# RECENT ADVANCES IN ANALYTICAL DYNAMICS - CONTROL, STABILITY AND DIFFERENTIAL GEOMETRY; PROCEEDINGS 

Matematiöki institut SANU

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## FOREWORD

It has been a long tradition of the Mathematical Institute of Serbian Academy of Sciences and Arts to organize occasionally symposia dedicated to some contemporary problems of both theoretical and applied mathematics and mechanics. Such an one-day Sumposium entitled: Recent advances in the analytical dynamics - control, stability and differential geometry, was held on April 4, 2001. Participation on the Symposium was on the invitation only, and 11 reviewing and contributing papers were presented. Within these Proceedings you will find full texts of the papers.

The Symposium served as a convenient opportunity to pay trubute to Professor Veljko A. Vujičić, academician of the International Academy of Nonlinear Sciences, on the occasion of his 70th anniversary. His curriculum vitae and the complete list of his publications is included in the Proceedings. The Symposium was also attended by academician Vladimir Matrosov from Russia who, in a concluded speach, emphasized the role Professor Vujičić had played in establishing the Yugoslav branch of the Academy of Nonlinear Sciences, and delivered diplomas to newly elected members of the Academy from Yugoslavia.

December 12, 2001
Chairman of the Department of Mechanics academician Vladan Djordjević


Veljko A.Vujičić

# The life and work of Veljko A. Vujičić, on occasion of his 70th anniversary 

Prof. Veljko Vujičić was born on March 29, 1929 in Nikšić. His father was Akim and his mother was Ljubica, maiden name Vučinić. He finished elementary schooling and graduated from the High School in Nikšić. He received his Diploma from the Anti-aircraft Military Academy in Zadar, in 1949, and went to Užice as an Instructor in anti-aircraft firing and ordnance. He graduated from the Department of Mechanics at the Faculty of Natural Sciences and Mathematics of Belgrade University in 1957. Two years later he was the first M. Sc. graduate at the Department and became Assistant.

During the academic year 1959/60 he was an affiliate of the Faculty of Mechanics and Mathematics at the University of Moscow where he worked together with well-known Professors N. G. Chetaev and V. V. Rumyantsev.

Professor V. Vujičić received his doctoral degree in 1961 from the Faculty of Natural Sciences and Mathematics after having defended his thesis entitled Motion and stability of motion of dynamically variable objects in front of the committee consisting of A. Bilimović, K. Voronjec, T. Anđelić and D. Rašković. The same year he was elected an assistant professor at the Faculty of Natural Sciences and Mathematics, in 1968 he was elected an associate professor and promoted to a full professor in 1974. He has taught: Dynamics of Bodies of Variable Mass, Vibration Theory, Analytical Mechanics, Statics, Tensor Calculus, Theory of Fields (at the Department in Zrenjanin), Theory of Stability and Control of Mechanical Systems. At the Faculty of Mechanical Engineering in Belgrade he has held a post-graduate course in Tensor Calculus, at the Faculty of Engineering Sciences in Titograd he has taught Statics, Resistance of Materials and Technical Dynamics while at the Faculty of Mining and Metallurgy in Kosovska Mitrovica he has held a course in Mechanics II. He was in charge for scientific seminars on Stability of Motion, Control of Mechanical Systems and Analytical Mechanics. Professor V. Vujičić was the adviser of 8 Doctoral and 18 Master theses. He has written two university textbooks (Statics and Vibration Theory) and translated two textbooks from Russian.

Since his first scientific research work, published in 1960 in the USSR Academy of Science magazine Applied Mathematics and Mechanics, Professor Veljko Vujičić has been reporting the results of his research works at international, foreign and national scientific meetings. More than 150 bibliographic units have been published in international and national journals. His four monographs stand out in particular as original contributions to science: Covariant Dynamics (Belgrade, 1981), Dynamics of Rheonomic Systems (Belgrade, 1990), Some Problems of Mechanics of Nonautonomous

Systems (with A.A. Martinyuk; Belgrade-Kiev, 1991) and Preprinciples of Mechanics (Belgrade, 1998).

International reference journals have published over 120 reviews of his research works and there are more than 150 citations of his results in different journals, monographs, university textbooks and PhD theses. His monographs have been presented in detail in certain foreign scientific journals.

In the extremely wide scope of Prof. Vujičić's scientific interest - from dynamics of bodies of variable mass, analytical mechanics, vibration theory, stability of motion and control of mechanical systems to tensor calculus and differential geometry - his most important scientific achievements are the following mutually interlaced fields: pioneer work in the geometrization of mechanics of objects, the weight of which is a function of time; advancement of analytical mechanics through original contributions published in already mentioned monographs; introduction of a general criterion on motion stability in the vibration theory and the theory of motion stability and, finally, introduction of preprinciples of mechanics, which imposes a new logical sistematization of mechanics and significant changes in conceiving the principles of mechanics.

If one word were needed to characterize creative enthusiasm of Prof. V. Vujičić it would be courage - courage to stand for atitudes different from the postulates of the standard analytical mechanics: in the early 1970s he claimed that the standard integral calculus destroys the tensor nature of geometric and dynamic objects (this is overcome by introducing a notion of absolute or covariant integral); in 1980s he modified, or, more precisely, he rearranged the analytical mechanics of rheonomic systems; finally, in this Department, not once were we the witnesses of his daring spirit subjecting even the generality of the law of mutual attraction of bodies to the test. If one notion were needed to describe the scientific aspiration of Prof. V. Vujičić, then, there is no doubt about it, this would be invariance - the invariant developping of the whole theory of mechanics, about which the respected Professor is going to talk at today's Symposium.

Prof. Veljko Vujičić was a founder and, for more than 20 years, the Editor-inChief of Theoretical and Applied MECHANICS, a scientific journal being issued by the Yugoslav Society of Mechanics and included so far in five international registers. Prof. V. Vujičić is a member of Editorial Boards of four national and one foreign scientific journal as well as of one international serial of scientific monographs.

In organizing scientific research work, Prof. Veljko Vujičić took part as Chairman of Department of Mechanics, Director of Division of Mathematics, Mechanics and Astronomy, Dean of the Faculty of Natural Sciences and Mathematics, then as Director of Division of Mechanics at the SASA (Serbian Academy of Sciences and Arts) Institute of Mathematics, Vice Director and Acting Director of the Institute. Professor V. Vujičić was the initiator and the president of the first Committee for mathematics and mechanics of the Republic Association for Science (RAS) of Serbia, President of the RAS Chamber of natural and engineering sciences, and researcher in charge for the macroproject in mechanics at the Republic level. He was also President of the Steering Committee and the first President of the Serbian Society of Mechanics, as well as President of Administrative Committee of the Yugoslav Society of Mechanics.

Professor Veljko Vujičić is an honourary member of the Yugoslav Society of Mechanics. He is a member of several scientific foreign and international associations
(GAMM, Tensor Society, AMS). He is also a correspondent member of the American Academy of Mechanics (since 1994), a member of the International Academy of Nonlinear Sciences with the main office in Moscow (since 1996) and a correspondent member of the European Academy of Sciences and Arts with the main office in Paris (since 1998).

For his fruitful and tireless scientific work, Professor Veljko Vujičić has been given two decorations of honour (the Order of Work with Golden Wreath and the Order of Work with Red Flag) and two medals of the City of Belgrade.

Finally, let me, the undersigned - who owes respected Professor Veljko Vujičić profound gratitude for his devoted, unassuming and generous initiation into scientific work almost three decades ago - wish dear Professor many happy returns, good health and many more years of fruitfull work.

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# ON ASYMPTOTICALLY STABLE NON-OSCILLATORY DYNAMIC SYSTEMS 

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#### Abstract

A condition under which the symmetric damped linear multi-degree-of-freedom system does not oscillate is stated in terms of the coefficients of system matrices without solving the spectrum of the entire system. This criterion is then generalized to a class of asymmetric systems. A simple two-degree of freedom example which illustrates a comparison with the exact result is given.


## Introduction and earlier results

Systems of interest here are linear viscously damped systems described by the differential equation

$$
\begin{equation*}
M \ddot{q}+B \dot{q}+C q=0 \tag{1}
\end{equation*}
$$

where $q$ is the $n$-dimensional position vector and $M, B$ and $C$ are the inertia, damping and stiffness matrices, assumed to be constant, real symmetric and positive definite ( $>0$ ). As $\mathrm{M}>0$, one can utilize the positive definite square root in a familiar way to transform the equation to the form

$$
\begin{equation*}
\ddot{x}+D \dot{x}+K x=0 \tag{2}
\end{equation*}
$$

where $x=M^{1 / 2} q, D=M^{-1 / 2} B M^{-1 / 2}, K=M^{-1 / 2} C M^{-1 / 2}$. When a solution to equation (2) is assumed to have the form $x=U \exp (\lambda t)$, the following algebraic problem arises

$$
\begin{equation*}
\left(\lambda^{2} I+\lambda D+K\right) U=0 \tag{3}
\end{equation*}
$$

where $U$ is an eigenvector of dimension n and $O$ is its eigenvalue. There are 2 n eigenvalues which are governed by the characteristic equation

$$
\begin{equation*}
\operatorname{det}\left(\lambda^{2} I+\lambda D+K\right)=0 \tag{4}
\end{equation*}
$$

It is well known that all roots of this equation have negative real parts, and hence the system (2) is asymptotically stable.

The problem considered here is now defined: Under which conditions every solution $x(t)$ of the differential equation (2) will be a non-oscillating function of time, i. e., when all associated eigenvalues of the system will be negative real numbers.

A necessary and sufficient condition, based on a well-known algebraic method, has recently been reported in [3]. Nevertheless, this criterion is somewhat cumbersome to use if the number of degrees of freedom is large, since it requires the knowledge of the coefficients in the characteristic equation and the inspection of the minors of a $2 \mathrm{n} \times 2 \mathrm{n}$ matrix. From a practical point of view, therefore it is of interest to find conditions expressed by the properties of the system matrices - for nonoscillation of the system. Such conditions may yield design constraints in terms of the physical parameters of the system.

This problem is well-known solved for a single-degree-of-freedom system described by the scalar equation $\ddot{x}+d \dot{x}+k x=0$, where $d$ and $k$ are positive. Necessary and sufficient condition for such system to be non-oscillatory (overdamped or critically damped) is $d \geq 2 k^{1 / 2}$. Criterion analogous to this can be given in the case of multi-degree-of-freedom systems, when damping is classical, i.e., if D commutes with K then the system (2) does not oscillate if and only if $D^{2}-4 K \geq 0$. For a general case of damping the sufficient conditions for non-oscillation are as follows:
(A) $\left(x^{T} D x\right)^{2}-4 x^{T} x x^{T} K x>0$ for all nonzero real n-vectors $x[6]$;
(B) $d_{m}>2 \sqrt{k_{M}}$, where $d_{m}$ and $k_{M}$ are the minimal and maximal eigenvalues of the matrix $D$ and $K$, respectively [1];
(C ) $D-2 \sqrt{k_{M}} I>0,[2]$;
(D) $D^{2}-2\left(K+k_{M} I\right)>0$, [4];
(E) $D^{2}-3 K-k_{m}^{-1} K^{2} \geq 0$, [3].

The inequality (A) is the classical definition of overdamped systems first used by Duffin [6]. This condition is not easy to verify numerically. More recently, Barkwell and Lancaster [1] have suggested a criterion, which is equivalent to (A), but it requires the determination of a parameter by trial and error. Conditions (B) and (C) are equivalent. This fact can be readily verified reducing the matrix $D$ to diagonal form by means of an orthogonal transformation. From $K \leq k_{M} I$, we have

$$
D^{2}-2\left(K+k_{M} I\right) \geq\left(D-2 \sqrt{k_{M}} I\right)\left(D+2 \sqrt{k_{M}} I\right)
$$

Thus, the condition (D) is weaker than (C), since whenever the later is satisfied, (D) is automatically satisfied. Note that neither condition (E) nor condition (D) implies the other one (see [3]).

An additional sufficient condition for non-oscillation was proposed in [7]:

$$
\text { (F ) } D-2 K^{1 / 2}>0
$$

This criterion has been accepted and used by many authors (see [2]). However, condition (F) does not guarantee real eigenvalues, as the example in [1] illustrated.

## A new sufficient condition for non-oscillation

Theorem 1 . The system described by equation (2), with $D=D^{T}>0$ and $K=K^{T}>0$, does not oscillate if

$$
\begin{equation*}
D^{2}-K-k_{M} I-D\left(K+k_{M} I\right)^{-1} K D>0, \tag{5}
\end{equation*}
$$

where $k_{M}$ is the largest eigenvalue of the matrix K .
Proof. According to [4], we consider the following 2n-dimensional conservative gyroscopic system

$$
\binom{\ddot{y}}{\hdashline \ddot{z}}+\left(\begin{array}{c:c}
0 & D  \tag{6}\\
\hdashline-D & 0
\end{array}\right)\binom{\dot{y}}{\dot{z}}-\left(\begin{array}{c:c}
K & 0 \\
\hdashline 0 & K
\end{array}\right)\binom{y}{z}=\binom{0}{\frac{0}{}}
$$

The eigenvalue problem associated with (6) is

$$
\left(\begin{array}{c:c}
s^{2} I-K & s D  \tag{7}\\
\hdashline-s D & s^{2} I-K
\end{array}\right)\binom{Y}{Z}=\binom{0}{0}
$$

Suppose that the system (6) is stable. Then, all eigenvalues of (7) are purely imaginary. If we substitute $s=i \omega, i=\sqrt{-1}$, in (7), we obtain

$$
\begin{equation*}
-\left(\omega^{2} I+K\right) Y+i \omega D Z=0 \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
-i \omega D Y-\left(\omega^{2} I+K\right) Z=0 \tag{9}
\end{equation*}
$$

From (8),

$$
\begin{equation*}
Z=-\frac{i}{\omega} D^{-1}\left(\omega^{2} I+K\right) Y \tag{10}
\end{equation*}
$$

and substitution of this expression into (9) leads

$$
\begin{equation*}
\left(\omega^{2} I+\omega D+K\right) D^{-1}\left(\omega^{2} I-\omega D+K\right) Y=0 \tag{11}
\end{equation*}
$$

Let $s=i \omega, \omega<0$, be an eigenvale of (7) with eigenvector $\left(Y^{T}, Z^{T}\right)^{T}$. From equations (3) and (11), we deduce that $\lambda=\omega$ is eigenvalue of (3) with eigenvector $X=D^{-1}\left(\omega^{2} I-\omega D+K\right) Y$. Consequently, the system (2) does not oscillate if the system (6) is stable.

According to [5], we introduce auxiliary function of the form

$$
\begin{equation*}
V(y, z, \dot{y}, \dot{z})=V_{1}(\dot{y}, z)+V_{1}(\dot{z},-y) \tag{12}
\end{equation*}
$$

with

$$
\begin{equation*}
V_{1}(\xi, \eta)=\xi^{T}\left(D^{2}-K-k_{M} I\right) \xi+2 \xi^{T} D K \eta+\eta^{T}\left(K^{2}+k_{M} K\right) \eta \tag{13}
\end{equation*}
$$

Here, $\xi$ and $\eta$ are n-dimensional real vectors and $k_{M}$ is the maximal eigenvalue of matrix $K$. The time derivative of $V$, along every solution of equation (2), becomes $\dot{V}=0$. On the other hand, $V_{1}(\xi, \eta)$, as well as the function (12), are positive definite if and only if the condition (5) holds. Thus, under (5), according to Liapunov's stability theorem, the system (6) is stable and, consequently, the system (2) does not oscillate.

There is another way of establishing this result. Indeed, the roots of (4) and eigenvalues of the state matrix

$$
A=\left(\begin{array}{c:c}
-D & -K \\
\hdashline I & 0
\end{array}\right)
$$

are the same. The matrix $A$ is asymmetrir, but it can be expressed as the product of two symmetric matrices, i.e.,
where

$$
A_{1}=A_{1}^{T}=\left(\begin{array}{c:c}
D^{2}-K-k_{M} I & D K \\
\hdashline K D & K^{2}+k_{M} K
\end{array}\right)^{-1}
$$

and

$$
A_{2}=A_{2}^{T}=\left(\begin{array}{c:c}
-D^{3}+K D+D K+k_{M} D & -D^{2} K+K^{2}+k_{M} K \\
\hdashline-K D^{2}+K^{2}+k_{M} K & -K D K
\end{array}\right) .
$$

Furthemore, it can be shown that $A_{1}>0$ under the condition (5). Then, matrix $A$ is symmetrizable and, consequently, all eigenvalues of $A$ are real. From this and the fact that the system is asymptotically stable we obtain the result stated in Theorem 1.

## Illustrative example

Consider the two-degree of freedom system shown in Fig. 1, where $c_{i}$ and $\beta_{i}$ stand for the spring constants and coefficients of viscous damping, respectively, and $q_{1}$ and $q_{2}$ are the displacements from equilibrium positions of masses $m_{1}$ and $m_{2}$. For simplicity, we take $c_{1}=c_{2}=c, \beta_{1}=\beta_{2}=\beta_{3}=\beta$ and $m_{1}=m_{2}=m$. The equations of motion for this system can be written in the form (2) with

$$
D=d\left(\begin{array}{ll}
1 & 0 \\
0 & 2
\end{array}\right) \text { and } K=\left(\begin{array}{cc}
2 & -1 \\
-1 & 1
\end{array}\right)
$$

where

$$
d=\frac{\beta}{\sqrt{m c}} .
$$

The condition for non-oscillation derived here is next applied to this system and the result is compared with the exact solution.


Fig. 1. The system of example

An elementary calculation shows that $k_{M}=2.618$. Condition (5) takes the form

$$
\left(\begin{array}{cc}
0.603 d^{2}-4.618 & 0.333 d^{2}+1 \\
0.333 d^{2}+1 & 3.078 d^{2}-3.618
\end{array}\right)>0
$$

which yields $d>2.958$. Consequently, if the parameter $d$ is now chosen to satisfy this inequality then the system will not oscillate when perturbed from equilibrium.

According to [3] (Proposition 1), system of this example does not oscillate if and only if

$$
H=\left(\begin{array}{llll}
4 & s_{1} & s_{2} & s_{3} \\
s_{1} & s_{2} & s_{3} & s_{4} \\
s_{2} & s_{3} & s_{4} & s_{5} \\
s_{3} & s_{4} & s_{5} & s_{6}
\end{array}\right) \geq 0
$$

where

$$
\begin{aligned}
& s_{1}=-3 d ; \quad s_{2}=5 d^{2}-6 ; \quad s_{3}=3 d\left(4-3 d^{2}\right) ; \quad s_{4}=17 d^{4}-24 d^{2}+14 \\
& s_{5}=d\left(-33 d^{4}+50 d^{2}-45\right) ; \quad s_{6}=65 d^{6}-108 d^{4}+114 d^{2}-36
\end{aligned}
$$

A necessary and sufficient condition for a matrix to be positive semidefinite is for each of its principal minors to be nonnegative. Applying this condition to H yields $d \geq 2.952$. Thus, for this example the criterion (5) gives a good result.

## Generalization to a class of asymmetric systems

In this section it is shown that Theorem 1 can be generalized for a class of asymmetric systems (i.e., the symmetry restrictions are not met by mass, damping and stiffness matrices) commonly known as the symmetrizable systems.

Assuming that the mass matrix $M$ is nonsingular, equation of motion can be written as

$$
\begin{equation*}
I \ddot{q}+\tilde{D} \dot{q}+\widetilde{K} q=0 \tag{14}
\end{equation*}
$$

where $\tilde{D}=M^{-1} B$ and $\widetilde{K}=M^{-1} C$ are both real asymmetric matrices. The symmetrizable systems are defined by Inman [8] as systems that have symmetrizable matrices $\widetilde{D}$ and $\widetilde{K}$, i.e., such that factorizations $\widetilde{D}=S_{1} S_{2}$ and $\widetilde{K}=S_{1} S_{3}$ are permissible, where $S_{1}$ is symmetric and positive definite, while $S_{2}$ and $S_{3}$ need only be symmetric. Additionaly, it is supposed that $\widetilde{D}$ and $\widetilde{K}$ have positive eigenvalues. Then, $S_{2}$ and $S_{3}$ are both positive definite matrices and the system described by (14) is asymptotically stable [8], and hence all roots of corresponding characteristic equation have negative real parts.

Using the transformation $q=S_{1}^{1 / 2} y$, equation (14) is reduced to

$$
\begin{equation*}
\ddot{y}+\hat{D} \dot{y}+\hat{K} y=0 \tag{15}
\end{equation*}
$$

where $\hat{D}=\hat{D}^{T}=S_{1}^{1 / 2} S_{2} S_{1}^{1 / 2}$ and $\hat{K}=\hat{K}^{T}=S_{1}^{1 / 2} S_{3} S_{1}^{1 / 2}$. Since $\hat{D}$ and $\hat{K}$ are both symmetric and positive definite, Theorem 1 can be applied to equation (15). This implies that if

$$
\begin{equation*}
\hat{D}^{2}-\hat{K}-\hat{k}_{M} I-\hat{D}\left(\hat{K}+\hat{k}_{M} I\right)^{-1} \hat{K} \hat{D}>0 \tag{16}
\end{equation*}
$$

then the system does not oscillate. Here $\hat{k}_{M}$ represents the largest eigenvalue of the matrix $\hat{K}$. From the factorizations of $\widetilde{D}$ and $\widetilde{K}$, we have

$$
\begin{equation*}
S_{2}=S_{1}^{-1} \tilde{D} \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{3}=S_{1}^{-1} \tilde{K} \tag{18}
\end{equation*}
$$

Substituting (17) and (18) into the expressions

$$
\left(S_{1}^{1 / 2} S_{2} S_{1}^{1 / 2}\right)^{2}-S_{1}^{1 / 2} S_{3} S_{1}^{1 / 2}-\hat{k}_{M} I-S_{1}^{1 / 2} S_{2} S_{1}^{1 / 2}\left(S_{1}^{1 / 2} S_{3} S_{1}^{1 / 2}+\hat{k}_{M} I\right)^{-1} S_{1}^{1 / 2} S_{3} S_{1} S_{2} S_{1}^{1 / 2}
$$

and

$$
\hat{K}=S_{1}^{1 / 2} S_{3} S_{1}^{1 / 2}
$$

results in

$$
\begin{equation*}
S_{1}^{-1 / 2}\left(\widetilde{D}^{2}-\tilde{K}-\hat{k}_{M} I-\tilde{D}\left(\tilde{K}+\hat{k}_{M} I\right)^{-1} \tilde{K} \tilde{D}\right) S_{1}^{1 / 2} \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{K}=S_{1}^{-1 / 2} \tilde{K} S_{1}^{1 / 2} \tag{20}
\end{equation*}
$$

from (19) and (20) we see that $\hat{k}_{M}$ is equal to the largest eigenvalue of the matrix $\widetilde{K}=M^{-1} C$, and that (16) is equivalent to condition that eigenvalues of the asymmetric matrix $\widetilde{D}^{2}-\widetilde{K}-\hat{k}_{M} I-\widetilde{D}\left(\widetilde{K}+\hat{k}_{M} I\right)^{-1} \widetilde{K} \widetilde{D}$ are real and positive, since similarity transformation preserves eigenvalues. Thus, the following result is derived.

Theorem 2. The symmetrizable system described by equation (14), where $\widetilde{D}=M^{-1} B$ and $\widetilde{K}=M^{-1} C$ have positive eigenvalues, does not oscillate if the eigenvalues of the matrix

$$
\begin{equation*}
\left(M^{-1} B\right)^{2}-M^{-1} C-\hat{k}_{M} I-M^{-1} B\left(C+\hat{k}_{M} M\right)^{-1} C M^{-1} B \tag{21}
\end{equation*}
$$

are real and positive.

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# DISCRETE CONTUNUUM METHOD 

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#### Abstract

This paper presents the discrete continuum method on examples of homogeneous discrete systems' with limited number of degrees of motion freedom dynamics. These systems are in the form of homogeneous chains and nets in space and plain. Material points of these nets and chains are tied by elastic, standard hereditary or creep elements. By introducing the trigonometric method for studying properties and equations of dynamics of discrete homogeneous continuums we set up the discrete continuum method for the study of dynamics of chain systems with hereditary or creeping connections. These systems' dynamics is described by a system of integro-differential equations of differential equations with fractional derivatives. A light standard creep element is defined by a constitutive relation of stress-strain state, for the creation of which fractional order derivatives were used.


Keywords: Discrete continuum, discrete hereditary system, discrete homogeneous chain, discrete homogeneous material net, elastic element, standard hereditary element, standard creep element, integro-differential relation, fractional derivatives order, Jules-Lissajous figure, trigonometric method, small vibrations.

## I. Introduction

The fast development of science of material (see Ref. [2], [19], [21], [23], [24], [25]) and experimental mechanics, of methods of numerical analysis, led to the creation of different models of real material bodies and methods for studying dynamics and processes which happen in them during the transduction of disturbance through deformable bodies. In the process of creating a real body model certain simplifications and approximations are done. There also exist different approaches to creating real body models (see Ref. [26], [27]). One such approach is represented by a model of discrete system of material points which are connected by certain ties, and the number of which
is then increased to create a continuum, the motion and deformable wave propagation of which was then described by using partial differential equations. And then, due to the impossibility of solving them analytical, the approximation method was used for the purpose. Methods of discretization of systems of partial differential equations and methods of physical discretization of continuum were used. Computers were used for obtaining numerical solutions. In this paper, we made a combination of known trigonometric method and discrete continuum model to obtain the new discrete continuum method for determining solutions of motion of systems of material points of homogeneous discrete continuum in analytical form.

In an attempt to make a selection of authors who gave significant copntrributions to the knowledge on deformable body dynamics we came to a conclusion that it would require an entire review paper, which is not the goal of this paper so we shall restrict ourselves to citing authors on whose papers we directly rely.

## 2. Discrete Continuums Models and Elements

In this paper we shall use three basic models of discrete continuum with light constraint elements between material particles. We shall define discrete continuum as a system of material particles which are interconnected by light standard constraint elements which have the ability to resist axial deformation under static and dynamic conditions.

Basic elements of discrete continuums are:
$I *$ Material particles with mass $m_{i j k}$, with each particle having three degrees of motion freedom, defined by following coordinates $x_{i j k}, y_{i j k}$ and $z_{i j k}$, when $i, j, k$ changes by kinds $i=1,2,3,4, \ldots, N_{x}$, coulomns $j=1,2,3,4, \ldots, N_{y}$ and orders $k=1,2,3,4, \ldots, N_{z}$.
$2 *$ Light standard constraint element of negligible mass in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions; Constitutive relation between restitution force $\mathbb{P}$ and elongation $y$ or $x$ can be written down in the form $f_{p s r}\left(\mathrm{P}, \dot{\mathrm{P}}, y, \dot{y}, \mathrm{D}, \mathrm{J}, n, c, \tilde{c}, \mu, c_{\alpha}, T, U, \ldots \ldots\right)=0$, where $D$ and $J$ are differential integral operators (see Ref. [6], [3], [5], [10], [11], [14], [15], [17], [20], [2]) which find their justification in experimental verifications of material behavior, while $n, c, \tilde{c}, \mu, c_{\alpha} \ldots$. are material constants, which are also determined experimentally.

For every single light standard constrain element of negligible mass, we shall define a specific law of dynamics. This means that we will define dynamics constitutive relation as determinants of forces and/or change of forces with distances and changes of distances in time, with accuracy up to constants which depend on the accuracy of their determination through experiment.

The accuracy of those constants laws and with them the equations of forces and elongations will depend not only on knowing the nature of object, but also on our having the knowledge necessary for dealing with very complex stress-strain relations (see Ref. [18], [13], [1]). In this paper we shall use three such light standard constraint elements, and they will be:

1* Light standard ideally elastic constraint element for which the stress-strain relation for the restitution force as the function of element axial elongation is given by a linear relation of the form

$$
\begin{equation*}
\mathrm{P}=-c y, \tag{1}
\end{equation*}
$$

where $c$ is a rigidity coefficient or an elasticity coefficient (see Ref. [7], [8]). In natural, non-stressed force and deformation of such elemnt are equal to zero.

2* Light standard hereditary constraint element (see Ref. [5], [6], [21], [24], [16]) for which the stress-strain relation for the restitution force as the function of element elongation is given by a relation:
2. a* in differential form:

$$
\begin{equation*}
\mathrm{D} P=C y \quad \text { or } \quad n \dot{P}(t)+P(t)=n c y(t)+\tilde{c} y(t) \tag{2}
\end{equation*}
$$

where, the following differential operators are introduced:

$$
\begin{equation*}
\mathrm{D}=n \frac{d}{d t}+1 \quad \text { and } \quad \mathrm{C}=n c \frac{d}{d t}+\tilde{c} . \tag{3}
\end{equation*}
$$

and $n$ is a relaxation time and $c, \tilde{c}$ are rigidity coefficints - momentary and prolonged one.

$$
\begin{align*}
& \text { 2. } \mathbf{b}^{*} \text { in integral form } \\
& P(t)=c\left[y(t)-\int_{0}^{1} R(t-\tau) y(\tau) d \tau\right], \tag{4}
\end{align*}
$$

where $\quad R(t-\tau)=\frac{c-\tilde{c}}{n c} e^{-\frac{1}{n}(t-\tau)} \quad$ is relaxation kernel (or resolvente).
2. c* in integral form

$$
\begin{equation*}
y(t)=\frac{1}{c}\left[P(t)+\int_{0}^{t} K(t-\tau) P(\tau) d \tau\right], \tag{6}
\end{equation*}
$$

where $\mathrm{K}(t-\tau)=\frac{c-\tilde{c}}{n c} e^{-\frac{\tilde{\varepsilon}}{n c}(t-\tau)}$ is kernel of rheology (or retardation).
3* Light standard creep constraint element (see ref. [1], [2], [3], [16], [27]) for which the stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives (see Ref. [4]) in the form

$$
\begin{equation*}
P(t)=-\left\{c_{0} x(t)+c_{\alpha} D_{i}^{\alpha}[x(t)]\right\} \tag{8}
\end{equation*}
$$

where $D_{1}^{\alpha}[0]$ is operator of the $\alpha^{\text {l/h }}$ derivative with respect to time $t$ in the following form:

$$
\begin{equation*}
\mathrm{D}_{t}^{\alpha}[x(t)]=\frac{d^{\alpha} x(t)}{d t^{\alpha}}=x^{(\alpha)}(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{1} \frac{x(\tau)}{(t-\tau)^{\alpha}} d \tau \tag{9}
\end{equation*}
$$

where $c, c_{\alpha}$ are rigidity coefficients- momentary and prolonged one, and $\alpha$ a rational number between 0 and $1,0<\alpha<1$.

In this paper we shall define discrete continuum as a system of material particles interconnected by light standard constrain elements (elastic, hereditary or creep) and which are, in natural state, on defined interdistances (when constraint
elements are unstressed). Discrete continuum is ideally elastic if it's material particles are interconnected by light standard ideally elastic constraint elements. Discrete continuum is a standard hereditary continuum if it's material particles are interconnected by light standard hereditary elements. Discrete continuum is a standard creep continuum if its material particles are interconnected by light standard creep elements.

We shall define discrete chain system as a system of discrete material particles which can move along a line and are interconnected by standard constraint elements. The chain is ideally elastic if material particles are interconnected by ideally elastic elements. The chain is standard hereditary if material particles are interconnected by standard hereditary elements. The chain is standard creep if material particles are interconnected by standard creep elements. The number of degrees of freedom of each of these chains is equal to the number of particles in it, since we hypothesize that each material particle moves in the direction of the chain.

If all material particles of the discrete continuum move in the same plane they have two degrees of motion freedom and are interconnected by light standard constraint elements, as it is shown in the picture 1 ., or 2 . such material system we shall call the plane material net. The net can be elastic, standard hereditary or standard hereditary plane material net depending on the type of constraint elements that interconnects its material particles. It is a plane discrete material surface through which we can follow the propagation of deformation waves, which can be elastic, hereditary or creep in two orthogonal directions.

If each material particle of discrete continuum has three degrees of motion freedom, and if they are interconnected by light standard elements into a space discrete material net (as it is shown in Figure 3.) than we call it the space material discrete net of spatial discrete continuum.

Further we introduce the hypotheses about the homogenity of discrete continual chain or discrete continual material net, about small deformations of light standard constraint elements, and that displacements of material particles are small.

Also we introduce the hypothesis that the homogeneous discrete continuum, chain or net, was in natural, non-stressed state, before the initial moment of motion observation i.e. that light standard constraint elements do not have a prehistory nor memory of stress-strain state. With these hypotheses we shall direct our research to the dynamics of chain-like and net-like homogeneous systems.

## 3. Homogeneous Elastic, Linear and Plane Discrete Continuum

In reference [22], on page 157, differential and frequency equations are derived for small oscillations of homogeneous and non-homogeneous systems', while the idea of trigonometric method for solving and describing the dynamics of homogeneous chainlike systems which are constrained on both ends, free on both ends or free on one end and constrained on the other is presented on pages 163 through 167 with appropriate solutions and expressions for dynamic coefficients of amplification. By using the trigonometric method we study small oscillations of homogeneous plane discrete net (see Figure 1.) of the elastic continuum with finite degrees of motion freedom. Figure 5.
shows a model of decomposition of a discrete net-like system around a material particle $m_{i j k}$.


Figure 1. Model of elastic plane discrete continuum - Plane material elastic net.

Differential equtions of mass particles of the ortogonal elastic material net (see Figure 5.) are:

$$
\begin{align*}
& m_{i, j, k} \ddot{x}_{i, j, k}=-X_{(i-1, i,), j, k}+X_{(i, i+1), j, k} \\
& m_{i, j, k} \ddot{y}_{i, j, k}=-Y_{i,(j-1, j), k}+Y_{i,(j, j+1), k}  \tag{10}\\
& m_{i, j, k} \ddot{z}_{i, j, k}=-Z_{i, j,(k-1, k)}+Z_{i, j,(k, k+1)}
\end{align*}
$$



Figure 2. Model of elastic space discrete continuum - Space material elastic net.

By using the corresponding restitution forces into ideally elastic constraint elements (1) in the form:

$$
\begin{align*}
& X_{(i, i+1), j, k}=-c_{(i, i+1), j, k}\left(x_{(i+1),, k, k}-x_{i, j, k}\right) \\
& Y_{i,(j, j+1), k}=-c_{i,(j, j+1), k}\left(y_{i,(j+1), k}-y_{i, j, k}\right)  \tag{11}\\
& Z_{i, j,(k, k+1)}=-c_{i, j,(k, k+1)}\left(z_{i, j,(k+1)}-z_{i, j, k}\right)
\end{align*}
$$

previous differential equtions (10) of mass particles of the ortogonal elastic material net can be rewritten in the following form:

$$
\begin{align*}
& \frac{m}{c} \ddot{x}_{i, j, k}-x_{(i-1), j, k}+2 x_{i, j, k}-x_{(i+1), j, k}=0 \\
& \frac{m}{c} \ddot{y}_{i, j, k}-y_{i,(j-1), k}+2 y_{i, j, k}-y_{i,(j+1), k}=0  \tag{12}\\
& \frac{m}{c} \ddot{z}_{i, j, k}-z_{i, j,(k-1)}+2 z_{i, j, k}-z_{i, j,(k+1)}=0
\end{align*}
$$



Figure 3. Model of hereditary plane discrete continuum - Plane material hereditary net

Now, we assume that the solutions of these differential equations (12) can be taken in the following form:

$$
\begin{align*}
x_{i, j, k} & =A_{i, j, k} \cos \left(\omega_{0, j, k}^{(x)} t+\alpha_{0, j, k}^{(x)}\right) \\
y_{i, j, k} & =B_{i, j, k} \cos \left(\omega_{i, 0, k}^{(y)} t+\alpha_{i, 0, k}^{(j)}\right)  \tag{13}\\
z_{i, j, k} & =C_{i, j, k} \cos \left(\omega_{i, j, 0}^{(z)} t+\alpha_{i, j, 0}^{(z)}\right)
\end{align*}
$$

By introducing following charakteristic numbers:

$$
\begin{equation*}
u_{0, j, k}^{(r)}=\frac{m\left(\omega_{0, j, k}^{(x)}\right)^{2}}{c}, \quad u_{i, 0, k}^{(v)}=\frac{m\left(\omega_{i, 0, k}^{(y)}\right)^{2}}{c}, \quad u_{i, j, 0}^{(z)}=\frac{m\left(\omega_{i, j, 0}^{(z)}\right)^{2}}{c} \tag{14}
\end{equation*}
$$

and prvious assumed solutions (13), into (12), these system differential equations is transformed into following system homogeneous algbraic equations:

$$
\begin{align*}
& -A_{(i-1), j, k}+\left(2-u_{0, j)}^{(x)}\right) A_{i, j, k}-A_{(i+1), j, k}=0 \\
& -B_{i,(j-1), k}+\left(2-u_{i, 0, k}^{(v)}\right) B_{i, j, k}-B_{i,(j+1), k}=0  \tag{15}\\
& -C_{i, j,(k-1)}+\left(2-u_{i, j, 0}^{(z)}\right) C_{i, j, k}-C_{i, j,(k+1)}=0
\end{align*}
$$

Now, we assume that the solutions of these homogeneous algebraic equations (15) can be taken in the following form (see Ref. [22] - the idea of trigonometric method for solving and describing the dynamics of homogeneous chain-like systems which are constrained on both ends, free on both ends or free on one end and constrained on the other is presented on pages 163 through 167 with appropriate solutions and expressions for dynamic coefficients of amplification):

$$
\begin{equation*}
A_{i, j, k}=N_{o, j k} \sin i \varphi_{0, j, k}^{(x)}, \quad B_{i, j, k}=M_{i, o, k} \sin j \theta_{i, 0, k}^{(r)}, \quad C_{i, j, k}=L_{i, j, 0} \sin k \gamma_{i, j, 0}^{(i)} \tag{16}
\end{equation*}
$$

By introducing previous assumed solutions into system (15) we obtain the following:

$$
\begin{equation*}
u_{0, j, k}^{(r) \cdot(s)}=2\left(1-\cos \varphi_{0, j, k}^{(r)(s)}\right), \quad u_{i, 0, k}^{(y)(r)}=2\left(1-\cos \theta_{i, 0, k}^{(y),(r)}\right), \quad u_{i, j, 0}^{(2) \cdot(p)}=2\left(1-\cos \gamma_{i, j, 0}^{(z)(p)}\right) \tag{17}
\end{equation*}
$$

where:
A* For the dynamics of homogeneous chain-like systems which are free on both ends we obtain the following:

$$
\begin{equation*}
\varphi_{0, j, k}^{(x) \cdot(s)}=\varphi_{s}=\frac{s \pi}{N_{x}}, \quad \theta_{i, 0, k}^{(j))(r)}=\theta_{r}=\frac{r \pi}{N_{y}}, \quad \gamma_{i, j, 0}^{(z) .(p)}=\gamma_{p}=\frac{p \pi}{N_{z}} \tag{18}
\end{equation*}
$$

B*For the dynamics of homogeneous chain-like systems which are free on one end and constrained on the other, we obtain the following:

$$
\begin{equation*}
\varphi_{s}=\frac{(2 s-1) \pi}{2 N_{x}+1}, \quad \theta_{r}=\frac{(2 r-1) \pi}{2 N_{y}+1}, \quad \gamma_{p}=\frac{(2 p+1) \pi}{2 N_{z}+1} \tag{19}
\end{equation*}
$$

C* Fo the dynamics of homogeneous chain-like systems which are constrained on both ends, we obtain the following:

$$
\begin{equation*}
\varphi_{s}=\frac{s \pi}{N_{x}+1}, \quad \theta_{r}=\frac{r \pi}{N_{y}+1}, \quad \gamma_{p}=\frac{p \pi}{N_{z}+1} \tag{20}
\end{equation*}
$$

In the previous expressions there are: $s=1,2,3, \ldots, N_{x} ; r=1,2,3, \ldots, N_{y} ; \quad p=1,2,3, \ldots, N_{z}$


Figure 4. Model of space hereditary discrete continuum - Space hereditary material net.


Figure 5. Model of decomposition of the space material discrete continuum and palne of forces of interactions between material particles $\mathrm{a}^{*}$ in plane $0 x y$ i $\mathrm{b}^{*}$ in plane $0 x z$

Corresponding, chracteristic numbers - eigen values of the small free oscilations, depending on boundary chain conditions of the ends, are:

$$
\begin{array}{lll}
u_{s}^{(x)}=4 \sin ^{2} \frac{s \pi}{2 N_{x}} ; & u_{s}^{(x)}=4 \sin ^{2} \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ; & u_{s}^{(x)}=4 \sin ^{2} \frac{s \pi}{2\left(N_{x}+1\right)} \\
u_{r}^{(y)}=4 \sin ^{2} \frac{r \pi}{2 N_{y}} ; & u_{s}^{(y)}=4 \sin ^{2} \frac{(2 r-1) \pi}{2\left(N_{y}+1\right)} ; & u_{(,)}^{(y)}=4 \sin ^{2} \frac{r \pi}{2\left(N_{y}+1\right)} \\
u_{p}^{(z)}=4 \sin ^{2} \frac{p \pi}{2 N_{z}} ; & u_{p}^{(z)}=4 \sin ^{2} \frac{(2 p-1) \pi}{2\left(N_{z}+1\right)} ; & u_{(p)}^{(z)}=4 \sin ^{2} \frac{p \pi}{2\left(N_{z}+1\right)} \tag{22}
\end{array}
$$

Natural circular frequencies of the small free oscilations, depending on boundary chain conditions of the ends, are:

$$
\begin{align*}
& \omega_{0, j, k}^{(x),(s)}=\omega_{s}^{(x)}=2 \sqrt{\frac{c}{m}} \sin \frac{s \pi}{2 N_{x}} ; \quad \omega_{s}^{(x)}=2 \sqrt{\frac{c}{m}} \sin \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \quad \omega_{s}^{(x)}=2 \sqrt{\frac{c}{m}} \sin \frac{s \pi}{2\left(N_{x}+1\right)} ; \\
& \omega_{i, 0, k}^{(y),(r)}=\omega_{r}^{(y)}=2 \sqrt{\frac{c}{m}} \sin \frac{r \pi}{2 N_{y}} ; \quad \omega_{r}^{(y)}=2 \sqrt{\frac{c}{m}} \sin \frac{(2 r-1) \pi}{2\left(N_{y}+1\right)} ; \quad \omega_{(,)}^{(y)}=2 \sqrt{\frac{c}{m}} \sin \frac{r \pi}{2\left(N_{y}+1\right)} ;  \tag{23}\\
& \omega_{i, j, 0}^{(z) \cdot(p)}=\omega_{p}^{(z)}=2 \sqrt{\frac{c}{m}} \sin \frac{p \pi}{2 N_{z}} ; \quad \omega_{p}^{(z)}=2 \sqrt{\frac{c}{m}} \sin \frac{(2 p-1) \tau}{2\left(N_{z}+1\right)} ; \quad \omega_{p}^{(z)}=2 \sqrt{\frac{c}{m}} \sin \frac{p \pi}{2\left(N_{z}+1\right)} ;
\end{align*}
$$

Corresponding amplitudes of the small free oscillations, depending on boundary chain conditions of the ends, are:

$$
\begin{array}{lll}
A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} ; & A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} ; & A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} \\
B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2 N_{y}} ; & B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j(2 r-1) \pi}{2\left(N_{y}+1\right)} ; & B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2\left(N_{y}+1\right)} ;  \tag{24}\\
C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2 N_{z}} ; & C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k(2 p-1) \pi}{2\left(N_{z}+1\right)} ; & C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2\left(N_{z}+1\right)} ;
\end{array}
$$

By following the idea of trigonometric method from reference [22] the law of small free oscillations of material particles in space material elastic net, depending on boundary chain conditions of the ends, can be written in the following form:

$$
\begin{align*}
& x_{i, j, k}=\sum_{s=1}^{s=N_{x}} A_{i, j, k}^{(s)} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right]=\sum_{s=1}^{s=N_{x}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right] ; \\
& x_{i, j, k}=\sum_{s=1}^{s=N_{x}} A_{i, j, k}^{(s)} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right]=\sum_{s=1}^{s=N_{t}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{s}+1\right)} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right] ;  \tag{25}\\
& x_{i, j, k}=\sum_{s=1}^{s=N_{i}} A_{i, j, k}^{(s)} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right]=\sum_{s=1}^{s=N_{s}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} \cos \left[\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right] ; \\
& y_{i, j, k}=\sum_{r=1}^{r=N_{Y}} B_{i, j, k}^{(r)} \cos \left[\omega_{r}^{(v)} t+\alpha_{r}^{(v)}\right]=\sum_{r=1}^{r=N_{y}} M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2 N_{y}} \cos \left[\omega_{r}^{(r)} t+\alpha_{r}^{(v)}\right] ; \\
& y_{i, j, k}=\sum_{r=1}^{r=N_{y}} B_{i, j, k}^{(r)} \cos \left[\omega_{r}^{(y)} t+\alpha_{r}^{(v)}\right]=\sum_{r=1}^{r=N_{y}} M_{i, 0, k}^{(r)} \sin \frac{j(2 r-1) \pi}{2\left(N_{y}+1\right)} \cos \left[\omega_{r}^{(v)} t+\alpha_{r}^{(v)}\right] ;  \tag{26}\\
& y_{i, j, k}=\sum_{r=1}^{r=N,} B_{i, j, k}^{(r)} \cos \left[\omega_{r}^{(v)} t+\alpha_{r}^{(y)}\right]=\sum_{r=1}^{r=N} M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2\left(N_{y}+1\right)} \cos \left[\omega_{r}^{(v)} t+\alpha_{r}^{(y)}\right] ; \\
& z_{i, j, k}=\sum_{p=1}^{p=N_{i}} C_{i, j, k}^{(p)} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right]=\sum_{p=1}^{p=N_{i}} L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2 N_{z}} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right] ; \\
& z_{i, j, k}=\sum_{p=1}^{p=N_{i}} C_{i, j, k}^{(p)} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right]=\sum_{p=1}^{p=N_{i}} L_{i, j, 0}^{(p)} \sin \frac{k(2 p-1) \pi}{2\left(N_{z}+1\right)} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right] ;  \tag{27}\\
& z_{i, j, k}=\sum_{p=1}^{p=N_{i}} C_{i, j, k}^{(p)} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right]=\sum_{p=1}^{p=N_{i}} L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2\left(N_{z}+1\right)} \cos \left[\omega_{p}^{(z)} t+\alpha_{p}^{(z)}\right] ;
\end{align*}
$$

where $N_{0, j, k}^{(s)}, M_{i, 0, k}^{(r)}, L_{i, j, 0}^{(p)}, \alpha_{s}^{(x)}, \alpha_{r}^{(r)}, \alpha_{p}^{(z)}$ are arbitrary integral constants, depending on initial conditions.

We can see that by introducing the hypothesis about small oscillations the space material discrete net can be presented by a system of orthogonal homogeneous discrete chain-like continuums. Motion of each material particle of the discrete continuum is than the resulting motion of three space orthogonal multifrequancy oscillations. Each of these oscillations is a result of collinear superposition of asynchronous collinear oscillations in the general case. In order to create a graphical presentation of the resulting motion of each of material particles it is sufficient to make an analysis of motion of a single material particle in the plane, which lies at the intersection of two orthogonal chains and to introduce the hypothesis that initial conditions of each of these chains are such that all material particles in the chain oscillate with a single frequency.

For example, if both chains intersecting in $m_{i j k}$, are constrained on both ends than the amplification of amplitude of the observed material particle is given in the following form:

$$
\begin{equation*}
\eta_{(i-1), j, k}^{(x)(s)}=\frac{A_{i, j, k}^{(s)}}{A_{1, j, k}^{(s)}}=\frac{\sin i \varphi_{s}}{\sin \varphi_{s}}=\frac{\sin i \frac{s \pi}{\left(N_{x}+1\right)}}{\sin \frac{s \pi}{\left(N_{x}+1\right)}} ; \quad \eta_{i,(j-1), k}^{(y, x)}=\frac{B_{i, j, k}^{(r)}}{B_{i, 1, k}^{(r)}}=\frac{\sin j \theta_{r}}{\sin \theta_{r}}=\frac{\sin j \frac{r \pi}{\left(N_{y}+1\right)}}{\sin \frac{r \pi}{\left(N_{y}+1\right)}} \tag{28}
\end{equation*}
$$

The resulting motion of the material particle $m_{i j k}$ in the plane material net is the result of superposition of two orthogonal asynchronous oscillations. Based on that we can write:

$$
\begin{equation*}
x_{i, k, k}^{(s)}(t)=A_{i, j, k}^{(s)} \frac{\sin i \frac{s \pi}{\left(N_{x}+1\right)}}{\sin \frac{s \pi}{\left(N_{x}+1\right)}} \cos \left(\omega_{s}^{(x)} t+\alpha_{s}^{(x)}\right) ; \quad y_{i, j, k}^{(r)}=B_{i, 1, k}^{(r)} \frac{\sin j \frac{r \pi}{\left(N_{y}+1\right)}}{\sin \frac{r \pi}{\left(N_{y}+1\right)}} \cos \left(\omega_{r}^{(s)} t+\alpha_{r}^{(y)}\right), \tag{29}
\end{equation*}
$$

as well as the following:

$$
\begin{align*}
y_{i, j, k}^{(r)} & =B_{i, 1, k}^{(r)} \frac{\sin j \theta_{r}}{\sin \theta_{r}}\left\{\cos \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \alpha_{s}^{(r)}+\alpha_{r}^{(r)}\right] \cos \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \operatorname{arc} \cos \frac{x_{i, j, k}^{(s)}}{A_{1, j, k, k}^{(s)} \frac{\sin i \varphi_{s}}{\sin \varphi_{s}}}\right]\right\}+  \tag{30}\\
& +B_{i, 1, k}^{(r)} \frac{\sin j \theta_{r}}{\sin \theta_{r}}\left\{+\sin \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\theta_{r}}{2}} \alpha_{s}^{(r)}+\alpha_{r}^{(s)}\right] \sin \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \arccos \frac{\left.\left.x_{x_{, j, k}^{(s)}}^{A_{1,, k, k}^{(s)} \frac{\sin i \varphi_{s}}{\sin \varphi_{s}}}\right]\right\}}{}\right]\right\}
\end{align*}
$$

The resulting path of the motion of the material particle $m_{i j k}$ in the plane material net we can write in the following form:

$$
\begin{array}{r}
{\left[\frac{y_{i, j, k}^{(s)}}{B_{i, 1, k}^{(r)} \frac{\sin j \theta_{r}}{\sin \theta_{r}}}\right]^{2}-2\left[\frac{y_{i, j, k}^{(s)}}{B_{i, 1, k}^{(r)} \frac{\sin j \theta_{r}}{\sin \theta_{r}}}\right] \cos \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \alpha_{s}^{(x)}+\alpha_{r,}^{(s)}\right] \cos \left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \arccos \frac{x_{i,, k}^{(s)}}{A_{i, j, k}^{(s)} \frac{\sin i \varphi_{s}}{\sin \varphi_{s}}}\right]+}  \tag{31}\\
\quad+\cos ^{2}\left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \arccos \frac{\left.x_{x_{i, k, k}^{(s)}}^{A_{l, j, k}^{(s)} \frac{\sin i \varphi_{s}}{\sin \varphi_{s}}}\right]=\sin ^{2}\left[\frac{\sin \frac{\theta_{r}}{2}}{\sin \frac{\varphi_{s}}{2}} \alpha_{s}^{(x)}+\alpha_{r}^{(p)}\right]}{}\right.
\end{array}
$$

Previous equation is equation of the different forms of Jules-Lissajous figures, presented in Figure 6, 7, 8 and 9 for different initial and boundary conditions of the motion of the material particle $m_{i j k}$ in the plane material net (Figures from Ref. [22]).


Figure 6. Examples of constructions of Jules-Lissajous figures (Figures from Ref. [22]).

## 4. Homogeneous Standard Hereditary, Linear and Plane Discrete Continuum

By using constitutive relation (2) for corresponding differential relations of stress-strain state of standard hereditary constraint elements in homogeneous standard hereditary discrete continuum (see Figure 4.) we can write the following relations:

$$
\begin{align*}
& n_{(i, i+1), j, k} \dot{X}_{(i, i+1), j, k}+X_{(i,+1), j, k}=n_{(i, i+1), j, k} c_{(i, i+1), j, k}\left(\dot{x}_{(i+1), j, k}-\dot{x}_{i, j, k}\right)+\tilde{c}_{(i, i+1), j, k}\left(x_{(i+1), j, k}-x_{i, j, k}\right) \\
& n_{i,(j, j+1), k} \dot{Y}_{i,(j, j+1), k}+Y_{i,(j, j+1), k}=n_{i,(j, j+1), k} c_{i,(j, j+1), k}\left(\dot{y}_{i,(j+1), k}-\dot{y}_{i, j, k}\right)+\tilde{c}_{i,(j, j+1), k}\left(y_{i,(j+1), k}-y_{i, j, k}\right)  \tag{32}\\
& n_{i, j,(k, k+1)} \dot{z}_{i, j,(k, k+1)}+Z_{i, j,(k, k+1)}=n_{i, j,(k, k+1)} c_{i, j,(k, k+1)}\left(\dot{z}_{i, j,(k+1)}-\dot{z}_{i, j, k}\right)+\tilde{c}_{i, j,(k, k+1)}\left(z_{i, j,(k+1)}-z_{i, j, k}\right)
\end{align*}
$$

We will further restrict ourselves to exploring only homogeneous space hereditary nets, an example of which is shown in Figure 4. These nets constitute of material particles of equal mass $m$ and which are interconnected by standard hereditary elements of equal relaxation times $n$ and rigidity coefficients $c, \tilde{c}$, both prolonged and temporary ones. Stress-strain states of these standard hereditary elements can be expressed by the following constitutive relations:

$$
\begin{align*}
& \mathrm{D} \mathrm{X}_{(i, i+1), j, k}=\mathrm{C}\left(x_{(i+1), j, k}-x_{i, j, k}\right) \\
& \mathrm{D} Y_{i,(, j, j+1), k}=\mathrm{C}\left(y_{i,(j+1), k}-y_{i, j, k}\right)  \tag{33}\\
& \mathrm{D} Z_{i, j,(k, k+1)}=\mathrm{C}\left(z_{i, j,(k+1)}-z_{i, j, k}\right)
\end{align*}
$$

in which differential operators have been introduced $\mathrm{D}=n \frac{d}{d t}+1 \quad$ i $\quad \mathrm{C}=n c \frac{d}{d t}+\tilde{c}$.


Figure 7. Graphical presentations of Jules-Lissajous figures for different initial and boundary conditions of the motion of the material particle $m_{i j k}$ in the plane material net (Figures from Ref. [22]).

Taking into account rheological relations towards the prior constitutive relations (33) for standard hereditary elements in direction of rows, columns and ranks, and applying the differential operator $D$ on initial equations of motion (10) we obtain the system of differential equations for small movements in the direction of main coordinates of a homogeneous space discrete material hereditary net. When constitutuive relations are taken into account the folowing system of derived differential equations is obtained:

$$
\begin{align*}
& m D \ddot{x}_{i, j, k}=-\mathrm{C}\left(x_{i, j, k}-x_{(i-1), j, k}\right)+\mathrm{C}\left(x_{(i+1), j, k}-x_{i, j, k}\right) \\
& m \mathrm{D} \ddot{y}_{i, j, k}=-\mathrm{C}\left(y_{i, j, k}-y_{i,(j-1), k}\right)+\mathrm{C}\left(y_{i,(j+1), k}-y_{i, j, k}\right)  \tag{34}\\
& m \mathrm{D} \ddot{z}_{i, j, k}=-\mathrm{C}\left(z_{i, j, k}-z_{i, j,(k-1)}\right)+\mathrm{C}\left(z_{i, j,(k+1)}-z_{i, j, k}\right)
\end{align*}
$$

Now, we assume that the solutions of these differential equations (34) can be taken in the following form:

$$
\begin{equation*}
x_{i, j, k}=A_{i, j, k} e^{\lambda_{0, j, k}^{(x)}}, \quad y_{i, j, k}=B_{i, j, k} e^{\lambda_{i, 0, k}^{(y)} t}, \quad z_{i, j, k}=C_{i, j, k} e^{\lambda_{i, j, 0}^{(z)}} \tag{35}
\end{equation*}
$$

Let untroduce following notations for binoms:

$$
\begin{equation*}
D^{\lambda}=n \lambda+1 \quad \text { and } \quad C^{\lambda}=n c \lambda+\tilde{c} \tag{36}
\end{equation*}
$$

with corresponding markings with the coresponding lambda numbers $\lambda$ : $\lambda_{0, j, k}^{(r)}$, $\lambda_{i, 0, k}^{(y)}, \lambda_{i, j, 0}^{(z)}$. Lets introduce the following notation

Previous characteristic expressions are in the form of:

$$
\begin{equation*}
u=\frac{m \lambda^{2} D^{\lambda}}{C^{\lambda}}=\frac{m \lambda^{2}\left(n_{0} \lambda+1\right)}{\left(n_{0} c \lambda+\tilde{c}\right)} . \tag{37*}
\end{equation*}
$$

By introducing previous characteristic expressions (37), and previous assumed solutions (35) into system of derived differential equations (34), system differential equations is transformed into following system homogeneous algebraic equations:

$$
\begin{align*}
& -A_{(i-1), j, k}+\left(2+u_{0, j, k}^{(x)}\right) A_{i, j, k}-A_{(i+1), j, k}=0 \\
& -B_{i,(j-1), k}+\left(2+u_{i, 0, k}^{(y)}\right) B_{i, j, k}-B_{i,(j+1), k}=0  \tag{38}\\
& -C_{i, j,(k-1)}+\left(2+u_{i, j, 0}^{(z)}\right) C_{i, j, k}-C_{i, j,(k+1)}=0
\end{align*}
$$

Now, we assume that the solutions of these homogeneous algebraic equations (38) can be taken in the following form (see Ref. [22] and previous Chapter):

$$
A_{i, j, k}=N_{o, j, k} \sin i \varphi_{0, j, k}^{(x)}, \quad B_{i, j, k}=M_{i, o, k} \sin j \theta_{i, 0, k}^{(v)}, \quad C_{i, j, k}=L_{i, j, 0} \sin k \gamma_{i, j, 0}^{(z)}
$$

By introduce previous assumed solutions (39) into systems (38) we obtain the following:

$$
\begin{equation*}
u_{0, j, k}^{(x),(s)}=-2\left(1-\cos \varphi_{0, j, k}^{(x),(s)}\right), u_{i, 0, k}^{(y),(r)}=-2\left(1-\cos \theta_{i, 0, k}^{(y),(r)}\right), u_{i, j, 0}^{(z),(p)}=-2\left(1-\cos \gamma_{i, j, 0}^{(z),(p)}\right) \tag{40}
\end{equation*}
$$

$A^{*}$ For the dynamics of homogeneous chain-like system which are free on both ends we obtain the following:

$$
\begin{equation*}
\varphi_{0, j, k}^{(x)(s)}=\varphi_{s}=\frac{s \pi}{N_{x}}, \quad \theta_{i, 0, k}^{(y) \cdot(r)}=\theta_{r}=\frac{r \pi}{N_{y}}, \quad \gamma_{i, j, 0}^{(\tau),(p)}=\gamma_{p}=\frac{p \pi}{N_{z}} ; \tag{41}
\end{equation*}
$$

B* For the dynamics of homogeneous chain-like system which are free on one end and constrained on the other, we obtain the following:

$$
\begin{equation*}
\varphi_{s}=\frac{(2 s-1) \pi}{2 N_{x}+1}, \quad \theta_{r}=\frac{(2 r-1) \pi}{2 N_{y}+1}, \quad \gamma_{p}=\frac{(2 p+1) \pi}{2 N_{z}+1} \tag{42}
\end{equation*}
$$

C* For the dynamics of homogeneous chain-like system which are constrained on both ends, we obtain the following:

$$
\begin{equation*}
\varphi_{s}=\frac{s \pi}{N_{x}+1}, \quad \theta_{r}=\frac{r \pi}{N_{y}+1}, \quad \gamma_{p}=\frac{p \pi}{N_{z}+1} . \tag{43}
\end{equation*}
$$

In the previous expressions there are: $s=1,2,3, \ldots, N_{x} ; r=1,2,3, \ldots, N_{y} ; \quad p=1,2,3, \ldots, N_{z}$.
Corresponding characteristic eigen-expressions-relations (40) of the small free mass particle motions-oscillations, depending on boundary chain conditions on the ends, are:

$$
\begin{array}{lll}
u_{s}^{(x)}=-4 \sin ^{2} \frac{s \pi}{2 N_{x}} ; & u_{s}^{(x)}=-4 \sin ^{2} \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ; & u_{s}^{(x)}=-4 \sin ^{2} \frac{s \pi}{2\left(N_{x}+1\right)} ; \\
u_{r}^{(r)}=-4 \sin ^{2} \frac{r \pi}{2 N_{y}} ; & u_{s}^{(y)}=-4 \sin ^{2} \frac{(2 r-1) \pi}{2\left(N_{y}+1\right)} ; & u_{(\cdot)}^{(v)}=-4 \sin ^{2} \frac{r \pi}{2\left(N_{y}+1\right)} ;  \tag{44}\\
u_{p}^{(z)}=-4 \sin ^{2} \frac{p \pi}{2 N_{z}} ; & u_{p}^{(z)}=-4 \sin ^{2} \frac{(2 p-1) \pi}{2\left(N_{z}+1\right)} ; & u_{(z)}^{(z)}=-4 \sin ^{2} \frac{p \pi}{2\left(N_{z}+1\right)} ;
\end{array}
$$

By using the previous characteristic relations, we can form sets of characteristic polynoms-equations of dynamical hereditary processes in material net, in the following form:

$$
\begin{align*}
& {\left[m \mathrm{D}^{\lambda_{0, j, k}^{(x)}}\left(\lambda_{0, j, k}^{(x)}\right)^{2}-u_{0, j, k}^{(x)} \mathrm{C}^{\lambda_{0}^{(x)}, k}\right]_{s}=0,} \\
& {\left[m \mathrm{D}^{\lambda_{i 0, k, k}^{(y)}}\left(\lambda_{i, 0, k}^{(y)}\right)^{2}-u_{i, 0, k}^{(y)} \mathrm{C}^{\lambda_{i, 0, k}^{(p)}}\right]_{r}=0,}  \tag{45}\\
& {\left[m \mathrm{D}^{\left.\lambda_{j, 0,0}^{(z)}\left(\lambda_{i, j, 0}^{(z)}\right)^{2}-u_{i, j, 0}^{(z)} \mathrm{C}^{\lambda_{i, j, 0}^{(2)}}\right]_{p}=0 .}\right.}
\end{align*}
$$

and by introducing into (45) characteristic expressions in the form $u=\frac{m \lambda^{2} D^{\lambda}}{C^{2}}=\frac{m \lambda^{2}\left(n_{0} \lambda+1\right)}{\left(n_{0} c \lambda+\tilde{c}\right)}$ we can write:

$$
\begin{equation*}
m \lambda_{s}^{2} D^{\lambda_{s}}+2 C^{\lambda_{s}}\left(\cos \varphi_{s}-1\right)=0, \quad s=1,2,3, \ldots, n, \tag{45*}
\end{equation*}
$$

or in the following form:

$$
\begin{equation*}
m n_{0} \lambda_{s}^{3}+m \lambda_{s}^{2}+2\left(n_{o} c \lambda_{s}+\tilde{c}\right)\left(\cos \frac{s \pi}{n+1}-1\right)=0, s=1,2,3, \ldots, n, \tag{**}
\end{equation*}
$$

of which roots are $\lambda_{r, s}, r=1,2,3 ; s=1,2,3 \ldots, n$, an there are $3 n$ for each of the chain orthogonal directions.

Corresponding amplitudes of own small motion for forms corresponding to characteristic numbers and depending on boundary conditions, are:

$$
A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} ;
$$

$$
\begin{array}{lll}
B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2 N_{y}} ; & B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j(2 r-1) \pi}{2\left(N_{y}+1\right)} ; & B_{i, j, k}^{(r)}=M_{i, 0, k}^{(r)} \sin \frac{j r \pi}{2\left(N_{y}+1\right)}  \tag{46}\\
C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2 N_{z}} ; & C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k(2 p-1) \pi}{2\left(N_{z}+1\right)} ; & C_{i, j, k}^{(p)}=L_{i, j, 0}^{(p)} \sin \frac{k p \pi}{2\left(N_{z}+1\right)}
\end{array}
$$

By following the idea of trigonometric method from reference [22], the law of small free motion (oscillations) of material particles in space material elastic net, depending of boundary chain conditions on the ends, can be written in the following type-form:

$$
\begin{align*}
& x_{i, j, k}=\sum_{s=1}^{s=N_{x}} A_{i, j, k}^{(s)} e^{\lambda_{i}^{(x)} t}=\sum_{s=1}^{s=N_{s}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{\lambda_{i}^{(x)} t} ; \\
& y_{i, j, k}=\sum_{s=1}^{s=N_{i}} B_{i, j, k}^{(s)} e^{\lambda_{j}^{(s)}}=\sum_{s=1}^{s=N_{i}} M_{i, 0, k}^{(s)} \sin \frac{i s \pi}{2 N_{y}} e^{\lambda_{j}^{(s)} t}  \tag{47}\\
& z_{i, j, k}=\sum_{s=1}^{s=N_{z}} C_{i, j, k}^{(s)} e^{\lambda_{j}^{(j)}, t}=\sum_{s=1}^{s=N_{z}} N_{i, j, 0}^{(s)} \sin \frac{i s \pi}{2 N_{z}} e^{\lambda_{j}^{(s)}, t}
\end{align*}
$$

We can see that by introducing the hypothesis about small motion (or oscillations) the space material discrete net can be presented by a system of orthogonal homogeneous discrete hereditary chain-like continuums. Motion of each material particle of the discrete hereditary continuum is than the resulting motion of three space orthogonal motions or multifrequancy oscillations. Each of these motions is a result of collinear superposition of collinear mods hereditary motions in the general case. In order to create a graphical presentation of the resulting motion of each of material particles it is sufficient to make an analysis of motion of a single material particle in the plane, which lies at the intersection of two orthogonal hereditary chains and to introduce the hypothesis that initial conditions of each of these chains are such that all material particles have axial disturbances.

## 4. Homogeneous Standard Creep of a Fractional Order Derivative, Linear and Plane Discrete Continuum

By using constitutive relation (8) for corresponding fractional order differential relations of stress-strain state of standard creep constraint elements in homogeneous standard creep discrete continuum (see Figure 4.) we can write the correspondig relations for each of these constraint elements. We will further restrict ourselves to exploring only homogeneous space creep material nets, an example of which is shown in Figure 4. These nets constitute of material particles of equal mass $m$ and which are interconnected by standard creep elements of equal rigidity coefficients $c, c_{\alpha \alpha}$, both prolonged and temporary ones. Stress-strain states of these standard creep elements can be expressed by the corresponding constitutive relations, in which fractional order differential operator have been introduced as $D_{\alpha}^{t}$ by (9).

Taking into account corresponding creep relations (8) for standard creep constraint elements in direction of rows, columns and ranks, with fractional order derivative constitutive relation, and applying on initial equations of motion (10), we
obtain the system of differential equations, with fractional order derivative, for small movements in the direction of main coordinates of a homogeneous space discrete material creep net. When constitutuive relations are taken into account, the folowing system of derived differential equations, with fractional order derivative, is obtained:

$$
\begin{align*}
m_{i, j, k} \ddot{x}_{i, j, k}+ & \left\{c_{0(i-1, i), j, k}\left[x_{i, j, k}(t)-x_{i-1, j, k}(t)\right]+c_{\alpha(i-1, i), j, k} D_{t}^{\alpha}\left[x_{i, j, k}(t)-x_{i-1, j, k}(t)\right]\right\}- \\
& \left\{c_{0(i, i+1), j, k}\left[x_{i+1, j, k}(t)-x_{i, j, k}(t)\right]+c_{\alpha(i, i+1), j, k} D_{t}^{\alpha}\left[x_{i+1, j, k}(t)-x_{i, j, k}(t)\right]\right\}=0 \\
m_{i, j, k} \ddot{y}_{i, j, k}+ & \left\{\left\{_{0 i,(j-1, j), k}\left[x_{i, j, k}(t)-x_{i, j-1, k}(t)\right]+c_{\alpha i,(j-1, j), k} D_{i}^{\alpha}\left[x_{i, j, k}(t)-x_{i, j-1, k}(t)\right]\right\}-\right. \\
& \left\{c_{0 i,(j, j+1), k}\left[x_{, j i+1, k}(t)-x_{i, j, k}(t)\right]+c_{\alpha i,(j, j+1), k} D_{t}^{\alpha}\left[x_{i, j+1, k}(t)-x_{i, j, k}(t)\right]\right\}=0  \tag{48}\\
m_{i, j, k} \ddot{z}_{i, j, k}+ & \left\{c_{0 i, j(k-1, k)}\left[x_{i, j, k}(t)-x_{i, j, k-1}(t)\right]+c_{\alpha i, j,(k-1, k)} D_{t}^{\alpha}\left[x_{i, j, k}(t)-x_{i, j, k-1}(t)\right]\right\}- \\
& \left\{c_{0 i, j,(k, k+1)}\left[x_{i, j, k+1}(t)-x_{i, j, k}(t)\right]+c_{\alpha i, j(k, k+1)} D_{t}^{\alpha}\left[x_{i, j, k+1}(t)-x_{i, j, k}(t)\right]\right\}=0
\end{align*}
$$

If we introduce parameters: $\omega_{0}^{2}=\frac{c_{o}}{m}, \omega_{\alpha}^{2}=\frac{c_{\alpha}}{m}$, under the hypothesis about the homogeneous creeping material space net, the previous system of differential equations with fractional order derivative, can be written in a simpler form:

$$
\begin{align*}
& \ddot{x}_{i, j, k}+\omega_{\alpha}^{2} D_{t}^{\alpha}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 \\
& \ddot{y}_{i, j, k}+\omega_{\alpha}^{\alpha} D_{1}^{\alpha}\left[-y_{i, j-1, k}(t)+2 y_{i, j, k}(t)-y_{i, j+1, k}(t)\right]+\omega_{0}^{2}\left[-y_{i, j-1, k}(t)+2 y_{i, j, k}(t)-y_{i, j+1, k}(t)\right]=0(49)  \tag{49}\\
& \ddot{z}_{i, j, k}+\omega_{\alpha}^{2} D_{t}^{\alpha}\left[-z_{i, j, k-1}(t)+2 z_{i, j, k}(t)-z_{i, j, k+1}(t)\right]+\omega_{0}^{2}\left[-z_{i, j, k-1}(t)+2 z_{i, j, k}(t)-z_{i, j, k+1}(t)\right]=0
\end{align*}
$$

or

$$
\begin{align*}
& \ddot{x}_{i, j, k}(t)+\omega_{\alpha}^{2}\left[-x_{i-1, j, k}^{(\alpha)}(t)+2 x_{i, j, k}^{(\alpha)}(t)-x_{i+1, j, k}^{(\alpha)}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 \\
& \ddot{y}_{i, j, k}(t)+\omega_{\alpha}^{2}\left[-y_{i, j-1, k}^{(\alpha)}(t)+2 y_{i, j, k}^{(\alpha)}(t)-y_{i, j+1, k}^{(\alpha)}(t)\right]+\omega_{0}^{2}\left[-y_{i, j-1, k}(t)+2 y_{i, j, k}(t)-y_{i, j+1, k}(t)\right]=0  \tag{49*}\\
& \ddot{z}_{i, j, k}(t)+\omega_{\alpha}^{2}\left[-z_{i, j, k-1}^{(\alpha)}(t)+2 z_{i, j, k}^{(\alpha)}(t)-z_{i, j, k+1}^{(\alpha)}(t)\right]+\omega_{0}^{2}\left[-z_{i, j, k-1}(t)+2 z_{i, j, k}(t)-z_{i, j, k+1}(t)\right]=0
\end{align*}
$$

If we are dealing with subsystems of the same types, but independent, we shall study only one of these subsystems:

$$
\begin{equation*}
\ddot{x}_{i, j, k}(t)+\omega_{\alpha}^{2}\left[-x_{i-1, j, k}^{(\alpha)}(t)+2 x_{i, j, k}^{(\alpha)}(t)-x_{i+1, j, k}^{(\alpha)}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 \tag{50}
\end{equation*}
$$

I* For the case, that $\alpha=0$, previous subsystem (50) of differential equations with fractional order derivatives, take the following form:

$$
\begin{equation*}
\ddot{x}_{i, j, k}(t)+\left(\omega_{\infty}^{2}+\omega_{0}^{2}\right)\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 \tag{51}
\end{equation*}
$$

Now, we assume that the solution of the previous equations can be taken in the following forms:

$$
\begin{equation*}
x_{i, j, k}(t)=A_{i, j, k} \cos (\omega t+\beta) \tag{52}
\end{equation*}
$$

and by untroduce into system (51), we can obtain the following system of algabraic equations:

$$
\begin{equation*}
-A_{i-1, j, k}+(2-u) A_{i, j, k}-A_{i+1, j, k}=0 \tag{53}
\end{equation*}
$$

where is:

$$
\begin{equation*}
u=\frac{\omega^{2}}{\omega_{\infty}^{2}+\omega_{0}^{2}} . \tag{54}
\end{equation*}
$$

This system (53) of algebraic equations is easy to solve by the use of trigonometric method (see Ref. [22]), and for various cases of boundary conditions on the end of the chain in rank (or column or row), we obtain the following corresponding assumed solutions:

$$
\begin{equation*}
A_{i, j, k}=N_{o, j, k} \sin i \varphi, \tag{55}
\end{equation*}
$$

where is $u_{s}^{(x)}=2\left(1-\cos \varphi_{s}\right)$; for free on both ends $-\varphi_{s}=\frac{s \pi}{N_{x}}$; for free on one and constrained on the other end - $\varphi_{s}=\frac{(2 s-1) \pi}{2 N_{x}+1}$; and constrained on bouth ends $\varphi_{s}=\frac{s \pi}{N_{x}+1}, s=1,2,3, \ldots, N_{x}$.

Corresponding, eigen values of the small creeping free motions, depending on boundary conditions of the ends, are:

$$
\begin{equation*}
u_{s}^{(x)}=4 \sin ^{2} \frac{s \pi}{2 N_{x}} ; \quad u_{s}^{(x)}=4 \sin ^{2} \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \quad u_{s}^{(x)}=4 \sin ^{2} \frac{s \pi}{2\left(N_{x}+1\right)} ; \tag{56}
\end{equation*}
$$

Natural circular frequencies of the small free oscilations, depending on boundary chain condistions of the ends, are:

$$
\begin{array}{ll}
\omega_{s}^{(x)}=2 \sqrt{\omega_{0}^{2}+\omega_{00}^{2}} \sin \frac{s \pi}{2 N_{x}} ; & \omega_{s}^{(x)}=2 \sqrt{\omega_{0}^{2}+\omega_{00}^{2}} \sin \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} \\
\omega_{s}^{(x)}=2 \sqrt{\omega_{0}^{2}+\omega_{00}^{2}} \sin \frac{s \pi}{2\left(N_{x}+1\right)} ;
\end{array}
$$

Corresponding amplitudes of the small free creep oscilations, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} ; \tag{58}
\end{equation*}
$$

Now, the law of small free creep oscilations of material particles in space material creep net, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
x_{i, j, k}(t)=\sum_{s=1}^{N_{s}} A_{i, j, k}^{s} \cos \left(\omega_{s} t+\beta_{s}\right) \tag{59}
\end{equation*}
$$

or:

$$
\begin{align*}
& x_{i, j, k}(t)=\sum_{s=1}^{N_{r}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{s}} \cos \left(\omega_{s} i+\beta_{s}\right) ; \\
& x_{i, j, k}(t)=\sum_{s=1}^{N_{s}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right)  \tag{59*}\\
& x_{i, j, k}(t)=\sum_{s=1}^{N_{*}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right)
\end{align*}
$$

II* For the case that $\alpha=1$ previous subsystem (50) of differential equations with fractional order derivatives, take the following form:

$$
\begin{equation*}
\ddot{x}_{i, j, k}(t)+\omega_{1}^{2}\left[-x_{i-1, j, k}^{(1)}(t)+2 x_{i, j, k}^{(1)}(t)-x_{i+1, j, k}^{(1)}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0, \tag{60}
\end{equation*}
$$

or:

$$
\begin{equation*}
\ddot{x}_{i, j, k}(t)+\omega_{1}^{2}\left[-\dot{x}_{i-1, j, k}(t)+2 \dot{x}_{i, j, k}(t)-\dot{x}_{i+1, j, k}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 \tag{60*}
\end{equation*}
$$

By introduce the following:

$$
\begin{equation*}
x_{i, j, k}(t)=\tilde{x}_{i, j, k}(t) e^{\frac{\omega_{1}^{2}}{2} t} \tag{61}
\end{equation*}
$$

previous subsystem takes form:

$$
\begin{equation*}
\ddot{\tilde{x}}_{i, j, k}(t)+\left(\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}\right)\left[-\tilde{x}_{i-1, j, k}(t)+2 \tilde{x}_{i, j, k}(t)-\tilde{x}_{i+1, j, k}(t)\right]=0 \tag{62}
\end{equation*}
$$

a* Under the assumption that $\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}>0$, the previous system can be solved in the same way as in the previous case, with the only difference being that instead of $\omega_{0}^{2}+\omega_{00}^{2}$ we should write $\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}>0$. Based on that we write that natural circular frequencies depending on boundary conditions are the following:

$$
\begin{array}{ll}
\omega_{s}^{(x)} & =2 \sqrt{\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}} \sin \frac{s \pi}{2 N_{x}} ;
\end{array} \omega_{s}^{(x)}=2 \sqrt{\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}} \sin \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ;
$$

Corresponding amplitudes of the small free creep oscilations, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} ; \quad A_{i, j, k}^{(s)}=N_{0 . j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} ; \tag{64}
\end{equation*}
$$

Now, correspondig solutions of the subsystem (62) for the law of small free creep oscilations of material particles in space material creep net, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
\tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{s}} A_{i, j, k}^{s} \cos \left(\omega_{s} t+\beta_{s}\right) \tag{65}
\end{equation*}
$$

or

$$
\begin{align*}
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{x}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} \cos \left(\omega_{s} i+\beta_{s}\right) ; \\
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{s}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right)  \tag{66}\\
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{x}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right)
\end{align*}
$$

Now, the law of small free creep oscilations of material particles in space material creep net, depending on boundary chain conditions of the ends, are:

$$
\begin{align*}
& x_{i, j, k}(t)=e^{-\frac{\omega_{0}^{2}}{2}} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}} \sum_{s=1}^{N_{1}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} \cos \left(\omega_{s} i+\beta_{s}\right) ; \\
& x_{i, j, k}(t)=e^{-\frac{\omega_{0}^{2}}{2}} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}} \sum_{s=1}^{N_{x}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \tau}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right) ;  \tag{67}\\
& x_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}} \sum_{s=1}^{N_{x}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} \cos \left(\omega_{s} t+\beta_{s}\right) ;
\end{align*}
$$

$\mathbf{b}^{*}$ Under the assumption that $\omega_{0}^{2}-\frac{\omega_{1}^{4}}{4}<0$, the previous system (62) can be rewritten in the following form:

$$
\begin{equation*}
\ddot{\tilde{x}}_{i, j, k}(t)-\left(\frac{\omega_{1}^{4}}{4}-\omega_{0}^{2}\right)\left[-\tilde{x}_{i-1, j, k}(t)+2 \tilde{x}_{i, j, k}(t)-\tilde{x}_{i+1, j, k}(t)\right]=0 \tag{68}
\end{equation*}
$$

By uitroduce into previous system equations (68) the following assumed solutions:

$$
\begin{equation*}
\tilde{x}_{i, j, k}=A_{i, j, k} e^{2 t} \tag{69}
\end{equation*}
$$

we can take the following system algebraic equations:

$$
\begin{equation*}
-A_{i-1, j, k}(t)+(2-u) A_{i, j, k}(t)-A_{i+1, j, k}(t)=0 \tag{70}
\end{equation*}
$$

where we introduce:

$$
\begin{equation*}
u=\frac{\lambda^{2}}{\frac{\omega_{1}^{4}}{4}-\omega_{0}^{2}} . \tag{71}
\end{equation*}
$$

We can determine the solution of the previous system (70) of homogeneous algebraic equations in the same way as in the previous cases so that we can immediately write eigenvalues of the auxiliary system depending on the boundary chain conditions of the ends:

$$
\begin{align*}
& \lambda_{s}^{(x)}= \pm 2 \sqrt{\frac{\omega_{1}^{4}}{4}-\omega_{0}^{2}} \sin \frac{s \pi}{2 N_{x}} ; \quad \quad \lambda_{s}^{(x)}= \pm 2 \sqrt{\frac{\omega_{1}^{4}}{4}-\omega_{0}^{2}} \sin \frac{(2 s-1) \pi}{2\left(N_{x}+1\right)} ; \\
& \lambda_{s}^{(x)}= \pm 2 \sqrt{\frac{\omega_{1}^{4}}{4}-\omega_{0}^{2}} \sin \frac{s \pi}{2\left(N_{x}+1\right)} ; \tag{72}
\end{align*}
$$

Corresponding amplitudes of the small free creep oscilations, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \tau}{2\left(N_{x}+1\right)} ; \quad A_{i, j, k}^{(s)}=N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} ; \tag{73}
\end{equation*}
$$

Now, correspondig solutions of the subsystem (68) for the law of small free creep oscilations of material particles in space material creep net, depending on boundary chain conditions of the ends, are:

$$
\begin{equation*}
\tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{x}} A_{i, j, k}^{s} e^{ \pm \lambda_{j}^{\left(s_{i}\right)}} \tag{74}
\end{equation*}
$$

or:

$$
\begin{align*}
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{1}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{ \pm \lambda_{i}^{\left.()_{i}\right)}} ; \\
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{1}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} e^{ \pm \lambda_{i}^{(r) t}}  \tag{75}\\
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{x}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} e^{ \pm \lambda_{i}^{\left.()_{t}\right)}}
\end{align*}
$$

or

$$
\begin{align*}
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{s}}\left[N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{\left.+\lambda_{x_{t}^{(s)} t}^{( }+\bar{N}_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{-\lambda_{s}^{(s) t}}\right]}\right.  \tag{76}\\
& \tilde{x}_{i, j, k}(t)=\sum_{s=1}^{N_{s}} \sin \frac{i s \pi}{2 N_{x}}\left[\bar{N}_{0, j, k}^{(s)} C h \lambda_{s}^{(x)} t+\tilde{N}_{0, j, k}^{(s)} \operatorname{Sh} \lambda_{s}^{(x)} t\right]
\end{align*}
$$

Now, the law of small free creep oscilations of material particles in space material creep net, depending on boundary chain conditions of the ends, are:

$$
\begin{align*}
& x_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2} t} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2} t} \sum_{s=1}^{N_{s}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{ \pm x_{i}^{(x)} t} \\
& x_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2} t} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2} t} \sum_{s=1}^{N_{s}} N_{0, j, k}^{(s)} \sin \frac{i(2 s-1) \pi}{2\left(N_{x}+1\right)} e^{ \pm\left(k_{i}^{(t) t} t\right.}  \tag{77}\\
& x_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2} t} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}} \sum_{s=1}^{N_{s}} N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2\left(N_{x}+1\right)} e^{ \pm \lambda_{i}^{(t)} t}
\end{align*}
$$

For example, for the special case of the boundary conditions, when homogeneous chain-like systems, are free on both ends, we obtain the solution in the following form:

$$
\begin{align*}
x_{i, j, k}(t) & =e^{-\frac{\omega_{1}^{2}}{2} t} \tilde{x}_{i, j, k}(t)=e^{-\frac{\omega_{1}^{2}}{2}}, \sum_{s=1}^{N_{x}}\left[N_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{+\lambda_{s}^{(x)} t}+\bar{N}_{0, j, k}^{(s)} \sin \frac{i s \pi}{2 N_{x}} e^{-\lambda_{s}^{(x)} t}\right]= \\
& =e^{-\frac{\omega_{1}^{2}}{2} t} \sum_{s=1}^{N_{x}} \sin \frac{i s \pi}{2 N_{x}}\left[\bar{N}_{0, j, k}^{(s)} C h \lambda_{s}^{(x)} t+\tilde{N}_{0, j, k}^{(s)} \operatorname{Sh} \lambda_{s}^{(x)} t\right] \tag{78}
\end{align*}
$$

III* For the case that $\alpha \in(0,1)$ previous subsystem (50) of differential equations with fractional order derivatives, take the following form:

$$
\begin{equation*}
\ddot{x}_{i, j, k}(t)+\omega_{\alpha}^{2}\left[-x_{i-1, j, k}^{(\alpha)}(t)+2 x_{i, j, k}^{(\alpha)}(t)-x_{i+1, j, k}^{(\alpha)}(t)\right]+\omega_{0}^{2}\left[-x_{i-1, j, k}(t)+2 x_{i, j, k}(t)-x_{i+1, j, k}(t)\right]=0 . \tag{79}
\end{equation*}
$$

We solve previous subsystem (79) through the use of Laplace's transformations. After conducting Laplace's transformations of the previous systems (79) of differential equations with fractional order derivative and having in mind that we introduced notations $L\left\{x_{i, j, k}(t)\right\}$ for Laplace's transformations, as well as that it is:

$$
\begin{equation*}
工\left\{\frac{d^{\alpha} x_{i, j, k}(t)}{d t^{\alpha}}\right\}=p^{\alpha} L\left\{x_{i, j, k}(t)\right\}-\left.\frac{d^{\alpha-1} x_{i, k, k}(t)}{d t^{\alpha-1}}\right|_{t=0}=p^{\alpha} L\left\{x_{i, j, k}(t)\right\} \tag{80}
\end{equation*}
$$

and also having in mind, that we accepted the hypothesis that the initial conditions of fractional order derivatives of the system are given through the use of: $\left.\frac{d^{a-1} x_{i, j, k}(t)}{d t^{\alpha-1}}\right|_{t=0}=0$ as well that is

$$
\begin{equation*}
L\left\{\frac{d^{2} x_{i, j, k}(t)}{d t^{2}}\right\}=p^{2} L\left\{x_{i, j, k}(t)\right\}-\left[p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right], \tag{81}
\end{equation*}
$$

where $x_{0 i, j, k}$ and $\dot{x}_{0 i, j, k}$ initial conditions of system material particles dynamics, we can write the following system equations with unknown Laplace's transforms:

$$
\begin{equation*}
p^{2} L\left\{x_{i, j, k}(t)\right\}+\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)\left\langle-L\left\{x_{i-1, j, k}(t)\right\}+2 L\left\{x_{i, j, k}(t)\right\}-L\left\{x_{i+1, j, k}(t)\right\}\right)=\left[p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right], \tag{82}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{p^{2}}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right.} L\left\{x_{i j, k}(t)\right\}+\left\langle-L\left\{x_{i-1, j, k}(t)\right\}+2 L\left\{x_{i, j, k}(t)\right\}-L\left\{x_{i+1, j, k}(t)\right\}\right\rangle=\frac{\left\lfloor p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right\rfloor}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)} . \tag{82*}
\end{equation*}
$$

By introduce the following notation

$$
\begin{equation*}
v=\frac{p^{2}}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)} \tag{83}
\end{equation*}
$$

previous system (82*) equations with unknown Laplace's transforms, we can write in following forms:

$$
\begin{equation*}
-L\left\{x_{i-1, j, k}(t)\right\}+(2+v) L\left\{x_{i, j, k}(t)\right\}-L\left\{x_{i+1, j, k}(t)\right\}=\frac{\left\lfloor p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right\rfloor}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)} . \tag{84}
\end{equation*}
$$

Determinant of the previous system is:

$$
\Delta_{(0, j, j k}^{(x)}=\left|\begin{array}{cccc}
2+v & -1 & &  \tag{85}\\
-1 & 2+v & & \\
& & 2+v & -1 \\
& & -1 & 2+v
\end{array}\right| \neq 0
$$

By untroducing the following notation:

$$
\begin{equation*}
h^{(x)}(p)=\frac{\left\lfloor p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right\rfloor}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)}, \tag{86}
\end{equation*}
$$

for the determinants $\Delta_{(x), j, k}$ we can write:

$$
\Delta_{(11), j, k}^{(x)}=\left|\begin{array}{cccc}
h(p) & -1 & &  \tag{87}\\
0 & 2+v & & \\
& & 2+v & -1 \\
& & -1 & 2+v
\end{array}\right| ; \Delta_{(2), j, j, k}^{(x)}=\left|\begin{array}{cccc}
2+v & h(p) & \\
-1 & 0 & & \\
& & 2+v & -1 \\
& & -1 & 2+v
\end{array}\right| \neq 0
$$

or

$$
\Delta_{(3)] j \mathrm{j}}^{(x)}=\left|\begin{array}{ccccccc}
2+v & -1 & h(p) & & & &  \tag{88}\\
-1 & 2+v & 0 & & & & \\
& -1 & 0 & -1 & & & \\
& & 0 & 2+v & -1 & & \\
& & & -1 & 2+v & -1 & \\
& & & & -1 & 2+v & -1 \\
& & & & & -1 & 2+v
\end{array}\right| \ldots
$$

By solving the following equations (84), we obtain particular solutions, the linear combination of which can be used for solving the given problems:

$$
\begin{equation*}
-\mathrm{L}\left\{x_{i-1, j, k}(t)\right\}+(2+v) \mathrm{L}\left\{x_{i, j, k}(t)\right\}-\mathrm{L}\left\{x_{i+1, j, k}(t)\right\}=\frac{\left\lfloor p x_{0 i, j, k}+\dot{x}_{0 i, j, k}\right\rfloor}{\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)} \tag{89}
\end{equation*}
$$

Let first analyze the solution and characteristic equation of the homogeneous system (84):

$$
\begin{equation*}
-L\left\{x_{i-1, j, k}(t)\right\}+(2+u) L\left\{x_{i, j, k}(t)\right\}-L\left\{x_{i+1, j, k}(t)\right\}=0 . \tag{90}
\end{equation*}
$$

Solution of such a system of homogeneous equations (90) from which we obtain a series of determinants, can be obtained through the use of the trigonometric
method or by obtaining recurent formulas. Let us make use of the trigonometric method, and for that reason we suppose solutions in the following forms:

$$
\begin{equation*}
L\left\{x_{i j, k}(t)\right\}=N_{i, j, k} \sin i \varphi, \tag{91}
\end{equation*}
$$

and applying into previous system (90) we have: $u=2(\cos \varphi-1)$ as well as the following characteristic equations:

$$
\begin{equation*}
p^{2}+2\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)\left(1-\cos \varphi_{s}\right)=0, \quad s=1,2,3, \ldots, n \tag{92}
\end{equation*}
$$

where $\varphi_{s}$ depends on the boundary conditions on the ends of the corresponding chains.
Based to the (92) the characteristic determinant of the system can be written in the following form:

$$
\begin{equation*}
\Delta_{n i, j, k}^{(x)}=\prod_{s=1}^{s=n}\left[p^{2}+2\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)\left(1-\cos \varphi_{s}\right)\right] \neq 0, \tag{93}
\end{equation*}
$$

from which we obtain a series of determinants, when we replace one of the columns with a column of free members on the right side of the system (89). Based on that, for given initial conditions for one of particle coordinates of material particles in chain, we can obtain the following determinants, corresponding to a certain column (and to an unknown Laplace's transformation of the coordinate):

$$
\begin{equation*}
\Delta_{(1 \chi, j, k}^{(x)}=h(p) \Delta_{(n-1), j, k}^{(x)} ; \quad \Delta_{(2 \chi, j, k}^{(x)}=h(p) \Delta_{(n-2) \mid, j, k}^{(x)} ; \quad \Delta_{(m, x, j, k}^{(x)}=h(p) \Delta_{(n-m), j, k, k}^{(x)} ; \ldots \ldots \ldots \tag{94}
\end{equation*}
$$

Based on these discoveries we can deduce the following expressions for the unknown Laplace's transformations of the desired coordinate of the material particle:

$$
\begin{array}{r}
L\left\{x_{i, j, k}(t)\right\}=\frac{p x_{0 i, j, k}+\dot{x}_{0 i, j, k}}{\cdot \omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}} \frac{1}{\prod_{s=n-(i-1)}^{s=n}\left[p^{2}+2\left(\omega_{\alpha}^{2} p^{\alpha}+\omega_{0}^{2}\right)\left(1-\cos \varphi_{s}\right)\right]}  \tag{95}\\
i=1,2,3, \ldots, n
\end{array}
$$

The previous solution can be written in the following form.

$$
\begin{equation*}
I\left\{x_{i, j, k}(t)\right\}=\frac{1}{\omega_{0}^{2}}\left(x_{0 i, j, k}+\frac{\dot{x}_{0 i, j, k}}{p}\right) \frac{1}{p}\left(\frac{1}{1+p^{\alpha} \frac{\omega_{\alpha}^{2}}{\omega_{0}^{2}}}\right) \frac{1}{\prod_{s=n-(i-1)}^{s=n}\left[1+2 \frac{\omega_{\alpha}^{2}}{p^{2}}\left(1-\cos \varphi_{s}\right)\left(p^{\alpha}+\frac{\omega_{0}^{2}}{\omega_{\varepsilon}^{2}}\right)\right]}, i=1,2,3, \ldots, n, \tag{96}
\end{equation*}
$$

After developing the binoms into series previous particular solution (96) takes the following form:

$$
\Sigma\left\{x_{i, j, k}(t)\right\}=\frac{1}{\omega_{0}^{2}}\left(x_{0 i, j, k}+\frac{\dot{x}_{0 i, j, k}}{p}\right) \frac{1}{p} \sum_{m=1}^{m=\infty} \frac{(-1)^{m} \omega_{0}^{2 m} p^{\omega n}}{\omega_{\alpha}^{2 m}} \prod_{s=n-(i-1)}^{s=n} \sum_{h=1}^{n=\infty} \frac{(-2)^{n} \omega_{\alpha}^{2 n}\left(1-\cos \varphi_{s}\right)^{n}}{p^{2 h}}\left(p^{\alpha}+\frac{\omega_{0}^{2}}{\omega_{\varepsilon}^{2}}\right)^{n},
$$

or:

$$
\begin{gather*}
L\left\{x_{i, k}(t)\right\}=\frac{1}{\omega_{0}^{2}}\left(x_{0 i, j, k}+\frac{\dot{x}_{0 i, j, k}}{p}\right) \frac{1}{p} \sum_{m=1}^{m=\infty} \frac{(-1)^{m} \omega^{2 m} p^{\alpha \pi n}}{\omega_{\alpha}^{2 m}} \prod_{s=n-(i-1)}^{s=n} \sum_{h=1}^{h=\infty} \frac{(-2)^{h} \omega_{\alpha}^{2 n}\left(1-\cos \varphi_{s}\right)^{n}}{p^{2 h}} \sum_{r=1}^{r=n}\binom{h}{r} \frac{p^{\alpha r} \omega_{0}^{2(h-r)}}{\omega_{\alpha}^{2(h-r)}} \\
i=1,2,3, \ldots, n \tag{98}
\end{gather*}
$$

And, now, finally the previous particular solutions can be written in the following forms:

$$
\begin{align*}
L\left\{x_{i, j, k}(t)\right\} & =x_{0 i, j, k} \sum_{m=1}^{m=\infty}(-1)^{m} \prod_{s=n-(i-1)}^{s=n} \sum_{h=1}^{h=\infty}(-2)^{h}\left(1-\cos \varphi_{s}\right) \sum_{r=1}^{h} r=\binom{h}{r} \frac{\omega_{0}^{2(h-r+m-1)}}{\omega_{\alpha}^{2(m-r)}} \frac{1}{p^{2 h-\alpha(m+r)+1}}+ \\
& +\dot{x}_{0 i, j, k} \sum_{m=1}^{m=\infty}(-1)^{m} \prod_{s=n-(i-1)}^{s=n} \sum_{h=1}^{h=\infty}(-2)^{h}\left(1-\cos \varphi_{s}\right) \sum_{r=1}^{h=h}\binom{h}{r} \frac{\omega_{0}^{2(h-r+m-1)}}{\omega_{\alpha}^{2(m-r)}} \frac{1}{p^{2 h-\alpha(m+r)+2}} \tag{99}
\end{align*}
$$

The last form of the Laplace's transform (99) of solution is suitable for determining the inverse Laplace's transofrmation, which is, at the same time, one of solutions of the observed problem.

When initial conditions for all material particles are different form 0 , it is necessary to create linear combinations of the obtained particular solution, and that should be too difficult. The solution itself is not difficult to determine by finding an inverse transformation for a certain Laplace transformation.

The problem of studying the stability of dynamics of discrete continuum with creep constraints is a separate problem which requires separate studying.

## 5. Concluding Remarks

This paper presented an analytical method for the study of discrete homogeneous continuum dynamics that uses the idea of trigonometric method form the reference [22], where it is applied to dynamics of homogeneous elastic oscillatory chains. The paper showed that such analytical method of discrete continuums is efficient when applied to the study of dynamics of discrete homogeneous systems with:
a) ideally elastic constraint elements;
b) standard hereditary rheological constraint elements, which are characterized by the existance of a relaxation kernel;
c) ideal creep elements whose stress-strain relation can be expressed by a derivative of fractional order.

The paper also has a more general character since it opens space for application on discrete composite continuum. Models of such composite continuum can be obtained by combining of homogeneous material chains of different material properties (elastic, hereditary or creep) into a net. That would be the subject of other studies, both analytical and corresponding numerical.

In this paper we focused only on the study of motion of material particles of discrete continuum under the conditions of initial disturbances relayed to the continuum, but the method is also applicable to the study of dynamics of discrete homogeneous continuums under the effect of forced excitations on all or only on some of discrete chains. Such model of discrete continuum creates the basis for the creation of a model of dynamical advancement of a crack on the atomic level through a discrete continuum by breaking of corresponding constraints with elastic, hereditary or creep relations between material particles.

The length of this paper also restricted us to exploring only a small number of ideas allowed by this method.

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# ON THE PLASTIC SPIN CONTROVERSY IN POLYCRISTAL PLASTICITY 

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#### Abstract

The paper deals with some geometrical issues essential for constitutive modelling of plastic behaviour of metals. Geometric and kinematic aspects of intragranular as well as intergranular plastic deformation of polycrystals are discussed. A special emphasis is given to noneuclidean interpretation of inelastic deformation of monocrystals. Constitutive equation for elastic strain is covered by the effective field homogenization method inside a representative volume element. Evolution equations for plastic stretching and plastic spin are shown to be mutually dependent. This is very important for reducing number of material scalar functions during experimental calibration of evolution equations.


## 1. Introduction

The principal objective of this work is to present a rational approach to inelasticity of polycrystalline materials in a simplified (yet safely grounded on a correct differential geometry of deformation) way which should serve primarily to fit multiaxial experimental results on austenitic steels like AISI 316H having face centered cubic lattice (cf. [14, 16]). For this sake it is essential to reduce the number of material constants to be found from the available experiments. In other words, the general desire is to make always evolution equations with minimal number of material constants even if these equations originate form very general functionals like in [15]. The evolution equations comprise of plastic stretching(often named by experimentalists as plastic strain rate tensor) as well as plastic spin. Some authors claim that this spin has a triggering role for localization behavior while some others like Aifantis and Dafalias [2] require independence of these two evolution equations which greatly complicates identification problem. We are going to discuss this issue in the sequel.

## 2. Geometric preliminaries

2.1. Some generalities for polycrystals. As a prerequisite, a correct geometric description of an inelastic deformation process analyzed is necessary. Consider a polycrystalline body in a real configuration $(k)$ with dislocations and an
inhomogeneous temperature field $T(X, t)$ (where $t$ stands for time and $X$ for the considered particle of the body) subject to surface tractions. Corresponding to ( $k$ ) there exists, usually, an initial reference configuration $(K) \equiv\left(k\left(t_{0}\right)\right)$ with (differently distributed) dislocations at a homogeneous temperature $T_{0}$ without surface tractions. Due to these defects such a configuration is not stress-free but contains an equilibrated residual stress (often named as "back-stress"). It is generally accepted that linear mapping function $\mathrm{F}(., t):(K) \rightarrow(k)$ is compatible second rank total deformation gradient tensor. Here time $t$ as scalar parameter allows for family of deformed configurations $(k)$. In the papers dealing with continuum representations of dislocation distributions configuration $(k)$ is imagined to be cut into small elements denoted by $(n)$, these being subsequently brought to the temperature of $(K)$ free of neighbors. The deformation tensor $\mathrm{F}_{E}(., t):(n) \rightarrow(k)$ obtained in such a way is incompatible and should be called the thermoelastic distortion tensor whereas ( $n$ )-elements are commonly named as natural state local reference configuTations (cf. for instance $[8,12,4]$ ). Of course, the corresponding plastic distortion tensor

$$
\begin{equation*}
\mathrm{F}_{P}(., t):=\mathrm{F}_{E}(., t)^{-1} \mathrm{~F}(., t) \mathrm{F}_{E}\left(., t_{0}\right), r \tag{I}
\end{equation*}
$$

is not compatible, whereas F is found by comparison of material fibres in $(K)$ and $(k)$ while $\mathrm{F}_{E}$ is determined by crystallographic vectors in $(n)$ and $(k)$. Multiplying above formula from the left and right hand side by $\mathrm{F}_{E}(., t)$ and $\mathrm{F}_{E}\left(., t_{0}\right)^{-1}$ we reach at Kröner's decomposition rule which is often wrongly named as Lee's decomposition formula. In fact, the formula is slightly modified in [19] to account for the initial value of $\mathrm{F}_{E}(., t)$. It is worthy of note that curl $\mathrm{F}(., t)^{-1} \neq \mathrm{O}$ and this incompatibility is commonly connected to an asymmetric second order tensor of dislocation density.

Let us imagine that a typical ( $n$ )-element (called in the sequel representative volume elementand denoted by $R V E$ ) is composed of $N$ monocrystal grains, such that each $\Lambda$-th grain has $N_{s}$ slip systems $\mathrm{A}_{\alpha \Lambda} \equiv \mathrm{s}_{\alpha \Lambda} \otimes \mathrm{n}_{\alpha \Lambda}, \alpha \in\left\{1, N_{s}\right\}$. For instance, for FCC crystals $N_{s}=12$. Here $\mathrm{s}_{\alpha \Lambda}$ is the unit slip vector and $\mathrm{n}_{\alpha \Lambda}$ is the unit vector normal to the slip plane. For convenience, let us introduce a third unit vector $\mathbf{z}_{\alpha \Lambda}$ normal to the considered slip plane (cf. [1]) with dyads $\mathbf{A}_{\alpha \Lambda}^{1} \equiv \mathbf{n}_{\alpha \Lambda} \otimes \mathbf{z}_{\alpha \Lambda}$ and $\mathrm{A}_{\alpha \Lambda}^{2} \equiv \mathrm{z}_{\alpha \Lambda} \otimes \mathrm{S}_{\alpha \Lambda}$ useful when either cross-slip or climb of dislocations has to be taken into account.

By comparing a $R V E$ in $(n(t))$ and $\left(n\left(t_{0}\right)\right)$ we may write a formula analogous to (1) for the microplastic distortion tensor

$$
\Pi_{\Lambda}:=\Pi_{\Lambda E} \Pi_{\Lambda P}
$$

whose components are the residual microelastic distortion tensor $\Pi_{\Lambda E}$ and microplastic distortion tensor $\Pi_{\Lambda P}$. By means of the polar decomposition $\Pi_{\Lambda E}=$ $\mathbf{R}_{\Lambda} \mathbf{U}_{\Lambda E}$ (with microrotation $\mathbf{R}_{\Lambda}^{T} \mathbf{R}_{\Lambda}=1$ ) and assumed isoclinicity of $\mathbf{A}_{\alpha \Lambda}$ in $(n(t))$ and $\left(n\left(t_{0}\right)\right)$ we may finally write

$$
\mathbf{U}_{\Lambda E}=\operatorname{diag}\left(1+\lambda_{k \Lambda}\right), k \in\{1,2,3\} \text { as well as } \Pi_{\Lambda P}=1+\sum_{\alpha} \gamma_{\alpha \Lambda} \mathbf{A}_{\alpha \Lambda}
$$

If a $R V E$ has the volume $\Delta V=\sum_{\Lambda} \Delta V_{\Lambda}$ and the microplastic deformation tensors for individual grains are

$$
\mathbf{C}_{\Pi \Delta}=\Pi_{\Lambda P}^{T} \mathbf{U}_{\Lambda E}^{2} \Pi_{\Lambda P} \equiv\left[1+\sum_{\alpha} \gamma_{\alpha \Lambda} \mathbf{A}_{\alpha \Lambda}^{T}\right] \mathrm{U}_{\Lambda E}^{2}\left[1+\sum_{\beta} \gamma_{\beta \Lambda} \mathbf{A}_{\beta \Lambda}\right] .
$$

then their volume average named macroplastic deformation tensor $\mathrm{C}_{P}:=\mathrm{F}_{P}^{T} \mathrm{~F}_{P}$ has the following form:

$$
\begin{equation*}
\mathbf{C}_{P}=\left\langle\mathbf{C}_{\Pi \Delta}\right\rangle=\left\langle\Pi_{\Lambda}^{\mathrm{T}} \Pi_{\Lambda}\right\rangle \equiv \frac{1}{\Delta V} \sum_{\Lambda} \Pi_{\Lambda}^{\mathrm{T}} \Pi_{\Lambda} \Delta V_{\Lambda}, \tag{2}
\end{equation*}
$$

Of course, macroelastic strain tensor $\mathbf{E}_{E}=\left(\mathbf{F}_{E}^{T} \mathrm{~F}_{E}-1\right) / 2 \equiv\left(\mathrm{C}_{E}-1\right) / 2$ will be used in the sequel as well.

Moreover, in the corresponding polar decomposition $\mathbf{F}_{P}=\mathbf{R}_{P} \mathbf{U}_{P}$ the macroplastic rotation tensor $\mathbf{R}_{P}$ is arbitrary [22] and might be fixed either to be a unit tensor or to have the Mandel's isoclinicity property (cf. [15] for details). For a definition of isoclinicity we should have to find average crystal directions in $R V E(t)$ and $R V E\left(t_{0}\right)$ and to make them equal. The first choice i.e., $\mathrm{R}_{P}=1$ seems more appropriate for polycrystals and we are going to accept it in the following. Therefore, the relationship

$$
\begin{equation*}
\mathrm{F}_{P}=\mathrm{U}_{P}=\mathrm{C}_{P}^{1 / 2} \tag{3}
\end{equation*}
$$

will greatly simplify macroplastic spin issue.
The above introduced microisoclinicity of grains permits the exact relationship for material time rate of microplastic distortion

$$
D \Pi_{\Lambda P}=\sum_{\alpha} \mathbf{A}_{\alpha \Lambda} D \gamma_{\alpha \Lambda}
$$

such that relationships for microplastic stretching and microplastic spin tensors read:

$$
\mathbf{D}_{\Pi \Delta}=\mathbf{R}_{\Lambda}^{T}\left(\mathbf{D}_{\Lambda P}+D \log \mathbf{U}_{\Lambda E}\right) \mathbf{R}_{\Lambda}, \quad \mathbf{W}_{\Pi \Delta}=\mathbf{R}_{\Lambda}^{T} \mathbf{W}_{\Lambda P} \mathrm{R}_{\Lambda}+\mathrm{W}_{\Lambda E},
$$

with $D \log \mathbf{U}_{\Lambda E}=\operatorname{diag}\left\{D \lambda_{k \Lambda}\left(1+\lambda_{k \Lambda}\right)^{-1}\right\}$ and following notations

$$
2 \mathbf{D}_{\Lambda P}=D \Pi_{\Lambda P} \Pi_{\Lambda P}^{-1}+\Pi_{\Lambda P}^{-T} D \Pi_{\Lambda P}^{T}, \quad 2 \mathbf{W}_{\Lambda P}=D \Pi_{\Lambda P} \Pi_{\Lambda P}^{-1}-\Pi_{\Lambda P}^{-T} D \Pi_{\Lambda P}^{T}
$$

The corresponding macroplastic stretching and macroplastic spin tensors follow now directly from (3) in the following form:

$$
\begin{equation*}
2 \mathbf{D}_{P}=D \mathrm{U}_{P} \mathrm{U}_{P}^{-1}+\mathrm{U}_{P}^{-1} D \mathrm{U}_{P}, \quad 2 \mathbf{W}_{P}=D \mathrm{U}_{P} \mathrm{U}_{P}^{-1}-\mathrm{U}_{P}^{-1} D \mathrm{U}_{P} . \tag{4}
\end{equation*}
$$

It is worthy of note that such a representation considerably reduces number of material constants if tensor representation based evolution equations for $D_{P}$ and $\mathrm{W}_{P}$ are chosen. Connection of the macroplastic stretching with 2 by means of $2 \mathbf{D}_{P}=\mathrm{U}_{P}^{-1} D \mathbf{C}_{P} \mathbf{U}_{P}^{-1}$ and is straightforward and is obtained from:
$D \mathbf{C}_{\Pi \Delta}=\left(\sum_{\alpha} D \gamma_{\alpha \Lambda} \mathbf{A}_{\alpha \Lambda}^{T}\right) \mathbf{U}_{\Lambda E}^{2} \Pi_{\Lambda P}+\Pi_{\Lambda P}^{T} \mathrm{U}_{\Lambda E}^{2} \sum_{\beta} D \gamma_{\beta \Lambda} \mathbf{A}_{\beta \Lambda}+\Pi_{\Lambda P}^{T} D \mathbf{U}_{\Lambda E}^{2} \Pi_{\Lambda P}$ by the averaging $D \mathrm{C}_{P}=\left\langle D \mathrm{C}_{\Pi \Delta}\right\rangle$.
2.2. Intergrain transition of slips and rotations. At the end of this section let us consider small microplastic homogeneous deformations $\delta \Pi_{\Gamma}$ and $\delta \Pi_{\Delta}$ (and rotations $\delta \mathrm{R}_{\Gamma}$ and $\delta \mathrm{R}_{\Delta}$ ) of two arbitrarily chosen but tightly connected adjacent $\Gamma$ and $\Delta$ grains inside a $R V E$. Requiring continuity across their (presumably plane) joint boundary with unit normal $\mathrm{n}_{\Gamma \Delta}$ it is possible to derive that the most general difference between them must have the form

$$
\begin{equation*}
\delta \Pi_{\Gamma}-\delta \Pi_{\Delta}=\mathrm{b}_{\Gamma \Delta} \otimes \mathrm{n}_{\Gamma \Delta} \tag{5}
\end{equation*}
$$

wherein the vector $\mathrm{b}_{\Gamma \Delta}$ is arbitrary. Since this vector has three arbitrary components the above restriction gives six scalar equations. Thus, if $\Pi_{\Gamma}, \Pi_{\Delta}, \delta \Pi_{\Gamma}$, $\delta \lambda_{k \Gamma}$ and $\delta \Pi_{\Delta}=\delta\left\{\mathbf{R}_{\Delta} \operatorname{diag}\left(1+\lambda_{k \Delta}\right) \Pi_{\Delta P}\right\}$ are known then the three Euler angles increments constituting $\delta \mathrm{R}_{\Gamma}$ as well as three principal residual microelastic stretches increments constituting $\delta \operatorname{diag}\left(1+\lambda_{k \Delta}\right)$ can be found from the above equation. Therefore, we may conclude that for tight plane boundaries among the grains inside a. RVE the following relationship

$$
\mathbf{R}_{\Delta}=\mathbf{R}_{\Delta}\left(\left\{\lambda_{k \Gamma}\right\},\left\{\Pi_{\Gamma P}\right\}\right), \text { for all } \Gamma \in\{1, N\} .
$$

holds. More generally, knowing all microplastic distortions $\Pi_{\Lambda P}, \Lambda \in\{1, N\}$ for each grain we have six unknowns, namely, the mentioned three Euler angles as well as three principal residual microelastic stretches. Finally, suppose that our $R V E$ has originally at $t=t_{0}$ shape of a cube consisting of $N=n^{3}$ octahedral sub-cubes. Then total number of intergrain boundaries inside this $R V E$ amounts to $3 n^{2}(n-1)$ and number of all (5)-equations equals to $18 n^{2}(n-1)$. Since the total number of Euler angles and residual microelastic stretches of all the grains constituting the considered $R V E$ is $6 n^{3}$ we see that for $n>2$ the number of available equations is always larger than the number of unknowns.
2.3. A note on strain measures and defect distribution. In the recent literature there is a vast number of diverse strain and stress measures. Aside of generally a.ccepted (Lagrangian and Eulerian) strain tensors, it is possible to introduce following Hill [6] generalized strain measures by

$$
\begin{equation*}
\varepsilon=\sum_{a} \frac{\lambda_{a}^{2 m}-1}{2 m} \nu^{a} \otimes \nu^{a} \tag{6}
\end{equation*}
$$

where $\lambda_{a}(a \in 1,2,3)$ are principal extension ratios (along principal directions) and $\nu_{0}^{a}$ are Lagrangian principal direction unit vectors appearing in the right stretch tensor $\mathbf{U}$ when the polar decomposition $\mathbf{F}=\mathbf{R U}$ is applied. For $m=1$ we get La.grangian strain tensor $2 \varepsilon=\mathrm{F}^{T} \mathrm{~F}-\mathbf{1} \equiv \mathbf{U}^{2}-\mathbf{1}, m=-1$ corresponds to Almansi strain tensor $2 \varepsilon=1-\mathrm{F}^{-1} \mathrm{~F}^{-T} \equiv \mathbf{1}-\mathrm{U}^{-2}$ whereas for $m=0 \Rightarrow \varepsilon=\ln \mathbf{U}$ and $m=1 / 2 \Rightarrow \varepsilon=\mathbf{U}-1$ Hill's logarithmic and Biot's strain tensors are acquired respectively.

All these tensors vanish in the absence of stretches i.e., when $\lambda_{1}=\lambda_{2}=$ $\lambda_{3}=1$. The similar formulae are available for elastic and plastic tensors as well.

However, only one property will be stressed here. By means of the relations

$$
\varepsilon_{E}=\sum_{a} \frac{\lambda_{E a}^{2 m}-1}{2 m} \nu_{E}^{a} \otimes \nu_{E}^{a}, \quad \varepsilon_{P}=\sum_{a} \frac{\lambda_{P a}^{2 m}-1}{2 m} \nu_{P}^{a} \otimes \nu_{P}^{a},
$$

following from (6) it is possible to show that

$$
\lambda_{a} \neq \lambda_{E_{a}} \lambda_{P a}
$$

since proper directions of total, elastic and plastic deformations do not correspond to the same material fibres (curves along same material points).

Let us restrict our attention to monocrystals for the moment. As a measure of imperfections of a crystal body with dislocations Burgers vector is used whose surface density is in fact dislocation density tensor (cf. eg. [12, 13]). This vector is defined in a customary way according to Frank definition. Namely, if a closed contour in ( $k$ )-configuration is mapped onto the natural state element $(n)$, then as a result the contour will not be closed any more. The opening i.e. the closure error is in fact the mentioned Burgers vector

$$
d \mathbf{b}_{n}=\mathbf{B} d \mathbf{s}_{n}
$$

where $d \mathbf{s}_{n}$ is an infinitesimal surface vector in $(n)$-configuration and $\mathbf{B}=\mathcal{E}: \mathcal{B}$ is the second rank dislocation density tensor formed by means of double inner product of Ricci tensor $\mathcal{E}$ with the third rank antisymmetric dislocation density tensor

$$
\mathcal{B}=\mathbf{F}_{E}^{-T}\left(\operatorname{curl} \mathbf{F}_{E}\right) \mathbf{F}_{E}^{-1}
$$

where the curl differential operator is antisymmetrized gradient of the considered tensor. In other words, for any vector a such a differential operation reads curla $\equiv$ $\operatorname{grad} \mathbf{a}-(\operatorname{grad} \mathbf{a})^{T}$. The natural state elements as broken pieces might be collected into a continuous global stress-free body only in some non-Euclidean space.

- The torsion tensor of such a space is equal to the above mentioned third rank dislocation density tensor (cf. also [12,21]) if a distant parallelism space is chosen as a representative space.
- As another possibility, introducing anholonomic coordinates in each of $(n)$-elements, calculating their anholonomic object and setting the corresponding torsion tensor to be equal to zero, we get the corresponding Riemann-Christoffel tensor different from zero. Such a situation means that compatibility conditions put on the tensor $\mathbf{F}_{E}^{T} \mathbf{F}_{E}$ are not satisfied.
- The third possibility is to introduce a non-metric connection i.e., to take that covariant derivative of metric tensor $\mathbf{F}_{E}^{T} \mathbf{F}_{E}$ in a global space of collected $(n)$-elements does not vanish.
Details of these three geometric descriptions are given in [11].


## 3. Evolution and constitutive equations

3.1. Hooke's law by homogenization approach. Let the microelastic strain of a $\Lambda$-grain inside a $R V E$ be denoted by $\mathbb{E}_{\Lambda E}$ such that the macroelastic Lagrangian strain $\mathbf{E}_{E}=\left(\mathbf{F}_{E}^{T} \mathbf{F}_{E}-\mathbf{1}\right) / 2$ is the volume average of them i.e.
$\mathrm{E}_{E}=\left\langle\mathrm{E}_{\Lambda E}\right\rangle$. It must be noted, however, that microelastic strain $\mathrm{E}_{\Lambda E}$ is different and much larger from residual microelastic strain $\left(\mathrm{U}_{\Lambda E}^{2}-1\right) / 2$.

If the microelastic strain $\mathrm{E}_{\Lambda E}$ is provoked by the corresponding microstress $\Sigma_{\Lambda}$, then its volume average reads $S=\left\langle\Sigma_{\Lambda}\right\rangle$ where the second Piola-Kirchhoff stress tensor $\mathrm{S}=\mathrm{F}_{E}^{-1} \mathrm{~T}_{E}^{-T}$ is calculated with respect to the local reference $(n)$ configuration. If Hooke's law for the $\Lambda$-grain has the form

$$
\Sigma_{\Lambda}=\mathcal{D}_{\Lambda}: \mathrm{E}_{\Lambda E}
$$

then its volume averaging throughout the $R V E$ gives the familiar equation of homogenization approach

$$
\left\langle\Sigma_{\Lambda}\right\rangle=\mathcal{D}_{\text {eff }}:\left\langle\mathrm{E}_{\Lambda E}\right\rangle \text {, i.e. } \quad \mathrm{S}=\mathcal{D}_{\text {eff }}: \mathrm{E}_{E}
$$

In [10] the author proposed the approach that for polycrystals the considered grain is understood as an inclusion in an infinite matrix composed by all the other grains. If instead of an infinite medium we employ this reasoning to the considered $R V E$ then a direct application of the Levin's expression for the effective elastic moduli fourth rank tensor may be written as follows (index $M$ stands for matrix while the notation $\langle\bullet\rangle_{\omega}$ means averaging by orientation only):

$$
\mathcal{D}_{e f f}=\mathcal{D}_{M}+[\mathcal{D}]\left(\mathcal{I}-\langle\mathcal{A} \mathcal{P}\rangle_{\omega}[\mathcal{D}]\right)^{-1}\langle\mathcal{A}\rangle_{\omega} \text {, where } \mathcal{D}_{M} \equiv\langle\mathcal{D}\rangle_{\omega}
$$

Here $\mathcal{I}_{a b c d}=\delta_{a c} \delta_{b d}+\delta_{a d} \delta_{b c}$ is the unit fourth rank tensor and

$$
\begin{aligned}
\mathcal{P}_{\Lambda}=\mathcal{S}_{\Lambda} \mathcal{D}_{M}^{-1} & \equiv-\int_{\Delta V_{\Lambda}} \mathcal{K}\left(x-x^{\prime}\right) d V^{\prime}, \text { with } \mathcal{K}_{a b c d}=\left(\partial_{a} \partial_{d} G_{a c}\right)_{(a b)}, \\
\mathcal{A}_{\Lambda} & =\left(\mathcal{I}+\mathcal{P}_{\Delta}[\mathcal{D}]\right)^{-1}, \text { with }[\mathcal{D}]=\mathcal{D}_{\Lambda}-\mathcal{D}_{M} .
\end{aligned}
$$

In the above $S_{\Lambda}$ is the Eshelby's tensor and $G$ is the Green's function for the considered anisotropic crystal. The above expressions may be used for an analytical determination of the effective elastic constants.
3.2. Evolution equation. According to the principle of inelastic memory introduced by the author in [15] the second Piola-Kirchhoff stress is given by a very general functional accounting for plastic strain as well as plastic strain rate history as follows:

$$
\mathrm{S}(t)=\mathcal{F}_{\tau=0}^{\infty}\left[\varepsilon_{P}(t-\tau), D \varepsilon_{P}(t-\tau)\right],
$$

where the plastic strain tensor could be for instance $\varepsilon_{P}=\mathbf{U}_{P}-1$. When this functional may be represented by a nonlinear function of plastic strain and plastic strain rate the plastic material is of differential type (cf. [15]). Solving such an equation in plastic strain rate we would obtain the following evolution equation

$$
\begin{equation*}
D \varepsilon_{P}=D \varepsilon_{P}\left(\mathrm{~S}, \varepsilon_{P}\right) \tag{7}
\end{equation*}
$$

in its standard form for isotropic materials. By means of tensor representation theory $[20,15]$ it can be explicitly written as follows (MacAuley bracket $\langle f\rangle=1$
if plastic deformation takes place and $\langle f\rangle=0$ inside each elastic range):

$$
\begin{align*}
D \mathbf{U}_{P}=\langle f\rangle & {\left[d_{1}(\chi) \mathbf{1}+d_{2}(\chi) \mathbf{U}_{P}+d_{3}(\chi) \mathbf{U}_{P}^{2}\right.}  \tag{8}\\
& +d_{4}(\chi) \mathbf{S}+d_{5}(\chi) \mathbf{S}^{2}+d_{6}(\chi)\left(\mathbf{S U}_{P}+\mathbf{U}_{P} \mathbf{S}\right) \\
& \left.+d_{7}(\chi)\left(\mathbf{S}^{2} \mathbf{U}_{P}+\mathbf{U}_{P} \mathbf{S}^{2}\right)+d_{8}(\chi)\left(\mathbf{S U}_{P}^{2}+\mathbf{U}_{P}^{2} \mathbf{S}\right)\right]
\end{align*}
$$

On the other hand due to antisymmetry of the plastic spin tensor, its most general representation obtained by means of a tensorial function $\mathbf{W}_{P}=W_{P}\left(\mathbf{S}, \varepsilon_{P}\right)$ similar to (7) $[20,15]$ reads:

$$
\begin{align*}
\mathbf{W}_{P}=\langle f\rangle & {\left[w_{1}(\chi)\left(\mathbf{S} \mathbf{U}_{P}-\mathbf{U}_{P} \mathbf{S}\right)+w_{2}(\chi)\left(\mathbf{S}^{2} \mathbf{U}_{P}-\mathbf{U}_{P} \mathbf{S}^{2}\right)\right.}  \tag{9}\\
& +w_{3}(\chi)\left(\mathbf{S U}_{P}^{2}-\mathbf{U}_{P}^{2} \mathbf{S}\right)+w_{4}(\chi)\left(\mathbf{U}_{P}^{2} \mathbf{S}^{2}-\mathbf{S}^{2} \mathbf{U}_{P}^{2}\right) \\
& \left.+w_{5}(\chi)\left(\mathbf{U}_{P}^{2} \mathbf{S} \mathbf{U}_{P}-\mathbf{U}_{P} \mathbf{S} \mathbf{U}_{P}^{2}\right)\right]
\end{align*}
$$

where the scalar coefficients in both evolution equations are functions of the set $\chi \equiv$ $\left\{\operatorname{tr} \mathbf{S}, \operatorname{tr} \mathbf{S}^{2}, \operatorname{tr} \mathbf{S}^{3}, \operatorname{tr} \mathrm{U}_{P}, \operatorname{tr} \mathbf{U}_{P}^{2}, \operatorname{tr} \mathrm{U}_{P}^{3}, \operatorname{tr} \mathrm{U}_{P} \mathrm{~S}, \operatorname{tr} \mathbf{U}_{P}^{2} \mathrm{~S}, \operatorname{tr} \mathrm{U}_{P} \mathrm{~S}^{2}, \operatorname{tr} \mathrm{U}_{P}^{2} \mathrm{~S}^{2}\right\} \equiv$ $\left\{1_{S}, 2_{S}, 3_{S}, 1_{U}, 2_{U}, 3_{U}, 1_{U S}, 2_{U S}, 3_{U S}, 4_{U S}\right\}$ of invariants. This issue will be discussed in more detail in the following subsection. Even if all the scalar coefficients $d_{1}, \ldots, d_{8}, w_{1}, \ldots, w_{5}$ are constants their determination from experiments is a very tedious work due to large number of them. However, if the macroplastic rotation is taken to be zero, then from $(4,8,9)$ the coefficients $w_{1}, \ldots, w_{5}$ depend on $d_{1}, \ldots, d_{8}$ as well as on the invariants from $\chi$ as follows. Suppose, for simplicity, that macroplastic deformation is isohoric i.e. that $\operatorname{det} \mathbf{U}_{P}=1$. Such an assumption is widely accepted if damage does not develop significantly during the considered process. Then, the Cayley-Hamilton theorem allows

$$
\mathbf{U}_{P}^{-1}=\mathrm{U}_{P}^{2}-1_{U} \mathrm{U}_{P}+\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) 1
$$

Inserting this formula into expressions of plastic stretching and plastic spin (4) and by making use of (8)-(9) after extensive but simple calculations yields the following very important restrictions on constitutive scalar functions for plastic spin tensor:

$$
\begin{align*}
& w_{1}=-1_{U} d_{4}-\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) d_{6}+d_{8}  \tag{10}\\
& w_{2}=-1_{U} d_{5}-\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) d_{7} \\
& w_{3}=d_{4}+\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) d_{8} \\
& w_{4}=d_{5}, \quad w_{5}=d_{6}+d_{7}+1_{U} d_{8} .
\end{align*}
$$

The same procedure allows the following relationship for constitutive scalar functions of plastic stretching:

$$
\begin{align*}
0 & =\frac{1}{2} 1_{U}^{2} d_{1}+1_{U} d_{2}+2_{U} d_{3}+\left[3_{U S}-1_{U} 1_{U S}+\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) 1_{S}\right] d_{4} \\
& +\left[4_{U S}-1_{U} 2_{U S}+\frac{1}{2}\left(1_{U}^{2}-2_{U}\right) 2_{S}\right] d_{5}+2 d_{6} 1_{U S}+d_{7} 2_{U S}+d_{8} 3_{U S} \tag{11}
\end{align*}
$$

which shows that only seven of the scalar functions entering into (8) are independent. In other words, the above relationship could be written as $d_{8}=d_{8}\left(d_{1}, \ldots, d_{7}\right)$. It should be noted that in the case when damage like creep or low cycle fatigue is important ingredient of the deformation process the restriction (11) does not hold any more whereas the equations (10) become more involved including as one of additional invariants $3_{U}$ as well.

## 4. Concluding remarks

The following general conclusions might be drawn from the above analysis:

- The analysis of geometric features of inelastic straining of polycrystals made possible considerable reduction of material constants to be determined from experiments.
- In the paper [4] the authors connected to the natural state elements magnetization vectors in such a way that they are isoclinic in $(n)$ and inhomogeneous in ( $k$ ) the inhomogeneity being responsible for magnetostrictive strains. Such an assumption is very much in accord with the above geometrical analysis.


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# SOME INTRINSIC FORMULAS FOR COMPLEX HYPERSURFACES OF COMPLEX SPACES FORMS 

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Abstract. We prove that the formulas (3.6) are valid for any complex hypersurfaces of a complex space form.

## 1. The object of the paper

Matsumoto [1] examined the intrinsic properties of minimal hypersurfaces in a flat space and showed that for most of them the second fundamental form can be expressed in terms of the curvature and Ricci tensors. On the other hand, it is well known [2], [4] than any complex hypersurface in a Kähler space is minimal and is itself Kähler manifold.

The object of this paper is to generalize the investigations of Matsumoto for the complex hypersurface of complex space forms. We did not find the expressions for the second fundamental forms. Instead, we find some formulas valid for any complex hypersurface of a complex space form.

## 2. Preliminaries

Let $(\tilde{M}, \tilde{g}, \tilde{J})$ be the Kähler manifold with the metric $\tilde{g}$, complex structure $\tilde{J}$, and $\operatorname{dim} \tilde{M}=2 n+2$. The complex hypersurface $M$ in $\tilde{M}$ is the submanifold in $\tilde{M}$ of real codimension 2 having the property that the complex structure $\tilde{J}$ leaves invariant the tangent space of $M$ at each point $p \in M$. The complex hypersurface is itself a Kähler manifold $(M, g, J)$, [2], [4], where $g$ is induced metric and $J$ is induced complex structure. With respect to the local coordinates, on $M$, we have

$$
g_{a b} J_{i}^{a} J_{j}^{b}=g_{i j}, \quad \nabla_{k} J_{i}^{h}=0
$$

where the indices $a, b, h, i, j, k, \ldots$ run over the range $1,2, \ldots, 2 n$ and $\nabla$ denote the operator of the covariant derivative with respect to the Levi-Civita connection. Denoting by $R_{i j k l}$ and $R_{i j}$ the components of the curvature and Ricci tensor

[^0]respectively, we have [3]
\[

$$
\begin{gather*}
R_{i j a b} J_{k}^{a} J_{l}^{a}=R_{i j k l} \\
J_{j}^{a} R_{a i}=-R_{a j} J_{i}^{a} \text { or } J_{j}^{a} R_{a}^{h}=R_{j}^{a} J_{a}^{h}  \tag{2.1}\\
J_{a}^{j} R^{a i}=-R^{a j} J_{a}^{i} \text { or } R^{a b} J_{a}^{i} J_{b}^{j}=R^{i j} \tag{2.2}
\end{gather*}
$$
\]

where $R^{i j}=R_{a b} g^{i a} g^{j b}$. Finally, if we put $F_{i j}=J_{i}^{a} g_{a j}$, we have $F_{i j}=-F_{j i}$.
The relations (2.1) and (2.2) can be generalized in the following way. Let us put

$$
\begin{aligned}
& {\underset{p}{i j}}^{R_{p-1}}{\underset{p-1}{ }}_{R_{i a}} R_{j b} g^{a b}, \quad \underset{1}{R_{i j}}=R_{i j}, \quad p=2,3, \ldots \\
& R_{p}^{i j}=\underset{p}{R_{a b} g^{i a} g^{j b}}
\end{aligned}
$$

Then, using (2.1) and (2.2), we easily find

$$
\begin{gather*}
\underset{p}{a}{\underset{p}{a j}}=-\underset{p}{R_{a j}} J_{i}^{a} \text { or } J_{j}^{a}{\underset{p}{a}}^{h}=\underset{p}{R_{j}}{ }^{a} J_{a}^{h}  \tag{2.3}\\
{\underset{p}{a j} J_{a}^{i}=-\underset{p}{R^{a i}} J_{a}^{j}}^{\text {or }} \underset{p}{R^{a b}} J_{a}^{i} J_{b}^{j}=\underset{p}{R^{i j}} \tag{2.4}
\end{gather*}
$$

Since $M$ is a complex hypersurface, the normal plane of $M$ is left invariant by the complex structure $\tilde{J}$ at each point $p \in M$. Thus, there exists, in each neighborhood $U$ of $p \in M$, two unit vector fields $N$ and $\tilde{J} N$, mutually orthogonal and normal to $M$. If $h$ and $k$ are the second fundamental forms corresponding to $N$ and $\bar{J} N$, and $h_{i j}$ and $k_{i j}$ are their components with respect to local coordinates, then [2], [4]

$$
\begin{array}{ll}
h_{i j}=h_{j i}, & k_{i j}=k_{j i}, \\
h_{i j}=k_{i a} J_{j}^{a} & k_{i j}=-h_{i a} J_{j}^{a}, \\
h_{i j}=-h_{a b} J_{i}^{a} J_{j}^{b}, & k_{i j}=-k_{a b} J_{i}^{a} J_{j}^{b} . \tag{2.7}
\end{array}
$$

Using (2.4) and (2.7), we find

$$
\begin{array}{ll}
h_{i j} g^{i j}=0 & k_{i j} g^{i j}=0, \\
h_{i j} R_{p}^{i j}=0 & k_{i j} R_{p}^{i j}=0, \tag{2.9}
\end{array}
$$

for each $p=1,2, \ldots$. The relations (2.8) show that any complex hypersurface of a Kähle manifold is minimal. In view of (2.6), we have

$$
\begin{equation*}
h_{i a} h_{j b} g^{a b}=k_{i a} k_{j b} g^{a b} \tag{2.10}
\end{equation*}
$$

Also, we shall use the notations

$$
\underset{p}{R}=\underset{p}{R_{a b}} g^{a b}, \quad p=1,2, \ldots ; \underset{1}{R}=R
$$

## 3. Main results

Now, let us suppose that $\tilde{M}$ is a complex space form of holomorphic sectional curvature $c$. Then the Gaus equation for the complex hypersurface has the form

$$
\begin{align*}
& \frac{c}{4}\left(g_{i m} g_{j l}-g_{i l} g_{j m}+F_{i m} F_{j l}-F_{i l} F_{j m}-2 F_{i j} F_{l m}\right)  \tag{3.1}\\
&=R_{i j l m}-\left(h_{i m} h_{j l}-h_{i l} h_{j m}+k_{i m} k_{j l}-k_{i l} k_{j m}\right) .
\end{align*}
$$

Transvecting (3.1) with $g^{i m}$ and using (2.6), (2.7), (2.8) and (2.10), we find

$$
\begin{equation*}
R_{j l}=\frac{(n+1) c}{2} g_{j l}-2 h_{a l} h_{b j} g^{a b} \tag{3.2}
\end{equation*}
$$

Thus $h_{s k} R^{s}{ }_{l}-h_{s l} R^{s}{ }_{k}=0$ and therefore

$$
\begin{aligned}
& h_{i j} h_{s k} R^{s}{ }_{l}=h_{i j} h_{s l} R^{s}{ }_{k}, \\
& h_{i l} h_{s k} R^{s}{ }_{j}=h_{i l} h_{s j} R^{s}{ }_{k},
\end{aligned}
$$

because of which we have

$$
\begin{equation*}
h_{i j} h_{s k} R^{s}{ }_{l}-h_{i l} h_{s k} R^{s}{ }_{j}=\left(h_{i j} h_{s l}-h_{i l} h_{s j}\right) R^{s}{ }_{k} . \tag{3.3}
\end{equation*}
$$

Putting into (3.3) $a$ instead $j$ and $b$ instead $l$, transvecting with $J_{j}^{a} J_{l}^{b}$ and using (2.6), we obtain

$$
h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b}-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b}=\left(k_{i j} k_{s l}-k_{i l} k_{s j}\right) R_{k}^{s}
$$

Summing this and (3.3), we find

$$
\begin{aligned}
& \left(h_{i j} h_{s l}-h_{i l} h_{s j}+k_{i j} k_{s l}-k_{i l} k_{s j}\right) R_{k}^{s} \\
& \quad=h_{i j} h_{s k} R_{l}^{s}-h_{i l} h_{s k} R_{j}^{s}+h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b}-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{k}
\end{aligned}
$$

This relation, in view of (3.1) and (2.1), can be rewritten as follows:

$$
\begin{align*}
R_{i s l j} R_{k}^{s} & -\frac{c}{4}\left[g_{i j} R_{l k}-g_{i l} R_{k j}+F_{j i} R_{k b} J_{l}^{b}-F_{l i} R_{k b} J_{j}^{b}+2 R_{k a} J_{i}^{a} F_{j l}\right] \\
& =h_{i j} h_{s k} R_{l}^{s}-h_{i l} h_{s k} R_{j}^{s}+h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b}-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b} \tag{3.4}
\end{align*}
$$

Transvecting (3.4) with $g^{l k}$, we get

$$
\begin{aligned}
R_{i a b j} R^{a b} & -\frac{c}{4}\left[g_{i j} R-R_{i j}+F_{j i} R_{k b} J_{l}^{b} g^{k l}-F_{l i} R_{k b} J_{j}^{b} g^{k l}+2 R_{k a} J_{i}^{a} F_{j l} g^{k l}\right] \\
& =h_{i j}\left(h_{s k} R^{s k}\right)-h_{i l} h_{s k} R_{j}^{s} g^{l k}+h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b} g^{l k}-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b} g^{l k}
\end{aligned}
$$

But, in view of (2.1), (2.5), (2.6) and (2.9), we have

$$
\begin{gathered}
R_{k b} J_{l}^{b} g^{k l}=0, \quad-F_{l i} R_{k b} J_{j}^{b} g^{k l}=R_{j i}, \\
R_{k a} J_{i}^{a} g_{l b} J_{j}^{b} g^{k l}=R_{i j}, \\
h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b} g^{l k}=0, \\
-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b} g^{l k}=-h_{i l} h_{k s} g^{k l} R_{j}^{s},
\end{gathered}
$$

because of which, the preceding relation reduces to

$$
R_{i a b j} R^{a b}-\frac{c}{4}\left(g_{i j} R+2 R_{i j}\right)=-2 h_{i a} h_{s b} g^{a b} R_{j}^{s}
$$

or, taking into account (3.2), to

$$
\begin{equation*}
R_{i a b j} R^{a b}-R_{2} R_{i j}+\frac{c}{2}\left(n R_{i j}-\frac{R}{2} g_{i j}\right)=0 \tag{3.5}
\end{equation*}
$$

Thus, we can state
Proposition 1. For any complex hypersurface of the complex space form of holomorphic sectional curvature $c$, the relation (3.5) holds.

Now we shall prove
Proposition 2. If for the complex hypersurface of the complex space form of holomorphic sectional curvature $c$, the relation

$$
\begin{equation*}
R_{i a b j} R_{p}^{a b}-\underset{p+1}{R_{i j}}+\frac{c}{2}\left(n \underset{p}{R_{i j}}-\frac{1}{2} g_{i j} R \underset{p}{R}\right)=0 \tag{3.6}
\end{equation*}
$$

is valid for the integer $p$, then it is also valid for the integer $p+1$.
Proof. Transvecting (3.1) with $R_{p}^{i m}$ and using (2.4) and (2.9), we obtain

$$
R_{i j l m} R_{p}^{i m}+\left[h_{i l} h_{j m}+k_{i l} k_{j m}\right] R_{p}^{i m}=\frac{c}{4}\left[g_{j l} R+2 R_{p}+\underset{p l}{ }\right]
$$

But, in view of (2.6), we have

$$
k_{i l} k_{j m}{\underset{p}{i m}=h_{a l} h_{j b} R_{p}^{a b}, ~}_{\text {in }}
$$

because of which the preceding relation reduces to

$$
\begin{equation*}
R_{a j l b} R_{p}^{a b}+2 h_{a l} h_{j b} R_{p}^{a b}=\frac{c}{4}\left(g_{j l} R+\underset{p}{\left.2 R_{j l}\right)}\right. \tag{3.7}
\end{equation*}
$$

According our assumption (3.6) holds. Thus, substituting $R_{a j l b} R_{p}^{a b}$ from (3.6) into (3.7), we get

$$
\begin{equation*}
2 h_{a l} h_{j b} R_{p}^{a b}=-\underset{p+1}{R}{ }_{j l}+\frac{(n+1) c}{2} \underset{p}{R_{j l}} . \tag{3.8}
\end{equation*}
$$

On the other hand, transvecting (3.4) with $\underset{p}{R^{l k}}$ we find

$$
\begin{align*}
& R_{i s l j} \underset{p+1}{R^{s l}}-\frac{c}{4}\left(g_{i j} \underset{p+1}{R}-\underset{p+1}{R} i j+F_{j i} R_{k b} J_{l}^{b}{\left.\underset{p}{l k}-F_{l i} R_{k b} J_{j}^{b} R_{p}^{k l}+2 F_{j l} R_{k a} J_{i}^{a} R_{p}^{k l}\right)}_{(3.9)}+h_{i j} h_{s k} \underset{p+1}{R^{s k}}-h_{i l} h_{s k} R_{j}^{s} R_{p}^{l k}+h_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b} R_{p}^{l k}-h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b} R_{p}^{l k} .\right.
\end{align*}
$$

In view of (2.1), (2.4), (2.5), (2.6) and (2.9),

$$
\begin{gathered}
R_{k b} J_{l}^{b} \underset{p}{R^{l k}}=0, \quad-R_{k b} J_{j}^{b} R_{p}^{k l}=\underset{p+1}{R} R_{i j} \\
F_{j l} R_{k a} J_{i}^{a}{\underset{p}{k l}=\underset{p+1}{R} R_{i j}}_{R_{i a} h_{s k} R_{b}^{s} J_{j}^{a} J_{l}^{b} R_{p}^{l k}=0,} \\
h_{i b} h_{s k} R_{a}^{s} J_{j}^{a} J_{l}^{b} R_{p}^{l k}=h_{i b} h_{s a} R_{j}^{a} R_{p}^{s b}
\end{gathered}
$$

Thus, (3.9) reduces to

$$
R_{i a b j} R_{p+1}^{R^{a b}}-\frac{c}{4}\left(g_{i j} \underset{p+1}{R}+2 \underset{p+1}{R} R_{i j}\right)=-2 h_{i b} h_{a s} R_{p}^{b s} R_{j}^{a}
$$

Finally, in view of (3.8), we get

$$
R_{i a b j} \underset{p+1}{R^{a b}}-\underset{p+2}{R} i j+\frac{c}{2}\left(\underset{p+1}{R} i j-\frac{c}{2} g_{i j} \underset{p+1}{R}\right)=0
$$

This completes the proof of Proposition 2.
As a consequence of the Propositions 1 and 2, we can state our main result:
ThEOREM. For any complex hypersurface of complex space form of holomorphic sectional curvature $c$, the relation (3.6) is valid for any integer $p=1,2, \ldots$.

REMARK. If the complex hypersurface is an Einstein space, all formulas (3.6) are trivially satisfied.

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# THE EIGENVALUES OF LAPLACIAN AND THE GEOMETRY OF MANIFOLDS 

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#### Abstract

Some estimations (lower and upper bounds) of the first eigenvalue $\lambda_{1}$ of the Laplace operator are very usefull in the consideration of the corresponding compact Riemannian manifold ( $M, g$ ) and its characterization. One may use its asymptotics to study the similar problems of the characterization of some manifolds. The main purpose of this paper is to present some of well known results in this context as well as to derive new ones.


## 1 Introduction

Let $M$ be a Riemannian manifold. The Laplace operator $\Delta$ acting on smooth functions may be defined such that we have

$$
\begin{equation*}
\Delta f:=-\operatorname{div}(\operatorname{grad} f), \quad f \in C^{\infty}(M) . \tag{1.1}
\end{equation*}
$$

For example, in Euclidean plane and Descartues orthonormal coordinates $(x, y)$ one can get

$$
\Delta f=-\left(\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}\right)
$$

A function $f_{0} \subset C^{\infty}(M)$ is the eigenfunction of the operator $\Delta$ with the corresponding eigenvalue $\lambda$ if $f_{0}$ and $\lambda$ satisfy the relation $\Delta\left(f_{0}\right)=\lambda \cdot f_{0}$. The properties of eigenfunctions and eigenvalues of $\Delta$ heavily depend on compactness or noncompactness of $M$. We review in this paper only some facts assuming that $M$ is compact. Some standard results in theory of partial differential equations imply

[^1]that there exist a countable set of eigenvalues $\lambda_{i}$ and for each $\lambda_{i}$ the finite dimensional family of eigenfunctions $f_{i}$, such that we have $\Delta\left(f_{i}\right)=\lambda_{i} f_{i}$. Moreover, $\lambda_{i}$ are positive and $\lambda_{i} \longrightarrow \infty$, when $i \longrightarrow \infty$. The collection of eigenvalues $\left\{\lambda_{i}\right\}$, together with their multiplicities is the spectrum of $\Delta$. We refer the books [2], [8] and the survey article $[7]$ for more details.

In this spirit we may consider more general operators. So we say a partial differential operator $D$ on $C^{\infty}(M)$ is of Laplace type if its leading symbol is positive definite and defines the Riemannian metric $g$ on $M$. Consequently, in some local coordinate system one may write $D=-\left(g^{i j} \partial_{i} \partial_{j}+A^{i} \partial_{i}+B\right), 1 \leq i, j \leq n=$ $\operatorname{dim} M$. Gilkey has established in [11] the unique presentation of $D$ in terms of some connection and the potential function.
0.1 Lemma. There exists an unique connection $\nabla_{D}$ on $C^{\infty}(M)$ and a unique potential function $E_{D} \in C^{\infty}(M)$ such that we have $D=-\left(\operatorname{Tr}\left(\nabla_{D}^{2}\right)+E_{D}\right)$.

This Lemma is shown very fruitfull in the framework of spectral geometry, especially using the heat equation method. We refer [12] for more details. The Laplace type differential operators appear very naturally in the context of differential geometry. To ilustrate it, let $\nabla$ be a torsion free connection on a smooth manifold $M, f \in C^{\infty}(M)$ and $\left(\operatorname{Hess}_{\nabla} f\right)(u, v):=u(v(f))-d f\left(\nabla_{u} v\right)$ be the Hessian. For a semi-Riemannian metric $g$ one defines the 2nd order operators $H(\nabla)$ and $D(g, \nabla)$

$$
\begin{align*}
& H(\nabla) f:=\left(\operatorname{Hess} \nabla+\frac{1}{m-1} \rho_{\nabla}\right) f, \\
& D f:=D(g, \nabla) f=-\operatorname{Tr}_{g}(H(\nabla) f) . \tag{1.2}
\end{align*}
$$

If $g$ is a Riemannian metric, then $D$ is of Laplace type. Pinkall, SchwenckSchellschmidt and Simon [21] have used the operator $H(\nabla)$ to study affine harmonic functions and an affine version of the Minkowski problem. In [5] the authors use the asymptotics of the heat equation corresponding to $D$ given by (1.2) to construct spectral invariants in affine and projective geometry.

The fundamental solution of the heat equation depends on the spectrum of the corresponding operator $D$, and consequently its asymptotics too. Moreover, the coefficients in its asymptotics depend on the geometry quantities of the manifold $M$. The spectrum of other operators in the framework of affine differential geometry and Weyl geometry have been studied in [4] and [6] respectively.

Hence we may put the following question: What is the geometric meaning of eigenvalues $\lambda_{i}$, whose existence has been established using the methods of mathematical analysis?

The 20th century was devoted to the answering this question. Let us mention that the first half of that century passed by guessing that $\lambda_{i}$ completely determine the geometry of $M$, as Weyl [27] proved that $\lambda_{i}$ determine the volume of $M$. Finding two nonisometric manifolds Milnor [18] has given the first negative answer. Later on, many other examples have been found, like Milnor's one, as well as many results related to some geometric quantities of $M$ determined by its spectrum.

To find the spectrum is a very hard problem and therefore one may use different methods to investigate the dependence of the spectrum on a geometry of manifold
$M$. There are also numerous results for the estimation of the first eigenvalues and the corresponding geometric characteristics which depend on these ones.

The main purpose of this paper is to give overview on these results for some types of manifolds, and to give some new characterizations of rank one symmetric spaces, especially an Euclidean space. The paper consists 5 sections. In 1 we deal with the basic notion and notations which we use throughout this paper. Estimations of eigenvalues of the Laplacian, defined on a Riemannian manifold are given in 2. The same problem for hypersurfaces in an affine space we consider in 3. We devote 4 to the first eigenvalues of the Laplacian for functions and 1 -forms. Finally, we consider in 5 the first eigenvalue of a small geodesic ball in a Riemannian manifold. We present some known results as well as some new ones which deal with characterizations of rank one symmetric spaces in terms of $\lambda_{1}$ and some additional assumptions.

## 2 Estimations of eigenvalues of Riemannian manifolds

Lower bounds are more difficult to get than upper bounds, as if we want to use the minimax principle we must find estimates which hold for any function. Lichnerowicz [17] has given the first estimation for $\lambda_{1}$ in a class of Riemannian manifolds ( $M, g$ ) satisfying some conditions in terms of Ricci tensor $\rho=\rho(g)$. Namele he has proved the following theorem.
2.1 Theorem. Let $(M, g)$ be a compact Riemannian manifold and suppose that $\rho \geq(n-1) k g$, with $k>0$. Then we have $\lambda_{1} \geq n k$.

Obata [20] has characterized Riemannian manifolds, isometric to a sphere, in terms of the Ricci tensor and the eigenvalue estimation.
2.2 Theorem. Let $(M, g)$ be a compact Riemannian manifold, and suppose that $\rho \geq(n-1) k g$, with $k>0$. Then $\lambda_{1}(M, g) \geq \lambda_{1}\left(S^{n}(k)\right)$. Furthermore, if the equality is achieved, $(M, g)$ is isometric to $S^{n}(k)$.

Reilly [23] has derived a formula for the integrals of the invariants of the Hessian of a function on a Riemannian manifold, and then, using this one, he has generalized Lichnérowicz and Obata Theorems.
2.3 Theorem. Let $(M, g)$ be a compact Riemannian manifold with nonempty boundary $N$. Assume that there is a constant $c^{2}>0$ such that $\rho \geq(n-1) c^{2} g$ and that the first mean curvature of $N$ in $M$ is nonpositive. Then the first eigenvalue $\lambda_{1}$ of $\Delta$ satisfies the inequality $\lambda_{1} \geq n c^{2}$. Moreover $\lambda_{1}=n c^{2}$ if and only if $M$ is isometric to a closed hemisphere of the Euclidean sphere $S^{n}\left(c^{2}\right)$ of radius $1 / c$.

To ilustrate a type of upper bound estimations for eigenvalues we give this one due to Cheng [10].
2.4 Theorem. Let $(M, g)$ be a compact Riemannian manifold with diameter $d$ and volume $V$, whose Ricci curvature is greater than $(n-1) k$. Set $i_{0}=$ $\left(c(n) d^{n} / V\right)^{1 /(n-1)}$, where $c(n)$ is the volume of the unit ball in $\mathbb{R}^{n}$. Then
(1) if $i \leq i_{0}$, then $\lambda_{i}(M, g) \leq 16 \frac{i^{2}}{d^{2}} A_{1}\left(\frac{d \sqrt{|k|}}{i}\right)$;
(2) if $i \geq i_{0}$, then $\lambda_{i}(M, g) \leq 16\left[\frac{(i+1) c(n)}{V}\right]^{2 / n} A_{2}\left[\left(\frac{\mid k k^{n / 2} V}{(i+1) c(n)}\right)^{1 / n}\right]$, where $A_{1}(x)=\left[\frac{\sinh (x / 2)}{\sinh (x / 4)}\right]^{n}$ and $A_{2}(x)=\left(\frac{\sinh x}{x}\right)^{2} A_{1}(x)$.
Let us remark to see why we must have different estimates for $i \leq i_{0}$ and $i \geq i_{0}$, one may check the explicit example of $(M, g)=S^{n-1}(\alpha) \times S^{1}(D)$, with $\alpha<D$.

## 3 Estimations of eigenvalues of hypersurfaces in an affine space

The first eigenvalue $\lambda_{1}$ for submanifolds of different spaces has been studied intensively through these decades. We refer [24], [1] for more details and other references. So far there are only few results on $\lambda_{1}=\lambda_{1}(M, g)$ in affine differential geometry. Simon has studied $\lambda_{1}$ for a hypersurface $x: M^{n} \longrightarrow \mathcal{A}^{n+1}$ with the Blaschke metric $g$. We present here some of his results. More details one can find in [24].
3.1 Theorem. Let $x$ be a hyperovaloid in $\mathcal{A}^{n+1}$ with the mean curvature $H$. Then $\lambda_{1}=\lambda_{1}(M, g)$ fulfills $0<\lambda_{1} \leq n \max H$; equality holds iff $x(M)$ is an ellipsoid.

Let $k_{1} \leq k_{2} \leq \cdots \leq k_{n}$ be eigenvalues of the equiaffine Weingarten field $B$ (equiaffine shape operator). They are called the equiaffine principal curvatures. The relations among $\lambda_{1}(M, g), H$ and $k_{1}$ are given in the following theorem.
3.2 Theorem. Let $x$ be a hyperovaloid with $B$ regular. Then

$$
\lambda_{1}(M, g) \geq \frac{n}{n-1} \min _{q \in M}\left(\frac{n}{2} H(q)+\frac{1}{2}(n-2) k_{1}(q)\right) \geq n \min _{q \in M} k_{1}(q)>0 .
$$

Equality $\lambda_{1}(M, g)=\frac{n}{n-1} \min _{q \in M}\left(\frac{n}{2} H(q)+\frac{1}{2}(n-2) k_{1}(q)\right)$ holds iff $x(M)$ is an ellipsoid.

Other estimates for $\lambda_{1}$ for the Dirichlet problem under special boundary conditions have been studied too.
3.3 Theorem. Let $M$ be compact with boundary $\partial M$. Assume one of the following conditions (1)-(3) to be fulfilled.
(1) $x$ is a graph, i.e. there exists $b \in V$ such that $\langle X, b\rangle \neq 0$ on $M$ (so $\langle X, b\rangle>0$ by the proper choice of the orientation of $b$ ) and $\langle X, b\rangle=0$ on $\partial M$.
(2) $x(M)$ is convex and $x(\partial M)$ is a p-shadow boundary with respect to parallel light $b \in V$.
(3) $x(M)$ is convex and $x(\partial M)$ is a c-shadow boundary with respect to the center $z_{0}$, and $H>0$ on $M$.
Then $n \max H \geq \lambda_{1} \geq n \min H$.
Equality on the left implies that $H=$ const on $M$. Equality on the right implies that $H=$ const and $f$ is a first eigenfunction of the Laplacian.

Let us mention that $f:=\langle X, b\rangle$ for (1) and (2), and $f:=\rho\left(z_{0}\right)-1 / H$ in case (3), where $\rho\left(z_{0}\right)(q):\left\langle X(q), z_{0}-x(q)\right\rangle$ is the affine support function.

## 4 First Eigenvalues of the Laplacian for Functions and 1-Forms

Let us recall that the Laplacian $\Delta$ may also acts on $p$-forms. Then we use the following formula

$$
\begin{equation*}
\Delta=d \delta+\delta d \tag{4.1}
\end{equation*}
$$

where $d \delta$ are the differential and the codifferential acting on $p$-forms. We notice that for functions ( 0 -forms) (4.1) and (1.1) are equivalent.

Let $\lambda_{1}$ be the first (nonzero) eigenvalue of $\Delta$ for functions, and $\mu_{1}$ the first eigenvalue of $\Delta$ for 1-forms. Kobayashi [14] has improved Lichnerowicz's estimate.
4.1 Theorem. Let $(M, g)$ be a compact Riemannian manifold of dimension $n$ with Ricci tensor $\rho \geq c g,(c>g)$. Then

$$
\frac{n c}{n-1} \leq \mu_{1}=\lambda_{1}
$$

and the equality holds if and only if $M$ is isometric to a sphere.
Let us recall that a complex vector field $v$ on a complex manifold is decomposed into the $(1,0)$-component $v^{\prime}$ and the $(0,1)$-component $v^{\prime \prime}$. A vector field $v$ is real if and only if $v^{\prime \prime}=\bar{v}^{\prime}$. We say that a real vector field $v$ is holomorphic if its ( 1,0 )-component $v^{\prime}$ is holomorphic.

We are ready now to state Kobayashi's theorem for Kähler manifolds, similar to the previous one for Riemannian spaces.
4.2 Theorem. Let $M$ be a compact Kähler manifold with Kähler metric $g$ and Ricci tensor $\rho \geq c g,(c>0)$. Then
(1) $2 c \leq \mu_{1}=\lambda_{1}$, and if the strict inequality $\mu_{1}<\lambda_{1}$ holds, then all eigen 1 -forms belonging to the eigenvalue $\mu_{1}$ must be $\delta$-closed.
(2) If $\mu_{1}=2 c$, then a real 1 -form $\alpha$ is an eigenform belonging to the eigenvalue $\mu_{1}$ if and only if
(i) the real vector field $v_{\alpha}$ corresponding to $\alpha$ is holomorphic;
(ii) $\rho\langle\alpha, \alpha\rangle=c|\alpha|^{2}$, where $\rho\langle\cdot, \cdot\rangle$ is the symmetric form defined by the Ricci tensor.
Moreover, the holomorphic vector fields corresponding to these eigenforms a form a Lie algebra.
(3) If $\lambda_{1}=2 c$, then a real function $f$ is an eigenfunction belonging to the eigenvalue $\lambda_{1}$ if and only if
(i) the vector field $J v_{d f}$ is a Killing vector field, i.e., a holomorphic infinitesimal isometry, where $J$ stands for the complex structure of $M$;
(ii) $\rho\langle d f, d f\rangle=c|d f|^{2}$. Moreover, the Killing vector fields corresponding to these eigenfunctions form a Lie algebra.

Nagano [19] determined the first eigenvalue of the Laplacian on all irreducible compact symmetric spaces. In particular, he proved the equality $\lambda_{1}=2 c$ for all irreducible compact Hermitian symmetric spaces, see also [25]. Consequently the condition $\lambda_{1}=2 c$ is not sufficient to characterize a complex projective space $P^{n} \mathbb{C}$ with Fubini-Study metric in terms of the first eigenvalue $\lambda_{1}$. Anyhow, Kobayashi [14] has characterized $P^{n} \mathbb{C}$ in the following way.
4.3 Corollary. A compact Kähler manifold $M$ of dimension $n$ is isometrically biholomorphic to $P^{n} \mathbb{C}$ with Fubini-S起y metric if and only if $\lambda_{1}=2 c$ with multiplicity $n(n+2)$.

5 The first eigenvalue of a small geodesic ball in a Riemnnian manifold
Let $(M, g)$ be a Riemannian manifold, and $M_{m}$ the tangent space at a point $m \in M$. We denote by $B_{m}^{\nabla}(\varepsilon)$ a metric ball of radius $\varepsilon$, centered at $m \in M$, i.e.

$$
B_{m}^{\nabla}(\varepsilon)=\left\{\exp _{m}(u)\left|u \in M_{m},|u| \leq \varepsilon\right\}\right.
$$

where $\exp : M_{m} \longrightarrow M$ is the exponential mapping of the Riemannian manifold at $m \in M$.

Let $\tau$ be the scalar curvature, $\rho$ the Ricci tensor and $R$ the Riemann curvature tensor. $|R|^{2},|\rho|^{2}$ are the quadratic curvature invariants of the Riemannian metric (see [2] for more details). Karp and Pinsky [13] have investigated the asymptotic behavior of $\lambda_{1}(\varepsilon, m)$, the first Dirichlet eigenvalue of the Laplacian $\Delta$ in a metric ball of radius $\varepsilon$, centered at $m \in M$. The coefficients are expressed in terms of the curvature tensor and its derivatives.
5.1 Theorem. When $\varepsilon \downarrow 0$ we have the expansion

$$
\lambda_{1}(\varepsilon, m)=\frac{c_{0}}{\varepsilon^{2}}+c_{1} \tau_{m}+\varepsilon^{2}\left[|R|^{2}-|\rho|^{2}+6 \Delta \tau\right]_{m}+O\left(\varepsilon^{4}\right)
$$

$c_{0}, c_{1}, c_{2}$ depend only on the dimension $n$ and satisfy $c_{0}>0 c_{1}<0, c_{2}<0$.
If $\omega_{n}$ denotes the $(n-1)$ dimensional measure of the unit sphere $S^{n-1}$. Then
coefficient of $|R|^{2}: \omega_{n} \int f_{0} f_{0}^{\prime} r^{n+2} d r[1 / 30 n(n+2)]$,
coefficient of $|\rho|^{2}: \omega_{n} \int f_{0} f_{0}^{\prime} r^{n+2} d r[-1 / 30 n(n+2)]$,
coefficient of $\tau^{2}$ : zero
coefficient of $\Delta \tau: \omega_{n} \int f_{0} f_{0}^{\prime} r^{n+2} d r[1 / 5 n(n+2)]$.
We note also that the first eigenfunction $f_{0}$ is positive and decreasing, and consequently the first and the last coefficients above are negative, proving that $c_{0}<0$.

Karp and Pinsky have derived from this theorem the following characterizaions of the Euclidean metrics.
5.2 Corollary. Let $n<6$. If for every $m \in M$ we have $\lambda_{1}(\varepsilon, m)=c_{0} / \varepsilon^{2}+$ $O\left(\varepsilon^{4}\right), \varepsilon \downarrow 0$, then $(M, g)$ is locally isometric to $\mathbb{R}^{n}$ with the standard Euclidean metric.
5.3 Corollary. If for every $m \in M$ we have $\lambda_{1}(\varepsilon, m)=c_{0} / \varepsilon^{2}+O\left(\varepsilon^{4}\right), \varepsilon \downarrow 0$, and the Ricci tensor is semidefinite (esp. if $g$ is an Einstein metric) then $(M, g)$ is locally isometric to $\mathbb{R}^{n}$ with the standard Euclidean metric.

Similar results are valid for comparision with other rank one symmetric spaces (see [13]).

In the remaining part of this section we derive new characterizations of the Euclidean metrics as well as rank one symmetric spaces of an arbitrary dimension, but under some assumption.
5.4 THEOREM. If for every $m \in M$ we have $\lambda_{1}(\varepsilon, m)=c_{0} / \varepsilon^{2}+O\left(\varepsilon^{4}\right), \varepsilon \downarrow 0$, and any of the following additional hypothesis
(i) $M$ is conformally flat;
(ii) $M$ is a Bochner flat Kähler manifold;
(iii) $M$ is a Kähler manifold with the complex conharmonic curvature tensor equal to zero;
(iv) $M$ is a product of surfaces; then $(M, g)$ is locally isometric to $\mathbb{R}^{n}$ with the standard Euclidean metric.

Proof. (i) If $n<6$ Theorems follows from Corollary 5.2. If $n \geq 6$ then the Weyl curvature tensor $C$ of $M$ satisfies

$$
|C|^{2}=|R|^{2}-\frac{4}{n-2}|\rho|^{2}+\frac{2}{(n-1)(n-2)} \tau^{2} .
$$

Theorem 5.1 implies $\tau=0$ and consequently $C=0$ gives $|R|^{2}=\frac{4}{n-2}$, which implies the required result.
(ii) Let $B$ be the Bochner curvature tensor for a $2 n$-dimensional Kähler manifold $(n>1)$. Then we have

$$
\begin{equation*}
|B|^{2}=|R|^{2}-\frac{8}{n+2}|\rho|^{2}+\frac{2}{(n+1)(n+2)} \tau^{2} \tag{5.1}
\end{equation*}
$$

Since $B=0$ if and only if $M$ is Bochner flat, one can use our assumption $B=0$, Theorem 5.1 and (5.1) to see $|R|^{2}=\frac{8}{n+2}|\rho|^{2}$, i.e. $|R|^{2}=0$ and hence $M$ is locally flat.
(iii) Let $H$ be the complex conharmonic curvature tensor for a $2 n$ dimensional Kähler manifold (see [22] and [3] for more details). Then we have

$$
\begin{equation*}
|H|^{2}=|R|^{2}-\frac{8}{n+2}|\rho|^{2}+\frac{n}{n+2} \tau^{2} . \tag{5.2}
\end{equation*}
$$

If $H=0$ then Theorem 5.1 and (5.2) imply $|R|^{2}=\frac{8}{n+2}|\rho|^{2}$ and consequently one can have the required result.
(iv) Let $M$ be the Riemannian product of the surfaces $M_{i}, i=1, \ldots, p$. Then we have $|R|^{2}=\sum\left|R_{i}\right|^{2}$ and $|\rho|^{2}=\sum\left|\rho_{i}\right|^{2}$, where $\left|R_{i}\right|^{2}=2\left|\rho_{i}\right|^{2}$. Consequently
$|R|^{2}=2|\rho|^{2}$, just as for surfaces. This relation, with Theorem 5.1, as before imply $M$ is locally flat.

Karp and Pinsky [13] have characterized the rank one symmetric spaces of dimension $n<6$ by the first eigenvalue of $\Delta$. To characterize rank one symmetric spaces of dimension $n \geq 6$ we need some additional assumption. Some of possible assumptions may be in terms of power series expansions of the $(n-1)$ - dimensional volume function $S_{m}(\varepsilon)$ of the geodesic ball $B_{m}^{\nabla}(\varepsilon)$, the total scalar curvature $\tau_{m}(\varepsilon)$ of the small geodesic sphere $G_{m}^{\nabla}(\varepsilon)$

$$
G_{m}^{\nabla}(\varepsilon)=\left\{\exp _{m}(u)\left|u \in M_{m},|u|=\varepsilon\right\}\right.
$$

or the total norm $|\sigma|_{m}^{2}(\varepsilon)$ of the second fundamental form of $G_{m}(\varepsilon)$, where

$$
\begin{aligned}
\tau_{m}(\varepsilon) & =\int_{G_{m}(\varepsilon)} \tilde{\tau}(p) d p \\
|\sigma|_{m}^{2}(\varepsilon) & =\int_{G_{m}(\varepsilon)}|\sigma|^{2}(p) d p
\end{aligned}
$$

Here $d p$ denotes the volume element of $G_{m}(\varepsilon)$ and $\tilde{\tau}$ its scalar curvature.
Chen and Vanhecke [9] have derived the following formulas for the previously mentioned functions. So we have

$$
\begin{align*}
& S_{m}(\varepsilon)=c_{n-1} \varepsilon^{n-1}\left\{1+A(n) \varepsilon^{2}+B(n) \varepsilon^{4}+O\left(\varepsilon^{6}\right)\right\}  \tag{5.3}\\
& \tau_{m}(\varepsilon)=c_{n-1} \varepsilon^{n-3}\left\{(n-1)(n-2)+C(n) \varepsilon^{2}+D(n) \varepsilon^{4}+O\left(\varepsilon^{6}\right)\right\} \tag{5.4}
\end{align*}
$$

$$
\begin{equation*}
|\sigma|_{m}^{2}(\varepsilon)=c_{n-1} \varepsilon^{n-3}\left\{(n-1)+E(n) \varepsilon^{2}+F(n) \varepsilon^{4}+O\left(\varepsilon^{6}\right)\right\} \tag{5.5}
\end{equation*}
$$

where
(5.a) $A(n)=-\frac{1}{6 n} \tau(m)$,
(5.b) $B(n)=\frac{1}{360 n(n+2)}\left(-3|R|^{2}+8|\rho|^{2}+5 \tau^{2}+18 \Delta \tau\right)(m)$,
(5.c) $C(n)=-\frac{(n-2)(n-3)}{6 n} \tau(m)$,
(5.d) $D(n)=\frac{1^{6 n}}{360 n(n+2)}\left(-3(n+2)(n+3)|R|^{2}+8\left(n^{2}+5 n+21\right)|\rho|^{2}+5\left(n^{2}-7 n-\right.\right.$ 6) $\left.\tau^{2}+18(n-2)(n-3) \Delta \tau\right)(m)$,
(5.e) $E(n)=-\frac{(n+3)}{6 n} \tau(m)$,
(5.f) $F(n)=\frac{6 n}{360 n(n+2)}\left(-3(n-13)|R|^{2}+8(n+12)|\rho|^{2}+5(n+7) \tau^{2}+18(n+\right.$ 7) $\Delta \tau)(m)$.

Let us recall that $c_{n-1}=n(\pi)^{n / 2} / \Gamma\left(\frac{n}{2}+1\right)$ is the volume of the unit sphere $S^{n-1}(1)$ in Euclidean space $\mathbb{R}^{n}$.

We subscript to distinguish invariants of different manifolds. Hence $\tau_{i}$ denotes the scalar curvature function on $M_{i}$, for example. Let $M_{1}$ be a rank one symmetric space. Then $\tau_{1},\left|R_{1}\right|^{2},\left|\rho_{1}\right|^{2}$ are constant so $\Delta \tau_{1}=0$.
5.5 ThEOREM. Suppose $\lambda_{1}\left(\varepsilon, m_{1}\right)=\lambda_{1}\left(\varepsilon, m_{2}\right)$ and any of the following additional hypothesis.
(i) $S_{m_{1}}(\varepsilon)=S_{m_{2}}(\varepsilon)$,
(ii) $\tau_{m_{1}}(\varepsilon)=\tau_{m_{2}}(\varepsilon)$,
(iii) $|\sigma|_{m_{1}}^{2}(\varepsilon)=|\sigma|_{m_{2}}^{2}(\varepsilon)$,
for all sufficiently small $\varepsilon$ and for all $m_{1} \in M_{1}$ and all $m_{2} \in M_{2}$.
Then $\tau_{1}=\tau_{2},\left|\rho_{1}\right|^{2}=\left|\rho_{2}\right|^{2},\left|R_{1}\right|^{2}=\left|R_{2}\right|^{2}$.
Proof. (i) Comparing the coefficients in the asymptotic expansions of $\lambda_{1}\left(\varepsilon, m_{i}\right)$, $i=1,2$ shows $\tau_{m_{1}}=\tau_{m_{2}},\left|R_{1}\right|^{2}-\left|\rho_{1}\right|^{2}+6 \Delta \tau_{1}=\left|R_{2}\right|^{2}-\left|\rho_{2}\right|^{2}+6 \Delta \tau_{2}$, for all $m_{1} \in M_{1}$ and $m_{2} \in M_{2}$. Therefore, $\tau_{1}$ and $\tau_{2}$ are constant and $\Delta \tau_{1}=\Delta \tau_{2}=0$. Consequently,

$$
\begin{equation*}
\left|R_{1}\right|^{2}-\left|\rho_{1}\right|^{2}=\left|R_{2}\right|^{2}-\left|\rho_{2}\right|^{2} \tag{5.6}
\end{equation*}
$$

Comparing the coefficients of $\varepsilon^{n+1}$ in (5.3), using (5.b) then yields

$$
\begin{equation*}
a_{1}\left(\left|R_{1}\right|^{2}-\left|\rho_{1}\right|^{2}\right)+2_{2}\left|R_{1}\right|^{2}=a_{1}\left(\left|R_{2}\right|^{2}-\left|\rho_{2}\right|^{2}\right)+a_{2}\left|R_{2}\right|^{2} \tag{5.7}
\end{equation*}
$$

where is simple to check $a_{2} \neq 0$. We combine now (5.6) and (5.7) to complete the proof.

The proof of (ii) and (iii) is similar to this one of (i).
The linear and quadratic curvature invariants do not determine the Riemannian geometry generally. One can find in [16], [15], [26], etc. some examples of spaces with different geometries, but the same previously mentioned invariants $\tau,|\rho|^{2}$ and $|R|^{2}$. But these invariants determine certain classes of rank-one symmetric spaces (for more details see for example [9]). Consequently, we may use Theorem 5.5 to prove the following Corollary.
5.6 Corollary. Let $M_{i}, i=1,2$ be as in Theorem 5.5.
(a) If $M_{1}$ has constant sectional curvature $c$, then $M_{2}$ has constant sectional curvature $c$.
(b) If $M_{i}$ are Kähler manifolds, and if $M_{1}$ is with constant holomorphic sectional curvature, then $M_{2}$ has the same constant holomorphic sectional curvature.
(c) Let the holonomy group of $M_{2}$ be a subgroup of $S p(n) \cdot S p(1)$ and let $M_{1}$ be $\mathbb{Q} P^{n}(\nu)$ or its noncompact dual. Then $M_{2}$ is locally isometric to $\mathbb{Q} P^{n}(\nu)$ or its noncompact dual.

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# STABILITY OF LINEAR CONTINUOUS SINGULAR SYSTEMS IN THE SENSE OF LYAPUNOV: AN OVERVIEW 

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#### Abstract

Singular systems are those whose dynamics is governed by a mixture of algebraic and differential equations. A brief survey of results concerning stability of these systems, in the sense of Lyapunov as well as the main features of this class of systems are also presented.


## Introduction

Singular systems are those whose dynamics is governed by a mixture of algebraic and differential equations.In that sense the algebrac equations represent the constraints to the solution of differential part.

These systems are also known as descriptor, semi-state and generalized systems arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology etc..

Consider the autonomous linear continuous singular systems (LCSS) represented, in the forced regime, by

$$
\begin{align*}
& E \dot{\mathbf{x}}(t)=A \mathbf{x}(t), \quad \mathbf{x}\left(t_{0}\right)=\mathbf{x}_{0}  \tag{1}\\
& \mathbf{y}(t)=C \mathbf{x}(t),
\end{align*}
$$

with matrix $E$ possibly singular, where $\mathbf{x}(t) \in \Re^{n}$ is generalized state-space vector and $\mathbf{u}(t) \in \Re^{\prime \prime \prime}$ is control variable. Matrices $A$ and $C$ are of appropriate dimensions and are defined over the field of real numbers.

System models in this form have some important advantages in comparison with the models in the normal form, e.g. when $E=I$ :

- These models preserve the sparsity of systems matrices (that is many entries of system matrices are equal to zero)
- There is a tight relation between the system physical variables and the variables in the model
- The structure of tihe physical system is well reflected in the model

There is a great simplicity in derivation of these equations and in this connection there is no necessity for the elimination of the unwanted (redudant) variables, as there is no request for the formulation of the state variables.
The complex nature of singular systems causes many difficultes in analytical and numerical treatment that do not appear when systems in the normal form are concerned. In that sense the questions of existence, solvability, uniqueness, and smothness are present and should be solved in satisfaction manner.

The survey of updated results for singular systems and the broad bibliography can be found in Bajic(1992), Campbell (1980, 1982), Lewis (1986, 1987), Debeljkovic at el. (1996.a, 1996.b, 1998) and in the two special issues of the journal Circuits, Systems and Signal Procesing (1986, 1989).

A specific nature of singular systems is well documented in the Fig.1.


Fig. 1.

## Preliminaries

Some fundamental questions of existence and uniqueness of singular system solutions. Consider the (LCSS) described by (1). When the matrix pencil $(s E-A)$ is regular. i.e. when:

$$
\begin{equation*}
\operatorname{det}(s E-A) \neq 0, s \in C, \tag{2}
\end{equation*}
$$

then solutions of (1) exist and they are unique for so-called consistent initial conditions $\mathbf{x}_{0}$ of $\mathbf{x}(t)$, and moreover, the closed form of these solutions is known.

Definition 1. Let matrices $E, A \in C^{1 \times x}$ and $t_{0} \in \Re$. We should say that vector $\mathrm{x}\left(t_{0}\right)$ $=\mathrm{x}_{0} \in \mathfrak{R}^{n}$ is consistent vector associated at $t_{0}$ if (1) has unique solution.
Eq. (1) je tractable at $t_{0}$, if it has unique solution for every consistent vector of initial condition $\mathbf{x}$, associated with $t_{0}$.

If $E \dot{\mathbf{x}}(t)=A \mathbf{x}(t)$ is tractable for any moment $t_{0} \in \Re$, then it is tractable for every moment $t \in \Re$, so one can say that eq. (1) is tractable.

Theorem 1. For given $E, A \in \Re^{n \times 2 n}$, eq. (1) is tractable if and only if there exist scalar $\lambda \in C$, such that matrix $(\lambda E-A)^{-1}$ exist.

## Solvabulity

According to the Fig.1, the singular system is regular, when the matrix pencil satisfy eq. (2). The regularity condition can be tested using the following theorem Yip, Sincovec (1981).

Theorem 2. The following expressions are equivalent:
a) The matrix pencil $(A, E)$ is solvable, e.g., $\operatorname{det}(s E-A) \neq 0$,
b) Let $X_{0}=\mathfrak{\aleph}(A)$

$$
\begin{equation*}
\mathbf{X}_{i}=\left\{\mathbf{x}(\mathrm{t}): A \mathbf{x}(\mathrm{t}) \in E \mathbf{X}_{i-1}\right\}, \tag{3}
\end{equation*}
$$

then:

$$
\begin{equation*}
\mathfrak{N}(E) \cap \mathbf{X}_{i}=\{0\}, \forall i=0,1, \ldots \tag{4}
\end{equation*}
$$

c) Let $Y_{0}=\mathbb{N}\left(A^{T}\right)$

$$
\begin{equation*}
\mathbf{Y}_{i}=\left\{\mathbf{x}(\mathrm{t}): A \mathbf{x}(\mathrm{t}) \in E \mathrm{Y}_{i-1}\right\}, \tag{5}
\end{equation*}
$$

then:

$$
\begin{equation*}
\mathfrak{N}\left(E^{\tau}\right) \cap \mathbf{Y}_{i}=\{\mathbf{0}\}, \forall i=0,1, . \tag{6}
\end{equation*}
$$

d) Matrix:

$$
\left.G(n)=\left[\begin{array}{cccc}
E & 0 & \cdots & 0  \tag{7}\\
A & E & \cdots & 0 \\
0 & A & \cdots & 0 \\
\vdots & & & \vdots \\
\vdots & & & E \\
0 & & \cdots & A
\end{array}\right]\right\} n+1
$$

has full row rank for $n=1,2, \ldots$
e) Matrix:

$$
F(n)=\left[\begin{array}{ccccc}
E & A & 0 & \cdots & 0  \tag{8}\\
0 & E & A & \cdots & 0 \\
0 & 0 & E & \cdots & 0 \\
\vdots & & & & \vdots \\
\vdots & & & & E \\
0 & & \cdots & & A
\end{array}\right]
$$

has full column rank for $n=1,2, \ldots$
A particular approach to the question of solvabilty can be achived using so-called shuffle algoritam, Luenberger (1978).

The procedure starts with forming the following matrix:

$$
E A
$$

If matrix $E$ is regular, procedure is finished and system is solvable. But if this is not case one can apply elementary row operations, so the matrix defined above can be reduced to the form:

$$
\begin{aligned}
& T A_{1} \\
& 0 A_{2}
\end{aligned}
$$

where matrix $T$ has a full rank.
Next "suffle" operation leads to the following matrix form:

$$
\begin{aligned}
& T A_{3} \\
& A_{4} 0
\end{aligned}
$$

If the matrix $A_{4}$ is regular, the system is solvable. The similar procedure is applicable for systems given by eq. (1).

## Consistent initial conditions

Having in mind the implicit character of eq. (1), with respect to $\dot{\mathbf{x}}(t)$, it is obvious, that not all initial conditions are permissible.

The problem of consistent initial conditions is not characteristic for the systems in the normal form, but it is basic one for the singular systems. One of the consequence of this problem is the fact that it is not yet possible to derive, in an algorithmically testable form of the necessary and sufficient conditions for the asymptotic stability of even linear singular systems with constant coefficients.

We will say that the initial condition $\mathbf{x}_{0}$ is consistent if there exist $a$ differentiable, continuous solution of eq. (1). The solution $\mathbf{x}(t)$ should be differentiable a finite number of times and it is real analytic on interval $t \geq 0$.

Discussion and generation of consistent initial conditions were treated by several authors. Some of these, most imortant, results are presented here.

Campbell et al. (1976) showed that $\mathbf{x}_{0}$ is a consistent initial condition for eq. (1) if and only if:

$$
\begin{equation*}
\left(I-\hat{E} \hat{E}^{D}\right) \mathbf{x}_{0}=0 \tag{9}
\end{equation*}
$$

or in equivalent notation:

$$
\begin{equation*}
W_{k}=\aleph\left(I-\hat{E} \hat{E}^{D}\right) \tag{10}
\end{equation*}
$$

where $\hat{E}^{D}$ is Drazin inverse of matrix $\hat{E}$ and

$$
\begin{equation*}
\hat{E}=(\lambda E-A)^{-1} E . \tag{11}
\end{equation*}
$$

where $A^{-1}()$ dentos the inverse image of () under operator $A$ and $\mathcal{N}()$ nullspace or kernell.

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions is the subspace sequence:

$$
\begin{gather*}
W_{0}=\Re^{n},  \tag{12}\\
W_{j+1}=A^{-1}\left(E W_{j}\right), j \geq 0,
\end{gather*}
$$

Lemma 1. The subspace sequence $\left\{W_{0}, W_{1}, W_{2}, \ldots.\right\}$ is nested in sense that:

$$
\begin{equation*}
W_{0} \supset W_{1} \supset W_{2} \supset W_{3} \supset \ldots . . \tag{13}
\end{equation*}
$$

Moreover:

$$
\begin{equation*}
\aleph(A) \subset W_{j}, \quad \forall j \geq 0 \tag{14}
\end{equation*}
$$

and there exist and integer $k \geq 0$ such that:

$$
\begin{equation*}
W_{k+1}=W_{k}, \tag{15}
\end{equation*}
$$

so that:

$$
\begin{equation*}
W_{k+j}=W_{k}, \quad \forall j \geq 1 \tag{16}
\end{equation*}
$$

If $k^{*}$ is the smallest such integer with this property, then:

$$
\begin{equation*}
W_{k} \cap \mathcal{\aleph}(E)=\{0\}, k \geq k^{*}, \tag{17}
\end{equation*}
$$

provided that $(\lambda E-A)^{-1}$ is invertibble for some $\lambda \in \Re$, Owens, Debeljkovic (1985).
In some circumstances, it is useful to introduce the linear nonsingular transformation of system governed by (1), in order to get the first canonical form of linear singular system as:

$$
\begin{gather*}
\dot{\mathbf{x}}_{1}(t)=A_{1} \mathbf{x}_{1}(t)+A_{2} \mathbf{x}_{2}(t),  \tag{18}\\
0=A_{3} \mathbf{x}_{1}(\mathrm{t})+A_{4} \mathbf{x}_{2}(\mathrm{t}) . \tag{19}
\end{gather*}
$$

Then, the set of consistent initial values is equal to the manifold, or in other words $x_{0}$ has to satisfy:

$$
\begin{equation*}
0=A_{3} \mathbf{x}_{10}+A_{4} \mathbf{x}_{20}, \tag{20}
\end{equation*}
$$

or in equivalent notation:

$$
M=\kappa\left(\left[\begin{array}{ll}
A_{3} & A_{4} \tag{21}
\end{array}\right]\right)
$$

## State response of linear singular system

Suppose that $E$ and $A$ are square matrices. Then the state response of autonomous system, described by eq. (1) is given with:

$$
\begin{equation*}
\mathbf{x}(t)=e^{-\hat{E}^{D} \hat{A}(t-t)} \hat{E} \hat{E}^{D} \mathbf{q}, \mathbf{q} \in \Re^{n} . \tag{22}
\end{equation*}
$$

It is obvious that vector of consistent initial conditions must satisfy:

$$
\begin{equation*}
\mathrm{x}_{0}=\hat{E} \hat{E}^{D} \mathrm{x}_{0} \tag{23}
\end{equation*}
$$

## Matrix transfer function

It can be shown that the matrix transfer function of linear singular system, in certain circumstances, can not be found. This problem is completely determined by the question of possible solvability of singular system.

If ones look at the nonautonomous linear singular system, with control matrix $B$ then the matrix transfer function is given with:

$$
\begin{equation*}
W(s)=C[s E-A]^{-1} B=C \frac{\operatorname{adj}[s E-A]}{\operatorname{det}(s E-A)} B, \tag{24}
\end{equation*}
$$

It is more than clear that only regular singular systems, see Fig.1. can have such description.

If singular system has no transfer function, i.e. it is irregular, it may still have a general description pairing, Dziurla, Newcomb (1987), that is the description of the form:

$$
\begin{equation*}
R(s) \mathbf{Y}(s)=Q(s) \mathbf{U}(s), \tag{25}
\end{equation*}
$$

where $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ are Laplace transforms of output and input, respectively and $R(s)$ and $Q(s)$ are polinomials of complex variable.

## Impulsive behavior of linear singular system

Linear singular systems can always have an impulsive solutions whenever the initial conditions are different from those determined by consistent initial conditions. In that case the free response of singular system exibits exponential motions and in addition contains some impulsive motions corresponding to the "infinite - frequency" modes, Verghese et al. (1981). This feature of singular systems is undesirable.

If the matrix $A_{4}$ is regular, the system is solvable and this fact guarantees that no impulsive motions occur in response to arbitrary initial conditions, since then singular system is reduced to its normal form. Conversely, one can show that if $A_{4}$ is singular, and system under the consideration is solvable then it exibits impulsive - free motions for certain initial conditions.

## Main results

Stability plays a central role in systems theory and control engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This section is concerned with the stability of equlibrium points in the sense of Lyapunov. As we treat the linear systems this is equivalent to the study of systems stability. The Lyapunov direct method is well exposed in a number of very well known references. Here we present some different and interesting approaches to this problem.

## Stability definitions

Definition 2. Eq.(1) is exponentially stable if one can find two positive constants $\alpha$, $\beta$ such that:

$$
\begin{equation*}
\|x(t)\| \leq \beta \exp (-\alpha t)\|x(t)\| \tag{26}
\end{equation*}
$$

for every solution of eq.(1), Pandolfi (1980).
Definition 3. The system given by eq.(1) will be termed asymptotically stable iff, for all consistent initial conditions $\mathbf{x}_{0}$, it has that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, Owens, Debeljkovic (1985).

Definition 4. We call system given by eq. (1) asymptotically stable if all roots of $\operatorname{det}(s E-A)$, i.e. all finite eigenvalues of this matrix pencil, are in the open left - half complex plane, and system under consideration is impulsivelz free if there is no $\mathrm{X}_{0}$ such that $\mathbf{x}(t)$ exibits discontinuous behavior in free regime. Lewis (1986).

Definition 5. The system given by eq. (1) is called asymptotically stable iff all finite eigenvalues $\lambda_{i}, i=1, \ldots, n_{1}$, of the matrix pencil $(\lambda E-A)$ have negative parts, Muller (1993).

Definition 6. The equilibrium $x=0$ of system given by eq.(1) is said to be stable if for every $\varepsilon>0$, and any $t_{0} \in J$, there exists a $\delta=\delta\left(\varepsilon, t_{0}\right)>0$, such that $\left\|x\left(t, t_{0}, \mathbf{x}_{0}\right)\right\|<\varepsilon$ hold for all $t \geq t_{0}$, whenever $\mathbf{x}_{0} \in W_{k}$ and $\left\|\mathbf{x}_{0}\right\|<\delta$, where $J$ denotes time interval such that $J=\left[t_{0},+\infty\right), t_{0} \geq 0$, Chen, Liu (1997).

Definition 6. The equilibrium $x=0$ of system given by eq. (1) is said to be unstable if there exist a $\varepsilon>0$, and $t_{0} \in J$, for any $\delta>0$, such that there exists a $t^{*} \geq t_{0}$, for which $\left\|x\left(t^{*}, t_{0}, \mathbf{x}_{0}\right)\right\| \geq \varepsilon$ holds, although $\mathbf{x}_{0} \in W_{k}$ and $\left\|\mathbf{x}_{0}\right\|<\delta$, Chen, Liu (1997).

Definition 7. The equilibrium $x=0$ of system given by eq. (1) is said to be attractive if for every $t_{0} \in J$, there exists an $\eta=\eta\left(t_{0}\right)>0$, such that $\lim _{t \rightarrow \infty} x\left(t, t_{0}, \mathbf{x}_{0}\right)=0$, whenever $\mathbf{x}_{0} \in W_{k}$ and $\left\|\mathbf{x}_{0}\right\|<\eta$, Chen, Liu (1997).

Definition 8. The equilibrium $x=0$ of singular system given by eq. (1) is said to be asymptotically stable if it is stable and attractive, Chen, Liu (1997).

Lemma 2. The equilibrium $x=0$ of linear singular system given by eq. (1) is asymptotically stable if and only if it is impulsive-free, and $\sigma(E, A) \subset C^{-}$Chen, Liu (1997).

Lemma 3. The equilibrium $x=0$ of system given by eq. (1) is asymptotically stable if and only if it is impulsive-free, and $\lim _{t \rightarrow \infty} x(t)=0$, Chen, Liu (1997).

## Stability theorems

Theorem 3. Eq. (1), with $A=I, I$ being identity matrix, is exponentially stable iff the eigenvalues of $E$ have nonpositive real parts.

Proof. State response of singular system, under the consideration, is given by eq. (22), with restriction on vector of consistent initial conditions, given by eq. (23).

If $E$ is written in diagonal form, then:

$$
e^{-\hat{E}^{D} \hat{A}\left(1-t_{0}\right)} \hat{E} \hat{E}^{D}=\left[\begin{array}{cc}
e^{M^{-1} t} & 0  \tag{27}\\
0 & 0
\end{array}\right] \mathbf{q},
$$

which decays exponentially when $\lambda \in \sigma(0)$ implies that $\operatorname{Re}(\lambda)<0$.
Because the eigenvalues of $M$ are those eigenvalues of $E$ which are not zero, it has completed the proof.

Theorem 4. Let $I_{\Omega}$ be the matrix which represents the operator on $\Re^{n}$ which is the identity on $\Omega$ and the zero operator on $\Lambda$. Eq. (1), with $A=I$, is stable if there exists an $n \times n$ matrix $P$, which is the solution of the matrix equation:

$$
\begin{equation*}
E^{T} P+P E=-I_{\Omega} \tag{28}
\end{equation*}
$$

with the following properties:
i) $P=P^{T}$,
ii) $\quad P q=0, q \in \Lambda$
iii) $\mathbf{q}^{T} P q>0, q \neq 0, q \in \Omega$.,
where:

$$
\begin{gather*}
\Omega=W_{k}=\aleph\left(I-E E^{D}\right),  \tag{29}\\
\Lambda=\aleph\left(E E^{D}\right), \tag{30}
\end{gather*}
$$

Proof. If eq. (28) has a solution $P$ as above, $E$ cannot have eigenvalues with positive real parts. Hence, eq. (1) is stable. Conversely, assume that eq. (1) is stable. Let $P$ be defined by:

$$
\begin{equation*}
\mathbf{q}^{T} \mathrm{Pq}=\int_{0}^{+\infty}\left\|\exp (E t) E^{D} \mathbf{q}\right\|^{2} \mathrm{dt} \tag{31}
\end{equation*}
$$

The integral is zero if $q \in \Lambda$ and is finite number if $q \in \Omega$. It is clear that matrix $P$ is solution of eq. (28) with the properties, a), b), c), Pandolfi (1980).

Theorem 5. The system given by eq. (1) is asymptotically stable if and only if
a) $A$ is invertible and
b) there exists a positive-definite, self-adjoint operator $P$ on $\mathfrak{R}^{n}$, such that:
c)

$$
\begin{equation*}
A^{T} P E+E^{T} P A=-Q \tag{32}
\end{equation*}
$$

where $Q$ is self-adjoint and positive in the sense that:

$$
\begin{equation*}
\mathbf{x}^{T}(t) Q \mathbf{x}(t)>0 \text { for all } x \in W_{k} . \backslash\{0\} \tag{33}
\end{equation*}
$$

Owens, Debeljkovic (1985).
Proof. To prove sufficiency, note that $W_{k} \cap \mathcal{\aleph}(E)=\{0\}$ indicates that:

$$
\begin{equation*}
V(\boldsymbol{x})=\mathbf{x}^{T}(t) E^{T} P E \mathbf{x}(t), \tag{34}
\end{equation*}
$$

is a positive-definite quadratic form on $W_{k}$. All smooth solutions $\mathbf{x}(t)$ evolve in $W_{k}$. so $V(\boldsymbol{x})$ can be used as a "Lyapunov function". Clearly, using the equation of motion (1), we have:

$$
\begin{align*}
\dot{V} & =\dot{\mathbf{x}}^{T}(t) E^{T} P E \mathbf{x}(t)+\mathbf{x}^{T}(t) E^{T} P E \dot{\mathbf{x}}(t) \\
& =(E \mathbf{x}(t))^{T} P E \mathbf{x}(t)+\mathbf{x}^{T}(t) E^{T} P E \dot{\mathbf{x}}(t) \\
& =(A \mathbf{x}(t))^{T} E^{T} P E \mathbf{x}(t)+\mathbf{x}^{T}(t) E^{T} P A \mathbf{x}(t) \\
& =\mathbf{x}^{T}(t) A^{T} P E \mathbf{x}(t)+\mathbf{x}^{T}(t) E^{T} P A \mathbf{x}(t) \\
& =-\mathbf{x}^{T}(t) Q \mathbf{x}(t) \leq-\lambda V, \tag{35}
\end{align*}
$$

where:

$$
\begin{equation*}
\lambda=\min \left\{\mathbf{x}^{T}(t) Q \mathbf{x}(t): V(x)=1, x \in W_{k^{*}}\right\} \tag{36}
\end{equation*}
$$

is strictly positive by eq. (33).
Clearly:

$$
\begin{equation*}
0 \leq V(x(t)) \leq V\left(x_{0}\right) e^{-\lambda t} \rightarrow 0(t \rightarrow \infty) \tag{37}
\end{equation*}
$$

so that $E x(t)$ and $x(t)$ tend to zero as $t \rightarrow \infty$ as required.
Theorem 6. The system given by eq. (1) is asymptotically stable if and only if
a) $A$ is invertible and
b) there exists a positive-definite, self-adjoint operator $P$ such that:

$$
\begin{equation*}
\mathbf{x}^{T}(t)\left(A^{T} P E+E^{T} P A\right) \mathbf{x}(t)=-\mathbf{x}^{T}(t) I \mathbf{x}(t) \text { for all } x \in W_{k} . \tag{38}
\end{equation*}
$$

Owens, Debeljkovic (1985).
Theorem 7. Let $(E, A)$ be regular and $(E, A, C)$ be observable. Then $(E, A)$ is impulsive free and asymptotically stable if and only if there exist a positive definite solution $P$ to:

$$
\begin{equation*}
A^{T} P E+E^{T} P A+E^{T} C^{T} C E=0 \tag{39}
\end{equation*}
$$

and if $P_{1}$ and $P_{2}$ are two such solutions, then $E^{T} P_{1} E=E^{T} P_{2} E$, Lewis (1986).
Theorem 8. If there are symmetric matrices $P, Q$ satisfying:

$$
\begin{equation*}
A^{T} P E+E^{T} P A=-Q \tag{40}
\end{equation*}
$$

and if:
$\mathrm{x}^{T} E^{T} P E \mathrm{x}>0 \forall \mathrm{x}=S_{1} \mathrm{y}_{1} \neq 0$,

$$
\begin{equation*}
\mathbf{x}^{T} Q \mathrm{x} \geq 0 \forall \mathrm{x}=S_{1} \mathrm{y}_{1} \tag{41}
\end{equation*}
$$

then system described by eq. (1) is asymptotically stable if:

$$
\operatorname{rank}\left[\begin{array}{c}
s E-A  \tag{43}\\
S_{1}^{T} Q
\end{array}\right]=n \forall s \in C
$$

and marginally stable if condition given by eq. (39) does not hold, Muller (1993).
Proof. Assume $P, Q$ according to eq. $(37,38)$, then by transformation:

$$
\begin{gather*}
\mathrm{R}=\left[\begin{array}{l}
R_{1} \\
R_{2}
\end{array}\right], \mathrm{S}=\left[\begin{array}{ll}
S_{1} & S_{2}
\end{array}\right],  \tag{44}\\
R E S=\left[\begin{array}{cc}
I_{1} & 0 \\
0 & N_{k}
\end{array}\right], R A S=\left[\begin{array}{cc}
A_{1} & 0 \\
0 & I_{2}
\end{array}\right], \tag{45}
\end{gather*}
$$

where the identity matrices $I_{1}$ and $I_{2}$ are of dimension $n_{1}$ and $n_{2}$ with $n_{1}+n_{2}=n$ and the $n_{2} \times n_{2}$ matrix $N_{k}$ is nilpotent of index $k$, one has:

$$
\begin{equation*}
A_{1}^{T} P_{1}+P_{1} A_{1}=-S_{1}^{T} Q S_{1}=-Q_{1}, \tag{46}
\end{equation*}
$$

with:

$$
\begin{equation*}
P_{1}=P_{1}^{T}>0, Q_{1}=Q_{1}^{T} \geq 0 \tag{47}
\end{equation*}
$$

Therefore system given by eq. (1) is stable in the sense of Lyapunov and is asymptotically stable iff:

$$
\operatorname{rank}\left[\begin{array}{c}
s I_{1}-A_{1}  \tag{48}\\
Q_{1}
\end{array}\right]=n_{1}, \forall s \in C
$$

So, it is necessary to show that condition:

$$
\operatorname{rank}\left[\begin{array}{c}
s E-A  \tag{49}\\
S_{1}^{T} Q
\end{array}\right]=n, \forall s \in C
$$

is equivalent to expression, given by eq. (48). By the transformation of eq. $(44,45)$ one has:

$$
\operatorname{rank}\left[\begin{array}{c}
s E-A  \tag{50}\\
S_{1}^{T} Q
\end{array}\right]=\operatorname{rank}\left[\begin{array}{cc}
s I_{1}-A_{1} & 0 \\
0 & s N_{k}-I_{2} \\
Q_{1} & Q_{12}
\end{array}\right]=n_{2}+\operatorname{rank}\left[\begin{array}{c}
s I_{1}-A_{1} \\
Q_{1}
\end{array}\right]
$$

showing the equivalence of eq.(48) and eq. (49).
Theorem 9. The equilibrium $x=0$ of system given by eq. (1) is asymptotically stable, if there exists an $n \times n$ symmetric positive definite matrix $P$, such that along solutions of system (1) the derivative of function $V(E x)=(E x)^{T} P(E x)$, is negative definite for the variates of Ex, Chen, Liu (1997)

Proof. First, the regularity of $(E, A)$ means that there exists $n \times n$ nonsingular matrices $U$ and $V$ such that:

$$
U E V=\left[\begin{array}{cc}
I_{1} & 0  \tag{51}\\
0 & N
\end{array}\right], U A V=\left[\begin{array}{cc}
A_{1} & 0 \\
0 & I_{2}
\end{array}\right]
$$

and eq. (1) is equivalent to:

$$
\begin{align*}
& \dot{\mathbf{z}}_{1}=A_{1} \mathbf{z}_{1}+0  \tag{52}\\
& N \dot{\mathbf{z}}_{2}=0+\mathrm{z}_{2}
\end{align*}
$$

here $Q\left(\begin{array}{ll}\mathbf{z}_{1} & \mathbf{z}_{2}\end{array}\right)^{T}=x, A_{1}$ is an $n_{1} \times n_{1}$ nonsingular matrix and $N$ is an $n_{2} \times n_{2}$ nilpotent matrix, $n_{1}+n_{2}=n$.

Next, $V(E x)$ is a negative definite quadratic form for the variates of $E x$ means that there exists an $n \times n$ symetric matrix $W$ with $E^{T} W E$ is positive semi definite and the rank of $E^{T} W E$ is equal to $r$, such that:

$$
\begin{equation*}
V(E x)=-(E \mathbf{x})^{T} W(E x) \tag{53}
\end{equation*}
$$

or:

$$
\begin{equation*}
A^{T} P E+E^{T} P A=-E^{T} W E \tag{54}
\end{equation*}
$$

Letting:

$$
\begin{align*}
& P=U^{T}\left[\begin{array}{ll}
P_{11} & P_{12} \\
P_{12}^{T} & P_{22}
\end{array}\right] U,  \tag{55}\\
& W=U^{T}\left[\begin{array}{ll}
W_{11} & W_{12} \\
W_{12}^{T} & W_{22}
\end{array}\right] U, \tag{56}
\end{align*}
$$

one has:

$$
\begin{align*}
P_{11} A_{1}+A_{1}^{T} P_{12} & =-W_{11} \\
P_{22} N+N^{T} P_{22} & =-N^{T} W_{22} N  \tag{57}\\
P_{12}+A_{1}^{T} P_{12} N & =-W_{12} N
\end{align*}
$$

where $P_{11}, P_{22}$ and $W_{1}$ are all positive definite matrices.
In the following it proves that $N=0$. Suppose that the form of nilpotent matrix $N$ is

$$
N=\left[\begin{array}{llll}
J_{1} & & &  \tag{58}\\
& \ddots & & \\
& & J_{i} & \\
& & & 0
\end{array}\right]
$$

where $J_{i}$ is a Jordan block matrix which the diagonal elements are all zero $(i=1, \ldots, s)$, then all elements of first row of both $N^{T} P_{22}$ and $N^{T} W_{22} N$ are zero, it is follows from the second formula of eq.(57) that al elements of first row $P_{22} N$ are zero. If $N=0$ is not true, without loss of generality it suposses that $J_{1} \neq 0$, then it can be deduced that the element of first row and first column of matrix $P_{22}$ is zero, this is not true since $P_{22}$ is positive definite.

Thus it has $N=0$, in other words, the linear singular system described by eq. (1) is impulse-free.

The positive definitity of matrix $W_{1}$ and the first formula of eq.(57) imply that $A_{1}$ is an asymptotically stable matrix. It follows from eq.(52) and $N=0$ that $\lim _{t \rightarrow+\infty} x=0$ hold from $x=Q\left(\begin{array}{ll}\mathbf{z}_{1} & \mathbf{z}_{2}\end{array}\right)^{T}$ and the conclusion of Theorem 9. follows from Lemma2.

Theorem 10. If there exist an $n x n$ symmetric, positive definite matrix $P$, such that along solutions of system given by eq. (1) the derivative of function $V(E x)=$ $(E x)^{T} P(E x)$ i.e. $\dot{V}(E x)$ is positive definite for all variates of $E x$, then the equilibrium $x$ $=0$ of system given by eq. (1) is unstable, Chen, Liu (1997).

Theorem 11. If there exist an $n \times n$ symmetric, positive definite matrix $P$, such that along solutions of system given by eq. (1) the derivative of function $V(E x)=$ $(E x)^{T} P(E x)$ i.e. $\dot{V}(E x)$ is negative semidefinite for all variates of $E x$, then the equilibrium $x=0$ of system given by eq. (1) is stable, Chen, Liu (1997).

## Numerical examples

Some numerical examples have been worked out to show excellent application of previous results. We shall illustrate some of these methods. No discussion is presented since the procedure and conclussions are obviouos.

Pandolfi (1980).

$$
\begin{aligned}
& {\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right] \dot{\mathbf{x}}(t)=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right] \mathbf{x}(t)} \\
& A_{\theta}^{-1}=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right], A=A_{0}^{-1} A_{b}=I \\
& E=A_{b}^{-1} E_{0}=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right] \\
& E^{T} P+P E=-I_{\Omega}=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right] \\
& P=P^{T}=\left[\begin{array}{ccc}
0.5 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 0 & 0
\end{array}\right] \\
& \mathfrak{N}(P)=\mathfrak{N}(E)=\operatorname{span}\left\{\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right\}, P \mathrm{q}=\mathbf{0}, \mathbf{q} \in \Lambda \\
& \mathfrak{R}(P)=\mathfrak{R}(E)=\operatorname{span}\left\{\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\}, \\
& \mathbf{q}^{T} P \mathbf{q}=0.5\left(q_{1}^{2}+q_{2}^{2}\right)>0, \forall \mathbf{q} \in \Omega, \mathbf{q} \neq 0 .
\end{aligned}
$$

So, system under consideration is stable in sense of Lyapunov.
2. Owens, Debeljkovic (1985).

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right] \dot{\mathbf{x}}(t)=\left[\begin{array}{ccc}
-2 & -2 & 0 \\
-2 & -3 & 0 \\
0 & 1 & 1
\end{array}\right] \mathbf{x}(t)} \\
& W_{k}=\left\{\mathbf{x}: x_{1} \in \mathfrak{R} ; x_{2} \in \mathfrak{R} ; x_{3}=-x_{2}\right\} .
\end{aligned}
$$

Martix $G$ can be addopted as:

$$
\begin{aligned}
& G=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{array}\right]=G^{T} \\
& \mathbf{x}^{T} G \mathbf{x}=x_{1}^{2}+x_{2}^{2}-2 x_{2} x_{3}+x_{3}^{2} \\
& \mathbf{x}^{T} \mathrm{Gx}=x_{1}^{2}+4 x_{2}^{2}>0, \forall \mathbf{x} \in W_{k} \backslash\{\boldsymbol{0}\} \\
& P=\left[\begin{array}{ccc}
6.75 & 0.25 & 0 \\
0.25 & 2.75 & 1 \\
0 & 1 & p_{33}
\end{array}\right]
\end{aligned}
$$

So, for $p_{33}>0,0135 P$ is symmetric and positive definite matrix and system under consideration is asymptotically stable.

Chen, Liu (1997).

$$
\begin{gathered}
{\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \dot{\mathbf{x}}(t)=\left[\begin{array}{ccc}
0 & 1 & 0 \\
1 & 1 & 0 \\
0.5 & 0 & -1
\end{array}\right] \mathbf{x}(t)} \\
P=\left[\begin{array}{ccc}
9 & -9 & 2 \\
-9 & 16 & -2 \\
2 & -2 & 8
\end{array}\right]=P^{T}>0 \\
V(E x)=(E x)^{T} P(E x)
\end{gathered}
$$

Using eq.(54), one can easily get:

$$
\begin{aligned}
\dot{V}(E x) & =-(E \mathbf{x})^{T} W(E \mathbf{x})= \\
& =-\left[\begin{array}{lll}
x_{1} & 0 & x_{3}
\end{array}\right]\left[\begin{array}{ccc}
16 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 16
\end{array}\right]\left[\begin{array}{c}
x_{1} \\
0 \\
x_{3}
\end{array}\right] \\
& =-16\left(x_{1}^{2}+x_{3}^{2}\right)
\end{aligned}
$$

so, the equilibrium point of system under consideration is asymptotically stable since the derivative of Lyapunov function, along the trajectories of system, is negative definite for the variates of $x_{1}$ and $x_{3}$ of $E x$.

## Conclusion

To assure asymptotical stability for linear singular systems it is not only enough to have the eigenvalues of matrix pair $(E, A)$ in the left half complex plane, but also to provide an impulse-free motion of system under consideration. Some different approaches have been shown in order to construct Lyapunov stability criterions of stability, asymptotic stability and unstabilty for this class of systems. Several numerical examples have, also, been worked out in order to illustrate the efficiency of methods proposed.

## Appendix A

## Drazin inverse

If $F$ is an $n \times n$ matrix, then $F^{D}$ is the unique solution of the equations:

$$
\begin{align*}
& F F^{D}=F^{D} F \\
& F^{D} F F^{D}=F  \tag{A1.}\\
& F^{D} F^{k+1}=F^{k}
\end{align*}
$$

where $k$ is the index of matrix $F, k=\operatorname{Ind}(F)$, defined to be the smallest non-negative integer such that:

$$
\begin{equation*}
\operatorname{rank} F^{j+1}=F^{j} \tag{A2.}
\end{equation*}
$$

With $\mathfrak{\aleph}(F)$ and $\mathfrak{R}(F)$ we will denote the kernel (null space) and range on any operator $F$, respectively, i.e.:

$$
\begin{gather*}
\mathcal{N}(F)=\left\{\mathbf{x}: F \mathbf{x}=0, \forall \mathbf{x} \in \mathfrak{R}^{n}\right\},  \tag{A3.}\\
\mathfrak{R}(F)=\left\{\mathbf{y} \in \mathfrak{R}^{m}, \mathbf{y}=F \mathbf{x}, \mathbf{x} \in \mathfrak{R}^{n}\right\},  \tag{A4.}\\
\operatorname{dim} \mathfrak{\aleph}(F)+\operatorname{dim} \Re(F)=n . \tag{A5.}
\end{gather*}
$$

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# ON THE GEOMETRICAL SENSE OF COVARIANT DIFFERENTIATION IN NON-EUCLIDEAN SPACE 

Zoran Drašković


#### Abstract

An approach to geometrical interpretation of the operation of covariant differentiation-where the role of operators of parallel transport in non-Euclidean space is also pointed out-is presented.


## Introduction

When considering the sense of the operation of covariant differentiation, either in Euclidean or in non-Euclidean spaces, the intention to provide a possibility of obtaining new tensor fields from the given one is usually underlined ${ }^{1}$. However, the fact that, on the one hand, this operation has a well defined geometrical sense (as a limit process), and, on the other hand, in non-Euclidean spaces is often introduced by analogy with the procedure in Euclidean space (and without stressing the possible geometrical interpretation), was the reason to point out a geometrical aspect of the operation of covariant differentiation in non-Euclidean spaces, too. But, first of all, we shall dwell

## On covariant differentiation in Euclidean space

It is well known that the expression for the covariant differentiation of a vector field $v=v^{i} g_{i}$ defined in a domain of Euclidean space reads ${ }^{2}$

$$
\begin{equation*}
\left.v_{, j}^{i}\right|_{P_{0}}=\left.\frac{\partial v^{i}}{\partial x^{j}}\right|_{P_{0}}+\left.\Gamma_{j k}^{i}\right|_{P_{0}} v^{k}\left(P_{0}\right), \tag{1}
\end{equation*}
$$

where $\Gamma_{j k}^{i}$ are Christoffel symbols of the second kind determined in curvilinear coordinates $x^{i}$ introduced in this space, $\boldsymbol{g}_{i}$ are the base vectors of these coordinates and $P_{0}$ is the point where the covariant differentiation is performed. It is well

[^2]known that the following equality (quoted in $\S 46$ of [6] when discussing the sense of the covariant differentiation)
\[

$$
\begin{equation*}
\left.\frac{\partial v}{\partial x^{j}}\right|_{P_{0}}=\left.v_{, j}^{i}\right|_{P_{0}} g_{i}\left(P_{0}\right) \tag{2}
\end{equation*}
$$

\]

holds, too. However, we can proceed in the following manner as well:

$$
\left.\left.\begin{array}{rl}
\left.\frac{\partial v}{\partial x^{j}}\right|_{P_{0}} & =\lim _{\Delta x^{j} \rightarrow 0} \frac{v(P)-v\left(P_{0}\right)}{\Delta x^{j}} \\
& =\lim _{\Delta x^{j} \rightarrow 0} \frac{v^{i}(P) g_{i}(P)-v^{i}\left(P_{0}\right) g_{i}\left(P_{0}\right)}{\Delta x^{j}} \\
& =\lim _{\Delta x^{j} \rightarrow 0} \frac{v^{k}(P) g_{. k}^{i}\left(P_{0}, P\right) g_{i}\left(P_{0}\right)-v^{i}\left(P_{0}\right) g_{i}\left(P_{0}\right)}{\Delta x^{j}} \\
& =g_{i}\left(P_{0}\right) \lim _{\Delta x^{j} \rightarrow 0} \frac{v^{k}(P) g_{. k}^{i}\left(P_{0}, P\right)-v^{i}\left(P_{0}\right)}{\Delta x^{j}}  \tag{3}\\
& =g_{i}\left(P_{0}\right) \lim _{\Delta x^{j} \rightarrow 0} \frac{\left[v^{k}(P)-v^{k}\left(P_{0}\right)\right] g_{. k}^{i}\left(P_{0}, P\right)+v^{i}\left(P_{0}\right)\left[g_{. k}^{i}\left(P_{0}, P\right)-\delta_{k}^{i}\right]}{\Delta x^{j}} \\
& =g_{i}\left(P_{0}\right)\left[\lim _{\Delta x^{j} \rightarrow 0} \frac{v^{k}(P)-v^{k}\left(P_{0}\right)}{\Delta x^{j}} \lim _{\Delta x^{j} \rightarrow 0} g_{. k}^{i}\left(P_{0}, P\right)\right. \\
& =g_{i}\left(P_{0}\right)\left[\left.\frac{\partial v^{k}(P)}{\partial x^{j}}\right|_{P_{0}} \delta_{k}^{i}+\left.v^{k}\left(P_{0}\right) \frac{\partial g_{. k}^{i}\left(P_{0}, P\right)}{\partial x^{j}}\right|_{P_{0}}\right] \\
& =g_{i}\left(P_{0}\right)\left[\left.\frac{\partial v^{i}(P)}{\partial x^{j}}\right|_{P_{0}}+v^{k}\left(P_{0}, P\right)-\delta_{k}^{i}\right. \\
\partial x_{. k}^{j}\left(P_{0}, P\right) \\
\partial x^{j}
\end{array}\right|_{P_{0}}\right] \quad \$ 7
$$

where $g_{. j}^{i}$ are the shifting operators ${ }^{3}$ ("Euclidean shifters"; [4], p. 806). In this manner, the necessity of parallel transport (from the "current" point $P$ to the point $P_{0}$ where the derivation is performed) of a vector considered in this limit is unambiguously pointed out - this is a geometrical aspect of the operation of covariant differentiation.

## On covariant differentiation in non-Euclidean spaces

It is also well known that the expression for the covariant differentiation of a vector field. $v=v^{\alpha} a_{\alpha}$ defined in a domain of Riemannian space, i.e., on a surface ${ }^{4}$ (if we dwell on the two-dimensional case), reads analogously to the expression (1)

$$
\begin{equation*}
\left.v_{, \beta}^{\alpha}\right|_{P_{0}}=\left.\frac{\partial v^{\alpha}}{\partial u^{\beta}}\right|_{P_{0}}+\left.\Gamma_{\beta \gamma}^{\alpha}\right|_{P_{0}} v^{\gamma}\left(P_{0}\right), \tag{4}
\end{equation*}
$$

[^3]where $u^{\alpha}$ are so-called surface coordinates and $\Gamma_{\beta \gamma}^{\alpha}$ are Christoffel symbols of the second kind determined for this surface in coordinates $u^{\alpha}$. Analogously to the relation (2), the following relation:
\[

$$
\begin{equation*}
\left.\frac{\partial v}{\partial u^{\beta}}\right|_{P_{0}}=\left.v_{, \beta}^{\alpha}\right|_{P_{0}} a_{\alpha}\left(P_{0}\right) \tag{5}
\end{equation*}
$$

\]

can be established, too.
However, an attempt to establish the corresponding limit in the following case:

$$
\begin{equation*}
\left.\frac{\partial v}{\partial u^{\beta}}\right|_{P_{0}}=\lim _{\Delta u^{\beta} \rightarrow 0} \frac{v(P)-v\left(P_{0}\right)}{\Delta u^{\beta}}, \tag{6}
\end{equation*}
$$

immediately imposes a question of the procedure of transport of the quantities $v(P)$ and $v\left(P_{0}\right)$ to the same point in order to compare, i.e., to subtract, them. Putting aside, for a moment, the essence of this transport, let us suppose the existence of operators $K_{\beta}^{\alpha}\left(P_{0}, P\right)$ such that ${ }^{5}$

$$
\begin{equation*}
\overline{\bar{v}}^{\alpha}\left(P_{0}\right)=K_{. \beta}^{\alpha}\left(P_{0}, P\right) v^{\beta}(P), \tag{7}
\end{equation*}
$$

and, in order to perform an inverse process, the existence of inverse operators ${ }^{6}$ $K_{\beta}^{\alpha}\left(P_{0}, P\right)$

$$
\begin{equation*}
v^{\alpha}(P)=K_{\beta}^{\cdot \alpha}\left(P_{0}, P\right) \overline{\bar{v}}^{\beta}(P) \tag{8}
\end{equation*}
$$

where ${ }^{7}$

$$
\begin{equation*}
K_{\gamma}^{\beta}\left(P_{0}, P\right) K_{\cdot \beta}^{\alpha}\left(P_{0}, P\right)=\delta_{\gamma}^{\alpha} \text { and } K_{\gamma}^{\beta}\left(P_{0}, P\right) K_{\alpha}^{\gamma}\left(P_{0}, P\right)=\delta_{\alpha}^{\beta} . \tag{9}
\end{equation*}
$$

Using these quantities we can proceed analogously to (3), thus obtaining

$$
\begin{equation*}
\left.\frac{\partial v}{\partial u^{\beta}}\right|_{P_{0}}=a_{\alpha}\left(P_{0}\right)\left[\left.\frac{\partial v^{\alpha}(P)}{\partial u^{\beta}}\right|_{P_{0}}+\left.v^{\gamma}\left(P_{0}\right) \frac{\partial K_{\cdot \gamma}^{\alpha}}{\partial u^{\beta}}\right|_{P_{0}}\right] . \tag{10}
\end{equation*}
$$

It is now clear that the manner of this transport, if we want to preserve the usual expression (4) for covariant differentiation of a vector field, must satisfy the following condition:

$$
\begin{equation*}
\left.\frac{\partial K_{. \gamma}^{\alpha}\left(P_{0}, P\right)}{\partial u^{\beta}}\right|_{P_{0}}=\left.\Gamma_{\beta \gamma}^{\alpha}\right|_{P_{0}}=\left.\Gamma_{\beta \gamma}^{\varepsilon}\right|_{P_{0}} \delta_{\varepsilon}^{\alpha} \tag{11}
\end{equation*}
$$

[^4]which (because of $\left.\left.K_{. \varepsilon}^{\alpha}\left(P_{0}, P\right)\right|_{P_{0}}=\delta_{. \varepsilon}^{\alpha}\right)$ can be rewritten in the form
\[

$$
\begin{equation*}
\left.\left[\frac{\partial K_{. \gamma}^{\alpha}\left(P_{0}, P\right)}{\partial u^{\beta}}-\Gamma_{\beta \gamma}^{\varepsilon}(P) K_{. \varepsilon}^{\alpha}\left(P_{0}, P\right)\right]\right|_{P_{0}}=0 \tag{12}
\end{equation*}
$$

\]

i.e., (bearing in mind that the transport is performed along a curve $K$ with parametric equations $u^{\alpha}=u^{\alpha}(t)$, so the composition with $d u^{\beta} /\left.d t\right|_{P_{0}}$ is possible) in the form

$$
\begin{equation*}
\left.\left[\frac{d K_{\gamma \gamma}^{\alpha}\left(P_{0}, P\right)}{d t}-\Gamma_{\beta \gamma}^{\varepsilon}(P) K_{\varepsilon}^{\alpha}\left(P_{0}, P\right) \frac{d u^{\beta}}{d t}\right]\right|_{P_{0}}=0 \tag{13}
\end{equation*}
$$

However, due to the arbitrary character of the points $P_{0}$ and $P$, we conclude that the system of functions $K_{. \beta}^{\alpha}$ should satisfy (in each point of the above mentioned curve) the following system of differential equations:

$$
\begin{equation*}
\frac{d v_{\gamma}}{d t}-\Gamma_{\beta \gamma}^{\varepsilon} v_{\varepsilon} \frac{d u^{\beta}}{d t}=0 \tag{14}
\end{equation*}
$$

i.e., that the system of functions $K_{\beta}^{\cdot \alpha}$ should satisfy (along this curve) the system of differential equations

$$
\begin{equation*}
\frac{d v^{\alpha}}{d t}+\Gamma_{\beta \varepsilon}^{\alpha} v^{\varepsilon} \frac{d u^{\beta}}{d t}=0 \tag{15}
\end{equation*}
$$

hence, bearing in mind that $\left|K_{. \beta}^{\alpha}\right| \neq 0$ and $\left|K_{\beta}^{\alpha}\right| \neq 0$, the system of functions $K_{\beta}^{\alpha}$ shall represent the fundamental system of solutions of the homogeneous system ${ }^{8}$ (14), i.e., (15).

On the other hand, it was pointed out in [13] that the fundamental system of solutions represents the operators of parallel transport ${ }^{9}$ along the curve in whose points the system (14), i.e., (15) is satisfied. Hence, if we want the covariant derivative of a vector field in this two-dimensional space to have the form (4), it follows that the operators $K$ introduced in (7) and (8) must be the operators of parallel ${ }^{10}$ transport with respect to the surface along the given curve on this surface.

## Conclusion

This not so rigorous ${ }^{11}$ deduction points out, in a natural ${ }^{12}$ way (i.e., by considering the limit process in the definition of the operation of covariant differentiation), the reasonableness of introducing the notion of operators of parallel ${ }^{13}$ transport in Riemannian spaces, too.

[^5]
## Appendix: Operators of parallel transport along parallels on a spherical surface

Although, on the one hand, the operators of parallel transport along a curve in Riemannian space were introduced a long time ago ${ }^{14}$ (but without being explicitly determined), while, on the other hand, the sense of introducing the operation of absolute integration in non-Euclidean spaces (postulated in [8,10]) remained longcontested, mainly due to the fact that the operators of parallel transport which appear in this case had not been determined in the general case ${ }^{15}$ - the fact that the fundamental system of solutions for the homogeneous system (14) or (15) always exists, i.e., that this fundamental system represents the shifting operators along a curve where the system (14) or (15) is satisfied, was pointed out in [13], with reference to [1].

However, the existence of a fundamental system of solutions for the system (15) along a given curve, i.e., the existence of shifting operators along this curve, does not necessarily mean it is easy to find them. From the following well-known example, we shall see that these operators were at hand (for the simpler cases, at least) for a long time, but without being recognized.

As mentioned above, the system of differential equations for determining the coordinates of a vector parallelly propagated along a curve on a surface reads

$$
\begin{equation*}
\frac{d v^{\alpha}}{d s}+\Gamma_{\beta \gamma}^{\alpha} \frac{d u^{\gamma}}{d s}=0 . \tag{A.1}
\end{equation*}
$$

But, in the case of transport along the $\varphi$-parallel ${ }^{16}$ of a spherical surface with the radius $a$ we have $u^{1} \equiv \varphi=s / a \cos \vartheta_{0}, u^{2} \equiv \vartheta=\vartheta_{0}=$ const and, bearing in mind that only the three coordinates of the Christoffel symbols of the second kind are non-zero in geographical coordinates ( $\Gamma_{12}^{1}=\Gamma_{21}^{1}=-\operatorname{tg} \vartheta_{0}$ and $\left.\Gamma_{11}^{2}=\sin \vartheta_{0} \cos \vartheta_{0}\right)$, this system reduces to

$$
\begin{align*}
d v^{1} / d \varphi & =v^{2} \operatorname{tg} \vartheta_{0}  \tag{A.2}\\
d v^{1} / d \varphi & =-v^{1} \sin \vartheta_{0} \cos \vartheta_{0} .
\end{align*}
$$

The characteristic equation of this system of differential equations reads

$$
\left|\begin{array}{cc}
-\lambda & \operatorname{tg} \vartheta_{0}  \tag{A.3}\\
-\sin \vartheta_{0} \cos \vartheta_{0} & -\lambda
\end{array}\right|=0 ;
$$

hence $\lambda= \pm \sin \vartheta_{0} i$ and the general solution may be written in the form (see [9], p. 531)

$$
\begin{align*}
& v^{1}=C_{1} \operatorname{tg} \vartheta_{0} \cos \left(\varphi \sin \vartheta_{0}\right)+C_{2} \operatorname{tg} \vartheta_{0} \sin \left(\varphi \sin \vartheta_{0}\right)  \tag{A.4}\\
& v^{2}=-C_{1} \sin \vartheta_{0} \sin \left(\varphi \sin \vartheta_{0}\right)+C_{2} \sin \vartheta_{0} \cos \left(\varphi \sin \vartheta_{0}\right) .
\end{align*}
$$

[^6]We shall find the constants $C_{1} \mathrm{i} C_{2}$ from the condition that $v^{1}=v_{0}^{1}$ and $v^{2}=v_{0}^{2}$ for $\varphi=\varphi_{0}$. We thus obtain ${ }^{17}$

$$
\begin{align*}
& v^{1}=v_{0}^{1} \cos \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]+v_{0}^{2} \frac{\sin \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]}{\cos \vartheta_{0}}  \tag{A.5}\\
& v^{2}=-v_{0}^{1} \cos \vartheta_{0} \sin \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]+v_{0}^{2} \cos \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]
\end{align*}
$$

and, bearing in mind that the solution of the system of differential equations (A.1) represents the coordinates of a vector parallelly propagated along a curve on the given surface, it follows that the quantities (cf. with the expressions (7) and (8))

$$
\begin{align*}
\left\{K_{\beta}^{\alpha}\left(P, P_{0}\right)\right\} & =\left\{\begin{array}{ll}
K_{i}^{1} & K_{\dot{2}}^{1} \\
K_{1}^{2} & K_{2}^{2}
\end{array}\right\}  \tag{A.6}\\
& =\left\{\begin{array}{cc}
\cos \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right] & \frac{\sin \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]}{\cos \vartheta_{0}} \\
-\cos \vartheta_{0} \sin \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right] & \cos \left[\left(\varphi-\varphi_{0}\right) \sin \vartheta_{0}\right]
\end{array}\right\}
\end{align*}
$$

are the coordinates of shifting operators along the parallel connecting the points $P_{0}$ and $P$ on the spherical surface ${ }^{18}$. Note that this form of the coefficients $K$ could have been anticipated by noticing that the two following solutions of the system (A.2)

$$
\begin{align*}
& \left\{\begin{array}{l}
v_{(1)}^{1} \\
v_{(1)}^{2}
\end{array}\right\}=\left\{\begin{array}{cc}
\operatorname{tg} \vartheta_{0} & \cos \left(\varphi \sin \vartheta_{0}\right) \\
-\sin \vartheta_{0} & \sin \left(\varphi \sin \vartheta_{0}\right)
\end{array}\right\}  \tag{A.7}\\
& \left\{\begin{array}{c}
v_{(2)}^{1} \\
v_{(2)}^{2}
\end{array}\right\}=\left\{\begin{array}{cc}
\operatorname{tg} \vartheta_{0} & \sin \left(\varphi \sin \vartheta_{0}\right) \\
\sin \vartheta_{0} & \cos \left(\varphi \sin \vartheta_{0}\right)
\end{array}\right\}
\end{align*}
$$

form its fundamental system of solutions, since $\operatorname{Det}\left\{\begin{array}{cc}v_{(1)}^{1} & v_{(2)}^{1} \\ v_{(1)}^{2} & v_{(2)}^{2}\end{array}\right\} \neq 0\left(\vartheta_{0} \neq 0\right)$ ! Now, when we have obtained the explicit expressions for the operators of parallel transport $K_{\beta}^{\alpha}$ along the parallels on a spherical surface, the covariant coordinates of a vector shifted on this surface from the point $P_{0}$ to the point $P$ (along the arc of the parallel connecting them) would be calculated according to the formula

$$
\begin{equation*}
\overline{\bar{v}}^{\alpha}(P)=K_{\beta}^{\alpha}\left(P_{0}, P\right) v^{\beta}\left(P_{0}\right) \tag{A.8}
\end{equation*}
$$

(where $v^{1} \equiv v^{\varphi}, v^{2} \equiv v^{\vartheta}$ ) and the process of parallel transport of a vector along such curves on a spherical surface can easily be represented graphically. Let us consider a vector field with the coordinates

$$
\begin{align*}
& v^{1}=v_{0}^{1}=0 \\
& v^{2}=v_{0}^{2}=\text { const } \neq 0 \tag{A.9}
\end{align*}
$$

[^7]and let us perform its propagation from the point $P_{0}$ to the point $P \equiv P_{0}$ along the parallel connecting them, i.e., along the closed curve. Figure 1 illustrates the wellknown fact that, if a vector is transported parallelly along a contour on a spherical surface, then we might not obtain the same vector upon return to the starting point (see e.g., p. 154 in [7] or p. 185 in [11]), i.e., that the parallel displacement with respect to a surface generally depends on the path.


Fig. 1

Furthermore, the graphical representation of the procedure of transport of the vector $v(P)$ to the point $P_{0}$, where the differentiation of the vector field is performed, seems to be interesting as well. If we refer to field (A.9) once again, the transport to be performed inside the limit (6) will occur as shown in Figure 2. However, the limit $(6)^{19}$ itself can be represented as well (see Fig. 3) - this is only the graphical illustration of the fact, shown in Figure 1, that the vectors obtained by parallel transport of the vector field normal to the parallel along which the transport is performed differ from the field value in the corresponding point; hence the limiting process converges to a value different from zero, i.e., $v, 1 \neq 0$ in this point ${ }^{20}$.


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# FLOATING STABILITY OF A SOLID BODY IN A FLUID 

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#### Abstract

The analysis of the stability of floating of a solid body is made on the basis of the theoretical formulas defined in shipbuilding. This theory is not equivalent with Liapunov's theory since the stability is estimated only for one generalized coordinate $\varphi$ - deviation of the symmetry plane of the ship from the vertical. The final conclusions on the stability are made on the basis of mutual positions of the mass center $C_{m}$ of a solid body and the metacenter $M$. In practice it was observed that the solid body in the course of floating can change its positions many times. The present paper analyzes particularly the singular cases $k=0$ and $k=1$, where $k=\rho_{m} / \rho$ - the ratio of an average density $\left(\rho_{m}\right)$ of a solid body and the density $(\rho)$ of the fluid. The paper provides the theoretical explanation why the solid body - the circular cylinder, changes its positions of a stable equilibrium by $\varphi=90^{\circ}$ immediately upon sinking to the bottom. Besides the stability of other bodies was investigated: square, cube, and a circular cone. The theoretical explanations are given under which conditions the stable, labile, and indifferent positions of carried out later on the mentioned bodies show the overlapping with the theory.


## 1. Introduction

The initial ideas for the present paper where the consequence of the three TV broadcasts (December 1999. BK, June 2000. Palma, June 2000. Palma). In these broadcasts the scientists from the USA and Yugoslavia had commented the consequences of the explosion of the volcano St. Helen, on $18^{\text {th }}$ May 1980. The volcano is in the State Washington and in its activity he expelled $2 \mathrm{~km}^{3}$ of solid substance, plus $100 \mathrm{~km}^{3}$ of gases $\left(\mathrm{SO}_{2}, \mathrm{CO}_{2}, \mathrm{H}_{2} \mathrm{O}, \mathrm{CO}\right)$. Apart from that $400 \mathrm{~km}^{2}$ of the forest was destroyed and partly thrown into the lake The Spirit at the foot of the volcano. The trees were floating on the surface of the lake for several days in the position shown in fig. 1a. Upon to 6 days some arbors prior to sinking to the bottom turned spontaneously their axis into a vertical position, fig. 1b.

The scientists commenting this phenomenon did not supply a real reply why this changed had occurred. This paper, in the author's opinion, gives a real theoretical basis for the above mentioned phenomena.

a) $\mathrm{H}>\overline{\mathrm{H}}_{\mathrm{kI}}$
b) $\mathrm{H}<\mathrm{H}_{\mathrm{kr}}$

Fig. 1. The transformation of an indifferent position of the balance
(a) by $\mathrm{H}>\mathrm{H}_{\mathrm{ks}}$ into a stable position (b) $\mathrm{H}<\mathrm{H}_{\mathrm{ks}}$

fig. 2.
The submarine sailing in the water

fig. 3.
The balloon in the air

$\mathrm{Q}=\mathrm{mg}-$ weight, $\mathrm{Z}_{0}=\mathrm{m} / \mathrm{pBL}, \quad \mathrm{I}_{\text {min }}=\mathrm{LB}^{3} / 12, \quad \mathrm{~V}_{\mathrm{P}}=\mathrm{BLZ}_{0}-$ submerged volume
Fig. 4. Floating of parallelepiped in the water

## Exposition of the paper

A. Let us observe primarily a simpler variant (A) when solid body is entirely submerged into a fluid (submarine in the water fig. 2 and the balloon in the air fig.3.)

In both cases the thrust force is:

$$
P=\rho g V_{p}
$$

and the weight:

$$
\mathrm{Q}=\mathrm{mg}
$$

The weight acts vertically in the direction of the decrease of the angle $\varphi$. That is why in this case the conditions of a stable floating reads:
For a stable floating of a solid body $|\varphi|<15^{\circ}$ entirely submerged into the fluid the center of gravity of mass $C_{m}$ must be under the center of gravity $C_{V_{p}}$ of the submerged volume. Apart from that the solid body can be in equilibrium ( $\mathrm{P}=\mathrm{mg}$ ), and it can move vertically and horizontally depending on the initial conditions and on the whether $\mathrm{P}>\mathrm{mg}$ or $\mathrm{P}<\mathrm{mg}$. This analysis takes into account only two forces: the weight and the thrust force. In the case that other forces are acting, too, (waves, engine, wind) the mechanical analysis is more complex.
(B) Let us consider now the floating of the solid body which is partly submerged into the fluid fig. 4. - variant (B)
The condition of a stable floating reads:
The solid body partly submerged into the fluid for $|\varphi|<15^{\circ}$ floats stabile, if the center of gravity of the mass $C_{m}$ is under the metacenter $M$. It should be kept in mind that for small deviations $|\varphi|<15^{\circ}$ the thrust force $\mathrm{P}=\rho g \mathrm{~V}_{\mathrm{P}}$ act vertically up. The force acts on the coating of the submerged body with the attacking point on the coating. The attacking line passes through the center of the thrusting force $\mathrm{P}=\mathrm{mg}$, through the symmetry plane $z O y$. It is easier to find this point for the inclined position $0<\varphi<15^{\circ}$. In the shipbuilding theory the stability condition is satisfied in two variants:

1. If $\mathrm{C}_{\mathrm{m}}$ is under $\mathrm{C}_{\mathrm{V}_{\mathrm{p}}}$ - a sufficient condition
2. If $\mathrm{C}_{\mathrm{m}}$ is above $\mathrm{C}_{\mathrm{V}_{\mathrm{p}}}$ and $\delta<\mathrm{r}$

In the first case the analysis is identical with (A). The second case is partially explained in literature $[1,2,3]$; the shortened version is provided here. The essence of the complete analysis is that the center of gravity of the mass $C_{m}$ in a stable floating must be under the metacenter M . Then the weighting force and the thrusting force will act as a couple for stabilizing the solid body. The complete calculations $[1,2,3]$ shows that in variant $\mathrm{B}-2^{\circ}$ three important points $\mathrm{C}_{\mathrm{m}}, \mathrm{C}_{\mathrm{V}_{\mathrm{p}}}$ and M have the following distances:

$$
\begin{equation*}
\delta=\overline{C_{m} C_{V p}} \quad r=\frac{I_{\min }\left(A^{*}\right)}{V_{P}}=\overline{M C_{V_{p}}} \tag{1}
\end{equation*}
$$

where: $I_{\text {min }}\left(A^{*}\right)=L B^{3} / 12$-the axial moment of the inertia of the surface $A^{*}=B L$, which represent the flat surface obtained by the cross section of the solid body coating with the horizontal surfase of the fluid. $\mathrm{V}_{\mathrm{p}}=\mathrm{BL} \cdot \mathrm{Z}_{0}$ - represents the submerged volume of the solid body.

This paper stresses particularly that the explanation of the authors of the books $[1,2,3]$, that $\mathrm{C}_{\mathrm{V}_{\mathrm{p}}}$ - center of the pressure is simultaneously the attacking point of force P $=\rho g V_{p}$ is not correct. The following explanation is correct: Its attacking point is at the bottom of the solid body coating, it acts vertically up through the metacenter M . If equation (1) is applied to parallelepiped

$$
\begin{gather*}
\delta=\frac{1}{2}\left(H-Z_{0}\right)=\frac{H}{2}(1-k), \quad 0<\frac{\rho_{m}}{\rho}=k<1  \tag{2}\\
r=\frac{L B^{3}}{12 L B Z_{o}}=\frac{B^{2}}{12 Z_{o}}=\frac{B^{2}}{12 k H} \tag{3}
\end{gather*}
$$

From the stability condition $\delta<r \Rightarrow$

$$
\begin{equation*}
H<B[6 k(1-k)]^{-1 / 2}=H_{k r}(k) \tag{4}
\end{equation*}
$$

We conclude that parallelepiped floating is stable if its height H is not too high. The singular points are: $\mathrm{k}=0$ and $\mathrm{k}=1$.

$$
\begin{equation*}
\lim _{k \rightarrow 0^{+}} H_{k k}(k)=\lim _{k \rightarrow 1^{-}}^{H_{k j}}(k)=\infty \tag{5}
\end{equation*}
$$

$$
\mathrm{k}_{1}=0.21 \quad \mathrm{k}_{3}=9 / 32=0.28
$$

$$
\mathrm{k}_{4}=1-\mathrm{k}_{3}
$$

$$
\mathrm{k}_{2}=0.79
$$

Fig. 5. The position of the stable floating of the square $B=H<L$ in the density function $\rho_{\mathrm{m}}=\mathrm{k} \rho$.

In this case the parallelepiped floats stabile in its position (fig. 4.) without any restrictions.
Of interest is the case $\mathrm{H}=\mathrm{B}$ when conditions of the stabile floating is obtained from inequality (4) in the form:

$$
\begin{align*}
& \mathrm{f}(\mathrm{k})=6 \mathrm{k}^{2}-6 \mathrm{k}+1>0  \tag{6}\\
& \mathrm{k}_{1,2}=\left(3 \pm 3^{1 / 2}\right) / 6, \quad \mathrm{k}_{1}=0.21, \quad \mathrm{k}_{2}=0.79 \tag{7}
\end{align*}
$$

The values $k_{3}$ and $k_{4}$ were calculated in the book [3], so that the square can have two more stable positions of equilibrium for $k_{1}<k<k_{3}$ and $k_{4}<k<k_{2}$, which are located between those shown in fig. 5 .

The experiment with a wooden square $\mathrm{H}=\mathrm{B}<\mathrm{L}$ shows exactly the same situations as fig. 5 . That is why it is not strange that the tree of small density $\rho_{\mathrm{m}}=\mathrm{k} \rho, \mathrm{k}$ $<\mathrm{k}_{1}$ upon a longer stay in the water floats stabile in 5 different positions. It changes those positions spontaneously depending on the rate of the absorption of the water and the change of its density in accordance with fig. 5 . The experiments with the cube ( $B=$ $H=L)$ show the same situation for $k<k_{1}$ and $k>k_{2}$, while for $k_{3}<k<k_{4}$ the cube


$$
\begin{gathered}
\mathrm{Q}=\mathrm{mg}=\mathrm{pgR} \mathrm{R}^{2} \pi \mathrm{H}, \rho_{\mathrm{m}}=\mathrm{k} \rho<\rho, \mathrm{Z}_{\mathrm{o}}=\mathrm{kH}, \quad \delta=\mathrm{H}(1-\mathrm{k}) / 2, \\
\mathrm{r}=\mathrm{I}_{\text {min }}\left(\mathrm{A}^{*}\right) / \mathrm{V}_{\mathrm{p}}=\mathrm{D}^{2} /\left(16 \mathrm{Z}_{0}\right), A^{*}=\mathrm{R}^{2} \pi
\end{gathered}
$$

Fig. 6. Floating of the cylinder in the water $\left(\mathrm{H}<\mathrm{H}_{\mathrm{kr}}\right)$
floats in such a way that it has a big diagonal on its vertical $D=B \cdot 3^{1 / 2}$. In the case $k_{1}<k$ $<\mathrm{k}_{3}$ and $\mathrm{k}_{4}<\mathrm{k}<\mathrm{k}_{2}$ the cube has the position of a stabile floating so that the big diagonal is neither vertical nor horizontal.

Let us consider now the floating of a circular cylinder of diameter 2 R , of height $H$, density $\rho_{m}=k \rho$, in the water of density $\rho$. The condition of a stabile floating $\delta<r$ leads to the inequality:

$$
\begin{equation*}
H<D[8 k(1-k)]^{-1 / 2}=H_{k r}(k) \tag{8}
\end{equation*}
$$

The singular values are $\mathrm{k}=0$ and $\mathrm{k}=1$. The floating will be stabile in the position displayed in fig. 6. if $\mathrm{H}<\mathrm{H}_{\mathrm{ks}}$. Apart from that:

$$
\begin{equation*}
\lim _{k \rightarrow 0^{+}}^{H_{k r}}(k)=\lim \underset{k \rightarrow 1^{-}}{H_{k k}}(k)=\infty \tag{9}
\end{equation*}
$$

Practically it means that the cylinder which was for a long time in the water upon sufficient amount of the absorbed water will have the density:

$$
\rho_{\mathrm{m}} \approx \rho \Rightarrow \mathrm{k} \approx 1^{-}
$$

The consequence of it is that it can float stabile in the position according to fig. 6. Let us analyze the real data trees:

| Type of the tree | $k$ |
| :---: | :---: |
| oak | $0.4-0.96$ |
| beech | $0.7-0.97$ |
| pine | $0.33-0.89$ |
| ebony | $1-1.26$ |

When the tree stays a longer time in the water its density increases according to the table below

| $\mathrm{N}^{\circ}$ of <br> days | k | $\mathrm{H}_{\mathrm{kr}}$ |
| :---: | :---: | :---: |
| 1 | 0.5 | 0.71 D |
| 2 | 0.6 | 0.72 D |
| 3 | 0.7 | 0.77 D |
| 4 | 0.8 | 0.88 D |
| 5 | 0.9 | 1.18 D |
| 6 | 0.98 | 2.53 D |
| 7 | 0.99 | 3.55 D |
| 8 | 0.999 | 11.2 D |

The first days the tree floats in the position (Fig. 1a), since the limiting height $\mathrm{H}_{\mathrm{kx}}$ is small. Upon $6-7$ days $H_{k T} \approx 3.55 \mathrm{D}$, some arbor turn already in the position as in fig. 1b. Upon 10 days a much larger number of trees turn, since $H_{k T}>11 D$; this condition is satisfied in a majority of arbors. The analysis carried out on the circular cone turned by its top downwards leads to the following inequality:

$$
\begin{equation*}
H<\frac{D}{2}\left[k^{1 / 3}\left(1-k^{1 / 3}\right)\right]^{1 / 2}=H_{k r}(k) \tag{10}
\end{equation*}
$$

the other values being:

$$
\mathrm{h}=\mathrm{H} \cdot \mathrm{k}^{1 / 3}, \quad \delta=3\left(1-\mathrm{k}^{1 / 3}\right) \mathrm{H} / 4, \quad r=\frac{3}{16} \frac{D^{2}}{H} k^{1 / 3}
$$

So for the special values of $k$ we get:

| k | $\mathrm{H}_{\mathrm{kr}}$ |
| :---: | :---: |
| 0.8 | 1.8 D |


| 0.9 | 2.64 D |
| :---: | :---: |
| 0.98 | 6.1 D |
| 0.99 | 8.63 D |

The experiments with a wooden circular cone show the overlapping with the calculation provided.

## Conclusion

A homogeneous circular cylinder of density $\rho_{\mathrm{m}}=\mathrm{k} \rho, \quad 0<\mathrm{k}<1$ floats in the liquid of density $\rho$ in the position shown in fig. 6. if $\mathrm{H}<\mathrm{H}_{\mathrm{kr}}$ - inequality (8). If the cylinder is of wood, which at length absorbs the water, then $k \rightarrow 1^{\text {, }}$, so that from (9) we see that $H_{k s}$ $\rightarrow \infty$. That is why a mass turning of arbors from the position, fig. 1a into position fig. 1 b . Afterwards the wooden cylinder sings at the bottom of the lake in a vertical position. This conclusion is valid if the initial density $0<\rho_{\mathrm{m}}<\rho \Leftrightarrow 0<k<1$, which is accurate in many cases (oak, beech pine, linden). In case of ebony $\mathrm{k}>1$, the tree sinks stabile with the center of gravity $C_{m}$ below $C_{V_{p}}$ (variant $A$ ).

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# SPECTRAL LINE SHAPES IN ASTROPHYSICS 

Milan S. Dimitrijević


#### Abstract

A review of astrophysical problems where data on spectral line shapes broadened by charged particle impacts are of interest is given. Also, the results of spectral line shapes reserches in Yugoslavia relevant to astrophysical problems have been reviewed and discussed.


1. Astrophysical problems where data on spectral line shapes broadened by charged particles are needed

From celestial objects out of our Solar system we receive only their radiation and only by analysis of this radiation we may derive their properties. One powerful tool for such analyses is spectroscopy and it is interesting how many facts we might obtain from spectral lines. We may determine for example the chemical composition, temperature, electron density, surface gravity... In comparison with laboratory plasma sources, plasma conditions in astrophysical plasmas are incredibly various. Consequently, broadening due to interaction between emitter and charged particles (Stark broadening) is of interest in astrophysics in plasmas of such extreme conditions like in the interstellar molecular clouds or neutron star atmospheres, which can not be obtained in laboratory.

In interstellar molecular clouds, typical electron temperatures are around 30 K or smaller, and typical electron densities are $2-15 \mathrm{~cm}^{-3}$. In such conditions, free electrons may be captured (recombination) by an ion in very distant orbit with principal quantum number $(n)$ values of several hundreds and deexcite in cascade to energy levels $n-1, n-2, \ldots$ radiating in radio domain. Such distant electrons are weakly bounded with the core and may be influenced by very weak electric microfield. Consequently, Stark broadening may be significant (Omont and Encrenaz, 1977). In interstellar ionized hydrogen clouds, electron temperatures are around 10000 K and electron density is of the order of $10^{4} \mathrm{~cm}^{-3}$ (Smirnov et al, 1984). Corresponding series of adjacent radio recombination lines originating from energy levels with high (up to several hundreds) $n$ values are influenced by Stark broadening.

For $T_{\text {eff }}>10^{4} \mathrm{~K}$, hydrogen, the main constituent of stellar atmospheres is mainly ionized, and among collisional broadening mechanisms for spectral lines, the dominant is the Stark effect. This is the case for white dwarfs and hot stars of $\mathrm{O}, \mathrm{B}$ and A type. Even in cooler star atmospheres as e.g. Solar one, Stark broadening may be important. For example, the influence of Stark broadening within a spectral series increases with the increase of the principal quantum number of the upper level (Dimitrijević and Sahal-Bréchot, 1984ab; 1985) and consequently, Stark broadening contribution may become significant even in the Solar spectrum (Vince and Dimitrijević, 1985; Vince et al, 1985ab).

If you look for the star with the largest importance of Stark broadening, among such stars are PG1159 stars, hot hydrogen deficient pre-white dwarfs, with effective temperatures ranging from $T_{e f f}=100000 \mathrm{~K}$ (for PG1424+535 and PG1707+427) to $T_{e f f}=140000 \mathrm{~K}$ (for PG1159-035 and PG1520+525) (Werner et al, 1991). All such stars have the similar high surface gravity $(\log g=7)$. The photospheres are dominated by helium and carbon with a significant amount of oxygen present $(\mathrm{C} / \mathrm{He}=0.5$ and $\mathrm{O} / \mathrm{He}=0.13)$ (Werner et al, 1991). Their spectra, strongly influenced by Stark broadening, are dominated by He II, C IV, O VI and N V lines.

The densities of matter and electron concentrations and temperatures in atmospheres of neutron stars are orders of magnitude larger than in atmospheres of white dwarfs, and are typical for stellar interiors. Surface temperatures for the photospheric emission are of the order of $10^{6}-10^{7} \mathrm{~K}$ and electron densities of the order of $10^{24} \mathrm{~cm}^{-3}$ (Madej, 1989; Paerels, 1997). In Paerels (1997) has been obtained that the final opacity profile of He-like iron resonant line is therefore given by a Voigt profile, with a total damping parameter equal to the sum of natural and Stark (electron - impact) broadening. With the improved sensitivity of space born X-ray instruments, spectral lines originating from neutron star atmospheres should be resolved in the near future. Since the characteristic density in the atmosphere is directly proportional to the acceleration of gravity at the stellar surface, measurement of the pressure broadening of absorption lines will yield a direct measurement of $M / R^{2}$, where $M$ and $R$ are the stellar mass and radius. When this is coupled with a measurement of the gravitational redshift (proportional to $M / R$ ) in the same, or any other, line or set of lines, the mass and radius can be determined separately. These mass and radius measurements do not involve the distance to star, which is usually poorly determined, or the size of the emitting area (Paerels, 1997).

To the increasing needs for Stark broadening data, have especially contributed the space born spectroscopic instruments and the development of computers. With the development of space astronomy, an extensive amount of spectroscopic information over large spectral regions of all kind of celestial objects has been and will be collected, stimulating the spectral-line-shape research. Moreover, due to drastical increase of accuracy and possibilities, even the spectra of trace elements obtain an increasing astrophysical interest.

Development of computers also stimulated the need for a large amount of atomic and spectroscopic data, enabling the research on more complex problems than before. Particularly large number of data is needed for example for opacity
calculations. An illustrative example might be the article on the calculation of opacities for classical Cepheid models (Iglesias et al, 1990), where $11996532 \mathrm{spec}-$ tral lines have been taken into account and where Stark broadening contribution has been calculated within the Modified semiempirical method of Dimitrijević and Konjević (1980).

Interesting investigations which become possible with the development of computer technology, are calculations of equivalent width changes with the age in starburst stellar clusters and galaxies. In Gonzales - Delgado et al (1999), the change of particular hydrogen and helium lines equivalent widths during 500 milion years, has been calculated and compared with observations of stellar clusters of the Large Magellanic Clouds, the super - star clusters in the starburst galaxy NGC 205, the nucleus of the dwarf elliptic galaxy NGC 205, and a luminous "E+A" galaxy. Calculations have been done in two steps. First, the population of stars of different spectral types, as a function of age are calculated, and then the profiles of the lines are synthesized by adding the different contributions from stars. For spectral line profiles synthesis the effects of natural, Stark, Van der Waals and thermal Doppler broadening have been taken into account.

## 2. Line shapes investigations in Yugoslavia and Serbia within the period march 1997-31 december 2000 and astrophysical significance of some of obtained results

From the first paper on spectral line shapes investigations published in 1962 by Vladis Vujnović, up to the 31 December 2000, 1427 (1222 by serbian authors) bibliographic items have been published by 179 Yugoslav authors, among them 152 from Serbia, 26 from Croatia and 1 living in France (Dimitrijević, 1990; 1991; 1994; 1997, 2001a). We will review here shortly, the principal investigated problems, according to analysis in Dimitrijević (2001a) for the period March 1997-31 December 2000.

Stark broadening of hydrogen and hydrogen-like emitter lines, has been studied in particularly for $H$ beta line. Its profile assymetry, shifts of central parts, theoretical Stark broadened profile, line shape in coaxial diode discharge, the influence of the fine structure to line shape in a glow discharge cathode fall region, the application of line shape for electron density diagnostics in the range $10^{20}-10^{21} \mathrm{~m}^{-3}$ and the effect of magnetic field on its emission from a T-tube plasma have been considered. Also a paper is devoted to the program for electron density determination from the experimental $H$ beta line profile. The influence of the fine structure to the Stark splitting of the H gamma line in an external electric field, characteristic line profile parameters of hydrogen Balmer lines in such a field, Stark broadening of He II Paschen alpha line shapes, and the use of atomic hydrogen line shapes for the excited hydrogen atoms temperature determination in a glow discharge have been considered as well.

Work on the experimental determination of Stark broadening parameters of nonhydrogenic atom and ion spectral lines has been continued during the considered period: Stark broadening of folowing atoms and ions has been investigated: Ar I,

Ar II, Ar IV, B II, C II, C III, C IV, F II, F III, He I, Kr I, Kr II, Kr III, N II, N III, N IV, N V, Ne I, Ne II, Ne III, Ne I V, O II, O III, O V, O VI, S IV, Si I. Moreover, ion broadening parameters have been determined for Ar I and C I. Also, the influence of ion dynamics, temperature dependence, departure from LS coupling and $\mathrm{Li}-$, $\mathrm{Be}-, \mathrm{B}$ - and C -isoelectronic sequences have been investigated, as well as the use of relative intensities of forbidden and allowed components of He I lines for electric field measurements and the use of non hydrogenic spectral line shapes for the electron density diagnostics of inductively coupled plasmas, fitting procedures for recovering the profile of spectral lines and deconvolution procedure,

Using the semiclassical perturbation approach (Sahal-Bré chot, 1969ab), the spectra of following elements have been investigated within the considered 1997 2000 period: Zn I, Ca I, Au I, Sr I, Mg II, Tl III, In III, Y III, Ne IV, Pb IV, P IV, V V, S V, F VI, Cl VII, O V II, Ar VIII, K VIII, Kr VIII, K IX, Ca IX, Ca X, Na X, Sc X, Mg XI, Si XI, Ti XI, Sc XI, Ti XII, Si XIII and V XIII.

When it is not possible to use the semiclassical perturbation approach with the appropriate accuracy due to the lack of reliable atomic data, the modified semiempirical method (Dimitrijević and Konjević, 1980) can be used and within the considered period have been obtained Stark broadening data for spectral lines of the following emitters: Au II, Bi III, Co II, Co III, Cu III, Eu II, Eu III, Ga III, Kr II, Kr III, La II, La III, Mn II, Na II, Nd II, Pd II, Ra II, Sc II, Sn II, Sr III, Ti II, Ti III, Tl II, V II, V III, V IV, Xe II, Y II, Zr II and Zr III.

A special attention has been paid in a number of papers to the investigation of regularities and systematic trends of Stark broadening parameters. Similarities of Stark broadening parameters within supermultiplet have been investigated as well as Stark parameters dependence on the emitter rest core charge (seeing by optical electron) within a transition array, ion off-resonances and isoelectronic sequences, Stark width regularities along the argon isonuclear sequence and within Ar II spectrum. By using regularities and systematic trends, Stark broadening parameters of a. number of ion lines have been predicted.

Astronomical aspects of spectral line shapes research were studied in a number of publications, as Balmer emission in Solar and AGN coronas, white dwarfs and stellar flares, modeling of double-peaked lines in AGN, Sy I and quasar spectra, line profiles variation in Mrk 817, spectral line investigation of active galactic nuclei, Lyman alpha forest and the total absorption cross-section of galaxies, QSO environment and associated damped Ly alpha galaxies, diffuse bands in interstellar spectra and fullerenes, contribution of interstellar matter to linewidths of Ca II lines in spectra of late type stars, line profile variability of non-radially pulsating Be stars, zirconium conflict in abundance determination, Stark broadening mechanism in stellar atmospheres, the chromospheric behaviour of photospheric Mn I 539.47 nm spectral lines, spectroscopic investigations during Solar eclipses, the influence of Belgrade Solar spectrograph's apparatus function on line profiles and solution of the non LTE transfer problem using the method of iteration factors.

## Astrophysical importance of our semiclassical results and of modified semiempirical approach

In spite of the fact that the most supnisticated theoretical method for the calculation of a Stark broadened line profile is the quantum mechanical strong coupling approach, due to its complexity and numerical difficulties, only a small number of such calculations exist (see e. g. references in Dimitrijević, 1996). As an example, the first calculation of Stark broadening parameters within the quantum mechanical strong coupling method for a nonhydrogen neutral emitter spectral lines is for Li I $2 s^{2} S-2 p^{2} P^{\circ}$ transition (Dimitrijević et al, 1981).

In a lot of cases such as e.g. complex spectra, heavy elements or transitions between highly excited energy levels, the more sophysticated quantum mechanical approach is very difficult or even practically impossible to use and, in such cases, the semiclassical approach remains the most efficient method for Stark broadening calculations.

In order to complete as much as possible Stark broadening data needed for astrophysical and laboratory plasma research and stellar opacities calculations we are making a continuous effort to provide Stark broadening data for a large set of atoms and ions. Using the computer code developed by Benett and Griem (1971) for neutrals and by Jones et al (1971, see also Griem, 1974) for singly charged ions and adapted by Dimitrijević for multiply charged ions, Stark broadening data for Br I, Ge I, Hg I, Pb I, Rb I, Cd I, Zn I, O II, O III, C III, C IV, N II, N III, N IV, S III, S IV, Cl III, Ti II and Mn II spectral lines have been obtained (see Dimitrijević, 1996, 2001b and references therein).

In a series of papers we have performed large scale calculations of Stark broadening parameters for a number of spectral lines of various emitters, within the semiclassical - perturbation formalism (Sahal-Bréchot, 1969ab), for transitions when a sufficiently complete set of reliable atomic data exists and a good accuracy of obtained results is expected. All innovations and optimizations of the computer code have been discussed several times (see e.g. Dimitrijević and Sahal-Bréchot, 1996a, 2000). Extensive calculations have been performed, up to now for a number of radiators, and consequently, Stark broadening parameters for: $79 \mathrm{He}, 62 \mathrm{Na}, 51 \mathrm{~K}$, $61 \mathrm{Li}, 25 \mathrm{Al}, 24 \mathrm{Rb}, 3 \mathrm{Pd}, 19 \mathrm{Be}, 270 \mathrm{Mg}, 31 \mathrm{Se}, 33 \mathrm{Sr}, 14 \mathrm{Ba}, 189 \mathrm{Ca}, 32 \mathrm{Zn}, 6$ $\mathrm{Au}, 48 \mathrm{Ag}, 28 \mathrm{CaII}, 30 \mathrm{Be} \mathrm{II}, 29 \mathrm{Li} \mathrm{II}, 66 \mathrm{Mg} \mathrm{II}, 64 \mathrm{Ba}$ II, 19 Si II, $3 \mathrm{Fe} \mathrm{II}, 2 \mathrm{Ni}$ II, 22 Ne II, 12 B III, 23 Al III, 10 Sc III, 27 Be III, 5 Ne III, 32 Y III, 20 In III, 2 Tl III, 2 Ne IV, 10 Ti IV, 39 Si IV, 90 C IV, 5 O IV, 114 P IV, 2 Pb IV, 19 O V, $30 \mathrm{NV}, 25 \mathrm{C}$ V, 51 P V, 34 S V, 26 V V, 30 O VI, 21 S VI, 2 F VI, 14 O VII, 10 F VII, 10 Cl VII, 20 Ne VIII, 4 K VIII, 9 Ar VIII, 6 Kr VIII, 4 Ca IX, 30 K IX, 8 Na IX, 57 Na X, $48 \mathrm{Ca} \mathrm{X}, 4 \mathrm{Sc} \mathrm{X}, 7 \mathrm{Al} \mathrm{XI}, 4 \mathrm{Si}$ XI, 18 Mg XI, 4 Ti XI, 10 Sc XI, 9 Si XII, 27 Ti XII, 61 Si XIII and 33 V XIII multiplets become available, while the data for particular lines of F I, B II, C III, N IV, Ar II, Ga II, Ga III, Cl I, Br I, I I, Cu I, Hg II, N III, F V and S IV also exist (references of corresponding articles may be found in Dimitrijević, 1996; 2001b)

Our semiclassical Stark broadening parameters, were used for different astrophysical problems (see the corresponding references in Dimitrijević, 2001b). Since
the helium has the largest cosmical abundance after hydrogen, it is naturally that our He Stark broadening data have been often used for different investigations in astrophysics. They have been used for the considerations of following astrophysical problems: non LTE model analysis of the interacting binary $\beta$ Lyrrae; variability investigations of Balmer lines in Ap stars; investigations of peculiar helium - strong stars $\delta$ Orionis C and HD 58260, the chemical composition of the northern double cluster $h$ and $\chi$ Persei and the loose association Cepheus OBIII; the role of blending in the He singlet lines formation in Bp star atmospheres; the critical analysis of the ultraviolet temperature scale and the effective temperature calibration of white dwarfs; the investigation of extreme helium star BD-90-4395; the ionization and excitation in cool giant stars; the constitution of the atmospheric layers and the extreme ultraviolet-spectrum of hot hydrogen rich white dwarfs; spectral properties of hot hydrogen rich white dwarfs with stratified $\mathrm{H} / \mathrm{He}$ model; radiative accelerations on iron; radiative acceleration of helium in the atmospheres of sdOB stars; research of stars with peculiar helium and noble gases abundances; a spectroscopic analysis of DAO and hot DA white dwarfs. They entered in a spectrum synthesis program for binary stars (Linnell and Hubeny, 1994) and have been used for atmosphere research, helium surface mapping and spectrum variability considerations of ET Andromedae, for the investigation of the He I $\lambda 10830 \AA$ formation mechanism in classical cepheides, for the consideration of hot white dwarfs in the ExtremeUltraviolet Explorer survey, for the search for forced oscillations in the eclipsing and spectroscopic binary V436 Persei-1 Persei, for investigations of helium abundance in He rich stars and white dwarfs; for a study of the effect of diffusion and mass-loss on the helium abundance in hot white dwarfs and subdwarfs, for the spectral analysis of the low gravity extreme helium stars LSS 4357, LS II+3305 and LSS 99 and the field horizontal - branch B-type star Feige 86, for comparison with theoretical results obtained within the Stark broadening theory of solar Rydberg lines in the far infrared spectrum, for a discussion of He I 2P-nD line formation in $\lambda$ Eridani, for a study of the atmospheric variations of the peculiar $\mathrm{B}(\mathrm{e})$ star HD 45677 (FS Canis Majoris), for a new method for fitting observations with synthetic spectra, for the consideration of the abundance of $\mathrm{He}_{3}$ isotope in HgMn star atmospheres, and investigation of the helium stratification in the atmospheres of magnetic helium peculiar stars.

Our semiclassical Stark broadening results which have the highest impact in astrophysics, concern ionized silicon spectral lines. Results of our semiclassical investigations (Lanz et al, 1988) have been used (see references in Dimitrijević, 2001a) for silicon abundance analyses with co-added DAO spectrograms, of the HgMn stars $\phi$ Herculis, 28 Herculis, HR 7664, $\nu$ Cancri, $\iota$ Coronae Borealis, HR 8349, $\pi$ Bootis, $v$ Herculis, HR 7361, HR 4072, HR 7775, B stars $\pi$ Ceti, 134 Tauri, 21 Aquilae, $\nu$ Capricorni, $\gamma$ Pegasi, $\iota$ Herculis, $\zeta$ Draconis, $\eta$ Lyrae, 8 Cygni, 22 Cygni, B and A stars $\gamma$ Geminorum, 7 Sextantis, HR 4817, HR 5780, HD 60825, Merak, $\pi$ Draconis, $\kappa$ Cephei, early A type stars 68 Tauri, 21 Lyncis, $\alpha$ Draconis, 2 Lyncis, $\omega$ Ursae Majoris, $\phi$ Aquilae, 29 Vulpeculae, $\sigma$ Aquarii normal F main sequence stars $\theta$ Cygni, $\iota$ Piscium, $\sigma$ Bootis, the metallic lined stars 15 Vulpeculae, 32 Aquarii, HR 4072B, 60 Leonis, 6 Lyrrae, silicon abundance analyses
with Complejo Astronomico el Leoncito REOSC echelle spectrograms of $\kappa$ Cancri, HR 7245, ksi Octantis, HR 4487, 14 Hydrae, 3 Centauri A, silicon abundance studies of CP stars HD 43819, HD 147550, $\chi$ Lupi, 21 Canum Venaticorum, HD 133029, HD 192913, silicon abundance determination for $\gamma$ Geminorum, HR 1397, HR 2154, HR 60825 and 7 Sextantis. Our data have also been used for a discussion on the future of stellar spectroscopy, investigation of blue stragglers of $M 67$, determination of the effective temperature of B-type stars from the Si II lines of the UV multiplet 13.04 at $130.5-130.9 \mathrm{~nm}$, analysis of the red spectrum of Ap stars, NLTE Analysis of subluminous O type hot subdwarf in the binary system HD 128220, a discussion of the role of spectral line Stark shifts for stellar chemical composition determination with the method of atmospheric model, a discussion of the nature of the F str $\lambda$ 4077 type stars and have been used for atmosphere research, He surface mapping and spectrum variability considerations of ET Andromedae.

Semiclassical Stark broadening data on N II, N III and N IV lines obtained in Dimitrijević and Konjevic̀ (1981a) have been used for the investigation of the chemical composition of the young open cluster NGC 6611 (Brown et al, 1986). Our data for Ga II (Dimitrijević and Artru, 1986) have been used for galium abundance analysis of $\kappa$ Cancri (Ryabchikova and Smirnov (1994), normal late B (Smith, 1996) and HgMn stars (Smith, 1995; 1996) and for a discussion on anomalous gallium line profiles in HgMn stars as a possible evidence for chemically stratified atmospheres (Dworetsky et al, 1998). Our semiclassical results for lithium (Dimitrijević and Sahal-Bréchot, 1991) have been used for a study of the non-LTE formation of Li I lines in cool stars (Carlson et al, 1994), Results from Dimitrijević et al (1991) for C IV have been used for the consideration of the influence of gravitational settling and selective radiative forces in PG 1159 stars (Unglaub and Bues, 1996), high resolution UV spectroscopy of two hot (pre-) white dwarfs (KPD 0005+5106 and RXJ $2117+3412$ ) with the Hubble Space Telescope (Werner et al, 1996), spectral energy - distribution and the atmospheric properties of the helium-rich white-dwarf MCT 0501-2858 (Vennes et al, 1998) and for an investigation of stellar masses, kinematics, and white dwarf composition for three close DA +dMe binaries (Vennes et al, 1999). Stark broadening data for N V spectral lines from Dimitrijević and Sahal-Bréchot (1992a) have been used for the spectral analysis of the planetary nebula K 1-27 (Rauch et al, 1994) and data for O VI in Dimitrijević and SahalBréchot (1992b) for spectral analysis of the multiple-shell planetary nebula LoTr4 (Rauch et al, 1996) and for very hot hydrogen - deficient central stars of both nebulae, as well as for the study of the EUV spectrum of the unique bare stellar core H1504+65 (Werner and Wolf, 1999). Our Stark-broadening parameters of ionized mercury spectral lines of astrophysical interest (Dimitrijević 1992), have been used for determination of Hg abundances in normal late- B and HgMn stars from coadded IUE spectra (Smith, 1997); our data for Ca II (Dimitrijević and SahalBréchot, 1993) for abundance analyses of the double-lined spectroscopic binary $\alpha$ Andromedae (Ryabchikova et al, 1999), and our data for Mg I (Dimitrijević and Sahal-Bréchot, 1996) for a non-LTE analysis of Mg I in the solar atmosphere (Zhao et al, 1998).

In a large number of cases, especially for more complex spectra of heavier atoms,
there is no enough data to apply semiclassical perturbation method in an adequate way. In order to minimize the needed atomic data and provide to astrophysicists and physicists an adequate method for Stark broadening data calculations in such a situation, we have developed the modified semiempirical approach (Dimitrijević and Konjević, 1980; 1981b; 1987; Dimitrijević and Kršljanin, 1986). In fact, if there are no perturbing levels strongly violating the assumed approximation, for e.g. the line width calculations, we need only the energy levels with $\Delta n=0$ and $\ell_{i f}=\ell_{i f} \pm 1$, since all perturbing levels with $\Delta n \neq 0$, needed for a full semiclasical investigation, are lumped together and approximately estimated. Here, $n$ is the principal and $\ell$ the orbital angular momentum quantum numbers of the optical electron and with $i$ and $f$ are denoted the initial and final state of the considered transition.

Due to the considerably smaller set of needed atomic data in comparison with the complete semiclassical (Sahal-Bréchot, 1969ab) method, the MSE method is particularly useful for stellar spectroscopy depending on very extensive list of elements and line transitions with their atomic and line broadening parameters where it is not possible to use sophysticated theoretical approaches in all cases of interest.

The MSE method is also very useful whenever line broadening data for a large number of lines are required, and the high precision of every particular result is not so important like e.g. for opacity calculations or plasma modeling. Moreover, in the case of more complex atoms or multiply charged ions the lack of the accurate atomic data needed for more sophysticated calculations, makes that the reliability of the semiclassical results decreases. In such cases the MSE method might be very interesting as well.

In order to complete as much as possible the needed Stark broadening data, Belgrade group (Milan S. Dimitrijević, Luka Č. Popović, Vladimir Kršljanin, Dragana Tankosić, Nenad Milovanović) used the modified semiempirical method to obtain the Stark width and in some cases shift data for a large number of spectral lines for the different atom and ion species. Up to now (see references in Dimitrijević, 2001b) spectral line Stark widths for:

6 Fe II, 4 Pt II, 16 Bi II, $12 \mathrm{Zn} \mathrm{II}, 8 \mathrm{Cd}$ II, 18 As II, $10 \mathrm{Br} \mathrm{II}, 18 \mathrm{SbII}, 8 \mathrm{III}$, 20 Xe II, 138 Ti II, 3 La II, $16 \mathrm{Mñ}$ II, 14 V II, 6 Eu II, $37 \mathrm{Kr} \mathrm{II}$,6 Y II, 6 Sc II, 4 Be III, 4 B III, 13 S III, 8 Au II; 8 Zr II, 53 Ra II, $3 \mathrm{Mn} \mathrm{III}$,10 Ga III, 8 Ge III, 4 As III, 3 Se III, 6 Mg III, 6 La III, 5 Sr III, 8 V III, 210 Ti III, 9 C III, 7 N III, 11 O III, 5 F III, 6 Ne III, 8 Na III, 10 Al III, 5 Si III, 3 P III, 16 Cl III, 6 Ar III, 30 Zr III, 2 B IV, Cu IV, 30 V IV, 14 Ge IV, 7 C IV, 4 N IV, 4 O IV, 2 Ne IV, 4 Mg IV, 7 Si IV, 3 P IV, 2 S IV, 2 Cl IV, 4 Ar IV, 3 C V, 50 O V, 12 F V, 9 Ne V, 3 Al V, 6 Si V, 11 N VI, 28 F VI, 8 Ne VI, 7 Na VI, 15 Si VI, 6 P VI, and 1 Cl VI transitions have been calculated. The shift data for $16 \mathrm{Bi} \mathrm{II}, 12 \mathrm{Zn} \mathrm{II}, 8 \mathrm{Cd}$ II, 18 As II, 10 Br II, 18 Sb II, 8 III, 20 Xe II, 5 Ar II, 6 Eu II, $14 \mathrm{~V} \mathrm{II}$,8 Au II, 14 Kr II and 138 Ti II transitions have been calculated. Moreover, 286 Nd II Stark widths have been calculated (Popović et al, 2001) within the symplified modified semiempirical approach (Dimitrijević and Konjević, 1987).

Calculations within the our modified semiempirical approach, for comparison with experimental data or testing of the theory have been performed also by other
authors (see references in Dimitrijević, 2001b) for Stark widths for $14 \mathrm{Al} \mathrm{I}$, II, 12 Al III, $1 \mathrm{CIV}, 1 \mathrm{~N}$ V, 1 O VI, 1 Ne VIII, 3 N III, $3 \mathrm{O} \mathrm{IV}, 3 \mathrm{~F}$ V, 2 Ne VI , 12 C IV, 4 C II, 5 N II, 3 O II, 4 F II, 3 Ne II, 1 N II, 8 S II, 2 Ne VII, 4 N III, 2 F V, 2 Ne III, 2 Ar III, 2 Kr III, 2 Xe III, 3 Si III, 3 Ne III, 2 Ar III, 2 Kr III, 2 Xe III transitions. Moreover, Stark widths and shifts for 2 Cl II and 6 Ar III lines have been calculated.

The modified semiempirical method and Stark broadening parameters calculated within this approach have been applied in astrophysics e.g. for the determination of carbon, nitrogen and oxygen abundances in early B-type stars (Gies and Lambert, 1992) magnesium, aluminium and silicon abundances in normal late-B and HgMn stars, from co-added IUE spectra (Smith, 1993) and elemental abundances in hot white dwarfs (Chayer et al, 1995a), investigations of abundance anomalies in stars (Michaud and Richer, 1992), elemental abundance analyses with DAO spectrograms for 15 - Vulpeculae and 32 - Aquarii (Bolcal et al, 1992), radiative acceleration calculation in stellar envelopes (Le Blanc and Michaud, 1995; Gonzales et al, 1995ab; Alecian et al, 1993; Seaton, 1997), consideration of radiative levitation in hot white dwarfs (Chayer et al, 1995ab), quantitative spectroscopy of hot stars (Kudritzki and Hummer, 1990), non - LTE calculations of silicon - line strengths in B - type stars (Lennon et al, 1986), stellar opacities calculations and study (Iglesias et al, 1990; Iglesias and Rogers, 1992; Rogers and Iglesias, 1992; 1995; 1999; Seaton, 1993; Mostovych et al, 1995), atmospheres and winds of hot stars investigations (Butler, 1995), investigation of Ga II lines in the spectrum of Ap stars (Lanz et al, 1993). Stark broadening data calculated within the modified semiempirical method entered in a critical overview of atomic data for stellar abundance analysis (Lanz et Artru, 1988), and a catalogue of atomic data for lowdensity astrophysical plasma (Golovatyj et al, 1997). The modified semiempirical method entered also in computer codes, as e. g. OPAL opacity code (Rogers and Iglesias, 1995), handbooks (Peach, 1996; Vogt, 1996) and monographs (Gray, 1992; Griem, 1997; Konjević, 1999).

Our Stark broadening results are of interest not only in astronomy but also for laboratory, fusion, laser produced and technological plasma diagnostics, analysis and modeling, for design of existing and development of new lasers, for optimization of light sources etc, and were also used and cited in such investigations. In order to make the application and usage of our Stark broadening data obtained within the semiclassical and modified semiempirical approaches more easier, we are organizing them now in a database BELDATA.

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# ACTIVE VIBRATION CONTROL OF THE BEAM AND PLATE STRUCTURE USING OPTIMAL LQ TRACKING SYSTEM DESIGN 

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#### Abstract

The possibilities of active control of the beam and plate structure were investigated in the work. The bases for the investigation of the structures' behavior under excitation by disturbances were the state space models obtained through the finite element analysis procedure. The aim of control was to reduce the vibrations of the mechanical structures caused by exciting forces with frequencies corresponding to eigenmodes of the plate. Different types of controllers were designed and through the verification of designed control laws performed by computer simulations, optimal tracking system based on LQ controller was adopted as the most acceptable solution. Designed controller successfully faced the disturbance and performed good behavior in the sense of oscillation magnitude reduction and stability margins.


Key words: clamped beam, plate structure, active vibration control, digital tracking system based on optimal LQ regulator.

## 1. Introduction

The paper treats the problem of an active control of considered mechanical structures in order to provide vibration reduction. Specific models regarding beam and plate structures were considered and vibration control based on optimal $L Q$ tracking system was applied to these particular cases. Considered mechanical structures were supposed to be acted upon by exciting periodic sine-type forces with frequencies corresponding to the eigenfrequencies of mechanical structures which represent critical cases because of the possibility of resonance and system destruction. Complete control design procedure was performed in the case of beam and plate structure and designed control laws were verified by simulations. Piezoelectric patches were used as sensors and actuators.

## 2. Control of the Clamped Beam

This part of the work concerns control design problem for the model of the beam clamped at one end. The basis for the investigation of the beam behavior under
excitation by disturbances is the state-space model obtained through the finite element analysis procedure ([1], [2], [3], [4]). Proposed controller fulfills the conditions of acceptable stability margins and provides desired magnitudes and frequencies of the beam end oscillations as an alternative to complete vibration damping which is not possible due to the system uncontrollability.
2.1 Plant Description and the State-Space Model. The plant considered is a clamped beam treated as active plate structure controlled by four piezoelectric patch actuators attached to the beam, two on the top and two on the bottom of the plate. Geometry of the plant as well as the plate, actuator and sensor properties are listed in Fig.1. At first step the plant was represented in the form of a finite element model with a mesh of 235 passive and 80 active Semiloof shell elements ([1], [2], [4]). On the basis of this mesh the eigenfrequencies and eigenmodes were calculated. Considered frequencies which are of interest for bending mode study cases are $f_{1}=17.2 \mathrm{~Hz}, f_{2}=108.6 \mathrm{~Hz}, f_{4}=302.9 \mathrm{~Hz}$ and $f_{6}=606.1 \mathrm{~Hz}$. Exciting forces $F(t)=A \sin \left(\omega_{i} t\right)$ exerted at the corner points of the beam end were chosen according to the eigenfrequencies of interest.


## Material:

Beam:
$\mathrm{E}=2.00 \cdot 10^{5} \mathrm{~N} / \mathrm{mm}^{2}$
$v=0.3$
$p=7.86 \cdot 10^{-9} \mathrm{Ns}^{2} / \mathrm{mm}^{4}$
$\mathrm{t}=2.0 \mathrm{~mm}$ (thickness)

Actuator/sensor:
$\mathrm{E}_{11}=\mathrm{E}_{22}=3.77 \cdot 10^{4} \mathrm{~N} / \mathrm{mm}^{2}$

$$
\mathrm{d}_{31}=2.1 \cdot 10^{-7} \mathrm{~mm} / \mathrm{V}
$$

$\kappa_{33}=3.36 \cdot 10^{-9} \mathrm{~F} / \mathrm{m}$
$\mathrm{G}_{12}=1.3 \cdot 10^{4} \mathrm{~N} / \mathrm{mm}^{2}$
$v=0.38$
$\rho=7.85 \cdot 10^{-9} \mathrm{Ns}^{2} / \mathrm{mm}^{4}$
$\mathrm{t}=0.4 \mathrm{~mm}$ (thickness)

Fig. 1
Plate model was modally reduced and transformed into the state-space model using the finite element software [1]. Via data exchange interface the model was exported into Matlab/Simulink as a software environment for controller design and testing. The controller design investigation starts with the continuous state-space model:

$$
\begin{align*}
& \dot{x}=A x+B u+E d \\
& y=C x+D u+F d \tag{1}
\end{align*}
$$

with appropriate state-space matrices:

$$
\begin{aligned}
& C=\left[\begin{array}{lllllll}
1.54877435 & -0.23953924 & 0.0893192 & 0.04332433 & 0 & 0 & 0 \\
0
\end{array}\right] \quad D=\left[\begin{array}{llll}
0 & 0 & 0 & 0
\end{array}\right] \quad F=\left[\begin{array}{ll}
0 & 0
\end{array}\right]
\end{aligned}
$$

2.2. Disturbance Impact Reduction using Optimal Digital LQ Tracking System. For the purpose of vibration reduction digital tracking system with additional dynamics based on optimal $L Q$ regulator ([5], [6], [7], [8]) was designed. Additional dynamics is introduced in order either to track specified reference input or to reject disturbances. In this case both applications were used.

First the controller which reduces the magnitude of beam end oscillations was designed using additional dynamics. Additional dynamics is determined in the statespace form on the basis of disturbance and/or reference input poles. In this case disturbance (excitation force) is sine function and thus the $s$-plane poles of disturbance are complex conjugate numbers $\lambda_{1,2}= \pm j \omega_{i}$, where $\omega_{i}=2 \pi f_{i}$ and $f_{i}(i=1,2,4,6)$ are eigenfrequencies for bending modes. On the basis of $z$-plane pole locations obtained by mapping, polynomial $\delta(z)$ was obtained in the form:

$$
\begin{equation*}
\delta(z)=\prod_{i}\left(z-\mathrm{e}^{\lambda_{i} \mathrm{~T}}\right)^{m_{i}} \stackrel{\operatorname{def}}{=} z^{s}+\delta_{1} z^{s-1}+\ldots+\delta_{s} . \tag{2}
\end{equation*}
$$

Here $s=2$ since there are two disturbance poles. Matrices of additional dynamics $\Phi_{a}$ and $\Gamma_{a}$ are determined on the basis of the coefficients of the polynomial $\delta(z)$ :

$$
\Phi_{a}=\left[\begin{array}{ccccc}
-\delta_{1} & 1 & 0 & \cdots & 0  \tag{3}\\
-\delta_{2} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_{s-1} & 0 & 0 & \cdots & 1 \\
-\delta_{s} & 0 & 0 & \cdots & 0
\end{array}\right], \quad \Gamma_{a}=\left[\begin{array}{c}
-\delta_{1} \\
-\delta_{2} \\
\vdots \\
-\delta_{s-1} \\
-\delta_{s}
\end{array}\right]
$$

Discrete-time design model $\left(\Phi_{d}, \Gamma_{d}\right)$ is formed as a cascade combination of additional dynamics $\left(\Phi_{a}, \Gamma_{a}\right)$ and discrete-time plant model $(\Phi, \Gamma)$ obtained for specified sampling time T :

$$
\begin{equation*}
x_{d}[k+1]=\Phi_{d} x_{d}[k]+\Gamma_{d} u[k] \tag{4}
\end{equation*}
$$

where

$$
x_{d}[k]=\left[\begin{array}{c}
x[k]  \tag{5}\\
x_{a}[k]
\end{array}\right], \quad \Phi_{d}=\left[\begin{array}{cc}
\Phi & 0 \\
\Gamma_{a} C & \Phi_{a}
\end{array}\right], \quad \Gamma_{d}=\left[\begin{array}{l}
\Gamma \\
0
\end{array}\right] .
$$

Feedback gain matrix $L$ of the optimal $L Q$ regulator is calculated on the basis of design model (4) in such a way that the feedback law $\mathbf{u}[k]=-\mathbf{L} \boldsymbol{x}_{d}[k]$ minimizes the performance index:

$$
\begin{equation*}
J=\frac{1}{2} \sum_{k=0}^{\infty}\left(x_{d}[k]^{T} Q x_{d}[k]+u[k]^{T} R u[k]\right) \tag{6}
\end{equation*}
$$

subject to the constraint equation (4) where $Q$ and $R$ are symmetric, positive-definite matrices. Feedback gain matrix $L$ is afterwards partitioned into submatrices $\mathbb{L}_{1}$ and $\mathbb{L}_{2}$ corresponding to the plant and additional dynamics, respectively. Partitioned feedback gain matrix is implemented in the control system as shown in Fig.2.


Fig. 2
Additional dynamics for the sine disturbance input is of the second order and thus the order of design model is 10 . Adopted sampling time is $\mathrm{T}=0.0001 \mathrm{~s}$. Through several simulation steps, matrices $Q$ and $R$ in the performance index (6) for $L Q$ regulator design were adopted to provide the response with acceptable trade-off between vibration magnitude and control effort.

$$
\begin{equation*}
Q=\operatorname{diag}(1000, \ldots 1000)_{10 \times 10}, \quad R=I(I \text { unity matrix } 4 \times 4) \tag{7}
\end{equation*}
$$




Fig. 4

Simulations were performed for sine disturbances, magnitude 0.01 and different frequencies corresponding to eigenfrequencies: $f_{1}, f_{2}, f_{4}$ and $f_{6}$. Obtained simulation results are shown in Fig. 3. Diagrams on the left-hand side represent the response i.e. displacement of the beam end in case when the controller is on from the very beginning of the simulation; right hand-side diagrams represent the response when the controller is switched on 0.5 s after the beginning of simulation. The time axis is set in seconds.

Simulation results show great improvement in comparison with the response without controller. The results are acceptable from the control point of view, since the
maximum voltage (for : $f_{1}=17.2 \mathrm{~Hz}$ ) is less than 25 V . Stability margins were also calculated and they are: upper gain margin for inputs $u_{1}$ and $u_{2}: 3.46 \mathrm{~dB}$, upper gain margin for $u_{3}$ and $u_{4}: 3.08 \mathrm{~dB}$, lower gain margin for all inputs: -30.1 dB and phase margin: $128^{\circ}$.

Another step forward in controller design was done by providing not only reduction of the vibration magnitude, but also the reduction of the frequency. Namely, optimal $L Q$ tracking system provides tracking of specified reference input which was also introduced in this step. On the other hand additional dynamics was determined for disturbance as well for reference input $0.01 \sin \left(2 \pi f_{1} t\right)$ modeling. Thus, design model of the $14^{\text {th }}$ order was obtained and through the simulation process, weighting matrices for optimal $L Q$ regulator design were adopted to be: $Q=\operatorname{diag}(1000, \ldots 1000)_{14 \times 14}, R=$ $\operatorname{diag}(100, \ldots 100)_{4 \times 4}$.

Simulation results in Fig. 4 show that for disturbance frequencies $f_{2}=108.6 \mathrm{~Hz}$ and $f_{4}=302.9 \mathrm{~Hz}$, controller provided oscillations of the beam end with less magnitudes and frequencies than without controller, while for disturbance frequency $f_{6}=606.1 \mathrm{~Hz}$ only the frequency was reduced. Left-hand side diagrams represent displacement with controller switched after 0.5 s after the beginning of simulation and right-hand side diagrams represent control voltage signal. The time axis is set in seconds.

## 3. Control of the plate structure

The possibility of active control of the plate structure was investigated in this part of the work. As a result of a finite element analysis procedure the MIMO state space model of the plate structure was obtained [1] as a base for the controller design. Proposed controller was obtained using the procedure for optimal tracking system design and the control law was verified using Matlab/Simulink simulations. Designed controller performs very good behavior in the sense of oscillation magnitude reduction and stability margins.
3.1 Plant Description and the State-Space Model. Active plate structure considered in this work was represented in the form of a finite element model with a mesh of 892 normal Semiloof shell elements and 8 active Semiloof shell elements [1]. The plant is controlled by four actuators placed on the top of the plate, while displacement is detected by four sensors placed on the bottom of the plate. The system is excited by disturbance force $F(t)=A \sin \left(\omega_{i} t\right)$ with different frequencies $\omega_{i}=2 \pi f_{i}$ corresponding to eigenfrequencies $f_{i}(i=1, \ldots, 5)$ of the plate. These eigenfrequencies correspond to bending modes of the plate. It should be noted that $f_{3,4}$ is double eigenfrequency. Geometry of the plant as well as the plate, actuator and sensor properties are listed in Fig. 5.

Plate model was modally reduced and transformed into the state-space model (1) using the finite element software [1]. Via data exchange interface the model was exported into Matlab/Simulink as a software environment for controller design and testing. Appropriate state-space matrices of the MIMO model (1) are listed below.

## actuator/sensor



## Material:

## Plate:

$E=2.06 \cdot 10^{5} \mathrm{~N} / \mathrm{mm}^{2}$
$v=0.3$
$\rho=7.86 \cdot 10^{-9} \mathrm{Ns}^{2} / \mathrm{mm}^{4}$
$\mathrm{t}=0.9 \mathrm{~mm}$ (thickness)

Actuator/sensor:

$$
\begin{aligned}
& \mathrm{E}_{11}=\mathrm{E}_{22}=3.77 \cdot 10^{4} \mathrm{~N} / \mathrm{mm}^{2} \\
& \mathrm{G}_{12}=1.3 \cdot 10^{4} \mathrm{~N} / \mathrm{mm}^{2} \\
& v=0.38 \\
& \rho=7.85 \cdot 10^{-9} \mathrm{Ns}^{2} / \mathrm{mm}^{4}
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{d}_{31}=2.1 \cdot 10^{-7} \mathrm{~mm} / \mathrm{V} \\
& \mathrm{~K}_{33}=3.36 \cdot 10^{-9} \mathrm{~F} / \mathrm{m} \\
& \mathrm{t}=0.4 \mathrm{~mm} \text { (thickness) }
\end{aligned}
$$

Fig. 5
$A=10^{4} \cdot\left[\begin{array}{cccccccc}0 & 0 & 0 & 0 & 0.000100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.000100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.000100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.000100 \\ -1.099966 & 0 & 0 & 0 & -0.000159 & 0 & 0 & 0 \\ 0 & -2.645441 & 0 & 0 & 0 & -0.000553 & 0 & 0 \\ 0 & 0 & -6.642902 & 0 & 0 & 0 & -0.000515 & 0 \\ 0 & 0 & 0 & -9.72129031 & 0 & 0 & 0 & -0.000131\end{array}\right]$
$C=\left[\begin{array}{rrrrrrrr}-0.221510 & 0.173010 & 0.100141 & -0.097934 & 0 & 0 & 0 & 0 \\ -0.268108 & 0.178599 & -0.091541 & 0.075256 & 0 & 0 & 0 & 0 \\ -0.256084 & -0.140939 & 0.115913 & 0.078263 & 0 & 0 & 0 & 0 \\ -0.268928 & -0.178315 & -0.092725 & -0.074636 & 0 & 0 & 0 & 0\end{array}\right] \quad D=\left[\begin{array}{llll}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right] \quad F=\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 0\end{array}\right]$

$$
B=10^{3} \cdot\left[\begin{array}{ccll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.102529 & -0.115996 & -0.112300 & -0.099174 \\
0.383279 & 0.362693 & -0.350380 & -0.370527 \\
0.715363 & -0.789294 & 0.770933 & -0.709136 \\
-1.286293 & 1.198274 & 1.158158 & -1.241578
\end{array}\right] \quad E=10^{4} \cdot\left[\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
-0.234125 \\
-0.446124 \\
0.706964 \\
1.017061
\end{array}\right]
$$

3.2 Controller Design Results. Continuous-time state space model was converted to discrete-time state space model using zero-order-hold mapping method with sampling interval $T=0.0001 \mathrm{~s}$.

For each disturbance corresponding to different eigenfrequencies, optimal $L Q$ tracking system was designed.

Additional dynamics was determined in the form of matrices $\Phi_{a}, \Gamma_{a}(3)$ on the basis of the coefficients of the polynomial $\delta(z)$, equation (2), obtained according to conjugate complex poles $\pm j \omega_{i}$ of disturbance. Since the plant is a multiple output system, additional dynamics has to be replicated into four parallel systems (one per each output) described by the state space matrices:

$$
\begin{equation*}
\bar{\Phi} \stackrel{\operatorname{def}}{=} \operatorname{diag}\left(\Phi_{a}, \Phi_{a}, \Phi_{a}, \Phi_{a}\right), \bar{\Gamma} \stackrel{\operatorname{def}}{=} \operatorname{diag}\left(\Gamma_{a}, \Gamma_{a}, \Gamma_{a}, \Gamma_{a}\right) . \tag{8}
\end{equation*}
$$

Discrete-time design model $\left(\Phi_{d}, \Gamma_{d}\right)$ is formed as a cascade combination of additional dynamics $(\bar{\Phi}, \bar{\Gamma})$ and discrete-time plant model $(\Phi, \Gamma)$ obtained for specified sampling time T :

$$
\begin{equation*}
x_{d}[k+1]=\Phi_{d} x_{d}[k]+\Gamma_{d} u[k] \tag{9}
\end{equation*}
$$

where

$$
\Phi_{d}=\left[\begin{array}{cc}
\Phi & 0  \tag{10}\\
\bar{\Gamma} C & \bar{\Phi}
\end{array}\right], \quad \Gamma_{d}=\left[\begin{array}{l}
\Gamma \\
0
\end{array}\right], \quad x_{d}=\left[\begin{array}{c}
x[k] \\
x_{a}[k]
\end{array}\right] .
$$

Feedback gain matrix L of the optimal $L Q$ regulator was obtained for design model and partitioned into submatrices $\mathrm{L}_{1}$ and $\mathrm{L}_{2}$ corresponding to the plant and additional dynamics, respectively [5], [6], [7], [8]. Partitioned feedback gain matrix was implemented in control system similar to the one shown in Fig. 2, only instead of matrices $\Phi_{a}$ and $\Gamma_{a}$, matrices $\bar{\Phi}$ and $\bar{\Gamma}$ respectively exist in the controller block of digital tracking system.

Simulation results obtained for each disturbance force corresponding to different bending-mode eigenfrequencies are presented in Fig. 6. Simulation diagrams represent four system outputs, i.e. sensor responses for disturbance $F(t)=A \sin \left(2 \pi f_{i} t\right)$, $A=0.01$ when the controller is switched on after 0.5 s . Insight in simulation diagrams of input control signals shows that the control can be achieved with relatively small control effort which corresponds to small voltage signals.


Fig. 6
Robustness of designed controller in terms of stability margins (upper gain margin UGM [dB], lower gain margin LGM [dB] and phase margin PM [ ${ }^{\circ}$ ]) is given in Table 1. It shows good stability margins while the plate oscillation magnitudes were drastically reduced as a result of controller application.

Table 1

|  | $f_{1}$ |  |  | $f_{2}$ |  |  | $f_{3,4}$ |  |  | $f_{5}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | UGM | LGM | PM | UGM | LGM | PM | UGM | LGM | PM | UGM | LGM | PM |
|  | 24.46 | -30.1 | 88 | 24.23 | -30.1 | 86 | 24.45 | -30.1 | 91 | 24.21 | -30.1 | 88 |
| $u_{2}$ | 24.54 | -30.1 | 88 | 24.56 | -30.1 | 86 | 24.69 | -30.1 | 91 | 24.54 | -30.1 | 88 |
| $u_{3}$ | 24.83 | -30.1 | 88 | 24.85 | -30.1 | 87 | 24.99 | -30.1 | 91 | 24.83 | -30.1 | 88 |
| $u_{4}$ | 24.46 | -30.1 | 88 | 24.48 | -30.1 | 86 | 24.72 | -30.1 | 91 | 24.46 | -30.1 | 88 |

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# ON NONCONSISTENT INVARIANCE IN ANALYTICAL MECHANICS 

Veljko A. Vujičić


#### Abstract

Fundamental thesis is proved in the paper that the principle of invariance is not applied in classical analytical mechanics consequently enough. Author of this reviewing article raises the invariance to the level of a preprinciple [12] and asics for its consequent application in all relations of analytical mechanics. This leads to significant modifications of basic relations, as well as in integral calculus. Important problems, which were treated in autor's papers and monographs, are pointed aut briefly.


## Introduction

The term indicated in the title of this review paper invariance means that the motion nor properties of the body motion depend upon the form of statement: the determined truth about motion, once it is written in some linguistic form, is equally contained in the written output of some other form or some other alphabet.

This preprinciple of invariance or independence of formalities allows for mass, as well as time, to be denoted by some other letters, let's say $\bar{m}$ and $\bar{t}$, which do not change the nature of numbers $m$ and $t$, and for which there must be $\bar{m}=m$ and $\bar{t}=t$ in the whole correspondence, [12]. The same stands for distance $\Delta r$. No matter where the origin of coordinates from which the position vector begins is chosen, let's say $\rho$, there is an equality $\Delta r=\Delta \rho$, so that distance $\Delta r$ does not depend on the form of writing. This is even more expressed in the coordinate form, in which the choice of forms is considerably larger, such as

$$
\Delta r=\sum_{i=1}^{3}\left(\Delta r^{i}\right) e_{i}=\Delta r^{i} e_{i}=\Delta y^{i} e_{i}=\Delta z^{i} \ni_{i}=\Delta \rho^{j} g_{j}=\cdots
$$

As such, all the three realities $m \in \mathbb{R}, t \in \mathbb{R}$ and $\Delta r \in R_{3}$ are invariants, $m$ and $t$ being scalar ones, while $\Delta r$ is a vector invariant.

Key words and phrases. Invariance, preprinciple, principle, covariant equations, invariant tensorial integral, invariant criterion of stability.

All other factors of the body motion can also be invariantly expressed in various coordinate systems.

## Coordinate Systems

The concept of coordinate system here implies an ordered set of real numbers and a set of mutually independent vectors that are called coordinate vectors. The coordinate vectors differ from the base ones only in the sense that the base ones are previously determined with respect to objects, while the coordinate ones are determined with respect to the base ones. If the coordinate ones are original, then they are base coordinate vectors. On the basis of the base vectors

$$
\begin{equation*}
e_{i}=\text { const } \tag{1}
\end{equation*}
$$

it is possible to introduce other coordinate systems $x=\left(x^{1}, x^{2}, x^{3}\right),\left(x^{i} \in \mathbb{R}\right)$ in which the material point's position is explicitly mapped while the velocity has a general invariant form.

Any other rectilinear coordinate system can be chosen as well, let's say $(z, \vartheta)$, whose directions change in time with respect to base system $(y, e)$. The two systems' ratio is determined by the relations:

$$
y^{i}=\gamma_{\alpha}^{i} z^{\alpha}, \quad e_{i}=\frac{\partial z^{\alpha}}{\partial y^{i}} \ni_{\alpha}=\bar{\gamma}_{i}^{\alpha} \ni_{\alpha}, \gamma_{\alpha}^{i} \bar{\gamma}_{i}^{\beta}=\delta_{\alpha}^{\beta}
$$

The velocity vector can be represented by the equation:

$$
\begin{aligned}
v & =\frac{d}{d t}\left(y^{i} e_{i}\right)=\dot{y}^{i} e_{i}=\left(\dot{\gamma}_{\alpha}^{i} z^{\alpha}+\gamma_{\alpha}^{i} \dot{z}^{\alpha}\right) \bar{\gamma}_{i}^{\beta} \ni_{\beta}= \\
& =\left(\dot{\gamma}_{\alpha}^{i} \bar{\gamma}_{i}^{\beta} z^{\alpha}+\delta_{\alpha}^{\beta} \dot{z}^{\alpha}\right) \ni_{\beta}=\left(\dot{z}^{\beta}+\omega_{\alpha}^{* \beta} z^{\alpha}\right) \ni_{\beta}=v^{\beta} \ni_{\beta}
\end{aligned}
$$

where $\omega_{\alpha}^{* \beta}=\dot{\gamma}_{\alpha}^{i} \bar{\gamma}_{i}^{\beta}=-\omega_{* \alpha}^{\beta}=-\gamma_{\alpha}^{i} \dot{\bar{\gamma}}_{i}^{\beta}$ are anti-symmetrical coefficients and $*$ denotes the empty place of an index. The projections of velocity vector $\dot{y}_{i}$ upon the axes of base vectors $e_{i}$, as scalar products of vector $\dot{v}_{i}$ and base vectors $e_{i}$, are equal to the velocity vector coordinates $\dot{y}^{i}$ :

$$
\dot{y}_{i}=\delta_{i j} \dot{y}^{j}
$$

while $v_{i}$ projections upon the axes of the coordinate vectors $g_{i}$ are linear homogeneous forms of the velocity vector coordinates:

$$
\begin{equation*}
v_{i}=g_{i j} v^{j}=g_{i j} \frac{D r^{j}}{d t}=g_{i j} \dot{x}^{j} \tag{2}
\end{equation*}
$$

where $g_{i j}(x)$ is metric tensor.

The velocity square, as a scalar invariant, can now be written in the following form:

$$
\begin{equation*}
v^{2}=\delta_{i j} \dot{y}^{i} \dot{y}^{j}=g_{i j} \dot{x}^{i} \dot{x}^{j}=g_{i j} \frac{D r^{i}}{d t} \frac{D r^{j}}{d t} . \tag{3}
\end{equation*}
$$

The previous relations can be related to the base vectors' covariant derivatives with respect to the coordinates

$$
\begin{equation*}
\nabla_{k} g_{j}=\frac{\partial g_{j}(x)}{\partial x^{k}}-\Gamma_{j k}^{i}(x) g_{i}(x)=0 \tag{4}
\end{equation*}
$$

which are very important for describing the base vectors and their changes in time. Just as relations

$$
\begin{equation*}
g_{i}=\frac{\partial y^{k}}{\partial x^{i}} e_{k}=\frac{\partial r}{\partial x^{i}}=g_{i}(x), \tag{5}
\end{equation*}
$$

establish the ratio between base vectors $e$ and the subsequently introduced coordinate $g$, so the covariant derivative $\nabla_{k} g_{j}$ stands in a direct relation with conditions

$$
\begin{equation*}
\frac{d e_{i}}{d t}=0 . \tag{6}
\end{equation*}
$$

The derivatives of relations (5) with respect to time, due to condition (6) are:

$$
\frac{d g_{i}}{d t}=\frac{\partial^{2} y^{k}}{\partial x^{j} \partial x^{i}} \dot{x}^{j} e_{k} .
$$

It is always possible to introduce such functions $\Gamma(x)$ so that it is

$$
\frac{\partial^{2} y^{k}}{\partial x^{j} \partial x^{i}}=\Gamma_{i j}^{\lambda} \frac{\partial y^{k}}{\partial x^{\lambda}}
$$

thus, it is obtained

$$
\begin{equation*}
\frac{d g_{i}}{d t}-g_{\lambda} \Gamma_{j i}^{\lambda} \frac{d x^{j}}{d t}=\frac{D g_{i}}{d t}=\nabla_{j} g_{i} \dot{x}^{j}=0 . \tag{7}
\end{equation*}
$$

These are the conditions which show that coordinate vectors $g_{i}$ are covariantly constant.

This clearly shows that the velocity vector coordinates are varied regarding various coordinate vectors. Due to the preprinciple of invariance as well as the casual definiteness of the statement about "natural derivative" from the definition of velocity, it is natural that the chosen coordinate vectors should be the ones that can be related to some base vectors (1), invariable in time.

Once base vectors $e_{i}$ are chosen, other oriented coordinate vectors $g_{i}$ can be chosen, including curvilinear ones, for which the natural derivatives (7) will be valid.

## Motion impulse

In accordance with the velocity definition and the above-given definition, the motion impulse can be written in the following way:

$$
\begin{align*}
p & =m v=m \dot{y}^{i} e_{i}=m \frac{D z^{i}}{d t} \ni_{i}=m v^{i} g_{i}= \\
& =m \frac{D r^{i}}{d t} g_{i}=m \frac{\partial r}{\partial x^{i}} \dot{x}^{i}=m \dot{x}^{i} g_{i} . \tag{8}
\end{align*}
$$

Further on, special emphasis will be put on $p_{i}$ projections of this vector upon coordinate directions $g_{i}$ :

$$
\begin{equation*}
p_{i}=p \cdot g_{i}=m g_{i j} \dot{x}^{j}=a_{i j} \dot{x}^{j}, \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{i j}=m g_{i j}=m \frac{\partial r}{\partial x^{i}} \cdot \frac{\partial r}{\partial x^{j}}=a_{j i}(m, x) \tag{10}
\end{equation*}
$$

is inertia tensor.
It should be noted that inertia tensor $a_{i j}(m, x)$ differs from the metric tensor $g_{i j}(x)$. The basic physical dimensions of the impulse vector are:

$$
\operatorname{dim} p=\mathrm{MLT}^{-1}
$$

but its coordinates and projections can also have other dimensions:
If $x$ coordinate is an angle, then it is:

$$
\operatorname{dim} p_{i}=\mathrm{ML}^{2} \mathrm{~T}^{-1} .
$$

Inertia tensor $a_{i j}$ sets up a relation between impulse and velocity at any position. Its essential content is mass which exists for every body or material point as well as in all coordinate systems.

System of Material Points and Finite Constraints. All the relations derived only for one material point of mass $m$, stand for every $\nu$-th material point of mass $m_{\nu}$. Such a system of $N$ material points will have $N$ vector equations of the form

$$
\begin{equation*}
m \frac{d v}{d t}=F+R . \tag{11}
\end{equation*}
$$

and $k$ constraints equations. Nothing more important than this changes. However, the manner of solving problems concerned with the system motion comprises some difficulties and innovations originating from the limitations of the applied mathematical apparatus as well as from mutual constraint of the material points that generate forces of a complex mathematical structure.

The simplest and thus, the most widely used, way of describing is the one with respect to base coordinate system $(y, \boldsymbol{e})$.

It is assumed that there are $N$ material points of mass $m_{\nu}(\nu=1, \ldots, N)$ whose position vectors $r_{\nu}=y_{\nu}^{i} e_{i}(i=1,2,3)$ and that they are connected by $k$ finite constraints

$$
\begin{equation*}
f_{\mu}\left(y_{\nu}^{1}, y_{\nu}^{2}, y_{\nu}^{3}\right)=f_{\mu}\left(y^{1}, \ldots, y^{3 N}\right)=0 \tag{12}
\end{equation*}
$$

where the following notations are introduced

$$
\begin{gather*}
y_{\nu}^{1}=: y^{3 \nu-2}, y_{\nu}^{2}=: y^{3 \nu-1}, y_{\nu}^{3}=: y^{3 \nu}  \tag{13}\\
m_{3 \nu-2} \equiv m_{3 \nu-1} \equiv m_{3 \nu} \tag{14}
\end{gather*}
$$

The constraints (12) must satisfy the velocities conditions

$$
\begin{equation*}
\dot{f}_{\mu}=\frac{\partial f_{\mu}}{\partial y^{\alpha}} \dot{y}^{\alpha}=0, \quad(\alpha=1, \ldots, k, k+1, \ldots, 3 N) \tag{15}
\end{equation*}
$$

as well as the acceleration conditions

$$
\begin{equation*}
\ddot{f_{\mu}}=\frac{\partial^{2} f_{\mu}}{\partial y^{\beta} \partial y^{\alpha}} \dot{y}^{\alpha} \dot{y}^{\beta}+\frac{\partial f_{\mu}}{\partial y^{\alpha}} \ddot{y}^{\alpha}=0 . \tag{16}
\end{equation*}
$$

These constraints are considered independent so that the determinant of the matrix $\left(\frac{\partial f_{\mu}}{\partial y^{\alpha}}\right)$ of the level $k$, is different from zero: The holonomic constraints equations can be written in the parametric form:

$$
\begin{equation*}
r_{\nu}=r_{\nu}\left(q^{0}, q^{1}, \ldots, q^{n}\right), n=3 N-k \tag{17}
\end{equation*}
$$

where $q=\left(q^{1}, \ldots, q^{n}\right)$ are independent generalized coordinates, while $q^{0}$ is a rheonomic coordinate satisfying equation

$$
\begin{equation*}
q^{0}-\tau(t)=0 \tag{18}
\end{equation*}
$$

The velocities of $\nu$-th material points, can be written in the following form:

$$
\begin{equation*}
v_{\nu}=\frac{\partial r_{\nu}}{\partial q^{0}} \dot{q}^{0}+\frac{\partial r_{\nu}}{\partial q^{1}} \dot{q}^{1}+\cdots+\frac{\partial r_{\nu}}{\partial q^{n}} \dot{q}^{n}=\frac{\partial r_{\nu}}{\partial q^{\alpha}} \dot{q}^{\alpha} \tag{19}
\end{equation*}
$$

where $\frac{\partial r_{\nu}}{\partial q^{\alpha}}(q)$ are coordinate vectors that will be marked by two-indices notation $g_{\nu \alpha}$; index $\nu$ denotes the number of the material point, while index $\alpha$ denotes the number of independent coordinate $q^{\alpha}(\alpha=0,1, \ldots, n)$.

For addition with respect to index $\nu$, we use addition sign $\sum_{\nu}$, while for addition with respect to the indices, coordinate $\alpha$ denotes iteration of the same letter in the same expression, as well as both the lower and the upper indices. Vector (19), as can be seen, has $n+1$ independent elementary vectors. Accordingly, impulse vector (10) of the $\nu$-th material point of mass $m_{\nu}$ of the observed system can also be represented by the formula

$$
\begin{equation*}
p_{\nu}=m_{\nu} v_{\nu}=m \frac{\partial r_{\nu}}{\partial q^{\alpha}} \dot{q}^{\alpha} \tag{20}
\end{equation*}
$$

Scalar multiplication by coordinate vectors $\frac{\partial r_{\nu}}{\partial q^{\beta}}$ gives vector $p_{\nu}$ projection upon the tangential direction of $q^{\beta}$ coordinate of the $\nu$-th material point. We will denote it by a two-indices letter:

$$
p_{\nu \beta}=m_{\nu} \frac{\partial r_{\nu}}{\partial q^{\alpha}} \cdot \frac{\partial r_{\nu}}{\partial q^{\beta}} \dot{q}^{\alpha} .
$$

Regarding the fact that $p_{\nu \beta}$ impulses are scalars, it is possible to sum them up:

$$
\begin{equation*}
p_{\beta}:=\sum_{\nu=1}^{N} p_{\nu \beta}=\sum_{\nu=1}^{N} m_{\nu} \frac{\partial r_{\nu}}{\partial q^{\alpha}} \cdot \frac{\partial r_{\nu}}{\partial q^{\beta}} \dot{q}^{\alpha}=a_{\alpha \beta} \dot{q}^{\alpha}, \tag{21}
\end{equation*}
$$

from which it can be seen that $a_{\alpha \beta}$ is an inertia tensor of the whole system:

$$
\begin{equation*}
a_{\alpha \beta}=\sum_{\nu=1}^{N} m_{\nu} \frac{\partial \boldsymbol{r}_{\nu}}{\partial q^{\alpha}} \cdot \frac{\partial \boldsymbol{r}_{\nu}}{\partial q^{\beta}}=a_{\alpha \beta}\left(m_{1}, \ldots, m_{N} ; q^{0}, q^{1}, \ldots, q^{n}\right) . \tag{22}
\end{equation*}
$$

If the masses are constant quantities, this tensor is written as a function of independent coordinates:

$$
\begin{equation*}
a_{\alpha \beta}=a_{\beta \alpha}\left(q^{0}, q^{1}, \ldots, q^{n}\right) . \tag{23}
\end{equation*}
$$

By means of important relations (21) the concept of generalized impulses of the material points' system is introduced. Therefore, the sum of the material points' impulse vector projections upon the coordinate direction of the $\beta$-th generalized coordinate is considered as the generalized impulse $p_{\beta}$. The generalized impulses appear as linear homogeneous forms of the generalized velocities, which is in accordance with the basic definition of impulse (8). Regarding the fact that the inertia tensor $a_{\alpha \beta}$ determinant is different from zero, it is possible to determine the generalized velocities $\dot{q}^{\alpha}$ as linear homogeneous combinations of the generalized impulses, namely:

$$
\begin{equation*}
\dot{q}^{\alpha}=a^{\alpha \beta} p_{\beta}, \tag{24}
\end{equation*}
$$

where $a^{\alpha \beta}$ is countervariant inertia tensor.
If the constraints do not explicitly depend upon the known functions of time $\tau$, there is no rheonomic coordinate $q^{0}$, so that in all the expressions, from (17), coordinates $q^{0}, \dot{q}^{0}$ and $p_{0}$ vanish. The impulse form (21) does not change, expect for the fact that indices $\alpha=0,1, \ldots, n$ do not assume values from 0 to $n$, but from 1 to $n$. In order to facilitate this distinction further on, let Greek indices $\alpha, \beta, \gamma, \delta$ assume values from 0 to $n,(\alpha, \beta, \gamma, \delta=0,1, \ldots, n)$, while the Latin ones take $i, j, k, l$ from 1 to $n(i, j, k, l=1,2, \ldots, n)$. Then it can be written [44]:

$$
v_{\nu}=\frac{\partial r_{\nu}}{\partial q^{0}} \dot{q}^{0}+\frac{\partial r_{\nu}}{\partial q^{i}} \dot{q}^{i} \quad p_{i}=a_{0 i} \dot{q}^{0}+a_{i j} \dot{q}^{j}, ~=a_{00} \dot{q}^{0}+a_{0 j} \dot{q}^{j} .
$$

## Covariant Differential Equations of the System's Motion

If equations of motion

$$
\begin{equation*}
m \frac{d v_{\nu}}{d t}=F_{\nu} \tag{25}
\end{equation*}
$$

are successively multiplied scalarly by coordinate vectors $\frac{\partial r_{\nu}}{\partial q^{\alpha}}$ respective to index $\nu$ and if they are added with respect to index $\nu$, the system of $n+1$ covariant equations of the motion, [4], [5], [12],

$$
\begin{equation*}
a_{\alpha \beta} \frac{D \dot{q}^{\beta}}{d t}=Q_{\alpha}, \quad(\alpha=0,1, \ldots, n) \tag{26}
\end{equation*}
$$

or

$$
\begin{aligned}
& a_{i \beta} \frac{D \dot{q}^{\beta}}{d t}=Q_{i} \quad(i=1, \ldots, n) \\
& a_{0 \beta} \frac{D \dot{q}^{\beta}}{d t}=Q_{0}^{*}+R_{0}=: Q_{0}
\end{aligned}
$$

## Analysis and Solutions of Relation of Motion

The integration of differential equations or of a system of differential equations of motion and of analyses of the solutions obtained for known parameters at some moment of time represents the knowledge about mechanical objects' motion. Very few real motions of the body and, especially, systems of bodies, can be described by finite general analytical solutions of differential equations. Many system models described in the related textbooks do not reflect faithfully the real motion of objects. Still, with great accuracy and with a fairly proper estimate of the error size, mechanics successfully solves problems of all mechanical motions accessible to human eye or even more than that. Many books have been written about it; besides, solutions of new problems are daily published. Still, only a few statements are considered here, namely, those based upon the preprinciple of invariance [12].

## Integrals of Material Point's Motion Impulse

For the material point of constant mass and the condition

$$
\begin{equation*}
F+R=0 \tag{27}
\end{equation*}
$$

it is obtained from equation (25) that the motion impulse vector is constant, that is,

$$
\begin{equation*}
p=m v(t)=c=\mathrm{const}=m v\left(t_{0}\right)=p_{0} \tag{28}
\end{equation*}
$$

At first sight, it seems to be the simplest first vector integral by which the problem of determining motion is solved:

$$
\begin{equation*}
r(t)=v\left(t_{0}\right) t+r\left(t_{0}\right) \tag{29}
\end{equation*}
$$

However, a view of relations (8) and (9), and especially of (21), as well as disagreement about the impulse coordinates, both require that this essential meaning should be much more clarified. Integral (28) satisfies and best explains the preprinciple of casual definiteness. With as much accuracy as mass and velocity are known at some moment $t_{0}$, motion impulse $p(t)$ can be determined under condition (27) at any other moment.

The preprinciple of invariance must be satisfied so that integral (28)-essential impulse $\boldsymbol{p}(t)$-could be sustained in this theory. If vector (28) is resolved in coordinate system $(y, e)$, as in (8), that is

$$
p=m v=m \dot{y}^{i} e_{i}=c^{i} e_{i}=m \dot{y}_{0}^{i} e_{i}
$$

and if it is scalarly multiplied by vector $e_{j}$, it is obtained that

$$
\begin{equation*}
p_{j}(t)=m \dot{y}_{j}=m \dot{y}_{j}\left(t_{0}\right)=p_{j}\left(t_{0}\right) . \tag{30}
\end{equation*}
$$

Allowing for parallel displacement of base vectors $e_{i}$, and thus of coordinate vectors $g_{k}=\partial y^{i} / \partial x^{k} e_{i}$ for free displacement of the point, vector (8), that is,

$$
\begin{equation*}
p=m \dot{x}^{k} g_{k}(x)=m \dot{x}^{K}\left(t_{0}\right) g_{K}\left(x_{0}\right) \tag{31}
\end{equation*}
$$

can be scalarly multiplied by vector $g(x)$. That is how projections of integral (28) upon coordinate directions $g_{l}(x)$ are obtained in the form

$$
\begin{equation*}
p_{l}(x, \dot{x})=a_{k l}(x) \dot{x}^{k}=a_{k l}\left(x_{0}, x\right) \dot{x}^{k}\left(t_{0}\right)=a_{K l} a^{K L} p_{L}=a_{l}^{L} p_{L} \tag{32}
\end{equation*}
$$

where capital letters in the index denote respective value at the initial moment of time, while the tensor

$$
\begin{equation*}
a_{K l}=m\left(\frac{\partial y}{\partial x^{k}}\right)_{0} \frac{\partial y}{\partial x^{l}}=m g_{K l}=m g_{K}\left(x_{0}\right) \cdot g_{l}(x) \tag{33}
\end{equation*}
$$

represents a bipoint inertia tensor. In the referential literature, tensor $g_{K l}$ can be found as "the tensor of parallel displacement".

In order to satisfy the preprinciple of invariance, integrals (30) and (32) should be directly obtained from the coordinate forms of motion equations

$$
\begin{equation*}
m \ddot{y}_{i}=Y_{i}+R_{i} \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{i j} \frac{D v^{i}}{d t}=X_{j}+R_{j} \tag{35}
\end{equation*}
$$

According to the preprinciple of invariance, this relation should also be valid with respect to the curvilinear coordinate system. This is confirmed by integrating the equations (35) for $X_{j}+R_{j}=0$. The covariant integral [1], [2] is

$$
\begin{equation*}
\hat{\int} a_{i j} D v^{i}=\hat{\int} D\left(a_{i j} v^{j}\right)=a_{i j} v^{j}-\mathcal{A}_{i}=0 \tag{36}
\end{equation*}
$$

where $\mathcal{A}_{i}$ is covariantly constant covector $\mathcal{A}_{i}=g_{i}^{K} p_{K}\left(t_{0}\right)$. Accordingly, integral (36) is integral (32)

$$
\begin{equation*}
p_{i}(t)=a_{i j} \dot{x}^{j}=a_{i J} \dot{x}^{J}=a_{i J} a^{J K} p_{K}=g_{i}^{K} p_{K}\left(t_{0}\right) \tag{37}
\end{equation*}
$$

Without pointing to the possibility of parallel displacement of covector $g_{i}$, impulses (33) can be translated from the system of $y$ coordinates into $x$ curvilinear coordinates. If $x$ coordinates are denoted by indices $k, l=1,2,3$, it will follow

$$
p_{j}(t)=p_{k} \frac{\partial x^{k}}{\partial y_{j}}=p_{j}\left(t_{0}\right)=p_{K}\left(t_{0}\right) \frac{\partial x^{K}}{\partial y^{j}}
$$

Multiplying by matrix $\frac{\partial y^{j}}{\partial x^{l}}$ it is obtained that

$$
p_{j}(t) \frac{\partial y^{j}}{\partial x^{l}}=p_{K}\left(t_{0}\right) \frac{\partial x^{K}}{\partial y^{j}} \cdot \frac{\partial y^{j}}{\partial x^{l}}=g_{l}^{K} p_{K}(t)=p_{l}(t)
$$

since it is

$$
g_{l}^{K}=\frac{\partial x^{K}}{\partial y^{j}} \frac{\partial y^{j}}{\partial x^{l}}
$$

Though the covariant integrals satisfy the preprinciple, of invariance such integration is not widely spread in mechanics due to the "difficulties" in determining tensor $g_{l}^{K}$. That is why the ordinary first integrals reduced to constants are looked for, instead of covariantly-constant integrals.

Let the differential equations of motion be written in the extended form:

$$
\begin{equation*}
a_{i j} \frac{D \dot{x}^{j}}{d t}=\frac{D a_{i j} \dot{x}^{j}}{d t}=\frac{D p_{i}}{d t}=\frac{d p_{i}}{d t}-p_{k} \Gamma_{i j}^{k} \frac{d x^{j}}{d t}=X_{i}+R_{i} \tag{38}
\end{equation*}
$$

For the conditions

$$
\begin{equation*}
X_{i}+R_{i}+p_{k} \Gamma_{i j}^{k} \dot{x}^{j}=0, \tag{39}
\end{equation*}
$$

that are different from conditions (27) the ordinary first integrals are obtained

$$
\begin{equation*}
p_{j}(t)=\mathrm{const}=p_{j}\left(t_{0}\right) \tag{40}
\end{equation*}
$$

with respect to coordinate system $(x, g)$. Therefore, it is the same as in the case of integral (30) in base coordinate system ( $y, e$ ). These integrals considerably differ from integral (37) and, therefore, from (30). That is why integrals (30) and (37) will be called covariant integrals. These ordinary integrals (40) destroy the tensor nature of the observed objects.

A shorter, clearer, more general and important difference of the first integrals of the impulses $p_{i}=c_{i}$ and the covariant integrals $p_{i}=\mathcal{A}_{i}$ shows integration of differential equations (30) under the condition that the generalized forces are $Q_{i}=0$. Let it be, for the time being, once again motion of one material point in curvilinear system of coordinates $x^{1}, x^{2}, x^{3}$, that is,

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial E_{k}}{\partial \dot{x}^{i}}-\frac{\partial E_{k}}{\partial x^{i}}=0, \quad(i=1,2,3) \tag{41}
\end{equation*}
$$

These equations can be written in the form

$$
\begin{equation*}
\frac{D}{d t} \frac{\partial E_{k}}{\partial \dot{x}^{i}}=0 \tag{42}
\end{equation*}
$$

From equations (41) for

$$
\frac{\partial E_{k}}{\partial x^{i}}=0
$$

integrals (40) are obtained, while from equations (42) covariant integrals (37) are obtained, since it is

$$
\frac{\partial E_{k}}{\partial \dot{x}^{i}}=p_{i} .
$$

Canonical differential equations of motion, as can be seen from

$$
\frac{d p_{i}}{d t}=-\frac{\partial H}{\partial x^{i}}+X_{i}, \quad(i=1,2,3)
$$

usually produce integral impulses of the type (40) under the condition that the right sides of these equations are equal to zero.

The distribution of the ordinary integral and of integral (40) is greater comparing to covariant integrals (37). The reason for this mostly lies in insufficiently developed calculation with vectors, that is, tensors. The advantage of ordinary integration is also reflected in the fact that, at smaller number of integral impulses
than that of impulse coordinates, constants can be determined depending on the given initial values of the observed impulse, for example,

$$
\begin{gathered}
p_{1}(t)=c_{1}=p_{1}\left(t_{0}\right) \text { and } p_{3}(t)=c_{3}=p_{3}\left(t_{0}\right) ; \\
p_{2} \neq \text { const }
\end{gathered}
$$

This advantage becomes prominent with the system of material points with constraints, and especially upon manifolds $T M$. Accuracy of both of them is proved, though at various conditions. The covariant integration is invariant with respect to the linear homogeneous transformations of the coordinate systems; thus, it reflects the tensor nature of the integrals. This is not the case with ordinary integration; neither is it in accordance with the preprinciple of invariance which points to the fact that the final results of the synthesis should be verified by comparing them to the respective results in coordinate systems $(y, e)$.

Dynamics is a science about real equilibrium and motions of material systems. However, every state of the mechanical system that corresponds to mathematically strict solutions of both the rest equations and the differential equations of motion is not being observed in reality. The general principle for choosing solutions that correspond to stable states in mechanics has not been given; instead, the character of science about idealized systems has been accepted and for every strict application to our nature-every time, on principle-solutions of the stability problems were looked for, [12].

## Invariant Criterion of Motion Stability

The concept of the invariant criterion implies general measurement standard in all the coordinate systems for estimating stability of some undisturbed mechanical system's motion. As such, it comprises stability of the equilibrium position and state, stability of stationary motions and, in general, of motion of mechanical systems whose disturbance equations are of coordinate shape

$$
\begin{equation*}
\frac{D \eta_{\alpha}}{d t}=\psi_{\alpha}(t, \eta, \xi) \tag{43}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{D \eta_{\alpha}}{d t} & =\psi_{\alpha}(t, \eta, \xi) \\
\frac{D \xi^{\beta}}{d t} & =a^{\alpha \beta} \eta_{\alpha} \tag{44}
\end{align*}
$$

If for the differential equations of disturbance (43) and (44) there is such a positively definitive function $W$ of disturbance $\xi^{0}, \ldots, \xi^{n}$ and time $t$ that the expression is

$$
\begin{equation*}
\frac{\partial W}{\partial t}+a^{\alpha \beta}\left(\Psi_{\alpha}+\frac{\partial W}{\partial \xi^{\alpha}}\right) \eta_{\beta} \leq 0 \tag{45}
\end{equation*}
$$

smaller or equal to zero, the undisturbed state of the mechanical system's motion is stable.

Proof. As can be seen from equation

$$
\begin{equation*}
\Psi_{\gamma}:=\sum_{\nu=1}^{N}\left(\boldsymbol{F}_{\nu}^{*}-\boldsymbol{F}_{\nu}\right) \cdot \frac{\partial r_{\nu}}{\partial q^{\gamma}}=\Psi_{\gamma}(\xi, \eta, t) \tag{46}
\end{equation*}
$$

functions $\Psi_{\alpha}$ for undisturbed motion $\xi^{\alpha}=0, \eta_{\alpha}=0$ are equal to zero, that is, $\Psi_{\alpha}(0,0, t)=0$.

The function

$$
\begin{equation*}
V=\frac{1}{2} a^{\alpha \beta} \eta_{\alpha} \eta_{\beta}+W(\xi, t) \tag{47}
\end{equation*}
$$

is positively definite, since it is

$$
a^{\alpha \beta}\left(q^{0}(t), q^{1}(t), \ldots, q^{n}(t)\right)
$$

a positively definite matrix of the functions upon $M^{n+1}$, while $W(\xi, t)$ is a positively definite function of disturbance $\xi^{\alpha}$. As a scalar invariant, $V$ is a tensor of zero order. That is why ordinary derivative $\frac{d V}{d t}$ is equal to the natural derivative

$$
\begin{equation*}
\frac{D V}{d t}=a^{\alpha \beta} \frac{D \eta_{\alpha}}{d t} \eta_{\beta}+\frac{\partial W}{\partial \xi^{\alpha}} \frac{D \xi^{\alpha}}{d t}+\frac{\partial W}{\partial t} \tag{48}
\end{equation*}
$$

which necessarily has to be smaller or identical to zero. By substitution of the natural derivatives from equations (43) and (44) in (48) it is obtained that

$$
\frac{D V}{d t}=\frac{\partial W}{\partial t}+a^{\alpha \beta} \Psi_{\alpha} \eta_{\beta}+\frac{\partial W}{\partial \xi^{\alpha}} a^{\alpha \beta} \eta_{\beta}
$$

and this, along with the criterion requirement, is reduced to

$$
\begin{equation*}
\frac{\partial W}{\partial t}+a^{\alpha \beta}\left(\Psi_{\alpha}+\frac{\partial W}{\partial \xi^{\alpha}}\right) \eta_{\beta} \leq 0 \tag{49}
\end{equation*}
$$

Therefore, the stability criterion is proved, [12].
If neither forces $F_{\nu}^{*}$ and $\boldsymbol{F}$ from relations (46) nor differences $F_{\nu}^{*}-F_{\nu}$ depend of time $t$ on position $r$ and velocity $v$, function $\Psi_{\gamma}$ will also be explicitly independent of $t$. Then function $W$ should also be looked for only in its dependence on disturbances $\xi^{0}, \xi^{1}, \ldots, \xi^{n}$, that is, $W=W\left(\xi^{0}, \xi^{1}, \ldots, \xi^{n}\right)$, so that expressions (45) and (49) are reduced to

$$
\begin{equation*}
a^{\alpha \beta}\left(\Psi_{\alpha}+\frac{\partial W}{\partial \xi^{\alpha}}\right) \eta_{\beta} \leq 0 \tag{50}
\end{equation*}
$$

If the mechanical system's constraints do not depend on time, $q^{0}, \xi^{0}, \eta_{0}, \Psi_{0}$, vanish, so that expression (45), that is (49), is reduced to

$$
\begin{equation*}
\frac{\partial W}{\partial t}+a^{i j}\left(\Psi_{i}+\frac{\partial W}{\partial \xi^{i}}\right) \eta_{j} \leq 0 \tag{51}
\end{equation*}
$$

while expression (50) is reduced to

$$
\begin{equation*}
a^{i j}\left(\Psi_{i}+\frac{\partial W}{\partial \xi^{i}}\right) \eta_{j} \leq 0 \tag{52}
\end{equation*}
$$

where $\Psi_{i}$ and $W$ do not depend on $\xi^{0}$ and $\eta^{0}$.
All the expressions of the previously given criterion for the equilibrium state stability appear as consequences of expression (49) if $\xi$ and $\eta$ are regarded as disturbances of equilibrium state $q$ and $p$.

## On Integrals of Covariant Equations of Disturbance

Covariant equations of motion

$$
\begin{equation*}
\frac{D p_{\alpha}}{d t}=Q_{\alpha} \tag{53}
\end{equation*}
$$

or differential equations of disturbance respective to them (43) in their extended form and in the general case have a very complex structure what makes their integration difficult. However, by applying the covariant integration some first covariantly constant integrals are obtained as a means of assessing the equilibrium state stability as well as undisturbed motion. As an addition to this assertion, the two recognizable and acceptable examples are presented here.

1. Let the generalized forces $Q_{\alpha}$ in equations (53) have a function of force $U\left(q^{0}, q^{1}, \ldots, q^{n}\right)$. Let's multiply each equation (53) by respective differential from equation $\dot{q}^{\alpha}=a^{\alpha \beta} p_{\beta}$ and add in the following way,

$$
a^{\alpha \beta} p_{\beta} D p_{\alpha}=Q_{\alpha} d q^{\alpha}=\frac{\partial U}{\partial q^{\alpha}} d q^{\alpha} .
$$

Since $D a^{\alpha \beta}=0$, it is $\frac{1}{2} D\left(a^{\alpha \beta} p_{\beta} p_{\alpha}\right)=d U$ and $\frac{1}{2} a^{\alpha \beta} p_{\beta} p_{\alpha}-U=C=$ const.
2. Let the right sides of covariant equations (43) be linear forms of disturbance from $\xi^{1}, \ldots, \xi^{n}$, that is, $\Psi_{i}=-g_{i j}\left(q^{1}(t), \ldots, q^{n}(t)\right) \xi^{j}$, where $g_{i j}$ as well as $a^{i j}\left(q^{1}, \ldots, q^{n}\right)$ are covariantly constant tensor. For the given disturbances, equations (43) and (44) can be written in the covariant form:

$$
\frac{D \eta_{i}}{d t}=g_{i j} \xi^{j}, \frac{D \xi^{i}}{d t}=a^{i j} \eta_{j} .
$$

By mutual complete multiplication and addition with respect to index $i$, as in the previous example with respect to $\alpha$, it follows $a^{i j} \eta_{j} D \eta_{i}=-g_{i j} \xi^{j} D \xi^{i}$. The
covariant integration gives $\frac{1}{2} a^{i j} \eta_{j} \eta_{i}-g_{i j} \xi^{j} \xi^{i}=\mathcal{A}$, where $\mathcal{A}$ is a constant, $D \mathcal{A}=$ $d \mathcal{A}=0$.

Therefore, by covariant or ordinary integration and the solution analysis or directly by applying criterion (45) or

$$
\frac{\partial W}{\partial t}+\left(Q_{i}+\frac{\partial W}{\partial q^{i}}\right) \dot{q}^{i} \leq 0 \quad(i=1, \ldots, n)
$$

the stability of undisturbed motion $\xi=0, \eta=0$ or that of the equilibrium state of system $q=q_{0}, p=0$ can be assessed.

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[^2]:    ${ }^{1}$ See e.g., [2], pp. 143 and 180.
    ${ }^{2}$ Einstein's summation convention for diagonally repeated indices is used; Latin indices have the range $1,2,3$, while Greek indices will have the range 1,2 .

[^3]:    ${ }^{3}$ The first index in $g_{. j}^{i}\left(P_{0}, P\right)$, either superscript or subscript, refers to the point determined by the first argument, while the second one refers to the point determined by the second argument.
    ${ }^{4}$ The vector at some point on surface is, by definition, the vector entirely lying in the tangent plane of the surface at this point (see [7], p. 144). Since $\boldsymbol{a}_{\alpha}=\partial \boldsymbol{r} / \partial u^{\alpha}$ ( $\boldsymbol{r}$ is the position vector of the mentioned point in the enveloping Euclidean space) are the vectors tangent to the surface, it follows that $v$ will also be a vector lying in the tangent plane of the surface.

[^4]:    ${ }^{5}$ The symbol " $=$ " denotes coordinates of a quantity transported to the corresponding point!
    ${ }^{6}$ This means that $\left|K_{\beta}^{\alpha}\left(P_{0}, P\right)\right| \neq 0$ and $\left|K_{\beta}^{\alpha}\left(P_{0}, P\right)\right| \neq 0$ !
    ${ }^{7}$ The first index in $K_{. \beta}^{\alpha}\left(P_{0}, P\right)$, either superscript or subscript, again refers to the point determined by the first argument, while the second one refers to the point determined by the second argument.

[^5]:    ${ }^{8}$ See e.g., p. 73 in [9].
    ${ }^{9}$ Mentioned in $[8,10,12]$ in connection with the introduction of the notion of an absolute integral of tensors in Riemannian spaces.
    ${ }^{10}$ This is in accordance with the statement that "the concept of absolute derivative is made to depend on the concept of parallel displacement of a given vector at one point on a curve $C$ to other points on $C^{\prime \prime}$ ([11], p. 178).
    ${ }^{11}$ The rigour was not necessary here since the existence of shifting operators with respect to the surface along a curve given on this surface was already pointed out in [13].
    ${ }^{12}$ Natural, in fact, in a measure in which we are capable of judging events (like the limit (6), for example) within a Riemannian space.
    ${ }^{13}$ Of course, the introduction of another procedure of transport of a vector over the surface would lead to another procedure of (covariant) differentiation in this Riemannian space.

[^6]:    ${ }^{14}$ E.g. as "parallel propagators" in [3], p. 59.
    ${ }^{15}$ See p. 1307 in [12]: "...the problem of the covariantly constant tensor [...A...] in Riemannian spaces is not solved generally".
    ${ }^{16}$ Geographical coordinates are in question!

[^7]:    ${ }^{17}$ Cf. e.g., with the expressions on p. 208 in [5]; see also p. 185 in [11].
    ${ }^{18}$ These coordinates differ from the ones obtained in [14] for the parallel transport along the geodesic lines! Only when the parallel transport along the equator is in question $\left(\vartheta=\vartheta_{0}=0\right)$, i.e., along the geodesic line, then the operators $K_{\beta}^{\alpha}$ in both cases reduce to the Kronecker $\delta$-symbols.

[^8]:    ${ }^{19}$ Its value in this case, as is well-known, is equal $\boldsymbol{v},\left.1\right|_{P_{0}}=\left.v_{1,1}^{\alpha}\right|_{P_{0}} a_{\alpha}\left(P_{0}\right)=-v_{0}^{2} \tan \vartheta_{0} a_{1}\left(P_{0}\right)$. ${ }^{20}$ It will be $v_{, 1}=0$ only in the case when $\vartheta=\vartheta_{0}=0$, i.e., when the parallel propagation along the equator is in question (since the transport is performed along the geodesic line)!

