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Theory and Applications of Gibbs Derivatives

Edited by

Paul L. Butzer and Radomir S. Stanković

IWGD-89

THEORY AND APPLICATIONS OF GIBBS DERIVATIVES

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Proceedings of the First International Workshop on Gibbs Derivatives held September 26-28, 1989 at Kupari-Dubrovnik, Yugoslavia

Editors
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PREFACE

The "First International Workshop on Gibbs Derivatives" was held on September 26 - 28, 1989, at Kupari-Dubrovnik, Yugoslavia. The organizing committee for this workshop consisted of Đuro Kurepa, Petar Vasić, Milić Stojić, Radomir S. Stanković, Paul L. Butzer, Franz Pichler, Claudio Moraga and Yaşushi Endow. The conference was attended by 15 mathematicians and engineers from eight countries, namely Austria, England, F.R. of Germany, Hungary, Japan, USA, USSR and Yugoslavia. It is most fortunate that the founder of Gibbs differentiation, John Edmund Gibbs, formerly of the National Physical Laboratory, Teddington, England, managed to be present.

The aim of the Workshop was to review results established in the theory and application of Gibbs derivatives since the introduction of the concept 22 years ago, to present new results in this area, as well as to stimulate further research.

This volume, presenting the proceedings of the workshop, includes 14 invited conference papers, one joint paper submitted at the workshop by two participants, and three papers submitted subsequently by colleagues who could, unfortunately, not participate, as well as a report devoted to new and unsolved problems based on a special problem session and as augmented by later communications. This report was kindly edited by Claudio Moraga.

The proceedings begin with comments on the literature on Gibbs derivatives based on a list of papers on the subject, as complete as possible, compiled by J.E. Gibbs and R.S. Stanković, accompanied by some remarks on the development of the subject and the major contributors.

The first, introductory paper by Franz Pichler is the author's view on the history of signal processing and the role of harmonic, particulary Fourier and Walsh analysis in this area.

The papers of these proceedings have been grouped into two parts. The first part contains those contributions that are mainly concerned with "continuous" Gibbs derivatives, considered in the setting of Walsh analysis or general harmonic analysis;

the topics are often counterparts or represent extensions of results from classical analysis. These papers deal, for example, with the connection between Newton-Leibniz and Gibbs differentiation, with further extensions of dyadic Gibbs differentiation, with term by term dyadic differentiation of Walsh series, with dyadic martingales, with Hörmander-type multiplier theorems on locally compact Vilenkin groups, with convergence properties of Walsh-Fourier integral operators. Applications of dyadic Gibbs differentiation in the theory of dyadic stationary processes and statistics are also considered.

The second part deals with papers devoted to "discrete" Gibbs derivates, in particular with their numerical evaluation, with their possible generalizations and extensions, with applications in image processing, linear system theory, logic design, and so on.

It will be observed that contributions by a larger part of the world's major representatives concerned with Gibbs differentiation will be found in these proceedings. Exceptions include those by further experts from China, Germany, USSR and Canada such as Wei-xing Cheng, Wei-yi Su, W.Splettstoesser, H.J.Wagner, V.A. Skorcov, and S.Cohn-Sfetcu. It is also regrettable that the number of papers dealing with concrete engineering applications of Gibbs derivatives was not greater.

The editors' warm thanks are due to all of the participants and contributors; they made the workshop the success it was; to the academic Svetozar Aljančić, the Chairman of the Scientific Council, and to Dr. Zoran Marković, the Director of the Institute of Mathematics in Belgrade who made it possible that these proceedings could appear in the series *Special Issues by the Institute of Mathematics*.

Paul L. Butzer — *Aachen*
Radomir S. Stanković — *Niš*
August 1990

OPENING REMARK

Dear participants of the First International Workshop on Gibbs derivatives, at Kupari-Dubrovnik, YU, 1989:09:26:2-4.

I am honoured and pleased to welcome you in Kupari at the Adriatic coast where we gathered at this symposium on derivatives.

The fundamental notion of derivatives of a function evolved slowly. The notion is tied to the geometrical notion of contact (tangent) and to that of instantaneus velocity (Newton).* It is instructive to notify that examples of constructions of tangents occured much prior than the concept of a tangent as a limit (Pascal and particularly Leibniz). Anyway, $f'(t)$ or $Df(t)$ is a limit. In this respect it is appropriate to quote the following:

"Ultimae rationes illae quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decresentium rationes semper appropinquant, et quas proprius assequi possunt quam pro data quavis differentia, nonquam vero transgredi, neque prius attingere quam quantitates diminuntur in infinitum",

or in English:

"Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities, decreasing without limit, approach, and which, though they can come nearer than any given difference, whatever, they can neither pass over not attain before the quantities have diminished indefinitely". (Newton, *Philosophiae Naturalis Principia*, End of Section I, London 1687, p.36[10-16]).

One is familiar with tremendous difficulties and controverses concerning infinitesimales tied with derivatives and differentials and how the last concepts are various, powerful in the theory as well as in Applications.

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* It was a great event when in 1934 was published a Newton's letter in which he pointed out that a Fermat's determination of a tangent provoked in him the idea of derivative.

The notion derivative was generalized in many directions. One knows e.g. the concept of fractional derivative $D^r f(x)$ for any real r>0 for any $f : R \to R$. Df was defined not only for $f : R \to R$ but also for $f : R^n \to R$ and for $f : C^n \to C$. So also one considered derivatives and Mathematical Analysis connected to semi-reals or $f : K \to K$ for any field K.

One approach was to consider the set K [[x]] of all formal infinite series $f := \sum_{n=0}^{\infty} f_n x^n$ with coefficients f_n belonging to K and to define $Df := \sum_{n=0}^{\infty} f_{n-1} x^n$ the approach worked irrespective as to wheter K is commutative or non commutative. The procedure is running without any trouble if the characteristic of K is 0. If the characteristic of K is >0 one had some difficulties in connection with Taylor's expansions; but the problem was settled.

The present Conference concerns a very interesting transfer of derivatives for functions defined on groups. In 1967 J.E. Gibbs introduced dyadic derivative of functions of some cyclic groups. Afterwards the procedure was extended and studied by various authors. I am glad to know that several of them are participating at the present conference: I welcome you, dear Colleagues.

The job is an interesting case of transfer (transplantation) of notions and researches from one structure, situation to other ones - a phenomenon which is of vital importance in Mathematics as well as in other activities. As a nice case of such considerations let be mentioned e.g. the notion of determinant over any field (J. Dieudonné 1943).

Dear Colleagues, I wish you fruitful work and enjoyable staying at this marvellous Adriatic coast of the ancient Dubrovnik Republik, one of the perls of the present Yugoslavia. Remember that mathematicians *M. Getaldić* (Dubrovnik 1568-Dubrovnik 1626) and *R.J. Bošković* (Dubrovnik 1711-Milano 1787) were born in this Republic and performed good services to their small country.

At the end I want to let know that the Conference would not take place without the financial help of the Zajednica za nauku i kulturu Srbije and the Matematički institut Beograd. The initiator and the main organizer of the Conference was docent Stanković Radomir; his main scientific contribution belongs to harmonic derivatives on groups.

I wish a good success of this Conference. I am convinced that in the future one shall have similar gatherings because the subject is useful, important, fertile and beautiful.

Kurepa Đuro R.

WHY IWGD-89?

A look at the bibliography of Gibbs derivatives

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It is always a very difficult task to measure or estimate, and to express quantitatively, the degree of interest in a specific branch of knowledge. But such measurement, however fraught with difficulty, may be a very worthwhile undertaking: for the level of interest, among the research community, in a given topic is usually highly correlated with its theoretical importance or its practical significance.

On the occasion of IWGD-89 we sall make bold to attack such a measurement problem in respect of Gibbs derivatives, a concept introduced into mathematics in 1967 [1]. For want of any more sophisticated criterion, we assume that the interest in (and hence the importance of) Gibbs derivatives is indicated by the number of publications (per year) in which the concept is considered or at least mentioned. We apply this criterion over the 22 years 1967-1988.

We began our analysis armed with a bibliography, consisting of 155 references. Information about the publications in the bibliography was obtained from the citations in the papers already available to us and from such abstracting journals as *Mathematical Reviewes* (USA), *Referativni Jurnal* (USSR), and *Zentralblatt für Mathematik* (West Germany). We are all too well aware that there must be a number of relevant publications that escaped our attention and are therefore not included in the bibliography. This caveat applies particularly, and with the highest probability, to the most recent papers; for the reviewing procedure of abstracting journals necessarily takes some time. Even so, we guess that the number of items missed can hardly exeed 5% of the total number of relevant items in the literature. This promisingly small estimate has encouraged us to make the analysis that follows. We shall be very grateful for any information helping us to improve or update our biblography of Gibbs derivatives. We take this opportunity to offer our sincere apologies to all authors of whose publications we have failed to do justice.

Figure 1 is a diagram showing the number of publications on Gibbs derivatives recorded in our bibliography for each of the years from 1969 to 1988 inclusive. Curve (a) relates to the total number of publications in each year. Bearing in mind the delay that always occurs before the appearance of a paper in a journal, we have introduced two further curves. Curve (b) refers to the results presented at conferences and symposia and published in the appropriate proceedings, as well as those appearing as internal reports, MSc and PhD dissertations, and the like. The number of papers appearing each year in journals or monographs is indicated by Curve (c).

Curve (a) exibits three important peaks. The first two peaks appear in 1973 and 1976, indicating clearly the period of greatest activity in Gibbs derivatives. The coincidence in time of the two main peaks in Curves (a) and (b) is in accordance with the fact that the most extensive considerations of concept in its infancy is usually chiefly confined to discussions at meetings and symposia. In this period the major contributions are from J.E. Gibbs, F. Pichler, and P.L. Butzer and his collaborators. Meanwhile the number of relevant papers in journals grows steadily, and between 1976 and 1977 Curve (c) crosses from below to above Curve (b), seldom to fall below it thereafter. The grand maximum of Curve (c) is attained in 1978. At about this time, contributions from the above-mentioned authors are joined by an especially noteworthy series of papers by C.W. Onneweer.

It is very important to notice that, after a period of apparently declining interest in Gibbs derivatives, Figure 1 shows recrudescence of interest in the last few years. A peak in all three curves appears in 1985, but we belived that great interest of that year has been maintained to this day, the decline apparent in Figure 1 being due merely to a lack of information about the most recent publications. Particularly noteworthy is that the articles on Gibbs derivatives during the last few years are due principally to a number of authors (Endow, Zelin He, Moraga, Stanković, etc.) who have not published on the subject before, who may therefore be regarded as a new generation of the family of researchers on Gibbs derivatives. The welcome circumstance that these authors come from a range of differnent countries (Japan, China, West Germany, Yugoslavia, etc.) surely points to a world-wide interest in Gibbs derivatives. In these facts we find a convincing answer to the question "Why IWGD-89?"

Let us mention in conclusion some further inferences that may be drawn from a detailed study of our bibliography.

It is interesting to notice the predilections of different authors as to the medium for presenting their results. Some, for example, Dr Gibbs, prefer to publish *on line*, as it were, thus mainly in conference proceedings and reports. Others, for instance, Prof. C. W. Onneweer, publish almost exclusively in journals. Other authors again, such as

Prof. Butzer and Prof. Pichler, make extensive use of both media. It is a special pleasure to emphasise the continual contributions from Prof. Bitzer during almost all the 22 years that the concept has been around.

If we simply count the number of publications by each author, the following picture emerges: J. E. Gibbs (27), P. L. Butzer (15), W. Splettstösser (13), H. J. Wagner (11), R. S. Stanković (11), C. W. Onneweer (10), W. Engles (10), F. Pichler (8), F. Schipp (8), J. Pál (8), S. Cohn-Sfetcu (5), etc.

We should like to make favourable mention of the number of colleagues initiated directly into the field, through co-authorship of publications, by J.E. Gibbs (6), and P.L. Butzer (6). The number of those initiated indirectly trough their numerous publications and in other ways is certainly much greater.

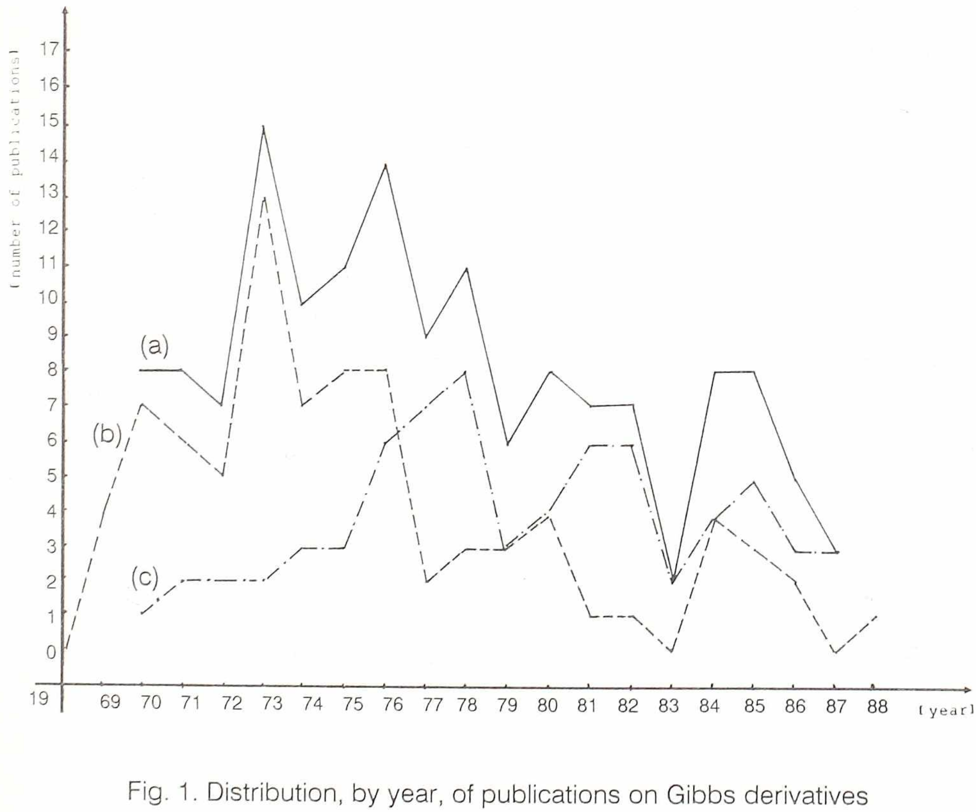

Fig. 1. Distribution, by year, of publications on Gibbs derivatives

REFERENCE

[1] J. E. Gibbs, Walsh spectrometry, a form of spectral analysis well suited to binary digital computation, NPL AIU Rept, 1967 January 13.

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Some Historical Remarks on The Theory of Walsh Functions and Their Applications in Information Engineering

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1. Introduction

After nearly 70 years since the introduction of a rather unusual complete set of orthogonal functions by the American mathematician Josepf L. Walsh into mathematics and after about 30 years of work on the application of this set of functions to Information Engineering it might be acceptable to make some historical remarks as seen from the subjective point of view of this author.

To some extent the set of Walsh functions are the simplest and at the same time for digital communication engineers most appealing set of orthogonal functions.

As we know, they take only the values +1 and -1 and they form with respect to multiplication an abelian group. Their electronic generation is most simple and they share with the sinusoidial functions, which are well established in the engineering field, many common properties. It is, therefore, of no surprise that engineers have for a long time had an interest to use Walsh function in mathematical modelling of signals and systems.

In the following, we describe some of the highlights concerning the different attempts to make Walsh functions useful in engineering applications. The author does not claim any originality of the exposition. Nor would he claim to be able to write in the strict manner of an historian of mathematics. The goal is to collect a few facts from the history of Walsh functions to show how the evolution of the concepts took place.

2. Early Contributions

When Joseph L. Walsh in 1923 invented his "Walsh functions" he did this for pure mathematical reasons. It was the time when - following Hilbert and Schmidt - of big interest was to construct infinite bases of functions for the Hilbert space $L_2(a,b)$. However, "Walsh functions" were already in use in open wire telephone transmission systems: To compensate cross talk between different 2-wire lines it is necessary to cross the wires in certain distances. In Germany the plan for doing this was known as the "Kreuzungsplan von Pinkert", named after a German telephone engineer in about 1880. Figure 1 shows as an example such a scheme of line-crossings for sixteen 2-wire lines.

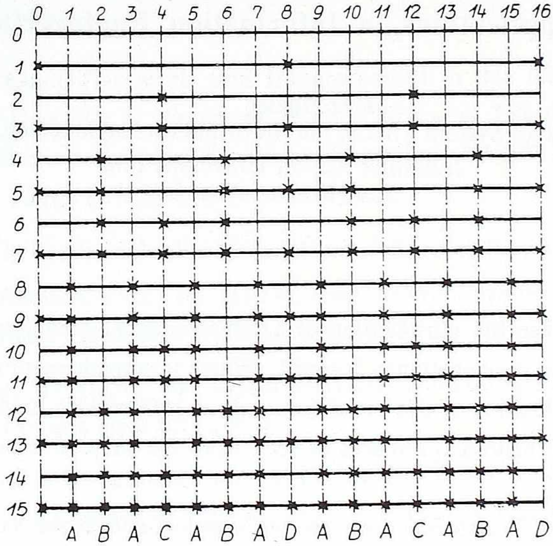

Fig. 1. Telephone line crossing according to Pinkert

In mathematics the Walsh functions found immediate interest and as one of the first investigators we have to mention the British mathematician R.E.A.C. Paley. When Norbert Wiener on his occasion of visiting Cambridge met Paley they both decided to do some common work on Walsh functions and their generalisation. They both attended the International Mathematical Congress in Zürich in 1932 and delivered a paper on "Characters of infinite Abelian groups".

The still existing abstract [1] shows that they had succeeded to develop a general theory for the harmonic-analysis of functions on abelian topological groups which includes as a special case the Walsh-Fourier analysis of functions on the dyadic group.

As it is known from Wiener´s autobiography, R.E.A.C. Paley - during the time of his visit at MIT - got killed by an avalanche. Norbert Wiener, to my knowledge did not continue to work alone in this area, although his popular scientific bestseller on "Cybernetics" [2] contains many facts which show that he still was in favour of the work done earlier together with R.E.A.C. Paley. In addition, his important book on the subject of "Filtering and Prediction" [3] shows also the strong influence of having a more general insight on the topic available. The best reference on this topic by Norbert Wiener is the article of Masani [4]. One can speculate that without the loss of R.E.A.C. Paley, Norbert Wiener would have elaborated a theory of communication based on Walsh functions.

The time for doing this by others came in the early sixties. We mention here the interesting PhD thesis of Frank E. Weiser [5] from 1964, an important paper by the Russion mathematicians Polyak and Shreider [6] and last but not least the earliest papers on Walsh functions and their application in communication engineering by Henning F. Harmuth [7], [8], [9].
From Weiser originates, for example, the concept of a "dyadic invariant linear system" which was many times reinvented later on (e.g. also by Pichler [10]). In the work of Polyak and Shreider we find already the important "multiplicity theorem", which determines the gaps in a Walsh-Fourier expansion, when a polynomial function is represented by it. This theorem was independently found later also by Liedl [11] and generalized by Weiß [12].
From Harmuth originates the fundamental idea of how to construct new type of communication systems which are based on the Walsh functions in the same manner as conventional systems are based on the use of sinusoidal functions. We will elaborate this idea in more detail in the next chapter.

3. Linear Systems on the Dyadic Group

After the invention of the telephone by Graham Bell in 1876 and the first unsuccessful attempts to transmit speech signals on cables over long distances, it became clear that a new theory for the synthesis of systems for cable telephone systems was needed. Oliver Heaviside in England gave the theoretical explanations, Michael Pupin in America put the ideas into practice. Today we have, the "Theory of electrical networks"[13] which covers many important problem areas as they are called in communication engineering. As a mathematical generalization we have today the theory of "Linear systems" [14], more exactly, the theory of ordinary linear differential- or difference equation systems (with constant coefficients), a theory which answers the relevant problems of communication- and control engineering.
Important concepts in Linear Systems Theory are "convolution", "frequency description", "time invariance", "filtering", "transfer function", "modulation", "Fourier-transform", "sampling theorem" and others.
All this concepts can be generalized and have been formulated also for the special case of "Linear Systems on the Dyadic Group". We mention some of the original contributions. Harmuth [8] formulated for the first time the concept of a "sequency band pass filter" by describing the (generalized) transfer function. He needed this concept to design a multi-channel multplexing system based on Walsh-Fourier representation of signals.
In the following Pichler [10] showed how such filters could be defined by dyadic convolution operators. Furthermore, Harmuth [9] introduced the concept of sequency-single-side band modulation. Pichler [10] proved the sampling theorem for sequency-limited signals. Furthermore, the theory of optimal Wiener filters was developed in Pichler [15].
Edmund Gibbs from the National Physical Laboratory Teddington, U.K., developed a complete theory of linear dyadic invariant systems (of finite kind) and he introduced the important concept of "logical differentiation" [16].

Differential-equations of this kind are the counterpart of ordinary classical differential equations for the case of linear dyadic invariant systems. Their solutions can be described by dyadic convolution operators in a similar manner such as the solutions of classical differential equations are described by the usual convolution operation. A sound mathematical treatment of the "Gibbs Derivative" and related equations was developed by Butzer [17], [18]. An important activity to bring researchers together internationally was started by the US Naval Research Laboratory in Washington D.C. in 1970. From 1970 to 1974 an annual conference on Walsh functions and their applications was held and the published conference proceedings are still providing valuable materials.

4. Later Developments and Current Stage

As was mentioned earlier, it cannot be a goal of this report to cover in any detail the development of Walsh function research and their possible applications in Information Engineering. However, we have to mention a few of the most important research activities. As we know, Walsh functions are of "digital nature" when considered as signals. However, linear dyadic systems are of "analog nature".

The question arises whether Walsh functions are of any importance also for "digital systems". The answer is yes and for a fundamental and early contribution we refer to the book of Karpovsky [19]. However, it seems that the descriptive power is more concentrated on digital systems without memory, that is to the class of two-valued boolean functions. Another topic of application is digital image processing. This has been considered as a possible application from the beginning, mainly because of the existence of the Fast Walsh Fourier-transform Algorithm (Welch [20]). Today spectral techniques based on Walsh functions are standard in digital image processing.

5. Conclusion

We have tried to mention some of the historical facts in the development of Walsh functions in respect to their application in Information Engineering. However, the high expectations which were set by Harmuth in the beginning, "that the Walsh functions are for digital systems just of that kind of importance as the sinusoidal functions have gained for analog systems" did not realize. The signals representing information in common communication and control systems demand in processing the use of strongly time-invariant systems (in case of speech) or shift- and rotational-invariant systems (in case of images). This means mathematically that the group R of real numbers and the group C of complex numbers are the "natural" domain for the definition of signals.

But for special tasks in signal processing and systems theory Walsh functions have proven to be a valuable mathematical tool.

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Part One
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LOCAL AND GLOBAL VIEWS OF DIFFERENTIATION

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Abstract: In contrast with the *formal analogy* between Gibbs (G) and Newton-Leibniz (NL) derivatives, ways in which these two concepts may be *seen intuitively as one* are considered in this paper. The natural definition of the G-derivative of a complex-valued function on a cyclic group is *global* (as a symmetric linear combination of difference quotients ranging over the whole group); that of the NL-derivative is *local* (as the limiting value of a difference quotient). Although the philosophies of these definitions appear diametrically opposed, they may be reconciled by Fourier considerations. In particular, a local definition of the G-derivative may be given, as the NL-derivative of an intuitively closely related function of a complex variable. Likewise, under suitable conditions, the NL-derivative may be defined globally, as the limit of a sequence of G-derivatives. The assimilation of the G- and NL-derivatives implied by these results appears to be contraindicative of any possibility of discovering an intuitive interpretation of the G-derivative distinct from that of the NL-derivative.

The relationship between Fourier methods and the local and global views of differentiation extends in a degenerate form to Fourier analysis in the dyadic field regarded as sequence space. In this case the global definition, in which each term of the G-derivative of a sequence is expressed as a linear combination of all the terms of the sequence, reduces to the local definition, which is equivalent to that of the sequence of first forward finite differences. Fourier analysis in the dyadic field is also of interest in requiring for consistency that divergent series of 0s and 1s be in certain circumstances summable modulo 2.

The paper ends with a brief account of the differentiation of dyadic functions, where the distinction between local and global definitions breaks down, and where no non-trivial Fourier analysis exists. The (partial) derivative is the analytic analogue of the algebraic concept of Boolean difference: it enters into quasi-Taylor series of two kinds, one with, in general, a countable infinity of terms, the other with an uncountable infinity. The latter presents an interesting summation problem.

1 INTRODUCTION

Since Gibbs differentiation was introduced in 1967, its claim to be called "differentiation" has been based almost exclusively on formal analogy. The functions differentiated have been principally, though not exclusively[1], complex-valued functions defined on various groups, for example, finite groups (abelian or otherwise[2]) and direct products of countably many cyclic groups. A prime example of what we mean by formal analogy is the fact that the characters of the domain group (supposed abelian) are the eigenfunctions of the G-differentiator (Gibbs differentiator). This fact not only exhibits the formal analogy between the Gibbs and Newton-Leibniz derivatives, but also provides a convenient means of defining the G-differentiator.

On the other hand, during the past 22 years, there has been little or no attempt, apart from a less than fully successful essay by Gibbs and Ireland (1974), to assimilate, heuristically or intuitively, the concepts of G-differentiation and NL-differentiation (Newton-Leibniz differentiation). There has indeed been little progress beyond the expression of a desire that an interpretation of the G-derivative might be found that would have the same intuitive appeal and practical usefulness that characterise the notion of rate of change as an interpretation of the NL-derivative.

In the present paper it will emerge that it may be too much to expect an intuitive interpretation of the G-derivative radically different from that of the NL-derivative. The two concepts are perhaps closer intuitively than has been supposed. We shall see, for example, that a definition of the G-derivative on a cyclic group may be obtained heuristically from the NL-derivative of an associated function of a complex variable, that this G-derivative is equal to a linear combination of difference quotients, and thus not unlike the NL-derivative expressed as the limit of a sequence of difference quotients; that the NL-derivative is expressible, under favourable conditions, as a limit of G-derivatives; and that the G-derivative of a function on a cyclic group is equal to the NL-derivative of a closely related periodic real function.

Bound up with the foregoing insights is a recognition that derivatives may be viewed, in general, either locally or globally, the transition between the two viewpoints being a matter of Fourier analysis. The G-derivative is naturally defined globally, but a local definition

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[1] Cohn-Sfetcu and Gibbs (1976), for example, define Gibbs differentiators on spaces of functions on a finite abelian group into a Galois field.

[2] Gibbs differentiation on finite non-abelian groups has been discussed by Stanković (1986a, 1986b, 1988).

can be obtained as the NL-derivative of a related function. On the other hand, the familiar definition of the NL-derivative is local, but, when appropriate conditions are met, a global definition may be given in terms of G-derivatives. There is thus a kind of contest between the local and the global views which, in the light of our considerations thus far, seems to result in dominance of the local definition for real and complex analysis and of the global definition for analysis on the likes of finite groups.

That the tension between the local and global views is not resolved quite so simply as the previous paragraph suggests is shown by a consideration of Fourier analysis in the dyadic field (Gibbs 1984), where a global definition of a derivative may be given showing some formal analogy with the NL-derivative, while the corresponding local definition mimics that of the first forward difference in the calculus of finite differences. The most summary account of Fourier analysis in the dyadic field would be incomplete without a discussion of the summability of divergent series of 0s and 1s modulo 2, which is therefore included.

We end this paper with an examination of differentiation on the space of functions from, and to, the dyadic field. Here there is no non-trivial Fourier analysis, the fundamental definition of the derivative is both local and global, and, for good measure, there are two kinds of Taylor series, one with a countable, the other with an uncountable set of terms.

2 HEURISTIC APPROACH TO THE GIBBS DIFFERENTIATOR

As concrete set underlying the cyclic group Z_q of order q we take the set

$$\Omega =_{\text{def}} \{\omega^0, \omega^1, \ldots, \omega^{q-1}\} \qquad (\omega =_{\text{def}} \exp(2\pi i/q))$$

of equidistantly spaced points of the unit circle $T =_{\text{def}} \{z : |z| = 1\}$, the group operation thus being complex multiplication. The characters of the group $Z_q = (\Omega, \cdot)$ are the T-valued homomorphisms X_k ($k \in P_q =_{\text{def}} \{0, 1, \ldots, q-1\}$), defined by

$$X_k(\omega^x) = \omega^{kx} \qquad (x \in P_q).$$

To obtain heuristically a definition of G-differentiation, we shall regard each of these characters as a restriction to Ω of an NL-differentiable function defined on the complex plane C. The arc of the unit circle intercepted in the positive sense between the identity ω^0 of Z_q and the arbitrary element ω^x ($x \in P_q$) is $s = 2\pi x/q$. A representative point setting out from ω^0 and travelling round the unit circle towards ω^x moves initially in the direction of the positive imaginary axis. The arc from ω^0 to ω^x may thus be "rectified" into the vector $t = is = 2\pi i x/q$. It is in terms of this argument t that we shall express X_k. Thus, for each $k \in P_q$,

$$X_k(\omega^x) = \omega^{kx} = \exp(2\pi ikx/q) = \exp kt \qquad (t \in 2\pi iq^{-1}P_q).$$

The function Ψ_k defined by $\Psi_k(t) = \exp kt \ (t \in K)$ is NL-differentiable and is, moreover, an extension of X_k. We equate the G-derivative DX_k of X_k at ω^x to the NL-derivative Ψ_k' of Ψ_k at t. Thus

$$(DX_k)(\omega^x) = \Psi_k'(t) = k\exp kt = kX_k(\omega^x).$$

The functions $\Psi_k \ (k \in Z)$ are in fact the characters of the quotient group $U =_{\mathrm{def}} R/(2\pi iZ)$ of equivalence classes of imaginary numbers modulo $2\pi i$. Concretely, U may be identified with the segment $[0, 2\pi)i$ of the non-negative imaginary axis, with addition modulo $2\pi i$. Apart from the real constant 1, U can be assimilated to a segment of the tangent at the point ω^0 to the unit circle of length equal to the circumference of the unit circle. This fact, among others, led the author to the present heuristic discussion, but it is sufficient for the purposes of this discussion to appreciate that the functions $\Psi_k \ (k \in P_q)$ extend the functions X_k to C and thus mediate their differentiation.

2.1 The G-derivative as a linear combination of difference quotients

If we extend the G-differentiator on Z_q from the X_k to the whole space C_q of bounded functions $Z_q \to C$, the concept of difference quotient associated with the NL-derivative will be found to re-emerge in an elegant form free, however, from any involvement with the limit concept. Such an extension may be readily effected by using the fact that an arbitrary bounded function may be expressed as a linear combination of the characters. This is a consequence of the fact that, with the inner product $(\cdot,\cdot)$ on C_q defined by

$$(f, g) = q^{-1}\sum_{x=0}^{q-1} f(\omega^x)g^*(\omega^x) \qquad (f, g \in C_q)$$

(the asterisk denotes complex conjugate), the set $\{X_k : k \in P_q\}$ forms an orthonormal basis for C_q:

$$(X_k, X_h) = 0 \quad (k \neq h), \qquad (X_k, X_h) = 1 \quad (k = h);$$

$$\text{if, for each } k \in P_q,\ (f, X_k) = 0, \text{ then } f = 0.$$

We can, then, expand an arbitrary $f \in C_q$ in terms of the X_k thus:

$$f(\omega^x) = \sum_{k=0}^{q-1}(f, X_k)X_k(\omega^x)$$

$$= q^{-1}\sum_{k=0}^{q-1}\sum_{\xi=0}^{q-1} f(\omega^\xi)X_k^*(\omega^\xi)X_k(\omega^x)$$

$$= q^{-1}\sum_{k=0}^{q-1}\sum_{\xi=0}^{q-1} f(\omega^\xi)\omega^{k(x-\xi)}.$$

On the assumption that the G-differentiator is linear, we have[3]

$$(Df)(\omega^x) = q^{-1} \sum_{k=0}^{q-1} k \sum_{\xi=0}^{q-1} f(\omega^{\xi}) \omega^{k(x-\xi)}$$

$$= q^{-1} \sum_{\xi=0}^{q-1} f(\omega^{x+\xi}) \sum_{k=0}^{q-1} k\omega^{-k\xi}.$$

In deriving the last equality, we used the group property $\omega^x \Omega = \Omega$, from which it follows that

$$\sum_{\xi=0}^{q-1} F(\omega^{\xi}) = \sum_{\xi=0}^{q-1} F(\omega^{x+\xi}) \qquad (F \in C_q).$$

The last expression of $(Df)(\omega^x)$ as a linear combination of the values of f at all points of Z_q takes a more intelligible form when the coefficients

$$q^{-1} \sum_{k=0}^{q-1} k\omega^{-k\xi} \qquad (\xi \in P_q)$$

are reduced using the easily-proved identities

$$q^{-1} \sum_{k=0}^{q-1} k\omega^{-k\xi} = \begin{cases} 2^{-1}(q-1) & (\xi = 0) \\ (\omega^{-\xi} - 1)^{-1} & (\xi \in P_q \backslash 0) \end{cases}$$

and

$$\sum_{\xi=1}^{q-1} (\omega^{-\xi} - 1)^{-1} = -2^{-1}(q-1).$$

Thus

$$(Df)(\omega^x) = q^{-1} f(\omega^x) \sum_{k=0}^{q-1} k + q^{-1} \sum_{\xi=1}^{q-1} f(\omega^{x+\xi}) \sum_{k=0}^{q-1} k\omega^{-k\xi}$$

$$= 2^{-1}(q-1) f(\omega^x) + \sum_{\xi=1}^{q-1} f(\omega^{x+\xi})(\omega^{-\xi} - 1)^{-1}$$

$$= -\sum_{\xi=1}^{q-1} (f(\omega^x)(\omega^{-\xi} - 1)^{-1} + \sum_{\xi=1}^{q-1} f(\omega^{x+\xi})(\omega^{-\xi} - 1)^{-1}$$

---

3 The first equality below yields an immediate proof that the mean value of Df is

$$q^{-1} \sum_{ksi}^{q} - 1 = 0(Df)(\omega^{ksi}) = 0.$$

The last expression below is consistent with Onneweer's (1977) definition (adopted also by Pál and Simon (1977a, 1977b)) of the Gibbs derivative for complex-valued functions on the direct product of countably many cyclic groups. The less-than-satisfactory definition given by Gibbs and Ireland (1974) failed this test of consistency, as Prof. Onneweer courteously hinted in the paper cited.

$$= \sum_{\xi=1}^{q-1} (f(\omega^{x+\xi}) - f(\omega^{x}))(\omega^{-\xi} - 1)^{-1}$$

$$= -\omega^{x} \sum_{\xi=1}^{q-1} \omega^{\xi}(f(\omega^{x+\xi}) - f(\omega^{x}))(\omega^{x+\xi} - \omega^{x})^{-1}.$$

Substituting X_k for f in this expression yields, as it should,

$$(DX_k)(\omega^x) = kX_k(\omega^x).$$

Apart from the factor[4] $-\omega^x\omega^\xi$, each term of our expression for Df agrees in form with the expression

$$f'(z) = \lim_{h \to 0} (f(z+h) - f(z))((z+h) - z)^{-1}$$

for the NL-derivative of a differentiable function $f: \mathrm{C} \to \mathrm{C}$, except of course for the absence of the limit and the notation for the argument values[5].

3 THE RELATIONSHIP BETWEEN GIBBS AND NEWTON-LEIBNIZ DERIVATIVES

The most striking feature of the G-differentiator is that it acts globally on its argument, in the sense that every point of Z_q except ω^x enters symmetrically into the expression for $(Df)(\omega^x)$, which is a linear combination, rather than a limit, of difference quotients. The NL-derivative $F'(z)$, on the other hand, is determined by the values of F on an arbitrarily small neighbourhood of z. In spite of this sharp distinction in behaviour, the concept of NL-derivative can be constructed, as we shall see, from that of the G-derivative.

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4 The factor $-\omega^x\omega^\xi$ has no immediately obvious intuitive interpretation, but this is perhaps only to be expected. The need for this factor is illustrated by the case in which f is the identity, defined by $f(\omega^x) = \omega^x (x \in P_q)$. In this case

$$(Df)(\omega^x) = \omega^x \sum_{\xi=1}^{q-1} (-\omega^\xi) = \omega^x.$$

We might expect, by over-strict analogy with NL-differentiation, that the derivative of the identity would be 1 everywhere. The expression $(f(\omega^{x+\xi} - f(\omega^x))(\omega^{x+\xi} - \omega^x)^{-1} = 1$, of course, but the coefficient $-\omega^\xi$ is required, to make the summation with respect to ξ yield 1 rather than $q - 1$. The remaining factor ω^x is needed to ensure that the mean value of Df vanishes:

$$q^{-1}\sum_{x=0}^{q-1} (Df)(\omega^x) = q^{-1}\sum_{x=0}^{q-1} \omega^x = 0.$$

Without the factor ω^x, the mean of Df would of course be 1.

5 The difference of notation arises from the fact that we took Z_q as a multiplicative group, while C is of course additive.

3.1 The NL-derivative of a periodic function as a limit of G-derivatives

To perform this construction we shall take a real periodic function F, of period 1, everywhere equal to its Fourier series

$$F(x)=\sum_{k=-\infty}^{\infty}\int_0^1 F(\xi)\exp 2\pi ik(x-\xi)d\xi$$

and everywhere differentiable term by term, so that

$$F'(x)=2\pi i\sum_{k=-\infty}^{\infty}k\int_0^1 F(\xi)\exp 2\pi ik(x-\xi)d\xi.$$

In view of the definition of the Riemann integral, this equality may be written

$$F'(x)=2\pi i\sum_{k=-\infty}^{\infty}k\lim_{q\to\infty}q^{-1}\sum_{\xi=0}^{q-1}F\left(\frac{\xi}{q}\right)\exp 2\pi ik\left(x-\frac{\xi}{q}\right).$$

We assimilate F to a function $f:T\to \mathrm{R}$ defined by

$$f(\exp 2\pi ix)=F(x)\qquad (x\in \mathrm{R}).$$

We define a sequence $(f_q)_{q=2}^{\infty}$ of functions $f_q:\mathrm{Z}_q\to\mathrm{R}$ as restrictions of f:

$$\omega_q=\exp\frac{2\pi i}{q},\qquad f_q(\omega_q^u)=f\left(\exp\frac{2\pi iu}{q}\right)=F\left(\frac{u}{q}\right)\qquad (q\in\{2,3,\ldots\},\ u\in P_q).$$

For each $x\in[0,1)$, there is a sequence $(u_q(x))_{q=2}^{\infty}$ of non-negative integers such that

$$\lim_{q\to\infty}\frac{u_q(x)}{q}=x.$$

In particular, we define $u_q(x)$ as $[qx]$, the greatest integer not exceeding qx. With these definitions,

$$\begin{aligned}F'(x)&=2\pi i\sum_{k=-\infty}^{\infty}k\lim_{q\to\infty}q^{-1}\sum_{\xi=0}^{q-1}f_q(\omega_q^{\xi})\omega_q^{k([qx]-\xi)}\\
&=2\pi i\lim_{q\to\infty}q^{-1}\sum_{k=-q+1}^{q-1}k\sum_{\xi=0}^{q-1}f_q(\omega_q^{\xi})\omega_q^{k([qx]-\xi)}\\
&=2\pi i\lim_{q\to\infty}q^{-1}\sum_{k=1}^{q-1}k\sum_{\xi=0}^{q-1}f_q(\omega_q^{\xi})(\omega_q^{k([qx]-\xi)}-\omega_q^{-k([qx]-\xi)})\\
&=-4\pi\Im\lim_{q\to\infty}q^{-1}\sum_{k=0}^{q-1}k\sum_{\xi=0}^{q-1}f_q(\omega_q^{\xi})\omega_q^{k([qx]-\xi)}\\
&=-4\pi\Im\lim_{q\to\infty}q^{-1}(Df_q)(\omega_q^{[qx]}),\end{aligned}$$

where $\Im$ denotes imaginary part.

3.2 The G-derivative on a cyclic group as an NL-derivative

The last formula[6] expresses the NL-derivative of F in terms of the limit of the G-derivatives of an associated sequence of functions f_q. The G-derivative of a function $f_q : Z_q \to C$ may likewise be expressed as the NL-derivative of an associated function $\phi_q : C \to C$. Indeed, in view of our heuristic approach to the G-differentiator on Z_q, this is trivial. We have only to define

$$U_q = \{2\pi i x/q : x \in P_q\} \subset U$$

and $\phi_q : U_q \to C$ by

$$\phi_q(2\pi i x/q) = f_q(\omega_q^x) = q^{-1} \sum_{k=0}^{q-1} \sum_{\xi=0}^{q-1} f_q(\omega_q^\xi)\omega_q^{k(x-\xi)}.$$

With $t =_{\text{def}} 2\pi i x/q$, we have

$$\phi_q(t) = q^{-1} \sum_{k=0}^{q-1} \sum_{\xi=0}^{q-1} f_q(\omega_q^\xi)\omega_q^{-k\xi} \exp kt,$$

which permits extension, in an obvious way, of ϕ_q to C. Then

$$(Df_q)(\omega_q^x) = q^{-1} \sum_{k=0}^{q-1} k \sum_{\xi=0}^{q-1} f_q(\omega_q^\xi)\omega_q^{-k\xi} \exp kt = \phi_q'(t).$$

4 LOCAL VERSUS GLOBAL DEFINITIONS OF DERIVATIVES

We have seen that the only natural definition of the G-differentiator on a cyclic group Z_q is of a *global* nature, in that the G-derivative of a function f on Z_q at a given point is expressed as a symmetric function of the values of f at all the other points of Z_q. To obtain a *local* definition of the G-derivative, as in Section 3.2, we have to do something quite artificial, namely, to construct a certain function $\phi : C \to C$ whose values coincide at the appropriate points of T with those of f, and then to NL-differentiate ϕ.

On the other hand, the natural definition of the NL-derivative of a function $F : R \to R$ (or $C \to C$) is undoubtedly the *local* definition as the limit of a difference quotient. A *global* definition can be given, but only in the case of functions that are not only differentiable but satisfy other conditions such as those imposed at the beginning of Section 3.1. As we have seen, where the global definition is available, it can be expressed in terms of the limit of a sequence of G-derivatives.

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[6] The reader may find it interesting to check, in the case of some familiar function F (periodic, of period 1) that this formula does indeed give the known expression for F'. For example, if $F(x) = \cos 2\pi x$, a routine calculation using the above formula yields $F'(x) = -2\pi \sin 2\pi x$.

In either case, of the G- or of the NL-derivative, it is Fourier analysis that enables the transition to be made between global and local definitions, or vice versa.

It may appear from the foregoing discussion that it is the simpler structure of the domain group Z_q that results in the natural definition being global, while the more sophisticated structures of the real and complex groups not only allow, through the limit concept, a local definition, but forbid, through a general failure of necessarily onerous conditions, the general use of a global definition.

As a counterexample to the above way of thinking, we shall define a G-differentiator on a structurally unsophisticated function space on which (consistent) global and local definitions may be given (without recourse to *conventional* Fourier theory), the *local* definition being the more natural.

4.1 Fourier analysis in the dyadic field

The space we have in mind is F, the space of functions $u: Z \to GF(2)$ such that for some integer m, for each $r < m$, $u(r) = 0$. The elements of F exhibit analogies with the complex numbers and are therefore called *dyadic numbers*. We call

$$M(u) =_{\text{def}} \sup\{m \in Z : \text{for each } r < m,\ u(r) = 0\}$$

the *modulus* of the dyadic number u. The function space F acquires the structure of a field if we define addition pointwise and multiplication convolutionwise:-

$$(u+v)(r) = u(r) + v(r),$$

$$(uv)(r) = \sum_{s \in Z} u(r-s)v(s) \qquad (u, v \in F, r \in Z).$$

Multiplication is well defined because (unless $u = 0$ or $v = 0$) both $M(u)$ and $M(v)$ exist and are finite, and so the summation is of a finite number of terms. (All additions and summations in $GF(2)$ are of course modulo 2.) The field F is commonly called the dyadic field.

Rather than specifying all the values of a function $u \in F$, as, for example,

$$u(4) = 1, \quad u(5) = 1, \quad u(r) = 0 \ \ (r < 4 \text{ or } r > 5),$$

we borrow the notation of real numbers written in the binary scale; thus

$$u = 0 \cdot 00011$$

defines the same function as the previous example. For brevity we also write

$$u = \overset{4}{1}1,$$

which may be pronounced "four crown one one".

The analogue in F of the Euler function $\exp 2\pi i\cdot$ in conventional (Fourier) analysis is the function

$$w = 0\cdot 1111\ldots \;= 0\cdot\dot{1} = \underset{1}{\dot{1}},$$

pronounced "dot one foot one". The integer powers of w are easily written down, because an analogue of the recursion formula

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

for binomial coefficients holds among the values of $w^n(r)$, namely

$$w^{n-1}(r) = w^n(r) + w^n(r+1).$$

Thus a partial table of integer powers of w looks like this:-

n	w^n
-3	1111
-2	101
-1	11
0	1
1	$0\cdot\dot{1}$
2	$0\cdot\dot{0}\dot{1}$
3	$0\cdot\dot{0}01\dot{1}$
4	$0\cdot\dot{0}00\dot{1}$

Inspection of this table suggests the (easily proved) the transposition formula

$$w^n(r) = w^{1-r}(1-n),$$

analogous to

$$\exp 2\pi inx = \exp 2\pi i(-x)(-n).$$

The set

$$W = \{w^n : n \in Z\}$$

of integer powers of w forms a basis for the space F, that is, each element of F may be expressed uniquely as a linear combination of elements of W:

$$u = \sum_{s\in Z} U(s)w^s \qquad (u \in F).$$

We define the *dyadic conjugate* $\overline{w^n}$ of each element w^n of W by

$$\overline{w^n}(r) = w^r(n) \qquad (n, r \in Z)$$

and hence, by linearity, the dyadic conjugate of each element of F :

$$\bar{u}(r) = \sum_{s \in Z} U(s)\overline{w^s}(r) = \sum_{s \in Z} U(s) w^r(s),$$

where the coefficients $U(s)$ are, of course, determined by

$$u = \sum_{s \in Z} U(s) w^s.$$

The operation of dyadic conjugation defined on the dyadic field is analogous to that of complex conjugation on the complex field, though it has a less easily visualised (if any) geometrical interpretation. By analogy with the inner product

$$(f, g) = \int_{x \in R} f(x) g^*(x)$$

of two functions $f, g \in L^2$, we define the (pseudo) *inner product* in F by

$$(u, v) = \sum_{s \in Z} u(s)\bar{v}(s).$$

The *Fourier transform* $\hat{u}$ of $u \in F$ is defined, by analogy with

$$\hat{f}(k) = (f, \exp 2\pi i k x) = \int_{x \in R} f(x) \exp(-2\pi i k x),$$

by

$$\hat{u}(r) = (u, w^r) = \sum_{s \in Z} u(s)\overline{w^r}(s) = \sum_{s \in Z} u(s) w^s(r).$$

The Fourier transform operator is self-inverse,

$$\hat{\hat{u}} = u,$$

from which it follows that

$$u = \sum_{s \in Z} \hat{u}(s) w^s.$$

We can therefore identify the coefficients $U(s)$ above with the coordinates of $\hat{u}$, and define the dyadic conjugation operator $\bar{\cdot}$ concisely by

$$\bar{u}(r) = \sum_{s \in Z} \hat{u}(s) w^r(s) = \sum_{s \in Z} w^r(s) \sum_{t \in Z} u(t) w^t(s).$$

The last expression may be compared with that of the analogue of Fourier's integral theorem,

$$u(r) = \sum_{s \in Z} w^s(r) \sum_{t \in Z} u(t) w^t(s).$$

4.2 A G-differentiator on the dyadic field

This expression may be used to give a global definition of a G-differentiator $\Delta : F \to F$. By analogy with $dx^n/dx = nx^{n-1}$, let us define Δ by

$$\Delta w^s = w^{s-1} \qquad (s \in Z)$$

and extend this by linearity to F, thus:

$$\begin{aligned}
(\Delta u)(r) &= \sum_{s\in Z} (\Delta w^s)(r) \sum_{t \in Z} u(t) w^t(s) \\
&= \sum_{s\in Z} w^{s-1}(r) \sum_{t\in Z} u(t) w^t(s) \\
&= \Big(w^{-1} \sum_{s\in Z} w^s \sum_{t\in Z} u(t) w^t(s) \Big)(r) \\
&= (w^{-1} u)(r) \\
&= \sum_{s=r}^{r+1} w^{-1}(r-s) u(s) \\
&= u(r+1) + u(r).
\end{aligned}$$

Bearing in mind that addition is the same as subtraction modulo 2 we see that the G-differentiator Δ is simply the first forward finite-difference operator on the sequence $(u_r)_{r=M(u)}^{\infty}$. So Gibbs differentiation subsumes the calculus of finite differences, at least in a limited sense! The local definition

$$(\Delta u)(r) = u(r+1) + u(r)$$

of Δ is evidently much simpler and more natural than the global definition

$$(\Delta u)(r) = \sum_{s\in Z} w^{s-1}(r) \sum_{t\in Z} u(t) w^t(s).$$

On the face of it, evaluating the latter expression entails a knowledge of the function u on the whole of Z, but in reality the properties of the powers of w ensure that only $u(r)$ and $u(r+1)$ are needed.

In Section 5 we shall meet G-differentiation on a space where no non-trivial Fourier theory appears to exist. Nevertheless a local definition of the G-derivative may be given and two kinds of Taylor series exist, one with a countable infinity of terms, the other with an uncountable infinity thereof.

4.3 Summability of divergent series of terms in $GF(2)$

It may have occurred to the reader that the inner product (u, v) may fail of definition for some ordered pair $(u, v) \in \mathrm{F}$ because neither u nor $\bar{v}$ terminates. For example, if $u = w = 0 \cdot \dot{1}$ and $v = 0 \cdot 1$, so that $\bar{v} = w$, then

$$(u, v) = \sum_{s \in \mathrm{Z}} u(s)\bar{v}(s) = \sum_{s=1}^{\infty} w(s),$$

and the summation is of a countably infinite set of 1s (modulo 2). It is in fact a convention of Fourier analysis in the dyadic field necessary for consistency that a divergent series

$$\sum_{r=1}^{\infty} u(r)$$

is summable if $\hat{u}$ terminates, and in this case

$$\sum_{r=1}^{\infty} u(r) = \sum_{s=1}^{\infty} \hat{u}(s).$$

The condition that $\hat{u}$ terminates is equivalent to the condition that, for some non-negative q, u is periodic, of period 2^q.

Further details of Fourier analysis in the dyadic field have been recorded by Gibbs (1984).

5 G-DIFFERENTIATION OF DYADIC FUNCTIONS

We define a *dyadic function*, on the analogy of a real function $\mathrm{R} \to \mathrm{R}$, as a function $f : \mathrm{F} \to \mathrm{F}$. To define differentiation on the space of such functions has long been a desideratum. The conventional idea of G-differentiation, restricted to functions $G \to \mathrm{C}$, where G is some suitable group, is open to criticism on the ground that it is only a half-generalisation tion of differentiation $\mathrm{C} \to \mathrm{C}$ —a generalisation only with respect to the domain of the functions. The critic who takes this position would prefer to see a generalisation affecting both domain and codomain in the same way, so that differentiation $\mathrm{R} \to \mathrm{R}$ and $\mathrm{C} \to \mathrm{C}$ is extended to differentiation of functions $\mathrm{K} \to \mathrm{K}$, where K is an arbitrary member of some not-too-restricted class of fields. In the first instance, he might be happy to see differentiation extended to functions $\mathrm{F} \to \mathrm{F}$.

The algebraic groundwork for such an extension already exists in the copious literature[7] of so-called boolean differential calculus, which applies to functions $\{0,1\}^n \to \{0,1\}$, where n is a positive integer. To convert this rudimentary calculus into a discipline in which non-trivial analytical considerations play a part, we have to allow n to become countably infinite, that is, to consider not finite-dimensional vectors of 0s and 1s but infinite sequences thereof. The domain thus becomes the set underlying F. We also have to recognise that a function $\{0,1\}^n \to \{0,1\}^p$ is essentially equivalent to a p-vector of functions $\{0,1\}^n \to \{0,1\}$. By this observation we are led to regard a function $\mathrm{F} \to \mathrm{F}$ as a sequence of functions $\mathrm{F} \to \Phi =_{\mathrm{def}} GF(2)$. Such is the programme that is partly carried out in a note by Gibbs (1979).

5.1 The proper differentiator

The initial idea in this development is that of the *proper differentiator* δ, defined on the space Λ of functions $\Phi \to \Phi$. We define $\delta : \Lambda \to \Lambda$ by

$$(\delta\phi)(\xi) = \phi(\xi) + \phi(\xi + 1) = \phi(0) + \phi(1) \qquad (\phi \in \Lambda, \xi \in \Phi).$$

where 0 and 1 denote the zero and unity of Φ, respectively. Notice that the proper derivative of an arbitrary function $\phi \in \Lambda$ is a constant function, equal to $1 \in \Lambda$ if the values of ϕ at 0 and 1 are distinct, and equal to $0 \in \Lambda$ if these values are the same. It follows that $\delta^2\phi = 0 \in \Lambda$.

The proper differentiator δ is thus a linear operator on Λ such that the derivative $\delta\iota$ of the identity function $\iota \in \Lambda$ is $1 \in \Lambda$ and the derivative $\delta\gamma$ of each constant function $\gamma \in \Lambda$ is $0 \in \Lambda$. These properties of δ are analogues of familiar properties of the real differentiator and of the dyadic differentiator[8]. Unlike[9] the dyadic differentiator, δ satisfies a simulacrum of the product rule

$$D(fg) = f\,Dg + g\,Df$$

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[7] See, for example, the lecture notes of Thayse (1981).

[8] The property $\delta_\iota = 1$ holds for the dyadic differentiator only in the vacuous sense that there is no identity function from the dyadic group to C.

[9] The (extended) dyadic differentiator obeys a product rule, in general, only if f and g are Walsh functions (Butzer, Engels, and Wipperfürth, 1986).

for real differentiation, namely[10],

$$\delta(\phi\psi) = \phi\delta\psi + \psi\delta\phi + \delta\phi\delta\psi .$$

The proper differentiator also satisfies

$$\sum_{\xi\in\Phi}(\delta\phi)(\xi) = 0$$

analogously to the corresponding property $\sum_{x=0}^{q-1}(Df)(\omega^x)$ of the G- differentiator for the functions $Z_q \to C$. The operator δ possesses the eigenfunction property only in a trivial sense. The only character of $GF(2)$ is the principal character $1 \in \Lambda$. The corresponding eigenvalue is $0 \in \Phi$ and

$$\delta 1 = 0 1 .$$

5.2 Partial proper differentiators

The dyadic field has been introduced in Section 4.1 above. Here we shall identify a dyadic number u with a sequence (u_r), where $u_r =_{def} u(r)$. With this convention the definitions of addition and multiplication in F are

$$(u+v)_r = u_r + v_r ,$$

$$(uv)_r = \sum u_{r-s} v_s .$$

Each $u \in$ F generates a subspace $F(u)$ of the linear space F defined by

$$F(u) = \{v \in F : (r \in Z) \text{ if } u_r = 0 \text{, then } v_r = 0\} .$$

We say that $u \in$ F is *terminated* iff, for some (unique) $N(u) \in Z$, $x_{N(u)-1} = 1$ and, for each $r \geq N(u)$, $u_r = 0$. Each terminated $u \in$ F generates a *finite* subspace $F(u)$ having at most $2^{N(u)-M(u)}$ elements.

The set $W = \{w^n : n \in Z\}$ was introduced in Section 4.1 as a basis for F; another, less sophisticated, basis is the set $\{d_s\}_{s\in Z}$ defined by

$$d_s(t) = 1 \qquad (s = t),$$

$$d_s(t) = 0 \qquad (s \neq t).$$

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[10] The generalisation of the product rule to the product of n functions $\phi_1, \phi_2, \dots, \phi_n$ may be written

$$(1+\delta)\prod_{i=1}^{n}\phi_i = \sum_{a_1\in\Phi}\sum_{a_2\in\Phi}\dots\sum_{a_n\in\Phi}\prod_{s=1}^{n}\delta^{a_s}\phi_s ,$$

where, of course, $\delta^0 =_{def} 1$ and $\delta^1 =_{def} \delta$.

Evidently $d_0 = 1 \in F$. We define $(r \in Z)$ the operator τ_r by

$$(\tau_r u)_s = u_{r+s} \qquad (u \in F,\ s \in Z).$$

The set of operators $\{\tau_r\}_{r \in Z}$ is connected with the basis $\{d_s\}_{s \in Z}$ by

$$\tau_r d_s = d_{s-r};$$

in particular,

$$\tau_r d_r = d_0.$$

Let L denote the space of functions $F \to F$. We define $(r \in Z)$ an operator $\partial_r : L \to L$ by

$$(\partial_r f)(u) = f(u) + f(u + d_r) \qquad (f \in L,\ u \in F).$$

The operator ∂_r is evidently a generalisation of the proper differentiator δ: we therefore call ∂_r the r-th partial proper differentiator.

5.3 Taylor series for dyadic functions

As we remarked before, a dyadic function (a function $F \to F$) may be regarded as a sequence of functions $F \to \Phi$. More precisely, if f is a dyadic function, we define, for each $r \in Z$, a function $f_r : F \to \Phi$ by

$$f_r(u) = (f(u))_r \qquad (u \in F).$$

The value of f_r at u may be expressed in terms of the value of f_r and its partial proper derivatives at an arbitrary point a of $F\ (a \neq u)$ by means of a generalised Taylor series. The corresponding finite expansion was given by Akers (1959). The full Taylor series may be written

$$f_r(u) = \left(\left(\sum_{v \in F(a+u)} \prod_{s \in Z} \partial_s^{(a+u)_s v_s}\right) f_r\right)(a).$$

Since such an equality holds for each $r \in F$, it follows at once that

$$f(u) = \left(\left(\sum_{v \in F(a+u)} \prod_{s \in Z} \partial_s^{(a+u)_s v_s}\right) f\right)(a).$$

If $a + u$ is terminated, then the subspace $F(a+u)$ is finite, and the series comprises at most $2^{N(a+u)-M(a+u)}$ terms[11]. If $a+u$ is not terminated, then the "series" comprises $2^{\aleph_0}$ terms, and its summation presents an interesting problem.

There is another series expression for $f(u)$ in terms of $f(a)$ and values of partial proper derivatives of f. It was given by Thayse (1971) as a finite expression for boolean functions. It has the advantage of having only a countable infinity of terms, and is therefore a series in the conventional sense. On the other hand, it can hardly be called a *Taylor* series (though we shall extend it that courtesy), for the partial proper derivatives that it uses are not all evaluated at the same point of F. The series may be written

$$f(u)=f(a)+\sum_{s=M(a+u)}^{\infty}(a+u)_s(\partial_s f)\left(a+\sum_{t=M(a+u)}^{s-1}(a+u)_t d_t\right).$$

As in the case of the Akers-Taylor series, the Thayse-Taylor series reduces to a finite expansion in the case that $a+u$ is terminated. It then comprises at most $1+N(a+u)-M(a+u)$ terms[12].

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11 A numerical example may help to clarify this unusual kind of Taylor expansion. Let $a=0\cdot 101\dot{1}$ and $u=0\cdot 110\dot{1}$. Suppose that a partial table of values of f is as follows:

	x	$f(x)$	
	$0\cdot 100\dot{1}$	$1\cdot 111$	
a	$0\cdot 101\dot{1}$	$1\cdot 101$	$f(a)$
u	$0\cdot 110\dot{1}$	$1\cdot 011$	$f(u)$
	$0\cdot 111\dot{1}$	$0\cdot 111$	

Then $a+u=0\cdot 011$, $F(a+u)=\{0\cdot 000, 0\cdot 001, 0\cdot 010, 0\cdot 011\}$ and the Taylor expansion for $f(u)$ in terms of $f(a)$ and the partial proper dervatives of f at a is

$$f(u)=\left(\left(\sum_{v\in F(a+u)}\prod_{s=2}^{3}\partial_s^{v_s}\right)f\right)(a)$$

$$=((1+\partial_2+\partial_3+\partial_2\partial_3)f)(a)$$

$$=1\cdot 101+1\cdot 010+0\cdot 010+1\cdot 110,$$

which sums, as it should, to $1\cdot 011$.

12 As a numerical example, let us take the same a, u, and f as in the previous footnote. Then

$$f(u)=f(a)+\sum_{s=2}^{3}(\partial_s f)\left(a+\sum_{t=2}^{s-1}(a+u)_t d_t\right)$$

$$=f(a)+(\partial_2 f)(a)+(\partial_3 f)(a+0\cdot 01)$$

$$=1\cdot 101+1\cdot 010+1\cdot 100,$$

which sums, likewise, to $1\cdot 011$.

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BACKGROUND TO AN EXTENSION OF GIBBS DIFFERENTIATION IN WALSH ANALYSIS

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Abstract

The aim of this paper is to deal with an extension of the standard dyadic derivative which is not restricted only to piecewise constant functions but covers piecewise polynomial functions, many very unsmooth functions, but also several classically smooth functions. This extension results by first equipping the sum defining the standard dyadic derivative with the multiplicative factor ± 1 and then applying Euler's summation process to it. The extension, which is therefore rather basic, is of such a nature that the characteristic properties that a "derivative" usually has are still preserved as far as possible. This new approach to dyadic differentiation was introduced by the authors together with Udo Wipperfürth in 1986. In contrast to our results there, the present approach is not given in the setting of (finite) Walsh-Fourier transforms but even in the original function space. However, it is restricted to integer order derivatives; it does not cover the fractional case.

The main theorems, dealing with integer order derivatives and anti-derivatives, together with their proofs, are new. This extension is therefore independent of the one of 1986.

1. Introduction

This paper does not attempt to survey recent results on dyadic differentiation but to discuss in detail the background to an extension of the dyadic derivative, one which is applicable not only to piecewise constant functions but also to piecewise polynomial functions, for example. The extension is of such a nature that the characteristic properties that a derivative usually possesses, in particular the standard dyadic derivative, are preserved as far as possible.

There exist various forms of dyadic differentiation, one which interacts with the Walsh functions $\psi_k(x)$ similarly as does the classical derivative with the exponential functions e^{ikx} : $D^{[1]}\psi_k = k\ \psi_k$ for $k = 0,1,\cdots$, $D^{[1]}$ being the dyadic derivative. One form is due to the initiator of this field of research, John E. Gibbs [16,17,18,19,20], other forms to Pichler [33,34], Wagner and Butzer [8,9,10,11], Pàl [30,31], Onneweer [27,28,29], He Zelin [23] and others ([36], [37], [39], [43] etc).

Here the functions in question are defined on the interval $[0,1]$ (or $[0,\infty)$), on the dyadic group or dyadic field. Since the dyadic derivative depends on the ordering of the Walsh functions, Onneweer [28] managed to come up with a derivative whose eigenvalues are independent of the particular enumeration of the $\psi_k(x)$ as well as to unify some of the various approaches to dyadic differentiation.

The purpose here is not to generalize the dyadic derivative in such directions but to try to modify its definition rather basically: first equip the sum defining the derivative with the multiplcative factor ± 1 and then apply Euler's summation process to it. The range of impact of the resulting derivative is not restricted to piecewise constants but also covers several classically smooth functions. However, this new derivative is especially applicable to rather unsmooth functions; for example, also to $x^n d(x)$, $d(x)$ being Dirichlet's function.

Let us add that this new approach to dyadic differentiation was first considered in our papers [4, 5] where, however, the derivative is not defined in the original function space, as below, but in the realm of Walsh transforms. So the present approach is more natural, and one which makes good use of Euler summability. In this respect our main theorems (namely Theorems 1 to 4), the proofs of which are somewhat intricate, are new. They actually complete results begun in Engels [12]. Further, the present matter is not oriented towards fractional order derivatives, the subject of [4, 5], but to derivatives and anti-derivatives of integer order $r \in \mathbb{N}$.

Section 2 deals with the basic concepts of Walsh analysis. Section 3 is concerned with the standard definition of the dyadic derivative and of anti-derivatives, the reason for their introduction, their role and advantages. Whereas Section 4 is devoted to the new, extended dyadic derivative and its basic properties, Section 5 deals with the deeper ones, including the counterpart of the fundamental theorem of the Newton-Leibniz calculus in the frame of Walsh analysis. Section 6 treats the extended derivative in the fractional order case, its properties, as well as compares the standard with the extended dyadic derivative. Next come several examples of functions which are ED-differentiable, listed in Section 7. The final section is then devoted to applications, namely to Fourier analysis and best approximation in the Walsh setting.

2. Preliminaries

Let $\mathbb{N} = \{1,2,3,\cdots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z} := \{0,\pm1,\pm2,\cdots\}$. Each $k \in \mathbb{N}_0$ has a unique dyadic expansion $k = \Sigma_{j=0}^{K} k_j 2^j$ with $k_j \in \{0,1\}$ and $2^K \le k < 2^{K+1}$, $K \in \mathbb{N}_0$. Likewise each $x \in [0,1)$ has a unique expansion $x = \Sigma_{j=0}^{\infty} x_j 2^{-(j+1)}$ with $x_j \in \{0,1\}$, the finite expansion being chosen in case x is a dyadic rational. The dyadic sum of $x \in [0,1)$ and $y \in [0,1)$ is defined by $x \oplus y = \Sigma_{j=0}^{\infty} h_j 2^{-(j+1)}$, where $h_j := (x_j+y_j)_{\text{mod } 2}$, if there does not exist a number j_0 such that $x_j \neq y_j$ for all $j \ge j_0$. Consequently dyadic addition $x \oplus y$ is defined for countably many $y \in [0,1)$ in case $x \in [0,1)$ is fixed. The Walsh functions $\psi_k(x)$ are defined (in Paley's enumeration) by $\psi_k(x) = (-1)^{\Sigma_{j=0}^{\infty} x_j k_j}$ for $x \in [0,1)$ and $k \in \mathbb{N}_0$. They possess the properties

(2.1) $\psi_k(x \oplus y) = \psi_k(x)\psi_k(y)$ $\qquad$ ($x \in [0,1)$ fixed, almost all $y \in [0,1)$),

(2.2) $$2k_j = 1 - \psi_k(2^{-j-1}) \qquad (k,j \in \mathbb{N}_0),$$

(2.3) $$\int_0^1 \psi_k(x)\ \psi_n(x)\ dx = \delta_{k,n} \quad (\text{= Kronecker delta}).$$

$X = X[0,1)$ will stand for the spaces $L^p(0,1)$, $1 \le p \le \infty$, for which the norms $\|f\|_p := \{\int_0^1 |f(x)|^p dx\}^{1/p}$ in case $1 \le p < \infty$, or $=$ ess $\sup_{x\in[0,1)} |f(x)|$ in case $p = \infty$, are finite. Here $f = g$ in the space X will mean $\|f-g\|_X = 0$.

The Walsh-Fourier series of $f \in X[0,1)$ and the respective coefficients $f^\wedge(k)$ are given by

(2.4) $$f(x) \sim \sum_{k=0}^{\infty} f^\wedge(k)\psi_k(x), \quad f^\wedge(k) := \int_0^1 f(u)\ \psi_k(u)\ du.$$

For $h \in [0,1)$ there holds

(2.5) $$|f^\wedge(k)| \le \|f\|_X, \quad [f(\cdot+h)]^\wedge(k) = \psi_k(h)f^\wedge(k) \qquad (k \in \mathbb{N}_0).$$

The dyadic convolution of $f \in X[0,1)$ and $g \in L^1(0,1)$ as well as the associated dyadic convolution theorem read

(2.6) $$(f * g)(x) := \int_0^1 f(x\oplus u)g(u)\ du \qquad (x \in [0,1))$$

$$[f * g]^\wedge(k) = f^\wedge(k)\ g^\wedge(k) \qquad (k \in \mathbb{N}_0).$$

The uniqueness theorem states for $f \in X[0,1)$,

(2.7) $$f^\wedge(k) = 0 \quad (k \in \mathbb{N}_0) \quad \text{iff} \quad f = 0.$$

For the 2^n-th partial sum $S_{2^n} f(x) := \Sigma_{k=0}^{2^n-1} f^\wedge(k)\ \psi_k(x)$ there holds

(2.8) $$\lim_{n\to\infty} \|S_{2^n}f - f\|_X = 0,$$

and for the associated Walsh-Fourier kernel $D_n(x) := \sum_{k=0}^{n-1} \psi_k(x)$,

(2.9) $$\|D_n\|_1 = O(\log n),\ \|D_{2^n}\|_1 = 1 \qquad (n \to \infty).$$

The Walsh-Fejér kernel, $F_n(x) := n^{-1} \sum_{k=0}^{n} D_k(x)$, has the property

(2.10) $$\|F_n\|_1 \le 2 \qquad (n \in \mathbb{N}).$$

This material is standard; see e.g. [41],[40].

3. Towards the Standard Concept of Dyadic Differentiation

Almost all classical orthogonal systems can be represented as solutions of differential equations. Obviously this cannot be true for the Walsh functions, since step-functions are not differentiable in the Newton-Leibniz sense (jumps !). From this point of view the question arises whether it is possible to define a concept of differentiation adaptable to Walsh functions. In classical Fourier analysis (e.g. [6]), with

(3.1) $$f(x) \sim \sum_{k=-\infty}^{\infty} f_F^{\wedge}(k)e^{ikx},\quad f_F^{\wedge}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)e^{-iku}\,du \qquad (k \in \mathbb{Z}),$$

the r-th derivative can be formally obtained by

(3.2) $$f^{(r)}(x) \sim \sum_{k=-\infty}^{\infty} (ik)^r f_F^{\wedge}(k)e^{ikx},$$

for which $D^r e^{ikx} \equiv (\frac{d}{dx})^r e^{ikx} = (ik)^r e^{ikx}$, $k \in \mathbb{Z}$, the factor ik being the eigenvalue of the classical operator D^1 of differentiation.

In 1972 Butzer and Wagner [8] introduced the dyadic differential operator $D^{[r]}$ which satisfied

(3.3)
$$D^{[r]}f(x) \sim \sum_{k=0}^{\infty} (k)^r f^{\wedge}(k)\, \psi_k(x)$$

for which $D^{[r]}\, \psi_k(x) = k^r\, \psi_k(x)$, $k \in \mathbb{N}_0$; it is the analogue of (3.2).

To see how to obtain $D^{[1]}$, let $k = \Sigma_{j=0}^{K} k_j\, 2^j$, taking k_j in the form (2.2). Then $k = \Sigma_{j=0}^{K}\{1-\psi_k(2^{-j-1})\}2^{j-1}$, and so, noting (2.1),

(3.4)
$$k\, \psi_k(x) = \sum_{j=0}^{\infty} 2^{j-1}\{\psi_k(x)-\psi_k(x\oplus 2^{-j-1})\} = \frac{1}{4}\sum_{j=0}^{\infty} \frac{\psi_k(x)-\psi_k(x\oplus 2^{-j-1})}{2^{-j-1}}.$$

This suggests defining $D^{[1]}$ in the original function space as follows: $f \in X$ is said to be *strongly dyadically differentiable in* X, if the sequence of functions

(3.5)
$$d_n f(x) := \sum_{j=0}^{n-1} 2^{j-1}\{f(x)-f(x\oplus 2^{-j-1})\}$$

converges in the norm of X; in this case the limit g of (3.5) is called the *strong dyadic derivative* of f and denoted by $D^{[1]}f = g$. Higher order derivatives of $f \in X$ are defined by $D^{[r]}f = D^{[1]}\,(D^{[r-1]}f)$, $r = 2,3,\cdots$. Pointwise dyadic derivatives, to be denoted by $f^{[r]}(x)$, are defined accordingly.

It turned out that $D^{[r]}$ is a linear, closed operator on X; similarly as in classical Newton analysis (e.g. [6]) there holds for $f \in X$:

(3.6) $D^{[r]}f = g$ exists with $g \in X \Longleftrightarrow$
there exists $g \in X : g^{\wedge}(k) = k^r\, f^{\wedge}(k)$, $k \in \mathbb{N}_0$.

Most results obtained in the past 20 years in Walsh-Fourier analysis have confirmed that this is probably the best concept of a dyadic derivative to use. Nevertheless, three main questions were posed and are still not all solved. *Firstly,* is it possible to give an interpretation of the dyadic derivative as for the classical derivative, which may be associated with the slope of a tangent to a curve, or with the speed of an object ? Thus, can one associate the dyadic derivative with basic geometric or physical notions? Although several attempts were made to solve this problem, there is still no intuitive interpretation of the dyadic derivative, which may perhaps lie in the setting of information theory and related fields, namely in fields which make use of dyadic Walsh analysis (see also below). Anyhow, there is a first, but rather abstract interpretation given by Gibbs and Ireland [20] in the realm of locally compact abelian groups. The *second* question was a possible comparison between dyadic differentiation and classical differentiation in terms of an analytical relationship; an answer to this very important question is also still lacking. This is perhaps due to the fact that dyadic differentiation and classical differentiation are of rather different nature. It is well known that a function being differentiable at some point in the classical sense needs only to be defined in a arbitrary small interval about that point; dyadic differentiation in contrast, - on account of dyadic addition in (3.5) - also takes into consideration points having a distance up to $x \oplus 2^{-1}$ from x. Moreover, classical differentiation is defined via the limit of *one* differential quotient; dyadic differentiation sums up infinitely many differential quotients of a particular kind, where addition is now dyadic addition. Nevertheless, a function whose derivative in either sense is equal to zero is always a constant function.

A *third* question raised, but solved, was the problem of describing presisely the class of functions which are dyadically differentiable. It soon became apparent that dyadic differentiability does not really apply to classically smooth functions. Thus, Skvorcov and Wade [38], who improved earlier results due to Bockarev [3], Butzer and Wagner [10] as well as Schipp [37], showed that if f is *continuous* on [0,1) in the classical sense and dyadically differentiable at all but countably many points $x \in [0,1)$, then f is a constant. Then in 1985 Engels [13] fully characterized the class of functions which are dyadically differentiable. In fact, a bounded function f on [0,1), which possesses a finite or a countably infinite number of discontinuities exclusively of first kind (namely jumps only) having at most a finite number of cluster points in [0,1), is dyadically differentiable on [0,1), if and only if f is a piecewise constant. Although this seems to be a rather restrictive condition, the standard dyadic derivative is especially adapted to functions that have only a few or short intervals of constancy. It is even applicable to functions which seem to be rather "exotic", like

$$(3.7)\qquad f(x) := 1 + \sum_{k=1}^{\infty} \frac{\psi_k(x)}{k^n} \qquad (x \in [0,1);\ n \geq 2).$$

Although this function has discontinuities at each dyadic rational, it is still dyadically differentiable, with $D^{[1]}f(x) = \Sigma_{k=1}^{\infty} k^{-n-1}\ \psi_k(x)$, $x \in [0,1)$. For a graph of this function for n = 2 see [14]. Anyhow, one must admit that piecewise constant functions play an essential role in digital signal processing, communication theory, digital filter design, binary digital circuits, multiplex systems (digital multiplexing) and coding with binary elements. This is due to fact that Walsh functions can only take on the values ±1, which play an important role in these areas. Also in this respect, in elementary particle physics one has even and odd parity, particles and antiparticles, positive and negative charges; recall the famous Pauli principle of quantum mechanics here. It is a well-known fact that the Walsh system can be applied in many fields which are connected with binary and digital processes [22,1,2]. So if one regards dyadic differentiation as a counterpart of the Newton-Leibniz derivative especially for working in the binary frame (note that 'binary' is closely related with 'dyadic'), it is not at all surprising that dyadic differentiation applies only to piecewise constants.

However, from the point of mathematics per se and signal analysis (see e.g. [7]), it is desirable to differentiate functions which are not only piecewise constants but piecewise polynomials, say. For this purpose the definition of the dyadic derivative would have to be modified. So the problem now is to enlargen the class of differentiable functions, without loosing the basic properties the standard dyadic derivative possesses.

A first possible attempt in this directions would consist in supplying the series in (3.5) with the multiplicative factor $(-1)^j$, thus to try to define the extended derivative, to be denoted by $D^{[[1]]}f$, in terms of the limit in the norm of X of the sequence

$$(3.8)\qquad \sum_{j=0}^{n} (-1)^j\ 2^{j-1}\{f(x) - f(x\oplus 2^{-j-1})\}.$$

The factor $(-1)^j$ seems to be especially compatible with the nature of the Walsh functions, which lie at the base of dyadic analysis, since these functions have the same jumping character; they only take on the values ±1.

Further, the built-in factor $(-1)^j$ seems to be only a very slight modification of (3.5), not disturbing the properties of dyadic differentiation.

In this case, defining the r-th derivative $D^{[[r]]}f$ in an obvious manner, its existence for $f \in X$ implies

$$(3.9)\qquad [D^{[[r]]}f]^{\wedge}(k) = (k^*)^n f^{\wedge}(k) \qquad (k \in \mathbb{N}_0),$$

where $k^* = k^*(k) \in \mathbb{Z}$ is now defined by

$$(3.10)\qquad k^* = \sum_{j=0}^{\infty} (-1)^j k_j 2^j,$$

where k_j are the binary coefficients of $k \in \mathbb{N}_0$.

In fact, taking the case r = 1 for simplicity, and noting (2.5) and (2.2),

$$\left|\left[\sum_{j=o}^{n} (-1)^j 2^{j-1}\{f(\cdot)-f(\cdot\oplus 2^{-j-1})\} - D^{[[1]]}f\right]^{\wedge}(k)\right|$$

$$= \left|\sum_{j=o}^{n} (-1)^j 2^{j-1}(2kj)f^{\wedge}(k) - [D^{[[1]]}f]^{\wedge}(k)\right|$$

$$\le \left\|\sum_{j=o}^{n} (-1)^j 2^{j-1}\{f(\cdot)-f(\cdot\oplus 2^{-j-1})\} - D^{[[1]]}f(\cdot)\right\|_X.$$

As the norm tends to zero for $n \to \infty$, there follows (3.9) if one defines k^* by (3.10).

Concerning the sequence of the k^*, as k increases they oscillate from positive to negative values, always satisfying

$$k^* = k^*(L) = \begin{cases} k, & L \text{ even} \\ -k, & L \text{ odd} \end{cases}, \quad \text{if } k = 2^L,\ L \in \mathbb{N}_0,$$

$$k^* = k^*(L) = \begin{cases} \text{positive}, & L \text{ odd} \\ \text{negative}, & L \text{ even} \end{cases}, \text{ if } 2^{L-1} < k < 2^L,\ L \in \mathbb{N},$$

and

$$\text{(3.11)} \qquad k/5 \leq |k^*| \leq k \qquad (k \in \mathbb{N}).$$

For more details and a table of the k^* for $0 \leq k \leq 71$ see [4]. For example, $0^* = 0$, $1^* = 1$, $2^* = -2$, $3^* = -1$, $4^* = 4$, $5^* = 5$, $6^* = 2$, $7^* = 3$, $8^* = -8$, $9^* = -7$.

However, not even piecewise linear functions are dyadically differentiable in the sense of (3.8). Taking, for example, the function

$$f(x) := x\,\psi_1(x) = \begin{cases} x, & x \in [0,1/2) \\ -x, & x \in [1/2,1), \end{cases}$$

then (3.8) does not converge in X-norm. Thus the possible extension (3.8) is not applicable to a wider class of functions. Furthermore, the converse of (3.9), as in the case of the operator $D^{[1]}f$ (recall (3.6)), is not necessarily true. Thus the assertion that $(k^*)^r f\hat{}(k) = g\hat{}(k)$, $k \in \mathbb{N}_0$, for some $g, f \in X[0,1)$, does not generally lead to the existence of $D^{[[r]]}f = g$.

Note that one could, at least formally, rewrite definition (3.8) in the setting of (3.3), thus define $D^{[[r]]}f$ via the X-norm in terms of

$$\text{(3.12)} \qquad \sum_{k=0}^{n} (k^*)^r\, f\hat{}(k)\, \psi_k(x).$$

For a short survey of dyadic differentiation as of 1981 see [40, pp. 653-657].

4. Extended Dyadic Derivative

As observed, the foregoing extension of the dyadic derivative is not a true one. Therefore let us try to apply a summability process to the series (3.8) or (3.12) in order to try to force them to convergence not only for piecewise constants. A suitable process is that connected with the name of Euler.

Definition 1: The series $\Sigma_{j=0}^{\infty} a_j$ is said to be *Euler-summable* to s, if

(4.1) $$\frac{1}{2^{n+1}} \sum_{\nu=0}^{n} \binom{n+1}{\nu+1} \sum_{j=0}^{\nu} a_j \equiv \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} a_j$$

tends to s for $n \longrightarrow \infty$.

The fact that both sides of (4.1) are equal to another seems to be somewhat astonishing but is easy to show [24, p. 232].

The left-hand side of (4.1), which might be the more convenient form of the definition, is usually written in the symmetric form

(4.2) $$\frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} \sum_{j=0}^{\nu} a_j .$$

(Regarding the conversion of one into the other see [24, p. 236]).

The series (4.2) is the particular case q = 1/2 of the general Euler-Knopp (E,q) transform, given by

(4.3) $$\sum_{\nu=0}^{n} \binom{n}{\nu} q^{\nu}(1-q)^{n-\nu} \sum_{j=0}^{\nu} a_j \qquad (q \in \mathbb{R}\backslash\{0,1\}).$$

The (E,q)-transform is regular in the sense that it sums convergent series to the same limit if and only if q is a real satisfying $0 < q \leq 1$. (For a proof in case q = 1/2 see Knopp [24, p. 232]). It is well-known that the Euler process is a very effective method of convergence; it has a wide region of summability. Although every (E,q)-summable series is Borel summable, there is a certain superiority of the (E,q)-process in comparison with that of Borel, especially since the (E,q)-transform permits adjunction of elements (translative) and is easier to handle. At any rate, the Cesàro-process and Euler-process are incomparable [24]. Anyhow, the (E,q)-process is one of the most powerful of practical limitation processes. For questions regarding Euler summability see in particular [24], [25, pp. 244 ff, 468 ff., 509 ff.], [35, pp. 56 ff, 92 ff], [32], [42, pp. 130 ff].

Definition 2: Let $f \in X$. If there is a $g \in X$ such that

$$\lim_{n\to\infty} \|d_n^{\mathcal{E}} f(\cdot) - g(\cdot)\|_X = 0, \tag{4.4}$$

where [1]

$$d_n^{\mathcal{E}} f(x) := \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \psi_k(u) \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\, 2^{j-1} \{f(x)-f(x\oplus 2^{-j-1})\} \tag{4.5}$$

$(x \in [0,1))$,

then g is called the first *strong extended dyadic (=ED) derivative* of f, denoted by $\mathcal{E}^{\{1\}}f = g$. ED-derivatives of higher order $r \in \mathbb{N}$ are defined by $\mathcal{E}^{\{r\}}f = \mathcal{E}^{\{1\}}(\mathcal{E}^{\{r-1\}}f)$. Set $X^{\{r\}} := \{f \in X[0,1);\ \mathcal{E}^{\{r\}}f \in X[0,1)\}$.

Lemma 1: *If for* $f \in X$ *there exists* $\mathcal{E}^{\{r\}}f \in X$ *for some* $r \in \mathbb{N}$, *then*

$$[\mathcal{E}^{\{r\}}f]^\wedge(k) = (k^*)^r f^\wedge(k) \qquad (k \in \mathbb{N}_0). \tag{4.6}$$

Proof: Let $r = 1$. Since $\int_0^1 \psi_k(u)\,\{f(u)-f(u\oplus 2^{-j-1})\}\,du = f^\wedge(k)\{1-\psi_k(2^{-j-1})\} = 2k_j\, f^\wedge(k)$, one has

$$[d_n^{\mathcal{E}} f]^\wedge(k) = \left\{\sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\, 2k_j\right\} f^\wedge(k). \tag{4.7}$$

Hence it follows by (2.5),

$$|[d_n^{\mathcal{E}} f]^\wedge(k) - [\mathcal{E}^{\{1\}}f]^\wedge(k)| \le \|d_n^{\mathcal{E}} f - \mathcal{E}^{\{1\}}f\|_X \to 0 \qquad (n \to \infty).$$

Now for arbitrary $k \in \mathbb{N}$ there is (by definition of the dyadic expension of k) $j_0 \in \mathbb{N}$ with $2^K \le j_0 < 2^{K+1}$ such that $k_j = 0$ for all $j > j_0$. So, recalling that the Euler process is regular,

---

[1] Note that one could also define $d_n^{\mathcal{E}}f(x)$, on account of (4.1), by $(2^{n+1})^{-1} \Sigma_{\nu=0}^{n} \binom{n+1}{\nu+1} \Sigma_{j=0}^{\nu} (-1)^j\, 2^{j-1}\{f(x)-f(x\oplus 2^{-j-1})\}$.

$$\lim_{n\to\infty}\left\{\sum_{\nu=0}^{n}\frac{1}{2^{\nu+1}}\sum_{j=0}^{j_0}\binom{\nu}{j}(-1)^j\,2^j\,k_j\right\}f^\wedge(k)=\left\{\sum_{j=0}^{j_0}(-1)^j\,2^j\,k_j\right\}f^\wedge(k)=k^*\,f^\wedge(k).$$

This yields (4.6) in case $r = 1$. The rest follows by induction.

The sums $d_n^{\mathcal{E}}f$ of (4.5) can also be represented as follows:

<u>Lemma 2:</u> *The sums* $d_n^{\mathcal{E}}f$, $f \in X[0,1)$, *are equal to*

$$(4.8)\quad d_n^D\,f(x) := \frac{1}{2^{n+1}}\sum_{\nu=0}^{n}\binom{n+1}{\nu+1}\sum_{j=0}^{\nu}(-1)^j\,2^{j-1}\{(f*D_\nu)(x)-(f*D_\nu)(x\oplus 2^{-j-1})\}$$

with $x \in [0,1)$, $D_\nu(x) := \Sigma_{m=0}^{\nu-1}\,\psi_m(x)$, $[D_\nu]^\wedge(k) = 1$, $k \in \mathbb{N}_0$ and $\nu > k$.

<u>Proof:</u> Taking the Walsh-Fourier coefficients of (4.8),

$$[d_n^D\,f]^\wedge(k) = \frac{1}{2^{n+1}}\sum_{\nu=0}^{n}\binom{n+1}{\nu+1}\sum_{j=0}^{\nu}(-1)^j\,2^{j-1}\,f^\wedge(k)\,[D_\nu]^\wedge(k)\,\{1-\psi_k(2^{-j-1})\}$$

$$= \left\{\frac{1}{2^{n+1}}\sum_{\nu=0}^{n}\binom{n+1}{\nu+1}\sum_{j=0}^{\nu}(-1)^j\,2^j\,k_j\right\}f^\wedge(k)$$

which is identical to the right side of (4.7) on account of the two equivalent forms of Def. 1. The uniqueness theorem then yields (4.8).

Let us denote the rth X-norm ED-derivative built up iteratively from the difference $d_n^D f$ by $\mathcal{E}_D^{\{r\}}f$. (When using the alternative definition of Euler summability, representation $d_n^D f$ is more complicated; see Lemma 5.1 of [4]).

The representation (4.8) in comparison with (4.5) is interesting in the sense that f has been replaced by the convolution $f*D_\nu$ which smoothens f.

The Walsh functions $\psi_k(x)$, known to be arbitrarily often differentiable in the sense of (3.5), turn out to possess the same property in the extended sense. Indeed, Lemma 3: *For the Walsh functions* $\psi_k(x)$ *one has*

$$\mathcal{E}^{\{r\}}\ \psi_k(x) = \psi_k^{\{r\}}(x) = (k^*)^r\ \psi_k(x) \qquad (r,k \in \mathbb{N}_0). \tag{4.9}$$

Proof: Let r = 1. The general case follows by induction.

$$\lim_{n\to\infty} \|d_n^{\mathcal{E}}\ \psi_k - k^*\ \psi_k\|_X = \lim_{n\to\infty} \left\| \left\{ \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\ 2^j k_j - k^* \right\} \psi_k \right\|_X$$

$$= \lim_{n\to\infty} \left| \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\ 2^j k_j - k^* \right| \|\psi_k\|_X,$$

which tends to zero for $n \to \infty$, similarly as in the proof of Lemma 1.

5. The Fundamental Theorem for the ED-Calculus

A basic role in this section will be played by the functions $W_r^*(x)$, defined via

$$W_r^{*\wedge}(k) = \begin{cases} 1, & k = 0 \\ (k^*)^{-r}, & k \in \mathbb{N} \end{cases} \qquad (r \in \mathbb{N}). \tag{5.1}$$

The Walsh-Fourier series of W_r^* so has the form

$$W_r^*(x) \sim 1 + \sum_{k=1}^{\infty} (k^*)^{-r}\ \psi_k(x). \tag{5.2}$$

Lemma 4. *For* $r \in \mathbb{N}$, *one has* $W_r^* \in L^1(0,1)$.

Proof: If $r \geq 2$, the series (5.2) is uniformly convergent by (3.11) and represents a function $g_r \in L^1(0,1)$ with coefficients equal to $g_r^\wedge(0) = 1$, $g_r^\wedge(k) = (k^*)^{-r}$ for $k \in \mathbb{N}$. Thus $g_r(x) = W_r^*(x)$ a.e. by the uniqueness theorem (2.7).

In case r = 1 set

$$W^*_{1,m}(x) := S_{2^m}(W^*_1)(x) = 1 + \sum_{k=1}^{2^m-1} W^{*\wedge}_1(k)\ \psi_k(x).$$

Abel's formula for partial summation with n > m yields

(5.3) $\quad W^*_{1,n}(x) - W^*_{1,m}(x)$

$$= \sum_{k=2^m}^{2^n-1} \frac{\psi_k(x)}{k^*} = \sum_{k=2^m}^{2^n-2} \left\{\frac{1}{(k+1)^*} - \frac{1}{k^*}\right\} D_{k+1}(x) - \frac{D_{2^m}(x)}{(2^m)^*} + \frac{D_{2^n}(x)}{(2^n-1)^*} .$$

Making use of (2.9),

$$\|W^*_{1,n} - W^*_{1,m}\|_{L^1} \le c \sum_{k=2^m}^{2^n-2} \left|\frac{k^*-(k+1)^*}{(k+1)^*k^*}\right| \log k + \left|\frac{1}{(2^m)^*}\right| + \left|\frac{1}{(2^n-1)^*}\right|$$

$$\le 25\ c \sum_{k=2^m}^{2^n-2} |k^* - (k+1)^*|\ \frac{\log k}{k^2} + \frac{1}{2^m} + \frac{5}{2^{n+1}} .$$

Now one can readily show (see [12, p. 70 f]) that for $k \in [2^s, 2^{s+1}-2]$, $m \le s \le n-1$,

$$\sum_{k=2^s}^{2^{s+1}-2} |k^* - (k+1)^*| \le \sum_{l=0}^{s} 2^{s-l-1} \left|\ \tfrac{1}{3}\ (1-(-2)^{l+2})\right| \le 2^{s+1} \sum_{l=0}^{s} 1 = 2^{s+1}(s+1).$$

This yields that

$$\sum_{k=2^m}^{2^n-2} |k^*-(k+1)^*|\ \frac{\log k}{k^2} \le \sum_{s=m}^{n-1} \frac{(s+1)\log 2}{4^s} \sum_{k=2^s}^{2^{s+1}-2} |k^* - (k+1)^*| + \sum_{s=m}^{n-2} \frac{(s+1)2^{s+1}\log 2}{4^s}$$

$$\le 2 \log 2 \sum_{s=m}^{n-1} (s+1)^2\ 2^{-s} + 2 \log 2 \sum_{s=m}^{n-2} (s+1)\ 2^{-s}.$$

Since the series $\Sigma_{s=0}^{\infty}(s+1)^2 2^{-s}$ and $\Sigma_{s=0}^{\infty}(s+1)2^{-s}$ converge, the right hand side tends to zero for $m,n \to \infty$, and so $W_{1,m}^*$ converges in $L^1(0,1)$-norm to a function $g_1 \in L^1(0,1)$ as $L^1(0,1)$ is complete. Since $W_{1,m}^{*\,\wedge}(k) = 1$ for $k = 0$, and $= (k^*)^{-1}$ for $1 \le k \le 2^m-1$, $W_{1,m}^{*\,\wedge}(k) = g_1^{\wedge}(k)$ for $k \in \mathbb{N}_0$. Thus $g_1(x) = W_1^*(x)$ a.e., and $W_1^* \in L^1(0,1)$.

The function $W_r^*(x)$ allows one to define an operator $I_{\{r\}}$, inverse to $\mathcal{E}^{\{r\}}$.

<u>Definition 3.</u> Let the operator $I_{\{r\}} : X[0,1) \to X[0,1)$ be given for $f \in X[0,1)$ by

$$(5.4)\qquad I_{\{r\}}f(x) := (W_r^* * f)(x) = \int_0^1 f(x \oplus u)\, W_r^*(u)\, du.$$

It is obvious that $I_{\{r\}}$ is linear and dyadically continuous.

<u>Lemma 5.</u> *If for* $f \in X[0,1)$ *there exists* $g \in X[0,1)$ *such that for* $r \in \mathbb{N}$

$$(5.5)\qquad (k^*)^r f^{\wedge}(k) = g^{\wedge}(k) \qquad (k \in \mathbb{N}_0),$$

then

$$(5.6)\qquad f(x) = (I_{\{r\}}g)(x) + f^{\wedge}(0) \quad \text{a.e..}$$

<u>Proof:</u> Since $f,g \in X[0,1)$, $[I_{\{r\}}g]^{\wedge}(k) = [W_r^* * g]^{\wedge}(k) = 0$ for $k = 0$, and $= (k^*)^{-r} g^{\wedge}(k) = (k^*)^{-r} f^{\wedge}(k)(k^*)^r$ for $k \in \mathbb{N}$. Thus $[I_{\{r\}}g + f^{\wedge}(0)]^{\wedge}(k) = f^{\wedge}(k)$ for $k \in \mathbb{N}_0$, yielding (5.6).

<u>Corollary 1.</u> *Let* $f \in X[0,1)$ *with* $f^{\wedge}(0) = 0$. *If for* $r \in \mathbb{N}$ *there exists* $\mathcal{E}^{\{r\}}f \in X[0,1)$, *then*

$$(5.7)\qquad (I_{\{r\}}(\mathcal{E}^{\{r\}}f))(x) = f(x) \quad \text{a.e.}$$

<u>Proof:</u> If $\mathcal{E}^{\{r\}}f \in X[0,1)$, then $[\mathcal{E}^{\{r\}}f]^{\wedge}(k) = (k^*)^r f^{\wedge}(k)$ by (4.6). Lemma 5 then gives

$$[I_{\{r\}}(\mathcal{E}^{\{r\}}f)]^\wedge(k) = [W_r^* * \mathcal{E}^{\{r\}}f]^\wedge(k) = (k^*)^{-r}(k^*)^r\, f^\wedge(k) \quad (k \in \mathbb{N}_0),$$

and so (5.7).

Now we wish to show that if $(k^*)^r f^\wedge(k) = g^\wedge(k)$, $k \in \mathbb{N}_0$, for some $g \in X[0,1)$, then there exists $\mathcal{E}^{\{r\}}f \in X[0,1)$. Further, if $f \in X[0,1)$, then also $\mathcal{E}^{\{r\}}(I_{\{r\}}f) = f$. For this purpose, let $W_r^{*(m)}$ be defined via

$$[W_r^{*(m)}]^\wedge(k) = \begin{cases} 0, & 0 \le k < 2^m \\ (k^*)^{-r}, & k \ge 2^m, \end{cases} \qquad (r \in \mathbb{N}) \tag{5.8}$$

so having Walsh-Fourier series $\Sigma_{k=2^m}^{\infty}(k^*)^{-r}\,\psi_k(x)$.

<u>Lemma 6.</u> *One has for* $r \in \mathbb{N}$, $m \longrightarrow \infty$,

$$\|W_r^{*(m)}\|_{L^1(0,1)} = \mathcal{O}(m^{2r}\, 2^{-mr}).$$

<u>Proof:</u> Applying Abel's partial summation once more to (5.3), one obtains

$$\sum_{k=2^m}^{2^n-1} \frac{1}{k^*}\psi_k(x) = \sum_{k=2^m}^{2^n-3} \left(\frac{1}{k+1}\left\{\sum_{i=1}^{k+1} D_i(x)\right\}(k+1)\right) \cdot$$

$$\cdot\left\{\left(\frac{1}{k^*} - \frac{1}{(k+1)^*}\right) - \left(\frac{1}{(k+1)^*} - \frac{1}{(k+2)^*}\right)\right\} - \left(\frac{1}{2^m}\left\{\sum_{i=1}^{2^m} D_i(x)\right\}2^m\right)\left\{\frac{(2^m+1)^* - (2^m)^*}{(2^m+1)^*\,(2^m)^*}\right\}$$

$$+ \left(\frac{1}{2^n-1}\left\{\sum_{i=1}^{2^n-1} D_i(x)\right\}(2^n-1)\right)\left\{\frac{(2^n-1)^* - (2^n-2)^*}{(2^n-2)^*\,(2^n-1)^*}\right\} - \frac{D_{2^m}(x)}{(2^m)^*} + \frac{D_{2^n}(x)}{(2^n-1)^*}$$

$$= \sum_{k=2^m}^{2^n-3} \frac{(k+1)}{(k+1)^*} F_{k+1}(x)\left\{\frac{(k+1)^*-k^*}{k^*} - \frac{(k+2)^*-(k+1)^*}{(k+2)^*}\right\}$$

$$- F_{2^m}(x)\,\frac{2^m}{(2^m)^*}\left\{\frac{(2^m+1)^* - (2^m)^*}{(2^m+1)^*}\right\} + F_{2^n-1}(x)\,\frac{(2^n-1)}{(2^n-1)^*}\left\{\frac{(2^n-1)^* - (2^n-2)^*}{(2^n-2)^*}\right\}$$

$$- \frac{D_{2^m}(x)}{(2^m)^*} + \frac{D_{2^n}(x)}{(2^n-1)^*}.$$

Now the sum on the right may be rewritten as

$$\sum_{k=2^m+1}^{2^n-2} \frac{(k+1)^*-k^*}{k^*(k+1)^*} \{(k+1)F_{k+1}(x) - kF_k(x)\}$$

$$+ F_{2^n-2}(x) \frac{2^n-2}{(2^n-2)^*} \left\{\frac{(2^n-2)^* - (2^n-3)^*}{(2^n-3)^*}\right\} + F_{2^m+1}(x) \frac{(2^m+1)}{(2^m+1)^*} \left\{\frac{(2^m+1)^* - (2^m)^*}{(2^m)^*}\right\}.$$

Now $|(2^m+1)^*-(2^m)^*| = |(2^m-2)^*-(2^m-1)^*| = 1$, as well as $|(2^m-3)^*-(2^m-2)^*| = 3$. Combining the results, and observing (3.11), (2.9) and (2.10),

$$\left\| \sum_{k=2^m}^{2^n-1} \frac{1}{k^*} \psi_k(\cdot)\right\|_{L^1(0,1)} \leq \sum_{k=2^m+1}^{2^n-2} \left|\frac{(k+1)^*-k^*}{k^*(k+1)^*}\right| \cdot \|(k+1)F_{k+1}(\cdot) - kF_k(\cdot)\|_{L^1(0,1)}$$

$$+ O(2^{-n}) + O(2^{-m}) + O(2^{-m}) + O(2^{-n}) + O(2^{-m}) + O(2^{-n}).$$

However, by (2.9), $\|D_{k+1}\|_1 = \|(k+1)F_{k+1} - kF_k\|_1 = O(\log k)$. So the sum on the right side is bounded by

$$O\left(\sum_{k=2^m+1}^{2^n-2} |(k+1)^* - k^*| \frac{\log k}{k^2}\right) = O\left(\sum_{j=m}^{\infty} j^2\, 2^{-j}\right) = O(m^2\, 2^{-m}) \qquad (m \to \infty),$$

the estimate following similarly as in the proof of Lemma 4. This completes the case $r = 1$.

If $r = 2$, then $\|W_2^{*(m)}\|_1 = \|W_1^{*(m)} * W_1^{*(m)}\|_1 \leq (\|W_1^{*(m)}\|_1)^2 = O(m^4 2^{-2m})$. The proof for $r \geq 3$ follows by induction.

It is important to observe that in standard dyadic analysis, the counterpart of Lemma 6 for the function $W_r^{(m)}$, for which $W_r^{(m)\wedge}(k) = (k)^{-r}$ for $k \in \mathbb{N}$, the order does not contain the factor m^{2r}, so that it reads $O(2^{-mr})$. This has far reaching implications, in particular in regard to applications, namely to Walsh-Fourier series and approximation theory in the Walsh-frame (see Sec. 8).

Now to the companion of Corollary 1, namely the other half of the fundamental theorem.

Lemma 7. *For* $f \in X[0,1)$ *with* $f^\wedge(0) = 0$ *there holds*

$$\mathcal{E}^{\{r\}}(I_{\{r\}}f) = f \qquad (r \in \mathbb{N}). \tag{5.11}$$

Proof: First take $r = 1$, and let us establish the identity

$$d_m^{\mathcal{E}}(I_{\{1\}}f)(x) - f(x) \tag{5.12}$$

$$= \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}(-1)^j 2^{j-1}\{(W_1^* * f)(x) - (W_1^* * f)(x\oplus 2^{-j-1})\} - f(x)$$

$$= S_{2^m} f(x) - f(x) + (f * F_m^*)(x),$$

where

$$F_m^*(x) := \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}(-1)^j 2^{j-1}\{W_1^{*(m)}(x) - W_1^{*(m)}(x\oplus 2^{-j-1})\}.$$

For this purpose we take the Walsh-Fourier coefficients of both sides of (5.12). For the left side,

$$[d_m^{\mathcal{E}}(I_{\{1\}}f]^\wedge(k) - f^\wedge(k)$$

$$= \left\{\sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}(-1)^j\, 2^j\, k_j\right\} [W_1^*]^\wedge(k) f^\wedge(k) - f^\wedge(k)$$

$$= \begin{cases} 0, & \text{if } 0 \le k < 2^m \\ f^\wedge(k)\{(k^*)^{-1} \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}(-1)^j\, 2^j\, k_j - 1\}, & \text{if } k \ge 2^m, \end{cases}$$

noting (5.8) and $f^\wedge(0) = 0$.

Concerning the right side,

$$[S_{2^m} f]^\wedge(k) - f^\wedge(k) = \begin{cases} 0\,, & 0 \le k < 2^m \\ -f^\wedge(k)\,, & 2^m \le k \end{cases},$$

$$[f * F_m^*]^\wedge(k) = [W_1^{*(m)}]^\wedge(k) f^\wedge(k) \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\, 2^j\, k_j$$

$$= \begin{cases} 0, & 0 \le k < 2^m \\ (k^*)^{-1} f^\wedge(k) \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j\, 2^j\, k_j, & 2^m \le k. \end{cases}$$

Since both sides are equal, the proof of (5.12) is complete.

To establish (5.11) it remains to show, an account of (2.8), that

$$\lim_{n \to \infty} \|f * F_m^*\|_X = 0. \tag{5.13}$$

Indeed, by Lemma 6,

$$\|F_m^*\|_{L^1(0,1)} \le \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}\, 2^j\, \|W_1^{*(m)}\|_{L^1(0,1)}$$

$$\le c\, 2^{-m} m^2 \sum_{\nu=0}^{m} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j}\, 2^j = c\, 2^{-m} m^2 \sum_{\nu=0}^{m} 2^{-\nu-1}\, 3^\nu$$

$$= c\, 2^{-m} m^2 \frac{1}{2} \left(\frac{(3/2)^{m+1} - 1}{3/2 - 1} \right) = \mathcal{O}\left(m^2 \left(\tfrac{3}{4}\right)^m\right) \qquad (m \to \infty).$$

Therefore (5.13) follows, since $f \in X[0,1)$ and $F_m^* \in L^1(0,1)$ yields that $f * F_m^* \in X[0,1)$.

The general case $r \ge 2$ now follows by induction. Since $W_{r+1}^* = W_r^* * W_1^*$, and the convolution operation is associative,

$\mathcal{E}^{\{r+1\}}(I_{\{r+1\}}f) = \mathcal{E}^{\{r+1\}}(W^*_{r+1} * f) = \mathcal{E}^{\{1\}}(\mathcal{E}^{\{r\}}(W^*_r * (W^*_r * f)))$

$= \mathcal{E}^{\{1\}}(W^*_r * f) = f$. This completes the proof of Lemma 7.

Combining Corollary 1 and Lemma 7 we now have the fundamental theorem.

Theorem 1. *Let* $f \in X[0,1)$ *with* $f\hat{\ }(0) = 0$.

a) *If there exists* $\mathcal{E}^{\{r\}}f \in X[0,1)$ *for some* $r \in \mathbb{N}$, *then* $I_{\{r\}}(\mathcal{E}^{\{r\}}f) = f$.

b) *There holds* $\mathcal{E}^{\{r\}}(I_{\{r\}}f) = f$.

Now to the existence of $\mathcal{E}^{\{r\}}f$ and the characterization of the class of ED-differentiable functions.

Theorem 2. *The following assertions are equivalent for* $f \in X[0,1)$ *and* $r \in \mathbb{N}$:

(i) $\mathcal{E}^{\{r\}}f = g$ *exists with* $g \in X[0,1)$;

(ii) *there exists* $g \in X[0,1)$ *with* $(k^*)^r f\hat{\ }(k) = g\hat{\ }(k)$, $k \in \mathbb{N}_0$;

(iii) *there exists* $g \in X[0,1)$ *with* $f = I_{\{r\}}g + f\hat{\ }(0)$.

Proof: The implication (i) $\Rightarrow$ (ii) follows by Lemma 1, (ii) $\Rightarrow$ (iii) by Lemma 5, and (iii) $\Rightarrow$ (i) by Lemma 7 with $[\mathcal{E}^{\{r\}}f]\hat{\ }(0) = 0$.

Now to the final, basic property of the operator $\mathcal{E}^{\{1\}}$.

Theorem 3. *The operator* $\mathcal{E}^{\{r\}} : X^{\{r\}}[0,1) \subset X[0,1) \to X[0,1)$, $r \in \mathbb{N}$, *is closed.*

Proof: The operator $\mathcal{E}^{\{r\}}$ is closed if and only if

(5.14;15) $$\lim_{n\to\infty} \|f_n - f\|_X = 0, \quad \lim_{n\to\infty} \|\mathcal{E}^{\{r\}}f_n - g\|_X = 0$$

imply $f \in X^{\{r\}}[0,1)$ and $g = \mathcal{E}^{\{r\}}f$. In fact, (5.15) implies by (2.5) and Lemma 1,

$$\lim_{n\to\infty} [\mathcal{E}^{\{r\}}f_n]\hat{\ }(k) = \lim_{n\to\infty} (k^*)^r f_n\hat{\ }(k) = g\hat{\ }(k) \qquad (k \in \mathbb{N}_0).$$

By (5.14) and Lemma 1 again, this yields $g^\wedge(k) = (k^*)^r f^\wedge(k)$, $k \in \mathbb{N}_0$. Thus $g = \mathcal{E}^{\{r\}}f \in X[0,1)$ by Theorem 2.

Let us observe that Lemma 7 with its proof, and the resulting Theorems 1 to 3, are new. They complete the ED-analysis begun in [12].

6. ED - Derivative of Fractional Orders

Let us now give a further extension of the dyadic derivative, one in which the order r in $\mathcal{E}^{\{r\}}f$ is extended to fractional $a \in \mathbb{R}$; the case for negative a will cover anti-differentiation.

In analogy with the classical derivative (3.2) or the classical dyadic case (i.e., (3.5) and (3.3)), it is advantageous to define this fractional ED-derivative in the transformed state. In this frame $D^{[[r]]}f$ of (3.8), (3.9) could also be defined by

(6.1) $$\lim_{n\to\infty} \Big\| \sum_{k=0}^{n} (k^*)^r f^\wedge(k)\, \psi_k(\cdot) - D^{[[r]]}f(\cdot)\Big\|_X = 0.$$

But as this extension did not turn out to be a true one, the Euler summability process will again be applied. This leads in case r is replaced by $a \in \mathbb{R}$ to

Defintion 3: Let $f \in X[0,1)$. If there exists $g \in X$ such that

(6.2) $$\lim_{n\to\infty} \|E_n^{\{a\}}f(\cdot) - g(\cdot)\|_X = 0,$$

(6.3) $$E_n^{\{a\}}f(x) := \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^a f^\wedge(j)\, \psi_j(x)$$

$$E_n^{\{0\}}f(x) := f(x) \qquad (x \in [0,1)),$$

then g is called the *strong ED-derivative of* f *in* X *of (fractional) order* a in case $a > 0$, and the *strong anti - ED-derivative of order* a of f in X in case $a < 0$. In both instances g will be denoted by $E^{\{a\}}f$.

The partial sums (6.3) of the Euler process applied to the Walsh-Fourier series

$$(6.4)\qquad \sum_{k=0}^{\infty} (k^*)^{\alpha} f^{\wedge}(k)\ \psi_k(x)$$

may be rewritten as a convolution integral, namely

Lemma 8. *For* f ∈ X[0,1) *there holds*

$$(6.5)\qquad E_n^{\{\alpha\}} f(x) = (e_n^{\{\alpha\}} * f)(x) \quad \text{a.e.},$$

$$(6.6)\qquad e_n^{\{\alpha\}}(x) := \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^{\alpha}\ \psi_j(x) \qquad (x \in [0,1)).$$

Indeed, noting (2.6), (2.1), and (2.4),

$$(e_n^{\{\alpha\}} * f)(x) = \sum_{\nu=0}^{n} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^{\alpha} \int_0^1 \psi_j(x \oplus u)\ f(u)\ du$$

$$= \sum_{\nu=0}^{n} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^{\alpha}\ \psi_j(x)\ f^{\wedge}(j)$$

which is the right side of (6.3). Another proof of (6.5) can be obtained by Fourier-Walsh transform techniques. In fact,

$$(6.7)\qquad [e_n^{\{\alpha\}}]^{\wedge}(k) = \sum_{\nu=0}^{n} 2^{-\nu-1} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^{\alpha}\ \delta_{k,j} = \left\{ \sum_{\nu=k}^{n} 2^{-\nu-1} \binom{\nu}{k} \right\} (k^*)^{\alpha}.$$

Thus

$$[E_n^{\{\alpha\}} f]^{\wedge}(k) = \left\{ \sum_{\nu=k}^{n} \frac{1}{2^{\nu+1}} \binom{\nu}{k} \right\} (k^*)^{\alpha}\ f^{\wedge}(k) \qquad (k \in \mathbb{N}_0)$$

On the other hand, the coefficients of (6.3) are

$$(6.8) \qquad [E_n^{\{\alpha\}}f]^\wedge(k) = \left\{ \sum_{\nu=k}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (j^*)^\alpha \, \delta_{k,j} \right\} f^\wedge(k).$$

The uniqueness theorem, i.e. (2.7), again yields (6.5).

The ED-derivative $E^{\{\alpha\}}f$ may also be defined in terms of the alternative definition given by the left side of (4.1), thus as a convolution via

$$(6.9) \qquad [E_n^{\{\alpha\}}]^\wedge(k) := \left[\frac{1}{2^{n+1}} \sum_{\nu=0}^{n} \binom{n+1}{\nu+1} \sum_{j=0}^{\nu} (-1)^j (j^*)^\alpha \, \psi_j(\cdot) \right]^\wedge(k)$$

$$= \frac{1}{2^{n+1}} \sum_{\nu=0}^{n} \binom{n+1}{\nu+1} \sum_{j=0}^{\nu} (-1)^j (j^*)^\alpha \, \delta_{k,j} = \frac{1}{2^{n+1}} \sum_{\nu=k}^{n} \binom{n+1}{\nu+1} (k^*)^\alpha.$$

This gives rise, in view of (6.7), to the binomial coefficient identity

$$\sum_{\nu=k}^{n} \frac{1}{2^\nu} \binom{\nu}{k} = \frac{1}{2^n} \sum_{\nu=k}^{n} \binom{n+1}{\nu+1} \qquad (0 \le k \le n \in \mathbb{N})$$

which is in fact a particular case of a more general identity due to H.W. Gould [21, p. 17].

Concerning fractional order derivatives in the dyadic situation, it was Onneweer [29] who first defined such derivatives in the dyadic frame. Closer to the present analysis is the approach by He Zelin [23]. In fact, he defined the derivative $T^{[\alpha]}f$ of $f \in X$, which coincides with $D^{[\alpha]}f$ of (3.3), (3.6) in case $\alpha = r \in \mathbb{N}$, in terms of

$$(6.10) \qquad \lim_{n\to\infty} \left\| \sum_{k=0}^{n} k^\alpha f^\wedge(k) \, \psi_k(\cdot) - T^{[\alpha]}f(\cdot) \right\|_X = 0.$$

The Walsh functions $\psi_k(x)$, $x \in [0,1)$, are also differentiable in the extended fractional sense of (6.3). Indeed, Lemma 9. *The* $\psi_k(x)$ *have the property*

$$E^{\{a\}}\,\psi_k(x) = \psi_k^{\{a\}}(x) = (k^*)^a\,\psi_k(x) \qquad (x \in [0,1);\ k \in \mathbb{N}_0,\ a \in \mathbb{R}). \tag{6.11}$$

The counterpart of one half of (3.6) and of (4.6) reads

Lemma 10. *If* $f \in X^{\{a\}} := \{f \in X[0,1);\ E^{\{a\}}f \in X[0,1\}$, $a \in \mathbb{R}$, *then*

$$[E^{\{a\}}f]^\wedge(k) = (k^*)^a\, f^\wedge(k) \qquad (k \in \mathbb{N}_0).$$

Lemma 11. *Let* $f \in X^{\{a\}}$. *Then*

$$E^{\{a\}}f = 0 \iff \begin{cases} f = \text{const.}, & \text{in case } a > 0 \\ f = 0, & \text{in case } a < 0. \end{cases}$$

Lemma 12. *If for* $f \in X[0,1)$ *there exists* $g \in X[0,1)$ *such that* $g^\wedge(k) = (k^*)^a f^\wedge(k)$, $k \in \mathbb{N}_0$ *for fixed* $a > 0$, *then*

$$f = g * e_\infty^{\{-a\}} + f^\wedge(0) = E^{\{-a\}}g + f^\wedge(0),$$

where

$$[e_\infty^{\{-a\}}]^\wedge(k) = \begin{cases} 1 & , \quad k = 0 \\ (k^*)^{-a} & , \quad k \geq 1 \end{cases} \qquad (a \in \mathbb{R}),$$

or

$$e_\infty^{\{-a\}}(x) = \underset{n\to\infty}{\text{str-lim}}\{1 + \sum_{k=1}^{n-1} (k^*)^{-a}\,\psi_k(x)\} \qquad (a > 0).$$

Comparing $e_\infty^{\{-a\}}(x)$ with $W_r^*(x)$ of (5.2), $e_\infty^{\{-a\}}(x)$ is actually equal to

$$W_a^*(x) \sim 1 + \sum_{k=1}^{\infty} (k^*)^{-a}\,\psi_k(x),$$

so that

$$E^{\{-r\}}f(x) = (e_\infty^{\{-r\}} * f)(x) = \int_0^1 f(x \oplus u)\, W_r^*(u)\,du.$$

Note that in Def. 3 the Euler process is actually superfluous for negative a, the case of anti-differentiation. So in that case,

$$E^{\{-a\}}f(x) = f^{\wedge}(0) + \sum_{k=1}^{\infty} (k^*)^{-a} f^{\wedge}(k) \psi_k(x) \qquad (a > 0).$$

The most important result here, quite similar to that given by Thm. 2, reads

Theorem 4. *Let* $f \in X[0,1)$ *and* $a > 0$. *The following three assertions are equivalent:*

(i) $E^{\{a\}}f = g \in X[0,1)$;

(ii) there exists $g \in X[0,1)$ with $g^{\wedge}(k) = (k^*)^a f^{\wedge}(k)$, $k \in \mathbb{N}_0$;

(iii) there exists $g \in X[0,1)$ with $f = E^{\{a\}}g + f^{\wedge}(0)$.

The fractional counterpart of the "Fundamental Theorem" states

Theorem 5. *Let* $f \in X$, $f^{\wedge}(0) = 0$, $a \in \mathbb{R}$. *Then*

$$E^{\{a\}}(E^{\{-a\}}f) = f.$$

If, in addition, $E^{\{a\}}f \in X[0,1)$, *then*

$$E^{\{a\}}(E^{\{-a\}}f) = E^{\{-a\}}(E^{\{a\}}f) = f.$$

A further result in this direction is that the operator $E^{\{a\}} : X^{\{a\}} \subset X[0,1) \to X[0,1)$ is linear and, as well, closed for $a > 0$.

For detailed proofs of Lemma 9-12 and Theorems 4,5 see [4]. See also the remark preceding Lemma 14 below.

From Theorems 2 and 4 one concludes immediately that the fractional order ED-derivative $E^{\{a\}}f$ of Def. 3 coincides with the integral order ED-derivative $E^{\{r\}}f$ of Def. 2 provided $a = r$; the latter is an extension of the former. Summing up our results, we have

Corollary 2: *Let* f ∈ X[0,1), r ∈ ℕ.

a) The following three assertions are equivalent to another:

(i) *there exists* g ∈ X[0,1) *with*

$$\lim_{n\to\infty} \| d_n^{\mathcal{E}}(\mathcal{E}^{\{r-1\}}f) - g\|_X = 0;$$

(ii) *there exists* g ∈ X[0,1) *with*

$$\lim_{n\to\infty} \| d_n^{D}(\mathcal{E}_D^{\{r-1\}}f) - g\|_X = 0;$$

(iii) *there exists* g ∈ X[0,1) *with*

$$\lim_{n\to\infty} \| e_n^{\{r\}} * f - g\|_X = 0.$$

b) The following two assertions are equivalent to each other:

(α) *there exists* g ∈ X[0,1) *such that*

$$I_{\{r\}}f(x) \equiv \int_0^1 f(x\oplus u)\, W_r^*(u)\, du = g(x);$$

(β) *there exists* g ∈ X[0,1) *such that*

$$\lim_{n\to\infty} \| e_n^{\{-r\}} * f - g\|_X = 0$$

or,

$$E^{\{-r\}}f = g.$$

Whereas assertions (i) and (ii) above are given in the original function space, statement (iii) is in the transformed space. Thus one has at least three ways (actually six if one uses the alternative forms of Def. 1) of defining the ED-derivative, a fact which turns out to be a very powerful tool in calculating various examples.

Note, however, that generally $[d_n\ \mathcal{E}^{\{r-1\}}f]^\wedge(k) \neq [E_n^{\{r\}}f]^\wedge(k)$ for each $n \in \mathbb{N}$, though the limits for $n \to \infty$ of both are equal, namely to $(k^*)^r f^\wedge(k)$ for each $k \in \mathbb{N}_0$.

In view of the regularity of the Euler process, a possible existence of $D^{[[r]]}f$ in the form (6.1) would yield that of $E^{\{a\}}f$ (or $\mathcal{E}^{\{r\}}f$) for $a = r \in \mathbb{N}$, and the two (three) would be equal to another. More important is the fact that it will turn out (see below) that every function which is dyadically differentiable in the classical sense (of (3.5)) is also ED-differentiable in the sense of Def. 2.

7. The ED-Derivative of Particular Functions

Let us now consider examples of functions that are ED-differentiable.

Firstly, piecewise constant functions satisfying most reasonable hypotheses (recall Sec. 2) can be shown to be pointwise ED-differentiable (of order 1) on [0,1) (see Thm. 3.1 in [5, II]). Since such functions are dyadically differentiable in the standard sense (and characterize such differentiability completely), there follows

Theorem 6. *Every function which is dyadically differentiable (according to (3.5)) is also ED-differentiable. Thus, if* $D^{[r]}f$ *exists so does* $\mathcal{E}^{\{r\}}f$.

However, the two derivatives, $D^{[r]}f$ and $\mathcal{E}^{\{r\}}f$, do not necessarily agree (since no regularity question is involved here). Thus, since $D^{[r]}\ \psi_k(x) = k^r\ \psi_k(x)$, one has $D^{[r]}\ \psi_3(x) = 3^r\ \psi_3(x)$ or $D^{[r]}\ \psi_{47}(x) = (47)^r\ \psi_{47}(x)$, but $E^{\{r\}}\ \psi_3(x) = (-1)^r\ \psi_3(x)$ and $E^{\{r\}}\ \psi_{47}(x) = (-37)^r\ \psi_{47}(x)$, according to Lemma 9.

Secondly, the continuous monomials $f_n(x) =: x^n$, $n \in \mathbb{N}_0$ are ED-differentiable. In fact, $\mathcal{E}^{\{1\}}f_1$ exists as an element in $L^1(0,1)$ (see Thm. 3.2 in [5, II]),

$$\mathcal{E}^{\{1\}}f_1(x) = \sum_{\nu=0}^{\infty} 2^{-\nu-4} \sum_{j=1}^{[\log_2\nu]} \binom{\nu}{2^j}(-1)^{j+1}\ \varphi_{2^j}(x)$$

where $\varphi_\kappa(x) = \psi_{2^\kappa}(x)$ is the Rademacher function of order κ. Likewise f_2 is ED-differentiable; this establishes the general case by reduction.

In particular, algebraic polynomials are ED-differentiable (of order 1) in $L^1(0,1)$. Thus the ED-derivative already has a wider range of applicability than the standard dyadic derivative. See Figure 1 in this respect.

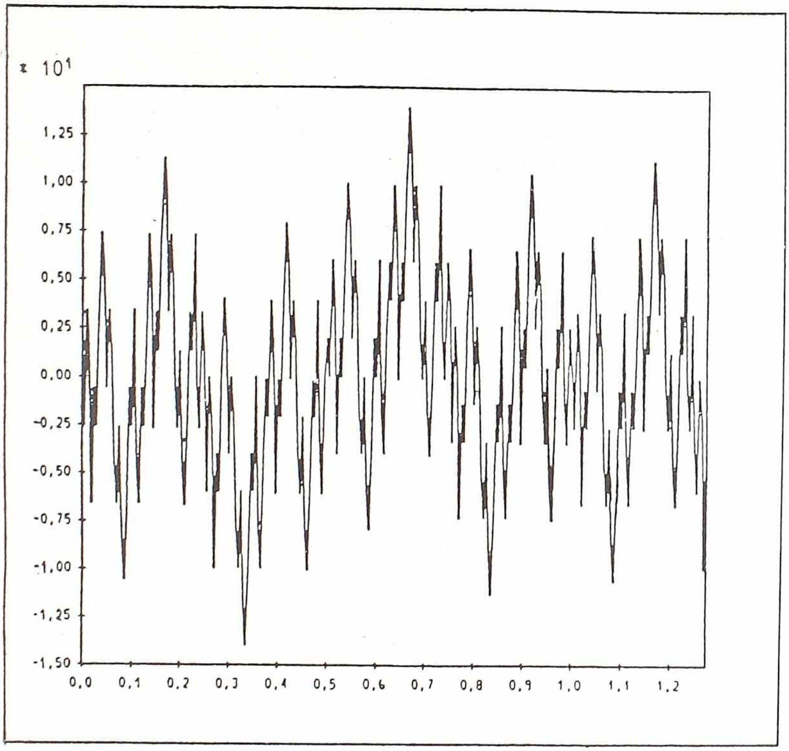

Fig. 1. Approximation of graph of $\mathcal{E}_n^{\{1\}} f_1(x)$ for n = 256.

Thirdly, "piecewise polynomials" such as $g_{n,k}(x) := x^n\, \psi_k(x)$, $n,k \in \mathbb{N}_0$, namely "polynomials" of order n having a finite number of jump discontinuities, are ED-differentiable (see Thm. 3.3 [5, II]). Thus, for $g_{1,1}(x) = x\, \psi_1(x)$,

$$E^{\{1\}} g_{1,1}(x) = \frac{\psi_1(x)}{2} \left\{ 1 + \sum_{\nu=1}^{\infty} 2^{-\nu-1} \sum_{j=1}^{[\log_2 \nu]-1} \binom{\nu}{2^{j+1}} \left[(-1)^j 2^j + 1\right] (-1)^{x_{j+1}} 2^{-j} \right\}$$

Further, $E^{\{\alpha\}} g_{1,1}(x)$ exists in $L^1(0,1)$ for all $0 < \alpha \leq 1$.

It can be shown that whereas (even) the derivative $T^{[\alpha]}g_{1,1}$ of (6.10) does exist for $0 < \alpha < 1$, it does not do so for $\alpha = 1$ (see Lemma 6.3 in [4, I]). See Figure 2 for the ED-derivative of $g_{1,1}(x)$.

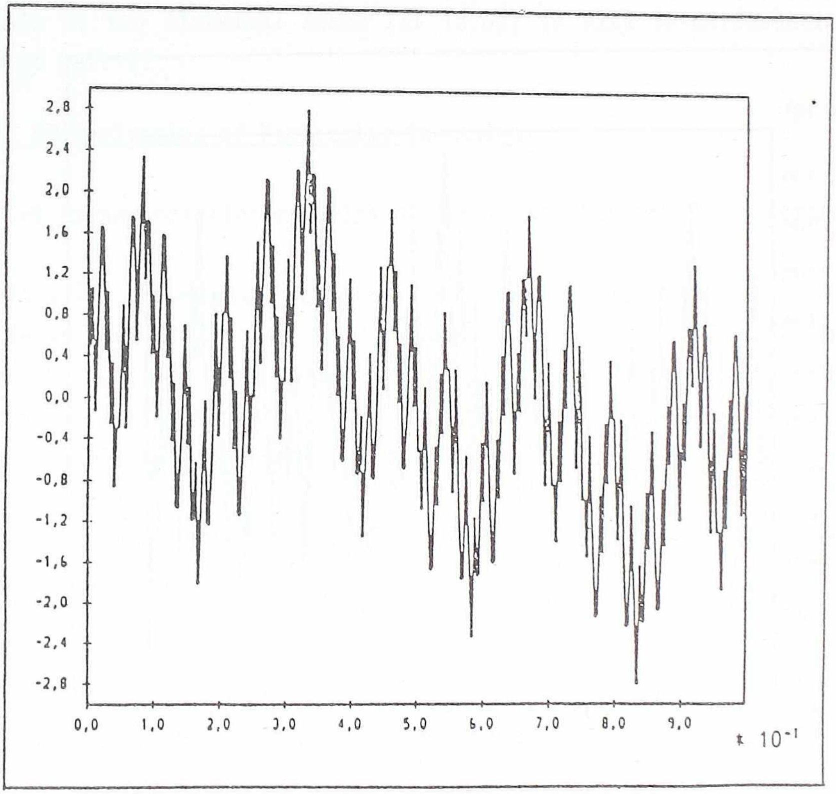

Fig. 2. Approximation of graph of ED-derivative $\mathcal{E}_n^{\{1\}}g_{1,1}(x)$ for n = 512.

Fourthly, even the Dirichlet function d(x), given by d(x) := 1 if $x \in [0,1)\backslash\mathbb{Q}$, and = 0, elsewhere, $\mathbb{Q}$ being the rationals, is $E^{\{\alpha\}}$-differentiably in the norm and pointwise sense for all $\alpha > 0$ with value zero (see Lemma 3.1 in [5, II]). Note that since d(x) has infinitely many discontinuities which lie dense in [0,1) (and do not have a finite number of cluster points in [0,1)), the result of our first example above is <u>not</u> applicable here.

Fifthly, the exotic, modified Dirichlet function x^n d(x), which is a "polynomial" of order n possessing infinitely many discontinuities which lie dense in [0,1), is ED-differentiable with $\mathcal{E}^{\{1\}}(x^n\, d(x)) = \mathcal{E}^{\{1\}}f_n(x)$ (Lemma 3.6 in [5, II]).

Sixthly, to ED-derivative of the Walsh series $W_r^*(x) := 1 + \sum_{k=1}^{\infty}(k^*)^{-r}\,\psi_k(x)$, $r \geq 2$. Here one can show (Lemma 3.2 [5, II]) that $E^{\{a\}}(W_r^*)(x) = W_{r-a}^*(x)-1$, $x \in [0,1)$ whenever $0 < a < r$.

Seventhly, whether classical functions such as e^x, $\sin x$, $\log(1+x)$ or even $\psi_k(x)e^x$, $\psi_k(x)\sin x$, $\psi_k(x)\log(1+x)$ are ED-differentiable is not yet certain. Even the famous van der Waerden function, $\mathrm{wae}(x) := \sum_{k=1}^{\infty} 4^{-k}\{4^k x\}$, $x \in \mathbb{R}$, where $\{x\}$ denotes the distance from x to the nearest integer, seems to be ED-differentiable (though it is differentiable nowhere in the classical sense).

Eigthly, consider the interesting function s(x), defined by

$$s(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\,\varphi_k(x) \qquad (x \in [0,1))$$

which is piecewise constant with infinitely many jump discontinuties in [0,1), so that it is ED-differentiable (according to the first example). In fact,

$$\mathcal{E}^{\{1\}}\, s(x) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{2^k} = 1 - 4x \qquad (x \in [0,1)).$$

Whereas this derivative has a closed representation, s(x) does not seem to. [Note that s(x) is continuous except at the dyadic rationals as well as differentiable a.e. in [0,1), both taken in the classical sense]. Further, $\mathcal{E}^{\{2\}}s(x)$ exists in view of the second example, and turns out to be

$$\mathcal{E}^{\{2\}}\, s(x) = -\,4\,\mathcal{E}^{\{1\}}\, f_1(x).$$

Conversely, s(x) can be regarded as the (first order) dyadic anti-derivative of 1-4x. In fact,

$$I_{\{1\}}(1-4x) = s(x) \qquad (x \in [0,1)).$$

Classical differentiation of $x-2x^2$ decreases its order from 2 to 1 to zero; ED-differentiation first increases the order of s(x), a piecewise constant, to the linear function 1-4x, and then passes on to $-4\mathcal{E}^{\{1\}}f_1$ (recall Fig. 1).

Let us try to compare the Newtonian-Leibnizian calculus with the ED-calculus on the basis of this example:

Newtonian Calculus			ED-Calculus		
	$x - 2x^2$			$s(x)$	
Differen-tiation ↓	$1 - 4x$	↑ Integration	Differen-tiation ↓	$1 - 4x$	↑ Anti-Differ-entiation
↓	-4	↑	↓	$-4\ \varepsilon^{\{1\}} f_1$	↑

For an approximation to the graph of s(x) see Figure 3.

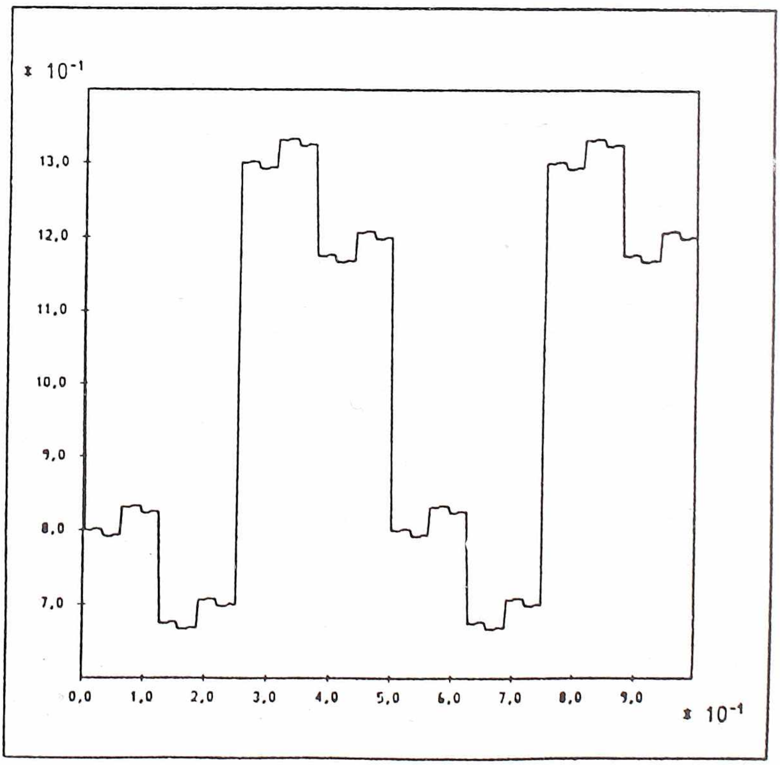

Fig. 3. The graph of the 40th partial sum of s(x). This is a very good approximation to s, differing from s by at most 4^{-40}

A possible interpretation of the anti-differentiation operator $I_{\{r\}} = E^{\{-r\}}$ for $r \in \mathbb{N}$, in particular of $I_{\{1\}}f = (f*W_1^*)(x)$, is also still lacking. It does not seem to be associated with the area under a curve, as is the classical integral. Further, $W_1^*(x)$ is probably *not* positive on $[0,1)$. In any case, $W_1^*(x) \geq -1$ on $[0,1)$. See also [5, p. 935 f] in this respect.

8. Applications to Walsh-Fourier Analysis and the Theory of Best Approximation

The dyadic modulus of continuity of $f \in X[0,1)$ is defined by

$$\omega(\delta;f;X) := \sup_{0\leq h<\delta} \|f(\cdot) - f(\cdot \oplus h)\|_X$$

and the dyadic Lipschitz class of order $\beta > 0$ by

$$\mathrm{Lip}(\beta;X) := \{f \text{ [illegible] } ;X) = \mathcal{O}(\delta^\beta),\ \delta \to 0\}.$$

Denoting by $\mathcal{P}_n$ the [illegible] polynomials of degree $\leq n$, i.e., of all $p_n(x) = \sum_{k=0}^{n-1} c_k\, \psi_k(x$ [illegible], then the best approximation of $f \in X[0,1)$ by $p_n \in \mathcal{P}_n$ is [illegible] by

$$E_n(f;X) := \inf_{p_n \in \mathcal{P}_n} \|f(\cdot) - p_n(\cdot)\|_X .$$

There exists a $p_n^* \in \mathcal{P}_n$ such that $E_n(f;X) = \|f - p_n^*\|_X$. Basic is the following identity which easily follows by applying the Walsh-Fourier transform to both sides together with the uniqueness theorem.

Lemma 13. *For* $f \in X^{\{r\}}$, $h \in [0,2^{-n})$, $n \in \mathbb{N}$, $r \in \mathbb{N}$, *one has*

$$f(x) - f(x\oplus h) = W_r^{*(n)} * (\mathcal{E}^{\{r\}}f(\cdot) - \mathcal{E}^{\{r\}}f(\cdot\oplus h))(x) \text{ a.e.}$$

Theorem 7. *Let* $f \in X^{\{r\}}$, $r \in \mathbb{N}$.

a) $\omega(\delta;f;X) = \mathcal{O}([\log_2 \delta^{-1}]^{2r}\, \delta^r\, \omega(\delta;\mathcal{E}^{\{r\}}f;X)) \qquad (\delta \to 0+).$

b) $\omega(\delta;f;X) = \mathcal{O}([\log_2 \delta^{-1}]^{2r}\, \delta^r \|\mathcal{E}^{\{r\}}f\|_X \qquad (\delta \to 0+).$

c) $f^\wedge(k) = \mathcal{O}([\log_2 k]^{2r}\, k^{-r}\, \omega(k^{-1};\mathcal{E}^{\{r\}}f:X)) \qquad (k \to \infty).$

d) If, in addition, $\mathcal{E}^{\{r\}}f \in \mathrm{Lip}(\beta;X)$, $\beta > 0$, then

$$f^{\wedge}(k) = \mathcal{O}([\log_2 k]^{2r} k^{-r-\beta}) \qquad (k \to \infty).$$

To prove a), one has by Lemmas 6 and 13,

$$\|f(\cdot) - f(\cdot \oplus h)\|_X \leq \|W_r^{*(n)}\|_1 \, \|\mathcal{E}^{\{r\}}f(\cdot) - \mathcal{E}^{\{r\}}f(\cdot \oplus h)\|_X$$

$$= \mathcal{O}\left(n^{2r}2^{-nr} \, \omega(\delta;\mathcal{E}^{\{r\}}f;X)\right)$$

which leads to the desired estimate for $2^{-n} < \delta < 2^{-n+1}$. Part b) is immediat by a), and c) follows by the well-known inequality $|f^{\wedge}(k)| \leq (1/2)\, \omega(k^{-1};f;X)$ $k \in \mathbb{N}_0$.

If one compares the above estimates with the corresponding ones of [10 for the standard dyadic derivative, one sees that in that case the orders ar better. In particular, if the standard dyadic derivative $D^{\{r\}}f$ belongs t $\mathrm{Lip}(\beta;X)$, then the counterpart of part d) reads $f^{\wedge}(k) = \mathcal{O}(k^{-r-\beta})$; it does no contain the multiplicative factor $(\log_2 k)^{2r}$, which corresponds to the facto m^{2r} in the basic Lemma 6. Nevertheless, the main order, namely $k^{-r-\beta}$ dominates the factor $(\log_2 k)^{2r}$ for large k.

Theorem 7 in case r is replaced by $\alpha \in \mathbb{R}^+$ is to be found in [4, 5] There, however, in the power 2r of the $\log_2$-function the r, thus the there, is missing. This remark applies in [5] to Theorems 4.1, 4.3, and 4.4 Lemmas 4.1, 4.3 and Corollary 4.1. This is due to the fact that the proof o Lemma 5.5 [4], used in these results, is not complete. In fact, the estimat (5.11) [4] should read

$$\|V_m^{\{\alpha\}} - V_\infty^{\{\alpha\}}\|_1 = \mathcal{O}(m^{\alpha}(\log_2 m)^{2\alpha}) \qquad (m \to \infty).$$

(Observe the additional α in the power of $\log_2 m$). In the terminology of th present paper it is the estimate

$$\|W_\alpha^{*(m)}\|_1 = \mathcal{O}(m^{\alpha}2^{-m\alpha})$$

established for $\alpha = r$ in Lemma 6.

In the present frame the Bernstein and Jackson-type inequalities read

Lemma 14. a) *For* $p_n \in \mathcal{P}_n$, $r \in \mathbb{N}$ *one has*

$$\|\mathcal{E}^{\{r\}} p_n\|_X \le C\, n^r \|p_n\|_X \qquad (n \in \mathbb{N}), \tag{8.1}$$

C being a constant independent of n, r, p_n and f.

b) *If* $f \in X^{\{r\}}$, *then*

$$E_n(f;X) = \mathcal{O}\left((\log_2 n)^{2r} \frac{1}{n^r} \|\mathcal{E}^{\{r\}} f\|_X\right) \qquad (n \to \infty). \tag{8.2}$$

Observe that the order in (8.1) does not match (8.2) in the standard sense in view of the additional factor $(\log_2 n)^{2r}$ in (8.2), which does not occur in standard dyadic analysis ([see e.g. 10, 41]). But inequality (8.1) has its normal form.

Now to the counterparts of the theorems of Jackson, Steckin and Lebesgue in approximation theory and the theory of Fourier series, respectively.

Theorem 8. a) *If* $f \in X^{\{r\}}$, $r \in \mathbb{N}$, *then*

$$E_n(f;X) = \mathcal{O}\left((\log_2 n)^{2r} \frac{1}{n^r}\, \omega(n^{-1}; \mathcal{E}^{\{r\}} f; X)\right) \qquad (n \to \infty).$$

In particular, if further $\mathcal{E}^{\{r\}} f \in \mathrm{Lip}(\beta;X)$, $\beta > 0$, *then*

$$E_n(f;X) = \mathcal{O}((\log_2 n)^{2r}\, n^{-r-\beta}) \qquad (n \to \infty). \tag{8.3}$$

b) *In the converse direction, if (8.3) holds, then* $\mathcal{E}^{\{j\}} f$ *exists, belongs to* $X[0,1)$ *for* $0 \le j \le r$, *and*

$$\|\mathcal{E}^{\{j\}} f - \mathcal{E}^{\{j\}} p_n^* \|_X = \mathcal{O}\left((\log_2 n)^{2r} \frac{1}{n^{r+\beta-j}}\right) \qquad (n \to \infty).$$

c) *If* $f \in X^{\{r\}}$, $r \in \mathbb{N}$, *then*

$$\|S_n f - f\|_X = \mathcal{O}\left((\log_2 n)^{2r} \frac{1}{n^r}\, \omega(\tfrac{1}{n}; \mathcal{E}^{\{r\}} f; X)\right) \qquad (n \to \infty).$$

In particular, if further $\mathcal{E}^{\{r\}}f \in \mathrm{Lip}(\beta;X)$, $\beta > 0$, *then*

$$\|S_n f - f\|_X = \mathcal{O}((\log_2 n)^{2r} n^{-r-\beta}).$$

Lemma 14 and Theorem 8 follows by the standard techniques and results in Walsh analysis when observing Theorem 7.

Again note the additional factor $(\log_2 n)^{2r}$ in the estimates of parts a), b) and c) above.

Concerning part b), a further open question is whether condition (8.3) implies that $\mathcal{E}^{\{r\}}$ satisfies some type of Lipschitz condition.

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THE GIBBS DERIVATIVE AND TERM BY TERM DIFFERENTIATION OF WALSH SERIES

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Abstract. The problem of term by term pointwise (Gibbs) differentiation of a Walsh series was introduced by Butzer and Wagner in 1975. They asked under what conditions on the coefficients a_k is a Walsh series term by term differentiable. This problem has proved to be an interesting one and we shall survey efforts made to resolve it. We shall also include many concrete examples.

The techniques developed in response to this problem can be divided into two broad classes: remainder techniques and interchange arguments. We begin by illustrating these techniques for the lacunary case where the whole thing is quite simple. We then look at solutions to the problem for general Walsh series, dividing the results according to whether global or local differentiability is obtained. We then look at the problem of term by term strong differentiability and report results in this direction for the first time. We close with several open questions concerning term by term differentiability and discuss their relationship to growth conditions for Walsh-Fourier coefficients and the Walsh-Fourier transform.

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§1. Introduction

We assume the reader is familiar with the Walsh-Paley system $w_0, w_1, \dots$ (a complete orthonormal system defined on the unit interval $[0,1)$ which contains the Rademacher functions), and with dyadic addition $\dot{+}$ (a commutative, associative, binary operation defined on $[0,1)$) which enjoys the relation

$$w_k(x \dot{+} y) = w_k(x)w_k(y) \tag{1}$$

for $k = 0, 1, \dots,$ $x, y \in [0,1)$, and $x \dot{+} y$ not a dyadic rational. We also assume that the reader knows elementary facts about Walsh-Fourier series, Walsh-Fourier coefficients, and the dyadic group. Sufficient information on these subjects can be found in Fine [12].

Each Walsh function takes on only the values $+1$ and -1 and has finitely many discontinuities. Consequently, classical differentiation cannot distinguish one Walsh function from another. Gibbs (see [13], for example) introduced a derivative which overcame this difficulty. Butzer and Wagner [8], [9] modified the Gibbs derivative and introduced this concept to the mathematical world as the dyadic derivative. The strong (dyadic) derivative was discussed in [8] and the pointwise dyadic derivative was discussed in [9]. Their definitions were essentially the following ones. Given a function f defined at points in $[0,1)$ set

$$d_n f(x) = \sum_{\ell=0}^{n-1} 2^{\ell-1} \left(f(x) - f(x \dot{+} 2^{-\ell-1})\right)$$

for $x \in [0,1)$ and $n = 1, 2, \dots$. Let $\mathbf{X}$ be a metrizable topological vector space of functions which are defined a.e. on $[0,1)$ such that the distance between two functions in $\mathbf{X}$ which equal each other a.e. is zero. The function f is said to be *strongly differentiable* in $\mathbf{X}$ if $f,\ d_n f \in \mathbf{X}$ for all $n \geq 1$ and

$$d^{[1]} f := \lim_{n\to\infty} d_n f$$

exists in $\mathbf{X}$. The function f is said to be *dyadically differentiable* at a point $x \in [0,1)$ if f is defined at x and $x \dot{+} 2^{-j}$ for $j = 1, 2, \dots$ and

$$f^{[1]}(x) := \lim_{n\to\infty} d_n(f, x)$$

exists and is finite. The function $d^{[1]} f$ is called the *strong derivative* of f in $\mathbf{X}$ and $f^{[1]}(x)$ is called the *dyadic derivative* of f at x. Butzer and Wagner showed that each Walsh function is always strongly differentiable and everywhere dyadically differentiable with

$$d^{[1]} w_k = w_k^{[1]} = k w_k \tag{2}$$

for $k = 1, 2, \dots$. They also showed that the Walsh functions are the non-trivial solutions of a first-order linear differential equation with respect to the strong derivative.

We shall say that a Walsh series

$$f(t) := \sum_{k=0}^{\infty} a_k w_k(t) \tag{3}$$

is *term by term strongly differentiable* in the space $\mathbf{X}$ if (3) converges in some sense to a function f which is strongly differentiable in $\mathbf{X}$ and satisfies

$$d^{[1]}f = \sum_{k=0}^{\infty} k a_k w_k,$$

in the topology of $\mathbf{X}$. Similarly, we say that f is *term by term (dyadically) differentiable* at a point $x \in [0,1)$ if f converges at x, and $x \dot{+} 2^{-j}$ for $j = 1, 2, \ldots$, is dyadically differentiable at x and

$$f^{[1]}(x) = \sum_{k=0}^{\infty} k a_k w_k(x).$$

We shall refer to the series $\sum_{k=1}^{\infty} k a_k w_k$ as the *derived series.*

The problem of term by term dyadic differentiation was introduced by Butzer and Wagner in [9]. They asked under what conditions on the coefficients a_k is the series f given by (3) term by term differentiable. This problem has proved to be an interesting one and we shall discuss efforts made to resolve it. Our discussion will be organized as follows. In §2 we examine the lacunary case. The next two sections contain a survey of known results concerning term by term pointwise differentiability: §3 deals with global differentiability and §4 deals with local differentiability. In §5 we discuss term by term strong differentiation and in §6 we raise some open problems and unanswered questions.

§2. The lacunary case

There are two basic techniques which have been used on the problem of term by term dyadic differentiation: the remainder term and the interchange argument. In this section we shall illustrate these two techniques in the simple case of lacunary Walsh series.

The first technique centers on a calculation which estimates the remainder term

$$\mathcal{R}_n(x) := d_n f(x) - \sum_{k=0}^{\infty} k a_k w_k(x).$$

It then remains to find hypotheses which imply that $\mathcal{R}_n(x)$ tends to zero as $n \to \infty$.

The second technique rests on finding hypotheses which allow an interchange of limit and summation. For example, the following result appeared in [23].

LEMMA 2.1. *Let $b_k^{(n)}$, b_k, and x_k be real numbers for $n, k = 0, 1, \ldots$, and suppose*

$$\sum_{k=0}^{\infty} x_k$$

converges to a finite real number. If $b_k^{(n)} \to b_k$ as $n \to \infty$ and there is an absolute constant $M > 0$ such that

$$\sum_{k=0}^{\infty} \left| b_k^{(n)} - b_{k+1}^{(n)} \right| \le M < \infty$$

for $n = 0, 1, \ldots$ *then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} b_k^{(n)} x_k = \sum_{k=0}^{\infty} b_k x_k$$

exist and are finite.

Such results are used to verify

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_k d_n w_k(x) = \sum_{k=0}^{\infty} a_k \lim_{n\to\infty} d_n w_k(x).$$

If f is defined at x and $x \dot{+} 2^{-\ell}$ for $\ell = 1, 2, \ldots$, then $d_n f(x) = \sum a_k d_n w_k(x)$. Thus we see by (2) that such an interchange implies term by term dyadic differentiation of the Walsh series (3).

The Walsh series f is called *lacunary* if there exist integers $k_1, k_2, \ldots$ and a number $q > 1$ such that

(4) $$k_{j+1}/k_j \geq q$$

for $j = 1, 2, \ldots$ and

$$f = \sum_{j=1}^{\infty} a_{k_j} w_{k_j}.$$

Since the Rademacher functions $r_0, r_1, \ldots$ satisfy

$$r_j = w_{2^j}$$

for $j = 0, 1, \ldots$ it is clear that every Rademacher series is a lacunary Walsh series. It is also clear that a Rademacher series

(5) $$g(t) := \sum_{j=0}^{\infty} c_j r_j$$

has $\sum_{j=1}^{\infty} 2^j c_j r_j(x)$ as its derived series.

The remainder term of (5) is easy to analyze. Suppose that g converges at x and $x \dot{+} 2^{-\ell-1}$ for $\ell = 0, 1, \ldots$. Fix $\ell \geq 0$. By (1) we have

$$\begin{aligned} g(x \dot{+} 2^{-\ell-1}) &= \sum_{j=0}^{\infty} c_j r_j(x \dot{+} 2^{-\ell-1}) \\ &= \sum_{j=0}^{\infty} c_j r_j(x) r_j(2^{-\ell-1}). \end{aligned}$$

But

$$r_j(2^{-\ell-1}) = \begin{cases} -1 & \text{for } j = \ell \\ 1 & \text{otherwise.} \end{cases}$$

Hence $g(x) - g(x \dot{+} 2^{-\ell-1}) = 2c_\ell r_\ell(x)$ for $\ell = 0, 1, \ldots$. It follows that

$$d_n g(x) = \sum_{\ell=0}^{n-1} 2^\ell c_\ell r_\ell.$$

In particular, $\mathcal{R}_n(x) \equiv \sum_{\ell=n}^{\infty} 2^\ell c_\ell r_\ell$ and we have proved the following result.

THEOREM 2.1. *If the Rademacher series g given (5) converges at x and $x \dot{+} 2^{-\ell-1}$ for $\ell = 0, 1, \ldots$ then g is term by term dyadically differentiable at x if and only if the derived series $\sum 2^j c_j r_j$ converges at x.*

This theorem is due to Onneweer [16]. In fact, Onneweer investigated term by term dyadic differentiation of Rademacher series on any compact group **G** which is the direct product of cyclic groups of order p_n for any sequence of primes $\{p_n\}$. He showed that a given Rademacher series which converges absolutely on **G** is term by term differentiable at a point $x \in$ **G** if and only if the derived series converges at x. This reduces to Theorem 2.1 in the case when $p_n \equiv 2$ for $n = 1, 2, \ldots$.

To illustrate the interchange argument we pass to the full lacunary case. For each pair of non-negative integers $n, k \geq 0$ let $\langle k \rangle_n$ represent the integer congruent to k modulo 2^n, i.e., let $p := \langle k \rangle_n$ be determined by $0 \leq p < 2^n$ and $k = \ell 2^n + p$ for some integer $\ell \geq 0$. Recall from Butzer and Wagner [9] that

$$d_n w_k = \langle k \rangle_n w_k \tag{6}$$

for $n, k = 0, 1, \ldots$.

Let $\sum_{j=0}^{\infty} a_{k_j} w_{k_j}$ be a lacunary Walsh series which converges at x and $x \dot{+} 2^{-\ell-1}$, for $\ell = 0, 1, \ldots$, whose derived series converges at x. Use (6) to write

$$d_n \left(\sum_{j=0}^{\infty} a_{k_j} w_{k_j} \right) = \sum_{j=0}^{\infty} \frac{\langle k_j \rangle_n}{k_j} k_j a_{k_j} w_{k_j}.$$

Since $\{k_j\}$ satisfies (4), an easy calculation verifies

$$\sum_{j=1}^{\infty} \left| \frac{\langle k_j \rangle_n}{k_j} - \frac{\langle k_{j+1} \rangle_n}{k_{j+1}} \right| \leq 1 + 2\left(\frac{q}{q-1} \right)$$

for $n = 1, 2, \ldots$. Thus by Lemma 2.1 we have proved the following.

THEOREM 2.2. *Suppose the Walsh series f given by (3) is lacunary. Let $x \in [0, 1)$ and suppose f converges at $t = x$ and $t = x \dot{+} 2^{-\ell-1}$ for $\ell = 0, 1, \ldots$. If the derived series converges at x then f is term by term dyadically differentiable at x.*

A generalization of this result will appear in [24]. Thus the problem of term by term dyadic differentiation of lacunary Walsh series is closed.

§3 GLOBAL DIFFERENTIABILITY

The first results on term by term dyadic differentiation were obtained by Butzer and Wagner [9]. By an interchange argument they proved

THEOREM 3.1. *If $\sum_{k=1}^{\infty} k|a_k| < \infty$ then the Walsh series f given by (3) is term by term differentiable everywhere on $[0, 1)$.*

If the coefficients are monotone the growth condition in Theorem 3.1 can be relaxed. In [23] we find an estimate of the remainder term which verifies the following.

THEOREM 3.2. *If $a_k \downarrow 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} |a_k| < \infty$ then (3) is term by term differentiable everywhere on* $(0,1)$.

Thus

$$f(t) = \sum_{k=1}^{\infty} k^{-\alpha} w_k(t)$$

is term by term differentiable everywhere on $[0,1)$ when $\alpha > 2$ and on $(0,1)$ when $\alpha > 1$. (This series is nowhere term by term differentiable when $\alpha = 1$ because the derived series fails to converge in this case).

Recall that a sequence $\{b_k\}$ is quasi-convex if

$$\sum_{k=1}^{\infty} (k+1)|b_{k+2} - 2b_{k+1} + b_k| < \infty.$$

Concerning a.e. term by term differentiability, Butzer and Wagner [9] proved

THEOREM 3.3. *If $\{a_k\}$ and $\{ka_k\}$ are quasi- convex and $ka_k \to 0$ and $k \to \infty$ then (3) is term by term differentiable a.e. on* $[0,1)$.

In connection with Theorem 3.3 Butzer and Wagner conjectured that term by term differentiability would still be possible if quasi- convexity were replaced by $ka_k \downarrow 0$ as $k \to \infty$. This conjecture was verified by Schipp [27] who estimated the remainder term with $d_n\left(\sum_{k=2^n}^{\infty} a_k w_k\right)$. He proved:

THEOREM 3.4. *If $ka_k \downarrow 0$ as $k \to \infty$ then (3) is term by term differentiable at any point $x \neq 2^{-i}$ for $i = 1, 2, \ldots$.*

It follows that

$$f(t) = \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{\alpha}} w_k(t)$$

is term by term differentiable for every $\alpha > 0$ at every point $t \neq 2^{-i}$ for $i = 1, 2, \ldots$.

A different form of the remainder term was used in [29] which resulted in the following. (Here and elsewhere the empty sum is defined to be zero).

THEOREM 3.5. *If*

$$2^n \left(|a_{2^n}| + |a_{2^{n+1}-1}| + \sum_{k=2^n}^{2^{n+1}-2} |a_k - a_{k+1}| \right) \to 0 \quad \text{as} \quad n \to \infty \tag{7}$$

and the derived series converges at some point $x \neq 2^{-i}, i = 1, 2, \ldots$ then (3) is term by term dyadically differentiable at x.

Notice that the hypotheses of Theorem 3.5 are met by coefficients which satisfy $ka_k \downarrow 0$ as $k \to \infty$. Thus Theorem 3.5 contains Theorem 3.4 Notice also that the hypotheses are met by coefficients which satisfy $a_k \geq a_{k+1}$ for $2^n \leq k < 2^{n+1} - 1$ and n large, and

$2^k a_k \to 0$ as $k \to \infty$. Thus Theorem 3.5 applies in situations where $ka_k \downarrow 0$ fails and where $\sum |a_k| = \infty$. For example the series

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=2^n}^{2^{n+1}-1} w_k(t)$$

is term by term differentiable at all points $x \neq 2^{-i}$ for $i = 1, 2, \ldots$.

A quasi-convex sequence which converges to zero is necessarily of bounded variation. Thus the conditions of Theorem 3.3 imply that $\{ka_k\}$ is of bounded variation. If we assume a little bit more we can once again obtain everywhere differentiability (see [23]):

THEOREM 3.6. *If $\{k^\alpha a_k\}$ is of bounded variation for some $\alpha > 1$ then (3) is term by term differentiable everywhere on $(0, 1)$.*

Thus

$$f(t) = \sum_{k=2}^{\infty} \frac{1}{k^\alpha (\log k)^\beta} w_k(t)$$

is everywhere term by term differentiable on $(0, 1)$ when $\alpha > 1$ and $\beta > 0$.

§4. LOCAL DIFFERENTIABILITY

The object is to identify conditions sufficient to conclude that f is term by term differentiable at a particular point. We have mentioned two results of this type above: Theorem 2.2 and Theorem 3.5.

In [23] an effort was made to get away from conditions which imply that $\{ka_k\}$ is of bounded variation. Using Lemma 2.1 and the interchange argument, it was shown that

THEOREM 4.1. *Let $x \in [0, 1)$. Suppose for some $\alpha \equiv \alpha(x) > 1$ that*

$$\sum_{k=1}^{\infty} k^\alpha a_k w_k(x)$$

converges to a finite real number. Then the Walsh series f given by (3) is term by term differentiable at x.

Thus

$$h(t) = \sum_{k=1}^{\infty} \frac{a_k}{k^\alpha} w_k(t) \tag{8}$$

is term by term differentiable for $\alpha > 1$ at any point where f converges. In particular, if $\{a_k\}$ are the Fourier coefficients of some $f \in L^p[0, 1)$ for $p > 1$ then (8) is a.e. term by term differentiable on $[0, 1)$.

Since a Walsh series with coefficients of bounded variation necessarily converges on $(0, 1)$ (a simple argument using Abel's transformation), it is clear that Theorem 4.1 contains Theorem 3.6. It is also clear for the case $\alpha > 2$ that Theorem 4.1 is a corollary of Theorem 3.1.

It turns out that the crucial properties for Theorem 4.1 are that $\{k^\alpha\}$ is monotone non-decreasing and its reciprocal is summable. Indeed, the following will appear in [25].

THEOREM 4.2. *Let $0 < \beta_0 \leq \beta_1 \leq \ldots$ be real numbers with $\sum_{k=0}^{\infty} 1/\beta_k < \infty$. If*

$$\sum_{k=0}^{\infty} a_k \beta_k w_k(x)$$

converges at some point $x \in [0,1)$ then (3) is term by term differentiable at x.

Thus if

$$\sum_{k=2}^{\infty} k(\log k)^{\alpha} a_k w_k(x) \tag{9}$$

converges to a finite real number for some $\alpha > 1$, then (3) is term by term differentiable at x.

One way to interpret Theorem 4.2 is that if the derived series converges fast enough at x then (3) is term by term differentiable at x. The following result, which will appear in [25], follows up this line of investigation.

THEOREM 4.3. *Let $\gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_k \geq \cdots \geq 0$ be real numbers. Let $x \in [0,1)$ and suppose that the function f given by (3) exists and is finite at $t = x + 2^{-j}$ for $j = 1, 2, \ldots$.*

a) *If*

$$\sum_{k=0}^{\infty} \gamma_k < \infty$$

and

$$\sum_{k=m}^{\infty} a_k w_k(x) = o(\gamma_m) \quad \text{as} \quad m \to \infty$$

then f is term by term differentiable at x.

b) *If*

$$\sum_{k=0}^{\infty} \frac{\gamma_k}{k} < \infty$$

and

$$\sum_{k=m}^{\infty} k a_k w_k(x) = o(\gamma_m) \quad \text{as} \quad m \to \infty$$

then f is term by term differentiable at x.

In particular, if

$$\sum_{k=m}^{\infty} a_k w_k(x) = o\left(\frac{1}{m(\log m)^{\alpha}}\right) \quad \text{as} \quad m \to \infty \tag{10}$$

for some $x \in [0,1)$ and $\alpha > 1$ then (3) is term by term differentiable at x.

It is interesting to note that Theorem 4.3 is a generalization of Theorem 4.2 (see [25]).

§5 Strong differentiation

The problem of term by term strong differentiation seemed closed by the initial work of Butzer and Wagner [8]. They showed that if $\mathbf{X}$ is a Banach space of integrable functions whose norm satisfies three conditions for all $f, g \in \mathbf{X}$:

$$\|\tau_j f\| = \|f\|, \tag{11}$$

where $\tau_j f(x) := f(x \dotplus 2^{-j})$ for $j = 1, 2, \ldots,$

$$\|f\|_1 \le \|f\| \quad , \tag{12}$$

and

$$\|f * g\| \le \|f\| \, \|g\|_1, \tag{13}$$

where $f * g(x) := \int_0^1 f(x \dotplus t) g(t)\, dt$, then the Walsh-Fourier series of an integrable f is term by term strongly differentiable in $\mathbf{X}$ if both $S[f]$ and the derived series $\sum_{k=0}^{\infty} k\hat{f}(k) w_k$ converge in the norm of $\mathbf{X}$. Their proof rested on characterizing the strong derivative in the following way. A function f is strongly differentiable in a Banach space $\mathbf{X}$ whose norm satisfies (11), (12), and (13) if and only if there exists a function $g \in \mathbf{X}$ whose Walsh-Fourier coefficients satisfy $\hat{g}(k) = k\hat{f}(k)$ for $k = 1, 2, \ldots.$

Notice when (12) holds, that any Walsh series which converges in $\mathbf{X}$ is necessarily a Walsh-Fourier series. Thus Butzer and Wagner showed

Theorem 5.1. *Let $\mathbf{X}$ be a Banach space which satisfies (11), (12), and (13). If the Walsh series f given by (3) and its derived series converge in $\mathbf{X}$ then f is term by term strongly differentiable in $\mathbf{X}$.*

If $\mathbf{X}$ is a Banach space which does not satisfy (11), (12), and (13) then this characterization no longer holds and the problem of term by term strong differentiation comes alive again.

A quasi-normed linear space is a vector space $\mathbf{X}$ together with a positive definite function $\|.\| : \mathbf{X} \to [0, \infty)$ such that

$$\|x + y\| \le \|x\| + \|y\|, \qquad \|-x\| = \|x\|,$$

$$\lim_{\alpha_n \to 0} \|\alpha_n x\| = 0, \quad \text{and} \quad \lim_{\|x_n\| \to 0} \|\alpha x_n\| = 0$$

for $x, x_n, y \in \mathbf{X}$ and scalars α_n. A quasi-normed linear space is called complete if its Cauchy sequences converge. (See Yosida [33], for example).

We shall call a quasi-normed linear space $\mathbf{X}$ *monotone (by means of Ψ)* if there is a finite-valued non-decreasing subadditive function Ψ on $[0, \infty)$ such that $\Psi(\alpha) \to 0$ as $\alpha \downarrow 0$,

$$\Psi(|\alpha\beta|) \le \Psi(|\alpha|)\Psi(|\beta|)$$

and

$$\|\alpha x\| \le \Psi(|\alpha|)\|x\|$$

for $x \in \mathbf{X}$ and scalars α and β. We shall call $\mathbf{X}$ *appropriate* (for dyadic differentiation) if $\mathbf{X}$ contains the Walsh functions, every element of $\mathbf{X}$ is a function defined a.e. on $[0,1)$, $\tau_j f \in \mathbf{X}$ for $f \in \mathbf{X}$ and $j = 1, 2, \dots$, and $\|f\| = 0$ if and only if $f = 0$ a.e. on $[0,1)$. Thus if the Walsh series f given by (3) converges in $\mathbf{X}$ and $\mathbf{X}$ is appropriate then $d_n f \in \mathbf{X}$ and by (6) we have that

$$d_n f = \sum_{k=1}^{\infty} \langle k \rangle_n a_k w_k$$

converges in $\mathbf{X}$. This allows one to use the interchange argument on any appropriate quasi-normed linear space where results such as Lemma 2.1 have been established. Such results were obtained in [25] for all monotone, appropriate complete quasi- normed linear spaces. This eventuated in the following analogues of Theorems 4.2 and 4.3.

THEOREM 5.2. *Let $\mathbf{X}$ be an appropriate complete quasi-normed linear space which is monotone by means of Ψ. Let $0 < \beta_0 \le \beta_1 \le \dots$ be real numbers with $\sum_{k=0}^{\infty} \Psi(1/\beta_k) < \infty$. If the Walsh series f given by (3) and the series*

$$\sum_{k=0}^{\infty} a_k \beta_k w_k$$

converge in $\mathbf{X}$ then f is term by term strongly differentiable in $\mathbf{X}$.

THEOREM 5.3. *Let $\mathbf{X}$ be an appropriate complete quasi-normed linear space which is monotone by means of Ψ. Let $\gamma_0 \ge \gamma_1 \ge \dots \ge \gamma_k \ge \dots \ge 0$ be real numbers, and suppose that the Walsh series f given by (3) converges in $\mathbf{X}$.*

a) *If*

$$\sum_{k=0}^{\infty} \Psi(\gamma_k) < \infty$$

and

$$\|\sum_{k=m}^{\infty} a_k w_k\| = o(\Psi(\gamma_m)) \qquad \text{as} \quad m \to \infty$$

then f is term by term strongly differentiable in $\mathbf{X}$.

b) *If*

$$\sum_{k=0}^{\infty} \Psi\left(\frac{\gamma_k}{k}\right) < \infty$$

and

$$\|\sum_{k=m}^{\infty} k a_k w_k\| = o(\Psi(\gamma_m)) \qquad \text{as} \quad m \to \infty$$

then f is term by term strongly differentiable in $\mathbf{X}$.

It should be pointed out that in this setting Theorem 5.2 is not a corollary of Theorem 5.3.

What kind of spaces satisfy the hypotheses of Theorems 5.2 and 5.3? Clearly every Banach space is monotone by means of $\Psi(\alpha) := \alpha$ and every Banach space which satisfies (11), (12), and (13) is appropriate. Thus Theorems 5.2 and 5.3 apply to the classical Banach spaces $L^p[0,1)$, $BMO[0,1)$ and $H^p[0,1)$ for $1 \le p < \infty$. Theorems 5.2 and 5.3 also apply (see [25]) to the quasi-normed linear spaces $L^p[0,1)$ for $0 < p < 1$ and to the block spaces $\mathcal{B}_q$, $q \ge 1$, which were introduced by Taibleson and Weiss [30]. In particular, (3) is term by term strongly differentiable in $L^p[0,1)$, $0 < p < 1$, when

$$m^{\alpha/p}\|\sum_{k=m}^{\infty} a_k w_k\|_p^p \to 0,$$

as $m \to \infty$ and is term by term strongly differentiable in a block space $\mathcal{B}_q$ for some $q \ge 1$ when

$$\frac{m^\alpha}{\log m}\|\sum_{k=m}^{\infty} a_k w_k\|_{\mathcal{B}_q} \to 0$$

as $m \to \infty$ for some $\alpha > 1$.

§6 Unanswered questions

Can the condition $\sum 1/\beta_k < \infty$ in Theorem 4.2 be relaxed? An affirmative answer would be provided if one could show that (3) is term by term differentiable at x when

$$(14) \qquad |\sum_{k=1}^{\infty} k \log k a_k w_k(x)| < \infty$$

Similarly concerning Theorem 4.3 the question arises, is (3) term by term differentiable at x when

$$\sum_{k=m}^{\infty} a_k w_k(x) = o(m \log m) \quad \text{as} \quad m \to \infty?$$

By Theorem 3.1, (3) is term by term differentiable on $[0,1)$ when the derived series converges absolutely on $[0,1)$. It is natural to ask the following question. If the derived series converges everywhere on $[0,1)$ is (3) term by term differentiable at some point in $[0,1)$? Perhaps (3) is term by term differentiable a.e. on $[0,1)$ or at least on a dense $\mathcal{G}_\delta$ set. An easier question along these lines is the following one of a probabilistic nature. If the derived series converges at x, is $\sum_{k=0}^{\infty} \pm a_k w_k(x)$ term by term differentiable for almost all choices of the signs $\pm$?

One of the curiosities of Walsh-Fourier analysis is that unlike the trigonometric case, the Walsh-Fourier coefficients of a smooth function cannot converge too rapidly to zero. Fine [12] was first to notice this phenomenon by showing that if f is absolutely continuous and its Walsh-Fourier coefficients satisfy $\hat{f}(k) = o(1/k)$ as $k \to \infty$ then f is constant on the interval $[0,1)$. Several authors have identified conditions on the Walsh-Fourier coefficients of a continuous function f which are sufficient to conclude f is constant. These conditions regularly show up as hypotheses of theorems about pointwise dyadic differentiability. For example, the conditions of Theorems 3.1 and 3.5 were first considered by Coury [10] in this context.

This connection between pointwise dyadic differentiability and constant continuous functions is no accident. In the first place, no non-constant continuous function can be dyadically differentiable at all but countably many points on $[0,1)$ (see [29]). Thus when Bočkarev [2] gives an example of a continuous non-constant function whose Walsh-Fourier coefficients satisfy

$$\hat{f}(k) = O\left(\frac{1}{k \log k}\right) \quad \text{as} \quad k \to \infty$$

he provides an example of a Walsh series f whose coefficients satisfy $a_k = O(1/(k \log k))$ as $k \to \infty$ which is not term by term differentiable at all but countably many points on $[0,1)$.

In the second place, Engels [11] has shown that if f is a bounded function which has at most countably many discontinuities on the interval $[0,1)$ (all exclusively of the first kind), and these discontinuity points have at most a finite number of cluster points, then f is dyadically differentiable at all but countably many points in $[0,1)$ if and only if f is piecewise constant on $[0,1)$. Thus it does not seem unreasonable to conjecture that any condition on the coefficients of f sufficient to conclude that f is constant when continuous is also sufficient to conclude that f is term by term differentiable, whether f is continuous or not.

Here are some special cases worth considering. Bočkarev ([1] and [2]) proved that if g is continuous and its Walsh-Fourier coefficients satisfy $|\hat{g}(k)| \le \delta_k$ for some sequence $\delta_k \downarrow 0$ with $\sum_{k=1}^{\infty} \delta_k < \infty$ then g is constant. Is (3) term by term dyadically differentiable at all but countably many points in $[0,1)$ when

$$|a_k| \le \delta_k \downarrow 0 \quad \text{as} \quad k \to \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \delta_k < \infty?$$

A less ambitious project is suggested by a corollary of Bočkarev's result. Is (3) term by term dyadically differentiable at all but countably many points in $[0,1)$ when

$$a_k = O\left(\frac{1}{k \log k^{\alpha}}\right) \quad \text{as} \quad k \to \infty \tag{15}$$

for some $\alpha > 1$? Notice that condition (14) is stronger than (15) for $\alpha = 1$ but weaker than (15) for any $\alpha > 1$. Thus an affirmative answer to the first question in this section would be an improvement of Bočkarev's result. Other growth conditions to check are provided by [31] and [32]. Specifically, is (3) term by term differentiable at all but countably many points in some interval $[a,b)$ when any one of the following conditions is satisfied?

$$\lim_{N \to \infty} 2^N \sum_{n=N+1}^{\infty} \sum_{\ell=1}^{2^{n-N-1}} \sum_{k=2^n+(2\ell-1)2^N}^{2^n+2\ell 2^N} a_k w_k \quad \text{converges everywhere on} \quad [a,b), \tag{16}$$

or

$$\sup_{N\geq 0} \left| \sum_{n=0}^{N} \sum_{k=2^n}^{2^{n+1}-1} 2^n a_k w_k \right| \quad \text{is finite valued on } [a, b).$$

Notice that condition (16) is stronger than condition (7) when a=0 and b=1. Thus an affirmative answer to this last question would extend Theorem 3.5 and would provide a local version of it as well.

With the exception of generalized Rademacher series (see Onneweer [16]), the problem of term by term differentiation of Vilenkin series has not been touched. All of the results cited in §3 go over to the special case of the group of integers of a p-series field, but the general problem is wide open. The requisite tool of a fundamental theorem for the bounded case is provided by Pál and Simon [22] and a several definitions have been provided and compared by Onneweer [17], [18].

Another problem which has not yet been examined is term by term dyadic differentiation of multiple Walsh series. Theoretical foundations of the dyadic derivative in several variables have been provided by Butzer and Engels [3], [4] and Móricz [15] has conditions sufficient to conclude a double Walsh series is the Walsh-Fourier series of some integrable function. Moreover, a fundamental theorem is now available in this setting (see [28]).

What about an analogue of these results on the dyadic field? The analogous problem would be what conditions on the Walsh transform F of a function f integrable on $\mathbf{R}^+$ imply that f is dyadically differentiable and that derivative is the Walsh transform of $F^{[1]}$? Again the background work has been done. The dyadic derivative on $\mathbf{R}^+$ was introduced long ago by Butzer and Wagner [7] and the fundamental theorem is well-known (see Pál [20], [21]).

Finally, Butzer, Engels, and Wipperfürth [5], [6] have introduced the ED-derivative. This is an extension of the dyadic derivative which is not only applicable to piecewise constant functions but also to piecewise polynomial functions. Since more functions are ED-differentiable than dyadically differentiable it is presumably easier to show a given Walsh series is term by term ED-differentiable than dyadically differentiable. Thus the growth conditions in the theorems cited above may be weakened for the ED derivative. Moreover, the class of series available for this setting includes power series as well as Walsh series. There is much work to do.

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MULTIPLE WALSH ANALYSIS

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ABSTRACT.This paper is a short summary of some results concerning multiple Walsh series.The connection between Walsh series,dyadic martingales and quasi-measures is investigated in one and two dimensional case. Inequalities for martingale maximale functions, martingale transforms and square functions are used to study norm and a.e. convergence of Walsh series.

Duality of dyadic Hardy-,BMO-,and VMO-spaces and atomic characterisation is discussed in one and two dimensional case.The dyadic differentiation and integral is extended to function of two variables and the fundamental theorem of the calculus is given.The method can be used to study a.e. Cesàro summability of Walsh-Fourier series in one and two variables.

1. DYADIC MARTINGALES

We shall denote the set of non-negative integers by $\mathbf{N}$,the set of positive integers by $\mathbf{P}$, the set of real numbers by $\mathbf{R}$, and the set of dyadic rationals in the unit interval $[0,1]$ by $\mathbf{Q}$. In particular, each element of $\mathbf{Q}$ has the form $p/2^n$ for some $p,n \in \mathbf{N}, 0 \le p \le 2^n$. Furhermore, let $\mathbf{I} := [0,1)$ be the unit interval.

For any set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^1 := \mathbf{X}$ and denote by $\mathbf{X}^2$ the cartesian product $\mathbf{X} \times \mathbf{X}$. Thus $\mathbf{N}^2$ is the collection of integral latice points in the first quadrant, and $\mathbf{I}^2$ is the unit square . We shall use the notation $z = (z_\alpha, \alpha \in A)$ to represent a collection z indexed by a set A. Thus a sequence and a double sequence will be represented in the form $(z_n, n \in \mathbf{N})$ and $(z_m, m \in \mathbf{N}^2)$, respectively, where $z_m := z_{p,q}$ for $m = (p,q) \in \mathbf{N}^2$.

We shall use the following partial ordering in $\mathbf{R}^2$. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}^2$ let $x \le y$ if $x_1 \le y_1$ and $x_2 \le y_2$, and set $|x| := |x_1| + |x_2|$. Furhermore, let $\wedge n := n$ if $n \in \mathbf{N}$, and $\wedge n := min\{n_1, n_2\}$ if $n = (n_1, n_2) \in \mathbf{N}^2$.

Fix $j \in \{1,2\}$ and denote the j-dimensional Lebesgue measure of any measurable subset Y of $\mathbf{I}^j$ by $|Y|$ .The $L^p(\mathbf{I}^j)$ norm of any function $f \in L^p(\mathbf{I}^j)$ will be denoted by $\|f\|_p$.

By a dyadic interval in $\mathbf{I}$ we mean one of the form $[p/2^n, (p+1)/2^n)$ for some $p,n \in \mathbf{N}, 0 \le p < 2^n$. Given $n \in \mathbf{N}$ and $x \in \mathbf{I}$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. Denote the collection of dyadic intervals by J.

Let J^2 denote the collection of dyadic intervals in $\mathbf{I}^2$,i.e., the sets of the form $I = I_1 \times I_2$,where $I_1, I_2 \in J$. Clearly, given $x = (x_1, x_2) \in \mathbf{I}^2$, the dyadic intervals in J^2 containing x are of the form

$$I_n(x) := I_{n_1}(x_1) \times I_{n_2}(x_2), \tag{1.1}$$

where $n := (n_1, n_2) \in \mathbf{N}^2$. The set of dyadic squares is denoted by

(1.2) $$\mathcal{Q} := \{I = K \times L \in \mathcal{J}^2 : |K| = |L|\}.$$

For each $n \in \mathbf{N}$ let $\mathcal{A}^n$ represent the atomic σ-algebra generated by the dyadic intervals $I \in \mathcal{J}$ of length 2^{-n}.Thus every element of $\mathcal{A}^n$ is a finite union of intervals of the form $[k/2^n, (k+1)/2^n)$.

The atomic σ-algebra generated by the two dimensional dyadic intervals of the form $I = K \times L$ with $|K| = 2^{-p}$ and $|L| = 2^{-q}$ will be denoted by $\mathcal{A}^{(p,q)}$.For $j \in \{1,2\}$ and $n \in \mathbf{N}^j$ let $L(\mathcal{A}^n)$ denote the set of $\mathcal{A}^n$-measurable function defined on $\mathbf{I}^j$. Set $\mathcal{A}^n_- := \mathcal{A}^{n-1}$ if $n \in \mathbf{N}$ and let

$$\mathcal{A}^n_- := \mathcal{A}^{(n_1-1, n_2-1)} \quad (n = (n_1, n_2) \in \mathbf{N}^2),$$

where $\mathcal{A}^{-1} := \mathcal{A}^0$ and $\mathcal{A}^{(-1,-1)} := \mathcal{A}^0, \mathcal{A}^{(-1,i)} = \mathcal{A}^{(0,i)}, \ \mathcal{A}^{(i,-1)} := \mathcal{A}^{(i,0)} \ (i \in \mathbf{N})$.

The conditional expectation of the function $f \in L^1(\mathbf{I}^j)$ $(j = 0, 1)$ with respect to $\mathcal{A}^n$ $(n \in \mathbf{N}^j)$ is denoted by $E_n f$ and can be given in the form

(1.3) $$(E_n f)(x) = \frac{1}{|I_n(x)|} \int_{I_n(x)} f \qquad (x \in I^j, n \in N^j).$$

Extending (1.3) we set

$$(E_{(\infty,k)} f)(u,v) := \frac{1}{|I_k(v)|} \int_{I_k(v)} f(u,s)ds \qquad ((u,v) \in \mathbf{I}^2, k \in \mathbf{N}).$$

A sequence of functions $\mathrm{f} = (f_n, n \in \mathbf{N}^j)$ defined on $\mathbf{I}^j$ is called a dyadic martingale if each f_n belongs to $L(\mathcal{A}^n)$ and

(1.4) $$E_n f_m = f_n \quad \text{for all} \quad n \le m \quad \text{and} \quad n, m \in \mathbf{N}^j.$$

If $0 < p \le \infty$, $f_n \in L^p(\mathbf{I}^j)$ $(n \in \mathbf{N}^j)$ and

$$\|\mathrm{f}\|_p := \sup_{n \in \mathbf{N}^j} \|f_n\|_p < \infty,$$

then the martingale f is called L^p-bounded.

Let $f \in L^1(\mathbf{I}^j)$ and define the sequence $\mathrm{f} = (f_n, n \in \mathbf{N}^j)$ by

(1.5) $$f_n := E_n f \quad (n \in \mathbf{N}^j).$$

It easy to see that f is a martingale.Martingales of this type are called regular. Moreover, if $f \in L^p(\mathbf{I}^j)$ for some $1 \le p < \infty$, then f is L^p-bounded and

$$\lim_{n \to \infty} \|f_n - f\|_p = 0.$$

Here and in the following the limit of a double sequence is taken in Pringsheim sense, i.e., for every $\epsilon > 0$ there exists an index $N \in \mathbf{N}^2$ such that $\|f_n - f\| < \epsilon$ if $n \geq N$. If $1 < p < \infty$ then the martingale f can be written in the form (1.5) with a function $f \in L^p(\mathbf{I}^j)$ if and only if f is L^p-bounded. In the case $p = 1$,f is of the form (1.5) with a function $f \in L^1(\mathbf{I}^j)$ if and only if the martingale f is uniformly integrable, i.e.,

(1.6) $$\sup_{n \in N^j} \int_{\{|f_n| > y\}} |f_n| \to 0 \quad \text{as} \quad y \to \infty.$$

This characterisation of regular martingales holds for all martingales indexed by an upward directed set (see *NEVEU [29]*).

Thus $f \mapsto \mathrm{f} := (E_n f, n \in \mathbf{N}^j)$ is a norm-preserving map from L^p onto the space of L^p-bounded martingales if $1 < p < \infty$ and consequently the two spaces can be identified. In a similar way, we can identify $L^1(\mathbf{I}^j)$ with the space of uniformly integrable martingales.

The martingale maximal function is defined by

(1.7) $$f^* := \sup_{n \in \mathbf{N}^j} |f_n|.$$

To define the martingale transform and square function introduce the martingale difference sequence in the one-dimensional case by

(1.8) $$d_0 := f_0, d_n := f_n - f_{n-1} \quad (n \in \mathbf{P}),$$

and in two-dimensional case by

(1.9) $$d_{0,0} := f_{0,0}, d_{k,0} := f_{k,0} - f_{k-1,0}, d_{0,k} := f_{0,k} - f_{0,k-1} \quad (k \in \mathbf{N})$$

$$d_n := f_{(n_1,n_2)} - f_{(n_1-1,n_2)} - f_{(n_1,n_2-1)} + f_{(n_1-1,n_2-1)} \quad (n = (n_1, n_2) \in \mathbf{N}^2).$$

Obviously,

$$f_n = \sum_{k \leq n} d_k.$$

Moreover,if $\alpha = (\alpha_n, n \in \mathbf{N}^j)$ and $\alpha_n \in \mathcal{A}_-^n$ $(n \in \mathbf{N}^j)$, then the sequence

(1.10) $$f_n^\alpha := \sum_{k \leq n} \alpha_k d_k, \quad \mathrm{f}^\alpha := (f_n^\alpha, n \in \mathbf{N}^j)$$

is also a martingale and it is called the transform of f by the sequence α.

We introduce a set of function sequences to define special martingale transforms.

To this end set

$$(1.11) \qquad \mathcal{T} :=$$

$$= \{\tau = (\tau_n, n \in \mathbf{N}^j) : \tau_n(x) \in \{0,1\}, \tau_n \in L(\mathcal{A}^n_-) \quad \text{and} \quad \tau_n \geq \tau_m \quad \text{if} \quad n \leq m\}.$$

The square function of the martingale f is the function

$$(1.12) \qquad Qf := \Big(\sum_{n \in \mathbf{N}^j} |d_n|^2 \Big)^{1/2}.$$

For $0 < p < \infty$ denote by $\mathbf{H}^p$ the set of martingales $\mathbf{f} = (f_n, n \in \mathbf{N}^j)$ for which

$$(1.13) \qquad \|\mathbf{f}\|_{\mathbf{H}^p} := \|f^*\|_p < \infty.$$

It is easy to see that if $p \geq 1$ then (1.13) implies that f is uniformly integrable and consequently $\mathbf{H}^p$ can be identified by a subspace of $L^1(\mathbf{I}^j)$.

We shall use the hibrid Hardy space $\mathbf{H}^\sharp$ based on the maximal function

$$(1.14) \qquad f^\sharp := \sup_{k \in \mathbf{N}} |E_{(\infty,k)} f| \quad (f \in L^1(\mathbf{I}^2)).$$

Thus $\mathbf{H}^\sharp$ will represent the collection of functions $f \in L^1(\mathbf{I}^2)$ satisfying

$$\|f\|_\sharp := \|f^\sharp\|_1 < \infty.$$

Clearly $\mathbf{H}^1 \subset \mathbf{H}^\sharp$, and if f is non-negative then f belongs to $\mathbf{H}^\mathbf{I}$ if and only if $f \in L\log^+ L$,i.e.,

$$\int_{\mathbf{I}^2} |f| \log^+ |f| < \infty.$$

It is convenient in one dimension to identify $\mathbf{H}^\sharp$ with $L^1(\mathbf{I})$.

For any subspace $Y \subseteq L^1(\mathbf{I}^j)$ denote by Y_0 the set of elements in Y satisfying

$$Y_0 := \{f \in Y : E_0 f = 0\} \quad (j = 1),$$

$$Y_0 := \{f \in Y : E_{(n,0)} f = E_{(0,n)} f = 0 \quad (n \in \mathbf{N})\} \quad (j = 2).$$

It was proved by *HARDY* (see *ZYGMUND [46]*), *PALEY [30]* (for $j = 1$) and by *CAIROLI [13]* (for $j = 2$) that the L^p-norm of f and f^* are equivalent (in notation $\|f\|_p \sim \|f\|_{\mathbf{H}^p}$), i.e.,there exist constants $A_p, B_p > 0$ such that for $1 < p < \infty$,

$$(1.14) \qquad A_p \|f\|_p \leq \|f^*\|_p \leq B_p \|f\|_p \quad (f \in L^p(\mathbf{I}^j)),$$

and (1.14) does not hold for $p = 1$.But the L^p-norn of f^* and Qf are equivalent for any $0 < p < \infty$,

$$(1.15) \qquad \|f^*\|_p \sim \|Qf\|_p \quad (0 < p < \infty).$$

For j=1 see *BURKHOLDER [7]*,for j=2 see *BROSSARD [5],[6]* ($p \leq 2$) and *METRAUX [26]* ($p > 1$).

From (1.14) and (1.15) it follows that if $|\alpha_n| \leq 1 \quad (n \in \mathbf{N}^j)$, then the martingale transform defined in (1.10) satisfies

$$\sup_{n \in \mathbf{N}^j} \|f_n^\alpha\|_p \leq C_p \|f\|_p \quad (1 < p < \infty)$$

with a constant $C_p > 0$ depending only on p.

In case $j = 1$ the dual of $\mathbf{H}_0^1$ is the space of functions with bounded mean oscillation,i.e.,the space BMO (see *GARSIA [18]*).A function $f \in L_0^2(\mathbf{I})$ belongs to BMO if

$$\|f\|_{BMO} := \sup_{n \in \mathbf{N}} \|(E_n(f - E_n f)^2)^{1/2}\|_\infty < \infty. \tag{1.17}$$

It is easy to see that the BMO-norm is equivalent to

$$\|f\| := \sup_{\tau \in \mathcal{T}} |\{\wedge\tau = 0\}|^{-1/2} \|f - f^\tau\|_2, \tag{1.18}$$

where $\wedge\tau := \inf_{n \in \mathbf{N}^j} \tau_n$ and $f := (E_n f, n \in \mathbf{N}^j)$,and $j = 1$.

The subspace of functions in BMO satisfying

$$\lim_{n \to \infty} \|(E_n(f - E_n f)^2)^{1/2}\|_\infty = 0 \tag{1.19}$$

is denoted by VMO, and it is called the space of functions with vanishing mean oscillation.It is easy to see that VMO is the closure of the set of dyadic step functions (i.e.,of $\cup_n L(\mathcal{A}^n)$) in BMO-norm.The dual of VMO is $\mathbf{H}_0^1$ (see *SCHIPP [34]*).

Definition (1.18) can be used to define the space BMO in the two dimensional case.Namely,let BMO be the set of functions in $L_0^2(\mathbf{I}^2)$ for which the norm defined in (1.18) is finite.The dual space of $\mathbf{H}_0^1$ is this BMO space (see *BERNARD [3]*).Moreover,if VMO is defined as the closure of $\cup_{n \in \mathbf{N}^2} L(\mathcal{A}^n)$ in BMO-norm,then the dual space of VMO is the Hardy space $\mathbf{H}_0^1$ (see *F. WEISZ [43]*).

A function β in $L^\infty(\mathbf{I})$ is called an ∞-atom if there is a dyadic interval $J \subseteq \mathbf{I}$ such that

$$\{\beta \neq 0\} \subseteq J \quad , \quad \|\beta\|_\infty \leq 1/|J| \quad , \quad \int_{\mathbf{I}} \beta = 0.$$

In the two-dimensional case it is convenient to use 2-atoms.A function $\alpha \in L_0^2(\mathbf{I}^2)$ is said to be a 2-atom,if there exist a sequence $\tau \in \mathcal{T}$ such that

$$\alpha_n = 0 \quad \text{on} \quad \text{the} \quad \text{set} \quad \{\tau_n = 1\} \quad (n \in \mathbf{N}^2),$$

$$\|\alpha^*\|_2 \leq |\{\wedge\tau = 0\}|^{-1/2},$$

where $\alpha_n := E_n\alpha \; (n \in \mathbf{N}^2)$.

Dyadic atoms characterize dyadic Hardy spaces in both cases.A function $f \in L_0^1(\mathbf{I}^j)$ belongs to $\mathbf{H}_0^1$ if and only if there exist dyadic atoms $\beta_n (n \in \mathrm{N})$ and a sequence $a = (a_n, n \in \mathrm{N})$ of real numbers such that

$$(1.20) \qquad f = \sum_{n \in \mathrm{N}} a_n \beta_n \quad \text{and} \quad \sum_{n \in \mathrm{N}} |a_n| < \infty.$$

Moreover,let

$$(1.21) \qquad |f| := \inf \sum_{n \in \mathrm{N}} |a_n|,$$

where the infimum is taken over all sequences satisfying (1.20).Then the norm $|\cdot|$ is equivalent to the $\mathbf{H}^1$-norm.

For the case $j = 1$ see *SCHIPP-WADE-SIMON-PÁL [36]*, for j=2 see *BERNARD [3]*, for atomic characterisation of $\mathbf{H}^p$ with $0 < p < 1$ and duality see *F.WEISZ [40-44]*.

Dyadic martingales are closely connected to quasi-measures and Walsh series.

2. WALS SERIES, MARTINGALES AND QUASI-MEASURES

Let $j \in \{1,2\}$ and denote $\mathfrak{R}^j$ be the algebra of sets generated by the dyadic intervals in $\mathbf{I}^j$. By a quasi-measure we shall mean a real-valued set function which is finitely additive on $\mathfrak{R}^j$. Clearly,the restriction of every finite Borel-measure on $\mathbf{I}^j$ to $\mathfrak{R}^j$ is a quasi-measure, but not conversely.

We shall denote the collection of quasi-measures on $\mathfrak{R}^j$ by $\mathbf{QM}^j$ .Let $\mathbf{VM}^j$ be the set of quasi-measures with (finite) bounded variation, let $\mathbf{BM}^j$ be the set of (finite) Borel-measures on $\mathbf{I}^j$, and denote by $\mathbf{AM}^j$ the set of absolutely continuous measures in $\mathbf{BM}^j$. Recall that the map $f \mapsto \nu^f$ defined by

$$(2.1) \qquad \nu^f(I) := \int_I f \qquad (I \in \mathfrak{R}^j, j = 1,2)$$

is $1-1$ from $L^1(\mathbf{I}^j)$ onto $\mathbf{AM}^j$. Moreover, if $\|\nu\|$ denotes the total variation of $\nu \in \mathbf{VM}^j$ then $\|\nu^f\| = \|f\|_1$,i.e.,the map in question is isometric.

Let r be the function defined on $\mathbf{I}$ by

$$(2.2) \qquad r(x) := \begin{cases} 1, & x \in [0,1/2) \\ -1, & x \in (1/2,1) \end{cases}$$

extended to $\mathbf{R}$ by periodicity of period 1. The Rademacher system $r := (r_n, n \in \mathrm{N})$ is defined by

$$(2.3) \qquad r_n(x) := r(2^n x) \qquad (x \in \mathbf{R}, n \in \mathrm{N}).$$

Given $n \in \mathrm{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \tag{2.4}$$

where $n_k = 0$ or 1 for $k \in \mathrm{N}$. The numbers n_k will be called the binary coefficients of n.

Given $x \in \mathrm{I}$ we shall call the expansion

$$x = \sum_{k \in N} x_k 2^{-(k+1)}, \tag{2.5}$$

where each $x_k = 0$ or 1,the dyadic expansion of x. If $x \in \mathrm{I} \setminus \mathrm{Q}$, then (2.5) is uniquely determined.By the dyadic expansion of $x \in \mathrm{Q}$, we shall always mean the one which terminates in 0's.The dyadic sum of $x, y \in \mathrm{I}$ in terms of the dyadic expansion of x and y is defined by

$$x \dot{+} y := \sum_{k \in N} |x_k - y_k| 2^{-(k+1)}. \tag{2.6}$$

For $x = (x_1, x_2), y = (y_1, y_2) \in \mathrm{I}^2$ set

$$x \dot{+} y := (x_1 \dot{+} y_1, x_2 \dot{+} y_2). \tag{2.7}$$

The Walsh(-Paley) system $w = (w_n, n \in \mathrm{N})$ was introduced by Paley as products of Rademacher functions in the following way.If $n \in \mathrm{N}$ has binary coefficients $(n_k, k \in \mathrm{N})$, then

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k}. \tag{2.8}$$

The double Walsh system $(w_n, n \in \mathrm{N}^2)$ is the Kronecker product system generated by the Walsh system,i.e.,

$$w_n(x) := (w_{n_1} \times w_{n_2})(x) := w_{n_1}(x_1) w_{n_2}(x_2) \tag{2.9}$$

$$(n = (n_1, n_2) \in \mathrm{N}^2, x = (x_1, x_2) \in \mathrm{I}^2)).$$

There is a direct connection between dyadic expansions and Walsh functions, namely

$$w_n(x) = (-1)^{<n,x>}, \tag{2.10}$$

where

$$< n, x > := \sum_{k \in N} n_k x_k \pmod 2 \quad (n \in \mathrm{N}, x \in \mathbf{I}).$$

This implies that the Walsh functions behave almost like characters with respect to dyadic addition,namely for a.e. $x, y \in \mathbf{I}^j$

$$(2.11) \qquad w_n(x \dot{+} y) = w_n(x) w_n(y) \quad (n \in \mathbf{N}^j).$$

If $\nu \in \mathrm{QM}^j$ then the Walsh-Fourier-Stieltjes coefficients of ν are defined by

$$(2.12) \qquad \hat{\nu}(n) := \int_{\mathbf{I}^j} w_n \, d\nu \qquad (n \in \mathbf{N}^j).$$

Since each Walsh function is constant on sufficiently small dyadic intervals ,this definition makes sense.

It is easy to prove that the map $\nu \mapsto \hat{\nu}$ is a $1-1$ function from QM^j onto the space of sequences

$$(2.13) \qquad \ell^j := \{x : x = (x_n, n \in \mathbf{N}^j), x_n \in \mathbf{R} \quad (n \in \mathbf{N}^j)\} \quad (j = 1, 2).$$

For $f \in L^1(\mathbf{I}^j)$ we shall denote by

$$(2.14) \qquad \hat{f}(n) := \int_{\mathbf{I}^j} f w_n \qquad (n \in \mathbf{N}^j)$$

the n-th Walsh-Fourier coefficient of f.It is clear that

$$(2.15) \qquad (\nu^f)\hat{} = \hat{f} \qquad (f \in L^1(\mathbf{I}^j)).$$

The dyadic convolution of $f, g \in L^1(\mathbf{I}^j)$ is defined by

$$(2.16) \qquad (f * g)(x) := \int_{\mathbf{I}^j} f(x \dot{+} y) g(y) dy \qquad (x \in \mathbf{I}^j)$$

and satisfies

$$(f * g)\hat{} = \hat{f} \cdot \hat{g}.$$

The Walsh-Fourier-Stieltjes series of $\nu \in \mathrm{QM}^j$ is the Walsh series

$$(2.17) \qquad S\nu := \sum_{n \in \mathbf{N}^j} \hat{\nu}(n) w_n \quad (j = 1, 2),$$

and the set of Walsh series will be denoted by $\mathcal{S}^j$. For $f \in L^1(\mathbf{I}^j)$ we set

$$(2.18) \qquad Sf := S\nu^f := \sum_{n \in \mathbf{N}^j} \hat{f}(n) w_n.$$

The rectangular partial sums of $S\nu$ are defined by

$$(2.19) \qquad S_n \nu := \sum_{k < n} \hat{\nu}(k) w_k \quad (n \in \mathbf{P}^j),$$

where in case $j = 2$ the inequality $k < n$ means that $k_1 < n_1$ and $k_2 < n_2$. If $n_1 = 0$ or $n_2 = 0$, then set $S_n\nu = 0$, and for $n = (n_1, n_2) \in \mathbf{N}^2$ denote

$$2^n := (2^{n_1}, 2^{n_2}). \tag{2.20}$$

It is easy to see that the sequence of 2^n-th partial sums ($n \in \mathbf{N}^j, j = 1, 2$) of any Walsh series is a dyadic martingale. Conversely, every dyadic martingale can be obtained in this way. Thus the investigation of 2^n partial sums of Walsh series leads to a study of dyadic martingales. Notice also that the map $\nu \mapsto (S_{2^n}\nu, n \in \mathbf{N}^j)$ is an isomorphism from QM^j onto the collection of dyadic martingales $\mathcal{M}^j$. The inverse of this map is of the form $(f_n, n \in \mathbf{N}^j) \mapsto \nu$ from $\mathcal{M}^j$ to QM^j, and can be given by

$$\nu(I) := \lim_{n\to 0} \int_I f_n \quad (I \in \mathfrak{R}^j). \tag{2.21}$$

In the case $j = 2$ the above limit is taken in the Pringsheim sense, i.e., $\wedge n := \min\{n_1, n_2\} \to \infty$.

Thus for each $j = 1, 2$ we have four pairwise isomorphic linear spaces QM^j, $\mathcal{M}^j$, ℓ^j and $\mathcal{S}^j$, and the isomorphism can be given by the Walsh system as follows:

i) $\nu \mapsto (S_{2^n}\nu, n \in \mathbf{N}^j)$ from QM^j onto $\mathcal{M}^j$

(2.22) ii) $\nu \mapsto \hat{\nu}$ from QM^j onto ℓ^j

iii) $\nu \mapsto S\hat{\nu}$ from QM^j onto $\mathcal{S}^j$.

It is easy to see that every set function $\nu \in \mathrm{QM}^j$ can be obtained from a function $F : \mathbf{Q}^j \mapsto \mathbf{R}$ in a simple way. Namely in the case $j = 1$ set

$$\nu(I) := \nu_F(I) := F(\beta) - F(\alpha) \quad (I = [\alpha, \beta) \in \mathfrak{R}), \tag{2.23}$$

and if $j = 2$ then let

$$\nu(I) := \nu_F(I) := F(\alpha_1, \alpha_2) - F(\alpha_1, \beta_2) - F(\beta_1, \alpha_2) + F(\beta_1, \beta_2) \tag{2.24}$$

for $I = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \in \mathcal{J}^2$ and $F : \mathbf{Q}^j \mapsto \mathbf{R}$.

The map $F \mapsto \nu_F$ is an isomorphism from the set of functions $\mathcal{F}^j$ defined on $\mathbf{Q}^j$ onto $\mathcal{M}^j$. Moreover, if F is the integral function of $f \in L^1(\mathbf{I}^j)$, i.e.,

$$F(x) := \int_{J_x} f \quad ,\text{where} \quad J_x := \{u \in \mathbf{I}^j : 0 \le u \le x\}, \tag{2.25}$$

then

$$\nu_F = \nu^f \quad \text{on} \quad \mathcal{A}^j. \tag{2.26}$$

The partial sums of any function $f \in L^1(\mathrm{I}^j)$ are martingale transforms of the regular martingales $\mathrm{f}_m := (E_n(fw_m), n \in \mathrm{N}^j)$ $(m \in \mathrm{N}^j)$, namely for each $m \in \mathrm{N}^j$

(2.27) $$S_m f = w_m \mathrm{f}_m^{\alpha^m},$$

where $\alpha_i^m := m_i$ for $j = 1$ and $\alpha_{(i_1,i_2)}^{(m^1,m^2)} := m_{i_1}^1 m_{i_2}^2$ for $j = 2$ and m_i, m_i^j $(i \in \mathrm{N})$ are the binary coefficients of m and m^j, respectively.

Applying inequality (1.16) we get that the operators

$$S_n : L^p(\mathrm{I}^j) \mapsto L^p(\mathrm{I}^j) \quad (n \in \mathrm{N}^j)$$

are uniformly bounded if $1 < p < \infty$, and consequently

$$\|f - S_n f\|_p \to \infty \quad \text{as} \quad n \to \infty$$

for all $f \in L^p(\mathrm{I}^j)$, $1 < p < \infty$ and $j = 1, 2$.

It is interesting that, in contrast with the trigonometric case, the operators

(2.28) $$T_n f := \sum_{k \le n, k_1 \le k_2} \hat{f}(k) w_k \quad (n = (n_1, n_2) \in \mathrm{N}^2, n_1 = n_2)$$

are uniformly bounded in $L^p(\mathrm{I}^2)$-norm if and only if $p = 2$. Moreover, the same holds for the operators

$$R_t^\beta f := \sum_{n_2 \le t - \beta n_1} \hat{f}(n) w_n \quad (t > 0)$$

for each $\beta > 0$ (see *HARRIS [20]*). This implies that the one dimensional trigonometric and Walsh systems are not equivalent bases in $L^p(\mathrm{I})$ if $p \neq 2$ (see *WO-SANG YOUNG [45]*).

3. WALSH-FOURIER COEFFICIENTS

For each linear normed subspace $\mathcal{L} \subset \ell^j$ denote by $\mathcal{L}_*$ the set of sequences $x = (x_n, n \in \mathrm{N}^j) \in \ell^j$ such that there exists a non-increasing sequence of non-negative numbers $\alpha = (\alpha_n, n \in \mathrm{N}^j) \in \mathcal{L}$ satisfying

(3.1) $$|x_n| \le \alpha_n \quad (n \in \mathrm{N}^j).$$

Introduce a norm in $\mathcal{L}$ by

(3.2) $$\|x\|_* := \inf\{\|\alpha\| : \alpha \in \mathcal{L} \quad \text{and} \quad (3.1) \quad \text{is} \quad \text{satisfied}\}.$$

Denote by

(3.3) $$\|x\|_p := (\sum_{n \in \mathrm{N}^j} |x_n|^p)^{1/p} \quad (p > 0),$$

and let ℓ^p be the set of sequences $x \in \ell$ satisfying $\|x\|_p < \infty$.

By the Hausdorff-Young inequality,

$$\|\hat{f}\|_q \leq \|f\|_p \quad (f \in L^p(\mathrm{I}^j), 1 \leq p \leq 2, 1/p + 1/q = 1).$$

For functions belonging to H^1 the following analogue of the Hardy inequality is true:

$$\sum_{n \in \mathrm{P}^j} |\hat{f}(n)|/n^* \leq C\|f\|_{\mathrm{H}^1} \quad (f \in \mathrm{H}^1),$$

where $n^* = n$ if $j = 1$ and $n^* = n_1 n_2$ if $n = (n_1, n_2) \in \mathrm{N}^j$ and $j = 2$.

For $j = 1$ see *LADHAWALA [23]*, for the case $j = 2$ see *F. WEISZ [44]*.

For the trigonometric system there is a direct relationship between smoothness of a function and how rapidly its Fourier coefficients tend to zero. This is not the case for Walsh-Fourier coefficients.

If F is an absolutely continuous function, i.e., if F is of the form (2.25) and if $\hat{f}(0) = 0$, then

$$\hat{F}(2^n + k) = (-1)^j 2^{-|n|-2j} \hat{f}(k) + o(2^{|n|}) \tag{3.4}$$

holds uniformly in $k \in \mathrm{N}^j$ as $\wedge n \to \infty$. Consequently, if for an absolutely continuous function $\hat{F}(m) = o(1/m)$ as $m \to \infty$ then F is constant. For $j = 1$ see *FINE [16]*, the case $j = 2$ is similar.

Next we investigate how rapidly the Walsh-Fourier coefficients of a non-constant continous function can decay.

If $F \in C([0,1]^j)$, then the Fourier coefficients of $\nu_F \in \mathrm{QM}^j$ can be expressed by the Fourier coefficients of F. Namely, if $j = 1$, $\nu = \nu_F, n \in \mathrm{N}$ and $2^i \leq n < 2^{i+1}$, then for all $k \geq i$

$$\hat{\nu}(n) = 2^k \sum_{m=k}^{\infty} \sum_{t=0}^{2^{m-k}-1} \hat{F}(2^m + t2^k + n)(w_{2^m + t2^k}(1 - \frac{1}{2^{m+1}}) - 1) \tag{3.5}$$

(see *BOCKAREV [4]*).

This implies the following theorem of *BOCKAREV*:

(3.7) *If $F \in C[0,1]$ and $\hat{F} \in \ell^1_*$, then F is constant.*

A similar identity and theorem holds in the two dimensional case for functions $F \in C_0([0,1]^2)$ (see *AMOODI [1]*).

4. DYADIC CALCULUS

The dyadic difference operator for function $h : \mathbf{I} \mapsto \mathbf{R}$ was introduced by *GIBBS [18]* in the following way:

$$(4.1) \qquad (d_n h)(t) := \sum_{j=0}^{n-1} 2^{j-1}(h(t) - h(t\dot{+}2^{-j-1}), \qquad (n \in \mathbf{P}).$$

It is easy to see that (2.10) implies

$$(4.2) \qquad (d_n w_m)(t) = m w_m(t) \quad (m \in \mathbf{N}, m \geq 2^n, t \in \mathbf{I}).$$

The partial dyadic derivative was introduced by *BUTZER* and *ENGELS* in [11] by the help of difference operators $\partial_{n,k}$.For $f : \mathbf{I}^2 \mapsto \mathbf{R}$ and $k = 1,2$ let

$$(\partial_{n,k} f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x\dot{+}e_j^k)),$$

where $e_j^1 := (2^{-(j+1)}, 0)$ and $e_j^2 := (0, 2^{-(j+1)})$.For this operator we have

$$\partial_{n,k} w_m = m_k w_m \qquad (m = (m_1, m_2) \in \mathbf{N}^2, m_k \leq 2^n, n \in \mathbf{N}, k = 1,2).$$

Another dyadic difference operator for function of two variables was introduced by *SCHIPP* and *WADE [35]* as follows: For $n = (n_1, n_2)$ and $f : I^2 \mapsto R$ let

$$(d_n f)(x) := \sum_{j<n} 2^{|j|-2}(f(x) - f(x\dot{+}e_{j_1}^1) - f(x\dot{+}e_{j_2}^2) + f(x\dot{+}e_{j_1}^1\dot{+}e_{j_2}^2)).$$

It is easy to see that for $f = g \times h$ and $n = (n_1, n_2)$

$$d_n f = d_{n_1} g \times d_{n_2} h,$$

and consequently

$$(4.4) \qquad d_n w_m = m' w_m \quad (m \leq 2^n, m, n \in \mathbf{N}^j).$$

The function $f : \mathbf{I}^j \mapsto \mathbf{R}$ is said to be differentiable at a point $t \in \mathbf{I}$ if $(d_n f)(t)$ converges, as $n \to \infty$, to some finite number $f^{[1]}(t)$. If for some $g \in L^1(\mathbf{I}^j)$

$$\lim_{n\to\infty} \|d_n f - g\|_p = 0,$$

then f is called strongly differentiable with strong derivative g. The strong derivative of f will be denoted by df.

The pointwise and strong partial derivatives can be defined in a similar way and will be denoted by ∂_k (see [11]).

Obviously,it follows from (4.4) that the Walsh functions $w_n\ (n \in \mathbf{N}^j)$ are dyadic differentiable and

$$dw_m = w_m^{[1]} = m^* w_m \quad , \quad \partial_k w_m = m_k w_m \quad (m = (m_1, m_2) \in \mathbf{N}^j, k = 1, 2).$$

The inverse operator of d ,i.e.,the dyadic antiderivative (or integral) can be given by the convolution

$$Jf := f * W^j \qquad (f \in L^1(\mathbf{I}^j)),$$

where W^j is the function whose Walsh-Fourier coefficients satisfy

$$\hat{W}^j(n) := \begin{cases} 1, & \text{if } \wedge n = 0 \\ 1/n^*, & \text{if } n \in \mathbf{P}^j \end{cases}.$$

It can be proved (see *BUTZER* and *WAGNER [9]*) that in the one dimensional case $W^1 \in L^1(\mathbf{I})$, and consequently

$$W^2 = W^1 \times W^2 \in L^1(\mathbf{I}^2).$$

The fundamental theorem for strong derivative was proved by *BUTZER* and *WAGNER [9]*:*For each $f \in L^1(\mathbf{I})$ with $\hat{f}(0) = 0$*

$$f = J(df) \quad and \quad d(Jf) = f.$$

The Hardy-Littlewood maximal function is defined by

$$(Mf)(x) := \sup_h \frac{1}{|I(x;h)|} \left| \int_{I(x;h)} f \right| \qquad (x \in \mathbf{I}^j),$$

where

$$I(x;h) := \{y \in \mathbf{I}^j : x \le y \le x + h\}.$$

The dyadic counterpart of this operator is given by

$$J^* f := \sup_{n \in \mathbf{N}^j} |d_n(Jf)|.$$

Then the following analogue of the Hardy-Littlewood maximal inequality is true: *The operator J^* is of weak-type with respect to $\mathbf{H}^1$, i.e., there is an absolute constant $C > 0$ such that*

$$|\{x \in \mathbf{I}^j : (J^* f)(x) > y\}| \le \frac{C}{y} \| \, |f| \, \|_1$$

for $y > 0$ and f with $|f| \in \mathbf{H}^1$.

This implies the fundamental theorem for the pointwise calculus: *If $|f| \in \mathbf{H}^1$, in particular, if $f \in L^1(\mathbf{I})$ or $f \in (L\log^+ L)(\mathbf{I}^2)$, then*

$$(d_n J)(f) \to f \quad a.e. \quad as \quad \wedge n \to \infty.$$

See *SCHIPP [31]* for $j = 1$ and *SCHIPP* and *WADE [35]* for $j = 2$. A similar inequality and convergence theorem holds for measures (see *SCHIPp-WADE-SIMON-PÁL [36]*).

For other results and applications concerning dyadic partial derivatives see *BUTZER* and *ENGELS [11],[12]*.

5. CESÀRO SUMMABILITY

The dyadic derivative is closely connected to $(C,1)$-summability of Walsh-Fourier series. Indeed, let

$$(5.1) \qquad D_n^1 := \sum_{k<n} w_k \quad (n \in \mathbf{N}) \quad , \quad K_n^1 := \sum_{k<n} (1 - \frac{k}{n}) w_k \quad (n \in \mathbf{P})$$

be the Walsh-Dirichlet and Walsh-Fejér kernel, respectively.

The two dimensional kernels are of the form

$$(5.2) \qquad D_n^2 = D_{n_1}^1 \times D_{n_2}^1 \quad , \quad K_n^2 = K_{n_1}^1 \times K_{n_2}^1 \quad (n = (n_1, n_2) \in \mathbf{N}^2),$$

and the partial sums and $(C,1)$ means can be expressed in the form

$$(5.3) \qquad S_n f = f * D_n^j, \quad \sigma_n f = f * K_n^j \quad (n \in \mathbf{N}^j, j = 1, 2).$$

It is easy to see that in the one dimensional case

$$K_n = D_n - \frac{1}{n} D_n^{[1]} \quad (n \in \mathbf{P}).$$

This identity can be used to study a.e $(C.1)$-summability of Walsh series.

Introduce the maximal operators

$$\sigma^* f := \sup_{n \in \mathbf{P}^j} |\sigma_n f|, \; \sigma f := \sup_{n \in \mathbf{N}^j} |\sigma_{2^n} f|, \; \sigma_\alpha f := \sup_{|n_1 - n_2| \le \alpha} |\sigma_{2^n} f|.$$

It can be proved that *there exists an absolute constant $A > 0$ such that*

$$|\{\sigma f > y\}| \le \frac{A}{y} \|f\|_{\mathbf{H}} \quad , \quad |\{\sigma^* f > y\}| \le \frac{A}{y} \| |f| \|_{\mathbf{H}}$$

for every $y > 0$ and f in $\mathbf{H}^j$ and f with $|f| \in \mathbf{H}^j$, respectively. Consequenty, if $|f| \in \mathbf{H}^j$ (in particular if $f \in L^1(\mathbf{I})$, or $f \in (L \log^+ L)(\mathbf{I}^2)$), then

$$\sigma_n \to f \quad a.e. \quad as \quad \wedge n \to \infty.$$

For $j = 1$ see *SCHIPP-WADE-SIMON-PÁL* [36], for $j = 2$ see *MÓRICZ-SCHIPP-WADE* [28].

The operator σ_α can be estimated by the maximal function with respect to squares:

$$\|\sigma_\alpha f\| \leq C_\alpha \| \sup_{n \in N} |E_{(n,n)} f| \|_1,$$

$$|\{\sigma_\alpha f > y\}| \leq \frac{C\alpha}{y} \|f\| \quad (y > 0).$$

Consequently,

$$\sigma_{2^n} f \to f \quad a.e \quad if \quad |n_1 - n_2| \leq \alpha \quad and \quad \wedge n \to \infty.$$

(See *MÓRICZ-SCHIPp-WADE* [28]).

In the proof of the above results *quasi-local operators* play an important role. For each $J \in \mathcal{J}^j$ and each $r \in \mathbf{N}$ let $J^r \in \mathcal{J}^j$ be defined by $J \subset J^r$ and $|J^r| = 2^{rj}|J|$. An operator T from $L^1(\mathbf{I}^j)$ into the set of measurable function on $\mathbf{I}^j$ will be called quasi-local if there exist constants $C > 0$ and $r \in \mathbf{N}$ such that

$$\int_{\mathbf{I}^j \setminus J^r} |Tf| \leq C\|f\|_1$$

for all $f \in L^1(\mathbf{I}^j)$ and $J \in \mathcal{J}^j$ which satisfy $\{f \neq 0\} \subseteq J$.

Let $(T_\gamma, \gamma \in \Gamma)$ be a collection of bounded sublinear operators on $L^1(\mathbf{I}^j)$ and set

$$Tf := \sup_{\gamma \in \Gamma} |T_\gamma f| \quad (f \in L^1(\mathbf{I}^j)).$$

If T is quasi-local and bounded as an operator on $L^\infty(\mathbf{I}^j)$, then there is a constant $C > 0$ such that

$$\|Tf\|_1 \leq C\| \sup_{k \in \mathbf{N}} |E_k^\diamond f| \|_1, \qquad |(Tf > y)| \leq \frac{C}{y}\|f\|_1 \quad (y > 0, f \in L^1(\mathbf{I}^j)),$$

where $E_k^\diamond = E_k$ if $j = 1$ and $E_k^\diamond := E_{(k,k)}$ if $j = 2$ for all $k \in \mathbf{N}$.

A connection between the one and two dimensional maximal function is given in the next result.

Let $(V_n^i, n \in \mathbf{N})$, $i = 0,1$ be a sequence of $L^1(\mathbf{I})$ functions. Define the one dimensional operators

$$T^i h := \sup_{k \in \mathbf{N}} |h * V_k^i| \quad (i = 0, 1, h \in L^1(\mathbf{I})),$$

and suppose there exist absolute constant A_0, A_1 such that

$$|\{T^0 h > y\}| \leq \frac{A_0}{y}\|h\|_1, \ \|T^1 h\| \leq A_1 \|h\|_H \quad (h \in L^1(\mathbf{I}), y > 0).$$

If $V_k^0 \geq 0$ for all $k \in \mathbf{N}$ and

$$Tf := \sup_{(n_1, n_2) \in \mathbf{N}^2} |f * (V_{n_1}^0 \times V_{n_2}^1)|,$$

then there holds

$$|\{Tf > y\}| \leq \frac{A_0 A_1}{y} \|f\|_1 \quad (f \in \mathbf{H}^1, y > 0).$$

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WALSH-FOURIER TRANSFORM AND DYADIC DERIVATIVE

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Abstract. In this paper we shall be concerned with the Walsh-Fourier transform which is a Fourier transform on the additive group of the dyadic field and with the dyadic derivative and integral defined on the dyadic field.

1. THE DYADIC FIELD

Let $\mathbf{F}$ denote the set of doubly infinite sequences

$$x = (x_n, n \in \mathbf{Z})$$

where $x_n = 0$ or 1 and $x_n \to 0$ as $n \to -\infty$. Denote the doubly infinite sequence whose entries are identically zero by 0. Thus to each $x \in \mathbf{F}$ with $x \neq 0$ there corresponds an integer $S(x) \in \mathbf{Z}$ such that

$$(1) \qquad x_{S(x)} = 1 \qquad \text{but} \qquad x_n = 0 \qquad \text{for} \qquad n < S(x).$$

Let $x = (x_n, n \in \mathbf{Z})$ and $y = (y_n, \in \mathbf{Z})$ be elements of $\mathbf{F}$. Define the sum of x and y by

$$(2) \qquad x + y := (|x_n - y_n|, \qquad n \in \mathbf{Z}).$$

Define the product of x and y by

$$(3) \qquad x \bullet y := (\xi_n, \quad n \in \mathbf{Z})$$

where for each $n \in \mathbf{Z}$

$$\xi_n := \sum_{i+j=n} x_i y_j \qquad (\text{mod} 2).$$

Notice that $(\mathbf{F}, +)$ is an abelian group, $(\mathbf{F}, \bullet)$ is an abelian semigroup, and in fact, $(\mathbf{F}, +, \bullet)$ is a commutative algebra over the finite field $\mathbf{Z}_2 := \{0, 1\}$. This algebra has an identity

$$(4) \qquad e_o := (\delta_{o,n}, \quad n \in \mathbf{Z})$$

and contains a subgroup

$$\mathbf{F}_o := \{x \in \mathbf{F} : x_n = 0 \qquad \text{for} \qquad n < 0\}$$

which is isomorphic to the dyadic group.

The algebra $\mathbf{F}$ is normed. Indeed, for $x = (x_n, \quad n \in \mathbf{Z}) \in \mathbf{F}$ define

$$|x| := \sum_{n \in \mathbf{Z}} x_n 2^{-n-1}. \tag{5}$$

It is easy to see that

$$|x| \geq 0, \qquad |x+y| \leq |x| + |y|, \qquad |x \bullet y| \leq |x||y|$$

for all $x, y \in \mathbf{F}$.

There is another norm, a non-Archimedean one, which can be defined on $\mathbf{F}$. Set $\|0\| := 0$ and for each $x \in \mathbf{F}$ with $x \neq 0$ set

$$\|x\| := 2^{-S(x)}, \tag{6}$$

where $S(x)$ is defined in (1). Notice that

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad \text{and} \quad \|x \bullet y\| = \|x\|\|y\| \tag{7}$$

for $x, y \in \mathbf{F}$. Also, by definition we have

$$\frac{1}{2}\|x\| \leq |x| \leq \|x\| \qquad (x \in \mathbf{F}).$$

Hence (6) is a norm on the algebra $\mathbf{F}$ which is equivalent to (5).

Let

$$\mathbf{B} := \{x \in \mathbf{F} : \|x\| = 1\}$$

denote the unit ball in $\mathbf{F}$. It is easy to see that $\mathbf{B}$ is a multiplicative subgroup of $\mathbf{F}$.

Define the usual closed system in $\mathbf{F}$ by

$$e_n := (\delta_{n,j}, \quad j \in \mathbf{Z}), \tag{8}$$

and observe that

$$e_n \bullet x = (x_{j-n}, \quad j \in \mathbf{Z})$$

for each $n \in \mathbf{Z}$ and $x = (x_j, \quad j \in \mathbf{Z}) \in \mathbf{F}$. Thus multiplication by e_n is a shift operator on $\mathbf{F}$. Clearly,

$$e_n \bullet e_m = e_{n+m} \qquad \text{and} \qquad \|e_n\| = 2^{-n} \qquad (n, m \in \mathbf{Z}).$$

Thus $\{e_n : \quad n \in \mathbf{Z}\}$ forms a 1-parameter subgroup of $\mathbf{F}$ and is algebraically isomorphic to $\mathbf{Z}$.

Let $x \in \mathbf{F}$, $x \neq \mathbf{0}$. Choose $n \in \mathbf{Z}$ such that $\|x\| = 2^{-n}$. By (7) we have

Hence $e_{-n} \bullet x$ is invertible, and it follows that x is invertible with

$$x^{-1} = (e_n \bullet e_{-n} \bullet x)^{-1} = (e_{-n} \bullet x)^{-1} \bullet e_{-n}.$$

Therefore $\mathbf{F}$ is a field. It is called the dyadic field and was introduced by *N.J.Fine* [2].

It is easy to obtain from (6) that addition and multiplication are continuous maps from $\mathbf{F} \times \mathbf{F}$ into $\mathbf{F}$. Moreover, the inequality

$$\|(e_o + x)^{-1} - e_o\| \leq \frac{\|x\|}{1 - \|x\|}$$

holds for all $x \in \mathbf{F}$ satisfying $\|x\| < 1$, so the map $x \mapsto x^{-1}$ is continuous from $\mathbf{F}^*$ into $\mathbf{F}$. Therefore the dyadic field is a topological field.

Define maps $\pi_n : \mathbf{F} \to \{0,1\}$ by

$$\pi_n(x) := \pi_n((x_j, \quad j \in \mathbf{Z})) := x_n \qquad (n \in \mathbf{Z}).$$

Define the integer part of $x \in \mathbf{F}$ by

$$[x] := (\ldots, x_{-2}, x_{-1}, 0, 0, \ldots)$$

i.e., $[x]$ is that element of $\mathbf{F}$ defined by

$$\pi_n([x]) = \begin{cases} 0, & n \geq 0 \\ x_n, & n < 0. \end{cases}$$

Characters of additive group $(\mathbf{F}, +)$ can be generated in the following way. For each $x, y \in \mathbf{F}$ define

(9) $$\psi_y(x) := (-1)^{\pi_{-1}(x \bullet y)}.$$

Since π_{-1} is linear, it is clear that each ψ_y is a character on $(\mathbf{F}, +)$, i.e., ψ_y is continuous on $\mathbf{F}$ and satisfies

$$\psi_y(x + x') = \psi_y(x)\psi_y(x') \qquad (x, x' \in \mathbf{F}).$$

It is also clear that

(10) $$\psi_y(x) = \psi_x(y)$$

and

(11) $$\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y) \qquad (x, y \in \mathbf{F}).$$

In particular, the group of characters of $(\mathbf{F}, +)$ is isomorphic to $(\mathbf{F}, +)$. It can be shown that the functions $\psi_y\,(y \in \mathbf{F})$ (the so-called generalized Walsh functions) exhaust the characters of the additive group $(\mathbf{F}, +)$ (see [2]).

Differentiation of functions defined on $\mathbf{F}$ can be defined as follows (see [1]). For each $n \in \mathbf{N}$ and each function f on $\mathbf{F}$ set

$$d_n f := \sum_{j=-n}^{n} 2^{j-1}(f - \tau_{e_j} f) \tag{12}$$

where

$$(\tau_h f)(x) := f(x+h) \qquad (x, h \in \mathbf{F}) \tag{13}$$

represents translation of f by an element $h \in \mathbf{F}$. If

$$f^{[1]}(x) := \lim_{n \to \infty} (d_n f)(x) \tag{14}$$

exists at some $x \in \mathbf{F}$, we shall say that f is dyadically differentiable at x and call $f^{[1]}(x)$ the pointwise dyadic derivative of f at x.

Similarly, if X is some Banach space of functions on $\mathbf{F}$ and if the limit

$$df := \lim_{n \to \infty} d_n f \tag{15}$$

exists in the norm of X, then we shall say that f is strongly dyadically differentiable in X and call df the strong dyadic derivative of f.

It can be shown (see [1]) that the additive characters ψ_y $(y \in \mathbf{F})$ are everywhere dyadically differentiable with

$$\psi_y^{[1]} = |y|\psi_y \qquad (y \in \mathbf{F}). \tag{16}$$

2. THE WALSH-FOURIER TRANSFORM

The additive group $(\mathbf{F}, +)$ is a locally compact abelian group and its unit ball

$$\mathbf{B} = \{x \in \mathbf{F} : \|x\| = 1\}$$

is compact. Hence there is a unique Haar measure μ on $\mathbf{F}$ which satisfies

$$\mu(\mathbf{B}) = 1.$$

The spaces $L^p_\mu(\mathbf{F})$ will be denoted by $L^p(\mathbf{F})$ for $1 \le p \le \infty$ and the corresponding norm by $\| \bullet \|p$.

Given $f \in L^1(\mathbf{F})$, the Walsh-Fourier transform of f is the function on $\mathbf{F}$ defined by

$$\hat{f}(y) := \int_{\mathbf{F}} f(x)\psi_y(x) d\mu(x) \qquad (y \in \mathbf{F}). \tag{17}$$

(see [2]). The map $f \mapsto \hat{f}$ is a linear map from $L^1(\mathbf{F})$ into $L^\infty(\mathbf{F})$. In fact, since

(18) $$\|\hat{f}\|_\infty \le \|f\|_1 \qquad (f \in L^1(\mathbf{F})),$$

it is clear that

(19) $$\lim_{n\to\infty} \hat{f}_n(y) = \hat{f}(y) \qquad (y \in \mathbf{F})$$

for any sequence $f_n\ (n \in \mathbf{N})$ which converges to f in $L^1(\mathbf{F})$-norm.

The Walsh-Fourier transform of an integrable function is continuous and bounded on $\mathbf{F}$. In fact, if $f \in L^1(\mathbf{F})$, then $\hat{f}$ is uniformly continuous on $\mathbf{F}$. Moreover, if $f \in L^1(\mathbf{F})$, and $h \in \mathbf{F}$, then

(20) $$(\tau_h f)\hat{} = \psi_h \cdot \hat{f}$$

and

(21) $$(\psi_h \cdot f)\hat{} = \tau_h \hat{f}.$$

The convolution of two functions f and g in $L^1(\mathbf{F})$ is defined by

(22) $$(f * g)(x) := \int_{\mathbf{F}} f(x+t)g(t)\,d\mu(t) \qquad (x \in \mathbf{F}).$$

By Fubini's theorem $f * g \in L^1(\mathbf{F})$ and

(23) $$\|f * g\|_1 \le \|f\|_1 \|g\|_1 \qquad (f, g \in L^1(\mathbf{F})).$$

Thus $L^1(\mathbf{F})$ is a Banach algebra under function addition and convolution.

The Walsh-Fourier transform takes convolution to pointwise multiplication: if $f, g \in L^1(\mathbf{F})$, then

(24) $$(f * g)\hat{} = \hat{f} \cdot \hat{g}.$$

The following multiplication formula is also true: if $f, g \in L^1(\mathbf{F})$, then

(25) $$\int_{\mathbf{F}} \hat{f}(y)g(y)\,d\mu(y) = \int_{\mathbf{F}} f(y)\hat{g}(y)\,d\mu(y).$$

We remark that these properties hold for any Fourier transforms defined on a locally compact abelian group.

The following result shows a connection between the Walsh-Fourier transform and the strong dyadic derivative in $L^1(\mathbf{F})$ (see [1], [3]):

Theorem 1. *If the function $f \in L^1(\mathbf{F})$ is strongly dyadically differentiable in $L^1(\mathbf{F})$, then*

$$(26) \qquad (df)^{\wedge}(y) = |y|\hat{f}(y) \qquad (y \in \mathbf{F}).$$

Thus, the Walsh-Fourier transform takes a derivative of f to a polynomial multiple of $\hat{f}$. The following result shows that the Walsh-Fourier transform takes a polynomial multiple of f to a derivative of $\hat{f}$ (see [3]).

Theorem 2. *Let $f \in L^1(\mathbf{F})$ and set*

$$g(x) := |x|f(x) \qquad (x \in \mathbf{F}).$$

If $g \in L^1(\mathbf{F})$, then $\hat{f}$ is pointwise dyadically differentiable on $\mathbf{F}$ and

$$(27) \qquad (\hat{f})^{[1]}(y) = \hat{g}(y) \qquad (y \in \mathbf{F}).$$

In the following we identify the dyadic field $\mathbf{F}$ with the set of non-negative real numbers $\mathbf{R}^+$ via the map $|\bullet|$ in (5). This identification takes the Haar measure μ to Lebesque measure on $\mathbf{R}^+$, the characters of $(\mathbf{F}, +)$ to generalized Walsh-Paley functions on $\mathbf{R}^+$. We shall denote dyadic addition by $\dot{+}$ on $\mathbf{R}^+$, but leave all other notations the same. We shall extend the Walsh-Fourier transform from $L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$ to $L^2(\mathbf{R}^+)$ in a special way using the eigenfunctions of the Walsh-Fourier transform (see [4], [5]).

For each $k \in \mathbf{N}$ define a function Ω_k on $\mathbf{R}^+$ by

$$(28) \qquad \Omega_k(x) := \begin{cases} \psi_k(x), & x \in [k, k+1) \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that Ω_k $(k \in \mathbf{N})$ are eigenfunctions of the Walsh-Fourier transform:

$$(29) \qquad \hat{\Omega}_k = \Omega_k \qquad (k \in \mathbf{N}).$$

For each $k \in \mathbf{N}$ set

$$\Omega_{k,0,1} := \Omega_k \qquad \text{and} \qquad \Omega_{k,0,-1} := 0.$$

If $k \in \mathbf{N}$, $n \in \mathbf{P}$ and $j \in \{-1, 1\}$, define a function $\Omega_{k,n,j}$ on $\mathbf{R}^+$ by

$$(30) \qquad \Omega_{k,n,j}(x) := \begin{cases} \frac{1}{\sqrt{2}}\psi_{k+n}(x), & x \in [k, k+1) \\ \frac{j}{\sqrt{2}}\psi_k(x), & x \in [k+n, k+n+1) \\ 0, & \text{otherwise.} \end{cases}$$

An easy computation gives that

$$(31) \qquad \hat{\Omega}_{k,n,j} = j\Omega_{k,n,j} \qquad (k, n \in \mathbf{N}, \quad j \in \{-1, 1\}).$$

Therefore the system

$$\Omega := \{\Omega_{k,n,j} : k, n \in \mathbf{N}, j \in \{-1, 1\}\} \tag{32}$$

is orthonormal in $L^2(\mathbf{R}^+)$ and consists entirely of eigenfunctions of the Walsh-Fourier transform. It can be shown that the system Ω is complete.

For each $f \in L^2(\mathbf{R}^+)$ let

$$c_{k,n,j}(f) := \int_{\mathbf{R}^+} f(x)\Omega_{k,n,j}(x)dx \qquad (k, n \in \mathbf{N}, j \in \{-1, 1\}) \tag{33}$$

represent the Ω-Fourier coefficients of f. Define the Walsh-Fourier transform of f to be the formal Walsh-Fourier transform of the Ω-Fourier series of f, that is, let

$$Ff := \sum_{k,n \in \mathbf{N}} (c_{k,n,1}(f)\hat{\Omega}_{k,n,1} + c_{k,n,-1}(f)\hat{\Omega}_{k,n,-1}). \tag{34}$$

This defines a function Ff for each $f \in L^2(\mathbf{R}^+)$. In fact, we have the following

Theorem 3. *If $f \in L^2(\mathbf{R}^+)$, then (34) converges in $L^2(\mathbf{R}^+)$-norm. Moreover,*

$$\|Ff\|_2 = \|f\|_2 \qquad (f \in L^2(\mathbf{R}^+), \tag{35}$$

and

$$F(Ff) = f. \tag{36}$$

To obtain a closed form for the Walsh-Fourier transform of functions from $L^2(\mathbf{R}^+)$, set

$$D_t(y) := \int_0^t \psi_y(x)dx \tag{37}$$

for $t, y \in \mathbf{R}^+$. This function is an analogue of the Walsh-Dirichlet kernel. We can prove the following (see [5])

Theorem 4. *If $f \in L^2(\mathbf{R}^+)$, then*

$$(Ff)(t) = \frac{d}{dt}(\int_{\mathbf{R}^+} f(y)D_t(y)dy) \tag{38}$$

for a.e. $t \in \mathbf{R}^+$.

This is an analogue of the classical result for the trigonometric Fourier transform on $\mathbf{R}$.

It is now easy to see that the following corollaries are true.

Corollary 1. *If $f \in L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$, then*

$$Ff = \hat{f}. \tag{39}$$

Corollary 2. *For $f \in L^2(\mathbf{R}^+)$ and $t \in \mathbf{R}^+$ define a function f_t on $\mathbf{R}^+$ by*

$$f_t(x) := \begin{cases} f(x), & x \in [0,t) \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$Ff = \lim_{t \to \infty} \hat{f_t} \tag{40}$$

in $L^2(\mathbf{R}^+)$-norm.

3. THE DYADIC INTEGRAL

In this section we shall be concerned with the inversion of the dyadic derivative on $\mathbf{R}^+$. *H.J. Wagner* [10] has defined for each $n \in \mathbf{Z}$ a function W_n whose Walsh-Fourier transform is the following function:

$$(FW_n)(y) = \begin{cases} 0, & y \in [0, 2^{-n}) \\ 1/y, & y \in [2^{-n}, +\infty). \end{cases} \tag{41}$$

Since $FW_n \in L^2(\mathbf{R}^+)$, this equality uniquely defines $W_n \in L^2(\mathbf{R}^+)$.

It can be shown that the W_n's are also integrable:

Theorem 5. *For each $n \in \mathbf{Z}$,*

$$W_n \in L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+).$$

Moreover,

$$W_n(x) = \lim_{k \to \infty} \int_{2^{-n}}^{2^k} \frac{1}{y} \psi_x(y)\,dy \tag{42}$$

for a.e. $x \in \mathbf{R}^+$ and $n \in \mathbf{Z}$, where this limit exists both pointwise and in the $L^1(\mathbf{R}^+)$-norm.

The functions W_n $(n \in \mathbf{N})$ provide a kernel for dyadic integration. Specifically, if for a function $f \in L^1(\mathbf{R}^+)$ there exists a function $g \in L^1(\mathbf{R}^+)$ such that

$$\lim_{n \to \infty} \|W_n * f - g\|_1 = 0, \tag{43}$$

then g is called the (strong) dyadic integral of f and will be denoted by If. This notion of the dyadic integral is due to *H.J. Wagner* [10]. He was given the following characterization of the dyadic integral: suppose that $f, g \in L^1(\mathbb{R}^+)$. Then

$$g = If$$

if and only if

(44) $$\hat{g}(y) = \begin{cases} 0, & y = 0 \\ 1/y\hat{f}(y), & y > 0. \end{cases}$$

This leads easily to one half of a fundamental theorem of dyadic calculus on $\mathbb{R}^+$ (see [10]).

Corollary 3. *If $f \in L^1(\mathbb{R}^+)$ is strongly dyadically differentiable and $\hat{f}(0) = 0$, then*

(45) $$f = I(df).$$

It is clear that not every $f \in L^1(\mathbb{R}^+)$ is dyadically integrable. A concrete example is given by $f := \chi[0,1)$ (see [10]). We remark that when replacing $\chi[0,1)$ by $\chi[0,1)\psi_m$ ($m \in \mathbf{P}$), it can be shown that these functions are dyadically integrable. Moreover, the following is true (see [6]):

Theorem 6. *Let*

(46) $$Y := \{\chi[0, 2^n)\psi_{2^{-n}m} : \quad n \in \mathbf{N}, m \in \mathbf{P}\}.$$

Then the linear hull of Y is L^1-dense in the set of dyadically integrable functions.

For the remainder of this section we shall investigate the strong and pointwise dyadic differentiability of dyadic integrals If. We begin by describing the difference functions $d_n(If)$ ($n \in \mathbf{N}$) of If (see [6]).

Theorem 7. *Suppose that $f \in L^1(\mathbb{R}^+)$ is dyadically integrable. Then*

(47) $$d_n(If) = d_n W_n * f$$

and

(48) $$(d_n W_n * f)(x) = \lim_{k \to \infty} \int_0^{2^k} \frac{\alpha_n(y)}{y} \hat{f}(y)\psi_x(y)\,dy$$

for $n \in \mathbf{N}$, where the limit exists both in L^1-norm and for a.e. $x \in \mathbb{R}^+$, where

(49) $$\alpha_n(y) := \sum_{j=-n}^{n} 2^{j-1}(1 - \psi_{2^{-(j+1)}}(y)) = \sum_{j=-n}^{n} y_j 2^{-j-1}$$

$$(y = \sum_{j=-\infty}^{\infty} y_j 2^{-j-1} \in \mathbf{R}^+, y_j \in \{0,1\}, n \in \mathbf{N}).$$

To estimate the functions $d_n W_n$ $(n \in \mathbf{N})$ we introduce the functions β_n on $\mathbf{R}^+$ by

(50) $$\beta_n(y) := \sum_{j=-n}^{0} y_j 2^{-j-1}$$

for $n \in \mathbf{N}$ and $y \in \mathbf{R}^+$. Notice that

(51) $$2^n \alpha_n(2^{-n} y) = \beta_{2n}(y)$$

for $n \in \mathbf{N}$ and $y \in \mathbf{R}^+$. We also introduce functions f_n, g_n by

(52) $$f_n(x) := 2^{-n} \sum_{j=-\infty}^{n} 2^j \sum_{i=j}^{n} D_{2^i}(x \dot{+} 2^{-j-1}),$$

$$g_n(x) := \sum_{j=-\infty}^{n} 2^j \sum_{i=n}^{\infty} 2^{-i} D_{2^i}(x \dot{+} 2^{-j-1})$$

for $n \in \mathbf{Z}$ and $x \in \mathbf{R}^+$.

Theorem 8. *For each $n \in \mathbf{N}$ and $x \in \mathbf{R}^+$ let*

(53) $$V_n(x) := \lim_{k \to \infty} \int_0^{2^k} \frac{\beta_n(y)}{y} \psi_x(y) dy.$$

Then V_n exists everywhere and

(54) $$(d_n W_n)(x) = 2^{-n} V_{2n}(2^{-n} x)$$

for $n \in \mathbf{N}$ and $x \in \mathbf{R}^+$. Moreover,

(55) $$|V_n| \le 10 f_o + g_o + \chi[0,1)|d_n W| \qquad (n \in \mathbf{N})$$

and

(56) $$\|V_n\|_1 = 0(1)$$

as $n \to \infty$, where $d_n W$ $(n \in \mathbf{N})$ was defined on $[0,1)$ as follows (see [8]):

(57) $$(d_n W)(x) := D_{2^n}(x) + \sum_{i=1}^{\infty} \sum_{k=0}^{2^n-1} \frac{k}{i2^n + k} \psi_{i2^n+k}(x) \qquad (x \in [0,1), n \in \mathbf{N}).$$

These estimates imply that every dyadic integral is strongly dyadically differentiable (see [6], [10]).

Theorem 9. *If $f \in L^1(\mathbf{R}^+)$ is dyadically integrable, then If is strongly dyadically differentiable and*

$$d(If) = f. \tag{58}$$

To investigate the pointwise dyadic differentiability of the dyadic integral we introduce the maximal operator

$$T^* f := \sup_{n \in \mathbf{N}} |d_n W_n * f| \qquad (f \in L^1(\mathbf{R}^+)). \tag{59}$$

Since by (54) and (56)

$$\|d_n W_n\|_1 = 0(1) \tag{60}$$

as $n \to \infty$, the operator T^* is of type (∞, ∞), i.e., there is a constant $C > 0$ such that

$$\|T^* f\|_\infty \le C\|f\|_\infty \qquad (f \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)). \tag{61}$$

The following result shows that T^* is of weak type $(1,1)$ (see [7]).

Theorem 10. *There is a constant $A > 0$ such that*

$$|\{x \in \mathbf{R}^+ : (T^* f)(x) > y\}| \le A \frac{\|f\|_1}{y} \tag{62}$$

for all $f \in L^1(\mathbf{R}^+)$ and $y > 0$.

From this theorem we can deduce the following

Corollary 4. *Suppose that $f \in L^1(\mathbf{R}^+)$ is dyadically integrable. Then If is a.e. dyadically differentiable on $\mathbf{R}^+$, and*

$$(If)^{[1]} = f \qquad \text{a.e} \tag{63}$$

We remark that it is easy to see that (63) is true for functions defined in (46) whose linear hull is L^1 dense in the set of dyadically integrable functions.

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ON THE A.E. DYADIC DIFFERENTIABILITY OF DYADIC INTEGRAL ON R+

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Abstract. In their paper [1] *P.L.Butzer* and *H.J.Wagner* have introduced the concept of *dyadic derivative* for functions defined on the dyadic field $\mathbf{R}^+$. Furthermore, *Wagner* [5] has defined the notion of *dyadic integral* as the inverse of the dyadic derivative and investigated the strong dyadic differentiability of dyadic integrals. In this connection see also our previous paper [2]. In this work we shall be concerned with the almost everywhere dyadic differentiability of dyadic integrals on $\mathbf{R}^+$.

1. THE DYADIC DERIVATIVE

Let $f : \mathbf{R}^+ \to \mathbf{C}$ be a function defined on $\mathbf{R}^+$, and let for every $n \in \mathbf{N}$

$$(1) \qquad d_n f := \sum_{j=-n}^{n} 2^{j-1}(f - \tau_{2^{-(j+1)}} f)$$

be the n^{th} dyadic difference function of f, where τ_h $(h \in \mathbf{R}^+)$ are the dyadic translation operators defined by

$$(2) \qquad (\tau_h f)(x) := f(x \dot{+} h) \quad (x, h \in \mathbf{R}^+)$$

($\dot{+}$ denotes the dyadic addition on $\mathbf{R}^+$).
If for some point $x \in \mathbf{R}^+$ the limit

$$(3) \qquad \lim_{n \to \infty} (d_n f)(x) =: f^{[1]}(x)$$

exists, then we say that f *is dyadically differentiable at the point* $x \in \mathbf{R}^+$ and $f^{[1]}(x)$ is the dyadic derivative of f at $x \in \mathbf{R}^+$. If $f \in L^1(\mathbf{R}^+)$ is an integrable function and there exists a function $g \in L^1(\mathbf{R}^+)$ for which

$$(4) \qquad \lim_{n \to \infty} \|d_n f - g\|_1 = 0$$

holds, then f is said to be *strongly dyadically differentiable* in $L^1(\mathbf{R}^+)$, and $Df := g$ is the strong dyadic derivative of f.

Let $n \in \mathbf{N}$, and define the function W_n by its *Walsh-Fourier transform* $\hat{W}_n$ as follows:

$$\hat{W}_n(y) := \begin{cases} 0, & y \in [0, 2^{-n}) \\ 1/y, & y \in [2^{-n}, +\infty). \end{cases} \tag{5}$$

Wagner has proved (see [5]) that there exists an unique function $W_n \in L^1(\mathbf{R}^+)$ for which (5) holds; moreover,

$$W_n(x) = \lim_{k \to \infty} \int_{2^{-n}}^{2^k} \frac{1}{y} \psi_x(y) dy \quad (x \in \mathbf{R}^+), \tag{6}$$

and the limit can be taken in the $L^1(\mathbf{R}^+)$-norm or in the pointwise sense. Here and in the sequel the symbols ψ_x $(x \in \mathbf{R}^+)$ denote the generalized Walsh functions.

In the following we introduce the *inverse operation of the dyadic derivative* by the following definition (see [5]): *if for a function $f \in L^1(\mathbf{R}^+)$ there exists a function $g \in L^1(\mathbf{R}^+)$ such that*

$$\lim_{n \to \infty} \|W_n * f - g\|_1 = 0, \tag{7}$$

then g is called the strong dyadic integral of f *and is denoted by If* ($*$ denotes the dyadic convolution).

For this notion of a dyadic integral *Wagner* proved that the following assertions are equivalent for $f, g \in L^1(\mathbf{R}^+)$:

(8)

i) $$g = If,$$

ii) $$\hat{g}(y) = \begin{cases} 0 & y = 0 \\ \hat{f}(y)/y & y > 0. \end{cases}$$

We remark that if $f \in L^1(\mathbf{R}^+)$, then If is not necessarily defined. For example, if $f := \chi[0,1)$ is the characteristic function of the interval $[0,1)$, then If is not defined (see [5]). Therefore, in the following we suppose that $f \in L^1(\mathbf{R}^+)$ and the dyadic integral $If \in L^1(\mathbf{R}^+)$ of f exists. In the paper [2] we proved that in this case for the dyadic difference functions of If we have

$$d_n(If) = d_n W_n * f \qquad (n \in \mathbf{N}). \tag{9}$$

Moreover, for the functions $d_n W_n$ $(n \in \mathbf{N})$ the following equality is true:

$$(d_n W_n)(x) = \lim_{k \to \infty} \int_0^{2^k} \frac{1}{y} \alpha_n(y) \psi_x(y) dy \quad (n \in \mathbf{N}, x \in \mathbf{R}^+), \tag{10}$$

where

(11)
$$\alpha_n(y) := \sum_{j=-n}^{n} 2^{j-1}(1-\psi_{2^{(j+1)}}(y)) = \sum_{j=-n}^{n} y_j 2^{-j-1}$$

$$\left(y = \sum_{j=-\infty}^{\infty} y_j 2^{-j-1} \in \mathbf{R}^+, \quad y_j \in \{0,1\}, n \in \mathbf{N}\right).$$

2. PRELIMINARIES

In the following our aim is to give an estimate for the functions $d_n W_n$ $(n \in \mathbf{N})$. To this end, we define $\beta_n(y)$ for $y \in \mathbf{R}^+$ and $n \in \mathbf{N}$ as follows:

(12)
$$\beta_n(y) := \sum_{j=-n}^{0} y_j 2^{-j-1}.$$

It is easy to see that

(13)
$$2^n \alpha_n(2^{-n} y) = \beta_{2n}(y) \qquad (y \in \mathbf{R}^+, n \in \mathbf{N}).$$

Let us introduce the functions V_n $(n \in \mathbf{N})$ by the following equality:

(14)
$$V_n(x) := \lim_{k\to\infty} \int_0^{2^k} \frac{1}{y}\beta_n(y)\psi_x(y)dy \qquad (x \in \mathbf{R}^+, n \in \mathbf{N}).$$

With the functions V_n $(n \in \mathbf{N})$ we can express the functions $d_n W_n$ $(n \in \mathbf{N})$ as follows (see [2]):

(15)
$$(d_n W_n)(x) = 2^{-n} V_{2n}(2^{-n}x) \qquad (x \in \mathbf{R}^+, n \in \mathbf{N}).$$

In the following we give an estimate for the functions V_n $(n \in \mathbf{N})$. For this purpose we introduce the functions f_n, g_n by

$$f_n(x) := 2^{-n} \sum_{j=-\infty}^{n} 2^j \sum_{i=j}^{n} D_{2^i}(x \dot{+} 2^{-j-1}),$$

(16)

$$g_n(x) := \sum_{j=-\infty}^{n} 2^j \sum_{i=n}^{\infty} 2^{-i} D_{2^i}(x \dot{+} 2^{-j-1})$$

for $n \in Z$ and $x \in \mathbf{R}^+$, where D_{2^i} $(i \in \mathbf{Z})$ denotes the 2^i-th Walsh-Dirichlet kernel on $\mathbf{R}^+$:

(17)
$$D_{2^i}(x) = \begin{cases} 2^i & x \in [0, 2^{-i}) \\ 0 & x \in [2^{-i}, +\infty). \end{cases}$$

From the definition of f_n and g_n it is easily follows that for every $n \in \mathbf{Z}$,

(18)
$$\|f_n\|_1 \le \sum_{j=-\infty}^{n} (n-j+1)2^{-n+j} = 4,$$

$$\|g_n\|_1 \le \sum_{j=-\infty}^{n} 2^{j-n+1} = 4.$$

Lemma 1. *For the functions V_n $(n \in \mathbf{N})$ the following estimate is true:*

(19)
$$|V_n| \le 10 f_0 + g_0 + \chi[0,1)|d_n W| \qquad (n \in \mathbf{N}),$$

where the functions $d_n W$ $(n \in \mathbf{N})$ were defined on $[0,1)$ in [3] as follows:

(20)
$$(d_n W)(x) := D_{2^n}(x) + \sum_{i=1}^{\infty} \sum_{k=0}^{2^n-1} \frac{k}{i2^n + k} \cdot \psi_{i2^n+k}(x) \quad (x \in [0,1), n \in \mathbf{N}).$$

Proof. First we notice that for $\beta_n(y)$, defined in (12), we have $\beta_n(y) = k$ for $y \in [i2^n + k, i2^n + k + 1)$, $i, n \in \mathbf{N}$, $0 \le k < 2^n$. Next define $0 \le j_n < 2^n$ for each $j \in \mathbf{N}$ by $j \equiv j_n (\mathrm{mod} 2^n)$. With these notations the function V_n can be written in the following form:

$$V_n(x) = \sum_{j=1}^{\infty} j_n \psi_j(x) \int_j^{j+1} \frac{\psi_{[x]}(y)}{y} dy =$$

(21)
$$= \sum_{j=1}^{\infty} j_n \psi_j(x) A_j(x) + \chi[0,1)(x) \cdot \sum_{j=1}^{\infty} \frac{j_n \psi_j(x)}{j} =:$$

$$=: V_n^{(1)}(x) + \chi[0,1)(x) \cdot (d_n W)(x),$$

where

(22)
$$A_j(x) := \int_j^{j+1} \left(\frac{1}{y} - \frac{1}{j}\right) \psi_{[x]}(y) dy$$

for $j \in \mathbf{P}$ and $x \in \mathbf{R}^+$. For each $j \in \mathbf{P}$ we have using integration by parts that

$$A_j(x) = \int_j^{j+1} \frac{1}{y^2}(J^{(1)}_{[x]}(y) - J^{(1)}_{[x]}(1))dy =$$

(23)
$$= \int_j^{j+1} (\frac{1}{y^2} - \frac{1}{j^2})(J^{(1)}_{[x]}(y) - J^{(1)}_{[x]}(1))dy +$$

$$+\frac{1}{j^2}(J^{(2)}_{[x]}(1) - J^{(1)}_{[x]}(1)) =: A^1_j(x) + A^2_j(x),$$

where

(24)
$$J^{(1)}_\ell(x) := \int_0^x \psi_\ell(t)dt,$$

$$J^{(2)}_\ell(x) := \int_0^x J^{(1)}_\ell(t)dt \qquad (\ell \in \mathbf{N}, x \in \mathbf{R}^+)$$

are the integral functions of ψ_ℓ and $J^{(1)}_\ell$ $(\ell \in \mathrm{N})$, respectively. It is easy to see that

$$|J^{(1)}_{[x]}(1)| \le f_0(x), \quad |J^{(2)}_{[x]}(\xi)| \le f_0(x)$$

for every $x, \xi \in \mathrm{R}^+$. Using the mean value theorem for integrals we see that for a suitable choice of $\xi \in (j, j+1)$,

$$|A^1_j(x)| = (\frac{1}{j^2} - \frac{1}{(j+1)^2}) \cdot |\int_\xi^{j+1} (J^{(1)}_{[x]}(y) - J^{(1)}_{[x]}(1))dy| \le \frac{4}{j^3} f_0(x).$$

Similarly we obtain,

$$|A^2_j(x)| \le \frac{1}{j^2} f_0(x).$$

Therefore for every $x \in \mathrm{R}^+$ we have

$$|V^1_n(x)| \le 4 f_0(x) \sum_{j=1}^{\infty} \frac{j_n}{j^3} +$$

$$+ |\sum_{j=1}^{2^n - 1} \frac{\psi_j(x)}{j}| \cdot |J^{(2)}_{[x]}(1) - J^{(2)}_{[x]}(1)| + f_0(x) \sum_{j=2^n}^{\infty} \frac{j_n}{j^2}.$$

Since

$$\sum_{j=1}^{\infty} \frac{j_n}{j^3} < \sum_{j=1}^{\infty} \frac{1}{j^2} < 2,$$

and

$$\sum_{j=2^n}^{\infty} \frac{j_n}{j^2} = \sum_{i=1}^{\infty} \sum_{k=0}^{2^n-1} \frac{k}{(i2^n+k)^2} \leq \sum_{i=1}^{\infty} \frac{2^{2n}}{i^2 2^{2n}} < 2,$$

we have for $x \in \mathbf{R}^+$

$$|V_n^1(x)| \leq 10 f_0(x) + |\sum_{j=1}^{2^n-1} \frac{\psi_j(x)}{j}| \cdot |J_{[x]}^{(2)}(1) - J_{[x]}^{(1)}(1)|.$$

It is easy to see that

$$|J_{[x]}^{(2)}(1) - J_{[x]}^{(1)}(1)| \leq \frac{1}{2} D_1(x) + \frac{1}{4} \sum_{j=0}^{\infty} 2^{-j} D_1(x \dot{+} 2^j) \qquad (x \in \mathbf{R}^+),$$

$$|\sum_{j=1}^{2^n-1} \frac{\psi_j(x)}{j}| \leq \sum_{k=0}^{\infty} 2^{-k+2} \bar{D}_{2^k}(x) \qquad (x \in \mathbf{R}^+, n \in \mathbf{N}),$$

where $\bar{D}_{2^k}$ $(k \in \mathbf{N})$ is the 1-periodic extension of D_{2^k} from $[0,1)$ to $\mathbf{R}^+$ (see [3]). Using these facts and the definition of g_0 we get the estimate

$$|\sum_{j=1}^{2^n-1} \frac{\psi_j(x)}{j}| \cdot |J_{[x]}^{(2)}(1) - J_{[x]}^{(1)}(1)| \leq g_0(x) \qquad (n \in \mathbf{N}, x \in \mathbf{R}^+).$$

Consequently, we obtain that

$$|V_n^1| \leq 10 f_0 + g_0$$

and

$$|V_n| \leq 10 f_0 + g_0 + \chi[0,1)|d_n W| \qquad (n \in \mathbf{N}).$$

Thus, the lemma is proved.

We remark that since

(25) $$\int_0^1 |d_n W| = 0(1) \qquad (n \to \infty)$$

(see [3]), from the lemma we have that

(26) $$\|V_n\|_1 = 0(1) \qquad (n \to \infty).$$

Using this fact it is easy to prove that the dyadic integral function $If \in L^1(\mathbf{R}^+)$ of $f \in L^1(\mathbf{R}^+)$ (if it exists) is strongly dyadically differentiable and

$$D(If) = f \tag{27}$$

(see [2], [5]).

3. POINTWISE DIFFERENTIABILITY OF THE DYADIC INTEGRAL

In the following we investigate the pointwise dyadic differentiability of dyadic integral functions. For this purpose we introduce the following maximal operator:

$$T^* f := \sup_{n \in \mathbf{N}} |d_n W_n * f| \qquad (f \in L^1(\mathbf{R}^+)). \tag{28}$$

Since from (26) and (15)

$$\|d_n W_n\|_1 = 0(1) \qquad (n \to \infty), \tag{29}$$

the maximal operator T^* is of type (∞, ∞), i.e., there is a constant $C > 0$ such that

$$\|T^* f\|_\infty \leq C\|f\|_\infty \qquad (f \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)). \tag{30}$$

The following result shows that T^* is of weak type $(1,1)$.

Theorem. *There is a constant $A > 0$ such that*

$$|\{x \in \mathbf{R}^+ : (T^* f)(x) > y\}| \leq A \frac{\|f\|_1}{y} \tag{31}$$

for all $f \in L^1(\mathbf{R}^+)$ and $y > 0$.

From this theorem easily follows the following

Corollary. *Suppose that $f \in L^1(\mathbf{R}^+)$ is dyadically integrable. Then the dyadic integral If is a.e. dyadically differentiable on $\mathbf{R}^+$ and*

$$(If)^{[1]} = f \qquad a.e. \tag{32}$$

In fact, the functions $\chi[0, 2^n)\psi_{2^{-n}m}$ $(n \in \mathbf{N}, m \in \mathbf{P})$ form a closed system in the space of dyadically integrable functions in $L^1(\mathbf{R}^+)$-norm and it is easy to check that for these functions (32) is true (see [2]).

For the proof of the Theorem we need the notion of quasi-locality for operators. Let us introduce the following notation:

$$L_0^1(\mathbf{R}^+) := \{f \in L^1(\mathbf{R}^+) : \hat{f}(0) = \int_{\mathbf{R}^+} f = 0\}.$$

The operator $T : L^1(\mathbf{R}^+) \to L_0^1(\mathbf{R}^+)$ is called quasi-local if $f \in L_0^1(\mathbf{R}^+)$ and $\{x \in \mathbf{R}^+ : f(x) \neq 0\} \subset I$ imply that Tf is integrable outside I and

$$\int_{\mathbf{R}^+ \setminus I} |Tf| \leq C\|f\|_1, \tag{33}$$

where I denotes a dyadic interval and $C > 0$ is a constant independent of f and I. For example, it is easy to see that the operators

$$(E_i^j f)(x) := (S_{2^i} f)(x \dotplus 2^{-j-1}) \qquad (i, j \in \mathbf{Z}, f \in L^1(\mathbf{R}^+), x \in \mathbf{R}^+) \tag{34}$$

are quasi-local (here $S_{2^i} f = f * D_{2^i}$, $i \in \mathbf{Z}$, denotes the 2^i-th partial integral of $f \in L^1(\mathbf{R}^+)$).

In the following, let

$$T_n : L^1(\mathbf{R}^+) \to L^1(\mathbf{R}^+) \qquad (n \in \mathbf{N})$$

be a sequence of bounded linear operators, and denote by T the maximal operator defined by

$$Tf := \sup_{n \in \mathbf{N}} |T_n f| \qquad (f \in L^1(\mathbf{R}^+)).$$

For the operator T we have the following

Lemma 2. *Suppose that T is quasi-local and of type (∞, ∞), i.e.,*

$$\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty =: A < +\infty.$$

Then T is of weak type $(1,1)$, i.e., for every $f \in L^1(\mathbf{R}^+)$ and $y > 0$ we have

$$|\{x \in \mathbf{R}^+ : (Tf)(x) > y\}| \leq \tilde{C} \frac{\|f\|_1}{y}, \tag{35}$$

where $\tilde{C} > 0$ is a constant independent of f and y.

Proof. We use for f a decomposition of Calderon-Zygmund-type: for every $y > 0$ the function f can be written in the form

$$f = g + h = g + \sum_{k \in \mathbf{N}} h_k,$$

where

$$h_k = \chi(I_k) h \qquad (k \in \mathbf{N}),$$

and

$$\text{i)} \ \|g\|_\infty \leq 2y, \qquad \text{ii)} \ \int_{\mathbf{R}^+} h_k = \int_{I_k} h = 0,$$

$$\|h_k\|_1 = \int_{I_k} |h| \leq 4y|I_k| \qquad (k \in \mathbf{N}),$$

iii) the dyadic intervals I_k $(k \in \mathbf{N})$ are pairwise disjoint and for the set

$$\Omega := \bigcup_{k\in \mathbf{N}} I_k \quad , \quad |\Omega| \leq \frac{\|f\|_1}{y}.$$

Since the operator T is quasi-linear, we deduce that

$$Tf \leq Tg + Th \quad , \quad \|Tg\|_\infty \leq A\|g\|_\infty \leq 2Ay.$$

Using these facts, we have that

$$|\{x \in \mathbf{R}^+ : (Tf)(x) > (1+2A)y\}| \leq |\{x \in \mathbf{R}^+ : (Th)(x) > y\}| \leq$$

$$\leq |\Omega| + |\{x \notin \Omega : (Th)(x) > y\}| \leq \frac{\|f\|_1}{y} + \frac{1}{y}\int_{\bar{\Omega}} Th,$$

where $\bar{\Omega} := \mathbf{R}^+ \setminus \Omega$ is the complement of Ω. From the quasi-linearity of T we get that

$$Th \leq \sum_{k\in \mathbf{N}} Th_k,$$

and the quasi-locality of T implies

$$\int_{\bar{\Omega}} Th_k \leq C\|h_k\|_1 \qquad (k \in \mathbf{N}).$$

Using these facts, we have that

$$\frac{1}{y}\int_{\bar{\Omega}} Th \leq \frac{1}{y}\sum_{k\in \mathbf{N}} \int_{\bar{\Omega}} Th_k \leq$$

$$\leq \frac{C}{y}\sum_{k\in \mathbf{N}} \|h_k\|_1 \leq \frac{C}{y}\sum_{k\in \mathbf{N}} 4y|I_k| \leq 4C\frac{\|f\|_1}{y}.$$

Moreover,

$$|\{x \in \mathbf{R}^+ : (Tf)(x) > (1+2A)y\}| \leq (1+4C)\frac{\|f\|_1}{y}$$

or, equivalently,

$$|\{x \in \mathbf{R}^+ : (Tf)(x) > y\}| \leq (1+4C)(1+2A)\frac{\|f\|_1}{y} \qquad (f \in L^1(\mathbf{R}^+), y > 0),$$

i.e., T is of weak type $(1,1)$. Thus, the lemma is proved.

Proof of Theorem. Since the maximal operator T^* is of type (∞,∞), we have only to show that T^* is quasi-local. Let us use for $n \in \mathbf{Z}$ and any function h defined on $\mathbf{R}^+$ the following notation:

$$h^{<n>}(x) := 2^{-n} h(2^{-n}x) \qquad (x \in \mathbf{R}^+). \tag{36}$$

Let f_n, g_n be defined by (16) and recall that

$$\|f_n\|_1 \quad , \quad \|g_n\|_1 \leq 4 \qquad (n \in \mathbf{Z}).$$

Notice for $i,j \in \mathbf{Z}$ that

$$2^{-n} D_{2^i}(2^{-n}x \dot{+} 2^{-j-1}) = D_{2^{i-n}}(x \dot{+} 2^{-(j-n)-1})$$

for all $x \in \mathbf{R}^+$. Consequently,

$$g_m^{<n>} = g_{m-n}, \quad f_m^{<n>} = f_{m-n} \qquad (m,n \in \mathbf{Z}). \tag{37}$$

Moreover, for $2^{s-1} \leq \ell < 2^s$, $s,\ell \in \mathbf{P}$ we have the following estimate for the Walsh-Fejér kernels K_ℓ defined on $[0,1)$ (see [3]):

$$|K_\ell| \leq \sum_{j=0}^{s-1} 2^{j-s} \sum_{i=j}^{s-1} (D_{2^i} + \tau_{2^{-j-1}} D_{2^i}) \leq \sum_{i=0}^{s} 2^{i-s} D_{2^i} + f_s. \tag{38}$$

For the functions $d_n W$ $(n \in \mathbf{N})$ defined on $[0,1)$ (see (20)) the following estimate is true for $n \in \mathbf{N}$, $x \in [0,1)$ ([3]):

$$|d_n W(x)| \leq 3D_{2^n}(x) + \tag{39}$$

$$+4 \cdot \sum_{j=0}^{n-1} 2^j \sum_{i=n}^{\infty} 2^{-i} \cdot D_{2^i}(x \dot{+} 2^{-j-1}) + 8 \cdot 2^{-n} \sum_{k=1}^{2^n} |K_k(x)| + 4K_{2^n}(x).$$

From this and (38) we deduce that

$$\chi[0,1)|d_n W| \leq 4(D_{2^n} + \sum_{s=0}^{n} 2^{s-n} D_{2^s}) + 8(\sum_{s=0}^{n} 2^{s-n} f_s + f_n).$$

Since

$$D_{2^s}^{<n>} = D_{2^{s-n}} \qquad (s,n \in \mathbf{Z}),$$

we conclude by (15) and Lemma 1 that

$$|d_n W_n| = |V_{2n}^{<n>}| \leq 10 f_{-n} + g_{-n} +$$
$$+4(D_{2^n} + \sum_{s=0}^{2n} 2^{s-2n} \cdot D_{2^{s-n}}) + 8(f_n + \sum_{s=0}^{2n} 2^{s-2n} f_{s-n}) \qquad (n \in \mathbf{N}).$$

Let us introduce the following maximal operators:

$$E^* f := \sup_{i \in \mathbf{Z}} |D_{2^i} * f|, \qquad G^* f := \sup_{i \in \mathbf{Z}} |g_i * f|,$$
$$F^* f := \sup_{i \in \mathbf{Z}} |f_i * f| \qquad (f \in L^1(\mathbf{R}^+)).$$

We have proved that for $f \in L^1(\mathbf{R}^+)$,

$$T^* f \leq 12 E^* |f| + G^* |f| + 34 F^* |f|.$$

Thus, the quasi-locality of T^* follows from the quasi-locality of E^*, G^* and F^*. This is trivial for E^*, and we prove the quasi-locality for G^* as follows (the proof of quasi-locality for F^* is similar): let $f \in L_0^1(\mathbf{R}^+)$ and $I \subset \mathbf{R}^+$ be a dyadic interval for which $\{x \in \mathbf{R}^+ : f(x) \neq 0\} \subset I$. If $|I| = 2^{-m}$, $m \in \mathbf{Z}$, then we define $\ell(I) := m$. Since for $x \notin I$ and $j \geq \ell(I) - 1$, $x \notin \tau_{2^{-j-1}}(I)$, we have that

$$(E_i^j f)(x) = (S_{2^i} f)(x \dot{+} 2^{-j-1}) = 0$$

if $i \leq \ell(I)$, $j \in \mathbf{Z}$ or $i \in \mathbf{Z}$, $j \geq \ell(I) - 1$. Thus

$$|(g_n * f)(x)| \leq \sum_{j=-\infty}^{\ell(I)-1} 2^j \sum_{i=\ell(I)+1}^{\infty} 2^{-i} |(E_i^j f)(x)|$$

for $n \in \mathbf{Z}, x \in \mathbf{R}^+$. Since the right side of this estimate is independent of n, it also holds for $(G^* f)(x)$. Using this,

$$\int_{\mathbf{R}^+ \setminus I} G^* f \leq \|f\|_1 \sum_{j=-\infty}^{\ell(I)-1} 2^j \sum_{i=\ell(I)+1}^{\infty} 2^{-i} \leq \|f\|_1,$$

i.e., the maximal operator G^* is quasi-local. Thus, the Theorem is proved.

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HÖRMANDER-TYPE MULTIPLIER THEOREMS ON LOCALLY COMPACT VILENKIN GROUPS

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Abstract. Let G be a locally compact Vilenkin group. We present a possible approach to a theory of dyadic differentiation on such groups. As an application we discuss several multiplier theorems for weighted Lebesgue and Hardy spaces on G, in which the assumption is a Hörmander-type condition for the multiplier. This condition is formulated in terms of the dyadic derivative of fractional order.

1. INTRODUCTION

For functions defined on $\mathbf{R}$ the multiplier theorem of Hörmander can be formulated most conveniently using the following notation, cf [5]. We say that a function m belongs to $M(s,\ell,\mathbf{R})$, $1 \leq s \leq \infty$ and $\ell \in \mathbf{N}$, if $f \in L^\infty(\mathbf{R})$ and if for all integers β with $0 \leq \beta \leq \ell$ we have

$$\sup\left\{R^{\beta s-1}\int_{R<|x|<2R}\left|D^\beta m(x)\right|^s dx : R>0\right\} < \infty.$$

In [2, Theorem 2.5] Hörmander proved his celebrated multiplier theorem, here stated for functions on $\mathbf{R}$ instead of on $\mathbf{R}^n$.

Theorem H. Let $1 < p < \infty$. If $m \in M(2, \ell, \mathbf{R})$ for some $\ell \in \mathrm{N}$ then there exists a $C > 0$ so that for all f in the Schwartz space $\mathcal{S}(\mathbf{R})$ we have $||(m\hat{f})\check{}||_p \leq C||f||_p$.

In 1979 Kurtz and Wheeden generalized Theorem H to weighted Lebesgue spaces, where the weight function either satisfied a Muckenhoupt A_p condition or else was a suitable power of $|x|$. In Theorem 3 of [5] they proved the following result, stated here again only for functions on $\mathbf{R}$ instead of on $\mathbf{R}^n$.

Theorem KW. Let $0 < p < 1$ and $1 < s \leq 2$. If $m \in M(s, 1, \mathbf{R})$ then there exists a $C > 0$ so that for all $f \in \mathcal{S}(\mathbf{R})$ we have

$$||(m\hat{f})\check{}||_{p,|x|^\alpha} \leq C||f||_{p,|x|^\alpha},$$

provided $-1 < \alpha < p - 1$.

Eight years later Muckenhoupt, Wheeden and Young extended the definition of the classes $M(s, \ell, \mathbf{R})$ to all positive real values of ℓ and then proved several additional generalizations of Theorems H and KW. Of particular interest here is the following result, cf [6, Section 4].

Theorem MWY. Let $1 < p < \infty$, $1 < s \leq \infty$ and $\lambda > \max(1/s, \frac{1}{2})$. If $m \in M(s, \lambda, \mathbf{R})$ then there exists a $C > 0$ so that for all $f \in \mathcal{S}(\mathbf{R})$ we have

$$||(m\hat{f})\check{}||_{p,|x|^\alpha} \leq C||f||_{p,|x|^\alpha},$$

provided $\max(-1, -p\lambda) < \alpha < \min(p - 1, p\lambda)$.

In view of the various recent attempts to develop a differentiation theory for functions defined on certain topological groups, like the dyadic group, the (compact) Vilenkin groups or their locally compact generalizations, one of the authors raised the question in [10] whether an analogue of the above-mentioned theorems holds if we define a Hörmander condition, $m \in M(s, \lambda)$, on such groups using the new definition of differentiability. A solution to this question was found recently by Kitada, who obtained a direct analogue of Theorem MWY for functions defined on a locally compact Vilenkin group. Kitada also proved a multiplier theorem for Hardy spaces on these groups, again using a Hörmander type assumption for the multipliers. Subsequently, this result was extended to power-weighted Hardy spaces by Onneweer and Quek and, independently, by Kitada. In this paper we shall discuss these results.

2. DIFFERENTIATION ON VILENKIN GROUPS

Throughout this paper G will denote a locally compact Vilenkin group. Thus, G is a locally compact abelian group containing a strictly decreasing sequence of open compact subgroups $(G_n)_{-\infty}^{\infty}$ such that

(i) $\sup\{ \text{ order } G_n/G_{n+1} : n \in \mathbf{Z}\} < \infty$,

(ii) $\cup_{-\infty}^{\infty} G_n = G$ and $\cap_{-\infty}^{\infty} G_n = \{0\}$.

Such groups are the locally compact version of the so-called Vilenkin groups, which were first described in 1947 by Vilenkin [14]. Examples of such groups are given in [1, §4.1.2]. Further examples are the additive group of the field of p-adic numbers or, more general, of a local field, see [13].

Let Γ denote the dual group of G and for each $n \in \mathbf{Z}$ let

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

Then $(\Gamma_n)_{-\infty}^{\infty}$ is a strictly increasing sequence of compact open subgroups of Γ and

(i)* order $(\Gamma_{n+1}/\Gamma_n) = \text{order}(G_n/G_{n+1})$,

(ii)* $\cup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\cap_{-\infty}^{\infty} \Gamma_n = \{\gamma_0\}$,

where $\gamma_0(x) = 1$ for all $x \in G$, see [1, §4.1.4]. Thus the system $\Gamma, (\Gamma_{-n})_{-\infty}^{\infty}$ is again a locally compact Vilenkin group.

We choose Haar measures μ on G and λ and Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $\mu(G_n) = (\lambda(\Gamma_n))^{-1} := (m_n)^{-1}$ for every $n \in \mathbf{Z}$. Furthermore, there exists a metric d on $G \times G$ defined by $d(x,x) = 0$ and $d(x,y) = (m_n)^{-1}$ if $x - y \in G_n \backslash G_{n+1}$. For $x \in G$ we set $||x|| = d(x,0)$. For every $\alpha \in \mathbf{R}$ we define the function v_α on G by $v_\alpha(x) = ||x||^\alpha$; the corresponding measure $v_\alpha d\mu$ will also be denoted by $d\mu_\alpha$. The Lebesgue spaces on G with respect to the measures $d\mu_\alpha$ will be denoted by $L^p_\alpha(G)$ or L^p_α, and for $f \in L^p_\alpha$, $0 < p < \infty$ and $\alpha \in \mathbf{R}$, we set

$$||f||_{p,\alpha} = \left(\int_G |f(x)|^p d\mu_\alpha(x)\right)^{1/p}.$$

If $\alpha = 0$ we write, as usual, L^p and $||f||_p$ instead of L^p_0 and $||f||_{p,0}$.

As a further generalization of the usual L^p-spaces we define the Herz spaces on G. Both here and elsewhere we shall use the notation χ_A for the characteristic function of a set A.

Definition (2.1). Let $0 < p, q < \infty$ and $\alpha \in \mathbf{R}$. A measurable function $f : G \to \mathbf{C}$ belongs to the Herz space $K(\alpha, p, q; G) = K(\alpha, p, q)$ if

$$||f||_{K(\alpha,p,q)} = \left(\sum_{\ell=-\infty}^{\infty} ((m_\ell)^{-\alpha} ||f\chi_{G_\ell \backslash G_{\ell+1}}||_p)^q\right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

Thus, $K(\alpha/p, p, p) = L^p_\alpha$ if $\alpha \in \mathbf{R}$ and $0 < p < \infty$. We can also define a metric δ on $\Gamma \times \Gamma$ and in this case we set $\|\gamma\| = \delta(\gamma, \gamma_0) = m_n$ if $\gamma \in \Gamma_{n+1} \backslash \Gamma_n$.

The symbols $\hat{}$ and $\check{}$ will denote the Fourier transform and the inverse Fourier transform, respectively. We have

$$(\chi_{G_n})\hat{} = \left(\lambda(\Gamma_n)\right)^{-1} \chi_{\Gamma_n} = (m_n)^{-1} \chi_{\Gamma_n} := F_n$$

and, hence,

$$(\chi_{\Gamma_n})\check{} = \left(\mu(G_n)\right)^{-1} \chi_{G_n} := \Delta_n.$$

We now briefly describe the spaces of test functions $\mathcal{S}(G)$ and distributions $\mathcal{S}'(G)$; for more details, see [13]. A function $f : G \to \mathbf{C}$ belongs to $\mathcal{S}(G)$ if there exists $k, \ell \in \mathbf{Z}$ so that supp $f \subset G_k$ and f is constant on the cosets of G_ℓ in G. A sequence $(f_n)_1^\infty$ in $\mathcal{S}(G)$ converges to $f \in \mathcal{S}(G)$ if there exist $k, \ell \in \mathbf{Z}$ so that every f_n and f has its support in G_k and is constant on the cosets of G_ℓ in G and if $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on G. Next, $\mathcal{S}'(G)$ is the space of continuous linear functionals on $\mathcal{S}(G)$. A sequence $(\phi_n)_1^\infty$ in $\mathcal{S}'(G)$ converges to $\phi \in \mathcal{S}'(G)$ if for all $f \in \mathcal{S}(G)$ we have $\lim_{n\to\infty} < \phi_n, f > = < \phi, f >$.

In 1979 Onneweer gave a definition for a Riesz-type derivative for functions defined on G, see Remark 3 in [7]. In a subsequent paper [8] this definition was extended, at least for functions on a local field, to derivatives of any order $\alpha > 0$. The definition given in [8] easily extends to groups like G or Γ.

Definition (2.2). Let $\alpha > 0$ and $f \in L^1_{\text{loc}}(G)$. For $x \in G$ and $m \in \mathbf{Z}$ define $E_{m,\alpha} f(x)$ by

$$E_{m,\alpha} f(x) = \sum_{\ell=-\infty}^{m-1} \left((m_{\ell+1})^\alpha - (m_\ell)^\alpha\right)(f - \Delta_\ell * f)(x).$$

(i) If $\lim_{m\to\infty} E_{m,\alpha} f(x)$ exists, the limit is called the pointwise derivative of order α of f at x, denoted by $f^{(\alpha)}(x)$.
(ii) If $f \in L^p(G), 1 \le p < \infty$, and if $\lim_{m\to\infty} E_{m,\alpha} f$ exits in $L^p(G)$, the limit is called the strong derivative in $L^p(G)$ of order α of f, denoted by $D^\alpha_p f$. We set $\mathcal{D}(D^\alpha_p) = \{f \in L^p : D^\alpha_p f \text{ exists}\}$.

Remark (2.3). Since $\Gamma, (\Gamma_{-n})_{-\infty}^\infty$ is also a locally compact Vilenkin group we can define similarly pointwise and strong derivatives for functions $g \in L^1_{\text{loc}}(\Gamma)$. In this case we may replace the $E_{m,\alpha} f(x)$ in Definition (2.2) by

$$\widetilde{E}_{m,\alpha}\, g(\gamma) = \sum_{j=-m+1}^{\infty} \left((m_{j-1})^{-\alpha} - (m_j)^{-\alpha}\right)(g - F_j * g)(\gamma).$$

The following results are easy to prove, cf [8, Theorem 1].

Proposition (2.4). Let $\alpha > 0$. Then
(a) for every $\gamma \in \Gamma$ and $x \in G$, $\gamma^{(\alpha)}(x)$ exists and $\gamma^{(\alpha)}(x) = ||\gamma||^{\alpha}\gamma(x)$,
(b) for every $k \in \mathbf{Z}$, $(\Delta_k)^{(\alpha)}(x)$ exists for every $x \in G$ and $D_p^{\alpha}\Delta_k$ exists for every p with $1 \leq p < \infty$. Moreover,

$$D_p^{\alpha}\Delta_k(x) = (\Delta_k)^{(\alpha)}(x) = \sum_{\ell=-\infty}^{k} (m_\ell)^{\alpha}(\Delta_\ell - \Delta_{\ell-1})(x).$$

Since every $f \in \mathcal{S}(G)$ is a finite linear combination of translates of some function Δ_k (k depending on f), we have

Corollary (2.5). Let $\alpha > 0$. Every $f \in \mathcal{S}(G)$ is differentiable, both pointwise and in L^p for every $1 \leq p < \infty$.

Remark (2.6). It follows immediately from Proposition (2.4) that the derivative of a function in $\mathcal{S}(G)$ does not necessarily belong to $\mathcal{S}(G)$. Thus, $\mathcal{S}(G)$ is not a suitable class of test functions to define distributional derivatives on G. It would be interesting to determine what, if any, choice for the class of test functions would lead to an interesting and useful theory of distributional derivatives on G.

We now explain why the derivative as defined in Definition (2.2) is, at least for $1 \leq p \leq 2$, a Riesz-type derivative on G. To do so, we recall the definition of the spaces $W(L^p, \alpha)$ from [8].

Definition (2.7). Let $\alpha > 0$ and $1 \leq p \leq 2$. Then

$$W(L^p, \alpha) = \{f \in L^p : \text{there exists a } g \in L^p \text{ such that } \hat{g}(\gamma) = ||\gamma||^{\alpha}\hat{f}(\gamma) \text{ for a.e. } \gamma \in \Gamma\}.$$

An argument like the one used to prove Theorem 3 in [8] yields

Theorem (2.8). If $\alpha > 0$ and $1 \leq p \leq 2$ then $\mathcal{D}(D_p^{\alpha}) = W(L^p, \alpha)$.

For $2 < p < \infty$ no such simple characterization of the elements of $\mathcal{D}(D_p^{\alpha})$ in terms of their Fourier transforms is known. In this case we have a different characterization; a proof for the case where G is the additive group of a local field may be found in [9, Theorem 1].

Theorem (2.9). Let $\alpha > 0$ and $f \in L^p(G)$ with $2 < p < \infty$. Then $D_p^\alpha f$ exists if and only there exists a $\phi \in L^p(G)$ so that for all $g \in \mathcal{D}(D_{p'}^\alpha)$, where $1/p + 1/p' = 1$, we have

$$\int_G f(x) D_{p'}^\alpha g(x) d\mu(x) = \int_G \phi(x) g(x) d\mu(x).$$

Moreover, $\phi = D_p^\alpha f$ in case either of these functions exists; ϕ is called the weak derivative of f of order α.

3. HÖRMANDER-TYPE MULTIPLIERS ON G

In this section we give an application of the differentiation theory developed in Section 2. Our application consists of two multiplier theorems, one for multipliers on power-weighted Lebesgue spaces, Theorem (3.6), the other for power-weighted Hardy spaces, Theorem (3.15). In both theorems the multiplier spaces $M(s, \lambda)$ are described in terms of strong derivatives of order λ in $L^s(\Gamma)$, at least for $1 < s \leq 2$. We begin by giving the definition, due to Kitada, of the Hörmander classes $M(s, \lambda)$ on a Vilenkin group G. Throughout this section, if $\phi \in L^\infty(\Gamma)$ and $k \in \mathbf{Z}$ we set $\phi_k = \phi \chi_{\Gamma_k}$ and $\phi^k = \phi_{k+1} - \phi_k$.

Definition (3.1). Let $\phi \in L^\infty(\Gamma)$. For $\lambda > 0$ and $j \in \mathbf{Z}$ we define $D^\lambda \phi^j$ by $D^\lambda \phi^j = (||x||^\lambda (\phi^j)^\check{})^\hat{}$. We say that $\phi \in M(s, \lambda)$, $1 \leq s \leq \infty$, if

$$B(\phi, s, \lambda) := ||\phi||_\infty + \sup\{(m_j)^{\lambda - 1/s} ||D^\lambda \phi^j||_s : j \in \mathbf{Z}\} < \infty.$$

Remark (3.2). If $1 \leq s \leq 2$ then $\phi \in M(s, \lambda)$ provided the functions ϕ^j, the restrictions of ϕ to $\Gamma_{j+1} \backslash \Gamma_j$, are strongly differentiable in $L^s(\Gamma)$ and satisfy the inequality $B(\phi, s, \lambda) < \infty$. Clearly, this is the direct analogue on Γ of the Hörmander condition $m \in M(s, \lambda, \mathbf{R})$ for functions on $\mathbf{R}$ as defined in the Introduction.

The following simple result will be used later on, cf [4, Lemma 4.2], see also [6, Theorem 2.12 (i)].

Lemma (3.3). Let $1 < s \leq \infty$, $1 \leq t < \infty$ and $0 < \sigma \leq \lambda$. If $\sigma - 1/t \leq \lambda - 1/s$ then $M(s, \lambda) \subset M(t, \sigma)$. Moreover, there exists a $C > 0$ so that for all $\phi \in M(s, \lambda)$ we have $B(\phi, t, \sigma) \leq C B(\phi, s, \lambda)$.

Definition (3.4). Let $1 \leq p < \infty$ and $\alpha \in \mathbf{R}$. A function $\phi \in L^\infty(\Gamma)$ is a multiplier on L^p_α, $\phi \in \mathcal{M}(L^p_\alpha)$, if there exists a $C > 0$ so that for all $f \in \mathcal{S}(G) \cap L^p_\alpha$ we have

$$||(\phi \hat{f})^\check{}||_{p,\alpha} \leq C ||f||_{p,\alpha}.$$

In [11] a proof is given of the following theorem for multipliers on L^p_α.

Theorem OQ1. Let $\phi \in L^\infty(\Gamma)$ and $1 < r < \infty$. Assume there exist constants $C, \epsilon > 0$ so that for all $k, n \in \mathbf{Z}$ and for all $y \in G_l \subset G_{n+1}$,

$$\left(\int_{G_n \backslash G_{n+1}} |(\phi_k)\check{}(x-y) - (\phi_k)\check{}(x)|^r d\mu(x)\right)^{1/r} \leq C(m_n)^{\epsilon+1/r'}(m_\ell)^{-\epsilon}.$$

Then $\phi \in \mathcal{M}(L^p_\alpha)$ for $1 < p < \infty$ and $\max(-1, -p/r') < \alpha < \min(p-1, p/r')$.

An application of Theorem OQ1 yields our first Hörmander-type multiplier theorem on locally compact Vilenkin groups.

Theorem (3.5). Let $1 < r < \infty$, $1 < s \leq \infty$, let $t = \min(2, s, r')$ and let $\lambda > 1/t$. If $\phi \in M(s, \lambda)$ then $\phi \in \mathcal{M}(L^p_\alpha)$ for $1 < p < \infty$ and $\max(-1, -p/r') < \alpha < \min(p-1, p/r')$.

Proof. Take any $k, n \in \mathbf{Z}$; consider the function ϕ_k and any $y \in G_\ell \subset G_{n+1}$. First assume $k \leq \ell$, so that $G_\ell \subset G_k$. Since supp $\phi_k \subset \Gamma_k$, $(\phi_k)\check{}$ is constant on the cosets of G_k and, hence, on the cosets of G_ℓ. Therefore, if $x \in G$ and $y \in G_\ell$ we have $(\phi_k)\check{}(x-y) = (\phi_k)\check{}(x)$. Consequently,

$$I(y) := \left(\int_{G_n \backslash G_{n+1}} |(\phi_k)\check{}(x-y) - (\phi_k)\check{}(x)|^r d\mu(x)\right)^{1/r} = 0.$$

If $k > \ell$ we set $\phi_k = \phi_\ell + \sum_{j=\ell}^{k-1} \phi^j$. By the preceding argument and Minkowski's inequality we see that

$$\begin{aligned} I(y) &\leq \sum_{j=\ell}^{k-1}\left(\int_{G_n \backslash G_{n+1}} |(\phi^j)\check{}(x-y)|^r d\mu(x)\right)^{1/r} + \sum_{j=\ell}^{k-1}\left(\int_{G_n \backslash G_{n+1}} |(\phi^j)\check{}(x)|^r d\mu(x)\right)^{1/r} \\ &\leq 2\sum_{j=\ell}^{\infty}\left(\int_{G_n \backslash G_{n+1}} |(\phi^j)\check{}(x)|^r d\mu(x)\right)^{1/r}, \end{aligned}$$

because if $y \in G_\ell \subset G_{n+1}$ then $x - y \in G_n \backslash G_{n+1}$ if and only if $x \in G_n \backslash G_{n+1}$. Therefore,

$$\begin{aligned} I(y) &\leq C\sum_{j=\ell}^{\infty}(m_n)^\lambda\left(\int_{G_n \backslash G_{n+1}} \left|\|x\|^\lambda(\phi^j)\check{}(x)\right|^r d\mu(x)\right)^{1/r} \\ &\leq C\sum_{j=\ell}^{\infty}(m_n)^{\lambda-1/r+1/t'}\left(\int_{G_n \backslash G_{n+1}} \left|\|x\|^\lambda(\phi^j)\check{}(x)\right|^{t'} d\mu(x)\right)^{1/t'} \\ &\leq C\sum_{j=\ell}^{\infty}(m_n)^{\lambda-1/r+1/t'}\|D^\lambda\phi^j\|_t \\ &\leq C(m_n)^{\lambda-1/r+1/t'}B(\phi, t, \lambda)\sum_{j=\ell}^{\infty}(m_j)^{1/t-\lambda}. \end{aligned}$$

Applying Lemma (3.3) and using the assumption that $\lambda > 1/t$, so that $\sum_{j=\ell}^{\infty}(m_j)^{1/t-\lambda} \leq C(m_\ell)^{1/t-\lambda}$, we obtain

$$\begin{aligned} I(y) &\leq C(m_n)^{\lambda-1/t+1/r'}B(\phi,s,\lambda)(m_\ell)^{1/t-\lambda} \\ &\leq C(m_n)^{\epsilon+1/r'}(m_\ell)^{-\epsilon}, \end{aligned}$$

with $\epsilon = \lambda - 1/t > 0$. Therefore, ϕ satisfies the assumptions of Theorem OQ1 and, hence, its conclusion.

The following result is an improvement of Theorem (3.5); it is the direct analogue on G of Theorem MWY, stated in the Introduction. Theorem (3.6) is due to Kitada, cf [3].

Theorem (3.6). Let $1 < s \leq \infty$ and $\lambda > \max(\frac{1}{2}, 1/s)$. If $\phi \in M(s,\lambda)$ then $\phi \in \mathcal{M}(L^p_\alpha)$ for $1 < p < \infty$ and $\max(-1,-p\lambda) < \alpha < \min(p-1,p\lambda)$.

Proof. If $\lambda \geq 1$ then $\lambda > 1/r'$ for all $r \in (1,\infty)$ and Theorem (3.5) implies that $\phi \in \mathcal{M}(L^p_\alpha)$ for $1 < p < \infty$ and $-1 < \alpha < p-1$ and, hence, for $\max(-1,-p\lambda) < \alpha < \min(p-1,p\lambda)$. If $\lambda < 1$ and $\max(-1,-p\lambda) < \alpha < \min(p-1,p\lambda)$ we can choose an $r \in (1,\infty)$ so that $\lambda > 1/r'$ and $-p\lambda < -p/r' < \alpha < p/r' < p\lambda$. Again, Theorem (3.5) implies that $\phi \in \mathcal{M}(L^p_\alpha)$.

Remark (3.7). In [3] Kitada also proved the analogue on G of two additional multiplier theorems of Muckenhoupt, Wheeden and Young for power-weighted Lebesgue spaces, namely an analogue of Theorems (1.2) and (5.1) in [6]. The precise statement of Kitada's results follows; a proof of these theorems will appear elsewhere.

Theorem (3.8). Let $1 < p < \infty$, $1 \leq s \leq \infty$ and $\lambda > \max(1/s, |1/p - \frac{1}{2}|)$. If $\phi \in M(s,\lambda)$ and $\max(-1,-p\lambda,-1+p(\frac{1}{2}-\lambda)) < \alpha < \min(p\lambda,-1+p(\frac{1}{2}+\lambda),-1+p(\lambda+1-1/s))$, $\alpha \neq p-1$, then there exists a $C > 0$ so that for all $f \in \mathcal{S}_0(G) := \{f \in \mathcal{S}(G) : \hat{f}(\gamma_0) = 0\}$ we have

$$\|(\phi\hat{f})\check{}\|_{p,\alpha} \leq C\|f\|_{p,\alpha}.$$

Theorem (3.9). Let $1 < p < \infty$, $2 < s \leq \infty$ and $\frac{1}{2} \geq \lambda > \max(1/s, |1/p - \frac{1}{2}|)$. If $\phi \in M(s,\lambda)$ and $\max(-p\lambda,-1+p(\frac{1}{2}-\lambda)) < \alpha < \min(p\lambda,-1+p(\lambda+\frac{1}{2}))$ then there exists a $C > 0$ so that for all $f \in \mathcal{S}(G)$ we have

$$\|(\phi\hat{f})\check{}\|_{p,\alpha} \leq C\|f\|_{p,\alpha}.$$

Before presenting our second multiplier theorem, which deals with power-weighted Hardy spaces on G, we first state the relevant definitions as given in [4] or [12].

Definition (3.10). Let $0 < p \leq 1$ and $\alpha > -1$. A function $a : G \longrightarrow \mathbf{C}$ is called a $(p,\infty)_\alpha$ atom if there exists a set $I = x_0 + G_n$ so that (i) supp $a \subset I$, (ii) $||a||_\infty \leq (\mu_\alpha(I))^{-1/p}$, (iii) $\int_G a(x)d\mu(x) = 0$.

Definition (3.11). Let $0 < p \leq 1$ and $\alpha > -1$. A distribution $f \in \mathcal{S}'(G)$ belongs to the weighted (atomic) Hardy space H^p_α if there exists a sequence $(\lambda_i)_1^\infty$ in ℓ^p and a sequence $(a_i)_1^\infty$ of $(p,\infty)_\alpha$ atoms so that

$$f = \sum_{i=1}^{\infty} \lambda_i a_i \qquad \text{in } \mathcal{S}'(G).$$

We set

$$||f||_{H^p_\alpha} = \inf \left(\sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p},$$

with the infimum taken over all such representations of f.

Both in [4] and in [12] a proof is given of the following theorem.

Theorem (3.12). Let $0 < p \leq 1$ and $\alpha > -1$. An $f \in \mathcal{S}'(G)$ belongs to H^p_α if and only if the function $f^*(x) := \sup_k |f * \Delta_k(x)|$ belongs to L^p_α. Moreover, $||f||_{H^p_\alpha} \sim ||f^*||_{p,\alpha}$.

Definition (3.13). Let $0 < p \leq 1$ and $-1 < \alpha \leq 0$. A function $\phi \in L^\infty(\Gamma)$ is a multiplier on H^p_α, $\phi \in \mathcal{M}(H^p_\alpha)$, if there exists a $C > 0$ so that for all $f \in H^p_\alpha \cap L^2$ we have

$$||(\phi\hat{f})\check{}\,||_{H^p_\alpha} \leq C||f||_{H^p_\alpha}.$$

In [12, Corollary (4.14)] the following multiplier theorem for power-weighted Hardy spaces was proved.

Theorem OQ2. Let $0 < p \leq 1$ and $1 \leq s < \infty$. If $\phi \in L^\infty(\Gamma)$ and if for some $\epsilon > 0$,

$$\sup_k (m_k)^{1/p-1+\epsilon} ||(\phi^k)\check{}\,||_{K(1/p-1/s+\epsilon,s,\infty)} < \infty,$$

then $\phi \in \mathcal{M}(H^p_\alpha)$ for $-1 + p/s < \alpha \leq 0$.

As an application of Theorem OQ2 we obtain the following Hörmander-type multiplier theorem for power-weighted Hardy spaces on G.

Theorem (3.14). Let $0 < p \leq 1$ and $1 < s \leq \infty$. If $\phi \in M(s,\lambda)$ for some $\lambda > 1/p - 1/\max(2,s')$ then $\phi \in \mathcal{M}(H^p_\alpha)$ for $-1 + p/\max(2,s') < \alpha \leq 0$.

Proof. (i) Assume $1 < s \leq 2$. Then $\lambda > 1/p - 1/s'$ and there exists an $\epsilon > 0$ so that $\lambda = 1/p - 1/s' + \epsilon$. Moreover,

$$\begin{aligned}
\sup_j (m_j)^{1/p-1+\epsilon} \|(\phi^j)^{\check{}}\|_{K(1/p-1/s'+\epsilon,s',\infty)} &\leq \sup_j (m_j)^{\lambda-1/s} \|(\phi^j)^{\check{}}\|_{K(\lambda,s',s')} \\
&= \sup_j (m_j)^{\lambda-1/s} \| \, \|x\|^\lambda (\phi^j)^{\check{}} \|_{s'} \\
&= \sup_j (m_j)^{\lambda-1/s} \|(D^\lambda \phi^j)^{\check{}}\|_{s'} \\
&\leq C \sup_j (m_j)^{\lambda-1/s} \|D^\lambda \phi^j\|_s \\
&\leq C B(\phi, s, \lambda) < \infty,
\end{aligned}$$

because $\phi \in M(s,\lambda)$. Thus Theorem OQ2 implies that $\phi \in \mathcal{M}(H^p_\alpha)$ for $-1+p/s' < \alpha \leq 0$.

(ii) If $2 < s \leq \infty$ then, according to Lemma (3.3), if $\phi \in M(s,\lambda)$ then $\phi \in M(2,\lambda)$, provided $\lambda > 0$. Thus if $\lambda > 1/p - 1/2$ then it follows immediately from part (i) that $\phi \in \mathcal{M}(H^p_\alpha)$ for $-1 + p/2 < \alpha \leq 0$. This completes the proof of the theorem.

In [4, Theorem 4.5] the following somewhat stronger result was obtained. As far as the authors know, no comparable result for power-weighted Hardy spaces on $\mathbf{R}^n$ is known.

Theorem (3.15). Let $0 < p \leq 1$ and $1 < s \leq \infty$. If $\phi \in M(s,\lambda)$ for some $\lambda > 1/p - 1/\max(2,s')$ then $\phi \in \mathcal{M}(H^p_\alpha)$ for $\max(-1, -\lambda p) < \alpha \leq 0$.

Proof. (i) Assume $1 < s \leq 2$. If $\lambda p \geq 1$ and $-1 < \alpha \leq 0$ we can choose t such that $1 < t \leq s$ and $-1 + p/t' < \alpha$. Since, according to Lemma (3.3), $\phi \in M(t,\lambda)$, it follows immediately from Theorem (3.14) that $\phi \in \mathcal{M}(H^p_\alpha)$. If $1 - p/s' < \lambda p < 1$ and $-\lambda p < \alpha \leq 0$ we can choose t so that $1 < t \leq s$ and $-\lambda p < -1 + p/t' < \alpha$ and it follows again from Theorem (3.14) that $\phi \in \mathcal{M}(H^p_\alpha)$. (ii) Assume $2 < s \leq \infty$. Since $\phi \in M(s,\lambda)$ implies that $\phi \in M(2,\lambda)$ we may conclude from part (i) that $\phi \in \mathcal{M}(H^p_\alpha)$ for $\max(-1, -\lambda p) < \alpha \leq 0$. This concludes the proof of the theorem.

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DYADIC DIFFERENTIABILITY CONDITIONS FOR DYADIC STATIONARY PROCESSES

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Abstract A mean dyadic differentiability condition for harmonizable dyadic stationary processes is given in terms of its spectral distribution function, and their derivatives are shown to be the same type of the processes. It is also shown that linear dyadic processes, a special class of the harmonizable dyadic stationary processes, are mean dyadic differentiable if and only if their kernels in the representation are strong dyadic differentiable.

INTRODUCTION

The notion of the dyadic (logical or Walsh) differentiation was initiated by Gibbs and Millard[1]. One of the remarkable things of the differentiation is that the Walsh functions are eigenfunctions of the dyadic differential operator , i.e., for every $x \in R_+$,

$$D_t\psi_x(t) = x\psi_x(t), \quad t \in R_+ \tag{1}$$

where D_t is the dyadic differential operator, $\psi_x(t)$ is the Walsh function, and $R_t = [0, \infty)$. A function $f(t)$ defined on R_+ is called dyadic differentiable at $t \in R_+$, if the limit

$$\lim_{n\to\infty} \sum_{k=-N}^{n} 2^{k-2}(f(t) - f(t \oplus 2^{-k})) \tag{2}$$

exists, where $\oplus$ is the dyadic addition and $N = N(t)$ is the integer such that $0 \leq t - 2^N < 2^N$. The limit is called the dyadic derivative of $f(t)$ and typified by $D_t f(t)$ or $f^{[1]}(t)$. If $f(t)$ is dyadic differentiable at every $t \in R_+$ then it is simply called dyadic differentiable. If $f^{[1]}(t)$

is also dyadic differentiable then its derivative is called the second dyadic derivative and is denoted by $f^{[2]}(t)$ or $D_t^2 f(t)$. The higher order derivatives are similarly defined.

A function $f(t)$ on R_+ is called dyadic (or W-) continuous at $t \in R_+$ if it satisfies that

$$f(t \oplus h) \to f(t), \tag{3}$$

as $h \to 0+$. if $f(t)$ is dyadic continuous at every $t \in R_+$ then it is simply called dyadic continuous. A continuous function is obviously dyadic continuous. The Walsh functions are dyadic continuous, since at every (dyadic rational) discontinuity point x, $x \oplus h > x$ for sufficiently small $h \in R_+$, and the Walsh functions are continuous on the right. We remark that a differentiable function is, as is well-known, continuous in the ordinary case, but there exists no such a simple relation in the dyadic case. Let us define a function $f(t)$ such that

$$\begin{aligned} f(t) &= 1 \quad t \in D_+, \\ &= 0 \quad t \notin D_+, \end{aligned} \tag{4}$$

where D_+ is a set of dyadic rational contained in R_+. Then it is easy to see that this function is dyadic differentiable but not dyadic continuous (this was pointed out by W. R. Wade at IWGD'89). The converse statement is neither ture. The function $f(t) = at + b$ $(a \neq 0)$ is clearly dyadic continuous, but not dyadic differentiable at any $t \in R_+$.

In other convergence sense of the limit in (2) the strong differentiability was discussed by Butzer and Wagner[2]. For $f(t) \in L^2(R_+)$ if there exists a function denoted by $f_M^{[1]}(t)$ such that

$$\int_0^\infty \left| \sum_{k=-N}^{n} 2^{k-2}[f(t) - f(t \oplus 2^{-k})] - f_M^{[1]}(t) \right|^2 dt \to 0, \tag{5}$$

as $n \to \infty$, then $f(t)$ is said to be strong dyadic differentiable, and $f_M^{[1]}(t)$ is called the strong dyadic derivative of $f(t)$. It can be shown that if both $f^{[1]}(t)$ and $f_M^{[1]}(t)$ exist then they are equal almost everywhere, i.e.,

$$f_M^{[1]}(t) = f^{[1]}(t) \quad a.a.t. \tag{6}$$

SECOND ORDER PROCESSES

A stochastic process $\{X(t), t \in R_+\}$ with finite seond moment is usually called a second order process. For a second order process $\{X(t), t \in R_+\}$ it is called mean dyadic (or W-) continuous at $t \in R_+$ if it is satisfied that

$$E|X(t \oplus h) - X(t)|^2 \to 0, \quad t \in R_+, \tag{7}$$

as $h \to 0+$. If it is mean dyadic continuous at every $t \in R+$, then it is simply called mean dyadic continuous. It is clear that mean continuous processes are mean dyadic continuous.

Now we introduce a concept of dyadic derivatives to stochastic processes. For a second order process $\{X(t), t \in R_+\}$ if

$$E\left|\sum_{k=m}^{n} 2^{k-2}[X(t) - X(t \oplus 2^{-k})]\right|^2 \to 0, \quad t \in R_+, \tag{8}$$

as $m,\ n \to \infty$, then it is called mean dyadic differentiable at $t \in R_+$. By the completeness of the Hilbert space with the inner product defined by $(X,Y) = EX\overline{Y}$, if it is mean dyadic differentiable there exists a random variable with second moment denoted by $X^{[1]}(t)$ such that

$$E\left|\sum_{k=-N}^{n} 2^{k-2}[X(t) - X(t \oplus 2^{-k})] - X^{[1]}(t)\right|^2 \to 0, \tag{9}$$

as $n \to \infty$. If a process is mean dyadic differentiable at every $t \in R_+$, then it is simply called mean dyadic differentiable, and the set of random variables $\{X^{[1]}(t), t \in R_+\}$ is a second order process and is called the mean dyadic derivative of the process. The higher order derivatives are analogously defined.

DYADIC STATIONARY PROCESSES

A second order process $\{X(t), t \in R_+\}$ is called a dyadic stationary (DS) process if it has a constant mean and satisfies that for every $s, t \in R_+$,

$$EX(s)\overline{X(t)} = EX(s \oplus \tau)\overline{X(t \oplus \tau)}, \quad a.a.\tau. \tag{10}$$

We remark that the equality in (10) is required to hold for not all but almost all τ. We assume throughout without loss of generality that $EX(t) = 0,\ t \in R_+$. A DS process is called W-harmonizable if it is represented by the stochastic integral,

$$X(t) = \int_0^\infty \psi_s(v)d\xi(v), \tag{11}$$

(in quadratic mean) in terms of the Walsh function $\psi_t(v)$ and an orthognal random measure $\xi(\cdot)$. Its covariance function $R(s,t) = EX(s)\overline{X(t)}$ is also expressed by

$$R(s,t) = \int_0^\infty \psi_s(v)\psi_t(v)dF(v), \tag{12}$$

where $F(\cdot)$ is the spectral distribution function with

$$dF(\cdot) = E|d\xi(\cdot)|^2. \tag{13}$$

For a W-harmonizability condition of DS processes the reader is refered to Endow[3,4]. We note that any W-harmonizable DS process is necessarily mean dyadic continuous. Acutually, since the Walsh function is dyadic continuous, it is easy to see that

$$E|X(t\oplus h) - X(t)|^2 \leq K\left[\int_0^\infty |\psi_{t\oplus h}(v) - \psi_t(v)|^2 dF(v)\right]^{1/2},$$

in which the right side converges to zero as $h \to 0+$, where K is a constant.

The following condition on mean dyadic differentiability of the process was given by Endow[5].

Theorem 1. *A W-harmonizable DS process $\{X(t), t \in R_+\}$ is r-th mean dyadic differentiable, if and only if its spectral distribution function satisfies that*

$$\int_0^\infty v^{2r} dF(v) < \infty, \quad r = 1, 2, 3, \ldots \tag{14}$$

The r-th mean dyadic derivative of the process is given by

$$X^{[t]}(t) = \int_0^\infty D_t^r \psi_t(v)d\xi(v) = \int_0^\infty v^r \psi_t(v)d\xi(v). \tag{15}$$

It follows from *Theorem 1* that the r-th mean dyadic derivative $\{X^{[r]}(t), t \in R_+\}$ is also a W-harmonizable DS process and its covariance function takes the form

$$R_{[r]}(s,t) = EX^{[r]}(s)\overline{X^{[r]}(t)} = \int_0^\infty v^{2r}\psi_s(v)\psi_t(v)dF(v). \tag{16}$$

Corollary 2. *If (14) holds, then*

$$D_s^p D_t^q R(s,t) = \int_0^\infty v^{p+q}\psi_s(v)\psi_t(v)dF(v), \tag{17}$$

where p and q are non-negative integers such that $p+q \leq 2r$.

Proof: Since the integral

$$\int_0^\infty v\psi_s(v)\psi_t(v)dF(v)$$

exists, it is easy to see that

$$\Big|\sum_{k=-N}^{n} 2^{k-2}[R(s,t) - R(s\oplus 2^{-k},t)] \;-\; \int_0^\infty v\psi_s(v)\psi_t(v)dF(v)\Big|$$

$$\leq \int_0^\infty \Big|\sum_{k=-N}^{n} 2^{k-2}(1-\psi_{2^{-k}}(v)) \;-\; v\Big| dF(v),$$

in which the integrand converges monotonously to zero, as $n \to \infty$. Hence we have shown (17) for $p+q=1$, and inductively we may have the conclusion.

The concept of the mean dyadic derivative of the DS processes is significant for both theory and applications. Among others Engels and Splettstössor[6] applied it to signal processes and gave an estimate of aliasing error resulting from the sampling of not necessarily squence-limited random signals.

LINEAR DYADIC PROCESS

Let η be an orthogonal random measure on $\mathcal{B}(R_+)$ with $E\eta(A) = 0$ and

$$E\eta(A)\overline{\eta(B)} \;=\; \sigma^2 \int_{A\cap B} dx, \tag{18}$$

where $A \cap B$ is the intersection of A and B in $\mathcal{B}(R_+)$, which is the σ-field of all Borel subsets of R_+. For $\Phi(t) \in L^2(R_+)$, define a stochastic process by the stochastic integral,

$$X(t) \;=\; \int_0^\infty \Phi(t\oplus s)d\eta(s), \quad t \in R_+, \tag{19}$$

in quadratic mean. The process $\{X(t), t \in R_+\}$ thus defined is called a linear dyadic (LD) process.

Now we define the Walsh transform in L^2-sense of an indicator function $\chi_A(x)$ of a bounded set $A \in \mathcal{B}(R_+)$;

$$J_A \;=\; \underset{T\to\infty}{\text{l.i.m.}} \int_0^T \chi_A(x)\psi_t(x)dx \;=\; \int_A \psi_t(x)dx. \tag{20}$$

With using $J_A(t)$, we define a random measure $\zeta(\cdot)$ by the integral,

$$\zeta(A) = \int_0^\infty J_A(t)d\eta(t), \tag{21}$$

which satisfies

$$E\zeta(A)\overline{\zeta(B)} = \sigma^2\int_0^\infty J_A(t)J_B(t)dt$$
$$= \sigma^2\int_0^\infty \chi_A(x)\chi_B(x)dx = \sigma^2\int_{A\cap B} dx. \tag{22}$$

Since for every function $f \in L^2(R_+)$ it can be shown that

$$\int_0^\infty F(t)d\eta(t) = \int_0^\infty f(x)d\zeta(x), \tag{23}$$

where

$$F(t) = \underset{T\to\infty}{\text{l.i.m.}}\int_0^T f(x)\psi_t(x)dx, \tag{24}$$

we will have that

$$X(t) = \int_0^\infty \Phi(t\oplus s)d\eta(s) = \int_0^\infty \psi_t(x)\phi(x)d\zeta(x). \tag{25}$$

Put

$$\xi(A) = \int_A \phi(x)d\zeta(x),$$

then $E\zeta(A) = 0$ and

$$E\zeta(A)\overline{\zeta(B)} = \sigma^2\int_{A\cap B}|\phi(x)|^2dx, \quad A, B \in \mathcal{B}(R_+). \tag{26}$$

Hence, rewriting (25) as

$$X(t) = \int_0^\infty \psi_t(x)d\xi(x), \tag{27}$$

we have shown the following[4].

Lemma 3. *Let $\{X(t), t \in R_+\}$ be an LD process expressed by (19). Then it is a W-harmonizable DS process expressed by (27) with the spectral density (26).*

The convese statemant was also shown by Endow[4].

Lemma 4. *A W-harmonizable DS process with spectral density is also an LD process.*

These results in discrete parameter case were given by Nagai[7]. Application of *Theorem 1* to an LD process represented by (27) gives the following result.

Theorem 5. *For an LD process, it is r-th mean dyadic differentiable if and only if $v^r\phi(v) \in L^2(R_+)$, $r = 1, 2, 3, \ldots$.*

Now suppose that $\Phi(t) \in L^2(R_+)$ is strong dyadic differentiable and its derivative is denoted by $\Phi_M^{[1]}(t)$. Since $\Phi_M^{[1]}(t) \in L^2(R_+)$ the stochastic integral

$$\int_0^\infty \Phi_M^{[1]}(t \oplus s)d\eta(s)$$

exists in quadratic mean. So

$$E\Big|\sum_{k=-N}^{n} 2^{k-2}[X(t) - X(t \oplus 2^{-k})] - \int_0^\infty \Phi_M^{[1]}(t \oplus s)d\eta(s)\Big|^2$$

$$= \sigma^2 \int_0^\infty \Big|\sum_{k=-N}^{n} 2^{k-2}[\Phi(t \oplus s) - \Phi(t \oplus s \oplus 2^{-k})] - \Phi_M^{[1]}(t \oplus s)\Big|^2 ds, \qquad (28)$$

in which the right side converges to zero as n tends to infinity. Conversely, if the process $\{X(t), t \in R_+\}$ is mean dyadic differentiable then the left side in the following equation converges to zero as m and n tend to infinity;

$$E\Big|\sum_{k=m}^{n} 2^{k-2}[X(t) - X(t \oplus 2^{-k})]\Big|^2$$

$$= \sigma^2 \int_0^\infty \Big|\sum_{k=m}^{n} 2^{k-2}[\Phi(s) - \Phi(s \oplus 2^{-k})]\Big|^2 ds. \qquad (29)$$

Since $L^2(R_+)$ is complete, there exists a function $\Psi(t) \in L^2(R_+)$ such that

$$\int_0^\infty \Big|\sum_{k=-N}^{n} 2^{k-2}[\Phi(s) - \Phi(s \oplus 2^{-k})] - \Psi(s)\Big|^2 ds \to 0,$$

as $n \to \infty$. This shows the strong differentiability of $\Phi(t)$. Hence we have the following.

Theorem 6. *For an LD process $\{X(t), t \in R_+\}$ represented by (19), it is mean dyadic differentiable if and only if $\Phi(t)$ is strong dyadic differentiable. The mean dyadic derivative is given by*

$$X^{[1]}(t) = \int_0^\infty \Phi_M^{[1]}(t \oplus s)d\eta(s). \qquad (33)$$

The dyadic derivative $\{X^{[1]}(t), t \in R_+\}$ *is also an LD process with* $EX^{[1]}(t) = 0,\ t \in R_+$ *and its covariance function is given by*

$$R_{[1]}(s,t) = \sigma^2 \int_0^\infty \Phi_M^{[1]}(s \oplus u)\Phi_M^{[1]}(t \oplus u)du. \tag{31}$$

Corollary 7. *If* $\Phi(t) \in R_+$ *in (19) is strong dyadic differentiable, then* $D_s D_t R(s,t)$ *exists and*

$$D_s D_t R(s,t) = R_{[1]}(s,t). \tag{32}$$

Proof: Since

$$R(s,t) = EX(s)\overline{X(t)} = \sigma^2 \int_0^\infty \Phi(s \oplus u)\Phi(t \oplus u)du,$$

it is easy that

$$\Big|\sum_{k=-N}^{n} 2^{k-2}[R(s,t) - R(s \oplus 2^{-k}, t)] - \sigma^2 \int_0^\infty \Phi_M^{[1]}(s \oplus u)\Phi(t \oplus u)du\Big|$$

$$\leq \Big[\int_0^\infty \Big|\sum_{k=-N}^{n} 2^{k-2}[\Phi(s \oplus u) - \Phi(s \oplus u \oplus 2^{-k})] - \Phi_M^{[1]}(s \oplus u)\Big|^2 du\Big]^{1/2} \Big[\int_0^\infty |\phi(u)|^2 du\Big]^{1/2},$$

in which the right side tends to zero as $n \to \infty$. Thus we have

$$D_s R(s,t) = \int_0^\infty \Phi_M^{[1]}(s \oplus u)\Phi(t \oplus u)du,$$

and similarly

$$D_s D_t R(s,t) = \int_0^\infty \Phi_M^{[1]}(s \oplus u)\Phi_M^{[1]}(t \oplus u)du.$$

This completes the proof.

We remark that successive applications of *Theorem 6* give us the higher order case; an LD process is r-th strong mean dyadic differentiable if and only if $\Phi(t)$ is r-th strong dyadic differentible. The r-th mean dyadic derivative $\{X^{[r]}(t), t \in R_+\}$ is also an LD process and is given by

$$X^{[r]}(t) = \int_0^\infty \Phi_M^{[r]}(t \oplus s)d\eta(s), \tag{33}$$

and its covariance function is also given by

$$R_{[r]}(s,t) = \sigma^2 \int_0^\infty \Phi^{[r]}(s \oplus u)\Phi^{[r]}(t \oplus u)du. \tag{34}$$

Hence, on account of *Theorem 5*, we have a following byproduct.

Corollary 8. *A function* $\Phi(t) \in L^2(R_+)$ *is* r*-th strong dyadic differentiable if and only if* $v^r\phi(v) \in L^2(R_+)$, *where* $\phi(v)$ *is the Walsh transform of* $\Phi(t)$ *in* L^2*-sense.*

Characteristic function of LD process

In this section we suppose that $\{X(t), t \in R_+\}$ is an LD process with the representation (19). We also suppose that the random measure $\eta(\cdot)$ has independent increments. It then has no singular points because of (18). In this case it is shown[8] that its characteristic function is expressed by

$$E\{iu\eta(A)\} = exp\Big\{\int_A \int_0^\infty [exp\{iux\} - iux - 1](1/x^2)G(dx \times ds)\Big\}, \tag{35}$$

where the Borel measure G is uniquely determined by the relation

$$-(d^2/du^2)lnE[exp\{iu\eta(A)\}] = \int_A \int_{-\infty}^\infty exp\{iux\}G(dx \times ds), \tag{36}$$

which reduces to

$$\sigma^2 dx = \int_{-\infty}^\infty G(dx \times ds). \tag{37}$$

Since an LD process with its random measure having independent increments is so-called a linear process[8], the characteristic function of the process and its derivative is expressed explicitely in terms of the Borel measure G as

$$Eexp\{i(\mathrm{u} \cdot \mathrm{X} + \mathrm{v} \cdot \mathrm{X}^{[1]})\}$$

$$= exp\Big\{\int_0^\infty \int_{-\infty}^\infty [exp\{ix(\mathrm{u} \cdot \Phi(s) + \mathrm{v} \cdot \Phi_M^{[1]}(s))\} - ix(\mathrm{u} \cdot \Phi(s) + \mathrm{v} \cdot \Phi_M^{[1]}(s)) - 1]$$

$$\times(1/x^2)G(dx \times ds)\Big\}, \tag{38}$$

provided that $\Phi(t) \in L^2(R_+)$ is strong dyadic differentiable, where

$$\mathrm{u} \cdot \mathrm{X} = \sum_{k=1}^m u_k X(t_k), \quad \mathrm{v} \cdot \mathrm{X}^{[1]} = \sum_{k=1}^n v_k X^{[1]}(\tau_k), \quad \mathrm{u} \cdot \Phi(s) = \sum_{k=1}^m u_k \Phi(t_k \oplus s),$$

and

$$\mathrm{v} \cdot \Phi_M^{[1]}(s) = \sum_{k=1}^n v_k \Phi_M^{[1]}(\tau_k \oplus s).$$

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APPLICATIONS OF THE GIBBS DERIVATIVES IN THE COMPUTATIONS OF STATISTICAL MOMENTS

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Abstract

In this paper, we review the application of Gibbs derivatives in the computation of statistical moments. On the one hand, the Gibbs derivatives are applied directly to the Walsh characteristic function to obtain the desired natural moments of the corresponding probability density function (PDF). The indicated characteristic function is defined as the statistical expectation value of the associated PDF. Although, consideration is primarily given of a single-variate PDF, the technique can be easily extended to accomodate the multivariate case .

In this connection, we also exploit the use of Walsh series expansions for the representation of a certain class of PDFs. Consequently, we can invoke the Gibbs derivative to attain useful expressions for the computations of the moments in terms of the resulting Walsh expansion coefficients.

The indicated techniques are further utilized in the computations of output moments of certain classes of instantatneous nonlinear transformations .In particular, direct utilization is made of the Walsh series expansions of probability distributions available at the input of such transformations.

1 Generalized Walsh Functions

One convenient method of defining a set of generalized Walsh functions is via the set of Rademacher functions [1]:

$$\varphi(n, x) = \varphi(0, 2^n x), \quad n = 1, 2, 3, \ldots \tag{1}$$

where the zeroth-order Rademacher function is defined by

$$\varphi(0, x) = \begin{cases} +1, & 0 \leq x < 1/2 \\ -1, & 1/2 \leq x < 1 \end{cases} \tag{2}$$

The resulting set of Walsh functions is subsequently defined by

$$\Psi(0, x) = 1, \ 0 \leq \times < 1 \tag{3}$$

and

$$\Psi(n, x) = \Pi_{i=0}^{N} \ [\varphi(i+1, x)]^{n_i}, \tag{4}$$

where the integer n is assumed to have the dyadic (binary) representation

$$n = \sum_{i=0}^{N} 2^i n_i, \quad n_i \varepsilon \{0, 1\}. \tag{5}$$

The generalized Walsh functions are, in turn, defined by [2]

$$\Psi(\sigma, x) = \Psi([\sigma], x)\Psi([x], \sigma); 0 \leq \sigma, x < \infty, \tag{6}$$

where $[\sigma]$ and [x] designate the largest integers in the real sequency and spatial variables, σ and x, respectively.

2 Moments and Walsh Characteristic Functions

In this section, it is intended to define a Walsh characteristic function (WCF) and subsequently apply a Gibbs derivative to compute the corresponding statistical moments.

For a random variable X, with a probability density function (PDF), f(x), the WCF is defined as the statistical expectation of the Walsh transform kernel $\Psi(\sigma, x)$. Hence [4]:

$$\begin{aligned} S_x(\sigma) &= E_x[\Psi(\sigma, x)] \\ &= \int_0^{\infty} f(x)\Psi(\sigma, x)dx. \end{aligned} \tag{7}$$

This characteristic function possesses the properties usually enjoyed by such a function; i.e.

$$\begin{aligned} i) \quad & S_x(0) = 1 \\ ii) \quad & | S_x(\sigma) | \leq 1. \end{aligned} \tag{8}$$

The relation connecting moments and a WCF is estabished through a dyadic derivative, commonly called a Gibbs derivative [3], and is defined, for a function $g(x)$, by

$$\frac{d}{dx} \oplus g(x) = \sum_{i=-\infty}^{\infty} 2^{i-2}[g(x) - g(x \oplus 2^{-i}] \tag{9}$$

The dyadic operator $\oplus$ indicates a modulo -2 sum operation, without-carry.

Applying the Gibbs derivative to a generalized Walsh function yields

$$\frac{d}{d\sigma} \oplus \Psi(\sigma, x) = x\Psi(\sigma, x). \tag{10}$$

Generalization of result (10) to higher-orders is rather staightforward. In particular, we obtain

$$\frac{d^r}{d\sigma^r} \oplus \Psi(\sigma, x) = x^r \Psi(\sigma, x). \tag{11}$$

Thus, applying the Gibbs derivative to (7) renders

$$\frac{d^r}{d\sigma^r} \oplus S_x(\sigma) = \int_0^\infty x^r f(x)\Psi(\sigma, x)dx. \tag{12}$$

Hence

$$\begin{aligned} \frac{d^r}{d\sigma^r} \oplus S_x(\sigma) |_{\sigma=0} &= \int_0^\infty x^r f(x)dx \\ &= m_r, \end{aligned} \tag{13}$$

yielding a moment of order r.

Explicit formulation for m_1 and m_2, for instance, are obtained using (9). This yields

$$m_1 = \sum_{i=-\infty}^{\infty} 2^i \frac{1}{4}[S_x(0) - S_x(2^{-i})] \tag{14}$$

and

$$m_2 = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{i+j} \; [\tfrac{1}{16} \; S_x(0) - S_x(2^i)$$
$$- \; S_x(2^{-j}) + S_x(2^{-i} \oplus 2^{-j})]. \qquad (15)$$

The multivariate case of a joint PDF can be handled in a similar fashion by exploiting partial Gibbs differentiation. For example, a bivariate PDF has the corresponding two-dimensional WCF:

$$S_{xy}(\sigma, \propto) = \int_{-0}^{\infty} \int_{0}^{\infty} f_{XY}(x,y)\Psi(\sigma,x)\Psi(\propto,y)dxdy. \qquad (16)$$

Hence, the two-dimensional r-th moments are evaluated by

$$\frac{\partial^r}{\partial\sigma^r} \frac{\partial^n}{\partial \propto^n} \oplus S_{xy}(\sigma, \propto) \mid_{\sigma=\propto=0} = \int_{0}^{\infty} \int_{0}^{\infty} x^r y^n f_{xy}(x,y)dxdy. \qquad (17)$$

An interesting result in this connection concerns Walsh transforms of dyadic derivatives of functions. The result is given by

$$F_r(\sigma) = W[\frac{d^r}{dx^r} \oplus f(x)] = \sigma^r F(\sigma), \quad r = 1, 2, 3, \ldots \qquad (18)$$

For illustration, we derive result (18) for the case r=1. For higher values of r, derivations proceed similarly.

Thus we consider

$$F_1(\sigma) = \int_{0}^{\infty} [\frac{d}{dx} \oplus f(x)]\Psi(\sigma,x)dx. \qquad (19)$$

Applying the definition of a Gibbs derivative as dictated by (9), then (19) becomes

$$F_1(\sigma) = \int_{0}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} 2^{n-2}[f(x) - f(x \oplus 2^{-n})] \right\} \Psi(\sigma,x)dx. \qquad (20)$$

Interchanging summation and integration in (20), and using the dyadic shift property of the Walsh transform; i.e

$$\begin{array}{ll} \text{If} & W[f(x)] = F(\sigma) \\ \text{then} & W[f(x \oplus \propto)] = F(\sigma)\Psi(\sigma,\propto), \end{array}$$

we obtain

$$F_1(\sigma) = \sum_{n=-\infty}^{\infty} 2^{n-2}[F(\sigma) - \Psi(\sigma, 2^{-n})F(\sigma)]. \quad (21)$$

For a real variable σ, the associated dyadic expansion is, in general, given by

$$\sigma = \sum_{n=-\infty}^{\infty} 2^{-n}\sigma_n, \quad \sigma_n \varepsilon \{0,1\}. \quad (22)$$

Furthermore, we have an established relation between the expansion coefficients σ_n and Walsh functions, specified by

$$\sigma_{1-n} = \frac{1}{2}[1 - \Psi(\sigma, 2^{-n})].$$

Hence, employing these latter results in (22) yields

$$\begin{aligned} F_1(\sigma) &= \sum_{n=-\infty}^{\infty} 2^{-n}\sigma_n[1 - \Psi(\sigma, 2^{-n})]F(\sigma) \\ &= \sigma F(\sigma). \end{aligned} \quad (23)$$

3 PDF Walsh Series Expansion

In order to exploit a Walsh series expansion [5], we assume a PDF $f(x)$. In this respect, we formulate the expansion,

$$f(x) = \sum_{n=0}^{\infty} a_n \Psi(n, x/x_0) \quad (24)$$

where the Walsh expansion coefficients are evaluated by

$$a_n = \frac{1}{x_0}\int_0^{x_0} f(x)\Psi(n,x)dx; n = 1,2,3,... \quad (25)$$

For convenience, we will, however, assume that $x_0 = 1$. In this case, comparison of (25) with (7) yields

$$a_n = S_x(n). \quad (26)$$

Furthermore, (25) gives

$$| a_n | \leq \int_0^1 | f(x) | \; | \Psi(n,x) | \, dx = 1. \quad (27)$$

It is possible to invoke the Gibbs derivative in application to (24) upon completion of the Walsh transform of the same. The WCF in this case is given by

$$S_x(\sigma) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} \Psi(n,x)\Psi(\sigma,x)dx. \tag{28}$$

Hence

$$m_r = \frac{d^r}{d_{\sigma^r}} \oplus S_x(\sigma) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} x^r \Psi(n,x)dx. \tag{29}$$

Obviously, we could alternatively arrive at (29) by computing moments directly from the PDF. In any case, it becomes of immediate interest, in connection with (29) to consider the integral:

$$I_n(r) = \int_0^1 x^r \Psi(n,x)dx. \tag{30}$$

One method of evaluating (30) is due to Slook [6], and renders the following difference evaluation type of a recursive relationship:

$$I_{2p+1}(r) = -I_{2p}(r) + 2^{-r} I_p(r); \quad p = 0, 1, 2, \ldots \tag{31}$$

This latter relation can be employed for computing the even-indexed I-integrals from the odd-indexed ones, and vice-versa. Essentially, it cuts the computation efforts to half.

Applications To Nonlinear Transformations

It is both of interest and significance to extend the previous techniques in application to certain classes of nonlinear transformations. We assume an instantaneous nonlinear transformation:

$$y = g(x): \quad 0 \leq x < 1,$$

and a corresponding Walsh series expansion

$$g(x) = \sum_{n=0}^{\infty} a_n \Psi(n,x), \quad 0 \leq x < 1, \tag{32}$$

where the Walsh expansion coefficients are evaluated by

$$a_n = \int_0^1 g(x)\Psi(n,x)dx.$$

For a positive integer r, moments m_r of the indicated nonlinear transformations are subsequently given by

$$m_r = \int_0^1 [g(x)]^r f(x) dx. \tag{33}$$

We now further assume that the input PDF $f_X(x)$ has a Walsh series expansion specified by

$$f_X(x) = \sum_{n=0}^{\infty} A_n \Psi(n, x), \quad 0 \leq x < 1 \tag{34}$$

Thus, using (33) and (34) in (32) yields

$$m_r = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} a_{n_1}, \ldots a_{n_r} A_{n_1 \oplus \ldots \oplus n_r}. \tag{35}$$

The operation $\oplus$ designates modulo-two (no-carry) summation

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CONVERGENCE PROPERTIES OF A CLASS OF WALSH-FOURIER INTEGRAL OPERATORS

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Abstract

This paper discusses the norm-convergence of a class of p-adic Walsh-convolution operators. The class is that of operators with positive parameter r with kernels that can be represented as the Walsh-Fourier transform of a quasi-convex function $g(x/r)$ vanishing at infinity. The paper proves that such kernels are approximate identity kernels and that the rate of convergence of the associated operator depends only on the behaviour of g and its first two derivatives at the origin. The paper gives an explicit and comprehensive expression estimating the rate of convergence.

1. Introduction

Although there has been as much interest in discussing Fourier integral operators on the linear group $\mathbb{R}$ as on the circle group $\mathbb{T}$ there has been relatively little discussion of Walsh–Fourier operators on $\mathbb{R}^+$ compared with the discussion of those on $[0,1)$.

In this paper we discuss the question of the convergence and the rate of convergence in the q–norm of a class of Walsh–Fourier integral operators on $L^q(\mathbb{R}^+)(1 \leqslant q \leqslant \infty)$. We discuss Walsh–Fourier convolution operators T of the class defined by

$$Tf = f * \int_0^{\infty} \eta(\frac{t}{\rho})\psi_{\bullet}(t)dt \tag{1}$$

where η is a quasi–convex function (see [1]) with $\eta(0) = 1$ and $\eta(\infty) = 0$, ψ_x is the p–adic Walsh function with index x and $*$ denotes p–adic Walsh convolution.

Following Fourier analysis nomenclature (see [1]) we distinguish a few special operators of this class as

W–Fejér if $\eta(t) = \begin{cases} 1 - t & (0 \leqslant t \leqslant 1) \\ 0 & (1 \leqslant t) \end{cases};$

W–Cauchy–Poisson if $\eta(t) = e^{-t} \quad (0 \leqslant t);$

W–Gauss–Weierstrass if $\eta(t) = e^{-t^2} \quad (0 \leqslant t);$

and W–Jackson if $\eta(t) = \begin{cases} 1 - 3t^2/2 + 3t^3/4 & (0 \leqslant t \leqslant 1) \\ (2 - t)^3/4 & (1 \leqslant t \leqslant 2), \\ 0 & (2 \leqslant t) \end{cases}$

where "W" denotes "Walsh type".

In Fourier analysis an important and easy way for some integral operators of proving that they are convergent is to prove that their kernels are approximate identity kernels (see [1]) but in Walsh analysis it is not as easy. In section 2 of this paper we prove that $\int_0^{\infty} \eta(t/\rho)\psi_{\bullet}(t)dt$ in (1) is an approximate identity kernel in $L^q(\mathbb{R}^+)$ $(1 \leqslant q \leqslant \infty)$, so solving the problem of the convergence of

(1). In section 3 we discuss the rate of convergence of (1), obtaining a comprehensive result that shows that for all operators of this type for which η has continuous second-order derivatives in a right neighborhood of 0 the rate of convergence depends only on the *orders* of $1-\eta(t)$, $t\eta'(t)$ and $t^2\eta''(t)$ as $t \to 0$.

Denoting the p-adic sum and difference operators by $\oplus$ and $\ominus$ we use the following notation and definitions:

$$D(a,x) = \int_0^a \psi_x(t)dt; \qquad F(a,x) = \int_0^a (1-\frac{t}{a})\psi_x(t)dt;$$

$$(f*g)(x) = \int_0^\infty f(x\ominus t)g(t)dt \ (f \in L^q(\mathrm{R}^+), g \in L(\mathrm{R}^+));$$

$$\hat{f}(x) = \int_0^\infty f(t)\bar{\psi}_x(t)dt \ (f \in L(\mathrm{R}^+)) \quad \text{and}$$

$$\omega(f,\delta) = \omega(q,f,\delta) = \sup_{0\leqslant t<\delta} \|f(\cdot\oplus t) - f(\cdot)\|_q$$

$$\text{where} \quad \|\cdot\|_q = \|\cdot\|_{L^q(\mathrm{R}^+)}.$$

2. The convergence of the operators

Theorem 1 *Let η be a quasi-convex function on R^+ with $\eta(0)=1$ and $\eta(\infty)=0$ and let ψ_x be the p-adic Walsh functions on R^+ then*

$$K_\rho(x) = \int_0^\infty \eta(\frac{t}{\rho})\psi_x(t)dt \quad (\rho > 0) \tag{1}$$

is an approximate identity kernel; that is K_ρ satisfies the following conditions:

I. $\displaystyle\int_0^\infty K_\rho(x)dx = 1;$

II. $\|K_\rho\|_1 < \infty;$

III. $\displaystyle\lim_{\rho\to\infty}\int_\delta^\infty |K_\rho(x)|dx = 0$ *for all* $\delta > 0$.

Proof We have proved in [3] that η has the following properties:

(a) $\displaystyle h(x) = \int_0^\infty \eta(t)\psi_x(t)dt = \int_0^\infty t d\eta'(t)\int_0^t (1-\frac{u}{t})\psi_x(u)du \in L(\mathrm{R}^+),$

(b) $\|h\|_1 \leqslant A_1 \cdot A_2$, where

$$A_1 = \sup_\rho \left\| \int_0^\rho (1-\frac{u}{\rho})\psi_\bullet(u)du \right\|_1 < \infty \quad \text{and} \quad A_2 = \int_0^\infty t|d\eta'(t)| < \infty,$$

(c) $\eta(x) = \hat{h}(x) = \int_0^\infty h(t)\overline{\psi}_x(t)dt.$

The conclusions I and II of Theorem 1 are easy to get from those properties.

We first notice that for each $\rho > 0$ $\tilde{\eta}(t) = \eta(t/\rho)$ is still a quasi–convex function on $\mathbf{R}^+$ and satisfies $\tilde{\eta}(0) = 1$ and $\tilde{\eta}(\infty) = 0$, so we have by (a) and (c)

$$K_\rho(x) = \int_0^\infty \eta(\frac{t}{\rho})\psi_x(t)dt = \int_0^\infty \tilde{\eta}(t)\psi_x(t)dt \ \in L(\mathbf{R}^+)$$

and

$$\int_0^\infty K_\rho(x)dx = \int_0^\infty K_\rho(x)\psi_0(x)dx = \tilde{\eta}(0) = 1$$

This proves conclusion I.

Letting $s = t/\rho$ we have by (a) and (b)

$$\|K_\rho\|_1 = \left\| \int_0^\infty sd\eta'(s) \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_\bullet(u)du \right\|_1 \leqslant \int_0^\infty s|d\eta'(s)| \left\| \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_\bullet(u)du \right\|_1 \leqslant A_1 \cdot A_2$$

and so this proves conclusion II.

We now come to prove III. Letting $s = t/\rho$ again

$$\begin{aligned} K_\rho(x) &= \int_0^\infty \eta(\frac{t}{\rho})\psi_x(t)dt \\ &= \int_0^\infty sd\eta'(s) \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_x(u)du \\ &= \int_0^d sd\eta'(s) \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_x(u)du + \int_d^\infty sd\eta'(s) \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_x(u)du, \end{aligned}$$

where d is an undetermined positive number, so

$$\begin{aligned} \int_\delta^\infty |K_\rho(x)|dx &\leqslant \int_0^d s|d\eta'(s)| \int_\delta^\infty dx \left| \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_x(u)du \right| \\ &+ \int_d^\infty s|d\eta'(s)| \int_\delta^\infty dx \left| \int_0^{\rho s} (1-\frac{u}{\rho s})\psi_x(u)du \right| \\ &= I_1 + I_2, \quad \text{say}. \end{aligned}$$

For every $\epsilon > 0$, by the property (b),

$$I_1 \leqslant \int_0^d s|d\eta'(s)| \sup_t \left\| \int_0^t (1 - \frac{u}{t})\psi_x(u)du \right\|_1 \leqslant A_2 \int_0^d s|d\eta'(s)| < \frac{\epsilon}{2}, \tag{2}$$

when d is chosen small enough.

We now estimate I_2. To do this we first estimate

$$J(t) = \int_\delta^\infty dx \left| \int_0^t (1 - \frac{u}{t})\psi_x(u)du \right|.$$

Let $t = \sum_{j=1}^{\infty} \alpha_j p^{n_j}$ *) where $\alpha_j \in \{1, 2, ..., p-1\}$ and $n_1 > n_2 > ... > n_j > ...$ and let j_0 be an integer such that

$$p^{-n_{j_0}} < \frac{\delta}{2} \tag{3}$$

and

$$p^{-n_{j_0+1}} \geqslant \frac{\delta}{2} \tag{4}$$

then by [4] and writing χ_S for the characteristic function of the set S

$$\begin{aligned} J(t) &= \int_\delta^\infty dx \left| \int_0^t (1 - \frac{u}{t})\psi_x(u)du \right| \\ &\leqslant \int_\delta^\infty \frac{B}{t} \sum_{j=1}^{\infty} \sum_{k=-n_j}^{\infty} \sum_{\ell=0}^{p-1} p^{-k-1+n_j} \chi_{[\ell p^k, \ell p^k + p^{-n_j})}(x)dx \quad \text{(for some constant B)} \\ &= \int_\delta^\infty \frac{B}{t} \sum_{j=1}^{j_0} \sum_{k=-n_j}^{\infty} \sum_{\ell=0}^{p-1} p^{-k-1+n_j} \chi_{[\ell p^k, \ell p^k + p^{-n_j})}(x)dx \\ &+ \int_\delta^\infty \frac{B}{t} \sum_{j=j_0+1}^{\infty} \sum_{k=-n_j}^{\infty} \sum_{\ell=0}^{p-1} p^{-k-1+n_j} \chi_{[\ell p^k, \ell p^k + p^{-n_j})}(x)dx \\ &= J_1(x) + J_2(x), \quad \text{say.} \end{aligned}$$

We get an estimate for J_2 as

$$J_2(t) \leqslant \frac{B}{t} \sum_{j=j_0+1}^{\infty} \sum_{k=-n_j}^{\infty} p^{-k} \leqslant \frac{B}{t} \sum_{j=j_0+1}^{\infty} p^{n_j} \leqslant \frac{B}{t} p^{n_{j_0+1}} \leqslant \frac{B}{t} \frac{2p}{p-1} \frac{1}{\delta} \tag{5}$$

We now estimate J_1. Let k_0 be an integer such that

$$p^{k_0} + p^{-n_{j_0}} < \delta \tag{6}$$

$$p^{k_0+1} + p^{-n_{j0}} \geqslant \delta \tag{7}$$

then for $0 \leqslant \ell \leqslant p-1$, $1 \leqslant j \leqslant j_0$ and $k < k_0$ we have $[\ell p^k, \ell p^k + p^{-n_j}) \subset [0, \delta)$ so

$$\begin{aligned} J_1 &= \frac{B}{t}\int_\delta^\infty \sum_{j=1}^{j_0}\sum_{k=-n_j}^{\infty}\sum_{\ell=0}^{p-1} p^{-k-1+n_j}\chi_{[\ell p^k, \ell p^k + p^{-n_j})}(x)dx \\ &\leqslant \frac{B}{t}\int_\delta^\infty \sum_{j=1}^{j_0}\sum_{k=k_0}^{\infty}\sum_{\ell=0}^{p-1} p^{-k-1+n_j}\chi_{[\ell p^k, \ell p^k + p^{-n_j})}(x)dx \\ &\leqslant \frac{B}{t}\sum_{j=1}^{j_0}\sum_{k=k_0}^{\infty} p^{-k-1} \\ &\leqslant \frac{B}{t} j_0 p \frac{p^{-k_0-1}}{1-\frac{1}{p}} \\ &= \frac{B j_0}{t}\frac{p^{-k_0+1}}{p-1}. \end{aligned} \tag{8}$$

It now follows by (3) and (7) that

$$pp^{k_0} + \frac{\delta}{2} > \delta$$

so

$$p^{-k_0} < \frac{2p}{\delta} \tag{9}$$

and by (3)

$$p^{-n_{j0}} < \frac{\delta}{2} \text{ so } -n_{j_0}\ln p < \ln\frac{\delta}{2} \text{ and so } n_{j_0} \geqslant -\frac{\ln\frac{\delta}{2}}{\ln p}$$

and so, by the definition of n_j,

$$j_0 - 1 \leqslant n_1 - n_{j_0} \leqslant \frac{\ln t}{\ln p} + \frac{\ln\frac{\delta}{2}}{\ln p}$$

and so

$$j_0 \leqslant \frac{\ln t}{\ln p} + \frac{\ln\frac{\delta}{2}}{\ln p} + 1. \tag{10}$$

Substituting (9) and (10) into (8), we get

$$J_1(t) \leqslant \frac{B}{t}\left(\frac{\ln t}{\ln p} + \frac{\ln\frac{\delta}{2}}{\ln p} + 1\right)\frac{p^2}{p-1}\frac{1}{\delta}$$

therefore.

$$\begin{aligned} I_2 &= \int_d^\infty s|d\eta'(s)|\int_\delta^\infty dx\left|\int_0^{\rho s}(1-\frac{u}{\rho s})\psi_x(u)du\right| \\ &\leqslant \int_d^\infty s|d\eta(s)|(J_1(\rho s) + J_2(\rho s)) \end{aligned}$$

$$
\begin{aligned}
&< \int_d^\infty s|d\eta'(s)|\left[\frac{B}{\rho s}\left(\frac{\ln t}{\ln p}+\frac{\ln\frac{\delta}{2}}{\ln p}+1\right)\frac{p^2}{p-1}\frac{1}{\delta}+\frac{B}{\rho s}\frac{2p}{p-1}\frac{1}{\delta}\right]\\
&\leqslant \int_d^\infty s|d\eta'(s)|\left[\frac{B}{\rho d}\left(\frac{\ln t}{\ln p}+\frac{\ln\frac{\delta}{2}}{\ln \rho}+1\right)\frac{p^2}{p-1}\frac{1}{\delta}+\frac{B}{\rho d}\frac{2p}{p-1}\frac{1}{\delta}\right]\\
&\leqslant \frac{\epsilon}{2},
\end{aligned}
\tag{11}
$$

when ρ is big enough.

Combining (2) and (11) we get the conclusion III of Theorem 1.

3. The rate of convergence

One sees that the kernel K_ρ of the operator T of (1.1) depends only on the function η. We shall show that if η is quasi– convex on $\mathbb{R}$ with $\eta(0) = 1$ and $\eta(\infty) = 0$ then the rate of convergence of T depends only on the behaviour of η near 0; in fact we shall show it depends only on the convergence rates of η, η' and η'' near 0. To be more precise we give a definition of "smooth convergence rate".

Definition If η has a continuous second derivative in $(0, \delta)$ for some $\delta > 0$ and

$$\lambda_1 = \sup\{a \mid 1 - \eta(t) = O(t^a) \ \text{ as } t \to 0\},$$

$$\lambda_2 = \sup\{b \mid t\eta'(t) = O(t^b) \ \text{ as } t \to 0\},$$

$$\lambda_3 = \sup\{c \mid t^2\eta''(t) = O(t^c) \ \text{ as } t \to 0\}$$

$$\text{and} \quad \lambda = \min\{\lambda_1, \lambda_2, \lambda_3\}$$

we say that $\eta(t)$ tends to 1 at the smooth convergence rate λ as $t \to 0$, or simply that *the smooth convergence rate of* η (at 0) is λ and write

$$O_s(\eta) = \lambda.$$

Because the operator T of (1.1) and its kernel $K_\rho(x) = \int_0^\infty \eta(t/\rho)\psi_x(t)dt$ depend only on η and (as we shall show) the convergence rate depends only on the number λ we also call the number λ the characteristic number of the operator T and its kernel.

---

*) If the expresion of t is of a finit form $t = \sum_{j=1}^{n} \alpha_j p^{n_j}$, the following J_2 becomes a finit sum or vanishes, that is easier to get our conclusion.

We give some examples.

1. $\eta_1(t) = \begin{cases} 1-t & 0 \leqslant t \leqslant 1 \\ 0 & t > 1. \end{cases}$

$1-\eta_1(t) = \overline{O}(t^1), t\eta'(t) = \overline{O}(t^1)$ and $t^2\eta''(t) = 0 = O(t^a)$ for every positive number a

so $\lambda = \min\{1, 1, \infty\} = 1$; that is, $O_s(\eta_1) = 1$.

2. $\eta_2(t) = e^{-t} \quad (t \geqslant 0)$.

$1-\eta_2(t) = \overline{O}(t), t\eta'(t) = \overline{O}(t)$ and $t^2\eta''(t) = \overline{O}(t)$

so $\lambda = \min\{1, 1, 1\} = 1$; that is, $O_s(\eta_2) = 1$.

3. $\eta_3(t) = e^{-t^2} \quad (t \geqslant 0)$;

$O_s(\eta_3) = 2$.

4. $\eta_4(t) = \begin{cases} 1 - \frac{3t^2}{2} + \frac{3}{4}t^3 & 0 \leqslant t \leqslant 1 \\ (2-t)^3/4 & 0 \leqslant t \leqslant 2; \\ 0 & t > 2 \end{cases}$

$O_s(\eta_4) = 2$.

5. $\eta_5(t) = 1 - e^{-1/t^2} \quad (t \geqslant 0)$;

$O_s(\eta_5) = \infty$.

Now we state the theorem on the rate of convergence.

Theorem 2 *Let η and K_ρ be as in Theorem 1; let $f \in L^q(\mathbb{R}^+)$ $(1 \leqslant q \leqslant \infty)$ and let $O_s(\eta) = \lambda$ $(0 < \lambda < \infty)$ then*

I. $$\|f - f * K_\rho\|_q = O(1)\left[\sum_{j=0}^{n} p^{(j-n)\lambda}\omega(f, p^{-j}) + p^{-n\lambda}\right]$$

where $p^{n-1} \leqslant \rho < p^n$;

II. *if* $f \in \mathrm{Lip}\alpha$ *then*

$$\|f - f * K_\rho\|_q = \begin{cases} O(\rho^{-\alpha}) & (\alpha < \lambda) \\ O(\rho^{-\lambda}\ln\ \rho) & (\alpha = \lambda) \\ O(\rho^{-\lambda}) & (\alpha > \lambda) \end{cases}$$

Proof First we do some preparatory work.

(i) If $O_s(\eta) = \lambda$ then there exists $\delta_1 > 0$ such that η has continuous second derivatives in $(0, \delta_1)$ and there exists a constant B_1 such that when $0 < t < \delta_1$ $|1 - \eta(t)| \leqslant B_1 t^\lambda$, $|t\eta'(t)| \leqslant B_1 t^\lambda$ and $|t''\eta'(t)| \leqslant B_1 t^\lambda$ so $|1 - \eta(t)| \leqslant B_1 t^\lambda$, $|t\eta'(t)| \leqslant B_1 t^\lambda$ and

$$\int_0^t u|d\eta(u)| \leqslant \int_0^t u\eta''(u)du \leqslant B_1 \int_0^t u^{\lambda-1}du \leqslant B_1 t^\lambda; \tag{1}$$

on the other hand there exists B_2 such that when $\delta_1 \leqslant t \leqslant 1$ $|1 - \eta(t)| \leqslant B_2\delta_1^\lambda \leqslant B_2 t^\lambda$, $|t\eta(t)| \leqslant B_2\delta_1^\lambda \leqslant B_2 t^\lambda$ and

$$\int_0^t u|d\eta'(t)| \leqslant B_2\delta_1^\lambda \leqslant B_2 t^\lambda \tag{2}$$

so by (1) and (2), when $0 < t < 1$, $|1 - \eta(t)| \leqslant Bt^\lambda$, $|t\eta(t)| \leqslant Bt^\lambda$ and

$$\int_0^t u|d\eta(u)| \leqslant Bt^\lambda \tag{3}$$

where $B = \max\{B_1, B_2\}$.

(ii) Integrating by parts twice we have

$$\int_a^b \eta(u)\psi_t(u)du = \int_a^b ud\eta'(u)\int_0^u (1 - \frac{v}{u})\psi_t(v)dv - \eta'(u)u\int_0^u (1 - \frac{v}{u})\psi_t(v)dv\Big|_{u=a}^b$$

$$+\eta(u)\int_0^u \psi_t(u)du\Big|_{u=a}^b. \tag{4}$$

Letting $a = p^j$ and $b = p^{j+1}$, and using $1 - \eta(t/\rho)$ instead of $\eta(t)$, we have

$$\int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right)\psi_t(u)du \in L^1(\mathbf{R}^+) \tag{5}$$

and

$$\left\|\int_{p^j}^{p^{j+1}}\left(1-\eta(\frac{u}{\rho})\right)\psi_t(u)du\right\|_1 \leqslant A_1\left(\int_{\frac{p^j}{\rho}}^{\frac{p^{j+1}}{\rho}} s|d\eta'(s)| - \frac{p^j}{\rho}\left|\eta'(\frac{p^j}{\rho})\right| + \frac{p^{j+1}}{\rho}\left|\eta'(\frac{p^{j+1}}{\rho})\right|\right)$$

$$+\left|\eta(\frac{p^j}{\rho})\right| + \left|\eta(\frac{p^{j+1}}{\rho})\right| = O(1)(p^{(j-n)\lambda}). \tag{6}$$

(iii) We recall that (see [2]) $\left|\int_0^u \psi_t(u)du\right| \leqslant \max\left(\frac{p}{2t},\frac{1}{2}\right)$ and so

$$\left|\int_0^u\left(1-\frac{v}{u}\right)\psi_t(v)dv\right| = \left|\frac{1}{u}\int_0^u du\int_0^u \psi_t(v)dv\right| \leqslant \max\left(\frac{p}{2t},\frac{1}{2}\right),$$

and

$$\left|\int_0^u\left(1-\frac{v}{u}\right)\psi_t(v)dv\right| \leqslant \left|\int_0^u 1dv\right| \leqslant u.$$

We also recall that (see [1] p.248) if $\eta(t)$ is quasi–convex and $\eta(\infty)=0$ then $a^2\eta'(a)\to 0$ (as $a\to 0$) and $b\eta'(b)\to 0$ (as $b\to\infty$).

In (4) using $1-\eta(u/\rho)$ instead of $\eta(u)$ and letting $b=1$ and $a\to 0$. we have

$$\int_0^1\left(1-\eta(\frac{u}{\rho})\right)\psi_t(u)du = -\int_0^{\frac{1}{\rho}} sd\eta'(s)\int_0^{\rho s}\left(1-\frac{v}{\rho s}\right)\psi_t(v)dv$$

$$+\eta'(\frac{1}{\rho})\frac{1}{\rho}F(1,t) - \eta(\frac{1}{\rho})D(1,t) \in L(\mathrm{R}^+) \tag{7}$$

and

$$\left\|\int_0^1\left(1-\eta(\frac{u}{\rho})\right)\psi_t(u)du\right\|_1 \leqslant A_1\left(\int_0^{\frac{1}{\rho}} s|d\eta'(s)| + \frac{1}{\rho}|\eta'(\frac{1}{\rho})|\right)$$

$$+\left|\eta(\frac{1}{\rho})\right| = O(1)p^{-n\lambda}. \tag{8}$$

Using $\eta(u/\rho)$ instead of $\eta(u)$ and letting $a=p^n$ and $b\to\infty$ we have

$$\int_{p^n}^{\infty}\eta(\frac{u}{\rho})\psi_t(u)du = \int_{\frac{p^n}{\rho}}^{\infty} sd\eta'(s)\int_0^{\rho s}\left(1-\frac{v}{\rho s}\right)\psi_t(v)dv$$

$$+\frac{p^n}{\rho}\eta'\left(\frac{p^n}{\rho}\right)F(p^n,t) - \eta\left(\frac{p^n}{\rho}\right)D(p^n,t) \in L(\mathrm{R}^+) \tag{9}$$

and

$$\left\|\int_{p^n}^{\infty}\eta\left(\frac{u}{\rho}\right)\psi_{\bullet}(u)du\right\|_1 \leqslant A_1\left(\int_{\frac{p^n}{\rho}}^{\infty} s|d\eta'(s)| + \frac{p^n}{\rho}\left|\eta'\left(\frac{p^n}{\rho}\right)\right|\right)$$

$$+\left|\eta\left(\frac{p^n}{\rho}\right)\right| = O(1). \tag{10}$$

(iv) Using some basic properties of the Walsh–Fourier transform, in particular that if f and $\int_0^\infty f(u)\psi_\bullet(u)du \in L(\mathsf{R}^+)$ then $\left(\int_0^\infty f(u)\psi_\bullet(u)du\right)^\wedge(x) = f(x)$ (see [2]), that if $f \in L^1(\mathsf{R}^+)$ and $\hat{f} = 0$ then $f = 0$ and that if $f, g \in L^1(\mathsf{R}^+)$ then $\widehat{f * g} = \hat{f}\hat{g}$, we have

$$\int_0^{p^j} \psi_t(u)du * \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_t(u)du = 0$$

and so

$$f * \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_t(u)du = \left(f - f * \int_0^{p^j} \psi_t(u)du\right) * \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_t(u)du.$$

Similarly

$$f * \int_{p^n}^{\infty} \eta(\frac{u}{\rho})\psi_t(u)du = \left(f - f * \int_0^{p^n} \psi_t(u)du\right) * \int_{p^n}^{\infty} \eta(\frac{u}{\rho})\psi_t(u)du$$

and so

$$\begin{aligned} \left\| f * \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_t(u)du \right\|_q &\leqslant \omega(f, p^{-j}) \left\| \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_\bullet(u)du \right\|_1 \\ &= O(1)p^{(j-n)\lambda}\omega(f, p^{-j}). \end{aligned} \tag{11}$$

and

$$\begin{aligned} \left\| f * \int_{p^n}^{\infty} \eta(\frac{u}{\rho})\psi_t(u)dn \right\|_q &\leqslant \omega(f, p^{-n}) \left\| \int_{p^n}^{\infty} \eta(\frac{u}{\rho})\psi_\bullet(u)du \right\|_1 \\ &= O(1)\omega(f, p^{-n}). \end{aligned} \tag{12}$$

Now we come to the conclusion of Theorem 2. Defining R by $R = \|f - f * K_\rho\|_q$ then by (8)–(12)

$$\begin{aligned} R &= \left\| f - f * \int_0^{\infty} \eta(\frac{u}{\rho})\psi_\bullet(u)du \right\|_q \\ &\leqslant \left\| f - f * \int_0^{p^n} \psi_\bullet(u)du \right\|_q + \left\| f * \int_0^{1} \left(1 - \eta(\frac{u}{\rho})\right) \psi_\bullet(u)du \right\|_q \\ &\quad + \left\| \sum_{j=0}^{n-1} f * \int_{p^j}^{p^{j+1}} \left(1 - \eta(\frac{u}{\rho})\right) \psi_\bullet(u)du \right\|_q + \left\| f * \int_{p^n}^{\infty} \eta(\frac{u}{\rho})\psi_\bullet(u)du \right\|_q \\ &\leqslant \omega(f, p^{-n}) + \|f\|_q \cdot O(1)p^{-n\lambda} + \sum_{j=0}^{n-1} O(1)p^{(j-n)\lambda}\omega(f, p^{-j}) + O(1)p^{-n\lambda} \\ &= O(1)\left(\sum_{j=0}^{n} p^{(j-n)\lambda}\omega(f, p^{-j}) + p^{-n\lambda}\right). \end{aligned} \tag{13}$$

That is conclusion I of Theorem 2. From (13) we prove conclusion II. If $f \in \mathrm{Lip}\alpha$ then $\omega(f, p^{-k}) = O(1)p^{-k\alpha}$.

For $\alpha < \lambda$,

$$R = O(1)\left(\sum_{j=0}^{n} p^{(j-n)\lambda}p^{-j\alpha} + p^{-n\lambda}\right) = O(1)\left(\frac{1}{p^{n\alpha}}\sum_{j=0}^{n} p^{(j-n)\lambda}p^{(n-j)\alpha} + p^{-n\lambda}\right)$$

$$= O(1)\left(\frac{1}{p^{n\alpha}}\sum_{j=0}^{n} p^{-(n-j)(\lambda-\alpha)} + p^{-n\alpha}\right) = O\left(\frac{1}{\rho^{\alpha}}\right).$$

For $\alpha = \lambda$

$$R = O(1)\left(\sum_{j=0}^{n} p^{(j-n)\lambda}p^{-k\alpha} + p^{-n\alpha}\right) = O(1)(n+1)p^{-n\lambda}$$

$$= O\left(\frac{n}{\rho^{\lambda}}\right) = O\left(\frac{\ln \rho}{\rho^{\lambda}}\right).$$

For $\alpha > \lambda$

$$R = O(1)\left(\sum_{j=0}^{n} p^{(j-n)\lambda}p^{-j\alpha} + p^{-n\alpha}\right)$$

$$= O(1)\frac{1}{p^{n\lambda}}\left(\sum_{j=0}^{n} p^{j(\lambda-\alpha)} + p^{-n\lambda}\right)$$

$$= O\left(\frac{1}{\rho^{\lambda}}\right).$$

This completes the proof of Theorem 2.

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Part Two
Discrete Gibbs Derivatives

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SOME REMARKS ON GIBBS DERIVATIVES ON FINITE DYADIC GROUPS

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ABSTRACT: In this paper we consider Gibbs derivatives on finite dyadic groups. We first discuss the matrix interpretation of partial dyadic Gibbs derivatives and disclose some properties of these operators. Defining the dyadic Gibbs derivative as a linear combination of these partial differential operators we derive a modified product rule for dyadic Gibbs derivatives.

In the second part of the paper we consider an application of partial dyadic Gibbs derivatives to Boolean functions. We treat some properties of these operators valid only in this case and derive a method for detection of symmetry and co-symmetry properties of Boolean functions.

1. INTRODUCTION

The discovery that information can be coded, measured and transmitted discretely [1,2] causes great interest in the application of the discrete algebraic structures in engineering practice and related scientific areas. Discrete functions are very powerful mathematical tools for studying the discrete structures. Among these, the Boolean functions defined as a mapping $f:\{0,1\}^n \in \{0,1\}$, $n \in N$, are most frequently used. This is a natural consequence of the fact that todays technology mainly provides us only with commercially available digital equipment based on circuits with two stable states.

It seems that the main concurrents to Boolean functions in future will be multi-valued logic (MVL) functions and fuzzy functions. The continuing advancement of technology guaranties that the application of these functions in the future will be adequately supported with the corresponding hardware components. A confirmation for such statements can found in the research efforts in these areas all over the world. Let us note the organization of annual symposiums on MVL functions conducted by IEEE and the organization of many research groups in these areas. Good information on these activities can be found in the Bulletin of MVL Technical Committee of IEEE appearing quarterly.

In this paper we will consider the application of dyadic Gibbs derivatives to Boolean functions regarded as the functions defined on the finite dyadic group. Our overall goal is to show that this operator can be efficiently used in the analysis of Boolean functions with the hope that the derived results will trace a way for a similar application of some generalizations of these operators (see, for example [3,4,5,6]) to MVL functions or possibly fuzzy functions.

In our consideration we start from the existing relationship between the Boolean difference and partial Gibbs derivatives and discuss some properties characteristic for these operators when they are applied to Boolean functions. Then we propose a method for detection of symmetry and co-symmetry properties of Boolean functions.

Note that the application of the dyadic Gibbs derivative in fault detection is considered by Edwards [8,9]. The difference with respect to our approach is that in the methods presented in [8,9] Boolean functions are mapped from {0,1} into the {1,-1} domain.

2. NOTATIONS AND DEFINITIONS

It is usual practice to consider the Boolean functions as functions defined on a Boolean algebra, representing in the same time the elements of another Boolean algebra under multiplication and modulo 2 addition defined pointwise. In this paper we will use a slightly modified approach which will enable the derivation of some results interesting both from the theoretical and computational point of view.

As we noted above the domain of Boolean functions is the set of all n-tuples $(x_1,\dots,x_n)$, $x_i\in\{0,1\}$. This set under pointwise addition modulo 2 forms an algebraic structure called the finite dyadic group of order n, which we denote by G_n. We denote by B_n the set of non-negative integers less than 2^n. Recall that there is one-to-one correspondence between the elements of B_n and those of G_n. More precisely, for each $x\in B_n$ there is a unique dyadic expansion of the form

$$x = \sum_{i=1}^{n} 2^{n-i} x_i, \qquad x_i\in B_1$$

and, conversely, every such expansion, that is each n-tuple from G_n, defines a unique element of B_n. This element is usually called the decimal index of the given n-tuple. Due to this a Boolean function can in fact be just as well regarded as being defined on B_n taking their values in $B_n\subset Z\subset C$, where Z is the set of integers and C the set of complex numbers. Also, recall that the set of all complex functions on B_n, denoted by $C(B_n)$, is an Abelian group under pointwise addition defined by

$$(f+g)(x)=f(x)+g(x), \qquad \forall f,g\in C(B_n),\ \forall x\subset B_n.$$

Enriched with multiplication by a scalar $\alpha \subset C$ defined by

$$(\alpha f)(x)=\alpha f(x), \ \forall f \in C(B_n),$$

$C(B_n)$ becomes a linear space admitting an inner product $\langle .,. \rangle$ defined by

$$\langle f,g \rangle = 2^{-n} \sum_{x=B_n} f(x)g^*(x), \qquad \forall f,g \in C(B_n).$$

where g^* denotes the complex conjugate of g.

Therefore, $C(B_n)$ exhibits a Hilbert space structure with norm

$$\|f\| = (\langle f,f \rangle)^{1/2} = 2^{-n}(\sum_{x=B_n} |f|^2)^{1/2}, \qquad \forall f \in C(B_n).$$

It follows that a Boolean function can be regarded as a particular element of this Hilbert space. Furthermore, $C(G)$ may be given the structure of a complex function algebra by introducing the pointwise product of functions through

$$(f \cdot g)(x)=f(x)g(x), \qquad \forall f,g \in C(B_n), \ \forall x \in B_n.$$

Another important operation in $C(B_n)$ is the convolution product defined by:

$$(f*g)(z) = \sum_{x=B_n} f(x)g(z \oplus x') = \sum_{x=B_n} f(z \oplus x')g(x), \ \forall z \in B_n, \ \forall f,g \in C(B_n),$$

where x' is the additive inverse of $x \in B_n$, and $\oplus$ denotes pointwise addition modulo 2 of the dyadic expansions of z and x'.

The dual object of G_n is the set of discrete Walsh functions $G'=\{\mathrm{wal}(x,w)\}$, $\forall x,w \in B_n$ [9], defined in a matrix form by

$$\mathbf{W} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes n},$$

where $\otimes n$ denotes the n-th power Kronecker product (see,for example

[10,11,12]). This set forms an orthonormal base in $C(B_n)$ so that the Walsh (Fourier) transform can be defined as

$$S_f(w) = \sum_{x=0}^{2^n-1} f(x)\mathrm{wal}(x,w), \tag{1}$$

$$f(x) = 2^{-n} \sum_{w=0}^{2^n-1} S_f(w)\mathrm{wal}(x,w). \tag{2}$$

For the thus defined Walsh transform the main properties of the classical Fourier transform hold. For example, the convolution theorem states that if $h=f*g$, $\forall h,f,g\in C(B_n)$, then

$$S_h(w) = 2^{-n} S_f(w)S_g(w), \tag{3}$$

and the reverse statement also holds.

Also, let us note the shift property which states

$$S_{f(x\oplus a)}(w) = \mathrm{wal}(a,w)S_f(w),$$

where $\oplus$ denotes the pointwise addition modulo 2 of the dyadic expansions of x and a.

3. DYADIC GIBBS DERIVATIVE

To make this paper self-contained we will first briefly discuss the definition and some properties of the Gibbs derivative on finite dyadic groups. Then we will derive some new properties related with the application of Gibbs derivatives to Boolean functions.

Definition 1. The partial dyadic Gibbs derivative of a function $f\in C(B_n)$ with respect to the variable x_i is defined by [13]:

$$(D_i f)(x_1,\ldots,x_n)=f(x_i\oplus 1)-f(x_i)=f(x_i')-f(x_i), \tag{4}$$

where $f(x_i')$ is short for $f(x_1,\ldots,x_i',\ldots,x_n)$, x_i' is the additive inverse of x_i and $\oplus$ denotes modulo 2 addition.

It is obvious that there is a strong relationship between the partial dyadic Gibbs derivative and the Boolean difference which is defined as

$$(\Delta_i f)(x_1,\ldots,x_n)=f(x_i\oplus 1)\oplus f(x_i).$$

More precisely, we immediately have

$$|D_i f|=\Delta_i f, \tag{5}$$

where $|a|$ is the absolute value of a.

Definition 2. The dyadic Gibbs derivative of a function $f\in C(B_n)$ is defined [13] by

$$Df(x_1,\ldots,x_n)= -2^{-1}\sum_{i=1}^{n} 2^{n-i}(D_i f)(x_1,\ldots,x_n). \tag{6}$$

It is obvious that in a matrix notation the partial dyadic Gibbs derivative can be represented by a 2^n by 2^n matrix $\mathbf{D}_i$ defined as:

$$\mathbf{D}_i = \bigotimes_{j=1}^{n} \mathbf{A}_i, \tag{7}$$

with

$$\mathbf{A}_j = \begin{cases} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, & j=i \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{otherwise}, \end{cases}$$

where $\otimes$ denotes the Kronecker product.

From this fact, according to Definition 2, the dyadic Gibbs derivative is represented by a 2^n by 2^n matrix representing the linear combination of sparse matrices $\mathbf{D}_i$,

$$D = -2^{-1} \sum_{i=1}^{n-1} 2^{n-i} D_i.$$

Comparing the matrices $\mathbf{D}_i$ with the corresponding matrices appearing in the factorization of the Walsh matrix **W** allowing the introduction of the fast Walsh transform [10,14], we infer a strong similarity between the matrix describing the partial dyadic Gibbs derivative D_i and the matrix describing the i-th step of the fast Walsh transform, which implies the similarity between the corresponding flow graphs. The only difference is that in our case the weights in all horizontal branches of the graph are equal to -1. It follows that the computation of the partial dyadic Gibbs derivative can be carried out by using this modified flow graph for computing the i-th step of the fast Walsh transform. To illustrate this statement we show in Fig.1a the matrix representation of partial dyadic Gibbs derivatives for n=3. The corresponding flow graphs are shown in Fig.1.*b*.

It is obvious from relation (8) that the computation of the dyadic Gibbs derivative Df of a given function $f \in C(B_n)$ can be carried out by summing the outputs of the flow graphs for the calculation of the partial derivatives multiplied respectively with the factors 2^{n-i}, i=1,...,n. The result obtained must be multiplied by the extra factor 2^{-1} introduced in Definition 2 to be consistent with earlier definitions of the dyadic Gibbs derivative [13]. This factor is omitted in the definition of the partial derivative for simplicity of their application in which we are primarily interested in this paper.

$$\mathbf{D}_1=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-1&0&0&0&1&0&0&0\\0&-1&0&0&0&1&0&0\\0&0&-1&0&0&0&1&0\\0&0&0&-1&0&0&0&1\\1&0&0&0&-1&0&0&0\\0&1&0&0&0&-1&0&0\\0&0&1&0&0&0&-1&0\\0&0&0&1&0&0&0&-1\end{bmatrix}$$

$$\mathbf{D}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-1&0&1&0&0&0&0&0\\0&-1&0&1&0&0&0&0\\1&0&-1&0&0&0&0&0\\0&1&0&-1&0&0&0&0\\0&0&0&0&-1&0&1&0\\0&0&0&0&0&-1&0&1\\0&0&0&0&1&0&-1&0\\0&0&0&0&0&1&0&-1\end{bmatrix}$$

$$\mathbf{D}_3=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-1&1&0&0&0&0&0&0\\1&-1&0&0&0&0&0&0\\0&0&-1&1&0&0&0&0\\0&0&1&-1&0&0&0&0\\0&0&0&0&-1&1&0&0\\0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&-1&1\\0&0&0&0&0&0&1&-1\end{bmatrix}$$

Fig.1.a. The partial dyadic Gibbs derivatives on the dyadic group of order 3

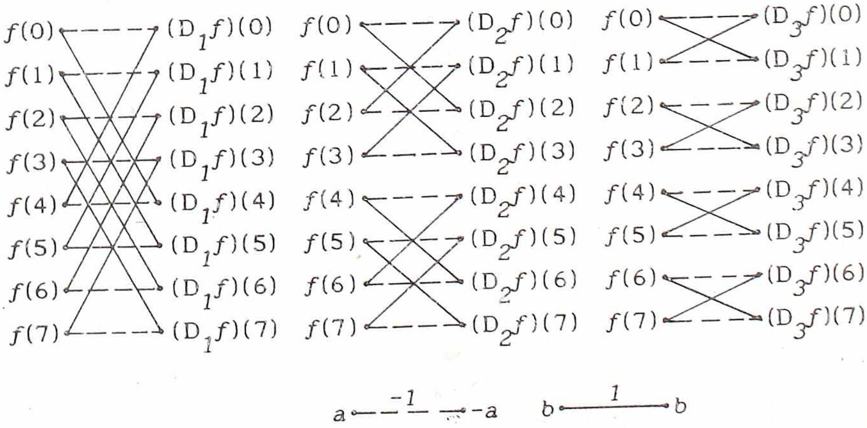

Fig.1b. The flow graphs for calculation of partial dyadic Gibbs derivatives on the dyadivc group of order 3

The main properties of the partial dyadic Gibbs derivatives are stated in the following theorem.

Theorem 1. Let $f\in C(B_n)$.

1. The function f is independent of its i-th argument iff $\forall x\in B_n$ there holds $(D_i f)(x)=0$.

2. $D_i(D_j f)=D_j(D_i f)$.

3. $D_i(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1 D_i f_1+\alpha_2 D_i f_2$, $\quad \alpha_1,\alpha_2\in C,\quad f_1,f_2\in C(B_n)$.

4. D_i commutes with the translation operator T_q on B_n defined by $T_q f(x)=f(x\oplus q)$, where $\oplus$ denotes the pointwise addition modulo 2 of the dyadic expansions for x and q,

$$D_i T_q f = T_q D_i f.$$

5. If S_f is the Fourier transform of $f\in C(B_n)$, then that of its partial dyadic Gibbs derivative $D_i f$ is given by

$$S_{D_i f}(w)=B_i(w)S_f(w),$$

where $B_i(w) = 2^{-1}(\mathrm{wal}(w,2^{n-i}) - 1)$.

6. Convolution property

$$D_i(f*g) = D_i f*g = f*D_i g \quad \forall f,g\in C(B_n),$$

where $*$ denotes the convolution.

7. The product rule

$$D_i(f\cdot g) = f\cdot(D_i g)+g\cdot(D_i f)+(D_i f)\cdot(D_i g),$$

where $\cdot$ denotes the pointwise multiplication.

Proof.1. According to Definition 1, the condition $(D_i f)(x_1,\ldots,x_n)=0$ simply means

$$f(x_i \oplus 1) = f(x_i),$$

and since $x_i \in \{0,1\}$ it follows that the function f takes the same value independently of the value of its i-th argument.

2. By definition,

$$\begin{aligned} D_j(f(x_i \oplus 1, x_j) - f(x_i, x_j)) &= f(x_i \oplus 1, x_j \oplus 1) - f(x_i \oplus 1, x_j) - f(x_i, x_j \oplus 1) + f(x_i, x_j) \\ &= D_i(f(x_i, x_j \oplus 1) - f(x_i, x_j)) \\ &= D_i(D_j f)(x). \end{aligned}$$

3. Linearity follows directly from Definition 1.

4. By definition,

$$\begin{aligned} D_i T_q f(x) &= f(x_1 \oplus q_1, \ldots, x_i \oplus 1 \oplus q_i, \ldots, x_n \oplus q_n) \\ &\quad - f(x_1 \oplus q_1, \ldots, x_i \oplus q_i, \ldots, x_n \oplus q_n) \\ &= T_q(f(x_1, \ldots, x_i \oplus 1, \ldots, x_n)) - T_q(f(x_1, \ldots, x_i, \ldots, x_n)) \\ &= T_q(f(x_1, \ldots, x_i \oplus 1, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)) \\ &= T_q D_i f(x). \end{aligned}$$

5. Starting from the definition of the Walsh transform and using the shift property,

$$\begin{aligned} S_{D_i f}(w) &= \sum_{x=0}^{2^n-1} (D_i f)(x) \mathrm{wal}(w,x) \\ &= \sum_{x=0}^{2^n-1} (f(x_i \oplus 1) - f(x_i)) \mathrm{wal}(w,x) \end{aligned}$$

$$= \sum_{x=0}^{2^n-1} f(x_i\oplus 1)\mathrm{wal}(w,x) - S_f(w)$$

$$= \mathrm{wal}(2^{n-i},w)S_f(w) - S_f(w) = B_i(w)S_i(w),$$

where $B_i(w)=(\mathrm{wal}(2^{n-i},w)-1)$.

As a direct consequence of this property we have

$$D_i(\mathrm{wal}(2^{n-i},x)) = -2\mathrm{wal}(2^{n-i},x).$$

6. The Property 6 follows directly from the definition of the partial dyadic Gibbs derivatives, Property 5 and the convolution property of the Walsh transform.

7. By definition,

$$(D_i(f\cdot g))(x)=f(x_i\oplus 1)\cdot g(x_i\oplus 1)-f(x_i)\cdot g(x_i).$$

Now, also by definition,

$$f\cdot D_i g+g\cdot D_i f+D_i f\cdot D_i g = f(x_i)\cdot(g(x_i\oplus 1)-g(x_i))$$

$$+g(x_i)\cdot(f(x_i\oplus 1)-f(x_i))$$

$$-(f(x_i\oplus 1)-f(x_i))\cdot(g(x_i\oplus 1)-g(x_i))$$

$$=f(x_i\oplus 1)\cdot g(x_i\oplus 1)-f(x_i)\cdot g(x_i)=Df\cdot g.$$

Note that the dyadic Gibbs derivative is a linear combination of the partial dyadic Gibbs derivatives and, therefore, all properties mentioned in Theorem 1 can be similarly formulated to hold for this differential operator. Since in this paper we are primarily interested in the application of partial derivatives, the derivation of these properties will not be discussed here. Let us only briefly consider the product rule for the dyadic Gibbs derivative.

It is known that the absence of a product rule in the form

$$D(f\cdot g) = f\cdot Dg + g\cdot Df,$$

is a property characterizing Gibbs derivatives generally. However, starting from Definition 2 and using the Property 7 from Theorem 1 we obtain, after a short calculation, the modified product rule:

$$D(f\cdot g)=g\cdot Df+f\cdot Dg-2^{-1}\sum_{i=1}^{n} 2^{n-i}(D_i f)\cdot(D_i g).$$

We illustrate this fact by considering the simplest case of the Gibbs derivative on the finite dyadic group of order 4. According to relation (7),

$$D(f\cdot g)=2^{-1}(2D_1+D_2)(f\cdot g) = 2^{-1}(2D_1(f\cdot g)+D_2(f\cdot g))$$

$$=2^{-1}(2g\cdot D_1f+2f\cdot D_1g+2(D_1f)\cdot(D_1g)+g\cdot D_2f+f\cdot D_2g+(D_2f)\cdot(D_2g))$$

$$=2^{-1}(g\cdot(2D_1+D_2)f+f\cdot(2D_1+D_2)g+2(D_1f)\cdot(D_1g)+(D_2f)\cdot(D_2g))$$

$$= g\cdot Df + f\cdot Dg - 2^{-1}(2(D_1f)\cdot(D_1g) + (D_2f)\cdot(D_2g)).$$

4. PARTIAL DYADIC GIBBS DERIVATIVES OF BOOLEAN FUNCTIONS

In this section we will discuss the application of the partial dyadic derivatives to Boolean functions. Boolean functions are a particular subset of functions from $C(B_n)$ and, hence,several new properties of these differential operators valid only in this case can be treated.

In what follows we need the following notations and definitions.

Any Boolean function f is uniquely determined by its truth vector $\mathbf{F}=[f(0),\ldots,f(2^n-1)]^T$. Each variable x_i can be regarded as a n-variable Boolean function, the truth vector $\mathbf{x}_i$ of which can be partitioned into 2^i segments having 2^{n-i} zeros or ones. We denote by x_i' the additive inverse of x_i, and by $f'(x)$ the complement of $f(x)$ defined by $f'(x)=0$ for $f(x)=1$ and $f'(x)=1$ for $f(x)=0$. Also, denote by $f_{i'}$ the Boolean function obtained from a given Boolean function f by replacing the variable x_i by x_i' i.e.,

$$f_{i'}(x_1,\ldots,x_i,\ldots,x_n) = f(x_i,\ldots,x_i',\ldots,x_n) .$$

Recall that the replacement of a variable by its additive inverse simply means the permutation of the function values in the truth vector.

By $\mathbf{0}$ and $\mathbf{1}$ we denote the zero and unit vector, i.e., the vector of order 2^n all elements of which are equal to 0 and to 1, respectively.

Theorem 2. Let $f(x_1,\ldots,x_n)$ be a Boolean function. Then,

1. $D_i f = -D_i f'$
2. $D_i f = -D_i f_{i'}$
3. $D_i(\mathbf{x}_j) = D_i(\mathbf{x}_j') = \mathbf{0}$
4. $D_i(\mathbf{x}_i) = \mathbf{1}-2\mathbf{x}_i$
5. $D_i(\mathbf{x}_i \oplus \mathbf{x}_j) = D_j(\mathbf{x}_i \oplus \mathbf{x}_j) = \mathbf{1}-2(\mathbf{x}_i \oplus \mathbf{x}_j)$
6. $D_i(f \cdot g) = 2^{-1}(D_i f + D_i g) - D_i(f \oplus g)$,

where $\oplus$ means pointwise addition modulo 2.

Note that the Properties 4 and 5 are slightly different from the corresponding relations for the Boolean difference (see, for example [12]), which is a natural consequence of the absolute value appearing in (5).

Proof.1. Since $f \oplus g = f + g - 2f \cdot g$, for $g=1$, $f \oplus 1 = 1-2f$, and hence

$$(D_i f')(x) = (D_i(f \oplus 1))(x) = (D_i(1-f))(x) = (D_i 1)(x) - (D_i f)(x) = -(D_i f)(x).$$

2. By definition $D_i f_{i'} = f(x_i' \oplus 1) - f(x_i')$. Since $x_i' = x_i \oplus 1$,

$$D_i f_{i'} = f(x_i \oplus 1 \oplus 1) - f(x_i \oplus 1) = f(x_i) - f(x_i \oplus 1) = -D_i f.$$

3. The variables x_j and x_j', considered as n-variable Boolean functions, obviously do not depend on the variable x_i, and the property follows directly from Property 1 of Theorem 1.

4. Since $\mathrm{wal}(2^{n-i}, x) = 1 - 2x_i$, using the relation (9) we have

$$D_i x_i = 2^{-1} D_i(1 - \mathrm{wal}(2^{n-i}, x)) = -2^{-1} D_i(\mathrm{wal}(2^{n-i}, x)) = 1 - 2x_i.$$

5. The Property 5 directly follows from Property 4 since there is a one-to-one correspondence between the modulo 2 sum of variables $x_i, \ldots, x_n$ regarded as n variable Boolean functions and Walsh functions (see, for example, [17]).

6. Since $x \oplus y = x + y - 2xy$, $\forall x, y \in B_1$, we immediately have that for Boolean functions f and g taking their values in B_1,

$$(D_i fg) = 2^{-1}((D_i f) + (D_i g) - D_i(f \oplus g)).$$

5. DETECTION OF SYMMETRY PROPERTIES OF BOOLEAN FUNCTIONS

In this section we will consider the application of partial dyadic Gibbs derivatives in the detection of symmetry properties of Boolean functions.

Some particular kinds of symmetries which could appear in a Boolean function are defined and studied in [15,16,17]. Using a similar approach the concept of complementary symmetries (co symmetries) is introduced in [18], and their relationship with symmetries is shown. The application of the symmetries and co-symmetries in logic design and functional decomposition is discussed in [16,17] and [18,19], respectively. These results show that the existence of symmetry or co-symmetry properties in a given Boolean function offers some good possibilities for the optimization of combinatorial network realizing this function.

In general, the answer to the question what kind of symmetries or co-symmetries a given Boolean function does have can be obtained only by means of a search procedure. Since there is a great number of possible combinations it is important to have some efficient procedures for detection of symmetry and co-symmetry properties. Such procedures are formulated both for symmetries and co-symmetries in terms of Walsh transform coefficients. Note that for symmetries an alternative approach based on the application of Boolean difference is also suggested [16]. Starting from these results we will show that partial dyadic Gibbs derivatives can be efficiently used in symmetry and co-symmetry properties detection. To make the paper self-contained we will first repeat some definitions using the notation from [18].

For a $x=(x_1,\ldots,x_n)$ we define the contex of x with respect to i,j in G_n as:

$$\varphi_{ij}(x)=\{\alpha_{ij} \mid \alpha_{ij}\in G_{n-2}\}$$

where $\alpha_{ij}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$.

The restriction of x with respect to $i,j\in G_n$ is given by:

$$x_{ab} = x \quad \text{with} \quad x_i = a,\ x_j = b, \quad \text{and} \quad a,b\in B_1.$$

Definition 3. Let f be a Boolean function.

1. f has equivalence symmetry in x_i and x_j ($E\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{11}) = f(x_{00}) \qquad :E\{x_i, x_j\}.$$

2. f has non-equivalence symmetry in x_i and x_j ($N\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{10}) = f(x_{01}) \qquad :N\{x_i, x_j\}.$$

3. f has partial symmetry in x_i relative to x_j ($P\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{11}) = f(x_{01}) \qquad :P\{x_i, x_j\}.$$

4. f has partial symmetry in x_i relative to x_j' ($P\{x_i,x_j'\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{00}) = f(x_{10}) \qquad :P\{x_i, x_j'\}.$$

5. f has equivalence co-symmetry in x_i and x_j ($EC\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{11}) = 1 \oplus f(x_{00}) \qquad :EC\{x_i, x_j\}.$$

6. f has non-equivalence co-symmetry in x_i and x_j ($NC\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{10}) = 1 \oplus f(x_{01}) \qquad :NC\{x_i, x_j\}.$$

7. f has partial co-symmetry in x_i relative to x_j ($PC\{x_i,x_j\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{11}) = 1 \oplus f(x_{01}) \qquad :PC\{x_i, x_j\}.$$

8. f has partial co-symmetry in x_i relative to x_j' ($PC\{x_i,x_j'\}$) iff for all elements of the contex of x with respect to i,j

$$f(x_{10}) = 1 \oplus f(x_{00}) \qquad :PC\{x_i, x_j'\}.$$

Note that no function can have both a symmetry and the corresponding co-symmetry with respect to exactly the same restriction in its domain. However, the existence of some symmetries or co-symmetries may be conditioned to the existence of other symmetries and co-symmetries. A study of this fact can be found in [19].

Let us now consider, for example, non-equivalence symmetry.

According to part 2 of Definition 3, we can say that to detect whether a given Boolean function f exhibits this property we actually need to compare some pairs of function values specified by the relation

$$f(x_{10}) = f(x_{01}).$$

Since any condition is allowed in the remaining function values, they can be moved from the search procedure. It is fairly obvious that this can be done by multiplying f by (x_i-x_j). Proceeding like this we in at same time associate to the remaining function values a scheme of + and - signs in such a way that the different sign is associated to those values which must be mutually equal if the given function has the examined symmetry property. Then we compare these function values by calculating the partial dyadic Gibbs derivative relative to x_i and x_j. Obtaining a zero vector will mean that the examined pairs of the function values are equal and we conclude that the given function has the non-equivalence symmetry. More precisely, by definition

$$\begin{aligned} D_j(D_i(x_i-x_j)f(x_i,x_j)&=((x_i\oplus 1)-(x_j\oplus 1))f(x_i\oplus 1,x_j\oplus 1)\\ &-(x_i-(x_j\oplus 1))f(x_i,x_j\oplus 1)-((x_i\oplus 1)-x_j)f(x_i\oplus 1,x_j)+(x_i-x_j)f(x_i,x_j), \end{aligned}$$

from which

$$D_j(D_i(x_i-x_j)f(x_i,x_j) = \begin{cases} f(x_{01})-f(x_{10}) & \text{for } x=x_{00} \\ f(x_{10})-f(x_{01}) & \text{for } x=x_{01} \\ -f(x_{01})+f(x_{10}) & \text{for } x=x_{10} \\ -f(x_{10})+f(x_{01}) & \text{for } x=x_{11}, \end{cases}$$

which is obviously equal to zero iff $f(x_{01})=f(x_{10})$, i.e, iff the equality defining non-equivalence symmetry is satisfied. Therefore, the check for detection of non-equivalence symmetry relative to the variables x_i and x_j is given by

$$D_j(D_i(x_i-x_j)f)(x) = 0, \quad \forall x \in B_n.$$

In the same manner we can treat the similar relations for the detection of the remaining symmetries described in Definition 3. The results obtained are given in Table 1.

Table 1

Symmetry	Definition	Test for detection
$E\{x_i,x_j\}$	$f(x_{00})=f(x_{11})$	$D_j(D_i((x_i'-x_j)f))(x)=0$
$N\{x_i,x_j\}$	$f(x_{10})=f(x_{01})$	$D_j(D_i((x_i-x_j)f))(x)=0$
$P\{x_i,x_j\}$	$f(x_{11})=f(x_{01})$	$x_j(D_if)(x)=0$
$P\{x_i,x_j'\}$	$f(x_{00})=f(x_{10})$	$x_j'(D_if)(x)=0$
$EC\{x_i,x_j\}$	$f(x_{11})=1\oplus f(x_{00})$	$D_j(D_i((x_i'-x_j)(x_ix_j\oplus f)))(x)=0$
$NC\{x_i,x_j\}$	$f(x_{10})=1\oplus f(x_{01})$	$D_j(D_i((x_i-x_j)(x_ix_j'\oplus f)))(x)=0$
$PC\{x_i,x_j\}$	$f(x_{11})=1\oplus f(x_{01})$	$x_jD_i(x_ix_j\oplus f)(x)=0$
$PC\{x_i,x_j'\}$	$f(x_{10})=1\oplus f(x_{00})$	$x_j'D_i(x_ix_j'\oplus f)(x)=0$

In a practical application the number of computations required by the tests shown in Table 1 can be considerably reduced by using the fact that $(D_i f)(x_i) = -(D_i f)(x_i')$, and that the truth vectors of auxiliary functions $x_i x_j$, $x_i' x_j$, $x_i - x_j$, $x_i' - x_j$ contain a lot of zeros. In this way the number of computations reduces to the 2^{n-2} subtractions. Since the number of pairs of function values which must be checked to examine whether a particular symmetry or co-symmetry does exist in a given function is equal to 2^{n-2} by definition, the tests implementable in exactly that number of calculations can be considered to be the optimal ones with this respect.

Illustrative example

To illustrate the preceding discussion let us consider the detection of non-equivalence symmetry $N\{x_1,x_2\}$ of a three variable Boolean function $f(x_1,x_2,x_3)$ regarded as a function on the finite dyadic group of order 3 and given by its truth vector $\mathrm{f} = [f(0),\ldots,f(2^n-1)]^T$.

Since $x_1 - x_2 = [0\ 0\ -1\ -1\ 1\ 1\ 0\ 0]^T$, we need to calculate $D_2(D_1 f^*)$ where $f^* = (x_1 - x_2)f$ is given by $f^* = [0\ 0\ -f(2)\ -f(3)\ f(4)\ f(5)\ 0\ 0]^T$;

it can be carried out by the flow graphs shown in Fig.2 obtained by the corresponding modification of the flow graphs for calculation of D_1 and D_2 shown in Fig 1b.

Since $(D_1 f)(x_1,x_2,x_3) = -(D_1 f)(x_1',x_2,x_3)$, the last part of the second step in this flow graph can be eliminated. Furthermore, since $(D_2 f)(x_1,x_2,x_3) = -(D_2 f)(x_1,x_2',x_3)$, another two subtractions can be eliminated. the reduced flow graph obtained in this way is shown in Fig.2b.

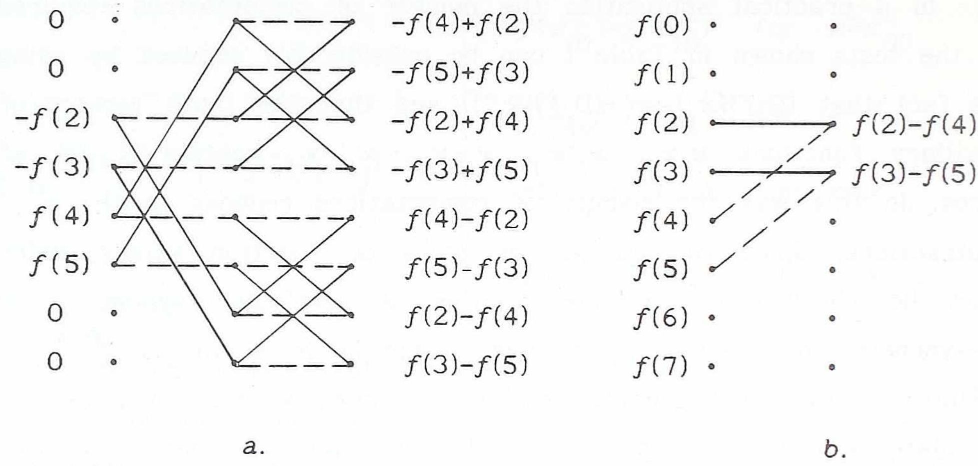

Fig.2. a. Flow graph for detection of $N\{x_1, x_2\}$ property of three variable Boolean funct ons
b. The flow graph of the reduced test

Conclusion

The Gibbs derivative on a finite dyadic group was the initiating concept, a generalization of which lead to a whole theory of Gibbs derivatives. Representing the dyadic Gibbs differential operator in terms of partial dyadic Gibbs derivatives, a modified product rule for dyadic Gibbs derivative is formulated. We hope that this result can be a starting point for the derivation of similar results for numerous generalizations of this operator.

By considering the application of partial dyadic Gibbs derivatives to Boolean functions, some new properties of these operators are discussed, and a method for detection of symmetry and co-symmetry properties of Boolean functions is proposed. We hope that together with Edward's results related with fault detection in combinatorial networks [8,9], the result derived here could represent a basis for further work on the application of Gibbs derivatives in logic design.

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EXPERIMENTS WITH 1-D AND 2-D SIGNALS USING GIBBS DERIVATIVES

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1. Introduction

The concept of a "logical derivative" has been introduced by the work of J.E. Gibbs [1] to provide for functions defined on the finite dyadic group D(n) a calculus similar to the differential calculus of real functions. Butzer [2], [3], Schipp [4] and others developed a complete theory of "logical differentiation" for real functions defined on the interval [0,1).

In this paper we will consider this type of differentiation (which we call Gibbs derivation) in the context of signal processing. We first discuss the concepts and tools which are used by us to be able to realize effective experiments with 1-D and 2-D signals. The subsequent section deals with the experiments we have performed. Finally we give an outlook on other experiments which seem to be promising.

2. Gibbs Derivation

For any real-valued function f defined on a finite dyadic group D(n) the Gibbs derivative $D(f) = f^{[1]}$ of f, is defined by

$$f^{[1]}(x) = \sum_{i=0}^{n-1}\{f(x) - f(x \oplus \Delta(i))\}2^{i-1} \tag{1}$$

where $\Delta(i) = (y_{n-1}, y_{n-2},\ldots,y_o)$ and $y_k = 1$ if $k = n-i-1$ and $y_k = 0$ for all $k \neq n-i-1$.

It is easy to show that the Gibbs derivative defines a linear operation which has as its eigenfunctions the Walsh function ψ_s as defined in the Paley-ordering. We have for all $s \in D(n)$

$$\psi_s^{[1]} = \lambda(s)\psi_s \qquad (2)$$

where for $s = (s_{n-1}, s_{n-2}, \ldots, s_o)$ we have $\lambda(s) = s_{n-1}2^{n-1} + s_{n-2}2^{n-2} + \ldots + s_o$.

Equation (2) shows that the Gibbs derivative is the result of processing f by a linear dyadic invariant system with transfer function H given by

$$H(s) = \lambda(s) \quad \text{for } s \in D(n). \qquad (3)$$

By this interpretation it is quite clear that a Gibbs derivation operator suppresses the Dc-values of a signal and realizes a gain-factor which increases linearly with the spectral parameter s.
On the other hand, we have also the opportunity to consider Gibb derivation as the result of the dyadic convolution operation with impulse response h given by

$$h(x) = \begin{cases} (2^n-1)/2 & \text{for } x = 00\ldots0 \\ -2^{i-1} & \text{for } x = \Delta(i) \\ 0 & \text{for all other } x. \end{cases} \qquad (4)$$

3. Theoretical Background for the Experiments

We have developed the method base system CAST.FOURIER which provides many important software tools to experiment with signals defined on finite Abelian Groups [5]. We will use CAST.FSM to investigate by experimentation the effects of the Gibbs derivation operator on 1-D and 2-D signals.
If f is a 1-D or 2-D signal we will compute $Df := f^{[1]}$ with CAST.FOURIER in the following way:
(1) in the first step we compute the Walsh transform S_f of f by applying the Fast Walsh-Transform algorithm available in CAST.FOURIER,
(2) in the second step we multiply S_f by the transfer function H of the Gibbs derivation operator D as given by (3),
(3) in the third step we apply the inverse Walsh-Transform Algorithm to the product $H \cdot S_f$ to compute Df.

We have to add the information how we want to define the Gibbs derivation operator for 2-D signals. To do this we first remember that the 2-D Walsh-functions $\psi_{s,t}$ are defined by
$\psi_{s,t}(x,y) := \psi_s(x)\ \psi_t(y)$ where $s,t \in D(n)$ and $(x,y) \in D(n) \times D(n)$.
For $f : D(n) \times D(n) \to R$ the Gibbs derivative $Df = f^{[1]}$ can be defined by

$$f^{[1]}(x,y) = \sum_{i=0}^{n-1}[f(x,y)-f(x\oplus\Delta(i),y)]\,2^{i-1} + \sum_{j=0}^{n-1}[f(x,y)-f(x,y\oplus\Delta(j))]\,2^{j-1} \quad (5)$$

This definition ensures that the 2-D Walsh-functions $\psi_{s,t}$ are - in analogy to the 1-D case - the eigenfunctions of D. We have

$$D\psi_{s,t} = (\lambda(s) + \lambda(t))\,\psi_{s,t}\ . \quad (6)$$

Besides experimenting with the Gibbs derivative we want also to make some experiments concerning the approximation of polynomial functions by Walsh-Fourier expansions. It has been observed by Polyak and Shreider [6] and independently also by Liedl [7] that in a Walsh-Fourier expansion of a polynomial function of degree k all Walsh-Fouriercoefficients c(s) of Walsh-functions ψ_s, which are generated by more than k Rademacher-functions, have the value zero. This is the essential meaning of the "multiplicity theorem" (the number $m(\psi_s)$ of Rademacher-functions in a Walsh-function ψ_s has been called the "multiplicity" of ψ_s).
The multiplicity-concept allows the definition of a special class of linear dyadic invariant filters, which have special properties for the approximation of signals, especially if these have a shape similar to a polynomial.
Finally as the last concept Gibbs derivations of a boolean function will be considered by CAST.FOURIER experiments. It can be proven that the following theorem holds for boolean functions $f : D(n) \rightarrow \{0,1\}$.
For any function $g : D(n) \rightarrow R$ let bool (g) denote the boolean function derived from g by

$$\mathrm{bool}(g)(x) := \begin{cases} 1 & \text{if } g(x) \geq 0 \\ 0 & \text{if } g(x) < 0\ . \end{cases} \quad (7)$$

Then for the Gibbs derivative $f^{[1]}$ of a boolean function f the following theorem is valid

$$\mathrm{bool}\,(f^{[1]}) = f\ . \quad (8)$$

Equation (8) shows that the Gibbs derivative of a boolean function f followed by the boolean threshold operation bool reproduces the boolean function. We will show this property also by CAST.FOURIER experiments in 1-D and 2-D cases of signals.

. CAST.FOURIER Experiments

1 Computation of Gibbs derivatives

s a first task we consider the computation of the Gibbs derivative of 1-D signals and 2-D signals

4.1.1 1-D signals

Figure 1(a) shows the discrete function sin : {0,1,...,63} → R given by sin(2π x /64). Figure 1(b) is the related Walsh-Fourier spectrum S_{sin}. We apply now the Gibbs derivative to the function sin and get as result the function shown in Figure 1(c). Figure 1(d) shows the computation of the second Gibbs derivative D^2 (sin) of the function sin.

4.1.2 2-D signals

Figure 2(a) shows the considered image "Spanish girl" sg. Figure 2(b) shows the computed Walsh Fourier spectrum (where a log-scaling is used in presenting amplitudes). Figure 2(c) is the Gibbs derivative D(sg) of sg. Figure 2(d) is the second Gibbs derivative D^2(sg) of sg. We see the effect of the Gibbs derivative in sharpening the edges of the image.

4.2 Multiplicity filtering

4.2.1 1-D signals

Figure 3(a) shows for the 1-D signal of Figure 1(a) the result of multiplicity low pass filtering of order 3 (all spectral components of multiplicity<3 passes). Figure 3(b) is the corresponding Walsh-Fourier spectrum. Figure 3(c) shows the signal when it passes a multiplicity low pass filter of order 4. We see that the signal is completely reconstructed (the function sin does not contain spectral components of order higher than 3).

4.2.2 2-D signals

Figure 4(a) and Figure 4(b) show the effect of filtering the image "Spanish girl" sp (of Figure 2(a)) by multiplicity low pass filters of order 6 and order 8, respectively.

4.3 Gibbs derivation of boolean functions

We consider an arbitrarily chosen boolean function f : $B^6 \rightarrow$ B according to Fig. 5(a). Figure 5(b) shows the Gibbs derivative g = D(f) of the boolean function. Figure 5(c) shows the result of the operation bool on g. We see that, as requested by (8), we get bool D(f) = f.

Finally Figure 6(a) shows a boolean image "cars". Figure 6(b) shows the Gibbs derivative D(cars) and Figure 6(c) shows bool (D(cars)). Again we see that we get the original image "cars" back; bool(D(cars)) = cars.

5. Conclusion

The present paper discussed some of the experiments performed with the interactive method bank system CAST.FOURIER. The experiment show that with the help of the tool "CAST.FOURIER" it is quite easy to make the effects of taking Gibbs derivatives of 1-D signals (for example representing speech information) and 2-D signals (representing images) better understood. Furthermore, it is possible to combine Gibbs derivation with other known transformations, e.g. with multiplicity filtering. It is our hope that with the performance of such and similar experiments we will be able to develop a new type of algorithms for signal processing, especially for the field of image processing.

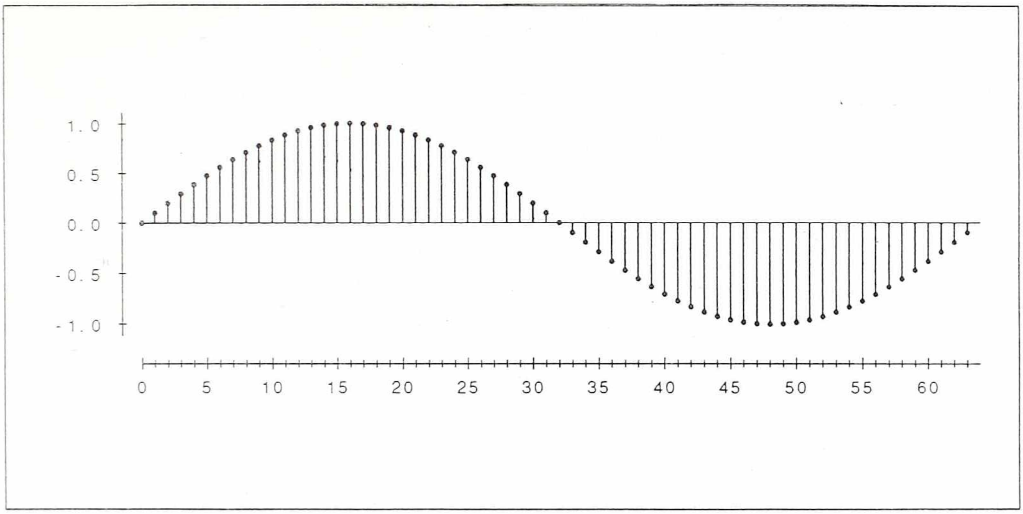
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Fig. 1(a)

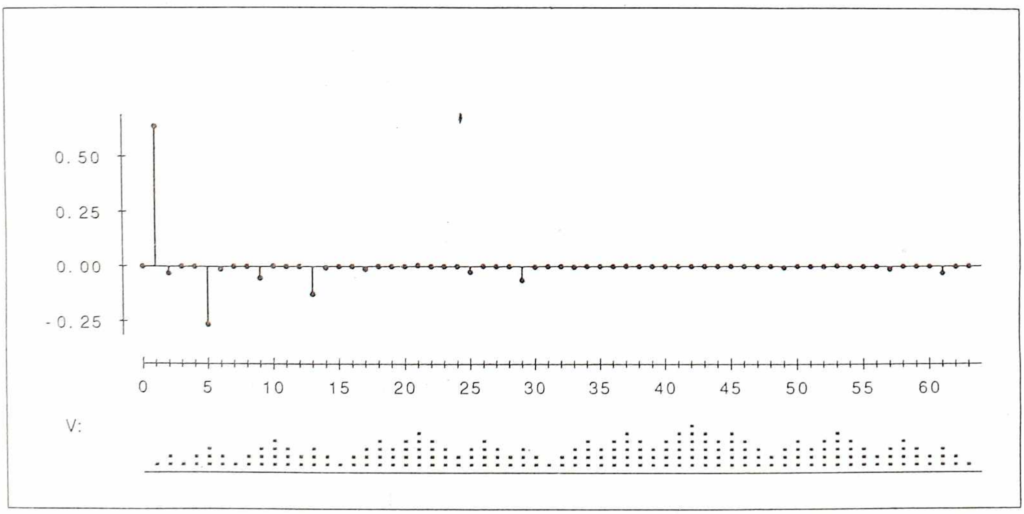
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Fig. 1(b)

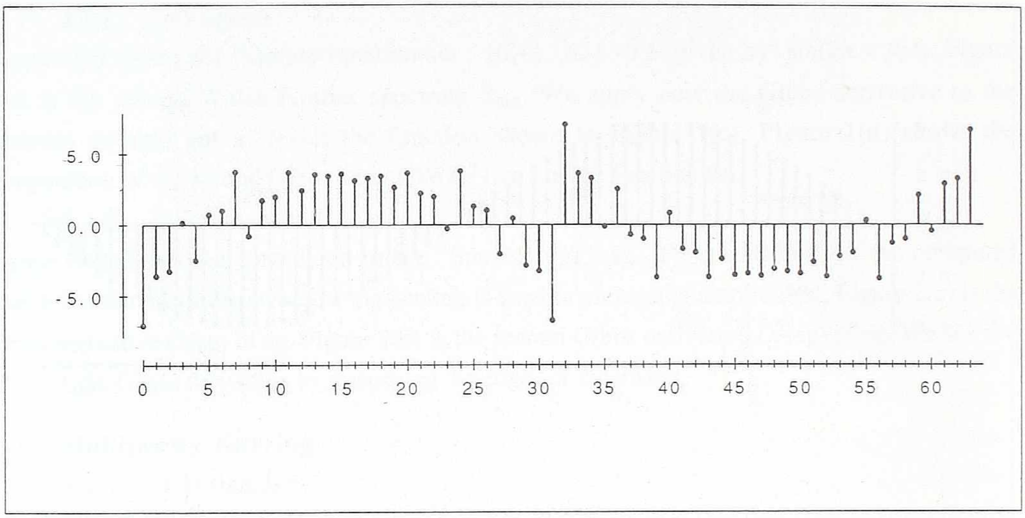
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Fig. 1(c)

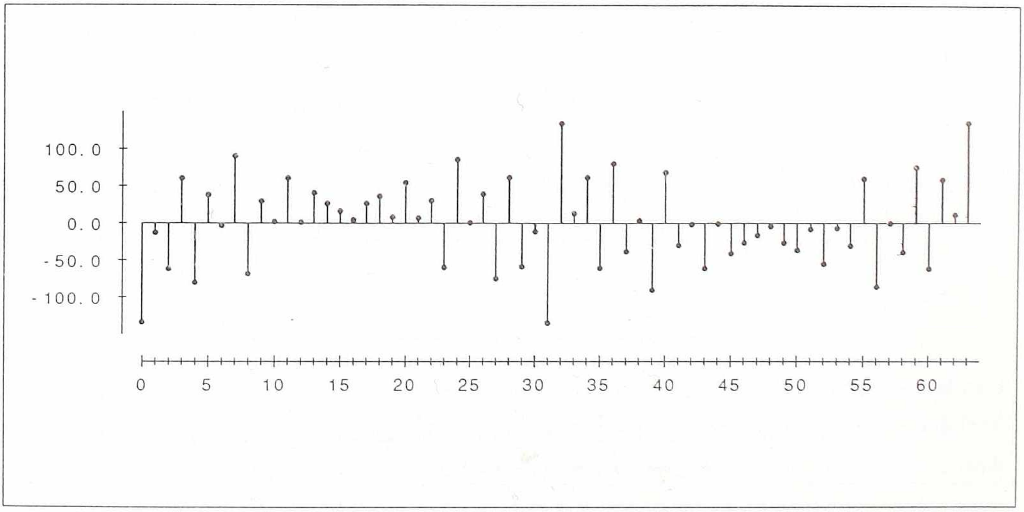
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Fig. 1(d)

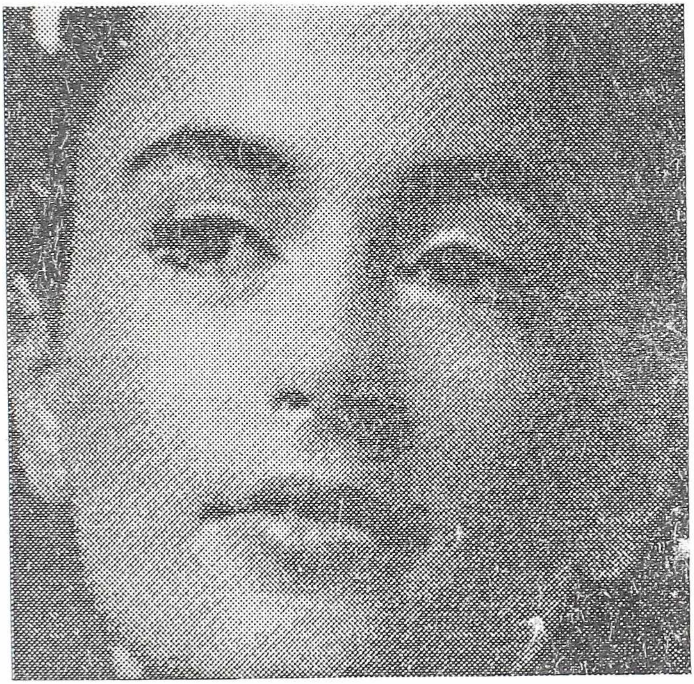

Fig. 2(a)

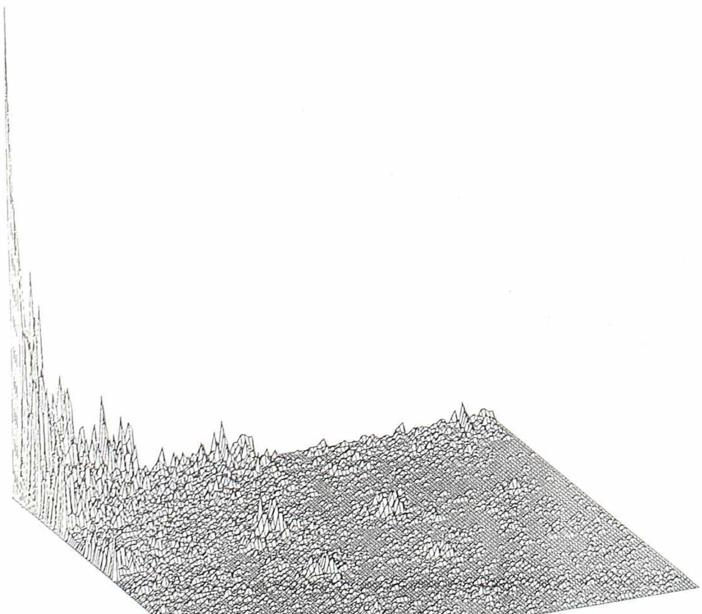

Fig. 2(b)

Fig. 2(c)

Fig. 2(d)

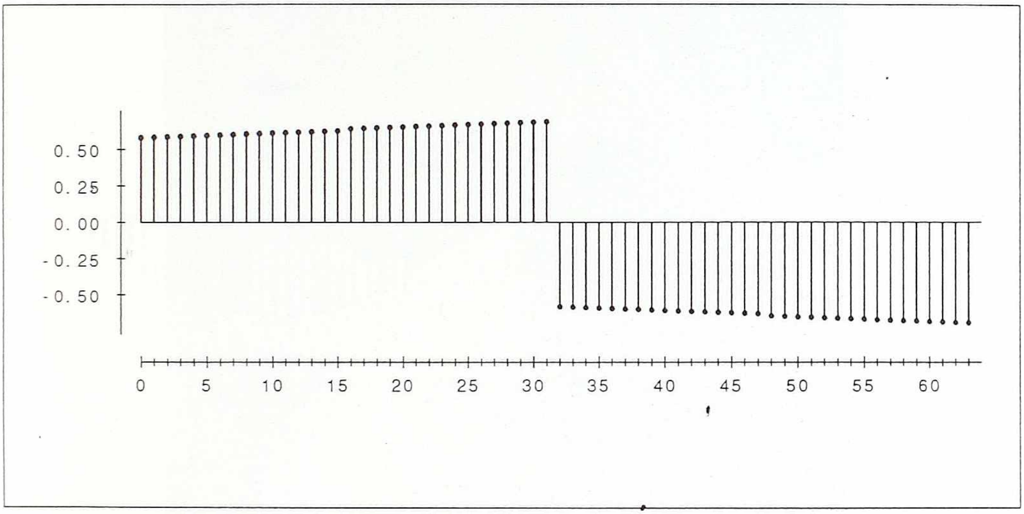
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Fig. 3(a)

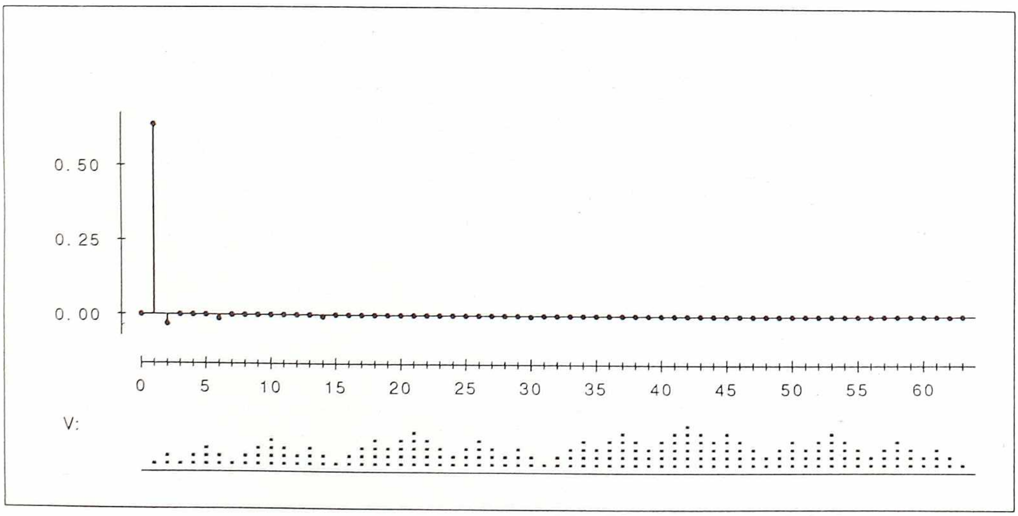
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Fig. 3(b)

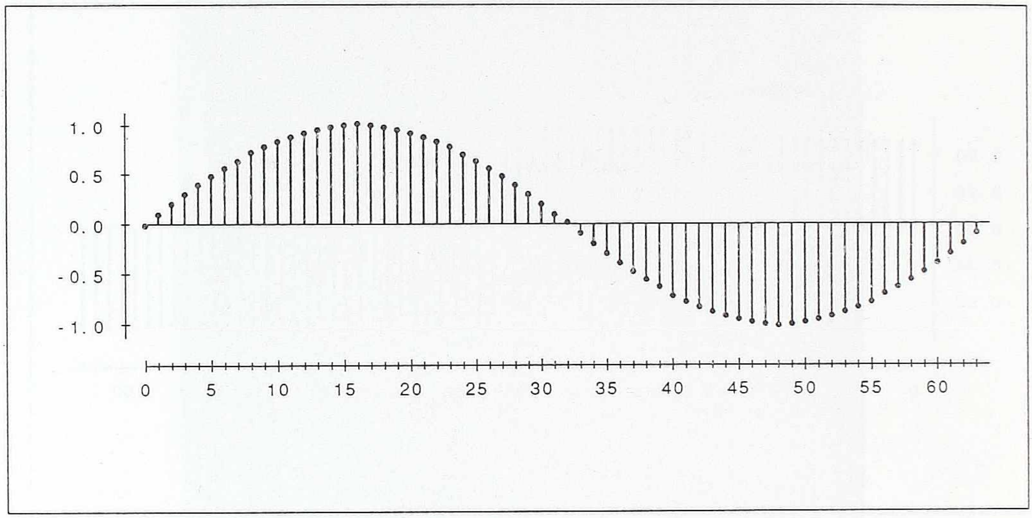

Fig. 3(c)

Fig. 4(a)

Fig. 4(b)

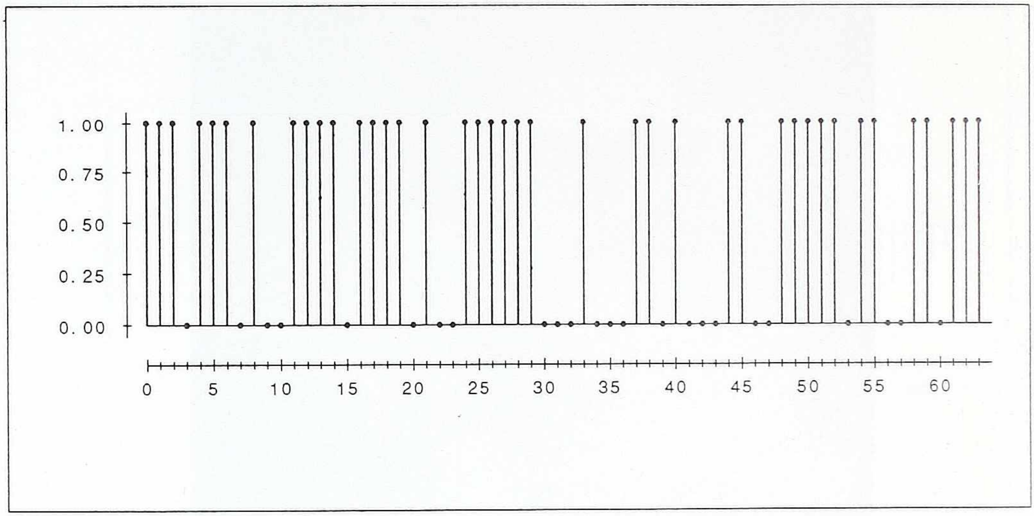
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Fig. 5(a)

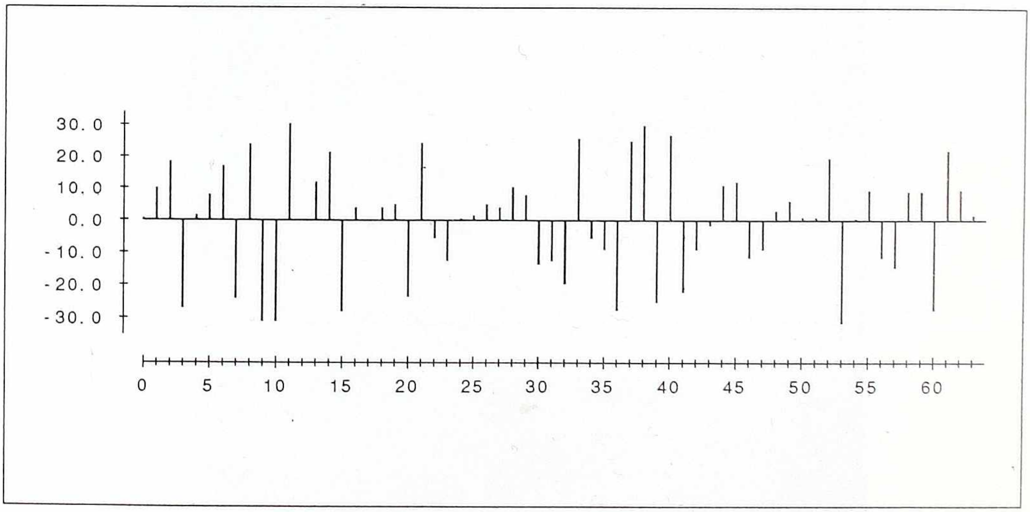
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Fig. 5(b)

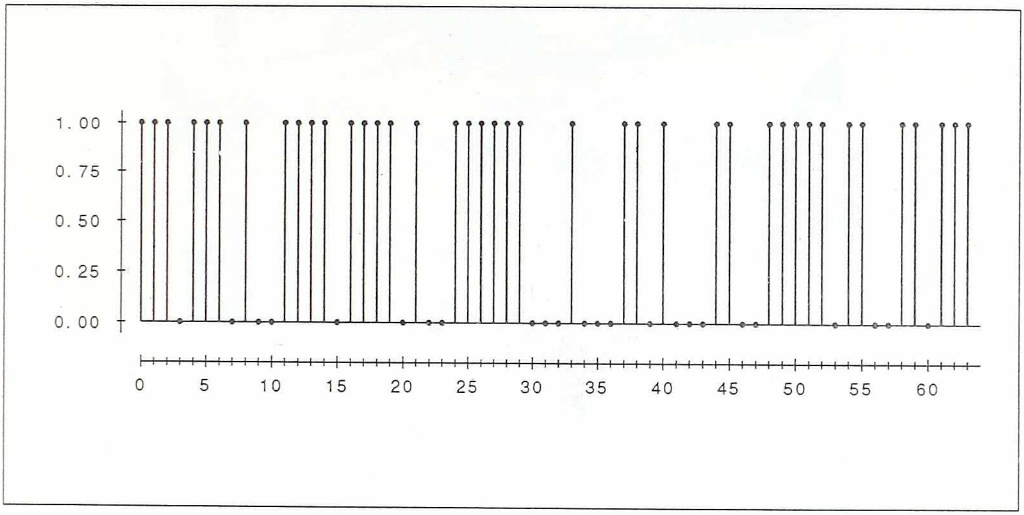

Fig. 5(c)

Fig. 6(a)

Fig. 6(b)

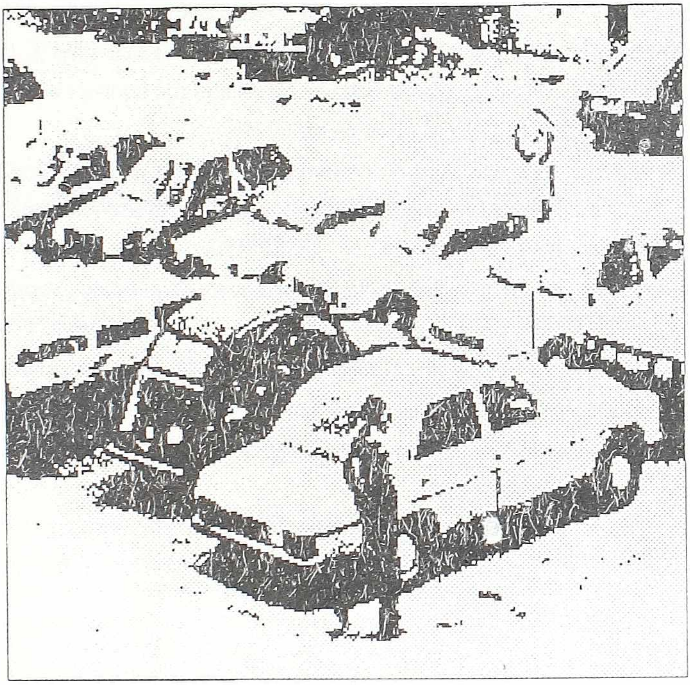

Fig. 6(c)

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VARIATIONS ON THE GIBBS' DERIVATIVES [1]

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Abstract

In this paper we introduce a *Variation* on the Gibbs' logic derivative . The main difference with respect to the Gibbs' definition and its various generalizations is that the factor $j=\sqrt{-1}$ is considered in the sequence domain. It is shown that the logic derivative as disclosed in this paper has some basic properties more analogous to those of the Newton-Leibnitz derivative. We are concerned here with the definition and properties of the logic derivative only for discrete one-dimensional sequences, however the results of this paper can be extended to the case where discrete sequences are multiple-dimensional. Furthermore we discuss discrete cyclic invariant systems which may be described by a linear logic difference equation. Finally, it is shown that there is a simple relationship between the logic derivative and the ordinary derivative of generalized functions.

[1] *Variation:* (Music). Repetition of a theme or melody with ornamental notes or modifications in rythm, tune, harmony or key. (Merriam-Webster Dictionary)

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I. Introduction

The concept of logic derivative was first presented by J. E, Gibbs [1,2,3] with the property that the Walsh functions are characteristic solutions of a linear operation equation just as that exponential functions emerge as the characteristic solutions of a linear ordinary differential equation. Gibbs called the linear operation *logic derivative*. Even though J.E. Gibbs and B. Ireland further developed this idea to derivatives on finite abelian groups [4], the logic derivative of Gibbs has remained associated to Walsh functions in many later publications. This is probably why, unaware of [4], Deng Weihang, Su weiyi, Ren Fuxian [5] and C.Moraga [6] introduced as a "generalization of Gibbs' derivative" a logic derivative in $\mathbb{Z}_p{}^n$ and showed that in this case the Chrestenson functions [7] also emerge as a solution of a characteristic value problem.

The discrete Chrestenson functions are defined by

$$CH_k(n) = \exp(j2\pi \sum_{s=0}^{m-1} k_s n_s / p) \qquad (1)$$

where $k, n = 0,1,\ldots,p^m-1;\ k_s, n_s \in \mathbb{Z}_p,\ j=\sqrt{-1}$

and $$k=\sum_{s=0}^{m-1} k_s p^{m-1-s} \qquad n=\sum_{s=0}^{m-1} n_s p^{m-1-s} .$$

According to the definition of the logic derivative for this case [4, 5, 6], we have

$$\left(CH_k(n)\right)^{(1)} = k\, CH_k(n) \qquad (2)$$

Now we consider the most simple and basic case where $m=1$ and $p=N > 2$ for the Chrestenson functions. In this case the Chrestenson functions are just the kernel functions of the discrete Fourier Transform (DFT):

$$\Phi_k(n) = e^{j2\pi kn/N} , \qquad (3)$$

where $k=k_s$ and $n=n_s$.

Therefore, for the logic derivative we have:

$$\left(e^{j2\pi kn/N} \right)^{(1)} = k\, e^{j2\pi kn/N} \qquad (4)$$

Comparing with the Newton-Leibnitz derivative of exponential functions:

$$(e^{j\omega t})' = j\omega e^{j\omega t}, \qquad (5)$$

there are some questions to be studied in order to find a (possibly local) relationship between the Gibbs derivative (or a *variation* thereof) and the Newton-Leibnitz derivative:

(1) Why does the factor j not affect the logic derivative of the exponential functions?

(2) It is known that the exponential functions exp(j2πkn/N) are periodic:

$$e^{j2\pi(k + N)n/N} = e^{j2\pi kn/N}$$

is its logic derivative also periodic?

In this paper we present a *variation* on the logic derivative of Gibbs in order to give an answer to the questions stated above.

II. A Variation on the Logic Derivative

Let f(n), n=0,1,...,N-1, be a one-dimensional complex sequence. f(n) can be expressed as

$$f(n)= \frac{1}{N}\sum_{k=0}^{N-1} F(k)e^{j2\pi kn/N} \qquad (6)$$

where F(k) is the discrete Fourier transform of f(n) and

$$F(k)= \sum_{n=0}^{N-1} f(n)e^{-j2\pi kn/N} \qquad (7)$$

Definition 1:

The logic derivative of a one-dimensional complex sequence f(n) is defined by

$$f^{(1)}(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)\left(e^{j2\pi kn/N}\right)^{(1)} \tag{8}$$

and
$$\left(e^{j2\pi kn/N}\right)^{(1)} = jk\, e^{j2\pi kn/N} \tag{9}$$

In the more general case of some periodic function with period N we consider $\ell = 0, 1, \ldots, Q$ with $Q > N$. We define

$$\left(e^{j2\pi \ell n/N}\right)^{(1)} = j[\ell]\, e^{j2\pi \ell n/N} \tag{10}$$

where $[\ell]$ stands for the residue of ℓ modulo N.

Obviously,

$$\left(e^{j2\pi kn/N}\right)^{(1)} = \left(e^{j2\pi (k+N)n/N}\right)^{(1)} = j[(k+N)]\, e^{j2\pi kn/N}$$

$$= jk\, e^{j2\pi kn/N}$$

since
$$[N] \equiv 0 \text{ modulo } N \tag{13}$$

Definition 2:

Let
$$d(n) = \frac{1}{N}\sum_{k=0}^{N-1} jk\, e^{j2\pi kn/N} \tag{14}$$

It is easy to prove that

$$f^{(1)}(n) = f(n) * d(n) = \sum_{\tau=0}^{N-1} f(\tau)\, d(n \ominus \tau) \tag{15}$$

where $\ominus$ stands for subtraction mod. N.

It follows that d(n) can be computed as shown below:

$$(1/N)\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{N-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \cdots & \varepsilon^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{N-1} & \varepsilon^{2(N-1)} & \cdots & \end{bmatrix} \begin{bmatrix} 0 \\ j \\ j2 \\ \cdot \\ j(N-1) \end{bmatrix} = j \begin{bmatrix} (N-1)/2 \\ 1/(\varepsilon^1 - 1) \\ 1/(\varepsilon^2 - 1) \\ \cdot \\ \cdot \\ 1/(\varepsilon^{N-1} - 1) \end{bmatrix}$$

where $\varepsilon = \exp(j2\pi/N)$.

Then we have:

$$d(n)=\begin{cases} j(N-1)/2 & \text{if} \quad n=0\ , \\ j/(\varepsilon^{n}-1) & \text{if} \quad n\neq 0 \end{cases} \tag{16}$$

III. Properties of the Variation on the Logic Derivative

P1. The logic derivative operation is linear:

$$\left(af_1(n) + bf_2(n)\right)^{(1)} = af_1^{(1)}(n) + bf_2^{(1)}(n) \tag{17}$$

where a, b are any complex numbers.

If $f(n) = \varphi(n) + j\psi(n)$, were $\varphi(n)$ and $\psi(n)$ are real valued sequences, we have:

$$f^{(1)}(n) = \varphi^{(1)}(n) + j\psi^{(1)}(n)$$

Proof: Set $a = 1+j0$ and $b = 0+j1$ in Eq. (17).

P2. The logic derivative of a constant is zero.

$$(c)^{(1)} = 0 \tag{18}$$

P3. Let $f(n) \longleftrightarrow F(k)$ denote that $F(k)$ is the discrete Fourier transform of $f(n)$, then $f^{(1)}(n) \longleftrightarrow jkF(k)$ and

$$(-jn)f(n) \longleftrightarrow F^{(1)}(k) \tag{19}$$

P4.

$$\left(\cos 2\pi kn/N\right)^{(1)} = \frac{1}{2}\left(e^{j2\pi kn/N} + e^{-j2\pi kn/N}\right)^{(1)}$$

$$= \frac{1}{2}\left(jke^{j2\pi kn/N} - jke^{-j2\pi kn/N}\right)$$

$$= -k\sin 2\pi kn/N \tag{20}$$

P5.

$$\left(\sin 2\pi kn/N\right)^{(1)} = \frac{1}{2j}\left(e^{j2\pi kn/N} - e^{-j2\pi kn/N}\right)^{(1)}$$

$$= \frac{1}{2j}\left(jke^{j2\pi kn/N} + jke^{-j2\pi kn/N}\right)$$

$$= k\cos 2\pi kn/N \tag{21}$$

Notice that properties P4 and P5 do not hold for Gibbs derivatives.

VI Cyclic Invariant Systems

Definition 3. The r-th derivative $f^{(r)}(n)$ of f(n) is defined by

$$(e^{j2\pi kn/N})^{(r)}=(jk)^r e^{j2\pi kn/N} \qquad (22)$$

$$f^{(r)}(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)(jk)^r e^{j2\pi kn/N} \qquad (23)$$

where f(n)<--->F(k).

The discrete unit impulse function δ(n) is defined by

$$\delta(n)= \begin{cases} 1 , & \text{if } n=0 \\ 0 , & \text{if } n\neq 0. \end{cases}$$

Obviously, δ(n)<--->1,

$$\delta^{(1)}(n) \longleftrightarrow jk \qquad (24)$$

and

$$\delta^{(r)}(n) \longleftrightarrow (jk)^r \qquad (25)$$

With eqs. (7) and (24) we obtain

$$\delta^{(1)}(n) = \frac{1}{N}\sum jk\, e^{j2\pi kn/N}$$

i.e.

$$d(n) = \delta^{(1)}(n) \qquad (26)$$

and

$$f(n) = f(n) * \delta(n)$$

$$f^{(1)}(n) = f(n) * \delta^{(1)}(n),$$

$$f^{(r)}(n) = f(n) * \delta^{(r)}(n).$$

Let F(k) <---> f(n). Then we have:

$$F(k \oplus k_o) = \sum_{k=0}^{N-1} f(n)\; e^{j2\pi(k \oplus k_o)n/N}$$

$$= e^{j2\pi k_o n/N} \sum_{k=0}^{N-1} f(n)\; e^{j2\pi k n/N}$$

$$= e^{j2\pi k_o n/N}\; F(k)$$

i.e.

$$F(k \oplus k_o) \;\langle\text{---}\rangle\; f(n)\, e^{j2\pi k_o n/N}$$

Using this shift operatiion in the sequency domain it is simple to show that for modulated signals we have the following property.

$$f(n) \sin 2\pi k_0 n/N \;\langle\text{---}\rangle\; \big(F(k \ominus k_0) - F(k \oplus k_0)\big)/2j$$

$$f(n) \cos 2\pi k_0 n/N \;\langle\text{---}\rangle\; \big(F(k \ominus k_0) + F(k \oplus k_0)\big)/2 \qquad (27)$$

Definition 4.

A finite discrete linear cyclic invariant system is a convolutional system represented as

x(n) ---> [ ] --->y(n)

where x(n) is the input function or source function and y(n) the output function or response function. The input and output function are related by a finite discrete linear derivative equation; i.e.,

$$b_q y^{(q)}(n) + b_{q-1} y^{(q-1)}(n) + \ldots + b_0 y(n) =$$

$$= a_r x^{(r)}(n) + a_{r-1} x^{(r-1)}(n) + \ldots + a_0 x(n), \qquad (28)$$

where $b_0 \neq 0$.

After computing the discrete Fourier transform for the equation we obtain

$$Y(k) \sum_{i=0}^{q} b_i (jk)^i = X(k) \sum_{i=0}^{r} a_i (jk)^i$$

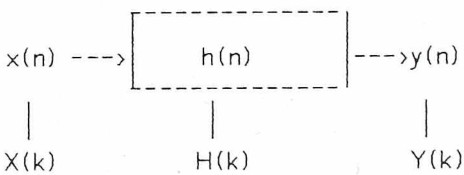
which can be rewritten as

$$Y(k)=H(k)X(k), \tag{29}$$

where

$$H(k)=\frac{\sum_{i=0}^{q} b_i(jk)^i}{\sum_{i=0}^{r} a_i(jk)^i}, \qquad x(n)\longleftrightarrow X(k) \quad \text{and} \quad y(n)\longleftrightarrow Y(k).$$

Let $h(n)\longleftrightarrow H(k)$. If $x(n)=\delta(n)$, then $y(n)=h(n)$. We recognize that $h(n)$ is the impulse response function, which only depends on the system. We consider $H(k)$ to be a representation of the system in the sequency domain and $h(n)$ the representation in original domain as shown below:



with $$y(n)=x(n)*h(n) \tag{30}$$

and $$Y(k)=X(k)H(k).$$

VII. Relationship between the Logic Derivative and the Ordinary Derivative of Generalized Functions

As earlier mentioned , the logic derivative is different from the ordinary derivative; but there is a relationship between the logic derivative and the ordinary derivative of generalized functions.

Let $X(\omega)$ be the Fourier transform of a function $x(t)$, and let $\tilde{X}(\omega)$ be the one of the sampled signal $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(n\tau)\delta(t-n\tau) \tag{31}$$

where $\delta(t-n\tau)$ represents a generalized delta function .

Define a function x_T, which consists of an infinite sum of functions x shifted by integer multiples of T:

$$x_T(t) = \sum_{n=-\infty}^{\infty} x(t - nT) \tag{32}$$

We construct now the discrete function x_T by specifying values of this function for $m\tau$, where $T = N\tau$ and $m = 0, 1, \ldots, N-1$. Then we have:

$$x_T(m\tau) = \sum_{n=-\infty}^{\infty} x\big((m + nN)\tau\big) \tag{33}$$

It is known that the Fourier Transform of a δ distribution is given by:

$$\mathfrak{F}\big(\delta(t - \nu\tau)\big) = e^{-j\omega\nu\tau} \tag{34}$$

(see e.g. [8], eq. 2-53)

It is simple then to prove that the Fourier Transform of $\tilde{x}(t)$ given in eq. (31) is:

$$\tilde{X}(\omega) = \sum_{\nu=-\infty}^{\infty} x(\nu\tau)\, e^{-j\omega\nu\tau} \tag{35}$$

Now we express the infinite sum in terms of periods of length N and get:

$$\tilde{X}(\omega) = \sum_{n=-\infty}^{\infty} \sum_{\nu=nN}^{(n+1)N-1} x(\nu\tau)\, e^{-j\omega\nu\tau}$$

which leads to the following, if we replace "ν - nN" by "m" :

$$\tilde{X}(\omega) = \sum_{m=0}^{N-1} \sum_{n=-\infty}^{\infty} x\big((m+nN)\tau\big) e^{-j\omega m\tau} e^{-j\omega nN\tau}$$

and using eq. (33) we obtain:

$$\tilde{X}(\omega) = \sum_{m=0}^{N-1} x_T(m\tau)\; e^{-j\omega m\tau}\; e^{-j\omega nN\tau}$$

Finally we let $\omega = 2\pi k/(N\tau)$, leading to:

$$\tilde{X}\big(2\pi k/(N\tau)\big) = \sum_{m=0}^{N-1} x_T(m\tau)\, e^{-j2\pi mk/N} \tag{36}$$

from where

$$x_T(m\tau) = N^{-1} \sum_{k=0}^{N-1} \tilde{X}\big(2\pi k/(N\tau)\big) e^{j2\pi km/N}$$

The logical derivative of $x_T(m\tau)$ is given by

$$x_T^{(1)}(m\tau) = N^{-1} \sum_{k=0}^{N-1} jk\ \tilde{X}(k2\pi/N\tau)\ e^{j2\pi km/N} \tag{37}$$

Suppose $\tilde{\varphi}(t)$ represents the ordinary derivative of the sampled function $\tilde{x}(t)$:

$$\tilde{\varphi}(t)=\partial(\tilde{x}(t))/\partial t = \Big(\sum_{n=-\infty}^{\infty} x(n\tau)\delta(t-n\tau)\Big)' = \sum_{n=-\infty}^{\infty} x(n\tau)\delta'(t-n\tau)$$

where $\delta'(t-n\tau)$ denotes the ordinary derivative of $\delta(t-n\tau)$.

Since $j\omega\tilde{X}(\omega)$ is the Fourier transform of $\tilde{\varphi}(t)$, we obtain

$$\varphi_T'(n\tau)=N^{-1}\sum_{k=0}^{N-1} j(k2\pi/N\tau)\tilde{X}(k2\pi/N\tau)\ e^{j2\pi kn/N} \tag{36}$$

where $\varphi_T(t) = \sum_{n=-\infty}^{\infty} \varphi(t-nT)$, and $\tilde{\varphi}(t)= \sum_{n=-\infty}^{\infty} \varphi(n\tau)\delta(t-n\tau)$. (37)

Therefore, from eqs. (34) and (37) follows that the logic derivative of the sequence of function values $x_T(n\tau)$, $n=0,1,\dots,N-1$, of a function $x_T(t)$ is equal to the sequence of values $\varphi_T(n\tau)$, $n=0,1,\dots,N-1$, of the function $\varphi_T(t)$, up tp a scaling factor $2\pi/N\tau$, whereas the sampled signal $\tilde{\varphi}_T(t)$ is the ordinary derivative of the sampled signal $\tilde{x}_T(t)$ of the function $x_T(t)$.

$$\varphi_T(n\tau) = \frac{2\pi}{N\tau} x_T^{(1)}(n\tau) \tag{38}$$

VIII Conclusions

We have shown that by introducing a *variation* on the Gibbs derivative it is possible to obtain some local relationships between Gibbs and Newton-Leibnitz derivatives. It becomes apparent, that for higher dimensional transforms there are many forms of introducing *variations*: the factor "jk" of eq. (9) will become $j\varkappa$, where $\varkappa$ is a *coding* on the components $k_1,\dots,k_r$ of k (assuming an r-dimensional case.)

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PROGRAM FOR EVALUATION GIBBS DERIVATIVES ON FINITE ABELIAN GROUPS

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ABSTRACT: In this paper we describe a program package for calculating the values of Gibbs derivatives on finite Abelian groups. We use the matrix representations of Gibbs derivatives which enable us to define FFT-like parallel algorithm for calculation. For programming realization we use the programming language Ada.

1.INTRODUCTION

Programs for implementation of FFT (the algorithm for efficient computation of DFT) are today a standard part of software equipment of any computer, ranged from PC to the large computer systems. These programs can be easily modified to be applicable for the calculation of the generalized Fourier transform on finite Abelian groups [1] as well as on finite non-Abelian groups [2].

Gibbs derivatives on finite groups (see, for example [3,4] for Abelian and [5] for non-Abelian case) form a class of differential operators closely related to the generalized Fourier transform on finite groups. To support their possible greater application in different areas, as for example in linear system theory [4,6,7], signal processing [8], logic design [9], some FFT-like algorithms for numerical evaluation of these operators are developed [10].

In this paper we give the programming realization of these algorithms, restricting ourselves to Gibbs derivatives on finite Abelian groups. For programming realization we use the programming language Ada which supports our intention to exploit a great parallelism inherent in the algorithms proposed in [10]. In this way we obtain highly parallel easily implementable algorithms and subsequently programs for the computation of the Gibbs derivative of a given function on a finite Abelian group.

Using these programs the complexity of computation of a Gibbs derivative is approximately equal to the complexity of computation of the Fourier transform on the given group [11].

2. NOTATION AND DEFINITIONS

To make this paper self-contained we will repeat briefly some notations and definitions from [10].

Let G be a finite Abelian group of order g. We associate to each group element one non-negative integer from the set $\{0,1,\ldots,g-1\}$ providing that 0 is associated to the group identity. In what follows the group elements will be identified with the non-negative integers associated with them.

We assume that G can be represented as a direct product of some subgroups $G_1,\ldots,G_n$ of orders $g_1,\ldots,g_n$, respectively, i.e.,

$$G = \mathop{\times}_{i=1}^{n} G_i , \qquad g = \prod_{i=1}^{n} g_i, \quad g_1 \le g_2 \le \ldots \le g_n . \tag{1}$$

The convention adopted above for denotation of group elements applies to the subgroups G_i as well. Due to this assumption each $x \in G$ can be uniquely represented as:

$$x = \sum_{i=1}^{n} a_i x_i, \qquad x_i \in G_i, \qquad x \in G,$$

with

$$a_i = \begin{cases} \prod_{j=i+1}^{n} g_j, & i=1,\dots,n-1 \\ 1, & i=n \end{cases}, \tag{2}$$

where g_j is the order of G_j.

The group operation $\circ$ of G can be expressed in terms of the group operations $\overset{i}{\circ}$ of the subgroups G_i, $i=1,\dots,n$ as:

$$x \circ y = (x_1 \overset{1}{\circ} y_1, x_2 \overset{2}{\circ} y_2, \dots, y_n \overset{n}{\circ} y_n), \quad x,y \in G, \quad x_i, y_i \in G_i. \tag{3}$$

The Gibbs derivative on finite groups is defined in [10] as follows

Definition 1. The Gibbs derivative $\mathbf{D}_g$ on a finite Abelian group G of order g is defined by:

$$\mathbf{D}_g = g^{-1}\, \chi\, \mathbf{G}\, \chi^*$$

where χ is the $(g \times g)$ matrix of group characters of G, $\mathbf{G}$ = diag(0,1, ...,$g-1$), and χ^* denotes the transpose conjugate of χ.

For numerical calculations it is more convenient to use the definition of the Gibbs derivative in terms of partial Gibbs derivatives, defined as follows.

Definition 2. Let G be representable in the form (2). The partial Gibbs derivative $\mathbb{D}_i$ with respect to the variable x_i is defined as:

$$\mathbb{D}_i = \bigotimes_{j=1}^{n} \mathbf{A}_j \tag{4}$$

with

$$\mathbf{A}_j = \begin{cases} \mathbf{D}_{g_j}, & j = i \\ \mathbf{I}_j, & j \neq i \end{cases}$$

where $\mathbf{I}_j$ is a $(g_j \times g_j)$ identity matrix, and $\otimes$ denotes the Kronecker product.

Definition 1. can be rewritten in terms of the partial derivatives as follows.

Definition 3. The Gibbs derivative $\mathbf{D}_g$ on a finite group G of order g is given by [10]:

$$\mathbf{D}_g = \sum_{i=1}^{n} a_i \mathbb{D}_i, \tag{5}$$

where the coefficients a_i are defined by (2).

3. FAST ALGORITHM

To obtain an efficient algorithm for the computation of the Gibbs derivative on finite Abelian groups it is convenient to start from the definition of the Gibbs derivative in terms of partial Gibbs derivatives, that is, from Definition 2.

According to this definition, the i-th partial Gibbs derivative is described by a sparse matrix $\mathbb{D}_i$ obtained as the Kronecker product of the matrix of the Gibbs derivative on the subgroup G_i and a number of identity matrices. As is noted in [10], there is a strong similarity between the matrix $\mathbb{D}_i$ and the matrix describing the i-th tep of the FFT on G. It follows that we can associate directly to $_i$ the flow graph of a fast algorithm similar to the flow graph of -th step of a FFT on G. Both flow graphs have identical forms. The veights in our flow graph are determined by non zero elements of $\mathbb{D}_i$ n a manner equal to that used in describing the FFT. This is best explained by some example.

Example: Let $G = Z_{18} = \{ 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15, 16,17; \circ \}$ be the group with the group operation defined in Table 1. The group representations of Z_{18} over the complex field are the Vilenkin Chrestenson functions shown in a matrix form in Fig.1.

The group Z_{18} can be considered as the product $Z_{18}=Z_2 \times Z_3 \times Z_3$, where $Z_2=\{ 0,1; \overset{2}{\circ} \}$ is the cyclic group of order 2 with modulo 2 addition as group operation, and $Z_3=\{ 0,1,2; \overset{3}{\circ}\}$ is the group of integers less than 3 with modulo 3 addition as group operation.

Therefore, any complex function f on Z_{18} can be considered as a three variable function $f(x_1,x_2,x_3)$, $x_1 \in Z_2$ and $x_2,x_3 \in Z_3$. The matrices $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{D}_3$ of the partial Gibbs derivatives relative to the variables x_1,x_2 and x_3 are shown at Fig. 2 a,b,c, respectively. The corresponding flow graphs are shown in Fig.3 a,b,c. The Gibbs derivative we calculate as $D_{18} = 9\mathbb{D}_1 + 3\mathbb{D}_2 + \mathbb{D}_3$, according to (5).

4. PARALLEL ALGORITHM

From Fig.3 it is obvious that the fast algorithms for calculating partial Gibbs derivatives, by virtue of their nature, are the highly parallel algorithms. The same applies to Gibbs derivative since it is expressed as a linear combination of partial derivatives.

Due to this fact we deduce from Definition 2. the following parallel algorithm implementable on a multiprocessor system.

PARALLEL ALGORITHM FOR CALCULATION OF GIBBS DERIVATIVE

Step 1. [Initialization].

i:=0; for i:=i+1 while i≤n repeat through step 16.

Step 2. [The g_i parallel arithmetic processors are actived];

Do through step 13 parallel

[k is sign of the k-the processor, $k \in \{1,2,\ldots g_i\}$]

Step 3. [The weight matrix of the i-th partial derivative is generated and transported in the processor's memories];

Compute $t_i^k(j) := w_i(k,j)$ for $j:=1,1,2,\ldots,g_i$.

Step 4. Compute $Ng := \prod_{j=1}^{i-1} g_j$ and $Gk := g/Ng/g_i$

Step 5. Let l := 1; jp := 0;

step 6. j0:=0; for j0:=j0+1 while j0 ≤ Ng repeat through step 13.

Step 7. j1:=0; for j1:=j1+1 while j1 ≤ Gk repeat through step 12.

Step 8. Let jp:=jp+1;

j2:=0; for j2:=j2+1 while j2 ≤ g_i repeat step 9.

Step 9. Let $b_i^k(j2) := f(jp)$; jp := jp + Gk;

Step 10. [Each processor computes one values of the partial derivative];

Compute $d_i^k(l) := \sum_{s=1}^{g_i} t_i^k(s) * b_i^k(s)$;

Step 11. [The partial derivative is multiplied by appropriate weight];

Compute $d_i^k(l) := a(i) * d_i^k(l)$;

Step 12. Let l:=l+1;

Step 13. Let jp:=jp+Gk;

Step 14. [The values of the Gibbs derivative are calculed by adding of partial derivatives];

Let Nn:=Gk*Ng; s:=1;

l:=0; for l:=l+1 while l ≤ Nn repeat through step 16.

Step 15. k:=0; for k:=k+1 while k ≤ g repeat step 16.

Step 16. Let $D_i(s) := D_{i-1}(s) + d_i^k(l)$; s:=s+1;

Step 17. [$D_n(s)$ for s:=1,2,...,g are the values of the Gibbs derivative]

End.

An implementation of the parallel algorithm described above on the multiprocessor system is illustrated in Fig 4. With this system it is assumed that the function f is given by a truth vector **f** = $(f_1,f_2,\ldots,f_g)$. This vector is stored initially in the memory of the control processor. At the end of the computation the values of the Gibbs derivative will be stored also in this memory.

The values of the Gibbs derivative are computed sequentially, step by step, the values of the first partial derivative first, then the values of the second derivative and finally the values of the n-th partial Gibbs derivative.

For the computation of the i-th partial derivative g_i, arithmetic processors are used. Each of the arithmetic processors is capable of performing some elementary transforms on the data stored in its own memory under the general control of the control processor. Those are the transforms defined by step 10. and step 11. In one cycle, the g_i terms of the function f are transported by the control processor to the entire ensemble of arithmetic processors, and stored in their local memories. Each arithmetic processor independently computes one value of the partial i-th Gibbs derivative by simply multiplying the stored values of f by the appropriate weight (also stored in local weight array t), and finally adding the resulting product.

When all the input terms have been processed the processors' memories have accumulated the intermediate results. This set of terms can then each be transformed independently and concurrently according to the step 11.

After performing the g_i independent Nn point transforms, the coefficients of the i-th partial Gibbs derivative are transported, in order to the control processor, in array **D** in its memory.

The control processor computes the values of the Gibbs derivative by simply adding the appropriate values of partial derivatives.

For the programming realization of the described parallel algorithm, the programing language Ada is used. The program is presented as Appendix 2. Note that the arithmetic processors are presented as array of parallel tasks [12].

5. CONCLUSION

This paper is concerned with the program for the evaluation of the values of the Gibbs derivative of order k of a complex function on a finite Abelian group. The program is based upon the matrix representation of Gibbs derivatives which enable the formulation of an efficient parallel algorithm.

The program is written in the programming language Ada, which has suitable mechanisms for the description of parallel tasks. In this way we obtain an efficient program suitable for implementation on multiprocessor systems.

We hope the results presented here will stimulate further application of Gibbs derivatives in those areas where their numerical evaluation is possibly required.

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APPENDIX 1.

Table 1.

∘	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14	12	16	17	15
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12	13	17	15	16
3	3	4	5	6	7	8	0	1	2	12	13	14	15	16	17	9	10	11
4	4	5	3	7	8	6	1	2	0	13	14	12	16	17	15	10	11	9
5	5	3	4	8	6	7	2	0	1	14	12	13	17	15	16	11	9	10
6	6	7	8	0	1	2	3	4	5	15	16	17	9	10	11	12	13	14
7	7	8	6	1	2	0	4	5	3	16	17	15	10	11	9	13	14	12
8	8	6	7	2	0	1	5	3	4	17	15	16	11	9	10	14	12	13
9	9	10	11	12	13	14	15	16	17	0	1	2	3	4	5	6	7	8
10	10	11	9	13	14	12	16	17	15	1	2	0	4	5	3	7	8	6
11	11	9	10	14	12	13	17	15	16	2	0	1	5	3	4	8	6	7
12	12	13	14	15	16	17	9	10	11	3	4	5	6	7	8	0	1	2
13	13	14	12	16	17	15	10	11	9	4	5	3	7	8	6	1	2	0
14	14	12	13	17	15	16	11	9	10	5	3	4	8	6	7	2	0	1
15	15	16	17	9	10	11	12	13	14	6	7	8	0	1	2	3	4	5
16	16	17	15	10	11	9	13	14	12	7	8	6	1	2	0	4	5	3
17	17	15	16	11	9	10	14	12	13	8	6	7	2	0	1	5	3	4

$$
\left[\begin{array}{ccccccccc|ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 \\
1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 \\
1 & 1 & 1 & e_1 & e_1 & e_1 & e_2 & e_2 & e_2 & 1 & 1 & 1 & e_1 & e_1 & e_1 & e_2 & e_2 & e_2 \\
1 & e_1 & e_1 & e_1 & e_2 & 1 & e_2 & 1 & e_1 & 1 & e_1 & e_2 & e_1 & e_2 & 1 & e_2 & 1 & e_1 \\
1 & e_2 & e_1 & e_1 & 1 & e_2 & e_2 & e_1 & 1 & 1 & e_2 & e_1 & e_1 & 1 & e_2 & e_2 & e_1 & 1 \\
1 & 1 & 1 & e_2 & e_2 & e_2 & e_1 & e_1 & e_1 & 1 & 1 & 1 & e_2 & e_2 & e_2 & e_1 & e_1 & e_1 \\
1 & e_1 & e_2 & e_2 & 1 & e_1 & e_1 & e_2 & 1 & 1 & e_1 & e_2 & e_2 & 1 & e_1 & e_1 & e_2 & 1 \\
1 & e_2 & e_1 & e_2 & e_1 & 1 & e_1 & 1 & e_2 & 1 & e_2 & e_1 & e_2 & e_1 & 1 & e_1 & 1 & e_2 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 & -1 & -e_1 & -e_2 & -1 & -e_1 & -e_2 & -1 & -e_1 & -e_2 \\
1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 & -1 & -e_2 & -e_1 & -1 & -e_2 & -e_1 & -1 & -e_2 & -e_1 \\
1 & 1 & 1 & e_1 & e_1 & e_1 & e_2 & e_2 & e_2 & -1 & -1 & -1 & -e_1 & -e_1 & -e_1 & -e_2 & -e_2 & -e_2 \\
1 & e_1 & e_2 & e_1 & e_2 & 1 & e_2 & 1 & e_1 & -1 & -e_1 & -e_2 & -e_1 & -e_2 & -1 & -e_2 & -1 & -e_1 \\
1 & e_2 & e_1 & e_1 & 1 & e_2 & e_2 & e_1 & 1 & -1 & -e_2 & -e_1 & -e_1 & -1 & -e_2 & -e_2 & -e_1 & -1 \\
1 & 1 & 1 & e_2 & e_2 & e_2 & e_1 & e_1 & e_1 & -1 & -1 & -1 & -e_2 & -e_2 & -e_2 & -e_1 & -e_1 & -e_1 \\
1 & e_1 & e_2 & e_2 & 1 & e_1 & e_1 & e_2 & 1 & -1 & -e_1 & -e_2 & -e_2 & -1 & -e_1 & -e_1 & -e_2 & -1 \\
1 & e_2 & e_1 & e_2 & e_1 & 1 & e_1 & 1 & e_2 & -1 & -e_2 & -e_1 & -e_2 & -e_1 & -1 & -e_1 & -1 & -e_2
\end{array}\right]
$$

$$e_1 = -\frac{1}{2}(1 - i\sqrt{3}), \qquad e_2 = -\frac{1}{2}(1 + i\sqrt{3})$$

Fig.1. The group representations of Z_{18} over C.

$$\mathbb{D}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a.

$$\mathbb{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & a & 0 & 0 & b \\ b & 0 & 0 & 1 & 0 & 0 & a & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & b & 0 & 0 & 1 & 0 & 0 & a \\ a & 0 & 0 & b & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 & 1 \end{bmatrix}$$

b.

$$\mathbb{D}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & 1 \end{bmatrix}$$

c.

$$a=\frac{1}{3}(e_1-1), \qquad b=\frac{1}{3}(e_2-1)$$

Fig.2. Matrices of the partial Gibbs derivatives

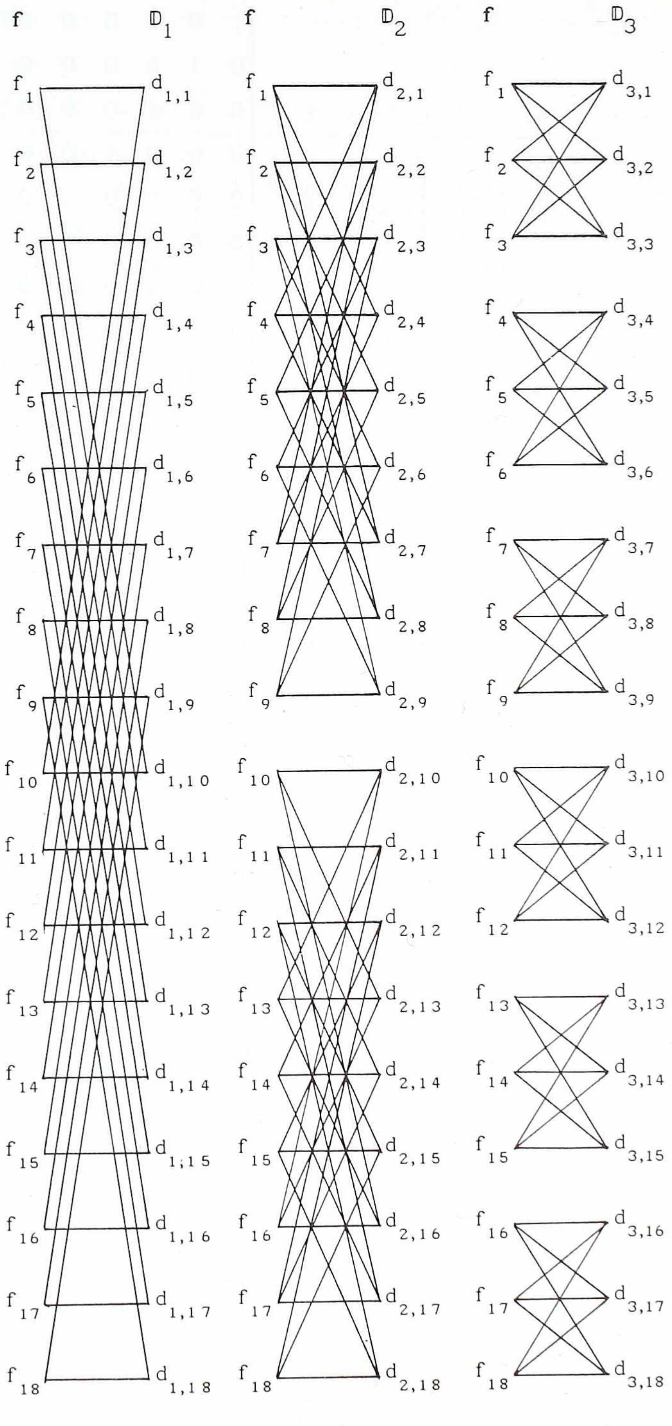

Fig.3. The flow graphs of partial Gibbs derivatives.

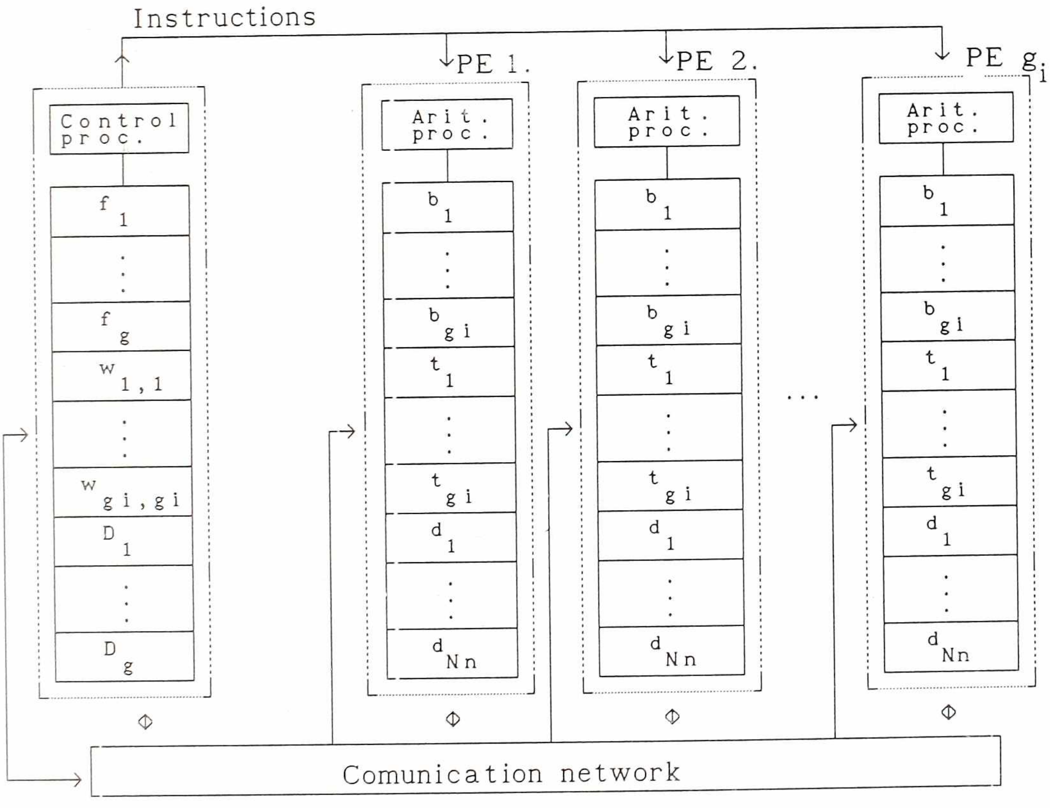

Fig. 4. A multiprocessor system implementation

APPENDIX 2.

```
procedure MAIN is
   use PACCOMPLEX
-- PACCOMPLEX is external package for complex calculs
   type COMPLEX is
      record
         R: FLOAT;
         Y: FLOAT;
      end record;
   task type PROCESSES is
      entry TIN (I1: in INTEGER, T1: in COMPLEX);
      entry FIN (I2: in INTEGER, F1: COMPLEX);
      entry BAD (G1: in INTEGER);
      entry DAD (I4: in INTEGER, A1: COMPLEX);
      entry DOUT(I5: in INTEGER, D1: out COMPLEX);
   end;
   PROC: array (INTEGER range <>) of PROCESSES;
   B,D,F: array (INTEGER range <>) of COMPLEX;
   G: array (INTEGER range <>) of INTEGER;
   W: array (INTEGER range <>, INTEGER range <>) of COMPLEX;
   DD,FPOM:COMPLEX;
   I,J1,J2,J3,L,N,NG,GI,GG,GK,JPOM,JPO: INTEGER;

   begin
      input(N); input(F); input(G);
      GG := 1;
      for I in 1..N loop
         GG := GG * G(I);
      end loop;
      for I  in 1..N loop
         GI := G(I);
         input(W);
         for J1 in 1..GI loop
            for J2 in 1..GI loop
               PROC(J2).TIN(J1,W(J2,J1));
            end loop;
         end loop;
         NG := 1;
         for J1 in 1..I-1 loop
            NG := NG * G(J1);
         end loop;
         GK := (GG/NG)/GI;
         JPOC:= 0;
         L := 1;
```

```
        for J0 in 1..NG loop
           for J1 in 1..GK loop
              JPOM := JPOC +J1;
              for J2 in 1..GI loop
                 FPOM := F(JPOM);
                 for J3 in 1..GI loop
                    PROC(J3).FIN(J2,FPOM);
                 end loop;
                 JPOM := JPOM + GK;
              end loop;
              for J2 in 1..GI loop
                 PROC(J2).BAD(GI);
              end loop;

              for J2 in 1..GI loop
                 PROC(J2).DAD(L,GK*GI);
              end loop;
              L := L+1;
           end loop;
           JPOC := JPOC + GK;
        end loop;
        L := 1;
        for J0 in 1..NG loop
           for J1 in 1..GK loop
              for J2 in 1..GI loop
                 PROC(J2).DOUT(J1,DD);
                 D(L) := D(L) + DD;
                 L := L+1;
              end loop;
           end loop;
        end loop;
     end loop;
end MAIN.
```

```
task body PROCESSES is
   use PACCOMPLEX;
   I,J,GL,A: INTEGER;
   POM: COMPLEX;
   T,D,B: array (INTEGER range <>) of COMPLEX;
begin
   loop
      select
         accept TIN(I1: in INTEGER, T1: in COMPLEX) do
            I := I+1;
            T(I) :=T1;
         end;
      or
         accept FIN (I2: in INTEGER, F1: in COMPLEX) do
            I := I2;
            B(I) := F1;
         end;
      or
         accept BAD(G1: in INTEGER) do
            POM := 0;
            GL := G1;
            for J in 1..GL loop
               POM := POM + B(J) * T(J);
            end loop;
         end;
      or
         accept DAD(I4: in INTEGER, A1: in COMPLEX) do
            I := I4;
            A := A1;
            D(I) := POM * A;
         end;
      or
         accept DOUT (I5: in INTEGER, D1: out COMPLEX) do
            I := I5;
            D1 := D(I);
         end;
      end select;

   end loop;
end PROCESSES.
```

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REAL GIBBS DERIVATIVES ON FINITE ABELIAN GROUPS

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Abstract: In this paper we define the partial and global differential operators on the space $\mathbb{C}(G)$ of all functions from a finite Abelian group G into the complex field $\mathbb{C}$. The main properties of these operators are studied.

Considering the multiple-valued (MV) functions as the functions belonging to $\mathbb{C}(G)$ taking their values in $\mathbb{Z}_{p^n} \subset \mathbb{Z} \subset \mathbb{C}$, it is shown that some symmetry properties of these functions can be detected using the partial derivatives introduced.

1. INTRODUCTION

In engineering practice the elements of certain functional spaces are usually employed as the mathematical models of signals. In this setting the signals are frequently identified with the functions representing their mathematical models. Several classes of signals can be distinguished with respect to their mathematical models.

Continuous signals are described by functions of continuous variables. Continuous signals of continuous amplitude are sometimes called analogue signals. The best known example is the signals modelled by functions mapping the real line $\mathbb{R}$ into itself, or into the complex field $\mathbb{C}$.

Signals representable by discrete functions are called discrete signals. Usually, the domain of their mathematical models is the set of integers $\mathbb{Z}$, or one of its subsets, for example, $\mathbb{Z}_{p^n}$, the set of non-negative integers less than some p^n.

Signals described by discrete functions taking their values in finite sets, i.e., quantized discrete signals, are called digital signals. Formally, the mathematical model of a digital signal is a function $f:\mathbb{S}\to\mathbb{L}$, where $\mathbb{S}$ and $\mathbb{L}$ are non-empty finite sets usually identified with subsets of the non-negative integers, i.e., $\mathbb{S}=\{0,1,\ldots,g-1\}$, $\mathbb{L}=\{0,1,\ldots,r-1\}$, $g,r\in N$. More generally, the domain $\mathbb{S}$ can be considered as a direct product of some finite sets $\mathbb{S}_i=\{0,1,\ldots.g_i-1\}$, $i=1,\ldots,n$, in which case f is an n-variable function $f(x_1,\ldots,x_n)$, $x_i\in\mathbb{S}_i$.

A network whose inputs and outputs are digital signals is called a digital network. Also, the input-output relations describing digital networks are expressed by digital functions, or alternatively, a digital network realizes a digital function.

It is commonly known that the problem of digital network design is greatly simplified if the discrete functions realized belong to some special class of digital functions; let us note symmetric or monotone functions as examples. For this reason the detection of peculiar properties of a given digital function, usually reported as the analysis of digital functions, is an important task. Two approaches prevalent today are analysis in the original domain and analysis in the spectral domain.

The first is based on some algebraic transformations and the difference operators for digital functions are one of the most powerful tools. Recall the Boolean difference [1] applied to Boolean functions, and difference operators for digital functions introduced in [2,3].

The second approach starts from the fact that the algebraic structure of a group can be imposed on the domain $\mathbb{S}$, and thus digital functions can be considered as functions on groups. For example, the Boolean functions can be considered as functions on finite dyadic groups [4], while MV functions are defined on the group $\mathbb{Z}_{p^n}$. It follows that mathematical tools provided by abstract harmonic analysis on groups can be applied to solve the problems of analysis and synthesis of digital functions. Let us mention as examples the discrete Walsh transform applied to Boolean functions and the Chrestenson transform applied to MV functions [6,7,8]. The methods based on discrete transform applications are commonly called spectral methods, and their efficiency is supported by fast transform algorithms [5].

In this paper we consider the digital functions as a subset of the complex functions on finite Abelian groups and we define an operator on $\mathbb{C}(G)$ (see definition below) possessing some useful properties of both the above-mentioned approaches when it is applied to digital functions. More precisely, we define a differential operator having the characters of the corresponding finite Abelian group as its eigenfunctions. In this way, the operator defined has some properties in common with difference operators for digital functions; on the other hand, since the operator is closely related to the generalized Fourier transform on finite Abelian groups, its computation is supported by a fast algorithm, comparable in complexity with well-known FFT algorithms.

2. NOTATIONS AND DEFINITIONS

Let G be a finite Abelian group of order g, and denote by $\mathbb{C}(G)$ the set of all functions mapping G into the complex field $\mathbb{C}$. The set $\mathbb{C}(G)$ is an Abelian group under pointwise addition defined by

$$(f+g)(x) = f(x) + g(x), \ \forall f,g\in\mathbb{C}(G), \ \forall x\in G.$$

Enriched with multiplication by a scalar defined by

$$(\alpha f)(x)=\alpha f(x), \forall f\in\mathbb{C}(G), \forall\alpha\in\mathbb{C},$$

$\mathbb{C}(G)$ is a linear space admitting an inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle f,g\rangle = g^{-1}\sum_{x\in G} f(x)\overline{g(x)}, \quad \forall f,g\in\mathbb{C}(G),$$

where the bar notation indicates complex conjugation.

Thus $\mathbb{C}(G)$ has a Hilbert space structure with norm

$$\|f\| = (\langle f,f\rangle)^{1/2} = (g^{-1}\sum_{x\in G} |f(x)|^2)^{1/2}, \quad \forall f\in\mathbb{C}(G).$$

Furthermore, $\mathbb{C}(G)$ may be given the structure of a complex function algebra by introducing the pointwise product of functions through

$$(fg)(x) = f(x)g(x), \quad \forall f,g\in\mathbb{C}(G), \ \forall x\in G.$$

Another important operation in $\mathbb{C}(G)$ is the convolution product defined by

$$(f*g)(z) = \sum_{x\in G} f(x)g(z\circ x') = \sum_{x\in G} f(z\circ x')g(x), \quad \forall z\in G, \ \forall f,g\in\mathbb{C}(G),$$

where x' is the additive inverse of x in G, i.e., $x\circ x'=e$, $(x')'=x$

The symbols $\circ$ and e represent the group operation and the identity of G.

Let us suppose that G is representable as a direct product of cyclic subgroups G_i of orders g_i, $i=1,\dots,n$, respectively, i.e.,

$$G = \mathop{\times}_{i=1}^{n} G_i \qquad g = \prod_{i=1}^{n} g_i \qquad g_1 \le g_2 \le \dots \le g_n. \tag{1}$$

In what follows the subgroups $G_1, G_2, \dots, G_n$ will be regarded as an n-tuple $(G_1, G_2, \dots, G_n)$.

Due to this assumption any function $f \in \mathbb{C}(G)$ can be considered as a function of n variables, i.e., $f(x) = f(x_1, \dots, x_n)$, $x \in G$, $x_i \in G_i$, $i \in \{1, \dots, n\}$.

Note that the group operation $\circ$ of G can be expressed in terms of the group operations $\overset{\circ}{_i}$ of subgroups G_i, $i \in \{1, \dots, n\}$ as:

$$x \circ y = (x_1 \overset{\circ}{_1} y_1,\ x_2 \overset{\circ}{_2} y_2,\ \dots,\ x_n \overset{\circ}{_n} y_n), \qquad x, y \in G, \qquad x_i, y_i \in G_i.$$

Also, each $x \in G$ can be defined in terms of the x_i as follows:

$$x = \sum_{i=1}^{n} a_i x_i, \qquad x_i \in G_i, \tag{2}$$

with

$$a_i = \begin{cases} \displaystyle\prod_{j=i+1}^{n} g_j, & i=1,\dots,n-1 \\ 1, & i=n \end{cases} \tag{3}$$

where g_j is the order of G_j.

Now, let us note that the digital functions can be considered as the particular functions belonging to $\mathbb{C}(G)$ taking their values from a subset $\mathbb{L}\subset\mathbb{Z}\subset\mathbb{C}$. In this case the variable $x\in G$ associated by (2), can be considered as the decimal index corresponding to the vector of variables $(x_1,\ldots,x_n)$, and the equality (2) is a generalization of the well-known p-ary or binary weighted coding applied to MV and Boolean functions, respectively.

The characters of G are defined as the homomorphisms of G into the unit circle [15,16], i.e., they are given by

$$\chi(w,x) = \chi((w_1,\ldots,w_n),(x_1,\ldots,x_n)) = exp(2\pi i \sum_{i=1}^{n} w_i x_i/g_i), \quad (4)$$

on the assumption that x is represented by (2) and a similar expression for w.

Moreover, the set of characters $\{\chi(w,x)\}$ under pointwise multiplication forms a group isomorphic to G. Also, $\{\chi(w,x)\}$ is a complete orthonormal set for $\mathbb{C}(G)$, i.e.,

$$\langle\chi(w,\cdot),\chi(k,\cdot)\rangle = \delta(w,k),$$
$$\langle f,\chi(w,\cdot)\rangle = 0, \ \forall w\in G, \quad \text{implies that } f=0.$$

Here δ is the Kronecker symbol.

Therefore, using the characters, the direct and inverse Fourier transform on $\mathbb{C}(G)$ are defined respectively by:

$$S_f(w) = \sum_{x=0}^{g-1} f(x)\overline{\chi}(w,x), \quad (5)$$

$$f(x) = g^{-1} \sum_{w=0}^{g-1} S_f(w)\chi(w,x).$$

For this Fourier transform the main properties of the classical Fourier transform hold. For example, the convolution theorem states that if $h=f*u$, $f,u\in\mathbb{C}(G)$, then

$$S_h = g^{-1}S_f S_u. \tag{6}$$

The reverse statement is also valid, i.e., if the Fourier transform of a function $h\in\mathbb{C}(G)$ can be represented as a componentwise product of the Fourier transforms S_f and S_u of some functions $f,u\in\mathbb{C}(G)$, then h is the convolution product of f and u, i.e., $h=f*u$.

4. DERIVATIVE

In this section the definition of a differential operator will be introduced and its properties will be studied.

The partial differential operator introduced in [3] can be slightly generalized as follows.

Definition 1. For a function $f\in\mathbb{C}(G)$ the partial derivative D_i with respect to the variable x_i is defined by

$$(\mathrm{D}_i f)(x) = g_i^{-1} \sum_{k=0}^{g_i-1} ((f(x\circ a_i k) - f(x)). \tag{7}$$

Although the modular arithmetic related to the group operation $\circ$ is implicitly included in (7), the partial derivative of a function may however be conveniently evaluated using conventional arithmetic operations only. This becomes apparent if (7) is expressed in a matrix form.

It is obvious that the partial real Gibbs derivative D_i can be described by a g by g matrix $\mathbf{D}_i$ defined by

$$\mathbf{D}_i = g^{-1} \bigotimes_{j=1}^{n} \mathbf{A}_j, \tag{8}$$

where $\otimes$ denotes the Kronecker product, and $\mathbf{A}_j$ is a g_j by g_j matrix given by:

$$\mathbf{A}_j = \begin{cases} \begin{bmatrix} -(g_j-1) & 1 & 1 & \cdots & 1 \\ 1 & -(g_j-1) & 1 & \cdots & 1 \\ \vdots & & \cdots & & \vdots \\ 1 & 1 & & & -(g_j-1) \end{bmatrix}, & j=i, \\ \mathbf{I}_j, & j\neq i, \end{cases}$$

with $\mathbf{I}_j$ the identity matrix of order g_j.

It follows that the partial real Gibbs derivative of a function $f\in\mathbb{C}(G)$ with respect to its i-th argument is given by

$$\underline{D}_i f = \mathbf{D}_i \underline{F}, \qquad (9$$

where $\mathbf{D}_i$ is given by (8), $\underline{F}$ is the truth vector of f, i.e., $\underline{F} = [f(0),\ldots,f(g-1)]^T$, and $\underline{D}_i f = [(D_i f)(0),\ldots,(D_i f)(g-1)]^T$.

From this matrix representation it is obvious that the values of $\underline{D}_i f$ can be calculated by using a fast flow graph having the same structure as the i-th step in the flow graph of the fast algorithm for a generalized Fourier transform on finite Abelian groups [9], except that the weights are different; here all of them are real and only g of them are not equal to 1. More precisely, the weights corresponding to the horizontal branches in the flow graph are equal to $-(g_i-1)$.

For an illustration of this statement, the matrix representations of the partial Gibbs derivatives on $\mathbb{Z}_{3^2}$ are given in Fig.1a. The flow graphs of the corresponding fast algorithm for calculation of the values of these derivatives of a function are shown in Fig.1b.

$$\underline{D}_1 = 3^{-1}\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\underline{D}_2 = 3^{-1}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \right)$$

Fig.1a. The partial real Gibbs derivatives on $\mathbb{Z}_{3^2}$.

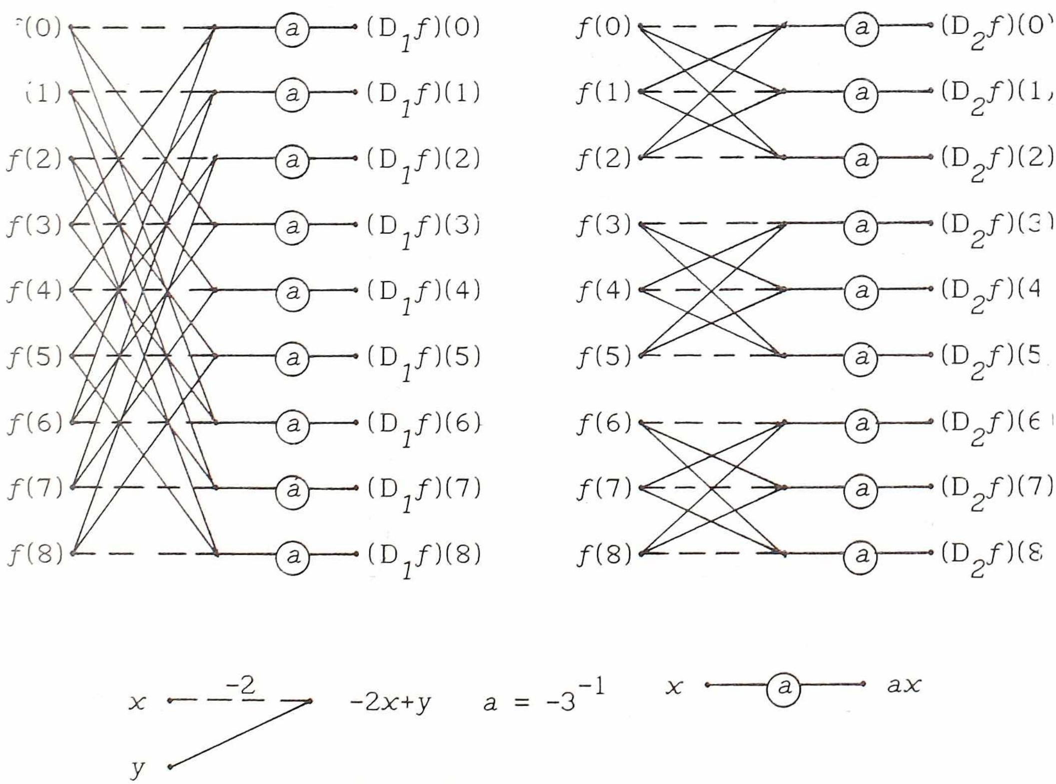

Fig.1b. The flow graphs of the fast algorithm for calculation of the derivatives shown in Fig.1a.

The partial derivative D_i can be considered as a generalization of the Boolean difference [1]. A justification for this statement can be found in the properties of the operator D_i stated in the following theorem.

Theorem 1. Let $f \in \mathbb{C}(G)$. Then,

1. f is independent of its i-th argument iff $D_i f = 0 \in \mathbb{C}(G)$.,

2. $D_i(D_j f) = D_j(D_i f)$,

3. $D_i(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 D_i f_1 + \alpha_2 D_i f_2$, $\quad \alpha_1, \alpha_2 \in \mathbb{C}$, $f_1, f_2 \in \mathbb{C}(G)$,

4. D_i commutes with the translation operator T_q on G defined by $T_q f(x) = f(x \circ q)$, $\forall q, x \in G$, i.e.,

$$D_i T_q f = T_q D_i f.$$

5. Let $f \in \mathbb{C}(G)$. Denote by $f_{i'}$ a function obtained from f by inverting its i-th variable, i.e.,

$$f_{i'}(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x_i', \ldots, x_n).$$

Then,

$$(D_j f_{i'})(x_1, \ldots, x_i, \ldots, x_n) = (D_j f)(x_1, \ldots, x_i', \ldots, x_n), \quad \forall j \in \{1, \ldots, n\}.$$

6. If S_f is the Fourier transform of an $f \in \mathbb{C}(G)$, then that of its partial real Gibbs derivative $D_i f$ is given by

$$S_{D_i f}(w) = S_{B_i}(w) S_f(w),$$

where

$$S_{B_i}(w) = g_i^{-1} \sum_{k=0}^{g_i - 1} (\chi(w, a_i k) - 1),$$

with a_i defined by (3).

Particularly, for $f(x)=\chi(\omega,x)$, we have $D_i(\chi(w,x)) = S_{B_i}(w)\chi(w,x)$, that is , the characters of G are the eigenfunctions of the partial Gibbs derivatives.

7. Convolution property:

$$D_i(f*g) = D_i f*g = f*D_i g, \quad \forall f,g\in\mathbb{C}(G),$$

where * denotes the convolution on G.

Proof. 1. Let us first suppose that $(D_i f)(x)=0$. According to the Definition 1, this condition can be expressed as

$$g_i^{-1}\sum_{k=0}^{g_i-1} (f(x\circ a_i k)(x) - f(x)) = 0,$$

or equivalently,

$$f(x_i \overset{\circ}{_i} 0)+f(x_i \overset{\circ}{_i} 1)+\ldots+f(x_i \overset{\circ}{_i} (g_i-1)) = g_i f(x), \tag{11}$$

where $f(x_i \overset{\circ}{_i} k)$ is a short notation for $f(x_1,\ldots,x_i \overset{\circ}{_i} k,\ldots,x_n)$.

Therefore, the condition $(D_i f)(x)=0$ actually describes a system of algebraic equations obtained for different values of $x_i\in G_i$. According to (8), this system can be expressed in matrix form as

$$\mathbf{A}_i\mathbf{F} = 0, \tag{12}$$

where $\mathbf{F}$ is a column matrix of order g_i given by $\mathbf{F} = [f(x_i=0),f(x_i=1),\ldots,f(x_i=g_i-1)]^T$, and $f(x_i=k)$ stands for $f(x_1,\ldots,x_{i-1},k,x_{i+1},\ldots,x_n)$.

Now, we will apply a series of g_i-1 identical linear transformations to the rows of matrix $\mathbf{A}_i$. The transformation consists of the replacement of a row of $\mathbf{A}_i$ by the row obtained by subtracting the row considered from its preceding row. In this way the system (12) is transformed into its equivalent system

$$\mathbf{Y}\mathbf{F} = 0,$$

where the elements y_{sr}, $s,r\in\{1,\ldots,g_i\}$, of $\mathbf{Y}$ are defined by

$$y_{1r} = \begin{cases} -(g_i-1), & r=1, \\ 1, & r=2,\ldots,g_i, \end{cases}$$

$$y_{sr} = \begin{cases} -g_i, & s=r-1, \\ g_i, & s=r, \qquad s=2,\ldots,g_i. \\ 0, & \text{otherwise}, \end{cases}$$

From the structure of $\mathbf{Y}$ it follows directly that this system is satisfied only if $f(x_i=0)=f(x_i=1)=\ldots=f(x_i=g_i-1)$. Conversely, if these equalities are satisfied, then $(D_if)(x) = 0$ follows directly, since the sum of elements in each row of $\mathbf{A}_i$ is equal to 0.

2. The proof follows directly after a routine calculation starting from the definition of partial real Gibbs derivatives and using the fact that $(x\circ a_ik)\circ a_jr = (x\circ a_jr)\circ a_ik)$ due to the associativity of the group operation $\circ$ of G.

3. Linearity follows directly from Definition 1.

4. The proof follows from Definition 1 and the definition of the translation (shift) operator T_q,

$$(D_i(T_qf))(x) = g_i^{-1}\sum_{k=0}^{g_i-1}(T_qf(x\bullet a_ik)-T_qf(x))$$

$$= g_i^{-1}\sum_{k=0}^{g_i-1}(f(x\circ a_ik\circ q) - f(x\circ q))$$

$$= g_i^{-1} \sum_{k=0}^{g_i-1} T_q(f(x \circ a_i k) - f(x)) = (T_q(D_i f))(x).$$

5. Let us recall that the additive inverse x_i' of an $x_i \in G_i$ if $x_i \neq 0$ can be expressed by $x_i' = g_i - x_i$. Now the property is proved in a manner similar to that used in proving the Property 4 for the particular choice of $q = (q_1, \ldots, q_i, \ldots, q_n)$ defined by

$$q_s = \begin{cases} g_i - x_i, & s = i \\ 0, & s \neq i \end{cases} \qquad s = 1, \ldots, n.$$

6. In proving this property we use the shift property of the Fourier transform on groups. The proof goes as follows.

By definition,

$$S_{D_i f}(w) = \sum_{x=0}^{g-1} (D_i f)(x)\bar{\chi}(w,x) = \sum_{x=0}^{g-1} (g_i^{-1}) \sum_{k=0}^{g_i-1} (f(x \circ a_i k) - f(x))\chi(\bar{w},x)$$

$$= g_i^{-1} \sum_{k=0}^{g_i-1} \left(\sum_{x=0}^{g-1} f(x \circ a_i k)\bar{\chi}(w,x) - \sum_{x=0}^{g-1} f(x)\bar{\chi}(w,x)\right)$$

$$= g_i^{-1} \sum_{k=0}^{g_i-1} (S_{f(x \circ a_i k)}(w) - S_f(w)) = S_{B_i}(w) S_f(w).$$

The second part of the statement follows directly by the definition of partial real Gibbs derivatives using the relation $\chi(w, x \circ a_i k) = \chi(w,x)\chi(w, a_i k)$.

7. We will prove this property in the spectral domain. From the Property 6 and the convolution theorem we have

$$S_{D_i(f*g)}(w) = S_{B_i}(w)S_{(f*g)}(w) = S_{B_i}(w)S_f(w)S_g(w)$$

$$= S_{D_i f}(w)S_g(w) = S_f(w)S_{B_i}(w)S_g(w)$$

$$= S_f(w)S_{D_i g}(w).$$

Now, let us consider in more detail the functions S_{B_i} in the spectral domain given by (10).

Recall that the matrix of characters of a group G representable in the form (1) is the Kronecker product of the matrices of characters of the subgroups G_i. Therefore, the first g_n rows of the matrix of characters of G can be considered as the periodic repetition of the matrix of characters of the subgroup G_n. Since by definition $a_n=1$ (relation (3)), the values of $S_{B_n}(w)$, $w=0,1,\ldots,g-1$, according to (10) can be obtained by the componentwise summation of the first g_n rows of the character matrix of G. The character matrices are symmetric and, because of the periodicity in the first g_n rows, this summation reduces to the repeated evaluation of the Haar integral of the characters of G_n. It is known that the Haar integral of any character is equal to zero except for the principal character, for which it is equal to the group order g_n. According to (10) we can conclude that

$$S_{B_n}(w) = \begin{cases} 0, & w = kg_n, \quad k = 0,1,\ldots,n \\ 1, & \text{otherwise} \end{cases}.$$

If the variable x_n is considered as a n-variable function on G represented by its truth vector $\underline{x}_n=[x(w)]^T$, $w = 0,1,\ldots,g-1$, then it is apparent that S_{B_n} takes the zero value at those w where the zero value appears in x_{-n}. This conclusion in a similar way can be extended for establishing a relation between the remaining functions S_{B_i} $i=1,\dots,n-1$ and the corresponding variables x_i. More precisely, the functions S_{B_i} can be considered as the characteristic functions associated with the variables x_i, since

$$S_{B_i}(w) = \begin{cases} 1, & x_i(w) \neq 0, \\ 0, & x_i(w) = 0, \end{cases} \qquad w = 0,1,\dots,g-1.$$

The Property 6 of Theorem 1 establishes a relation between the Fourier transform of a function f and that of its partial real Gibbs derivative. The relating function B_i is therefore purposely defined in the transform domain by the relation (10). Using the inverse Fourier transform, we obtain

$$B_i(x) = \begin{cases} g_i^{-1}(g_i-1) & \text{for } x = 0 \\ -g_i^{-1} & \text{for } x = a_i k,\ k=1,\dots,g_i-1, \\ 0 & \text{otherwise} \end{cases} \qquad i=1,\dots,n.$$

It follows from the Property 6 and the convolution theorem that

$$(D_i f)(x) = (B_i * f)(x). \tag{13}$$

Thus, the matrix $\mathbf{D}_i$ defined by (8) is a convolution matrix. Therefore, for the calculation of the values of the partial real Gibbs derivatives of a given function the fast convolution algorithm on groups can be used [17].

Using the partial derivatives and the relation (2) we can define the global real Gibbs derivative on finite Abelian groups.

Definition 2. The real Gibbs derivative $\mathrm{D}f$ of a function $f\in\mathbb{C}(G)$ is defined by

$$(\mathrm{D}f)(x) = \sum_{i=1}^{n} a_i(\mathrm{D}_i f)(x), \tag{14}$$

where the coefficients a_i are defined by (3).

This derivative resembles very much the Gibbs differential operators on groups introduced in [10,13]. The main distinction is in the values of the coefficients of the terms in the summation (14), which will be discussed later in more detail. The chief properties of the derivative D are given in the following theorem.

Theorem 2. Let $f\in\mathbb{C}(G)$. Then,

1. $\mathrm{D}f = 0$ iff $f = \text{const.}\in \mathbb{C}(G)$,

2. $\mathrm{D}(\alpha_1 f_1+\alpha_2 f_2) = \alpha_1\mathrm{D}f_1 + \alpha_2\mathrm{D}f_2$, $\alpha_1,\alpha_2\in\mathbb{C}$, $f_1,f_2\in\mathbb{C}(G)$,

3. $\mathrm{D}(f*g) = \mathrm{D}f*g = f*\mathrm{D}g$, $\forall f,g\in\mathbb{C}(G)$,

4. If the Fourier transform of f is S_f, then that of $\mathrm{D}f$ is given by

$$S_{\mathrm{D}f}(w) = S_{\mathrm{C}}(w)S_f(w),$$

where

$$S_{\mathrm{C}}(w) = \sum_{i=1}^{n} a_i S_{\mathrm{B}_i}(w),$$

with the coefficients a_i determined by (3).

Thus the characters of G are the eigenfunctions of D, i.e., the characters $\chi(w,x)$ can be obtained as the solutions of the linear first order differential equation

$$\mathrm{D}f(x) - S_{\mathrm{C}}(w)f(x) = 0.$$

Proof. According to Definition 2, the real Gibbs derivative is a linear combination of the partial real Gibbs derivatives. Hence, the properties 1 to 4 can be proved in a manner similar to that used in proving the corresponding properties of the partial real Gibbs derivatives stated in Theorem 1. The proof of Theorem 2 is therefore omitted entirely.

It is interesting to consider the relationship of the real Gibbs derivative with some existing relevant operators.

In order to achieve computational efficiency we define our derivative as an operator acting on $\mathbb{C}(G)$. This means that when it is applied to digital functions we practically need only integer arithmetic. In this way we avoid not only the necessity for modular arithmetic appearing in dealing with digital functions in the original domain, but also the necessity for complex arithmetic when the digital functions are processed by spectral means. The price paid is the absence of some other interesting properties characteristic of some other relevant operators. For example, an advantage of the Gibbs derivative [10] is that it can apparently be extended, at least heuristically, to the real (Newton-Leibniz) derivative [10]. Unlike the differential operators introduced by Thayse for dealing with digital functions [2], our derivative, as well as the partial derivative, of a digital function is not a digital function, because, in general, the values of Df obviously belong not to $\mathbb{L}$ but to $\mathbb{Z}$.

The Property 4. of Theorem 2 suggests that there is a strong relationship between our real Gibbs derivative and the Gibbs derivative on finite Abelian groups [10]. This relationship becomes apparent if we compare the matrix representations of these operators.

According to Definition 2, the matrix $\mathbf{D}$ describing the derivative D is a linear combination of matrices $\mathbf{D}_i$ defined by (8).

On the other hand, the Gibbs derivative D_G on finite Abelian groups can be expressed in a matrix form by

$$\mathbf{D}_G = g^{-1}[\chi]\mathbf{A}[\bar{\chi}],$$

where $[\chi]$ is a g by g matrix of characters of a finite Abelian group G of order g, and $\mathbf{A} = \mathrm{diag}(0,1,\ldots,g-1)$ [11,12].

Comparing the matrices $\mathbf{D}$ and $\mathbf{D}_G$ we have

$$\mathbf{D} = \mathrm{Re}(\mathbf{D}_G),$$

where equality stands componentwise and Re(Q) denotes the real part of Q. Therefore, the elements of the matrix $\mathbf{D}$ are equal to the real parts of the corresponding elements of the matrix $\mathbf{D}_G$. Here we find the justification for the name *the real Gibbs derivative* of our differential operator. Note that if the group G has real-valued characters, as is the case with the finite dyadic group, the matrices $\mathbf{D}$ and $\mathbf{D}_G$ coincide and, hence, the derivative D is equal to the Gibbs derivative. It follows that in the case of groups having complex-valued characters, the real Gibbs derivative can exhibit some advantages in application to real-valued functions, because with this operator complex arithmetic is not required.

4. APPLICATIONS

It is apparent that, owing to the properties mentioned in Theorem 1, the possible applications of the partial differential operator D_i can be found in the same areas where the Boolean difference is already applied in the case of Boolean functions. One of these areas is certainly the detection of appropriately defined symmetry properties of digital functions. In order to show this we introduce the following definition, which can be considered as an extension of the corresponding definition for Boolean functions introduced by Hurst [7]. For simplicity we consider only MV functions obtained for $g_1 = g_2 = \ldots = g_n = p$.

Definition 3. For a chosen $k\in\{0,1,\dots,p-1\}$ a given MV function f: $\{0,1,\dots,p-1\}^n \longrightarrow \{0,1,\dots,p-1\}$ exhibits k-th single-variable symmetry in x_i with respect to x_j, which we express as $(k\mathrm{SVS}x_i,x_j)$, if for each $x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_n$ we have

$$f(x_i=0) = f(x_i=1) = \dots = f(x_i=p-1), \tag{15}$$

for each $x_j\in\{0,1,\dots,p-1\}\setminus\{k\}$.

We will show that a possible approach to the detection of this kind of symmetries in MV functions can be formulated in terms of partial real Gibbs derivatives in a manner analogous to the corresponding application of Boolean difference in the case of switching functions.

Let us consider the function

$$f'(x)=z_j(D_i f)(x) \tag{16}$$

where z_j is a MV function on G defined by the requirement $x_j\oplus z_j=k$; $\oplus$ stands for modulo p addition.

Note first that the condition (15) is a weaker version of the condition for a MV function to be independent of its i-th argument (Property 1 of Theorem 1), since a restriction on its j-th argument is imposed.

Now. let us suppose that $f'(x)=0$. This condition is equivalent to a system of p^n-p^{n-1} linear algebraic equations, because by definition $z_j=0$ for $x_j=k$.

Note that the subset of variables $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ can take $(p-1)p^{n-2}$ different combinations of values because to the restriction imposed on the variable x_j. We denote these combinations by $c(s)$, $s\in\{0,1,\dots,(p-1)p^{n-2}-1\}$.

The system of linear algebraic equations under consideration can be partitioned into $p^{-1}(p^n-p^{n-1})$ disjoint subsystems of p equations each, so that each subsystem corresponds to a particular $c(s)$. According to (8), these subsystems are given by

$$z_j \mathbf{A}_i \mathbf{F}_i^s = 0_p, \quad s=0,1,\ldots,(p-1)p^{n-2}-1,$$

with $\mathbf{F}_i^s=[f(x_i=0,c(s)),f(x_i=1,c(s)),\ldots,f(x_i=p-1,c(s))]^T$, and 0_p is the zero vector of order p.

By solving these subsystems of equations in a manner like that used in proving Property 1 of Theorem 1, we conclude that if $f'(x)=0$, then the function f satisfies the condition (15). Conversely, if we suppose that the condition (15) is satisfied by the function f, then it easily follows that $f'(x)=0$, since the sum of the elements in each row of the matrix $\mathbf{D}_i$ describing the i-th partial real Gibbs derivative is equal to zero.

In this way we actually prove that the necessary and sufficient condition for existence of $(k\text{SVS}x_i,x_j)$ property in a given MV function, and consequently, a test for detection of this property, is given by $f'(x)=0$.

ILLUSTRATIVE EXAMPLE

To illustrate the application of the partial real Gibbs derivatives for detection of the symmetry properties defined above, let us consider the method of detection of $(2\text{SVS}x_1,x_3)$ of a function f: $\{0,1,2\}^3 \rightarrow \{0,1,2\}$ given by its truth vector shown in Table. 1. It follows from Definition 3 that a three-variable three-valued function will have $(2\text{SVS}x_1,x_3)$ if

$$\left.\begin{array}{l} f(0,x_2,0) = f(1,x_2,0) = f(2,x_2,0) \\ f(0,x_2,1) = f(1,x_2,1) = f(2,x_2,1) \end{array}\right. \qquad \forall x_2 \in \{0,1,2\}.$$

To detect this property we calculate the partial real Gibbs derivative on x_1, $(\mathbf{D}_1 f)(x)$, using the matrix $\mathbf{D}_1$ given by

$$\mathbf{D}_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting vector is multiplied componentwise by the vector $\underline{z}_3$ obtained from the truth vector $\underline{x}_3$ of the variable x_3 considered as a 3-variable function on G according to the requirement $x_3 \oplus z_3 = 2$ which must be satisfied by the corresponding elements of the vectors $\underline{x}_3$ and $\underline{z}_3$. The vector $\underline{z}_3$ is also shown in Table 1. After the calculation is carried out we obtain a zero vector and, hence, we conclude that the given function exhibits the property $(2SVSx_1,x_3)$.

Table 1.

	$x_1x_2x_3$	f	D_1f	z_3	z_3D_1f
0.	0 0 0	$f(000)$	0	2	0
1.	0 0 1	$f(001)$	0	1	0
2.	0 0 2	$f(002)$	$-2f(002)+f(102)+f(202)$	0	0
3.	0 1 0	$f(010)$	0	2	0
4.	0 1 1	$f(011)$	0	1	0
5.	0 1 2	$f(012)$	$-2f(012)+f(112)+f(212)$	0	0
6.	0 2 0	$f(020)$	0	2	0
7.	0 2 1	$f(021)$	0	1	0
8.	0 2 2	$f(022)$	$-2f(022)+f(122)+f(222)$	0	0
9.	1 0 0	$f(000)$	0	2	0
10.	1 0 1	$f(001)$	0	1	0
11.	1 0 2	$f(102)$	$f(002)-2f(102)+f(202)$	0	0
12.	1 1 0	$f(010)$	0	2	0
13.	1 1 1	$f(011)$	0	1	0
14.	1 1 2	$f(112)$	$f(012)-2f(112)+f(212)$	0	0
15.	1 2 0	$f(020)$	0	2	0
16.	1 2 1	$f(021)$	0	1	0
17.	1 2 2	$f(122)$	$f(022)-2f(122)+f(222)$	0	0
18.	2 0 0	$f(000)$	0	2	0
19.	2 0 1	$f(001)$	0	1	0
20.	2 0 2	$f(202)$	$f(002)+f(102)-2f(202)$	0	0
21.	2 1 0	$f(010)$	0	2	0
22.	2 1 1	$f(011)$	0	1	0
23.	2 1 2	$f(212)$	$f(012)+f(112)-2f(212)$	0	0
24.	2 2 0	$f(020)$	0	2	0
25.	2 2 1	$f(021)$	0	1	0
26.	2 2 2	$f(222)$	$f(022)+f(122)-2f(222)$	0	0

CONCLUSION

The partial as well as the global real Gibbs derivatives on finite Abelian groups are introduced in this paper. These operators are related to the Fourier transform on groups in the same way as the ordinary Newton-Leibniz derivative is related to the classical Fourier transform on the real line. At the same time these operators have some properties characteristic of difference operators introduced in the area of MV functions. We hope that these properties offer some further applications of Gibbs derivatives in logic design and related areas as is suggested in Section 4.

Taking only real values, the differential operators introduced may be useful in application to the processing of real-valued signals. The efficiency of their application is further supported by fast algorithms for numerical calculation which can be easily derived by slightly modifying the existing fast Fourier transform algorithms on groups, or directly using the fast convolution algorithms, as is suggested by relations (8) and (13), respectively.

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GENERALIZED DISCRETE GIBBS DERIVATIVES AND RELATED LINEAR EQUATIONS

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Abstract: In this paper there is given a generalization of the discrete Gibbs derivative on finite intervals using the concept of generalized convolution product. In this way there is obtained a family of discrete differential operators involving some known Gibbs derivatives as particular examples.

The linear discrete differential equations with operators, introduced and studied in this paper, are analysed. The determination of the Moore-Penrose inverse of the generalized discrete Gibbs differential operators enables one to obtain the general as well as the minimum-norm least-square solutions of these equations.

1. INTRODUCTION

Spectral techniques take up very great role among the methods for digital signal processing. The discrete analogs of the Fourier transform in finite rings or Galois fields, which represent the mathematical foundations of discrete spectral analysis, are developed in the context of algebra and number theory. Their practical applications are supported by the advent of numerical methods for signal processing.

Pollard [31] and Nicholson [26] developed the algebraic theory of the Fourier transform on finite rings and fields. For more details on these subjects and the other discrete transforms see, for example [21,35]. A good interpretation of these results in the context of some engineering applications is given in [12].

We know from Fourier transform theory that the following holds,

$$F(f(t \pm s)) = \exp(\pm i\omega s)F(f(t)), \tag{1}$$

$$F\left(\frac{d^n}{dt^n}f(t)\right) = (i\omega)^n F(f(t)), \tag{2}$$

where F denotes the Fourier transform operator. Relation (1) is the basis for a method of algebraization of differential, difference,and difference-differential equations [7,28,36].

Properties (1) and (2) and their applications suggest that it is a very important task to disclose and study some shift and differential operators suitable for discrete spectral analysis.

The first step in this direction was made by Gibbs who introduced a so-called dyadic derivative for complex functions on finite dyadic groups [15]. Later, some generalizations of this operator were given by Gibbs and his associates [15-20], as well as by a number of other researchers [8,27,32,33]. Some applications of Gibbs derivatives are met in approximation theory [8], statistics [29], and linear system theory [6,10,23].

All Gibbs differential operators mentioned above satisfy the properties analogous to these described by (1) and (2). Moreover, the introduction and the study of the systems described by Gibbs differential equations is based on these properties [10,25,30]. Some interesting applications of these systems are also suggested [1,29].

In this paper there is given a generalization of discrete Gibbs derivatives on finite intervals using the generalized convolution product introduced in [3]. In this way there is obtained a family of discrete differential operators involving some known Gibbs derivatives as particular examples.

The linear discrete differential equations with operators, introduced and studied in this paper, are analysed. The determination of the Moore-Penrose inverse of the generalized discrete Gibbs differential operators enables one to obtain the general as well as minimum-norm least-square solutions of these equations.

2. PRELIMINARIES

In this section we will introduce some notations and we will review some definitions and statements needed for our further considerations.

Let $\mathbb{P}$ denote an arbitrary field and let G be a non-empty ordered set, which could be identified without loss of generality with the subset of first non-negative integers, i.e., $G=\{0,1,\dots,n-1\}$, $n\in\mathbb{N}$. Also, denote by $\mathbb{P}(G)$ the linear space of all functions mapping G into $\mathbb{P}$.

Denote by $\mathbf{A}$ and $\mathbf{B}$ two invertible $(n\times n)$ matrices. The columns of the matrices $\mathbf{A}$ and $\mathbf{B}$ we denote by $\{\mathbf{a}(0),\dots,\mathbf{a}(n-1)\}$ and $\{\mathbf{b}(0),\dots,\mathbf{b}(n-1)\}$, respectively. Each column $\mathbf{a}(k)=[a(0,k),\dots,a(n-1,k)]^T, k\in G$ can be considered as the vector defining the values of a particular function from $\mathbb{P}(G)$. The same applies to the columns of the matrix $\mathbf{B}$. Now, we have that the ordered sets $\{\mathbf{a}(0),\dots,\mathbf{a}(n-1)\}$ and $\{\mathbf{b}(0),\dots,\mathbf{b}(n-1)\}$ form two different bases in the linear space $\mathbb{P}(G)$.

Furthermore, we define the pointwise product of functions $\mathbf{f}=[f(0),\dots,f(n-1)]^T$, $\mathbf{g}=[g(0),\dots,g(n-1)]^T$ through

$$(f\circ g)(i) = f(i)g(i),\ \forall f,g\in\mathbb{P}(G),\quad \forall i\in G.$$

For the given $\mathbf{A}=[a(i,j)]$, $\mathbf{B}=[b(i,j)]$, $(i,j)\in G$, and for each $k\in G$, the k-th $\mathbf{A},\mathbf{B}$-shift, of a function $f\in\mathbb{P}(G)$, denoted by $(S^k_{\mathbf{A},\mathbf{B}}f)(i) = f(i\ominus_{\mathbf{A},\mathbf{B}}k)$, is defined in [3] as the function $S^k_{\mathbf{A},\mathbf{B}}\mathbf{f}=[(S^k_{\mathbf{A},\mathbf{B}}f)(0),\ldots,(S^k_{\mathbf{A},\mathbf{B}}f)(n-1)]^T$, where

$$(S^k_{\mathbf{A},\mathbf{B}}f)(i) = \sum_{j=0}^{n-1} \beta^{(i)}(j,k)f(j), \qquad i\in G, \tag{3}$$

$$\beta^{(i)}(j,k) = \sum_{p=0}^{n-1} a^{(-1)}(i,p)b(p,j)a(p,k),$$

where $a^{(-1)}(i,p)$ is the (i,p)-th element of $\mathbf{A}^{-1}$.

In matrix notation the $\mathbf{A},\mathbf{B}$-shift of $\mathbf{f}$ is given by

$$S^k_{\mathbf{A},\mathbf{B}}\mathbf{f} = \mathbf{A}^{-1}(\mathbf{a}(k)\circ\mathbf{B})\mathbf{f}.$$

Note that for the particular choice of the matrices $\mathbf{A}$ and $\mathbf{B}$ the $\mathbf{A},\mathbf{B}$-shift reduces to some known shift operators. For example, if $\mathbf{B}=\mathbf{A}$ the $\mathbf{A},\mathbf{B}$-shift reduces to the so-called $\mathbf{A}$-shift defined and studied first in [2]. Note that the $\mathbf{A}$-shift for functions taking their values in commutative rings was reintroduced later in [22] in the study of discrete analogues of the generalized shift operators [13,24]. Furthermore, if both $\mathbf{A}$ and $\mathbf{B}$ are equal to the discrete Fourier transformation matrix, the $\mathbf{A},\mathbf{B}$-shift reduces to the cyclic shift. Also, for $\mathbf{A}=\mathbf{B}=\mathbf{W}$, with $\mathbf{W}$ being the Walsh matrix, $\mathbf{A},\mathbf{B}$-shift reduces to the dyadic shift.

Lemma 1. If $\mathbf{A}$ and $\mathbf{B}$ are biorthogonal, i.e.,

$$\sum_{j=0}^{n-1} a(j,s)b(p,j) = \begin{cases} 0, & p \neq s \\ 1, & p = s \end{cases},$$

then
$$a(i \underset{A,B}{\ominus} k,s) = a^{(-1)}(i,s)a(s,k), \quad \forall i,k,s\in G \tag{4}$$

$$a^{(-1)}(i \underset{A,B}{\ominus} k,s) = a^{(-1)}(i,s)a(s,k), \quad \forall i,k,s\in G. \tag{5}$$

Proof. From (3) it follows that

$$a(i \underset{A,B}{\ominus} k,s) = (S^k_{A,B}a(s))(i) = \sum_{j=0}^{n-1} \beta^{(i)}(j,k)a(j,s), \quad \forall i,k,s\in G,$$

i.e.,

$$a(i \underset{A,B}{\ominus} k,s) = \sum_{j=0}^{n-1}\sum_{p=0}^{n-1} a^{(-1)}(i,p)b(p,j)a(p,k)a(j,s)$$

from which by using the biorthogonality of **A** and **B** the relation (4) follows. The relation (5) can be proved similarly.

Using the **A**,**B**-shift the generalized convolution (**A**,**B**-convolution) of two functions $f,g\in P(G)$, is defined [3] as

$$(f \underset{A,B}{*} g)(i)= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \beta^{(i)}(j,k)f(j)g(k), \quad \forall i\in G. \tag{6}$$

i.e.,

$$(f \underset{A,B}{*} g)(i) = \sum_{j=0}^{n-1} g(i \underset{A,B}{\ominus} j)f(j). \tag{7}$$

In matrix notation this convolution is given by

$$\mathbf{f} \underset{A,B}{*} \mathbf{g} = \mathbf{C}_{A,B}(\mathbf{g})\mathbf{f}$$

where the generalized convolution matrix $\mathbf{C}_{A,B}(\mathbf{g})$ is an $(n\times n)$ matrix whose elements are given by $c(i,j)=g(i \underset{A,B}{\ominus} j)$, $i,j\in G$.

From (6) it follows [3] that

$$f \underset{A,B}{*} g = A^{-1}(Bf \circ Ag). \tag{8}$$

Note that as for A,B-shift, the A,B-convolution for B=A reduces to the so-called A-convolution defined and studied in [2] (see, also [4]). Furthermore, for the particular choice of the matrices **A** and **B**, the A,B-convolution reduces to the convolution appearing in the theory of discrete orthogonal transforms on finite groups. For example, if both **A** and **B** are equal to the discrete Fourier transformation matrix, the A,B-convolution reduces to the cyclic convolution. Also, for A=B=W with **W** being the Walsh matrix, A,B-convolution reduces to the dyadic convolution.

It is shown [3] that A,B-convolution satisfies the following properties.

Lemma 2. Let $f,g,h \in \mathbb{P}(G)$. Then,

1. $f \underset{A,B}{*} (g+h) = f \underset{A,B}{*} g + f \underset{A,B}{*} h$

2. $f \underset{A,B}{*} (g \underset{A,B}{*} h) = (f \underset{A,B}{*} g) \underset{A,B}{*} h,$

3. $f \underset{A,B}{*} g = A^{-1}B(g \underset{B,A}{*} f),$

4. $A(f \underset{A,B}{*} g) = Bf \circ Ag.$

Obviously, the Property 4 from Lemma 2 is the generalization of the well-known convolution theorem for Fourier transform which states that the spectrum of the convolution of two functions is equal to the product of the spectra of these functions. It is important to note that the A,B-convolution enables one to formulate the convolution theorem for any discrete transform on G. For example, the convolution theorem for discrete Haar transform was considered in [5].

Lemma 3. Let $f,g\in\mathbb{P}(G)$,. Then,

1. $$\mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{f}+\mathbf{g}) = \mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{f}) + \mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{g}),$$

2. $$\mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{f}) = \mathbf{A}^{-1}(diag(\mathbf{Af}))\mathbf{B},$$

3. $$\mathbf{C}_{\mathbf{A},\mathbf{D}}(\mathbf{g} \underset{\mathbf{A},\mathbf{B}}{*} \mathbf{h}) = \mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{g})\mathbf{C}_{\mathbf{B},\mathbf{D}}(\mathbf{h}).$$

Proof. 1. The proof of 1. is easily obtained by using the definition of convolution matrix, and hence it is omitted.

2. In order to prove 2. we will start from the property 4 of Lemma 2, from which by using (7) we have

$$\mathbf{A}(\mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{g})\mathbf{f}) = (\mathrm{diag}(\mathbf{Ag}))\mathbf{Bf},$$

i.e.,

$$\mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{g})\mathbf{f} = \mathbf{A}^{-1}(\mathrm{diag}(\mathbf{Ag}))\mathbf{Bf},$$

which proves 2.

3. Using 2, from property 4 of Lemma 2 we have

$$\mathbf{C}_{\mathbf{A},\mathbf{D}}(\mathbf{f} \underset{\mathbf{A},\mathbf{B}}{*} \mathbf{g}) = \mathbf{A}^{-1}(\mathrm{diag}(\mathbf{Bf}\circ\mathbf{Ag}))\mathbf{D},$$

i.e.,

$$\mathbf{C}_{\mathbf{A},\mathbf{D}}(\mathbf{f} \underset{\mathbf{A},\mathbf{B}}{*} \mathbf{g}) = \mathbf{A}^{-1}(\mathrm{diag}(\mathbf{Ag}))\mathbf{B}\mathbf{B}^{-1}(\mathrm{diag}(\mathbf{Bf}))\mathbf{D} = \mathbf{C}_{\mathbf{A},\mathbf{B}}(\mathbf{g})\mathbf{C}_{\mathbf{B},\mathbf{D}}(\mathbf{f}).$$

If the field $\mathbb{P}$ is such that the conditions for the existence of a Fourier transform are satisfied (see, for example [21]), then the Fourier transform of $f\in\mathbb{P}(G)$ relative to $\mathbf{A}$ could be defined by

$$\hat{f}_{\mathbf{A}}(i) = \sum_{j=0}^{n-1} a(i,j)f(j), \quad \forall i\in G.$$

As in the theory of discrete transforms, the spectral coefficients $\hat{f}_{\mathbf{A}}(i)$ will be called the $\mathbf{A}$-spectrum of f.

The inverse Fourier transform is given by

$$f(i) = \sum_{j=0}^{n-1} a^{(-1)}(i,j)\hat{f}_A(j), \qquad i\in G.$$

In matrix notation this Fourier transform and its inverse may be written as $\hat{\mathbf{f}}_A=\mathbf{Af}$, and $\mathbf{f}=\mathbf{A}^{-1}\hat{\mathbf{f}}_A$, respectively.

Property 2 from Lemma 3 enables us to calculate the Moore-Penrose inverse $C^+_{A,B}$ of $C_{A,B}$. More precisely, since $C_{A,B}(\mathbf{f})=\mathbf{A}^{-1}(diag(\mathbf{Af}))\mathbf{B}$, it follows immediately (see, for example [9, p.97]):

$$C^+_{A,B}(\mathbf{f}) = \mathbf{B}^{-1}(\mathrm{diag}\hat{\mathbf{f}}^+_A)\mathbf{A},$$

where

$$\hat{f}^+_A(i) = \begin{cases} \hat{f}^{-1}_A(i) & \text{for } \hat{f}_A(i) \neq 0 \\ 0 & \text{for } \hat{f}_A(i) = 0 \end{cases}.$$

The relations 1,2, and 3 from Lemma 3 and relation (9) involve as a particular case the results from [6] obtained for both **A** and **B** equal to the Chrestenson matrix.

3. DISCRETE GIBBS DIFFERENTIAL OPERATORS

In this section we will introduce the definition of the discrete Gibbs differential operators relative to two a priori chosen bases in $\mathbb{P}(G)$. Also, the concept of the Gibbs anti-derivative will be introduced.

Definition 1. For a function $f\in\mathbb{P}(G)$, the **A**,**B**-Gibbs derivative relative to the given bases **A** and **B** is defined by

$$f^{(1)}(i) = (f \underset{A,B}{*} h)(i) = \sum_{k=0}^{n-1} h(i \underset{A,B}{\ominus} k)f(k), \ \forall i\in G, \quad (10)$$

with

$$h(i) = \sum_{j=0}^{n-1} j a^{(-1)}(i,j).$$

In matrix notation $\mathbf{h}=\mathbf{A}^{-1}\mathbf{V}$, where $\mathbf{V}=[0,1,\ldots,n-1]^{T}$, and therefore, the $\mathbf{A},\mathbf{B}$-Gibbs derivative is given by

$$\mathbf{f}^{(1)} = \mathbf{f} \underset{\mathbf{A},\mathbf{B}}{*} \mathbf{h},$$

where $\mathbf{f}^{(1)} = [f^{(1)}(0),\ldots,f^{(1)}(n-1)]^{T}$.

From this representation and Property 2 of Lemma 3 we have

$$\mathbf{f}^{(1)} = D_{\mathbf{A},\mathbf{B}}\mathbf{f} = \mathbf{A}^{-1}(\mathrm{diag}\mathbf{V})\mathbf{B}\mathbf{f},$$

i.e., the $\mathbf{A},\mathbf{B}$-Gibbs differential operator is given in matrix form by

$$D_{\mathbf{A},\mathbf{B}} = \mathbf{A}^{-1}(\mathrm{diag}\mathbf{V})\mathbf{B}.$$

The derivatives of higher orders we define recursively as $\mathbf{f}^{(m)} = (\mathbf{f}^{(m-1)})^{(1)}$, which yields

$$\mathbf{f}^{(m)} = \mathbf{A}^{-1}(\mathbf{B}\mathbf{f}\circ(\mathbf{V}\circ\mathbf{B}\mathbf{A}^{-1})^{m-1}\mathbf{V}),$$

i.e.,

$$\mathbf{f}^{(m)} = \mathbf{A}^{-1}(\mathrm{diag}(((\mathrm{diag}\mathbf{V})\mathbf{B}\mathbf{A}^{-1})^{m-1}\mathbf{V}))\mathbf{B}\mathbf{f}.$$

The $\mathbf{A},\mathbf{B}$-Gibbs derivative defined in this way includes some of well known discrete differential operators. For example, if $\mathbf{B}=\mathbf{A}$ and G is considered as an Abelian group this operator reduces to the operators studied in [32]. Recall that these operators include the results from [15,17,33] obtained for $\mathbf{A}$ to be equal to the Walsh, Chrestenson and Haar matrix, respectively. Notice that the result obtained for $\mathbf{A}$ and $\mathbf{B}$ to be equal to the Chrestenson matrix was rediscovered in [23]. However, unlike to the existing results, the Definition 1 holds also for nonorthogonal bases. Recall that for a given nonsingular matrix $\mathbf{A}$, the matrix $\mathbf{C}$ whose eigenfunctions are the columns of **A**, can be determined by a procedure given in [5]. It can be shown that for the matrix **C** obtained in this way holds $\mathbf{C} = D_{A,A}$.

The relationship of the dyadic Gibbs derivative with Ritt-Kolchin derivatives was discussed in [11], while its relationship with Boolean difference was considered in [14].

The representation of the **A**,**B**-Gibbs derivative as a convolution operator suggests an application of the fast convolution algorithms for the numerical evaluation of this operator of a given function.

As we noted above, **A**,**B**-Gibbs derivative introduced here can be considered as a generalization of some known discrete differential operators. A justification for this statement can be found in the following theorem.

Theorem 1. Let $f\in\mathbb{P}(G)$. Then,

1. If **A** is orthogonal, and $\mathbf{A}^{-1}$ and **B** are biorthogonal matrices, then $D_{A,B}\mathbf{a}(p)=p\mathbf{a}^{-1}(p)$, where $\mathbf{a}^{-1}(p)$ is the p-th column of $\mathbf{A}^{-1}$. Similarly, $D_{A^{-1},B}\,\mathbf{a}(p)=p\mathbf{a}(p)$.

2. $(D_{A,B}\mathbf{f})\hat{}_A = \mathbf{V}\circ\hat{\mathbf{f}}_B$

3. $D_{A,B}(\mathbf{f} \ast_{A,B} \mathbf{g}) = \mathbf{f} \ast_{A,B} (D_{A,B}\mathbf{g})$,

4. $D_{B,A}(S^k_{A,B}\mathbf{f}) = \mathbf{A}^{-1}\mathbf{B}S^k_{B,A}(D_{A,B}(\mathbf{B}^{-1}\mathbf{A}\mathbf{f}))$.

Proof.

1. By (10) it follows that

$$(D_{A,B}\mathbf{a}(p))(i) = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} ja^{(-1)}(i \ominus_{A,B} k,j)a(k,p),$$

and by using (5)

$$(D_{A,B}a(p))(i) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} ja^{(-1)}(j,k)a(k,p).$$

Since A is orthogonal,

$$(D_{A,B}a(p))(i) = pa^{(-1)}(i,p).$$

The proof of the second part of the statement goes similarly.

2. The Fourier transform of A,B-Gibbs derivative of a function $f \in \mathbb{P}(G)$ is by definition

$$(D_{A,B}f)^{\wedge}_{A} = A(D_{A,B}f) = A(f \underset{A,B}{*} h) = Bf \circ Ah.$$

Since $h = A^{-1}V$ it follows that

$$(D_{A,B}f)^{\wedge}_{A} = Bf \circ V.$$

3. The proof of property 3 is easily obtained by using the Property 2 from Lemma 2, and hence it is omitted.

4. From (3) and (10) we have

$$D_{A,B}(S^{k}_{B,A}f) = A^{-1}(\mathrm{diag}V)(\mathrm{diag}b(k))Af,$$

i.e.,

$$D_{A,B}(S^{k}_{B,A}f) = A^{-1}BB^{-1}(\mathrm{diag}b(k))AA^{-1}(\mathrm{diag}V)BB^{-1}Af,$$

from which the statement follows directly.

Now we will consider the problem of the determination of the values of a function from the values of its A,B-Gibbs derivatives.

Definition 2. The A,B-Gibbs anti-derivative of a function $f \in \mathbb{P}(G)$ is defined by

$$(I_{A,B}f)(i) = (f \underset{A,B}{*} h^{-1})(i) = \sum_{k=0}^{n-1} h^{-1}(i \underset{A,B}{\ominus} k)f(k),$$

where

$$h^{-1}(i) = \sum_{j=1}^{n-1} j^{-1}a^{(-1)}(i,j).$$

In matrix notation $\mathbf{h}^{-1} = \mathbf{A}^{-1}\mathbf{V}^{-1}$, where $\mathbf{V}^{-1} = [0,1,2^{-1},\ldots,(n-1)^{-1}]^T$, and therefore, the **A**,**B**-Gibbs anti-derivative is given by $\mathbf{I}_{A,B}\mathbf{f} = \mathbf{f} \underset{A,B}{*} \mathbf{h}^{-1}$.

The **A**,**B**-Gibbs anti-derivatives of higher orders we define recursively as in the case of **A**,**B**-Gibbs derivative, which finally yields that the **A**,**B**-Gibbs anti-derivative of order m is given by

$$I^m_{A,B}\mathbf{f} = \mathbf{A}^{-1}\mathrm{diag}((\mathrm{diag}\mathbf{V}^{-1}\mathbf{B}\mathbf{A}^{-1})^{m-1}\mathbf{V}^{-1})\mathbf{B}\mathbf{f}.$$

Starting from the definition of **A**,**B**-Gibbs derivative and anti-derivative we have the following statement.

Theorem 2. Let $f \in \mathbb{P}(G)$. Then,

1. If $\hat{f}_A(0) = 0$, then $I_{A,B}(D_{B,A}f)(i) = f(i)$.

2. If $\hat{f}_B(0) = 0$, then $D_{B,A}(I_{A,B}f)(i) = f(i)$.

Proof.

1. By Property 2 of Theorem 1 we have

$$\mathbf{A}(I_{A,B}(D_{B,A}\mathbf{f})) = \mathbf{B}(D_{B,A}\mathbf{f})\circ\mathbf{A}\mathbf{h}^{-1} = \mathbf{A}\mathbf{f}\circ(\mathbf{V}\circ\mathbf{A}\mathbf{h}^{-1}).$$

Since $\mathbf{V}\circ\mathbf{A}\mathbf{h}^{-1} = \mathbf{V}\circ\mathbf{V}^{-1} = [0,1,1,\ldots,1]^T$ and, by assumption $\hat{f}_A(0)=0$, the statement is valid.

2. The proof can be obtained in a manner analogous to that used for proving the Statement 1, and therefore, it is omitted.

Note that this theorem for **B**=**A**=**W**, where **W** is the Walsh matrix, represents the discrete analogue of the corresponding result given in [8] for the continuous Walsh functions.

4. GIBBS DISCRETE DIFFERENTIAL EQUATIONS

In this section linear discrete differential equations with **A**,**B**-Gibbs differential operators will be introduced and studied. The determination of the Moore-Penrose inverse of the **A**,**B**-Gibbs differential operators enables one to obtain the general solution as well as the minimum-norm least-square solution of these equations.

Definition 3. For a given function $x \in \mathbb{P}(G)$ the relation

$$\sum_{i=0}^{r-1} d(i)y(t \underset{A,B}{\ominus} i) = \sum_{i=0}^{s-1} e(i)x(t \underset{A,B}{\ominus} i), \quad r,s \leq n, t \in G, \tag{11}$$

will be called an **A**,**B**-difference equation.

Using (7) and Property 3 from Lemma 2 the equation (11)can be written as

$$C_{B,A}(\mathbf{d})\mathbf{y} = C_{B,A}(\mathbf{e})\mathbf{x}, \tag{12}$$

where $\mathbf{d}=[d(0),\ldots,d(n-1)]^T$, and similarly for **y**, **e**, **x**.

To solve this equation we will use some results from the theory of generalized inverses (see, [9]). In this way and using (9) we derive the following theorem.

Theorem 3.

1. An equation of the form (11) is consistent if and only if

$$(1 - \hat{d}_B \circ \hat{d}_B^{+})\hat{e}_B \hat{x}_A = 0 \tag{13}$$

where 1 is the unit vector of order *n*, i.e., the vector of order *n* whose elements are all equal to 1.

The general solution of a consistent A,B-shift equation is given by

$$y = A^{-1}(\hat{d}_B^{+} \circ \hat{e}_B \circ \hat{x}_A) + (I - A^{-1}(\hat{d}_B^{+} \circ \hat{d}_B \circ A))T, \tag{14}$$

where T is an arbitrary matrix.

2. The minimum-norm least-square solution of equation (11) is given by

$$y = A^{-1}(\hat{d}_B \circ \hat{e}_B \circ \hat{x}_A). \tag{15}$$

Proof. To prove Theorem 3 we will use the method for the determination of the general solution of a system of linear equations based on the application of generalized inverses [9].

1. From Theorem 6.3.2 proved in [9 p.97] we have that the equation (11) written in the matrix form (12) is consistent if and only if

$$C_{B,A}(d)C^{+}_{B,A}(d)C_{B,A}(e)x = C_{B,A}(e)x.$$

By using the Property 2 of Lemma 3 and the relation (9) we have after a short calculation that the consistency condition is given by (13).

By using the second part of Theorem 2.1.1 from [9, p.28] we have that the general solution of the equation (12) is given by

$$y = C^{+}_{B,A}(d)C_{B,A}(e)x + (I - C^{+}_{B,A}(d)C_{B,A}(e))T,$$

where T is an arbitrary matrix, which after a short calculation yields the relation (14).

2. By using Theorem 6.3.2., from [9, p.97] the minimum-norm least-square solution of equation (11), written in the matrix form (12) is given by

$$y = C^{+}_{B,A}(d)C_{B,A}(e)x.$$

Using the property 2 of Lemma 3 and the relation (9) we immediately have (15).

Definition 4. For a given function $x \in P(G)$ a relation of the form

$$\sum_{i=0}^{r-1} p_i D^{i}_{A,B} y = \sum_{i=0}^{r-1} q_i D^{i}_{A,B} x, \tag{16}$$

will be called the **A,B**-Gibbs differential equation.

Note that the equation (16) can be transformed into a n **A,B**-shift equation with constant coefficients by using the convolution representation of **A,B**-Gibbs derivative.

Using (7) and (10) the equation (16) can be written as

$$E_{A,B}(\mathbf{p})\mathbf{B}\mathbf{y} = E_{A,B}(\mathbf{q})\mathbf{B}\mathbf{x},$$

where

$$E_{A,B}(\mathbf{p}) = \sum_{i=0}^{r-1} p_i \operatorname{diag}((\mathbf{V} \circ \mathbf{B}\mathbf{A}^{-1})^{i-1}\mathbf{V})),$$

and $\mathbf{p}=[p_0,\dots,p_{n-1}]^T$ and similarly for $\mathbf{q}$.

The solution of this equation is given in the following theorem obtained by the use of Moore-Penrose inverse.

Theorem 4.

1. An **A,B**-Gibbs discrete differential equation is consistent if and only if

$$(I-E_{A,B}(p)E^{+}_{A,B}(p))E_{A,B}(q)\hat{x}_{B}=0.$$

The general solution of a consistent **A**,**B**-Gibbs discrete differential equation is given by

$$y=B^{-1}E^{+}_{A,B}(p)E_{A,B}(q)\hat{x}_{B}+B^{-1}(I-E^{+}_{A,B}(p)E_{A,B}(p))T,$$

where **T** is an arbitrary matrix.

2. The minimum-norm least-square solution of an **A**,**B**-Gibbs differential equation is given by

$$y=B^{-1}E_{A,B}(p)E_{A,B}(q)\hat{x}_{B}.$$

Proof. The proof of Theorem 4 is obtained by using the Theorem 6.3.2 from [9, p.97] in a manner similar to that used in proving Theorem 3, and therefore, it is omitted.

Note that if **B**=**A**, the requirement which must be satisfied for **A**,**B**-Gibbs differential equation to be consistent reduces to

$$\left(I-\left(\sum_{i=0}^{r-1} p_i V^{\circ i}\right)\left(\sum_{i=0}^{r-1} p_i V^{\circ i}\right)^{+}\right)\left(\sum_{i=0}^{r-1} q_i V^{\circ i}\right)\hat{x}_A=0,$$

and the general solution is given by

$$y=A^{-1}\left(\left(\sum_{i=0}^{r-1} a(i)V^{\circ i}\right)^{+}\circ\left(\sum_{i=0}^{r-1} b(i)V^{\circ i}\right)\circ\hat{x}_A\right)$$

$$+\ A^{-1}\left(I-\left(\sum_{i=0}^{r-1} a(i)V^{\circ i}\right)^{+}\left(\sum_{i=0}^{r-1} a(i)BV^{\circ i}\right)\right)T,$$

where $V^{\circ i}$ denotes the pointwise product of **V** by itself taken i times, and **T** is an arbitrary matrix.

In this case the minimum-norm least-square solution is given by

$$y=A^{-1}\left(\left(\sum_{i=0}^{r-1} p_i\, Q^{\circ i}\right)^{+}\circ \sum_{i=0}^{r-1} q_i Q^{\circ i}\circ \hat{x}_A\right).$$

Theorem 4 includes Theorem 4.1 and Theorem 5.2 from [6] as the particular examples obtained for **A** and **B** be equal to the Chrestenson matrix.

CONCLUSION

In this paper there is given a generalization of Gibbs differential operators on finite intervals. Unlike the existing results the definition given in this paper holds also for the nonorthogonal bases. At the same time, the known Gibbs differential operators on finite Abelian groups are involved as the particular examples.

The discrete differential equations relative to the operators introduced are discussed. The condition of consistency is determined and in that case the minimum-norm least-square solutions are given.

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GIBBS DERIVATIVES ON FINITE NON-ABELIAN GROUPS

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Abstract: In this paper we give an overview of the theory of Gibbs differentiation on finite not necessarily Abelian groups. We also present some new results in this area. We consider the concept of the Gibbs derivative and anti-derivative on finite non-Abelian groups and discuss the corresponding Gibbs differential equations. The recent results reviewed here are related to the introduction of partial Gibbs derivatives and to the formulation of the fast algorithms for numerical evaluation of Gibbs derivatives on finite groups. We also suggest the application of the Gibbs derivatives considered here in linear system theory and logic design.

1. INTRODUCTION

In engineering practice the elements of some functional spaces are used for mathematical modeling of signals. Usually one deals with functions on the real line $\mathbb{R}$ or on the set of integers $\mathbb{Z}$ or on some of its finite subsets. In a unique setting these functions can be regarded as functions on some locally compact Abelian groups. Powerful tools are developed for the processing of signals defined in such a way. These tools are based on some operators in the functional spaces mentioned. However, there are real-life signals and systems which are naturally modeled as functions and, respectively, relations between functions on finite non-Abelian groups. As noted in [1], some relevant examples are a problem of pattern recognition for two colored pictures, which may be considered as a problem of realization of a function defined on the group of binary matrices, a problem of synthesis of rearrangeable switching networks whose outputs depend on the permutation of input terminals [2,3], a problem of interconnecting telephone lines, etc. An application of non-Abelian groups in linear system theory can be found in the approximation of a linear time invariant system by a system whose input and output are functions defined on non-Abelian groups [4].

Another interesting application of non-Abelian groups is proposed in [5], where a general model of a suboptimal Wiener filter over a group is defined. It is shown that with respect to some criteria the use of a non-Abelian group may be more advantageous than the use of an Abelian group. For example, in some cases the use of various non-Abelian groups results in improved statistical performance of the filter as compared to DFT.

Therefore, there is more than academic interest in extending signal processing methods to functions defined on finite non-Abelian groups. In this paper we consider a particular area of abstract harmonic analysis. We are interested in extending the ideas of Gibbs differentiation to functions defined on a finite not necessarily Abelian group G into the field P, which could be the complex field $\mathbb{C}$ or a finite one admitting the existence of a Fourier transform. The space of such functions we denote by $P(G)$.

In the paper we give first a short survey of the author's work in this area, and then we report on some new related results. We first consider the definition of the Gibbs differential operator in the space $P(G)$. This operator is an extension of the Gibbs derivative on finite Abelian groups [6,7] obtained by the replacement of the group characters by irreducible unitary group representations. We also consider the problem of determination of the values of a function from the values of its Gibbs derivative. In this order the Gibbs anti-derivative on finite non-Abelian groups is defined. The concept of partial Gibbs derivatives is introduced. The recent reslts on matrix representation on Gibbs derivatives and partial Gibbs derivatives are reported and discussed. Upon these results is based the formulation of the fast algorithm for numerical evaluation of Gibbs derivatives on finite non-Abelian groups.

Using Gibbs derivatives the associated differential equations with constant coefficients are defined and solved. These equations can be considered as the input-output-state relations of a subclass of linear translation invariant systems on finite non-Abelian groups [8]. An application of partial Gibbs derivatives for the detection of some properties of functions from $P(G)$ is suggested.

2. NOTATIONS AND DEFINITIONS

Let G be a finite not necessarily Abelian group of order g. We associate in a unique manner to each group element one non-negative integer from the set $\{0,1,\ldots,g-1\}$ providing that 0 is associated to the group identity. In what follows the group elements will be identified with the non-negative integers associated to them.

For the field P we will assume henceforth:

1. *char* $P = 0$, or *char* P does not divide g.
2. P is the so-called splitting field for G.

Recall that the complex field is the splitting field for any finite group.

Let K be the number of equivalence classes of irreducible representations of G over P. Each such equivalence class contains just one unitary representation. We shall denote the K irreducible unitary representations of G in some fixed order by R_0, $R_1,\dots,R_{K-1}$. The set $\Gamma = \{R_0,R_1,\dots,R_{K-1}\}$ is the dual object for G. We denote by $R_w(x)$ the value of R_w at $x\in G$. Note that $R_w(x)$ stands for a non-singular $(r_w \times r_w)$ matrix with elements $R_w^{(i,j)}(x)\in P$, $x\in G$, $i,j=1,2,\dots,r_w$. It is well known that the functions $R_w^{(i,j)}(x)$, $w=0,1,\dots,K-1$, $i,j=1,\dots,r_w$ form an orthogonal system in the space $P(G)$. Therefore, the direct and inverse Fourier transform of a function $f\in P(G)$ are defined respectively by

$$S_f(w) = r_w g^{-1} \sum_{u=0}^{g-1} f(u)R_w(u^{-1}), \tag{1}$$

$$f(x) = \sum_{w=0}^{K-1} \mathrm{Tr}(S_f(w)R_w(x)), \tag{2}$$

where Tr A denotes the trace of A.

Here and in the sequel we shall assume without explicitly stating it that all arithmetical operations are carried out in the field P.

Note that Tr R_w is called the character of R_w. Recall that if g is an Abelian group then all its representations are one-dimensional, and therefore they reduce to the characters. In that case the set Γ under componentwise multiplication forms a multiplicative group isomorphic to G.

For the Fourier transform as defined above, the main properties characteristic for the classical Fourier transform are satisfied. Let us mention linearity, translation of arguments, Wiener-Kintchine, Plancherel and Poisson theorems. For example, the translation (shift) operator T on g is defined by

$$(T^{\tau} f)(x) = f(x\tau), \; x,\tau\in G, \tag{3}$$

and the following property holds

$$S_{f(x\tau)}(w) = R_w(\tau)S_{f(x)}(w).$$

Note that the convolution theorem is valid in only one direction. More precisely, if h is a function obtained as the convolution on G of two functions f, and g, which we denote by $h=f*g$, then

$$r_w g^{-1} S_h(w) = S_f(w)S_g(w).$$

Obviously, the reverse statement analogous to the second part of the convolution theorem in classical Fourier analysis, can hardly be formulated since the dual object Γ does not have a group structure so that convolution on Γ is not defined.

Computation of the Fourier and inverse Fourier transform can be carried out using fast algorithms [1] representing the generalization of the famous FFT.

3. GIBBS DIFFERENTIAL CALCULUS

The Gibbs differential operators on Abelian groups are defined as linear operators having the group characters as their eigenfunctions (see, for example [7,9,10,11]). Since the group characters are the kernels of Fourier transforms on locally compact Abelian groups, it is very convenient to characterize the Gibbs derivatives by Fourier transform coefficients. Moreover, the strong relationship between the Gibbs derivatives and Fourier coefficients is sometimes used as the starting point for introducing the Gibbs derivatives on some groups (see, for example [11,12,13]). Using the same approach Gibbs derivatives on finite non-Abelian groups are defined in terms of Fourier coefficients as follows [14].

Definition 1. The Gibbs derivative Df of a function $f \in P(G)$ whose Fourier transform is S_f is defined by

$$(Df)(x) = \sum_{w=0}^{K-1} w\mathrm{Tr}(S_f(w)R_w(x)). \tag{4}$$

As it is noted in [14] this definition is unique only by virtue of the fixed order which we adopted for the elements of Γ. If a different notation were adopted, then (4), though unchanged in appearance, would define a distinct differentiator. This phenomenon is nothing new; it is already present in the definition of the dyadic Gibbs derivative [6], which depends upon the order assumed for the Walsh functions (the characters of the dyadic group). The same statement applies to all other Gibbs derivatives on different groups.

In what follows the Gibbs derivative will be denoted by Df or, alternatively, by $f^{(1)}$.

Following the approach used in [15] for the dyadic derivative, and lately in [16] for Gibbs derivatives on finite Abelian groups, an interpretation of the Gibbs derivative defined by (4) can be given as follows [14].

Define the partial sum $f_p(x)$, $p \le K$, by

$$f_p(x) = \sum_{w=0}^{p-1} \mathrm{Tr}(S_f(w)R_w(x)). \tag{5}$$

Define also the Fejér sum as

$$\sigma_q(x) = q^{-1} \sum_{p=1}^{q} f_p(x). \tag{6}$$

Substituting (5) into (6) we have, after a simple calculation,

$$f(x) - \sigma_K(x) = K^{-1} \sum_{w=0}^{K-1} w\mathrm{Tr}(S_f(w)R_w(x)).$$

The left member of this equality is the error in the approximation of f by its Fejér sum $\sigma_K(x)$. Hence, the Gibbs derivative on a finite non-Abelian group g can be interpreted as this error multiplied by K. Note that the corresponding result for the infinite dyadic group is given in [17].

The main properties of the Gibbs derivative are analogous to the corresponding properties of the classical Newton-Leibniz derivative, and they are given in the following theorem.

Theorem 1. If $f \in P(G)$, then

1. $D(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Df_1 + \alpha_2 Df_2$, $\alpha_1, \alpha_2 \in P$, $f_1, f_2 \in P(G)$.

2. $Df = 0 \in P$ iff f is a constant function.

3. If the Fourier transform of f is S_f, then that of $f^{(1)}$ is given by

$$S_{f^{(1)}}(w) = wS_f(w), \quad w=0,1,\ldots,K-1.$$

This property can be interpreted as the fact that the set $\{R_w^{(i,j)}(x)\}$ is the set of eigenfunctions of the Gibbs derivative, i.e.,

$$DR_w^{(i,j)}(x) = wR_w^{(i,j)}(x).$$

From there, due to the linearity of Gibbs derivative, we have further

$$D\mathrm{Tr}R_w(x) = w\mathrm{Tr}R_w(x).$$

4. From Property 3 it easily follows that

$$D(f_1 * f_2) = (Df_1) * f_2 = f_1 * (Df_2), \quad f_1, f_2 \in P(G)$$

where * denotes the convolution on G.

5. The Gibbs derivative commutes with the translation (shift) operator T, i.e.,

$$D(T^{\tau} f) = T^{\tau}(Df), \ \forall \tau \in G.$$

It is known that Gibbs differential operators do not obey the product rule. The same applies to the Gibbs derivative considered here, i.e., it is false that for each f_1 and f_2

$$D(f_1 f_2) = f_1(Df_2) + (Df_1)f_2.$$

The Gibbs derivative can be extended to an arbitrary complex order k by way of the definition of the delta function:

$$\delta(x) = g^{-1} \sum_{w=0}^{K-1} r_w \mathrm{Tr} R_w(x).$$

The δ-function thus defined has the property

$$\delta(x) = \begin{cases} 1, & x=0 \\ 0, & x \neq 0 \end{cases}.$$

The Gibbs derivative of order k of the δ-function is obtained by a direct generalization of the property (7),

$$\delta^{(k)}(x) = g^{-1} \sum_{w=0}^{K-1} w^k r_w \mathrm{Tr} R_w(x).$$

Using Property 4 from Theorem 1 we have:

$$(D^k f)(x) = ((D^k \delta) * f)(x) = \sum_{w=0}^{K=1} w^k \mathrm{Tr}(S_f(w) R_w(x)). \qquad (8)$$

4. GIBBS ANTI-DERIVATIVE

In this section we consider a method for the determination of the values of a function from the values of its Gibbs derivatives.

It is obvious from (8) that the Gibbs derivative can be considered as a convolution operator on $P(G)$. More precisely, if we introduce a function W_k defined by its Fourier transform coefficients as:

$$S_{W_k}(w) = \begin{cases} 0, & w = 0 \\ r_w g_w^{-1\ k} I_{r_w}, & w = 1,\ldots,K-1 \end{cases}$$

where I_{r_w} is the $(r_w \times r_w)$ identity matrix, then according to (8) the Gibbs derivative of order k of a function $f \in P(G)$ is given by

$$(Df)(x) = (W_k * f)(x).$$

From here we immediately deduce the concept of the Gibbs anti-derivative.

Let us introduce a function W_{-k} defined in the transform domain by:

$$S_{W_{-k}}(w) = \begin{cases} 1, & w = 0 \\ r_w w^{-k} I_{r_w}, & w=1,2,\ldots,K-1 \end{cases} \tag{9}$$

Note that by an application of the inverse Fourier transform

$$W_{-k}(x)=1+\sum_{w=1}^{K-1} w^{-k} r_w \mathrm{Tr}(R_w(x)).$$

Observe that functions of this kind for the particular case of the dyadic group were apparently first investigated by Watari [18], in the frame of Walsh-Fourier multiplier theory. Such functions were later used in [9] for the same purposes as those considered here. To be consistent with these particular definitions we omitted the factor g^{-1} the appearance of which could be expected according to the convolution theorem.

Using the function W_{-k} we introduce an inverse operator which will be called the Gibbs anti-derivative.

Definition 2. For a function $f \in P(G)$ the Gibbs anti-derivative of order k, denoted by I^k, is defined as

$$(I^k f)(x) = (W_{-k} * f)(x).$$

The Gibbs anti-derivative can be considered as a Fourier multiplier operator, thus having all properties characteristic for these operators. Therefore, there is no need for any particular consideration of these properties here.

Having the concept of the Gibbs anti-derivative we can deduce a theorem which shows how to determine the values of a function f from the values of its Gibbs derivative of order k.

Theorem 2. Let $f \in P(G)$ be such that $S_f(0) = 0$. Then,

$$f(x) = g^{-1} I^k (D^k f)(x) \tag{10}$$

or equivalently

$$f(x) = g^{-1} D^k (I^k f)(x). \tag{11}$$

Here the factor g^{-1} appears at the right hand side of the equalities (10) and (11) since we omit it in the definition (9).

Note that Theorem 2 can be regarded as a kind of counterpart of the so-called fundamental theorem for dyadic analysis due to Butzer and Wagner [9]. Moreover, as it is noted in [19], the theorems of this kind are a kind of counterpart of the fundamental theorem of the Newton-Leibniz calculus in the frame of abstract harmonic analysis on groups.

5. PARTIAL GIBBS DERIVATIVES

In this section we shall be concerned with the partial Gibbs derivatives for functions from $P(G)$.

We assumed that a finite not necessarily Abelian group G of order g can be represented as a direct product of some subgroups $G_1,\ldots,G_n$ of orders $g_1,\ldots,g_n$, respectively, i.e.,

$$G = \mathop{\times}_{i=1}^{n} G_i, \quad g = \prod_{i=1}^{n} g_i, \quad g_1 \leq g_2 \leq \ldots \leq g_n. \tag{12}$$

The convention adopted in Section 2 for the notation of group elements applies to the subgroups G_i as well. Due to this assumption each $x \in g$ can be uniquely represented as

$$x = \sum_{i=1}^{n} a_i x_i, \quad x_i \in G_i, \quad x \in G,$$

with

$$a_i = \begin{cases} \prod_{j=i+1}^{n} g_j, & i=1,\ldots,n-1 \\ 1, & i=n, \end{cases}$$

where g_j is the order of G_j.

The group operation $\circ$ of G can be expressed in terms of the group operations $\overset{i}{\circ}$ of the subgroups G_i, $i=1,\dots,n$ as

$$x\circ y = (x_1 \overset{1}{\circ} y_1,\ x_2 \overset{2}{\circ} y_2,\dots,x_n \overset{n}{\circ} y_n), \quad x,y\in G, \quad x_i,y_i\in G_i. \qquad (13)$$

Note that if the group G is representable in the form (12), then its irreducible unitary representations can be obtained as the Kronecker product of the irreducible unitary representations of subgroups G_i, $i=1,\dots,n$. Therefore, the cardinality K of Γ can be expressed as

$$K = \prod_{i=1}^{n} K_i,$$

where K_i is the cardinality of the dual object Γ_i of G_i.

Now, for a given group G of the form (12), the index w of each unitary irreducible representation R_w can be represented as

$$w = \sum_{i=1}^{n} b_i w_i, \qquad w_i = 0,1,\dots,K_i-1\ ,\ w = 0,1,\dots,K-1$$

with

$$b_i = \begin{cases} \prod_{j=i+1}^{n} K_j\ , & i=1,\dots,n-1 \\ 1, & i=n \end{cases}. \qquad (14)$$

Using the notation introduced, a given function $f(x)$, $x\in G$ can be considered as a function of several variables, $f(x_1,\dots,x_n)$, where $x_i\in G_i$. For this function the partial Gibbs derivative with respect to the variable x_i is defined [20] as follows.

Starting from Definition 1 and using some well-known properties of group characters we easily have:

$$(\mathrm{D}f)(x) = \sum_{w=0}^{K-1} w\mathrm{Tr}(r_w g^{-1} \sum_{u=0}^{g-1} f(u)\mathrm{R}_w(u^{-1}\circ x)).$$

From the invariance under translation property of the Haar integral

$$\sum_{y\in G} g(y) = \sum_{y\in G} g(z\circ y),\ \forall z\in G,\ g\in P(G)$$

we obtain

$$(\mathrm{D}f)(x) = g^{-1}\sum_{u=0}^{g-1} f(u\circ x) \sum_{w=0}^{K-1} wr_w\mathrm{Tr}(\mathrm{R}_w(u^{-1}).$$

Now we have the following definition.

Definition 3. The partial Gibbs derivative $(\Delta_i f)(x)$ at a point $x=(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)\in G$ with respect to the i-th variable x_i of a function $f\in P(G)$ is defined as the Gibbs derivative $(\mathrm{D}f_i)(x_i)$, at x_i, of the function $f_i\in P(G)$ defined by $f_i(y) = f((x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n))$.

Thus,

$$(\mathrm{D}_i f)(x) = (\mathrm{D}f_i)(x_i)$$

$$= g_i^{-1} \sum_{u_i=0}^{g_i-1} f(x_1,\ldots,x_{i-1},u_i^{\,i}\circ x_i,x_{i+1},\ldots,x_n) \sum_{w=0}^{K_i-1} wr_w^i\mathrm{Tr}\mathrm{R}_w^i(u_i^{-1}),$$

where g_i is the order of G_i , K_i denotes the number of nonequivalent irreducible representations of G_i, and r_w^i is the dimension of the representation R_w^i of G_i.

Actually, the partial Gibbs derivative D_i thus defined is the restriction on G_i of the Gibbs derivative on G. It follows that the partial Gibbs derivatives have properties similar to those of the Gibbs derivative.

Theorem 3. Let $f\in P(G)$. Then,

1. $D_i f = 0$ iff f is a constant on G_i, i.e., iff f has the same value for $\forall x_i \in G_i$. Moreover, $D_i c=0$ for any constant $c\in P(G)$.

2. $D_i(c_1 f_1 + c_2 f_2) = c_1 D_i f_1 + c_2 D_i f_2$, $c_1, c_2 \in P$, $f_1, f_2 \in P(G)$.

3. $D_i(D_j f) = D_j(D_i f)$.

4. If the Fourier transform of $f\in P(G)$ is S_f, then that of $D_i f$ is given by

$$S_{D_i f}(w) = B_i(w) S_f(w), \qquad w=0,1,\ldots,K-1$$

where $B_i(w) = [0,1,\ldots,K_i-1,0,1,\ldots,K_i-1,\ldots,0,1,\ldots,K_i-1]^T$.

5. $D_i(f*g) = D_i f*g = f D_i g$.

6. MATRIX REPRESENTATION OF GIBBS DERIVATIVES

The Gibbs derivatives as defined by Definition 1 are conveniently characterized by Fourier coefficients, a fact explicitly expressed also in Property 3 of Theorem 1.Therefore, to get a matrix representation of these differential operators it is very appropriate to start from the matrix representation of the Fourier transform on G. Since this transform is given in terms of the irreducible unitary representations which are square matrices with elements in P, we need the following definitions of generalized matrix multiplication.

Definition 4. Let **A** be a $(m\times n)$ matrix with elements $a_{ij}\in P$, $i\in\{0,1,\ldots,m-1\}$, $j\in\{0,1,\ldots,n-1\}$, and let **[B]** be a $(n\times r)$ matrix whose elements $\underline{b}_{jk}$, $j\in\{0,1,\ldots,n-1\}$, $k\in\{0,1,\ldots,r-1\}$ are $(p\times p)$ matrices with element in P. We define the product $\mathbf{A}\odot[\mathbf{B}]$ as a $(m\times r)$ matrix **[Y]** whose elements $\underline{y}_{ik}$, $i\in\{0,1,\ldots,m-1\}$, $k\in\{0,1,\ldots,r-1\}$ are $(p\times p)$ matrices with elements in P given by

$$\underline{y}_{ik}=\sum_{j=0}^{n-1} a_{ij}\underline{b}_{jk}.$$

The product $[\mathbf{B}]\odot\mathbf{A}$ is defined similarly.

Definition 5. Let **[Z]** be a $(m\times n)$ matrix whose elements $\underline{z}_{ij}$, $i\in\{0,1,\ldots,m-1\}$, $j\in\{0,1,\ldots,n-1\}$ are $(p\times p)$ matrices with elements in P, and let **[B]** be a $(n\times r)$ matrix whose elements $\underline{b}_{jk}$, $j\in\{0,1,\ldots,n-1\}$, $k\in\{0,1,\ldots,r-1\}$ are $(p\times p)$ matrices with elements in P. The product of matrices **[Z]** and **[B]** is a $(m\times r)$ matrix $\mathbf{Y} = [\mathbf{Z}]\circ[\mathbf{B}]$ whose elements $y_{ik}\in P$ are given by

$$y_{ik}=\sum_{j=1}^{n} \mathrm{Tr}(\underline{z}_{ij}\underline{b}_{jk}).$$

Using the matrix operations introduced the Fourier transform pair defined by (1) and (2) can be expressed as follows.

Let $f\in P(G)$ be given as a vector $\mathbf{f}=[f(0),\ldots,f(g-1)]^{\mathrm{T}}$. Then its Fourier transform is given by

$$[\mathbf{S}_f]=g^{-1}[\mathbf{R}^{-1}]\odot\mathbf{f},$$

where $[\mathbf{S}_f]=[\mathbf{S}_f(0),\ldots,\mathbf{S}_f(\mathrm{K}-1)]^{\mathrm{T}}$, and $[\mathbf{R}^{-1}]=\underline{b}_{sq}$ with $\underline{b}_{sq}=\mathrm{r}_s\mathrm{R}_s^{-1}(q)$, $s\in\{0,1,\ldots,\mathrm{K}-1\}$, $q\in\{0,1,\ldots,g-1\}$.

The inverse Fourier transform is given by

$$f = [R]\circ[S_f],$$

where $[R] = [a_{ij}]$ with $a_{ij}=R_j(i)$, $i\in\{0,1,\ldots,g-1\}$, $j=\{0,1,\ldots,K-1\}$.

Using these definitions relation (4) defining the Gibbs derivative can be rewritten in a matrix form as follows.

Definition 6. The matrix $\mathbf{D}_g$ describing the Gibbs derivative on a given group g of order g is given by

$$\mathbf{D}_g = g^{-1}[R]\circ G\odot[R^{-1}],$$

where $[R]$ is the matrix of unitary irreducible representations of G over P, i.e., $[R] = [a_{ij}]$ with $a_{ij}R_j(i)$, $i\in\{0,1,\ldots,g-1\}$, $j\in\{0,1,\ldots,K-1\}$, G is a diagonal (K×K) matrix given by $G=\text{diag}(0,1,\ldots,K-1)$, and $[R^{-1}]=[b_{sq}]$ with $b_{sq}=r_sR_s^{-1}(q)$, $s\in\{0,1,\ldots,K-1\}$, $q\in\{0,1,\ldots,g-1\}$.

The matrix $\mathbf{D}_g^k$ describing the Gibbs derivative of order k can be represented in the same form substituting G by $G^k=\text{diag}(0,1,2^k,\ldots,(K-1)^k)$.

The matrix representation of the partial Gibbs derivatives follows directly since they are defined as the restriction of the Gibbs derivative to the corresponding subgroups.

Definition 7. Let G be representable in the form (12). The partial Gibbs derivative Δ_i with respect to the variable x_i is defined as:

$$\Delta_i = \bigotimes_{j=1}^{n} \mathbf{A}_j, \tag{15}$$

with

$$\mathbf{A}_j = \begin{cases} \mathbf{D}_{g_j}, & j=i \\ \mathbf{I}_j, & j\neq i \end{cases},$$

where $\mathbf{I}_j$ is a $(g_j\times g_j)$ identity matrix, and $\otimes$ denotes the Kronecker product.

Now, Definition 6 can be rewritten in terms of the partial Gibbs derivatives as follows.

Definition 8. The matrix $\mathbf{D}_g$ of the Gibbs derivative on a group G of order g is given by

$$\mathbf{D}_g = \sum_{i=1}^{n} b_i \underline{\Delta}_i,$$

where the coefficients b_i are defined by (14).

7. FAST ALGORITHM FOR EVALUATION OF GIBBS DERIVATIVES

To obtain a fast and efficient algorithm for the evaluation of the values of the Gibbs derivative of a given function we further exploit the relationship of this differential operator with the Fourier transform on groups. Recall that for the calculation of the Fourier transform on groups some fast algorithms were developed as a generalization of the world famous fast Fourier transform (FFT) for the calculation of the discrete Fourier transform (DFT) regarded as a particular example of the Fourier transforms on groups. See [1] for fast Fourier transform on non-Abelian groups.

For the representation of the numerical procedure for evaluation of Gibbs derivatives we use flow-graphs. Note that the matrix $\underline{\Delta}_q$ of the q-th partial Gibbs derivative on a given group G is strongly similar to the matrix describing q-th step of the fast Fourier transform on the same group. Therefore, we can associate easily to each matrix $\underline{\Delta}_q$ a flow-graph for the calculation of the product $\underline{\Delta}_q \mathbf{f}$ in a manner equal to that used with FFT. Recall that a flow-graph consists of the input and output nodes connected adequately with the branches. It is determined by the structure of the matrix $\underline{\Delta}_q$ the nodes of which will be mutually connected similarly as in the case of FFT. More precisely, the output node j will be connected with the input node i iff the (i,j)-th element d_{ij} of the matrix $\underline{\Delta}_q$ describing the q-th partial derivative is not equal to zero, else these nodes will be disconnected. As in the case of FFT to each branch a weighting coefficient is associated. The weight associated to the branch connecting the output node i with the input node j is equal to d_{ij}.

Having the fast algorithms for the computation of partial Gibbs derivatives available, the fast algorithms for computation of Gibbs derivative of a function $f \in P(G)$ are obtained according to the Definition 8 simply by summing the output nodes of the flow-graphs for calculation of the partial Gibbs derivatives multiplied by the weight coefficients b_i defined by (14).

Note that the algorithm thus obtained is quite suitable for a parallel implementation. First, the partial Gibbs derivatives with respect to all variables can be evaluated simultaneously. The vectors thus obtained need be multiplied componentwise by the corresponding factors b_i and subsequently added componentwise which requires only one step in parallel implementation. Secondly, a kind of parallelism is inherent in the flow-graphs describing the calculation of partial Gibbs derivatives since they consist of only one basic operation, similar to the so-called butterfly operation used with FFT, which is applied simultaneously to some different subsets of input nodes.

8. EQUATIONS WITH GIBBS DERIVATIVES

Relation (8) introduces Gibbs derivatives of an arbitrary complex order. The Gibbs derivative of a positive integer order could be defined recursively as $D^{n+1}f = D(D^n f)$, $n=1,2,\ldots$. This allows linear Gibbs discrete differential equations with constant coefficients to be defined and solved. These equations can be considered as a particular case of the generalized linear equations studied in [21].

Definition 9. A linear Gibbs discrete differential equation with constant coefficients is an equation of the form

$$\sum_{k=0}^{n} a_k y^{(k)} = \sum_{k=0}^{m} b_k f^{(k)}, \tag{16}$$

where a, b are real numbers, $f \in P(G)$ and y is the required solution.

As in the case of ordinary differential equations we get the general solution, y, of the equation (16) as the sum of the solution y_{zi} of the homogeneous equation and the particular solution y_{zs} of the inhomogeneous equation, i.e.,

$$y = y_{zi} + y_{zs}. \tag{17}$$

In order to find y_{zi} one looks for the roots of the characteristic equation of (16) given by

$$\sum_{k=0}^{n} a_k z^k = 0.$$

Now, we have the following theorem.

Theorem 4. If the roots $\{z_i\}$, $i=0,1,\ldots,n$ of the characteristic equation are distinct and belong to the set $\{0,1,\ldots,K-1\}$, then the homogeneous solution is [8]:

$$y_{zi}(x) = \sum_{i=0}^{n} \sum_{j,k=1}^{i} C_{jk}^{z_i} R_{z_i}^{(j,k)}(x),$$

where constants $C_{jk}^{z_i}$ depend on the boundary conditions.

There are some important differences encountered in solving linear Gibbs discrete differential equations with constant coefficients compared to solving ordinary differential equations. A homogeneous equation of order n does not always have n linearly independent solutions. The following statements are, in a way, often taken for granted, however, we could not find a proof for these statements, anywhere.

If t of the roots of the characteristic equation are repetitions of the other roots, then the number of linearly independent solutions of a linear Gibbs discrete differential equation of order k is $\sum_{i=0}^{k-t} r_{z_i}^2$, provided that each root of the characteristic equation is in the set $\{0,1,\ldots,K-1\}$.

If s of the roots are not in this set, then the number of linearly independent solutions of the given equation is $\sum_{i=0}^{k-s-t} r_{z_i}^2$.

This is not any peculiarity of the case considered here. A corresponding statement for logical differential equations with Gibbs derivatives on dyadic groups is given in [10]. Moreover, it seems that an analogous statement is valid more generally, as it is noted without proof in [21].

To get the particular solution of (16) we apply the Fourier transform on both sides of (16), and with Property 3 of Theorem 1 we obtain:

$$\sum_{k=0}^{n} a_k w^k S_y(w) = \sum_{k=0}^{m} b_k w^k S_f(w). \tag{18}$$

From there, provided that equation (18) is compatible, that is, $S_f(0) = 0$ for all $w \in \{z_i\}$, $i=o,\ldots,n$, we have

$$S_y(w) = \frac{P}{Q} S_f(w), \text{ where } P = \sum_{k=0}^{m} b_k w^k, \quad Q = \sum_{k=0}^{n} a_k w^k.$$

Introducing the notation

$$H(w) = r_w g^{-1} P/Q, \tag{19}$$

we have

$$S_y(w) = r_w^{-1} g H(w) S_f(w). \tag{20}$$

From (13) by using the convolution property, the inverse Fourier transform produces the particular solution

$$y_{zs}(x) = \sum_{u=0}^{g-1} h(u) f(xu^{-1}). \tag{21}$$

Now, we have that (16) has a general solution y of the form

$$y(x) = \sum_{i=0}^{n} \sum_{j,k=1}^{m} C_{jk}^{z_i} R_{z_i}^{(j,k)}(x) + \sum_{u=0}^{g-1} h(u) f(xu^{-1}). \tag{22}$$

9. APPLICATION OF GIBBS DERIVATIVES IN LINEAR SYSTEM THEORY

In linear system theory two general classes of systems are usually distinguished according to their input and output signals: the continuous systems and the discrete systems.

The input and output signals of continuous systems are modeled by functions defined on the real line $\mathbb{R}$, while those of discrete systems are modeled by functions on the set of integers $\mathbb{Z}$ or on one of its finite subsets. Both classes of systems can be uniquely considered as systems defined on some particular locally compact Abelian group. In this setting, using some other groups as the domain of definition of input and output signals, several new classes of systems can be considered. Let us note as examples the dyadic systems [22-29], which are defined on the dyadic groups and the **p**-adic systems defined on the group $\mathbb{Z}_{p^n}$ of integers less than some p^n [11,29].

As we noted in the Introduction, in reality there are the signals which are naturally modeled by functions on some non-Abelian groups. Therefore, it was interesting to introduce linear systems on non-Abelian groups [4,5,8,30]. The study of these systems proves useful since the use of such systems in some practical applications offers some essential advantages over the application of the corresponding systems on Abelian groups. A confirmation for such a statement can be found, for example, in [5]. Here we want to point out that a subclass of systems on finite non-Abelian groups can be described by Gibbs discrete differential equations. We start from the following consideration.

In its most abstract form a system is defined as a triplet (U,Y,s) where U and Y are sets of mappings and s is a binary (input-output) relation in $U\times Y$. Defined likewise, the system is much too abstract, providing a model which is hardly tractable at all. A concrete system can be obtained by imposing certain structures on the input and output sets, as well as on the relation s itself. Here we do this by the following definition.

Definition 10. A scalar linear system A over a finite not necessarily Abelian group G is defined as a triplet $(P(G),\ P(G),{*})$ where the input output relation $*$ is the convolution product on G,

$$y = h{*}f, \qquad f,h,y\in P(G),$$

i.e.,

$$y(\tau) = \sum_{x=0}^{g-1} h(x)f(\tau x^{-1}), \qquad x,\tau\in G. \tag{23}$$

So, an ordered pair $(f,y)\in P(G)\times P(G)$ is exactly then an input-output pair of A if f and y fulfill equation (23). The function $h\in P(G)$ is the impulse response of A.

It is easy to show that the system A is invariant against the translation of input functions. By this we mean that if y is the output to f, then $T^{\tau}y$ is the output to $T^{\tau}f$, for all $\tau \in G$. Therefore, we denote the system A as a linear translation invariant (LTI) system [8].

It is apparent that when G is the dyadic group, Definition 10 reduces to the dyadic systems introduced by Pichler [25] and further studied in [27,28]. If G is the group $\mathbb{Z}_{p^n}$ we obtain the systems studied by Moraga [11] and Cohn-Sfetcu [29].

The dyadic and p-adic systems are closely related with Gibbs differentiators on the dyadic group and on $\mathbb{Z}_{p^n}$, respectively (see [27] for dyadic and [11] for p-adic systems). A corresponding relationship can be established between LTI systems described by Definition 10 and the Gibbs differential operators on finite non-Abelian groups.

First of all, let us note that relation (8) shows that the Gibbs differentiator D^k of order k is a LTI system having an impulse response h given by $h=\delta^{(k)}$ (see, [8]).

The Gibbs discrete differential equation (16) can be interpreted as an input-output relation of a system A belonging to a linear combination of Gibbs derivatives on a finite non-Abelian group. Note that systems of this kind on dyadic groups are considered in [23,27].

According to (17), the general output function of this system is represented as the sum of the zero-input response of the system y_{zi} and the zero-state response y_{zs} has a form identical to (23).

Therefore, we infer that the scalar linear system A associated with (16), is a LTI system for which (22) represents an input-output-state relation, and h is the impulse response of A to the unit impulse $\delta(x)$. Since h is the inverse Fourier transform of H(w), we have that the transfer function of A is given by (19).

10. APPLICATION OF PARTIAL GIBBS DERIVATIVES IN LOGIC DESIGN

Logic design is a scientific discipline concerned with the design of so-called logical networks whose input-output relations are given in terms of digital functions. Recall that a n-variable digital function is defined as a mapping $f:\{S_1\times\ldots\times S_n\}\to L$, where S_i, $i=1,\ldots,n$ and L are finite sets. As examples we can mention the Boolean functions defined by $f:\{0,1\}^n\to\{0,1\}$ and $f:\{0,1,\ldots,p-1\}^n\to\{0,1,\ldots,p-1\}$, ($p$-prime), respectively.

It appears very useful for logic design purposes to impose some algebraic structure to the sets S and L. For example, the Boolean functions can be considered as a subset of complex functions on finite dyadic groups, while the multiple valued functions can be considered as a subset of complex functions on $\mathbb{Z}_{p^n}$. Due to this approach the methods and results from abstract harmonic analysis on groups, known as spectral techniques [31], can be applied in the study of digital functions.

Another useful approach for analysis and synthesis of digital functions is based on the application of some differential and difference operators. Let us note the Boolean difference applied to Boolean functions and the difference operator introduced in [32] applied to multiple valued functions.

We believe that Gibbs derivatives, being a class of differential operators closely related with Fourier transforms and thus exhibiting good properties of both approaches mentioned above, can be applied advantageously in logic design. Here we suggest the application of partial Gibbs derivatives for detection of some properties of digital functions.

Recall that the procedure for synthesis of a logical network realizing a given digital function can be greatly simplified if the given function exhibits some particular properties, i.e., if it belongs to some special class of digital functions for which efficient design methods may be known. Therefore, the analysis of properties of digital functions is a very important task. One of the basic problems in this analysis is to detect whether a given function depends or not on some of its variables. This problem is closely related with the detection of some symmetry properties of digital functions. It is known that for the solution of this problem differential operators as well as spectral methods can be efficiently used [31,32]. For example, in the case of Boolean functions we have that a given function $f(x_1,\ldots,x_n)$ is independent of the variable x_i iff the Boolean difference with respect to x_i is equal to zero (see, for example [31]).As it is suggested in [31] a corresponding statement can be formulated in a similar form in terms of Gibbs derivatives on finite dyadic groups. In the case of multiple-valued functions a differential operator with the same property is defined in [32]. Now, we want to develop a similar result for functions mapping a finite non-Abelian group G which can be represented in the form (12) into a finite field P.

The fact that a function $f(x_1,\ldots,x_n)$ does not depend on its variable x_i can be viewed as the fact that $f(x_1,\ldots,x_n)$ has the constant value for $\forall x_i \in G_i$. From this observation and the Property 1 from Theorem 3 we deduce that the test for detecting whether a given function is independent on one of its variables can be expressed by the following statement.

A function $f(x_1,\ldots,x_n)$ is independent of its variable x_i iff $\forall x_i \in G$ there holds $D_i f(x_1,\ldots,x_i) = 0$.

CONCLUSION

This paper presents some basic concepts of the general theory of Gibbs differentiation on finite non-Abelian groups: introduction and justification of the Gibbs derivative, its main properties, the corresponding anti-differentiation operator, a kind of counterpart of the fundamental theorem of differential calculus in the frame of abstract harmonic analysis on groups, and partial Gibbs derivatives. All these concepts are discussed in the general setting of the space of functions mapping a finite not necessarily Abelian group into a field which could be the complex field or a finite field admitting the existence of a Fourier transform on G.

The definition of the Gibbs derivative in terms of partial Gibbs derivatives is given. We also consider the matrix representation of Gibbs and partial Gibbs derivatives and we discussed the fast algorithm for computation of Gibbs derivatives based upon this representation.

Discrete Gibbs differential equations with constant coefficients are defined and solved. It is shown that these equations can be used as mathematical models of a subclass of convolution systems on finite non-Abelian groups.

An application of partial Gibbs derivatives in logic design is suggested.

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TRENDS AND OPEN QUESTIONS

Claudio Moraga (Ed.)

In this chapter we summarize comments and remarks of contributors of this volume, pointing out important aspects of their contributions which may be considered for further research. Readers are sincerely invited to join efforts and work in the suggested areas. The authors would be very glad to know of new results motivated by comments and questions given below.

The formal analogy that exists between Newton-Leibniz and Gibbs differentiation has continued to inspire an extensive development of novel forms of analysis. Looking back, we see the impressive structure of abstract harmonic analysis, the analysis of complex-valued functions on locally compact Abelian (LCA) groups and on compact non-Abelian groups, which curiously lacked, before 1967, a derivative concept. Abstract harmonic analysis was developed as a generalisation of Fourier analysis by replacing complex-valued functions on a specific LCA group, the real group, by complex-valued functions on an arbitrary LCA group. That this route was taken explains, at least to some extent, the long-continued absence of a derivative concept. The notion of differentiation is closely connected with that of Fourier analysis, to be sure, but the two do not necessarily arise simultaneously in the mind, as is shown also by the earlier history of analysis.

Classical analysis may be said to have begun with the formulation of the differential calculus by Newton and Leibniz, though of course their work was foreshadowed by that of earlier mathematicians. It was only considerably later that Fourier introduced the immensely important ideas associated with his name. So in classical analysis the concept of differentiation considerably pre-dated that of Fourier analysis, which came as a bolt from the blue, not as something obviously missing from the calculus of Fourier's time. If Fourier had pre-dated Newton and Leibniz, the derivative might have been defined by means of Fourier considerations, but one may wonder what could have lead to the introduction of such a definition, and whether it would then have been clear that the derivative could be used to solve practical problems concerning rate of change. But we may think that a similar inversion of events has occurred in abstract harmonic analysis, where Fourier considerations were extended to a more general topological situation many years before differentiation was introduced in the same context. Perhaps, then, it is no wonder that we are somewhat at a loss to interpret the Gibbs derivative in an intuitive way analogous to the interpretation of the Newton-Leibniz derivative as rate-of-change. Possibly the truth is that differentiation must be regarded as a purely mathematical concept that just happens, in the special case of real and complex analysis, to be intuitively significant and useful.

On the other hand, it may be that further generalisation is all that is needed to provide differentiation with a brighter future in the realm of applications. An oddity of abstract harmonic analysis is that the domain of the functions considered has undergone, thanks to group theory and topology, a profound generalisation, while the co-domain remains almost exclusively restricted to the complex field. Doubtless there are excellent technical reasons for this state of affairs, but it may be that some good would come of remembering that the differential calculus started with a study of real functions, that is, of functions whose domain and co-domain are essentially identical. The simplest and most appealing of the post-Gibbs calculi is the dyadic calculus, which applies to complex-valued functions defined on the dyadic group. It was the proving ground of the idea of differentiation on groups, and has received the most extensive study. It could again be used as a guinea-pig, this time to examine the possibility of reverting to the symmetry of the Newtonian situation, and taking both domain and co-domain (of the functions to be differentiated) as the dyadic field, whose elements it is convenient to call *dyadic numbers.* Such functions, on the analogy of real functions, are naturally called *dyadic functions.*

This particular test-bed is specially convenient in view of the existence already of an algebraic analogue of the proposed calculus for dyadic functions, an analogue called Boolean difference calculus. All that remains is to put in the usual topology of the dyadic group and the new "dyadic analysis" emerges. (The name *pure dyadic analysis* has been suggested to distinguish the new discipline from the well-established dyadic analysis of complex-valued functions on the dyadic group.) There may be some surprises in store, however: for example, the Taylor series induced from Boolean difference calculus are of two kinds, one with a countable, the other with an uncountable infinity of terms, corresponding respectively to the cardinalities of the integers and the dyadic group. The latter "series" presents an interesting problem in summation technique, except in the trivial case where all but a finite set of terms are zero. The former series, which is a series in the conventional sense, also has a point of difficulty, that the "expansion" is not, as in the conventional Taylor series, about a fixed point of the domain. These problems may well be overcome, may indeed suggest interesting new modes of analysis, if the case of summability of divergent series of 0s and 1s modulo 2 is anything to go by. This problem, let alone its solution, long seemed destined to remain in cloud-cuckoo-land, but a context in so-called Fourier analysis in (not *on*) the dyadic field enables such a series to be summed, provided that it is periodic of a period that is some non-negative integer power of 2.

Some unsolved problems concerning the extended dyadic derivative are summarized below.

A function $f \in L^p(0,1)$, $1 \le p \le \infty$, is said to be strongly dyadically differentiable in the extended sense in $L^p \equiv L^p(0,1)$, if the sequence of functions

$$(1) \qquad \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^j \, 2^{j-1} \{ f(x) - f(x \oplus 2^{-j-1}) \}$$

converges in the norm of L^p; in this case the limit of (1) is called the (first) strong extended dyadic (= ED) derivative of f, and denoted by $\mathcal{E}^{\{1\}}f$. In this respect, a series $\sum_{j=0}^{\infty} a_j$ is said to be Euler-summable to s if

$$\frac{1}{2^{\nu+1}} \sum_{\nu=0}^{n} \binom{n+1}{\nu+1} \sum_{j=0}^{\nu} a_j \equiv \sum_{\nu=0}^{n} \frac{1}{2^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} a_j$$

tends to s for $n \to \infty$. So (1) deals with the case $a_j := (-1)^j 2^{j-1} \{ f(x) - f(x \oplus 2^{-j-1}) \}$, with $\oplus$ denoting dyadic addition. In comparison with the standard dyadic derivative, (1) results by adding the multiplicative factor $(-1)^j$ to the difference and then by applying Euler's summation process to the resulting sum. For details here see [1].

Now W. Engels (1985) [3] showed that a function f, bounded on [0,1) and possessing a finite or countably infinite set of discontinuities of first kind (having at most a finite number of cluster points in [0,1)) is pointwise dyadically differentiable in the standard sense iff f is a piecewise constant on [0,1). Since every function which is dyadically differentiable in the standard sense is also differentiable in the extended sense, the basic question that arises is the nature of the functions that are ED-differentiable.

In this respect the authors [2] showed that the monomials x^n, $n \in \mathbb{N}_0$ are ED-differentiable, thus so are algebraic polynomials. Also "piecewise polynomials" such as $x^n \psi_k(x)$, $n,k \in \mathbb{N}_0$, namely "polynomials" of order n having a finite number of jump discontinuities, are ED-differentiable ($\psi_k(x)$ denoting the kth Walsh function in Paley's enumaration). Further, even extremely unsmooth functions such as Dirichlet's function d(x) or even $x^n d(x)$ are ED-differentiable.

Some of the specific open problems are the following:

(i) Are the classical functions such as e^x, sin x, log(1+x) or even $\psi_k(x)e^x$, $\psi_k(x)$ sin x, $\psi_k(x)$ log(1+x) ED-differentiable ?

(ii) Is the famous van der Waerden function wae(x) := $\sum_{k=1}^{\infty} 4^{-k} \{4^k x\}$, $x \in \mathbb{R}$, known to be <u>nowhere</u> differentiable in the classical Newton-Leibniz sense, ED-differentiable? Here {x} denotes the distance from x to the nearest integer.

(iii) The ED-derivative of $f_1(x) := x$ turns out to be

$$\mathcal{E}^{\{1\}}\left(f_1\right)(x) = \sum_{\nu=0}^{\infty} 2^{-\nu-4} \sum_{j=1}^{[\log_2 \nu]} \binom{\nu}{2^j} (-1)^{j+1} \varphi_{2^j}(x) \ ,$$

$\varphi_\kappa(x) = \psi_{2^\kappa}(x)$ being the Rademacher function of order κ.

The concrete question is whether this double sum has a closed representation. (Is it possibly a quadratic function?) If so, it would yield a closed expression of the <u>second</u> extended dyadic-derivative $\mathcal{E}^{\{2\}}s(x)$, where $s(x) := \sum_{k=0}^{\infty} (-1)^k 4^{-k} \psi_k(x)$, $x \in [0,1)$, since $\mathcal{E}^{\{2\}}s(x) = -4\mathcal{E}^{\{1\}}f_1(x)$. In fact, one does not seem to have a closed form for s(x) either. More generally, further information concerning closed representations of Euler summation processes of specific sums of Rademacher functions would be of interest.

Of further basic importance is the associated operator of anti-differentiation of order $r \in \mathbb{N}$, $I_{\{r\}}: L^p \to L^p$, defined for $f \in L^p$ by $I_{\{r\}}f(x) := \int_0^1 f(x \oplus u) W_r^*(u)du$, where $W_r^*(x) \sim 1 + \sum_{k=1}^{\infty} \left(k^*\right)^{-r} \psi_k(x)$, and $k^* = \sum_{j=0}^{\infty} (-1)^j k_j 2^j$, k_j being the binary coefficients of $k \in \mathbb{N}_0$ (note the factor $(-1)^j$). In this respect, if $f \in L^p$, $f\hat{}(0) = 0$, then $\mathcal{E}^{\{r\}}(I_{\{r\}}f) = f$, $r \in \mathbb{N}$. And $I_{\{r\}}(\mathcal{E}^{\{r\}}f) = f$ when $\mathcal{E}^{\{r\}}f \in L^p$. This is the counterpart of the fundamental theorem of the Newton-Leibniz calculus in the ED-setting. In this respect there is the following question:

(iv) What is a possible interpretation of the operator $I_{\{r\}} = \mathcal{E}^{\{-r\}}$ for $r \in \mathbb{N}$, in particular of $I_{\{1\}}$?. It does not seem to be associated with the area under a curve, as is the classical integral. The interpretation may perhaps be given in terms of more intuitive concepts that may occur in those sciences which make use of the Walsh analysis. Further, is there perhaps a possible comparison between dyadic anti-differentiation and classical integration in terms of an analytical relationship? See, e.g. [2, p. 935 f.] here. Returning to the function s(x) above, an intuitive interpretation of the formula $I_{\{1\}}(1-4f_1)(x) = s(x)$, $x \in [0,1)$ would be of interest.

In matters interpretation, J.E. Gibbs [4] has just given a very interesting and contemplative discussion of the question. He feels that the intuitive interpretation of the dyadic derivative is not radically different from that of the classical Newton derivative. For example, he obtains the definition of the (standard) dyadic derivative on a cyclic group heuristically from the Newton derivative of an associated function of a complex variable. Nevertheless, for real functions $f: [0,1) \to \mathbb{R}$ the question is by no means fully solved, particularly in the case of the ED-derivative.

Defining the best approximation of $f \in L^p$ by $p_n \in \mathcal{P}_n$ (= set of all Walsh polynomials of degree $\leq n$) by

$$E_n(f;L^p) := \inf_{p_n \in \mathcal{P}_n} \| f(\cdot) - p_n(\cdot) \|_p ,$$

then if $\mathcal{E}^{\{r\}}f$ belongs to the dyadic Lipschitz class Lip(β ; L^p) of order $\beta > 0$,

(2) $$E_n(f;L^p) = O((\log_2 n)^{2r} n^{-r-\beta}) \qquad (n \to \infty).$$

The converse direction is not fully solved. Indeed,

(v) If (2) holds, then $\mathcal{E}^{\{j\}}f$ exists and belongs to L^p for each $0 \leq j \leq r$. The question now is whether $\mathcal{E}^{\{r\}}f$ satisfies some type of Lipschitz condition.

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W. R. Wade

The following are some important open questions with respect to the paper *"The Gibbs derivative and term by term differentiation of Walsh series".* For notation and background information, please refer to the corresponding chapter of the present Proceedings.

(i) If $\sum_{k=2}^{\infty} k \log k \, a_k \, w_k(x)$ converges at x, is $f(x) = \sum_{k=0}^{\infty} a_k \, w_k(x)$ term by term dyadically differentiable ?

(ii) If $\sum_{k=m}^{\infty} a_k \, w_k(x) = o(m \log m)$ as $m \to \infty$, is $f(x) = \sum_{k=0}^{\infty} a_k \, w_k(x)$ term by term dyadically differentiable ?

(iii) If $\sum_{k=1}^{\infty} k \, a_k \, w_k(x)$ converges for all $x \in [0,1)$, is $f(x) = \sum_{k=0}^{\infty} a_k \, w_k(x)$ term by term dyadically differentiable at some point in [0,1) ? On some dense $\mathcal{G}_\delta$ set in [0,1]? Almost everywhere on [0,1]?

(iv) If $|a_k| \le \delta_k$, $\delta_k \downarrow 0$ and $\sum_{k=1}^{\infty} \delta_k < \infty$, is $f(x) = \sum_{k=1}^{\infty} k \, a_k \, w_k(x)$ term by term dyadically differentiable at all but countably many points in [0,1)?

(v) If $a_k = O(k^{-1} (\log k)^{-\alpha})$ as $k \to \infty$, for some $\alpha > 1$, is $f(x) = \sum_{k=1}^{\infty} k \, a_k \, w_k(x)$ term by term dyadically differentiable at all but countably many points in [0,1)?

Please recall that other unanswered questions were introduced and discussed in section 6 of our main chapter.

F. Schipp, J. Pàl, P. Simon

We would like to comment on some unsolved problems in dyadic analysis.

(i) W. Splettstösser and W. Ziegler have introduced a concept of dydic derivative for the Haar system on $\mathbb{R}^+$ (see *Proceedings of the A. Haar Memorial Conference*, 1985, Vol II). They have proved the fundamental theorem of dyadic calculus in the sense of norm convergence. The question concerning a.e. differentiability of an integral function is open.

(i i) It is known (see N. Fujii, *Proc. Amer. Math. Soc.* **77**, 111-116, (1979)) that the L^1-norm of the maximal function of (C,1) means for Walsh-Fourier series can be estimated from above by the dyadic H^1-norm of the function. It is an open question whether the above norms are equivalent (for nonnegative functions the equivalence is true).

(i i i) Let $\mathcal{E}$ be a set of positive measures in [0,1). Does there exist a Walsh polynomial such that its maximum norm is less than 1 and the maximal function of its Walsh-Fourier series greater than $C\cdot\log(1/|\mathcal{E}|)$ on the set $\mathcal{E}$, where $C>0$ is an absolute constant[2].

T. Kitada

The following questions refer to the paper *"Hörmander-type multiplier Thecrems on locally compact Vilenkin groups"*, co-authored by T. Kitada and C.W. Onneweer. Please see the corresponding chapter of the present book for notation and contextual information.

Theorem MWY holds for $\lambda = s = 1$. This is a generalisation of the Marcinkiewicz multiplier theorem on $\mathbb{R}$ and was proved by D. Kurtz (*Trans. Amer. Math. Soc.* **259**, 235-254, (1980)) with more general weight functions.

It is interesting to obtain a (locally compact) Vilenkin group analogue of the above theorem using our dyadic derivatives (or any other derivatives) on such groups.

For instance, does Theorem 3.6 hold for $\lambda = s = 1$? Even for the unweighted case we do not know whether this is true or not, but it is easily seen that $\varphi \in M(1,1)$ does not in general imply that $\varphi \in M(H^1, L^1)$.

He Zelin

Please refer to the paper *"Convergence Properties of a class of Walsh-Fourier Integral Operators"*, co-authored by He Zelin and D. Mustard in these Proceedings. I would like to present the three following questions open for further study:

(i) Let $f \in L^q(\mathbb{R}^+)$ $(1 \le q \le \infty)$, $\eta(t)$ is quasi-convex on $\mathbb{R}^+$, $\eta(0)=1$, $\eta(\infty)=0$ and $O_s(\eta)=\lambda$ $(0< \lambda <\infty)$.

Does $I_\rho = \| f - f*\int_0^\infty \eta(t/\rho)\psi_0(t)dt \|_q = O(\rho^{-\alpha})$ $(\alpha<\lambda)$ imply $f \in \text{Lip}\,\alpha$, i.e.

$$\omega(f,\delta) = \sup_{0\le h<\delta} \| f(x\oplus h) - f(x) \|_q \le A\delta^\alpha \ ?$$

If $I_\rho = O(\rho^{-\lambda} \ln\rho)$ or $O(\rho^{-\lambda})$, what kind of conclusion could be made about f ?

(ii) Let $f \in L^q(\mathbb{R}^+)$ $(1 \le q \le \infty)$, $\eta(t)$ is quasi-convex on $\mathbb{R}^+$, $\eta(0)=1$ and $\eta(\infty)=0$.

If there is a λ such that for any $f \in \text{Lip}\,\alpha$ $\| f - f*\int_0^\infty \eta(t/\rho)\psi_0(t)dt \|_q = O(\rho^{-\alpha})$ if $\alpha<\lambda$, $O(\rho^{-\lambda} \ln\rho)$ if $\alpha=\lambda$ or $O(\rho^{-\lambda})$ if $\alpha>\lambda$, can we conclude that $O_s(\eta)=\lambda$?

(iii) It is normal that we try to obtain in Walsh analysis important and useful results similar to that in Fourier analysis. Conversely, there are some significant results in Walsh analysis, for which we need to set up their analogues in Fourier analysis. To set up a result in Fourier analysis similar to theorem 2 of our contribution in these Proceedings is precisely one of these questions. A formal statement of the problem is the following:

Let $f \in \text{Lip}(L^q,\alpha)$ $(1 \le q \le \infty)$, η is quasi-convex on $\mathbb{R}$, $\eta(0)=1$, $\eta(\pm\infty)=0$ and $O_s(\eta)=\lambda$ $(0< \lambda <\infty)$. We would like to have:

$$\| f - f*\int_{-\infty}^{\infty} \eta(u/\rho)e^{i\cdot u}\, du \|_q = \begin{cases} O(\rho^{-\lambda}) & (\alpha<\lambda) \\ O(\rho^{-\lambda} \ln\rho\,) & (\alpha=\lambda) \\ O(\rho^{-\lambda}) & (\alpha>\lambda) \end{cases}.$$

Prove whether this is true or not.

R. Stanković

In our paper *"Some remarks on Gibbs derivatives on finite dyadic groups"* in these Proceedings we introduce a modified product rule for Gibbs derivatives (Theorem 1, Property 7). As it is well known the lack of a product rule is an important problem for Gibbs derivatives. It seems to me that the result mentioned above could be used to obtain a modified product rule for Gibbs derivatives on finite (both Abelian and non-Abelian) groups. The same applies to the real Gibbs derivative.

The application of partial Gibbs derivatives on finite groups in fault detection in (non necessarily binary) logical networks and communication channels is an open area. Possibly partial Gibbs derivatives on finite non-Abelian groups should also be considered.

Another open question is the extension of the theory of Gibbs differential calculus on finite non-Abelian groups to infinite non-Abelian groups.

As in the case of Abelian groups, we know that the Gibbs derivative of order $k \in \mathbb{N}$ on a finite non-Abelian group may be considered as the mathematical model of a particular linear shift invariant system whose impulse response is equal to the k-th Gibbs derivative of the δ-function. Moreover, the discrete Gibbs differential equations with constant coefficients may be regarded as the input-output relations of a subclass of linear systems on finite non-Abelian groups. It is important to estimate how large is that class of systems and to give a strong characterization of it, i.e. determine which conditions have to be satisfied so that a convolution system on a finite non-Abelian group can be represented by a discrete Gibbs differential equation. The inverse problem, that of translating a system given by a discrete Gibbs differential equation into a convolution system seems also worthy of consideration.

Another open problem is the analysis of observability and controllability of the considered subclass of convolution systems on finite non-Abelian groups, based on properties of the Gibbs derivatives.

The problem of approximating a given discrete time-invariant system by a system on a suitably chosen non-necessarily Abelian group has been reported in the literature, but as far as I know the problem is far from having been completely solved. It is interesting to consider whether the Gibbs derivatives can be used to provide a casual, Markovian, step-by-step description of the group model.

C. Moraga

In the paper *"Real Gibbs derivatives on finite Abelian groups"* co-authored by R. Stanković and C. Moraga in these Proceedings, we introduced a real Gibbs derivative which actually corresponds to the real part of the *classical* Gibbs derivative. This immediately suggested two questions for further research:

(i) What are the properties of the imaginary part of the Gibbs derivative? Recall that in the paper *"Variations on the Gibbs derivative"* co-authored by Zhang Gongli and C. Moraga we introduced *ad hoc* a complex-valued Gibbs derivative to preserve some aspects of the Newton-Leibniz derivative of trigonometric functions.

(i i) Is it possible to define a new kind of derivative that relates to the Gibbs derivative in a similar way as the Hartley transform relates to the Fourier Transform?

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