who suggested many linguistic improvements; Dragan Blagojević provided valuable technical help with the preparation of the manuscript.

We would like to thank Prof. H.J. Keisler for the useful discussions during our stay at the University of Wisconsin, Madison, USA.

Kragujevac, 1996.

Miodrag Rašković  
Radosav Đorđević

## CHAPTER 1

### FIRST NOTIONS OF NONSTANDARD MEASURE THEORY

Nonstandard analysis was invented in 1960 by Abraham Robinson, who used ideas from logic to construct an enrichment  ${}^*\mathbb{R}$  of the reals  $\mathbb{R}$  containing infinite elements and infinitesimals. This enrichment, which inherits all the properties of  $\mathbb{R}$ , has become known as the nonstandard model of the reals. Robinson gave a simple but ingenious solution of the famous problem of the consistent foundations of Leibnitz' Analysis, a problem which stood open for three hundred years.

Nonstandard techniques have been used to obtain new results in various fields, such as Banach spaces, differential equations, measure theory, algebraic number theory, and practically all mathematical fields where infinite objects appear. Robinson found a way to embed every mathematical structure  $\mathfrak{A}$  – for example the standard measure space  $\langle A, \mathcal{F}, \mu \rangle$ , in an extension  ${}^*\mathfrak{A}$  which contain “ideal” elements. A suitable extension  ${}^*\mathfrak{A}$  can be obtained algebraically by taking an ultrapower  ${}^*\mathfrak{A} = \mathfrak{A}^I/\mathcal{U}$ , where  $\mathcal{U}$  is an ultrafilter on the subsets of the index set  $I$ .

Our aim in this chapter is to give a concise survey of the methods of nonstandard measure theory and its standard applications. Most of these applications such as the extension of measures, stem from Loeb's discovery of a way to construct new rich standard measure spaces, called Loeb spaces. We adopt the superstructure approach, and we begin this chapter with a description of the superstructure  $V(S)$  which contains all the “interesting” properties and structures on  $S$ .

#### 1.1. The nonstandard universe ${}^*V(S)$

We will briefly state the definitions and main facts about nonstandard analysis. We assume that all mathematical objects can be ultimately defined as sets.

Let  $S$  be a set of urelements large enough for our purposes; i.e., if  $x \in S$ , then  $x$  has no members and let  $S$  be big enough to ensure that  $V(S)$  contains all mathematical objects of interest. The *superstructure over  $S$*  is the iterated power set  $V(S)$  defined by

$$V_0(S) = S, \quad V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S)), \quad V(S) = \bigcup_{n < \omega} V_n(S).$$

If  $x \in V(S)$ , the *rank* of  $x$  is the smallest  $n$  such that  $x \in V_n(S)$ . For instance, let  $f: S \rightarrow S$  be a function, i.e., a set of ordered pairs. Then  $f$  is in  $V_3(S)$  since  $\{a\}$  and  $\{a, b\}$  are in  $V_1(S)$ ; so an ordered pair  $(a, b) \in V_2(S)$ .

By a *bounded formula* we mean a first-order formula all of whose quantifiers are bounded, that is, of the forms  $(\forall x)(x \in y \rightarrow \dots)$  and  $(\exists x)(x \in y \ \& \ \dots)$ . Bounded quantifiers are abbreviated by  $(\forall x \in y)$  and  $(\exists x \in y)$ .

A structure  $*S$  and a map  $*$ :  $V(S) \rightarrow V(*S)$  is a *nonstandard model* for  $S$  if the following principles hold of it:

- (E) The *extension principle*, which says that  $*S$  is a proper extension of  $S$  and  $* \upharpoonright S$  is the identity;
- (T) The *transfer principle*, which says that the map  $*$  preserves the truth of bounded formulas; that is, for all bounded formulas  $\varphi(x_1, \dots, x_n)$  and  $a_1, \dots, a_n \in V(S)$ ,  $\varphi(a_1, \dots, a_n)$  holds in  $V(S)$  iff  $\varphi(*a_1, \dots, *a_n)$  holds in  $V(*S)$ ;
- (S) The  $\omega_1$ -*saturation principles*, that is, if  $A_0 \supseteq A_1 \supseteq \dots$  are nonempty sets in  $*V_n(S)$ , then  $\bigcap_{m < \omega} A_m$  is nonempty, which allows the construction of “ideal” elements.

If in an application one wants to study a specific mathematical structure, it is often convenient to let  $S$  contain all the auxiliary structures one may need such as  $\mathbb{R}$ ,  $\mathbb{C}$ , topological spaces, measure spaces etc.

We will consider the case  $S = \mathbb{R}$  of the reals in some more detail. By the transfer principle, the algebraic axioms and the axioms for an ordered field hold in an enrichment  $*\mathbb{R}$  of the reals. For instance, we know that

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(x < y \rightarrow (x + z) < (y + z));$$

hence

$$(\forall x \in *\mathbb{R})(\forall y \in *\mathbb{R})(\forall z \in *\mathbb{R})(x * < y \rightarrow (x * + z) * < (y * + z))$$

since  $< \in V_3(\mathbb{R})$  and  $+ \in V_5(\mathbb{R})$ . Also, if  $x, y \in \mathbb{R}$  and  $x < y$ , then  $x * = x * < y * = y$ ; i.e.,  $* <$  extends  $<$ , etc. For this reason, the ordered field  $\langle *\mathbb{R}, +, \times, * <, 0, 1 \rangle$  properly extends the real ordered field. At the same time every real function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  or set  $X \subseteq \mathbb{R}^n$  is extended to a function  $*f: (*\mathbb{R})^n \rightarrow *\mathbb{R}$  or a set  $*X \subseteq (*\mathbb{R})^n$  such that the properties of  $f$  and  $X$  are inherited by  $*f$  and  $*X$ . It is customary to drop the prefix  $*$  from  $+$ ,  $\times$ ,  $* <$  and similarly for other basic functions and relations such as  $|\cdot|$ ,  $*$ , etc.

The following definition makes sense for any ordered field extending  $\mathbb{R}$ .

**1.1.1. Definition.** Let  $x, y \in *\mathbb{R}$ . Then

- (1)  $x$  is *finite* if  $|x| < n$  for some  $n \in \mathbb{N}$ ;
- (2)  $x$  is *infinite* if  $|x| > n$  for all  $n \in \mathbb{N}$ ;
- (3)  $x$  is *infinitesimal* if  $|x| < 1/n$  for all  $n \in \mathbb{N}$ .

By the extension principle,  $*\mathbb{R} \setminus \mathbb{R} \neq \emptyset$ ; so we can choose  $x \in *\mathbb{R} \setminus \mathbb{R}$ . If  $x$  is infinite, then  $x^{-1}$  is non-zero infinitesimal. If  $x$  is finite, then there is a unique  $^\circ x = \text{st}(x) = \sup\{\alpha \in \mathbb{R} : \alpha < x\}$ , called the *standard part* of  $x$ , with  $x - ^\circ x$  is non-zero infinitesimal and  $(x - ^\circ x)^{-1}$  is infinite. If  $x \in \mathbb{R}$  is infinite, there is a unique  $H \in *\mathbb{N} \setminus \mathbb{N}$  with  $H \leq x < H + 1$ , so the set  $*\mathbb{N}$  is an important subset of

$*\mathbb{R}$  which has infinite members. We call  $*\mathbb{R}$  the *set of hyperreal numbers* and  $*\mathbb{N}$  the *set of non-negative hyperintegers*.

The reals are the ordered field with the least upper bound property. The so-called *completeness axiom* of  $\mathbb{R}$  states:

$$(\forall A \in V_1(\mathbb{R})) \left( (\exists x \in \mathbb{R})(\forall y \in A)(y \leq x) \rightarrow \right. \\ \left. \rightarrow (\exists x_0 \in \mathbb{R}) \left( (\forall y \in A)(y \leq x_0) \wedge (\forall x \in \mathbb{R}) \left( (\forall y \in A)(y \leq x) \rightarrow (x_0 \leq x) \right) \right) \right).$$

By the transfer principle, we get the same statement for all  $A \in *V_1(\mathbb{R})$  by so starring things but, in fact,  $V(*\mathbb{R})$  does not equal  $*V(\mathbb{R})$ . For this reason, we are really interested in  $\bigcup_{n \in \mathbb{N}} *V_n(\mathbb{R})$ . The following important definition is needed.

**1.1.2. Definition.** An object  $A \in V(*S)$  is called *internal* if  $A \in *B$  for some  $B \in V(S)$ . Other objects in  $V(*S)$  are *external*.

So,  $*V(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} *V_n(\mathbb{R})$  is the collection of all internal objects in  $V(*\mathbb{R})$ . The notion of internal sets is the most important concept in nonstandard analysis since these are the objects we learn about via the transfer principle. For this reason, it is important to know how to construct new internal sets from old ones, and how one recognizes internal sets when one meets them. This idea is made general as follows.

**1.1.3. Internal Definition Theorem.** Let  $B, A_1, \dots, A_n$  be internal sets and let  $\varphi$  be a bounded formula. Then the set

$$A = \{ x \in B : \varphi(x, A_1, \dots, A_n) \text{ holds in } V(*S) \}$$

is internal.

*Proof.* Choose  $k \in \mathbb{N}$  so large that  $B, A_1, \dots, A_n \in *V_k(S)$  and transfer the true statement:

$$(\forall B, A_1, \dots, A_n \in V_k(S)) (\exists A \in V_k(S)) (\forall x \in V_k(S)) \\ \left( x \in A \leftrightarrow (x \in B \wedge \varphi(x, A_1, \dots, A_n)) \right). \quad \square$$

Many interesting internal sets are embedded standard sets, where a set  $A$  is *standard* if  $A = *B$  for some  $B \in V(S)$ . We shall list several elementary, but important, properties of internal sets without proofs.

- 1.1.4. Theorem.** (1) If  $A \in V(S)$ , then  $*A$  is internal.  
 (2) A set  $A$  is internal if and only if  $A \in B$  for some standard set  $B$ .  
 (3) If  $A$  is internal and  $B \in A$ , then  $B$  is internal.  
 (4) If  $A_1, A_2, \dots$  is a sequence of internal objects, then  $\bigcup_{n \in \mathbb{N}} A_n$  is internal if and only if  $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^m A_n$  for some  $m \in \mathbb{N}$ .

Let us go back now to the example of  $\mathbb{R}$ . The transfer of the completeness property for  $\mathbb{R}$  tells us that any internal subset of  $*\mathbb{R}$  which is bounded above (in  $*\mathbb{R}$ ) has a least upper bound. So we conclude that  $\mathbb{R}$  itself is an external subset

of  ${}^*\mathbb{R}$ . Also,  $\mathbb{N}$  and  $\{x \in {}^*\mathbb{R} : {}^o x = r\}$  for  $r \in \mathbb{R}$  are external sets. By the  $\omega_1$ -saturation principle, which states that if  $A_0 \supseteq A_1 \supseteq \dots$  are non-empty internal sets in  $V({}^*\mathbb{R})$ , then  $\bigcap_{m \in \mathbb{N}} A_m$  is non-empty, the intersection of a countable strictly decreasing chain of internal sets is always external. An analogous result holds for unions, and it follows that every countable set is external.

Important among internal sets are those that are hyperfinite. These are the internal sets corresponding to finite sets in  $V(\mathbb{R})$ .

**1.1.6. Definition.** An internal set  $A$  is *hyperfinite* or *starfinite* if there is an internal bijection  $f: \{1, 2, \dots, H\} \rightarrow A$  for some  $H \in {}^*\mathbb{N}$ . The cardinality  $H$  is called the *internal cardinality* of  $A$  and is denoted by  $|A|$ .

The importance of infinite hyperfinite sets lies in the fact that they behave formally like finite sets. For example, if  $A$  is hyperfinite and  $H = |A|$ , then  ${}^*\mathcal{P}(A)$  is hyperfinite and  $|{}^*\mathcal{P}(A)| = 2^H$ , where  ${}^*\mathcal{P}(A)$  denotes the set of all internal subsets of  $A$ . Given a hyperfinite set  $A \subseteq {}^*\mathbb{R}$ , we may form  $\sum_{a \in A} a$ ,  $\max\{a : a \in A\}$ , and so forth.

## 1.2. Loeb measure

Nonstandard measure theory has been used in recent years to establish new standard results such as the extension of measures, stochastic differential equations, adapted probability distributions, etc. The technical basis for these developments is the Loeb construction which takes a finitely additive, internal measure and turns it into a real-valued,  $\sigma$ -additive measure. We restrict attention to Loeb measures and their use in a wide variety of fields.

We shall begin with some basic notions from measure theory. A *finite additive measure space with finite measure* is a triple  $\langle A, \mathcal{F}, \mu \rangle$  where  $\mathcal{F}$  is a field of subsets of  $A$ ,  $\mu: \mathcal{F} \rightarrow \mathbb{R}_+$  and for  $X, Y \in \mathcal{F}$ ,  $\mu(X \cup Y) = \mu(X \setminus Y) + \mu(Y \setminus X) + \mu(X \cap Y)$ . A triple  $\langle A, \mathcal{F}, \mu \rangle$  is a *measure space* if, in addition,  $\mathcal{F}$  is a  $\sigma$ -field and  $\mu$  is countably additive; i.e., whenever  $X_0 \subseteq X_1 \subseteq \dots$  are  $\mu$ -measurable, then  $\mu(\bigcup_{n \in \mathbb{N}} X_n) = \lim_{n \rightarrow \infty} \mu(X_n)$ . So starrng things, we get  $\langle {}^*A, {}^*\mathcal{F}, {}^*\mu \rangle$ , which is an example of a so-called *internal measure space*. This means that

- (a)  ${}^*\mathcal{F} \subseteq {}^*\mathcal{P}({}^*A)$  is an internal algebra (not  $\sigma$ -algebra) of subsets of  ${}^*A$ ; i.e., if  $(A_n)_{n \leq H}$  is an internal sequence of internal sets from  ${}^*\mathcal{F}$ , for  $H \in {}^*\mathbb{N}$ , then  $\bigcup_{n \leq H} A_n \in {}^*\mathcal{F}$ ;
- (b)  ${}^*\mu: {}^*\mathcal{F} \rightarrow {}^*\mathbb{R}_+$  is an internal  ${}^*\sigma$ -additive measure (not  $\sigma$ -additive) on  ${}^*\mathcal{F}$ ; i.e., if  $(A_n)_{n \leq H}$  is an internal sequence of disjoint sets in  ${}^*\mathcal{F}$ , for  $H \in {}^*\mathbb{N}$ , then  ${}^*\mu(\bigcup_{n \leq H} A_n) = \sum_{n \leq H} {}^*\mu(A_n)$ .

But there are many internal measure spaces that are not derived from a standard measure space as  $\langle {}^*A, {}^*\mathcal{F}, {}^*\mu \rangle$  above is. This notion is made general as follows.

**1.2.1. Definition.** An *internal measure space*  $\langle A, \mathcal{F}, \mu \rangle$  consists of an internal set  $A$ , an internal algebra  $\mathcal{F} \subseteq {}^*\mathcal{P}(A)$  of subsets of  $A$  and internal  ${}^*\sigma$ -additive function  $\mu: \mathcal{F} \rightarrow {}^*\mathbb{R}_+$ .

The obstacle to developing powerful applications of nonstandard methods in measure theory was that the measure  ${}^*\mu$  constructed above is only finitely additive. This obstacle can be surmounted by the construction of the Loeb measure, which is

a simple application of the  $\omega_1$ -saturation principle. The idea is to take an internal measure space, which will in general be only finitely additive and need not be derived from any standard space, and to construct a new standard,  $\sigma$ -additive measure space from it.

Let  $\langle A, \mathcal{F}, \mu \rangle$  be an internal finitely additive measure space. Define a finitely additive measure  ${}^\circ\mu$  on  $\mathcal{F}$  by  ${}^\circ\mu(X) = {}^\circ(\mu(X))$ .

**1.2.2. Definition.** The *Loeb space* of  $\langle A, \mathcal{F}, \mu \rangle$  is the unique countably additive measure space  $\langle A, L(\mathcal{F}), L(\mu) \rangle$  such that  $L(\mathcal{F})$  is the  $\sigma$ -algebra generated by  $\mathcal{F}$  and  $L(\mu)(X) = {}^\circ\mu(X)$  for all  $X \in \mathcal{F}$ .

**1.2.3. Theorem (Loeb).** Let  $\langle A, \mathcal{F}, \mu \rangle$  be an internal finitely additive measure space with  ${}^\circ\mu(A) < \infty$ . Then the Loeb space of  $\langle A, \mathcal{F}, \mu \rangle$  exists and is unique.

*Proof.* The existence of a measure  $\nu$  on the  $\sigma$ -algebra  $\sigma(\mathcal{F})$  generated by  $\mathcal{F}$ , is obtained by appealing to the *Caratheodory extension theorem* which requires the following to be verified: if  $(X_n)_{n \in \mathbb{N}}$  is an increasing sequence from  $\mathcal{F}$  with  $\bigcup_{n \in \mathbb{N}} X_n = X \in \mathcal{F}$ , then  $\lim_{n \rightarrow \infty} {}^\circ\mu(X_n) = {}^\circ\mu(X)$ . In our case, this condition is trivially satisfied since the Theorem 1.1.5. tells us that  $X = X_m$  for some  $m \in \mathbb{N}$ . The Loeb  $\sigma$ -algebra  $L(\mathcal{F})$  is the completion of  $\sigma(\mathcal{F})$  with respect to the measure  $\nu$ , and the Loeb measure  $L(\mu)$  is the extension of  $\nu$  to  $L(\mathcal{F})$ . In fact,  $L(\mathcal{F})$  is the family of all subsets  $X$  of  $A$  such that

$$\sup \{ {}^\circ\mu(Y) : Y \in \mathcal{F} \text{ \& } Y \subseteq X \} = \inf \{ {}^\circ\mu(Z) : Z \in \mathcal{F} \text{ \& } X \subseteq Z \},$$

and the Loeb measure  $L(\mu)$  is given by

$$L(\mu)(X) = \sup \{ {}^\circ\mu(Y) : Y \in \mathcal{F} \text{ \& } Y \subseteq X \} \text{ for all } X \in L(\mathcal{F}). \quad \square$$

Loeb measure sets are very close to internal sets.

**1.2.4. Theorem.** If  $X \in L(\mathcal{F})$  and  $L(\mu)(X) < \infty$ , then there is an internal  $Y \in \mathcal{F}$  such that  $L(\mu)(X \Delta Y) = 0$ .

*Proof.* Choose  $Y_n \in \mathcal{F}$  such that  $X \subseteq Y_{n+1} \subseteq Y_n$  and  $L(\mu)(Y_n) - L(\mu)(X) < 1/n$  for each  $n \in \mathbb{N}$ . Choose an internal sequence  $(Y_n)_{n \in {}^*\mathbb{N}}$  which extends  $(Y_n)_{n \in \mathbb{N}}$ . We get the set

$$M = \{ n \in {}^*\mathbb{N} : Y_n \in \mathcal{F}, (\forall k \in {}^*\mathbb{N})(k < n \rightarrow Y_{k+1} \subseteq Y_k), L(\mu)(X) < \mu(Y_n) + 1/n \}.$$

The set  $M$  is internal, by the Internal Definition Theorem, and includes  $\mathbb{N}$ ; so it contains an infinite  $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ . Let  $Y = Y_H$ . Then  $L(\mu)(X) < \mu(Y) + 1/H$ ; so  $L(\mu)(X) \leq L(\mu)(Y)$ . The sets  $Y_n$  are arbitrarily  $L(\mu)$ -close to  $X$  and  $Y \subseteq Y_n$  for each  $n \in \mathbb{N}$ , it follows that  $L(\mu)(X) \geq L(\mu)(Y)$ . Therefore  $L(\mu)(X \Delta Y) = 0$ .  $\square$

From this proof, we conclude that every Loeb measurable set can be approximated above and below by internal measurable sets.

Important among Loeb spaces are those that are hyperfinite. When  $A$  is hyperfinite it is usual to assume that  $\mathcal{F} = {}^*\mathcal{P}(A)$ . The simplest example of this is the *internal counting probability measure*  $P_A$  on  $A$  given by  $P_A(X) = |X|/|A|$  for

an internal  $X \subseteq A$ . The associated Loeb space is the *uniform Loeb probability space on  $A$* . The uniform Loeb measure can be used to give nice constructions of some standard measures such as Lebesgue measure and Wiener measure. We shall see that for the Lebesgue measure. Lebesgue measure and integration on  $[0, 1]$  can be defined from the simple counting probability measure on a hyperfinite set  $A = \{0, 1/H, 2/H, \dots, 1\}$  for  $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ . It is easy to check that the measure  $\lambda$  on  $[0, 1]$  defined by  $\lambda(B) = L(P_A)(\text{st}^{-1}(B))$  is a Lebesgue measure. This example illustrates a technique used often in nonstandard measure theory. In order to construct a standard measure, first find a discrete version of it on a hyperfinite space; then apply the Loeb construction to get a  $\sigma$ -additive measure, and finally push the Loeb measure down to a standard space by means of the standard part map.

Now we shall list several basic results regarding measurability in the Loeb space  $\langle A, L(\mathcal{F}), L(\mu) \rangle$  of a standard measure space  $\langle A, \mathcal{F}, \mu \rangle$ .

**1.2.5. Lifting Theorem.** *Let  $\overline{\mathbb{R}}$  be the extended real numbers  $\mathbb{R} \cup \{-\infty, \infty\}$ . Then  $f: A \rightarrow \overline{\mathbb{R}}$  is Loeb measurable if and only if there is an internal  $\mathcal{F}$ -measurable function  $F: A \rightarrow {}^*\mathbb{R}$  such that  ${}^\circ F(a) = f(a)$  for  $L(\mu)$ -almost all  $a \in A$ .*

*Proof.* Let  $F: A \rightarrow {}^*\mathbb{R}$  be an  $\mathcal{F}$ -measurable function. Then, for any real  $r$  we have  $\{a : {}^\circ F(a) \leq r\} = \bigcap_{n \in \mathbb{N}} \{a : F(a) \leq r + 1/n\}$ , so  ${}^\circ F$  is Loeb measurable. Thus, if  $f(a) = {}^\circ F(a)$  for  $L(\mu)$ -almost all  $a \in A$ , then  $f$  is also Loeb measurable.

Conversely, let  $f: A \rightarrow \overline{\mathbb{R}}$  be a Loeb measurable and let  $X_n = \{a : f(a) \leq q_n\} \in L(\mathcal{F})$  for all rationals  $q_n$ ,  $n \in \mathbb{N}$ . Choose internal sets  $Y_n \in \mathcal{F}$  such that  $L(\mu)(X_n \Delta Y_n) = 0$  and  $Y_n \subseteq Y_m$  whenever  $q_n \leq q_m$  by the Theorem 1.2.4. Let  $(Y_n)_{n \in {}^*\mathbb{N}}$  be a sequence which extends  $(Y_n)_{n \in \mathbb{N}}$  and let

$$M = \{n \in {}^*\mathbb{N} : Y_n \in \mathcal{F} \text{ \& } (q_k \leq q_m \rightarrow Y_k \subseteq Y_m, \text{ for } k, m \leq n)\}.$$

The set  $M$  is an internal extension of  $\mathbb{N}$ . Let  $H$  be an infinite element of this set. Since  $\{q_1, \dots, q_H\}$  is hyperfinite and  $Y_n \in \mathcal{F}$  for all  $n \leq H$ , define an internal  $\mathcal{F}$ -measurable function  $F: A \rightarrow {}^*\mathbb{R}$  such that for every  $n \leq H$ ,  $F(a) \leq q_n$  iff  $a \in Y_n$ . Then  ${}^\circ F(a) = f(a)$  for all  $a \notin \bigcup_{n \in \mathbb{N}} (X_n \Delta Y_n)$ .  $\square$

A function  $F$  is called a *lifting* of  $f$ . The following diagram commutes for  $L(\mu)$ -almost all  $a \in A$ .

$$\begin{array}{ccc} A & \xrightarrow{F} & {}^*\mathbb{R} \\ i \downarrow & & \downarrow {}^\circ \\ A & \xrightarrow{f} & \overline{\mathbb{R}} \end{array}$$

The lifting  $F$  of  $f$  produced in the Theorem 1.2.5. has a hyperfinite range.

**1.2.6. Theorem.** *Let  $f$  be bounded Loeb measurable function with bounded lifting  $F$ . Then  $\int f dL(\mu) = {}^\circ \int F d\mu$ .*

*Proof.* First, we consider a simple Loeb measurable function  $g \leq f$  which has finite range  $\{r_1, \dots, r_n\}$ . By using the Theorem 1.2.4. on the sets  $g^{-1}\{r_i\}$ , we obtain a lifting  $G$  of  $g$  with range  $\{r_1, \dots, r_n\}$  and  $\int g dL(\mu) = {}^\circ \int G d\mu$  from the definition of the integrals. Take a bounded lifting  $F_1$  of  $f$  with  $G \leq F_1$ , we get  $\int g dL(\mu) = {}^\circ \int G d\mu \leq {}^\circ \int F_1 d\mu = {}^\circ \int F d\mu$ , since  $|F(a) - F_1(a)| \leq 1/n$  for almost

all  $a \in A$  and for any  $n \in \mathbb{N}$ . Hence  $\int f dL(\mu) \leq^\circ \int F d\mu$ , and the reverse inequality follows by considering  $-f$  and  $-F$ .  $\square$

If  $\langle A_1, \mathcal{F}_1, \mu_1 \rangle$  and  $\langle A_2, \mathcal{F}_2, \mu_2 \rangle$  are two internal finitely additive measure spaces, we have the internal product space  $\langle A_1 \times A_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2 \rangle$ , where  $\mathcal{F}_1 \times \mathcal{F}_2$  is the internal product algebra generated by  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . This space has as a corresponding Loeb space  $\langle A_1 \times A_2, L(\mathcal{F}_1 \times \mathcal{F}_2), L(\mu_1 \times \mu_2) \rangle$ . Also we can construct from the Loeb spaces of  $\langle A_1, \mathcal{F}_1, \mu_1 \rangle$  and  $\langle A_2, \mathcal{F}_2, \mu_2 \rangle$  the standard product space  $\langle A_1 \times A_2, L(\mathcal{F}_1) \times L(\mathcal{F}_2), L(\mu_1) \times L(\mu_2) \rangle$ . Anderson and Hoover showed that  $\overline{L(\mathcal{F}_1) \times L(\mathcal{F}_2)} \subsetneq L(\mathcal{F}_1 \times \mathcal{F}_2)$ , where  $\overline{L(\mathcal{F}_1) \times L(\mathcal{F}_2)}$  denotes the completion of the product  $\sigma$ -algebra with respect to the product measure, and that  $L(\mu_1 \times \mu_2)$  agrees with  $L(\mu_1) \times L(\mu_2)$  on the  $\overline{L(\mathcal{F}_1) \times L(\mathcal{F}_2)}$ . The standard Fubini Theorem can be extended for the product of two Loeb spaces.

**1.2.7. Keisler's Fubini Theorem.** *Let  $f: A_1 \times A_2 \rightarrow \overline{\mathbb{R}}$  be  $L(\mu_1 \times \mu_2)$ -measurable function. Then:*

- (1) *For  $L(\mu_2)$ -almost all  $a_2 \in A_2$ , the function  $f(\cdot, a_2): A_1 \rightarrow \overline{\mathbb{R}}$  is  $L(\mu_1)$ -measurable;*
- (2) *If  $f$  is integrable, then*
  - (a) *for  $\mu_2$ -almost all  $a_2$ , the function  $f(\cdot, a_2)$  is integrable over  $A_1$ ,*
  - (b) *the function  $g(a_2) = \int f(a_1, a_2) dL(\mu_1)$  is integrable over  $A_2$ ,*
  - (c)  *$\int g(a_2) dL(\mu_2) = \int f(a_1, a_2) dL(\mu_1 \times \mu_2)$ .  $\square$*

### 1.3. Loeb construction of Daniell integral

Recall that Loeb's construction of a measure by nonstandard analysis, presented above, corresponds to the Lebesgue integral. By a functional approach in [41], Loeb has developed a Daniell type extension of his own theory. We shall begin with some basic notions about the Daniell integral. A *vector lattice of  $\mathbb{R}$ -valued functions on a set  $A$*  is a vector subspace  $\mathcal{F}$  of all  $\mathbb{R}$ -valued functions on  $A$  such that  $f \wedge g \stackrel{\text{def}}{=} \min\{f, g\} \in \mathcal{F}$  and  $f \vee g \stackrel{\text{def}}{=} \max\{f, g\} \in \mathcal{F}$  for any  $f, g \in \mathcal{F}$ . A triple  $\langle A, \mathcal{F}, I \rangle$  where  $I$  denotes a positive linear functional on  $\mathcal{F}$  which has the monotone convergence property; i.e., if  $f_1 \leq f_2 \leq \dots$  are elements from  $\mathcal{F}$  with  $\lim_{n \rightarrow \infty} f_n = f$  and  $f \in \mathcal{F}$ , then  $I(f) = \lim_{n \rightarrow \infty} I(f_n)$ , called a *Daniell integral*.

So starring notions above, let  $A$  be an internal set in a  $\omega_1$ -saturated enrichment of a structure containing the real numbers  $\mathbb{R}$ , let  $L$  be an internal vector lattice of  ${}^*\mathbb{R}$ -valued functions on  $A$  and let  $I$  be an internal positive linear functional on  $L$ . Also, the set  $L_0$  of all internal and external  ${}^*\mathbb{R}$ -valued functions  $f$  on  $A$  such that for any  $\varepsilon \in \mathbb{R}_+$  there is a  $g \in L$  with  $|f| \leq g$  and  $I(g) < \varepsilon$ , and the set  $L_1$  of all  $\mathbb{R}$ -valued functions  $h$  on  $A$  such that  $h = \varphi + \psi$ , where  $\varphi \in L$ ,  ${}^\circ I(|\varphi|) < +\infty$  and  $\psi \in L_0$ , are vector lattices. A well-defined positive linear functional  $J$  on  $L_1$  is obtained by  $J(h) = {}^\circ I(\varphi)$  for any  $h \in L_1$ . By the  $\omega_1$ -saturation principle it follows that the monotone convergence property holds for  $J$  on  $L_1$ .

The triple  $\langle A, L, I \rangle$  is related to  $\langle A, L_1, J \rangle$  in a similar way as an internal measure space is related to its Loeb space. For example,  $h \in L_1$  if and only if for each  $\varepsilon \in \mathbb{R}_+$  there exist functions  $\varphi_1$  and  $\varphi_2$  in  $L$  such that  $\varphi_1 \leq h \leq \varphi_2$ ,  ${}^\circ I(|\varphi_1|) < +\infty$ ,  $I(\varphi_2 - \varphi_1) < \varepsilon$  and, hence,  ${}^\circ I(\varphi_1) \leq J(h) \leq {}^\circ I(\varphi_2) + \varepsilon$  (see the Theorem 1.2.4).

We shall make an extensions of  $L_1$  and  $J$  as follows. Let  $L^+$  be the set of functions  $g: A \rightarrow \mathbb{R}_+$  such that  $g \wedge f \in L_1$  for each  $f \geq 0$  in  $L_1$ . Let  $\bar{J}(g) = \sup_{0 \leq f \in L_1} J(g \wedge f)$  for any  $g \in L^+$ . Let  $\bar{L} = \{g : g \vee 0 \in L^+ \text{ \& } -g \vee 0 \in L^+ \}$  and  $\bar{J}(g) = \bar{J}(g \vee 0) - \bar{J}(-g \vee 0)$  if at least one of the values  $\bar{J}(g \vee 0)$  and  $\bar{J}(-g \vee 0)$  is finite. We note that  $L_1 \subseteq \bar{L}$ ,  $\bar{J} \upharpoonright L_1 = J$ ,  $1 \in L^+$  and if  $\varphi \in L$ , then  ${}^\circ\varphi \in \bar{L}$ . Let  $\mathcal{L}_1$  be the set of all  $B \subseteq A$  such that the characteristic function  $\chi_B$  of  $B$  is in  $L_1$  and, similarly, let  $\mathcal{F}$  be the set of all  $B \subseteq A$  such that  $\chi_B$  is in  $L^+$ .

The Daniell integral  $\langle A, L^+, \bar{J} \rangle$  of non-negative extended real-valued functions on  $A$  produces a standard measure space  $\langle A, \mathcal{F}, \mu \rangle$ , where  $\mu(B) = \bar{J}(\chi_B)$  for each  $B \in \mathcal{F}$ , such that  $\bar{J}(g) = \int_A g d\mu$  for any  $g \in L^+$ . In the case when  $1 \in L_1$ , we get  $\mathcal{F} = \mathcal{L}_1$  and  $\mu(A) < +\infty$ . This special case will be stressed below.

**1.3.1. Theorem (Loeb).** *Let  $\langle A, L, I \rangle$  be a Daniell integral on  $A$  in an  $\omega_1$ -saturated nonstandard universe such that  $1 \in L$ . Then there is a complete countable additive real-valued measure  $\mu$  on  $A$  such that  $\int {}^\circ\varphi d\mu = {}^\circ I(\varphi)$  for each finitely bounded  $\varphi \in L$ . If, in addition,  $I(1) = 1$ , then there is a probability measure  $\mu$ .  $\square$*

We shall finish this chapter with a remark which connects the Loeb measure from 1.2. to the Daniell type extension of measure.

**1.3.2. Remark.** When  $L$  is the lattice of  $\mathcal{F}$ -simple functions, i.e., functions which have finite range, on an internal measure space  $\langle A, \mathcal{F}, \nu \rangle$ , then  $L_1$  is the lattice of Loeb integrable functions and the measure  $\mu$  above is equal to the Loeb measure  $L(\nu)$ . The measure  $L(\nu)$  can also be obtained by applying the Riesz Representation Theorem to the functional  $f \mapsto {}^\circ \int {}^*f d\nu$ .

## ADMISSIBLE SETS AND $L_{\omega_1\omega}$ LOGIC

In this part we give a brief introduction to admissible set theory and to logic  $L_{\omega_1\omega}$ . For details see [3] and [30].

### 2.1. Consistency property

Let  $L_{\omega\omega}$  be the ordinary first-order logic. We shall define a first-order language as any set  $L$  of constant symbols, function symbols and relation symbols. The logical symbols of  $L_{\omega\omega}$  are the so-called connectives  $\wedge$  (and),  $\vee$  (or) and  $\neg$  (negation), then the equality sign  $=$ , quantifiers  $\forall$  (universal) and  $\exists$  (existential). We also have  $\omega_1$  variables  $v_0, v_1, \dots, v_\alpha, \dots$ , ( $\alpha < \omega_1$ ). The language for the logic  $L_{\omega_1\omega}$  is the same, but in  $L_{\omega_1\omega}$  the disjunction symbol  $\vee$  and conjunction symbol  $\wedge$  may be applied to countable or finite sets of formulas as well as to pairs of formulas.

**2.1.1. Definition.** The set of all formulas of the logic  $L_{\omega_1\omega}$  is the least set  $F$  such that:

- (1) Each atomic formula of  $L_{\omega_1\omega}$  is in  $F$ ;
- (2) If  $\varphi \in F$  and  $\alpha < \omega_1$ , then  $\neg\varphi \in F$ ,  $\exists v_\alpha\varphi \in F$  and  $\forall v_\alpha\varphi \in F$ ;
- (3) If  $\Phi$  is a finite or countable, non-empty, subset of  $F$ , then  $\bigvee \Phi \in F$  and  $\bigwedge \Phi \in F$ .

We introduce the abbreviations  $\rightarrow$  and  $\leftrightarrow$ , in the usual way, for implication and equivalence.

A *sentence* is a formula with no free variables. We shall be concerned only with formulas having finitely many free variables. We use  $\varphi(x_1, \dots, x_n)$  to denote a formula whose free variables are included in the set  $\{x_1, \dots, x_n\}$ .

A *model* is every structure  $\mathfrak{A} = \langle A, \mathcal{R}^\mathfrak{A}, \mathcal{F}^\mathfrak{A}, C^\mathfrak{A} \rangle$ , where  $A$  is a non-empty set (the domain of  $\mathfrak{A}$ ),  $\mathcal{R}^\mathfrak{A}$  is a set of relations over  $A$ ,  $\mathcal{F}^\mathfrak{A}$  is a family of operations over  $A$  and  $C^\mathfrak{A}$  is a set of constants of  $\mathfrak{A}$ .

The *satisfaction notation*  $\mathfrak{A} \models \varphi[a_1, \dots, a_n]$  means that  $(a_1, \dots, a_n)$  satisfies  $\varphi(x_1, \dots, x_n)$  in  $\mathfrak{A}$ . A formula is said to be *valid*,  $\models \varphi$ , if and only if  $\varphi$  is satisfied in every model for  $L$  by every interpretation of the free variables of  $\varphi$ .

For each formula  $\varphi$  of  $L_{\omega_1\omega}$ , the formula  $\varphi \neg$  is defined inductively as follows:

- (a) If  $\varphi$  is atomic, then  $\varphi \neg$  is  $\neg\varphi$ ;
- (b)  $(\neg\varphi) \neg$  is  $\varphi$ ;
- (c)  $(\bigwedge_{\varphi \in \Phi} \varphi) \neg$  is  $\bigvee_{\varphi \in \Phi} \neg\varphi$ ;
- (d)  $(\bigvee_{\varphi \in \Phi} \varphi) \neg$  is  $\bigwedge_{\varphi \in \Phi} \neg\varphi$ ;
- (e)  $(\forall x\varphi) \neg$  is  $\exists x\neg\varphi$ ;
- (f)  $(\exists x\varphi) \neg$  is  $\forall x\neg\varphi$ .

Now we shall give a list of axioms and rules of inference for  $L_{\omega_1\omega}$ . In what follows,  $x, y, z$  are arbitrary variables,  $\varphi, \psi, \tau$  are arbitrary formulas of  $L_{\omega_1\omega}$ , and  $\Phi$  is an arbitrary countable set of formulas of  $L_{\omega_1\omega}$ .

**2.1.2. Definition.** The list of all axioms of the logic  $L_{\omega_1\omega}$  is the following:

- (1)  $\varphi \rightarrow (\psi \rightarrow \varphi)$ ;
- (2)  $(\varphi \rightarrow (\psi \rightarrow \tau)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \tau))$ ;
- (3)  $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$ ;
- (4)  $(\neg\varphi) \leftrightarrow (\varphi \neg)$ ;
- (5)  $\bigwedge \Phi \rightarrow \varphi$ , where  $\varphi \in \Phi$ ;
- (6)  $\forall x\varphi(x, \vec{y}) \rightarrow \varphi(t, \vec{y})$ , where  $t$  is a term which is free for  $x$  in  $\varphi(x, \vec{y})$  and  $\varphi(t, \vec{y})$  is obtained by replacing each free occurrence of  $x$  by  $t$ ;
- (7)  $x=x$ ;
- (8)  $x=y \rightarrow y=x$ ;
- (9)  $x=y \wedge y=z \rightarrow x=z$ ;
- (10)  $\varphi(x, \vec{y}) \wedge t=x \rightarrow \varphi(t, \vec{y})$ , where  $\varphi(x, \vec{y})$  and  $\varphi(t, \vec{y})$  are as in (6) above.

**2.1.3. Definition.** Rules of inference for  $L_{\omega_1\omega}$ :

- (1) From  $\varphi, \varphi \rightarrow \psi$ , infer  $\psi$ ;
- (2) From  $\psi \rightarrow \varphi(x, \vec{y})$ , infer  $\psi \rightarrow \forall x\varphi(x, \vec{y})$ , where  $x$  does not occur free in  $\psi$ ;
- (3) From  $\psi \rightarrow \varphi$ , for all  $\varphi \in \Phi$ , infer  $\psi \rightarrow \bigwedge \Phi$ .

The set of theorems of  $L_{\omega_1\omega}$  is the least set of formulas of  $L_{\omega_1\omega}$  which contains all the axioms and is closed under the rules of inference. The abbreviation  $\vdash_{L_{\omega_1\omega}} \varphi$  means that  $\varphi$  is a theorem of  $L_{\omega_1\omega}$ .

In order to prove the next theorem we introduce, for each formula  $\varphi$  of  $L_{\omega_1\omega}$ , the set of subformulas, denoted by  $\text{sub}(\varphi)$ , which is defined inductively as follows:

If  $\varphi$  is an atomic formula, then  $\text{sub}(\varphi) = \{\varphi\}$ ,

$$\begin{aligned} \text{sub}(\neg\varphi) &= \text{sub}(\varphi) \cup \{\neg\varphi\}, \\ \text{sub}(\forall x\varphi) &= \text{sub}(\varphi) \cup \{\forall x\varphi\}, \\ \text{sub}(\exists x\varphi) &= \text{sub}(\varphi) \cup \{\exists x\varphi\}, \\ \text{sub}(\bigwedge \Phi) &= \bigcup_{\varphi \in \Phi} \text{sub}(\varphi) \cup \{\bigwedge \Phi\}, \\ \text{sub}(\bigvee \Phi) &= \bigcup_{\varphi \in \Phi} \text{sub}(\varphi) \cup \{\bigvee \Phi\}. \end{aligned}$$

It is clear that for every formula  $\varphi$ , the set  $\text{sub}(\varphi)$  is finite or countable. Moreover, if  $\varphi$  has only finitely many free variables, so does every  $\psi \in \text{sub}(\varphi)$ .

The following important definition and the subsequent theorem, due to Makkai, are the principal tools for building models.

Let  $C = \{c_n : n \in \omega\}$  be a countable set of new constant symbols, and let  $M = L \cup C$ . Let  $M_{\omega_1\omega}$  be infinitary logic corresponding to  $M$ . By a *basic term* we mean either a constant symbol or a term of the form  $F(c_{i_1}, \dots, c_{i_n})$ , where  $c_{i_1}, \dots, c_{i_n} \in C$  and  $F$  is a function symbol of  $L$ .

**2.1.4. Definition.** A consistency property of  $M_{\omega_1\omega}$  is a set  $S$  of countable sets of sentences of  $M_{\omega_1\omega}$  such that the following holds for each  $s \in S$ :

- (c<sub>0</sub>)  $\emptyset \in S$  and if  $s \subseteq s' \in S$ , then  $s \cup \{\varphi\} \in S$ , for each  $\varphi \in s'$ ;
- (c<sub>1</sub>) If  $\varphi \in s$ , then  $\neg\varphi \notin s$ ;
- (c<sub>2</sub>) If  $\neg\varphi \in s$ , then  $s \cup \{\varphi\} \in S$ ;
- (c<sub>3</sub>) If  $\bigwedge \Phi \in s$ , then for all  $\varphi \in \Phi$ ,  $s \cup \{\varphi\} \in S$ ;
- (c<sub>4</sub>) If  $(\forall v\varphi(v)) \in s$ , then for every  $c \in C$ ,  $s \cup \{\varphi(c)\} \in S$ ;
- (c<sub>5</sub>) If  $\bigvee \Phi \in s$ , then for some  $\varphi \in \Phi$ ,  $s \cup \{\varphi\} \in S$ ;
- (c<sub>6</sub>) If  $(\exists v\varphi(v)) \in s$ , then for some  $c \in C$ ,  $s \cup \{\varphi(c)\} \in S$ ;
- (c<sub>7</sub>) Let  $t$  be any basic term and  $c, d \in C$ :
  - (i) If  $(c=d) \in s$ , then  $s \cup \{d=c\} \in S$ ;
  - (ii) If  $\varphi(t), (c=t) \in s$ , then  $s \cup \{\varphi(c)\} \in S$ ;
  - (iii) For some  $e \in C$ ,  $s \cup \{e=t\} \in S$ .

**2.1.5. Model Existence Theorem.** Let  $S$  be a consistency property for  $M_{\omega_1\omega}$ . For each  $s_0 \in S$ , there is a model  $\mathfrak{A} \models \bigwedge s_0$ .

*Proof.* It is easy to see that for  $s_0 \in S$  there is the least set of formulas  $X$  such that:

- $s_0 \subseteq X$ ;
- If  $c, d \in C$ , then  $(c=d) \in X$ ;
- If  $(\neg\varphi) \in X$ , then  $(\varphi) \in X$ ;
- If  $t$  is a term,  $d \in C$  and  $\varphi(t) \in X$ , then  $\varphi(d) \in X$ ;
- $X$  is closed under subformulas.

Let  $Y$  be the set of all sentences in  $X$ . Obviously  $Y$  is countable, and so put  $Y = \{\varphi_0, \varphi_1, \dots\}$ . Let  $T = \{t_0, t_1, \dots\}$  be the set of all basic terms of  $L$ .

We define a sequence  $\langle s_n : n < \omega \rangle$  of elements of  $S$  as follows.  $s_0 \in S$  is given. Given  $s_n$  choose  $s_{n+1}$  to satisfy the following conditions:

- (1)  $s_n \subseteq s_{n+1}$ ;
- (2) If  $s_n \cup \{\varphi_n\} \in S$ , then  $\varphi_n \in s_{n+1}$ ;
- (3) If  $s_n \cup \{\varphi_n\} \in S$  and  $\varphi_n = \bigvee \Phi$ , then for some  $\psi \in \Phi$ ,  $\psi \in s_{n+1}$ ;
- (4) If  $s_n \cup \{\varphi_n\} \in S$  and  $\varphi_n = \exists x\varphi(x)$ , then for some  $c \in C$ ,  $\varphi(c) \in s_{n+1}$ ;
- (5) For some  $c \in C$ ,  $(c=t_n) \in s_{n+1}$ .

Now let  $s_\omega = \bigcup_{n < \omega} s_n$ . For  $c, d \in C$ , let

$$c \sim d \text{ if and only if } (c=d) \in s_\omega.$$

It follows from (c<sub>7</sub>) that  $\sim$  is an equivalence relation. Let  $[c]$  denotes the equivalence class of the constant symbol  $c$ . Let  $\mathfrak{A}$  be the model with universe  $A = \{[c] : c \in C\}$ . By (c<sub>7</sub>),  $\sim$  is a congruence relation. So define relations  $R^{\mathfrak{A}}([c_1], \dots, [c_n])$  holding if and only if  $R(c_1, \dots, c_n) \in s_\omega$ , and similarly for functions.

By induction on the complexity of terms, using  $(c_1)$  and  $(c_7)$ , it can be shown that any atomic sentence or negated atomic sentence in  $s_\omega$  holds in  $\mathfrak{A}$ . Finally, by induction on the complexity of sentences and using rules  $(c_2)$ – $(c_6)$  one shows that every sentence in  $s_\omega$  holds in  $\mathfrak{A}$ . Therefore  $\mathfrak{A}$  is a model of  $s_0$ .  $\square$

**2.1.6. Corollary (Extended Version).** *Let  $S$  be a consistency property and let  $T$  be a countable set of sentences of  $M_{\omega_1\omega}$ , such that for all  $s \in S$  and  $\varphi \in T$ ,  $s \cup \{\varphi\} \in S$ . Then for all  $s \in S$ ,  $s \cup T$  has a model.*  $\square$

The following important theorem is a consequence of the Model Existence Theorem.

**2.1.7. Completeness Theorem.** *If  $\varphi$  is a sentence of  $L_{\omega_1\omega}$ , then  $\vdash_{L_{\omega_1\omega}} \varphi$  if and only if  $\models \varphi$ .*

*Proof.* One shows that the set  $S = \{s : s \text{ is a finite set of sentences of the logic } M_{\omega_1\omega} \text{ each containing only finitely many constants from } C \text{ and not } \vdash_{L_{\omega_1\omega}} \neg \bigwedge s\}$  is a consistency property, and then appeal to the Extended Version of the Model Existence Theorem.  $\square$

## 2.2. Admissible sets

Now, we are going to introduce the notion of an admissible set. This weak set theory plays an important role in formalizing the syntax of a probability logic. First we start with the set theory notation. By a formula of set theory we shall mean a formula in the first-order logic with only one binary relation symbol  $\in$ . We shall use abbreviations (i.e., bounded quantifiers)  $(\forall x \in y)\varphi$  for  $(\forall x)(x \in y \rightarrow \varphi)$  and  $(\exists x \in y)\varphi$  for  $(\exists x)(x \in y \wedge \varphi)$ .

### 2.2.1. Definition.

- (a) The set of all  $\Delta_0$ -formulas is the least set  $\Delta_0$  such that:
  - (i) Atomic formulas and their negations belong to  $\Delta_0$ ;
  - (ii) If  $\varphi, \psi$  are in  $\Delta_0$ , so are  $(\varphi \wedge \psi)$  and  $(\varphi \vee \psi)$ ;
  - (iii) If  $\varphi$  is in  $\Delta_0$ , then so are  $(\forall x \in y)\varphi$  and  $(\exists x \in y)\varphi$  for all variables  $x$  and  $y$ .
- (b) The set of all  $\Sigma$ -formulas is the least set  $\Sigma$  such that:
  - (i)  $\Delta_0 \subseteq \Sigma$ ;
  - (ii)  $\Sigma$  is closed under  $\bigvee, \bigwedge, \forall x \in y, \exists x \in y$  and  $\exists$ .
- (c) The set of all  $\Pi$ -formulas is the least set  $\Pi$  such that:
  - (i)  $\Delta_0 \subseteq \Pi$ ;
  - (ii)  $\Pi$  is closed under the  $\bigvee, \bigwedge, \forall x \in y, \exists x \in y$  and  $\forall$ .
- (d) The formula  $\varphi$  is a  $\Delta$ -formula if and only if, there are a  $\Sigma$ -formula  $\varphi_1$  and a  $\Pi$ -formula  $\varphi_2$  such that  $\varphi \leftrightarrow \varphi_1 \leftrightarrow \varphi_2$ .

The Kripke-Platek set theory (denoted by  $KP$ ) consists of the universal closures of the following formulas:

*Extensionality:*  $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y$ ;

*Foundation:*  $\exists x\varphi(x) \rightarrow \exists x[\varphi(x) \wedge (\forall y \in x)\neg\varphi(y)]$  for all formulas  $\varphi(x)$  in which  $y$  does not occur free;

*Pair:*  $\exists z(x \in z \wedge y \in z)$ ;

*Union:*  $\exists z(\forall y \in x)(\forall t \in y)(t \in z)$ ;

$\Delta_0$  *separation:*  $\exists z\forall t[t \in z \leftrightarrow t \in x \wedge \varphi(t)]$  for all  $\Delta_0$ -formulas in which  $z$  does not occur free;

$\Delta_0$  *collection:*  $(\forall t \in x)\exists y\varphi(t, y) \rightarrow \exists z(\forall t \in x)(\exists y \in z)\varphi(t, y)$  for all  $\Delta_0$ -formulas in which  $z$  does not occur free.

Recall that a set  $x$  is *transitive* if and only if  $z \in y$  and  $y \in x$  implies  $z \in x$ .

**2.2.2. Definition.** A set  $\mathcal{A}$  is said to be an admissible set if and only if

- (1)  $\mathcal{A}$  is non-empty and transitive;
- (2)  $\mathcal{A}$  is a model of  $KP$ .

A very important example is the admissible set of hereditary countable sets  $HC = \{x : TC(x) < \omega_1\}$ , where  $TC(x)$  is the least transitive set which includes  $x$ .

We agree that each variable  $v_\alpha$  is the ordered pair  $\langle v, \alpha \rangle$  and each logical symbol or a symbol of  $L$  is a natural number. Atomic formulas are finite sequences of symbols. The formula  $\neg\varphi$  is the ordered pair  $\langle \neg, \varphi \rangle$ ,  $\bigwedge \Phi$  is  $\langle \wedge, \Phi \rangle$  and  $\forall v_\alpha \varphi$  is  $\langle \forall, v_\alpha, \varphi \rangle$  ( $\forall$  and  $\exists$  are treated in the same way). It is of crucial importance that each formula of  $L_{\omega_1\omega}$  is a set.

If we denote the set of all formulas of the logic  $L_{\omega_1\omega}$  with  $L_{\omega_1\omega}$ , then for any admissible set  $\mathcal{A}$ , let  $L_{\mathcal{A}} = L_{\omega_1\omega} \cap \mathcal{A}$ . We shall call  $L_{\mathcal{A}}$  a *fragment* of  $L_{\omega_1\omega}$ . It is easy to show that the following conditions are satisfied:

- (1)  $\omega \subseteq \mathcal{A}$ ;
- (2) If  $\theta(x_1, \dots, x_n)$  is an atomic formula of  $L$  and  $a_i \in \mathcal{A}$  for  $i = 1, 2, \dots, n$ , then  $\theta(a_1, \dots, a_n) \in L_{\mathcal{A}}$ ;
- (3)  $L_{\mathcal{A}}$  is closed under  $\neg$  and finite  $\bigwedge$  and  $\bigvee$ ;
- (4) If  $\varphi \in L_{\mathcal{A}}$ , then  $\text{sub}(\varphi) \subseteq \mathcal{A}$ ;
- (5) If  $\varphi \in L_{\mathcal{A}}$  and  $x \in \mathcal{A}$ , then  $\forall x\varphi \in L_{\mathcal{A}}$  and  $\exists x\varphi \in L_{\mathcal{A}}$ ;
- (6) If  $\Phi$  is a countable set of formulas and  $\Phi \in \mathcal{A}$ , then  $\bigwedge \Phi \in L_{\mathcal{A}}$  and  $\bigvee \Phi \in L_{\mathcal{A}}$ . Moreover,  $\{\neg\varphi : \varphi \in \Phi\} \in \mathcal{A}$ ,  $\{\varphi\neg : \varphi \in \Phi\} \in \mathcal{A}$  and  $\{\exists x\varphi : \varphi \in \Phi\} \in \mathcal{A}$  for  $\Phi \in \mathcal{A}$ ;
- (7) For every countable set  $\Phi \subseteq L_{\omega_1\omega}$ , there is a least countable fragment  $L_{\mathcal{A}}$  such that  $\Phi \subseteq L_{\mathcal{A}}$ .

We shall call  $\varphi$  a *theorem* of  $L_{\mathcal{A}}$  (and write  $\vdash_{L_{\mathcal{A}}} \varphi$ ) if and only if it has a proof in  $L_{\omega_1\omega}$  such that every formula of the proof belongs to  $L_{\mathcal{A}}$ . From the Model Existence Theorem one derives the Completeness Theorem for  $L_{\mathcal{A}}$ .

**2.2.3. Completeness Theorem.** If  $\varphi$  is a sentence of a countable fragment  $L_{\mathcal{A}}$  of  $L_{\omega_1\omega}$ , then  $\vdash_{L_{\mathcal{A}}} \varphi$  if and only if  $\models \varphi$ .  $\square$

**2.2.4. Definition.** We say that a set  $X \subseteq \mathcal{A}$  is  $\Sigma$  on  $\mathcal{A}$  if and only if there is a  $\Sigma$ -formula  $\varphi(x, \vec{y})$  and elements  $c_1, \dots, c_n \in \mathcal{A}$  such that

$$X = \{a \in \mathcal{A} : \langle \mathcal{A}, \in \rangle \models \varphi[a, \vec{c}]\}.$$

Finally, we shall formulate the following very important theorems.

**2.2.5. Barwise Completeness Theorem.** Let  $\mathcal{A} \subseteq HC$  be an admissible set. Then the set of all valid sentences  $\varphi \in L_{\mathcal{A}}$  is  $\Sigma$  on  $\mathcal{A}$ .  $\square$

**2.2.6. Barwise Compactness Theorem.** *Let  $\mathcal{A} \subseteq HC$  be a countable admissible set. Let  $T$  be a set of sentences of  $L_{\mathcal{A}}$  which is  $\Sigma$  on  $\mathcal{A}$ . If every  $T_0 \subseteq T$  which is an element of  $\mathcal{A}$  has a model, then  $T$  has a model.  $\square$*

**2.2.7. Corollary** (Extended Completeness Theorem). *Let  $L_{\mathcal{A}}$  be a countable admissible fragment. Let  $T$  be a set of sentences of  $L_{\mathcal{A}}$  which is  $\Sigma$  on  $\mathcal{A}$ . Then the set  $\{\varphi \in L_{\mathcal{A}} : T \models \varphi\}$  is  $\Sigma$  on  $\mathcal{A}$ .  $\square$*

## CHAPTER 3

### $L_{AP}$ LOGIC AND COMPLETENESS THEOREMS

In this chapter we shall introduce the logic  $L_{AP}$ . This logic is similar to the finitary logic  $L_A$ , except that quantifiers  $P\vec{x} \geq r$  ( $\vec{x}$  is a finite sequence of variables) are used instead of usual  $\forall x$  and  $\exists x$ . A model of this logic is a classical model with a probability measure on the universe, such that each relation is measurable. The formula  $(P\vec{x} \geq r)\varphi(\vec{x})$  means that the set  $\{\vec{x} : \varphi(\vec{x})\}$  has probability  $\geq r$ . In this part we study the completeness problem for this logic.

#### 3.1. Syntax

We assume that  $\mathcal{A}$  is an admissible set such that  $\mathcal{A} \subseteq \text{HC}$  and  $\omega \in \mathcal{A}$  (see Chapter 2).

Let  $L$  be a countable,  $\Sigma$ -definable set of finitary relation and constant symbols (no function symbols). We need the following *logical symbols*:

- (1) The parentheses  $(, )$ .
- (2) The variables  $v_0, v_1, \dots, v_n, \dots, n \in \mathbb{N}$ .
- (3) The connectives  $\neg$  and  $\bigwedge$ .
- (4) The quantifiers  $(P\vec{x} \geq r)$ , where  $r \in \mathcal{A} \cap [0, 1]$ .
- (5) The equality symbol  $=$  (optional).

Let us note that because of the absence of the universal quantifier and of function symbols the equality relation plays only a minor role in  $L_{AP}$ .

**3.1.1. Definition.** The formulas of  $L_{AP}$  are defined as follows:

- (1) An atomic formula of first-order logic is a formula of  $L_{AP}$ ;
- (2) If  $\varphi$  is a formula of  $L_{AP}$ , then  $\neg\varphi$  is a formula of  $L_{AP}$ ;
- (3) If  $\Phi \in \mathcal{A}$  is a set of formulas of  $L_{AP}$  with only finitely many free variables, then  $\bigwedge \Phi$  is a formula of  $L_{AP}$ ;
- (4) If  $\varphi$  is a formula of  $L_{AP}$ , then  $(P\vec{x} \geq r)\varphi$  is a formula of  $L_{AP}$ .

We shall assume that  $L_{AP} \subseteq \mathcal{A}$  and denote  $L_{AP}$ , where  $\mathcal{A} = \text{HC}$ , by  $L_{\omega_1 P}$ . Thus,  $L_{AP} = \mathcal{A} \cap L_{\omega_1 P}$ . All other syntactical notions are defined similarly as in the  $L_A$  case.

**3.1.2. Definition.** We shall use the following abbreviations:

- (a)  $(P\vec{x} < r)\varphi$  for  $\neg(P\vec{x} \geq r)\varphi$ ;
- (b)  $(P\vec{x} \leq r)\varphi$  for  $(P\vec{x} \geq 1 - r)\neg\varphi$ ;
- (c)  $(P\vec{x} > r)\varphi$  for  $\neg(P\vec{x} \geq 1 - r)\neg\varphi$ ;
- (d) The connectives  $\vee$ ,  $\rightarrow$  and  $\leftrightarrow$  are defined as usual.

The quantifier  $(P\vec{x} \geq 1)$  is a weak analog of  $(\forall \vec{x})$ , while  $(P\vec{x} > 0)$  is a strong analog of  $(\exists \vec{x})$ .

### 3.2. Models

Let  $\langle A, \mathcal{F}, \mu \rangle$  be a probability space such that each singleton is measurable. Then, for each  $n \in \mathbb{N}$ , one shows (see [34]) that  $\langle A, \mathcal{F}^{(n)}, \mu^{(n)} \rangle$  is a probability space, where  $\mathcal{F}^{(n)}$  is the  $\sigma$ -algebra generated by the measurable rectangles and the diagonal sets, and  $\mu^{(n)}$  is the restriction of the completion of  $\mu^n$  to  $\mathcal{F}^{(n)}$ .

Now we introduce several notions of probability models.

**3.2.1. Definition.** A probability model for  $L$  is a structure

$$\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu \rangle_{i \in I, j \in J}$$

where  $\langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}} \rangle_{i \in I, j \in J}$  is a classical model,  $\mu$  is a countably additive probability measure on  $A$  such that each singleton is measurable, each  $n$ -placed relation  $R_i^{\mathfrak{A}}$  is  $\mu^{(n)}$ -measurable, and each  $c_j^{\mathfrak{A}} \in A$ .

Now we define a wider but very important class of models.

**3.2.2. Definition.** A graded probability model for  $L$  is a structure

$$\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$$

such that:

- (1)  $\langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}} \rangle_{i \in I, j \in J}$  is a classical model;
- (2) Each  $\mu_n$  is a countably additive probability measure on  $A^n$ ;
- (3) For all  $m, n \in \mathbb{N}$ ,  $\mu_{m+n}$  is an extension of the product measure  $\mu_m \times \mu_n$ ;
- (4) Each  $\mu_n$  is invariant under permutations; that is, whenever  $\pi$  is a permutation of  $\{1, 2, \dots, n\}$  and  $B \in \text{dom}(\mu_n)$ , if

$$\pi B = \{ (a_{\pi(1)}, \dots, a_{\pi(n)}) : (a_1, \dots, a_n) \in B \},$$

then  $\pi B \in \text{dom}(\mu_n)$  and  $\mu_n(\pi B) = \mu_n(B)$ ;

- (5)  $\langle \mu_n \mid n \in \mathbb{N} \rangle$  has the Fubini property: If  $B$  is  $\mu_{m+n}$ -measurable, then
  - (a) for each  $\vec{x} \in A^m$ , the section  $B_{\vec{x}} = \{ \vec{y} \in A^n : (\vec{x}, \vec{y}) \in B \}$  is  $\mu_n$ -measurable,
  - (b) the function  $f(\vec{x}) = \mu_n(B_{\vec{x}})$  is  $\mu_m$ -measurable,
  - (c)  $\int f(\vec{x}) d\mu_m = \mu_{m+n}(B)$ ;
- (6) Each atomic formula with  $n$  free variables is measurable with respect to  $\mu_n$ .

Note that (6) in the above definition ensures that the equality relation is  $\mu_2$ -measurable; so by (5) singletons are  $\mu_1$ -measurable.

It is easy to see that each probability model  $\mathfrak{A}$  is a graded probability model with  $\mu_n = \mu_1^{(n)}$ .

**3.2.3. Definition.** A Borel probability model for  $L$  is a structure

$$\mathfrak{A} = \langle [0, 1], R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu \rangle_{i \in I, j \in J}$$

such that:

- (1)  $\langle [0, 1], R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}} \rangle_{i \in I, j \in J}$  is a classical model;
- (2)  $\mu$  is a Borel measure on  $A$ ;
- (3) Each  $n$ -placed relation  $R_i^{\mathfrak{A}}$  is  $\mu^n$ -measurable and each  $c_j^{\mathfrak{A}} \in A$ .

**3.2.4. Definition.** (a) A finite probability model is a probability model  $\mathfrak{A}$  whose universe  $A$  is finite, and whose measure  $\mu(B) = |B|/|A|$  is the counting measure.

(b) A  $*$ -finite probability model is a probability model which is finite in the nonstandard sense.

(c) A hyperfinite probability model is a probability model  $\mathfrak{A}$  such that the universe  $A$  is an infinite  $*$ -finite (hyperfinite) set and  $\mu$  is the Loeb measure determined by the  $*$ -counting measure on  $A$ .

(d) A hyperfinite graded model is a graded probability model whose universe  $A$  is a hyperfinite set and each  $\mu_n$  is the Loeb measure determined by the  $*$ -counting measure on  $A^n$ .

Let  $\mathfrak{A}$  be one of the models defined above and let  ${}^n\mu$  denotes either  $\mu^{(n)}$  or  $\mu_n$ . The satisfaction relation  $\mathfrak{A} \models \varphi[\vec{a}]$ , for  $\varphi(\vec{x}) \in L_{AP}$  and  $\vec{a} \in A^m$ , is defined recursively in the same way as it was for  $L_A$  except for the quantifier clause:

$$\mathfrak{A} \models (P\vec{y} \geq r)\varphi(\vec{x}, \vec{y})[\vec{a}] \quad \text{iff} \quad {}^n\mu\{\vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\} \geq r.$$

The following lemma is needed to show that the satisfaction relation has the intended meaning for  $L_{AP}$ .

**3.2.5. Lemma.** For each probability model  $\mathfrak{A}$ , formula  $\varphi(\vec{x}, \vec{y}) \in L_{AP}$  and sequence  $\vec{a} \in A^m$ , the set  $\{\vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\}$  is  ${}^n\mu$ -measurable.

In the case of graded models the proof of the above Lemma follows from the clause (5) of the definition, while in the other cases it follows from the following theorem.

**3.2.6. Fubini Theorem.** Let  $\mu$  be a probability measure such that each singleton is measurable, and let  $B \subseteq A^{m+n}$  be  $\mu^{(m+n)}$ -measurable. Then

- (1) Every section  $B_{\vec{x}} = \{\vec{y} \in A^n : (\vec{x}, \vec{y}) \in B\}$  is  $\mu^{(n)}$ -measurable;
- (2) The function  $f(\vec{x}) = \mu^{(n)}(B_{\vec{x}})$  is  $\mu^{(m)}$ -measurable;
- (3)  $\mu^{(m+n)}(B) = \int f(\vec{x}) d\mu^{(m)}$ .  $\square$

The model-theoretic notions of the  $L_{AP}$ -equivalence, the isomorphism and of  $L_{AP}$ -elementary submodels are defined as usual and are denoted by  $\mathfrak{A} \equiv_{AP} \mathfrak{B}$ ,  $\mathfrak{A} \cong \mathfrak{B}$  and  $\mathfrak{A} \prec_{AP} \mathfrak{B}$ , respectively.

### 3.3. Expressive power of the language

We give the following list of examples.

- (1) There is a countable set of measure one if and only if

$$(Px \geq 1)(Py > 0) \ x=y.$$

- (2) There are no singletons of positive measure if and only if

$$(Px \geq 1)(Py \geq 1) \ x \neq y.$$

The *random variable* is a measurable function  $X: A \rightarrow \mathbb{R}$ . By the Fubini Theorem one can represent the random variable  $X$  by some binary relation  $R(x, y)$  in a probability model  $\mathfrak{A}$  with  $X(x) = \mu\{y : R(x, y)\}$ . In the following examples, let the language  $L$  have binary relation symbols  $R, R_n, n \in \mathbb{N}$ . We denote the corresponding random variables by  $X, X_n, n \in \mathbb{N}$ .

- (3)  $X(v) \geq r$  iff  $(Pu \geq r)R(v, u)$ .

- (4)  $|X_1(v) - X_2(v)| \leq r$  iff

$$\bigwedge_{q \in \mathbb{Q}} (X_1(v) \geq q \rightarrow X_2(v) \geq q - r) \wedge (X_2(v) \geq q \rightarrow X_1(v) \geq q - r).$$

- (5)  $X_n \rightarrow X$  almost surely (a.s.) iff

$$(Pv \geq 1) \bigwedge_n \bigvee_m \bigwedge_{k \geq m} |X_k(v) - X(v)| \leq \frac{1}{n}.$$

- (6)  $X_n \rightarrow X$  in probability iff

$$\bigwedge_n \bigvee_m \bigwedge_{k \geq m} \left( Pv \geq 1 - \frac{1}{n} \right) |X_k(v) - X(v)| \leq \frac{1}{n}.$$

- (7)  $X_1$  and  $X_2$  have the same distribution iff

$$\bigwedge_{q \in \mathbb{Q}} \bigwedge_{r \in \mathbb{Q}} (Pv \geq r)(X_1(v) \geq q) \leftrightarrow (Pv \geq r)(X_2(v) \geq q).$$

### 3.4. Proof theory

We now give a list of axioms and rules of inference for  $L_{AP}$ . In what follows,  $\varphi, \psi$  are arbitrary formulas of  $L_{AP}$ ,  $\Phi \in \mathcal{A}$  is an arbitrary set of formulas of  $L_{AP}$  and  $r, s \in \mathcal{A} \cap [0, 1]$ .

**3.4.1. Definition.** The axioms of the weak  $L_{AP}$  are the following:

- (W<sub>1</sub>) All axioms of  $L_{\mathcal{A}}$  without quantifiers;
- (W<sub>2</sub>) *Monotonicity:*  $(P\vec{x} \geq r)\varphi \rightarrow (P\vec{x} \geq s)\varphi$ , where  $r \geq s$ ;
- (W<sub>3</sub>)  $(P\vec{x} \geq r)\varphi(\vec{x}) \rightarrow (P\vec{y} \geq r)\varphi(\vec{y})$ ;
- (W<sub>4</sub>)  $(P\vec{x} \geq 0)\varphi$ ;
- (W<sub>5</sub>) *Finite additivity:*
  - (a)  $(P\vec{x} \leq r)\varphi \wedge (P\vec{x} \leq s)\psi \rightarrow (P\vec{x} \leq r + s)(\varphi \vee \psi)$ ;
  - (b)  $(P\vec{x} \geq r)\varphi \wedge (P\vec{x} \geq s)\psi \wedge (P\vec{x} \leq 0)(\varphi \wedge \psi) \rightarrow (P\vec{x} \geq r + s)(\varphi \vee \psi)$ ;
- (W<sub>6</sub>) *The Archimedean property:*

$$(P\vec{x} > r)\varphi \leftrightarrow \bigvee_{n \in \mathbb{N}} \left( P\vec{x} \geq r + \frac{1}{n} \right) \varphi.$$

**3.4.2. Definition.** The axioms for graded  $L_{AP}$  consist of the axioms for weak  $L_{AP}$  plus the following set of Hoover's axioms:

(H<sub>1</sub>) *Countable additivity*:

$$\bigwedge_{\Psi \subseteq \Phi} (P\vec{x} \geq r) \bigwedge \Psi \rightarrow (P\vec{x} \geq r) \bigwedge \Phi,$$

where  $\Psi$  ranges over the finite subsets of  $\Phi$ .

(H<sub>2</sub>) *Symmetry*:

$$(Px_1 \dots x_n \geq r)\varphi \leftrightarrow (Px_{\pi(1)} \dots x_{\pi(n)} \geq r)\varphi,$$

where  $\pi$  is a permutation of  $\{1, \dots, n\}$ .

(H<sub>3</sub>) *Product independence*:

$$(P\vec{x} \geq r)(P\vec{y} \geq s)\varphi \rightarrow (P\vec{x}\vec{y} \geq r \cdot s)\varphi,$$

where all variables in  $\vec{x}, \vec{y}$  are distinct.

**3.4.3. Definition.** The axioms for the full  $L_{AP}$  consist of the axioms for graded  $L_{AP}$  plus the following Keisler's axiom<sup>1</sup>:

(K) *Product measurability*:

$$(P\vec{x} \geq 1)(P\vec{y} > 0)(P\vec{z} \geq r)(\varphi(\vec{x}, \vec{z}) \leftrightarrow \varphi(\vec{y}, \vec{z}))$$

for each  $r < 1$ , where all variables in  $\vec{x}, \vec{y}, \vec{z}$  are distinct.

**3.4.4. Definition.** The axioms for the atomic  $L_{AP}$  consist of the axioms for the full  $L_{AP}$  plus the following axiom:

$$(A) \quad (Px \geq 1)(Py > 0) x=y.$$

**3.4.5. Definition.** The axioms for the atomless  $L_{AP}$  consist of the axioms for the full  $L_{AP}$  plus the following axiom:

$$(AL) \quad (Px \geq 1)(Py \geq 1) x \neq y.$$

**3.4.6. Definition.** The rules of inference for all of the above logics are:

$$(MP) \quad \text{Modus Ponens: } \frac{\varphi, \varphi \rightarrow \psi}{\psi}.$$

$$(C) \quad \text{Conjunction: } \frac{\varphi \rightarrow \psi, \psi \in \Phi}{\varphi \rightarrow \bigwedge \Phi}.$$

$$(G) \quad \text{Generalization: } \frac{\varphi \rightarrow \psi(\vec{x})}{\varphi \rightarrow (P\vec{x} \geq 1)\psi(\vec{x})}, \quad \text{where } \vec{x} \text{ is not free in } \varphi.$$

The notions of a deduction, theorem, satisfaction, validity etc. are defined in the usual way.

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<sup>1</sup>Usually we drop prefix full in the above notion.

**3.4.7. Proposition.** *The following are theorems of graded  $L_{AP}$ :*

- (1)  $(P\vec{x} \leq 1)\varphi$ ;
- (2)  $(P\vec{x} > r) \bigvee \Phi \leftrightarrow \bigvee_{\Phi_0 \subseteq \Phi} (P\vec{x} > r) \bigvee \Phi_0$ , where  $\Phi_0$  is finite;
- (3)  $(P\vec{x} \geq r)\varphi \leftrightarrow \bigwedge_n (P\vec{x} \geq r - 1/n)\varphi$ ;
- (4)  $(P\vec{x} \leq r)\varphi(\vec{x}) \wedge (P\vec{y} \leq s)\psi(\vec{y}) \rightarrow (P\vec{x}\vec{y} \leq rs)(\varphi(\vec{x}) \wedge \psi(\vec{y}))$ ;
- (5)  $(P\vec{x} \geq r)\varphi(\vec{x}) \rightarrow (P\vec{x}\vec{y} \geq r)\varphi(\vec{x})$ ;
- (6)  $(P\vec{x}\vec{y} \geq r + s - rs)\varphi \rightarrow (P\vec{x} \geq r)(P\vec{y} \geq s)\varphi$ ;
- (7)  $(P\vec{x} \geq 1)(P\vec{y} \geq 1)\varphi \leftrightarrow (P\vec{x}\vec{y} \geq 1)\varphi$ ;
- (8)  $(P\vec{x} \geq 1)(\varphi(\vec{x}) \rightarrow \psi(\vec{x})) \rightarrow ((P\vec{x} \geq r)\varphi(\vec{x}) \rightarrow (P\vec{x} \geq r)\psi(\vec{x}))$ ;
- (9)  $\bigwedge_n \bigvee_m (P\vec{y} < 1/n)((P\vec{x} \geq r - 1/m)\varphi(\vec{x}, \vec{y}) \wedge \neg(P\vec{x} \geq r)\varphi(\vec{x}, \vec{y}))$ ;
- (10)  $\bigwedge_n \bigvee_{\Phi_0 \subseteq \Phi} (P\vec{x} < 1/n)(\bigwedge \Phi_0(\vec{x}) \wedge \neg \bigwedge \Phi(\vec{x}))$ , where  $\Phi_0$  is finite and  $\bigwedge \Phi(\vec{x}) \in \mathcal{A}$ ;
- (11)  $\bigwedge_{1 \leq i \leq n-1} (P\vec{x} \geq r_i)(P\vec{y} \in [s_i, s_{i+1}])\varphi(\vec{x}, \vec{y}) \rightarrow (P\vec{x}\vec{y} \geq \sum_i r_i s_i)\varphi(\vec{x}, \vec{y})$ ,  
where  $s_1 \leq \dots \leq s_n$ ,  $s_1, r_1, \dots, s_n, r_n \in [0, 1]$ ;
- (12)  $\bigwedge_{1 \leq i \leq n-1} (P\vec{x} \leq r_i)(P\vec{y} \in [s_i, s_{i+1}])\varphi(\vec{x}, \vec{y}) \rightarrow (P\vec{x}\vec{y} \leq \sum_i r_i s_{i+1})\varphi(\vec{x}, \vec{y})$ ,  
where  $0 = s_1 \leq \dots \leq s_n = 1$ .  $\square$

### 3.5. Consistency properties and weak models

In order to prove more important completeness results we need the following notions and theorems. These notions are slightly different from those given in Chapter 2 for the logic  $L_{\mathcal{A}}$ .

**3.5.1. Definition.** A weak model for  $L_{AP}$  is a structure

$$\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$$

such that  $\langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}} \rangle_{i \in I, j \in J}$  is a classical model, each  $\mu_n$  is a finitely additive probability measure on  $A^n$  with each singleton measurable, and with the set<sup>2</sup>

$$\{ \vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}] \}$$

$\mu_n$ -measurable for each  $\varphi(\vec{x}, \vec{y}) \in L_{AP}$  and each  $\vec{a}$  in  $A$ .

It is easy to see that the notion of a weak model is more general than previously introduced notions of probability models.

<sup>2</sup> With the natural definition of satisfaction.

Let us introduce some convenient notation.  $\varphi \neg$  is defined as follows:

$$\begin{aligned} \varphi \neg &= \neg \varphi, \quad \varphi \text{ atomic}; & (\bigvee_n \varphi_n) \neg &= \bigwedge_n \neg \varphi_n; \\ (\neg \varphi) \neg &= \varphi; & ((P\vec{x} \geq r)\varphi) \neg &= (P\vec{x} > 1 - r)\neg \varphi; \\ (\bigwedge_n \varphi_n) \neg &= \bigvee_n \neg \varphi_n; & ((P\vec{x} > r)\varphi) \neg &= (P\vec{x} \geq 1 - r)\neg \varphi. \end{aligned}$$

It is easy to see that for a countable set of sentences  $\Phi \subseteq L_{AP}$  there is a countable admissible set  $\mathcal{B}$ , such that  $\Phi \subseteq L_{BP} \subseteq L_{AP}$ . So, without loss of generality, we can suppose in this chapter that  $\mathcal{A}$  is a countable admissible set<sup>3</sup>.

Let  $C$  be a countable set of new constant symbols, and let  $K = L \cup C$ . Then we form the logic  $K_{AP}$  corresponding to  $K$  and we introduce a notion of a consistency property.

**3.5.2. Definition.** A consistency property for  $L_{AP}$  is a set  $S$  of countable sets  $s$  of sentences of  $K_{AP}$  which satisfies the following conditions for each  $s \in S$ :

- (C<sub>1</sub>) (Triviality rule)  $\emptyset \in S$ ;
- (C<sub>2</sub>) (Consistency rule) Either  $\varphi \notin s$  or  $\neg \varphi \notin s$ ;
- (C<sub>3</sub>) ( $\neg$ -rule) If  $\neg \varphi \in s$ , then  $s \cup \{\varphi \neg\} \in S$ ;
- (C<sub>4</sub>) ( $\bigwedge$ -rule) If  $\bigwedge \Phi \in s$ , then for all  $\varphi \in \Phi$ ,  $s \cup \{\varphi\} \in S$ ;
- (C<sub>5</sub>) ( $\bigvee$ -rule) If  $\bigvee \Phi \in s$ , then for some  $\varphi \in \Phi$ ,  $s \cup \{\varphi\} \in S$ ;
- (C<sub>6</sub>) ( $P$ -rule) If  $(P\vec{x} > 0)\varphi(\vec{x}) \in s$ , then for some  $\vec{c} \in C$ ,  $s \cup \{\varphi(\vec{c})\} \in S$ ;
- (C<sub>7</sub>) If  $\varphi(\vec{x}) \in K_{AP}$  is an axiom, then
  - (a)  $s \cup \{(P\vec{x} \geq 1)\varphi(\vec{x})\} \in S$ ,
  - (b)  $s \cup \{\varphi(\vec{c})\} \in S$ , where  $\vec{c} \in C$ .

**3.5.3. Model Existence Theorem.** If  $S$  is a consistency property, then any  $s_0 \in S$  has a weak model.

*Proof.* Let  $\varphi_0, \varphi_1, \varphi_2, \dots$  be an enumeration of the sentences of  $K_{AP}$ . We shall construct a sequence  $s_0 \subseteq s_1 \subseteq s_2 \subseteq \dots$  of elements of  $S$  as follows.  $s_0$  is given. Given  $s_n$  choose  $s_{n+1}$  to satisfy the following conditions:

- (1)  $s_n \subseteq s_{n+1}$ ;
- (2) If  $s_n \cup \{\varphi_n\} \in S$ , then  $\varphi_n \in s_{n+1}$ ;
- (3) If  $s_n \cup \{\varphi_n\} \in S$ ,  $\varphi_n = \bigvee \Phi$ , then for some  $\theta \in \Phi$ ,  $\theta \in s_{n+1}$ ;
- (4) If  $s_n \cup \{\varphi_n\} \in S$ ,  $\varphi_n = (P\vec{x} > 0)\psi(\vec{x})$ , then for some  $\vec{c} \in C$ ,  $\psi(\vec{c}) \in s_{n+1}$ .

We now define a model  $\mathfrak{A}$  of  $s_0$ . Let  $s_\omega = \bigcup_{n < \omega} s_n$ . Let  $T$  be the set of constants of  $K_{AP}$ . For  $c, d \in T$ , let  $c \sim d$  iff  $c = d \in s_\omega$ . Then,  $\sim$  is an equivalence relation. Let  $[c]$  denote the equivalence class of the constant  $c$ . Let  $\mathfrak{A}$  have the universe set  $A = \{[c] : c \in T\}$ . If  $R$  is an  $n$ -placed relation symbol and  $c_1, \dots, c_n \in C$ , then

$$\mathfrak{A} \models R([c_1], \dots, [c_n]) \quad \text{iff} \quad R(c_1, \dots, c_n) \in s_\omega.$$

Define  $\mu_n$  on the subsets of  $A^n$  definable by formulas of  $L_{AP}$  with parameters from  $A$ , by

$$\mu_n\{\vec{a} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\} = \sup\{r : (P\vec{x} \geq r)\varphi(\vec{x}, \vec{c}) \in s_\omega\}.$$

<sup>3</sup> In this way we avoid an important notation of countable fragment.

It is not difficult to show that everything is well-defined,  $\mu_n$ 's are finitely additive probability measures, and it is routine to check that

$$\mathfrak{A} \models \varphi[[c_1], \dots, [c_n]] \quad \text{iff} \quad \varphi(c_1, \dots, c_n) \in s_\omega.$$

Therefore  $\mathfrak{A}$  is a weak model of  $s_\omega$ , and hence a model of  $s_0$ .  $\square$

**3.5.4. Weak Completeness Theorem.** *A set  $T$  of sentences of  $L_{AP}$  has a weak model if and only if  $T$  is consistent in weak  $L_{AP}$ .*

*Proof.* Let  $S$  be the set of all countable sets  $s$  of sentences of  $K_{AP}$  such that only finitely many  $c \in C$  occur in  $s$ , and not  $\vdash_{K_{AP}} \neg \bigwedge s$ . We claim that  $S$  is a consistency property. We check that  $S$  satisfies  $(C_6)$ .

Let  $(P\vec{x} > 0)\varphi(\vec{x}) \in s$  but for all  $\vec{c} \in C$ ,  $s \cup \{\varphi(\vec{c})\} \notin S$ . Take a  $\vec{c} \in C$  which does not occur in  $s$ . Then  $\vdash_{K_{AP}} \neg \bigwedge (s \cup \{\varphi(\vec{c})\})$ , hence  $\vdash_{K_{AP}} \neg (\bigwedge s \wedge \varphi(\vec{c}))$  and  $\vdash_{K_{AP}} \bigwedge s \rightarrow \neg \varphi(\vec{c})$ . Let  $\vec{y}$  be a tuple of variables not occurring in  $s$ . Then replacing  $\vec{c}$  by  $\vec{y}$  in the proof and using axioms and rule  $(G)$  we get  $\vdash_{K_{AP}} \bigwedge s \rightarrow \neg (P\vec{y} > 0)\varphi(\vec{y})$  and hence  $\vdash_{K_{AP}} \neg \bigwedge s$ . A contradiction.  $\square$

**3.5.5. Corollary.** *If  $\varphi$  is a sentence of  $L_{AP}$ , then  $\vdash_{K_{AP}} \varphi$  iff  $\models \varphi$ .  $\square$*

### 3.6. Graded models

Hoover has found an example of a graded probability model which is not  $L_{AP}$ -equivalent to any ordinary probability model (see [34]). Nevertheless, a very important class of hyperfinite graded models can be obtained from a counting measure.

**3.6.1. Theorem** (Keisler). *Let  $A$  be a  $*$ -finite set. For each  $n$ , let  $\mu_n$  be the internal counting measure on  $A^n$ , and let  $L(\mu_n)$  be the corresponding Loeb measure. Then  $\langle L(\mu_n) \mid n \in \mathbb{N} \rangle$  has the Fubini property. Hence, if each  $n$ -ary relation of  $\mathfrak{A}$  is  $L(\mu_n)$ -measurable, then  $\mathfrak{A}$  is a graded probability model.  $\square$*

It follows from the above that the following result obtained by Hoover (see [22]) is quite important.

**3.6.2. Graded Completeness Theorem.** *Every countable set  $T$  of sentences which is consistent in graded  $L_{AP}$  has a graded model.*

*Proof.* Let  $V(S)$  be a superstructure over  $S$  and  $\mathbb{R} \cup A \subseteq S$ . We suppose that a formula  $\varphi(\vec{x}, \vec{a})$  with parameters from  $A$ , a weak model  $\mathfrak{A}$  of  $T$ , and the relation  $\models$  are represented by sets in  $V(S)$ . Then  ${}^*\varphi(\vec{x}, \vec{a})$  and  ${}^*\mathfrak{A}$  are sets in the nonstandard universe  $V({}^*S)$ , and  ${}^*\models$  is an internal relation. If the context is clear we write simply  $\models$ .

$\langle ({}^*A)^n, L(\mu_n) \rangle$  is a probability space by Loeb's theorem (see [39]). The model  $\langle {}^*A, {}^*R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, L(\mu_n) \rangle_{i \in I, j \in J, n \in \mathbb{N}}$  (shortly  $\langle {}^*\mathfrak{A}, L(\mu_n) \rangle$ ) is graded because of the fact that the weak model  $\mathfrak{A}$  is a model for graded  $L_{AP}$ .

The main step in our proof is to show that for each  $\varphi(\vec{x}) \in L_{AP}$  and  $\vec{a} \in A$

$$\langle \mathfrak{A}, \mu_n \rangle \models \varphi[\vec{a}] \quad \text{iff} \quad \langle {}^*\mathfrak{A}, L(\mu_n) \rangle \models \varphi[\vec{a}]. \quad (*)$$

To prove  $(*)$ , we prove by induction on formulas that for  $\varphi(\vec{x}, \vec{y}) \in L_{AP}$ ,  $\vec{a} \in A$

$$\begin{aligned} L(\mu_n) \{ \{ \vec{e} \in ({}^*A)^n : \langle {}^*\mathfrak{A}, L(\mu_n) \rangle \models \varphi(\vec{e}, \vec{a}) \} \Delta \\ \Delta \{ \vec{e} \in ({}^*A)^n : \langle {}^*\mathfrak{A}, \mu_n \rangle \models \varphi(\vec{e}, \vec{a}) \} \} = 0. \end{aligned}$$

From now on we shall suppress parameters from  $A$ .

The nontrivial steps in our induction are conjunction and quantification.

Case 1.  $\varphi(\vec{x}) = \bigwedge_n \psi_n(\vec{x})$ .

Then  $*(\bigwedge_n \psi_n) = \bigwedge_{n \in \mathbb{N}} * \psi_n \neq \bigwedge_{n \in \mathbb{N}} \psi_n$ . By Proposition 3.4.7. (10) we have:

$$t = L(\mu_n) \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models * (\bigwedge_m \psi_m(\vec{e})) \Delta \bigwedge_m * \psi_m(\vec{e}) \} = 0.$$

The induction step follows by the triangle argument and the induction hypothesis:

$$\begin{aligned} L(\mu_n) \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models * (\varphi(\vec{e})) \Delta \varphi(\vec{e}) \} &\leq \\ &\leq t + \sum_m L(\mu_m) \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models * (\psi_m(\vec{e})) \Delta \psi_m(\vec{e}) \} = 0. \end{aligned}$$

Case 2.  $\varphi(\vec{x}) = (P\vec{y} \geq r) \psi(\vec{x}, \vec{y})$ .

Then we have  $*(P\vec{x} \geq r) \psi(\vec{x}) \leftrightarrow * \mu_n \{ \vec{a} : * \psi(\vec{a}) \} \geq r$  and

$$(P\vec{x} \geq r) \psi(\vec{x}) \leftrightarrow (\forall n \in \mathbb{N}) (* \mu_n \{ \vec{a} : * \psi(\vec{a}) \} \geq r - 1/n).$$

By the triangle argument:

$$\begin{aligned} &L(\mu_n) \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models * \varphi(\vec{e}) \Delta \varphi(\vec{e}) \} \leq \\ &\leq L(\mu_n) \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models * ((P\vec{y} \geq r) \psi(\vec{e}, \vec{y})) \Delta (P\vec{y} \geq r) * \psi(\vec{e}, \vec{y}) \} + \\ &+ L(\mu_n) \{ \{ \vec{e} \in (*A)^n : * \langle \mathfrak{A}, L(\mu_m) \rangle \models (P\vec{y} \geq r) * \psi(\vec{e}, \vec{y}) \} \Delta \\ &\Delta \{ \vec{e} \in (*A)^n : \langle * \mathfrak{A}, L(\mu_m) \rangle \models (P\vec{y} \geq r) \psi(\vec{e}, \vec{y}) \} \}. \end{aligned}$$

The first term is 0 by Proposition 3.4.7. (9). By applying the induction hypothesis:

$$\begin{aligned} &L(\mu_{n+m}) \{ \{ (\vec{e}, \vec{c}) \in (*A)^{n+m} : * \langle \mathfrak{A}, \mu_k \rangle \models * \psi(\vec{e}, \vec{c}) \} \Delta \\ &\Delta \{ (\vec{e}, \vec{c}) \in (*A)^{n+m} : \langle * \mathfrak{A}, L(\mu_k) \rangle \models \psi(\vec{e}, \vec{c}) \} \} = 0. \end{aligned}$$

So, for all  $\vec{e}$ 's but a set of  $L(\mu_n)$ -measure 0 we have:

$$\begin{aligned} &L(\mu_m) \{ \{ \vec{c} \in (*A)^m : * \langle \mathfrak{A}, \mu_k \rangle \models * \psi(\vec{e}, \vec{c}) \} \Delta \\ &\Delta \{ \vec{c} \in (*A)^m : \langle * \mathfrak{A}, L(\mu_k) \rangle \models \psi(\vec{e}, \vec{c}) \} \} = 0. \end{aligned}$$

So, for all  $\vec{e}$ 's but a set of  $L(\mu_n)$ -measure 0 we have:

$$L(\mu_m) \{ \vec{c} \in (*A)^m : * \langle \mathfrak{A}, \mu_k \rangle \models * \psi(\vec{e}, \vec{c}) \} \geq r$$

iff

$$L(\mu_m) \{ \vec{c} \in (*A)^m : \langle * \mathfrak{A}, L(\mu_k) \rangle \models \psi(\vec{e}, \vec{c}) \} \geq r.$$

Hence the second term in the inequality is 0.

Now the theorem follows from (\*).  $\square$

### 3.7. Probability models

Now, we shall prove the main completeness theorem. First, we need the following two lemmas.

**3.7.1. Lemma.** *Let  $\lambda$ ,  $\nu$  and  $\mu$  be probability measures on  $A$ ,  $B$  and  $A \times B$  such that  $\lambda \times \nu \subseteq \mu$ . Let  $T$  be  $\mu$ -measurable. Then, for each  $\varepsilon > 0$ , there is a finite union  $M$  of  $\lambda \times \nu$ -measurable rectangles such that  $\mu(T \Delta M) < \varepsilon$  iff there is a  $\lambda \times \nu$ -measurable set  $N$  such that  $\mu(T \Delta N) = 0$ .  $\square$*

The main idea of the proof is to use the Monotone Class Theorem (see [31]).

**3.7.2. Rectangle Approximation Lemma.** *Let  $\mathfrak{A}$  be a graded probability structure satisfying axiom (K). Then for each  $\varepsilon > 0$  and a formula  $\varphi(\vec{x})$  of  $L_{AP}$ , there are finitely many formulas  $\psi_{ij}(\vec{y}, x_j)$ , where  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , such that*

$$\mathfrak{A} \models (P\vec{y} > 0)(P\vec{x} > 1 - \varepsilon)(\varphi(\vec{x}) \leftrightarrow \bigvee_{i=1}^m \bigwedge_{j=1}^n \psi_{ij}(\vec{y}, x_j)). \quad \square$$

The lemma says that any definable set  $\varphi(\vec{x})$  in  $\mathfrak{A}$  can be approximated within  $\varepsilon$  by a finite union of definable rectangles, and this can be done uniformly in parameters  $\vec{y}$  from a set of positive measure. In the proof, axiom (K) is used  $n$  times.

**3.7.3. Soundness and Completeness Theorem for full  $L_{AP}$ .** *A set of sentences  $T$  of the full  $L_{AP}$  has a probability model if and only if  $T$  is consistent in the full  $L_{AP}$ .*

*Sketch of the proof.* In order to prove the Soundness the only difficulty lies in checking the validity of the axiom (K). By Proposition 3.4.7. (3) it suffices to show that for each  $p, q < 1$ , the formula

$$(P\vec{x} \geq p)(P\vec{y} > 0)(P\vec{z} \geq q)(\varphi(\vec{x}, \vec{z}) \leftrightarrow \varphi(\vec{y}, \vec{z}))$$

is valid. This can be proven using Lemma 3.7.1. and Fubini's Theorem.

Let us prove the second part of the theorem. Let  $\mathfrak{A}$  be a graded model of  $T$ . Use the Rectangle Approximation Lemma in order to find an ordinary probability model  $\mathfrak{B}$  such that  $\mathfrak{B}$  is  $L_{AP}$ -equivalent to  $\mathfrak{A}$ . Models  $\mathfrak{A}$  and  $\mathfrak{B}$  have the same universe, constants and measures. For each  $R^{\mathfrak{A}}$  and  $\varepsilon > 0$  there is a finite union  $M$  of  $\mu^n$ -measurable rectangles such that  $\mu_n(M \Delta R^{\mathfrak{A}}) < \varepsilon$ . Then, by Lemma 3.7.1. there is a  $\mu^n$ -measurable (and also  $\mu^{(n)}$ -measurable)  $R^{\mathfrak{B}}$  such that  $\mu_n(R^{\mathfrak{A}} \Delta R^{\mathfrak{B}}) = 0$ .

By induction on  $\varphi$  we can show that

$$\mathfrak{A} \models \varphi[\vec{a}] \quad \text{iff} \quad \mathfrak{B} \models \varphi[\vec{a}]$$

for  $\mu_n$ -almost all  $\varphi$ . It follows that  $\mathfrak{B} \models T$ .  $\square$

By usual arguments<sup>4</sup> one can obtain the following theorem.

**3.7.4. Barwise Completeness Theorem.** *The set of all valid sentences of  $L_{AP}$  is  $\Sigma$  on  $\mathcal{A}$ .  $\square$*

<sup>4</sup> Similar as for the logic  $L_A$ .

### 3.8. Atomic and atomless models

In this part we assume that  $L$  has no constants symbols.

**3.8.1. Definition.** Let  $\mathfrak{A}$  be a probability model. An element  $a \in A$  is an atom if  $\{a\}$  has positive measure.  $\mathfrak{A}$  is atomic if every element is an atom.

Atomic models are countable.

**3.8.2. Completeness Theorem for Atomic  $L_{AP}$ .** A countable set of sentences  $T$  of the atomic  $L_{AP}$  has a model if and only if  $T$  is consistent in the atomic  $L_{AP}$ .

*Sketch of the proof.* Suppose  $T$  is consistent in the atomic  $L_{AP}$ . By the Weak Completeness Theorem there is a weak model  $\mathfrak{A}$  for  $T$  in which all theorems of atomic  $L_{AP}$  hold. From Proposition 3.4.7. the following is deducible (from axiom (A)) in the atomic  $L_{AP}$

$$\bigwedge_m \bigvee_n (Px > 1 - 1/m)(Py \geq 1/n) \ x=y.$$

It follows that  $A$  has a finite subset of  $\mu_1$ -measure greater than  $1 - 1/m$ .

Let  $\mathfrak{B}$  be a model with the same universe and relations as  $\mathfrak{A}$ , and with a probability measure  $\mu$  defined on all subsets of  $A$  by

$$\mu(X) = \sup\{\mu_1(Y) : Y \subseteq X, Y \text{ finite}\}.$$

It is easy to see that  $\mathfrak{B} \equiv_{L_{AP}} \mathfrak{A}$ .

Let  $N$  be a set of atoms of  $\mathfrak{A}$ . Then  $N$  generates a model  $\mathfrak{N}$  such that  $\mathfrak{N} \prec_{L_{AP}} \mathfrak{B}$ . The atomic model  $\mathfrak{N}$  is the required model of  $T$ .  $\square$

For atomic models (without constants) we can show that the ordinary quantifiers can be defined in terms of probability quantifiers in the following way:

$$\mathfrak{A} \models (\forall x)\varphi(x, \vec{y}) \leftrightarrow (Px \geq 1)\varphi(x, \vec{y}) \quad \text{and} \quad \mathfrak{A} \models (\exists x)\varphi(x, \vec{y}) \leftrightarrow (Px > 0)\varphi(x, \vec{y}).$$

If we consider the general case where  $L$  has constant symbols, then we must either add new axioms of the type  $(Px > 0) x=c$ , or define  $\mathfrak{A}$  to be atomic if every element of  $A$  is either of positive measure or is equal to a constant symbol. The proof of the completeness proceeds in a similar way.

Now we shall assume that  $L$  has only finitely many constant symbols.

Quite different from the atomic  $L_{AP}$  is the atomless  $L_{AP}$  logic (see the Definition 3.4.5). Of course, its models are atomless and uncountable.

The next theorem says that we can reduce atomless models to hyperfinite models. For the proof we need the Law of Large Numbers and the Normal Form Theorem, facts which we shall mention in the next section.

**3.8.3. Existence Theorem for Hyperfinite Models.** Let  $\mathfrak{A}$  be an atomless probability model for  $L$ , and let  $H$  be an infinite  $\ast$ -finite set. Then there exists a hyperfinite probability model  $\mathfrak{N}$  with universe  $H$  which is  $L_{AP}$ -equivalent to  $\mathfrak{A}$ .

The completeness theorem is an easy consequence.

**3.8.4. Completeness Theorem for Atomless  $L_{AP}$ .** A countable set  $T$  of sentences of the atomless  $L_{AP}$  has a hyperfinite probability model if and only if  $T$  is consistent in the atomless  $L_{AP}$ .  $\square$

### 3.9. Model theory for $L_{AP}$ logic

In this part we shall give a short overlook of the model theory of  $L_{AP}$ . All statements are given without proofs.

First, we state a model-theoretic form of the law of large numbers with no first-order analog. The main consequence of this law is the Normal Form Theorem. Further, we discuss hyperfinite models which play the role in  $L_{AP}$  that saturated models play in classical model theory. These models are used to prove (see [34]) the Robinson consistency and the Craig interpolation theorem for  $L_{AP}$ . Finally, we consider questions about the Hanf and Löwenheim numbers.

The results in this section hold for all graded probability structures.

To state the laws of large numbers we need the notion of a finite sample of  $\mathfrak{A}$ .

**3.9.1. Definition.** Let  $\langle \mathfrak{A}, \mu_n \rangle$  be a graded probability model for  $L$ , and let  $\vec{a}_k = (a_1, \dots, a_k) \in A^k$  be a  $k$ -tuple of elements of  $A$ . The *finite sample*  $\mathfrak{A}(\vec{a}_k)$  is the probability model whose universe is the union of  $\{a_1, \dots, a_k\}$  and the constants of  $\mathfrak{A}$ , whose first-order part is a submodel of  $\langle \mathfrak{A}, \mu_n \rangle$ , and whose measure  $\nu$  is given by

$$\nu(S) = \frac{|\{m \leq k : a_m \in S\}|}{k}.$$

**3.9.2. Definition.** A *finite existential formula* of  $L_{AP}$  is a formula of the form

$$(P\vec{x}_1 > r_1) \dots (P\vec{x}_n > r_n)\psi,$$

where  $\psi$  is a finite quantifier-free formula of  $L$ .

A *finite universal formula* of  $L_{AP}$  is a formula of the form

$$(P\vec{x}_1 \geq r_1) \dots (P\vec{x}_n \geq r_n)\psi,$$

where  $\psi$  is as before.

**3.9.3. Theorem.** Let  $\langle \mathfrak{A}, \mu_n \rangle$  be a graded probability model for  $L$  and let  $\varphi$  be a finite existential sentence of  $L_{AP}$  such that  $\langle \mathfrak{A}, \mu_n \rangle \models \varphi$ . Let  $\mu_{\mathbb{N}}$  be the completion of the measure on  $A^{\mathbb{N}}$  determined by the  $\mu_n$ 's. Then:

- (1) Weak Law of Large Numbers for  $L_{AP}$ :

$$\lim_{k \rightarrow \infty} \mu_k \{ \vec{a}_k \in A^k : \mathfrak{A}(\vec{a}_k) \models \varphi \} = 1.$$

- (2) Strong Law of Large Numbers for  $L_{AP}$ : For  $\mu_{\mathbb{N}}$  almost all sequences  $\vec{a} \in A^{\mathbb{N}}$ ,  $\mathfrak{A}(\vec{a}_k) \models \varphi$  for all but finitely many  $k \in \mathbb{N}$ .  $\square$

**3.9.4. Normal Form Theorem** (Hoover [23]). Every formula  $\varphi(\vec{x})$  of graded  $L_{\omega_1 P}$  is equivalent to a countable Boolean combination of formulas of the form  $(P\vec{y} \geq r)\psi(\vec{x}, \vec{y})$ , where  $\psi(\vec{x}, \vec{y})$  is a finite conjunction of atomic formulas of  $L$ .  $\square$

**3.9.5. Corollary.** Let  $\langle \mathfrak{A}, \mu_n \rangle$  and  $\langle \mathfrak{B}, \nu_n \rangle$  be graded probability structures for  $L$ . The following are equivalent:

- (1)  $\langle \mathfrak{A}, \mu_n \rangle \equiv_{\omega_1 P} \langle \mathfrak{B}, \nu_n \rangle$ ;
- (2)  $\langle \mathfrak{A}, \mu_n \rangle \equiv_{AP} \langle \mathfrak{B}, \nu_n \rangle$ ;
- (3)  $\langle \mathfrak{A}, \mu_n \rangle \models \varphi$  iff  $\langle \mathfrak{B}, \nu_n \rangle \models \varphi$ , for each sentence  $\varphi$  of  $L_{AP}$  in the normal form of the Theorem 3.9.4.  $\square$

Before we proceed further on we recommend the reader to read again Definition 3.2.4.

The following list of definitions and theorems is given in [34] with more details.

**3.9.6. Definition.** Let  $\langle \mathfrak{A}, \mu_n \rangle$  be a hyperfinite graded model. A *lifting* of  $\langle \mathfrak{A}, \mu_n \rangle$  is a  $*$ -finite probability model  $\langle \mathfrak{B}, \nu_n \rangle$  such that  $\mathfrak{B}$  has the same universe and constants as  $\mathfrak{A}$ , and for each atomic formula  $\varphi(\vec{x})$ , the set

$$\{ \vec{a} : \langle \mathfrak{A}, \mu_n \rangle \models \varphi[\vec{a}] \text{ iff } \langle \mathfrak{B}, \nu_n \rangle \models \varphi[\vec{a}] \}$$

has  $\mu_n$ -measure one.

**3.9.7. Theorem.** (1) Every infinite  $*$ -finite probability model is a lifting of some hyperfinite graded model.

(2) Every hyperfinite graded model has a lifting.  $\square$

**3.9.8. Definition.** Let  $\langle \mathfrak{A}, \mu \rangle$  and  $\langle \mathfrak{B}, \nu \rangle$  be probability models. An almost sure isomorphism from  $\langle \mathfrak{A}, \mu \rangle$  to  $\langle \mathfrak{B}, \nu \rangle$  (in symbols,  $h: \langle \mathfrak{A}, \mu \rangle \cong \langle \mathfrak{B}, \nu \rangle$  a.s.) is a bijection  $h: A \rightarrow B$  such that:

- (i)  $h$  is an isomorphism on the probability spaces,  $h: \langle A, \mu \rangle \cong \langle B, \nu \rangle$ ;
- (ii) For each atomic formula  $\varphi(\vec{x})$ ,

$$\langle \mathfrak{A}, \mu^{(n)} \rangle \models \varphi[\vec{a}] \text{ iff } \langle \mathfrak{B}, \nu^{(n)} \rangle \models \varphi[h\vec{a}]$$

almost surely in  $\mu^{(n)}$ .

**3.9.9. Theorem.** Let  $\langle \mathfrak{A}_1, \mu_n^1 \rangle$  and  $\langle \mathfrak{A}_2, \mu_n^2 \rangle$  be hyperfinite probability models with the same universe  $A$ . Then, the following are equivalent:

- (1) There is an  $h: \langle \mathfrak{A}_1, \mu_n^1 \rangle \cong \langle \mathfrak{A}_2, \mu_n^2 \rangle$  a.s.;
- (2)  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are  $L_{AP}$ -equivalent;
- (3) There is an internal  $h$  such that  $h: \langle \mathfrak{A}_1, \mu_n^1 \rangle \cong \langle \mathfrak{A}_2, \mu_n^2 \rangle$  a.s.  $\square$

This theorem is false for hyperfinite graded models (see [34]).

The following three results are due to Hoover (see [22],[25]). Below, by an  $L_{AP}$  logic we mean either the graded  $L_{AP}$  or the full  $L_{AP}$ .

**3.9.10. Robinson Consistency Theorem for  $L_{AP}$ .** Let  $L_1$  and  $L_2$  be two languages with  $L_0 = L_1 \cap L_2$ . Let  $\langle \mathfrak{A}_1, \mu_n^1 \rangle, \langle \mathfrak{A}_2, \mu_n^2 \rangle$  be probability models for  $L_1, L_2$  respectively, whose reducts  $\langle \mathfrak{A}_1 \upharpoonright L_0, \mu_n^1 \rangle, \langle \mathfrak{A}_2 \upharpoonright L_0, \mu_n^2 \rangle$  are  $L_{0AP}$ -equivalent. Then there exists a probability model  $\langle \mathfrak{B}, \nu_n \rangle$  for  $L_1 \cup L_2$  such that

$$\langle \mathfrak{B} \upharpoonright L_1, \nu_n \rangle \equiv_{L_1 AP} \langle \mathfrak{A}_1, \mu_n^1 \rangle \text{ and } \langle \mathfrak{B} \upharpoonright L_2, \nu_n \rangle \equiv_{L_2 AP} \langle \mathfrak{A}_2, \mu_n^2 \rangle. \quad \square$$

**3.9.11. Craig Interpolation Theorem for  $L_{AP}$ .** Let  $L_0 = L_1 \cap L_2$  and let  $\varphi_1 \in L_{1,AP}$ , and  $\varphi_2 \in L_{2,AP}$  be sentences such that  $\models \varphi_1 \rightarrow \varphi_2$ . Then there is a sentence  $\varphi_0 \in L_{0,AP}$  such that  $\models \varphi_1 \rightarrow \varphi_0$ , and  $\models \varphi_0 \rightarrow \varphi_2$ .  $\square$

The following theorem is much stronger than the previous one.

**3.9.12. Almost Sure Interpolation Theorem.** Let  $L_0 = L_1 \cap L_2$  and suppose that  $L_0$  has a well-ordering in  $\mathcal{A}$ . Let  $\varepsilon > 0$  and let  $\varphi_1(\vec{x}) \in L_{1,AP}$  and  $\varphi_2(\vec{x}) \in L_{2,AP}$  be formulas such that

$$\models (P\vec{x} \geq 1 - \varepsilon)(\varphi_1(\vec{x}) \rightarrow \varphi_2(\vec{x})).$$

Then, for every  $\delta > \sqrt[3]{\varepsilon} + \sqrt{\varepsilon}$ , there is a formula  $\varphi_0 \in L_{0,AP}$  such that

$$\models (P\vec{x} \geq 1 - \delta)(\varphi_1(\vec{x}) \rightarrow \varphi_0(\vec{x})) \quad \text{and} \quad \models (P\vec{x} \geq 1 - \delta)(\varphi_0(\vec{x}) \rightarrow \varphi_2(\vec{x})).$$

Moreover, if

$$\models (P\vec{x} \geq 1)(\varphi_1(\vec{x}) \rightarrow \varphi_2(\vec{x})),$$

then there is a formula  $\varphi_0(\vec{x}) \in L_{0,AP}$  such that

$$\models (P\vec{x} \geq 1)(\varphi_1(\vec{x}) \rightarrow \varphi_0(\vec{x})) \quad \text{and} \quad \models (P\vec{x} \geq 1)(\varphi_0(\vec{x}) \rightarrow \varphi_2(\vec{x})). \quad \square$$

Now, we shall be dealing with Löwenheim-Skolem theorems. It is easy to see that for a given probability structure we can produce  $L_{\omega_1 P}$ -equivalent models of arbitrarily large cardinality by simply adding a set of new elements of measure zero. So, for this reason we introduce the following notation.

**3.9.13. Definition.** We say that a probability model  $\mathfrak{A}$  is *reasonable* if every set of measure one has the same cardinality as  $A$  does.

**3.9.14. Downward Löwenheim-Skolem Theorem.** Let  $\langle \mathfrak{A}, \mu_n \rangle$  be a reasonable probability model of power at least  $\lambda$ , where  $\lambda = \lambda^\omega$ . Then, for every set  $X \subseteq A$  of power  $\leq \lambda$ ,  $\mathfrak{A}$  has a reasonable  $L_{\omega_1 P}$ -elementary submodel  $\langle \mathfrak{B}, \nu_n \rangle$  of power  $\lambda$  with  $X \subseteq B$ .  $\square$

**3.9.15. Upward Löwenheim-Skolem Theorem.** Every uncountable reasonable probability model  $\langle \mathfrak{A}, \mu_n \rangle$  has reasonable  $L_{\omega_1 P}$ -elementary extensions of arbitrarily large cardinality.  $\square$

The compactness theorem fails for  $L_{AP}$ .

**3.9.16. Example.** Let  $T$  be the set of sentences containing  $(Px > 0)R(x)$  and  $(Px \leq 1/n)R(x)$ , for  $n = 1, 2, \dots$ . Then, every finite subset of  $T$  has a model, but  $T$  itself has no model.

However, the compactness theorem holds for some special classes of sentences.

**3.9.17. Definition.** The set of universal conjunctive formulas of  $L_{\omega_1 P}$  is the least set containing all quantifier-free formulas and which is closed under arbitrary  $\bigwedge$ , finite  $\bigvee$ , and quantifiers  $(P\vec{x} \geq r)$ .

**3.9.18. Finite Compactness Theorem.** *Let  $T$  be a set of universal conjunctive sentences of  $L_{AP}$ . If every finite subset of  $T$  has a graded model, then  $T$  has a graded model.*  $\square$

The above theorem does not hold for probability models. However, one can prove the following Barwise-type theorem for a countable admissible set  $\mathcal{A}$ .

**3.9.19. Barwise Compactness Theorem.** *Let  $T$  be a  $\Sigma$ -definable set of sentences of  $\mathcal{A}$ . If every  $T_0 \subseteq T$  such that  $T_0 \in \mathcal{A}$  has a model, then  $T$  has a model.*  $\square$

### 3.10. Finitary probability logic

In the case when  $\omega$  is not an element of  $\mathcal{A}$  each formula of  $L_{AP}$  is finite. Here, we assume that the rationals are defined in such a way that  $\mathbb{Q} \subseteq \mathcal{A}$ . So  $L_{AP}$  has at least quantifiers  $(P\vec{x} \geq r)$ ,  $r \in \mathbb{Q} \cap [0, 1]$ .

The Archimedean axiom ( $W_6$ ) must be replaced by a new infinite rule of inference

$$\frac{\psi \rightarrow (P\vec{x} \geq r)(P\vec{y} \geq s - 1/n)\varphi, \quad n \in \mathbb{N}}{\psi \rightarrow (P\vec{x} \geq r)(P\vec{y} \geq s)\varphi}.$$

The infinitary axiom ( $H_1$ ) and the infinite conjunction rule become finite. Hoover [22] has shown that the weak, graded and full completeness theorems hold in this case.

### 3.11. Random variables

Now, we discuss models with random variables. We have already seen that structures with random variables are very important in probability theory.

**3.11.1. Definition.** An  $n$ -fold *random variable* on a probability space  $\langle A, \mathcal{F}, \mu \rangle$  is a  $\mu^{(n)}$ -measurable function  $X: A^n \rightarrow \mathbb{R}$ .

Fix a language  $L = \{X_i, c_j, R_k : i \in I, j \in J, k \in K\}$ .

**3.11.2. Definition.** The auxiliary language of  $L$  is the language  $L^X$  which has the same constant symbols  $c_j$  and relation symbols  $R_k$  of  $L$ , but has new relation symbols  $[X_i(\vec{x}) \geq r]$  and  $[X_i(\vec{x}) \leq r]$  for each  $i \in I$  and  $r \in \mathbb{Q}$ .

**3.11.3. Definition.** We shall use the following abbreviations:

$$[X(\vec{x}) < r] \text{ for } \neg[X(\vec{x}) \geq r] \quad \text{and} \quad [X(\vec{x}) > r] \text{ for } \neg[X(\vec{x}) \leq r].$$

The language  $L_{AP}(\mathbb{R})$  has the same set of formulas as  $L_{AP}^X$ . The axioms and rules of inference are as in  $L_{AP}^X$ , plus four new axioms, where  $r, s \in \mathbb{Q}$ :

- (RV<sub>1</sub>)  $[X_i(\vec{x}) \geq r] \rightarrow [X_i(\vec{x}) \geq s]$ , where  $r \geq s$ ;
- (RV<sub>2</sub>)  $[X_i(\vec{x}) > r] \leftrightarrow \bigvee_n [X_i(\vec{x}) \geq r + 1/n]$ ;
- (RV<sub>3</sub>)  $[X_i(\vec{x}) \geq r] \leftrightarrow \bigwedge_n [X_i(\vec{x}) \geq r - 1/n]$ ;
- (RV<sub>4</sub>)  $\bigvee_n ([X_i(\vec{x}) \geq -n] \wedge [X_i(\vec{x}) \leq n])$ .

**3.11.4. Definition.** A random variable model for  $L$  is a structure  $\langle \mathfrak{A}, \mu \rangle$ , where  $\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, R_k^{\mathfrak{A}} \rangle_{i \in I, j \in J, k \in K}$  is a first-order structure,  $\mu$  is a probability measure on  $A$ ,  $X_i^{\mathfrak{A}}$  is an  $n_i$ -fold random variable,  $c_j^{\mathfrak{A}} \in A$ ,  $R_k^{\mathfrak{A}}$  is  $\mu^{(n_i)}$ -measurable, and each singleton is measurable.

The satisfaction relation is defined as one expects:

$$\langle \mathfrak{A}, \mu \rangle \models [X_i(\vec{x}) \geq r][\vec{a}] \quad \text{iff} \quad X_i^{\mathfrak{A}}(\vec{a}) \geq r.$$

**3.11.5. Soundness and Completeness Theorem for  $L_{AP}(\mathbb{R})$ .** A countable set  $T$  of sentences of  $L_{AP}(\mathbb{R})$  has a random variable model if and only if it is consistent in  $L_{AP}(\mathbb{R})$ .

*Proof.* The soundness is trivial. Suppose  $T$  is consistent. Let  $\Sigma$  be the set of sentences of the form  $(P\vec{x} \geq 1)\psi$ , where  $\psi$  is one of the  $RV$  axioms. It is easy to see that  $T \cup \Sigma$  is consistent and has probability model  $\mathfrak{A}'$ . We can get random variable model  $\mathfrak{A}$  by defining

$$X_i^{\mathfrak{A}}(\vec{a}) = \sup \{ r \in \mathbb{Q} : \mathfrak{A}' \models [X_i(\vec{x}) \geq r][\vec{a}] \}.$$

Using  $\Sigma$  one shows that  $X_i^{\mathfrak{A}}$  is uniquely determined and almost surely finite.  $\square$

## LOGICS WITH INTEGRALS AND CONDITIONAL EXPECTATIONS

The development of probability model theory has engendered the need for the study of logics with a stronger expressive power than that of the logic  $L_{AP}$ . The logic  $L_{Af}$ , introduced by Keisler in [34] as an equivalent of the logic  $L_{AP}$ , allows us to express many properties of random variables in an easier way. In this logic the quantifiers  $\int \dots dx$  are incorporated instead of the quantifiers  $(Px \geq r)$ .

The completeness proof for  $L_{Af}$  used the Loeb construction of the Daniell integral as given in Chapter 1. We shall consider the logic  $L(f)_A$  which is a modification of the logic  $L_{Af}$  introduced by Hoover in [27]. The main reason is that a well-known measure-theoretic construction produces a model whose universe is an analytic subset of the real line, with a measure on the relative Borel sets, directly from the weak model, without using methods of nonstandard analysis.

The logics  $L_{Af}$  and  $L(f)_A$  are not rich enough to express probabilistic notions involving conditional expectations of random variables with respect to  $\sigma$ -algebras, such as martingale, Markov process, Brownian motion, stopping time, optional stochastic process, etc. These properties can be naturally expressed in a language with both integral quantifiers and conditional expectation operators. The logics  $L_{AE}$  and  $L_{ad}$  introduced by Keisler in [32], are appropriate for the study of random variables and stochastic processes. The model theory of these logics has been developed further by Hoover in [22], [23], Keisler in [34] and Fajardo in [14], [15].

In this chapter we shall consider some basic model-theoretic properties of  $L_{Af}$ ,  $L(f)_A$ ,  $L_{AE}$  and  $L_{ad}$ .

### 4.1. Logic with integral operators

The logic  $L_{Af}$  presented in this section is essentially equivalent to the logic  $L_{AP}$ . The main reason for developing a new logic and, more exactly, an extension of  $L_{Af}$  is that many properties of random variables and stochastic processes are usually easily expressible using integrals instead of probability quantifiers (see 4.4.8). First, we review the pertinent notions as given in [34].

Let  $L$  be an  $\Sigma$ -definable set of finitary relation and constant symbols which has countably many variables  $v_1, v_2, \dots$ ,  $n$ -ary connective  $\mathbb{F}$  for each continuous

function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n \geq 0$  such that  $F \upharpoonright \mathbb{Q}^n \in \mathcal{A}$ , and quantifiers  $\int \dots dx$  which build terms with bound variables.

**4.1.1. Definition.** The set of terms of  $L_{\mathcal{A}f}$  is the least set such that:

- (1) Each atomic term  $\mathbf{1}(R(\vec{x}))$  or  $\mathbf{1}(x = y)$  for each atomic formula of the first-order logic  $L_{\mathcal{A}}$ , is a term;
- (2) Each real number  $r \in \mathcal{A} \cap \mathbb{R}$  is a term;
- (3) If  $\tau$  is a term and  $x$  is a variable, then  $\int \tau dx$  is a term;
- (4) If  $\tau_1, \dots, \tau_n$  are terms, then  $\mathbb{F}(\tau_1, \dots, \tau_n)$  is a term.

We point out that the variable  $x$  is bounded in  $\int \tau dx$ . A *closed term* is a term with no free variable.

**4.1.2. Definition.** The set of formulas of  $L_{\mathcal{A}f}$  is the least set such that:

- (1) Each atomic formula  $\tau \geq 0$  for each term  $\tau$  of  $L_{\mathcal{A}f}$ , is a formula;
- (2) If  $\varphi$  is a formula, then so is  $\neg\varphi$ ;
- (3) If  $\Phi$  is a set of formulas with finitely many free variables and  $\Phi \in \mathcal{A}$ , then  $\bigwedge \Phi$  is a formula.

Therefore, the logic  $L_{\mathcal{A}f}$  has no quantifiers, but it does have integral operators which eliminate quantifiers from  $L_{\mathcal{A}P}$ , similarly as Skolem functions do in the classical logic. As in the first-order logic, a *sentence* is a formula with no free variables.

**4.1.3. Definition.** An interpretation of  $L_{\mathcal{A}f}$  in a probability structure  $\langle \mathfrak{A}, \mu \rangle$  is a function which assigns to each term  $\tau(\vec{x})$ , where  $\vec{x} = x_1, \dots, x_n$  is a tuple of distinct variables, a function  $\tau^{\mathfrak{A}}: A^n \rightarrow \mathbb{R}$  such that:

- (1)  $\mathbf{1}(R(\vec{x}))^{\mathfrak{A}}(\vec{a}) = \begin{cases} 1, & \text{if } R^{\mathfrak{A}}[\vec{a}] \\ 0, & \text{if } \neg R^{\mathfrak{A}}[\vec{a}] \end{cases}$ ,  $\mathbf{1}(x = y)^{\mathfrak{A}}(a, b) = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$ ;
- (2)  $r^{\mathfrak{A}} = r$ ;
- (3)  $(\int \tau(x, \vec{y}) dx)^{\mathfrak{A}}(\vec{a}) = \int \tau^{\mathfrak{A}}(b, \vec{a}) d\mu(b)$ ;
- (4)  $\mathbb{F}(\tau_1, \dots, \tau_m)^{\mathfrak{A}}(\vec{a}) = F(\tau_1^{\mathfrak{A}}(\vec{a}), \dots, \tau_m^{\mathfrak{A}}(\vec{a}))$ .

For every probability structure  $\langle \mathfrak{A}, \mu \rangle$  there exists an interpretation of  $L_{\mathcal{A}f}$  in  $\langle \mathfrak{A}, \mu \rangle$  because each function  $\tau^{\mathfrak{A}}$  is bounded and  $\mu^{(n)}$ -measurable.

The *satisfaction relation*  $\mathfrak{A} \models \varphi[\vec{a}]$  for a formula  $\varphi(\vec{x})$  of  $L_{\mathcal{A}f}$  and  $\vec{a} \in A^n$  is defined recursively exactly as for the first-order logic except for atomic formulas: If  $\tau(\vec{x}) \geq 0$  is atomic, then

$$\mathfrak{A} \models (\tau(\vec{x}) \geq 0)[\vec{a}] \quad \text{iff} \quad \tau^{\mathfrak{A}}(\vec{a}) \geq 0.$$

The following set of axioms and rules of inference of  $L_{\mathcal{A}f}$ , motivated by the results from the PhD thesis of Rodenhäusen [61], was introduced by Keisler in [34]. These axioms express natural properties of integrals.

- (E<sub>1</sub>) All axiom schemas for  $L_{\mathcal{A}}$  without quantifiers, where the atomic formula  $\mathbf{1}(x = y) = 1$  is incorporated instead of  $x = y$ ;
- (E<sub>2</sub>)  $\tau = 0 \vee \tau = 1$ , for each atomic term  $\tau$ ;

(E<sub>3</sub>) *Order axioms.*

$$(i) \ r \geq r,$$

$$(ii) \ \tau \geq r \rightarrow \tau \geq s, \text{ when } r \geq s;$$

(E<sub>4</sub>)  $\langle \tau_1, \dots, \tau_m \rangle \in S \rightarrow \mathbb{F}(\tau_1, \dots, \tau_m) \in [a, b]$ , for each rational closed rectangle  $S \subseteq \mathbb{R}^m$  with  $F(S) = [a, b]$ ;

(E<sub>5</sub>) *Integral axioms.*

$$(i) \ \int r \, dx = r,$$

$$(ii) \ \int \tau(x) \, dx = \int \tau(y) \, dy,$$

$$(iii) \ \iint \tau(x, y) \, dx dy = \iint \tau(x, y) \, dy dx,$$

$$(iv) \ \int (r \cdot \sigma + s \cdot \tau) \, dx = r \cdot \int \sigma \, dx + s \cdot \int \tau \, dx;$$

(E<sub>6</sub>) *Archimedean axiom.*

$$\tau > r \leftrightarrow \bigvee_n (\tau \geq r + 1/n);$$

(E<sub>7</sub>) *Product measurability.* For each  $m \in \mathbb{N}$ ,

$$\bigvee_k \int \mathbb{F}_k \left( 1 - \int \mathbb{F}_k \left( \int |\tau(\vec{x}, \vec{z}) - \tau(\vec{y}, \vec{z})| \, d\vec{z} \right) d\vec{y} \right) d\vec{x} \geq 1 - \frac{1}{m},$$

$$\text{where } F_k(a) = \begin{cases} 0, & \text{if } a \leq 1/k, \\ \text{linear}, & \text{if } 1/k \leq a \leq 2/k, \\ 1, & \text{if } a \geq 2/k. \end{cases}$$

The rules of inference for  $L_{\mathcal{A}f}$  are:

(F<sub>1</sub>)  $\varphi, \varphi \rightarrow \psi \vdash \psi$  (*Modus Ponens*);

(F<sub>2</sub>)  $\{\varphi \rightarrow \psi : \psi \in \Psi\} \vdash \varphi \rightarrow \bigwedge \Psi$  (*Conjunction*);

(F<sub>3</sub>)  $\varphi \rightarrow (\tau(x, \vec{y}) \geq 0) \vdash \varphi \rightarrow (\int \tau(x, \vec{y}) \, dx \geq 0)$ , where  $x$  is not free in  $\varphi$  (*Generalization*).

The proof of the Completeness Theorem for  $L_{\mathcal{A}f}$  falls into three parts.

First, a Henkin construction is used to get a *weak integral model*  $\langle \mathfrak{A}, I \rangle$  of a countable consistent set  $T$  of sentences of  $L_{\mathcal{A}f}$ , where  $\mathfrak{A}$  is a first-order model and  $I$  is what may be called an  $\mathcal{A}$ -Daniell integral on  $\mathfrak{A}$ ; i.e.,  $I$  is a function on the set of terms of  $L_{\mathcal{A}f}$  with at most one free variable and parameters from  $A$  such that:

$$(1) \ I(r) = r;$$

$$(2) \ I(r \cdot \sigma + s \cdot \tau) = r \cdot I(\sigma) + s \cdot I(\tau);$$

$$(3) \ \text{If } \tau^{\mathfrak{A}}(b, \vec{a}) \geq 0 \text{ for all } b \in A, \text{ then } I(\tau(x, \vec{a})) \geq 0.$$

In a weak integral model of  $T$  the logic  $L_{\mathcal{A}f}$  can be interpreted, for instance  $(\int \tau(x, \vec{y}) \, dx)^{\mathfrak{A}}(\vec{a}) = I(\tau(x, \vec{a}))$  so that the axioms are satisfied. Then we pass to an  $\omega_1$ -saturated nonstandard universe and form the *internal weak integral structure*  $\langle {}^*\mathfrak{A}, {}^*I \rangle$ .

Second, the Loeb construction of Daniell integral as given in 1.3. produces a probability measure  $\mu$  on  ${}^*A$  such that for each term  $\tau(x)$  with parameters from  ${}^*A$ , the standard part of  ${}^*I(\tau)$  is the integral of the standard part of  $\tau^{{}^*\mathfrak{A}}$  with respect

to  $\mu$ . Similarly, we can construct a probability measure  $\mu_n$  on  $*A^n$  using iterated integrals.

Finally, the axiom  $(E_7)$ , which is essentially a translation of the axiom  $(B_4)$  for  $L_{AP}$ , is satisfied almost everywhere in the graded model  $\langle *A, \mu_n \rangle$ . This yields a probability model of  $T$  as given in 3.1.

**4.1.4. Soundness and Completeness Theorem for  $L_{Af}$ .** *A countable set of sentences of  $L_{Af}$  has a probability model iff it is consistent.*  $\square$

By induction on the complexity of terms we prove that, for each term  $\tau(\vec{x})$  of  $L_{Af}$  and for each  $r \in \mathbb{R} \cap \mathcal{A}$  there is a formula  $\varphi_{\tau,r}$  of  $L_{AP}$  such that in every probability structure  $\langle A, \mu \rangle$  for each  $\vec{a} \in A^n$ , the following is true:

$$\tau^A(\vec{a}) > r \quad \text{iff} \quad A \models \varphi_{\tau,r}[\vec{a}].$$

The key idea is to formalize the definition of an integral in terms of a measure, using the fact that  $\omega \in \mathcal{A}$ , so that the appropriate limits can be expressed in  $L_{AP}$ .

If  $\tau(\vec{x})$  is an atomic term  $\mathbf{1}(R(\vec{x}))$ , then  $\varphi_{\tau,r} = \begin{cases} x = x, & \text{if } r < 0 \\ R(\vec{x}), & \text{if } 0 \leq r < 1. \\ x \neq x, & \text{if } r \geq 1 \end{cases}$ . Similarly

we define  $\varphi_{\tau,r}$  when  $\tau(\vec{x})$  is the atomic term  $\mathbf{1}(x = y)$ . Let  $\tau(\vec{x})$  be a term of the form  $\mathbb{F}(\tau_1, \dots, \tau_m)$  and let  $F^{-1}(r, +\infty)$  be a countable union of open rectangles  $(r_{1i}, s_{1i}) \times \dots \times (r_{mi}, s_{mi})$ ,  $i < \omega$ . Then, define

$$\varphi_{\tau,r} = \bigvee_i \bigwedge_{1 \leq j \leq m} \left( \varphi_{\tau_j, r_{ji}} \wedge \bigvee_{q < s_{ji}, q \in \mathbb{Q}} \neg \varphi_{\tau_j, q} \right).$$

Let  $\tau(\vec{x})$  be a term of the form  $\int \sigma(\vec{x}, y) dy$ . Then, define

$$\varphi_{\tau,r} = \bigvee_{n < \omega} \bigvee_{\substack{r_1, q_1, \dots, r_n, q_n \in \mathbb{Q} \\ r_i \in [0, 1], \sum r_i \cdot q_i > r}} \bigwedge_{1 \leq j \leq n} (Py \geq r_j) \varphi_{\sigma, q_j}(\vec{x}, y).$$

The function which assigns to each term  $\tau$  of  $L_{Af}$  and  $r \in \mathbb{R} \cap \mathcal{A}$  a formula  $\varphi_{\tau,r}$  of  $L_{AP}$  is  $\Sigma$ -definable. Hence, for each formula  $\varphi(\vec{x})$  of  $L_{APf}$  which has all the symbols and formation rules of  $L_{AP}$  and  $L_{Af}$  in common, there is a formula  $\psi(\vec{x})$  of  $L_{AP}$  such that  $L_{APf} \models \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$ . Also, one can formalize the definition of a measure in terms of an integral, so that there is a formula  $\theta(\vec{x})$  of  $L_{Af}$  such that  $L_{APf} \models \varphi(\vec{x}) \leftrightarrow \theta(\vec{x})$ . It is trivial to verify that for any sentence  $\varphi$  of  $L_{Af}$ , we have

$$L_{APf} \models \varphi \quad \text{iff} \quad L_{Af} \models \varphi.$$

Also, for any sentence  $\psi$  of  $L_{AP}$ , we have

$$L_{APf} \models \psi \quad \text{iff} \quad L_{AP} \models \psi.$$

Thus,  $L_{APf}$  is a *conservative definitional extension* of both  $L_{AP}$  and  $L_{Af}$ , which shows that these logics are equivalent in a strong sense, and which allows us to convert a theorem about  $L_{AP}$  to a similar theorem about  $L_{Af}$  and vice versa. So, the Barwise Completeness and Compactness Theorems, the Finite Compactness Theorem with the set of sentences of  $L_{Af}$  of the form  $\tau \in [r, s]$  replacing the set of universal conjunctive sentences of  $L_{AP}$ , the Normal Form Theorem, the Existence Theorem and the Uniqueness Theorem for hyperfinite models, the Robinson Consistency and the Craig Interpolation Theorems also hold for  $L_{Af}$ .

## 4.2. Logic with integrals related to analytic models

This section contains a modification  $L(f)_A$  of the logic  $L_{A^f}$  which was introduced by Hoover in [27]. It improves  $L_{A^f}$  in that it works for languages with function symbols. Measure-theoretic problems about function symbols are avoided because the corresponding structure for  $L(f)_A$  is an analytic subset of  $\mathbb{R}$  with a measure on the relative Borel sets. Recall that a subset  $A$  of  $\mathbb{R}$  is an *analytic set* if it is derived from a family  $\mathcal{F}$  of Borel subsets of  $\mathbb{R}$  by the Souslin operation; i.e.,  $A = \bigcup_{\alpha} \bigcap_n F_{\alpha|n}$ , where  $\alpha$  ranges over all infinite sequences of positive integers,  $\alpha|n$  is the finite sequence  $\alpha(1), \dots, \alpha(n)$  and  $F_{\alpha|n} \in \mathcal{F}$ . In other words,  $A$  is an image under a Borel function of a Borel set.

We shall describe the syntax and the semantics of  $L(f)_A$  in more detail, because these definitions allow further constructions of related measure theory on a weak models without using nonstandard measure constructions.

Hoover in [27] defines a language of  $L(f)_A$  as any countable set  $L$  of relation symbols from  $L^2$  which are intended to take values in  $\{0, 1\}$ , relation symbols from  $L^{\mathbb{R}}$  which are intended to take values in  $[0, 1]$  and function and constant symbols from  $L^f$ . All these sets are  $\Sigma$  on  $\mathcal{A}$ , they are pairwise disjoint and each of them may be empty. Also,  $L$  has countably many variables  $v_1, v_2, \dots$ , connectives  $\cdot, +, \min, \max, \sup$  and quantifiers  $\int \cdots dx$ . The quantifier  $\int \cdots dx$  being considered to bind variable  $x$ .

*Terms* are defined as in the first-order logic.

**4.2.1. Definition.** The set of real-valued formulas of  $L(f)_A$  is the least set such that:

- (1) Each atomic formula,  $R(\tau_1, \dots, \tau_n)$  or  $r$ , is a real-valued formula, where  $R \in L^{\mathbb{R}} \cup L^2$  is  $n$ -ary,  $\tau_1, \dots, \tau_n$  are terms and  $r \in \mathbb{R} \cap \mathcal{A}$ ;
- (2) If  $\varphi$  and  $\psi$  are real-valued formulas, then  $\varphi \cdot \psi$ ,  $\varphi + \psi$ ,  $\min(\varphi, \psi)$  and  $\max(\varphi, \psi)$  are real-valued formulas;
- (3) If  $\Phi$  is a set of real-valued formulas with finitely many free variables and  $\Phi \in \mathcal{A}$ , then  $\min(\sup \Phi, b)$  is a real-valued formula for each  $b \in \mathbb{R}$ ;
- (4) If  $\varphi$  is a real-valued formula and  $x$  is a variable, then  $\int \varphi dx$  is a real-valued formula.

**4.2.2. Definition.** The set of two-valued formulas (propositions) of  $L(f)_A$  is the least set such that:

- (1) If  $\varphi$  and  $\psi$  are real-valued formulas, then  $\varphi = \psi$  and  $\varphi \leq \psi$  are propositions;
- (2) If  $p$  is a proposition, then so is  $\neg p$ ;
- (3) If  $S$  is a countable set of propositions with finitely many free variables and  $S \in \mathcal{A}$ , then  $\bigvee S$  is a proposition.

A *sentence* is a proposition with no free variables.

We mention an important property of the syntax: if  $p$  is a proposition, then it is possible to introduce a new real-valued formula  $\mathbf{1}(p)$ , the *indicator* of  $p$ , recursively by:

$$\begin{aligned}
\mathbf{1}(\varphi \leq \psi) &= 1 - \min \left( \sup_{n < \omega} n \cdot (\varphi - \psi), 1 \right), \\
\mathbf{1}(\varphi = \psi) &= \min \left( \mathbf{1}(\varphi \leq \psi), \mathbf{1}(\psi \leq \varphi) \right), \\
\mathbf{1}(\neg p) &= 1 - \mathbf{1}(p), \\
\mathbf{1}(\bigvee S) &= \min \left( \sup_{p \in S} \mathbf{1}(p), 1 \right).
\end{aligned}$$

For short we shall write  $\int \varphi d\vec{x}$  instead of  $\int \cdots \int \varphi dx_1 \dots dx_n$ , and  $P\{\vec{x} : p(\vec{x})\}$  instead of  $\int \mathbf{1}(p) d\vec{x}$ .

A probability model  $\langle \mathfrak{A}, \mu \rangle$  is said to be *analytic* if  $A$  is an analytic subset of  $\mathbb{R}$  and for each  $n$ -ary relation symbol  $R$ , if  $R \in L^{\mathbb{R}}$  ( $R \in L^2$ ), then  $R^{\mathfrak{A}}$  is a  $\mu^n$ -Borel measurable function from  $A^n$  to  $[0, 1]$  (from  $A^n$  to  $\{0, 1\}$ ), and for each  $n$ -ary function symbol  $f \in L^f$ ,  $f^{\mathfrak{A}}$  is a  $\mu^n$ -measurable function from  $A^n$  to  $A$ . While the equality relation plays only a minor role in  $L_{AP}$  or  $L_{Af}$ , a fact which stems from the absence of the universal quantifier and of function symbols, in  $L(f)_{\mathcal{A}}$  this is more restrictive. For instance, a probability model  $\langle \mathfrak{A}, \mu \rangle$  of a language  $L$  which does not have the equality, may not be expandable to a model of  $L$  with the equality added, since it must be  $\mu^2$ -measurable.

**4.2.3. Definition.** (a) An interpretation of terms of  $L(f)_{\mathcal{A}}$  in a probability model  $\langle \mathfrak{A}, \mu \rangle$  is a function which assigns to each term  $\tau(\vec{x})$  a function  $\tau^{\mathfrak{A}}: A^n \rightarrow A$  as in the first-order logic.

(b) An interpretation of the real-valued formulas of  $L(f)_{\mathcal{A}}$  in  $\langle \mathfrak{A}, \mu \rangle$ , is a function which assigns to a formula  $\varphi(\vec{x})$  a function  $\varphi^{\mathfrak{A}}: A^n \rightarrow \mathbb{R}$  such that:

- (1)  $r^{\mathfrak{A}} = r$ ;
- (2)  $R(\tau_1, \dots, \tau_m)^{\mathfrak{A}}(\vec{a}) = R^{\mathfrak{A}}(\tau_1^{\mathfrak{A}}(\vec{a}), \dots, \tau_m^{\mathfrak{A}}(\vec{a}))$ ;
- (3)  $(\varphi \cdot \psi)^{\mathfrak{A}}(\vec{a}) = \varphi^{\mathfrak{A}}(\vec{a}) \cdot \psi^{\mathfrak{A}}(\vec{a})$ ;
- (4)  $(\varphi + \psi)^{\mathfrak{A}}(\vec{a}) = \varphi^{\mathfrak{A}}(\vec{a}) + \psi^{\mathfrak{A}}(\vec{a})$ ;
- (5)  $\min(\varphi, \psi)^{\mathfrak{A}}(\vec{a}) = \min(\varphi^{\mathfrak{A}}(\vec{a}), \psi^{\mathfrak{A}}(\vec{a}))$ ;
- (6)  $\max(\varphi, \psi)^{\mathfrak{A}}(\vec{a}) = \max(\varphi^{\mathfrak{A}}(\vec{a}), \psi^{\mathfrak{A}}(\vec{a}))$ ;
- (7)  $\min(\sup \Phi, b)^{\mathfrak{A}}(\vec{a}) = \min(\sup_{\varphi \in \Phi} \varphi^{\mathfrak{A}}(\vec{a}), b)$ ;
- (8)  $(\int \varphi(x, \vec{y}) dx)^{\mathfrak{A}}(\vec{a}) = \int \varphi^{\mathfrak{A}}(b, \vec{a}) d\mu(b)$ .

(c) An interpretation of propositions of  $L(f)_{\mathcal{A}}$  in  $\langle \mathfrak{A}, \mu \rangle$  is a truth function defined inductively by:

- (1)  $(\varphi = \psi)^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  iff  $\varphi^{\mathfrak{A}}(\vec{a}) = \psi^{\mathfrak{A}}(\vec{a})$ ;
- (2)  $(\varphi \leq \psi)^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  iff  $\varphi^{\mathfrak{A}}(\vec{a}) \leq \psi^{\mathfrak{A}}(\vec{a})$ ;
- (3)  $(\neg p)^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  iff  $p^{\mathfrak{A}}(\vec{a}) = \text{FALSE}$ ;
- (4)  $(\bigvee S)^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  iff  $p^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  for some  $p \in S$ .

As a consequence of this definition we have that  $p^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  iff  $\mathbf{1}(p)^{\mathfrak{A}}(\vec{a}) = 1$  and  $p^{\mathfrak{A}}(\vec{a}) = \text{FALSE}$  iff  $\mathbf{1}(p)^{\mathfrak{A}}(\vec{a}) = 0$ . Each function  $\varphi^{\mathfrak{A}}$  is bounded and  $\mu^n$ -measurable, so for every probability model  $\langle \mathfrak{A}, \mu \rangle$  there exists an interpretation of  $L(f)_{\mathcal{A}}$  in  $\langle \mathfrak{A}, \mu \rangle$ .

The *satisfaction relation* is defined by:  $\mathfrak{A} \models p[\vec{a}]$  iff  $p^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$ . A proposition  $p(\vec{x})$  is *valid* in  $\langle \mathfrak{A}, \mu \rangle$  if and only if  $p^{\mathfrak{A}}(\vec{a}) = \text{TRUE}$  for each  $\vec{a} \in A^n$ .

Now we shall list the axioms and rules of inference for  $L(f)_{\mathcal{A}}$  as given in [27].

The *axioms* for  $L(f)_{\mathcal{A}}$  are:

- (G<sub>1</sub>) All axiom schemas for  $L_{\mathcal{A}}$  without quantifiers, where the propositions of  $L(f)_{\mathcal{A}}$  are substituted into the schemas;
- (G<sub>2</sub>) All finitary, quantifier free theorems of the theory of real closed fields with max and min, and with axioms added
 
$$\bigwedge_n (\varphi \leq \psi + 1/n) \rightarrow (\varphi \leq \psi),$$
 the diagram of ordered field of reals;
- (G<sub>3</sub>) *Atomic formula axioms.*
  - (i)  $0 \leq R \wedge R \leq 1$ , for each atomic formula with  $R \in L^{\mathbb{R}}$ ,
  - (ii)  $R = 0 \vee R = 1$ , for each atomic formula with  $R \in L^2$ ;
- (G<sub>4</sub>)  $\min(\sup \Phi, b) \leq \psi \leftrightarrow \bigwedge_{\varphi \in \Phi} \min(\varphi, b) \leq \psi$ ;
- (G<sub>5</sub>) *Identity and equality axioms* (if  $L$  has  $=$ ).
  - (i)  $(\tau = \tau) = 1$ ,
  - (ii)  $(\tau = \sigma) = (\sigma = \tau)$ ,
  - (iii)  $\min(\sigma = \tau, \tau = \pi) \leq \sigma = \pi$ ,
  - (iv)  $\min(\sigma_1 = \tau_1, \dots, \sigma_n = \tau_n) \leq f(\sigma_1, \dots, \sigma_n) = f(\tau_1, \dots, \tau_n)$ ,  
for  $f \in L^f$ ,
  - (v)  $\min(\sigma_1 = \tau_1, \dots, \sigma_n = \tau_n, R(\sigma_1, \dots, \sigma_n)) \leq R(\tau_1, \dots, \tau_n)$ ,  
for  $R \in L^2 \cup L^{\mathbb{R}}$ ;
- (G<sub>6</sub>) *Integral axioms.*
  - (i)  $\int \varphi(x) dx = \int \varphi(y) dy$ ,
  - (ii)  $\int \varphi \cdot \psi dx = \varphi \cdot \int \psi dx$ , where  $x$  not free in  $\varphi$ ,
  - (iii)  $\int \varphi + \psi dx = \int \varphi dx + \int \psi dx$ ,
  - (iv)  $\iint \varphi dx dy = \iint \varphi dy dx$ ;
- (G<sub>7</sub>)  $\int \min(\sup \Phi, b) dx = \min \left( \sup_{\text{finite } \Phi_0 \subseteq \Phi} \int \min(\sup \Phi_0, b) dx, b \right)$ ;
- (G<sub>8</sub>)  $P\{ \vec{x} : P\{ \vec{y} : \int |\varphi(\vec{x}, \vec{z}) - \varphi(\vec{y}, \vec{z})| d\vec{z} < \varepsilon \} > 0 \} = 1$ , for every real  $\varepsilon > 0$ .

The *rules of inference* for  $L(f)_{\mathcal{A}}$  are:

- (H<sub>1</sub>)  $\varphi, \varphi \rightarrow \psi \vdash \psi$  (*Modus Ponens*);
- (H<sub>2</sub>)  $p(x) \vdash p(\tau)$  (*Instantiation*);
- (H<sub>3</sub>)  $\varphi \leq \psi \vdash \int \varphi dx \leq \int \psi dx$  (*Generalization*).

The Soundness Theorem for  $L(f)_{\mathcal{A}}$  holds because all the axioms represent known properties of integrals. The proof of completeness falls into two parts. The first step is a Henkin construction of a weak integral model  $\langle \mathfrak{A}, I \rangle$  exactly in the same way as before, so we omit it. By translating the definition of integral in terms of a measure using the fact that  $\omega \in \mathcal{A}$ , and vice versa, it is not difficult to see that for any proposition  $p(\vec{x})$  of  $L(f)_{\mathcal{A}}$ , there is a formula  $\varphi(\vec{x})$  of  $L_{AP}(\mathbb{R})$  (see 3.11.) such that for any probability structure  $\langle \mathfrak{A}, \mu \rangle$  and  $\vec{a} \in A^n$  the following holds:  $\mathbf{1}(p)^{\mathfrak{A}}(\vec{a}) = 1$  iff  $\mathfrak{A} \models \varphi[\vec{a}]$ . Hence,  $L(f)_{\mathcal{A}}$  and  $L_{AP}(\mathbb{R})$  (if  $L^{\mathbb{R}}$  is empty, then  $L_{AP}$  instead of  $L_{AP}(\mathbb{R})$ ) are mutually interpretable. It follows that  $L(f)_{\mathcal{A}}$  is essentially equivalent to  $L_{AP}(\mathbb{R})$ . But yet the Completeness Theorem for  $L(f)_{\mathcal{A}}$  is stronger than those in 3.11, because it is valid for languages which may have function symbols, and

because, by using a well-known measure construction related to von Neumann in [63], we can get a product measurable probability model directly from the weak integral model. We point out that a considerable amount of measure construction is already contained in the syntax. The main notions and properties of the measure theory on a weak integral model  $\langle \mathfrak{A}, I \rangle$  are described as follows.

**4.2.4. Definition.** (a) The mean of  $X$ , denoted by  $E[X]$ , is  $(\int X d\vec{x})^{\mathfrak{A}}$ , where  $X: A^n \rightarrow \mathbb{R}$  is an  $\mathcal{A}$ -measurable bounded function; i.e.,  $X$  is  $\varphi^{\mathfrak{A}}$  for some real-valued formula  $\varphi(\vec{x})$  of  $L(f)_{\mathcal{A}}$ , and  $\vec{x} = x_1, \dots, x_n$  contains all variables free in  $\varphi$ .

(b) If  $\mathcal{F}$  is a finite algebra of  $\mathcal{A}$ -measurable sets, where  $F \subseteq A^\infty$  is  $\mathcal{A}$ -measurable if and only if its indicator  $\mathbf{1}(F)$  is an  $\mathcal{A}$ -measurable function, then the *conditional expectation* of the  $\mathcal{A}$ -measurable bounded function  $X$  with respect to  $\mathcal{F}$ , denoted by  $E[X|\mathcal{F}]$ , is the  $\mathcal{A}$ -measurable function

$$E[X|\mathcal{F}](a) = \begin{cases} E[X \cdot \mathbf{1}(F)] / E[\mathbf{1}(F)], & \text{if } E[\mathbf{1}(F)] > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $F$  is the atom of  $\mathcal{F}$  such that  $a \in F$ .

The following measure-theoretic result says that each  $\mathcal{A}$ -measurable function is approximated by a product measurable one already in a weak integral model, instead in a graded probability model, which was the case in 3.7. The proof is taken from [27] in order to stress the importance of the above syntax.

**4.2.5. Rectangle Approximation Lemma.** *If  $\varphi(\vec{x})$  is a real-valued formula of  $L(f)_{\mathcal{A}}$ ,  $\vec{x} = x_1, \dots, x_n$  and  $\varepsilon > 0$ , then there are finitely valued formulas  $\psi_{ij}(x_j)$  of  $L(f)_{\mathcal{A}}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$  such that  $E[|\varphi(\vec{x}) - \sum_i \prod_j \psi_{ij}(x_j)|^{\mathfrak{A}}] < \varepsilon$ , where  $\langle \mathfrak{A}, I \rangle$  is a given weak integral model.*

*Proof.* It will be enough to prove by induction on  $n$  that for any real-valued formula  $\varphi(\vec{y}, \vec{z})$  of  $L(f)_{\mathcal{A}}$  there exists  $m$  and finitely valued formulas  $\psi_i(\vec{y})$  and  $\theta_i(\vec{z})$  of  $L(f)_{\mathcal{A}}$ ,  $1 \leq i \leq m$  such that  $E[|\varphi(\vec{y}, \vec{z}) - \sum_i \psi_i(\vec{y}) \cdot \theta_i(\vec{z})|^{\mathfrak{A}}] < \varepsilon$ .

If  $\vec{y} = y_1, \dots, y_k$ , then we can make  $A^k$  to be a pseudo-metric space by defining  $d(\vec{a}, \vec{c}) = (\int |\varphi(\vec{x}, \vec{z}) - \varphi(\vec{y}, \vec{z})| d\vec{z})^{\mathfrak{A}}(\vec{a}, \vec{c})$ . From the axiom  $(G_8)$  it follows that for almost all  $\vec{a} \in A^k$ , the open  $\varepsilon/2$ -ball at  $\vec{a}$ , denoted by  $B_{\vec{a}, \varepsilon/2}$ , has a positive measure. From the axiom  $(G_7)$  it follows that there is some  $\delta > 0$  such that  $P\{\vec{a} : P(B_{\vec{a}, \varepsilon/2}) > \delta\} > 1 - \varepsilon$ , so we can choose a maximal subset  $\{\vec{a}_1, \dots, \vec{a}_m\}$  of  $\{\vec{a} : P(B_{\vec{a}, \varepsilon/2}) > \delta\}$  such that  $B_{\vec{a}_i, \varepsilon/2} \cap B_{\vec{a}_j, \varepsilon/2} = \emptyset$  for each  $1 \leq i < j \leq m$ . Let  $\psi_i(\vec{y})$  be the  $\{0, 1\}$ -valued formula  $\mathbf{1}(B_{\vec{a}_i, \varepsilon} \setminus \bigcup_{j>i} B_{\vec{a}_j, \varepsilon})$ , and let  $\bar{\theta}_i(\vec{z})$  be the formula  $\varphi(\vec{a}_i, \vec{z})$ . By the integral axioms we obtain

$$\begin{aligned} E[|\varphi(\vec{y}, \vec{z}) \cdot \psi_i(\vec{y}) - \psi_i(\vec{y}) \cdot \bar{\theta}_i(\vec{z})|^{\mathfrak{A}}] &< \varepsilon \cdot E[\psi_i(\vec{y})^{\mathfrak{A}}], \\ E[|\varphi(\vec{y}, \vec{z}) - \sum_i \psi_i(\vec{y}) \cdot \bar{\theta}_i(\vec{z})|^{\mathfrak{A}}] &< \varepsilon \cdot (1 + \|\varphi\|), \end{aligned}$$

where  $\|\varphi\|$  denotes the bound of  $\varphi^{\mathfrak{A}}$ . If we replace, by induction, each  $\bar{\theta}_i$  by a finitely valued formula  $\theta_i$  which approximates it, we finally obtain

$$E[|\varphi(\vec{y}, \vec{z}) - \sum_i \psi_i(\vec{y}) \cdot \theta_i(\vec{z})|^{\mathfrak{A}}] < \varepsilon(2 + \|\varphi\|). \quad \square$$

Now we shall sketch a measure construction starting with a weak integral model  $\langle \mathfrak{A}, \mu \rangle$  assuming that  $L$  does not contain the equality and  $L^f$  is empty.

Let  $\varphi_1, \varphi_2, \dots$  be an enumeration of the formulas of  $L(f)_A$  such that the free variables of  $\varphi_n$  are among  $v_1, \dots, v_n$ . For each  $n$ , let  $\mathcal{F}_n$  be the smallest algebra of subsets of  $A$  such that for each finitely valued formula  $\psi(v_1)$  appearing in the Rectangle Approximation Lemma for  $\varphi = \varphi_1, \dots, \varphi_n$  and  $\varepsilon = 2^{-2n}$ , the bounded function  $\psi(v_1)^{\mathfrak{A}}$  is  $\mathcal{F}_n$ -measurable, i.e.,  $E[\psi(v_1)^{\mathfrak{A}} | \mathcal{F}_n] = \psi(v_1)^{\mathfrak{A}}$  a. s. Let  $A_n$  be the set of atoms of  $\mathcal{F}_n$ , and let  $\mathcal{F}_n^{\#}$  be the finite algebra on  $A^\infty$  generated by sets of the form  $F_1 \times \dots \times F_n \times A^\infty$ , where  $F_1, \dots, F_n \in A_n$ . Let  $\mathcal{B}$  be a Boolean algebra of subsets of  $[0, 1]$ . A measure preserving homomorphism of Boolean algebras  $T_n: \mathcal{F}_n \rightarrow \mathcal{B}$ , which is defined recursively by

- (1)  $T_0(\emptyset) = \emptyset$ ,  $T_0(A) = [0, 1]$ ;
- (2) If atoms  $F_1, \dots, F_k$  of  $\mathcal{F}_{n+1}$  partition an atom  $F$  of  $\mathcal{F}_n$ , and if intervals of length  $E[\mathbf{1}(F_1)], \dots, E[\mathbf{1}(F_k)]$  partition  $T_n(F)$ , let  $T_{n+1}(F_1), \dots, T_{n+1}(F_k)$  be these intervals,

naturally induces a unique measure preserving map  $T_n^{\#}: \mathcal{F}_n^{\#} \rightarrow \mathcal{B}^\infty$  and, also, a unique mean preserving map  $T'_n$  from  $\mathcal{F}_n^{\#}$ -measurable functions  $A^\infty \mapsto \mathbb{R}$  into Borel functions  $[0, 1]^\infty \mapsto \mathbb{R}$ . Let  $h_{\varphi, n} = E[\varphi^{\mathfrak{A}} | \mathcal{F}_n^{\#}]$ ,  $g_{\varphi, n} = T'_n(h_{\varphi, n})$  and  $g_\varphi = \liminf g_{\varphi, n}$  for a real-valued formula  $\varphi$  of  $L(f)_A$ .

Let  $\lambda$  be a Lebesgue measure on the Borel subsets of  $[0, 1]$ , and let  $R^{\mathfrak{B}}(a_1, \dots, a_n)$  be defined by:

- (i)  $R^{\mathfrak{B}}(a_1, \dots, a_n) = \begin{cases} 1, & \text{if } g_{R(\vec{x}_{\vec{a}})}(a_{i_1}, \dots, a_{i_k}) = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } R \in L^2,$
- (ii)  $R^{\mathfrak{B}}(a_1, \dots, a_n) = g_{R(\vec{x}_{\vec{a}})}(a_{i_1}, \dots, a_{i_k}), \quad \text{for } R \in L^{\mathfrak{R}},$

where  $a_{i_1}, \dots, a_{i_k}$  are all distinct elements of  $\{a_1, \dots, a_n\}$  and  $\vec{x}_{\vec{a}} = x_1, \dots, x_k$  is a sequence of variables for each  $a_{i_1}, \dots, a_{i_k}$ . The main reason for that interpretation of  $L$  into the domain  $B \subseteq [0, 1]$  is that the diagonal set

$$\{\vec{a} \in [0, 1]^n : a_i = a_j \text{ for some } 1 \leq i < j \leq n\}$$

has  $\lambda^n$ -measure 0, and that the definition does not depend on the particular choice of the string of  $\vec{x}_{\vec{a}}$ , as long as distinct variables and distinct  $a_{i_1}, \dots, a_{i_k}$  are in one-to-one correspondence.

The following is the main theorem of this section.

**4.2.6. Completeness Theorem.** *Every consistent set  $T$  of sentences of  $L(f)_A$  which is  $\Sigma$  on  $A$  has an analytic probability model.*

*Proof.* By the above construction of the “strong” model  $\mathfrak{B}$  starting from a weak model  $\mathfrak{A}$  of  $T$ , for  $L$  without equality and  $L^f$  empty, it will be enough to prove, inductively, that  $\varphi^{\mathfrak{B}} = g_\varphi$  a. s. for any real-valued formula  $\varphi$  of  $L(f)_A$ , since it implies that  $\varphi^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$  for any real-valued sentence  $\varphi$  of  $L(f)_A$ . Let  $W$  be the set of real-valued formulas which satisfy  $\varphi^{\mathfrak{B}} = g_\varphi$  a. s. Then  $W$  contains all atomic formulas, and  $W$  is closed under  $+$ ,  $\cdot$ ,  $\min$  and  $\max$ . If  $\Phi \subset W$  and  $\Phi \in \mathcal{A}$ , then for each  $\varepsilon > 0$  we may choose a finite subset  $\Phi_0$  of  $\Phi$ , hence  $\min(\sup \Phi_0, b) \in W$ , such that  $E[|\min(\sup \Phi, b) - \min(\sup \Phi_0, b)|^{\mathfrak{A}}] < \varepsilon$  (axiom  $(G_7)$ ). By Jensen’s inequality

for conditional expectation, we obtain

$$\begin{aligned} E|g_{\min(\sup \Phi, b), n} - g_{\min(\sup \Phi_0, b), n}| &= E|h_{\min(\sup \Phi, b), n} - h_{\min(\sup \Phi_0, b), n}| \\ &\leq E[|\min(\sup \Phi, b) - \min(\sup \Phi_0, b)|^{\mathfrak{A}}] < \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  and  $n \rightarrow \infty$ ,

$$\begin{aligned} \min(\sup \Phi, b)^{\mathfrak{B}} &= \min(\sup \Phi_0, b)^{\mathfrak{B}} \\ &= g_{\min(\sup \Phi_0, b)} = g_{\min(\sup \Phi, b)} \quad \text{a. s.} \end{aligned}$$

and hence  $\min(\sup \Phi, b) \in W$ . If  $\varphi \in W$ , then

$$\begin{aligned} h_{\int \varphi dx, n} &= E\left[\left(\int \varphi dx\right)^{\mathfrak{A}} | \mathcal{F}_n^{\#}\right] = E\left[\int \varphi^{\mathfrak{A}} dx | \mathcal{F}_n^{\#}\right] \\ &= \int E[\varphi^{\mathfrak{A}} | \mathcal{F}_n^{\#}] dx = \int h_{\varphi, n} dx \quad \text{by } (G_6)(iv). \end{aligned}$$

Thus, we obtain  $(\int \varphi dx)^{\mathfrak{B}} = \int g_{\varphi} dx = \int g_{\varphi} dx = g_{\int \varphi dx}$ ; i.e.,  $\int \varphi dx \in W$ .

By using the interpretation of  $L$  into  $\mathfrak{B}$ , it is not difficult to see that even so-called *special Horn proposition* of  $L(f)_{\mathcal{A}}$ , i.e., a proposition of the form  $\min(R_1(\vec{x}), \dots, R_k(\vec{x}), Q(\vec{x})) \leq S(\vec{x})$  where  $R_1, \dots, R_k \in L^2$ ,  $S, Q \in L^2 \cup L^{\mathbb{R}}$  and if  $S \in L^2$  then  $Q \in L^2$ , which is valid in  $\mathfrak{A}$  is also valid in  $\mathfrak{B}$ . This fact plays an important role in the following measure construction which, also, starts from a weak model  $\mathfrak{A}$  of  $T$ , assuming that  $L$  contains the equality and, perhaps, function symbols.

Let  $E$  be a new two-valued binary relation symbol which is interpreted as the equality in a weak model  $\mathfrak{A}$  for  $L^E = (L \setminus \{=\}) \cup \{E\}$ . Let  $\bar{L}$  be the relational language without equality such that  $\bar{L}^2 \cup \bar{L}^{\mathbb{R}}$  contains all relation symbols  $R_{\tau_1, \dots, \tau_n}(\vec{x}_1, \dots, \vec{x}_n)$  for all terms  $\tau_1(\vec{x}_1), \dots, \tau_n(\vec{x}_n)$  and all relations  $R$  from  $L^E$ , where no variable is repeated in  $\vec{x}_1, \dots, \vec{x}_n$ . The corresponding model  $\bar{\mathfrak{A}}$  for  $\bar{L}$  is obtained by:

$$R_{\tau_1, \dots, \tau_n}^{\bar{\mathfrak{A}}}(\vec{a}_1, \dots, \vec{a}_n) = R^{\mathfrak{A}}(\tau_1^{\mathfrak{A}}(\vec{a}_1), \dots, \tau_n^{\mathfrak{A}}(\vec{a}_n)).$$

Hence, there is a model  $\langle \bar{\mathfrak{B}}, \lambda \rangle$  for  $\bar{L}$  which is  $\bar{L}(f)_{\mathcal{A}}$ -equivalent to  $\bar{\mathfrak{A}}$ . Let  $S$  be a set of constants  $c_{\tau(\vec{a})}$  for each variable free term  $\tau(\vec{a})$  which contains only elements listed in  $\vec{a}$  ( $S$  is essentially  $[0, 1]^{\infty}$ ). The relation  $\equiv$ , defined by

$$c_{\tau_1(\vec{a})} \equiv c_{\tau_2(\vec{e})} \quad \text{iff} \quad E_{\tau_1, \tau_2}^{\bar{\mathfrak{A}}}(\vec{a}, \vec{e}) = 1,$$

is an equivalence relation on  $S$  because the corresponding special Horn propositions valid in  $\mathfrak{A}$  are valid in  $\bar{\mathfrak{A}}$ , too. For instance, propositions of the form  $\min(E_{\tau_1, \tau_2}, E_{\tau_2, \tau_3}) \leq E_{\tau_1, \tau_3}$  are used to prove the transitivity of  $\equiv$ . Let  $D = S / \equiv$ ,

$$\begin{aligned} R^{\mathfrak{D}}([c_{\tau_1(\vec{a}_1})], \dots, [c_{\tau_n(\vec{a}_n})]) &= R_{\tau_1, \dots, \tau_n}^{\bar{\mathfrak{A}}}(\vec{a}_1, \dots, \vec{a}_n) \quad \text{for } R \in L^2 \cup L^{\mathbb{R}}, \\ f^{\mathfrak{D}}([c_{\tau_1(\vec{a}_1})], \dots, [c_{\tau_n(\vec{a}_n})]) &= [c_{f(\tau_1, \dots, \tau_n)(\vec{a}_1, \dots, \vec{a}_n)}] \quad \text{for } f \in L^f. \end{aligned}$$

We use a propositions of the form  $\min(E_{\tau_1, \sigma_1}, \dots, E_{\tau_n, \sigma_n}, R_{\tau_1, \dots, \tau_n}) \leq R_{\sigma_1, \dots, \sigma_n}$  and  $\min(E_{\tau_1, \sigma_1}, \dots, E_{\tau_n, \sigma_n}) \leq E_{f(\tau_1, \dots, \tau_n), f(\sigma_1, \dots, \sigma_n)}$  to prove that all the relation and function symbols are well interpreted on the set of equivalence classes.

The Lebesgue measure  $\lambda$  induces a probability measure  $\mu$  on the Borel sets of  $S$  defined by  $\mu(B) = \lambda(B \cap [0, 1])$ , and hence the probability measure  $\nu$  on  $D$  by  $\nu(U) = \mu(V)$ , where  $U \subseteq D$  and  $V = \{c_{\tau(\vec{a})} : [c_{\tau(\vec{a})}] \in U\}$ . Then, the model  $\langle \mathfrak{D}, \nu \rangle$  is the image of the Borel space  $\langle S, \mu \rangle$  under a Borel function; i.e., an analytic probability model which is  $\overline{L}(f)_{\mathcal{A}}$ -equivalent to  $\overline{\mathfrak{B}}$ .  $\square$

An analytic probability model produces a hyperfinite probability model which, in common with the previous theorem, asserts that the Existence and Uniqueness Theorems for hyperfinite models also hold for  $L(f)_{\mathcal{A}}$ . As before, the hyperfinite models play the role for  $L(f)_{\mathcal{A}}$  that saturated models play in classical model theory. Hence these models could be used to prove the Robinson Consistency and Craig Interpolation Theorems for  $L(f)_{\mathcal{A}}$ .

### 4.3. Logic with conditional expectation operators

The most interesting structures for probability theory are the structures of the form  $\langle A, X^{\mathfrak{A}}, \mathcal{F}, \mu \rangle$ , where  $X^{\mathfrak{A}}$  is a random variable defined on  $A$  with values in  $\mathbb{R}$ , and  $\mathcal{F}$  is a  $\sigma$ -algebra of  $\mu$ -measurable sets. This section contains some model theoretic results for the logic  $L_{AE}$  which extends  $L_{\mathcal{A}f}$ . It has a new operator  $E[\cdot|\cdot]$ , which allows one to express in a natural way many basic probability theory notions involving conditional expectation of random variables with respect to  $\sigma$ -algebras.

Let  $L$  be the  $\mathcal{A}$ -recursive set  $\{X_i : i \in I\} \cup \{c_j : j \in J\}$  of random variable symbols and constant symbols. The logic  $L_{AE}$  has countably many individual variables  $v_1, v_2, \dots$ , first-order connectives  $\neg, \wedge$ ,  $n$ -ary connective  $\mathbb{F}$  for each  $n < \omega$  and continuous function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $F \upharpoonright \mathbb{Q}^n \in \mathcal{A}$ , integral quantifiers  $\int \dots dx$  and conditional expectation quantifiers  $E[\cdot|\cdot]$ .

**4.3.1. Definition.** The set of terms and formulas of  $L_{AE}$  is defined exactly the same as for  $L_{\mathcal{A}f}$  except that:

- (1) Each atomic term  $[X_i(\vec{x}) \upharpoonright r]$  or  $\mathbf{1}(x = y)$  is a term, where  $\vec{x}$  is a tuple of constants or variables and  $r \in \mathbb{Q}_+$ ;
- (2) If  $\tau(\vec{x}, y)$  is a term with  $y$  not in  $\vec{x}$  and  $z$  is a variable, then  $E[\tau(\vec{x}, y)|y](z)$  is a term.

The notion of a free occurrence of a variable in a term  $\tau$  of  $L_{AE}$  is introduced by induction on the complexity of  $\tau$  in exactly the same way as for  $L_{\mathcal{A}f}$  except that a variable  $v$  is *free* in  $E[\sigma(\vec{x}, y)|y](z)$  if and only if  $v$  is  $z$  or  $v$  is not  $y$  and it is free in  $\sigma$ . A *closed term (sentence)* is a term (formula) with no free variables. We shall divide the set of free variables of  $L_{AE}$  into two parts: *E-free* and *E-bound*.

- 4.3.2. Definition.**
- (a) A variable  $v$  is *E-free* in  $[X_i(\vec{x}) \upharpoonright r]$  iff  $v$  is in  $\vec{x}$ ;
  - (b)  $v$  is *E-free* in  $\mathbf{1}(x = y)$  iff  $v$  is  $x$  or  $v$  is  $y$ ;
  - (c)  $v$  is *E-free* in  $\mathbb{F}(\tau_1, \dots, \tau_n)$  iff  $v$  is *E-free* in  $\tau_i$  for some  $i$ ;
  - (d)  $v$  is *E-free* in  $\int \tau dx$  iff  $v$  is not  $x$  and  $v$  is *E-free* in  $\tau$ ;
  - (e)  $v$  is *E-free* in  $E[\tau(\vec{x}, y)|y](z)$  iff  $v$  is not  $z$  and  $v$  is in  $\vec{x} \cup \{y\}$ ;
  - (f) A variable  $v$  is *E-bound* in  $\tau$  iff  $v$  is free in  $\tau$  and it is not *E-free* in  $\tau$ .

This distinction is based on the interpretation of  $L_{AE}$  in a conditional expectation structure. So, we shall mention these notions.

**4.3.3. Definition.** A *conditional expectation model* for  $L$  is a structure of the form  $\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, \mu, \mathcal{F} \rangle$ , where  $\mu$  is a probability measure on  $A$  such that each singleton is measurable,  $\mathcal{F}$  is a  $\sigma$ -algebra of  $\mu$ -measurable sets and a random variable  $X_i^{\mathfrak{A}}: A^n \rightarrow \mathbb{R}$  is a real-valued  $\mu^{(n)} / \mathcal{B}$ -measurable function; i.e., for every Borel subset  $B$  of  $\mathbb{R}$ ,  $(X_i^{\mathfrak{A}})^{-1}(B)$  is  $\mu^{(n)}$ -measurable.

**4.3.4. Definition.** An interpretation of  $L_{AE}$  in a conditional expectation structure  $\langle \mathfrak{A}, \mu, \mathcal{F} \rangle$  is a function which assigns to each term  $\tau(x_1, \dots, x_n)$  a function  $\tau^{\mathfrak{A}}: A^n \rightarrow \mathbb{R}$  defined exactly the same as for  $L_{Af}$  except that:

- (1)  $[X_i(\vec{x}) \upharpoonright r]^{\mathfrak{A}}(\vec{a}) = \begin{cases} -r, & \text{if } X_i^{\mathfrak{A}}(\vec{a}) \leq -r, \\ X_i^{\mathfrak{A}}(\vec{a}), & \text{if } -r \leq X_i^{\mathfrak{A}}(\vec{a}) \leq r, \\ r, & \text{if } r \leq X_i^{\mathfrak{A}}(\vec{a}); \end{cases}$
- (2)  $(E[\tau(\vec{x}, y)|y](z))^{\mathfrak{A}}(\vec{a}, b) = E[\tau^{\mathfrak{A}}(\vec{a}, \cdot)|\mathcal{F}](b)$  for each  $\vec{a} \in A^n$  and  $\mu$ -almost all  $b \in A$ , where  $E[\tau^{\mathfrak{A}}(\vec{a}, \cdot)|\mathcal{F}]: A \rightarrow \mathbb{R}$  denotes the conditional expectation of  $\tau^{\mathfrak{A}}(\vec{a}, \cdot): A \rightarrow \mathbb{R}$  with respect to  $\mathcal{F}$ ; that is, a  $\mathcal{F}$ -measurable function  $f: A \rightarrow \mathbb{R}$  such that for all  $B \in \mathcal{F}$ ,  $\int_B \tau^{\mathfrak{A}}(\vec{a}, \cdot) d\mu = \int_B f d\mu$ .

As a consequence of the well known Radon-Nikodym Theorem, it follows that the conditional expectation  $E[\tau^{\mathfrak{A}}(\vec{a}, \cdot)|\mathcal{F}]$  of a bounded  $\mu$ -measurable function  $\tau^{\mathfrak{A}}(\vec{a}, \cdot)$  with respect to  $\mathcal{F}$  exists and is almost surely unique. In fact,  $E[\tau^{\mathfrak{A}}(\vec{a}, \cdot)|\mathcal{F}]$  is the Radon-Nikodym derivative of the measure  $\nu(B) = \int_B \tau^{\mathfrak{A}}(\vec{a}, \cdot) d\mu$  for  $B \in \mathcal{F}$ , with respect to the measure  $\mu \upharpoonright \mathcal{F}$ . The conditional expectation operators are interpreted in such a manner that the interpretation over  $E$ -bound variables produces an  $\mathcal{F}$ -measurable random variable, but the interpretation over  $E$ -free variables produces a random variable which is not  $\mathcal{F}$ -measurable.

The atomic terms are truncated, and so each term is interpreted by a bounded and integrable random variable. Moreover, there is a uniform bound  $\|\tau\|$  on the interpretation of each term  $\tau$  of  $L_{Af}$  in arbitrary model  $\mathfrak{A}$  such that  $|\tau^{\mathfrak{A}}(\vec{a})| \leq \|\tau\|$ , for all  $\vec{a} \in A^n$ ; it is defined inductively by:

$$\begin{aligned} \|[X_i(\vec{x}) \upharpoonright r]\| &= r, & \|\mathbf{1}(x = y)\| &= 1, \\ \|r\| &= r, & \left\| \int \tau dx \right\| &= \|\tau\|, \\ \|E[\tau|y](z)\| &= \|\tau\|, & \|\mathbb{F}(\tau_1, \dots, \tau_n)\| &= \sup_{|s_i| \leq \|\tau_i\|} |F(s_1, \dots, s_n)|. \end{aligned}$$

The logic  $L_{AE}$  has all the axiom schemas and rules of inference of  $L_{Af}$  without  $(E_2)$  as well as the following axioms:

$(I_1)$  *Atomic terms axioms.*

- (i)  $\mathbf{1}(x = y) = 0 \vee \mathbf{1}(x = y) = 1$ ,
- (ii)  $[X_i(\vec{x}) \upharpoonright r] = \min(r, \max(-r, [X_i(\vec{x}) \upharpoonright s]))$ ,  $0 \leq r \leq s$ ,
- (iii)  $\bigvee_k [X_i(\vec{x}) \upharpoonright k + 1] \leq k$ ,
- (iv)  $\bigvee_k \int [X_i(\vec{x}) \upharpoonright k + 1] - [X_i(\vec{x}) \upharpoonright k] d\vec{x} \leq 1/m$ , for each  $m \in \mathbb{N}$ ;

(I<sub>2</sub>) *Conditional expectation axioms.*

- (i)  $E[\tau(\vec{x}, y)|y](z) = E[\tau(\vec{x}, u)|u](z)$  where  $y$  and  $u$  do not occur in  $\vec{x}$ ,
- (ii)  $\int E[\tau(\vec{x}, y)|y](y) \cdot \sigma(y) dy = \int E[\tau(\vec{x}, y)|y](y) \cdot E[\sigma(y)|y](y) dy$ .

The logic  $L_{AE}$  can be extended to a logic with two or more conditional expectation operators. Particularly interesting are the so-called discrete stochastic processes structures  $\mathfrak{A} = \langle A, X_n, \mathcal{F}_n, \mu \rangle_{n \in \mathbb{N}}$  where  $X_n: A \rightarrow \mathbb{R}$  is a random variable and  $\mathcal{F}_n$  is  $\sigma$ -algebra of  $\mu$ -measurable sets such that  $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$  for each  $n \in \mathbb{N}$ ; they may be covered by a logic  $L_{A\{E_n: n \in \mathbb{N}\}}$  with countably many conditional expectation operators. The interpretation of  $L_{A\{E_n: n \in \mathbb{N}\}}$  in  $\mathfrak{A}$  is anticipated, as follows:

$$(E_n[\tau(\vec{x}, y)|y](z))^{\mathfrak{A}}(\vec{a}, b) = E[\tau^{\mathfrak{A}}(\vec{a}, \cdot)|\mathcal{F}_n](b)$$

for each  $\vec{a} \in A^k$  and  $\mu$ -almost all  $b \in A$ .

Axioms and rules of inference for  $L_{A\{E_n: n \in \mathbb{N}\}}$  are those of  $L_{AE}$ , allowing each conditional expectation operator  $E_n$  to play the role of  $E$ , together with the following axiom:

(I<sub>3</sub>)  $E_n[\tau(\vec{x}, y)|y](y) = E_{n+1}[E_n[\tau(\vec{x}, y)|y](y)|y](y)$ , for each  $n \in \mathbb{N}$ .

By adding a new random variable symbol for each term with conditional expectation operator and using the Completeness Theorem for  $L_{Af}$ , we shall prove the Completeness Theorem for  $L_{A\{E_n: n \in \mathbb{N}\}}$ .

**4.3.5. Soundness and Completeness Theorem for  $L_{A\{E_n: n \in \mathbb{N}\}}$ .** Let  $T$  be a countable set of sentences of  $L_{A\{E_n: n \in \mathbb{N}\}}$ . Then,  $T$  has a generalized conditional expectation model  $\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, \mathcal{F}_n, \mu \rangle_{i \in I, n \in \mathbb{N}}$  with  $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$  for each  $n \in \mathbb{N}$  if and only if  $T$  is consistent in  $L_{A\{E_n: n \in \mathbb{N}\}}$ .

*Proof.* As usual, the proof of soundness is easy. For the other direction, let  $T$  be a consistent set of sentences in  $L_{A\{E_n: n \in \mathbb{N}\}}$ . Let  $K$  be the extension of  $L$  formed by adding a new random variable symbol  $X_\tau^n$  for each term  $\tau$  of the form  $E_n[\sigma(\vec{x}, y)|y](z)$ . A term  $\tau$  of  $L_{A\{E_n: n \in \mathbb{N}\}}$  can be translated into a term  $\mathcal{T}(\tau)$  of the logic  $K_{Af}(\mathbb{R})$  which is like  $K_{Af}$ , except that instead of the atomic terms  $\mathbf{1}(R(\vec{x}))$  this logic possesses the atomic terms  $[X_i(\vec{x}) \upharpoonright r]$ , in the following way:

- (1) If  $\tau$  is an atomic term, then  $\mathcal{T}(\tau)$  is  $\tau$ ;
- (2) If  $\tau$  is  $E_n[\sigma(\vec{x}, y)|y](z)$ , then  $\mathcal{T}(\tau)$  is  $[X_\tau^n(\vec{x}, z) \upharpoonright k]$ , where  $k \geq \|\tau\|$ ;
- (3) The translation  $\mathcal{T}$  commutes with the integral quantifiers and with each connective  $\mathbb{F}$ .

The translation of a formula  $\varphi$  of  $L_{A\{E_n: n \in \mathbb{N}\}}$  into a formula  $\mathcal{T}(\varphi)$  of  $K_{Af}(\mathbb{R})$  is then defined so that  $\mathcal{T}$  commutes with  $\neg$  and  $\wedge$ .

Let  $S$  be the theory consisting of all translations of sentences of  $T$ , all translations of theorems of  $L_{A\{E_n: n \in \mathbb{N}\}}$  and the sentence

$$\int |[X_\tau^n(\vec{x}) \upharpoonright k] - [X_\tau^n(\vec{x}) \upharpoonright l]| d\vec{x} = 0, \quad k, l \geq \|\tau\|.$$

Then,  $S$  is consistent in  $K_{Af}(\mathbb{R})$ .

We next use the Henkin construction to obtain a weak integral random variable model  $\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, X_\tau^{n\mathfrak{A}}, I \rangle$  of  $S$ . It follows by Loeb construction of Daniell integral,

that the internal weak structure  $*\mathfrak{A}$  produces a random variable probability model  $\overline{\mathfrak{A}} = \langle *A, X_i^{\overline{\mathfrak{A}}}, X_\tau^{n\overline{\mathfrak{A}}}, \mu \rangle$  of  $S$ , where

$$X_i^{\overline{\mathfrak{A}}}(\vec{a}) = \sup \{ r : [X_i(\vec{x}) \upharpoonright r]^{*\mathfrak{A}}(\vec{a}) \geq r \},$$

and  $X_\tau^{n\overline{\mathfrak{A}}}$  is defined exactly the same as  $X_i^{\overline{\mathfrak{A}}}$ . Let  $\mathcal{F}_n$  be the  $\sigma$ -algebra on  $*A$  generated by the sets  $\{ b \in *A : X_\tau^{n\overline{\mathfrak{A}}}(\vec{a}, b) \geq r \}$ , where  $\tau$  has the form  $E_n[\sigma(\vec{x}, y)|y](z)$  with  $z$   $E$ -bound in  $\tau$ ,  $\vec{a}$  is in  $*A$  and  $r \in \mathbb{R}$ . Let  $\mathfrak{B}$  be the reduct of  $\overline{\mathfrak{A}}$  to  $L$ . It follows from  $(I_3)$  that  $\mathcal{F}_n$ 's form an increasing sequence of  $\sigma$ -algebras, and it follows from  $(I_2)$  that the mapping  $\tau^{\mathfrak{B}} = \mathcal{T}(\tau)^{\overline{\mathfrak{A}}}$  is an interpretation of  $L_{\mathcal{A}\{E_n:n \in \mathbb{N}\}}$  in the generalized conditional expectation model  $\langle \mathfrak{B}, \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ . Using the axioms, it can be shown by induction on complexity of terms and formulas that a sentence  $\varphi$  of  $L_{\mathcal{A}\{E_n:n \in \mathbb{N}\}}$  holds in  $\mathfrak{B}$  if and only if  $\mathcal{T}(\varphi)$  holds in  $\overline{\mathfrak{A}}$ , and hence  $\mathfrak{B}$  is a model of  $T$ .  $\square$

The Existence and Uniqueness Theorems for  $L_{AE}$ -hyperfinite models hold (see [15]). The hyperfinite models for  $L_{AE}$  can be considered as a probability logic notion analogous to that of saturated models for the first-order logic, and hence the Robinson Consistency and Craig Interpolation Theorems hold for  $L_{AE}$ , too. The next section contains these properties for adapted probability logic  $L_{ad}$ , introduced as a two sorted form of the logic  $L_{AE}$ , which is appropriate for the study of continuous time stochastic processes. So, we shall consider previous notions and properties, in more detail, only for  $L_{ad}$ .

#### 4.4. Adapted probability logic

While the logic  $L_{AE}$  is concerned mainly with random variable structures, the adapted probability logic, denoted by  $L_{ad}$  and introduced by Keisler in [34] as a two sorted form of  $L_{AE}$ , is a logic adequate for a study of stochastic processes, and hence adapted probability spaces. The properties of stochastic integration and stochastic differential equations obtained just after the introduction of the Loeb measure and Anderson's nonstandard construction of Brownian motion in [2], can be naturally expressed in a language which has two sorts of variables, one for elements of the probability space and the other for real time, and two types of quantifiers corresponding to the integral and to the conditional expectation at time  $t$ .

Let  $\langle A, \mu \rangle$  be a probability space and let  $\beta$  be the Borel measure on the real unit interval  $[0, 1]$ . Recall that a *stochastic process* on  $A$  is a  $\mu \otimes \beta$ -measurable function  $X: A \times [0, 1] \rightarrow \mathbb{R}$ , representing a random quantity changing over the time which is represented by  $[0, 1]$ .

**4.4.1. Definition.** An adapted space is a structure  $\langle A, \mu, \mathcal{F}_t \rangle_{t \in [0, 1]}$ , where  $\mu$  is a probability measure on  $A$  and  $\{ \mathcal{F}_t : t \in [0, 1] \}$  is an increasing and right continuous filtration of  $\sigma$ -algebras of  $\mu$ -measurable sets; i.e., for each  $t \in [0, 1]$ ,  $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ .

Now we shall review the syntax of  $L_{ad}$ . Let  $L$  be an  $\mathcal{A}$ -recursive set  $\{ X_i : i \in I \}$  of stochastic processes symbols. The logic  $L_{ad}$  has just one *probabilistic variable*  $x$ , and countably many *time variables*  $t_1, t_2, \dots$ , first-order connectives  $\neg, \wedge, n$ -ary

function connective  $\mathbb{F}$  for each continuous function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $F \upharpoonright \mathbb{Q}^n \in \mathcal{A}$ , an integral quantifier symbol  $\int$  and a conditional expectation symbol  $E[\cdot|\cdot]$ .  $L_{ad}$  has no equality symbol.

**4.4.2. Definition.** The set of terms of  $L_{ad}$  is the least set such that:

- (1) Each atomic term  $[X_i(x, t) \upharpoonright r]$  for each  $r \in \mathbb{Q}_+$  is a term of  $L_{ad}$ ;
- (2) Each time variable  $t$  is a term;
- (3) Each real  $r \in \mathbb{R} \cap \mathcal{A}$  is a term;
- (4) If  $\tau$  is a term, then  $\int \tau dx$ ,  $\int \tau dt$  and  $E[\tau|s](x, t)$  are terms;
- (5) If  $\tau_1, \dots, \tau_n$  are terms, then  $\mathbb{F}(\tau_1, \dots, \tau_n)$  is a term.

The set of *formulas* of  $L_{ad}$  is defined exactly the same as for  $L_{AF}$  or  $L_{AE}$ . The notion of a free occurrence of variable in a term  $\tau$  of  $L_{ad}$  is introduced inductively by the complexity of  $\tau$  as usual, except that in  $E[\tau|s](x, t)$  all occurrences of  $s$  are bound, and  $x$  and  $t$  are free, but in  $E[\tau|s](x, s)$  the last occurrence of  $s$  is free, the others are bound. A *closed term (sentence)* is a term (formula) with no free variables.

A model relevant to our exposition consists of an adapted probability space and a family of continuous time stochastic processes on the space.

**4.4.3. Definition.** An adapted model for  $L$  is a structure of the form

$$\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, \mu, \mathcal{F}_t \rangle_{i \in I, t \in [0, 1]}$$

such that  $\langle A, \mu, \mathcal{F}_t \rangle_{t \in [0, 1]}$  is an adapted space, and each  $X_i^{\mathfrak{A}}: A \times [0, 1] \rightarrow \mathbb{R}$  is a stochastic process on  $A$ .

Thus, the adapted model for  $L$  is just the expansion of an adapted space to the language  $L$ .

**4.4.4. Definition.** An interpretation of  $L_{ad}$  in an adapted model  $\mathfrak{A}$  is a function which assigns to each term  $\tau(x, t_1, \dots, t_n)$  a function  $\tau^{\mathfrak{A}}: A \times [0, 1]^n \rightarrow \mathbb{R}$  such that:

- (1)  $[X_i(x, t) \upharpoonright r]^{\mathfrak{A}}(a, b) = \min(r, \max(-r, X_i^{\mathfrak{A}}(a, b)))$ ;
- (2)  $t^{\mathfrak{A}}(b) = b$ , for  $b \in [0, 1]$ ;
- (3)  $r^{\mathfrak{A}} = r$ ;
- (4)  $\left( \int \tau(x, \vec{t}) dx \right)^{\mathfrak{A}}(\vec{b}) = \int \tau^{\mathfrak{A}}(\cdot, \vec{b}) d\mu$ ;
- (5)  $\left( \int \tau(x, \vec{t}, s) ds \right)^{\mathfrak{A}}(a, \vec{b}) = \int \tau^{\mathfrak{A}}(a, \vec{b}, \cdot) d\beta$ ;
- (6)  $(E[\tau(x, s, t_1, \dots, t_n)|s](x, p))^{\mathfrak{A}}: A \times [0, 1]^{n+1} \rightarrow \mathbb{R}$  is  $\mathcal{F} \otimes \beta^n$ -measurable function<sup>1</sup> which satisfies the following conditions:
  - (a)  $(E[\tau(x, s, \vec{t})|s](x, p))^{\mathfrak{A}}(a, \vec{b}, c) = E[\tau^{\mathfrak{A}}(\cdot, \vec{b})|\mathcal{F}](a, c)$  for each  $n$ -tuple  $\vec{b}$  in  $[0, 1]$ , and for  $\mu \otimes \beta$ -almost all  $(a, c) \in A \times [0, 1]$ ,

<sup>1</sup>  $\mathcal{F}$  is so-called optional  $\sigma$ -algebra on  $A \times [0, 1]$  with respect to the filtration  $\{\mathcal{F}_t : t \in [0, 1]\}$  generated by the  $\mu \otimes \beta$ -measurable sets  $B$  such that the section  $B_t = \{x : (x, t) \in B\}$  belongs to  $\mathcal{F}_t$  and  $B_t = \cap_{s>t} B_s$  for each  $t \in [0, 1]$ .

- (b)  $\left(E[\tau(x, s, \vec{t})|s](x, p)\right)^{\mathfrak{A}}(a, \vec{b}, c) = E[\tau^{\mathfrak{A}}(\cdot, c, \vec{b})|\mathcal{F}_c](a)$  for each  $n$ -tuple  $\vec{b}$  and  $c$  in  $[0, 1]$ , and for  $\mu$ -almost all  $a \in A$ ;
- (7)  $\mathbb{F}(\tau_1, \dots, \tau_k)^{\mathfrak{A}}(a, \vec{b}) = F\left(\tau_1^{\mathfrak{A}}(a, \vec{b}), \dots, \tau_k^{\mathfrak{A}}(a, \vec{b})\right).$

The conditional expectation operators are interpreted in such a manner that any two interpretations of a term  $\tau(x, \vec{t})$  for all  $\vec{b}$  in  $[0, 1]$  agree at  $\tau(x, \vec{b})^{\mathfrak{A}}$  for  $\mu$ -almost all  $a \in A$ . The process  $X_i$  may be unbounded but the atomic terms  $[X_i(x, t) \upharpoonright r]$  truncate  $X_i$  at  $r$ , and consequently the interpretations of the terms built from the atomic terms are bounded, and hence integrable stochastic processes.

The *satisfaction relation*  $\mathfrak{A} \models \varphi[a, \vec{b}]$ , for a formula  $\varphi(x, \vec{t})$  of  $L_{ad}$  and  $a \in A$ ,  $\vec{b} \in [0, 1]^n$ , is defined as usual.

Rodenhausen in [61] gave a set of axioms and rules of inference for  $L_{ad}$  and proved a completeness theorem. In [34] Keisler gave a simpler proof of the Completeness Theorem by first proving the completeness of the sublogic  $L_d$  of distributions, which has only integrals and no conditional expectation quantifiers. A structure for  $L_d$  has the form  $\mathfrak{A} = \langle A, X_i^{\mathfrak{A}}, \mu \rangle_{i \in I}$ , where  $X_i^{\mathfrak{A}}$  is a stochastic process on  $A$ . The axiom schemas for  $L_d$  are similar to a two sorted form of the logic  $L_{Af}$ .

Now we shall list the axioms for  $L_d$  as given in [35].

(J<sub>1</sub>) Each instance of a tautology of propositional logic;

(J<sub>2</sub>)  $\bigwedge \Phi \rightarrow \varphi$ , where  $\varphi \in \Phi$ ;

(J<sub>3</sub>) *Atomic term axioms.*

$$(1) [X_i(x, t) \upharpoonright r] = \min(r, \max(-r, [X_i(x, t) \upharpoonright s])), \quad \text{when } 0 \leq r \leq s,$$

$$(2) \bigvee_k |[X_i(x, t) \upharpoonright k + 1]| \leq k,$$

$$(3) \bigvee_k \iint |[X_i(x, t) \upharpoonright k + 1] - [X_i(x, t) \upharpoonright k]| \, dx dt \leq 1/m, \quad \text{for each } m \in \mathbb{N};$$

(J<sub>4</sub>) *Order axioms.*

$$(1) r \geq r,$$

$$(2) \tau \geq r \rightarrow \tau \geq s, \quad \text{when } r \geq s,$$

$$(3) 0 \leq t \wedge t \leq 1;$$

(J<sub>5</sub>)  $\langle \tau_1, \dots, \tau_k \rangle \in S \rightarrow \mathbb{F}(\tau_1, \dots, \tau_k) \in [a, b]$ , for each rational closed rectangle  $S \subseteq \mathbb{R}^k$  with  $F(S) = [a, b]$ ;

(J<sub>6</sub>) *Integral axioms.*

$$(1) \int r \, dx = r \quad \text{and} \quad \int r \, dt = r,$$

$$(2) \int \mathbb{F}(t) \, dt = r, \quad \text{for each continuous function } F: [0, 1] \rightarrow \mathbb{R} \text{ such that } \int_0^1 F(s) \, ds = r,$$

$$(3) \int \tau(t, \dots) \, dt = \int \tau(s, \dots) \, ds,$$

$$(4) \iint \tau(s, t, \dots) \, ds dt = \iint \tau(s, t, \dots) \, dt ds,$$

$$(5) \int (r \cdot \tau + q \cdot \sigma) \, dx = r \cdot \int \tau \, dx + q \cdot \int \sigma \, dx,$$

$$(6) \int (r \cdot \tau + q \cdot \sigma) \, dt = r \cdot \int \tau \, dt + q \cdot \int \sigma \, dt,$$

$$(7) \bigwedge_{m \in \mathbb{N}} \bigvee_n \iint |\tau(x, s, \dots) - \tau(x, t, \dots)| \cdot \max(0, 1 - |s - t| \cdot n) \, ds dt dx \leq \frac{1}{m \cdot n}.$$

Axioms for the logic  $L_{ad}$  are those of  $L_d$  together with the following axiom:

(J<sub>7</sub>) *Conditional expectation axioms.*

- (1)  $E[\tau(s, \dots)|s](x, p) = E[\tau(t, \dots)|t](x, p),$
- (2)  $\int E[\tau|s](x, s) \cdot \sigma \, dx = \int E[\tau|s](x, s) \cdot E[\sigma|s](x, s) \, dx,$
- (3)  $s \leq t \rightarrow E[\tau|s](x, s) = E[E[\tau|s](x, s)|t](x, t).$

The *rules of inference* for both the logics  $L_d$  and  $L_{ad}$  are:

- (K<sub>1</sub>) (*Modus Ponens*)  $\varphi, \varphi \rightarrow \psi \vdash \psi;$
- (K<sub>2</sub>) (*Conjunction*)  $\{\varphi \rightarrow \psi : \psi \in \Psi\} \vdash \varphi \rightarrow \bigwedge \Psi;$
- (K<sub>3</sub>) (*Generalization*)

- (1)  $\varphi \rightarrow (\tau(x, s, \vec{t}) \geq 0) \vdash \varphi \rightarrow \int \tau(x, s, \vec{t}) \, dx \geq 0,$  where  $x$  is not free in  $\varphi,$
- (2)  $\varphi \rightarrow (\tau(x, s, \vec{t}) \geq 0) \vdash \varphi \rightarrow \int \tau(x, s, \vec{t}) \, ds \geq 0,$  where  $s$  is not free in  $\varphi.$

All the axioms represent known properties of stochastic processes and adapted probability spaces, and hence the Soundness Theorems for  $L_d$  and  $L_{ad}$  hold. We shall mention a proof of the Completeness Theorem for  $L_d$ , which contains a weak model-strong model construction as for  $L_{Af}$ , and all exceptions will be indicated.

**4.4.5. Completeness Theorem for  $L_d$ .** *A countable set  $\Phi$  of sentences consistent in  $L_d$  has a  $L_d$ -model.*

*Proof.* First, a Henkin construction is used to get a weak integral structure  $\mathfrak{A} = \langle A, T, X_i^{\mathfrak{A}}, G, H \rangle_{i \in I}$  of  $\Phi$ , where  $A$  and  $T$  are non-empty sets,  $G$  is a function from the terms of  $L_d$  with parameters from  $T$  and only  $x$  free into  $\mathbb{R}$ , and  $H$  is a function from the terms with parameters from  $A \cup T$  with only  $t$  free into  $\mathbb{R}$ .<sup>2</sup> The logic  $L_d$  can be interpreted in  $\mathfrak{A}$ . We point out the integral cases only:

$$\begin{aligned} \left( \int \tau(x, \vec{t}) \, dx \right)^{\mathfrak{A}}(\vec{b}) &= G(\tau(x, \vec{b})) \quad \text{and} \\ \left( \int \tau(x, s, \vec{t}) \, ds \right)^{\mathfrak{A}}(a, \vec{b}) &= H(\tau(a, s, \vec{b})) \quad \text{for } a \in A \text{ and } \vec{b} \text{ in } T. \end{aligned}$$

Then, we form the internal weak model  $^*\mathfrak{A}$  in an  $\omega_1$ -saturated nonstandard universe. The Loeb construction of Daniell integral produces a probability measure  $\mu$  on  $^*A$  such that  $^\circ(^*G(\tau)) = \int ^\circ(\tau^{*\mathfrak{A}}) \, d\mu$ , for each term  $\tau(x)$  of  $L_d$  with parameters from  $^*T$ , and a probability measure  $\delta$  on  $^*T$  such that for each term  $\tau(s)$  with parameters from  $^*A \cup ^*T$ ,  $^\circ(^*H(\tau)) = \int ^\circ(\tau^{*\mathfrak{A}}) \, d\delta$ . The measure  $\delta$  is a lifting of the Borel measure  $\beta$  on  $[0, 1]$  since, from the integral axioms, for every Borel set  $B \subseteq [0, 1]$ ,  $\delta(\text{st}^{-1}(B)) = \beta(B)$ , where the “standard map”  $\text{st}: ^*T \rightarrow [0, 1]$  is defined by  $\text{st}(b) = \sup\{r \in \mathbb{R} : ^*\mathfrak{A} \models (t \geq r)[b]\}$ . Let  $X_i^{*\mathfrak{A}}: ^*A \times I \rightarrow \mathbb{R}$  be a  $\mu \otimes \beta$ -measurable map defined by

$$X_i^{*\mathfrak{A}}(a, \text{st}(b)) = \sup\{r \in \mathbb{R} : ^*\mathfrak{A} \models [X_i(x, t) \upharpoonright r](a, b)\}.$$

By induction on the complexity of formulas, it follows that  $\langle ^*A, X_i^{*\mathfrak{A}}, \mu \rangle$  is  $L_d$ -equivalent to  $\mathfrak{A}$ , and hence it is a model of  $\Phi$  as required.  $\square$

<sup>2</sup>  $G$  and  $H$  may be called an  $\mathcal{A}$ -Daniell integrals on  $\mathfrak{A}$ .

We point out that the above proof avoids graded probability structures and constructions involving product measures, because  $L_d$  has only one probability variable.

**4.4.6. Completeness Theorem for  $L_{ad}$ .** A countable set  $\Phi$  of sentences consistent in  $L_{ad}$  has an adapted model.

*Proof.* The proof is similar to that for  $L_{AE}$  with only minor changes in the notation and constructions done. Let  $K$  be the extension of  $L$  formed by adding a new stochastic process symbol  $X_\tau$  for each term  $\tau$  of the form  $E[\sigma(x, p, \vec{t})|p](x, s)$ . Let  $\mathcal{T}$  be the translation from  $L_{ad}$  into  $K_d$  defined in the same way as in 4.3.5. Let  $\Sigma$  be the theory consisting of all translations of  $\Phi$ , all translations of theorems of  $L_{ad}$ , and the sentence  $\iint \iint |[X_\tau(x, s, \vec{t}) \upharpoonright k] - [X_\tau(x, s, \vec{t}) \upharpoonright l]| dx ds d\vec{t} = 0, k, l \geq n$ , which states that  $X_\tau$  is almost everywhere bounded by  $n$  – the least natural number which is an upper bound for  $\|\tau\|$ . Then,  $\Sigma$  is consistent in  $K_d$ . So, let  $\mathfrak{A}_K = \langle A, X_i^\mathfrak{A}, X_\tau^\mathfrak{A}, \mu \rangle$  be a  $K_d$ -model of  $\Sigma$ . For each  $s \in [0, 1]$ , let  $\mathcal{F}_s$  be the  $\sigma$ -algebra of subsets of  $A$  generated by the sets  $\{a \in A : X_\tau^\mathfrak{A}(a, s, \vec{b}) \geq r\}$ , where  $\tau$  has the form  $E[\sigma(x, p, \vec{t})|p](x, s)$ ,  $\vec{b}$  is a tuple of elements of  $[0, 1]$ , and  $r \in \mathbb{R}$ . Let  $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s, \mathcal{G}_1 = \mathcal{F}_1$ . It follows from  $(J_7)(3)$  that  $\mathcal{F}_s$  is increasing in  $s$ , and hence that  $\{\mathcal{G}_t : t \in [0, 1]\}$  is increasing and right continuous. Let  $\mathfrak{A}_L = \langle A, X_i^\mathfrak{A}, \mu \rangle$  be the reduct of  $\mathfrak{A}_K$  to  $L$ . Then,  $\mathfrak{A} = \langle \mathfrak{A}_L, \mathcal{G}_t \rangle_{t \in [0, 1]}$  is an adapted model. It follows from  $(J_7)(2)$  that  $\tau^\mathfrak{A} = (\mathcal{T}(\tau))^{\mathfrak{A}_K}$  is an interpretation of  $L_{ad}$  in  $\mathfrak{A}$ . Finally, it follows by induction that  $\mathfrak{A}$  is an adapted model of  $\Phi$ .  $\square$

Now we mention how to axiomatize in  $L_{ad}$  some basic notions of the general theory of stochastic processes such as stopping time and optional stochastic process. We also present all notions that are relevant to our exposition, and completeness theorems will be mentioned without proofs (see Fajardo [14]).

A random variable  $S: A \rightarrow [0, 1]$  is a *stopping time* with respect to the filtration  $\{\mathcal{F}_t : t \in [0, 1]\}$  if for all  $t \in [0, 1]$ ,  $\{a \in A : S(a) < t\} \in \mathcal{F}_t$ . So, we suppose that  $L$  has a random variable symbol  $S(x)$  as a non-logical symbol, and that the definition of terms of  $L_{ad}$  is modified in the obvious way with  $[S(x) \upharpoonright r]$  in place of the atomic terms  $[X_i(x, t) \upharpoonright r]$ . A structure  $\mathfrak{A} = \langle A, S^\mathfrak{A}, \mu, \mathcal{F}_t \rangle_{t \in [0, 1]}$  is a *stopping time model* for  $L$  if  $S^\mathfrak{A}$  is a stopping time with respect to the filtration  $\{\mathcal{F}_t : t \in [0, 1]\}$ .

The definition of stopping time immediately suggests a natural set of axioms in  $L_{ad}$  for a stopping time model. If we add the axioms

$$(M_1) \quad \bigwedge_{r \in \mathbb{Q}_+} \left( \int \min(0, [S(x) \upharpoonright r]) dx \geq 0 \wedge \int (\min(1, [S(x) \upharpoonright r]) - [S(x) \upharpoonright r]) dx \geq 0 \right);$$

$$(M_2) \quad \bigwedge_{r \in \mathbb{Q}_+} \bigvee_k \int \mathbb{E}_k \left( \int |E[\min([S(x) \upharpoonright r], t)|t](x, t) - \min([S(x) \upharpoonright r], t)|dx \right) dt \geq 1,$$

$$\text{where } F_k(a) = \begin{cases} 1, & \text{if } a \leq 0 \\ \text{linear}, & \text{if } 0 \leq a \leq 1/k, \\ 0, & \text{if } 1/k \leq a \end{cases} \text{ to the list of axioms for } L_{ad},$$

then we obtain the following theorem.

**4.4.7. Soundness and Completeness Theorem for Stopping Time.** A countable set  $T$  of sentences of  $L_{ad}$  has a stopping time model iff  $T \cup \{M_1, M_2\}$  is consistent in  $L_{ad}$ .  $\square$

Let  $\langle A, \mu, \mathcal{F}_t \rangle_{t \in [0,1]}$  be an adapted space. The  $\sigma$ -algebra  $\mathcal{F}$  on  $A \times [0, 1]$  generated by the  $\mu \otimes \beta$ -measurable sets  $B$  such that the section  $B_t = \{a : (a, t) \in B\}$  belongs to  $\mathcal{F}_t$  and  $B_t = \bigwedge_{s>t} B_s$ , for each  $t \in [0, 1]$ , is called the *optional  $\sigma$ -algebra* on  $A \times [0, 1]$  with respect to the filtration  $\{\mathcal{F}_t : t \in [0, 1]\}$ . A stochastic process  $X: A \times [0, 1] \rightarrow \mathbb{R}$  is *optional* if it is measurable with respect to  $\mathcal{F}$ . A structure  $\mathfrak{A} = \langle A, X^{\mathfrak{A}}, \mu, \mathcal{F}_t \rangle$  is an *optional model* for  $L$  if  $X^{\mathfrak{A}}$  is an optional stochastic process.

**4.4.8. Completeness Theorem for Optional Process.** *Let  $T$  be a countable set of sentences in  $L_{ad}$ . If  $T$  is consistent in  $L_{ad}$  with added sentence<sup>3</sup>*

$$\bigwedge_{r \in \mathbb{Q}_+} \bigvee_k \int \mathbb{F}_k \left( \int |E[[X(x, t) \upharpoonright r] | t](x, t) - [X(x, t) \upharpoonright r]| dx \right) dt \geq 1,$$

*then  $T$  has an optional model.*  $\square$

In a similar way, it is possible to axiomatize in  $L_{ad}$  notions of adapted and predictable stochastic processes, martingales, etc (see Fajardo [14]). As an indication of the expressive power of the logic  $L_{ad}$ , we shall describe notions of martingales, Markov processes and Brownian motion by  $L_{ad}$ -sentences. We shall write  $\tau(x, s) = \sigma(x, s)$  a. s. for  $\int |\tau(x, s) - \sigma(x, s)| dx \leq 0$ .

**4.4.9. Example.** (a) A stochastic process  $\tau$  is a martingale if

$$s \leq t \rightarrow \tau(x, s) = E[\tau(x, t) | s](x, s) \text{ a. s.};$$

(b)  $\tau$  is a Markov process with continuous transition function  $F \mapsto T_F$  if

$$s \leq t \rightarrow E[\mathbb{F}(\tau(x, t)) | s](x, s) = \mathbb{T}_F(s, t, \tau(x, s)) \text{ a. s.};$$

(c)  $\tau$  is a Brownian motion if:

$\tau$  is a martingale,

$$s = 0 \rightarrow \tau(x, s) = 0 \text{ a. s.},$$

$$s \leq t \rightarrow E[(\tau(x, t) - \tau(x, s))^2 | s](x, s) = t - s \text{ a. s.},$$

$$s \leq t \rightarrow E[\mathbb{F}(\tau(x, t) - \tau(x, s)) | s](x, s) = \int \mathbb{F}(\tau(x, t) - \tau(x, s)) dx \text{ a. s.}$$

We assume throughout the last part of this section that the language  $L$  has only countably many stochastic process symbols and no constant symbols, and we consider only adapted spaces  $\langle A, \mu, \mathcal{F}_t \rangle_{t \in [0,1]}$  with the additional property that  $\mathcal{F}_1$  is the set of all  $\mu$ -measurable sets. The following two sorts of the elementary equivalence relation in  $L_{ad}$  are important.

**4.4.10. Definition.** Two adapted probability models  $\mathfrak{A}$  and  $\mathfrak{B}$  for  $L_{ad}$  are weakly equivalent, denoted by  $\mathfrak{A} \equiv^w \mathfrak{B}$ , if they satisfy the same sentences of  $L_{ad}$ .

<sup>3</sup> where  $F: R \rightarrow R$  is defined as in  $(M_2)$ .

**4.4.11. Definition.**  $\mathfrak{A}$  and  $\mathfrak{B}$  are strongly equivalent, denoted by  $\mathfrak{A} \equiv^s \mathfrak{B}$ , if they satisfy the same formulas of  $L_{ad}$  with parameters from  $[0, 1]$ ; that is, for each tuple  $\vec{a}$  from  $[0, 1]$  and a formula  $\varphi(\vec{t})$  in which  $x$  is not free,  $\mathfrak{A} \models \varphi[\vec{a}]$  iff  $\mathfrak{B} \models \varphi[\vec{a}]$ .

The difference is that two models are weakly equivalent if they satisfy the same formulas of  $L_{ad}$  for almost all real times, and are strongly equivalent if they satisfy the same formulas for all real times. The notions such as martingale, Markov process and Brownian motion, are preserved under strong equivalence but not under weak equivalence, so we shall be particularly interested in the strong equivalence relation. The following definition is motivated by the corresponding notions from the classical model theory.

**4.4.12. Definition.** (a) An adapted space  $\mathfrak{A} = \langle A, \mu, \mathcal{F}_t \rangle$  is universal if every adapted model for  $L$  is strongly equivalent to an expansion  $\langle A, X_i^{\mathfrak{A}}, \mu, \mathcal{F}_t \rangle$  of  $\mathfrak{A}$ .

(b) An adapted space  $\mathfrak{A}$  is homogeneous if for any two expansions  $\mathfrak{B}$  and  $\mathfrak{C}$  of  $\mathfrak{A}$  such that  $\mathfrak{B} \equiv^s \mathfrak{C}$  there exists a bijection<sup>4</sup>  $h: A \rightarrow A$  which preserves the measure  $\mu$ , maps each  $\mathcal{F}_t$  onto itself, and for all  $t \in [0, 1]$  and each  $i \in I$ ,  $X_i^{\mathfrak{B}}(a, t) = X_i^{\mathfrak{C}}(ha, t)$ , for  $\mu$ -almost all  $a \in A$ .

(c) An adapted space  $\mathfrak{A}$  is saturated if whenever  $L \subseteq L'$ ,  $\mathfrak{B}$  is an expansion of  $\mathfrak{A}$  to  $L$ ,  $\mathfrak{C}$  is an adapted model for  $L$  strongly equivalent to  $\mathfrak{B}$ , and  $\mathfrak{C}'$  is an expansion of  $\mathfrak{C}$  to  $L'$ , there exists an expansion  $\mathfrak{B}'$  of  $\mathfrak{B}$  to  $L'$  such that  $\mathfrak{B}' \equiv^s \mathfrak{C}'$ .

The main result in [28] states that every adapted Loeb space which admits a Brownian motion is saturated, where an adapted space  $\langle A, \mu, \mathcal{F}_t \rangle_{t \in [0, 1]}$  is called adapted Loeb space if for some internal  $\ast$ -adapted space  $\langle A, \nu, \mathcal{G}_s \rangle_{s \in {}^\ast[0, 1]}$ , the measure  $\mu$  is the completion of the Loeb measure of  $\nu$ , and  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by  $\bigcup_{s=t} \mathcal{G}_s \cup (\text{null sets of } \mu)$ . An adapted Loeb space which admits a Brownian motion exists by Anderson's construction from [2], and hence saturated adapted spaces do exist.

Now we introduce a hyperfinite adapted spaces constructed in an  $\omega_1$ -saturated nonstandard universe. Let  $T = \{0, 1/H, 2/H, \dots, 1\}$ , for  $H \in {}^\ast\mathbb{N} \setminus \mathbb{N}$ , be a hyperfinite time line, and let  $A$  be a hyperfinite set with at least two elements. Thus, the set  $A^T$  of all internal functions from  $T$  into  $A$  is a hyperfinite set. For each  $s \in {}^\ast[0, 1]$ , the internal equivalence relation  $\approx_s$  on  $A^T$  is defined by

$$f \approx_s g \text{ iff } f(t) = g(t), \text{ for all } t \in T \text{ such that } t \leq s.$$

**4.4.13. Definition.** The adapted space  $\langle \Omega, \mu, \mathcal{F}_t \rangle_{t \in [0, 1]}$  is a hyperfinite adapted space if  $\Omega = A^T$ ,  $\mu$  is the Loeb counting measure on  $\Omega$ , and  $\mathcal{F}_t$  is the  $\sigma$ -algebra of all  $\mu$ -measurable sets  $B$  such that: if  $f \in B$  and  $g \approx_s f$  for all  $s \in T$  with  ${}^s s = t$ , then  $g \in B$ .

So, a hyperfinite adapted space consists of the set of all internal paths mapping a hyperfinite time line into a hyperfinite set, with the Loeb counting measure, and the family of  $\sigma$ -algebras of all events which depend only on the path for times with standard part at most  $t$ , where  $t \in [0, 1]$ . A hyperfinite adapted

<sup>4</sup> the so-called strong adapted isomorphism from  $\mathfrak{B}$  to  $\mathfrak{C}$ .

model  $\mathfrak{A} = \langle \Omega, X_i^{\mathfrak{A}}, \mu, \mathcal{F}_t \rangle_{i \in I, t \in [0,1]}$  for  $L$  agrees with the integral and the conditional expectation operators. For instance, if  $X^{\mathfrak{A}}: \Omega \times [0, 1] \rightarrow \mathbb{R}$  is a stochastic process on  $\mathfrak{A}$  and if an internal function  $Y: \Omega \times T \rightarrow {}^*\mathbb{R}$  is a lifting of  $X^{\mathfrak{A}}$ ,<sup>5</sup> then  $\sum_{k=1}^H Y(f, k/H) \cdot 1/H$  is a lifting of the integral  $\int_0^1 X^{\mathfrak{A}}(f, s) ds$ . Similarly, if  $X: \Omega \rightarrow \mathbb{R}$  is a bounded random variable and  $Y: \Omega \rightarrow {}^*\mathbb{R}$  is a finitely bounded lifting of  $X$ , then for every sufficiently large  $s \in T$  with  ${}^\circ s = t$ ,  $\sum_{g \in (f|s)} X(g) / |(f|s)|$  is a lifting of  $E[X|\mathcal{F}_t](f)$ , where  $(f|s)$  is an internal subset of  $\Omega$  defined by  $(f|s) = \{g \in \Omega : g(t) = f(t) \text{ for all } t \in T \text{ with } t \leq s\}$ , and  $|(f|s)|$  denotes the internal cardinality of  $(f|s)$ .

The main result in [36] states that a hyperfinite adapted spaces have homogeneity properties which enable them to play a role analogous to the one that saturated models play in the first-order logic. As a consequence of the homogeneity of hyperfinite adapted spaces, we shall mention the Robinson consistency and the Craig interpolation properties for  $L_{ad}$ .

**4.4.14. Robinson Consistency Theorem.** *Let  $L^1$  and  $L^2$  be two languages with  $L^0 = L^1 \cap L^2$ , and let  $\mathfrak{A}^1$  and  $\mathfrak{A}^2$  be two adapted probability models for  $L^1$  and  $L^2$  respectively whose reductions  $\mathfrak{A}^1 \upharpoonright L^0$  and  $\mathfrak{A}^2 \upharpoonright L^0$  are strongly equivalent. Then there exists an adapted model  $\mathfrak{B}$  for  $L^1 \cup L^2$  such that  $\mathfrak{B} \upharpoonright L^1 \equiv^s \mathfrak{A}^1$  and  $\mathfrak{B} \upharpoonright L^2 \equiv^s \mathfrak{A}^2$ .*

*Proof.* There are hyperfinite adapted models  $\mathfrak{B}^1$  and  $\mathfrak{B}^2$  such that  $\mathfrak{B}^1 \equiv^s \mathfrak{A}^1$  and  $\mathfrak{B}^2 \equiv^s \mathfrak{A}^2$ . So,  $\mathfrak{B}^1 \upharpoonright L^0 \equiv^s \mathfrak{B}^2 \upharpoonright L^0$ , and by homogeneity we can take the strong adapted isomorphism from  $\mathfrak{B}^1 \upharpoonright L^0$  to  $\mathfrak{B}^2 \upharpoonright L^0$  to be the identity map. Thus, the common expansion  $\mathfrak{B}$  of  $\mathfrak{B}^1$  and  $\mathfrak{B}^2$  satisfies the required properties.  $\square$

**4.4.15. Craig Interpolation Theorem.** *Let  $\varphi$  be an  $L_{ad}^1$ -sentence and let  $\psi$  be an  $L_{ad}^2$ -sentence such that  $\models \varphi \rightarrow \psi$ . Then, there is a  $L_{ad}^0$ -sentence  $\theta$  such that  $\models \varphi \rightarrow \theta$  and  $\models \theta \rightarrow \psi$ , where  $L^0 = L^1 \cap L^2$ .*

*Proof.* If there is no such  $\theta$ , then by a Henkin construction there are adapted models  $\mathfrak{A}^1$  of  $\varphi$  and  $\mathfrak{A}^2$  of  $\neg\psi$  such that  $\mathfrak{A}^1 \upharpoonright L^0 \equiv^s \mathfrak{A}^2 \upharpoonright L^0$ . By Robinson consistency,  $\varphi \wedge \neg\psi$  has an adapted model, this is a contradiction to  $\models \varphi \rightarrow \theta$ .  $\square$

A number of applications of hyperfinite adapted and adapted Loeb structures to standard probability theory can be found in [26], [28], [36].

<sup>5</sup>  ${}^\circ Y(f, t) = X^{\mathfrak{A}}(f, {}^\circ t)$  almost surely in  $\mu \otimes \beta$ .

## LOGIC WITH INFINITARY MEASURE

The subject of this chapter is to introduce the logic  $L_{AM}$ , which, instead of a probability measure, has a  $\sigma$ -finite one. We give a method how to transfer results from  $L_{AP}$  to the new logic (see [47]). The logic  $L_{AM}$  is like probability logic except that quantifiers  $(M\vec{x} \geq r)$ ,  $(M\vec{x} > r)$ ,  $(M\vec{x} \leq r)$  and  $(M\vec{x} < r)$  (for  $r \in \{s \in \mathbb{R} : s \geq 0\} \cup \{\infty\}\}$ ) are used instead of probability quantifiers  $(P\vec{x} \geq r)$ . Of course, we can not express here other types of quantifiers by  $(M\vec{x} \geq r)$ , as it was possible in the probability case.

We suppose that the equality and a countable sequence  $\{R_n : n \in \mathbb{N}\}$  of unary relational symbols are logical symbols.

Now, we give a list of axioms and rules of inference.

The axioms for  $L_{AM}$  are as follows:

(L) All axioms of  $L_A$  without quantifiers;

(H<sub>1</sub>) Axioms about monotonicity

$$(M\vec{x} \geq 0)\varphi(\vec{x}),$$

$$(M\vec{x} > r)\varphi(\vec{x}) \rightarrow (M\vec{x} \geq r)\varphi(\vec{x}), \quad (M\vec{x} < r)\varphi(\vec{x}) \rightarrow (M\vec{x} \leq r)\varphi(\vec{x}),$$

$$(M\vec{x} \geq r)\varphi(\vec{x}) \rightarrow (M\vec{x} \geq s)\varphi(\vec{x}), \quad (M\vec{x} \leq s)\varphi(\vec{x}) \rightarrow (M\vec{x} \leq r)\varphi(\vec{x}), \quad s \leq r;$$

(H<sub>2</sub>) Axioms about measure quantifiers

$$(M\vec{x} \geq r)\varphi(\vec{x}) \rightarrow (M\vec{y} \geq r)\varphi(\vec{y}), \quad (M\vec{x} > r)\varphi(\vec{x}) \rightarrow (M\vec{y} > r)\varphi(\vec{y}),$$

$$(M\vec{x} \leq r)\varphi(\vec{x}) \rightarrow (M\vec{y} \leq r)\varphi(\vec{y}), \quad (M\vec{x} < r)\varphi(\vec{x}) \rightarrow (M\vec{y} < r)\varphi(\vec{y});$$

(H<sub>3</sub>) Axioms of finite additivity

$$((M\vec{x} \leq r)\varphi(\vec{x}) \wedge (M\vec{x} \leq s)\psi(\vec{x})) \rightarrow (M\vec{x} \leq r+s)(\varphi(\vec{x}) \vee \psi(\vec{x})),$$

$$\begin{aligned} & \left( (M\vec{x} \geq r)\varphi(\vec{x}) \wedge (M\vec{x} \geq s)\psi(\vec{x}) \wedge (M\vec{x} \leq 0)(\varphi(\vec{x}) \wedge \psi(\vec{x})) \right) \rightarrow \\ & \rightarrow (M\vec{x} \geq r+s)(\varphi(\vec{x}) \vee \psi(\vec{x})); \end{aligned}$$

(A) The Archimedean property

$$(M\vec{x} > r)\varphi(\vec{x}) \leftrightarrow \bigvee_n (M\vec{x} \geq r + 1/n)\varphi(\vec{x}), \quad r \neq \infty$$

$$(M\vec{x} < r)\varphi(\vec{x}) \leftrightarrow \bigvee_n (M\vec{x} \leq r - 1/n)\varphi(\vec{x});$$

(N<sub>1</sub>) Axiom of countable additivity

$$\bigwedge_{\varphi \in \Phi} (M\vec{x} < \infty) \varphi(\vec{x}) \rightarrow \left( \bigwedge_{\Psi \subseteq \Phi} (M\vec{x} \geq r) \wedge \Psi \rightarrow (M\vec{x} \geq r) \wedge \Phi \right), \quad r \neq \infty,$$

where  $\Psi$  ranges over the finite subsets of  $\Phi$ ;

(N<sub>2</sub>) Axioms of symmetry ( $\pi$  is a permutation of  $\{1, \dots, n\}$ )

$$(Mx_1 \dots x_n \geq r) \varphi \leftrightarrow (Mx_{\pi 1} \dots x_{\pi n} \geq r) \varphi,$$

$$(Mx_1 \dots x_n \leq r) \varphi \leftrightarrow (Mx_{\pi 1} \dots x_{\pi n} \leq r) \varphi;$$

(N<sub>3</sub>) Axiom of product independence

$$(M\vec{x} \geq r)(M\vec{y} \geq s) \varphi(\vec{x}, \vec{y}) \rightarrow (M\vec{x}\vec{y} \geq rs) \varphi(\vec{x}, \vec{y}),$$

where all variables in  $\vec{x}, \vec{y}$  are distinct;

(N<sub>4</sub>) Axiom of product measurability

$$(M\vec{x}\vec{y}\vec{z} \geq s^k) R_n^k(\vec{x}, \vec{y}, \vec{z}) \rightarrow$$

$$\rightarrow (M\vec{x} \geq s^t)(M\vec{y} > 0)(M\vec{z} \geq r^l) (\varphi(\vec{x}, \vec{z}) \leftrightarrow \varphi(\vec{y}, \vec{z})) \wedge R_n^k(\vec{x}, \vec{y}, \vec{z}), \quad s \geq r,$$

where  $R_n^m(\vec{x})$  is an abbreviation for the conjunction  $R_n(x_1) \wedge \dots \wedge R_n(x_m)$  and  $k, t, l$  are lengths of the sequences  $\vec{x}\vec{y}\vec{z}, \vec{x}, \vec{z}$  respectively and  $k = 2t + l$ ;

( $\sigma$ ) Axiom of  $\sigma$ -finiteness

$$(Mx < \infty) R_i(x) \wedge (Mx > i) R_i(x) \wedge (Mx \leq 0) (R_i(x) \wedge \neg R_{i+1}(x)), \quad i \in \mathbb{N};$$

(C) Axiom for constants

$$\bigvee_{i \in \mathbb{N}} R_i(d) \quad \text{for each constant } d.$$

The rules of inference for  $L_{AM}$  are as follows:

$$(MP) \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \text{Modus Ponens};$$

$$(CO) \quad \frac{\varphi \rightarrow \psi, \psi \in \Psi}{\varphi \rightarrow \wedge \Psi} \quad \text{Conjunction};$$

$$(GE) \quad \frac{\psi \rightarrow \varphi(\vec{x})}{\psi \rightarrow (M\vec{x} \geq \infty) \varphi(\vec{x})}, \quad \frac{\psi \rightarrow \varphi(\vec{x})}{\psi \rightarrow (M\vec{x} \leq 0) \neg \varphi(\vec{x})} \quad \text{Generalization};$$

$$(\sigma_0) \quad \frac{\varphi \rightarrow (M\vec{x} \leq r) (\psi(\vec{x}) \wedge R_i^n(\vec{x})), \quad i \in \mathbb{N}}{\varphi \rightarrow (M\vec{x} \leq r) \psi(\vec{x})} \quad \sigma\text{-finiteness}.$$

Let  $\langle A, \mathcal{B}, \mu \rangle$  be a  $\sigma$ -finite measure space such that each singleton is measurable. Then  $\langle A^n, \mathcal{B}^{(n)}, \mu^{(n)} \rangle$  is a measure space, such that  $\mathcal{B}^{(n)}$  is the  $\sigma$ -algebra generated by the measurable rectangles and the diagonal sets  $\{\vec{x} \in A^n : x_k = x_s\}$ , and measure  $\mu^{(n)}$  is the restriction of the completion of  $\mu^n$  to  $\mathcal{B}^{(n)}$ .

**5.1. Definition.** A measure model for  $L$  is an ordered pair  $\langle \mathfrak{A}, \mu \rangle$ , where  $\mathfrak{A}$  is an ordinary model and  $\mu$  is a  $\sigma$ -finite measure on  $A$  such that each singleton is measurable and each relation  $R^{\mathfrak{A}} \subseteq A^m$  is  $\mu^{(m)}$ -measurable.

Before we prove the completeness theorem we shall show that a measure model  $\mathfrak{A}$  is completely determined by its restrictions  $\mathfrak{A}_i$  on  $R_i$ .

Let  $L_{Am}$  be a logic similar to  $L_{AP}$  except that real numbers occurring in quantifiers are from the interval  $[0, m]$ . Specially, for  $m = 1$ ,  $L_{A1} = L_{AP}$ .

The following definition is important.

**5.2. Definition.** The set of bounded formulas (that is  $B$ -formulas) of  $L_{AM}$  is the least set such that:

- (1) Each atomic formula is a  $B$ -formula;
- (2) A conjunction and a negation of  $B$ -formulas is a  $B$ -formula;
- (3) If  $\varphi$  is a  $B$ -formula then, for  $r \in \mathbb{R}$  and  $i \in \mathbb{N}$ ,  $(M\vec{x} \leq r)(\varphi(\vec{x}) \wedge R_i^n(\vec{x}))$  and  $(M\vec{x} > r)(\varphi(\vec{x}) \wedge R_i^n(\vec{x}))$  are  $B$ -formulas.

**5.3. Lemma.** For each formula  $\varphi$  of  $L_{AM}$  there is a  $B$ -formula  $\tau$  of  $L_{AM}$  such that  $\models \varphi \leftrightarrow \tau$ .

*Sketch of the proof.* The proof is by induction on the complexity of formulas.

Let  $\psi(\vec{x}, \vec{y})$  be a  $B$ -formula such that  $\models \theta(\vec{x}, \vec{y}) \leftrightarrow \psi(\vec{x}, \vec{y})$ . In the quantifier step we use the following equivalences:

$$\begin{aligned}
(M\vec{x} \leq r)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigwedge_{i \geq 0} (M\vec{x} \leq r)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})), \quad \text{for } r \in \mathbb{R}; \\
(M\vec{x} > r)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigvee_{i \geq 0} (M\vec{x} > r)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})); \\
(M\vec{x} \geq r)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigwedge_{n > 0} \bigvee_{i \geq 0} (M\vec{x} > r - 1/n)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})); \\
(M\vec{x} < r)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigvee_{n > 0} \bigwedge_{i \geq 0} (M\vec{x} \leq r - 1/n)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})); \\
(M\vec{x} \leq \infty)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigwedge_{i \geq 0} \bigvee_{n > 0} (M\vec{x} \leq n)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})); \\
(M\vec{x} > \infty)\theta(\vec{x}, \vec{y}) &\leftrightarrow (M\vec{x} > 0)(x \neq x); \\
(M\vec{x} \geq \infty)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigwedge_{n > 0} \bigvee_{i \geq 0} (M\vec{x} > n)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})); \\
(M\vec{x} < \infty)\theta(\vec{x}, \vec{y}) &\leftrightarrow \bigvee_{n > 0} \bigwedge_{i \geq 0} (M\vec{x} \leq n)(\psi(\vec{x}, \vec{y}) \wedge R_i^n(\vec{x})). \quad \square
\end{aligned}$$

We say that  $\bigwedge_{m \geq 0} \varphi_m$  is a *monotone conjunction* if  $\models \varphi_{m+1} \rightarrow \varphi_m$  for each  $m \geq 0$ , and  $\bigvee_{m \geq 0} \varphi_m$  is a *monotone disjunction* if  $\models \varphi_m \rightarrow \varphi_{m+1}$  for each  $m \geq 0$ .

**5.4. Definition.** The set of monotone bounded formulas ( $MB$ -formulas) of  $L_{AM}$  is the least set such that:

- (1) Each finite  $B$ -formula is a  $MB$ -formula;
- (2) A monotone conjunction of  $MB$ -formulas is a  $MB$ -formula;
- (3) A monotone disjunction of  $MB$ -formulas is a  $MB$ -formula.

Similarly, as in [32], we can prove the following lemma.

**5.5. Lemma.** Each  $B$ -formula  $\varphi$  is equivalent to some  $MB$ -formula  $\psi$ .

Now we produce a sequence of finite measure models

$$\langle \mathfrak{A}_i, \mu_i \rangle = \langle \mathfrak{A} \upharpoonright R_i, \mu \upharpoonright R_i \rangle, \quad i \in \mathbb{N}$$

which approximates the given measure model  $\langle \mathfrak{A}, \mu \rangle$ .

**5.6. Definition.** Let  $\langle \mathfrak{A}, \mu \rangle$  be a measure model and  $\{R_i : i \in \mathbb{N}\}$  a  $\sigma$ -cover of  $A$ . For each  $i \in \mathbb{N}$  we define a finite model  $\langle \mathfrak{A}_i, \mu_i \rangle = \langle \mathfrak{A} \upharpoonright R_i, \mu \upharpoonright R_i \rangle$  such that:

- (a)  $A_i = A \cap R_i^{\mathfrak{A}}$ ;
- (b) For each  $n$ -placed relation  $S$ ,  $S^{\mathfrak{A}_i} = S^{\mathfrak{A}} \cap (R_i^{\mathfrak{A}})^n$ ;
- (c) For each constant symbol  $c$ , if  $c^{\mathfrak{A}} \in R_i^{\mathfrak{A}}$  then  $c^{\mathfrak{A}_i} = c^{\mathfrak{A}}$ , otherwise  $c$  is not interpreted in  $\mathfrak{A}_i$ ;
- (d)  $\mu_i = \mu \upharpoonright \text{dom}(\mu) \cap R_i^{\mathfrak{A}}$ .

**5.7. Definition.** A weak measure model for  $L_{AM}$  is a model

$$\langle \mathfrak{A}, \mu \rangle = \langle \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}} \rangle_{i \in I, j \in J}, \mu_n \rangle_{n \in \mathbb{N}}$$

such that each  $\mu_n$  is a finitely additive  $\sigma$ -finite measure on  $A^n$  with each singleton measurable, and each set  $\{\vec{e} \in A^n : \langle \mathfrak{A}, \mu \rangle \models \varphi[\vec{a}, \vec{e}]\}$  is  $\mu_n$ -measurable for all  $\varphi(\vec{x}, \vec{y}) \in L_{AM}$  and  $\vec{a}$  in  $A^m$ .

Now we prove a completeness result.

**5.8. Weak Completeness Theorem for  $L_{AM}$ .** *Let  $A$  be countable. A set  $T$  of sentences is consistent in  $L_{AM}$  if and only if  $T$  has a weak measure model.*

*Sketch of the proof.* The soundness part is easy. To prove the hard part let  $C$  be a countable set of new constants, and let  $L' = L \cup C$ . By a Henkin construction,  $T$  can be extended to a maximal  $L'_{AM}$ -consistent set  $\Gamma$  of sentences with the following properties:

- (1) If  $\Phi \subseteq \Gamma$  and  $\bigwedge \Phi \in L'_{AM}$ , then  $\bigwedge \Phi \in \Gamma$ ;
- (2) If  $(M\vec{x} \geq 0)\varphi(\vec{x}) \in \Gamma$ , then  $\varphi(\vec{c}) \in \Gamma$  for some  $\vec{c} \in C^n$ ;
- (3) If  $(M\vec{x} > 0)\varphi(\vec{x}) \in \Gamma$ , then  $(M\vec{x} > 0)(\varphi(\vec{x}) \wedge R_i^n(\vec{x})) \in \Gamma$  for some  $i \in \mathbb{N}$ .

Let  $C^*$  be the set of all constants. In the usual way  $\Gamma$  induces a classical model  $\mathfrak{A} = \langle A, R_i, c_j \rangle$  for  $L'$ , where  $A = \{c^{\mathfrak{A}} : c \in C^*\}$ .

The axioms insure that a finitely additive  $\sigma$ -finite measure  $\mu_n$  can be well defined in the following way:

$$\mu_n\{\vec{c}^{\mathfrak{A}} : \varphi(\vec{c}, \vec{e}) \in \Gamma\} = \sup\{r : (M\vec{x} \geq r)\varphi(\vec{x}, \vec{e}) \in \Gamma\}.$$

Then  $\langle \mathfrak{A}, \mu_n \rangle$  is a weak model and it is easy to show that  $\langle \mathfrak{A}, \mu_n \rangle \models \Gamma$  and hence  $\langle \mathfrak{A}, \mu_n \rangle \models T$ .  $\square$

We say that  $\langle \mathfrak{A}, \mu \rangle$  is  $\Gamma$ -elementary equivalent to  $\langle \mathfrak{B}, \lambda \rangle$  ( $\langle \mathfrak{A}, \mu \rangle \equiv_{\Gamma} \langle \mathfrak{B}, \lambda \rangle$ ) provided that for each  $\varphi \in \Gamma \subseteq L_{\omega_1 M}$

$$\langle \mathfrak{A}, \mu \rangle \models \varphi \quad \text{iff} \quad \langle \mathfrak{B}, \lambda \rangle \models \varphi.$$

For  $\Gamma = L_{AM}$  ( $\Gamma = L_{AM_m}$ ) we write simply  $\langle \mathfrak{A}, \mu \rangle \equiv \langle \mathfrak{B}, \lambda \rangle$  ( $\langle \mathfrak{A}, \mu \rangle \equiv_m \langle \mathfrak{B}, \lambda \rangle$ ).

**5.9. Theorem.** *Let  $\langle \mathfrak{A}, \mu \rangle$  and  $\langle \mathfrak{B}, \lambda \rangle$  be measure models of the same similarity type and let  $\Gamma_m = \{(M\vec{x} \geq r)\psi : r \leq m, \psi \text{ is conjunction of atomic formulas}\}$ . Then*

$$\langle \mathfrak{A}, \mu \rangle \equiv_m \langle \mathfrak{B}, \lambda \rangle \quad \text{iff} \quad \langle \mathfrak{A}, \mu \rangle \equiv_{\Gamma_m} \langle \mathfrak{B}, \lambda \rangle.$$

*Proof.* For each formula  $\varphi \in L_{AM}$  let  ${}^m\varphi$  be a formula from  $L_{AP}$  given in the following way: each occurrence of  $(M\vec{x} \geq r)$  is replaced by  $(M\vec{x} \geq r/m^n)$  (where  $n$  is the length of a sequence  $\vec{x}$ ) and similarly for other quantifiers. Moreover, for each finite model  $\langle \mathfrak{A}, \mu \rangle$  with  $\mu(A) = m$ , let  $\langle \mathfrak{A}, (1/m) \cdot \mu \rangle$  be the corresponding probability model. It can be easily seen that

$$\langle \mathfrak{A}, \mu \rangle \models \varphi \quad \text{iff} \quad \langle \mathfrak{A}, (1/m) \cdot \mu \rangle \models {}^m\varphi.$$

From this fact and the Normal Form Theorem (see [23]) the equivalence follows.  $\square$

**5.10. Lemma.** *For any weak model  $\langle \mathfrak{A} \upharpoonright R_i, \mu \upharpoonright R_i \rangle$  there is a finite measure model  $\langle \mathfrak{B}_i, \lambda_i \rangle$  such that  $\langle \mathfrak{A} \upharpoonright R_i, \mu \upharpoonright R_i \rangle \equiv_{m_i} \langle \mathfrak{B}_i, \lambda_i \rangle$ , where  $m_i = \mu(R_i)$ .*

*Proof.* The proof follows immediately from the Completeness Theorem for  $L_{AP}$  and the previous theorem.  $\square$

**5.11. Theorem.** *Let  $\langle \mathfrak{A}, \mu \rangle$  be a weak measure model. Then there is a measure model  $\langle \mathfrak{N}, \eta \rangle$  such that  $\langle \mathfrak{A}, \mu \rangle \equiv \langle \mathfrak{N}, \eta \rangle$ .*

*Proof.* Let  $\{ \langle \mathfrak{B}_i, \lambda_i \rangle : i \in \mathbb{N} \}$  be the sequence as in the previous lemma. Let us define a sequence  $\{ \langle \mathfrak{N}_i, \eta_i \rangle : i \in \mathbb{N} \}$  in the following way. Let  $\langle \mathfrak{N}_0, \eta_0 \rangle = \langle \mathfrak{B}_0, \lambda_0 \rangle$ . Suppose that  $\langle \mathfrak{N}_n, \eta_n \rangle$  is defined and let:

- (1)  $N_{n+1} = N_n \cup (B_{n+1} \setminus B_n)$ ;
- (2) If  $S$  is a  $k$ -placed relation symbol, then

$$S^{\mathfrak{N}_{n+1}} = S^{\mathfrak{N}_n} \cup (S^{\mathfrak{B}_{n+1}} \cap (N_{n+1}^{(k)} \setminus N_n^{(k)}));$$

- (3) For each  $c \in L$ ,

$$c^{\mathfrak{N}_{n+1}} = \begin{cases} c^{\mathfrak{N}_n}, & \text{if } c^{\mathfrak{N}_n} \in N_n, \\ c^{\mathfrak{B}_n}, & \text{if } c^{\mathfrak{B}_{n+1}} \in B_{n+1} \setminus B_n, \\ \text{not interpreted,} & \text{otherwise;} \end{cases}$$

- (4)  $\text{dom}(\eta_{n+1}^{(k)}) = \{ U \cup V : U \in \text{dom}(\eta_n^{(k)}), V \in \text{dom}(\lambda_{n+1}^{(k)}) \cap (B_{n+1} \setminus B_n) \}$  and  $\eta_{n+1}^{(k)}(U \cup V) = \eta_n^{(k)}(U) + \lambda_{n+1}^{(k)}(V)$ .

Let us define finally a model  $\langle \mathfrak{N}, \eta \rangle$  in the following way:

- (a)  $N = \bigcup_{n \geq 0} N_n$ ;
- (b)  $S = \bigcup_{n \geq 0} S^{\mathfrak{N}_n}$ , for a  $k$ -placed  $S \in L$ ;
- (c)  $c^{\mathfrak{N}} = c^{\mathfrak{N}_n}$ , for some  $n \geq 0$ ;
- (d)  $\eta^{(k)} = \sup_{n \geq 0} \eta_n^{(k)}$ .

By definition of  $\langle \mathfrak{N}, \eta \rangle$  and Theorem 5.9. it follows that

$$\langle \mathfrak{N} \upharpoonright N_n, \eta \upharpoonright N_n \rangle \equiv_{\eta_1(N_n)} \langle \mathfrak{B}_n, \lambda_n \rangle.$$

By lemmas 5.3, 5.5. and 5.10. we have finally:  $\langle \mathfrak{A}, \mu \rangle \equiv \langle \mathfrak{N}, \eta \rangle$ .  $\square$

The Completeness Theorem is an immediate consequence of the previous theorem and the Weak Completeness Theorem 5.8.

**5.12. Completeness Theorem for  $L_{AM}$ .** *A countable set  $T$  of sentences is consistent in  $L_{AM}$  if and only if it has a measure model.*  $\square$

Using the above method one can prove many other model-theoretic results as for example: Löwenheim-Skolem theorems, Robinson Consistency and Craig Interpolation.

## BIPROBABILITY LOGICS

This chapter contains logics appropriate for the study of structures with two measures.

An *absolutely continuous biprobability space* is a structure  $\langle A, \mathcal{F}, \mu_1, \mu_2 \rangle$  with two probability measures  $\mu_1$  and  $\mu_2$  on  $A$  defined on the same  $\sigma$ -algebra  $\mathcal{F}$  of measurable sets, which satisfy the following: for each  $B \in \mathcal{F}$ , if  $\mu_2(B) = 0$ , then  $\mu_1(B) = 0$ . We say that  $\mu_1$  is absolutely continuous with respect to  $\mu_2$ , and we write  $\mu_1 \ll \mu_2$  for short. It is well known that the condition  $\mu_1 \ll \mu_2$  is equivalent to the condition: for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for each  $B \in \mathcal{F}$ , if  $\mu_2(B) < \delta$ , then  $\mu_1(B) < \varepsilon$ .

A *singular biprobability space* is a structure  $\langle A, \mathcal{F}, \mu_1, \mu_2 \rangle$  with two probability measures  $\mu_1$  and  $\mu_2$  on  $A$  such that  $\mu_1$  is singular with respect to  $\mu_2$ ; that is, there are sets  $B$  and  $C$  from the same  $\sigma$ -algebra  $\mathcal{F}$  of measurable sets for  $\mu_1$  and  $\mu_2$ , such that  $A = B \cup C$ ,  $B \cap C = \emptyset$ , and  $\mu_1(B) = 0$ ,  $\mu_2(C) = 0$ . We write  $\mu_1 \perp \mu_2$  for short. It is trivial that if  $\mu_1 \perp \mu_2$ , then  $\mu_2 \perp \mu_1$ .

### 6.1. Absolutely continuous biprobability logics

First, we shall introduce a biprobability logic with two types of probability quantifiers linked by absolute continuity.

Let  $\mathcal{A}$  be a countable admissible set and  $\omega \in \mathcal{A}$ . The logic  $L_{AP_1P_2}^a$  is similar to the standard probability logic. The only difference is that two types of probability quantifier ( $P_1\vec{x} \geq r$ ) and ( $P_2\vec{x} \geq r$ ) are allowed.

An *absolutely continuous biprobability model for  $L$*  is a structure  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$ , where  $\mathfrak{A}$  is a classical structure without operations for  $L = \{R_i, c_j : i \in I, j \in J\}$ , and  $\mu_1, \mu_2$  are two types of probability measures such that  $\mu_1 \ll \mu_2$ .

The satisfaction relation is defined recursively exactly the same as for  $L_{AP}$ , and the quantifiers are interpreted in a natural way; that is,

$$\langle \mathfrak{A}, \mu_1, \mu_2 \rangle \models (P_i\vec{x} \geq r)\varphi(\vec{x}, \vec{y})[\vec{c}] \quad \text{iff} \quad \mu_i^{(n)}\{\vec{a} \in A^n : \langle \mathfrak{A}, \mu_1, \mu_2 \rangle \models \varphi(\vec{a}, \vec{c})\} \geq r,$$

for  $i = 1, 2$ , where the measure  $\mu_i^{(n)}$  is the restriction of the completion of  $\mu_i^n$  to the  $\sigma$ -algebra generated by the measurable rectangles and the diagonal sets  $\{\vec{a} \in A^n : a_i = a_j\}$ .

Axioms and rules of inference are those of  $L_{AP}$ , as listed in 3.4, allowing both  $P_1$  and  $P_2$  to play the role of  $P$ , together with the following axioms:

(U<sub>1</sub>) *Axioms of continuity of probability quantifiers.*

$$(i) \bigwedge_n \bigvee_m (P_i \vec{y} < 1/n) (P_j \vec{x} \in [r - 1/m, r]) \varphi(\vec{x}, \vec{y}), \quad i, j = 1, 2,$$

$$(ii) \bigwedge_n \bigvee_m (P_i \vec{y} < 1/n) (P_j \vec{x} \in (r, r + 1/m]) \varphi(\vec{x}, \vec{y});$$

(U<sub>2</sub>) *Axiom of absolute continuity.*

$$\bigwedge_{\varepsilon \in \mathbb{Q}_+} \bigvee_{\delta \in \mathbb{Q}_+} \bigwedge_n \bigwedge_{\varphi \in \Phi_n} ((P_2 \vec{x} < \delta) \varphi(\vec{x}) \rightarrow (P_1 \vec{x} < \varepsilon) \varphi(\vec{x})),$$

where  $\Phi = \bigcup_n \Phi_n$ ,  $\Phi_n = \{\varphi \in \Phi : \varphi \text{ has } n \text{ free variables}\}$  and  $\Phi, \Phi_n \in \mathcal{A}$ .

We shall prove that this axiomatization is complete with respect to the class of absolutely continuous biprobability models by means of a combination of a consistency property argument like that of Keisler [34] and a weak-middle-strong model construction.

The Soundness Theorem for  $L_{AP_1 P_2}^a$  holds because all the axioms represent known properties of absolutely continuous measures.

**6.1.1. Soundness Theorem.** *Every sentence of  $L_{AP_1 P_2}^a$  which has an absolutely continuous biprobability model is consistent.*  $\square$

In order to prove the Completeness Theorem for  $L_{AP_1 P_2}^a$  we introduce two sorts of auxiliary models.

**6.1.2. Definition.** A weak biprobability model for  $L_{AP_1 P_2}^a$  is a structure  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle_{n \in \mathbb{N}}$  such that each  $\mu_n^i$  is a finitely additive probability measure on  $A^n$  with each singleton measurable, and the set  $\{\vec{a} \in A^n : \langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle \models \varphi[\vec{a}, \vec{c}]\}$  is  $\mu_n^i$ -measurable for each formula  $\varphi(\vec{x}, \vec{y})$  of  $L_{AP_1 P_2}^a$  and tuple  $\vec{c}$  in  $A$ .

**6.1.3. Definition.** A middle biprobability model for  $L_{AP_1 P_2}^a$  is a weak structure  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle$  such that the following is true: for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for each formula  $\varphi(\vec{x}, \vec{y})$  of  $L_{AP_1 P_2}^a$  and  $\vec{c} \in A^m$ , if  $\mu_n^2\{\vec{a} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\} < \delta$ , then  $\mu_n^1\{\vec{a} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\} < \varepsilon$ .

We point out that a middle model is a weak model in which the axiom of absolute continuity holds uniformly; that is, given  $\varepsilon$ , the same  $\delta$  works for all  $\varphi$ . In both cases, the satisfaction relation is defined naturally.

By using the Henkin construction as usual, one can prove the weak completeness theorem for  $L_{AP_1 P_2}^a$ . In the next theorem we shall use a weak model to construct a middle model.

**6.1.4. Middle Completeness Theorem.** *Every sentence  $\varphi_0$  of  $L_{AP_1 P_2}^a$  which is consistent with the axioms has a middle biprobability model.*

*Proof.* The axiom of absolute continuity can be extended to all formulas of  $L_{AP_1 P_2}^a$  by means of the following construction.

Let  $K = L \cup C$  be the language introduced in the Henkin construction, where  $C$  is a set of new constant symbols and  $C \in \mathcal{A}$ . Let  $M$  be a language with three kinds of variables:  $X, Y, Z, \dots$  variables for sets,  $x, y, z, \dots$  variables for urelements, and  $r, s, t, \dots$  variables for reals from  $[0, 1]$ .

Predicates of our language are  $\leq$  for reals,  $E_n(\vec{x}, X)$ ,  $n \geq 1$  and  $\vec{x} = x_1, \dots, x_n$  with canonical meaning  $(x_1, \dots, x_n) \in X$ , and  $\mu_i(X, r)$ ,  $i = 1, 2$ .

Function symbols are  $+$  and  $\cdot$  for reals.

Constant symbols are  $X_\varphi$  for each formula  $\varphi$  of  $K_{\mathcal{A}P_1P_2}^a$ , and  $\bar{r}$  for each real number  $r \in [0, 1] \cap \mathcal{A}$ .

Let  $T$  be the theory of  $M_{\mathcal{A}}$  together with the following list of formulas:

- (1) *Axiom of well-definedness.*  
 $(\forall X) \bigwedge_{n < m} \neg(\exists \vec{x}, \vec{y}) (E_m(\vec{x}, \vec{y}, X) \wedge E_n(\vec{x}, X)), \quad \text{where } \{\vec{x}\} \cap \{\vec{y}\} = \emptyset;$
- (2) *Axiom of extensionality.*  
 $(\forall \vec{x}) (E_n(\vec{x}, X) \leftrightarrow E_n(\vec{x}, Y)) \leftrightarrow X = Y;$
- (3) *Axioms of satisfaction.*
  - (a)  $(\forall \vec{x}) (E_n(\vec{x}, X_\varphi) \leftrightarrow \bigwedge_{\psi \in \Phi} E_n(\vec{x}, X_\psi)), \quad \text{for } \varphi = \bigwedge_{\psi \in \Phi} \psi,$
  - (b)  $(\forall \vec{x}) (E_n(\vec{x}, X_{\neg\varphi}) \leftrightarrow \neg E_n(\vec{x}, X_\varphi)),$
  - (c)  $(\forall \vec{x}) (E_n(\vec{x}, X_\varphi) \leftrightarrow$   
 $\left( (\exists 1X) (\mu_i(X) \geq \bar{r} \wedge (\forall \vec{y}) (E_{n+m}(\vec{x}, \vec{y}, X_\psi) \leftrightarrow E_m(\vec{y}, X))) \right)),$   
for  $\varphi = (P_i \vec{x} \geq r)\psi$  and  $\mu_i(X) \geq \bar{r} \leftrightarrow (\exists s) (s \geq \bar{r} \wedge \mu_i(X, s)),$
  - (d)  $(\forall \vec{x}) ((\forall \vec{y}) E_{n+m}(\vec{y}, \vec{x}, X_R) \leftrightarrow E_{n+m}(\vec{c}, \vec{x}, X_R)),$   
for each predicate  $R(\vec{c}, \vec{x}) \in \mathcal{A};$
- (4) *Axioms of measure.*
  - (a)  $(\forall X) (\exists 1r) \mu_i(X, r), \quad i = 1, 2,$
  - (b)  $(\forall X) (\forall Y) \left( (\mu_i(X, r) \wedge \mu_i(Y, s) \wedge \neg \bigvee_{n \geq 1} (\exists \vec{x}) (E_n(\vec{x}, X) \wedge E_n(\vec{x}, Y))) \rightarrow \right.$   
 $\left. \rightarrow (\exists Z) \left( \bigwedge_{n \geq 1} (\forall \vec{x}) (E_n(\vec{x}, Z) \leftrightarrow (E_n(\vec{x}, X) \vee E_n(\vec{x}, Y))) \wedge \mu_i(Z, r + s) \right) \right);$
- (5) *Axiom of absolute continuity.*  
 $(\forall \varepsilon > 0) (\exists \delta > 0) (\forall X) (\mu_2(X) < \delta \rightarrow \mu_1(X) < \varepsilon);$
- (6) *Axioms for an Archimedean field* (for real numbers);
- (7) *Axioms which are transformations of axioms of  $K_{\mathcal{A}P_1P_2}^a$ .*  
 $(\forall \vec{x}) E_n(\vec{x}, X_\varphi), \quad \text{where } \varphi \text{ is an axiom of } K_{\mathcal{A}P_1P_2}^a;$
- (8) *Axiom of realizability of a sentence  $\varphi_0$ .*  
 $(\forall x) E_1(x, X_{\varphi_0}).$

A weak structure  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle$  for  $K_{\mathcal{A}P_1P_2}^a$  can be transformed to a standard structure  $\mathfrak{B} = \langle B, P, F, E_n^{\mathfrak{B}}, \mu_1^{\mathfrak{B}}, \mu_2^{\mathfrak{B}}, +, \cdot, \leq, X_\varphi^{\mathfrak{B}}, r \rangle_{n \leq 1, \varphi, r}$  for  $M_{\mathcal{A}}$ , where  $B = A$ ,  $P \subseteq \bigcup_{n \geq 1} \mathcal{P}(A^n)$ ,  $F = F' \cap [0, 1]$ ,  $F' \subseteq \mathbb{R}$  a field,  $E_n^{\mathfrak{B}} \subseteq A^n \times P$ ,  $\mu_i^{\mathfrak{B}}: P \rightarrow F$ ,  $+, \cdot: F^2 \rightarrow F$ ,  $\leq \subseteq F^2$  and  $X_\varphi^{\mathfrak{B}} \in P$  by taking  $X_\varphi^{\mathfrak{B}} = \{ \vec{a} \in A^n : \mathfrak{A} \models \varphi[\vec{a}] \}$  and  $P = \{ X_\varphi^{\mathfrak{B}} : \varphi \in K_{\mathcal{A}P_1P_2}^a \}$ .

The theory  $T$  is  $\Sigma_1$ -definable over  $\mathcal{A}$  and each  $T_0 \subseteq T$ ,  $T_0 \in \mathcal{A}$  has a standard model because the axiom

$$\bigwedge_{\varepsilon \in \mathbb{Q}_+} \bigvee_{\delta \in \mathbb{Q}_+} \bigwedge_{n \geq 1} \bigwedge_{\varphi \in (T'_0)_n} ((P_2 \vec{x} < \delta) \varphi(\vec{x}) \rightarrow (P_1 \vec{x} < \varepsilon) \varphi(\vec{x})),$$

where  $T_0 \subseteq T'_0$  is the closure for substitution of constant symbols from  $C'$  and disjunction,  $T'_0 \in \mathcal{A}$  and  $(T'_0)_n = \{ \varphi \in T'_0 : \varphi \text{ has } n \text{ free variables} \}$ , holds in the weak model. It follows by means of the Barwise Compactness Theorem 2.2.5. that  $T$  has a standard model  $\mathfrak{B}$ . The model  $\mathfrak{B}$  can be transformed to a middle model  $\langle \mathfrak{B}, \bar{\mu}_n^1, \bar{\mu}_n^2 \rangle$  of  $\varphi_0$  by taking:  $R^{\mathfrak{B}} = \{ \vec{a} : E_n^{\mathfrak{B}}(\vec{a}, X_R) \}$  and  $\bar{\mu}_n^i(X) = r$  iff  $\mu_i^{\mathfrak{B}}(X, r)$ , for  $i = 1, 2$ . This completes the proof.  $\square$

Finally, a middle model is used to construct a “strong” model, i.e., an absolutely continuous biprobability model.

**6.1.5. Completeness Theorem.** *Every sentence  $\varphi$  of  $L_{AP_1P_2}^a$  consistent with the axioms has an absolutely continuous biprobability model.*

*Proof.* The proof makes use of the Loeb-Hoover-Keisler construction described in Chapter 3. The only remark is that the absolute continuity which holds uniformly in  $\varphi$  in the middle model, implies the absolute continuity of internal sets in the nonstandard superstructure, and from that we get absolute continuity of Loeb measurable sets, because they can be approximated by internal ones.  $\square$

Also, the theorem above holds for  $\Sigma_1$  definable theories, as can be proved using an argument similar to the one used above. But the absolutely continuous biprobability structure  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  cannot be axiomatized so that the extended completeness theorem holds in its full form. The following theory  $S$  is an example of a countable consistent theory in  $L_{AP_1P_2}^a$  which does not have an absolutely continuous biprobability model.

**6.1.6. Example.** Let  $L = \{R_1(x), R_2(x), \dots\}$  be a  $\Delta_1$  definable set of unary predicates which is not a subset of an element of  $\mathcal{A}$ .<sup>1</sup> Let  $\varphi_1, \varphi_2, \dots$  be an enumeration of all formulas from  $L_{AP_1P_2}^a$ , and let  $S_n(x)$  be the first predicate not occurring in  $\varphi_1, \dots, \varphi_n$ . If such a predicate does not exist, then  $L \subseteq \text{TC}(\varphi_1) \cup \dots \cup \text{TC}(\varphi_n) \in \mathcal{A}$ , and hence  $L \in \mathcal{A}$  as a  $\Delta_1$  definable set, but this obviously contradicts our assumption about  $L$ .

Let  $S$  be the theory

$$\{(P_1xy \geq 1)x \neq y, (P_2xy \geq 1)x \neq y\} \cup \{(P_2x < 1/2^n)S_n(x) : n \in \omega\} \cup \\ \cup \{(P_1x \geq 1/2)(S_1(x) \wedge \dots \wedge S_n(x)) : n \in \omega\},$$

where the first two formulas are added to eliminate the equality from all formulas. Let  $A$  be a unit interval  $[0, 1]$ ,  $\mu_1$  a Lebesgue measure on  $[0, 1]$  and  $\mu_2(B) = \int_B f d\mu_1$ , where  $B \subseteq [0, 1]$  and  $f: [0, 1] \rightarrow \mathbb{R}$  is defined by

$$f(x) = \begin{cases} 0, & \text{if } x < 1/2, \\ 2, & \text{if } x \geq 1/2. \end{cases}$$

If  $B_i = [0, 1/2 + 1/2^{i+2})$ , then  $\mu_1(B_i) \geq 1/2$  and  $\mu_2(B_i) < 1/2^i$ , for  $i \geq 1$ . For every Boolean atom  $A_{i_1 \dots i_n}^{k_1 \dots k_n} = B_{i_1}^{k_1} \cap \dots \cap B_{i_n}^{k_n}$  (where  $B_n^k = B_n$  for  $k = 1$ , and  $B_n^k = A \setminus B_n$  for  $k = -1$ ) we have that

$$\mu_2(A_{i_1 \dots i_n}^{k_1 \dots k_n}) = 0 \quad \text{implies} \quad \mu_1(A_{i_1 \dots i_n}^{k_1 \dots k_n}) = 0.$$

We interpret the predicates as follows:  $R_n^{\mathfrak{A}} = \begin{cases} B_n, & \text{if } R_n = S_m \text{ for some } m, \\ B_1, & \text{otherwise.} \end{cases}$

<sup>1</sup> The only way a  $\Delta_1$  definable set can fail to belong to  $\mathcal{A}$  is that it is not a subset of an element of  $\mathcal{A}$ .

It can be easily seen that  $\mu_1 \ll \mu_2$  is not true and that the theory  $S$  and all axioms for  $L_{\mathcal{AP}_1 P_2}^a$  are satisfied except perhaps the axiom of absolute continuity. Since only finitely many predicates  $S_n(x)$  can occur in an element of  $\mathcal{A}$ , it follows that the axiom of absolute continuity holds as well. Thus,  $S$  is a countable consistent theory in  $L_{\mathcal{AP}_1 P_2}^a$  which obviously does not have any absolutely continuous biprobability model.  $\square$

It is well known that the compactness fails for a large class of logics. The Example 3.9.16. shows that  $L_{\mathcal{AP}}$ , and hence  $L_{\mathcal{AP}_1 P_2}^a$  cannot satisfy the full compactness. So, this is a reason why we are looking for a part of  $L_{\mathcal{AP}_1 P_2}^a$  satisfying finite compactness property.

**6.1.7. Definition.** The set of universal conjunctive formulas of  $L_{\mathcal{AP}_1 P_2}^a$  is the least set containing all quantifier-free formulas and closed under arbitrary  $\wedge$ , finite  $\vee$ , and the quantifiers  $(P_1 \vec{x} \geq r)$  and  $(P_2 \vec{x} \geq r)$ .

**6.1.8. Definition.** A graded absolutely continuous biprobability model for  $L$  is a structure  $\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_n^1, \mu_n^2 \rangle_{i \in I, j \in J, n \in \mathbb{N}}$  such that:

- (1) Each  $\mu_n^k$  is a countable additive probability measure on  $A^n$ ;
- (2) Each  $n$ -placed relation  $R_i^{\mathfrak{A}}$  is  $\mu_n^k$ -measurable and the identity relation is  $\mu_n^2$ -measurable;
- (3)  $\mu_n^k \times \mu_m^k \subseteq \mu_{n+m}^k$ ;
- (4) Each  $\mu_n^k$  is preserved under permutations of  $\{1, 2, \dots, n\}$ ;
- (5)  $\{\mu_n^k : n \in \mathbb{N}\}$  has the Fubini property, that is, if  $B$  is  $\mu_{m+n}^k$ -measurable, then
  - (a) for each  $\vec{a} \in A^m$ , the section  $B_{\vec{a}} = \{\vec{c} : (\vec{a}, \vec{c}) \in B\}$  is  $\mu_n^k$ -measurable,
  - (b) the function  $f(\vec{a}) = \mu_n^k(B_{\vec{a}})$  is  $\mu_m^k$ -measurable,
  - (c)  $\int f(\vec{a}) d\mu_m^k = \mu_{m+n}^k(B)$ ;
- (6)  $\mu_n^1 \ll \mu_n^2$ , for each  $n \in \mathbb{N}$ .

The proof of the theorem below is based on the following three constructions: ultraproducts, Loeb measure and embedding of biprobability logic into a first-order logic.

**6.1.9. Finite Compactness Theorem.** *Let  $T$  be a set of universal conjunctive sentences of  $L_{\mathcal{AP}_1 P_2}^a$ . If every finite subset of  $T$  has a graded biprobability model, then  $T$  has a graded biprobability model.*

*Proof.* Let us suppose that each finite subset  $\Psi \subseteq T$  has a model  $\mathfrak{A}_\Psi$ . By 6.1.4. we can suppose that  $\mathfrak{A}_\Psi$  is a middle model. Take an ultraproduct  $^*\mathfrak{A} = \prod_D \mathfrak{A}_\Psi$  such that, for each  $\varphi \in T$ , almost every  $\mathfrak{A}_\Psi$  satisfies  $\varphi$ . Form a graded biprobability model  $\bar{\mathfrak{A}}$  from  $^*\mathfrak{A}$  by Loeb construction. Then, by induction, it follows that every universal conjunctive formula true in almost all  $\mathfrak{A}_\Psi$  holds in  $\bar{\mathfrak{A}}$ , too. In the language  $M_{\mathcal{A}}$  (see 6.1.4) the condition of absolute continuity can be expressed by the first-order sentence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall X)(\mu_2(X) < \delta \rightarrow \mu_1(X) < \varepsilon).$$

So, by the ordinary Los's Theorem and Loeb construction, this sentence holds in  $^*\mathfrak{A}$  and  $\bar{\mathfrak{A}}$ , too.  $\square$

The proof above does not work for biprobability models  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  ( $\mu_i^{(n)} = \mu_n^i$ ,  $i = 1, 2$ ), because the product measurability axiom is not universal conjunctive.

A biprobability logic  $L_{\mathcal{A}f_1f_2}^a$  with two types of integral operators is introduced in the obvious way as an extension of  $L_{\mathcal{A}f}$  appropriate for a study of properties of random variables in an absolutely continuous biprobability space. The logics  $L_{\mathcal{A}P_1P_2}^a$  and  $L_{\mathcal{A}f_1f_2}^a$  have a common conservative definitional extension, which allows us to convert a theorem about  $L_{\mathcal{A}P_1P_2}^a$  to a similar theorem about  $L_{\mathcal{A}f_1f_2}^a$  and vice versa. Although the Barwise Completeness Theorem for  $L_{\mathcal{A}f_1f_2}^a$  can be proved as a consequence of the Completeness Theorem, in order to point out an expressive power of this logic, we shall present a direct proof of it.

Let  $K$  be the extension of  $L$  formed by adding a countably many new unary relation symbols  $[X \geq r]$  and  $[X \leq r]$  for  $r \in \mathbb{Q}$ . Absolutely continuous biprobability models for  $K_{\mathcal{A}f_1f_2}^a$  are of the form  $\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, [X \geq r]^{\mathfrak{A}}, [X \leq r]^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_1, \mu_2 \rangle$ , where  $X^{\mathfrak{A}}: A \rightarrow \mathbb{R}_+$  is the Radon-Nikodym derivative of  $\mu_1$  with respect to  $\mu_2$ .

Axioms and rules of inference for  $K_{\mathcal{A}f_1f_2}^a$  are those of  $L_{\mathcal{A}f}$ , as listed in 4.1, allowibg both  $f_1$  and  $f_2$  to play the role of  $f$ , together with the following axioms:

(V<sub>1</sub>) *Axioms of continuity of integral operators.* ( $i, j = 1, 2$ )

$$(a) \bigwedge_n \bigvee_m \bigvee_k \int_i \mathbb{G}_k \left( \int_j \tau(\vec{x}, \vec{y}) d\vec{x} \right) d\vec{y} < 1/n,$$

$$\text{where } G_k(a) = \begin{cases} 1, & \text{if } r - 1/m + 1/k \leq a \leq r - 2/k \\ 0, & \text{if } a \leq r - 1/m \text{ or } a \geq r - 1/k \\ \text{linear,} & \text{otherwise,} \end{cases}$$

$$(b) \bigwedge_n \bigvee_m \bigvee_k \int_i \mathbb{H}_k \left( \int_j \tau(\vec{x}, \vec{y}) d\vec{x} \right) d\vec{y} < 1/n,$$

$$\text{where } H_k(a) = \begin{cases} 1, & \text{if } r + 2/k \leq a \leq r + 1/m - 1/k \\ 0, & \text{if } a \leq r + 1/k \text{ or } a \geq r + 1/m \\ \text{linear,} & \text{otherwise;} \end{cases}$$

(V<sub>2</sub>) *Axioms of random variable.* ( $i = 1, 2$ )

$$(a) \bigvee_k \int_i \mathbb{J}_k(\mathbf{1}[X < r](x), \mathbf{1}[X \geq s](x)) dx \geq 1, \quad (r \geq s)$$

$$\text{where } J_k(a, b) = \begin{cases} 1, & \text{if } a \geq 2/k \text{ or } b \geq 2/k \\ 0, & \text{if } a \leq 1/k \text{ and } b \leq 1/k \\ \text{linear,} & \text{otherwise,} \end{cases}$$

$$(b) \bigvee_p \bigvee_k \int_i \mathbb{J}_k(\mathbf{1}[X \leq r](x), \sum_{n=1}^p \mathbf{1}[X \geq r + 1/n](x)) dx \geq 1,$$

$$(c) \bigvee_p \bigvee_k \int_i \mathbb{J}_k(\sum_{n=1}^p \mathbf{1}[X < r - 1/n](x), \mathbf{1}[X \geq r](x)) dx \geq 1,$$

$$(d) \bigvee_p \bigvee_k \int_i \mathbb{L}_k(\sum_{n=1}^p \mathbf{1}[X \geq -n](x), \sum_{n=1}^p \mathbf{1}[X \leq n](x)) dx \geq 1,$$

$$\text{where } L_k(a, b) = \begin{cases} 1, & \text{if } a \geq 2/k \text{ and } b \geq 2/k \\ 0, & \text{if } a \leq 1/k \text{ or } b \leq 1/k \\ \text{linear,} & \text{otherwise;} \end{cases}$$

(V<sub>3</sub>) *Radon-Nikodym axioms.* ( $n \in \mathbb{N}$ ,  $s_0, \dots, s_n \in \mathbb{Q}$ )

$$(a) \left( \bigwedge_{0 \leq i \leq n-1} \int_2 \tau(x) \cdot \mathbf{1}[X < s_{i+1}](x) \cdot \mathbf{1}[X \geq s_i](x) dx \geq r_i \right) \rightarrow \left( \int_1 \tau(x) dx \geq \sum_{i=0}^{n-1} r_i \cdot s_i \right), \quad (s_0 \leq s_1 \leq \dots \leq s_n)$$

$$\begin{aligned}
(b) \quad & \left( \bigwedge_{0 \leq i \leq n-1} \int_2 \tau(x) \cdot \mathbf{1}[X < s_{i+1}](x) \cdot \mathbf{1}[X \geq s_i](x) dx \leq r_i \right) \rightarrow \\
& \rightarrow \left( \int_1 \tau(x) \cdot \mathbf{1}[X \leq m](x) dx \leq \sum_{i=0}^{n-1} r_i \cdot s_{i+1} \right), \\
& (s_0 \leq s_1 \leq \dots \leq s_n = m, m \in \mathbb{N}).
\end{aligned}$$

**6.1.10. Theorem.** *Every sentence consistent in  $K_{\mathcal{A}f_1f_2}^a$  has an absolutely continuous biprobability model.*

*Proof.* Let  $\varphi$  be a sentence consistent in  $K_{\mathcal{A}f_1f_2}^a$ . The Henkin-Loeb-Keisler construction as before produces a biprobability model  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  of  $\varphi$ . Then put  $X^{\mathfrak{A}} = \sup\{r : \mathfrak{A} \models [X \leq r][a]\}$ ,  $a \in A$ . The Radon-Nikodym axioms quaranties that  $\mu_1(B) = \int_B X^{\mathfrak{A}} d\mu_2$  holds for any internal measurable, and hence for any measurable, set  $B$ . Thus, a model  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  of  $\varphi$  is an absolutely continuous biprobability model.  $\square$

A biprobability model  $\mathfrak{A}$  for  $L_{\mathcal{A}f_1f_2}^a$  can be expanded to a model for  $K_{\mathcal{A}f_1f_2}^a$  simply by taking  $X^{\mathfrak{A}} = d\mu_1 / d\mu_2$  and defining  $[X \leq r]^{\mathfrak{A}}(a)$  iff  $X^{\mathfrak{A}}(a) \leq r$ . The set of all valid  $K_{\mathcal{A}f_1f_2}^a$ -sentences is  $\Sigma_1$  definable, for example by the formula

$$\exists P(P \text{ is a proof of } \varphi).$$

It remains to show that the set of all  $K_{\mathcal{A}f_1f_2}^a$ -sentences which do not contain symbols of the type  $[X \geq r]$  or  $[X \leq r]$  is  $\Sigma$  definable. This is not difficult if one knows that the “transitive closure” function TC is  $\Sigma_1$  definable, and that the set above is defined by the formula

$$\exists z(z = \text{TC}(\varphi) \ \& \ (\forall t \in z) \neg S(t)),$$

where  $S(t)$  is a  $\Delta_0$  formula which defines the set of all symbols of the type  $[X \geq r]$  and  $[X \leq r]$ . As a consequence of the preceding, we obtain

**6.1.11. Barwise Completeness Theorem.** *The set of all valid sentences of the logic  $L_{\mathcal{A}f_1f_2}^a$  is  $\Sigma_1$  definable over  $\mathcal{A}$ .  $\square$*

The section 4.2. contains a modification  $L(f)_{\mathcal{A}}$  of  $L_{\mathcal{A}f}$  which improves  $L_{\mathcal{A}f}$  in that it works for languages with function symbols. A measure-theoretic construction produces a model whose universe is an analytic subset of the real line with a measure on the relative Borel sets, directly from a weak model without using methods of nonstandard analysis.

Now we shall point out a corresponding results for the extension  $L(f_1, f_2)_{\mathcal{A}}^a$  of  $L(f)_{\mathcal{A}}$ . The only difference between them is that two types of integral operators  $\int_1 \dots dx$  and  $\int_2 \dots dx$  are allowed. An analytic absolutely continuous biprobability model  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  for  $L$  is defined in the obvious way, and the quantifiers are interpreted by

$$\left( \int_k \varphi(x, \vec{y}) dx \right)^{\mathfrak{A}}(\vec{c}) = \int \varphi^{\mathfrak{A}}(a, \vec{c}) d\mu_k(a), \quad k = 1, 2.$$

Axioms and rules of inference are those of  $L(f)_{\mathcal{A}}$ , as listed in 4.2, allowing both  $\int_1$  and  $\int_2$  to play the role of  $\int$ , together with the following axioms:

*Axioms of continuity of integral operators.* ( $i, j = 1, 2$ )

$$(W_1) \bigwedge_n \bigvee_m P_i \{ \vec{y} : P_j \{ \vec{x} : p(\vec{x}, \vec{y}) \} \in [r - 1/m, r] \} < 1/n;$$

$$(W_2) \bigwedge_n \bigvee_m P_i \{ \vec{y} : P_j \{ \vec{x} : p(\vec{x}, \vec{y}) \} \in (r, r + 1/m] \} < 1/n;$$

(recall that we use abbreviations  $P_k \{ \vec{x} : p(\vec{x}) \} = \int_k \mathbf{1}(p) d\vec{x}$ .)

*Axioms of absolute continuity.*

$$(W_3) \bigwedge_{\varepsilon \in \mathbb{Q}_+} \bigvee_{\delta \in \mathbb{Q}_+} \bigwedge_n \bigwedge_{\varphi \in \Phi_n} (|\int_2 \varphi d\vec{x}| < \delta \rightarrow |\int_1 \varphi d\vec{x}| < \varepsilon), \quad \text{where } \Phi = \bigcup_n \Phi_n,$$

$\Phi_n = \{ \varphi \in \Phi : \varphi \text{ is a real-valued formula with } n \text{ free variables} \}$   
and  $\Phi, \Phi_n \in \mathcal{A}$ ;

$$(W_4) \int_1 (\int_2 \varphi dx) dy = \int_2 (\int_1 \varphi dy) dx,$$

which improves the Fubini axiom  $(G_6)(iv)$  in 4.2. and expresses the natural property of integrals for absolutely continuous measures.

We shall prove that this axiomatization is complete with respect to the class of biprobability models, by combining a consistency property argument, like that of Keisler [34], and a weak-middle-strong model construction. All exceptions will be indicated.

The Generalized Radon-Nikodym Theorem<sup>2</sup>, and the Fubini Theorem prove that each function  $\varphi(x, y)^{\mathfrak{A}} : A^2 \rightarrow \mathbb{R}$  is compatible with absolutely continuous measures  $\mu_1$  and  $\mu_2$ ; that is,

$$\iint \tau(x, y)^{\mathfrak{A}} d\mu_1(x) d\mu_2(y) = \iint \tau(x, y)^{\mathfrak{A}} d\mu_2(y) d\mu_1(x).$$

Thus, we obtain the following theorem.

**6.1.12. Soundness Theorem.** *Every sentence of  $L(\int_1, \int_2)_{\mathcal{A}}^a$  which has an analytic absolutely continuous biprobability model is consistent.*  $\square$

Two relevant models for  $L(\int_1, \int_2)_{\mathcal{A}}^a$  are as follows.

**6.1.13. Definition.** A weak model for  $L(\int_1, \int_2)_{\mathcal{A}}^a$  is a structure  $\langle \mathfrak{A}, I_1, I_2 \rangle$ , where  $\mathfrak{A}$  is a first-order structure for  $L$  and  $I_k$  ( $k = 1, 2$ ) is a positive linear real function defined on the set of real-valued formulas with at most one free variable  $x$  and parameters from  $A$  ( $I_k$  may be called an  $\mathcal{A}$ -Daniell integral on  $\mathfrak{A}$ ).

**6.1.14. Definition.** A middle model for  $L(\int_1, \int_2)_{\mathcal{A}}^a$  is a weak model  $\mathfrak{A}$  such that the axiom of absolute continuity holds uniformly in any real-valued formula  $\varphi$ ; that is, for each  $\varepsilon > 0$  there is  $\delta > 0$  such that for each  $\varphi(x, \vec{y})$  and  $\vec{a} \in A^n$ , if  $|I_2(\varphi(x, \vec{a}))| < \delta$ , then  $|I_1(\varphi(x, \vec{a}))| < \varepsilon$ .

In both cases, for a real-valued formula  $\varphi$ , we define  $\varphi^{\mathfrak{A}}$  inductively as for biprobability models, except that at the integral step we define

$$\left( \int_k \varphi(x, \vec{y}) dx \right)^{\mathfrak{A}}(\vec{a}) = I_k(\varphi(x, \vec{a})), \quad k = 1, 2.$$

<sup>2</sup> In a biprobability space  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  with  $\mu_1 \ll \mu_2$  there is a measurable function  $f : A \rightarrow R_+$  such that  $\int g d\mu_1 = \int f \cdot g d\mu_2$  for each  $\mu_1$ -integrable function  $g : A \rightarrow R$ .

First, we shall suppose that  $L$  does not have equality and that  $L^f$  is empty (see 4.2). The proof of the following theorem is a “translation” of the proof of Theorem 6.1.4. into the syntax of  $L(\int_1, \int_2)_A^a$ .

**6.1.15. Middle Completeness Theorem.** *Let  $T$  be a set of sentences of  $L(\int_1, \int_2)_A^a$  such that  $T$  is  $\Sigma_1$  on  $\mathcal{A}$  and is consistent with the axioms. Then there is a middle model in which every proposition of  $T$  is valid.*

*Proof.* Let  $K = LUC$  be the language introduced in the Henkin construction of a weak model  $\langle \mathcal{A}, I_1, I_2 \rangle$  of  $T$ , and let  $M$  be a language with four kind of variables: for sets, urelements and reals as given in 6.1.4, and  $F, G, H, \dots$  variables for functions  $A^n \mapsto \mathbb{R}$ ,  $n \geq 0$ . Predicates are  $\leq$  for reals,  $E_n^s(\vec{x}, X)$  ( $n \geq 1$  and  $\vec{x} = x_1, \dots, x_n$ ) for sets,  $E_{n+1}^f(\vec{x}, r, F)$  ( $n \geq 0$ ) for functions<sup>3</sup>, and  $I_k(F, r)$  ( $k = 1, 2$  and  $F: A^0 \rightarrow \mathbb{R}$  or  $F: A^1 \rightarrow \mathbb{R}$ ). Function symbols are  $+$ ,  $\cdot$ ,  $\max$ ,  $\min$  and  $\sup$  for reals. Constant symbols are  $X_p$  for each proposition  $p$  of  $K(\int_1, \int_2)_A^a$ ,  $F_\varphi$  for each real-valued formula  $\varphi$ , and  $\bar{r}$  for each real number  $r \in \mathbb{R} \cap \mathcal{A}$ .

Let  $S$  be the following theory of  $M_A$ :

(Z<sub>1</sub>) *Axioms of well-definedness.*

- (a)  $(\forall X) \bigwedge_{n < m} \neg(\exists \vec{x}, \vec{y}) (E_m^s(\vec{x}, \vec{y}, X) \wedge E_n^s(\vec{x}, X))$ , where  $\{\vec{x}\} \cap \{\vec{y}\} = \emptyset$ ,
- (b)  $(\forall F) \bigwedge_{n < m} \neg(\exists \vec{x}, \vec{y}, r, s) (E_{m+1}^f(\vec{x}, \vec{y}, r, F) \wedge E_{n+1}^f(\vec{x}, s, F))$ ,
- (c)  $(\forall F) (\forall \vec{x}, r, s) ((E_{n+1}^f(\vec{x}, r, F) \wedge E_{n+1}^f(\vec{x}, s, F)) \rightarrow r = s)$ ;

(Z<sub>2</sub>) *Axioms of extensionality.*

- (a)  $(\forall \vec{x}) (E_n^s(\vec{x}, X) \leftrightarrow E_n^s(\vec{x}, Y)) \leftrightarrow X = Y$ ,
- (b)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F) \leftrightarrow E_{n+1}^f(\vec{x}, r, G)) \leftrightarrow F = G$ ;

(Z<sub>3</sub>) *Axioms of real-valued formulas.*

- (a)  $E_{0+1}^f(\bar{r}, F_\varphi)$ , if  $\varphi$  is atomic formula  $r$ ,
- (b)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_\varphi) \leftrightarrow E_n^s(\vec{x}, X_{R(v_1, \dots, v_n)=r}))$ , if  $\varphi$  is  $R(v_1, \dots, v_n)$ ,
- (c)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_{\varphi+\psi}) \leftrightarrow$   
 $\leftrightarrow (\exists s, t) (E_{n+1}^f(\vec{x}, s, F_\varphi) \wedge E_{n+1}^f(\vec{x}, t, F_\psi) \wedge r = s + t))$ ,
- (d)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_{\varphi \cdot \psi}) \leftrightarrow$   
 $\leftrightarrow (\exists s, t) (E_{n+1}^f(\vec{x}, s, F_\varphi) \wedge E_{n+1}^f(\vec{x}, t, F_\psi) \wedge r = s \cdot t))$ ,
- (e)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_{\min\{\varphi, \psi\}}) \leftrightarrow$   
 $\leftrightarrow (\exists s, t) (E_{n+1}^f(\vec{x}, s, F_\varphi) \wedge E_{n+1}^f(\vec{x}, t, F_\psi) \wedge r = \min\{s, t\}))$ ,
- (f)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_{\max\{\varphi, \psi\}}) \leftrightarrow$   
 $\leftrightarrow (\exists s, t) (E_{n+1}^f(\vec{x}, s, F_\varphi) \wedge E_{n+1}^f(\vec{x}, t, F_\psi) \wedge r = \max\{s, t\}))$ ,
- (g)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_{\min(\sup \Phi, s)}) \leftrightarrow$   
 $\leftrightarrow (\bigwedge_{\varphi \in \Phi} (\exists r_\varphi) E_{n+1}^f(\vec{x}, r_\varphi, F_\varphi) \wedge r = \min\{\sup_{\varphi \in \Phi} r_\varphi, \bar{s}\}))$ ,
- (h)  $(\forall \vec{x}, r) (E_{n+1}^f(\vec{x}, r, F_\varphi) \leftrightarrow$   
 $\leftrightarrow (\exists G) ((\forall y, s) (E_{1+1}^f(y, s, G) \leftrightarrow E_{n+1+1}^f(\vec{x}, y, s, F_\psi)) \wedge I_k(G, r)))$ ,  
 if  $\varphi$  is  $\int_k \psi(\vec{v}, v_0) dv_0$ ;

<sup>3</sup> With meaning  $F(x_1, \dots, x_n) = r$ .

(Z<sub>4</sub>) *Axioms of satisfaction.*

- (a)  $(\forall \vec{x})(E_n^s(\vec{x}, X_{\varphi=\psi}) \leftrightarrow (F_\varphi(\vec{x}) = F_\psi(\vec{x})))$ ,<sup>4</sup>
- (b)  $(\forall \vec{x})(E_n^s(\vec{x}, X_{\varphi \leq \psi}) \leftrightarrow (F_\varphi(\vec{x}) \leq F_\psi(\vec{x})))$ ,
- (c)  $(\forall \vec{x})(E_n^s(\vec{x}, X_{\neg p}) \leftrightarrow \neg E_n^s(\vec{x}, X_p))$ ,
- (d)  $(\forall \vec{x})(E_n^s(\vec{x}, X_{\bigvee_i p_i}) \leftrightarrow \bigvee_i E_n^s(\vec{x}, X_{p_i}))$ ;

(Z<sub>5</sub>) *Axiom of integral operators  $I_k$  ( $k = 1, 2$ ).*

- (a)  $(\forall F)(\bigwedge_{n \geq 2} \neg(\exists \vec{x}, r) E_{n+1}^f(\vec{x}, r, F) \leftrightarrow (\exists s) I_k(F, s))$ ,
- (b)  $(\forall r) I_k(F_r, r)$ ,
- (c)  $(\forall F, G, r, s)(I_k(r \cdot F + s \cdot G) = r \cdot I_k(F) + s \cdot I_k(G))$ ,<sup>5</sup>
- (d)  $(\forall F)((\forall x)(\exists r \geq 0) E_{1+1}^f(x, r, F) \rightarrow (\exists s \geq 0) I_k(F, s))$ ;

(Z<sub>6</sub>) *Axiom of absolute continuity.*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall F)(|I_2(F)| < \delta \rightarrow |I_1(F)| < \varepsilon);$$

(Z<sub>7</sub>) *Axioms for an Archimedean field and all theorems of the theory of real closed field with max and min (for real numbers);*

(Z<sub>8</sub>) *Axioms which are transformations of axioms of  $K(\int_1, \int_2)_A^a$ .*

$$(\forall \vec{x}) E_n^s(\vec{x}, X_p), \text{ where } p \text{ is an axiom of } K(\int_1, \int_2)_A^a;$$

(Z<sub>9</sub>) *Axioms of realizability of all sentences  $p$  in  $T$ .*

$$(\forall x) E_1^s(x, X_p).$$

A weak structure  $\langle \mathfrak{A}, I_1, I_2 \rangle$  for  $K(\int_1, \int_2)_A^a$  can be transformed to a standard structure

$$\mathfrak{B} = \langle B, P, Q, F, E_n^{s\mathfrak{B}}, E_{m+1}^{f\mathfrak{B}}, I_1^{\mathfrak{B}}, I_2^{\mathfrak{B}}, \leq, +, \cdot, \min, \max, \sup, X_p^{\mathfrak{B}}, F_\varphi^{\mathfrak{B}}, \bar{r} \rangle$$

for  $M_A$ , where  $B = A$ ,  $P \subseteq \bigcup_{n \geq 1} \mathcal{P}(A^n)$ ,  $Q \subseteq \bigcup_{n \geq 0} \mathcal{P}(A^n \times \mathbb{R})$ ,  $F = F' \cap [0, 1]$ ,  $F' \subseteq \mathbb{R}$  a field,  $X_p^{\mathfrak{B}} \in P$  for a proposition  $p$  and  $F_\varphi^{\mathfrak{B}} \in Q$  for a real-valued formula  $\varphi$ . We can take  $X_p^{\mathfrak{B}} = \{ \vec{a} \in A^n : \mathfrak{A} \models p[\vec{a}] \}$ ,  $P = \{ X_p^{\mathfrak{B}} : p \in K(\int_1, \int_2)_A^a \}$ ,  $F_\varphi^{\mathfrak{B}}(\vec{a}) = \varphi^{\mathfrak{A}}(\vec{a})$  for  $\vec{a} \in A^n$ ,  $Q = \{ F_\varphi^{\mathfrak{B}} : \varphi \in K(\int_1, \int_2)_A^a \}$  and  $I_k^{\mathfrak{B}}(F_\varphi^{\mathfrak{B}}) = I_k(\varphi)$  for a real-valued formula  $\varphi$  with at most one free variable.

The theory  $S$  is  $\Sigma_1$  definable over  $\mathcal{A}$  and  $S_0 \subseteq S$ ,  $S_0 \in \mathcal{A}$  has a standard model because the axiom of absolute continuity ( $W_3$ ) holds in the weak model. It follows by means of the Barwise Compactness Theorem (see 6.1.4) that  $S$  has a standard model  $\mathfrak{B}$ , which can be transformed to a middle model  $\langle \overline{\mathfrak{B}}, \bar{I}_1, \bar{I}_2 \rangle$  of  $T$  by taking:

$$\begin{aligned} R^{\overline{\mathfrak{B}}}(\vec{x}) &= r \quad \text{iff} \quad E_{m+1}^{f\mathfrak{B}}(\vec{x}, r, F_R) \quad \text{for } R \in L^{\mathbb{R}} \cup L^2, \\ \bar{I}_k(\varphi(x, \vec{a})) &= I_k^{\mathfrak{B}}(F_{\varphi(x, \vec{a})}) \quad \text{for } \vec{a} \in B^n \text{ and } k = 1, 2. \quad \square \end{aligned}$$

Now, we shall prove the version of the Rectangle Approximation Lemma, which is stated for middle models instead of weak models as in 4.2.

<sup>4</sup> We use  $F_\varphi(\vec{x}) = r$  iff  $E_{n+1}^f(\vec{x}, r, F_\varphi)$  for short.

<sup>5</sup>  $I_k(F) = r$  iff  $I_k(F, r)$ .

**6.1.16. Rectangle Approximation Lemma.** *Let  $\langle \mathfrak{A}, I_1, I_2 \rangle$  be a middle model for  $L(\int_1, \int_2)_{\mathfrak{A}}^a$ ,  $\varphi(\vec{x})$  a real-valued formula,  $\vec{x} = x_1, \dots, x_n$  and  $\varepsilon > 0$ . Then, there are finite-valued formulas  $\psi_{ij}(x_j)$ , ( $i = 1, \dots, m$ ;  $j = 1, \dots, n$ ) such that*

$$E_k \left[ \left| \varphi(\vec{x}) - \sum_i \prod_j \psi_{ij}(x_j) \right|^{\mathfrak{A}} \right] < \varepsilon \quad \text{for } k = 1, 2.$$

*Proof.* We may define  $E_k[X]$ , where  $X$  is  $\varphi^{\mathfrak{A}}$  for some formula  $\varphi$ , by means of iterated “Daniell’s integral”  $I_k$ . It follows from the axiom of absolute continuity that there is a sequence  $0 < \delta_n \leq \delta_{n-1} \leq \dots \leq \delta_1 \leq \delta_0 = \varepsilon$  of real numbers such that for each real-valued formula  $\varphi(x, \vec{a})$ , if  $|I_2(\varphi(x, \vec{a}))| < \delta_i$ , then  $|I_1(\varphi(x, \vec{a}))| < \delta_{i-1}$  ( $i = 1, 2, \dots, n$ ). For  $\delta_n$  and  $E_2[X]$  there are finite-valued formulas  $\psi_{ij}(x_j)$  such that  $E_2[|\varphi(\vec{x}) - \sum_i \prod_j \psi_{ij}(x_j)|^{\mathfrak{A}}] < \delta_n$  (see 4.2.5). Then, it follows from axioms  $(W_3)$  and  $(W_4)$  that  $E_1[|\varphi(\vec{x}) - \sum_i \prod_j \psi_{ij}(x_j)|^{\mathfrak{A}}] < \delta_0 = \varepsilon$ .  $\square$

We use a middle model to construct an analytic absolutely continuous biprobability model. Measures  $\mu_n^1$ , and  $\mu_n^2$  on  $\mathcal{F}_n^{\#}$  (see 4.2. for  $E = E_2$ ) are defined by  $\mu_n^k(F) = E_k[\mathbf{1}(F)]$ ,  $k = 1, 2$ . The axiom of absolute continuity in middle model gives  $\mu_n^1 \ll \mu_n^2$  for each  $n$ . Let  $f_n: A^{\infty} \rightarrow \mathbb{R}_+$  be a  $\mathcal{F}_n^{\#}$ -measurable function such that  $\mu_n^1(F) = \int_F f_n d\mu_n^2$  for each  $F \in \mathcal{F}_n^{\#}$ ; i.e.,  $f_n$  is the Radon-Nikodym derivate of  $\mu_n^1$  with respect to  $\mu_n^2$ . Let  $\bar{f}_n = T'_n(f_n)$  and  $\bar{f} = \liminf \bar{f}_n$ . The model  $\langle \mathfrak{B}, \lambda_1, \lambda_2 \rangle$  is an analytic absolutely continuous biprobability model, where the first-order structure  $\mathfrak{B}$  is defined as in 4.2,  $\lambda_2^{\infty}$  is Lebesgue measure on the Borel sets of  $T_n^{\#}(\mathcal{F}_n^{\#})$ ,  $\lambda_1^{\infty}(B) = \int_B \bar{f} d\lambda_2^{\infty}$  for each Borel set  $B$  of  $T_n^{\#}(\mathcal{F}_n^{\#})$  and  $\lambda_k = \lambda_k^{\infty} \upharpoonright [0, 1]$  for  $k = 1, 2$ .

**6.1.17. Analytic Completeness Theorem.** *Let  $T$  be a set of sentences of  $L(\int_1, \int_2)_{\mathfrak{A}}^a$ , such that  $T$  is  $\Sigma_1$  definable over  $\mathfrak{A}$  and consistent with the axioms of  $L(\int_1, \int_2)_{\mathfrak{A}}^a$ . Then there is an analytic absolutely continuous biprobability model of  $T$ .*

*Proof.* The proof for  $L$  without equality with  $L^f$  empty is immediate from the construction above and the following facts:

$$\begin{aligned} h_{f_1} \psi dx, n &= E_2 \left[ \left( \int_1 \psi dx \right)^{\mathfrak{A}} \middle| \mathcal{F}_n^{\#} \right] \\ &= \sum_i E_2 \left[ \mathbf{1}(F_i) \cdot \left( \int_1 \psi dx \right)^{\mathfrak{A}} \right] \cdot \mathbf{1}(F_i) / E_2[\mathbf{1}(F_i)] \quad \text{a.s.} \\ &= \int_1 \sum_i E_2[\mathbf{1}(F_i) \cdot \psi^{\mathfrak{A}}] \cdot \mathbf{1}(F_i) / E_2[\mathbf{1}(F_i)] dx \quad \text{by the axiom } (W_4) \\ &= \int_1 \sum_{i,j} E_2[(F_i \times G_j) \cdot \psi^{\mathfrak{A}}] \cdot \mathbf{1}(F_i \times G_j) / E_2[\mathbf{1}(F_i \times G_j)] dx \\ &= \int_1 h_{\psi, n} dx, \end{aligned}$$

where the atoms of  $\mathcal{F}_n^{\#}$  are written as  $F_i \times G_j$ ,  $1 \leq i, j \leq m$  and the  $G_j$ ’s depend

on  $x$  only and  $F_i$ 's do not depend on  $x$ . Then, for  $\varphi = \int_1 \psi dx$ ,

$$\begin{aligned} \varphi^{\mathfrak{B}} &= \int \psi^{\mathfrak{B}} d\lambda_1^\infty = \int g_\psi \bar{f} d\lambda_2^\infty \\ &= \liminf \int g_{\psi,n} \bar{f}_n d\lambda_2^\infty = \liminf T'_n \left( \int h_{\psi,n} f_n d\mu_n^2 \right) \\ &= \liminf T'_n \left( \int h_{\psi,n} d\mu_n^1 \right) = \liminf T'_n (h_{f_1} \psi dx, n) \\ &= \liminf g_{f_1 \psi dx, n} = g_\varphi \quad \text{a. s.} \end{aligned}$$

By using special Horn propositions of  $L(\int_1, \int_2)_A^a$  and formally closing a model under functions as in 4.2, we get the proof for  $L$  with equality and perhaps, function symbols. The only difference is that two absolutely continuous measures are allowed in each step of this construction.  $\square$

## 6.2. Singular biprobability logics

In this section, we shall consider biprobability logics related to a singular biprobability space. These logics have two types of probability quantifiers in exactly the same way as the logics from 6.1. do, except that these quantifiers are linked by the singularity relation.

The following Los-Marczewski Theorem will be an important step in the construction of “middle” biprobability models. For the proof, see [42] and [50].

**6.2.1. Los-Marczewski Theorem.** *Let  $\mathcal{F}$  be a field of subsets of a set  $M$ , and let  $\mu$  be a finitely additive probability measure on  $\mathcal{F}$ . Let  $A \subseteq M$  be such that  $A \notin \mathcal{F}$ , and let  $\mathcal{G}(\mathcal{F}, A)$  be the smallest field on  $M$  containing  $\mathcal{F}$  and  $A$ . Then there exists a finitely additive probability measure  $\bar{\mu}$  on  $\mathcal{G}(\mathcal{F}, A)$  which is an extension of  $\mu$ .*

Moreover, if  $a$  is any real number between

$$\begin{aligned} \mu_i(A) = \sup \left\{ \frac{1}{k} \left( \sum_{j=1}^n \mu(A_j) - \sum_{j=1}^m \mu(B_j) \right) : A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{F}, \right. \\ \left. k \cdot \chi_A + \sum_{j=1}^m \chi_{B_j} \geq \sum_{j=1}^n \chi_{A_j} \right\} \end{aligned}$$

and

$$\begin{aligned} \mu_e(A) = \inf \left\{ \frac{1}{k} \left( \sum_{j=1}^n \mu(A_j) - \sum_{j=1}^m \mu(B_j) \right) : A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{F}, \right. \\ \left. \sum_{j=1}^n \chi_{A_j} \geq \sum_{j=1}^m \chi_{B_j} + k \cdot \chi_A \right\}, \end{aligned}$$

then there is a finitely additive probability measure  $\bar{\mu}$  on  $\mathcal{G}(\mathcal{F}, A)$  such that  $\bar{\mu}(A) = a$  and  $\bar{\mu}$  is an extension of  $\mu$  from  $\mathcal{F}$  to  $\mathcal{G}(\mathcal{F}, A)$ . ( $\chi_A$  is the characteristic function of the set  $A$ .)

First, we consider the logic  $L_{\mathcal{A}P_1P_2}^s$  with quantifiers  $(P_1 \vec{x} \geq r)$  and  $(P_2 \vec{x} \geq r)$ . The syntax of the logic  $L_{\mathcal{A}P_1P_2}^s$  is introduced in exactly the same way as for  $L_{\mathcal{A}P_1P_2}^a$ . There is a difference only in the semantics.

A *singular biprobability model* for  $L$  is a structure  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$ , where  $\mathfrak{A}$  is a classical structure for  $L$  without operations and  $\mu_1, \mu_2$  are two types of probability measures such that  $\mu_1 \perp \mu_2$ . The quantifiers are interpreted in the usual way.

The logic  $L_{\mathcal{AP}_1 P_2}^s$  has all the axiom schemas and rules of inference of  $L_{\mathcal{AP}_1 P_2}^a$  together with the following axiom of *singularity* incorporated instead of the axiom of absolute continuity:

$$(P_i x = 0)((P_1 y > 0)(x = y) \wedge (P_2 y > 0)(x = y)), \quad i = 1, 2.$$

Now the auxiliary models are introduced as follows.

A *weak model* for  $L_{\mathcal{AP}_1 P_2}^s$  is defined exactly the same as for  $L_{\mathcal{AP}_1 P_2}^a$ .

**6.2.2. Definition.** A middle model for  $L_{\mathcal{AP}_1 P_2}^s$  is a weak structure  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle$  such that the following is true: there is a set  $B \subseteq A$  such that  $\mu_1^1(B) = 0$  and  $\mu_1^2(B) = 1$ .

In both cases, the satisfaction relation is defined naturally.

The Henkin construction as in [34] proves the Weak Completeness Theorem for  $L_{\mathcal{AP}_1 P_2}^s$ . A middle model will be obtained from a weak model by using the Los-Marczewski Theorem.

**6.2.3. Middle Soundness and Completeness Theorem.** A theory  $T$  of  $L_{\mathcal{AP}_1 P_2}^s$  is consistent if and only if it has a middle model in which each theorem of  $L_{\mathcal{AP}_1 P_2}^s$  is true.

*Proof.* The nontrivial part is to prove that if  $T$  is consistent then  $T$  has a middle model.

Let  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle$  be a weak model of  $T$  in which each theorem of  $L_{\mathcal{AP}_1 P_2}^s$  is true, and let  $Z_0, Z_1, \dots$  enumerate  $\text{dom} \mu_1^1 = \text{dom} \mu_1^2$ . Choose sets  $U$  and  $S$  in the following way:

$$\begin{aligned} U_0 &= \{ a \in A : \mu_1^1(\{a\}) > 0 \}, \quad S_0 = \{ a \in A : \mu_1^2(\{a\}) > 0 \}, \\ U_{n+1} &= U_n \cup \{x\}, \quad \text{where } x \in Z_n \setminus (U_n \cup S_n) \quad (\text{if there is any}), \\ S_{n+1} &= S_n \cup \{x\}, \quad \text{where } x \in Z_n \setminus (U_{n+1} \cup S_n) \quad (\text{if there is any}). \end{aligned}$$

If  $U = \bigcup_{n \geq 0} U_n$  and  $S = \bigcup_{n \geq 0} S_n$ , then  $U \cap S = \emptyset$  and each set  $Z_n$  with a positive measure  $\mu_1^1$  ( $\mu_1^2$ ) has a nonempty intersection with  $U$  ( $A \setminus U$ ).

Let us show that  $(\mu_1^1)_e(U) = 1$ . Let  $A_1, \dots, A_n, B_1, \dots, B_m$  be a members from  $\text{dom} \mu_1^1$  such that

$$\sum_{i=1}^n \chi_{A_i} \geq \sum_{i=1}^m \chi_{B_i} + k \cdot \chi_U.$$

Let  $C_s = \{ a \in A : \sum_{i=1}^n \chi_{A_i}(a) = s \}$ ,  $s = 0, 1, \dots, n$ , and let  $D_s = \{ a \in A : \sum_{i=1}^m \chi_{B_i}(a) = s \}$ ,  $s = 0, 1, \dots, m$ . It is easy to see that  $\mu_1^1(D_s) = 0$  for  $s > n - k$ ,  $\mu_1^1(C_s) = 0$  for  $s < k$ , and hence  $\mu_1^1(D_{n-k}) + \dots + \mu_1^1(D_{n-k-i}) \leq$

$\mu_1^1(C_n) + \dots + \mu_1^1(C_{n-i})$  for  $i = 0, 1, \dots, n - k$ . It follows that

$$\begin{aligned} & \sum_{i=1}^n \mu_1^1(A_i) - \sum_{i=1}^m \mu_1^1(B_i) = \sum_{s=1}^n s \cdot \mu_1^1(C_s) - \sum_{s=1}^m s \cdot \mu_1^1(D_s) \\ &= \sum_{s=k}^n s \cdot \mu_1^1(C_s) - \sum_{s=1}^{n-k} s \cdot \mu_1^1(D_s) = \sum_{s=k}^n s \cdot \mu_1^1(C_s) - \sum_{s=k}^n (s-k) \cdot \mu_1^1(D_{s-k}) \\ &\geq \sum_{s=k}^n s \cdot \mu_1^1(C_s) - \sum_{s=k}^n (s-k) \cdot \mu_1^1(C_s) = \sum_{s=k}^n k \cdot \mu_1^1(C_s) = k \cdot \sum_{s=0}^n \mu_1^1(C_s) = k. \end{aligned}$$

So,  $(\mu_1^1)_e(U) = 1$ . Similarly it can be shown that  $(\mu_1^2)_e(A \setminus U) = 1$ . Hence by the Los-Marczewski Theorem, measures  $\mu_1^1$  and  $\mu_1^2$  can be extended so that  $\mu_1^1 \subseteq \overline{\mu_1^1}$ ,  $\mu_1^2 \subseteq \overline{\mu_1^2}$  and  $\mu_1^1 \perp \mu_1^2$ . This completes the proof.  $\square$

The proof of the Completeness Theorem for  $L_{\mathcal{AP}_1 P_2}^s$  uses the above construction and the Loeb-Hoover-Keisler construction as given in Chapter 3, so we shall omit it.

**6.2.4. Soundness and Completeness Theorem.** A theory  $T$  of  $L_{\mathcal{AP}_1 P_2}^s$  is consistent if and only if  $T$  has a singular biprobability model.  $\square$

A graded model for  $L_{\mathcal{AP}_1 P_2}^s$  is a graded biprobability structure  $\langle \mathfrak{A}, \mu_n^1, \mu_n^2 \rangle$  such that  $\mu_n^1 \perp \mu_n^2$  for each  $n \in \mathbb{N}$ .

As a consequence of the ultraproducts, the Loeb measure construction and the embedding of biprobability logic into a first-order logic  $M_A$  (see 6.1.4), in which the condition of singularity can be expressed by the first-order sentence

$$(\exists X)(\mu_2(X) = 0 \ \& \ \mu_1(X) = 1),$$

we have that the Finite Compactness Theorem (see 6.1.9) for a graded models of  $L_{\mathcal{AP}_1 P_2}^s$  holds, too.

Analogously, the biprobability logic  $L_{\mathcal{A}f_1 f_2}^s$  with two types of integral operators related to a singular biprobability space can be introduced in the obvious way. The logics  $L_{\mathcal{AP}_1 P_2}^s$  and  $L_{\mathcal{A}f_1 f_2}^s$  have a common conservative definition extension as usual.

It is easier to prove the Barwise Completeness Theorem for  $L_{\mathcal{A}f_1 f_2}^s$  than for  $L_{\mathcal{AP}_1 P_2}^s$  because it is enough to add just one more relation symbol to the language  $L$  to get a logic for which the Completeness Theorem can be proved as usually. Let  $K$  be the extension of  $L$  formed by adding just one new unary relation symbol  $R$ . The axioms and rules of inference for  $K_{\mathcal{A}f_1 f_2}^s$  are those of  $L_{\mathcal{A}f_1 f_2}$  together with the axiom

$$\int_1 \mathbf{1}(R(x)) \, dx = 0 \ \& \ \int_2 \mathbf{1}(R(x)) \, dx = 1.$$

As usual, a Henkin-Loeb-Keisler construction produces a model  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  for  $K_{\mathcal{A}f_1 f_2}^s$  with two probability measures. The only simple remark is that for the set  $B = \{a \in A : R^{\mathfrak{A}}(a)\}$  we obtain  $\mu_1(B) = 0$  and  $\mu_2(B) = 1$ ; i.e.,  $\mu_1 \perp \mu_2$ .

A singular biprobability model  $\langle \mathfrak{A}, \mu_1, \mu_2 \rangle$  for  $L_{\mathcal{A}f_1 f_2}^s$  can be expanded to a model of  $K_{\mathcal{A}f_1 f_2}^s$  simply by taking  $R^{\mathfrak{A}}(a)$  iff  $a \in B$ , where  $B$  is the set given by the condition  $\mu_1 \perp \mu_2$ . Thus we proved the following theorem.

**6.2.5. Barwise Completeness Theorem.** *The set of all valid sentences of the logic  $L_{\mathcal{A}f_1f_2}^s$  is  $\Sigma_1$  definable over  $\mathcal{A}$ .  $\square$*

The modification  $L(f_1, f_2)_{\mathcal{A}}^s$  of  $L_{\mathcal{A}f_1f_2}^s$ , which improves former logics in that it works for languages with function symbols, is related to analytic singular biprobability models.

The logic  $L(f_1, f_2)_{\mathcal{A}}^s$  has all the axiom schemas and rules of inference of  $L(f_1, f_2)_{\mathcal{A}}^a$  together with the axiom of singularity

$$\int_k \mathbf{1} \left( \int_1 x = y \, dy > 0 \right) \cdot \mathbf{1} \left( \int_2 x = y \, dy > 0 \right) dx = 0, \quad k = 1, 2$$

incorporated instead of the axioms  $(W_3)$  and  $(W_4)$ .

A weak model for  $L(f_1, f_2)_{\mathcal{A}}^s$  is defined exactly the same as for  $L(f_1, f_2)_{\mathcal{A}}^a$ .

**6.2.6. Definition.** A middle model for  $L(f_1, f_2)_{\mathcal{A}}^s$  is a weak model structure  $\langle \mathfrak{A}, I_1, I_2 \rangle$  such that the following is true: there is a set  $A_1 \subseteq A$  such that the characteristic function  $\chi_{A_1}$  is  $\varphi(x, \vec{a})^{\mathfrak{A}}$  for some real-valued formula  $\varphi$  of  $L(f_1, f_2)_{\mathcal{A}}^s$  with at most one free variable  $x$  and parameters  $\vec{a}$  from  $A$ , and  $I_1(\chi_{A_1}) = 1$  and  $I_2(\chi_{A_1}) = 0$ .

In both cases, the interpretation of integrals is defined by

$$\left( \int_k \varphi(x, \vec{y}) \, dx \right)^{\mathfrak{A}} (\vec{a}) = I_k(\varphi(x, \vec{a})), \quad k = 1, 2.$$

**6.2.7. Analytic Completeness Theorem.** *Every set  $T$  of sentences of the logic  $L(f_1, f_2)_{\mathcal{A}}^s$  which is  $\Sigma_1$  definable over  $\mathcal{A}$  and consistent with the axioms, has an analytic singular biprobability model.*

*Proof.* The identity relation “=” belongs to  $L$ ; maybe,  $L^f \neq \emptyset$ , and in this case it is easy to see that in each step of Hoover’s construction from 4.2.6. the singularity of measures is preserved. Hence we assume that our language  $L$  is relational.

The Henkin construction produces a weak model  $\mathfrak{A} = \langle A, R^{\mathfrak{A}}, I_1, I_2 \rangle$  of  $T$ . Let  $\mathcal{F} = \{ B \subseteq A : \chi_B \in \text{dom}(I_1) = \text{dom}(I_2) \}$  be a field of subsets of  $A$ . Define finitely additive probability measures  $\nu_1, \nu_2$  on  $\mathcal{F}$  by  $\nu_k(B) = I_k(\chi_B)$ , where  $B \in \mathcal{F}$  and  $k = 1, 2$ . Then, for each  $a \in A$ , the singleton  $\{a\}$  belongs to  $\mathcal{F}$  because  $\chi_{\{a\}} = (x = a)^{\mathfrak{A}}$ , the set  $B = \{a \in A : \nu_1(a) > 0 \text{ \& } \nu_2(a) > 0\}$  belongs to  $\mathcal{F}$  because  $\chi_B = (\mathbf{1}(\int_1 x = y \, dy > 0) \cdot \mathbf{1}(\int_2 x = y \, dy > 0))^{\mathfrak{A}}$  and  $\nu_k(B) = 0$  for  $k = 1, 2$  by the axiom of singularity. By construction from 6.2.3, the measures  $\nu_1, \nu_2$  can be extended so that  $\nu_1 \subseteq \bar{\nu}_1, \nu_2 \subseteq \bar{\nu}_2$  and  $\bar{\nu}_1 \perp \bar{\nu}_2$ . Then we construct a middle model  $\bar{\mathfrak{A}} = \langle A, R^{\bar{\mathfrak{A}}}, \bar{I}_1, \bar{I}_2 \rangle$  of  $T$  by

$$\text{dom}(\bar{I}_k) = \text{dom}(I_k) \cup \{ \chi_C : C \in \bar{\mathcal{F}} \setminus \mathcal{F} \} \quad \text{and} \quad \bar{I}_k(\chi_C) = \bar{\nu}_k(C),$$

for each  $C$  from the extension  $\bar{\mathcal{F}}$  of  $\mathcal{F}$ . The singularity of finitely additive measures in the middle model will be preserved in analytic biprobability model of  $T$ .

Let  $A_1 \subseteq A$  be an element of  $\overline{\mathcal{F}}$  such that  $\overline{\nu}_1(A_1) = 1$  and  $\overline{\nu}_2(A_2) = 1$ , where  $A_2 = A \setminus A_1$ , and let  $\mathfrak{A}_k = \langle A_k, R^{\mathfrak{A}} \cap A_k, J_k \rangle$  for  $k = 1, 2$  be the weak integral model, where  $\text{dom}(J_k) = \{ f \upharpoonright A_k : f \in \text{dom}(\overline{I}_k) \}$  and  $J_k(f \upharpoonright A_k) = \overline{I}_k(f)$ . Using functions  $\varphi^{\mathfrak{A}} \upharpoonright A_k : A_k^n \rightarrow \mathbb{R}$  (the value of  $\varphi(\vec{x})$  in  $\mathfrak{A}_k$ ) for each real-valued formula  $\varphi(\vec{x})$  of  $L(\int_1, \int_2)_A^s$ ,  $\vec{x} = x_1, \dots, x_n$ , and Hoover's construction from 4.2. on the set of those functions, we can build an analytic model  $\mathfrak{M}_k = \langle [0, 1]_k, R^{\mathfrak{M}_k}, \lambda_k \rangle$ , where  $[0, 1]_k$  is a copy of the unit interval and  $\lambda_k$  is the Lebesgue measure on the algebra of Borel sets of  $[0, 1]$  (see 4.2,  $E = E_k$ ), which is elementary equivalent to  $\langle \mathfrak{A}_k, J_k \rangle$  for  $k = 1, 2$ . Then the model

$$\mathfrak{M} = \langle [0, 1]_1 \sqcup [0, 1]_2, R^{\mathfrak{M}_1} \sqcup R^{\mathfrak{M}_2}, \overline{\lambda}_1, \overline{\lambda}_2 \rangle,$$

where  $\sqcup$  denotes the disjoint union, and where  $\overline{\lambda}_1 \upharpoonright \mathcal{B}([0, 1]_2) \equiv 0$ ,  $\overline{\lambda}_2 \upharpoonright \mathcal{B}([0, 1]_1) \equiv 0$ ,  $\overline{\lambda}_k \upharpoonright \mathcal{B}([0, 1]_k) \equiv \lambda_k$  for  $k = 1, 2$ , is an analytic biprobability model of  $T$  which is elementary equivalent to  $\langle \overline{\mathfrak{A}}, \overline{I}_1, \overline{I}_2 \rangle$ .  $\square$

## MONADIC LOGIC WITH BOTH FIRST-ORDER AND PROBABILITY QUANTIFIERS

As it was already mentioned the first-order and the probability quantifiers cannot be put together in general (see Chapter 3). There are, of course, some exceptions as for example the following logic.

Let  $\mathcal{A} \subseteq \text{HC}$  be an admissible set which contains  $\omega$  and let  $L$  be a nonempty  $\mathcal{A}$ -recursive language which contains only unary relation symbols.

**7.1. Definition.** The set of formulas of  $(L_{\omega P\exists})_{\mathcal{A}P}$  is the least set such that:

- (1) Each atomic formula of the first-order logic without equality symbol is a formula of  $(L_{\omega P\exists})_{\mathcal{A}P}$ ;
- (2) If  $\varphi$  is a formula, then  $\neg\varphi$  is a formula;
- (3) If  $\Phi \in \mathcal{A}$  is a set of formulas, then  $\bigwedge \Phi$  is a formula;
- (4) If  $\varphi$  is a finite formula, then  $(\exists v_n)\varphi$  is a formula;
- (5) If  $\varphi$  is a formula and  $r \in \mathcal{A} \cap [0, 1]$ , then  $(P\vec{x} \geq r)\varphi$  is a formula.

The notion of a probability model for  $L$  and of the satisfaction relation are defined in the usual way.

The axioms for  $(L_{\omega P\exists})_{\mathcal{A}P}$  are the axioms for  $L_{\mathcal{A}P}$  logic (see 3.4) together with the usual first-order axioms. The rules of inference are the rules for  $L_{\mathcal{A}P}$  with the usual first-order generalization added.

**7.2. Lemma.** *Each  $(L_{\omega P\exists})_{\mathcal{A}P}$  sentence is  $(L_{\omega P\exists})_{\mathcal{A}P}$ -equivalent to a  $\sigma$ -Boolean combination of finite sentences.*

The proof can be obtained in a similar way as the proof of the Normal Form Theorem from [23]. So, we omit it.

The notion of a weak model  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  can be introduced as in 3.5.

**7.3. Lemma.** *A sentence of  $(L_{\omega P\exists})_{\mathcal{A}P}$  is consistent if and only if it has a weak model in which each theorem of  $(L_{\omega P\exists})_{\mathcal{A}P}$  is true.*

It is easy to see that Hoover's modification of Henkin's argument (see [22]) would work here.

**7.4. Lemma.** Let  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  be a weak model,  $\varphi(\vec{x}, \vec{y})$  a finite  $(L_{\omega P \exists})_{HCP}$ -formula and  $\vec{e} \in A^m$ . Then there is a quantifier free formula  $\Phi(\vec{x})$  such that:

$$\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega} \models (\forall \vec{x}) (\varphi(\vec{x}, \vec{e}) \leftrightarrow \Phi(\vec{x})).$$

*Proof.* We use induction on the complexity of  $\varphi$ . If  $\varphi$  is atomic the statement is trivial. The inductive step when  $\varphi$  is a propositional combination of formulas of smaller rank is also trivial. Suppose now that  $\varphi$  is of the form  $(P\vec{z} \geq r)\psi(\vec{x}, \vec{z})$ . By the inductive assumption we may assume that  $\psi$  is a finite quantifier free formula. Further, suppose that  $\vec{x}$  is  $(x_0, x_1, \dots, x_n)$  and that all relation symbols which occur in  $\psi$  are  $R_0, R_1, \dots, R_k$ . Now define:

$$\Gamma(v) = \left\{ \bigwedge \{ R_i^{f(i)}(v) : 0 \leq i \leq k \} : f \in 2^{k+1} \right\},$$

where  $R_i^{f(i)}(v) = \begin{cases} R_i(v), & \text{if } f(i) = 0 \\ \neg R_i(v), & \text{if } f(i) = 1 \end{cases}$ . Let  $\Sigma(\vec{x})$  be the set of all formulas of the form

$$\bigwedge \{ \Phi_i(x_i) : 0 \leq i \leq n \}, \quad \Phi_i(x_i) \in \Gamma(x_i)$$

for which there exists  $a_0, a_1, \dots, a_n \in A$  with:  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega} \models \Phi_i[a_i]$  for  $0 \leq i \leq n$ , and  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega} \models (P\vec{z} \geq r)\psi(\vec{a}, \vec{z})$ . Finally let  $\Phi(\vec{x})$  be the formula  $\bigvee \Sigma(\vec{x})$ . It is straightforward to check that the following holds:

$$\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega} \models (\forall \vec{x}) (\varphi(\vec{x}, \vec{e}) \leftrightarrow \Phi(\vec{x})).$$

The case when  $\varphi$  is of the form  $(\exists \vec{z})\psi(\vec{x}, \vec{z})$  can be dealt with in the same way as the previous one, so the claim of the lemma is established.  $\square$

**7.5. Corollary.** Let  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  be a weak probability model.

- (a) If  $B \subseteq A^n$  is definable by a finite formula, with parameters from  $A$ , then  $B$  is  ${}^n\mu_1$ -measurable; here by  ${}^n\mu_1$  we denote the finitely additive  $n$ -product of  $\mu_1$ 's.
- (b) If  $B \subseteq A^n$  is definable by a formula, possibly infinite with parameters from  $A$ , and  $\mu_n$  is  $\sigma$ -additive then  $B$  is  $\mu_1^n$ -measurable.  $\square$

Thus, the corollary allows us to identify  $\langle \mathfrak{A}, \mu_1 \rangle$  with  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  when only finite formulas are considered.

**7.6. Corollary.** Let  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  be a weak probability model. Then for every finite  $(L_{\omega P \exists})_{HCP}$ -formula  $\varphi(\vec{x}, \vec{y})$  with parameters from  $A$ , the set

$$\{ {}^n\mu_1 \{ \vec{c} \in A^n : \langle \mathfrak{A}, \mu_1 \rangle \models \varphi(\vec{c}, \vec{a}) \} : \vec{a} \in A^m \}$$

is finite.  $\square$

**7.7. Completeness Theorem for  $(L_{\omega P\exists})_{AP}$ .** A sentence  $\varphi$  of  $(L_{\omega P\exists})_{AP}$  is consistent if and only if it has a probability model.

*Proof.* The nontrivial part is to prove that  $\not\models \varphi$  implies  $\not\models \varphi$ , so suppose  $\not\models \varphi$ . By Lemma 7.3. there is a weak model  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$  which is a model for  $\neg \varphi$  and every axiom. By Lemma 7.2. it is enough to find a probability model  $\langle \mathfrak{B}, \nu \rangle$  which is a model for all finite  $(L_{\omega P\exists})_{AP}$  sentences which hold in  $\langle \mathfrak{A}, \mu_n \rangle_{n \in \omega}$ . To do that we shall use a method from [48]. Let  $K = L \cup C$  ( $(K_{\omega P\exists})_{AP}$ ) be the language (logic) introduced in Hoover's construction [22], where  $C$  is a countable set of new constant symbols and  $C \in \mathcal{A}$ .

Now, we introduce a language  $M$  with three sorts of variables. Let  $X, Y, Z, \dots$  be variables for sets,  $x_0, x_1, \dots$  variables for urelements and  $r, s, \dots$  variables for reals from  $[0, 1]$ . We suppose that predicates of our language are  $E_n(x_0, x_1, \dots, x_{n-1}, X)$  for  $n \geq 1$  (with a canonical meaning  $(x_0, x_1, \dots, x_{n-1}) \in X$ ) and  $\mu(X, r)$  (with a meaning  $\mu(X) = r$ ). For each finite  $(K_{\omega P\exists})_{HCP}$ -formula we have a constant symbol  $A_\varphi$  for a set, for each real number  $r \in [0, 1]$  a constant symbol  $\mathbf{r}$ , and a set  $D$  of new constant symbols of the cardinality of the continuum. Functional symbols are  $+$  and  $\cdot$  for reals.

Let  $T$  be the first order theory with the following list of axioms:

- (1) All axioms from (1) to (4) taken from the page 59 (Chapter 6);
- (2) Axioms which are transformations of finite axioms of  $(K_{\omega P\exists})_{HCP}$ :

$$(\forall \vec{x}) E_n(\vec{x}, A_\varphi) \quad \text{where } \varphi \text{ is a finite axiom;}$$

- (3) Sets of axioms which ensure  $\sigma$ -additivity of the extended measure:

$$\{ E_n(\vec{d}, A_\varphi) \} \cup \{ \neg E_n(\vec{d}, A_{\varphi_m}) : m \in \omega \},$$

where  $\{ \varphi_m : m \in \omega \}$  is a sequence of finite formulas,  $\vec{d}$  is a tuple of different constant symbols from  $D$  and all such tuples for a different sequences of formulas are pairwise disjoint,  $\{ \{ \vec{a} \in A^n : \langle \mathfrak{A}, \mu \rangle \models \varphi_m[\vec{a}] \} : m \in \omega \}$  is a monotone increasing sequence of subsets of  $A^n$ ,  $\langle \mathfrak{A}, \mu \rangle \models (\forall \vec{x})(\varphi_m(\vec{x}) \rightarrow \varphi(\vec{x}))$  and

$$\begin{aligned} & \mu(\{ \vec{a} \in A^n : \langle \mathfrak{A}, \mu \rangle \models \varphi[\vec{a}] \}) > \\ & > \sup \{ \mu(\{ \vec{a} \in A^n : \langle \mathfrak{A}, \mu_1 \rangle \models \varphi_m[\vec{a}] \}) : m \in \omega \}; \end{aligned} \quad (*)$$

- (4) Axioms of field (for real numbers) with a diagram for  $+$  and  $\cdot$ .

Let a standard model of the first-order logic for  $M$  be:

$$\mathfrak{M} = \langle M, B, F, E_n^{\mathfrak{M}}, \mu^{\mathfrak{M}}, +, \cdot, d^{\mathfrak{M}}, A_\varphi^{\mathfrak{M}}, r \rangle_{n \geq 1, \varphi \in S, r \in F, d \in D}$$

(for short  $\mathfrak{M} = \langle M, B, F, A_\varphi \rangle_S$ ), where  $B \subseteq \bigcup_{n \geq 1} \mathcal{P}(M^n)$ ,  $F = F' \cap [0, 1]$ ,  $F' \subseteq \mathbb{R}$  is a field,  $E_n^{\mathfrak{M}} \subseteq M^n \times B$ ,  $\mu^{\mathfrak{M}} : B \rightarrow F$ ,  $+, \cdot : F^2 \rightarrow F$ ,  $d^{\mathfrak{M}} \in M$ ,  $A_\varphi^{\mathfrak{M}} \in B$  and  $S \subseteq \{ \varphi \in (L_{\omega P\exists})_{HCP} : \varphi \text{ is finite} \}$ .

We claim that  $T$  is consistent. To prove the claim it is enough, by compactness, to show that all finite subtheories of  $T$  are consistent.

First, note that a weak model can be transformed to a probability model by taking:

$$A_\varphi^{\mathfrak{M}} = \{ \vec{a} \in M^n : \langle \mathfrak{A}, \mu \rangle \models \varphi[\vec{a}] \}, \quad B = \{ A_\varphi : \varphi \in (K_{\omega P \exists})_{HCP} \text{ is finite} \},$$

and arbitrarily interpreting constants from  $D$ . In this way we can get a model of a fixed finite subtheory of  $T$ .

Let  $T'$  be a finite subtheory of  $T$  and let  $\varphi, \{ \varphi_n : n \in \omega \}$  be as in the axiom (3). Pick some  $m \in \omega$  such that  $\neg E_n(\vec{d}, A_{\varphi_k}) \in T'$  for all  $k \geq m$ . By (\*) we may choose  $\vec{d}^{\mathfrak{M}} \in \{ \vec{a} \in M^n : \langle \mathfrak{A}, \mu_1 \rangle \models \varphi[\vec{a}] \} \setminus \bigcup_{i < \omega} \{ \vec{a} \in M^n : \langle \mathfrak{A}, \mu_1 \rangle \models \varphi_i[\vec{a}] \}$ . Thus we get a model for  $T'$ .

Since every finite subtheory  $T' \subseteq T$  has a model, by compactness, we conclude that  $T$  has a model, say  $\mathfrak{M}$ . Now we can transform our model  $\mathfrak{M}$  to a probability model with a first order part  $\mathfrak{B}$ . For a relation symbol  $R$  of the language  $L$  define  $R^{\mathfrak{B}} = \{ \vec{x} \in M^n : E_n^{\mathfrak{M}}(\vec{x}, A_R) \}$ , and a finitely additive measure  $\bar{\mu}$  on the ring  $\{ A_\varphi : \varphi \text{ is finite} \}$  by:  $\bar{\mu}(A_\varphi) = r$  iff  $\mu(A_\varphi, r)$  holds in  $\mathfrak{M} = \langle M, \dots \rangle$ .

Note that axioms (1) ensure  $\bar{\mu}$  to map  $\{ A_\varphi : \varphi \text{ is finite} \}$  into the reals. Axiom (3) allows us to apply Karatheodory's Theorem to the measure space

$$\langle M, \{ A_\varphi : \varphi \text{ is finite} \}, \bar{\mu} \rangle.$$

Thus  $\bar{\mu}$  can be extended to a  $\sigma$ -additive measure  $\nu$  on the  $\sigma$ -ring which extends  $\{ A_\varphi : \varphi \text{ is finite} \}$ . Let  $\nu$  be the  $\sigma$ -additive extension of  $\bar{\mu}$ . It is straightforward to check that  $\langle \mathfrak{B}, \nu \rangle$  is a probability model which satisfies the same finite  $(L_{\omega P \exists})_{AP}$  sentences as  $\langle \mathfrak{A}, \mu_1 \rangle$  does. That finishes the proof of the theorem.  $\square$

## SECOND-ORDER PROBABILITY LOGIC

Sentences from measure theory such as “for each closed interval  $[a, b]$ , a monotone function from  $[a, b]$  to  $\mathbb{R}$  has a finite derivative for *almost all*  $x$  in  $[a, b]$ ” forces us to take all subsets of a structure into consideration. So we shall allow a second kind of variable ranging over subsets of the universe and its cartesian products. In this chapter we introduce the second-order probability logic  $L_{APV}^2$  which possesses the probability quantifiers  $(P\vec{x} \geq r)$  on individual variables and the ordinary quantifiers  $(\forall X)$  and  $(\exists X)$  on set variables.

As usual, let  $\mathcal{A}$  be a countable admissible set and  $\omega \in \mathcal{A}$ . Let  $L$  be a countable  $\mathcal{A}$ -recursive set of individual constants  $c_i, i \in I$  and for each  $n \geq 1$   $n$ -ary set (predicate) constants  $P_j^n, j \in J_n$ . The logic  $L_{APV}^2$  has countably many individual variables  $x_1, x_2, \dots$ , for each  $n \geq 1$  countably many set (predicate) variables  $X_1^n, X_2^n, \dots$ , propositional connectives  $\neg$  and  $\wedge$ , ordinary quantifiers  $(\forall X^n)$  and  $(\exists X^n)$  on the set variables and probability quantifiers  $(P\vec{x} \geq r)$  on the individual variables.

The set of *formulas* of  $L_{APV}^2$  is defined similarly to the one for infinitary second-order logic without equality, except that instead of the ordinary quantifiers  $(\forall x)$  and  $(\exists x)$  on the individual variables, now we use the probability quantifiers  $(P\vec{x} \geq r)$ , where  $\vec{x}$  is a finite sequence of individual variables and  $r \in [0, 1] \cap \mathcal{A}$ , with the restriction that no ordinary quantifier on the set variable may occur within the scope of a probability quantifier. A *sentence* is a formula with no free variables.

**8.1. Definition.** A second-order probability structure for  $L$  is a sequence

$$\mathfrak{A} = \langle A, \{A_n : n \in \mathbb{N}\}, c_i^{\mathfrak{A}}, R_j^m, \mu \rangle_{i \in I, m \in \mathbb{N}, j \in J_m}$$

consisting of a universe  $A$  of individuals, for each  $n \geq 1$  a second-order universe  $A_n$  of  $n$ -ary relations, individual constants  $c_i^{\mathfrak{A}} \in A$ , set constants  $R_j^m \in A_m$  and probability measure  $\mu$  on  $A$  such that each  $R_j^m$  is  $\mu^m$ -measurable.

The *satisfaction relation*  $\mathfrak{A} \models \varphi[a_1, \dots, a_k, S_1, \dots, S_l]$  for a formula  $\varphi(\vec{x}, \vec{X})$  of  $L_{APV}^2$ , a tuple  $\vec{a}$  in  $A$  and  $S_i \in A_{n_i}$ , for  $i = 1, 2, \dots, l$ , is defined recursively exactly as for the second-order logic except for the probability quantifier by

$$\begin{aligned} \mathfrak{A} \models (P\vec{x} \geq r)\varphi(\vec{x}, \vec{y}, X_1^{n_1}, \dots, X_l^{n_l})[\vec{c}, S_1, \dots, S_l] \\ \text{iff } \mu^n\{\vec{a} : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}, S_1, \dots, S_l]\} \geq r. \end{aligned}$$

By induction from a “diagonal” form of the Fubini Theorem it follows that the set  $\{\vec{a} : \mathfrak{A} \models [\vec{a}, \vec{c}, S_1, \dots, S_l]\}$  is  $\mu^n$ -measurable, showing that the satisfaction relation has the intended meaning for  $L_{APV}^2$ . The main reason for the restriction in the definition of formulas of  $L_{APV}^2$ , that is, no ordinary quantifier on the set variable may occur within the scope of a probability quantifier, is that a projection of a measurable set need not be measurable.

Now we shall list the *axiom schemas* for  $L_{APV}^2$ .

- (S<sub>1</sub>)  $\varphi \rightarrow (\psi \rightarrow \varphi)$ ;
- (S<sub>2</sub>)  $(\varphi \rightarrow (\psi \rightarrow \tau)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \tau))$ ;
- (S<sub>3</sub>)  $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$ ;
- (S<sub>4</sub>)  $(\forall X^n)\varphi \rightarrow S_B^{X^n}\varphi$ , where  $B$  is an  $n$ -ary set variable or a set constant and  $S_B^{X^n}\varphi$  denotes the formula obtained by replacing each free occurrence of  $X^n$  in  $\varphi$  by  $B$ , and  $B$  is free for  $X^n$  in  $\varphi$ ; that is,  $\varphi$  and  $S_B^{X^n}\varphi$  have exactly the same number of bound variables;
- (S<sub>5</sub>)  $(\forall X^n)(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\forall X^n)\psi)$ , where  $X^n$  has no free occurrence in  $\varphi$ ;
- (S<sub>6</sub>) All axioms of  $L_{AP}$ ;
- (S<sub>7</sub>) All axioms for the measure on the set constants ( $n \geq 1$ ):
  - (a)  $(\forall X^n)((P\vec{x} \geq r)X^n(\vec{x}) \rightarrow (P\vec{x} \geq s)X^n(\vec{x}))$ , where  $r \geq s$ ;
  - (b)  $(\forall X^n)((P\vec{x} \geq r)X^n(\vec{x}) \rightarrow (P\vec{y} \geq r)X^n(\vec{y}))$ ;
  - (c)  $(\forall X^n)(P\vec{x} \geq 0)X^n(\vec{x})$ ;
  - (d)  $(\forall X^n)(\forall Y^n)((P\vec{x} \leq r)X^n(\vec{x}) \wedge (P\vec{x} \leq s)Y^n(\vec{x})) \rightarrow (P\vec{x} \leq r+s)(X^n(\vec{x}) \vee Y^n(\vec{x}))$ ;
  - (e)  $(\forall X^n)(\forall Y^n)((P\vec{x} \geq r)X^n(\vec{x}) \wedge (P\vec{x} \geq s)Y^n(\vec{x}) \wedge (P\vec{x} \leq 0)(X^n(\vec{x}) \wedge Y^n(\vec{x}))) \rightarrow (P\vec{x} \geq r+s)(X^n(\vec{x}) \vee Y^n(\vec{x}))$ ;
  - (f)  $(\forall X^n)((P\vec{x} > r)X^n(\vec{x}) \leftrightarrow \bigvee_{k \in \mathbb{N}}(P\vec{x} \geq r + 1/k)X^n(\vec{x}))$ ;
  - (g)  $\bigwedge_{K \subseteq M}(P\vec{x} \geq r) \wedge \bigwedge_{i \in K} X_i^n(\vec{x}) \rightarrow (P\vec{x} \geq r) \wedge \bigwedge_{i \in M} X_i^n(\vec{x})$ , where  $M \subseteq \mathbb{N}$  and  $K$  ranges over the finite subsets of  $M$ ;
  - (h)  $(\forall X^n)((P\vec{x} \geq r)X^n(\vec{x}) \leftrightarrow (Px_{\pi 1} \cdots x_{\pi n} \geq r)X^n(\vec{x}))$ , where  $\pi$  is a permutation of  $\{1, 2, \dots, n\}$ ;
  - (j)  $(\forall X^{n+m})((P\vec{x} \geq r)(P\vec{y} \geq s)X^{n+m}(\vec{x}, \vec{y}) \rightarrow (P\vec{x}, \vec{y} \geq r \cdot s)X^{n+m}(\vec{x}, \vec{y}))$ ;
  - (k) For each  $r < 1$ ,
    - $(\forall X^{n+m})(P\vec{x} \geq 1)(P\vec{y} > 0)(P\vec{z} \geq r)(X^{n+m}(\vec{x}, \vec{z}) \leftrightarrow X^{n+m}(\vec{y}, \vec{z}))$ , provided all variables in  $\vec{x}, \vec{y}$  and  $\vec{z}$  are distinct;
- (S<sub>8</sub>) *Comprehension axiom schema.*
  - $(\exists X^n)(P\vec{x} \geq 1)(X^n(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ , where  $\varphi(\vec{x})$  is a formula of  $L_{AP}$ .

The *rules of inference* for  $L_{APV}^2$  are as follows:

- (T<sub>1</sub>) (*Modus Ponens*)  $\varphi, \varphi \rightarrow \psi \vdash \psi$ ;
- (T<sub>2</sub>) (*Conjunction*)  $\{\varphi \rightarrow \psi : \psi \in \Psi\} \vdash \varphi \rightarrow \bigwedge \Psi$ ;
- (T<sub>3</sub>) (*Generalization*)
  - (i)  $\varphi \rightarrow \psi(\vec{x}) \vdash \varphi \rightarrow (P\vec{x} \geq 1)\psi(\vec{x})$ , provided  $\vec{x}$  is not free in  $\varphi$  and  $\psi(\vec{x})$  is a formula of  $L_{AP}$ ;
  - (ii)  $\varphi \rightarrow \psi(X^n) \vdash \varphi \rightarrow (\forall X^n)\psi(X^n)$ , provided  $X^n$  is not free in  $\varphi$ .

Each axiom of one of the forms  $(S_1)$ – $(S_7)$  is valid in a second-order probability structure  $\mathfrak{A}$ , and the rules of inference yield formulas valid in  $\mathfrak{A}$  when applied to formulas valid in  $\mathfrak{A}$ . The comprehension axiom schema  $(S_8)$  may fail to hold in  $\mathfrak{A}$ . To remedy this we make the following definition.

**8.2. Definition.** A second-order probability model for  $L$  is a second-order probability structure  $\mathfrak{A}$  for  $L$  such that every instance of the comprehension axiom schema  $(S_8)$  is valid in  $\mathfrak{A}$ .

In order to prove the main result, Completeness Theorem for second-order probability models, we shall introduce two sorts of auxiliary models.

**8.3. Definition.** A weak second-order probability model for  $L$  is a structure

$$\mathfrak{A} = \langle A, \{A_n : n \in \mathbb{N}\}, c_i^{\mathfrak{A}}, R_j^m, \mu_k \rangle_{i \in I, m, k \in \mathbb{N}, j \in J_m}$$

such that  $\mu_k$  is a finitely additive probability measure on  $A^k$ , each  $R_j^m$  is  $\mu_m$ -measurable, the set  $\{\vec{a} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\}$  is  $\mu_n$ -measurable for each formula  $\varphi(\vec{x}, \vec{y})$  of  $L_{AP}$  and each tuple  $\vec{c}$  in  $A$ , and every instance of the comprehension axiom schema is valid in  $\mathfrak{A}$ .

**8.4. Definition.** A graded second-order probability model for  $L$  is a structure  $\mathfrak{A}$  such that:

- (1) Each  $\mu_k$  is a probability measure on  $A^k$ ;
- (2) Each  $R_j^m$  is  $\mu_m$ -measurable;
- (3) If  $B$  is  $\mu_m$ -measurable, then  $B \times A^k$  is  $\mu_{m+k}$ -measurable;
- (4) Each  $\mu_k$  is preserved under permutations of  $\{1, 2, \dots, k\}$ ;
- (5) The sequence  $\langle \mu_k : k \in \mathbb{N} \rangle$  has the Fubini property, that is, if  $B$  is  $\mu_{m+k}$ -measurable, then
  - (i) for each  $\vec{a} \in A^m$ , the section  $B_{\vec{a}} = \{\vec{c} : (\vec{a}, \vec{c}) \in B\}$  is  $\mu_k$ -measurable,
  - (ii) the function  $f(\vec{a}) = \mu_k(B_{\vec{a}})$  is  $\mu_m$ -measurable,
  - (iii)  $\int f(\vec{a}) d\mu_m = \mu_{m+k}(B)$ ;
- (6) Every instance of the comprehension axiom schema is valid in  $\mathfrak{A}$ .

In both cases the satisfaction relation is defined naturally.

**8.5. Soundness and Completeness Theorem.** Let  $T$  be a set of sentences of  $L_{APV}^2$ . Then,  $T$  has a second-order probability model if and only if  $T$  is consistent in  $L_{APV}^2$ .

*Proof.* As usual, the proof of soundness is easy. For the other direction, let  $T$  be a consistent set of sentences in  $L_{APV}^2$ . Let  $K$  be the extension of  $L$  formed by adding a countable set  $C$  of new individual constants and, for each  $n \geq 1$ , a countable set  $D_n$  of new  $n$ -ary set constants such that  $C, D_n \in \mathcal{A}$ . Let  $C'$  be the set of individual constants of  $K$  and, for each  $n \geq 1$ ,  $D'_n$  be the set of  $n$ -ary set constants of  $K$ .

By the Henkin construction,  $T$  can be extended to a maximal  $K_{APV}^2$ -consistent set  $\Delta$  of sentences with the following witness properties:

- (a) If  $\Phi \subseteq \Delta$  and  $\bigwedge \Phi \in K_{APV}^2$ , then  $\bigwedge \Phi \in \Delta$ ;

- (b) If  $\varphi(\vec{c}) \in \Delta$  for all  $\vec{c}$  in  $C$ , then  $(P\vec{x} \geq 1)\varphi(\vec{x}) \in \Delta$ , where  $\varphi(\vec{x})$  is a formula of  $L_{AP}$ ;  
(c) If  $S_B^{X^n}\varphi \in \Delta$  for all  $B \in D_n$ , then  $(\forall X^n)\varphi \in \Delta$ .

We shall define a weak second-order probability model

$$\mathfrak{A} = \langle A, \{A_n : n \in \mathbb{N}\}, c^{\mathfrak{A}}, R_P, \mu_m \rangle_{m \in \mathbb{N}, c \in C', P \in D'_k, k \in \mathbb{N}}$$

as follows:

- (1)  $A = \{c^{\mathfrak{A}} : c \in C'\}$ ;
- (2)  $R_P = \{(c_1^{\mathfrak{A}}, \dots, c_k^{\mathfrak{A}}) : P(c_1, \dots, c_k) \in \Delta\}$ , for each  $P \in D'_k$ ;
- (3)  $A_k = \{R_P : P \in D'_k\}$ , for each  $k \geq 1$ ;
- (4)  $\mu_n\{\vec{c}^{\mathfrak{A}} : \varphi(\vec{c}, \vec{e}) \in \Delta\} = \sup\{r : (P\vec{x} \geq r)\varphi(\vec{x}, \vec{e}) \in \Delta\}$ , for each formula  $\varphi(\vec{x}, \vec{y})$  of  $L_{AP}$  and each tuple  $\vec{e}$  in  $C'$ ;
- (5)  $\mu_n(R_P) = \sup\{r : (P\vec{x} \geq r)P(\vec{x}) \in \Delta\}$ .

Then for each sentence  $\varphi$  of  $K_{AP\forall}^2$  we have:

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \varphi \in \Delta.$$

In an  $\omega_1$ -saturated nonstandard universe, we shall form the internal structure  $^*\mathfrak{A} = \langle ^*A, \{\sigma A_n : n \in \mathbb{N}\}, ^*c^{\mathfrak{A}}, ^*R_P, ^*\mu_m \rangle$ , where  $\sigma A_n = \{^*B : B \in A_n\}$ . By the transfer principle and Loeb construction, the structure

$$\widehat{\mathfrak{A}} = \langle ^*A, \{\sigma A_n : n \in \mathbb{N}\}, ^*c^{\mathfrak{A}}, ^*R_P, \widehat{\mu}_m \rangle,$$

where  $\widehat{\mu}_m$  is the Loeb measure of  $\mu_m$ , is a graded second-order probability model of  $\Delta$ . At last, by Keisler's construction as given in Chapter 3, this structure induces an ordinary second-order probability model of  $\Delta$ , and hence a model of  $T$ .  $\square$

The equality for the individual variables can be defined by:

$$x = y \quad \text{iff} \quad (\forall X^1)(X^1(x) \leftrightarrow X^1(y)).$$

So, in the second-order probability logic axioms of equality of the individual variables become theorems.

We shall finish this chapter with a remark related to biprobability cases.

**8.6. Remark.** It is possible to introduce a second-order biprobability logics  $L_{AP_1P_2\forall}^{2a}$  and  $L_{AP_1P_2\forall}^{2s}$  for absolutely continuous and singular case as given in Chapter 6. The only difference is that two types of probability quantifiers  $(P_1\vec{x} \geq r)$  and  $(P_2\vec{x} \geq r)$  on the individual variables are allowed.

The set of axioms  $(S_1)$ – $(S_7)$ , allowing both  $P_1$  and  $P_2$  to play the role of  $P$ , together with the following axioms

$$\begin{aligned} (AC) \quad & \bigwedge_{\varepsilon \in \mathbb{Q}_+} \bigvee_{\delta \in \mathbb{Q}_+} (\forall X^n)((P_2\vec{x} < \delta)X^n(\vec{x}) \rightarrow (P_1\vec{x} < \varepsilon)X^n(\vec{x})), \\ (C) \quad & (\exists X^n)\left((P_1\vec{x} \geq 1)(X^n(\vec{x}) \leftrightarrow \varphi(\vec{x})) \wedge (P_2\vec{x} \geq 1)(X^n(\vec{x}) \leftrightarrow \varphi(\vec{x}))\right), \\ & \text{where } \varphi(\vec{x}) \text{ is a formula of } L_{AP_1P_2}, \end{aligned}$$

is complete with respect to the class of second-order absolutely continuous biprobability models introduced in the obvious way.

Also, the set of axioms  $(S_1)$ - $(S_7)$ ,  $(C)$  together with the following *axiom of singularity*

$$(\exists X^n)((P_1 \vec{x} \leq 0)X^n(\vec{x}) \wedge (P_2 \vec{x} \geq 1)X^n(\vec{x})),$$

is complete with respect to the class of second-order singular biprobability models.

The Completeness Theorem for  $L_{AP_1 P_2 \forall}^{2a}$  and  $L_{AP_1 P_2 \forall}^{2s}$  is easier to prove because it is possible to avoid a "middle" second-order biprobability models; that is, yet on the "weak" models we have absolute continuity or singularity of the measures.

## INTUITIONISM AND PROBABILITY OPERATORS

In this chapter, unlike the others, we have to deal with intuitionism, the most important non-classical logic. Informations about intuitionism can be found in [21], [43]. Especially, for infinite intuitionistic logic, see [1], [46]. The second characteristic of this chapter is a probability operator which we apply to either classical or intuitionistic propositional logic.

### 9.1. Infinitary intuitionistic logic with both ordinary and probability quantifiers

The main result of this section is the completeness theorem for an infinitary intuitionistic logic with both ordinary and probability quantifiers with respect to Kripke semantics.

Our aim is to build an intuitionistic logic with probability quantifiers. However, we do not want to miss ordinary quantifiers; therefore we include them in our logic, too. In our logic (denoted by  $L_{API}$ ), which allows conjunction and disjunction of countable sets of sentences, all connectives and quantifiers are treated intuitionistically. Therefore, all of the following three classically valid equivalences are not valid in our logic:

$$\begin{aligned}(Px < r)\varphi &\leftrightarrow \neg(Px \geq r)\neg\varphi, \\(Px \leq r)\varphi &\leftrightarrow (Px \geq 1 - r)\neg\varphi, \\(Px > r)\varphi &\leftrightarrow \neg(Px \geq 1 - r)\neg\varphi.\end{aligned}$$

Let us say briefly something about Kripke models in the classical case. A Kripke model consists of an ordered tree  $\langle T_0, \leq \rangle$  with a single root  $t_0$ , and at each node  $t$  of the tree we have a classical model  $\mathfrak{A}_t$ . The nodes satisfy the following condition:  $s \leq t$  implies  $\mathfrak{A}_s \subseteq \mathfrak{A}_t$  (i.e., for each  $n$ -ary relation  $R$ ,  $R^{\mathfrak{A}_s} \subseteq R^{\mathfrak{A}_t} \cap A_s^n$ ). In essence, a classical model for predicate calculus is attached to each node, with the restriction that true atomic sentences are preserved in extensions. We can interpret this kind of model as representing the growth of knowledge. We want to extend this intuition to  $L_{API}$  logic. So, in each node of  $T_0$  we have a triple  $\langle \mathfrak{A}, \mu_i^t, \mu_o^t \rangle$ , where  $\mu_i^t$  is a

superadditive inner measure and  $\mu_o^t$  is a subadditive outer measure. Also,  $\mu_i^t$  is increasing and  $\mu_o^t$  is decreasing; i.e.,  $\mu_i^s \leq \mu_i^t \leq \mu_o^t \leq \mu_o^s$  for  $s \leq t$ .

The logic  $L_{API}$  is infinitary for two reasons.

First, this logic is much more expressive than the first-order version, so many important notions of probability theory can be expressed, as for example: convergence almost surely, convergence in probability, random variables  $X_1$  and  $X_2$  have the same distribution etc. (see [34]).

Secondly, in order to prove the Completeness Theorem we need infinity in a similar way as in the classical case. The full finiteness<sup>1</sup> can be ensured only for some logics with a finite number of quantifiers of the type  $(Px \geq r)$ ,  $(Px > r)$ ,  $(Px \leq r)$ ,  $(Px < r)$ . The infinitary intuitionistic logic has already been treated in [1] and [46], of course for ordinary quantifiers.

We shall assume throughout this part that  $\mathcal{A} \subseteq \text{HC}$  is an admissible set such that  $\omega, \mathbb{Q} \in \mathcal{A}$ .

We shall assume, also, that  $L$  is a  $\Sigma$ -definable set of finitary relation and constant symbols (no function symbols).

The logic has the following logical symbols:

- (a) A countable list of individual variables  $v_n$ ,  $n \in \mathbb{N}$ ;
- (b) The connectives  $\wedge, \vee, \rightarrow, \neg$ ;
- (c) The quantifiers  $(Px \geq r)$ ,  $(Px > r)$ ,  $(Px \leq r)$  and  $(Px < r)$ , where  $x$  is a variable and  $r \in \mathcal{A} \cap [0, 1] \cap \mathbb{Q}$ ;
- (d) Quantifiers  $(\forall x)$  and  $(\exists x)$ ;
- (e) The equality symbol  $=$ .

**9.1.1. Definition.** The set of formulas of  $L_{API}$  is the least set  $F$  such that:

- (1) Each atomic formula of first-order logic is a formula of  $L_{API}$ ;
- (2) If  $\varphi$  and  $\psi$  are formulas of  $L_{API}$ , then  $\neg\varphi$  and  $\varphi \rightarrow \psi$  are formulas of  $L_{API}$ ;
- (3) If  $\Phi \in \mathcal{A}$  is a set of formulas of  $L_{API}$  with only finitely many free variables, then  $\bigwedge \Phi$  and  $\bigvee \Phi$  are formulas of  $L_{API}$ ;
- (4) If  $\varphi$  is a formula of  $L_{API}$  and  $(Px \geq r)$ ,  $(Px > r)$ ,  $(Px \leq r)$  and  $(Px < r)$  are quantifiers of  $L_{API}$ , then  $(Px \geq r)\varphi$ ,  $(Px > r)\varphi$ ,  $(Px \leq r)\varphi$  and  $(Px < r)\varphi$  are formulas of  $L_{API}$ ;
- (5) If  $\varphi$  is a formula of  $L_{API}$  and  $(\forall x)$  and  $(\exists x)$  are quantifiers of  $L_{API}$ , then  $(\forall x)\varphi$  and  $(\exists x)\varphi$  are formulas of  $L_{API}$ .

**9.1.2. Definition.** The Axioms for  $L_{API}$  are as follows.

(A<sub>1</sub>) All axioms of finitary intuitionistic first-order logic:

$$\begin{array}{ll}
 \varphi \rightarrow (\varphi \wedge \varphi); & \varphi \rightarrow (\varphi \vee \psi); \\
 (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi); & (\varphi \vee \psi) \rightarrow (\psi \vee \varphi); \\
 (\varphi \rightarrow \psi) \rightarrow ((\varphi \wedge \theta) \rightarrow (\psi \wedge \theta)); & \varphi \rightarrow (\psi \rightarrow \varphi); \\
 ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \theta)) \rightarrow (\varphi \rightarrow \theta); & \neg\varphi \rightarrow (\varphi \rightarrow \psi); \\
 ((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi)) \rightarrow \neg\varphi; & (\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi; \\
 ((\varphi \rightarrow \theta) \wedge (\psi \rightarrow \theta)) \rightarrow ((\varphi \vee \psi) \rightarrow \theta). & 
 \end{array}$$

<sup>1</sup> Finite formulas and finite rules of inference.

(A<sub>2</sub>) *Infinitary axioms:*

$$\begin{array}{ll}
\bigwedge \Phi \rightarrow \varphi, \text{ where } \varphi \in \Phi; & \varphi \rightarrow \bigvee \Phi, \text{ where } \varphi \in \Phi; \\
\varphi \vee (\bigvee \Phi) \leftrightarrow \bigvee_{\psi \in \Phi} (\varphi \vee \psi); & \varphi \wedge (\bigwedge \Phi) \leftrightarrow \bigwedge_{\psi \in \Phi} (\varphi \wedge \psi); \\
\varphi \vee (\bigwedge \Phi) \leftrightarrow \bigwedge_{\psi \in \Phi} (\varphi \vee \psi); & \varphi \wedge (\bigvee \Phi) \leftrightarrow \bigvee_{\psi \in \Phi} (\varphi \wedge \psi); \\
\bigwedge_{\psi \in \Phi} (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \bigwedge \Phi); & \bigwedge_{\psi \in \Phi} (\psi \rightarrow \varphi) \leftrightarrow (\bigvee \Phi \rightarrow \varphi).
\end{array}$$

(A<sub>3</sub>) *Monotonicity* ( $r \geq s$ ):

$$\begin{array}{ll}
(Px \geq r)\varphi \rightarrow (Px \geq s)\varphi; & (Px \leq s)\varphi \rightarrow (Px \leq r)\varphi; \\
(Px > r)\varphi \rightarrow (Px \geq r)\varphi; & (Px < r)\varphi \rightarrow (Px \leq r)\varphi.
\end{array}$$

(A<sub>4</sub>) *Quantifiers axioms:*

$$\begin{array}{ll}
(Px \geq r)\varphi(x) \rightarrow (Py \geq r)\varphi(y); & (Px \leq r)\varphi(x) \rightarrow (Py \leq r)\varphi(y); \\
(Px > r)\varphi(x) \rightarrow (Py > r)\varphi(y); & (Px < r)\varphi(x) \rightarrow (Py < r)\varphi(y).
\end{array}$$

(A<sub>5</sub>) *Finite additivity:*

$$\begin{array}{l}
((Px \leq r)\varphi \wedge (Px \leq s)\psi) \rightarrow ((Px \leq r + s)(\varphi \vee \psi)); \\
((Px \geq r)\varphi \wedge (Px \geq s)\psi \wedge (Px \leq 0)(\varphi \wedge \psi)) \rightarrow ((Px \geq r + s)(\varphi \vee \psi)).
\end{array}$$

(A<sub>6</sub>)  $(Px \geq 0)\varphi$ .

(A<sub>7</sub>) *The Archimedean property:*

$$\begin{array}{l}
(Px > r)\varphi \leftrightarrow \bigvee_{n \in \mathbb{N}} (Px \geq r + 1/n)\varphi; \\
(\bigwedge_{n \in \mathbb{N}} (Px \geq r - 1/n)\varphi) \leftrightarrow (Px \geq r)\varphi.
\end{array}$$

(A<sub>8</sub>)  $(\forall x)\varphi \rightarrow (Px \geq 1)\varphi$ .

**9.1.3. Definition.** The Rules of Inference for  $L_{API}$  are as follows:

$$\begin{array}{ll}
(R_1) \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi} & \text{Modus Ponens;} \\
(R_2) \quad \frac{\varphi \rightarrow \psi, \psi \in \Phi}{\varphi \rightarrow \bigwedge \Phi} & \text{Conjunction;} \\
(R_3) \quad \frac{\psi \rightarrow \varphi, \psi \in \Phi}{\bigvee \Phi \rightarrow \varphi} & \text{Disjunction;} \\
(R_4) \quad \frac{\varphi \rightarrow \psi(x)}{\varphi \rightarrow (\forall x)\psi(x)} & \text{Generalization, (provided } x \text{ is not free in } \varphi).
\end{array}$$

**9.1.4. Definition.** A Kripke model is a (complex) structure

$$\mathfrak{K} = \langle \langle \mathfrak{A}_t, \mu_o^t, \mu_i^t \rangle, \langle T_0, \leq \rangle \rangle_{t \in T_0},$$

where:

- (a)  $\langle T_0, \leq \rangle$  is an ordered tree with the least element  $t_0$ ;
- (b)  $\mathfrak{A}_t \subseteq \mathfrak{A}_s$ , for  $t \leq s$ ;
- (c)  $\mu_i^t \leq \mu_i^s \leq \mu_o^s \leq \mu_o^t$ , for  $t \leq s$ ;
- (d)  $\mu_o^t$  is subadditive, i.e.,  $\mu_o^t(A) + \mu_o^t(B) \geq \mu_o^t(A \cup B)$ ;
- (e)  $\mu_i^t$  is superadditive, i.e.,  $\mu_i^t(A) + \mu_i^t(B) \leq \mu_i^t(A \cup B)$  for  $\mu_o^t(A \cap B) = 0$ .

**9.1.5. Definition.** Let  $\mathfrak{K}$  be a Kripke model. The relation  $t \Vdash \varphi$ , where  $t \in T_0$  and  $\varphi$  is a formula, is defined by induction on the formation of  $\varphi$  as follows:

- (1) If  $\varphi$  is an atomic formula, then  $t \Vdash \varphi[\vec{a}]$  iff  $\mathfrak{A}_t \models \varphi[\vec{a}]$ ;
- (2) If  $\Phi$  is a set of formulas, then
  - $t \Vdash \bigwedge \Phi$  iff for each  $\varphi \in \Phi$ ,  $t \Vdash \varphi$ ,
  - $t \Vdash \bigvee \Phi$  iff for some  $\varphi \in \Phi$ ,  $t \Vdash \varphi$ ;
- (3) If  $\varphi$  is  $\psi \rightarrow \theta$ , then  $t \Vdash \varphi$  iff for each  $s \geq t$ , if  $s \Vdash \psi$  then  $s \Vdash \theta$ ;
- (4) If  $\varphi$  is  $\neg\psi$ , then  $t \Vdash \varphi$  iff for no  $s \geq t$ ,  $s \Vdash \psi$ ;
- (5) If  $\varphi$  is  $(\forall x)\psi(x)$ , then  $t \Vdash \varphi$  iff for each  $s \geq t$  and each  $a \in A_s$ ,  $s \Vdash \psi[a]$ ;
- (6) If  $\varphi$  is  $(\exists x)\psi(x)$ , then  $t \Vdash \varphi$  iff there is  $a \in A_t$  such that  $t \Vdash \psi[a]$ ;
- (7) If  $\varphi$  is  $(Px \geq r)\psi$ , then  $t \Vdash \varphi$  iff  $\mu_i^t\{a \in A_t : t \Vdash \psi[a]\} \geq r$ , and similary for other quantifiers.

Finally,  $\mathfrak{K} \models \varphi$  iff  $t_0 \Vdash \varphi$ .

In order to prove the Completeness Theorem we need the notion of *saturated theory*.

**9.1.6. Definition.** The theory  $\Delta$  is saturated if:

- (i)  $Cn(\Delta) = \Delta$ , i.e.,  $\Delta \vdash \varphi$  iff  $\varphi \in \Delta$ ;
- (ii)  $\bigwedge \Phi \in \Delta$  iff for each  $\varphi \in \Phi$ ,  $\varphi \in \Delta$ ;
- (iii)  $\bigvee \Phi \in \Delta$  iff for some  $\varphi \in \Phi$ ,  $\varphi \in \Delta$ ;
- (iv)  $(\exists x)\varphi(x) \in \Delta$  iff there is a  $c \in \text{Ind}(\Delta)$  such that  $\varphi(c) \in \Delta$ .

Here,  $\text{Ind}(\Delta)$  is the set of all constants which appear in some formula of  $\Delta$ .

We now derive the necessary Rasiowa-Sikorski type result. The proof is very much in the spirit of the elementary proof of the Rasiowa-Sikorski Lemma for Boolean algebras, except that certain simplifications for Boolean algebras are not permissible in a more general setting.

**9.1.7. Lemma.** If  $T \not\models \varphi \rightarrow \psi$ , then there is a saturated theory  $\Delta \supseteq T$  such that  $\varphi \in \Delta$  and  $\psi \notin \Delta$ .

*Proof.* Let  $\psi_1, \psi_2, \psi_3, \dots$  be a sequence of all formulas from  $L_{API}$ , such that  $\psi_{3n+1} = \bigvee_j \varphi_j^{3n+1}$ ,  $\psi_{3n+2} = \bigwedge_j \varphi_j^{3n+2}$  and  $\psi_{3n+3} = (\exists x_j)\varphi_j^{3n+3}(x_j)$ .

We shall recursively construct sequences  $T \subseteq T_0 \subseteq T_1 \subseteq \dots$  and  $A_0 \subseteq A_1 \subseteq \dots$ . At each stage we shall have  $T_n \setminus T$  and  $A_n$  finite, and if  $\alpha^n = \bigwedge(T_n \setminus T)$  and  $\beta^n = \bigvee A_n$  of stage  $n$ , then  $T \not\models \alpha^n \rightarrow \beta^n$ . Let  $T_0 = T \cup \{\varphi\}$  and  $A_0 = \{\psi\}$ .

A typical stage falls into one of the following six cases.

Case (a) We consider  $n = 3k$ ,  $\psi_{n+1} = \bigvee_j \varphi_j^{n+1}$  and  $T \not\vdash (\bigvee_j \varphi_j^{n+1}) \wedge \alpha^n \rightarrow \beta^n$ . Now, since  $\alpha^n \wedge (\bigvee_j \varphi_j^{n+1}) \leftrightarrow \bigvee_j (\alpha^n \wedge \varphi_j^{n+1})$  it follows that for some  $j_0 < \omega$ ,  $T \not\vdash (\alpha^n \wedge \varphi_{j_0}^{n+1}) \rightarrow \beta^n$  and we put  $A_{n+1} = A_n$  and  $T_{n+1} = T_n \cup \{\varphi_{j_0}^{n+1}\}$ .

Case (b) We consider  $n = 3k$ ,  $\psi_{n+1} = \bigvee_j \varphi_j^{n+1}$  and  $T \vdash (\bigvee_j \varphi_j^{n+1}) \wedge \alpha^n \rightarrow \beta^n$ . In this case put  $T_{n+1} = T_n$  and  $A_{n+1} = A_n \cup \{\bigvee_j \varphi_j^{n+1}\}$ . We must verify that  $T \not\vdash \alpha^n \rightarrow \beta^n \vee (\bigvee_j \varphi_j^{n+1})$ . If not then

$$\begin{aligned} T &\vdash \alpha^n \rightarrow \beta^n \vee (\bigvee_j \varphi_j^{n+1}), \\ T &\vdash (\alpha^n \wedge \alpha^n) \rightarrow ((\beta^n \vee (\bigvee_j \varphi_j^{n+1})) \wedge \alpha^n), \\ T &\vdash \alpha^n \rightarrow (\beta^n \wedge \alpha^n) \vee \beta^n, && \text{by assumption} \\ T &\vdash \alpha^n \rightarrow \beta^n && \text{contradiction.} \end{aligned}$$

Case (c) We consider  $n = 3k+1$ ,  $\psi_{n+1} = \bigwedge_j \varphi_j^{n+1}$  and  $T \not\vdash \alpha^n \wedge (\bigwedge_j \varphi_j^{n+1}) \rightarrow \beta^n$ . In this case put  $T_{n+1} = T_n \cup \{\bigwedge_j \varphi_j^{n+1}\}$  and  $A_{n+1} = A_n$ .

Case (d) We consider  $n = 3k+1$ ,  $\psi_{n+1} = \bigwedge_j \varphi_j^{n+1}$  and  $T \vdash \alpha^n \wedge (\bigwedge_j \varphi_j^{n+1}) \rightarrow \beta^n$ . In this case we claim that for some  $j_0$ ,  $T \not\vdash \alpha^n \rightarrow (\beta^n \vee \varphi_{j_0}^{n+1})$ . If not then

$$\begin{aligned} T &\vdash \alpha^n \rightarrow (\beta^n \vee \varphi_j^{n+1}) && \text{for each } j, \\ T &\vdash \alpha^n \rightarrow \bigwedge_j (\beta^n \vee \varphi_j^{n+1}), \\ T &\vdash \alpha^n \rightarrow (\beta^n \vee (\bigwedge_j \varphi_j^{n+1})), \\ T &\vdash \alpha^n \rightarrow ((\beta^n \vee (\bigwedge_j \varphi_j^{n+1})) \wedge \alpha^n), \\ T &\vdash \alpha^n \rightarrow ((\beta^n \wedge \alpha^n) \vee ((\bigwedge_j \varphi_j^{n+1}) \wedge \alpha^n)), \\ T &\vdash \alpha^n \rightarrow ((\beta^n \wedge \alpha^n) \vee \beta^n), && \text{by assumption} \\ T &\vdash \alpha^n \rightarrow \beta^n && \text{contradiction.} \end{aligned}$$

Now put  $T_{n+1} = T_n$  and  $A_{n+1} = A_n \cup \{\varphi_{j_0}^{n+1}\}$ .

Case (e) We consider  $n = 3k+2$ ,  $\psi_{n+1} = (\exists x_j) \varphi_j^{n+1}(x_j)$  and suppose that  $T \not\vdash (\alpha^n \wedge (\exists x_j) \varphi_j^{n+1}(x_j)) \rightarrow \beta^n$ . Then we claim that  $T \not\vdash (\alpha^n \wedge \varphi_j^{n+1}(c_k)) \rightarrow \beta^n$  for some  $c_k$ . If not

$$\begin{aligned} T &\vdash (\forall x_j) ((\alpha^n \wedge \varphi_j^{n+1}(x_j)) \rightarrow \beta^n), && x_j \text{ new variable} \\ T &\vdash (\alpha^n \wedge (\exists x_j) \varphi_j^{n+1}(x_j)) \rightarrow \beta^n && \text{contradiction.} \end{aligned}$$

Let us put  $A_{n+1} = A_n$  and  $T_{n+1} = T_n \cup \{\varphi_j^{n+1}(c_k)\}$ .

Case (f) We consider  $n = 3k+2$ ,  $\psi_{n+1} = (\exists x_j) \varphi_j^{n+1}(x_j)$  and now suppose that  $T \vdash (\alpha^n \wedge (\exists x_j) \varphi_j^{n+1}(x_j)) \rightarrow \beta^n$ . We claim that  $T \not\vdash \alpha^n \rightarrow (\beta^n \vee (\exists x_j) \varphi_j^{n+1}(x_j))$ .

If not

$$\begin{array}{ll}
T \vdash \alpha^n \rightarrow (\beta^n \vee (\exists x_j) \varphi^{n+1}(x_j)), & \\
T \vdash \alpha^n \rightarrow ((\beta^n \wedge \alpha^n) \vee ((\exists x_j) \varphi^{n+1}(x_j) \wedge \alpha^n)), & \\
T \vdash \alpha^n \rightarrow ((\beta^n \wedge \alpha^n) \vee \beta^n), & \text{by assumption} \\
T \vdash \alpha^n \rightarrow \beta^n & \text{contradiction.}
\end{array}$$

Now put  $T_{n+1} = T_n$  and  $A_{n+1} = A_n \cup \{(\exists x_j) \varphi^{n+1}(x_j)\}$ .

We shall show that a theory  $\Delta = \{\psi : (\exists n \in \mathbb{N})(T \vdash \alpha^n \rightarrow \psi)\}$  satisfies the desired conditions of the Lemma. It is easy to see that  $T \subseteq \Delta$ ,  $\varphi \in \Delta$  and  $\Delta \cap (\bigcup_{n \in \mathbb{N}} A_n) = \emptyset$ .

We shall show that  $\Delta$  is deductively closed for two rules only (for other rules it is similar).

**Rule 1.** Suppose that  $\varphi, \varphi \rightarrow \psi \in \Delta$ . Then there is  $\alpha^n$  such that  $T \vdash \alpha^n \rightarrow \varphi$  and  $T \vdash \alpha^n \rightarrow (\varphi \rightarrow \psi)$ . So by rules and axioms:

$$\begin{array}{l}
T \vdash (\alpha^n \wedge \varphi) \rightarrow \psi, \\
T \vdash (\alpha^n \vee (\alpha^n \wedge \varphi)) \rightarrow \psi, \\
T \vdash \alpha^n \rightarrow (\alpha^n \vee (\alpha^n \wedge \varphi)), \\
T \vdash \alpha^n \rightarrow \psi \quad \text{i.e., } \psi \in \Delta.
\end{array}$$

**Rule 2.** Suppose that  $\varphi \rightarrow \psi \in \Delta$  for each  $\psi \in \Phi$ . So by construction of the theory  $\Delta$ ,  $\bigwedge_{\psi \in \Phi} (\varphi \rightarrow \psi) \in \Delta$ . By axiom  $\varphi \rightarrow \bigwedge \Phi \in \Delta$ .

The properties (2), (3) and (4) of the Definition 8.1.6. follow by construction.  $\square$

**9.1.8. Corollary.** (1) If  $T \not\vdash \varphi$ , then there is a saturated theory  $\Delta$  such that  $T \subseteq \Delta$  and  $\Delta \not\vdash \varphi$ .

(2) Let  $\Delta$  be a saturated theory. Then  $\varphi \rightarrow \psi \in \Delta$  iff for each saturated theory  $\Delta' \supseteq \Delta$ , if  $\varphi \in \Delta'$ , then  $\psi \in \Delta'$ .

(3) Let  $\Delta$  be a saturated theory. Then  $(\forall x_j) \varphi(x_j) \in \Delta$  iff for each saturated theory  $\Delta' \supseteq \Delta$  and each  $a \in \text{Ind}(\Delta')$ ,  $\varphi(a) \in \Delta'$ .

Now, we are going to prove our main result.

**9.1.9. Extended Completeness Theorem.** If  $\varphi$  is a sentence and  $T$  is a theory of  $L_{API}$ , then  $T \vdash \varphi$  if and only if  $T \models \varphi$ .

*Proof.* The trivial part is to prove that  $T \vdash \varphi$  implies  $T \models \varphi$ . In order to prove the theorem in the other direction we shall make a Kripke model  $\mathfrak{K}$  such that  $\mathfrak{K} \not\models \varphi$  and  $\mathfrak{K} \models T$ , where  $T \not\vdash \varphi$ . By corollary (1) there is a saturated theory  $\Delta_0$  such that  $T \subseteq \Delta_0$  and  $\Delta_0 \not\vdash \varphi$ .

Let  $\mathfrak{K}$  has the tree  $T_{\Delta_0} = \{\Delta \supseteq \Delta_0 : \Delta \text{ saturated}\}$ .

Also, let  $\mathfrak{A}_\Delta = \langle \text{Ind}(\Delta), R_j^\Delta, \dots \rangle$  be a model which corresponds to saturated theory  $\Delta$ .

If  $R_j$  is an  $n$ -ary relation symbol and  $c_1, \dots, c_n \in \text{Ind}(\Delta)$ , then  $R_j^\Delta(c_1, \dots, c_n)$  iff  $R_j(c_1, \dots, c_n) \in \Delta$ .

Define  $\mu_i^\Delta$  by  $\mu_i^\Delta\{c \in \text{Ind}(\Delta) : \varphi(c) \in \Delta\} = \sup\{r : (Px \geq r)\varphi \in \Delta\}$  and define  $\mu_o^\Delta$  by  $\mu_o^\Delta\{c \in \text{Ind}(\Delta) : \varphi(c) \in \Delta\} = \inf\{r : (Px \leq r)\varphi \in \Delta\}$ . It is easy to see that  $\mu_i^\Delta\{c \in \text{Ind}(\Delta) : \varphi(c) \in \Delta\} \geq r$  iff  $(Px \geq r)\varphi \in \Delta$  and similarly for  $(Px > r)$ ,  $(Px \leq r)$  and  $(Px < r)$ . From this and the properties of a saturated theory, it follows that  $\Delta \models \varphi$  iff  $\varphi \in \Delta$ .

Finally,  $\mathfrak{K} = \langle \langle \mathfrak{A}_\Delta, \mu_o^\Delta, \mu_i^\Delta \rangle, \langle T_{\Delta_0}, \leq \rangle \rangle_{\Delta \in T_{\Delta_0}}$  is the model we wanted.  $\square$

## 9.2. Logic of possibility

In this section we shall study a conservative extension of the propositional logic. We introduce the logic of possibility  $LP$  which is complete with respect to “natural” models, and is decidable. This part is taken from [52] and the main result is a solution of the problem proposed in [13, pp. 90]. Both papers were written independently. At the end we shall give a predicate extension.

**9.2.1. Definition.** Let  $\mathcal{P} = \{p_1, p_2, \dots\}$  be the list of propositional letters and let  $S$  be a finite subset of  $[0, 1]$  which contains 0 and 1. The set of all formulas of  $LP$  logic is the least set  $X$  such that:

- (a) Each propositional formula  $\varphi$  of  $L$  is in  $X$ ;
- (b) If  $\varphi$  is a formula of propositional logic, then  $P_r(\varphi) \in X$ , where  $r \in S$ ;
- (c) If  $A, B \in X$  and  $A, B$  are not propositional formulas, then  $\neg A, A \wedge B \in X$ .

**9.2.2. Remark.** Infinite  $S$  in the above definition does not make big difference, we only need more complicated list of axioms.

**9.2.3. Remark.** We denote propositional formulas by  $\varphi, \psi, \dots$  and  $LP$  formulas by  $A, B, \dots$ .

Rules of inferences are *Modus Ponens* and the following rule:  $\frac{\varphi}{P_1(\varphi)}$ .

Axioms for  $LP$  are all axioms of propositional logic plus the following:

- (1)  $P_0(\varphi)$ ;
- (2)  $P_r(\varphi) \rightarrow P_s(\varphi)$ , where  $r \geq s$ ;
- (3)  $(P_r(\varphi) \wedge P_s(\psi) \wedge P_1(\neg\varphi \vee \neg\psi)) \rightarrow P_{\min\{1, r+s\}}(\varphi \vee \psi)$ ;
- (4)  $(P_{1-r}(\neg\varphi) \wedge P_{1-s}(\neg\psi)) \rightarrow P_{\max\{0, 1-(r+s)\}}(\neg\varphi \wedge \neg\psi)$ ;
- (5)  $\neg P_{1-r}(\neg\varphi) \leftrightarrow P_{r^+}(\varphi)$ , where  $r^+ = \min\{s \in S : s \geq r\}$ .

Notions of proof, theorem etc. are defined in the usual way except that we must take care of limited application of our rules.

Let  $W \subseteq \mathcal{P}$  and let  $[\varphi]_W = \{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\}$  be a spectar of  $\varphi$ .

**9.2.4. Definition.** A model for  $LP$  logic is a measure space

$$\mathcal{W} = \langle W, \{[\varphi]_W : \varphi \in \text{For}_{\mathcal{P}}\}, \mu \rangle,$$

where  $\mu$  is a finitely additive measure.

**9.2.5. Definition.** The satisfaction relation  $\Vdash$  is defined in the following way:

$$\begin{aligned} \mathcal{W} \Vdash \varphi & \text{ iff } (\forall \mathfrak{A} \in W) \mathfrak{A} \models \varphi, \\ \mathcal{W} \Vdash P_r(\varphi) & \text{ iff } \mu\{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\} \geq r, \\ \mathcal{W} \Vdash A \wedge B & \text{ iff } \mathcal{W} \Vdash A \text{ and } \mathcal{W} \Vdash B, \\ \mathcal{W} \Vdash \neg A & \text{ iff } \mathcal{W} \not\Vdash A. \end{aligned}$$

**9.2.6. Completeness Theorem.** *Let  $T$  be a set of formulas of  $LP$ . Then,  $T$  is consistent if and only if  $T$  has a model.*

*Proof.* In order to prove the hard part of our theorem, let us suppose that  $T$  is a consistent theory and let  $\text{st}(T)$  be the set of all propositional consequences of  $T$ . Let  $A_1, A_2, \dots$  be an enumeration of all formulas of  $LP$  which are not from propositional logic.

Let  $\Sigma_0 = \text{st}(T) \cup \{P_1(\varphi) : \varphi \in \text{st}(T)\} \cup T \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \dots$  be a sequence such that:

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\}, & \text{if } \Sigma_n \cup \{A_n\} \text{ is consistent,} \\ \Sigma_n \cup \{\neg A_n\}, & \text{otherwise.} \end{cases}$$

It is easy to show that a theory  $\Sigma_\omega = \bigcup_n \Sigma_n$  is consistent.

Let  $W = \{\mathfrak{A} : \mathfrak{A} \models \text{st}(T)\}$  be a universe and let

$$\mu\{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\} = \max\{r : P_r(\varphi) \in \Sigma_\omega\}$$

be a finitely additive measure of our model. Then we can prove by induction that  $\mathcal{W} \Vdash A$  iff  $A \in \Sigma_\omega$ . In particular,  $\mathcal{W} \Vdash T$ .  $\square$

**9.2.7. Decidability Theorem.** *The logic  $LP$  is decidable.*

*Proof.* If a formula  $\varphi$  is propositional, then obviously it is decidable. If a formula  $A$  is not propositional, let  $p_1, p_2, \dots, p_n$  be the list of all propositional letters that occur in  $A$ , and let  $Q_1, Q_2, \dots, Q_m$  be the list of all formulas of the type  $P_r(\varphi_k)$  that occur in  $A$ .

It is easy to see that the formula  $A$  is a propositional combination  $\beta$  of the formulas of the type  $P_r(\varphi)$  taken as propositional letters.

Let  $\bigvee \{ (Q_1^{\varepsilon(1)} \wedge \dots \wedge Q_m^{\varepsilon(m)}) : \varepsilon \in {}^m 2, A(\varepsilon) = \top \}$  be a disjunctive normal form of the formula  $A$ , where

$$Q_j^{\varepsilon(j)} = \begin{cases} P_{r_j}(\varphi_{k_j}), & \text{if } \varepsilon(j) = 0, \\ \neg P_{r_j}(\varphi_{k_j}), & \text{if } \varepsilon(j) = 1. \end{cases}$$

The formula  $A$  is not a contradiction if and only if some formula  $Q_1^{\varepsilon(1)} \wedge \dots \wedge Q_m^{\varepsilon(m)}$  is not a contradiction.

For each  $Q_j = P_{r_j}(\varphi_{k_j})$ , let  $\bigvee \{ p_1^{\tau(1)} \wedge \dots \wedge p_n^{\tau(n)} : \tau \in {}^n 2, \varphi_{k_j}(\tau) = \top \}$  be a disjunction normal form of the formula  $\varphi_{k_j}$ .

Then, the formula  $A$  is not a contradiction if and only if there is a valuation  $\varepsilon \in {}^m 2$  such that  $A(\varepsilon) = \top$  and the following system of equations and inequalities

$$\begin{aligned} \sum_{\tau \in {}^n 2} \mu(p_1^{\tau(1)} \wedge \cdots \wedge p_n^{\tau(n)}) &= 1, \\ \mu(p_1^{\tau(1)} \wedge \cdots \wedge p_n^{\tau(n)}) &\geq 0, \quad \text{where } \tau \in {}^n 2, \\ \sum \{ \mu(p_1^{\tau(1)} \wedge \cdots \wedge p_n^{\tau(n)}) : \tau \in {}^n 2, \varphi_{k_1}(\tau) = \top \} &\begin{cases} \geq r_1, & \text{if } \varepsilon(1) = 0 \\ < r_1, & \text{if } \varepsilon(1) = 1 \end{cases}, \\ &\vdots \\ \sum \{ \mu(p_1^{\tau(1)} \wedge \cdots \wedge p_n^{\tau(n)}) : \tau \in {}^n 2, \varphi_{k_m}(\tau) = \top \} &\begin{cases} \geq r_m, & \text{if } \varepsilon(m) = 0 \\ < r_m, & \text{if } \varepsilon(m) = 1 \end{cases}. \end{aligned}$$

is consistent.

For short, we write  $\mu(\varphi)$  instead of  $\mu([\varphi]_W)$ .

We can conclude that the problem of decidability is reduced to an easy problem of linear programming, which can be positively solved.  $\square$

**9.2.8. Remark.** Instead of propositional, we can start with predicate logic and build “predicate” version of Logic of Possibility (see [52]).

Let  $\text{DNF}(A)$  be a disjunctive normal form of  $A$ . Note that we use in the proof the complete disjunctive normal form of the probabilistic formula  $A$ , but it is not necessary. We can proceed with arbitrary  $\text{DNF}(A)$ , in fact, a “clever” algorithm would use a minimal  $\text{DNF}(A)$ . The only reason for using the complete  $\text{DNF}(A)$  is uniformity of the notation.

A decision procedure based on the previous theorem could be described in the following way:

```

BEGIN
  DNF(A) := disjunctive normal form (A);
  FOR every disjunct D form DNF(A) DO
    BEGIN
      SYSTEM(D) := generate system (D);
      IF there is a solution of SYSTEM(D)
      THEN
        BEGIN
          write ('A is satisfiable');
          exit
        END;
      END;
    END;
  write ('A is not satisfiable');
END.
```

**9.2.9. Example.** We implemented a theorem prover (in fact a satisfiability checker) based on the given procedure. The figure 1 shows a program report for the formula  $P_{0,5}(p \rightarrow q) \wedge \neg P_{0,65}(p \rightarrow q) \wedge P_{0,8}(p) \wedge P_{0,6}(q)$ .

Input formula:  
 $V\{0, 5\}(p \text{ then } q) \text{ and not } (V\{0, 65\}(p \text{ then } q))$   
 and  $V\{0, 8\}(p) \text{ and } V\{0, 6\}(q)$ .  
 Formula is successfully translated.  
 During the translation the following coding is done:  
 $p$  is coded as  $R_2$ .  
 $q$  is coded as  $R_3$ .  
 First disjunct of the formula:  
 $V\{0.500\}(R_2 \text{ THEN } R_3) \text{ AND NOT } [V\{0.650\}(R_2 \text{ THEN } R_3)]$   
 AND  $V\{0.800\}(R_2) \text{ AND } V\{0.600\}(R_3)$   
 $X0$  corresponds to:  $!R_2()!R_3()$   
 $X1$  corresponds to:  $!R_2()R_3()$   
 $X2$  corresponds to:  $R_2()R_3()$   
 $X3$  corresponds to:  $R_2()!R_3()$   
 First disjunct in the extended format:  
 $V\{0.500\}(X0 + X1 + X2) \text{ AND NOT } V\{0.650\}(X0 + X1 + X2)$   
 AND  $V\{0.800\}(X2 + X3) \text{ AND } V\{0.600\}(X1 + X2)$   
 Generated system:  
 (In)equalities that correspond to propositional part

| $X0$  | $X1$  | $X2$  | $X3$  |    |         |
|-------|-------|-------|-------|----|---------|
| 1.000 | 1.000 | 1.000 | 1.000 | =  | + 1.000 |
| 1.000 | 0.000 | 0.000 | 0.000 | >= | + 0.000 |
| 0.000 | 1.000 | 0.000 | 0.000 | >= | + 0.000 |
| 0.000 | 0.000 | 1.000 | 0.000 | >= | + 0.000 |
| 0.000 | 0.000 | 0.000 | 1.000 | >= | + 0.000 |

(In)equalities that correspond to probabilistic part

| $X0$  | $X1$  | $X2$  | $X3$  |    |         |
|-------|-------|-------|-------|----|---------|
| 1.000 | 1.000 | 1.000 | 0.000 | >= | + 0.500 |
| 1.000 | 1.000 | 1.000 | 0.000 | <  | + 0.650 |
| 0.000 | 0.000 | 1.000 | 1.000 | >= | + 0.800 |
| 0.000 | 1.000 | 1.000 | 0.000 | >= | + 0.600 |

The system has a solution.  
 The formula is satisfiable.

FIGURE 1

As can be seen, the only disjunct of the formula is the formula itself. There are two primitive propositions:  $p$  and  $q$ . For shortness, program denoted combinations of  $p^\tau \wedge q^\tau$  by  $x_0, x_1, x_2, x_3$ . Next, every propositional formula is presented in a disjunctive normal form, and a system of linear (in) equalities is generated. Since the system is solvable, the examined formula is satisfiable.

Now we discuss an extension of *LPP*. Namely, Fagin, Halpern and Megiddo gave a probabilistic logic in [13], where linear combinations of the so-called weight terms are basis for probabilistic formulas. For example,  $t \geq c$  is a basic weight formula, where  $t = a_1 W(\varphi_1) + \dots + a_k W(\varphi_k)$ ,  $\varphi_i$ 's are propositional formulas,  $a_i$ 's and  $c$  are rational numbers, and weight terms  $W(\varphi_i)$ 's denote probability of  $\varphi_i$ 's. A basic weight formula is satisfied in model  $\mathcal{W}$  if  $a_1 \mu([\varphi_1]_{\mathcal{W}}) + \dots + a_k \mu([\varphi_k]_{\mathcal{W}}) \geq c$ . Weight formulas are Boolean combinations of basic weight formulas. Obviously, a formula of the form  $P_s(\varphi)$  corresponds to a basic weight formula of length 1; i.e., to  $W(\varphi) \geq s$ . They proved Gödel Completeness Theorem (i.e.,  $\vdash \varphi$  iff  $\models \varphi$ ).

Now, we shall sketch how to extend *LPP* to cover their logic. Suppose we have a slightly different language that allows basic weight formulas and their Boolean

combinations (this logic we denote by *ELPP*). The axioms for *LPP* are rewritten according to the new notation.

*Taut.* All instances of propositional tautologies.

- Prob.* P1.  $W(\varphi) \geq 0, \quad 1 \geq W(\varphi);$   
 P2.  $W(\varphi) \geq r \rightarrow W(\varphi) \geq s, \quad r \geq s;$   
 P3.  $(W(\varphi) \geq r \wedge W(\psi) \geq s \wedge W(\varphi \wedge \psi) \leq 0) \rightarrow W(\varphi \vee \psi) \geq r + s;$   
 P4.  $(W(\varphi) \leq r \wedge W(\psi) \leq s) \rightarrow W(\varphi \vee \psi) \leq r + s.$

Additionally, we include in *ELPP* a complete axiom system *Ineq.* for reasoning about linear inequalities.

- Ineq.* I1.  $W(\varphi) \geq W(\varphi);$   
 I2.  $(a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) \geq c) \leftrightarrow$   
 $\quad \leftrightarrow (a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) + 0 \cdot W(\varphi_{k+1}) \geq c);$   
 I3.  $(a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) \geq c) \rightarrow$   
 $\quad \rightarrow (a_{j_1} W(\varphi_{j_1}) + \dots + a_{j_k} W(\varphi_{j_k}) \geq c),$   
 $\quad \text{if } j_1, \dots, j_k \text{ is a permutation of } 1, \dots, k;$   
 I4.  $((a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) \geq c) \wedge (a'_1 W(\varphi_1) + \dots + a'_k W(\varphi_k) \geq c'))$   
 $\quad \rightarrow ((a_1 + a'_1) W(\varphi_1) + \dots + (a_k + a'_k) W(\varphi_k) \geq c + c');$   
 I5.  $(a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) \geq c) \leftrightarrow$   
 $\quad \leftrightarrow (da_1 W(\varphi_1) + \dots + da_k W(\varphi_k) \geq dc) \text{ if } d > 0;$   
 I6.  $(t \geq c) \vee (t \leq c), \quad t \text{ is term};$   
 I7.  $(t \geq c) \rightarrow (t > d), \quad c > d.$

The rules of inferences are

- (1) *Modus Ponens*:  $\frac{\varphi, \varphi \rightarrow \psi}{\psi};$   
 (2)  $\frac{\varphi}{W(\varphi) \geq 1};$   
 (3) *Archimedean rule*:  $\frac{A \rightarrow t \geq s - \frac{1}{n} \text{ for } n \in \omega}{A \rightarrow t \geq s}.$

We can formulate completeness theorem with respect to the same class of models that we have used so far, and prove it using the same ideas as in the last theorem.

**9.2.10. Completeness Theorem for *ELPP*.** *Let  $T$  be a set of *ELPP*-formulas.  $T$  is consistent iff  $T$  has a model.*

*Proof.* We follow the proof of the Theorem 8.2.7. The only new case concerns basic weight formulas and we solve it in the next lemma.  $\square$

**9.2.11. Lemma.** *Let  $\Sigma$ ,  $\mu$  and  $\mathcal{W}$  be defined as in the last theorem. Then,  $\mathcal{W} \models t \geq c$  iff  $t \geq c \in \Sigma$ .*

*Proof.* Suppose  $A \equiv a_1 W(\varphi_1) + \dots + a_k W(\varphi_k) \geq c$  and  $\mathcal{W} \models A$ . Also, suppose that  $\mu([\varphi_1]_{\mathcal{W}}) = s_1, \dots, \mu([\varphi_k]_{\mathcal{W}}) = s_k$ . For each natural  $n$  there are rational

numbers  $r_1, \dots, r_k$  such that

$$r_i + 1/n \geq \mu([\varphi_i]_W) \geq r_i \quad \text{for } i = 1, \dots, k.$$

Then  $W(\varphi_i) \geq r_i$ ,  $r_i + 1/n \geq W(\varphi_i) \in \Sigma$ , and by the axioms of *Ineq*,

$$t \geq \sum_{i=1}^k a_i r_i, \quad \sum_{i=1}^k a_i (r_i + 1/n) \geq t \in \Sigma.$$

By the Archimedean rule and the facts that  $\sum_{i=1}^k a_i s_i \geq c - (a_1 + \dots + a_k)/n$ , it follows that  $t \geq c \in \Sigma$ .

For the other direction, suppose  $A \in \Sigma$ . The trivial case  $A = W(\varphi) \geq s$  is solved in the last theorem. Assume that for every  $B$  with less than  $k$  weight terms, if  $B \in \Sigma$  then  $\mathcal{W} \Vdash B$ . Let  $A \equiv a_1 W(\varphi_1) + \dots + a_n W(\varphi_k) \geq c$  and  $s_k = \sup\{r : W(\varphi_k) \geq r \in \Sigma\}$ . For each natural number  $n$  there is rational number  $r_k$  such that  $r_k + 1/n \geq \mu([\varphi_k]_W) \geq r_k$ . It follows that  $r_k + 1/n \geq W(\varphi_k)$ ,  $W(\varphi_k) \geq r_k \in \Sigma$ . Then

$$\sum_{i=1}^{k-1} a_i W(\varphi_i) \geq \sum_{i=1}^{k-1} a_i s_i + a_k (s_k - r_k) - \frac{a_k}{n} \in \Sigma.$$

By the Archimedean rule  $\sum_{i=1}^{k-1} a_i W(\varphi_i) \geq c - a_k s_k \in \Sigma$ . Using inductive assumption

$$\sum_{i=1}^{k-1} a_i \mu([\varphi_i]_W) \geq c - a_k s_k, \quad \text{i.e.} \quad \sum_{i=1}^k a_i \mu([\varphi_i]_W) \geq c.$$

It follows  $\mathcal{W} \Vdash A$ .  $\square$

### 9.3. A probabilistic validity measure in intuitionistic propositional logic

Work on this subject was directly inspired by H.J. Keisler's treatment of probability quantifiers (see [32] and [34]). In this part we present an extension of the Heyting propositional logic by probability operators. The main aim was to describe syntax and semantics enabling an adequate expression of the following statement: "Probability of truthfulness of  $A$  is greater (less) or equal to  $r$ ", denoted by  $P_r A$  ( $P^r A$  respectively). The next paragraph deals with our formal system. Afterwards we define the corresponding Kripke-type semantics and algebraic models followed by the completeness results.

**9.3.1. Formal system.** The language of our system consists of usual symbols for logical connectives, propositional letters, parentheses and two types of probability operators:  $P^r$  and  $P_r$  for each  $r \in S \subset [0, 1]$ , where  $S$  is finite. The set of formulas is the smallest set containing propositional letters which is closed under formation rules: if  $A$  and  $B$  are formulas, then  $(\neg A)$ ,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ ,  $P^r A$  and  $P_r A$  are formulas.

Let  $\mathcal{H}$  be the Heyting propositional calculus. Then by  $\mathcal{H}_P$  we denote the logic having the following axioms:

- (0) All axioms from  $(A_1)$  to  $(A_8)$  of the part 9.1,  
 (1) *Monotonicity*

$$\begin{aligned} P^r A &\rightarrow P^s A, & \text{for } r \leq s \\ P_r A &\rightarrow P_s A, & \text{for } r \geq s, \end{aligned}$$

- (2)  $P^1 A$  and  $P_0 A$ ,  
 (3) *Finite semiadditivity*

$$\begin{aligned} P^r A \wedge P^s B &\rightarrow P^t (A \vee B), & \text{where } t = \min\{1, r + s\} \\ P_r A \wedge P_s B \wedge P^0 (A \wedge B) &\rightarrow P_t (A \vee B), & \text{where } t = \min\{1, r + s\}, \end{aligned}$$

and inference rules:

$$\frac{A \quad A \rightarrow B}{B} \text{ (MP)} \quad \text{and} \quad \frac{A}{P_1 A} \text{ (G)}.$$

The axioms (1), (2) and (3) and the inference rule (G), concerning the probability operators are called the *probability measure conditions*.

**9.3.2. Kripke type semantics.** In this paragraph, we are going to define the Kripke type model corresponding to the logic  $\mathcal{H}_P$ .

**9.3.2.1. Definition.** By an  $\mathcal{H}_P$ -frame we mean a structure of the form

$$\langle W, R, (\mu^x)_{x \in W}, (\mu_x)_{x \in W} \rangle$$

where

- (1)  $\langle W, R \rangle$  is an  $\mathcal{H}$ -frame, and  
 (2)  $\mu^x$  and  $\mu_x$  are partial functions from the power set of  $W$  to  $S \subset [0, 1]$  such that the following conditions are satisfied:

*Monotonicity:* For every  $x, y \in W$  and every measurable  $X \subseteq W$ , if  $xRy$ , then  $\mu^x(X) \geq \mu^y(X)$  and  $\mu_x(X) \leq \mu_y(X)$ ;

*Subadditivity:* For every  $x \in W$  and all measurable  $X, Y \subseteq W$ ,

$$\mu^x(X \cup Y) \leq \mu^x(X) + \mu^x(Y);$$

*Superadditivity:* For every  $x \in W$  and all measurable  $X, Y \subseteq W$ , if  $\mu^x(X \cap Y) = 0$ , then

$$\mu_x(X \cup Y) \geq \mu_x(X) + \mu_x(Y).$$

Note that the properties considered above correspond to the probability measure conditions.

An  $\mathcal{H}_P$ -frame together with a valuation, i.e. a mapping  $\nu$  from the set of propositional variables to the power set of  $W$ , such that for every propositional variable  $Q$  and all  $x, y \in W$ , if  $xRy$  and  $x \in \nu(Q)$ , then  $y \in \nu(Q)$ , will be denoted by

$$\mathcal{M} = \langle W, R, (\mu^x)_{x \in W}, (\mu_x)_{x \in W}, \nu \rangle$$

and called an  $\mathcal{H}_P$ -model.

Let  $\mathcal{M}$  be an  $\mathcal{H}_P$ -model. Then, for any  $x \in W$  and for any formula  $A$ , we define the relation  $x \models A$ , “ $A$  holds in the world  $x$  in  $\mathcal{M}$ ”, by induction on the complexity of the formula  $A$ , as follows:

$$\begin{aligned} x \models Q & \text{ iff } x \in \nu(Q), \\ x \models B \wedge C & \text{ iff } x \models B \text{ and } x \models C, \\ x \models B \vee C & \text{ iff } x \models B \text{ or } x \models C, \\ x \models B \rightarrow C & \text{ iff for all } y \in W, \text{ if } xRy \text{ and } y \models B, \text{ then } y \models C, \\ x \models \neg B & \text{ iff for all } y \in W, \text{ if } xRy, \text{ then not } y \models B, \\ x \models P^r B & \text{ iff } \mu^x\{y : y \models B\} \leq r, \\ x \models P_r B & \text{ iff } \mu_x\{y : y \models B\} \geq r, \end{aligned}$$

where we suppose that, for every formula  $A$ , the set  $\{y : y \models A\}$  is both  $\mu^x$  and  $\mu_x$ -measurable, for any  $x \in W$ . We say that  $A$  holds in a model  $\mathcal{M}$ , denoted by  $\mathcal{M} \models A$ , if and only if for any  $x \in W$ ,  $x \models A$ . We simply write  $\models A$  and say that  $A$  is  $\mathcal{H}_P$ -valid if and only if for every  $\mathcal{H}_P$ -model  $\mathcal{M}$ ,  $\mathcal{M} \models A$ . In the sequel of this paper we shall use abbreviations  $\mu^x(B)$  and  $\mu_x(B)$  for  $\mu^x\{y : y \models B\}$  and  $\mu_x\{y : y \models B\}$ , respectively.

Now, following a usual procedure, we are going to present the main steps of the proof that  $\mathcal{H}_P$  is sound and complete with respect to the class of  $\mathcal{H}_P$ -models.

**9.3.2.2. Intuitionistic Heredity Lemma.** *In every  $\mathcal{H}_P$ -model, for any formula  $A$  and any  $x, y \in W$ , if  $x \models A$  and  $xRy$ , then  $y \models A$ .*

*Proof.* By induction on the complexity of  $A$ . Let us consider just the cases with the probability operators.

(1) If  $A = P^r B$ ,  $xRy$  and  $x \models A$ , i.e.,  $x \models P^r B$ , then by definition of  $\models$ , we have  $\mu^x(B) \leq r$ . But having in mind the monotonicity of  $\mu^x(X)$ , for  $xRy$ , we have  $\mu^x(B) \geq \mu^y(B)$ , as well. Consequently:  $\mu^y(B) \leq r$ , i.e.,  $y \models A$ .

(2) Similarly, for  $A = P_r B$ ,  $xRy$  and  $x \models A$ , by definition of satisfiability relation and monotonicity of  $\mu^x(X)$ , we have  $r \leq \mu_x(B) \leq \mu_y(B)$ , i.e.,  $y \models A$ .  $\square$

Let  $Cn(\Gamma) = \{A : \Gamma \vdash_{\mathcal{H}_P} A\}$  be a deductive closure of any set of formulas  $\Gamma$ . We say that the set of formulas  $\Gamma$  is a saturated theory if and only if  $\Gamma$  is  $\mathcal{H}_P$ -consistent (i.e., there is a formula  $A$  such that  $\text{not } \Gamma \vdash_{\mathcal{H}_P} A$ ), deductively closed (i.e.,  $\Gamma = Cn(\Gamma)$ ) and has disjunction property (i.e., for any formula  $A$  and  $B$ , if  $A \vee B \in \Gamma$ , then  $A \in \Gamma$  or  $B \in \Gamma$ ).

**9.3.2.3. Lemma.** *A maximal consistent extension of any set of formulas is a saturated theory.*

*Proof.* Let  $\Gamma$  be a consistent set of formulas such that  $\text{not } \Gamma \vdash A$ . Further, let  $S = \{\Pi : \Gamma \subseteq \Pi \text{ and } \text{not } \Pi \vdash A\}$ . System  $S$  is a nonempty set of consistent extensions of  $\Gamma$  closed under unions of nonempty chains and, consequently, by Zorn's Lemma,  $S$  has a maximal element  $\Delta$ .  $\Delta$  is a consistent extension of  $\Gamma$ , obviously. We prove that  $\Delta$  is deductively closed. Let us suppose that there is a formula  $B$ , such that  $\Delta \vdash B$ , but  $B \notin \Delta$ . Then, since  $\Delta$  is a proper subset of  $\Delta \cup \{B\}$  and

$\Delta$  is a maximal consistent extension of  $\Gamma$ , we have that  $\Delta \cup \{B\} \vdash A$ , wherefrom  $\Delta \vdash A$ , which is a contradiction. We can also prove that  $\Delta$  has the disjunction property. Let us suppose that there are formulas  $B$  and  $C$ , such that  $B \vee C \in \Delta$ ,  $B \notin \Delta$  and  $C \notin \Delta$ . Then, since  $\Delta$  is a maximal consistent extension of  $\Gamma$ , we have  $\Delta \cup \{B\} \vdash A$  and  $\Delta \cup \{C\} \vdash A$ , wherefrom, by the deduction theorem, we infer  $\Delta \vdash B \rightarrow A$  and  $\Delta \vdash C \rightarrow A$ , and, consequently,  $\Delta \vdash B \vee C \rightarrow A$ . But, since  $B \vee C \in \Delta$ , we get a contradiction:  $\Delta \vdash A$ . It follows that  $\Delta$  is a saturated theory extending the set  $\Gamma$ .  $\square$

**9.3.2.4. Corollary.** *If not  $\Gamma \vdash A$ , then there exists a saturated theory  $\Delta$  extending  $\Gamma$ , such that not  $\Delta \vdash A$ .*

The next usual step is to introduce the canonical  $\mathcal{H}_P$ -model. Let  $X^c$  be the class of all saturated theories with respect to  $\mathcal{H}_P$ , and for any  $\Gamma, \Delta \in X^c$ ,

$$\begin{aligned} \Gamma \in R^c \Delta &\stackrel{\text{def}}{\iff} \Gamma \subseteq \Delta, \\ \nu^c(Q) &\stackrel{\text{def}}{=} \{\Gamma : Q \in \Gamma\}, \\ \mu^{c\Gamma}(A) &= \mu^{c\Gamma}\{\Delta : \Delta \models A\} \stackrel{\text{def}}{=} \sup\{t : t = \inf\{s : P^s A \in \Delta\}, \Gamma \subseteq \Delta\}, \\ \mu_\Gamma^c(A) &= \mu_\Gamma^c\{\Delta : \Delta \models A\} \stackrel{\text{def}}{=} \inf\{t : t = \sup\{s : P_s A \in \Delta\}, \Gamma \subseteq \Delta\}. \end{aligned}$$

Then the structure

$$\langle X^c, R^c, (\mu^{c\Gamma})_{\Gamma \in X^c}, (\mu_\Gamma^c)_{\Gamma \in X^c}, \nu^c \rangle$$

will be called the canonical  $\mathcal{H}_P$ -model. Hereafter, for shortness, we shall drop the superscript  $c$ , when it is clear that we work in the canonical model.

**9.3.2.5. Lemma.** *The canonical  $\mathcal{H}_P$ -model is an  $\mathcal{H}_P$ -model.*

*Proof.* First, we prove that  $\langle X, R, (\mu^\Gamma)_\Gamma, (\mu_\Gamma)_\Gamma \rangle$  is an  $\mathcal{H}_P$ -frame.  $\langle X, R \rangle$  is supposed to be an  $\mathcal{H}_P$ -frame. For each  $\Gamma \in X$ , both maps  $\mu^\Gamma(A)$  and  $\mu_\Gamma(A)$  are monotone in the sense of our definition: if

$$\begin{aligned} \mu^\Gamma(A) &= \mu^\Gamma\{\Pi : \Pi \models A\} = \sup\{t : t = \inf\{s : P^s A \in \Pi\}, \Gamma \subseteq \Pi\} = r_1, \\ \mu^\Delta(A) &= \mu^\Delta\{\Pi : \Pi \models A\} = \sup\{t : t = \inf\{s : P^s A \in \Pi\}, \Delta \subseteq \Pi\} = r_2 \end{aligned}$$

and  $\Gamma \subseteq \Delta$ , we have  $r_1 \geq r_2$ .

If  $\Gamma \in X$ , then, having in mind the probability measure conditions, as well as that

$$\begin{aligned} \mu\{y : y \models A \vee B\} &= \mu(A \vee B) = \mu((A) \vee (B)) = \mu(\{y : y \models A\} \cup \{y : y \models B\}), \\ \mu\{y : y \models A \wedge B\} &= \mu(A \wedge B) = \mu((A) \wedge (B)) = \mu(\{y : y \models A\} \cap \{y : y \models B\}), \end{aligned}$$

where  $\mu$  may be  $\mu^x$  or  $\mu_x$ , then

$$\mu^\Gamma(A \vee B) \leq \mu^\Gamma(A) + \mu^\Gamma(B)$$

and, for  $\mu^\Gamma(A \wedge B) = 0$ ,

$$\mu_\Gamma(A \vee B) \geq \mu_\Gamma(A) + \mu_\Gamma(B). \quad \square$$

**9.3.2.6. Lemma.** *In the canonical  $\mathcal{H}_P$ -model, for every formula  $A$  and every  $\Gamma \in X$ ,*

$$\Gamma \models A \quad \text{iff} \quad A \in \Gamma.$$

*Proof.* By induction on the complexity of  $A$ . We present only the part of proof related to the probability operators.

(1) If  $A = P^r B$  and  $\Gamma \models P^r B$ , then, by definition, we have

$$\sup \{ t : t = \inf \{ s : P^s B \in \Pi \}, \Gamma \subseteq \Pi \} \leq r$$

and, consequently,  $P^r B \in \Gamma$ .

(2) If  $A = P_r B$  and  $\Gamma \models P_r B$ , then, by definition  $\mu_\Gamma \{ \Pi : \Pi \models B \} \geq r$ , i.e.,

$$\inf \{ t : t = \sup \{ s : P_s B \in \Pi \}, \Gamma \subseteq \Pi \} \geq r.$$

In such a case, if  $t = \sup \{ s : P_s B \in \Gamma \}$ , we have  $P_t B \in \Gamma$ ,  $t \geq r$  and, consequently,

$$\inf \{ t : t = \sup \{ s : P_s B \in \Pi \}, \Gamma \subseteq \Pi \} \geq r,$$

i.e.,  $P_r \in \Gamma$ .  $\square$

**9.3.2.7. Completeness and Soundness Theorem.** *For any formula  $A$ ,  $A$  is provable in  $\mathcal{H}_P$  if and only if  $A$  is  $\mathcal{H}_P$ -valid.*

*Proof.* The soundness part can be proved by induction on the length of the proof of  $A$  in  $\mathcal{H}_P$ , and we omit it.

On the other hand, the completeness part is an immediate consequence of the preceding lemmas.  $\square$

**9.3.3. Algebraic models.** A pseudo-Boolean (or Heyting) probabilistic algebra (PBPA) is a structure

$$\langle B, \leq, (p^r)_{r \in S}, (p_s)_{s \in S} \rangle,$$

such that  $\langle B, \leq \rangle$  is a partial ordering with a least element denoted by 0 and greatest element denoted by 1, and for any  $x, y \in B$ :

- (1) there is a least upper bound of  $x$  and  $y$ , denoted  $x \cup y$ ;
- (2) there is a greatest lower bound of  $x$  and  $y$ , denoted  $x \cap y$ ;
- (3) there is a pseudo-complement of  $x$  relative to  $y$ , denoted  $x \rightarrow y$  and defined as the largest  $z \in B$  such that  $x \cap z \leq y$  (The pseudo-complement  $\sim x$  of  $x$  can be defined as  $x \rightarrow 0$ );
- (4)  $p^r x \leq p^s x$ , for  $r \leq s$ ;
- (5)  $p_r x \leq p_s x$ , for  $r \geq s$ ;
- (6)  $1 \leq p^1 x$ ;
- (7)  $1 \leq p_0 x$ ;
- (8)  $p^r x \cap p^s y \leq p^t (x \cup y)$ , where  $t = \min \{ 1, r + s \}$ ;
- (9)  $p_r x \cap p_s y \cap p^0 (x \cap y) \leq p_t (x \cup y)$ , where  $t = \min \{ 1, r + s \}$ ;
- (10)  $x \leq p_1 x$ .

A *PBPA* model is a mapping  $h$  from the set of all formulas to some *PBPA*, such that for any formulas  $A$  and  $B$ :

$$\begin{aligned} h(A \wedge B) &= h(A) \cap h(B), & h(\neg A) &= \sim h(A), \\ h(A \vee B) &= h(A) \cup h(B), & h(P^r A) &= p^r h(A), \\ h(A \rightarrow B) &= h(A) \rightarrow h(B), & h(P_r A) &= p_r h(A). \end{aligned}$$

$h$  is a model of  $A$ , if  $h(A) = 1$ , and  $h$  is a model of a theory  $T$  when  $h$  is a model of each  $A \in T$ .

Alternatively, a model  $h$  can be taken to be a mapping from the propositional variables into *PBPA* and then extend  $h$  to arbitrary formulas according to the above conditions.

The following Extended Completeness Theorem may be obtained.

**9.3.3.1. Theorem.** *A formula  $A$  is provable in a theory  $T$  if and only if every *PBPA* model of  $T$  is a *PBPA* model of  $A$ .*

*Proof.* We give a brief outline of the proof. The trivial part is to prove that  $T \vdash A$  implies  $T \models A$ . The opposite direction is a Lindenbaum-Tarski algebra construction. First note that the relation “ $A \rightarrow B$  and  $B \rightarrow A$  are provable in  $T$ ” is an equivalence relation. The function which maps each sentence into its own equivalence class is a *PBPA* model and each sentence of  $T$  is mapped into 1. If  $A$  is not provable in  $T$ , then  $A$  is not mapped into 1.  $\square$

## CYLINDRIC PROBABILITY ALGEBRAS

In this chapter we shall introduce the notion of a cylindric probability algebra, which can be considered as a common algebraic abstraction from a geometry associated with basic set-theoretic notions on the one hand, and the theory of deductive systems of probability logic on the other. These two sources are connected because models of deductive systems of probability logic give rise in a natural way to probability structures within set-theoretical algebras. As is well known, the theory of Boolean algebras is related to the sentential calculus, and the theory of cylindric algebras to the first-order predicate logic. The theory of cylindric probability algebras, designed to provide an apparatus for an algebraic study of graded probability logic, will be presented analogously to Boolean algebras and cylindric algebras.

First we shall describe the cylindric probability set algebras.

Let  $\langle A, \mu_n \rangle_{n < \omega}$  be a graded probability space and let  $\mu_\omega$  be the completion of the measure on  $A^\omega$  determined by the  $\mu_n$ 's. Let  $\langle K \rangle$  be a tuple of distinct integers corresponded to a finite subset  $K = \{k_1, \dots, k_n\}$  of  $\omega$ . For each  $\langle K \rangle$  and  $r \in [0, 1]$ , we define a unary operation  $C_{\langle K \rangle}^r$  on subsets of  $A^\omega$  by setting, for any  $X \subseteq A^\omega$ ,

$$C_{\langle K \rangle}^r(X) = \{y \in A^\omega : \mu_n\{(x_{k_1}, \dots, x_{k_n}) : x \in X \ \& \ (j \notin K \rightarrow x_j = y_j)\} \geq r\}.$$

It follows from the Fubini property that for any  $\mu_\omega$ -measurable set  $X$ , the section  $\{(x_{k_1}, \dots, x_{k_n}) : x \in X \ \& \ (j \notin K \rightarrow x_j = y_j)\}$  is  $\mu_n$ -measurable for each  $y \in A^\omega$ , and also that  $C_{\langle K \rangle}^r(X)$  is  $\mu_\omega$ -measurable. Thus, the operations  $C_{\langle K \rangle}^r$  are well-defined. By means of the operation  $C_{\langle K \rangle}^r$  we obtain the cylinders generated by translating only the section of  $X$  whose measure is not less than  $r$ , parallelly to the  $k_1, \dots, k_n$ -axis of the space  $A^\omega$ . So, these operations will be called *probability cylindrifications*. Also, for each permutation  $\sigma$  of  $\omega$  such that  $\sigma = id$  almost surely (a.s.), i.e.,  $\sigma \upharpoonright K^c = id$ , where  $K^c$  is a complement of a finite subset  $K$  of  $\omega$ , we define an unary operation  $S_\sigma$  on subsets of  $A^\omega$  by setting, for any  $X \subseteq A^\omega$ ,

$$S_\sigma X = \{(x_{\sigma(1)}, x_{\sigma(2)}, \dots) : x \in X\}.$$

For any  $\mu_\omega$ -measurable set  $X$ , the set  $S_\sigma X$  is  $\mu_\omega$ -measurable, too. These operations will be called *permutations*. We do not need the full class of permutations since each infinite formula of  $L_{AP}$  has only finitely many free variables.

Let  $\mathcal{A}$  be a countable admissible set and  $\omega \in \mathcal{A}$ . In advance we suppose that a fixed indexation by hereditarily countable sets (from  $\mathcal{A} \subseteq \text{HC}$ ) is given. So, let  $\mathbb{A} = \{A_i : i \in I\}$  and  $I \subseteq \mathcal{A}$ . We shall say that a Boolean set algebra  $\langle \mathbb{A}, \cup, \cap, \sim, \emptyset, A \rangle$ , where  $\mathbb{A} \subseteq \mathcal{P}(A)$ , is  $\mathcal{A}$ -complete if for any  $\{A_j : j \in J\} \subseteq \mathbb{A}$ , where  $J \subseteq I$  and  $J \in \mathcal{A}$ , the expression  $\bigcup_{j \in J} A_j \in \mathbb{A}$ . The abstract notion of a  $\mathcal{A}$ -complete Boolean algebra  $\langle A, +, \cdot, -, 0, 1 \rangle$  can be introduced in the obvious way.

**10.1. Definition.** A cylindric probability set algebra ( $CPS_\mu$  for short) is a structure of the form

$$\langle \mathbb{A}, \cup, \cap, \sim, \emptyset, A^\omega, C_{\langle K \rangle}^r, S_\sigma, D_{pq} \rangle_{K, \sigma, r \in [0,1], p, q < \omega}$$

in which  $\mathbb{A}$  is a collection of subsets of  $A^\omega$  closed under all operations of the  $\mathcal{A}$ -complete Boolean algebra, all probability cylindric operations and all permutations, having all diagonal hyperplanes  $D_{pq} = \{x \in A^\omega : x_p = x_q\}$  as distinguished members.

Second, the theory of cylindric probability algebras is rooted in probability logic, so we shall now describe the relationship between the graded probability logic  $L_{AP}$  and cylindric probability algebras.

The set  $Form_L$  of all formulas of  $L_{AP}$  is closed under the operations  $\vee, \wedge, \neg$ , and  $(P\vec{v} \geq r)$ , where  $\vec{v}$  is a finite sequence  $v_{k_1}, \dots, v_{k_n}$  of variables and  $r \in [0, 1]$ , and closed under the substitutions  $s_\sigma$  of variables. This set contains as distinguished elements expressions  $F$  (false),  $T$  (true) and  $v_p = v_q$  for any  $p, q < \omega$ . The structure

$$\mathfrak{Form}_L = \langle Form_L, \vee, \wedge, \neg, F, T, (P\vec{v} \geq r), s_\sigma, v_p = v_q \rangle$$

is the so-called *free algebra of formulas* of  $L_{AP}$ .

Let  $\Sigma$  be any set of sentences of  $L_{AP}$  and let  $\equiv_\Sigma$  be the relation on  $Form_L$  defined by

$$\varphi \equiv_\Sigma \psi \quad \text{iff} \quad \Sigma \vdash \varphi \leftrightarrow \psi.$$

If  $\Sigma \vdash \varphi \leftrightarrow \psi$ , then  $\Sigma \vdash (P\vec{v} \geq r)\varphi \leftrightarrow (P\vec{v} \geq r)\psi$  and  $\Sigma \vdash s_\sigma \varphi \leftrightarrow s_\sigma \psi$ , and hence the relation  $\equiv_\Sigma$  is a congruence relation on  $\mathfrak{Form}_L$  in the sense of the general theory of algebras. Let  $\varphi^\Sigma$  be the set of all formulas equivalent to  $\varphi$  under  $\Sigma$ . By applying a well-known algebraic procedure, we shall construct the quotient algebra

$$\mathfrak{Form}_L / \equiv_\Sigma = \langle Form_L / \equiv_\Sigma, \vee^\Sigma, \wedge^\Sigma, \neg^\Sigma, F^\Sigma, T^\Sigma, (P\vec{v} \geq r)^\Sigma, s_\sigma^\Sigma, (v_p = v_q)^\Sigma \rangle,$$

which will be called a *cylindric probability algebra of formulas of  $L_{AP}$  associated to the set of sentences  $\Sigma$* , where  $Form_L / \equiv_\Sigma$  is the set of all equivalence classes  $\varphi^\Sigma$ , and for  $\Phi \subseteq Form_L$ ,  $\Phi \in \mathcal{A}$ ,

$$\begin{aligned} \bigvee_{\varphi \in \Phi} \varphi^\Sigma &= \left( \bigvee \Phi \right)^\Sigma, & (P\vec{v} \geq r)^\Sigma \varphi^\Sigma &= ((P\vec{v} \geq r)\varphi)^\Sigma, \\ \bigwedge_{\varphi \in \Phi} \varphi^\Sigma &= \left( \bigwedge \Phi \right)^\Sigma, & s_\sigma^\Sigma \varphi^\Sigma &= (s_\sigma \varphi)^\Sigma, \\ \neg^\Sigma \varphi^\Sigma &= (\neg \varphi)^\Sigma. \end{aligned}$$

Thus, the basic metalogical problems are reduced to algebraic problems concerning the associated algebra of formulas. For instance, the problem of the type "Is a given sentence  $\varphi$  a theorem of  $\Sigma$ ?" is reduced to "Does the equation  $\varphi^\Sigma = T^\Sigma$  hold in  $\mathfrak{Form}_L / \equiv_\Sigma$ ?"

Let  $\mathfrak{A} = \langle A, R^\mathfrak{A}, c^\mathfrak{A}, \mu_n \rangle$  be a graded probability model for  $\Sigma$  and let  $\varphi$  be any formula of  $L_{AP}$ . Then, for  $\varphi^\mathfrak{A} = \{a \in A^\omega : \mathfrak{A} \models \varphi[a]\}$ , we obtain

$$\begin{aligned} C_{\langle K \rangle}^r(\varphi^\mathfrak{A}) &= \{a \in A^\omega : \mu_n\{(b_{k_1}, \dots, b_{k_n}) : \mathfrak{A} \models \varphi[b] \ \& \ (j \notin K \rightarrow b_j = a_j)\} \geq r\} \\ &= \{a \in A^\omega : \mathfrak{A} \models (P\vec{v} \geq r)\varphi[a]\} \\ &= ((P\vec{v} \geq r)\varphi)^\mathfrak{A}, \end{aligned}$$

where  $\vec{v}$  is  $v_{k_1}, \dots, v_{k_n}$  and  $K = \{k_1, \dots, k_n\}$ , and

$$\begin{aligned} S_\sigma(\varphi^\mathfrak{A}) &= \{(a_{\sigma(1)}, a_{\sigma(2)}, \dots) : \mathfrak{A} \models \varphi[a]\} \\ &= \{(a_{\sigma(1)}, a_{\sigma(2)}, \dots) : \mathfrak{A} \models s_\sigma \varphi[a_{\sigma(1)}, a_{\sigma(2)}, \dots]\} \\ &= (s_\sigma \varphi)^\mathfrak{A}. \end{aligned}$$

Thus the collection  $\mathbb{A}$  of all subsets  $\varphi^\mathfrak{A}$  is closed under all operations of the  $\mathcal{A}$ -complete Boolean set algebra, under all probability cylindrifications and under all permutations. Also, all diagonal hyperplanes  $D_{pq}$  belong to  $\mathbb{A}$ . We thus obtain a "natural" mapping  $f$  from the set  $Form_L / \equiv_\Sigma$ , of all equivalence classes  $\varphi^\Sigma$ , onto the collection  $\mathbb{A}$  of all sets  $\varphi^\mathfrak{A}$ , which has the following obvious properties:

- |                                                                                                                 |                                                                                                    |
|-----------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------|
| (1) $f(\bigvee_{\varphi \in \Phi}^\Sigma \varphi^\Sigma) = \bigcup_{\varphi \in \Phi} \varphi^\mathfrak{A}$ ;   | (5) $f(T^\Sigma) = A^\omega$ ;                                                                     |
| (2) $f(\bigwedge_{\varphi \in \Phi}^\Sigma \varphi^\Sigma) = \bigcap_{\varphi \in \Phi} \varphi^\mathfrak{A}$ ; | (6) $f((P\vec{v} \geq r)^\Sigma \varphi^\Sigma) = C_{\langle K \rangle}^r(\varphi^\mathfrak{A})$ ; |
| (3) $f(\neg^\Sigma \varphi^\Sigma) = \sim \varphi^\mathfrak{A}$ ;                                               | (7) $f(s_\sigma^\Sigma \varphi^\Sigma) = S_\sigma(\varphi^\mathfrak{A})$ ;                         |
| (4) $f(F^\Sigma) = \emptyset$ ;                                                                                 | (8) $f((v_p = v_q)^\Sigma) = D_{pq}$ .                                                             |

Thus we arrive at a cylindric probability set algebra obtained by a homomorphic transformation from the cylindric probability algebra of formulas. So, the algebraic problem "Does the equation  $\varphi^\Sigma = T^\Sigma$  hold in  $\mathfrak{Form}_L / \equiv_\Sigma$ ?" reduces to a set problem "Does the set equation  $\varphi^\mathfrak{A} = A^\omega$  hold in the probability set algebra?"

The abstract notion of a *cylindric probability algebra* (CPA for short) is defined by equational axioms which hold in all cylindric probability set algebras, where the stress is on the axioms of graded probability logic. Changing the notation introduced for cylindric probability set algebras and cylindric probability algebras of formulas, we shall consider algebraic structure  $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, C_{\langle K \rangle}^r, S_\sigma, d_{pq} \rangle$ , where  $\langle A, +, \cdot, -, 0, 1 \rangle$  is an  $\mathcal{A}$ -complete Boolean algebra,  $C_{\langle K \rangle}^r$  is a unary operation on  $A$  called probability cylindrification,  $S_\sigma$  is a unary operation on  $A$  called permutation, and  $d_{pq}$  is distinguished element of  $A$  for any  $p, q < \omega$ . We suggest the following set of postulates of a cylindric probability algebra  $\mathbf{A}$ :

- (CP<sub>1</sub>) (i)  $C_{\langle \emptyset \rangle}^r x = x$ ,  
(ii)  $C_{\langle K \rangle}^r 0 = 0$ , where  $r > 0$ ;
- (CP<sub>2</sub>)  $C_{\langle K \rangle}^0 x = 1$ ;
- (CP<sub>3</sub>)  $C_{\langle K \rangle}^r x \leq C_{\langle K \rangle}^s x$ , where  $r \geq s$ ;
- (CP<sub>4</sub>)  $C_{\langle K \rangle}^r (x + \sum_{j \in J} C_{\langle L \rangle}^s C_{\langle M \rangle}^t y_j) = C_{\langle K \rangle}^r x + \sum_{j \in J} C_{\langle L \rangle}^s C_{\langle M \rangle}^t y_j$ , where  $J \in \mathcal{A}$  and  $K \subseteq L \cup M$ ;
- (CP<sub>5</sub>) (i)  $C_{\langle K \rangle}^r x \cdot C_{\langle K \rangle}^s y \leq C_{\langle K \rangle}^{\max\{0, r+s-1\}} (x \cdot y)$ ,  
(ii)  $C_{\langle K \rangle}^r x \cdot C_{\langle K \rangle}^s y \cdot C_{\langle K \rangle}^1 - (x \cdot y) \leq C_{\langle K \rangle}^{\min\{r+s, 1\}} (x + y)$ ;
- (CP<sub>6</sub>)  $C_{\langle K \rangle}^r - x = -\sum_{m>0} C_{\langle K \rangle}^{1-r+1/m} x$ ;
- (CP<sub>7</sub>)  $\prod_{J_2 \subseteq J_1} C_{\langle K \rangle}^r \prod_{j \in J_2} x_j \leq C_{\langle K \rangle}^r \prod_{j \in J_1} x_j$ , where  $J_2$  ranges over the finite subsets of  $J_1$  and  $J_1 \in \mathcal{A}$ ;
- (CP<sub>8</sub>)  $C_{\langle K \rangle}^r x = C_{\langle \pi(K) \rangle}^r x$ , where  $\pi$  is a permutation of  $\{1, \dots, n\}$  and  $\langle \pi(K) \rangle$  is  $k_{\pi 1}, \dots, k_{\pi n}$ ;
- (CP<sub>9</sub>)  $C_{\langle K \rangle}^r C_{\langle L \rangle}^s x \leq C_{\langle K \rangle, \langle L \rangle}^{r \cdot s} x$ , where  $K \cap L = \emptyset$ ;
- (CP<sub>10</sub>) (i)  $S_{id} x = x$ ,  
(ii)  $S_\sigma S_\tau x = S_{\sigma \circ \tau} x$ ,  
(iii)  $S_\sigma \sum_{j \in J} x_j = \sum_{j \in J} S_\sigma x_j$ , where  $J \in \mathcal{A}$ ,  
(iv)  $S_\sigma - x = -S_\sigma x$ ;
- (CP<sub>11</sub>) (i)  $S_\sigma C_{\langle K \rangle}^r C_{\langle L \rangle}^s x = S_\tau C_{\langle K \rangle}^r C_{\langle L \rangle}^s x$ , where  $\sigma \upharpoonright (K \cup L)^c = \tau \upharpoonright (K \cup L)^c$ ,  
(ii)  $C_{\langle K \rangle}^r S_\sigma x = S_\sigma C_{\langle \sigma^{-1}(K) \rangle}^r x$  where  $\langle \sigma^{-1}(k) \rangle$  is  $\sigma^{-1}(k_1), \dots, \sigma^{-1}(k_n)$ ;
- (CP<sub>12</sub>) (i)  $d_{pp} = 1$ ,  
(ii)  $x \cdot d_{pq} \leq S_\sigma x$ , where  $\sigma(p) = q$ ,  
(iii)  $S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)}$ ,  
(iv)  $C_{\langle K \rangle}^1 d_{pq} = d_{pq}$ , where  $p, q \notin K$ .

The axioms  $CP_2$ ,  $CP_3$ ,  $CP_5$ - $CP_9$  express respectively non-negativity, monotonicity, finite additivity, the Archimedean property, countable additivity, symmetry and product independence of probability measures  $\{\mu_n : n < \omega\}$  (see [34]), and hence these axioms hold in any  $CPS_\mu$ -algebra. By routine checking of remaining axioms we shall prove the following theorem.

**10.2. Theorem.** *Every cylindric probability set algebra is a cylindric probability algebra.*

*Proof.* Let  $\mathbb{A}$  be a  $CPS_\mu$ -algebra. We shall check only the less trivial axioms.

Any set  $Z$  of the form  $Z = \bigcup_{j \in J} C_{\langle L \rangle}^s C_{\langle M \rangle}^t Y_j$  is a  $\langle L \rangle, \langle M \rangle$ -cylinder; so  $CP_4$  follows from  $C_{\langle K \rangle}^r Z = Z$ , for  $r > 0$  and  $K \subseteq L \cup M$ .

The axioms  $CP_{10}$  and  $CP_{11}$  follow from the definition of  $S_\sigma$ . We only remark that  $C_{\langle K \rangle}^r C_{\langle L \rangle}^s X$  is a  $\langle K \rangle, \langle L \rangle$ -cylinder; hence,  $S_\sigma C_{\langle K \rangle}^r C_{\langle L \rangle}^s X = S_\tau C_{\langle K \rangle}^r C_{\langle L \rangle}^s X$  for  $\sigma \upharpoonright (K \cup L)^c = \tau \upharpoonright (K \cup L)^c$ .

At last, we check  $CP_{11}$  (iv). If  $y \in C_{\langle K \rangle}^1 D_{pq}$ , and  $p, q \notin K$ , then

$$\mu_n \{ (x_{k_1}, \dots, x_{k_n}) : x_p = x_q \text{ \& } (j \notin K \rightarrow y_j = x_j) \} \geq 1.$$

Thus,  $y_p = x_p = x_q = y_q$ ; i.e.,  $y \in D_{pq}$ . The converse follows from

$$\{ (x_{k_1}, \dots, x_{k_n}) : x_p = x_q \text{ \& } (j \notin K \rightarrow y_j = x_j) \} = A^n,$$

since  $y \in D_{pq}$  and  $p, q \notin K$ .  $\square$

Similarly, by routine checking, we obtain the following.

**10.3. Theorem.** *Every cylindric probability algebra of formulas is a cylindric probability algebra.*

Now we shall prove several elementary properties of the probability cylindrification operations.

**10.4. Theorem.** (1)  $C_{\langle K \rangle}^r 1 = 1$ .

(2)  $C_{\langle K \rangle}^r C_{\langle L \rangle}^s x = C_{\langle L \rangle}^s x$ , where  $(r > 0 \text{ or } r = s = 0) \text{ and } K \subseteq L$ .

(3)  $C_{\langle K \rangle}^r x = x$  iff  $C_{\langle K \rangle}^s - x = -x$ , where  $r > 0$  and  $s > 0$ .

(4)  $C_{\langle K \rangle}^r (x + -C_{\langle L \rangle}^s y) = C_{\langle K \rangle}^r x + -C_{\langle L \rangle}^s y$ , where  $K \subseteq L$ .

(5)  $C_{\langle K \rangle}^r (x \cdot C_{\langle L \rangle}^s y) = C_{\langle K \rangle}^r x \cdot C_{\langle L \rangle}^s y$ , where  $(r > 0 \text{ or } r = s = 0) \text{ and } K \subseteq L$ .

(6)  $C_{\langle K \rangle}^r x \cdot -C_{\langle K \rangle}^r y \leq \sum_{m>0} C_{\langle K \rangle}^{1/m} (x \cdot -y)$ .

(7) If  $x \leq y$ , then  $C_{\langle K \rangle}^r x \leq C_{\langle K \rangle}^r y$ .

(8)  $C_{\langle K \rangle}^r x + C_{\langle K \rangle}^r y \leq C_{\langle K \rangle}^r (x + y)$ .

(9)  $C_{\langle K \rangle}^r x \cdot C_{\langle K \rangle}^r y \geq C_{\langle K \rangle}^r (x \cdot y)$ .

(10) If  $C_{\langle K \rangle}^r x_j = x_j$  for  $r > 0$ ,  $j \in J$  and  $J \in \mathcal{A}$ , then  $C_{\langle K \rangle}^r \sum_{j \in J} x_j = \sum_{j \in J} x_j$ .

(11)  $-C_{\langle K \rangle}^r x \leq C_{\langle K \rangle}^{1-r} - x$ .

(12)  $C_{\langle K \rangle}^1 x \cdot C_{\langle K \rangle}^1 y = C_{\langle K \rangle}^1 (x \cdot y)$ .

(13)  $C_{\langle K \rangle}^1 C_{\langle L \rangle}^1 x = C_{\langle K \rangle, \langle L \rangle}^1 x$ , where  $K \cap L = \emptyset$ .

*Proof.* (1): Putting  $x = 0$  in  $CP_6$  we obtain  $C_{\langle K \rangle}^r 1 = -\sum_{m>0} C_{\langle K \rangle}^{1-r+1/m} 0 = 1$ , from  $CP_1$ .

(2): Immediate by  $CP_2$  and  $CP_4$  (putting  $x = 0$ ,  $J$  is a singleton and  $M = \emptyset$ ).

(3): If  $C_{\langle K \rangle}^r x = x$ , then, for any  $s > 0$ , we obtain

$$C_{\langle K \rangle}^s - x = -\sum_{m>0} C_{\langle K \rangle}^{1-s+1/m} x = -\sum_{m>0} C_{\langle K \rangle}^{1-s+1/m} C_{\langle K \rangle}^r x = -x$$

from (2) and  $CP_6$ . The converse follows by symmetry.

(4): Immediate by (3) and  $CP_4$ .

(5): For  $r > 0$  we obtain

$$\begin{aligned}
 C_{\langle K \rangle}^r(x \cdot C_{\langle L \rangle}^s y) &= C_{\langle K \rangle}^r - (-x + -C_{\langle L \rangle}^s y) \\
 &= -\sum_{p>0} C_{\langle K \rangle}^{1-r+1/p}(-x + -C_{\langle L \rangle}^s y) \quad \text{by } CP_6 \\
 &= -\left(\sum_{p>0} C_{\langle K \rangle}^{1-r+1/p} - x + -C_{\langle L \rangle}^s y\right) \quad \text{by (4)} \\
 &= \left(-\sum_{p>0} C_{\langle K \rangle}^{1-r+1/p} - x\right) \cdot C_{\langle L \rangle}^s y \\
 &= C_{\langle K \rangle}^r x \cdot C_{\langle L \rangle}^s y \quad \text{by } CP_6.
 \end{aligned}$$

(6): For any  $r \in [0, 1]$  we obtain

$$\begin{aligned}
 C_{\langle K \rangle}^r x \cdot -C_{\langle K \rangle}^r y &= C_{\langle K \rangle}^r x \cdot \sum_{m>0} C_{\langle K \rangle}^{1-r+1/m} - y \quad \text{by } CP_6 \\
 &= \sum_{m>0} C_{\langle K \rangle}^r x \cdot C_{\langle K \rangle}^{1-r+1/m} - y \\
 &\leq \sum_{m>0} C_{\langle K \rangle}^{1/m}(x \cdot -y) \quad \text{by } CP_5 \text{ (i).}
 \end{aligned}$$

(7): If  $x \leq y$ , then  $x \cdot -y = 0$ . Hence,

$$C_{\langle K \rangle}^r x \cdot -C_{\langle K \rangle}^r y \leq \sum_{m>0} C_{\langle K \rangle}^{1/m}(x \cdot -y) = 0$$

from (6) and  $CP_1$ ; i.e.,  $C_{\langle K \rangle}^r x \leq C_{\langle K \rangle}^r y$ .

(8),(9): Immediate by  $x \leq x + y$ ,  $y \leq x + y$ ,  $x \cdot y \leq x$ ,  $x \cdot y \leq y$  and (7).

(10): Putting  $x = 0$  in  $CP_4$ , we obtain

$$C_{\langle K \rangle}^r \sum_{j \in J} x_j = C_{\langle K \rangle}^r \sum_{j \in J} C_{\langle K \rangle}^r x_j = \sum_{j \in J} C_{\langle K \rangle}^r x_j = \sum_{j \in J} x_j.$$

(11): By  $CP_6$  and  $CP_3$ , we obtain

$$-C_{\langle K \rangle}^r x = \sum_{m>0} C_{\langle K \rangle}^{1-r+1/m} - x \leq \sum_{m>0} C_{\langle K \rangle}^{1-r} - x = C_{\langle K \rangle}^{1-r} - x.$$

(12): By  $CP_5$  (i) we have  $C_{\langle K \rangle}^1 x \cdot C_{\langle K \rangle}^1 y \leq C_{\langle K \rangle}^1(x \cdot y)$ . The reverse inequality is an instance of (9).

(13): It follows from  $CP_9$  and  $CP_3$  that for each  $m, n > 0$  there is  $p > 0$  such that

$$C_{\langle K \rangle}^{1/m} C_{\langle L \rangle}^{1/n} - x \leq C_{\langle K \rangle, \langle L \rangle}^{1/(mn)} - x \leq C_{\langle K \rangle, \langle L \rangle}^{1/p} - x.$$

Further, we obtain  $C_{\langle K \rangle, \langle L \rangle}^1 x = -\sum_{q>0} C_{\langle K \rangle, \langle L \rangle}^{1/q} - x \leq -C_{\langle K \rangle}^{1/m} C_{\langle L \rangle}^{1/n} - x$  for each  $m > 0$ . Hence

$$C_{\langle K \rangle, \langle L \rangle}^1 x \leq \prod_{m>0} -C_{\langle K \rangle}^{1/m} C_{\langle L \rangle}^{1/n} - x = C_{\langle K \rangle}^1 - C_{\langle L \rangle}^{1/n} - x$$

for each  $n > 0$ . Finally, for  $M$  ranging over the finite subsets of  $N$ , we have:

$$\begin{aligned}
 C_{\langle K \rangle, \langle L \rangle}^1 x &\leq \prod_{n>0} C_{\langle K \rangle}^1 - C_{\langle L \rangle}^{1/n} - x \\
 &= \prod_{M \subseteq N} \prod_{n \in M} C_{\langle K \rangle}^1 - C_{\langle L \rangle}^{1/n} - x \\
 &= \prod_{M \subseteq N} C_{\langle K \rangle}^1 \prod_{n \in M} -C_{\langle L \rangle}^{1/n} - x \quad \text{by (12)} \\
 &\leq C_{\langle K \rangle}^1 \prod_{n>0} -C_{\langle L \rangle}^{1/n} - x \quad \text{by } CP_7 \\
 &= C_{\langle K \rangle}^1 C_{\langle L \rangle}^1 x. \quad \text{by } CP_6
 \end{aligned}$$

The reverse inequality is an instance of  $CP_9$ .  $\square$

Also, we shall prove several necessary properties of the permutation operations and diagonal elements.

- 10.5. Theorem.** (1)  $\sum_{m>0} C_{\langle K \rangle}^{1/m} d_{pq} = d_{pq}$ , where  $p, q \notin K$ .  
(2)  $C_{\langle K \rangle}^r (d_{pq} \cdot x) = d_{pq} \cdot C_{\langle K \rangle}^r x$ , where  $r > 0$  and  $p, q \notin K$ .  
(3)  $S_\sigma C_{\langle K \rangle}^r C_{\langle L \rangle}^s x = C_{\langle K \rangle}^r C_{\langle L \rangle}^s x$ , where  $\sigma \upharpoonright (K \cup L)^c = id$ .  
(4)  $S_\sigma 0 = 0$  and  $S_\sigma 1 = 1$ .  
(5)  $d_{pq} = d_{qp}$ .  
(6)  $d_{pq} \cdot d_{qr} \leq d_{pr}$ .  
(7)  $x \cdot d_{pq} = S_\sigma x \cdot d_{pq}$ , where  $\sigma(p) = q$  and  $\sigma \upharpoonright \{p, q\}^c = id$ .

*Proof.* (1): By  $CP_{12}$  (iv) and (2) of the Theorem 10.4., we obtain

$$C_{\langle K \rangle}^{1/m} d_{pq} = C_{\langle K \rangle}^{1/m} C_{\langle K \rangle}^1 d_{pq} = C_{\langle K \rangle}^1 d_{pq} = d_{pq}$$

for each  $m < \omega$ . Hence,  $\sum_{m>0} C_{\langle K \rangle}^{1/m} d_{pq} = d_{pq}$ .

- (2): Immediate by  $CP_{12}$  (iv) and (5) of 10.4.  
(3): We have

$$\begin{aligned}
 S_\sigma C_{\langle K \rangle}^r C_{\langle L \rangle}^s x &= S_{id} C_{\langle K \rangle}^r C_{\langle L \rangle}^s x \quad \text{by } CP_{11} \text{ (i)} \\
 &= C_{\langle K \rangle}^r C_{\langle L \rangle}^s x \quad \text{by } CP_{10} \text{ (i)}.
 \end{aligned}$$

(4): For a permutation  $\sigma$  and  $K = \{k_1, \dots, k_n\} \subseteq \omega$  such that  $\sigma \upharpoonright K^c = id$ , we obtain

$$S_\sigma 0 = S_\sigma C_{\langle K \rangle}^1 0 = C_{\langle K \rangle}^1 0 = 0$$

by  $CP_1$  and (3). Also,  $S_\sigma 1 = S_\sigma - 0 = -S_\sigma 0 = 1$  by  $CP_{10}$  (iv).

(5): Let  $\sigma$  be a permutation of  $\omega$  such that  $\sigma(p) = q$  and  $\sigma \upharpoonright \{p, q\}^c = id$  for  $p \neq q$ . Then

$$d_{pq} = d_{pq} \cdot d_{pq} \leq S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)} = d_{qp}$$

by  $CP_{12}$ . The converse follows by symmetry.

(6): Let  $\sigma$  be a permutation such that  $\sigma(q) = r$  and  $\sigma \upharpoonright \{q, r\}^c = id$ . For  $p \neq q \neq r \neq p$  we obtain

$$d_{pq} \cdot d_{qr} \leq S_\sigma d_{pq} = d_{\sigma(p)\sigma(q)} = d_{pr}$$

by  $CP_{12}$ .

(7): Let  $\sigma$  be a permutation such that  $\sigma(p) = q$  and  $\sigma \upharpoonright \{p, q\}^c = id$ . By  $x \cdot d_{pq} \leq S_\sigma x$  we obtain  $x \cdot d_{pq} \leq S_\sigma x \cdot d_{pq}$ . The reverse inequality is a consequence of the previous inequality and the axiom  $CP_{10}$  as follows:

$$S_\sigma x \cdot d_{pq} \leq S_\sigma S_\sigma x \cdot d_{pq} = S_{\sigma \circ \sigma} x \cdot d_{pq} = S_{id} x \cdot d_{pq} = x \cdot d_{pq}. \quad \square$$

Some algebraic notions such as subalgebras, homomorphisms, ideals and free algebras, can be modified using specific properties of cylindric probability algebras. At this moment we shall mention two of them.

**10.6. Definition.** A homomorphism from  $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, C_{\langle K \rangle}^r, S_\sigma, d_{pq} \rangle$  into  $\mathbf{B} = \langle B, +', \cdot', -', 0', 1', C_{\langle K \rangle}^{r'}, S_{\sigma'}, d_{pq}' \rangle$  is a function  $f$  mapping  $A$  into  $B$  such that (for all  $x \in A$ ,  $\{x_j : j \in J\} \subseteq A$  and  $J \in \mathcal{A}$ ):

- |                                                          |                                                                        |
|----------------------------------------------------------|------------------------------------------------------------------------|
| (1) $f(\sum_{j \in J} x_j) = \sum_{j \in J}' f(x_j)$ ;   | (5) $f(1) = 1'$ ;                                                      |
| (2) $f(\prod_{j \in J} x_j) = \prod_{j \in J}' f(x_j)$ ; | (6) $f(C_{\langle K \rangle}^r x) = C_{\langle K \rangle}^{r'} f(x)$ ; |
| (3) $f(-x) = -' f(x)$ ;                                  | (7) $f(S_\sigma x) = S_{\sigma'} f(x)$ ;                               |
| (4) $f(0) = 0'$ ;                                        | (8) $f(d_{pq}) = d_{pq}'$ .                                            |

We shall write  $\mathbf{A} \cong \mathbf{B}$  for short if there is an isomorphism of  $\mathbf{A}$  and  $\mathbf{B}$ , where the term *isomorphism* has the obvious meaning.

**10.7. Definition.** An ideal in a cylindric probability algebra  $\mathbf{A}$  is a non-empty set  $\mathcal{I} \subseteq A$  such that the following conditions hold:

- (1)  $\mathcal{I}$  is a Boolean ideal of  $\mathbf{A}$ ; i.e.,
  - (a)  $0 \in \mathcal{I}$ ,
  - (b) If  $\{a_j : j \in J\} \subseteq \mathcal{I}$  and  $J \in \mathcal{A}$ , then  $\sum_{j \in J} a_j \in \mathcal{I}$ ,
  - (c) If  $x \in \mathcal{I}$  and  $y \leq x$ , then  $y \in \mathcal{I}$ ;
- (2) For any finite  $K \subseteq \omega$  and  $r \in (0, 1]$ , if  $x \in \mathcal{I}$ , then  $C_{\langle K \rangle}^r x \in \mathcal{I}$ .
- (3) For any permutation  $\sigma$  of  $\omega$  such that  $\sigma = id$  a.s., if  $x \in \mathcal{I}$ , then  $S_\sigma x \in \mathcal{I}$ .

It follows from (6) of Theorem 10.4. that an ideal  $\mathcal{I}$  of a cylindric probability algebra  $\mathbf{A}$  determines a congruence relation  $\sim = \{(x, y) : x \cdot -y + y \cdot -x \in \mathcal{I}\}$ . So, we define a new quotient algebra  $\mathbf{A}/\mathcal{I} = \langle A/\mathcal{I}, \hat{+}, \hat{\cdot}, \hat{-}, \hat{0}, \hat{1}, \hat{C}_{\langle K \rangle}^r, \hat{S}_\sigma, \hat{d}_{pq} \rangle$  by  $\langle A/\mathcal{I}, \hat{+}, \hat{\cdot}, \hat{-}, \hat{0}, \hat{1} \rangle = \langle A, +, \cdot, -, 0, 1 \rangle / \mathcal{I}$ ,  $\hat{C}_{\langle K \rangle}^r[x] = [C_{\langle K \rangle}^r x]$ ,  $\hat{S}_\sigma[x] = [S_\sigma x]$  for all equivalence class  $[x] \in A/\mathcal{I}$ , and  $\hat{d}_{pq} = [d_{pq}]$ . It is not difficult to see that  $\mathbf{A}/\mathcal{I}$  is a CPA and that there is a natural homomorphism from  $\mathbf{A}$  onto  $\mathbf{A}/\mathcal{I}$ .

As in the case of arbitrary algebras, there is a natural correspondence between homomorphisms and ideals in cylindric probability algebras.

**10.8. Theorem.** If  $f$  is a homomorphism from  $\mathbf{A}$  onto  $\mathbf{B}$  and  $\mathcal{I} = \{x \in A : f(x) = 0'\}$ , then  $\mathcal{I}$  is an ideal of  $\mathbf{A}$ , and  $\mathbf{B} \cong \mathbf{A}/\mathcal{I}$ .  $\square$

The following theorem which gives a connection between cylindric probability algebras of formulas and ideals, can be proved using the theory of universal algebras, so we shall omit the proof.

**10.9. Theorem.** Let  $\mathcal{I}$  be an ideal in  $\mathfrak{Form}_L/\equiv_\Sigma$  and let  $\Delta$  be the set of all sentences  $\varphi$  of a graded probability logic  $L_{AP}$  such that  $(\neg\varphi)^\Sigma \in \mathcal{I}$ . Then  $\Sigma \subseteq \Delta$  and  $(\mathfrak{Form}_L/\equiv_\Sigma)/\mathcal{I}$  is isomorphic to  $\mathfrak{Form}_L/\equiv_\Delta$ .  $\square$

Cylindric probability algebras of formulas have some special properties not possessed by other  $CPA$ 's, so we shall mention an adaptable notion important for our purposes.

**10.10 Definition.** A dimension set  $\Delta x$  of an element  $x$  of a cylindric probability algebra  $\mathbf{A}$  is the set of all indices  $k < \omega$  such that  $C_k^1 x \neq x$  (we write  $C_k^1$  instead of  $C_{\langle\{k\}\rangle}^1$ ). A cylindric probability algebra  $\mathbf{A}$  is *locally finite-dimensional* if  $\Delta x$  is finite for all  $x \in A$ .

It is easy to see that  $k \in \Delta x$  if and only if  $C_k^r x \neq x$  for any  $r > 0$ . Every formula  $\varphi$  of  $L_{AP}$  has only finitely many free variables. If  $v_k$  is a variable not occurring in  $\varphi$ , then  $\models (Pv_k > 0)\varphi \leftrightarrow \varphi$ . As a consequence, for any given set  $\Sigma$  of sentences there are at most finitely many indices  $k < \omega$  such that  $\varphi$  is not equivalent with  $(Pv_k > 0)\varphi$  under  $\Sigma$ . Thus, any cylindric probability algebra of formulas  $\mathfrak{Form}_L/\equiv_\Sigma$  is locally finite-dimensional.

If  $f$  is a homomorphism from  $\mathbf{A}$  into  $\mathbf{B}$  and  $x \in A$ , then  $\Delta f(x) \subseteq \Delta x$  because for each  $k < \omega$ , if  $k \notin \Delta x$ , then  $C_k^1 x = x$ , and hence  $C_k^1 f(x) = f(C_k^1 x) = f(x)$ , i.e.,  $k \notin \Delta f(x)$ .

Now we shall prove several elementary properties of  $\Delta$ .

**10.11. Theorem.** (1)  $\Delta 0 = \Delta 1 = \emptyset$ .

(2)  $\Delta(\sum_{j \in J} x_j) \subseteq \bigcup_{j \in J} \Delta x_j$ ,  $J \in \mathcal{A}$ .

(3)  $\Delta(\prod_{j \in J} x_j) \subseteq \bigcup_{j \in J} \Delta x_j$ ,  $J \in \mathcal{A}$ .

(4)  $\Delta - x = \Delta x$ .

(5)  $\Delta C_{\langle K \rangle}^r x \subseteq \Delta x \setminus K$ .

(6)  $\Delta S_\sigma x \subseteq \sigma(\Delta x)$ .

(7)  $\Delta d_{pq} \subseteq \{p, q\}$ .

*Proof.* (1): Immediate by  $CP_1$  (ii) and (1) of the Theorem 10.4.

(2): If  $k \notin \bigcup_{j \in J} \Delta x_j$ , then

$$C_k^1 \sum_{j \in J} x_j = C_k^1 \sum_{j \in J} C_k^1 x_j = \sum_{j \in J} C_k^1 x_j = \sum_{j \in J} x_j$$

from (10) of 10.4. Thus  $k \notin \Delta(\sum_{j \in J} x_j)$ .

(3): Similary to (2).

(4): If  $k \notin \Delta x$ , then  $C_k^1 x = x$ , so  $C_k^1 - x = -x$  from (3) of 10.4, i.e.,  $k \notin \Delta - x$ .

The converse follows by symmetry.

(5): Let  $k$  be any integer such that  $k \notin \Delta x \setminus K$ . If  $k \in K$ , then  $C_k^1 C_{\langle K \rangle}^r x = C_{\langle K \rangle}^r x$  by (2) of 10.4. If  $k \notin \Delta x \cup K$ , then  $C_k^1 C_{\langle K \rangle}^r x = C_k^1 C_{\langle K \rangle}^r C_k^1 x = C_{\langle K \rangle}^r C_k^1 x = C_{\langle K \rangle}^r x$  by  $CP_4$ . So  $k \notin \Delta C_{\langle K \rangle}^r x$ .

(6): If  $k \notin \sigma(\Delta x)$ , then  $\sigma^{-1}(k) \notin \Delta x$  and, hence,  $C_k^1 S_\sigma x = S_\sigma C_{\sigma^{-1}(k)}^1 x = S_\sigma x$  by  $CP_{11}$  (ii). So  $k \notin \Delta S_\sigma x$ .

(7): Immediate by  $CP_{12}$  (iv).  $\square$

The following result shows that the algebras of the form  $\mathfrak{Form}_L / \emptyset$  (the set  $\Sigma$  is empty) have a certain freeness property.

**10.12. Theorem.** *Let  $L = \{R_i, c_j : i \in I_0, j \in J_0\}$  be a set of finitary relation and constant symbols and let  $\mathbf{A}$  be a cylindric probability algebra. Let  $f$  be a function from  $\{R_i : i \in I_0\}$  into  $\mathbf{A}$  such that  $\Delta f(R_i) \subseteq n_i$  for each  $n_i$ -ary relation  $R_i$ . Then there is a homomorphism  $g$  from  $\mathfrak{Form}_L / \emptyset$  into  $\mathbf{A}$  such that  $g(R_i(v_1, \dots, v_{n_i})^\emptyset) = f(R_i)$  for each  $i \in I_0$ .*

*Proof.* By induction on the complexity of formulas of graded probability logic we define a function  $h: \text{Form}_L \rightarrow \mathbf{A}$  which satisfies, if  $\vdash \varphi$ , then  $h(\varphi) = 1$ , as follows:

- (1) Let  $\varphi$  be an atomic formula  $R_i(v_{k_1}, \dots, v_{k_{n_i}})$ . Let  $j_1, \dots, j_{n_i}$  be the first  $n_i$  integers in  $\omega \setminus \{1, \dots, n_i, k_1, \dots, k_{n_i}\}$  and let  $\sigma, \tau$  be a permutations of  $\omega$  such that  $\sigma = \binom{1, \dots, n_i}{j_1, \dots, j_{n_i}}$ ,  $\tau = \binom{j_1, \dots, j_{n_i}}{k_1, \dots, k_{n_i}}$ ,  $\sigma \upharpoonright \{1, \dots, n_i, j_1, \dots, j_{n_i}\}^c = id$  and  $\tau \upharpoonright \{k_1, \dots, k_{n_i}, j_1, \dots, j_{n_i}\}^c = id$ . We define

$$h(\varphi) = S_\tau S_\sigma f(R_i);$$

- (2)  $h(v_p = v_q) = d_{pq}$ ;
- (3)  $h(\neg \varphi) = -h(\varphi)$ ;
- (4)  $h(\bigvee \Phi) = \sum_{\varphi \in \Phi} h(\varphi)$ ,  $\Phi \in \mathcal{A}$ ;
- (5)  $h(\bigwedge \Phi) = \prod_{\varphi \in \Phi} h(\varphi)$ ,  $\Phi \in \mathcal{A}$ ;
- (6)  $h((P\vec{v} \geq r)\varphi) = C_{\langle K \rangle}^r h(\varphi)$ .

First, by induction on the complexity of formulas of graded probability logic, we shall prove the *substitution* property:

$$h(s_\sigma \varphi) = S_\sigma h(\varphi). \quad (\text{S})$$

We present only the part of proof related to the atomic formulas and probability quantifiers.

If  $\varphi$  is an atomic formula  $R_i(v_{k_1}, \dots, v_{k_{n_i}})$ , then  $s_\sigma \varphi$  is  $R_i(v_{\sigma(k_1)}, \dots, v_{\sigma(k_{n_i})})$ . So, for  $j_1, \dots, j_{n_i} \in \omega \setminus \{1, \dots, n_i, k_1, \dots, k_{n_i}, \sigma(k_1), \dots, \sigma(k_{n_i})\}$  and a permutations  $\tau$  and  $\rho$  such that  $\tau = \binom{1, \dots, n_i}{j_1, \dots, j_{n_i}}$  and  $\rho = \binom{j_1, \dots, j_{n_i}}{k_1, \dots, k_{n_i}}$ , we obtain

$$h(s_\sigma \varphi) = S_{\sigma \circ \rho} S_\tau f(R_i) = S_\sigma S_\rho S_\tau f(R_i) = S_\sigma h(\varphi)$$

from  $CP_{10}$ .

If  $\varphi$  is  $v_p = v_q$ , then  $s_\sigma \varphi$  is  $v_{\sigma(p)} = v_{\sigma(q)}$ . So,

$$h(s_\sigma \varphi) = d_{\sigma(p)\sigma(q)} = S_\sigma d_{pq} = S_\sigma h(\varphi)$$

from  $CP_{12}$ .

Let  $\varphi$  be  $(Pv_{l_1}, \dots, v_{l_m} \geq r)\psi(v_{k_1}, \dots, v_{k_n}, v_{l_1}, \dots, v_{l_m})$  and let  $\sigma$  be a permutation of  $\omega$  such that  $L \cap (K \cup \sigma(K)) = \emptyset$ , where  $L = \{j_1, \dots, j_m\}$  and  $K = \{k_1, \dots, k_n\}$ . Then

$$h(s_\sigma \varphi) = C_{\langle L \rangle}^r h(s_\sigma \psi) = C_{\langle L \rangle}^r S_\sigma h(\psi) = S_\sigma C_{\langle L \rangle}^r h(\psi) = S_\sigma h(\varphi)$$

from  $CP_{11}$  (ii) since  $\sigma \upharpoonright L = id$ .

Second, by induction on the complexity of formulas, as previous, we shall prove the *dimension* property:

$$\text{if } v_k \text{ does not occur free in } \varphi, \text{ then } k \notin \Delta h(\varphi). \quad (D)$$

Also, we present only the part of proof related to the atomic formulas and probability quantifiers.

If  $\varphi$  is an atomic formula  $R_i(v_{k_1}, \dots, v_{k_{n_i}})$  and  $k \notin \{k_1, \dots, k_{n_i}\}$ , then

$$\begin{aligned} C_k^1 h(\varphi) &= C_k^1 S_\tau S_\sigma f(R_i) = S_\tau C_{\tau^{-1}(k)}^1 S_\sigma f(R_i) \\ &= S_\tau S_\sigma C_{\sigma^{-1}(\tau^{-1}(k))}^1 f(R_i) = S_\tau S_\sigma f(R_i) = h(\varphi), \end{aligned}$$

from  $CP_{11}$  since  $(\sigma \circ \tau)^{-1}(k) \notin \{1, \dots, n_i\}$  and  $\Delta f(R_i) \subseteq \{1, \dots, n_i\}$ . So, we obtain  $k \notin \Delta h(\varphi)$ .

If  $\varphi$  is  $(Pv_{l_1}, \dots, v_{l_m} \geq r)\psi(v_{k_1}, \dots, v_{k_n}, v_{l_1}, \dots, v_{l_m})$  and  $k \notin \{k_1, \dots, k_n\}$ , then

$$\begin{aligned} \Delta h(\varphi) &= C_{\langle L \rangle}^r h(\psi) \\ &\subseteq \Delta h(\psi) \setminus L \quad \text{by (5) of Theorem 10.11.} \\ &\subseteq \{k_1, \dots, k_n\} \quad \text{by induction hypothesis.} \end{aligned}$$

Hence,  $k \notin \Delta h(\varphi)$ .

Now we shall prove that each logical axiom of a graded probability logic  $L_{AP}$  is in the set

$$\Gamma = \{\varphi \in \text{Form}_L : h(\varphi) = 1\}.$$

First suppose that  $\varphi$  is an axiom of  $L_A$  without quantifiers. Using permutations, we prove that  $h(\varphi) = 1$  as in classical theory of cylindric algebras (see [18]). We present only the part of proof related to the identity axiom.

Let  $\varphi$  be  $v_p = v_q \rightarrow (\psi \rightarrow \theta)$ , where  $\theta$  is a formula obtained from an atomic formula  $\psi$  by replacing an occurrence of  $v_p$  in  $\psi$  by  $v_q$ . We may assume that  $p \neq q$ . There are two cases. If  $\psi$  is  $v_p = v_r$ , then  $\theta$  is  $v_q = v_r$ . Hence,  $h(\varphi) = -d_{pq} + -d_{pr} + d_{qr} = -(d_{pq} \cdot d_{pr}) + d_{qr} \geq -d_{qr} + d_{qr} = 1$  from (6) of the Theorem 10.5. If  $\psi$  has the form  $R_i(v_{k_1}, \dots, v_{k_{j-1}}, v_p, v_{k_{j+1}}, \dots, v_{k_{n_i}})$  and  $q \notin \{k_1, \dots, k_{j-1}, p, k_{j+1}, \dots, k_{n_i}\}$ , then  $\theta$  is  $R_i(v_{k_1}, \dots, v_{k_{j-1}}, v_q, v_{k_{j+1}}, \dots, v_{k_{n_i}})$ . Let  $\lambda_1, \dots, \lambda_{n_i}$  be the first  $n_i$  integers in  $\omega \setminus \{1, \dots, n_i, k_1, \dots, k_{n_i}, p, q\}$ . Let  $\sigma, \tau, \rho$  and  $\chi$  be permutations such that:

$$\begin{aligned} \sigma &= \begin{pmatrix} 1, \dots, n_i \\ \lambda_1, \dots, \lambda_{n_i} \end{pmatrix} & \text{and } \sigma \upharpoonright \{1, \dots, n_i, \lambda_1, \dots, \lambda_{n_i}\}^c &= id, \\ \tau &= \begin{pmatrix} \lambda_1, \dots, \lambda_j, \dots, \lambda_{n_i} \\ k_1, \dots, p, \dots, k_{n_i} \end{pmatrix} & \text{and } \tau \upharpoonright \{k_1, \dots, p, \dots, k_{n_i}, \lambda_1, \dots, \lambda_{n_i}\}^c &= id, \\ \rho &= \begin{pmatrix} \lambda_1, \dots, \lambda_j, \dots, \lambda_{n_i} \\ k_1, \dots, q, \dots, k_{n_i} \end{pmatrix} & \text{and } \rho \upharpoonright \{k_1, \dots, q, \dots, k_{n_i}, \lambda_1, \dots, \lambda_{n_i}\}^c &= id, \end{aligned}$$

and  $\chi(p) = q$  and  $\chi \upharpoonright \{p, q\}^c = id$ . We remark that  $\chi \circ \tau \circ \sigma \upharpoonright \{1, \dots, n_i\} = \rho \circ \sigma \upharpoonright \{1, \dots, n_i\}$ . So,  $S_{\chi \circ \tau \circ \sigma} f(R_i) = S_{\rho \circ \sigma} f(R_i)$  since  $\Delta f(R_i) \subseteq \{1, \dots, n_i\}$ . Thus,

$$\begin{aligned}
 h(\varphi) &= -d_{pq} + -S_\tau S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) \\
 &= -(d_{pq} \cdot S_\tau S_\sigma f(R_i)) + S_\rho S_\sigma f(R_i) \\
 &\geq -S_\chi S_\tau S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) && \text{by } CP_{12} \text{ (ii)} \\
 &= -S_\rho S_\sigma f(R_i) + S_\rho S_\sigma f(R_i) && \text{by } CP_{11} \text{ (i)} \\
 &= 1.
 \end{aligned}$$

Next, suppose that  $\varphi$  is an axiom of the graded probability logic  $L_{AP}$ .

If  $\varphi$  is the Monotonicity axiom  $(P\vec{v} \geq r)\psi \rightarrow (P\vec{v} \geq s)\psi$  for  $r \geq s$ , then

$$h(\varphi) = -C_{\langle K \rangle}^r h(\psi) + C_{\langle K \rangle}^s h(\psi) = 1$$

by  $CP_3$ .

If  $\varphi$  is  $(Pv_{k_1}, \dots, v_{k_n} \geq r)\psi(v_{k_1}, \dots, v_{k_n}) \rightarrow (Pv_{l_1}, \dots, v_{l_n} \geq r)\psi(v_{l_1}, \dots, v_{l_n})$ , then for  $\sigma = \binom{l_1, \dots, l_n}{k_1, \dots, k_n}$  and  $M = K \setminus L$  we obtain

$$\begin{aligned}
 h(\varphi) &= -C_{\langle K \rangle}^r h(s_\sigma \psi) + C_{\langle L \rangle}^r h(\psi) \\
 &= -C_{\langle K \rangle}^r S_\sigma h(\psi) + C_{\langle L \rangle}^r h(\psi) && \text{by (S)} \\
 &= -S_\sigma C_{\langle L \rangle}^r h(\psi) + C_{\langle L \rangle}^r h(\psi) && \text{by } CP_{11} \text{ (ii)} \\
 &= -S_\sigma C_{\langle L \rangle}^r C_{\langle M \rangle}^1 h(\psi) + C_{\langle L \rangle}^r C_{\langle M \rangle}^1 h(\psi) && \text{by (D)} \\
 &= 1
 \end{aligned}$$

from (3) of Theorem 10.5. since  $\sigma \upharpoonright (L \cup M)^c = \sigma \upharpoonright (L \cup K)^c = id$ .

Similarly, other logical axioms of  $L_{AP}$  are in  $\Gamma$ .

It is easy to see that  $\Gamma$  is closed under Modus Ponens and Conjunction rule.

If  $\varphi \rightarrow \psi(v_{k_1}, \dots, v_{k_n}) \in \Gamma$ , provided  $v_{k_1}, \dots, v_{k_n}$  is not free in  $\varphi$ , then

$$\begin{aligned}
 h(\varphi \rightarrow (P\vec{v} \geq 1)\psi) &= -h(\varphi) + C_{\langle K \rangle}^1 h(\psi) \\
 &= -C_{\langle K \rangle}^1 h(\varphi) + C_{\langle K \rangle}^1 h(\psi) && \text{by (D) and (13) of 10.4.} \\
 &= C_{\langle K \rangle}^1 (-C_{\langle K \rangle}^1 h(\varphi) + h(\psi)) && \text{by (3),(4) of 10.4.} \\
 &= C_{\langle K \rangle}^1 (-h(\varphi) + h(\psi)) && \text{by (D)} \\
 &= 1 && \text{by (1) of 10.4.}
 \end{aligned}$$

So,  $\varphi \rightarrow (P\vec{v} \geq 1)\psi \in \Gamma$ . Thus each logical theorem of  $L_{AP}$  is in  $\Gamma$ .

It follows that  $\vdash \varphi \leftrightarrow \psi$  implies  $h(\varphi) = h(\psi)$ . It is easily checked that the function  $g: \mathfrak{Form}_L / \equiv_\emptyset \rightarrow \mathbf{A}$  defined by  $g(\varphi^\emptyset) = h(\varphi)$  for any  $\varphi \in \text{Form}_L$ , is desired homomorphism.  $\square$

Finally, we prove that Boolean Representation Theorem of locally finite dimensional cylindric probability algebras holds.

**10.13. Boolean Representation Theorem.** *If  $\mathbf{A}$  is a locally finite dimensional cylindric probability algebra and  $|A| > 1$ , then there is a homomorphism of  $\mathbf{A}$  onto a cylindric probability set algebra.*

*Proof.* First, we prove that any locally finite-dimensional cylindric probability algebra  $\mathbf{A}$  is isomorphic to a cylindric probability algebra of formulas  $\mathfrak{Form}_L / \equiv_\Sigma$  for some  $L$  and  $\Sigma$ . For each  $a \in A$ , the set  $\Delta a$  is finite, i.e.,  $\Delta a \subseteq \{1, \dots, n\}$ ,  $n < \omega$ . Let  $R_a$  be an  $n$ -ary relation symbol introduced for  $a \in A$ , and let  $f$  be a function from  $L = \{R_a : a \in A\}$  into  $A$  defined by  $f(R_a) = a$ . Thus,  $\Delta f(R_a) \subseteq \{1, \dots, n\}$ . It follows from Theorem 10.12. that there exists a homomorphism  $g$  from  $\mathfrak{Form}_L / \equiv_\emptyset$  onto  $\mathbf{A}$  such that  $g(R_a^\emptyset) = f(R_a) = a$ . Let  $\mathcal{I} = \{\varphi^\emptyset : g(\varphi^\emptyset) = 0\}$  be an ideal of  $\mathfrak{Form}_L / \equiv_\emptyset$ , and let  $\Sigma$  be a set of sentences  $\varphi$  of  $L_{AP}$  such that  $(\neg\varphi)^\emptyset \in \mathcal{I}$ . Then,  $\Sigma$  is consistent, since  $|A| > 1$ . We obtain  $(\mathfrak{Form}_L / \equiv_\emptyset) / \mathcal{I} \cong \mathfrak{Form}_L / \equiv_\Sigma$  from Theorem 10.9. and  $\mathbf{A} \cong (\mathfrak{Form}_L / \equiv_\emptyset) / \mathcal{I}$  from Theorem 10.8. If  $\mathfrak{A}$  is a model of  $\Sigma$ , then we have a “natural” homomorphism from  $\mathfrak{Form}_L / \equiv_\Sigma$  onto the cylindric probability set algebra

$$\langle \{\varphi^\mathfrak{A} : \varphi \in \text{Form}_L\}, \cup, \cap, \sim, \emptyset, A^\omega, C_{\langle K \rangle}^r, S_\sigma, D_{pq} \rangle. \quad \square$$

In our opinion, many other problems of the classical theory of cylindric algebras such as representation and decision problems, can be considered for a cylindric probability algebras too.

## BIBLIOGRAPHY

- [1] **Aczel P.M.G.**, *Saturated intuitionistic theories*, in: *Contributions to Mathematical Logic* (H.A. Schmidt, K.J. Schute and H.J. Thiele, eds.), North-Holland, Amsterdam, 1968, pp. 1–11.
- [2] **Anderson R.M.**, *A non-standard representation of Brownian motion and Itô integration*, Israel J. Math. **25** (1976), 15–46.
- [3] **Barwise J.**, *Admissible Sets and Structures*, Springer-Verlag, Berlin, 1975.
- [4] **Barwise J.**, *An introduction to first-order logic*, in: *Handbook of Mathematical Logic* (J. Barwise, ed.), North-Holland, Amsterdam, 1977, pp. 6–46.
- [5] **Bhaskava Rao M., K.P.S. Bhaskava Rao**, *Theory of Charges*, Academic Press, 1983.
- [6] **Chang C.C., H.J. Keisler**, *Model Theory*, North-Holland, Amsterdam, 1973.
- [7] **Cutland N.**, *Nonstandard measure theory and its applications*, Bull. London Math. Soc. **15** (1983), 529–589.
- [8] **Dalen D. van**, *Logic and Structures*, Springer-Verlag, Berlin, 1980.
- [9] **Đorđević R.S.**, *Barwise completeness theorems for logics with integrals*, Publ. Inst. Math. (Beograd) (N.S.) **49** (1991), 1–5.
- [10] **Đorđević R.S.**, *Analytic completeness theorem for absolutely continuous bi-probability models*, Z. Math. Logik Grundlag. Math. **38** (1992), 241–246.
- [11] **Đorđević R.S.**, *Analytic completeness theorem for singular biprobability logic*, Math. Logic Quart. **39** (1993), 228–230.
- [12] **Đorđević R.S.**, *Logics with two types of integral operators*, Publ. Inst. Math. (Beograd) (N.S.) **54** (1993), 18–24.
- [13] **Fagin R., J.Y. Halper, N. Megiddo**, *A logic for reasoning about probabilities*, in: *Proceedings 3rd IEEE Symposium on Logic in Computer Science*, 1988, pp. 277–291.

- [14] **Fajardo S.**, *Completeness theorems for the general theory of stochastic processes*, in: *Proceeding of the 6th Latin American Symposium on Math. Logic*, Caracas, 1983, pp. 174–194.
- [15] **Fajardo S.**, *Probability logic with conditional expectation*, *Ann. Pure Appl. Logic* **28** (1985), 137–161.
- [16] **Halmos P.**, *Measure Theory*, Van Nostrand, Princeton, New York, 1950.
- [17] **Henkin L.**, *The completeness of the first-order functional calculus*, *J. Symbolic Logic* **14** (1949), 159–166.
- [18] **Henkin L., D. Monk, A. Tarski**, *Cylindric Algebras*, North-Holland, Amsterdam, 1971.
- [19] **Henson C.W.**, *Analytic sets, Baire sets and the standard part map*, *Canad. J. Math.* **31** (1979), 663–672.
- [20] **Henson C.W.**, *Unbounded Loeb measures*, *Proc. Amer. Math. Soc.* **74** (1979), 143–150.
- [21] **Heyting A.**, *Intuitionism*, North-Holland, Amsterdam, 1976.
- [22] **Hoover D.**, *Probability logic*, *Ann. Math. Logic* **14** (1978), 287–313.
- [23] **Hoover D.**, *A normal form theorem for  $L_{\omega_1 P}$  with applications*, *J. Symbolic Logic* **43** (1982), 605–624.
- [24] **Hoover D.**, *Synonymity, generalized martingales and subfiltrations*, *Ann. Probab.* **12** (1984), 703–713.
- [25] **Hoover D.**, *A probabilistic interpolation theorem*, *J. Symbolic Logic* **46** (1985), 708–713.
- [26] **Hoover D.**, *A characterization of adapted distributions*, *Ann. Probab.* **15** (1987), 1600–1611.
- [27] **Hoover D.**, *An analytic completeness theorem for logics with probability quantifiers*, *J. Symbolic Logic* **52** (1987), 802–816.
- [28] **Hoover D., H.J. Keisler**, *Adapted probability distributions*, *Trans. Amer. Math. Soc.* **286** (1984), 159–201.
- [29] **Keisler H.J.**, *Logic with the quantifier “there exist uncountably many”*, *Ann. Math. Logic* (1970), 1–93.
- [30] **Keisler H.J.**, *Model Theory for Infinitary Logic*, North-Holland, Amsterdam, 1971.
- [31] **Keisler H.J.**, *The monotone class theorem in infinitary logic*, *Proc. Amer. Math. Soc.* **64** (1977), 129–134.
- [32] **Keisler H.J.**, *Hyperfinite model theory*, in: *Logic Colloquium '76*, North-Holland, Amsterdam, 1977, pp. 5–110.
- [33] **Keisler H.J.**, *An infinitesimal approach to stochastic analysis*, *Mem. Amer. Math. Soc.* **48** (1984), no. 297.
- [34] **Keisler H.J.**, *Probability quantifiers*, in: *Model Theoretic Logics* (J. Barwise and S. Feferman, eds.), Springer-Verlag, Berlin, 1985, pp. 509–556.

- [35] **Keisler H.J.**, *A completeness proof for adapted probability logic*, Ann. Pure Appl. Logic **31** (1986), 61–70.
- [36] **Keisler H.J.**, *Hyperfinite models of adapted probability logic*, Ann. Pure Appl. Logic **31** (1986), 71–86.
- [37] **Kreisel G., J.L. Krivine**, *Elements of Mathematical Logic*, North-Holland, Amsterdam, 1967.
- [38] **Lindström T.**, *A Loeb measure approach to theorems of Prohorov, Sazanov and Groos*, Trans. Amer. Math. Soc. **269** (1982), 521–534.
- [39] **Loeb P.**, *Conversion from nonstandard to standard measure space and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.
- [40] **Loeb P.**, *Weak limits of measures and the standard part map*, Proc. Amer. Math. Soc. **77** (1979), 128–135.
- [41] **Loeb P.**, *A functional approach to nonstandard measure theory*, Contemp. Math. **26** (1984), 251–261.
- [42] **Łos J., E. Marczewski**, *Extension of measures*, Fund. Math. **36** (1949), 267–276.
- [43] **Marković Z.**, *Intuitionistic Model Theory*, Ph.D. thesis, Univ. of Pennsylvania, 1978.
- [44] **Mijajlović Ž.**, *An Introduction to Model Theory*, Institute of Mathematics, Novi Sad, 1987.
- [45] **Monk J.D.**, *Mathematical Logic*, Springer-Verlag, Berlin, 1976.
- [46] **Nadel M.E.**, *Infinitary intuitionistic logic from a classical point of view*, Ann. Math. Logic **14** (1978), 159–195.
- [47] **Rašković M.**, *Model theory for  $L_{AM}$  logic*, Publ. Inst. Math. (Beograd) (N.S.) **37** (1985), 17–22.
- [48] **Rašković M.**, *Completeness theorem for biprobability models*, J. Symbolic Logic **51** (1986), 586–590.
- [49] **Rašković M.**, *Weak completeness theorem for  $L_{AP\forall}$  logic*, Zb. Rad. (Kragujevac) **8** (1987), 69–72.
- [50] **Rašković M.**, *Completeness theorem for singular biprobability models*, Proc. Amer. Math. Soc. **102** (1988), 389–392.
- [51] **Rašković M.**, *A completeness theorem for an infinitary intuitionistic logic with both ordinary and probability quantifiers*, Publ. Inst. Math. (Beograd) (N.S.) **50** (1991), 7–13.
- [52] **Rašković M.**, *Classical logic with some probability operators*, Publ. Inst. Math. (Beograd) (N.S.) **53** (1993), 1–3.
- [53] **Rašković M., B. Boričić**, *A probabilistic validity measure in intuitionistic propositional logic*, Math. Balkanica (to appear).
- [54] **Rašković M., R. Đorđević**, *Finite compactness theorem for biprobability logics*, Math. Balkanica **5** (1991), 12–14.

- [55] Rašković M., R. Đorđević, *Second order probability logic*, Math. Balkanica **6** (1992), 105–108.
- [56] Rašković M., R. Đorđević, *Cylindric probability algebras* (to appear).
- [57] Rašković M., R. Živaljević, *Barwise completeness theorems for some bi-probability logics*, Z. Math. Logik Grundlag. Math. **32** (1986), 133–135.
- [58] Rašković M., P. Tanović, *Completeness theorem for a monadic logic with both first-order and probability quantifiers*, Publ. Inst. Math. (Beograd) (N.S.) **47** (1990), 1–4.
- [59] Robbin J.W., *Mathematical Logic*, W.A. Benjamin, New York, 1969.
- [60] Robinson A., *Non-standard Analysis*, North-Holland, Amsterdam, 1966.
- [61] Rodenhausen H., *The Completeness Theorem for Adapted Probability Logic*, Ph.D. Thesis, Heidelberg, 1982.
- [62] Stroyan K.D., A. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1967.
- [63] Von Neumann J., *Einige sätze uber messbare abbildungen*, Ann. Math. **33** (1932), 574–586.

## INDEX OF SYMBOLS

### *logical*

|                                                  |    |                                                   |   |
|--------------------------------------------------|----|---------------------------------------------------|---|
| $\neg$ (negation) . . . . .                      | 9  | $\in$ (membership) . . . . .                      | 1 |
| $\wedge$ (conjunction) . . . . .                 | 3  | $=$ (equality) . . . . .                          | 1 |
| $\vee$ (disjunction) . . . . .                   | 9  | $\emptyset$ (empty set) . . . . .                 | 2 |
| $\rightarrow$ (implication) . . . . .            | 2  | $\subseteq$ (inclusion) . . . . .                 | 2 |
| $\leftrightarrow$ (equivalence) . . . . .        | 3  | $\cup$ (union of) . . . . .                       | 1 |
| $\forall$ (universal quantification) . . . . .   | 2  | $\cap$ (intersection of) . . . . .                | 2 |
| $\exists$ (existential quantification) . . . . . | 2  | $\setminus$ (difference of) . . . . .             | 2 |
| $\models$ (true) . . . . .                       | 9  | $\Delta$ (symmetric difference of) . . . . .      | 5 |
| $\vdash$ (provable) . . . . .                    | 12 | $\times$ (Cartesian product) . . . . .            | 7 |
| $\cong$ (isomorphism) . . . . .                  | 17 | $\mathcal{P}$ (power set of) . . . . .            | 1 |
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| $\equiv$ (elementary equivalence) . . . . .      | 17 | $(\cdot, \cdot)$ (ordered pair) . . . . .         | 2 |
| $\ll$ (absolute continuity) . . . . .            | 57 | $\langle \dots \rangle$ (ordered tuple) . . . . . | 4 |
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### *special*

|                                     |     |                                    |     |                                |    |
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