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Slavik V. Jablan

Theory of Symmetry and Ornament

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PREFACE

This work represents an attempt at a comparative analysis of the theory of discrete and visually presentable continuous symmetry groups in E^2 or in $E^2 \setminus \{O\}$: Symmetry Groups of Rosettes (Chapter 1), Friezes (Chapter 2), Ornaments (Chapter 3), Similarity Symmetry Groups in E^2 (Chapter 4), Conformal Symmetry Groups in $E^2 \setminus \{O\}$ (Chapter 5) and ornamental motifs found in ornamental art that satisfy the afore mentioned forms of symmetry.

In each chapter symmetric forms are treated from the theory of groups point of view: generators, abstract definitions, structures, Cayley diagrams, data on enantiomorphism, form of the fundamental region,... The analysis of the origin of corresponding symmetry structures in ornamental art: chronology of ornaments, construction problems, visual characteristics, and their relation to geometric-algebraic properties of the discussed symmetry is given. The discussions are followed by illustrations, such as Cayley diagrams and ornaments. Many of ornaments date from prehistoric or ancient cultures. In choosing their samples, chronology was respected as much as possible. Therefore, most of the examples date from the earliest periods — Paleolithic, Neolithic and the period of ancient civilizations. The problems caused by various datings of certain archaeological excavation sites have been solved by compromise, by quoting the different dates. The problem of symbols used in literature for denoting the symmetry groups has been solved in the same way.

The extension of the theory of symmetry to antisymmetry and colored symmetry was made only to facilitate a more detailed analysis of the symmetry groups by the desymmetrization method.

The surprisingly early appearance of certain symmetry structures in ornamental art of the Paleolithic and Neolithic led to attempts to interpret the causes of this phenomenon. Among the explanations we can note the existence of models in nature, and constructional possibilities. As the universal criterion, the principle of visual entropy was applied — maximal visual and constructional simplicity and maximal symmetry.

Somewhat different in concept is the chapter on Conformal Symmetry in $E^2 \setminus \{O\}$. As opposed to the other chapters, where the chronological priority of ornaments "as the oldest aspect of higher mathematics given implicitly" was stressed, in this chapter the emphasis is on the path leading from the theory of symmetry (i.e., the derivation, classification and analysis of conformal symmetry groups) toward ornaments understood as the visual interpretations of abstract geometric-algebraic structures. Such an approach is becoming increasingly more important, since it makes possible the use of visually presented symmetry groups in all fields of science where there is a need for the visualization of symmetry structures (Crystallography, Solid State Physics, Chemistry, Quantum Physics, Particle Physics,...). Also, by applying a comparative, multidisciplinary analysis and by establishing the existence of parallelism between the theory of symmetry and ornamental art, the research field of ornamental design can be enlarged. By connecting the theory of symmetry and the theory of visual perception, more precise aesthetic criteria in fine arts may be created. The possibilities to apply these criteria when analyzing works of art (painting, sculptures,...) could form the subject of a new study.

The closing chapter, The Theory of Symmetry and Ornamental Art, is an attempt at a survey synthesizing the relationship between the theory of symmetry and ornamental art, and a summary of the conclusions derived from individual chapters. Written as a compendium, this chapter could be considered as an independent entity.

The bibliography has been divided into two parts: one represents work in the field of the theory of symmetry and disciplines related to it, and the other work related to ornamental art.

I am especially thankful to Dr Dragomir Lopandić, Professor at the Faculty of Natural Sciences and Mathematics in Belgrade, under whose inspirational guidance this study came to life and to all the others who have helped give this study its final form.

A first version of this work was completed in 1981 and published by APXAIA (Belgrade, 1984). In the present version essential changes have been made in the discussion of the color- symmetry desymmetrizations, according to recent results in the field of colored symmetry. Certain definitions that were not sufficiently precise have been corrected and replaced by new ones. Important other contributions to the theory of symmetry, which have been made since 1981, have been also included, either in full, or through concise references to the original works. The symbols for the symmetry groups of friezes and antisymmetry have been simplified. The author is grateful to Professors H.S.M. Coxeter, B. Grünbaum, A.F. Palistrant, H. Stachel and W. Jank for their remarks, advice and suggestions, that were of immense value for the final version of the text.

Belgrade, 1989

Chapter 1

INTRODUCTION

The symmetry existing in nature and its reflection in human artifacts has been present, from the earliest times, in all that has been done by man. Visual structures, that are the common elements of geometry and painting, were often arranged according to the laws of symmetry. However, the relatively independent development of geometry and painting resulted in the formation of two different languages. Even when talking about the same object (such as the laws of symmetry in the visual organization of a painting, that are most explicitly expressed within ornamental art) these languages use quite different terms. In fine art, the expression "symmetry" preserved for centuries the meaning it had in Greek aesthetics: in its wider sense it indicated harmony, accord, regularity, while in the more narrow sense it was identified with mirror symmetry in a vertical reflection line. The descriptive language used in most discussions on ornaments, drifted apart from the exact language of geometry.

With the development of natural sciences (Crystallography, Chemistry, Physics,...) symmetry structures have become an important area of geometric studies; here the key words are transformation groups, invariance, isometry,... When painting and sculpture were differentiated from the decorative arts, began the period of a relative stagnation of ornamental art, which acquired a rather subordinate role and remained on the margins of the dominant aesthetics. On the other hand, the dynamic progress of the mathematical theory of symmetry caused the fact that the first more significant incitement for the study of ornamental art came from mathematicians (A. Speiser, 1927).

The approach to the classification and analysis of ornaments based on symmetries was enriched by the contributions of different authors (E. Müller, 1944; A.O. Shepard, 1948; J. Garrido, 1952; N.V. Belov, 1956a; L. Fejes Tóth, 1964; D.W. Crowe, 1971, 1975; D. Washburn, 1977;...). In these works the descriptive language was replaced by more precise geometric-crystallographic terminology, and the theory of symmetry was established as a powerful tool for the study of ornamental art. In time, symmetry analysis of ornamental art became a reliable method, used mainly to study ancient ornamental art or that of primitive peoples.

The new approach to ornamental art leads to many new questions: which aspects of symmetry, when and where, appear in the history of ornamental art; which are the dominant forms; how to classify colored ornaments; etc. Following attempts to provide answers to all these and similar questions of the "how", "where" and "when" type, the question "why" arises naturally: why is man creating ornaments at all, why do some ornaments appear earlier or more often,... The questions in the first group do not penetrate the field of aesthetics and the psychology of visual perception, and so the language of geometry is almost sufficient for their discussion. In contrast, the second group of questions points to the necessity for a more profound understanding of the links between visuality and symmetry and also to the necessity to compare the language of the theory of symmetry with that of the theory of visual perception. Therefore, besides the question about the classification of ornamental motifs, the chronology of ornamental art, problems of colored ornaments etc., one of the aims of this work is to study the possibilities of translating geometric properties into the language of visuality, and vice versa. When discussing the numerous problems that ornamental art raises, special attention is paid to its roots. They are to be found in the ornamental art of the prehistoric period, which represents the most complete record of the beginnings of human understanding of regularity. In turn, regularity is the underlying basis of all scientific knowledge, so that in contemporary science, visualization of symmetry structures often represents the simplest way of their modeling and interpreting. This is an additional stimulus to strengthen the ties between science and art.

Knowledge of the terminology of the theory of symmetry is necessary for its application to the study of ornamental art. This is the reason for giving the basic technical terms and methods in the Introduction. The remaining mathematical terms with which the amateur reader is less familiar, are given later, in the relevant chapters.

1.1. Geometry and Its Basic Terms

We take as the basis of every *geometry* the set of *undefined elements* (*point*, *line*, *plane*) which constitute *space*, the set of *undefined relations* (*incidence*, *intermediacy*, *congruence*) and the set of basic apriori assertions: *axioms* (postulates). All other elements and relations are defined by means of these primitive concepts, while all other assertions (*theorems*) are derived as *deductive consequences* of primitive propositions (axioms). So that, the character of space (this is, its geometry) is determined by the choice of the initial elements and their mutual relations expressed by axioms. The axioms of the usual approaches to geometry can be divided into a number of groups: *axioms of incidence*, *axioms of order*, *axioms of continuity*, *axioms of congruence* and *axioms of parallelism*. Geometry based on the first three groups of axioms is called "*ordered geometry*", while geometry based on the first four groups of axioms is called "*absolute geometry*"; to the latter corresponds the *n-dimensional absolute space* denoted by S^n.

With respect to congruence, we distinguish the *analytic procedure* with the introduction of *space metric* and the *synthetic procedure*, also called non-metric. The justification for the name "absolute geometry" is derived from the fact that the system of axioms introduced makes possible a branching out into the *geometry of Euclid* and that of Lobachevsky (hyperbolic geometry). This is achieved by adding the axiom of parallelism. By accepting the 5th postulate of Euclid (or its equivalent, Playfair's axiom of parallelism: "For each point A and line a there exists in the plane (a, A) at most one line p which is incident with A and disjoint from a", where line p is said to be *parallel* to a) we come to Euclidean geometry. By accepting Lobachevsky's axiom of parallelism, which demands presence of at least two such lines, we come to non-Euclidean hyperbolic geometry, i.e. the *geometry of Lobachevsky* and that of space L^n. In particular, for $n = 0$ all these spaces are reduced to a point, and for $n = 1$ to a line; their specific characteristics come to full expression for $n = 2$, and we distinguish the absolute (S^2), the Euclidean (E^2), and the hyperbolic plane (L^2). If there is no special remark, then the terms "plane" and "space" refer to the Euclidean spaces E^2 and E^3 respectively. By a similar extension of the set of axioms, ordered geometry supplemented with two axioms of parallelism becomes *affine geometry* (H.S.M. Coxeter, 1969).

1.2. Transformations and Symmetry Groups

A function m is a *mapping* of a set A to a set B if for every element $a \in A$ there exists exactly one element $b \in B$ such that $m(a) = b$. The mapping m is *one-to-one* if $m(a) = m(a')$ implies $a = a'$, and it is *onto* if $m(A) = B$, where $m(A) = \{m(a) \mid a \in A\}$. A *transformation* is a mapping of a space to itself that is both one-to-one and onto, i.e. it is a one-to-one correspondence from the set of points in the space onto itself (H.S.M. Coxeter, 1969; G. E. Martin, 1982). If we denote a transformation of space by t, then for each point P which we call the *original* there exists exactly one point Q, the *image* of point P derived by transformation t and we write $t(P) = Q$. Each point Q of the space is the image of some point P derived by transformation t, where to equal images correspond equal originals. Points P, Q in the given order are called *homologous points* of transformation t.

A *figure* f is any non-empty subset of points of space. A figure f is called *invariant* with respect to a transformation S if $S(f) = f$; in this case the transformation S is called a *symmetry* of the figure f, or an *element of symmetry* of the figure f. The *identity transformation* of space is the transformation E under which every point of space is invariant, i.e. $E(P) = P$ holds for each point P of the space. The identity transformation is a symmetry of any given figure. Any figure whoose set of symmetries consists only of the identity transformation E is called *asymmetric*; any other figure is called *symmetric*. For example, the capital letters A,B,C,D,E,K,M,T,U,V,W,Y are mirror-symmetric, H,I,O,X doubly mirror-symmetric and point-symmetric, N,S,Z point-symmetric, and F,G,J,P,Q,R asymmetric. The letters b d or p q form the mirror symetric pairs, and b q or p d the point-symmetric pairs.

For every two transformations S_1, S_2 of the same space we define the *product* S_1S_2, as the composition of the transformations: $S_1S_2(P) = S_2(S_1(P))$. In other words, by product we mean the successive action of transformations S_1, S_2. As a symbol for the composition $S \ldots S$, where S occurs n times, we use S^n, i.e. the n-th power of the transformation S. The *order of the transformation* S is the minimal n ($n \in N$) for which $S^n = E$ holds. If there is no finite number n which satisfies the given relation, then the transformation S is called a *transformation of infinite order*. If $n = 2$, then the transformation S is called an *involution*. If transformations S_1 and S_2 are such that $S_1S_2 = E$, then S_1 is called the *inverse* of S_2, and vice versa. We denote this relationship as $S_1 = S_2^{-1}$ and $S_2 = S_1^{-1}$. For an involution S we have $S = S^{-1}$, and for the product of two transformations $(S_1S_2)^{-1} = S_2^{-1}S_1^{-1}$ holds.

A transformation t which maps every line l onto a line $t(l)$ is a *collineation*. An *affine transformation* (or linear transformation) is a collineation of the plane that preserves parallels.

As a *binary operation* $*$ we understand any rule which assigns to each ordered pair (A, B) a certain element C written as $A * B = C$, or in the short form, $AB = C$. A structure $(G, *)$ formed by a set G and a binary operation $*$ is a *group* if it satisfies the axioms:

a_1) (closure): for all $A_1, A_2 \in G$, $A_1A_2 \in G$ is satisfied;

a_2) (associativity): for all $A_1, A_2, A_3 \in G$, $(A_1A_2)A_3 = A_1(A_2A_3)$ is satisfied;

a_3) (existence of neutral element): there exists $E \in G$ that for each $A_1 \in G$ the equality $A_1E = A_1$ is satisfied;

a_4) (existence of inverse element): for each $A_1 \in G$ there exists $A_1^{-1} \in G$ so that $A_1^{-1}A_1 = E$ is satisfied.

If besides a_1–a_4) also holds

a_5) (commutativity): for all $A_1, A_2 \in G$, $A_1A_2 = A_2A_1$ is satisfied, the group is *commutative* or *abelian*.

The *order of a group G* is the number of elements of the group; we distinguish *finite* and *infinite groups*. The *power* and the *order of a group element* are defined analogously to the definition of the power and the order of a transformation.

A figure f is said to be an *invariant* of the group of transformations G if it is invariant with respect to all its transformations, i.e. if $A_1(f) = f$ for every $A_1 \in G$. All symmetries of a figure f form a group, that we call the *group of symmetries* of f and denote by G_f. For example, all the symmetries of a square (Figure 1.1a) form the non-abelian group, consisting of identity transformation E, reflections R, R_1, R_1RR_1, RR_1R, and rotations RR_1, $(RR_1)^2$, R_1R — the symmetry group of square D_4. The order of reflections is 2, the order of rotations RR_1, R_1R is 4, and the order of half-turn $(RR_1)^2$ is 2. This group consists of 8 elements, so it is of order 8. The elements of the same group, expressed as products of reflection R and rotation S of order 4 are: identity E, reflections R, RS, RS^2, SR, and rotations S, S^2 and S^3. Instead of a square, we may consider the plane tiling having the same symmetry (Figure 1.1b).

A subset H of group G, which by itself constitutes a group with the same binary operation, is called a *subgroup* of group G if and only if (*iff*) for all $A_1, A_2 \in H$, $A_1A_2^{-1} \in H$. Subgroups $H = G$ and $H = \{E\}$ of each group G are called *trivial*, while the other subgroups are *nontrivial subgroups* of the group G. In the symmetry group of square, identity transformation E

and rotations S, S^2, S^3 form the subgroup of the order 4 — the rotational subgroup of square $\mathbf{C}_4$.

Groups $(G_1, *)$ and $(G_2, \circ)$ are called *isomorphic* if there exists a one-to-one and onto mapping i of elements of the group G_1 onto elements of the group G_2, so that for all $A_1, A_2 \in G_1$, $i(A_1 * A_2) = i(A_1) \circ i(A_2)$ holds; the mapping i is called an *isomorphism*. For example, by the mapping $i(R) = R$, $i(R_1) = RS$ is defined the isomorphism of the symmetry group of square generated by reflections R, R_1, with the same group generated by reflection R and rotation S. Any isomorphism of a group G with itself is called an *automorphism*.

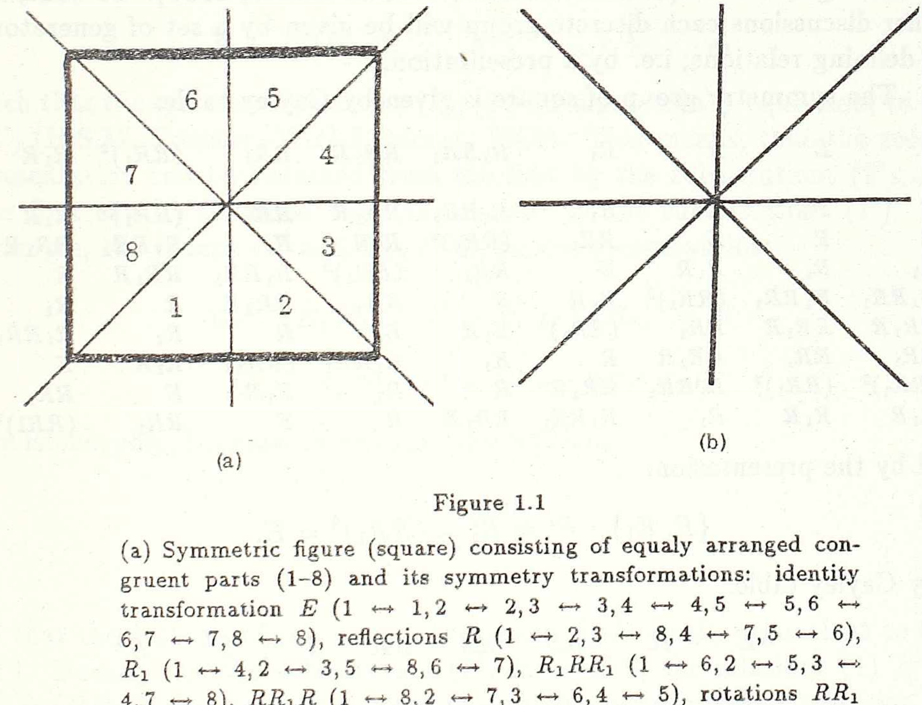


Figure 1.1

(a) Symmetric figure (square) consisting of equaly arranged congruent parts (1–8) and its symmetry transformations: identity transformation E ($1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3, 4 \leftrightarrow 4, 5 \leftrightarrow 5, 6 \leftrightarrow 6, 7 \leftrightarrow 7, 8 \leftrightarrow 8$), reflections R ($1 \leftrightarrow 2, 3 \leftrightarrow 8, 4 \leftrightarrow 7, 5 \leftrightarrow 6$), R_1 ($1 \leftrightarrow 4, 2 \leftrightarrow 3, 5 \leftrightarrow 8, 6 \leftrightarrow 7$), R_1RR_1 ($1 \leftrightarrow 6, 2 \leftrightarrow 5, 3 \leftrightarrow 4, 7 \leftrightarrow 8$), RR_1R ($1 \leftrightarrow 8, 2 \leftrightarrow 7, 3 \leftrightarrow 6, 4 \leftrightarrow 5$), rotations RR_1 ($1 \to 7, 2 \to 8, 3 \to 1, 4 \to 2, 5 \to 3, 6 \to 4, 7 \to 5, 8 \to 6$), R_1R ($1 \to 3, 2 \to 4, 3 \to 5, 4 \to 6, 5 \to 7, 6 \to 8, 7 \to 1, 8 \to 2$) and half-turn $(RR_1)^2$ ($1 \leftrightarrow 5, 2 \leftrightarrow 6, 3 \leftrightarrow 7, 4 \leftrightarrow 8$). The order of the symmetry group of square $\mathbf{D}_4$ is equal to the number of congruent parts (8); (b) plane tiling with the same symmetry.

Instead of representing the group in the traditional way, by means of its *Cayley table*, which offers a listing of all the elements of the group and their compositions (products), complete information about the group is given more effectively and concisely by a *group presentation* (i.e. abstract, generating definition): a set of generators and defining relations. The group of transformations G is *discrete* if for each point P of the space in which the group G acts there is a positive distance $d = d(P)$ such that no image of P (distinct from P) under an element of G is at distance less than d from P. The set $\{S_1, S_2, \ldots, S_m\}$ of elements of a discrete group G is called a *set of generators* of G if every element of the group can be expressed as a finite product of their powers (including negative powers). Relations $g_k(S_1, S_2, \ldots, S_m) = E$, $k = 1, 2, \ldots, s$, are called *defining relations* if all other relations which S_1, $S_2, \ldots$, S_m satisfy are algebraic consequences of the defining relations (H.S.M. Coxeter, W.O.J. Moser, 1980). So that, in further discussions each discrete group will be given by a set of generators and defining relations, i.e. by a presentation.

The symmetry group of square is given by Cayley table:

	E	R	R_1	R_1RR_1	RR_1R	RR_1	$(RR_1)^2$	R_1R
E	E	R	R_1	R_1RR_1	RR_1R	RR_1	$(RR_1)^2$	R_1R
R	R	E	RR_1	$(RR_1)^2$	R_1R	R_1	R_1RR_1	RR_1R
R_1	R_1	R_1R	E	RR_1	$(RR_1)^2$	R_1RR_1	RR_1R	R
R_1RR_1	R_1RR_1	$(RR_1)^2$	R_1R	E	RR_1	RR_1R	R	R_1
RR_1R	RR_1R	RR_1	$(RR_1)^2$	R_1R	E	R	R_1	R_1RR_1
RR_1	RR_1	RR_1R	R	R_1	R_1RR_1	$(RR_1)^2$	R_1R	E
$(RR_1)^2$	$(RR_1)^2$	R_1RR_1	RR_1R	R	R_1	R_1R	E	RR_1
R_1R	R_1R	R_1	R_1RR_1	RR_1R	R	E	RR_1	$(RR1)^2$

and by the presentation:

$$\{R, R_1\} \quad R^2 = R_1^2 = (RR_1)^4 = E,$$

or by Cayley table:

	E	R	RS	RS^2	SR	S	S^2	S^3
E	E	R	RS	RS^2	SR	S	S^2	S^3
R	R	E	S	S^2	S^3	RS	RS^2	SR
RS	RS	S^3	E	S	S^2	RS^2	SR	R
RS^2	RS^2	S^2	S^3	E	S	SR	R	RS
SR	SR	S	S^2	S^3	E	R	RS	RS^2
S	S	SR	R	RS	RS^2	S^2	S^3	E
S^2	S^2	RS^2	SR	R	RS	S^3	E	S
S^3	S^3	RS	RS^2	SR	R	E	S	S^2

and by the presentation:

$$\{S,R\} \quad S^4 = R^2 = (RS)^2 = E.$$

Two groups G_1 and G_2 which are given with their presentations:

$$G \quad \{S_1, S_2, \ldots, S_m\} \quad g_k(S_1, S_2, \ldots, S_m) = E \quad k = 1,2,\ldots,s \qquad (1)$$

$$G_1 \quad \{S'_1, S'_2, \ldots, S'_n\} \quad h_l(S'_1, S'_2, \ldots, S'_n) = E \quad l = 1,2,\ldots,t \qquad (2)$$

are isomorphic iff there exist relations:

$$S'_j = S_j(S_1, S_2, \ldots, S_m) \quad j = 1,2,\ldots,n \qquad (1')$$

$$S_i = S_i(S'_1, S'_2, \ldots, S'_n) \quad i = 1,2,\ldots,m \qquad (2')$$

such that the systems of relations (1), (1') are algebraically equivalent to (2), (2') (H.S.M. Coxeter, W.O.J. Moser, 1980). This means, that the second presentation can be obtained from the first by the substitutions (2'), and the first can be obtained from the second by the substitutions (1'). For example, the groups G_1 and G_2, given by the presentations:

$$G_1 \quad \{R, R_1\} \quad R^2 = R_1^2 = (RR_1)^4 = E \qquad (1)$$

$$G_2 \quad \{S, R\} \quad S^4 = R^2 = (RS)^2 = E \qquad (2)$$

are isomorphic, because there exist the relations:

$$S = RR_1 \qquad (1')$$

$$R_1 = RS \qquad (2')$$

so that the systems of relations (1), (1') are algebraically equivalent to (2), (2'). Namely, by the substitution (2') $R_1 = RS$, the relations (1) $R^2 = R_1^2 = (RR_1)^4 = E$ are transformed into algebraicly equivalent relations

$$R^2 = (RS)^2 = (RRS)^4 = E \quad \Longleftrightarrow \quad S^4 = R^2 = (RS)^2 = E \qquad (2)$$

and by the substitution (1') $S = RR_1$, the relations (2) are transformed into algebraicly equivalent relations

$$(RR_1)^4 = R^2 = (RRR_1)^2 = E \quad \Longleftrightarrow \quad R^2 = R_1^2 = (RR_1)^4 = E \quad (1).$$

Their isomorphism, defined by the mapping $i(R) = R$, $i(R_1) = RS$ is also simply visible from the corresponding Cayley tables.

By "*structure of the group*" we understand its isomorphism with some of the basic, well known groups (e.g., cyclic group C_n, dihedral group $D_n,\ldots$) or with a direct product of such groups. The cyclic group C_n is given by the presentation: $\{S\}\quad S^n = E$, and the dihedral group D_n can be given by two isomorphic presentations: $\{R, R_1\}\quad R^2 = R_1^2 = (RR_1)^n = E$ or $\{S, R\}\quad S^n = R^2 = (RS)^2 = E$. Hence, the structure of the symmetry group of square is D_4, and the structure of its rotational subgroup is C_4.

For groups G and G_1, $G \cap G_1 = \{E\}$, given by presentations (1), (2) we define the *direct product* $G \times G_1$ as the group with the set of generators $\{S_1, S_2, \ldots, S_m, S_1', S_2', \ldots, S_n'\}$, the set of defining relations of which is, besides the relations (1), (2), made up of relations $S_iS_j' = S_j'S_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. For each group G we can discuss the possibility of it being decomposed, i.e. represented as the direct product of its nontrivial subgroups. A group which allows such a decomposition we call *reducible*, otherwise it is called *irreducible*. For example, the direct product of two cyclic groups, C_3 given by the presentation $\{S\}\quad S^3 = E$ and C_2 given by $\{T\}\quad T^2 = E$ is the group $\{S, T\}\quad S^3 = T^2 = E\quad ST = TS$. By the substitution $U = ST$, this results in the presentation $\{U\}\quad U^6 = E$, so $C_3 \times C_2 \cong C_6$, showing that the group C_6 is reducible.

The term "decomposition" can be used in another sense. Each group can be decomposed according to its subgroup H:

$$G = g_1H \cup g_2H \cup \cdots \cup g_nH \cup \ldots$$

where $g_iH = \{g_ih \mid g_i \in G, h \in H\}$. The expression g_iH is called the *left coset* which corresponds to element g_i with respect to subgroup H. Analogously, there is the possibility of the right decomposition of group G according to subgroup H. If the above decompositions are finite, the number of cosets is called the *index* of the subgroup H in the group G; in the case of infinite decomposition we say that H is a *subgroup of infinite index*. We should also note the property that every two cosets are either disjoint or identical, and that the order of the group is equal to the product of the order of the subgroup H and its index. From this results the statement that the order of a subgroup is a divisor of the order of the group. A subgroup H of a group G is called a *normal subgroup* if $gH = Hg$ holds for every element $g \in G$. For example, for the symmetry group of square G and its rotatational subgroup H holds the decomposition $G = H \cup RH$, and $gH = Hg$ holds for every element $g \in G$, so H is the normal subgroup of index 2 in G. The order of H is 4, and order of G (8) is the product of the order of H (4) and index of H in G (2).

According to those basic geometric-algebraic assumptions, we can consider as the subject of this study the analysis of plane figures — ornamental motifs and their invariance with respect to symmetry groups.

The set of points $G(P) = \{g(P) \mid g \in G\}$, obtained from a point P by all transformations of the group G, is called the *orbit* of P with respect to G; it is the set of points equivalent to point P (or the transitivity class of P) with respect to the group G. Analogously we can also define the orbit (or transitivity class) of any figure f with respect to the group G and denote it by $G(f)$. A point P which is invariant with respect to a transformation S, i.e. a point for which $S(P) = P$, is also called *singular*. A figure f is invariant with respect to a transformation S if $S(f) = f$. A point P is a singular (invariant) point of a group G if it is a singular (invariant) point of all transformations of G. A point which is not an invariant point of a transformation S is also called a *point in general position* with respect to the transformation S. A point is said to be a point in general position with respect to a group of transformations G if it is in general position with respect to all the transformations of the group G, i.e. if it is not an invariant point of any transformation of the group G. For example, the singular (invariant) point of the symmetry group of square is the center of square. The points belonging to the mirror-reflection lines are the invariant points of the corresponding reflections. All other plane points, are the points in general position with respect to the symmetry group of square (Figure 1.1).

The orbit of some point P in general position with respect to the discrete group of transformations G makes possible a schematic interpretation of the group G: a *Cayley diagram* or a *graph of the group G* — a visual model of discrete group of transformations G. To each vertex of the graph corresponds exactly one element of the group, and to each edge corresponds one transformation. The edges which connect the homologous points of the same transformation are denoted by the same type of line (full, broken, dotted). The non-oriented edges correspond to the involutions. For any other, oriented edge, the motion in the direction of the arrow indicates the multiplication by the corresponding transformation from the right, and the motion in the opposite direction of the arrow corresponds to multiplication by the inverse of the corresponding transformation on the right. A Cayley diagram is a *connected graph*, i.e. there exists a path which connects every two vertexes of the graph. It represents the direct visual interpretation of the presentation of the group, since to every closed cycle there corresponds one defining relation (1). A *complete graph* is considered to be the graph in which every two vertexes are directly linked by the edge (Figure 1.2).

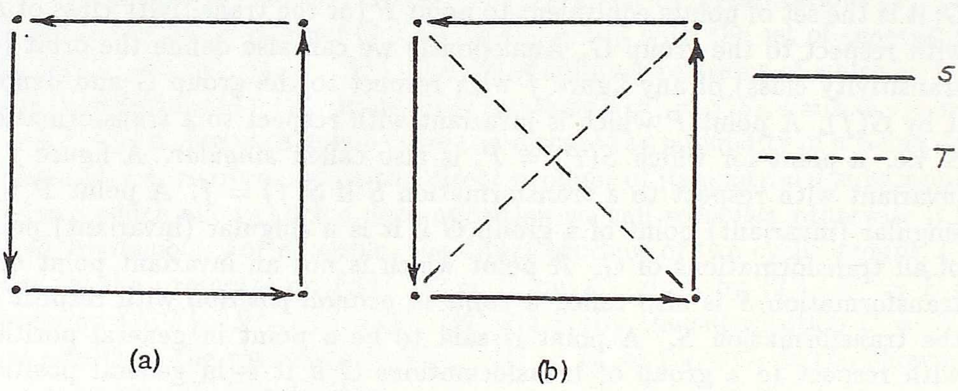


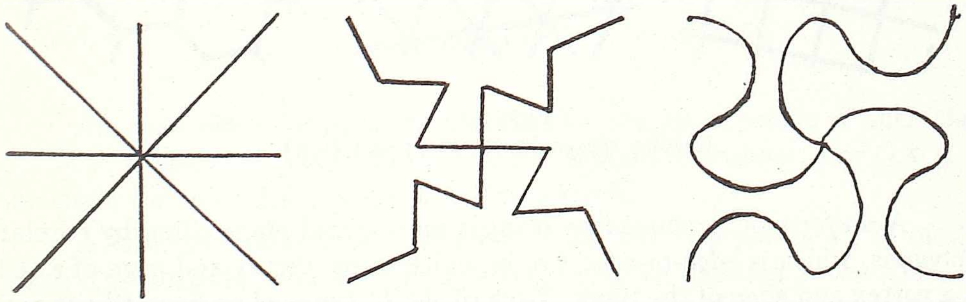
Figure 1.2

(a) Graph of the group C_4 given by the presentation $\{S\}$ $S^4 = E$;
(b) the complete graph of the same group.

For a discrete group G it is possible to define a *fundamental region* of G. A fundamental region F is a figure which satisfies the following conditions:

a) for each point P of the space where the group of transformations G acts, there exists $S \in G$ that $P \in S(F)$;

b) for each $S \in G \setminus \{E\}$ holds $int(F) \cap int(S(F)) = \emptyset$. If $Cl(F)$ is the closure of F, the orbit $G(Cl(F))$ represents a tiling of the space on which the group G acts. A space *tiling* or *tessellation* is a countable family of closed sets $T = \{T_1, T_2, \ldots\}$ covering space without gaps or overlaps. More explicitly, the union of the sets $T_1, T_2, \ldots$, which are known as the *tiles* of T, is to be the whole space, and the interiors of the sets T_i are to be pairwise disjoint (B. Grünbaum, G.C. Shephard, 1987). Since a fundamental region F has no points which are equivalent under any transformation of the group G, unless they are on the boundary, each internal point of F is a point in general position with respect to the group G. Regarding the extent of the fundamental region we distinguish between groups with bounded and unbounded fundamental regions. A discrete group of transformations G usually does not determine uniquely the fundamental region, or the induced tiling $G(Cl(F))$. Therefore, it is of interest to inquire about the different possible shapes of the fundamental region. In the tiling $G(Cl(F))$ the intersection of tiles of any finite set of tiles (containing at least two distinct tiles) may be empty or may consist of a set of isolated points (vertices) and arcs (edges). When discussing variations of the form of the fundamental region F we distinguish between two aspects of change: the change in the number of vertices and edges of the fundamental region F, and the change of the form of the edges (arcs) themselves in which the number of vertices and edges remains unchanged. As the result of the action of the symmetry groups we have *tile-transitive* or *isohedral tilings*. Their tiles belong to the same class of transitivity $G(Cl(F))$, since for every two tiles of $G(Cl(F))$ there exists a transformation of group G which maps one tile onto the other (Figure 1.3).



(a) (b)

Figure 1.3

(a) Isohedral plane tiling corresponding to the symmetry group D_4; (b) two isohedral plane tilings with different shape of the fundamental region, corresponding to its rotational symmetry subgroup C_4.

If the symmetry group G_T contains also transformations which map any vertex of tiling T onto any other vertex, i.e. if the vertices make up one class of transitivity, the tiling is said to be isogonal. By a *flag* in a tiling we mean a triple (V, E, T) consisting of a vertex V, an edge E and a tile T which are mutually incident. A tiling is called *regular* if its symmetry group is transitive on the flags of the tiling. In particular, for the symmetry groups of ornaments there exist exactly three regular tilings (regular tessellations) by means of regular polygons. Each of them can be denoted by a *Schläfli symbol* $\{p, q\}$ denoting regular p-gons, where q of them are incident with each vertex of the regular tessellation: $\{4,4\}$, $\{3,6\}$, $\{6,3\}$. A *dual* of regular tiling $\{p, q\}$ is the regular tiling $\{q, p\}$ (Figure 1.4).

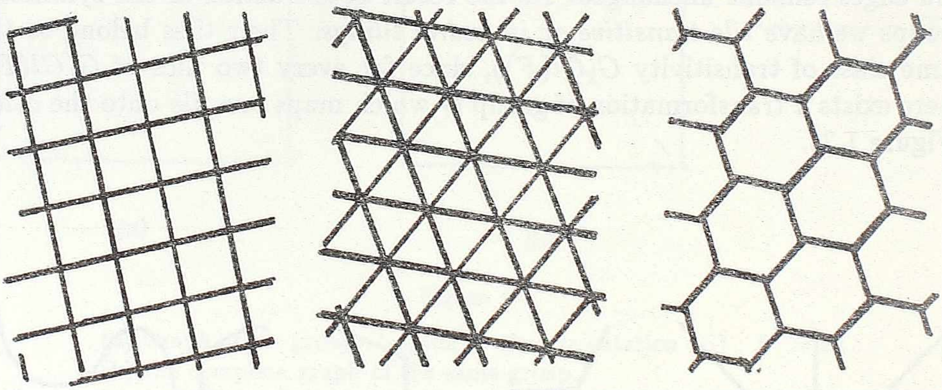


Figure 1.4
Regular tilings $\{4,4\}$, $\{3,6\}$ and $\{6,3\}$.

A *uniform* or *Archimedean tiling* is an isogonal plane tiling by regular polygons, which is *edge-to-edge*, i.e. in which every vertex and edge of a tile is a vertex and edge of the tiling. Each of the 11 types of uniform tilings can be denoted by the symbol $(p_1^{q_1} p_2^{q_2} \ldots p_n^{q_n})$ where $p_1, p_2, \ldots, p_n$ denote regular p-gons, and $q_1, q_2, \ldots, q_n$ the number of adjacent regular p-gons of the same type which are incident with one vertex. Besides regular tessellations $(3^6) = \{3,6\}, (6^3) = \{6,3\}$ and $(4^4) = \{4,4\}$ the family of uniform tilings consists of $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), $(4.6.12)$ and (4.8^2) (J. Kepler, 1619) (Figure 1.5). The Archimedean tiling $(3^4.6)$ occurs in two enantiomorphic forms — "left" and "right".

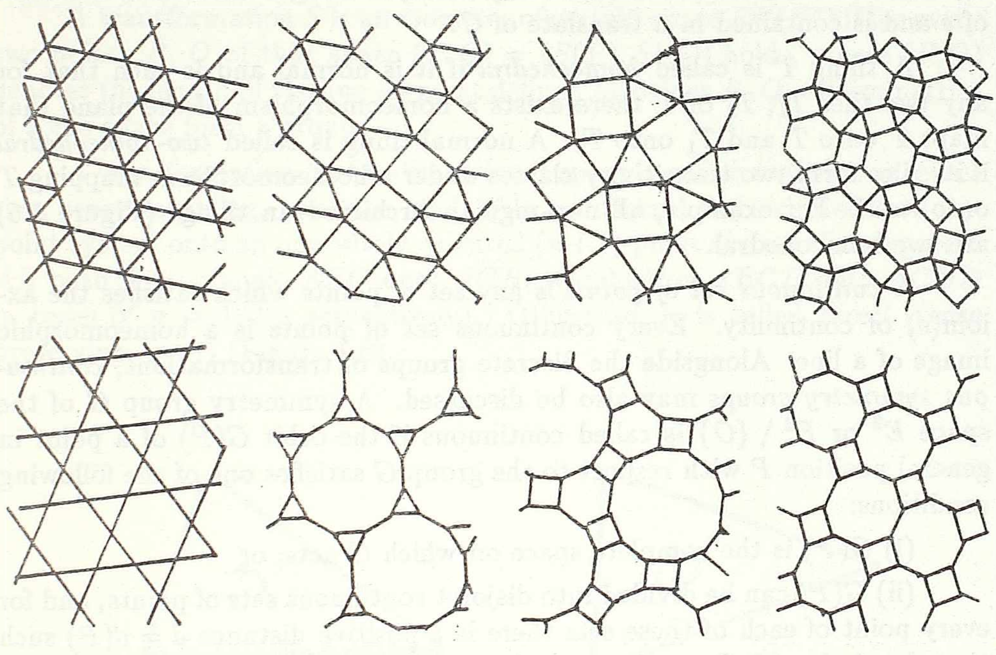


Figure 1.5

An *open circle* (or open circular disk) is the set of points X such that $OX < r$, where O is a fixed point and r is a positive number. For $OX \leq r$, the circle (circular disk) is called a *closed circle*.

A transformation t is *continuous* if for any two points P, Q of the plane it is possible to make $t(P)$ and $t(Q)$ as close together as we wish, by taking P and Q sufficiently close, and *bicontinuous* if both t and t^{-1} are continuous. A *homeomorphism* or *topological transformation* is any bicontinuous transformation. The open (closed) *topological disk* is any plane set which is homeomorphic image of an open (closed) circle.

A tiling T is *normal* if:

a) every tile of T is a topological disk;

b) the intersection of every two tiles of T is a connected set, i.e. does not consist of two closed and disjoint subsets;

c) the tiles of T are uniformly bounded, i.e. there exist circles c and C, with fixed radiuses, such that every tile T_i of tiling T contains a translate of c and is contained in a translate of C.

A tiling T is called *homeohedral* if it is normal and is such that for any two tiles T_1, T_2 of T there exists a homeomorphism of the plane that maps T onto T and T_1 onto T_2. A normal tiling is called *two-homeohedral* if its tiles form two transitivity classes under a homeomorphism mapping T onto itself. For example, all non-regular Archimedean tilings (Figure 1.5) are two-homeohedral.

A *continuous set of points* is any set of points which satisfies the axiom(s) of continuity. Every continuous set of points is a homeomorphic image of a line. Alongside the discrete groups of transformations, *continuous symmetry groups* may also be discussed. A symmetry group G of the space E^2 or $E^2 \setminus \{O\}$ is called continuous if the orbit $G(P)$ of a point in general position P with respect to the group G satisfies one of the following conditions:

(i) $G(P)$ is the complete space on which G acts; or

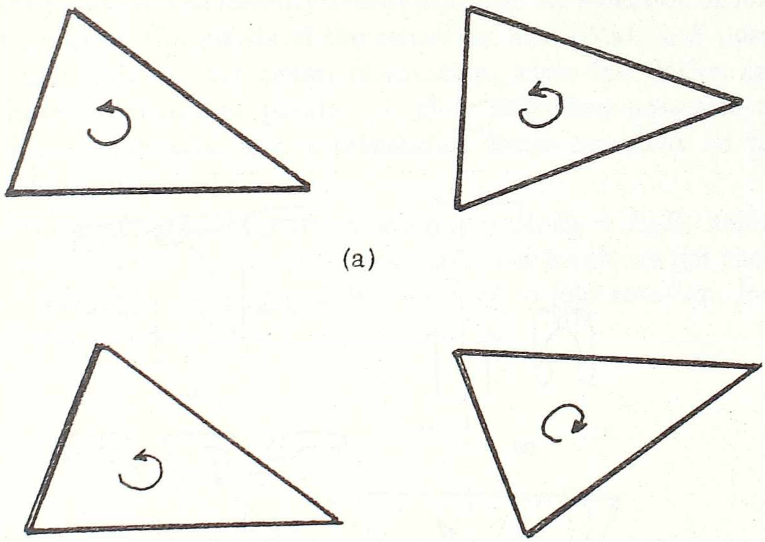
(ii) $G(P)$ can be divided into disjoint continuous sets of points, and for every point of each of these sets there is a positive distance $d = d(P)$ such that the circle $c(P, d)$ contains no points of any other of the sets mentioned. By the terms "continuous group of translations, rotations, central dilatations and dilative rotations" we mean that all translations along one line, all rotations around one center, all central dilatations with a common center, and all dilative rotations with a common center and with a fixed angle, are elements of such a group. In particular, the continuous symmetry groups of ornaments, depending on whether they satisfy condition (i) or (ii), are called the symmetry groups of *continua* or *semicontinua*.

1.3. Classification of Symmetry Transformations and Groups

As the basis for the *classification of the symmetry groups* G three elements were taken into consideration: the types of symmetries (isometries, similarity symmetries, conformal symmetries) that occur in G, the space on which the group G acts, and the sequence of maximal included proper subspaces, invariant with respect to the group G. According to this, the *Bohm symbols* (J. Bohm, K. Dornberger-Schiff, 1966) are used for the categorization of the groups of isometries. Symbols of the same type are applied to the similarity symmetry and conformal symmetry groups. For example, the symmetry group of square $\mathbf{D}_4$ acts in plane and possesses only one invariant point, so it belongs to the category G_{20} — the symmetry groups of rosettes.

A transformation S is an *isometry* of certain space E^n (S^n) if for every two points P, Q of that space $(P,Q) = (S(P), S(Q))$ holds, where (P,Q) denotes the length of the line segment defined by points P, Q. All isometries of some space form a group.

A transformation S of n-dimensional space is called *indirect* (or reflective, sense reversing, opposite, odd) if it transforms any oriented $(n+1)$-point system onto an oppositely oriented $(n+1)$-point system (line segment AB onto BA, triangle ABC onto ACB, tetrahedron $ABCD$ onto $ACBD$ in cases of $n = 1, 2, 3$ respectively). Otherwise, it is called *direct* (sense preserving, even) (Figure 1.6).



(a)

(b)

Figure 1.6
(a) Direct and (b) indirect plane isometry.

As an elementary isometric transformation we can take the *reflection*, non-identical isometry of space E^n (S^n) for which, every point of its subspace E^{n-1} (S^{n-1}) is an invariant point. In particular, for $n = 1$ we have *point reflection*, for $n = 2$ *line reflection* (or simply — reflection), for $n = 3$ *plane reflection*, involutional indirect isometries. According to the fundamental theorem on *minimal* or *canonic representation* of an isometric transformation of space E^n (S^n), which states that every isometry of this space can be presented as a composition of maximum $n + 1$ (plane) reflections, it is possible to classify the isometries of different spaces.

The classification of isometric transformations and corresponding symmetry groups is common for spaces E^n, S^n, L^n for $n < 2$, while for $n \geq 2$ different possibilities of relations of disjoint lines, which are defined by the axiom of parallelism, condition specific differences. This work exclusively discusses Euclidean spaces.

In the space E^2 (plane) we distinguish the following isometric transformations (Figure 1.7):

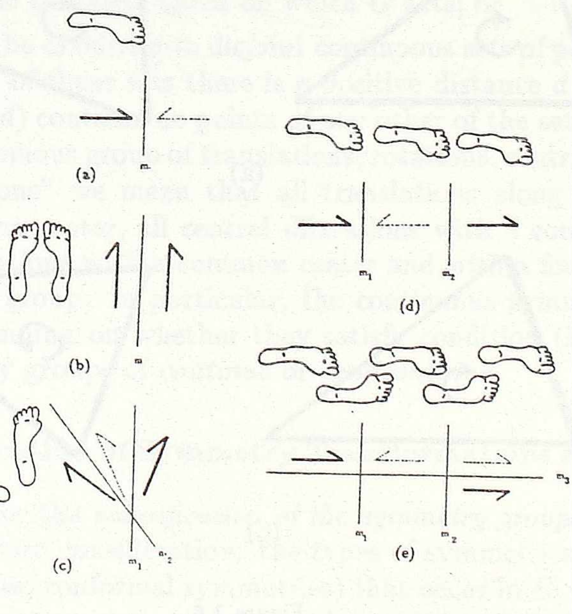


Figure 1.7

(a) Identity transformation; (b) reflection; (c) rotation; (d) translation; (e) glide reflection.

1) *identity transformation* E, with the minimal reflectional representation of the length 2 ($R^2 = E$);

2) *reflection* R;

3) *rotation* $S = R_1 R_2$, the product of two reflections in the reflection lines crossing in the invariant point (center of rotation). The oriented angle of rotation is equal to twice the angle between the reflection lines R_1, R_2;

4) *translation* $X = R_1 R_2$, the product of two reflections with parallel reflection lines, such that the translation vector is perpendicular to them and equal to twice the oriented distance between the reflection lines R_1, R_2;

5) *glide reflection* $P = R_3 X = X R_3 = R_1 R_2 R_3$, the commutative product of a translation X and a reflection R_3 with the reflection line parallel to the translation axis.

With respect to the invariant figures, all the points of the plane E^2 are invariant points of the identity transformation E, reflection R maintains the invariance of all the points of the reflection line, rotation S possesses a single invariant point — the center of rotation, while translation and glide reflection have no invariant points. A glide reflection possesses a single invariant line — the axis, and a translation keeps invariant all the lines parallel to the translation axis.

In the case of rotation, if the relation $S = R_1 R_2 = R_2 R_1$ holds, i.e. if the reflection lines R_1, R_2 are perpendicular, as a result we get the special involutional rotation — *central reflection* Z (two-fold rotation, half-turn, point-reflection) (Figure 1.8).

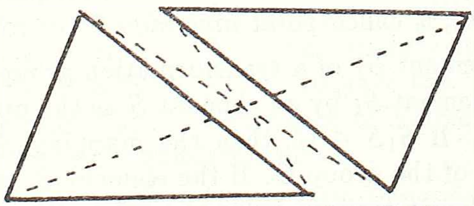


Figure 1.8
Central reflection Z.

When orientation is considered we distinguish *direct transformations* (or sense preserving transformations): identity transformation E, translation, rotation, and *indirect transformations* (sense reversing transformations): reflection and glide reflection. Since direct transformations are the product of an even, and the indirect ones of an odd number of reflections, we can call them respectively *even* and *odd transformations.*

If a symmetry transformation S can be represented as a composition $S = S_1 \ldots S_n$ such that $S_iS_j = S_jS_i$, $i,j = 1,\ldots,n$, we can call it a *complex* or *composite transformation* while the transformations $S_1,\ldots,S_n$ we call the *dependent transformations* or dependent elements of symmetry. We will use such approach whenever we are interested to learn to what degree the dependent elements of symmetry influence the characteristics of the composite transformation, and whether they have lost or preserved their geometric and visual characteristics during it. For example, a glide reflection is such commutative composition of translation and reflection, with reflection line parallel to the translation axis.

An analogous procedure makes possible the classification of isometries of the space E^3, where each isometry can be represented as the composition of four plane reflections at the most. Besides the transformations of the space E^2 afore mentioned with the line reflections substituted by plane reflections, as the new transformations of the space E^3 we have two more transformations. They are a direct isometry — *twist* (screw), the commutative composition of a rotation and a translation, the canonic representation of which consists of four plane reflections and indirect isometry — *rotatory reflection*, the commutative composition of a rotation and a plane reflection in the plane perpendicular to the rotation axis, the canonic representation of which consists of three plane reflections. In particular, the involutional rotatory reflection, which is the composition of three plane reflections of which every two commute, is called *point inversion* Z (or rotatory inversion).

For every element S_1 of a transformation group G we can define the *conjugate* of the element S_1 by an element S as the product $S^{-1}S_1S$, which we denote by S_1^S. If $S_1S \in G$, then the mapping S_1 onto S_1^S represents an automorphism of the group G. If the element S by means of which this automorphism is being realized belongs to the group G, such an automorphism is called an *internal automorphism.* Any other automorphism of a group G is called an *external automorphism.* An important characteristic of a conjugate is that the order of the conjugate S_1^S is equal to the order of the element S_1. If a figure f is invariant under a transformation S_1, then $S(f)$ is the figure invariant under transformation S_1^S. The conjugate of a reflection R with invariant reflection line p, derived by isometry S, is the reflection R^S with the invariant reflection line $S(p)$. Hence we can conclude that the isometry S_1 and all its conjugates S_1^S derived by different isometries S constitute one *class of equivalence*, i.e. the class of isometries having the same name, which means that (internal) automorphism of a group of isometries G transforms reflections onto reflections, rotations onto rotations, etc. The properties of the (internal) automorphisms are frequently used when proving theorems on isometric transformations and the other symmetry transformations. For example, by g^R, $g \in G$, is defined an internal automorphism of the symmetry group of square G, given by presentation $\{S, R\}$ $S^4 = R^2 = (RS)^2 = E$: $E^R = E$, $R^R = R$, $(RS)^R = SR$, $(SR)^R = RS$, $S^R = S^3$, $(S^2)^R = S^2$, $(S^3)^R = S$. In the same way, it is defined an external automorphism of the rotational group of square H, given by presentation $\{S\}$ $S^4 = E$: $S^R = S^3$, $(S^2)^R = S^2$, $(S^3)^R = S$, where the reflection line of reflection R contains the center of four-fold rotation S. Hence, external automorphisms are very efficient tool for extending symmetry groups.

Since the product of direct transformations is a direct transformation, and the inverse of a direct transformation is a direct transformation, each group of transformations G, which contains at least one indirect transformation has a subgroup of the index 2, denoted by G^+, which consists of direct transformations of the group G. For example, the rotational subgroup of sqare H satisfies this condition regarding the symmetry group of square, so $H = G^+$, $[G : H] = 2$. All direct isometries of the space E^n can be identified as movements of a material object in the space E^n, as opposed to indirect isometries which do not have such a physical interpretation (e.g., a plane reflection does not represent motion in E^3).

For a figure f with the symmetry group G_f, which consists only of direct symmetries, it is possible to have the *enantiomorphism* — *enantiomorphic modifications* of a figure f, i.e. to have the "left" and "right" form of the figure f (Figure 1.9). The existence of indirect symmetries of a figure f implies the absence of enantiomorphism.

Since reflections have a role of elementary isometric transformations, while all other isometries are their finite compositions, of special interest will be symmetry *groups generated by reflections* — groups, a set of generators of which consists exclusively of reflections. Since every reflection keeps invariant each point of the reflection line, the fundamental region of these groups will possess a fixed shape, will not allow variations and will have rectilinear edges. All symmetry groups will be subgroups of groups generated by reflections. In the case of conformal symmetry groups, along with reflections, circle inversions have the analogous function. For example, the symmetry group of square is the group generated by reflections, with the fundamental region of the fixed shape (Figure 1.3a).

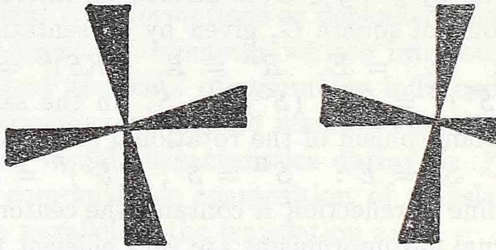


Figure 1.9
"Left" and "right" rosette with the symmetry group C_4, consisting of direct symmetries.

The results of composition of plane isometries are different *categories of groups* of isometries of the space E^2, represented by Bohm symbols as: G_{210} — symmetry groups of *finite friezes*, G_{20} — symmetry groups of *rosettes*, G_{21} — symmetry groups of *friezes* and G_2 — symmetry groups of *ornaments*. Because of the relation $G_{210} \subset G_{20}$, in this work we will discuss only the categories G_{20}, G_{21}, G_2, while the category G_{210} will be discussed within the category G_{20}. The *definitions of symmetry groups* will be derived directly from Bohm symbols: symmetry groups of rosettes are groups of isometries of the space E^2 (plane) with an invariant 0-dimensional subspace (point), symmetry groups of friezes are groups of isometries of the space E^2 with an invariant 1-dimensional subspace (line) and without invariant points, while symmetry groups of ornaments are groups of isometries of the space E^2 without invariant subspaces (points, lines). The groups of the category G_n are called the *space groups*, the groups of the category G_{n1} the *line groups*, and the groups of the category G_{n0} the *point groups* of the space E^n. With symmetry groups of friezes G_{21} and symmetry groups of ornaments G_2, a group contains one or two generating translations respectively, so that each of these groups has a *translational subgroup.* A *lattice* is the orbit of a point with respect to a discrete group of translations. For the friezes it is a linear series of equidistant points while for ornaments we get a *plane lattice* or simply a lattice. Five different symmetry types of plane lattices bear the name of *Bravais lattices*; the points of these lattices are defined by five different isohedral tessellations, which consist of parallelograms, rhombuses, rectangles, squares or regular hexagons. To Bravais lattices correspond the *crystal systems* of the same names (Figure 1.10).

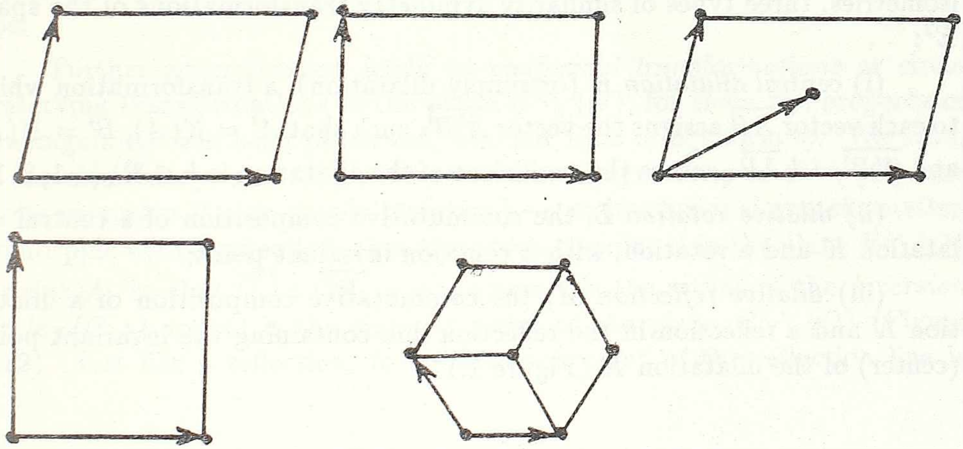


Figure 1.10
Five plane Bravais lattices.

Because the symmetry groups of friezes G_{21} are groups of isometries of the plane E^2 with an invariant line, they cannot have rotations of an order greater than 2.

For the symmetry groups of ornaments G_2 so-called *crystallographic restriction* holds, according to which symmetry groups of ornaments can have only rotations of the order n=1,2,3,4,6. The term "*crystallographic groups*" is used for all groups which satisfy this condition, despite the category they belong to.

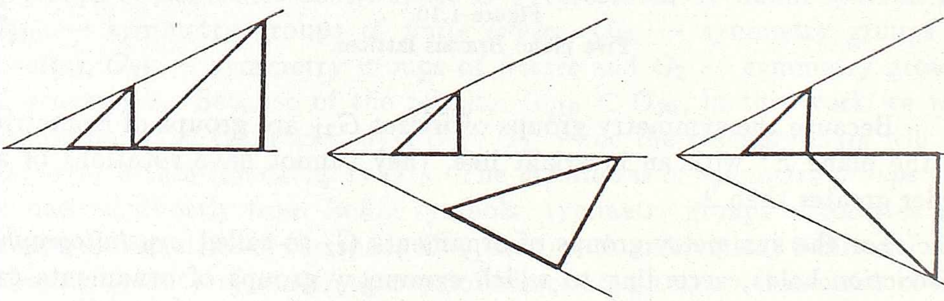
In isometry groups all distances between points under the effect of symmetries remain unchanged and the *congruence* of homologous figures is preserved. Consequently, the same holds for all other geometric properties of such figures, so that the *equiangularity* (the congruence of the angles of homologous figures) and their *equiformity* (the same form of homologous figures) are the direct consequences of isometrism.

The next class of symmetry groups we shall consider are the *similarity symmetry groups*. A *similarity transformation* of the space E^n is a transformation which to each line segment of length AB assigns a line segment of the length kAB whereby k is a real positive number, the *coefficient of similarity*. In particular, if $k = 1$ then a similarity transformation is an isometry. According to the theorem on the existence of an invariant point of every similarity transformation which is not an isometry, there are, besides isometries, three types of similarity symmetry transformations of the space E^2:

(i) *central dilatation* K (or simply dilatation), a transformation which to each vector $\overrightarrow{AB}$ assigns the vector $\overrightarrow{A'B'}$, such that $A' = K(A), B' = K(B)$ and $\overrightarrow{A'B'} = k\overrightarrow{AB}$, where the coefficient of the dilatation is $k \in \Re\backslash\{-1, 0, 1\}$;

(ii) *dilative rotation* L, the commutative composition of a central dilatation K and a rotation, with a common invariant point;

(iii) *dilative reflection* M, the commutative composition of a dilatation K and a reflection in the reflection line containing the invariant point (center) of the dilatation K (Figure 1.11).



(a) (b) (c)

Figure 1.11

(a) Dilatation; (b) dilative rotation; (c) dilative reflection.

Those transformations are, in the given order, isomorphic with the isometries of the space E^3: translation, twist and glide reflection. They make possible the extension of the symmetry groups of rosettes G_{20} by the external automorphism, having as the result *similarity symmetry groups* S_{20} that we will, thanks to the existence of the invariant point, call the *similarity symmetry groups of rosettes.*

Dilatations K and dilative rotations L are direct, while dilative reflections M are indirect transformations. They all possess the properties of equiangularity and equiformity. All other aspects of similarity symmetry groups (the problems of enantiomorphism, fundamental regions, tessellations,...) will be discussed analogously to the case of isometry groups.

Further generalization leads to *conformal transformations* or circle preserving transformations of the plane $E^2 \setminus \{O\}$; for them the property of equiangularity has been preserved, but not that of equiformity. We have, as the elementary transformation of conformal symmetry in $E^2 \setminus \{O\}$, the *circle inversion* R_I (or simply inversion) — an involutional transformation isomorphic with a reflection, that gives to each point A in the plane $E^2 \setminus \{O\}$ a point A_1 so that $\overrightarrow{OA} \circ \overrightarrow{OA_1} = r^2$, where r is the radius of the *inversion circle* $c(O,r)$ and O is the singular point of the plane $E^2 \setminus \{O\}$ (Figure 1.12). Just like a reflection, for which each point of the reflection line is

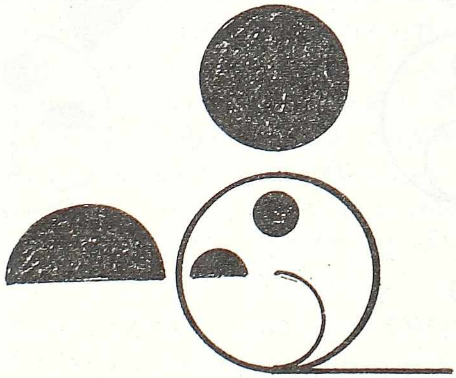


Figure 1.12
Circle inversion.

invariant, an inversion maintains invariant each point of the inversion circle. By discussing a line as a circle with an infinite radius (and treating as circles, at the same time and under the same term, circles and lines) it is possible to identify reflections with circle inversions. In such a case, all circle inversions (including line reflections) and their compositions, can be discussed as *circle preserving transformations*, i.e. transformations mapping circles (including lines) onto circles.

Besides the circle inversion R_I, by composing it with isometries maintaining invariant the circle line c of the inversion circle $c(O,r)$ — with a reflection with reflection line containing the circle center O or with a rotation with the rotation center O, we have two more conformal transformations: (i) *inversional reflection* $Z_I = R_I R = R R_I$, the involutional transformation, the commutative composition of a reflection and a circle inversion; (ii) *inversional rotation* $S_I = S R_I = R_I S$, the commutative composition of a rotation and a circle inversion (Figure 1.13).

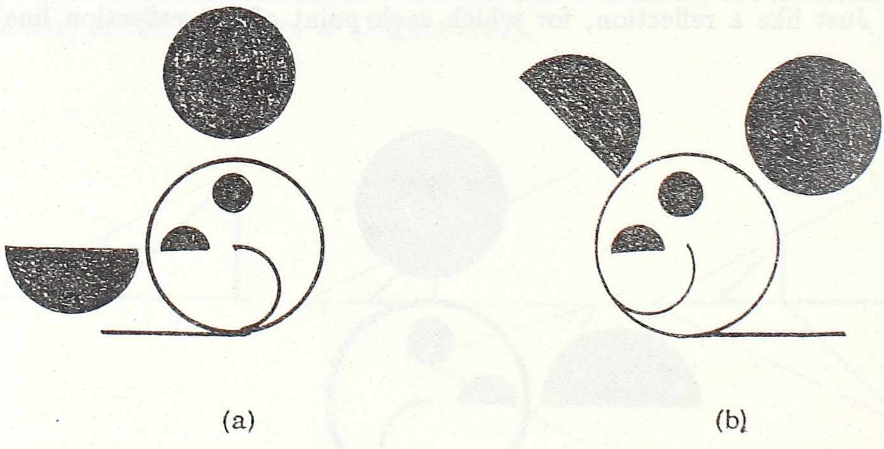


Figure 1.13
(a) Inversional reflection; (b) inversional rotation.

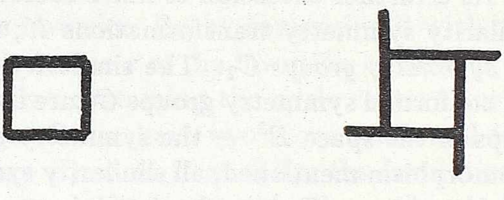
Those three conformal symmetry transformations, besides isometries and similarity symmetry transformations, constitute the finite and infinite *conformal symmetry groups* C_{21} ,C_2 — *conformal symmetry groups of rosettes* in $E^2 \setminus \{O\}$.

As an extension of the symmetry groups of rosettes G_{20} we have the *finite conformal symmetry groups* C_{21} isomorphic with the symmetry groups of *tablets* G_{320}. As a further extension of finite conformal symmetry groups C_{21} by the similarity symmetry transformations K, L, M, we get the *infinite conformal symmetry groups* C_2. The similarity symmetry groups S_{20} and the infinite conformal symmetry groups C_2 are isomorphic with the line symmetry groups of the space E^3 — the symmetry groups of *rods* G_{31}. In line with the isomorphism mentioned, all similarity symmetry and conformal symmetry transformations offer a reflectional (canonic) representation by, at most, four reflections (reflections and circle inversions). By applying this isomorphism, ornamental motifs which correspond to the similarity symmetry and conformal symmetry groups, satisfy one more scope of painting: adequate interpretation of space objects in the plane. The plane structures obtained are called *generalized projections* of the symmetry groups of tablets G_{320} and rods G_{31}.

1.4. Visual Interpretations of Symmetry Groups

All discrete symmetry groups can be visually modeled by adequate ornamental motifs (patterns, tilings...) which for centuries have been an important part of applied art. Besides different symbolic or schematic visual interpretations of symmetry groups (such as Cayley diagrams, *tables of graphic symbols of symmetry elements*, where the different symmetry transformations are denoted by graphic symbols: rotations by oriented regular polygons, reflections by full lines, glide reflections by dotted ones, etc.), they are an important aspect of "imaginative geometry" ("anshauliche geometrie" of D. Hilbert) — geometry of everyday life, and its relations with art (Figure 1.14).

When trying to translate the meanings of geometric properties of symmetry transformations and of symmetry groups into the visual sphere, one can note the links between the geometric-algebraic properties of transformations and the different visual parameters (stationariness, dynamism,...). A survey of geometric characteristics and their visual interpretations, which are relevant for such a study, is given in each chapter of this book. Alongside the elements already mentioned: presentations of symmetry groups, Cayley diagrams, data on enantiomorphism, also the *orientability*: *polarity*, *non-polarity* and *bipolarity* of different lines, invariants of symmetry transformations will be discussed.



(a) (b)

Figure 1.14

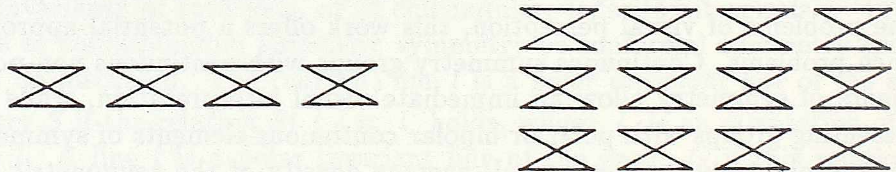
Graphic notation of the symmetry groups (a) $\mathbf{D}_4$; (b) $\mathbf{C}_4$.

In addition to the orientability of invariant lines — *axes*, also the orientability of *radial rays* — half-lines invariant to some dilatation K, of circles — invariants of rotations, or of equiangular, logarithmic spirals — invariants of corresponding similarity symmetry or conformal symmetry groups will be discussed. The (curved) line l is a *polar invariant line* of the symmetry S if the relation $S(\overrightarrow{l}) = \overrightarrow{l}$ holds, where $\overrightarrow{l}$ is an orientation of the line l. A line l is a polar invariant line of the group G if this relation is satisfied for all the elements of the group G. A (curved) invariant line l of the symmetry S is *non-polar* if the relation $S(\overrightarrow{l}) = -\overrightarrow{l}$ is satisfied, where $-\overrightarrow{l}$ denotes the oppositely oriented line $\overrightarrow{l}$. A line invariant with respect to the group G is non-polar if there exists at least one indirect symmetry $S \in G$ which satisfies the condition $S(\overrightarrow{l}) = -\overrightarrow{l}$. A non-polar (curved) line l, invariant of the symmetry S is *bipolar* if S is a direct symmetry. A non-polar (curved) invariant line of the group G is bipolar if the set $S_i = \{S \mid S(\overrightarrow{l}) = -\overrightarrow{l}, S \in G\}$ contains only direct transformations.

In the visual sense, the term "polarity" can be connected with the dynamism of ornamental motifs corresponding to discrete symmetry groups. For continuous symmetry groups it is immediately linked with the term *visual presentability*. As opposed to the discrete symmetry groups which can always be visually interpreted by means of ornamental motifs, the continuous symmetry groups will not always offer an adequate visualization. So, for example, for the continuous line group of translations, its visualization is not possible without introducing a supplementary symbol (e.g., the symbol $\Longrightarrow$, a suitable symbol of such a translation). Not having a previous agreement or convention about its meaning, it is not possible to give to the observer an adequate visual interpretation of this symmetry group, comprehensible without further explanation. Under the term "visual presentability" we understand the visual modeling of symmetry groups which offers to observer the complete visual information on the observed symmetry (in the sense of objective, geometric symmetry), without needing for an additional explanation. In visual arts, apart from the objective, geometric symmetry, very important are the effects of "visual forces" (R. Arnheim, 1965, 1969). They are, for example, the upward tendency of a vertical line, the visual effect of the "ascending" and "descending" diagonal, the "left" and "right" orientation. These subjective visual factors, having a great influence on the visual perception of symmetry and representing a subject of study in the psychology of visual perception, are not discussed under the term "visual presentability", which refers only to objective, geometric symmetry and its visual perception. Although a detailed analysis of the subjective, visual factors of symmetry is omitted, mainly because of the complexity of the problems of visual perception, this work offers a potential approach to such problems. Continuous symmetry groups with continuous non-polar elements of symmetry allow an immediate visual interpretation, while for representing groups with polar or bipolar continuous elements of symmetry we can apply *textures* — an equal, average density of the asymmetric figures arranged along the invariant polar or bipolar line, in accordance with the given continuous symmetry group (A.V. Shubnikov, N.V. Belov et al., 1964). So, for example, continuous line group of translations can be interpreted by means of textures as the series ,, , ,,,, ,,, ,,, , ,, In contrast to physical interpretations of all continuous symmetry groups which can be obtained by motion or some other physical effect, the domain of the visual presentability of continuous symmetry groups, if textures are not applied, is limited by the objective stationariness of ornamental art works to the continuous groups with non-polar continuous elements of symmetry.

1.5. Construction Methods. Desymmetrizations.

By considering and comparing the development of construction methods for the derivation of ornamental structures in art and geometry, one can note a few common approaches. After considering regularities on which the simplest ornamental motifs (rosettes, friezes) are based, mostly on originals existing in nature, and after discovering the first elementary constructions, a way was opened for the creation of ornamental motifs. This was usually achieved beginning from "*local symmetry*" — from the one fundamental region and regularly arranged neighboring fundamental regions, and resulting in the "*global symmetry*" — complete ornamental filling in of the plane. Such a procedure represents, in fact, a series of extensions and dimensional transitions, leading directly or indirectly from the point groups — the symmetry groups of rosettes G_{20}, over the line groups — the symmetry groups of friezes G_{21}, to the plane groups — the symmetry groups of ornaments G_2. In such a case, substructures (rosettes, friezes) are called *generating substructures* (Figure 1.15). A similar procedure can be traced for the similarity symmetry groups S_{20} and conformal symmetry groups C_{21} and C_2, derived as extensions of isometric point groups — the symmetry groups of rosettes G_{20}.



(a) (b)

Figure 1.15

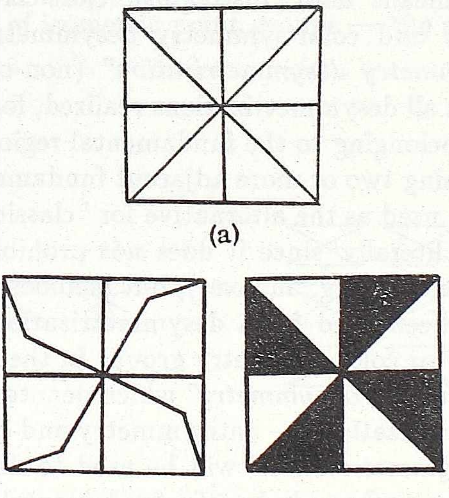
Derivation of (a) frieze mm; (b) ornament pmm from generating rosette with the symmetry group $\mathbf{D}_2$.

However, an almost equal role in the formation of different ornamental motifs belongs to the reverse *desymmetrization* procedure, a way which mainly leads from the *maximal symmetry groups* generated by reflections, characterized by a high degree of visual and constructional simplicity, to their subgroups. The results obtained are subgroups belonging to the same category as a group undergoing desymmetrization, or its subgroups with invariant subspace(s) of lower dimension(s) (e.g., the symmetry groups of friezes G_{21} as line subgroups of the symmetry groups of ornaments G_2).

At first restricted to the maximal groups of symmetry generated by reflections, to the regular tessellations or Bravais lattices, the desymmetrization method in painting becomes in time, firstly thanks to the use of colors, an efficient procedure for deriving all symmetry groups as subgroups of wider groups. Under the term "desymmetrization" of certain symmetry group we understand this as the procedure beginning with the elimination of corresponding symmetries and resulting in the derivation of certain subgroup H of the given group. In line with this, every desymmetrization is defined by the group G and its subgroup H, i.e. by the group/subgroup symbol G/H. The reverse procedure, resulting in some supergroup of the given group G, is called *symmetrization* (group extension).

Within the desymmetrization method, we can, depending on the desymmetrization means used, distinguish classical-symmetry (non-colored), antisymmetry and color-symmetry desymmetrizations. Under the term "*classical-symmetry desymmetrization*" (non-colored desymmetrization) we will discuss all desymmetrizations realized, for example, by using an asymmétric figure belonging to the fundamental region, or by deleting their boundaries and joining two or more adjacent fundamental regions, etc. The term "non-colored" used as the alternative for "classical-symmetry", should not be understood literally, since it does not prohibit the use of colors or some of their equivalents (e.g., indexes), but includes as well, all other cases where colors have been used for a desymmetrization without resulting in some antisymmetry or color-symmetry group. In the same sense we will use the term "*classical theory of symmetry*" which denotes the theory of symmetry without its generalizations — antisymmetry and colored symmetry. The term "*external desymmetrization*" will be used to denote a desymmetrization achieved by varying boundaries of a fundamental region (Figure 1.16b).

Let e_1 be an *antiidentity transformation* which satisfies the relations: $e_1^2 = E \quad e_1 S = S e_1$, where S is any symmetry transformation. The transformation $S' = e_1 S$ is then called an *antisymmetry transformation.* As the interpretation of the transformation e_1, it is possible to accept the alternating change of any bivalent quality, geometric or not, which commutes with symmetries, e.g., the color change black-white, change of electricity charges +, -, etc. A group which besides symmetry transformations contains antisymmetry transformations is called an *antisymmetry group.* As the basis for deriving antisymmetry groups we take some symmetry group G which we call a *generating group of antisymmetry* (or simply a generating group). By replacing the symmetries (generators) of the group G by antisymmetries (antigenerators) we obtain, as a result, an antisymmetry group G' which, depending upon whether the antiidentity transformation e_1 is the element of the group G' or not, is called a *senior* (neutral, gray) or a *junior* (black-white) *antisymmetry group* respectively. Every senior antisymmetry group has the form $G' = G \times \{e_1\} = G \times C_2$, where the group generated by e_1 is denoted by $\{e_1\}$. All junior groups are isomorphic with their generating group G. Every junior antisymmetry group is uniquely defined by the generating group G and by its subgroup H of the index 2. From there originated the group/subgroup symbols G/H of junior antisymmetry groups, where the relationship $G/H \cong C_2$ holds (Figure 1.16c). Since all (normal) subgroups of the index 2 of the generating group G can be obtained knowing junior



(a)

(b) (c)

Figure 1.16

(a) Generating rosette with the symmetry group D_4; (b) its external desymmetrization D_4/C_4; (c) antisymmetry group D_4/C_4.

antisymmetry groups derived from G, antisymmetry is included in the desymmetrization method. Besides a large field of application in Physics, various interpretations of the antiidentity transformation as a geometric transformation which commutes with all the symmetries of the generating group, make possible the dimensional transition from the symmetry groups of the n- dimensional space to those of the $(n+1)$-dimensional space. For example, the symmetry groups of *bands* G_{321} can be derived by using antisymmetry from the symmetry groups of friezes G_{21}, the symmetry groups of *layers* G_{32} from the symmetry groups of ornaments G_2, etc. Corresponding black-white antisymmetry plane motifs (so-called *Weber diagrams* or *antisymmetry mosaics*) can be understood as adequate visual interpretations of the symmetry groups of bands G_{321} or layers G_{32}, where the transformation e_1 — color change black-white is identified with the plane reflection in the invariant plane of the generating frieze or ornament (Figure 1 .17).

The first antisymmetry ornamental motifs are found in Neolithic ornamental art with the appearance of two-colored ceramics and for centuries have represented a suitable means for expressing the dualism, internal dynamism, alternation, with a distinct space component — a suggestion of the relationships "in front–behind", "above–below", "base–ground",...

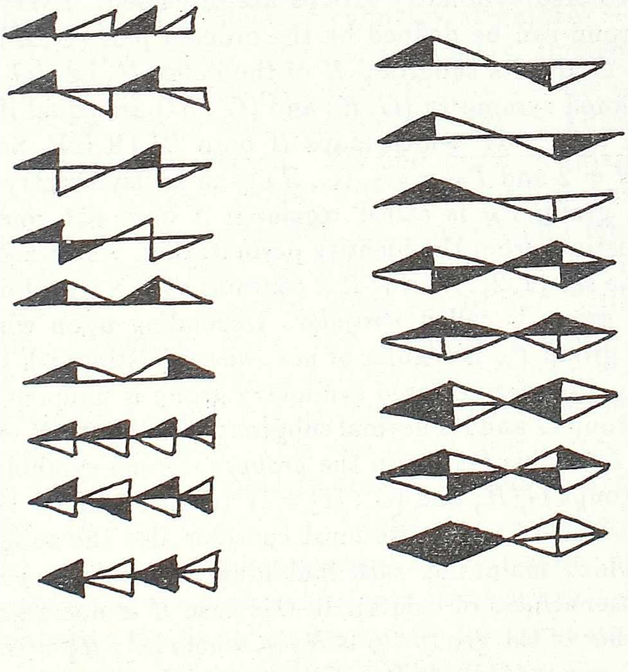
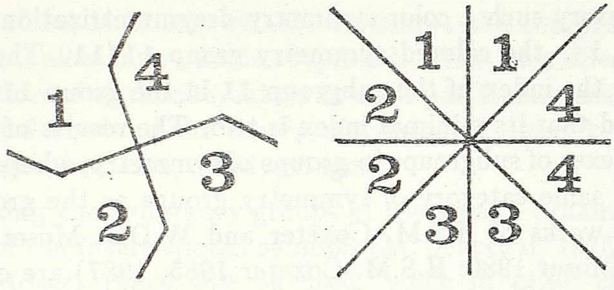


Figure 1.17
Weber diagrams of bands.

The next generalization of antisymmetry is the polyvalent, *colored symmetry* with the number of "colors" $N \geq 3$, where each color is denoted by the corresponding index $1, 2, \ldots, N$. A *permutation* of the set $\{1, 2, \ldots, N\}$ is any one-to-one mapping of this set onto itself. Let P_N be a subgroup of the *symmetric permutation group* S_N (or simply symmetric group), i.e. of the group of all the permutations of the set $\{1, 2, \ldots, N\}$, $c \in P_N$ and $cS = Sc$, where S is a symmetry transformation, an element of the symmetry group G. Then $S^* = cS$ is called a *colored symmetry transformation*. A *color permutation* c can be interpreted as a change of any polyvalent quality which commutes with symmetries $S \in G$. A *colored symmetry group* is a group which besides symmetry transformations contains colored symmetry transformations (or colored symmetries). By analogy to antisymmetry groups, the symmetry group G is called a *generating group of colored symmetry*. The colored symmetry group G^* derived from G is called a *junior colored symmetry group* iff it is isomorphic with G. In this work only junior colored symmetry groups are discussed. Every junior colored symmetry group can be defined by the ordered pair (G, H) which consists of the group G and its subgroup H of the index N, i.e. $[G : H] = N$. Two groups of colored symmetry (G, H) and (G', H') are equal if there exists an isomorphism $i(G) = G'$ which maps H onto H' (R.L.E. Schwarzenberger, 1984). For $N = 2$ and $P_N = C_2$, (G, H) is an antisymmetry group. A color permutation group P_N is called *regular* if it does not contain any transformation, distinct from the identity permutation, which keeps invariant an element of the set $\{1, 2, \ldots, N\}$. If it contains such a transformation, a color permutation group is called *irregular*. Depending upon whether the color permutation group P_N is regular or not, we can distinguish two cases. For a regular group P_N every colored symmetry group is uniquely defined by the generating group G and its normal subgroup H of index N — the symmetry subgroup of G^*. This results in the *group/subgroup* symbols of the colored symmetry groups G/H, and $[G : H] = N$ (Figure 1.18a). For the *irregular group* P_N, besides G and H we must consider also the subgroup H_1 of the group G^*, which maintains each individual index (color) unchanged (i.e. group of stationariness of colors). In this case H is not a normal subgroup of G. The order of the group P_N is NN_1, where $[G : H] = N, [H : H_1] = N_1$ and quotient group $G/H_1 \cong P_N$. To denote such colored symmetry groups, the symbols $G/H/H_1$ are used (Figure 1.18b).



(a) (b)

Figure 1.18
(a) Colored symmetry group C_4/C_1; (b) $D_4/D_1^-/C_1$.

By interpreting "colors" as physical polyvalent properties commuting with every transformation of the generating symmetry group, it is possible to extend considerably the domain of the application of colored ornaments treated as a way of modeling symmetry structures — subjects of natural science (Crystallography, Physics, Chemistry, Biology...). As an element of creative artistic work, although being in use for centuries, colored symmetry can be, taking into consideration the abundance of unused possibilities, a very inspiring region. We find proof of this in the works of M.C. Escher (M.C. Escher, 1971a, b). On the other hand, the various applications of colors in ornaments, e.g., ornamental motifs based on the use of colors in a given ratio, by which harmony — balance of colors of different intensities — is achieved, have yet to find their mathematical interpretation. Accepting "color" as a geometric property, and colored transformations as geometric transformations which commute with the symmetries of the generating group, has opened up a large unexplored field for the theory of colored symmetry. This was made clear in the recent works discussing multi-dimensional symmetry groups, curvilinear symmetries, etc. (A.M. Zamorzaev, Yu.S. Karpova, A.P. Lungu, A.F. Palistrant, 1986).

The results of the theory of antisymmetry and colored symmetry can be used also for obtaining the *minimal indexes of subgroups* in the symmetry groups. As opposed to the finite groups, where for the index of the given subgroup there is exactly one possibility, in an infinite group the same subgroup may have different indexes. For example, considering a frieze with the symmetry group 11, generated by a translation X, and its colorings by $N = 2, 3, 4, \ldots$ colors, where the group of color permutations is the cyclic group C_N of the order N, generated by the permutation $c = (123\ldots N)$, the result of every such a color-symmetry desymmetrization is the symmetry group **11**, i.e. the colored symmetry group **11/11**. Therefore, we can conclude that the index of the subgroup **11** in the group **11** is any natural number N and that its minimal index is two. The results of computing the (minimal) indexes of subgroups in groups of symmetry, where the subgroups belong to the same category of symmetry groups as the groups discussed, based on the works of H.S.M. Coxeter and W.O.J. Moser (H.S.M. Coxeter, W.O.J. Moser 1980; H.S.M. Coxeter 1985, 1987) are completed with the results obtained by using antisymmetry and colored symmetry. They are given in the corresponding tables of (minimal) indexes of subgroups in the symmetry groups. Besides giving the evidence of all subgroups of the symmetry groups, these tables can serve as a basis for applying the desymmetrization method, because the (minimal) index is the (minimal) number of colors necessary to achieve the corresponding antisymmetry and color-symmetry desymmetrization. For denoting subgroups which are not normal, italic indexes are used (e.g., *3*).

It is not necessary to set apart antisymmetry from colored symmetry, since antisymmetry is only the simplest case of colored symmetry ($N = 2$), but their independent analysis has its historical and methodical justification, because bivalence is the fundamental property of many natural and physical phenomena (electricity charges +, -, magnetism S, N, etc.) and of human thought (bivalent Aristotelian logic). In ornamental art, examples of antisymmetry are mainly consistent in the sense of symmetry, while consistent use of colored symmetry is very rare, especially for greater values of N.

1.6. Symbols of Symmetry Groups

When denoting symmetry groups and their generalizations, antisymmetry and colored symmetry groups, we always come across the unpleasant task of trying, at least to some extent, to reconcile and bring to accord the different sources and symbols used in literature. Most of symbols come from the work of crystallographers, some from the mathematicians who were engaged in studies of the theory of symmetry, while some chapters (e.g., that on conformal symmetry) demand the introduction of new symbols. Since only lately there have been attempts to make uniform the symbols of symmetry groups, positive results are mainly achieved with the symmetry groups of ornaments G_2 (International symbols). In the other cases, a great number of authors, with their original results introduced together new or modified symbols. Therefore, it is unavoidable to accept the compromise solution and quote several alternative kinds of symbols. Also, this offers possibilities for the application of optimal symbols in each particular case, since for the different practical needs of the theory of symmetry, every kind of symbols has its advantages, but also, disadvantages.

For denoting the symmetry groups of friezes and ornaments, the simplified version of the International symbols (M. Senechal, 1975; H.S.M. Coxeter, W.O.J. Moser, 1980) will be used, while in other cases the non-coordinate symbols, used by Soviet authors (A.V. Shubnikov, V.A. Koptsik, 1974) will be indicated also. The *symbols of antisymmetry and colored symmetry groups* will be given in the group/subgroup notation ($G/H, G/H/H_1$) (A.V. Shubnikov, V.A. Koptsik, 1974; A.M. Zamorzaev, 1976; H.S.M. Coxeter, 1985, 1987; V.A. Koptsik, J.N. Kotzev, 1974).

The *International symbols* are coordinate symbols of symmetry groups. For the symmetry groups of friezes and ornaments, the first coordinate denotes the translational subgroup p (c with the rhombic lattice) while the other coordinates are symbols of glide reflections g and reflections m perpendicular to the corresponding coordinate axis and symbols of the rotation axis n collinear with the corresponding coordinate axis.

The *non-coordinate symbols* of symmetry groups are mainly used in the works of Soviet authors, in which (a) denotes a translation, ($\tilde{a}$) a glide reflection, n the order of a rotation, the absence of symbols between elements — collinearity (incidence) of relevant elements of symmetry (denoted in the original works by the symbol •), while the symbol : denotes perpendicularity of relevant symmetry elements. For example, the symmetry groups D_4 and C_4 will be denoted, respectively, by 4 and 4m.

Given at the beginning of each chapter is a survey of the geometric-algebraic characteristics of the groups of symmetry discussed: presentation, group order, group structure, reducibility, form of the fundamental region, enantiomorphism, polarity (non-polarity, bipolarity), group-subgroup relations, table of minimal indexes of subgroups in groups, Cayley diagrams. Further on are discussed the antisymmetry and color-symmetry desymmetrizations, construction methods, questions related to continuous groups and to different problems of algebraic-geometric properties of symmetry groups, which directly influence the different visual parameters.

1.7. Geometric-Visual Analysis of Symmetry Groups

As an illustration of the methodological approach used in this work, we will give the example of one symmetry group and its complete comparative analysis from the point of view of the theory of symmetry and ornamental art.

Let the discrete group of isometric transformations of the plane E^2, generated by a glide reflection P and a reflection R_1, be given by the presentation:

$$\{P, R_1\} \quad R_1^2 = (R_1P)^2 = E.$$

The first defining relation $R_1^2 = E$ shows that the reflection R_1 is an involution, i.e. that $R_1 = R_1^{-1}$ while from the other relation follows:

$$(R_1P)^2 = E \Rightarrow R_1PR_1 = P^{-1} \Rightarrow R_1^{-1}PR_1 = P^{-1} \Rightarrow P^R = P^{-1}.$$

The glide reflection P is a transformation without invariant points, with an invariant line– the axis $\vec{l}$ of the glide reflection. The conjugate of the transformation P derived by reflection R_1 is also a glide reflection with the invariant line $R_1(\vec{l})$. Since the axis of the glide reflection P^{-1} is the line $-\vec{l}$, from the previous relations we conclude that $R_1(\vec{l}) = -\vec{l}$; consequently, it follows that the reflection line of the reflection R_1 is perpendicular to the axis of the glide reflection P, and that its axis $\vec{l}$ is non-polar (since there exists an indirect transformation, the reflection R_1, which transforms it onto the oppositely oriented line $-\vec{l}$).

The group discussed possesses an invariant space E^2 — the plane, an invariant subspace E^1 — a line, and has no invariant points. Therefore it belongs to the category of symmetry groups of friezes G_{21} — the line groups of the plane E^2 (S^2) without invariant points. Distinguishing between the spaces E^2, S^2, L^2 is not necessary because we are dealing with the line groups. Because this group is generated by a glide reflection P perpendicular to the reflection R_1, its crystallographic symbol will be pmg, or in short form mg (M. Senechal, 1975). Within the crystallographic symbol pmg, p denotes the presence of a translation $X = P^2$, i.e. the translational subgroup 11=$\{X\}$; the symbol m denotes a reflection R_1 perpendicular to this translation, and the symbol g denotes the glide reflection P. In the short symbol mg, the translation symbol p is omitted.

Since the set P, R_1 is a generator set of the group mg, after concluding that the reflection line of R_1 is perpendicular to the axis of the glide reflection P, we can construct an appropriate ornamental motif, the visual model of the frieze symmetry group mg. This is achieved by applying the transformations P and R_1 to the chosen asymmetric figure, which belongs to a fundamental region of the symmetry group mg (Figure 1.19).

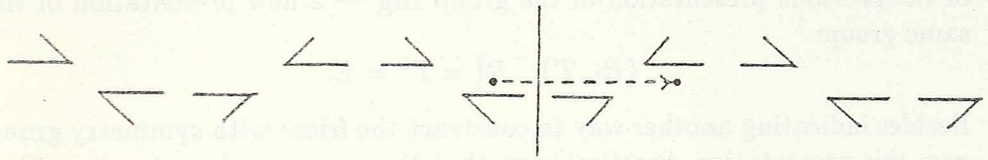


Figure 1.19

Representing the glide reflection P as the commutative composition $X_1R_3 = R_3X_1$ of a translation $X_1 = R_1R_2$ (composition of reflections R_1, R_2 with parallel reflection lines) and a reflection R_3 with the reflection line parallel to the axis of the translation X_1, we come to the conclusion that the product $R_1P = R_1R_1R_2R_3 = R_2R_3$ is the commutative composition of perpendicular reflections R_2, R_3, i.e. a half-turn T. The conjugates of reflection R_1 and half-turn T, derived by the powers of the glide reflection P,

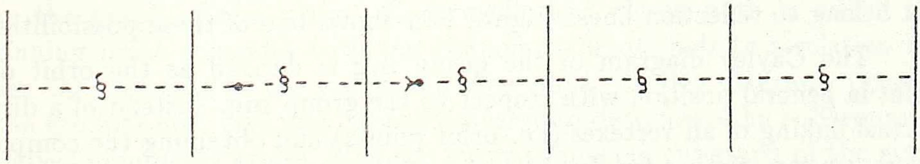


Figure 1.20

define respectively the set of reflections with equidistant reflection lines parallel to the reflection line of R_1, and the set of rotations of the order 2, where the distance between the neighboring reflection lines (rotation centers) is $| \overrightarrow{X_1} |$. So that, we come to the diagrammatic interpretation of the group **mg** — a table of graphic symbols of symmetry elements, where the axis of the glide reflection is indicated by the dotted line and by the vector of translation, reflection lines by solid lines, and centers of rotations of the order 2 by the symbol § (Figure 1.20).

Using the substitution $T = R_1P$ we come to an algebraic equivalent of the previous presentation of the group **mg** — a new presentation of the same group:

$$\{R_1, T\} \quad R_1^2 = T^2 = E.$$

Besides indicating another way to construct the frieze with symmetry group **mg**, this presentation directly shows that the group **mg** has structure D_∞. Namely, group D_n has the presentation:

$$\{S_1, S_2\} \quad S_1^2 = S_2^2 = (S_1S_2)^n = E.$$

If S_1S_2 is an element of infinite order, we obtain the group D_∞ having the following presentation:

$$\{S_1, S_2\} \quad S_1^2 = S_2^2 = E$$

which is isomorphic with the group **mg**.

Instead of the asymmetric figure, which under the action of the group **mg** gives the frieze pattern, by considering the orbit of the closure of a fundamental region of the group **mg** we obtain the corresponding frieze tiling. The fundamental region of the group **mg** and all other frieze symmetry groups, is unbounded and allows the variation of all boundaries which do not belong to reflection lines. Figure 1.21 shows two of these possibilities.

The Cayley diagram of the group **mg** is derived as the orbit of a point in general position with respect to the group **mg**. Instead of a direct mutual linking of all vertexes (i.e. orbit points) and obtaining the complete graph, we can, aiming for simplification, link only the homologous points of the group generators. By denoting with the broken oriented line the glide reflection P, and with the dotted non-oriented line the reflection R_1, we get the Cayley diagram which corresponds to the first presentation of the group **mg** (Figure 1.22a).

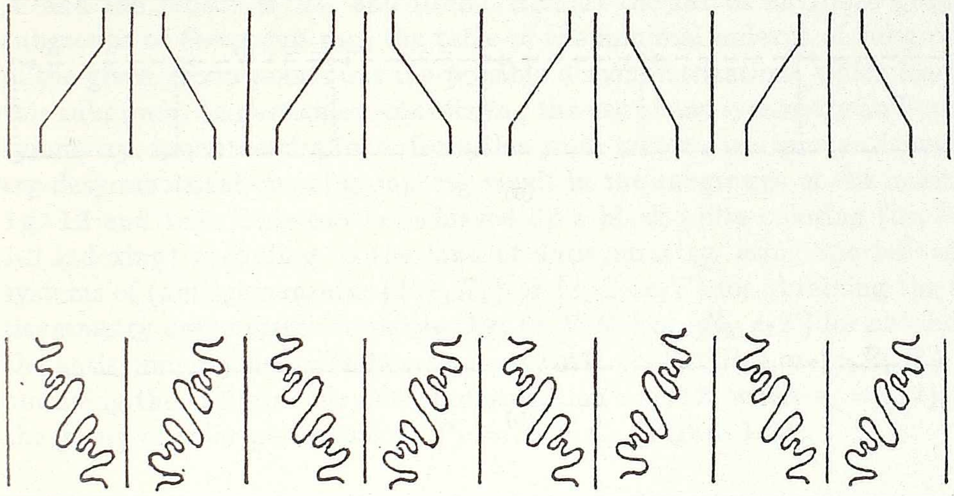
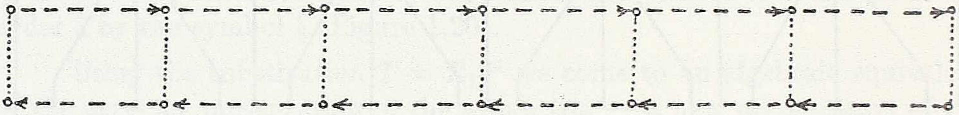


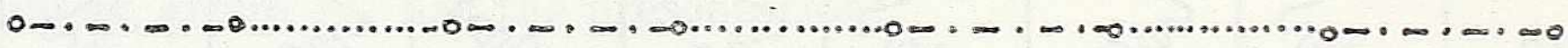
Figure 1.21

By an analogous procedure we come to the graph which corresponds to its second presentation with the generator set $\{R_1, T\}$, where a half-turn is indicated with the dot-dash line (Figure 1.22b).

Let us note also, that the defining relations can be read off directly from the graph of the group. Each cycle, i.e. closed path in which the beginning point coincides with the endpoint, corresponds to a relation between the elements of the group and vice versa. Cayley diagrams (graphs of the groups) may also very efficiently serve to determine the subgroups of the given symmetry group. Namely, every connected subgraph of the given graph satisfying the following condition determines a certain subgroup of the group discussed, and vice versa. The condition in question is: an element (transformation) is included in the subgraph either wherever it occurs, or not at all (i.e. it is deleted). Of course, to be able to determine all the subgroups of a given group, it is necessary to use its complete graph as the basis for defining the subgraphs.



(a)



(b)

Figure 1.22

Since in the group mg there are indirect isometries, this group does not give enantiomorphic modifications. For the groups consisting only of direct symmetries, the enantiomorphic modifications can be obtained by applying the "left"(e.g., b) and "right" (d) form of an elementary asymmetric figure. For example, for the group **11**, generated by a translation X, this results in the enantiomorphic friezes: bbbbbbbbbbbbbbbbbbbbbbbbbbb and ddddddddddddddddddddddddddd. The translation axis l of the group mg is non-polar, because there exists an indirect transformation, the reflection R_1 for which the relation $R_1(\overrightarrow{l}) = -\overrightarrow{l}$ holds. Rotations of the order 2 in the group mg are polar because each circle c drawn around the center of rotation of the order 2 is invariant only with respect to this rotation and to the identity transformation E, so that the group C_2 (2) (generated by the half-turn T) of transformations preserving the circle c invariant, a rosette subgroup C_2 (2) of the group mg, consists of direct transformations. Besides the rosette subgroups C_2 (2), the group mg has also the rosette subgroups D_1 (m), namely the one generated by the reflection R_1, or by its conjugates.

The group mg contains as subgroups the following symmetry groups of friezes: p1 (11) generated by the translation $X = P^2$, p1g (1g) generated by the glide reflection P, pm1 (m1) generated by the translation X and the reflection R_1, and itself. Besides the list of all frieze groups, subgroups of the group mg, the table of the minimal indexes of subgroups of the given group points out the possible desymmetrizations which lead to this subgroup. In particular, considering the use of antisymmetry and color-symmetry desymmetrizations, from this table we can see that antisymmetry desymmetrizations of group mg result in the subgroups of the index 2: 1g, 12 and m1. This can be achieved by a black-white coloring (or, e.g., 1-2 indexing) according to the laws of antisymmetry, using the following systems of (anti)generators: $\{P, e_1R_1\}$ or $\{e_1R_1, e_1T\}$ for obtaining the antisymmetry desymmetrization mg/1g; $\{e_1P, R_1\}$ or $\{R_1, e_1T\}$ for obtaining the antisymmetry desymmetrization mg/m1; $\{e_1P, e_1R_1\}$ or $\{e_1R_1, T\}$ for obtaining the antisymmetry desymmetrization mg/12, where $e_1 = (12)$, i.e. the group of color permutations $P_N = P_2 = C_2$ (Figure 1.23).

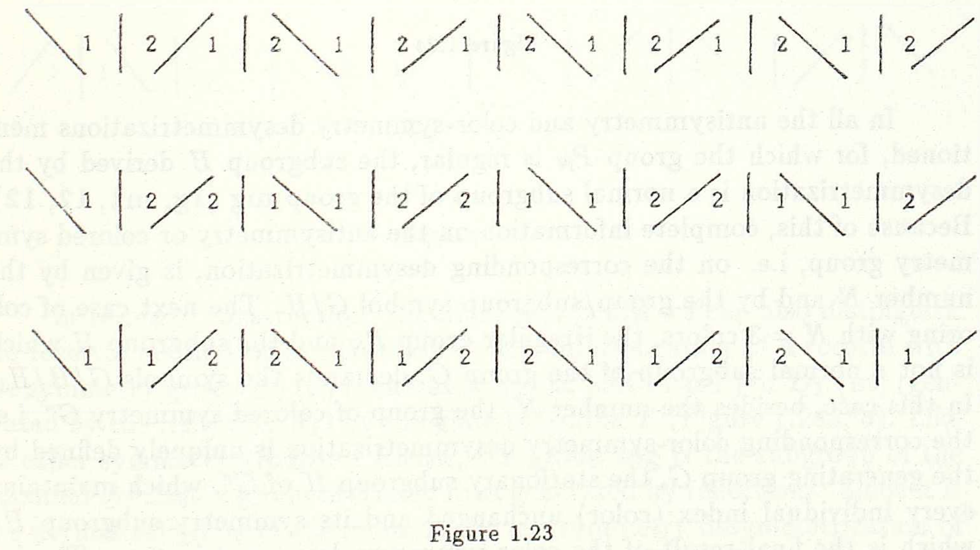


Figure 1.23

The junior antisymmetry groups obtained can be understood also as adequate visual interpretations of the symmetry groups of bands G_{321} — as the Weber diagrams of the symmetry groups of bands **p2₁11**, **pm11** and **p112** respectively. In this case the alternation of colors white-black is understood in the sense "above-under" the invariant plane of the frieze, i.e. as the identification of the antiidentity transformation e_1 with the plane reflection in the invariant plane of the group **mg**. The seven generating symmetry groups of friezes G_{21}, seven senior antisymmetry groups and seventeen junior antisymmetry groups correspond to the 31 groups of symmetry of bands, offering complete information on their presentations and structures.

Using $N = 4$ colors and the system of colored generators $\{c_1 P, c_2 R_1\}$ or $\{c_2 R_1, c_1 c_2 T\}$, we get the color-symmetry desymmetrization **mg/11**, where $c_1 = (12)(34)$ and $c_2 = (13)(24)$; hence, the group of color permutations is $P_N = P_4 = C_2 \times C_2 = D_2$ (Figure 1.24).

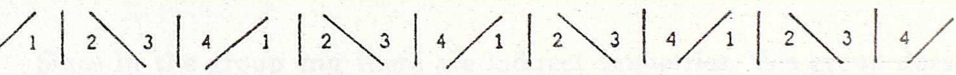


Figure 1.24

In all the antisymmetry and color-symmetry desymmetrizations mentioned, for which the group P_N is regular, the subgroup H derived by the desymmetrization is a normal subgroup of the group **mg** (**1g**, **m1**, **12**, **12**). Because of this, complete information on the antisymmetry or colored symmetry group, i.e. on the corresponding desymmetrization, is given by the number N and by the group/subgroup symbol G/H. The next case of coloring with $N = 3$ colors, the irregular group P_N and the subgroup H which is not a normal subgroup of the group G, demands the symbols $G/H/H_1$. In this case, besides the number N, the group of colored symmetry G^*, i.e. the corresponding color-symmetry desymmetrization is uniquely defined by the generating group G, the stationary subgroup H of G^*, which maintains every individual index (color) unchanged and its symmetry subgroup H_1 which is the final result of the color-symmetry desymmetrization. The index of the subgroup H in the group G is equal to N and the product of the index of the subgroup H_1 in group H and the number N is equal to the order of the group of color permutations P_N, i.e. $[G:H] = N$, $[H:H_1] = N_1$, and the order of the group P_N is NN_1.

As an example of the irregular case we can use the color-symmetry desymmetrization of the group **mg** obtained by $N = 3$ colors, i.e. by the system of colored generators: $\{c_1P, c_2R_1\}$ or $\{c_2R_1, c_1c_2T\}$, which results in the color-symmetry desymmetrization **mg/mg/1g**, where $c_1 = (123)$, $c_2 = (23)$, $P_N = P_3 = D_3$ and [mg:mg]=*3*, [mg:1g]=2. This color-symmetry desymmetrization **mg/mg/1g**, $N = 3$ is shown on Figure 1.25a, while the stationary subgroup H (**mg**) which maintains each individual index (color) unchanged is singled out on Figure 1.25b. All cases of subgroups which are not normal subgroups of the given group are denoted in the tables of (minimal) indexes of subgroups in groups by italic indexes (e.g., [mg:mg]=*3*).

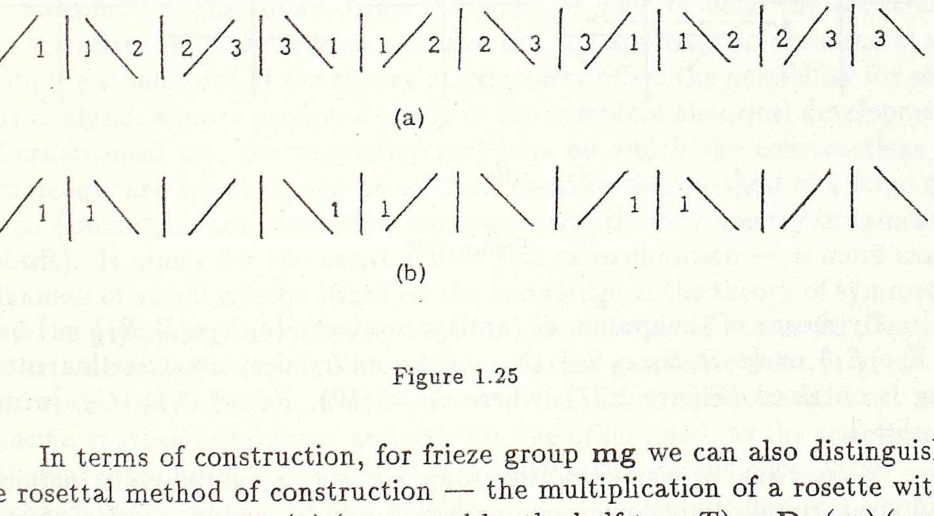
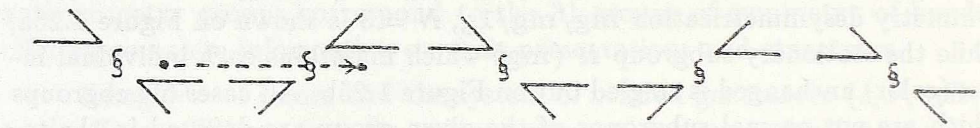


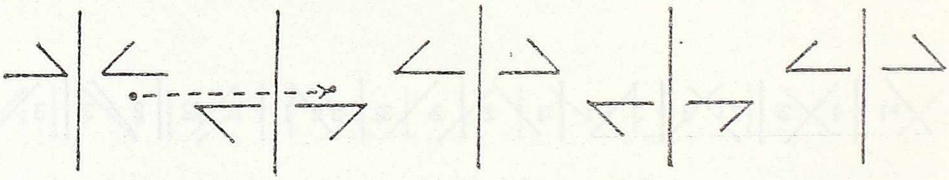
Figure 1.25

In terms of construction, for frieze group **mg** we can also distinguish the rosettal method of construction — the multiplication of a rosette with the symmetry group $\mathbf{C}_2$ (**2**) (generated by the half-turn T) or $\mathbf{D}_1$ (**m**) (generated by the reflection R_1) by the glide reflection P (Figure 1.26a, b). Like all other symmetry groups of friezes, the group **mg** is the subgroup of the maximal symmetry group of friezes **mm** generated by reflections. Since it is the normal subgroup of the index 2, the antisymmetry desymmetrization of the generating group **mm** with a set of generators $\{X, R, R_1\}$ or $\{R, R_1, R_2\}$

where X is the translation, R the reflection in translation axis line, and R_1, R_2 reflections with reflection lines perpendicular to the translation axis, can be used.



(a)



(b)

Figure 1.26

By means of the system of (anti)generators: $\{e_1X, e_1R, R_1\} = \{e_1X, e_1R, e_1R_1\}$ or $\{e_1R, R_1, e_1R_2\}$ the antisymmetry desymmetrization mm/mg is obtained (Figure 1.27), where $e_1 = (12)$, $P_N = P_2 = C_2$, [mm : mg] = 2.

Many visual properties of the group mg, e.g., a relative constructional and visual simplicity of corresponding friezes conditioned by a high degree of symmetry, specific balance of the stationariness conditioned by the presence of reflections, by the non-polarity of the glide reflection axis, by the absence of enantiomorphism, and the dynamism conditioned by the presence of glide reflection and by polar, oriented rotations, are the direct consequences of the algebraic-geometric characteristics mentioned. Also, the different possibilities that the group mg offers, e.g., the possibilities for antisymmetry and color-symmetry desymmetrizations, the ways of varying the form of the fundamental region, construction possibilities etc., become evident after the analysis of this symmetry group of friezes from the point of view of the theory of symmetry.

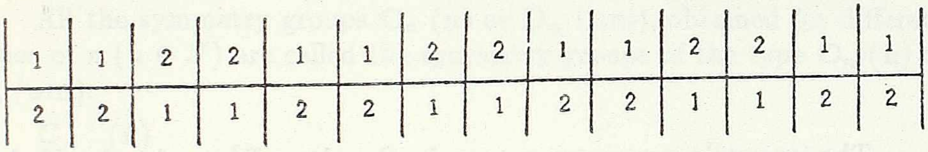


Figure 1.27

Even such a concise illustration of the connections between the theory of symmetry and ornamental art raises the question of the place that ornamental art has today, from the point of view of both the artist and the scientist. When analyzing works of art, the approach to ornamental art from the standpoint of the theory of symmetry offers the possibility for serious analysis, a more profound study of the complete historical development of ornamental art, the regularity and laws on which the constructions of ornaments are founded, and an efficient classification method of a large domain (isometric, non-isometric, antisymmetry, color-symmetry ornamental motifs). It opens for the artist a new field of exploration — a more exact planning of visual effects, based on the knowledge of the theory of symmetry and the psychology of visual perception. An example of successful creativity, artistic imagination and knowledge of exact geometric rules, is given by the work of M.C.Escher, which points to the future of ornamental art as a specific synthesis of science and art. On the other hand, to the scientists of different disciplines, the theory of symmetry offers various possibilities — to archaeologists an efficient and reliable method of classification and comparative analysis; to theorists of art the basis for working out exact aesthetic criteria; to crystallographers, physicists and chemists an obvious model of symmetry structures. Last, but not least, to mathematicians ornamental art, as the treasury of the implicit mathematical knowledge of humankind, represents an inspiring field, rich with questions seeking an answer.

Chapter 2

THEORY OF ISOMETRIC SYMMETRY GROUPS IN E^2 AND ORNAMENTAL ART

The isometric symmetry groups in the plane E^2 can be classified according to the spaces invariant with respect to the action of transformations of the groups in question. Bohm symbols have been used to denote the corresponding categories of symmetry groups (J. Bohm, K. Dornberger-Schiff, 1966). In the symbol $G_{nst\ldots}$, the first subscript n represents the maximal dimension of space in which the transformations of the symmetry group act, while the following subscripts $st\ldots$ represent the maximal dimensions of subspaces that are invariant with respect to the action of transformations of the symmetry group and that are properly included in each other. These symbols represent also the definitions of the corresponding categories of isometric symmetry groups in E^2: the symmetry groups of finite friezes G_{210}, rosettes G_{20}, friezes G_{21}, and ornaments G_2. In line with the relation $G_{210} \subset G_{20}$, and to simplify things, the category G_{210} will not be discussed individually but within the category G_{20}.

Antisymmetry and colored symmetry, the extensions of the theory of symmetry, will be used only for a more detailed analysis of the symmetry groups in E^2.

2.1. Symmetry Groups of Rosettes G_{20}

In S^2 and in E^2 the 0-dimensional, point discrete symmetry groups of rosettes G_{20} are the *cyclic group* $\mathbf{C}_n$ (n) and the *dihedral groups* $\mathbf{D}_n$ (nm). Also visually presentable is the continuous symmetry group of rosettes $\mathbf{D}_\infty$ (∞m). Here, and in the sequel, we shall indicate the symbol of each symmetry group first according to G.E. Martin (1982), followed (in parentheses) by Shubnikov's notation (A.V. Shubnikov, V.A. Koptsik, 1974). By $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm), $\mathbf{C}_\infty$ (∞), $\mathbf{D}_\infty$ (∞m) are denoted the symmetry groups of rosettes G_{20}, distinct from the abstract groups, denoted by C_n, D_n, C_∞, D_∞, and given by the presentations:

C_n $\{S_1\}$ $S_1^n = E$

D_n $\{S_1, S_2\}$ $S_1^2 = S_2^2 = (S_1S_2)^n = E$

C_∞ $\{S_1\}$

D_∞ $\{S_1, S_2\}$ $S_1^2 = S_2^2 = E$

All the symmetry groups $\mathbf{C}_n$ (n) or $\mathbf{D}_n$ (nm), obtained for different values of n ($n \in N$) are called the symmetry groups of the type $\mathbf{C}_n$ (n) or $\mathbf{D}_n$ (nm).

$\mathbf{C}_n$ (n)

Presentation: $\{S\}$ $S^n = E$

Order: n ($n \in N$)

Structure: C_n

Reducibility: If $n = km$, with $(k, m) = 1$, then $C_n = C_k \times C_m$; if $n = p$, with p – a prime number, then C_n is irreducible.

Form of the fundamental region: unbounded, allows variation of the shape of its boundaries.

Enantiomorphism: enantiomorphic modifications exist.

Polarity of rotations: rotations are polar.

$\mathbf{D}_n$ (nm)

Presentations: $\{S, R\}$ $S^n = R^2 = (SR)^2 = E$

$\{R, R_1\}$ $R^2 = R_1^2 = (RR_1)^n = E$ $(R_1 = RS)$

Order: $2n$ ($n \in N$)

Structure: D_n

Reducibility: If $n = 4m + 2$, then $D_n = C_2 \times D_{2m+1} = \{S^{2m+1}\} \times \{S^2, R\} = \{Z\} \times \{S^2, R\}$; in other cases D_n is irreducible.

Form of the fundamental region: unbounded, of a fixed shape, with rectilinear boundaries.

Enantiomorphism: there are no enantiomorphic modifications.

Polarity of rotations: rotations are non-polar.

$\mathbf{D}_\infty$ (∞m)

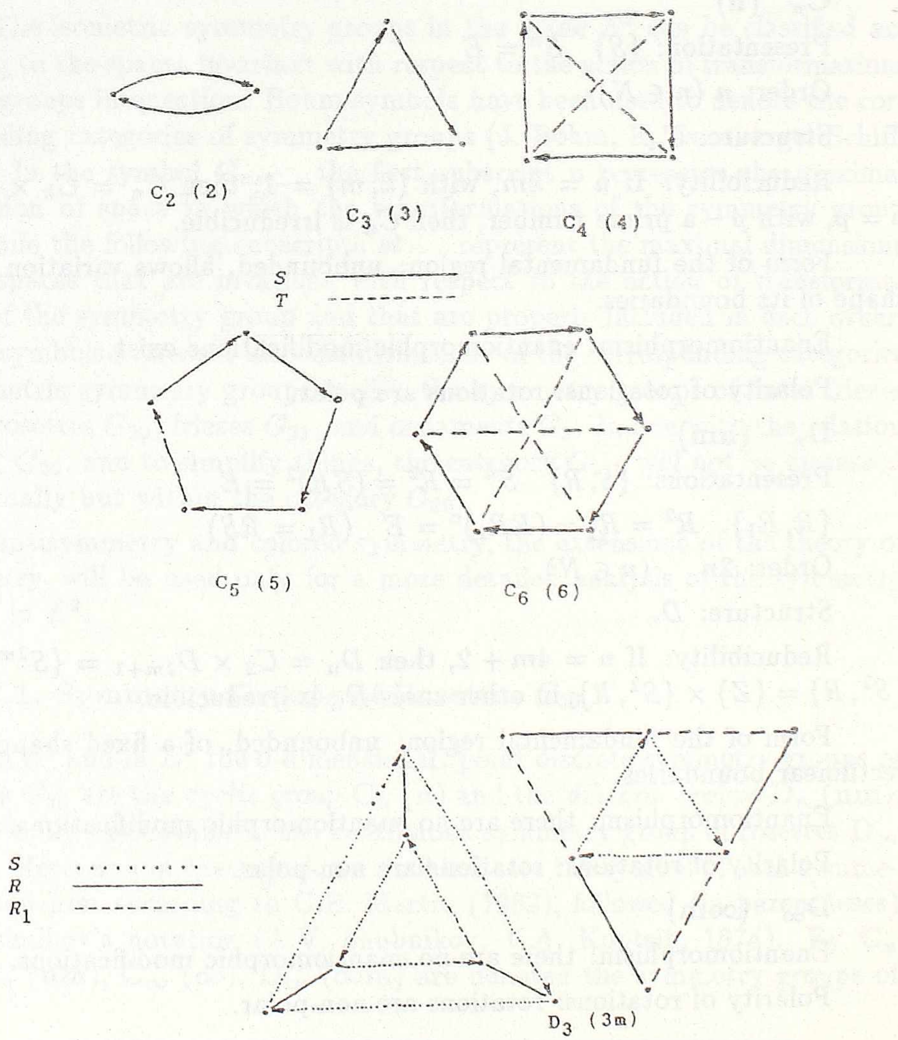
Enantiomorphism: there are no enantiomorphic modifications.

Polarity of rotations: rotations are non-polar.

Group-subgroup relations: $[D_n{:}C_n]=2$ $[D_{km}{:}D_m]=k$ (in particular $[D_{2m}{:}D_2]=m$) $[C_{km}{:}C_m]=k$ (in particular $[C_{2m}{:}C_2]=m$)

The above survey of characteristics of the groups C_n (n) and D_n (nm) is based on I. Grossman, W. Magnus (1964), W. Magnus, A. Karras, S. Solitar (1966), L.C. Biedenharn, W. Brouver, W.T. Sharp (1968), A.V. Shubnikov, V.A. Koptsik (1974), H.S.M. Coxeter, W.O.J. Moser (1980).

Cayley diagrams (Figure 2.1):



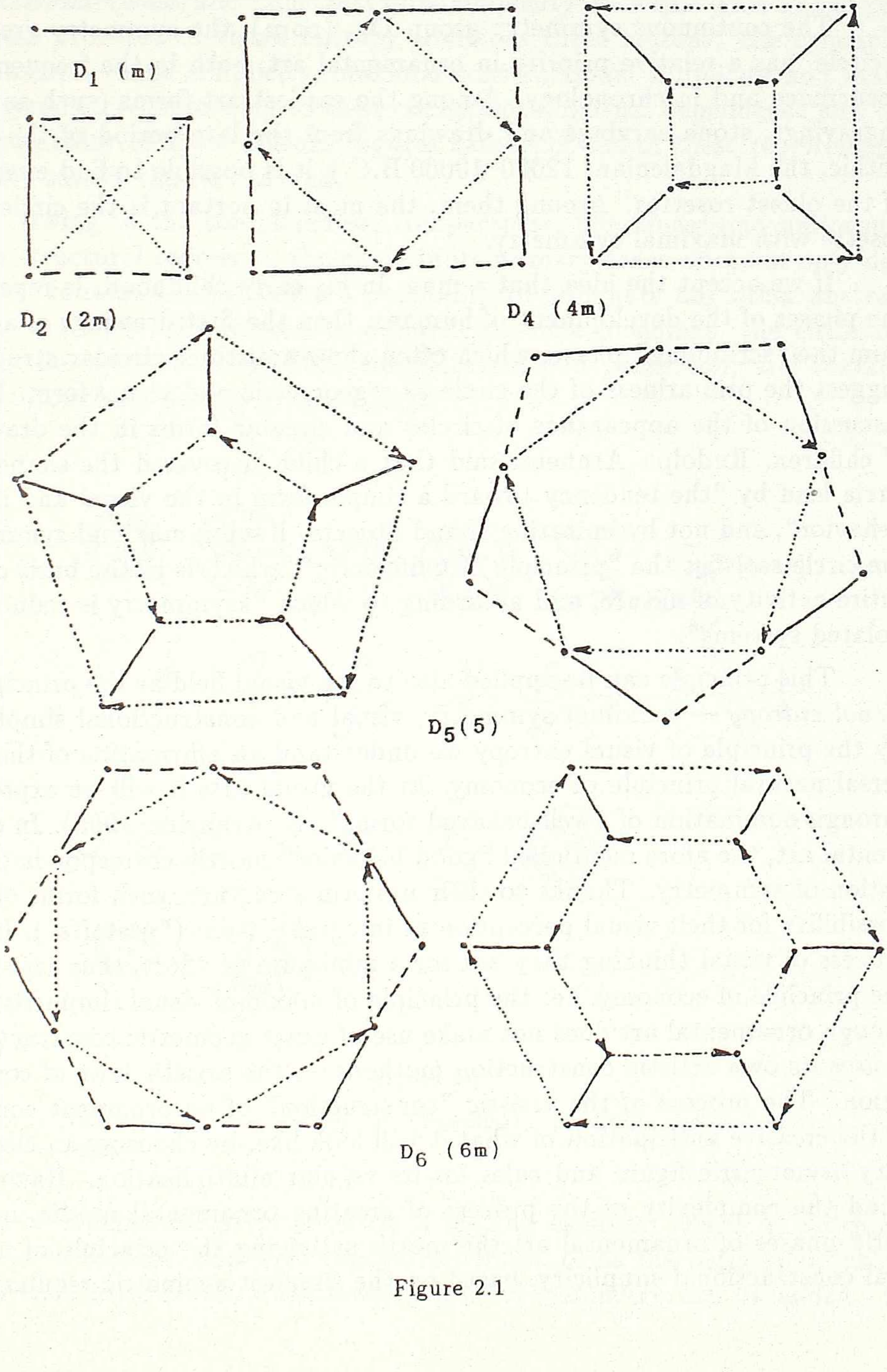


Figure 2.1

2.2. Rosettes and Ornamental Art

The continuous symmetry group $\mathbf{D}_\infty$ (∞m), the symmetry group of a circle, has a relative priority in ornamental art, both in the frequency of occurrence and in chronology. Among the earliest art forms (such as bone engravings, stone carvings and drawings from the late period of the Paleolithic, the Magdalenian, 12000–10000 B.C.), it is possible to find examples of the oldest rosettes. Among them, the most important is the circle — a rosette with maximal symmetry.

If we accept the idea that a man, in his early childhood, is repeating the phases of the development of humans, then the first drawings of a child from the "scribbling" phase, which often show a distinct circular structure, suggest the primariness of the circle as a geometric and visual form. In his discussion of the appearance of circles and circular forms in the drawings of children, Rudolph Arnheim said that a child discovered the shape of a circle lead by "the tendency toward a simple form in the visual and motor behavior", and not by imitating round objects. Having maximal symmetry, the circle satisfies the "principle of uniformity", which is in the basis of the entire activity of nature, and according to which "asymmetry is reduced in isolated systems".

This principle can be applied also to the visual field as the principle of *visual entropy* — maximal symmetry, visual and constructional simplicity. By the principle of visual entropy we understand an affirmation of the universal natural principle of economy. In the visual arts it will be expressed through domination of "well-behaved forms" (R. Arnheim, 1965). In ornamental art, the afore mentioned "good behavior" mostly corresponds to the notion of symmetry. Thanks to their uniform structure, such forms offer a possibility for their visual perception as integral entities ("gestalts"). In the process of visual thinking they ask for a minimum of effort, thus satisfying the principle of economy, i.e. the principle of maximal visual simplicity. Although ornamental art does not make use of exact geometric constructions, it uses its own artistic construction methods — the artistic laws of composition. The process of the artistic "construction" of an ornament consists of the creative anticipation of what it will look like, by choosing an elementary asymmetric figure and rules for its regular multiplication. Having in mind the complexity of the process of creating ornamental motifs, in the early phases of ornamental art the motifs satisfying the principle of maximal constructional simplicity, based on the simplest geometric regularities, were used first. Forms with maximal symmetry often satisfy the principles of maximal visual and constructional simplicity as well. This is the reason the principle of visual entropy combines three notions: the principles of maximal visual simplicity, maximal constructional simplicity and maximal symmetry. According to their connections, mutual dependence and the inseparability of these notions, figures satisfying one of these requirements usually satisfy the rest as well.

Owing to the completeness, compactness, boundness and uniformity of its structural segments, the circle in its primary sense does not only designate "roundness" but offers a possibility to designate any other abstract form — a unit. Associated to different possible meanings, the circle becomes the symbol of the Sun, completeness and perfection, and remains that throughout history (Figure 2.2).

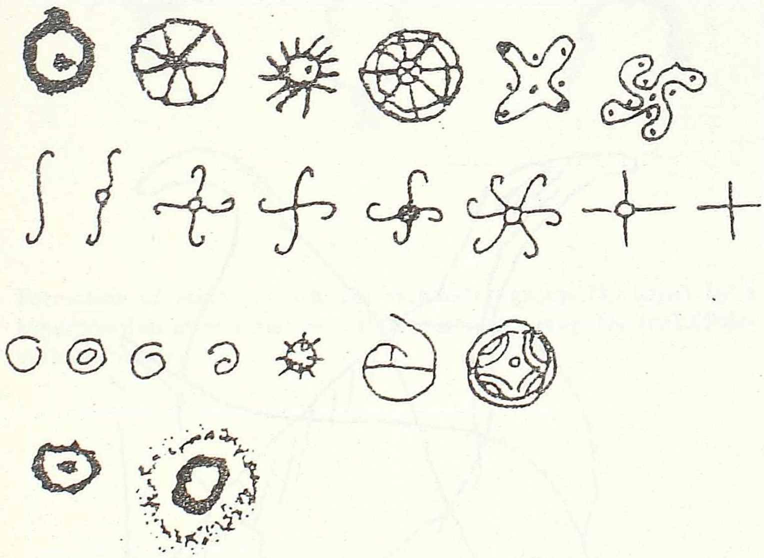


Figure 2.2

Variations of the Sun symbol (Paleolithic, Neolithic and Bronze Age).

In ornamental art, the circle a leading element, either as independent or in combination with concentric circles, or as the basis upon which some concentric rosette of a lower degree of symmetry can be added. In such a desymmetrization, the newly formed rosette — the result of the superposition — possesses the symmetry of the rosette that has caused the desymmetrization. In this process, the circle plays the role of the neutral element.

Among the symmetry groups of the type $\mathbf{D}_n$ (nm), the group $\mathbf{D}_1$ (m) is most frequent in ornamental art. Usually, it is presented by a vertical or horizontal line segment, to which in the geometric sense corresponds symmetry group $\mathbf{D}_2$ (2m), and in the visual sense group $\mathbf{D}_1$ (m). Discussing a vertical line segment, S. Langfeld (R. Arnheim, 1965, pp. 20) says: "If one is asked to bisect a perpendicular line without measuring it, one almost invariably places the mark too high. If a line is actually bisected, it is with difficulty that one can convince itself that the upper half is not longer then the lower half.". Therefore, from the visual point of view, a vertical or horizontal line segment possesses the symmetry group $\mathbf{D}_1$ (m). The primariness of these two directions is governed by their meaning in the

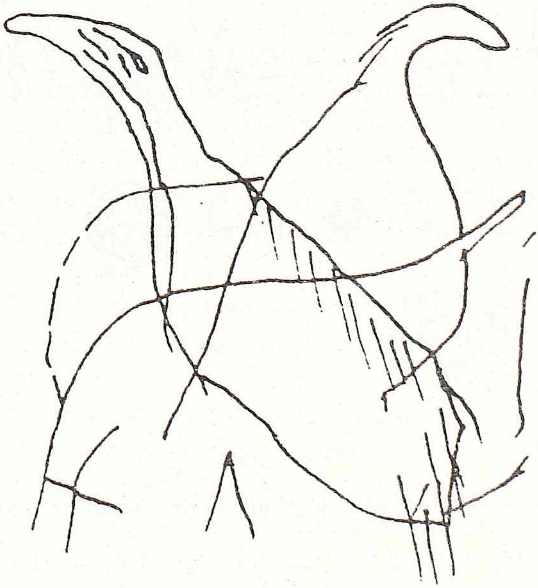


Figure 2.3

The rosette with symmetry $\mathbf{D}_1$ (m) (Paleolithic, El Pendo, Spain).

physical world. The symbolic meanings of a vertical line are: a man walking upright, growth, upright trees, firmness, activity, while the meanings of a horizontal line are: earth, matter, passiveness, rest, sleep, death,... Most of them result from the physiological organization of man, its perpendicular attitude toward the base and its plane symmetry. This is the reason, in drawings, reflections are usually vertical, and very seldom horizontal (Figure 2.3–2.8).

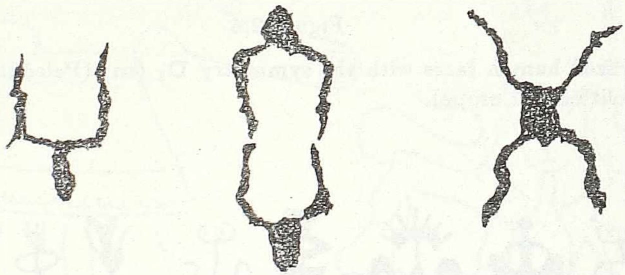


Figure 2.4

Formation of rosettes with the symmetry group $\mathbf{D}_2$ (2m) by a superposition of rosettes with the symmetry group $\mathbf{D}_1$ (m) (Paleolithic, France).

Figure 2.5

Examples of rosettes with the symmetry group $\mathbf{D}_1$ (m) (Paleolithic, France and Spain).

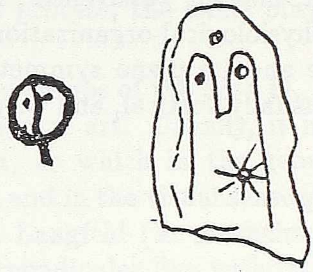


Figure 2.6

Stylized human faces with the symmetry D_1 (m) (Paleolithic and Neolithic of Europe).

Figure 2.7

Stylized human figures with the symmetry D_1 (m) (Paleolithic and Neolithic of Italy and Spain).

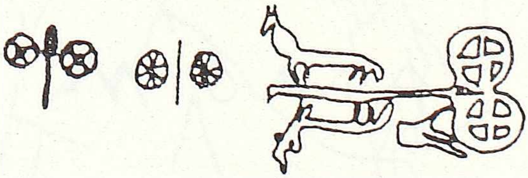


Figure 2.8

Stylized vehicle motifs with the symmetry D_1 (m).

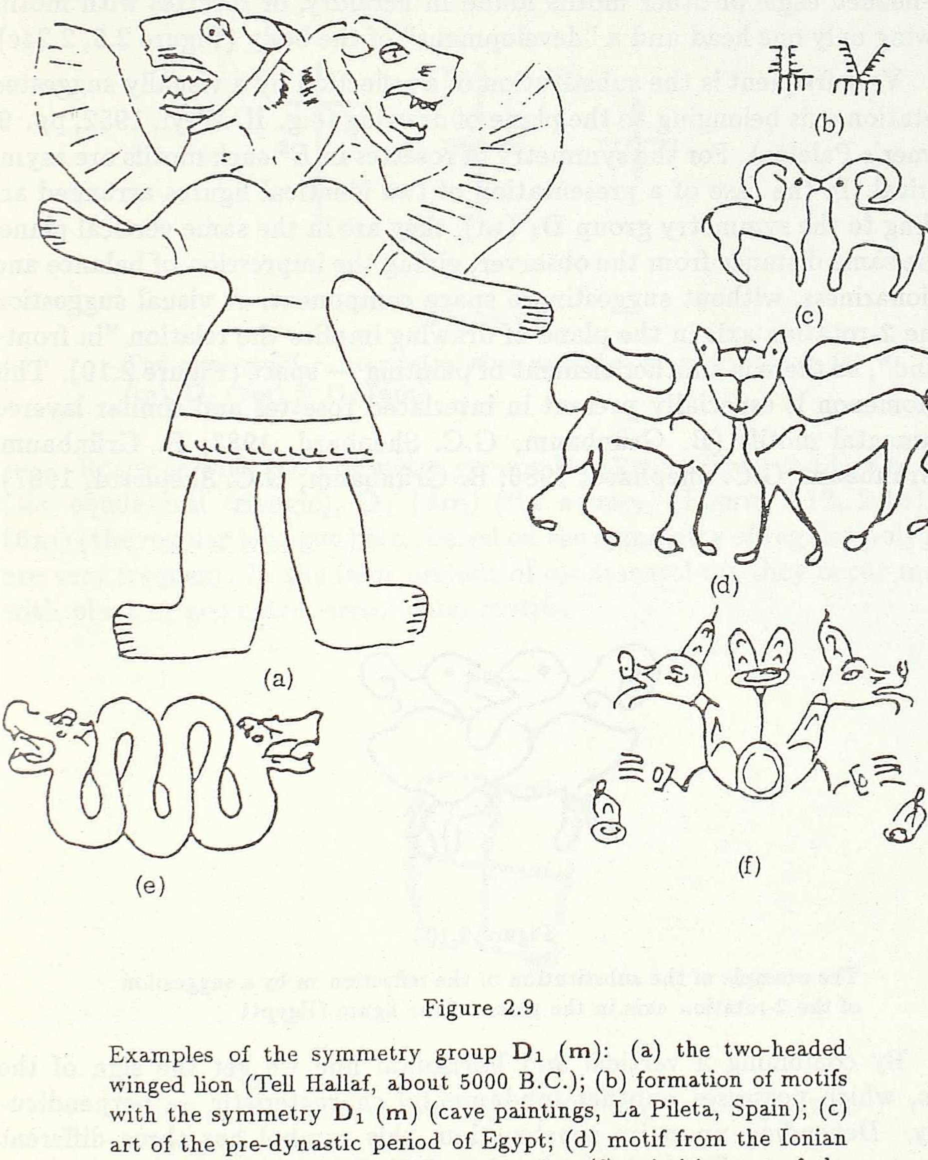


Figure 2.9

Examples of the symmetry group $\mathbf{D}_1$ (m): (a) the two-headed winged lion (Tell Hallaf, about 5000 B.C.); (b) formation of motifs with the symmetry $\mathbf{D}_1$ (m) (cave paintings, La Pileta, Spain); (c) art of the pre-dynastic period of Egypt; (d) motif from the Ionian amphora; (e) two-headed Mayans snake; (f) primitive art of the American Indians.

Rosettes with the symmetry group D_1 (m) have a dominant position in ornamental art. In the visual sense, the best illustrations of the importance of mirror symmetry are different forms of two-headed mythological animals: the two-headed snake of the Maya Indians, Amphisbaena, the two-headed eagle or other motifs found in heraldry, or rosettes with motifs showing only one head and a "development" of the body (Figure 2.9, 2.24c).

Very frequent is the substitution of a reflection by a visually suggested 2-rotation axis belonging to the plane of drawing (e.g. H. Weyl, 1952, pp. 9, Nermer's Palette). For the symmetry of rosettes in E^2 such motifs are asymmetrical. In the case of a presentation of two identical figures arranged according to the symmetry group D_1 (m), they are in the same vertical plane, at the same distance from the observer, giving the impression of balance and stationariness, without suggesting a space component. A visual suggestion of the 2-rotation axis in the plane of drawing implies the relation "in front–behind", so there is another element of painting — space (Figure 2.10). This phenomenon is especially present in interlaced rosettes and similar layered ornamental motifs (B. Grünbaum, G.C. Shephard, 1983; B. Grünbaum, Z. Grünbaum, G.C. Shephard, 1986; B. Grünbaum, G.C. Shephard, 1987).

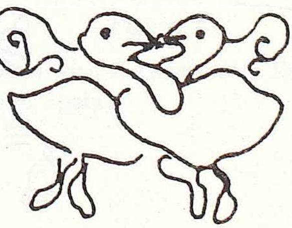


Figure 2.10

The example of the substitution of the reflection m by a suggestion of the 2-rotation axis in the plane of the figure (Egypt).

By combining a vertical and horizontal line we get the sign of the cross, which possesses another fundamental characteristic — perpendicularity. Depending upon its construction, this symbol has three different symmetry aspects: D_1 (m), D_2 (2m) and D_4 (4m) (Figure 2.11).

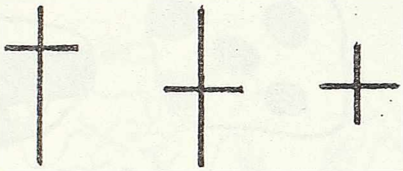


Figure 2.11

The cross motif — rosettal symbol with the symmetry group $\mathbf{D}_1$ (m), $\mathbf{D}_2$ (2m) or $\mathbf{D}_4$ (4m).

Rosettes with the symmetry groups $\mathbf{D}_2$ (2m) (Figure 2.13), $\mathbf{D}_3$ (3m) (the equilateral triangle), $\mathbf{D}_4$ (4m) (the square) (Figure 2.12, 2.13), $\mathbf{D}_6$ (6m) (the regular hexagon) etc., based on the symmetry of regular polygons, are very frequent. In the later periods of ornamental art they occur mostly with plant or geometric ornamental motifs.

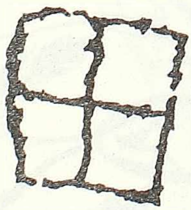


Figure 2.12

The rosette with the symmetry group $\mathbf{D}_4$ (4m) occurring in Paleolithic art.

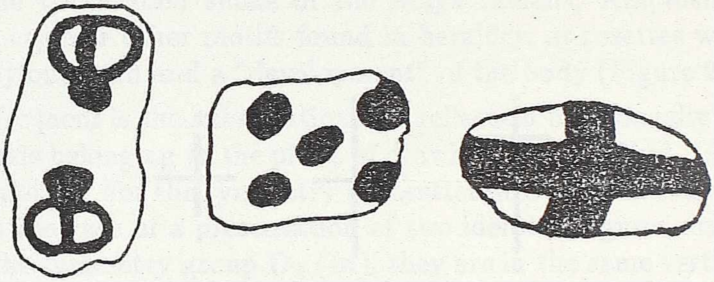


Figure 2.13

The rosettes with the symmetry groups D_2 (2m), D_4 (4m) (Paleolithic, Maz d'azil).

Although the principle of crystallographic restriction (n=1,2,3,4,6) is not respected in ornamental art, rosettes with the symmetry groups 1(m), 2(m), 3(m), 4(m), 6(m) prevail over rosettes with the symmetry group 5(m), 7(m), 9(m), etc., probably because of a simpler construction of the corresponding regular polygons. For practical reasons, rosettes with rotations of a higher order occur very seldom (Figure 2.14–2.20).

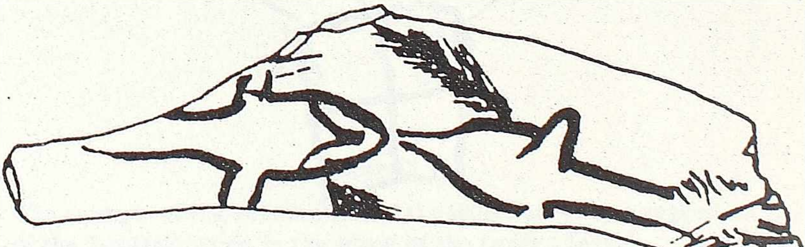


Figure 2.14

The rosette with the symmetry group C_2 (2) (Paleolithic, Magdalenian, around 10000 B.C., Laugerie Basse, France).

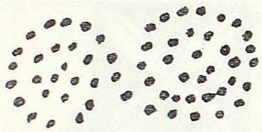


Figure 2.15

The spiral rosette with the symmetry group C_2 (2) (Paleolithic, Magdalenian, around 10000 B.C., Mal'ta, USSR).

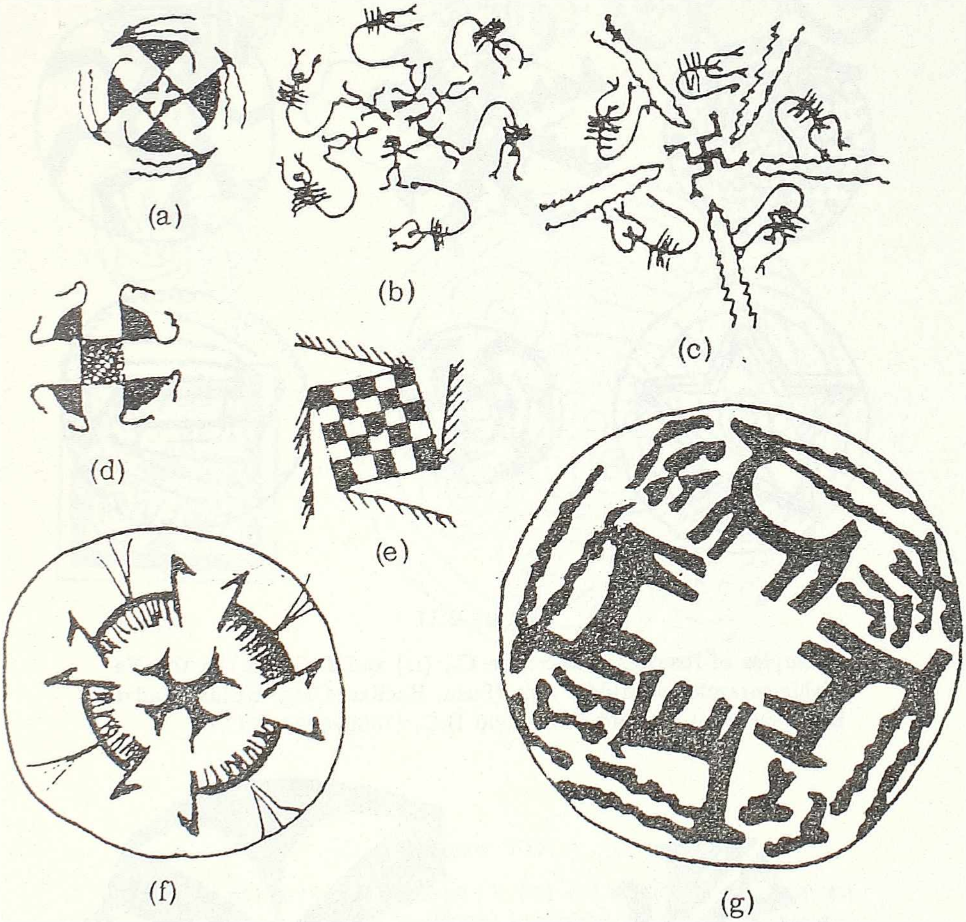


Figure 2.16

Examples of rosettes of the type C_n (n) and D_n (nm) in the Neolithic ceramics of Middle Asia: (a) C_4 (4); (b) C_6 (6); (c) C_5 (5); (d) D_4 (4m); (e) C_4 (4); (f) D_4 (4m); (g) C_4 (4) ((a)–(d), (g) Samara; (e), (f) Susa; around 5500–5000 B.C.).

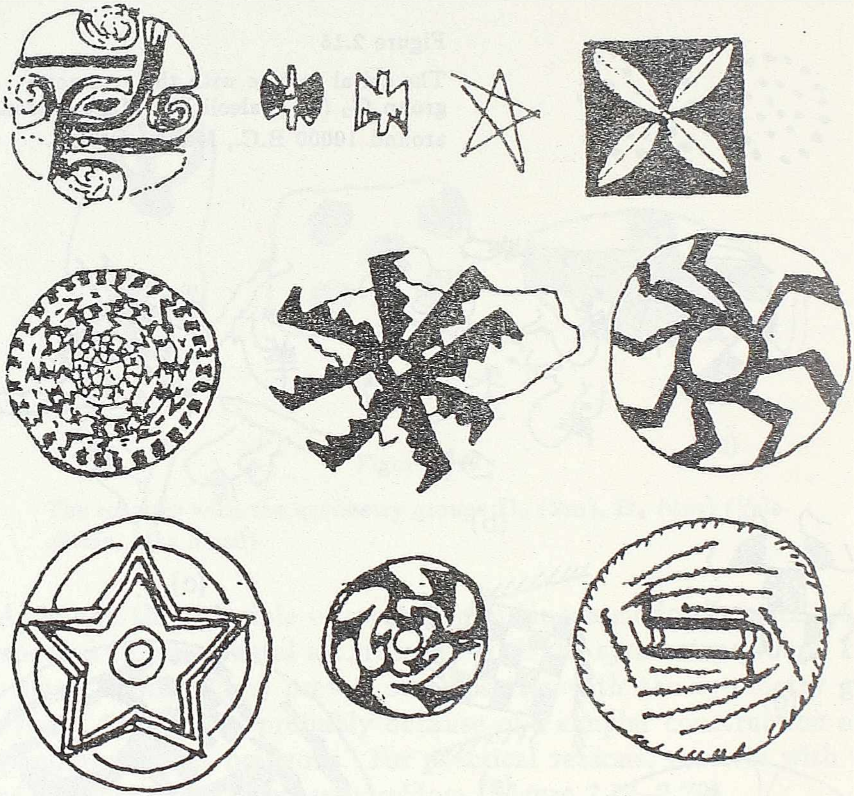
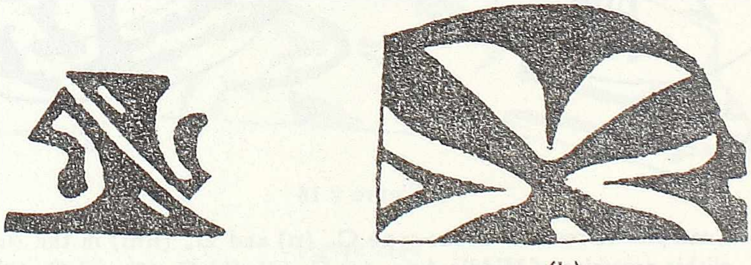


Figure 2.17

Examples of rosettes of the type C_n (n) and D_n (nm) in the Neolithic ceramics of Middle Asia (Susa, Hacilar, Catal Hüjük, Hallaf, Eridu culture), around 6000–4500 B.C. (7500–5000 B.C.?).



(a) (b)

Figure 2.18

Rosettes with the symmetry group C_2 (2) and D_4 (4m): (a) Hacilar, about 6000 B.C.; (b) Aznabegovo-Vrshnik, Yugoslavia, around 5000 B.C.

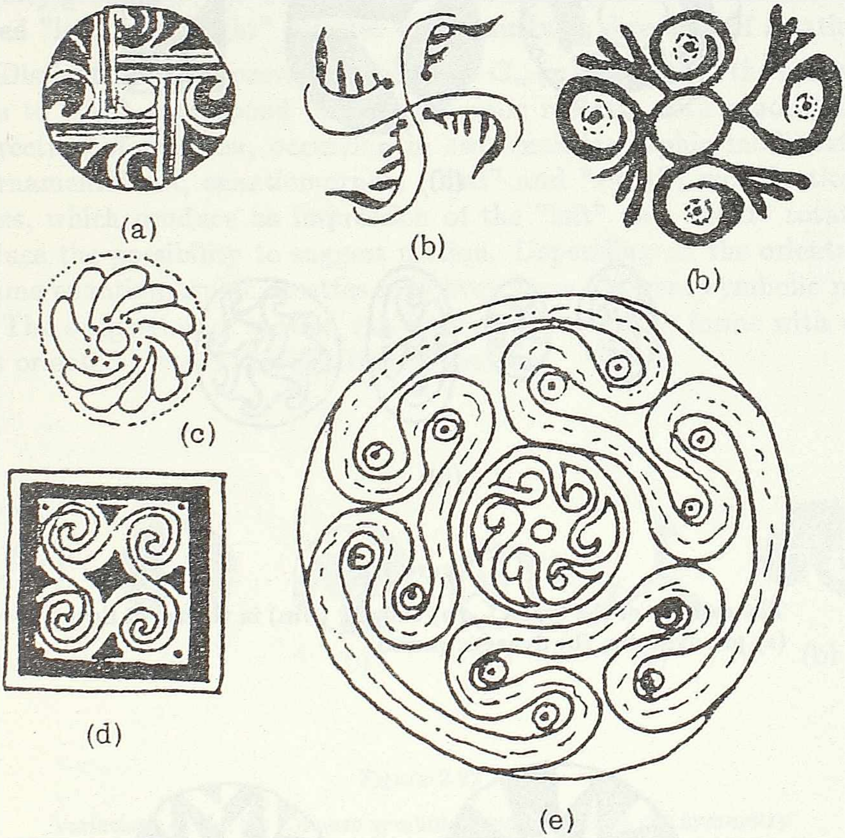
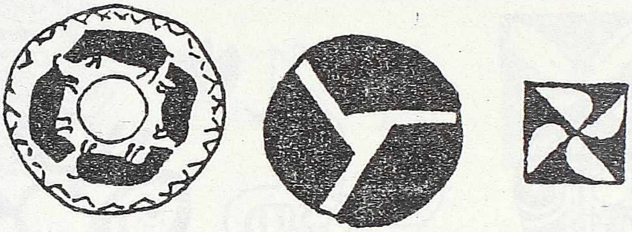


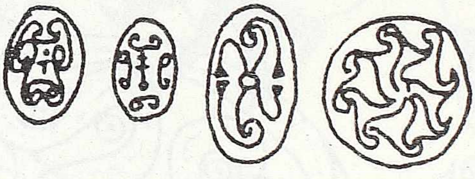
Figure 2.19

Examples of rosettes of the type C_n (n) and D_n (nm) in European ancient art: (a) C_4 (4), Neolithic, Rumania; (b) C_4 (4), Knossos, Crete, around 2500 B.C.; (c) C_{10} (10), Mycenae; (d) C_4 (4), Crete; (e) D_3 (3m), Mycenae.

In deriving rosettes with the symmetry group C_n (n) from rosettes with the symmetry group D_n (nm), the desymmetrization method can be used. The relationship $[D_n{:}C_n]= 2$ holds. Even from the Neolithic, for obtaining rosettes with the symmetry group C_n (n), the antisymmetry group D_n/C_n has been used (Figure 2.21, 2.27).



(a)



(b)

Figure 2.20

The rosettes of the type C_n (n) and D_n (nm) in the art of Egypt: (a) pre-dynastic; (b) dynastic period.

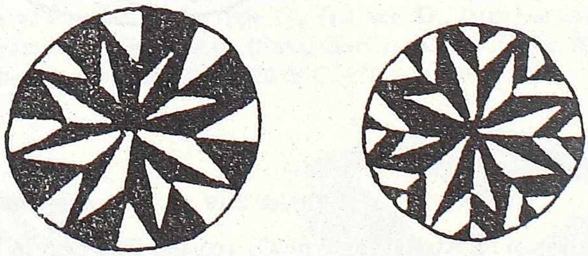


Figure 2.21

Examples of antisymmetry rosettes with the antisymmetry group D_8/C_8, that in the classical theory of symmetry are treated as rosettes with the symmetry group C_8 (8) (Hajji Mohammed, around 5000 B.C.).

Owing to the reflections, rosettes with the symmetry group D_n (nm) produce a visual impression of balance and stationariness, where one of the reflections is supposed to be vertical. According to the principle of visual entropy — the tendency toward a high degree of symmetry — rosettes with the symmetry group $\mathbf{D}_n$ (nm) will be more frequent than rosettes with the symmetry group $\mathbf{C}_n$ (n). They have no enantiomorphic modifications — the oriented "left" and "right" rosettes with a singular direction of rotations.

Distinct from the previous type, type $\mathbf{C}_n$ (n) consists of the symmetry groups to which correspond visually dynamic rosettes with a polar singular direction of rotation, occurring in two enantiomorphic modifications. For ornamental art, enantiomorphic ("left" and "right") modifications of rosettes, which produce an impression of the "left" and "right" rotations, introduce the possibility to suggest motion. Depending on the orientation, the same enantiomorphic rosettes may even have different symbolic meanings. The suggestion of motion can be stressed by using forms with acute angles oriented toward the direction of rotation.

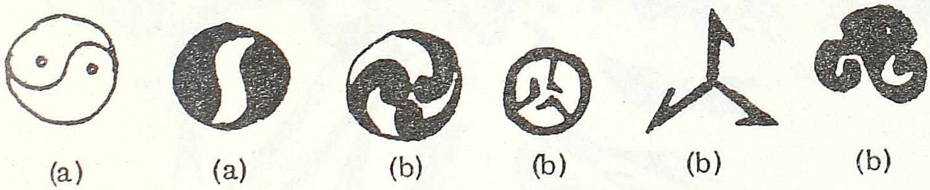


Figure 2.22

Variations of the (a) Chinese symbol "yang-yin" with the symmetry group C_2 (2) and (b) the triquetra motif with the symmetry group C_3 (3).

Typical examples of rosettes with the symmetry group C_n (n) are the triquetra (C_3 (3)) (Figure 2.22b), the swastika (C_4 (4)) (Figure 2.23) and similar motifs represented in different civilizations (e.g. Babylonian, Chinese, Aegean and Mayan) a symbol of the Sun, and lastly the Chinese symbol "yang-yin" (C_2 (2)) symbolizing a dynamic balance between the male and female principle (Figure 2.22a), etc.

In the early phases of ornamental art, these symbols were used mostly in their simplest form. Further development led toward more complexity, and the enrichment and variation of basic elementary asymmetric figures belonging to a fundamental region, which were multiplied by symmetry transformations. For the symmetry groups of the type C_n (n), a variety of rosettes can be achieved by using a curvilinear fundamental region, while with the symmetry groups of the type D_n (nm), generated by reflections, the fundamental region must be rectilinear.

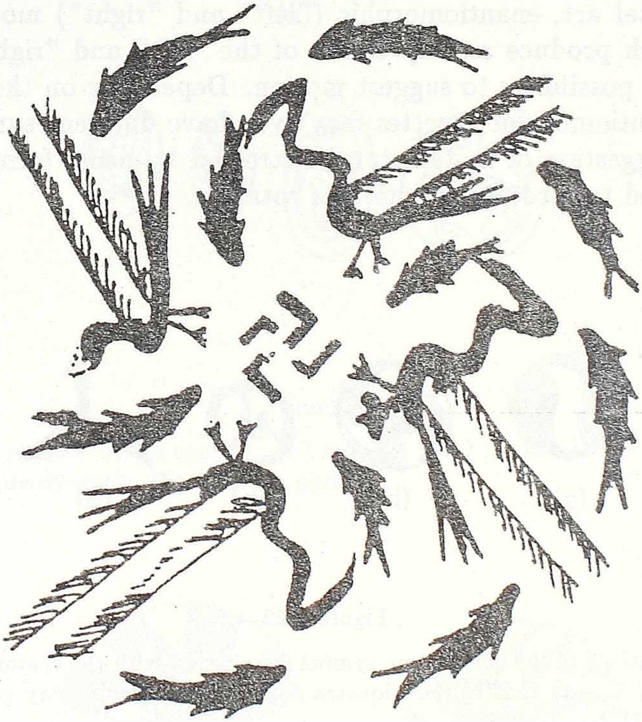
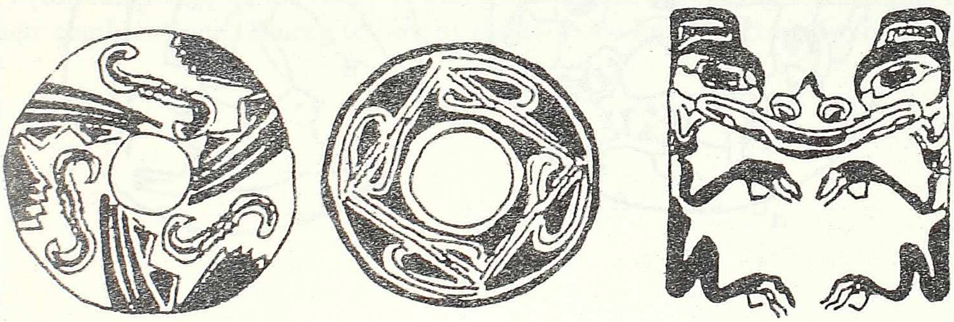


Figure 2.23

The rosette with the symmetry group C_4 (4) (ceramics of the Neolithic of Middle Asia, around 6000 B.C.).

Even in early ornamental art (e.g., in the ceramics of the American Indians before Columbus) with its very complicated geometric ornaments, there are no deviations from the strict principle of symmetry (Figure 2.24a, b).

Interesting examples are derived by superposing concentric rosettes with different symmetry groups. In such a case, the symmetry of the system is the symmetry of the least symmetrical rosette belonging to the composition or some of its subgroups, usually non-trivial (Figure 2.19e, 2.23).



(a) (b) (c)

Figure 2.24

Examples of rosettes with the symmetry group (a) C_3 (3); (b) C_4 (4); (c) D_1 (m) in the art of American Indians. The rosette (c) represents an example of the "development" of the body, that results in the symmetry D_1 (m).

In time, the symbolic meanings of rosettes, which in the beginning of ornamental art had played a role as a specific means of communication, were also lost. This way, rosettes became only decorations, and they remained so till modern times (Figure 2.25).

In the modern age, aiming for the simplest possible means of communication — mainly visual — the modern designer has developed a whole system of signs (traffic signs, trade marks, etc.) which have the symmetry of rosettes. Also, by a multidisciplinary approach uniting ornamental art, the theory of symmetry, and the sciences which need for the visual modeling of rosettal symmetry structures (Crystallography, Physics, Chemistry,...), rosettes gained new meanings.

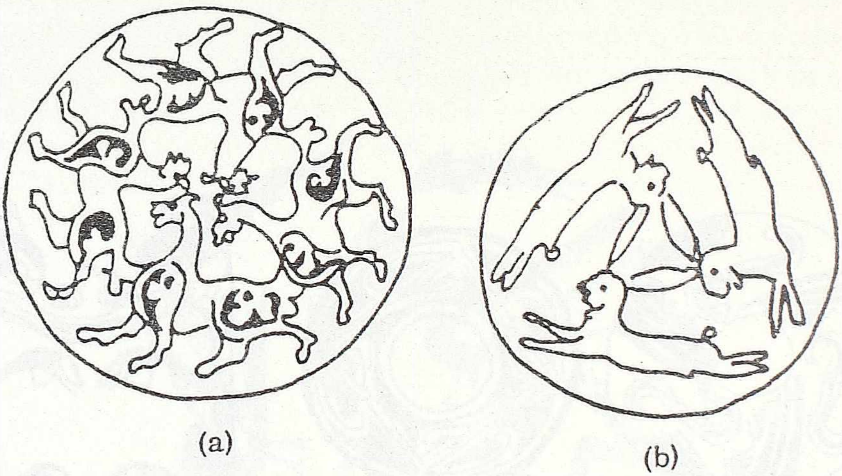


Figure 2.25

Examples of rosettes with the symmetry group (a) C_4 (4), 15th century; (b) C_3 (3), with the dominant decorative component.

* * *

Rosettes date back to the period of Paleolithic art and represent one of the oldest examples of the human aim to express regularity and symmetry. For the mathematical theory of symmetry, they are the simplest basis for an adequate mathematical treatment of ornaments, based on the theory of symmetry, and the record of the historic development going from visuality to the mathematical abstraction. Even to those acquainted with the theory of symmetry, rosettes remain the most evident visual illustration of the visually presentable symmetry groups C_n (n), D_n (nm), D_∞ (∞m). By analyzing rosettes from the point of view of the theory of symmetry, it is possible to note the common characteristics of rosettes and create a link between presentations of the symmetry groups of rosettes, their structures and the visual properties of corresponding rosettes. By using the principle of visual entropy, it is possible to establish relations between the maximal constructional and visual simplicity and maximal symmetry on the one side, and the period of origin, frequency of occurrence and variety of rosettes on the other. Because of this, we have the early appearance and dominance of rosettes that satisfy this principle.

A survey on the symmetry groups of rosettes and the group-subgroup relations can serve as a basis for the construction of rosettes by the desymmetrization method. These relations are schematically shown in Figure 2.26, where an arrow designates the group-subgroup relation, and an attached symbol the index of the subgroup in the group. These relations determine the possibilities available to the classical-symmetry, antisymmetry (for subgroups of the index 2) and color-symmetry desymmetrizations or their combinations, aiming to obtain rosettes of a lower degree of symmetry.

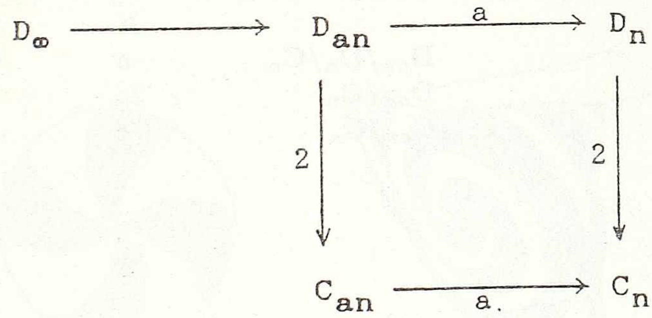


Figure 2.26

In the table of antisymmetry desymmetrizations, symbols of antisymmetry groups are given in the group/subgroup notation G/H (V.A. Koptsik, 1966; A.M. Zamorzaev, 1976; H.S.M. Coxeter, 1985). The group/subgroup notation G/H gives information on the generating symmetry group G and its (normal) subgroup H of the index 2, derived by the antisymmetry desymmetrization (Figure 2.27). The relation $G/H \cong C_2$ holds.

The table of antisymmetry desymmetrizations of symmetry groups of rosettes G_{20}:

$$\mathbf{D}_{2n}/\mathbf{D}_n$$

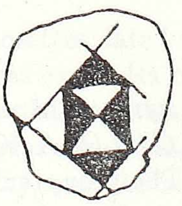
$$\mathbf{D}_n/\mathbf{C}_n$$

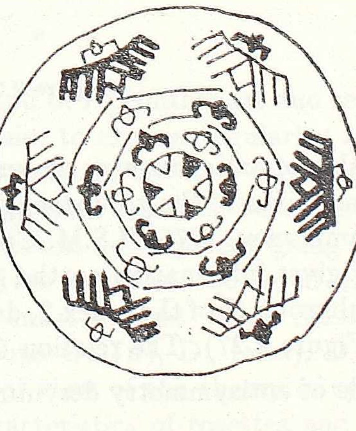
$$\mathbf{C}_{2n}/\mathbf{C}_n$$

In the table of color-symmetry desymmetrizations the symbols of colored symmetry groups G^* are given in the notation $G/H/H_1$ (A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant, 1978; V.A. Koptsik, J.N. Kotzev, 1974). In the symbol $G/H/H_1$, the first datum denotes the generating symmetry group G, the second gives the stationary subgroup H of the colored symmetry group G^*, which consists of transformations maintaining an individual index (color) unchanged, while the third denotes the symmetry subgroup H_1 of the colored symmetry group G^*. The subgroup H_1 is the result of the color-symmetry desymmetrization. A number N ($N \geq 3$) is the number of "colors" used to derive the colored symmetry group. For $H = H_1$, i.e. iff H is a normal subgroup of the group G, the symbol $G/H/H_1$ is reduced to the symbol G/H.

The table of color-symmetry desymmetrizations of symmetry groups of rosettes G_{20}:

	N
$\mathbf{D}_{an}/\mathbf{D}_n/\mathbf{C}_n$	a
$\mathbf{D}_{an}/\mathbf{C}_n$	$2a$
$\mathbf{C}_{an}/\mathbf{C}_n$	a


(a)


(b)

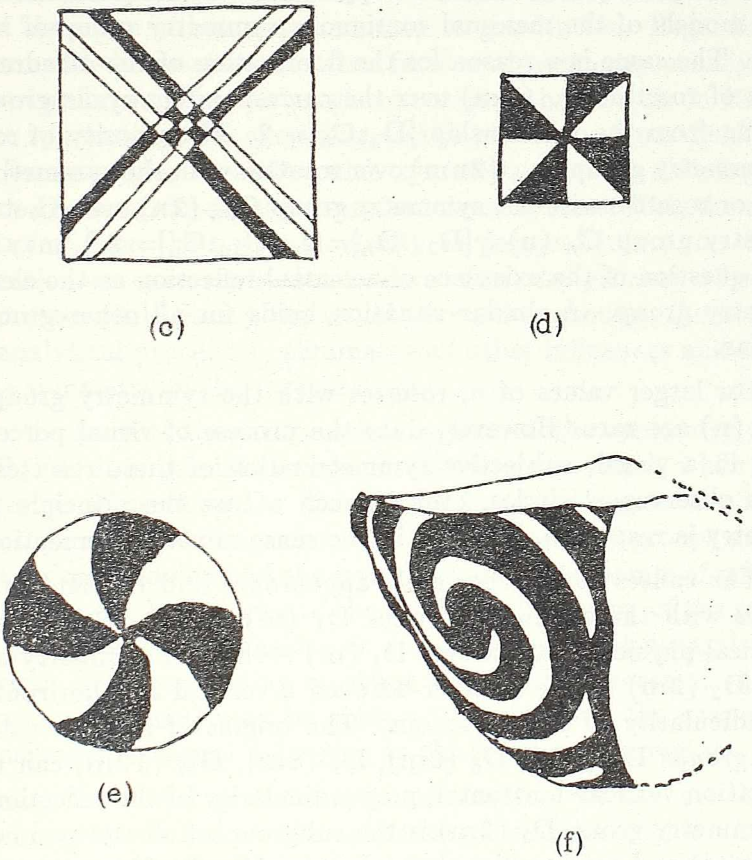


Figure 2.27

Antisymmetry rosettes in Neolithic ornamental art: (a) D_4/D_2, Danilo, Yugoslavia, about 3500 B.C.; (b) D_6/D_3, Near East; (c) D_4/C_4, Near East; (d) D_4/C_4, Middle East; (e) C_8/C_4, Middle East; (f) C_4/C_2, Dimini, Greece.

So that, by color-symmetry desymmetrizations of the symmetry groups of rosettes, it is possible to obtain exclusively symmetry groups of the type C_n (n).

Group-subgroup relations can also serve as an indicator of the frequency of occurrence of certain rosettes in ornamental art, in line with the tendency toward maximal visual simplicity and symmetry. The full expression of the principle of visual entropy is the very frequent use of circles — visual models of the maximal continuous symmetry group of rosettes D_∞ (∞m). The same is a reason for the domination of the dihedral symmetry groups of rosettes D_n (nm) over the corresponding cyclic groups C_n (n), resulting from the relationship $[D_n:C_n]= 2$. The priority of rosettes with the symmetry group D_{2n} (2nm) over rosettes with the symmetry group D_n (nm), or rosettes with the symmetry group C_{2n} (2n) over rosettes with the symmetry group C_n (n) ($[D_{2n}:D_n]= 2$, $[C_{2n}:C_n]= 2$), may be reduced to the question of the existence of a central reflection as the element of the symmetry group. A similar situation holds for all other group-subgroup relations.

For larger values of n, rosettes with the symmetry group D_n (nm) or C_n (n) are rare. However, since the process of visual perception often results in a visual, subjective symmetrization of these rosettes, perceived by the observer as circles, even in such a case the principle of maximal symmetry is respected, but only in the sense of visual perception.

The causes of the very early appearance and frequent occurrence of rosettes with the symmetry groups D_1 (m) and D_2 (2m) are mainly of a physical-physiological nature: D_1 (m) — human symmetry and binocularity, D_2 (2m) — the relation between a vertical and horizontal line and perpendicularity of the reflections. The origins of rosettes with the symmetry groups D_4 (4m), D_6 (6m), D_8 (8m), D_{12} (12m) can be found in the relation vertical–horizontal, perpendicularity of the reflections (because the symmetry group D_2 (2m) is the subgroup of all the symmetry groups mentioned) and constructional simplicity, while the frequent use of rosettes with the symmetry groups 3(m), 5(m),... results from their constructional simplicity. A considerable influence is the existence of models in nature: D_1 (m) — the symmetry of almost all living beings, D_2 (2m) — vertical and horizontal line, D_5 (5m) — a starfish, D_6 (6m) — a honeycomb, D_∞ (∞m) — all circular forms found in nature, etc.

The geometric basis of rosettes with the symmetry group D_n (nm) or C_n (n) is the construction of regular polygons, which is possible iff the number of the sides of the polygon is of the form: $2^m p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are prime Fermat numbers, i.e. prime numbers of the form $2^{2^n}+1$, $m \in N \cup \{0\}$ and $n \in N$.

The visual impression produced by a realistic rosette will be formed in the interaction between the symmetry group of the rosette itself and the visual, subjective factors of symmetry. Some of them are the human plane symmetry and binocularity, the symmetry of the limited part of the plane to which the rosette belongs and the symmetry group $\mathbf{D}_2$ (2m) conditioned by the fundamental natural directions — vertical and horizontal, i.e. by the action of the sense of balance connected to gravitation. Regarding the symmetry group of the rosette, the other factors of symmetry usually occur as desymmetrization factors, although sometimes they result in its visual, subjective symmetrization according to the principle of maximal visual simplicity. For example, a rosette with symmetry slightly differing from the symmetry $\mathbf{D}_1$ (m), the observer sees as mirror-symmetrical, the rosette slightly deviating from the perpendicularity as perpendicular, etc. Such a desymmetrization or symmetrization occurs after the primary visual impression is formed, while during the visual perception process, the observer, through an analytical procedure, eliminates all other influences and aims to recognize the symmetry of the rosette itself.

Due to the form of the fundamental region, at the symmetry groups of rosettes $\mathbf{C}_n$ (n) it is possible to use curvilinear boundaries, while at the symmetry groups of rosettes $\mathbf{D}_n$ (nm) a fundamental region has a fixed shape and rectilinear boundaries, because those symmetry groups are generated by reflections and demand the invariance of all the points of reflection lines. By changing the boundaries of the fundamental region of the symmetry group $\mathbf{C}_n$ (n) we may emphasize or decrease the visual dynamism and realize a variety of corresponding rosettes. A variety of rosettes with the symmetry group $\mathbf{D}_n$ (nm) may be achieved by changing the shape of an elementary asymmetric figure belonging to the fundamental region.

Analyzing the visual characteristics of rosettes, we can use data on the polarity of rotations and on the existence of enantiomorphic modifications, this directly indicating the dynamic or static visual impression that the given rosette will suggest. Rosettes with polar, oriented rotations will produce a dynamic, while rosettes with non-polar, non-oriented rotations will produce a static impression. The polarity of rotations of the symmetry groups of rosettes will depend on the existence of an indirect transformation — reflection as the element of the symmetry group. Rosettes with the symmetry group $\mathbf{C}_n$ (n) will be dynamic rosettes with polar, oriented rotations, and those with the symmetry group $\mathbf{D}_n$ (nm) will be static rosettes with non-polar rotations. For n — an even natural number, a central reflection is the element of the discrete symmetry groups $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm) and of the continuous symmetry group $\mathbf{D}_\infty$ (∞m). For $n = 4m+2$, decomposition $C_{4k+2} = C_2 \times C_{2m+1} = \{Z\} \times \{S^2\}$, $D_{4m+2} = C_2 \times D_{2m+1} = \{Z\} \times \{S^2, R\}$

holds, directly indicating to the corresponding subgroups and to the existence of the subgroup $\mathbf{C}_2$ (2) generated by the central reflection Z.

All the symmetry groups of the category G_{210}, namely $\mathbf{C}_1$ (1), $\mathbf{D}_1$ (m), $\mathbf{C}_2$ (2) and $\mathbf{D}_2$ (2m), are included in the category G_{20}. The group $\mathbf{C}_2$ (2) is the subgroup of the group $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm), and the group $\mathbf{D}_2$ (2m) is the subgroup of the group $\mathbf{D}_n$ (nm) iff n is an even natural number.

The visual impression produced by a certain rosette will be influenced also by the enantiomorphism — the "left" and "right" orientation of rosettes, representing an important part of the general problem of orientation in nature (H. Weyl, 1952; R. Arnheim, 1965).

From the point of view of ornamental art, of special interest are the symmetry groups $\mathbf{D}_n$ (nm) with n - an even natural number. Since one of their subgroups is the symmetry group $\mathbf{D}_2$ (2m), there is the possibility to place the rosette with the symmetry group $\mathbf{D}_n$ (nm) in such a position that the perpendicular reflections of the subgroup $\mathbf{D}_2$ (2m) coincide with the fundamental natural perpendiculars — vertical and horizontal lines.

A complete table survey of subgroups of the given symmetry group presents, in the geometric and also in the visual sense, the evidence of their symmetry substructures. Possibilities for their visual recognition depend on the nature of the substructures themselves, their impressiveness, visual simplicity, dynamism or stationariness, the nature of the rosette to which they belong and on the position of the substructure regarding the visual dominants — reflections, vertical and horizontal lines, etc. Necessary data can be also supplied by a survey giving decompositions of symmetry groups of rosettes.

Cayley diagrams are visual interpretations of symmetry groups of rosettes, giving complete information on symmetry groups. According to the established connection between the geometric-algebraic properties of the symmetry groups of rosettes and their visual models, many important visual characteristics of a rosette are implied by the structure of their symmetry group.

In the development of the theory of symmetry very important have been the visual interpretations of symmetry groups: ornaments, graphic symbols of symmetry elements and Cayley diagrams. Visual examples have been the motives for further analysis and discussion on the corresponding symmetry groups. In a modern science, instances of a reversed process — from abstract groups to their visual models — are frequent, especially in those cases where theory precedes the practice.

The discussion of the visual characteristics of rosettes given in this work, based on the theory of symmetry, can be used also in ornamental design, for the construction of new rosettes through anticipating their visual properties, and as a basis for exact aesthetic analyses. Also, they can be applied in all scientific fields in needing of visual interpretations of symmetry groups.

2.3. Symmetry Groups of Friezes G_{21}

In the plane S^2 and E^2 there are seven discrete, one-dimensional, line groups of isometries, the symmetry groups of friezes G_{21}: **p11 (11)**, **p1g (1g)**, **p12 (12)**, **pm1 (m1)**, **p1m (1m)**, **pmg (mg)**, **pmm (mm)** and two visually presentable continuous symmetry groups of friezes: $\mathbf{p_0m1=pm_01}$ $(\mathbf{m_01})$ and $\mathbf{p_0mm=pm_0m}$ $(\mathbf{m_0m})$.

To denote them, we have used the simplified version of International Two-dimensional Symbols (M. Senechal, 1975; H.S.M. Coxeter, 1985). Here, the first symbol represents an element of symmetry perpendicular to the direction of the translation, while the second denotes an element of symmetry parallel or perpendicular (exclusively for 2-rotations) to the direction of the translation.

Presentations and structures:

11	$\{X\}$				C_∞
1g	$\{P\}$				C_∞
12	$\{X,T\}$	$T^2=(TX)^2=E$			D_∞
	$\{T,T_1\}$	$T^2=T_1^2=E$	$(T_1=TX)$		
m1	$\{X,R_1\}$	$R_1^2=(R_1X)^2=E$			D_∞
	$\{R_1,R_2\}$	$R_1^2=R_2^2=E$	$(R_2=R_1X)$		
1m	$\{X,R\}$	$R^2=E$	$RX=XR$		$C_\infty\times D_1$
mg	$\{P,R_1\}$	$R_1^2=(R_1P)^2=E$			D_∞
	$\{R_1,T\}$	$R_1^2=T^2=E$	$(T=R_1P)$		
mm	$\{X,R,R_1\}$	$R^2=R_1^2=(R_1X)^2=E$	$RX=XR$	$RR_1=R_1R$	$D_\infty\times D_1$
	$\{R,R_1,R_2\}$	$R^2=R_1^2=R_2^2=E$	$RR_1=R_1R$	$RR_2=R_2R$	$(R_2=R_1X)$

Form of the fundamental region: unbounded, allows variation of the boundaries that do not belong to reflection lines.

Enantiomorphism: **11**, **12** possess enantiomorphic modifications, while in other cases the enantiomorphism does not occur.

Polarity of rotations: polar rotations — **12**, **mg**; non-polar rotations — **mm**, $\mathbf{m_0m}$.

Polarity of translations: polar translations — **11, 1g, 1m**; bipolar translations — **12**; non-polar translations — **m1, mg, mm, m$_0$1, m$_0$m.**

The table of minimal indexes of subgroups in groups:

	11	1g	12	m1	1m	mg	mm
11	2						
1g	2	3					
12	2		2				
m1	2			2			
1m	2	2			2		
mg	4	2	2	2			*3*
mm	4	4	2	2	2	2	2

All the discrete symmetry groups of friezes are subgroups of the group **mm** generated by reflections and given by the presentation:

$$\mathbf{mm}\ \{R, R_1, R_2\}\ R^2 = R_1^2 = R_2^2 = E\ \ RR_1 = R_1R\ \ RR_2 = R_2R\ \ D_\infty \times D_1$$

$R_1,\ T = RR_2$	generate	**mg**	D_∞
$R,\ X = R_1R_2$	generate	**1m**	$C_\infty \times D_1$
$R_1,\ X$	generate	**m1**	D_∞
$X,\ T = RR_1$	generate	**12**	D_∞
$P = RR_1R_2$	generates	**1g**	C_∞
X	generates	**11**	C_∞

The survey of the characteristics of the symmetry groups of friezes relies on the work of A.V. Shubnikov, V.A. Koptsik, 1974; H.S.M. Coxeter, W.O.J. Moser, 1980.

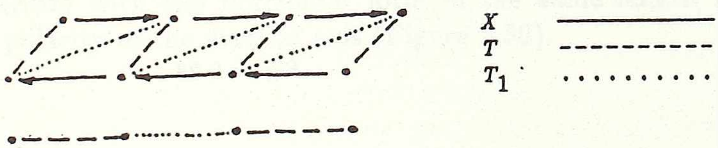
The first derivation of the symmetry groups of friezes as the line subgroups of the symmetry groups of ornaments G_2 and their complete list, was given by G. Pólya (1924), P. Niggli (1926) and A. Speiser (1927).

Cayley diagrams (Figure 2.28):

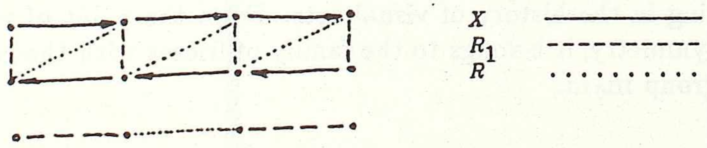
11

1g

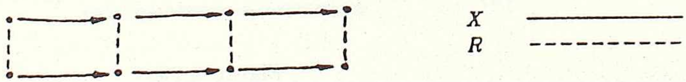
12



m1



1m



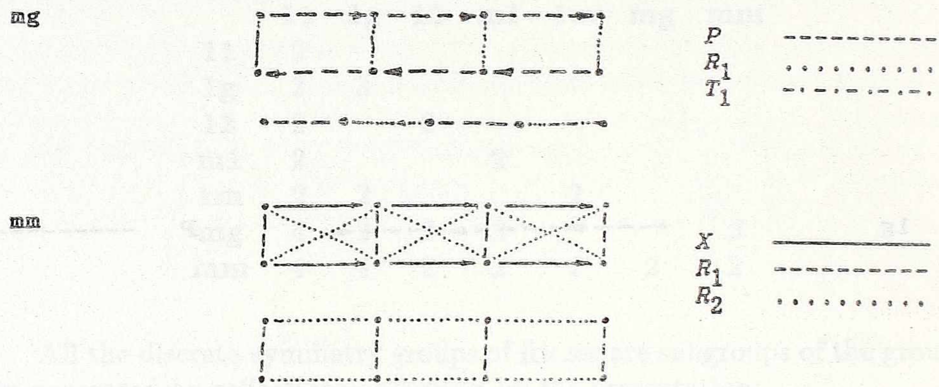


Figure 2.28

2.4. Friezes and Ornamental Art

A straight line is, most probably, the first and simplest frieze occurring in the history of visual arts. From the point of view of the theory of symmetry, it belongs to the family of friezes with the continuous symmetry group m_0m.

Figure 2.29

Continuous visually presentable symmetry groups of friezes (a) m_0m and (b) m_01.

Recent investigations of the process of visual perception have discovered that in a visual sense, the perceived symmetry of a line will depend on its position.

A horizontal line has the symmetry $\mathbf{m_0 m}$ — maximal continuous symmetry group of friezes (Figure 2.29a).

Regarded from the visual point of view, a vertical line has a polar, oriented continuous translation axis and the continuous symmetry group $\mathbf{p_0 1m}$ ($\mathbf{0m}$). The phenomenon of a visual, subjective desymmetrization of a vertical line is probably implied by gravitation. It is possible that the tendency of a vertical line to go upward results from the objective narrowing of vertical objects in the upper part; from the visual convergence of parallel lines with high objects observed from a lower position; and from the habit conditioned by a constant application of the central perspective, where the center of the perspective is usually placed in the upper part of the picture. A vertical line has the "directed tension", so that along with the objective symmetry $\mathbf{m_0 m}$, occurs its visual, subjective desymmetrization and the reduction to the symmetry $\mathbf{0m}$. The tendency to accept a vertical form as longer in comparison with the horizontal form of the same length is conditioned by the polarity of the vertical axis (Figure 2.30).

Figure 2.30

Illustration of the visual effect of polarity of the vertical line (the vertical and horizontal line segments are of the same length).

A similar visual, subjective desymmetrization — a reduction to a lower degree of symmetry and the polarity of the axis — occurs also with an "ascending" and "descending" diagonal. This phenomenon can be also treated as the difference between the "right" and "left" (H. Weyl, 1952, pp. 16; R. Arnheim, 1965, pp. 22), in the theory of visual perception discussed by H. Wölfflin, M. Gaffron, A. Dean and S. Cobb. Along with different physiological interpretations, there is an interesting Weyl's question about whether this phenomenon is connected to writing from the left to right, and whether it is expressed by nations writing in the opposite direction. M.Gaffron tried to explain it by the dominance of the left brain cortex containing brain centers for several activities in right-handed people, i.e. in the dominance of righthandness. About this, S. Cobb says (R. Arnheim, 1965, pp. 23): "Many fanciful ideas have been put forward, from the theory that the left hemisphere has a better blood supply than the right, to the heliocentric theory that the right hand dominates because man originated north of the equator and, looking at the sun, was impressed with the fact that great things move toward the right! Thus right became the symbol of rectitude and dexterity, and things on the left were sinister. It is an interesting observation that about 70 per cent of human foetuses lie in the uterus in the "left occiput posterior" position, i.e. facing to the right. No one has ever found out whether or not these become the right majority of babies. Probably the dominance of right-handness is due to chance in heredity.".

All the continuous symmetry groups of friezes have physical interpretations (A.V. Shubnikov, V.A. Koptsik, 1974). In the visual arts, besides visual models of the continuous symmetry groups of friezes $m_0 1$ and $m_0 m$ (Figure 2.29) with the non-polar translation axis, adequate visual interpretations of the other continuous symmetry groups of friezes with the polar or bipolar translation axis, can be realized, in the sense of the objective symmetry, exclusively by textures. Textures can be realized by equal average density of the same asymmetric elementary figures arranged along the singular direction in accordance with the corresponding continuous symmetry group. For a schematic visual interpretation of these groups by using the graphic symbols of symmetry elements, certain supplementary symbols (e.g., arrows) are indispensable (B. Grünbaum, G.C. Shephard, 1987).

The oldest examples of friezes are found in the art of the Paleolithic (Magdalenian, 12000–10000 B.C.) and Neolithic. In fact, examples of all seven discrete and two visually presentable symmetry groups of friezes are known from the Magdalenian period.

Despite a relative variety of motifs in ornamental art, there is also a repetition of ornamental motifs — basic, elementary patterns occurring in different parts of Europe, Asia and Africa, where the late Paleolithic and Neolithic cultures were formed. Since the possibilities for communication between distant areas were remote, we can assume that common ornamental motifs result from similar or the same models found in nature by prehistoric peoples and from the laws of symmetry.

Most frequently, friezes are the result of a translational repetition of different motifs, where the symmetry of the original motif — a rosette — determines the symmetry of the frieze itself, in the sense of composition of the translational group **11** and the symmetry group of the rosette or by an artistic schematization of natural objects possessing by themselves the symmetry of friezes.

The origins of friezes are visible in cave drawings and engravings on stones or bones from the earliest period — late Paleolithic.

Representing a herd of deer (Figure 2.31), prehistoric man had abstracted a motif, almost reducing it to a translational repetition of a pair of horns, i.e. to the frieze with the symmetry group **11**. A similar process, from the motif of dance to a frieze with the symmetry group **m1**, from the motif of harpoon to a frieze with the symmetry group **1m**, from the motif of waves to a frieze with the symmetry group **12** or **mg**, shows the original symbolic meanings friezes carried. Thus, friezes became one of the first visual communication means.

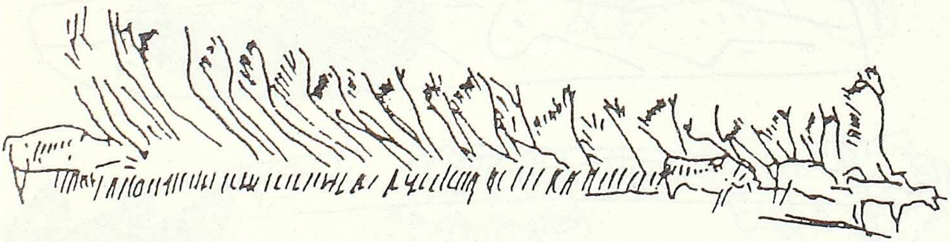


Figure 2.31

The origin of friezes with the symmetry group **11** by the stylization of natural models (Paleolithic, Altamira cave, Spain).

Also, the symmetry of friezes based on repetition, have made possible a symbolic representation of certain periodic natural phenomena — the turn of day and night, the daily and annual revolving of the Sun — where friezes played the role of a calendar. This is witnessed by friezes originating from primitive art and having even today precisely defined symbolic meanings and adequate names (Figure 2.54).

The symmetry group of friezes 11 is the result of a periodic, translational repetition of an asymmetric figure. The form of the fundamental region can be arbitrary. Owing to the polarity of the frieze axis and the sheer repetition of the motif, besides a possibility for figurative representation, there is also a possibility for a geometric-symbolic representation of directed phenomena. Hence, directed tension resulting from the polarity of the axis creates an impression of "motionless motion", thus forming the time component of painting. In combination with overlapping which makes possible the suggestion of the perspective in the sense "in front–behind", friezes have been frequently used in Egyptian art, both decorative and painting, and in the art of cultures tending to the "objective", natural axonometric presentation of spatial groups in motion (Figure 2.32–2.35).

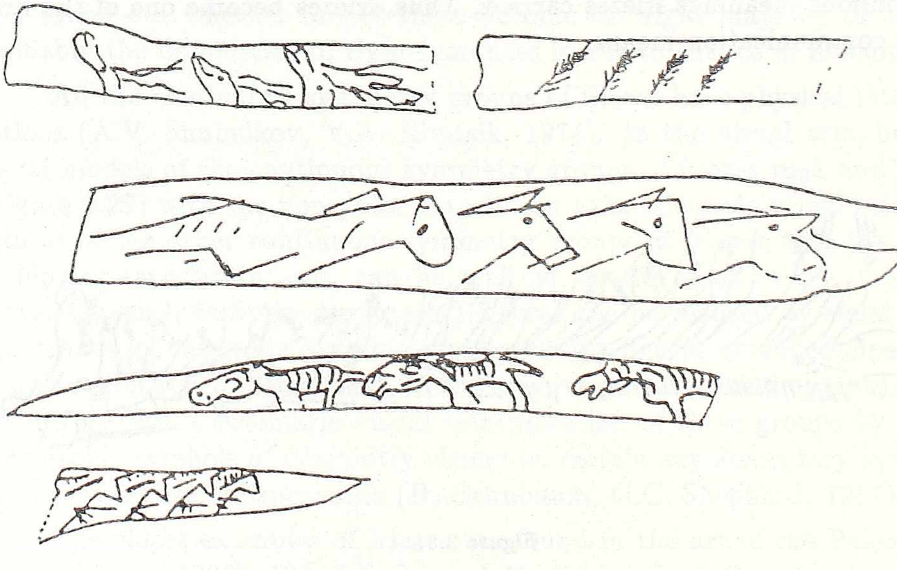


Figure 2.32

Examples of friezes with the symmetry group 11 in Paleolithic art.

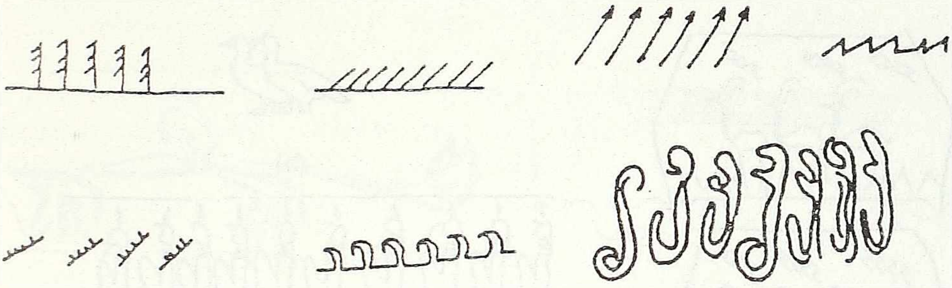


Figure 2.33

Examples of friezes with the symmetry group **11** and the formation of geometric ornamental motifs by stylization and schematization of natural models (Paleolithic).

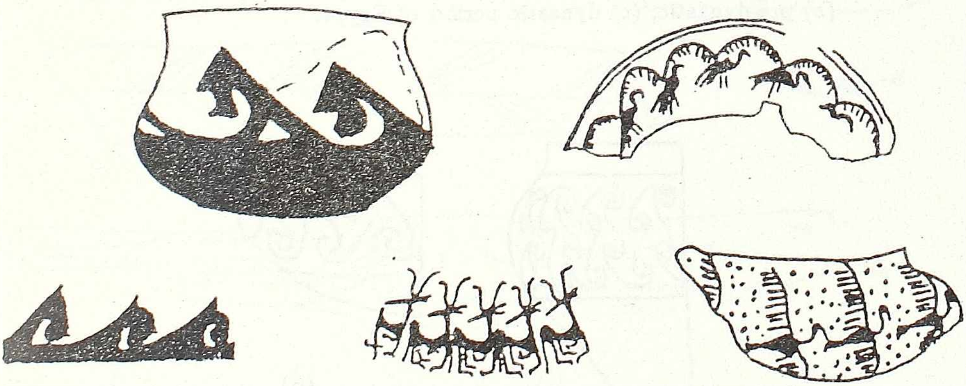


Figure 2.34

Examples of friezes with the symmetry group **11** in Neolithic art (Hallaf ceramics, around 5500–4500 B.C.).

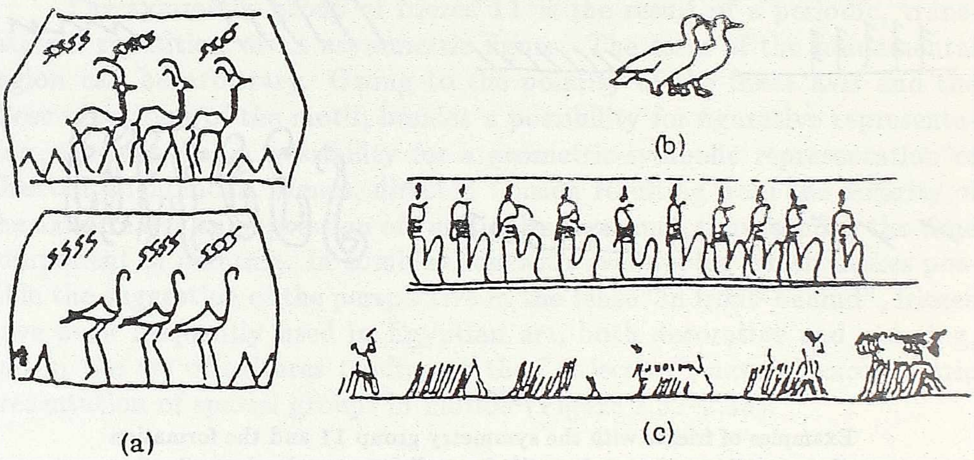


Figure 2.35

Examples of friezes with the symmetry group 11 in the art of (a), (b) pre-dynastic; (c) dynastic period of Egypt.

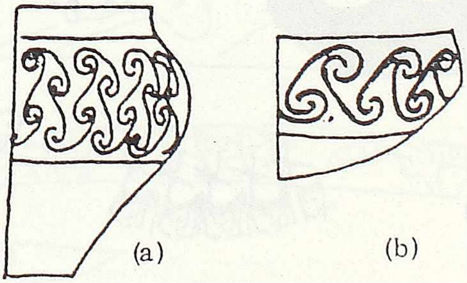


Figure 2.36

Examples of friezes with the symmetry group 1g in the Neolithic art of Yugoslavia: (a) Butmir, around 3500 B.C.; (b) Adriatic zone, around 3000–2000 B.C.

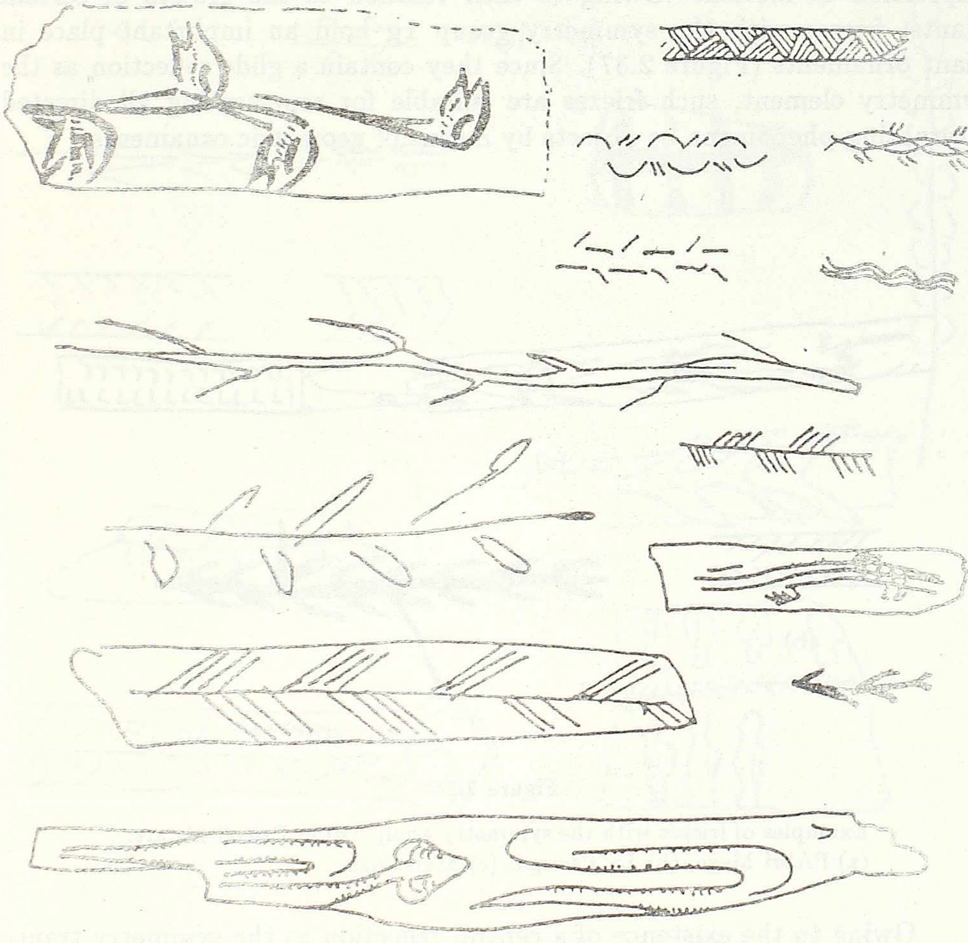


Figure 2.37

Examples of friezes with the symmetry group 1g in the art of the late Paleolithic and early Neolithic (Fontarnaud-Lugasson, Laugerie Haute, Le Placard, Marsoulas, around 15000–8000 B.C.).

A similar dynamic effect is produced by friezes with the symmetry groups **1g** (Figure 2.36, 2.37) and **1m** (Figure 2.38–2.40) which have a polar axis. By using different elementary asymmetric figures belonging to the fundamental region and different directed forms with acute angles oriented toward the direction of the axis, it is possible to intensify the already existing impression of motion. Owing to their relation to the growth of certain plants, friezes with the symmetry group **1g** hold an important place in plant ornaments (Figure 2.37). Since they contain a glide reflection as the symmetry element, such friezes are suitable for representing all directed alternating phenomena or objects by means of geometric ornaments.

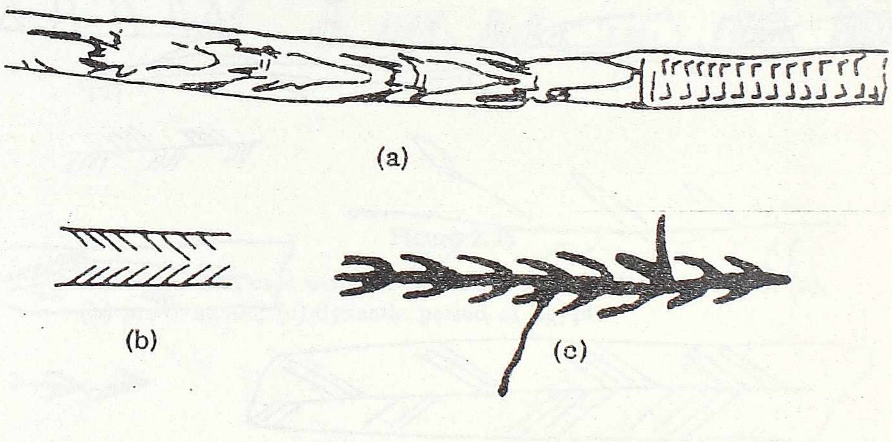


Figure 2.38

Examples of friezes with the symmetry group **1m** in Paleolithic art: (a) l'Abri Mege; (b) La Pasiega; (c) Marsoulas.

Owing to the existence of a central reflection as the symmetry transformation of the group **12**, the frieze axis is bipolar, so friezes with the symmetry group **12** offer the possibility for registering oppositely directed elementary asymmetric figures along the singular direction, i.e. two oppositely directed friezes with the symmetry group **11**. Friezes with the symmetry group **12** occur in many cultures (in the Neolithic, Egyptian, Aegean, etc.), with the application of spiral motifs (Figure 2.41–2.45).

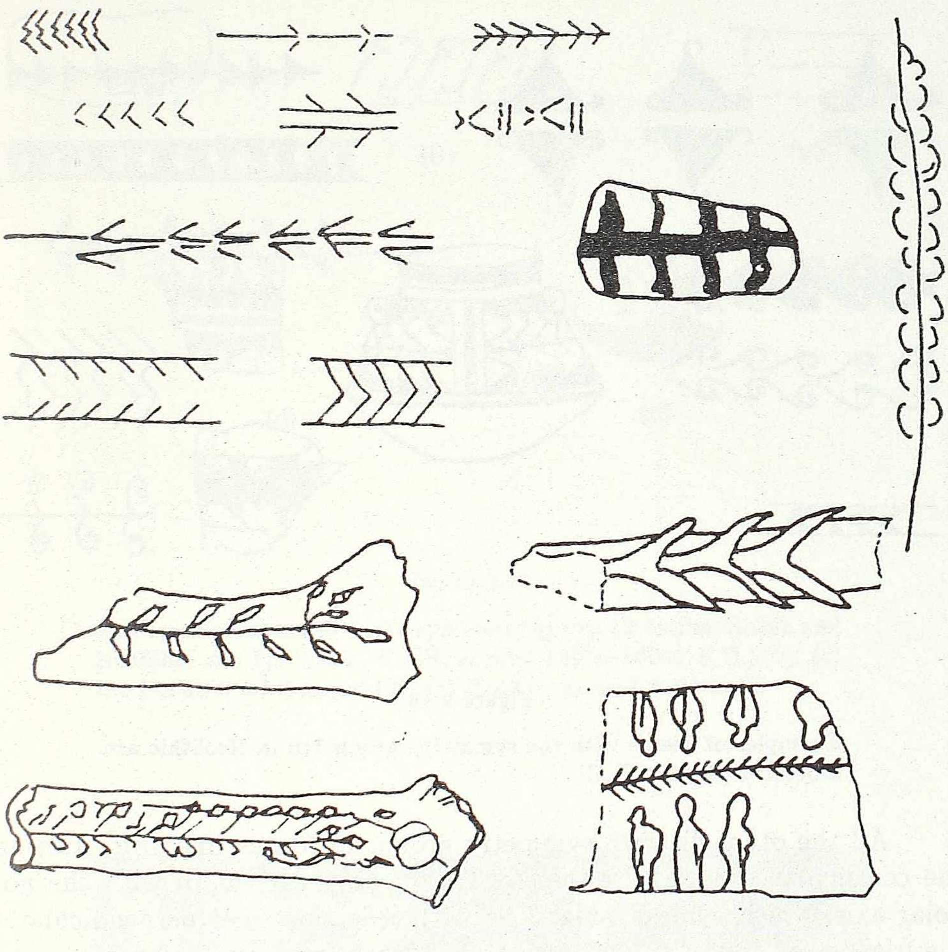


Figure 2.39

Friezes with the symmetry group **1m** in Paleolithic art (Maz d'azil, La Madlene, Barma Grande, Laugerie Base, around 10000 B.C.).

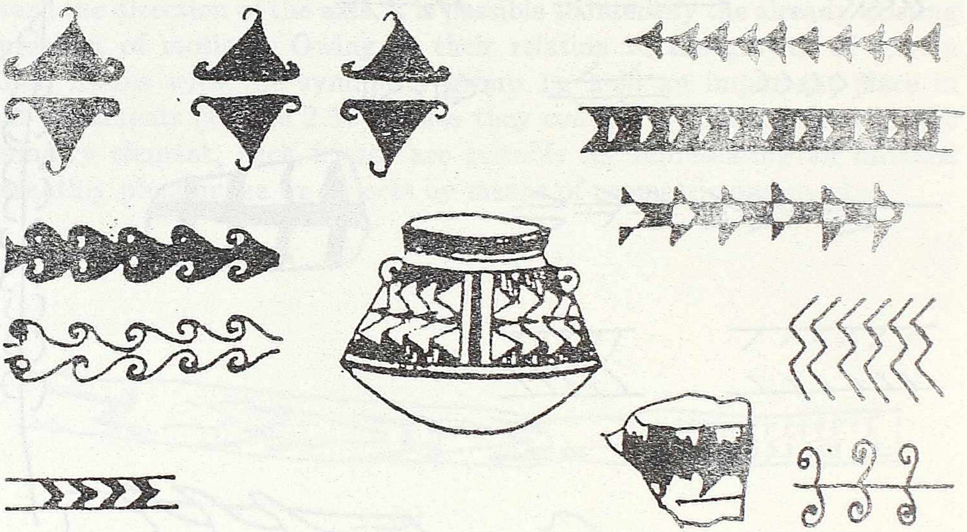


Figure 2.40

Examples of friezes with the symmetry group 1m in Neolithic art, around 6000–3000 B.C.

All the other discrete symmetry groups of friezes **m1**, **mg**, **mm** and the continuous symmetry groups of friezes $\mathbf{m_0 1}$ and $\mathbf{m_0 m}$ with the non-polar axis, which contain reflections with reflection lines perpendicular to the frieze axis, create an impression of stationariness and balance.

Friezes with the symmetry group **m1** were frequently used in the prehistoric period with the motif of the cult dance "kolo". The origin and development of this motif can be seen in the stone drawings from the period of cave painting (Figure 2.46). This motif, very characteristic and suitable for the analysis of the history of ornamental art, underwent a considerable stylization in the Neolithic period (Figure 2.47). During the Neolithic, by losing its symbolic meaning and by being enriched by new elements, the motif of dance and the corresponding friezes were reduced to sheer decorativeness. All types of ornaments underwent this process (Figure 2.48).

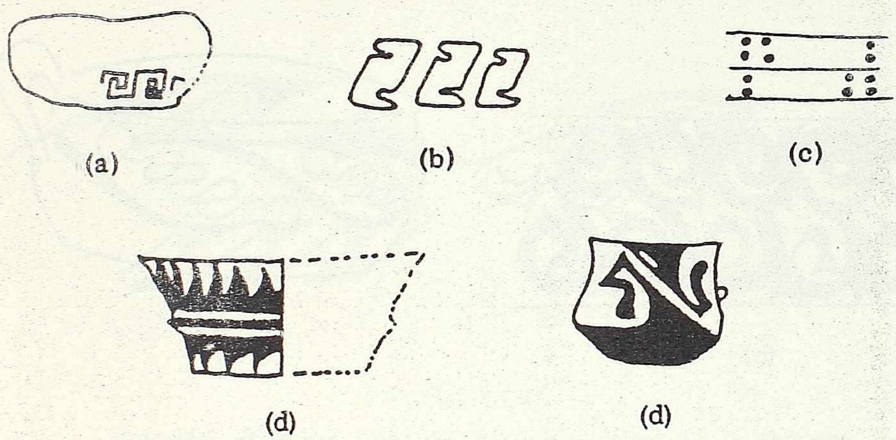
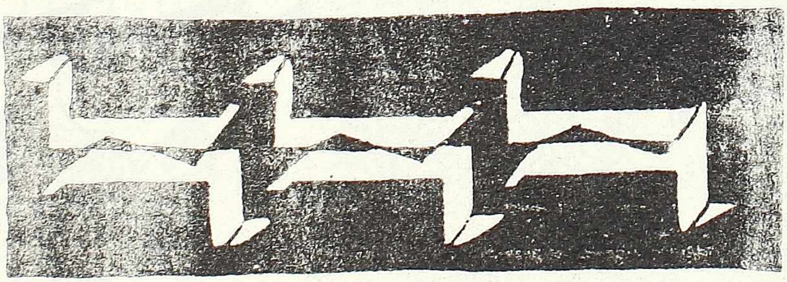
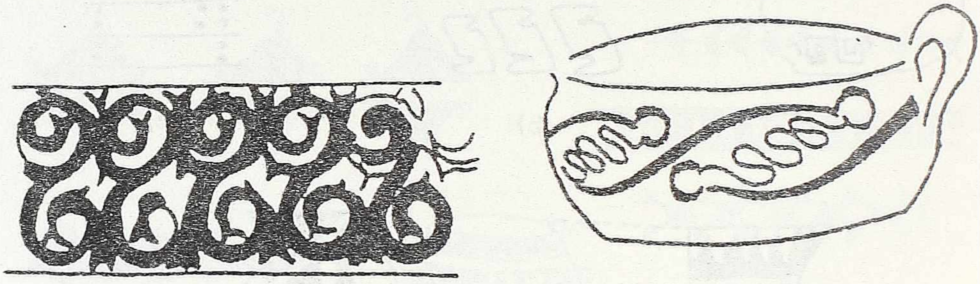


Figure 2.41

Examples of friezes with the symmetry group **12** in Paleolithic and Neolithic art: (a) Mezin, USSR, around 12000–10000 B.C.; (b), (c) the Neolithic of Europe; (d), (e) Hacilar, around 5300 B.C.



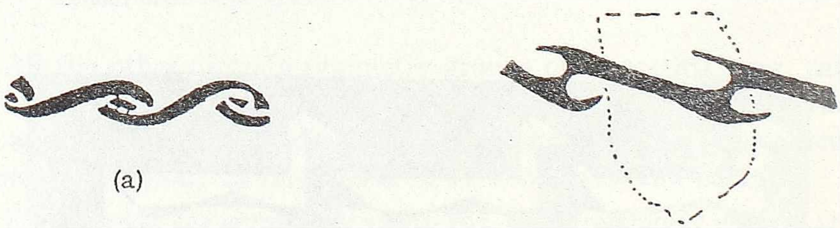
(a)



(b) (c)

Figure 2.42

Examples of friezes with the symmetry group 12: (a) Bakun, Iran, around 5000–4000 B.C.; (b) Malta, around 3000 B.C.; (c) Crete, around 3000–2500 B.C.



(a) (b)

Figure 2.43

Friezes with the symmetry group 12 in the Neolithic art of Yugoslavia: (a) Aznabegovo-Vrshnik, around 5000 B.C.; (b) Hvar, around 2500 B.C.

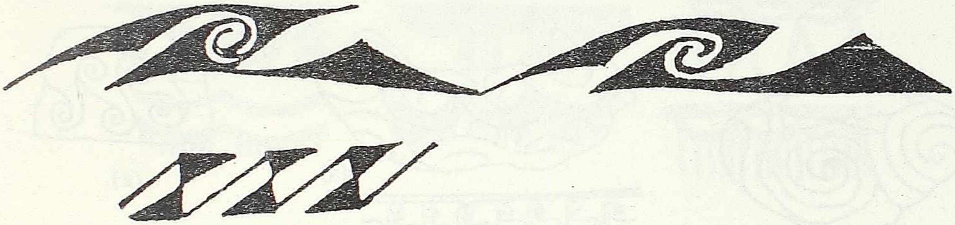
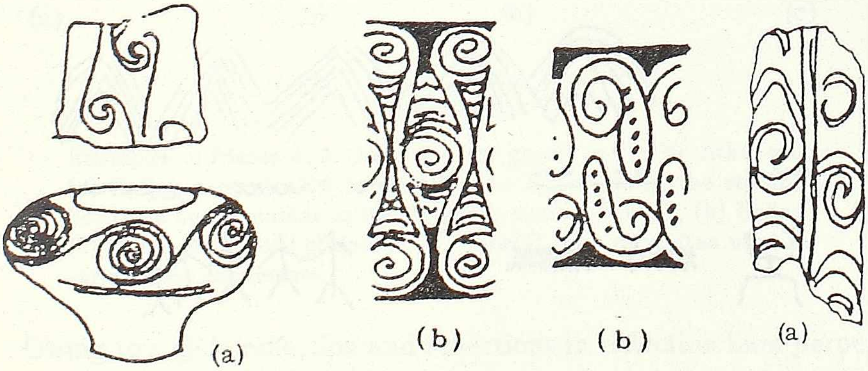


Figure 2.44

Examples of friezes with the symmetry group **12** in the pre-Columbian art of America (Mexico).



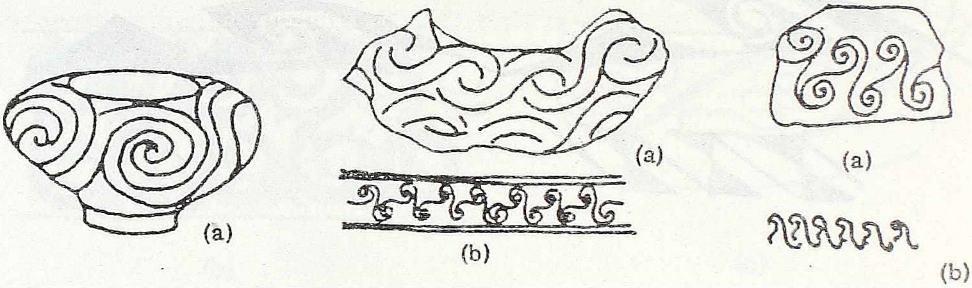


Figure 2.45

Friezes with the symmetry group **12** with the application of spiral motifs: (a) Neolithic art, Butmir, Yugoslavia; (b) Egypt.

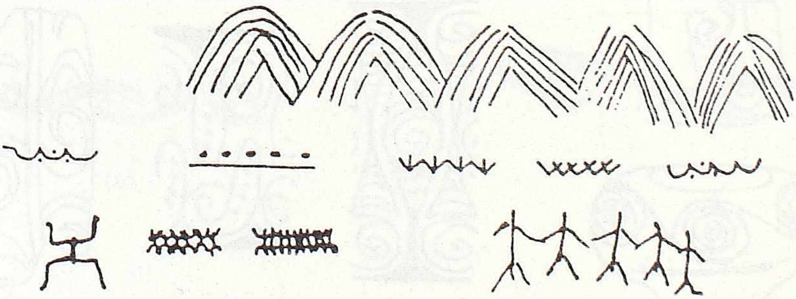


Figure 2.46

Examples of friezes with the symmetry group **m1** in the late Paleolithic (Magdalenian) and early Neolithic.

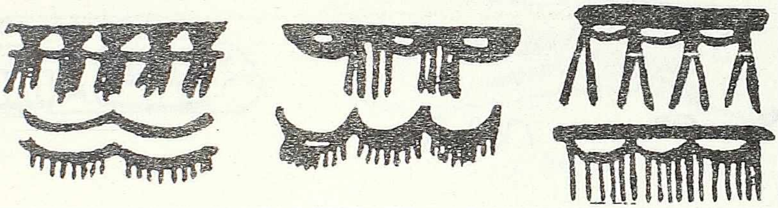


Figure 2.47

The formation and development of friezes with the symmetry group **m1**, with the "kolo" motif, in prehistoric art.

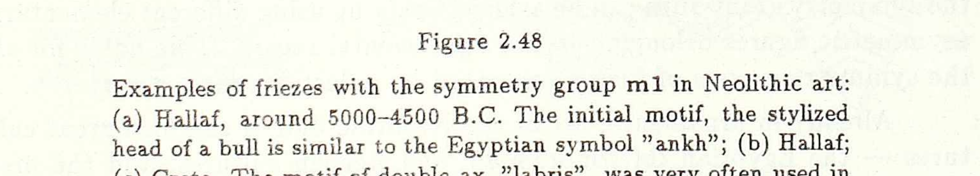


Figure 2.48

Examples of friezes with the symmetry group **m1** in Neolithic art: (a) Hallaf, around 5000–4500 B.C. The initial motif, the stylized head of a bull is similar to the Egyptian symbol "ankh"; (b) Hallaf; (c) Crete. The motif of double ax, "labris", was very often used in early Greek ornaments.

Owing to a glide reflection and reflections in reflection lines perpendicular to the frieze axis, friezes with the symmetry group **mg**, among geometric ornaments occur as symbols of regular alternating phenomena (Figure 2.49–2.51). Different variations of these friezes have in primitive art the following meanings (Figure 2.54a, b, c, d): "Up and down", "The daily motion of the Sun", "The Sun above and below the water level (the horizon)", "Breathing", "Water", "The rhythm of water".

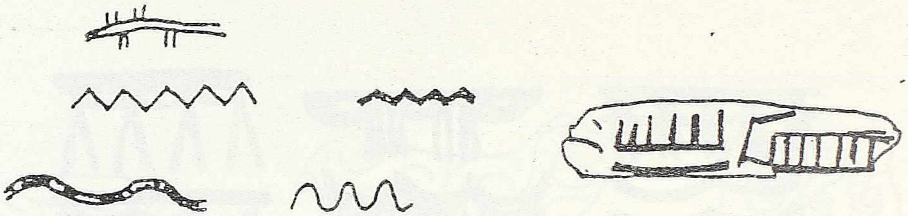


Figure 2.49

Examples of friezes with the symmetry group mg in Paleolithic art.

Friezes with the symmetry group mm (Figure 2.52, 2.53) often symbolize an even flow of time, years and similar phenomena with a high degree of symmetry (Figure 2.54f, g, h). It is interesting that in such cases time is considered as non-polar. Owing to the maximal degree of symmetry and to the fact that it contains all the other discrete symmetry groups of friezes as subgroups, besides having a significant independent function, the symmetry group mm will be a basis from which all the other discrete symmetry groups of friezes can be derived by means of the desymmetrization method. Having in mind the fixed shape of the fundamental region — rectilinear, perpendicular, unbounded fundamental region — a variety of friezes with the symmetry group mm can be achieved only by using different elementary asymmetric figures belonging to the fundamental region. This holds for all the symmetry groups of friezes generated by reflections (m1, mm).

Already in ornamental art of the Neolithic and of the first great cultures — the Egyptian, Mesopotamian, and Aegean cultures, and the pre-Columbian culture in America, etc. — by introducing new ornamental motifs and by enriching existing ones, the variety of friezes is achieved (Figure 2.55). Like for rosettes, superpositions of friezes are frequent. Some primary symbolic meanings have been gradually replaced by new ones. The application of different motifs unavoidably leads to decorativeness. The empirical perception of the properties of friezes and the regularities they are based on, resulted in new friezes constructed by using the construction rules comprehended, thus opening the way to artistic imagination and creation-play.

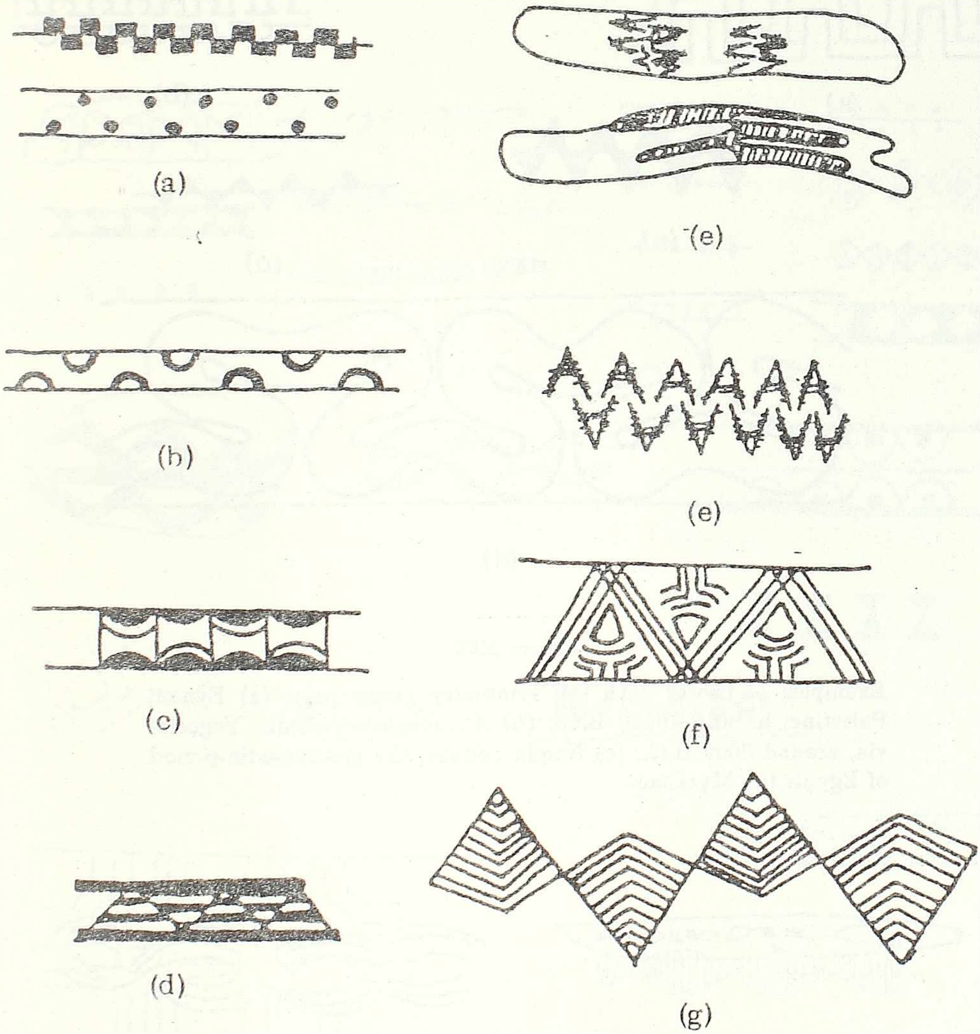


Figure 2.50

Examples of friezes with the symmetry group mg in Neolithic art: (a) the early Neolithic of Europe; (b) Catal Hüjük, around 6400–5800 B.C.; (c) Hallaf, around 6000 B.C. (7600–6900 B.C.?); (d) Hasuna, Iraq; (e) Magelmose, 7500–6500 B.C.; (f) Pakistan around 3000 B.C.; (g) the pre-dynastic period of Egypt, around 4200–3600 B.C.

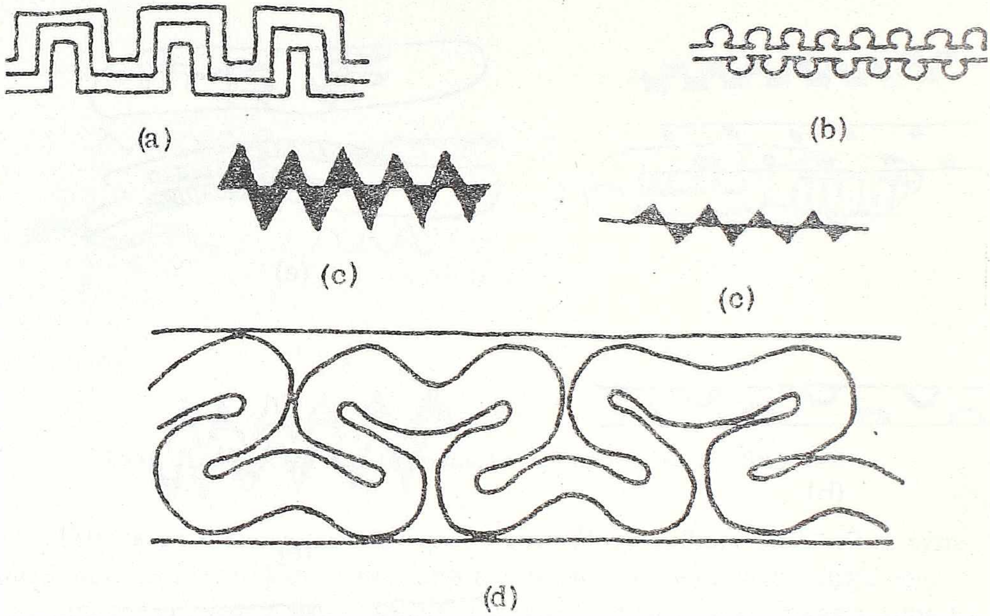


Figure 2.51

Examples of friezes with the symmetry group mg: (a) Eynan, Palestine, around 10000 B.C.; (b) Aznabegovo-Vrshnik, Yugoslavia, around 5000 B.C.; (c) Naqda culture, the pre-dynastic period of Egypt; (d) Mycenae.

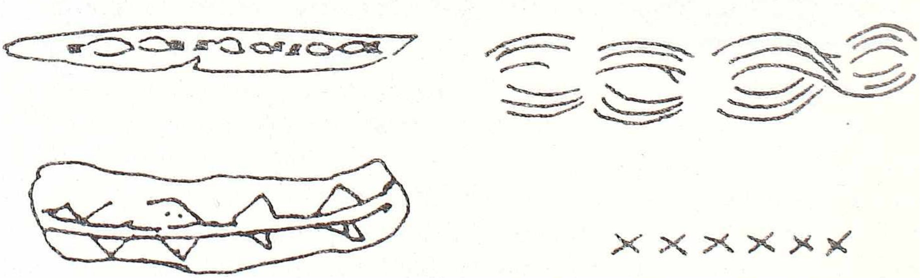


Figure 2.52

Friezes with the symmetry group mm in Paleolithic art.

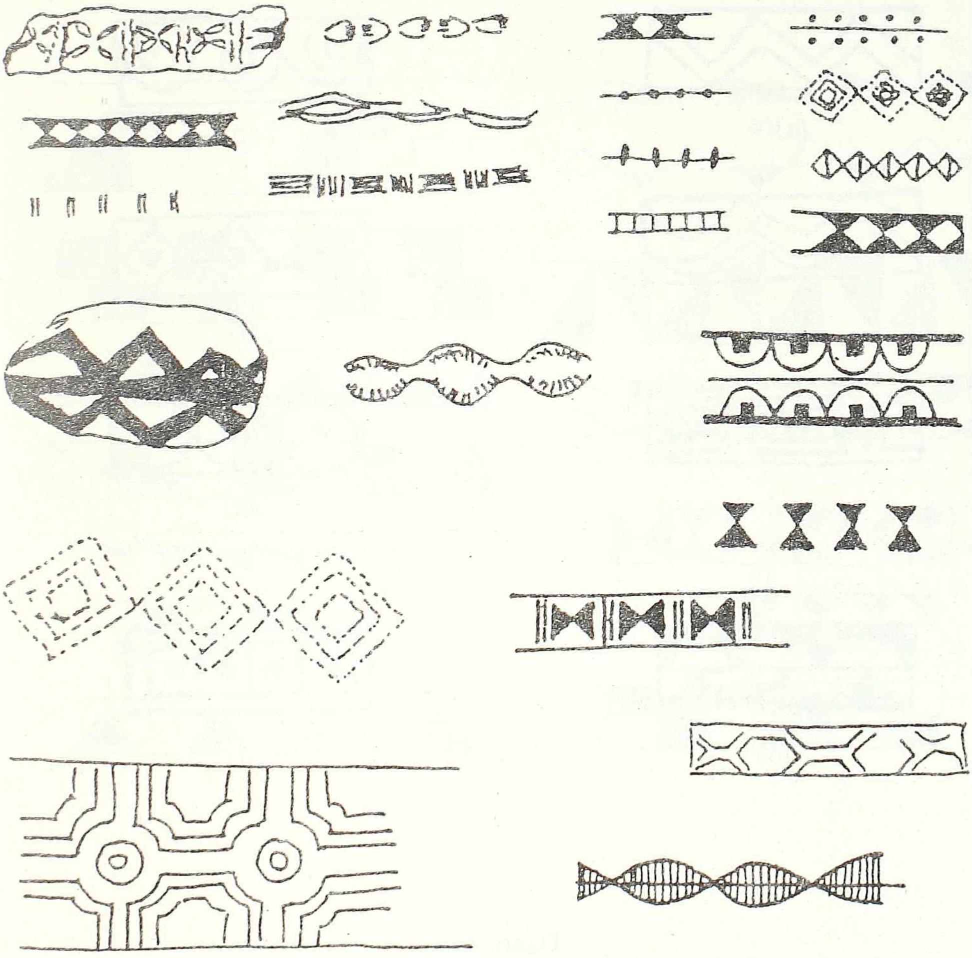


Figure 2.53

Friezes with the symmetry group mm in the Paleolithic (Magdalenian, around 10000 B.C.) and Neolithic.

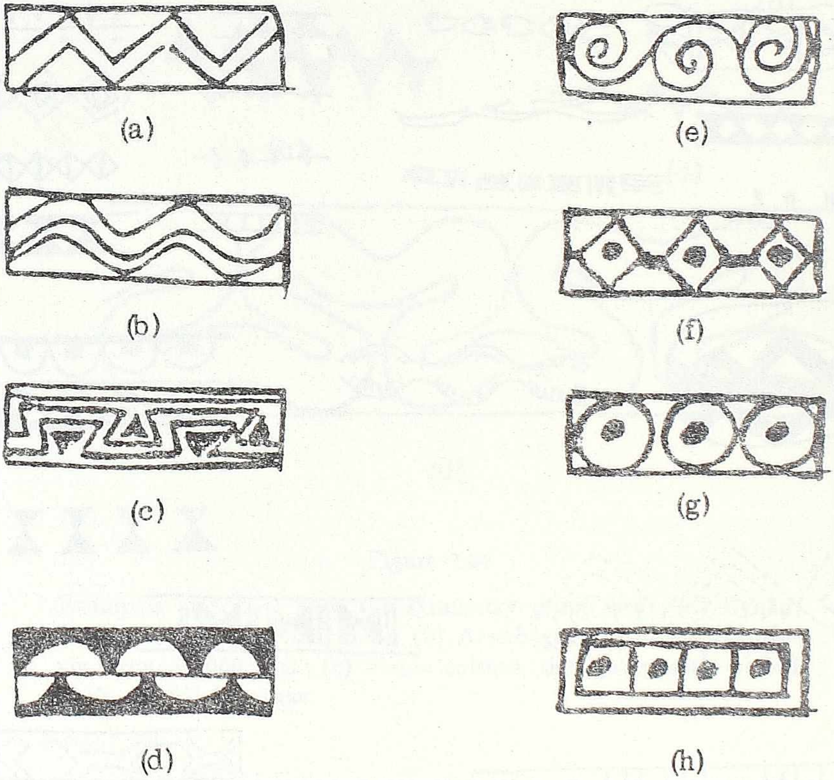


Figure 2.54

Examples of friezes in the art of primitive peoples, that possess precise symbolic meanings and the corresponding names: (a) "Up and down", "The Sun", Water", "Breathing"; (b) "The rhythm of water" (Congo); (c) The Sun above and below water (horizon)" (Pueblo Indians); (d) "Days of the full Moon" (Celebes); (e) "Endless running of the years" (Celebes); (f), (g), (h) "The continual motion of the Sun" (Fiji).

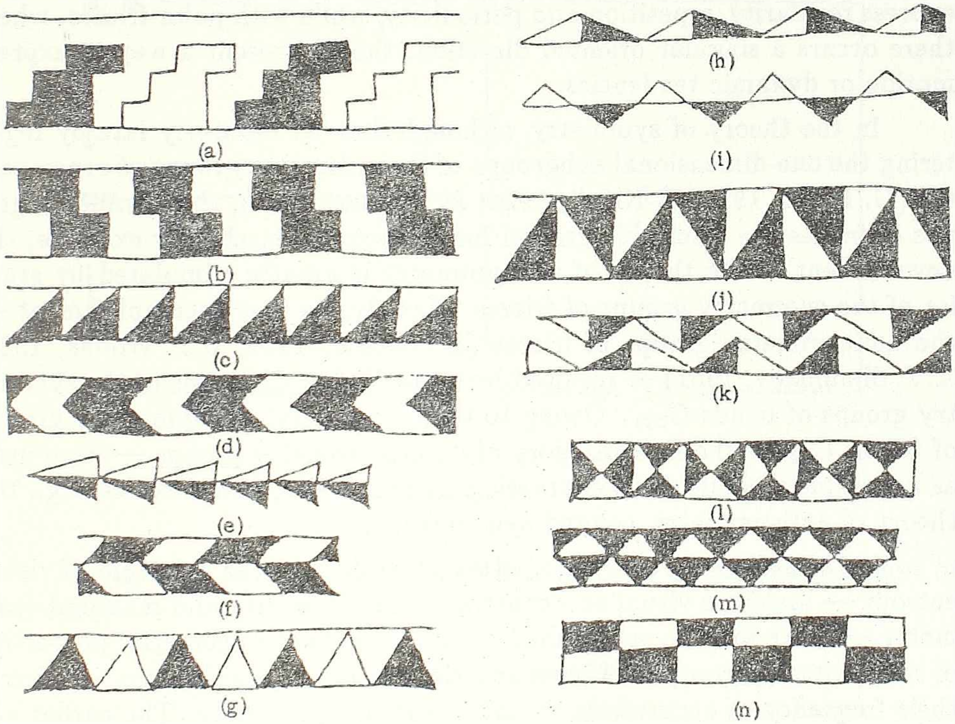


Figure 2.55

Antisymmetry friezes in Neolithic ornamental art: (a) **11/11**, Greece, around 3000 B.C.; (b) **12/12**, Greece; (c) **12/11**, Near East, around 5000 B.C.; (d) **1m/1m**, Near East, around 5000 B.C.; (e) **1m/11**, Near East; (f) **1m/11**, Anadolia, around 5000 B.C.; (g) **m1/m1**, Near East; (h) **m1/11**, Near East; (i) **mg/11**, Greece; (j) **mg/1g**, Near East, around 5000 B.C.; (k) **mg/12**, Anadolia; (l) **mm/mm**, Tell el Hallaf, around 4900–4500 B.C.; (m) **mm/m1**, Hacilar, about 5500–5200 B.C.; (n) **mm/mg**, Near East.

* * *

Examples of all seven discrete symmetry groups of friezes **11**, **1g**, **12**, **m1**, **1m**, **mg**, **mm** and two continuous visually presentable symmetry groups of friezes $\mathbf{m_01}$ and $\mathbf{m_0m}$, represented in bone engravings or cave drawings, date from Paleolithic art. In ornamental art, friezes are a way to express regularity, repetition and periodicity, while with polar friezes, where there occurs a singular oriented direction, they represent a way to express motion or dynamic tendencies.

In the theory of symmetry, although derived relatively late by registering the one-dimensional subgroups of the symmetry groups of ornaments G_2 (G. Pólya, 1924; P. Niggli, 1924; A. Speiser, 1927), the symmetry groups of friezes are a suitable ground for different research. For example, the development of the theory of antisymmetry is greatly stimulated by studies of the symmetry groups of friezes. Namely, its first accomplishment — the antisymmetry groups of friezes (H. Heesch, 1929; H.J. Woods, 1935; A.V. Shubnikov, 1951) — resulted from the Weber diagrams of the symmetry groups of bands G_{321}. Owing to their simplicity, the symmetry groups of friezes G_{21} — the first category of infinite isometry groups — were used as a suitable medium for constructing and analyzing new theories, e.g., the theory of antisymmetry, colored symmetry, etc.

In the same way as with rosettes, according to the principle of visual entropy — maximal visual and constructional simplicity and maximal symmetry — there may be established a relation between geometric properties of the symmetry groups of friezes and their visual interpretations — friezes, their frequency of occurrence, period of origin and variety. The earlier appearance and dominance of friezes satisfying this principle, is evident.

The table of the group-subgroup relations points to the possibility to apply the desymmetrization method for the derivation of the symmetry groups of friezes. According to the relations (Figure 2.56) and the tables of the (minimal) indexes of subgroups in groups, the classical-symmetry, antisymmetry (for subgroups of the index 2) and color-symmetry desymmetrizations or their combinations can be used aiming to obtain the symmetry groups of friezes of a lower degree of symmetry.

A survey of the antisymmetry desymmetrizations of friezes is given in the corresponding table. Symbols of antisymmetry groups G' are given in the group/subgroup notation G/H, offering information on the generating symmetry group G and its subgroup H of the index 2 — symmetry subgroup H of the group G', which is the final result of the antisymmetry desymmetrization. So that, by antisymmetry desymmetrizations it is possible to obtain all subgroups of the index 2 of the given symmetry group (H.S.M. Coxeter, 1985).

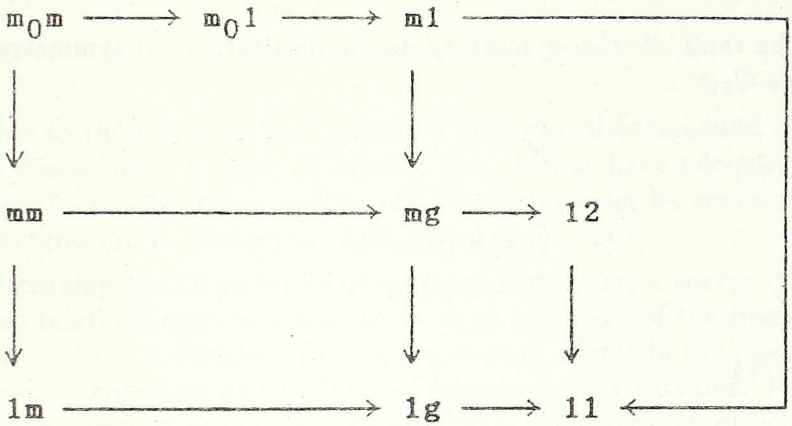


Figure 2.56

The table of antisymmetry desymmetrizations of symmetry groups of friezes G_{21}:

11/11	m1/m1	mg/m1	mm/mm
	m1/11	mg/12	mm/mg
1g/11		mg/1g	mm/1m
	1m/1m		mm/m1
12/12	1m/1g		mm/12
12/11	1m/11		

The table of the color-symmetry desymmetrizations relies on the works of J.D. Jarratt, R.L.E. Schwarzenberger (1980) and H.S.M. Coxeter (1987). By color-symmetry desymmetrizations, the symmetry groups of friezes 11, 1g, 1m, m1 and 12, may be obtained. Complete information on the color-symmetry desymmetrizations of friezes is given in the corresponding table. Each of the infinite classes of colored symmetry groups is denoted by a symbol $G/H/H_1$; its first datum represents the generating symmetry group G, the second its stationary subgroup H consisting of transformations of the colored symmetry group G^* that maintain each individual index (color) unchanged, and the third the symmetry subgroup H_1 of the group G^*. The symmetry group H_1 is the final result of the color-symmetry desymmetrization. A number N ($N \geq 3$) is the number of "colors" used to obtain the particular color-symmetry group. For $H = H_1$, i.e. iff H is a normal subgroup of the group G, the symbol $G/H/H_1$ is reduced to the symbol G/H.

The table of color-symmetry desymmetrizations of symmetry groups of friezes G_{21}:

11/11	N
1g/11	$N = 0 \pmod 2$
1g/1g	$N = 1 \pmod 2$
m1/m1/11	N
m1/11	$N = 0 \pmod 2$
12/12/11	N
12/11	$N = 0 \pmod 2$
1m/1m	N
1m/1g	$N = 0 \pmod 2$
1m/11	$N = 0 \pmod 2$
mg/mg/1g	$N = 1 \pmod 2$
mg/m1/11	$N = 0 \pmod 2$
mg/12/11	$N = 0 \pmod 2$
mg/1g	$N = 2 \pmod 4$
mg/11	$N = 0 \pmod 4$
mm/mm/1m	N
mm/m1/11	$N = 0 \pmod 2$ ($N \geq 6$)

mm/mg/1g	$N = 0 \pmod 2$
mm/12/11	$N = 0 \pmod 2$ $(N \geq 6)$
mm/1m	$N = 0 \pmod 2$
mm/11	$N = 0 \pmod 4$
mm/1g	$N = 0 \pmod 4$
mm/m1	$N = 4$
mm/12	$N = 4$

Due to the objective stationariness of works of ornamental art, continuous friezes with a polar or bipolar axis do not have adequate visual interpretations. The visual models of these groups can be constructed by using textures (A.V. Shubnikov, N.V. Belov et al., 1964).

According to the principle of maximal symmetry, a survey of group-subgroup relations may, in a way, serve as an indicator of the frequency of occurrence of the particular symmetry groups of friezes in ornamental art, and also for recognition and evidence of symmetry substructures. The influence of the principle of maximal symmetry on the frequency of occurrence of certain symmetry groups of friezes in ornamental art comes to its full expression for the most common frieze — the straight line, which represents a visual illustration of the maximal continuous symmetry group of friezes m_0m and in the frequent occurrence of friezes with the symmetry groups mm, mg, etc.

The origin of friezes arose also out of the existence of natural models, so that friezes with the symmetry group 12 or mg may be considered as stylized forms of waves; motifs with the symmetry 1g or 1m, that are found in arrangements of leaves in many plants, served as a source of many friezes; while the importance of mirror symmetry in nature caused friezes with the symmetry group m1, 1m or mm. This especially refers to the maximal discrete symmetry group of friezes mm, which contains as subgroups all the other discrete symmetry groups of friezes and the symmetry group of rosettes D_2 (2m) and expresses, in the visual sense, the relation "vertical-horizontal" and the quality of perpendicularity. Besides natural objects, the origin of friezes is also to be found in the periodic character of many natural phenomena (the turn of day and night, the turn of the seasons, the phases of the moon, the tides etc.). Therefore, friezes represent a record of the first human attempts to register periodic natural phenomena, i.e. the first calendars. In time, with their symbolic meanings clearly defined, friezes became a visual communication means: each frieze contains a message, i.e. its meaning is harmonized with its visual form. This can be proved by the preserved names of friezes in primitive art. Later, with the development of all other communication means, friezes lose their original symbolic function, to be partially or completely replaced by a decorative one.

The visual impression produced by a real frieze results from the interaction between its symmetry group, human mirror symmetry and binocularity, the symmetry group of a limited part of the plane to which the frieze belongs, and the symmetry group D_2 (2m) caused by the fundamental natural directions — the vertical and horizontal line. Regarding the first, the other symmetry groups mentioned have a role of desymmetrization or symmetrization factors. Besides the objective elements of symmetry, the visual impression is influenced by the subjective elements referring to the physiological-psychological properties of the visual perception (e.g., perception of the "right" diagonal as "ascending" and the "left" as "descending", etc.), so that this dependence may be very complex. Aiming to perceive and recognize the objective, geometric symmetry, the observer must eliminate these secondary, subjective visual factors.

A fundamental region of the symmetry group of friezes **m1** or **mm** is rectilinear, with boundaries incident withthe reflection lines. All the other friezes with partly or completely curved boundaries of the fundamental region, offer a change of its shape. Aiming to increase the variety of friezes with the symmetry group **m1** or **mm** possessing a rectilinear fundamental region, these possibilities are reduced to the use of different elementary asymmetric figures belonging to the fundamental region.

Data on the polarity of friezes and enantiomorphism, in the visual sense refer to the dynamic or static impression created by them. Friezes with the symmetry groups **11**, **1g**, **1m** with a polar, oriented singular direction will produce a dynamic effect. Since all friezes with the polar axis may have a curvilinear fundamental region, their visual dynamism may be emphasized by choosing an acuteangular fundamental region with the acute angle oriented toward the direction of the axis or by choosing an acuteangular elementary asymmetric figure belonging to the fundamental region and directed in the same way. The symmetry group of friezes **12** that contains a central reflection and possesses the bipolar axis, offers a similar possibility: recognition of two oppositely oriented polar friezes with the symmetry group **11**, which produce the visual impression of two-way motion.

Since the discrete symmetry groups **m1**, **mg** and **mm** and the continuous groups $\mathbf{m_0 1}$ and $\mathbf{m_0 m}$ contain reflections with reflection lines perpendicular to the frieze axis, friezes corresponding to them belong to a family of friezes with a non-polar axis. Enantiomorphic modifications of friezes with the discrete symmetry groups **1g**, **m1**, **1m**, **mg** and **mm** and continuous visually presentable symmetry groups $m_0 1$ and mm, which contain indirect isometries — reflections or glide reflections — does not occur.

Data on the polarity of friezes and enantiomorphism may be a basic indicator of the static or dynamic visual properties of friezes. Non-polar friezes produce static, while polar friezes produce a dynamic visual impression. Another component of a dynamic visual impression produced by a certain frieze, may be the presence of a glide reflection, suggesting the impression of alternating motion. The enantiomorphism is the "left" or "right" orientation of a frieze — the existence of only "left" or "right " homologous asymmetric elementary figures or fundamental regions. According to the principle of maximal symmetry, those symmetry groups of friezes with a high degree of symmetry, mg and mm, prevail. Among them, more frequent are static friezes with the symmetry group mm. Their distinctive stationariness results from the fact that each of them may be placed in such a position that the reflection lines corresponding to the rosettal subgroup D_2 (2m) coincide with the fundamental natural directions — the vertical and horizontal line. For the maximal continuous symmetry group of friezes $m_0 m$ that may be visually modeled by a straight (horizontal) line, a similar argument hold.

A table survey of subgroups of the symmetry groups of friezes and their decompositions (reducibility) offers complete evidence of their symmetry substructures. Through its use, a visual recognition of friezes and rosettes that a particular frieze contains, will be simplified. Certainly, static substructures with a higher degree of symmetry may be easily perceived and visually recognized, but low-symmetry substructures demand its use. Besides the visual simplicity of substructures, possibilities for their visual recognition will be caused by all the other elements taking part in the formation of a visual impression: the visual qualities of suprastructure, visual simplicity, stationariness or dynamism, the relation of substructures to vertical and horizontal line, to the surrounding, to the observer, etc.

Cayley diagrams of the symmetry groups of friezes are another suitable visual interpretation. Besides pointing out characteristics of generators, relations that consist of the presentation of the corresponding symmetry group and its structure, they indicate the visual qualities of the corresponding friezes. In the tables of the graphic symbols of symmetry elements, similar such information may be given.

The symmetry groups of friezes G_{21} are the simplest category of the infinite groups of isometries. In the development of the generalizations of the theory of symmetry — antisymmetry and colored symmetry, they had a significant role. Since visual models are the most obvious interpretation of abstract geometric-algebraic structures, among infinite discrete symmetry groups, friezes are the simplest and most suitable medium for analyzing such generalizations.

Through knowledge of the geometric-algebraic properties of the symmetry groups of friezes, the visual qualities of the corresponding friezes may be anticipated directly from the presentations and structures of their symmetry groups. This opens a large field for ornamental design; for the planning of visual effects produced by friezes; and for aesthetic analyses based on exact grounds. The presence of generators of infinite order in the symmetry groups of friezes and their possible identification with time, results in the occurrence of the time component, representing for ornamental art the possibility to suggest motion.

2.5. Symmetry Groups of Ornaments G_2

In the plane E^2 there is 17 discrete symmetry groups without invariant lines or points, the crystallographic symmetry groups of ornaments: **p1**, **p2**, **pm**, **pg**, **pmm**, **pmg**, **pgg**, **cm**, **cmm**, **p4**, **p4m**, **p4g**, **p3**, **p3m1**, **p31m**, **p6**, **p6m**, two visually presentable symmetry groups of semicontinua **p**$_{10}$**1m** (**s1m**), **p**$_{10}$**mm** (**smm**) and also one visually presentable symmetry group of continua **p**$_{00}$**∞m** (**s**$^{\infty\infty}$). The simplified International Symbols by Hermann and Maugin (H.S.M. Coxeter, W.O.J. Moser, 1980, pp. 40) are used to denote the discrete symmetry groups of ornaments, while the symbols introduced by A.V. Shubnikov, V.A. Koptsik (1974), B. Grünbaum, G.C. Shephard (1983) are used to denote the continuous symmetry groups of ornaments — symmetry groups of semicontinua and continua.

A complete survey of the presentations, structures, possible decompositions and Cayley diagrams of the 17 discrete symmetry groups of ornaments can be found in the monograph by H.S.M. Coxeter and W.O.J. Moser: *Generators and Relations for Discrete Groups* (1980, pp. 40–51). In the same book one can find discussion of all the symmetry groups of ornaments treated as subgroups of the maximal symmetry groups of ornaments **p4m** and **p6m**, generated by reflections (pp. 51–52), the table surveys of group-subgroup relations and minimal indexes of subgroups of the symmetry groups of ornaments (pp. 136, Table 4).

Presentations and structures:

p1	$\{X,Y\}$	$XY=YX$		$C_\infty \times C_\infty$
	$\{X,Y,Z\}$	$XYZ=ZYX=E$	$(Z=X^{-1}Y^{-1})$	

p2 $\{X,Y,T\}$ $XY=YX$ $T^2=(TX)^2=(TY)^2=E$

$\{T_1,T_2,T_3\}$ $T_1^2=T_2^2=T_3^2=(T_1T_2T_3)^2=E$ $(T_1=TY, T_2=XT, T_3=T)$

$\{T_1,T_2,T_3,T_4\}$ $T_1^2=T_2^2=T_3^2=T_4^2=T_1T_2T_3T_4=E$ $(T_4=T_1T_2T_3=T_1X)$

pm $\{X,Y,R\}$ $XY=YX$ $R^2=(RX)^2=E$ $RYR=Y$ $D_\infty \times C_\infty$

$\{R,R_1,Y\}$ $R^2=R_1^2=E$ $YR=RY$ $YR_1=R_1Y$ $(R_1=RX)$

pg $\{X,Y,P\}$ $XY=YX$ $P^2=E$ $XPX=P$ $<2,2,\infty>$

$\{P,Q\}$ $P^2=Q^2$ $(Q=PX)$

cm $\{P,Q,R\}$ $P^2=Q^2$ $R^2=E$ $RPR=Q$

$\{P,R\}$ $R^2=E$ $RP^2=P^2R$

$\{R,S\}$ $R^2=E$ $(RS)^2=(SR)^2$ $(S=PR)$

pmm $\{R,R_1,R_2,Y\}$ $R^2=R_1^2=R_2^2=(RR_1)^2=(R_1R_2)^2=(R_2Y)^2=E$

$YR=RY$ $YR_1=R_1Y$ $D_\infty \times D_\infty$

$\{R_1,R_2,R_3,R_4\}$

$R_1^2=R_2^2=R_3^2=R_4^2=(R_1R_2)^2=(R_2R_3)^2=(R_3R_4)^2=(R_4R_1)^2=E$

$(R_1=R, R_3=R_1, R_4=R_2Y)$

pmg $\{P,Q,R\}$ $P^2=Q^2$ $R^2=(RP)^2=(RQ)^2=E$

$\{R,T_1,T_2\}$ $R^2=T_1^2=T_2^2=E$ $T_1RT_1=T_2RT_2$ $(T_1=PR, T_2=QR)$

pgg $\{P,Q,T\}$ $P^2=Q^2$ $T^2=E$ $TPT=Q^{-1}$ $(\infty,\infty \mid 2,2)$

$\{P,O\}$ $(PO)^2=(P^{-1}O)^2=E$ $(O=PT)$

cmm $\{R_1,R_2,R_3,R_4,T\}$ $T^2=E$ $TR_1T=R_3$ $TR_2T=R_4$

$R_1^2=R_2^2=R_3^2=R_4^2=(R_1R_2)^2=(R_2R_3)^2=(R_3R_4)^2=(R_4R_1)^2=E$

$\{R_1,R_2,T\}$ $R_1^2=R_2^2=T^2=(R_1R_2)^2=(R_1TR_2T)^2=E$

p4 $\{T_1,T_2,T_3,T_4,S\}$ $T_1^2=T_2^2=T_3^2=T_4^2=T_1T_2T_3T_4=E$

$S^4=E$ $S^{-i}T_4S^i=T_i$ $i=1,2,3$ $[4,4]^+$

$\{S,T\}$ $S^4=T^2=(ST)^2=E$ $(T=T_4)$

p4m $\{R_1,R_2,R_3,R_4,R\}$ $R^2=E$ $RR_1R=R_4$ $RR_2R=R_3$

$R_1^2=R_2^2=R_3^2=R_4^2=(R_1R_2)^2=(R_2R_3)^2=(R_3R_4)^2=(R_4R_1)^2=E$ $[4,4]$

$\{R,R_1,R_2\}$ $R^2=R_1^2=R_2^2=(RR_1)^4=(R_1R_2)^2=(R_2R)^4=E$

p4g $\{R_1,R_2,R_3,R_4,S\}$ $S^4=E$ $S^{-i}R_4S^i=R_i$ $i=1,2,3$

$R_1^2=R_2^2=R_3^2=R_4^2=(R_1R_2)^2=(R_2R_3)^2=(R_3R_4)^2=(R_4R_1)^2=E$ $[4^+,4]$

$\{S,R\}$ $R^2=S^4=(RS^{-1}RS)^2=E$ $(R=R_4)$

p3 $\{X,Y,Z,S_1\}$ $XYZ=ZYX$ $S_1^3=E$ $S_1^{-1}XS_1=Y$

$S_1^{-1}YS_1=Z$ $S_1^{-1}ZS_1=X$ $\triangle^+$

$\{S_1,S_2,S_3\}$ $S_1^3=S_2^3=S_3^3=S_1S_2S_3=E$ $(S_2=S_1X, S_3=X^{-1}S_1)$

$\{S_1,S_2\}$ $S_1^3=S_2^3=(S_1S_2)^3=E$

p31m $\{S_1,S_2,R\}$ $S_1^3=S_2^3=(S_1S_2)^3=E$ $R^2=E$ $RS_1R=S_2^{-1}$ $[3^+,6]$

$\{R,S\}$ $R^2=S^3=(RS^{-1}RS)^3=E$ $(S=S_1)$

p3m1 $\{S_1, S_2, R\}$ $S_1^3 = S_2^3 = (S_1S_2)^3 = E$ $R^2 = E$ △
$RS_1R = S_1^{-1}$ $RS_2R = S_2^{-1}$
$\{R_1, R_2, R_3\}$ $R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 = E$
$(R_1 = RS_2, R_2 = S_1R, R_3 = R)$

p6 $\{S_1, S_2, T\}$ $S_1^2 = S_2^2 = (S_1S_2)^2 = E$ $T^2 = E$ $TS_1T = S_2$ $[3,6]^+$
$\{S, T\}$ $S^3 = T^2 = (ST)^6 = E$ $(S = S_1)$

p6m $\{R_1, R_2, R_3, R\}$ $R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 = E$
$R^2 = E$ $RR_1R = R_3$ $RR_2R = R_2$ $[3,6]$
$\{R, R_1, R_2\}$ $R^2 = R_1^2 = R_2^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = E$

All the discrete symmetry groups of ornaments are subgroups of the groups generated by reflections **p4m** and **p6m**, given by the presentations:

p4m $\{R, R_1, R_2\}$ $R^2 = R_1^2 = R_2^2 = (RR_1)^4 = (RR_1)^2 = (R_2R)^4 = E$ $[4,4]$

$R_1,\ S = RR_2$	generate	p4g	$[4^+,4]$
$S,\ T_1 = R_1R_2$	generate	p4	$[4,4]^+$
$T_1,\ R_2,\ R_1 = R_2S$	generate	cmm	
$P = R_1S,\ O = SR_1$	generate	pgg	
$R,\ P$	generate	cm	
$P,\ Q = RPR$	generate	pg	
$T_1,\ T_2 = S^2,\ R_4 = RR_1$	generate	pmg	
$R_1,\ R_2,\ R_4,\ R_3 = SR$	generate	pmm	$D_\infty \times D_\infty$
$T_1,\ T_2,\ T_3 = RT_1R$	generate	p2	
$R_1,\ R_3,\ Y = R_2R_4$	generate	pm	$D_\infty \times C_\infty$
$Y,\ X = R_1R_3$	generate	p1	$C_\infty \times C_\infty$

p6m $\{R, R_1, R_2\}$ $R^2 = R_1^2 = R_3^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = E$ $[3,6]$

$R,\ S = R_1R_2$	generate	p31m	$[3^+,6]$
$S,\ T = R_2R$	generate	p6	$[3,6]^+$
$R_1,\ R_2,\ R_3 = RR_1R$	generate	p3m1	△
$S_1 = R_1R_2,\ S_2 = R_2R_3$	generate	p3	$△^+$

Form of the fundamental region: bounded, offers a change of boundaries that do not belong to reflection lines. The groups generated by reflections **pmm**, **p3m1**, **p4m**, **p6m** do not offer any change of the shape of a fundamental region.

Number of edges of the fundamental region: **pm, pmm** — 4;
p4m — 3,4;
p1, pg, p3 — 4,6;
p4, p4g — 3,4,5;
p31m, p6m — 3,4,6;
p2, pmg, pgg, cm, cmm, p6 — 3,4,5,6.

Enantiomorphism: **p1, p2, p3, p4, p6** possesses enantiomorphic modifications, while in the other cases the enantiomorphism does not occur.

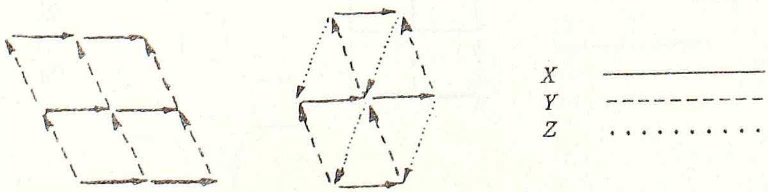
Polarity of rotations: polar rotations — **p2, pmg, pgg, p3, p4, p6**; non-polar rotations — **pmm, cmm, p3m1, p4m, p6m**. The symmetry group **p4g** contains polar 4-rotations and non-polar 2-rotations, and the symmetry group **p31m** contains polar and non-polar 3-rotations.

Polarity of generating translation axes:
both axes are polar — **p1, pg,p3, p31m**;
both axesare bipolar —**p2, pgg, p4, p4g, p6**;
one axis is polar, the other non-polar — **pm, cm**;
one axisis non-polar, the other bipolar — **pmg, cmm**;
both axes are non-polar — **pmm, p4m, p3m, p6m**.

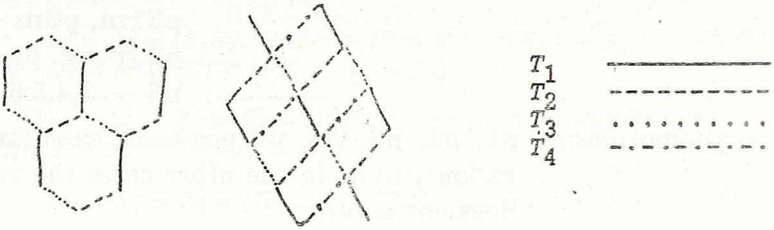
The first studies on the symmetry groups of ornaments G_2 were undertaken by C. Jordan (1868/69), but he did not succeed in discovering all the existing 17 symmetry groups. Namely, he omitted the group **pgg**, discovered by L. Sohncke (1874), who, on the other hand, omitted three other groups. The complete list of the discrete symmetry groups of ornaments was given by E.S. Fedorov (1891b).

Cayley diagrams (Figure 2.57):

p1



p2



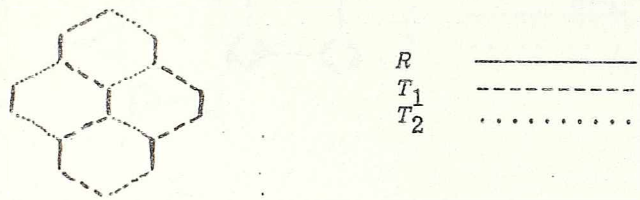
pm

pg

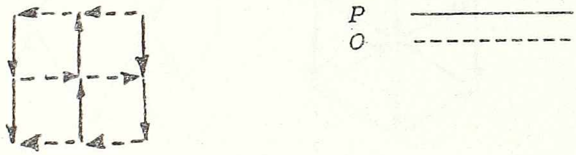
cm

pmm

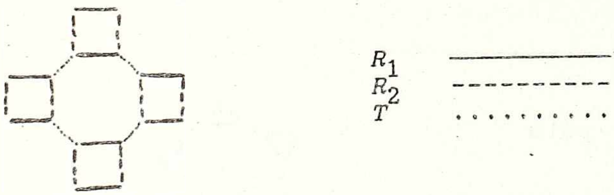
pmg



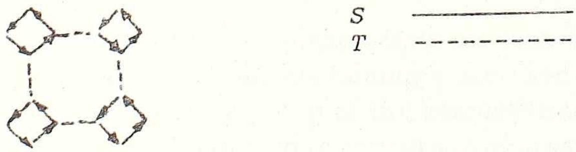
pgg



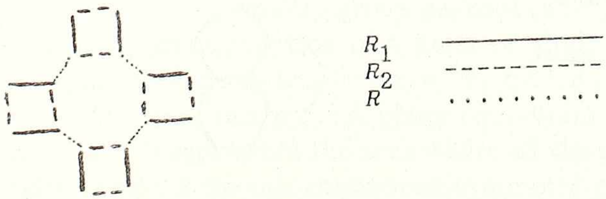
cmm



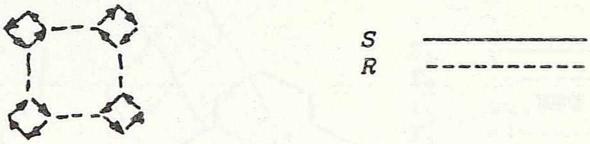
p4



p4m

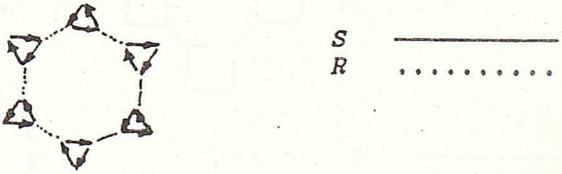


p4g

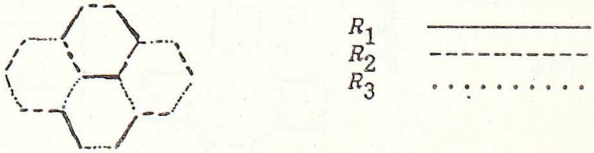


p3

p31m



p3m1



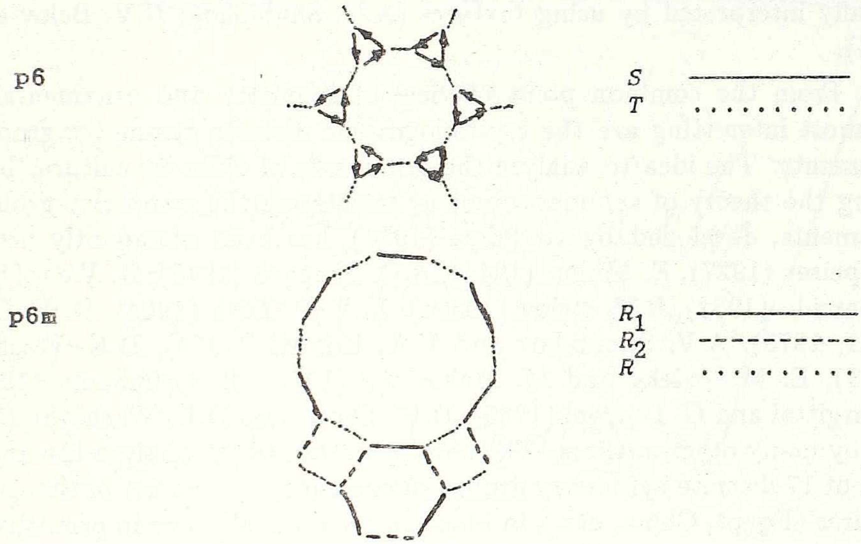


Figure 2.57

2.6. Ornaments and Ornamental Art

A plane is called *homogeneous* iff all the plane points are translationaly equivalent, and *isotropic* iff all directions containing some fixed point of the plane are equivalent. The symmetry group of the homogeneous and isotropic plane E^2 is the maximal continuous symmetry group of ornaments $p_{\infty\infty}\infty m$ ($s^{\infty\infty\infty}$). The continua with the symmetry group $p_{\infty\infty}mm$ ($s^{\infty\infty}$) can be understood as the result of the multiplication of a point or circle — a rosette with the continuous symmetry group D_∞ (∞m) — by means of the continuous symmetry group of translations $p_{\infty\infty}$. A plane continuum with the symmetry group $p_{\infty\infty}\infty m$ ($s^{\infty\infty\infty}$) represents the area where all the other plane symmetry groups exist. Apart from this continuous symmetry group of ornaments, also visually presentable are the symmetry groups of semicontinua $p_{10}1m$ ($s1m$) and $p_{10}mm$ (smm). They are derived, respectively, as the extensions of the visually presentable continuous symmetry groups of friezes $m_0 1$ and $m_0 m$ (Figure 2.29) by a translation perpendicular to the frieze axis. In ornamental art, the symmetry groups of semicontinua usually are presented by adequate systems of parallel lines, constructed by such a procedure. The other continuous symmetry groups of ornaments may be visually interpreted by using textures (A.V. Shubnikov, N.V. Belov et al., 1964).

From the common point of view of geometry and ornamental art, the most interesting are the crystallographic discrete symmetry groups of ornaments. The idea to analyze the ornaments of different cultures by applying the theory of symmetry and by registering the symmetry groups of ornaments, developed by G. Pólya (1924), has been abundantly used by A. Speiser (1927), E. Müller (1944), A.O. Shepard (1948), H. Weyl (1952), J. Garrido (1952), N.V. Belov (1956a), L. Fejes Tóth (1964), D.W. Crowe (1971, 1975), A.V. Shubnikov and V.A. Koptsik (1974), D.K. Washburn (1977), E. Makovicky and M. Makovicky (1977), B. Grünbaum (1984b), I. Hargittai and G. Lengyel (1985), D.W. Crowe and D.K. Washburn (1985) and by many other authors. The quoted works mainly analyze the appearance of 17 discrete symmetry groups of ornaments in the art of the ancient cultures (Egypt, China, etc.), in Moorish ornamental art or in primitive art. How difficult it is to exhaust all the symmetry possibilities for plane ornaments and to discover all the symmetry groups of ornaments, is illustrated by the fact that many nations, even those with a rich ornamental tradition, in their early art do not have such examples (B. Grünbaum, 1984b; B. Grünbaum, Z. Grünbaum, G.C. Shephard, 1986).

In the mathematical theory of symmetry, the first complete list of the discrete symmetry groups of ornaments was given by E.S. Fedorov (1891b), although this problem was, even before that, the subject of study of many important mathematicians.

Therefore, the fact that examples of most of the discrete symmetry groups of ornaments, given in bone and stone engravings or drawings, date from the Paleolithic ornamental art is very surprising. In the Neolithic there came the further development of ornamental art, mainly related to the decoration of ceramics. Neolithic ornamental art is characterized by paraphrasing, variation, enrichment of already existent ornaments and by the discovery of those symmetry groups of ornaments which remained undiscovered.

Ornaments with the symmetry group **p1** (Figure 2.58, 2.59) for the first time occur in Paleolithic art (Figure 2.58a, b). The origin of these ornaments, obtained by multiplying an asymmetric figure by means of a discrete group of translations, may be interpreted also as a translational

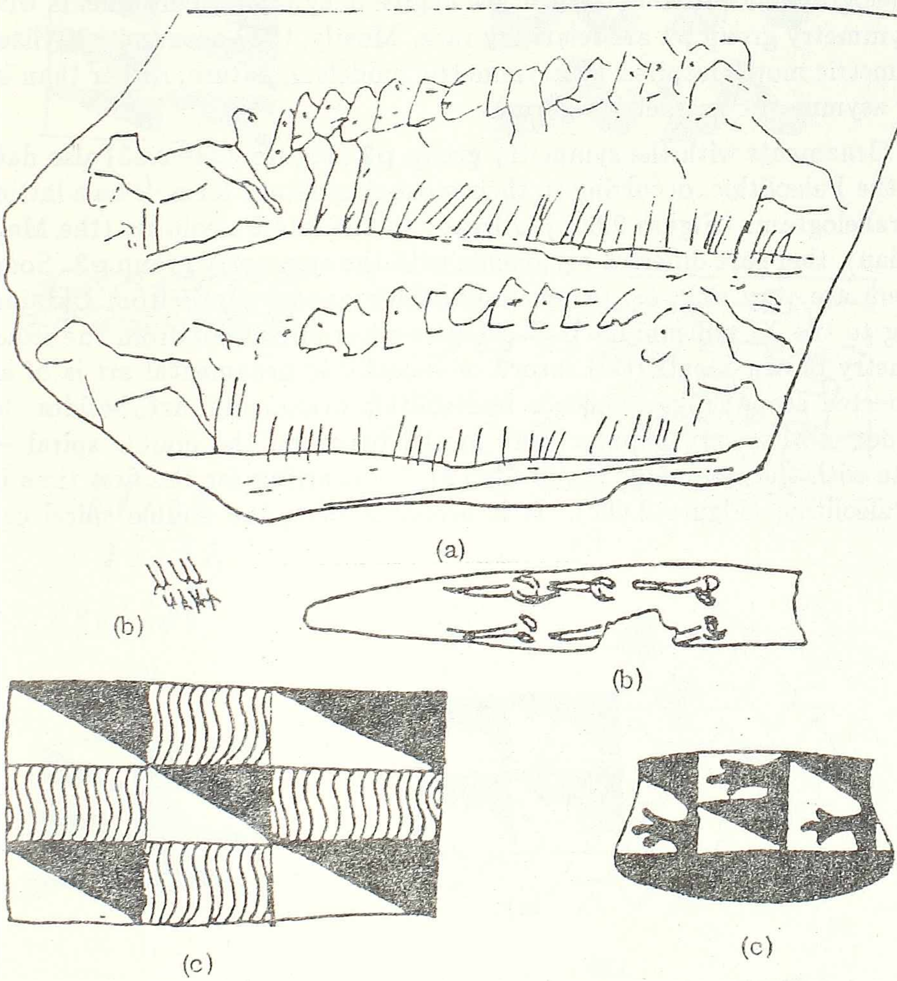
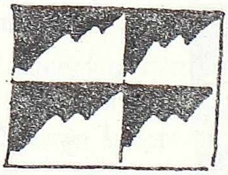


Figure 2.58

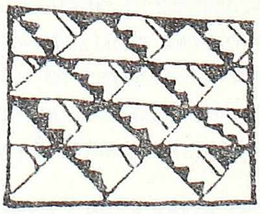
Examples of ornaments with the symmetry group **p1** in Paleolithic and Neolithic art: (a) Chaffaud cave, Paleolithic (Magdalenian); (b) Paleolithic bone engravings, around 10000 B.C.; (c) Hacilar, ceramics, around 5700–5000 B.C.

repetition of a frieze with the symmetry group 11, already existent in Paleolithic ornamental art. Both of the axes of generating translations are polar. Since the symmetry group p1 does not contain indirect isometries, the enantiomorphism occurs. A fundamental region usually has an arbitrary parallelogramic form. Due to their low degree of symmetry, ornaments with the symmetry group p1 are relatively rare. Mostly, they occur with stylized asymmetric motifs inspired by asymmetric models in nature, rather than by using asymmetric geometric figures.

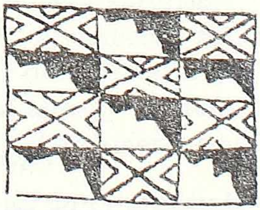
Ornaments with the symmetry group p2 (Figure 2.60–2.63) also date from the Paleolithic, occurring in their most elementary form — as a lattice of parallelograms (Figure 2.60b). Already in the late Paleolithic (the Magdalenian), there are different ornaments with the symmetry group p2. Some of them are very rich, as, for example, the meander motif from Ukraine, dating to the XI millennium B.C. Despite some deviations from the exact symmetry of ornaments, this record of Paleolithic ornamental art is of an unexpected scope (Figure 2.60a). In Neolithic ornamental art, besides the meander motifs, very popular were motifs based on the double spiral — rosette with the symmetry group C_2 (2) — occurring for the first time in the Paleolithic (Figure 2.60c). It is probable that, the double spiral can



(a)



(a)



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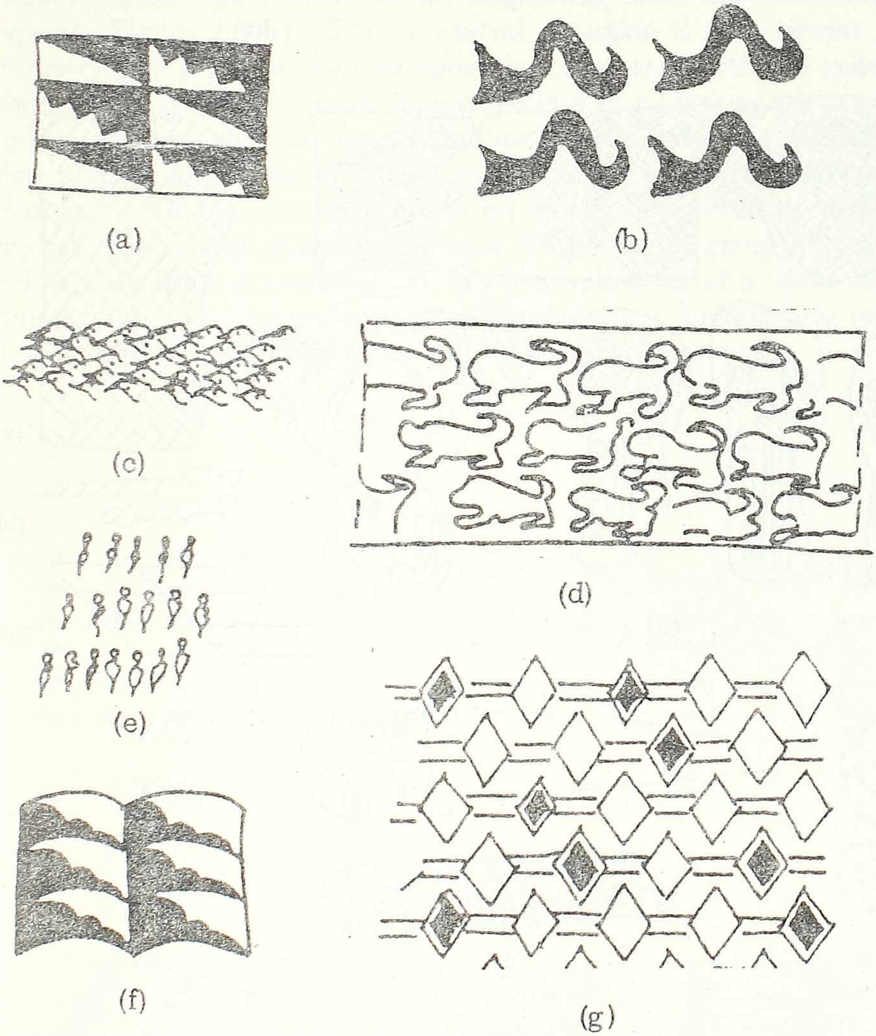


Figure 2.59

Examples of ornaments with the symmetry group **p1**: (a) Hacilar, around 5700–5000 B.C.; (b) Velushko-Porodin, Yugoslavia, around 5000 B.C.; (c) Western Pakistan, around 3000 B.C.; (d) Naqda culture, Egypt, around 3600–3200 B.C.; (e) the pre-dynastic period of Egypt; (f) art of pre-Columbian America, Nasca, Peru; (g) the ornament "Warms", the primitive art of Africa.

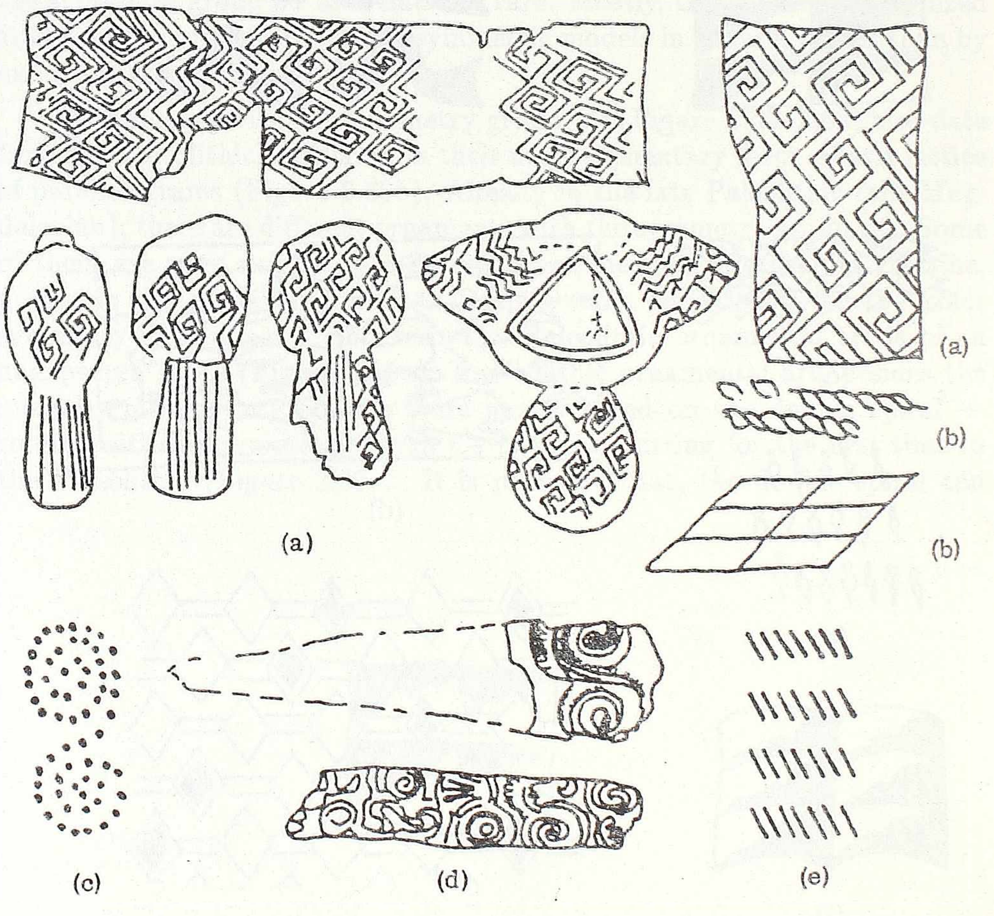
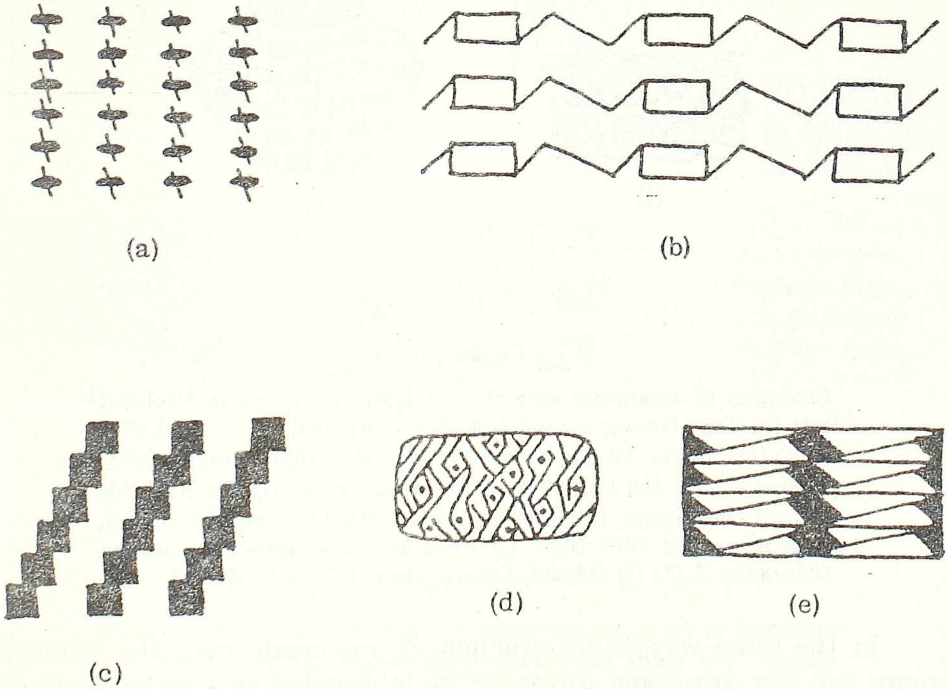


Figure 2.60

Examples of ornaments with the symmetry group p2 in Paleolithic art, around 12000–10000 B.C.: (a) Mezin, USSR; (b) the Paleolithic of Western Europe; (c) the motif of the double spiral, Mal'ta, USSR; (d) the application of the motif of the double spiral, Arudy, Isturiz; (e) the Paleolithic art of Europe.

be found in ornamental art of all the Neolithic cultures, often with wave motifs. Layered patterns (B. Grünbaum, G.C. Shephard, 1987) with the symmetry group **p2** (Figure 2.62a) originating from the Paleolithic, are frequent (Figure 2.60b). Since a central reflection is the element of the symmetry group **p2**, in the corresponding ornaments both the generating translation axes will be bipolar. Therefore, in a visual sense, such ornaments produce an impression of two-way motion. Enantiomorphic modifications exist. Since in nature the symmetry group C_2 (2) occurs relatively seldom, ornaments with the symmetry group **p2** mostly are geometric ones. The simplest construction of ornaments with the symmetry group **p2**, probably used in Paleolithic ornamental art, is a multiplication of a frieze with the symmetry group **12** by a non-parallel translation. A fundamental region is often triangular and offers the use of curvilinear boundaries.



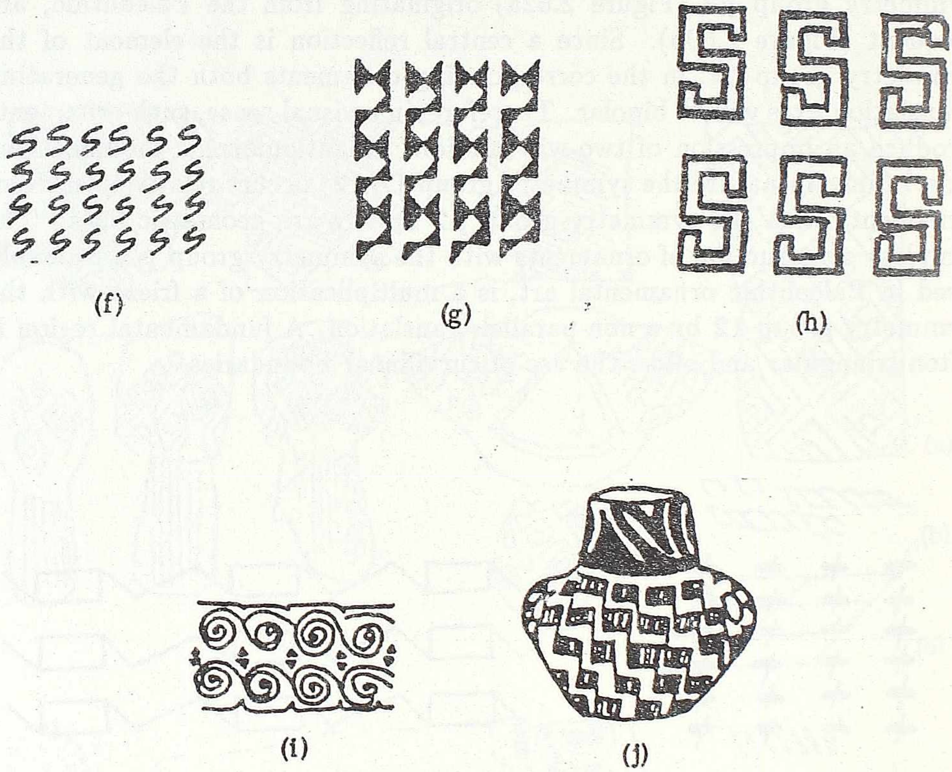


Figure 2.61

Examples of ornaments with the symmetry group **p2** in Neolithic art: (a) Tepe Guran, around 5800 B.C.; (b) Siyalk II, around 4000 B.C.; (c) Samara, around 5500 B.C.; (d) Catal Hüjük, around 6400–5800 B.C.; (e) the Neolithic of the Middle East; (f) the Neolithic of Iran and Egypt; (g) Neolithic of the Middle East; (h) Dimini, Greece, around 6000 B.C.; (i) Neolithic, Czechoslovakia, around 5000–4000 B.C.; (j) Odzaki, Greece, around 6100–5800 B.C.

In the same way, a construction of ornaments with the symmetry groups **pm**, **pg**, **pmg** and **pmm** can be interpreted as a multiplication of the corresponding friezes by means of a translation perpendicular to the frieze axis. All the afore mentioned symmetry groups of friezes originated from Paleolithic ornamental art.

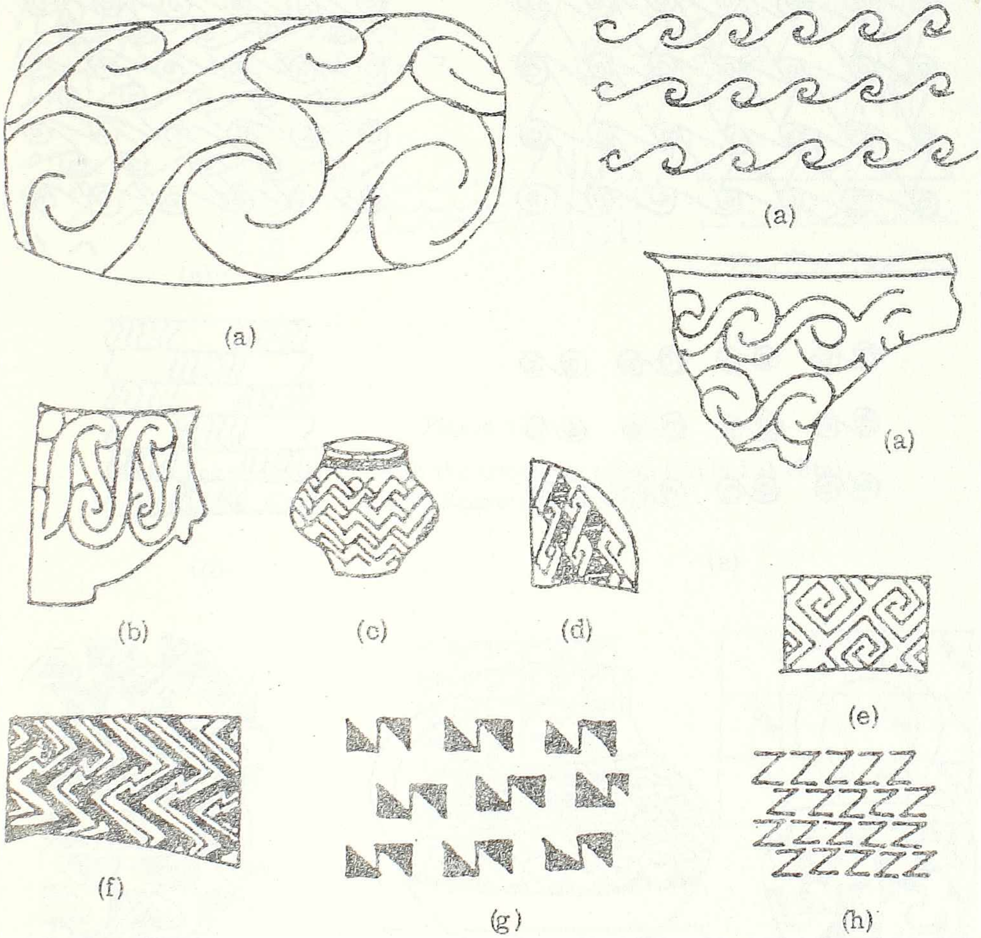


Figure 2.62

Examples of ornaments with the symmetry group p2 in Neolithic art: (a) Butmir II, Yugoslavia, around 3000 B.C.; (b) Adriatic zone, around 3000–2000 B.C.; (c) Starchevo, Yugoslavia, around 5000 B.C.; (d) Danilo, Yugoslavia, around 4000 B.C.; (e) Vincha II, Yugoslavia, around 4500–4000 B.C.; (f) Adriatic zone, around 3000–2000 B.C.; (g) Lendel culture, Hungary, around 2900 B.C.; (h) Neolithic, Italy, around 3700–2700 B.C.

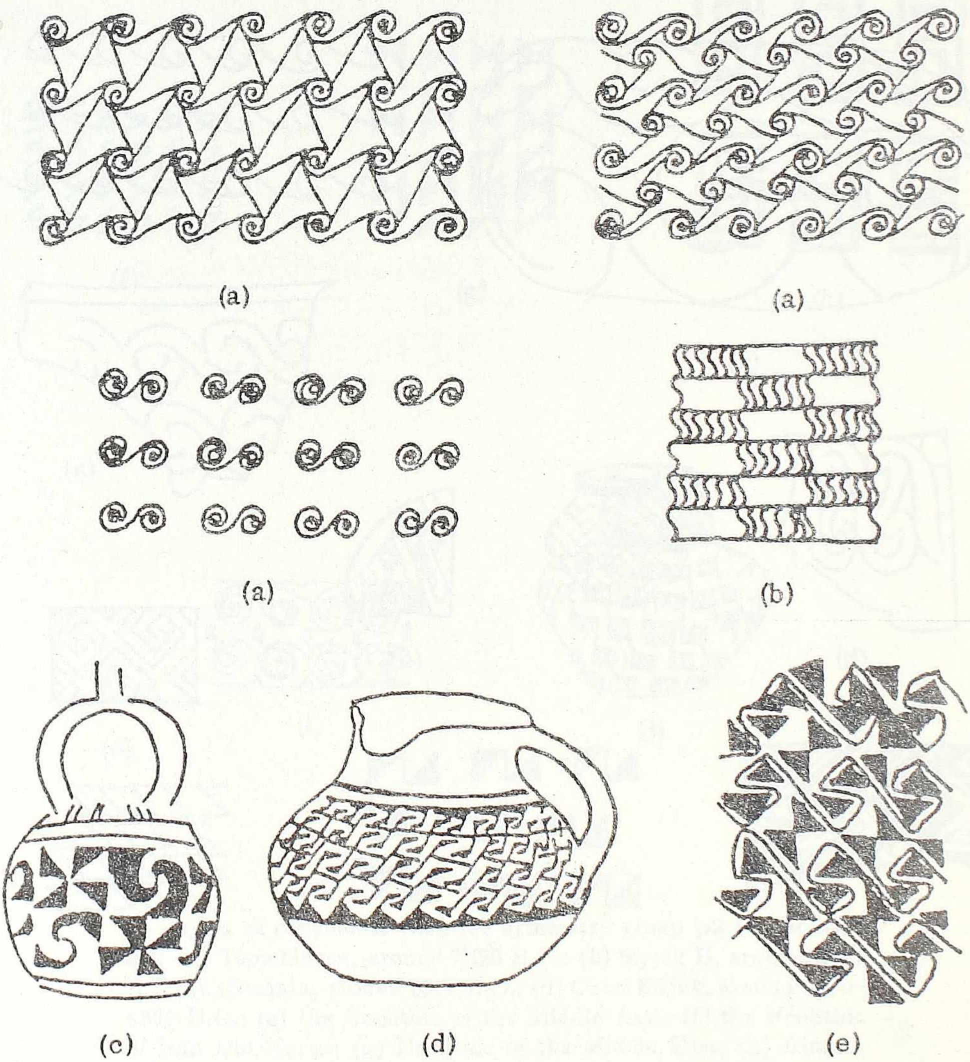


Figure 2.63

Ornaments with the symmetry group p2 in ornamental art: (a) application of a double spiral, rosette C_2 (2) in the ornaments of the Aegean cultures and Egypt; (b) Knossos; (c) the art of the pre-Columbian period, Peru; (d) the art of the Pueblo Indians; (e) the art of primitive peoples, Indonesia.

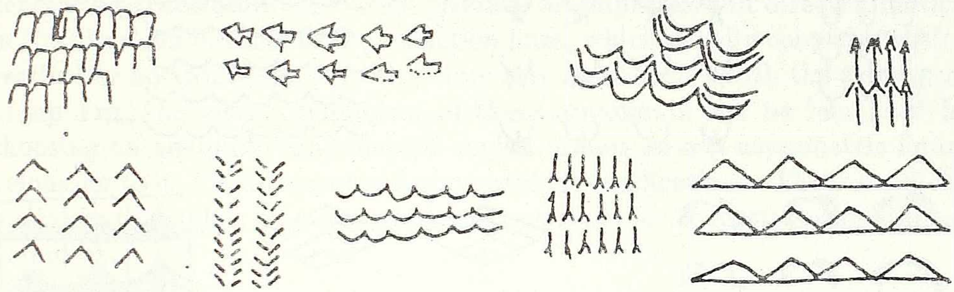


Figure 2.64

Examples of ornaments with the symmetry group pm in Paleolithic art (Ardales, Gorge d'Enfer, Romanelli caves).

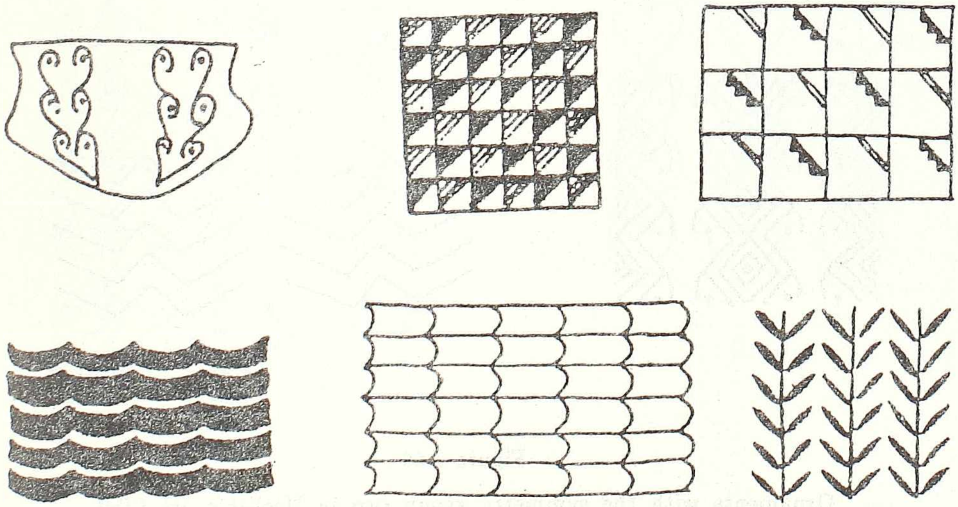


Figure 2.65

Examples of ornaments with the symmetry group pm in the Neolithic art of the Middle East (Hacilar, Tell el Hallaf, around 6000 B.C.).

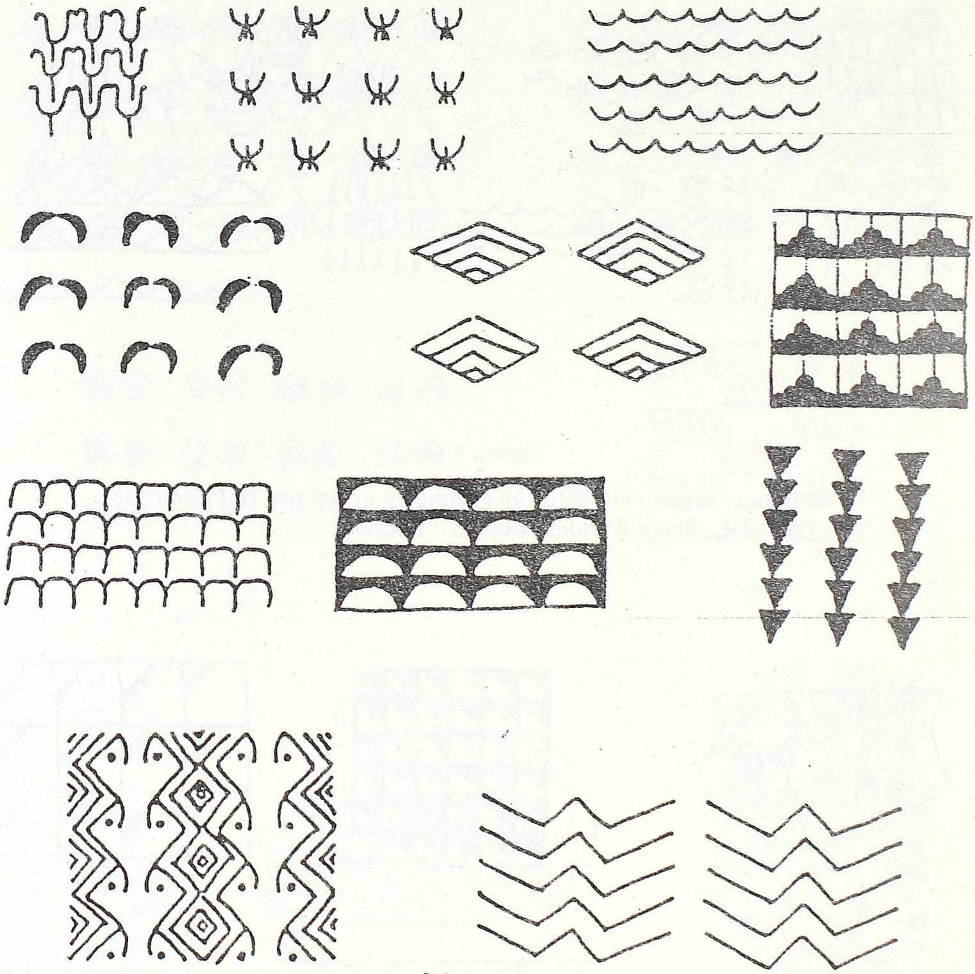
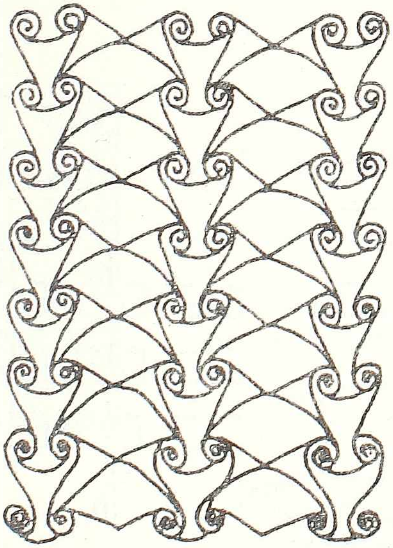


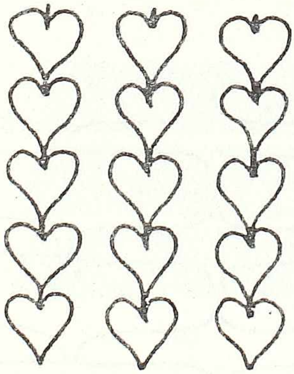
Figure 2.66

Ornaments with the symmetry group pm in Neolithic art (Tell Arpachiyah, around 6000 B.C.; Siyalk II, Eridu culture, around 5800 B.C.; Starchevo, around 5500 B.C.; Hacilar, around 5700–5000 B.C.; Namazga I, around 4000 B.C.).

From the Paleolithic onward, ornaments with the symmetry group **pm** (Figure 2.64–2.67) are frequently used in ornamental art, as "geometric", "plant" and "animal" ornaments. By having one polar and one non-polar generating translation axis, they produce an impression of directed motion in the direction parallel to the reflection lines, which usually coincides with a vertical or horizontal line. In the same way as in friezes with the symmetry group **1m**, the visual dynamism of these ornaments can be increased by choosing an adequate fundamental region or elementary asymmetric figure belonging to it. There are no enantiomorphic modifications. The form of the fundamental region may be arbitrary, but requires at least one rectilinear boundary.



(a)



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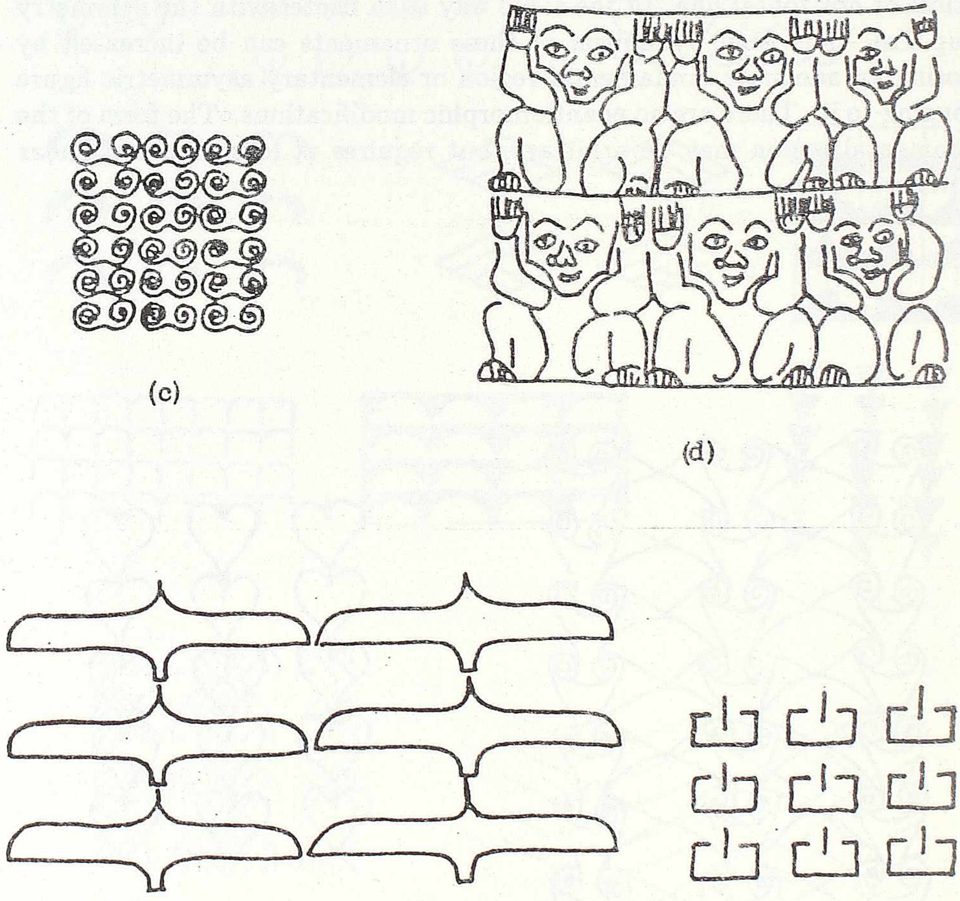
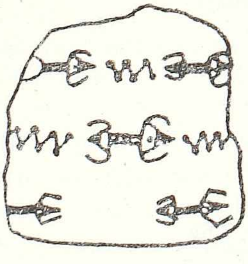


Figure 2.67

Ornaments with the symmetry group pm: (a) Egypt, 18th dynasty; (b) Bubastis, Egypt, 1250 B.C.; (c) Troy, around 1500 B.C.; (d) the primitive art of the Eskimos, Alaska, around 1825; (e) Japan; (f) the Mittla palace, the pre-Columbian period of America.

In ornamental art, friezes with the symmetry group **1g** occur relatively seldom, so the same holds for ornaments with the symmetry group **pg** (Figure 2.68, 2.69). The appearance of friezes with the symmetry group **1g** in Paleolithic ornamental art (e.g., in stylized plant motifs) offers some evidence to believe that also ornaments with the symmetry group **pg** originate from the Paleolithic. In Neolithic ornamental art this symmetry group mostly occurs in geometric ornaments, while in the pre-dynastic period of Egypt and Mesopotamia it was frequently used with zoomorphic motifs. In ornaments with the symmetry group **pg** both generating translation axes are polar. Since the symmetry group **pg** contains indirect isometries — glide reflections — enantiomorphic modifications do not occur. In the visual sense, ornaments with the symmetry group **pg** produce a visual impression of one-way alternating motion.

(a)



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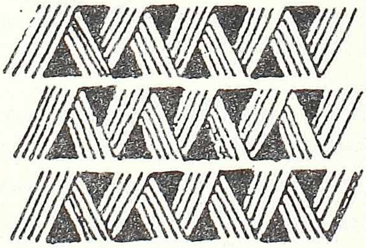

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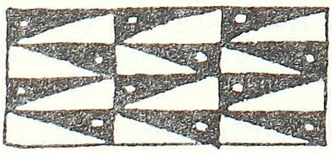

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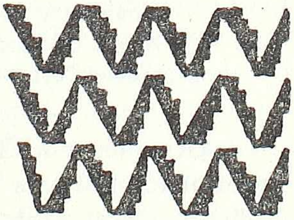

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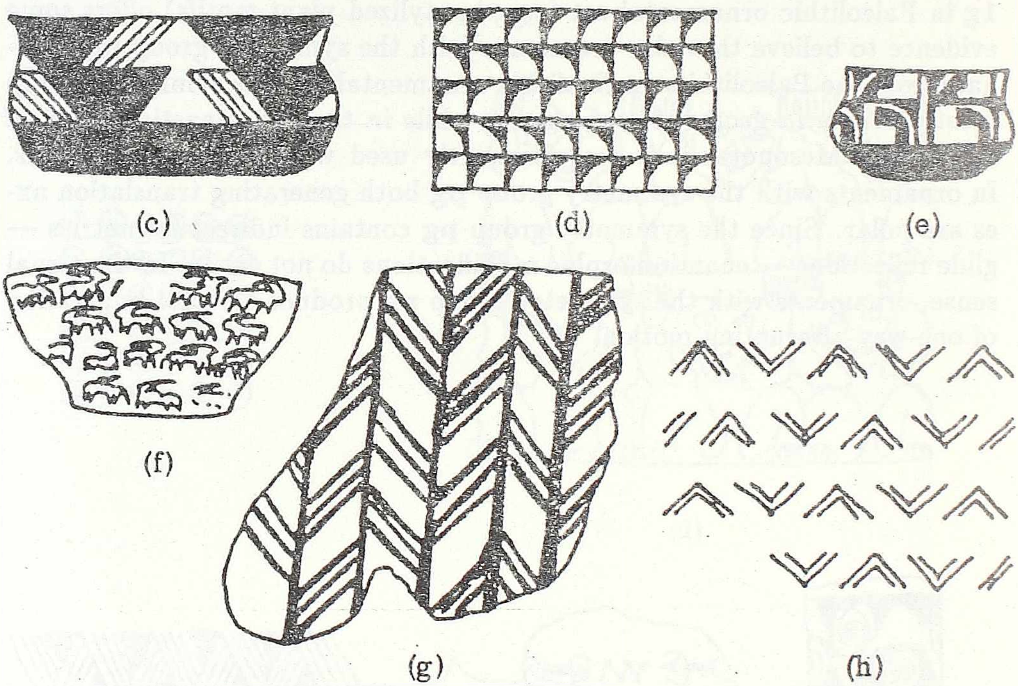


Figure 2.68

Examples of ornaments with the symmetry group pg: (a) Nuzi ceramics, Minoan period; (b) Mussian, Elam, around 5000 B.C.; (c) Hallaf, around 6000 B.C. (7600–6900 B.C.?); (d) Eridu culture, around 4500–4200 B.C.; (e) Hacilar, around 5700–5000 B.C.; (f) Naqda culture, Egypt, around 3600–3200 B.C.; (g) Adriatic zone, around 3000–2000 B.C.; (h) Iran, around 5000 B.C.

Ornaments with the symmetry group pmg (Figure 2.70–2.73), and corresponding friezes with the symmetry group mg, originate from the Paleolithic. Most probably, the symmetry group pmg is one of the oldest symmetry groups of ornaments used in ornamental art. Their first and most frequent visual interpretations are as stylized motifs of waves. The symmetry group pmg is the most frequent symmetry group of ornaments in Paleolithic and Neolithic ornamental art throughout the world.

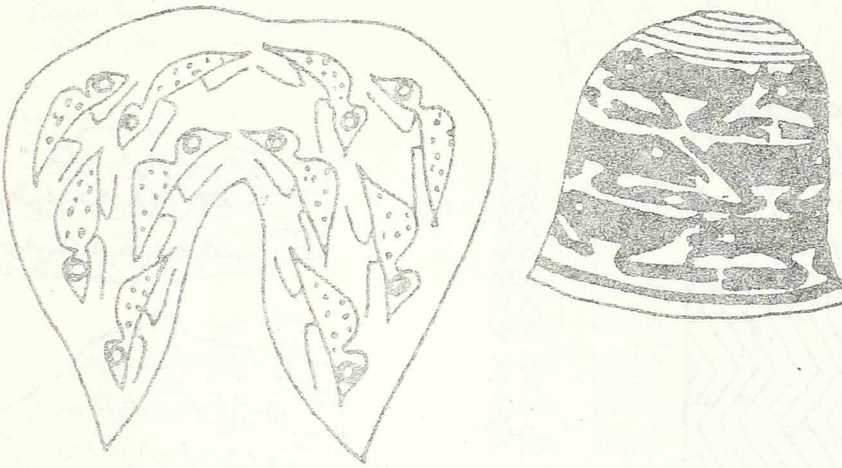


Figure 2.69

Examples of ornaments with the symmetry group pg in the ornamental art of Africa.

Regarding the frequency and variety of corresponding ornaments, only the symmetry group of ornaments p2 can be compared with it. It is probably impossible to find a culture and its prehistoric ornamental art without ornaments with the symmetry group pmg. Also, by having many various ornaments (Figure 2.71), it offers the possibility to analyze the connections between different Paleolithic and Neolithic cultures, distant both in space and time, by the similarity of the motifs they used. The symmetry group pmg contains reflections with the reflection lines perpendicular to one generating translation axis and parallel to the other, and central reflections, so that the first generating translation axis will be non-polar and the second bipolar. Enantiomorphic modifications do not exist. Owing to the bipolarity of the other generating translation axis and to glide reflections that produce the visual impression of two-way alternating motion, ornaments with the symmetry group pmg will be dynamic ones. On the other hand, as their static component, the reflections produce the impression of balance. A fundamental region is usually rectangular, and requires one rectilinear boundary that belongs to a reflection line. In Paleolithic and Neolithic ornamental art, ornaments with the symmetry group pmg mostly occur as geometric ornaments with stylized "water" and meander motifs. Later, the symmetry group pmg occurs in plant or even zoomorphic ornaments, where

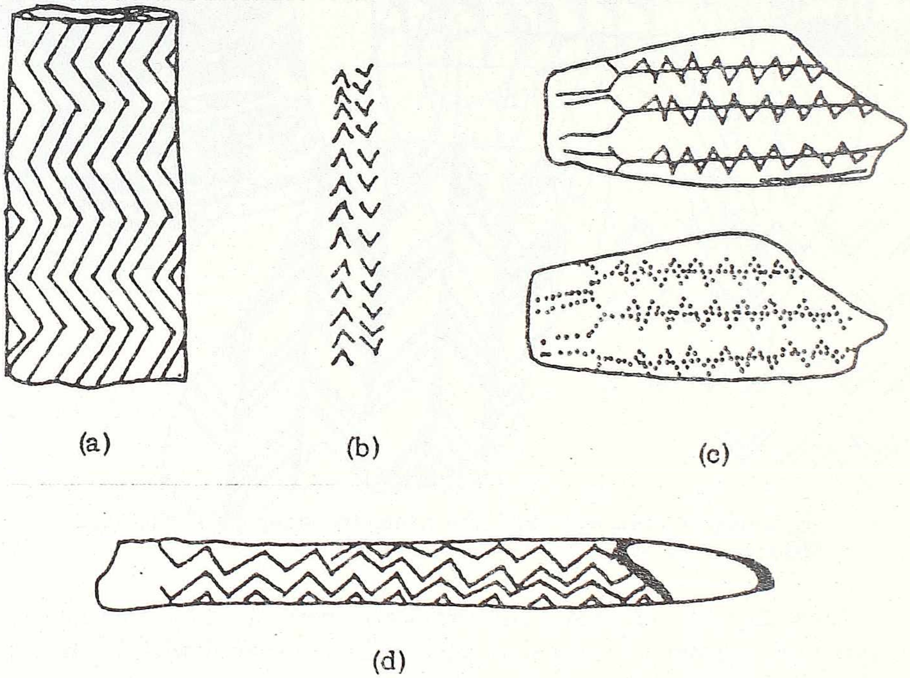
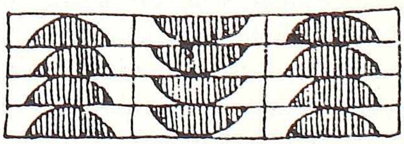


Figure 2.70

Examples of ornaments with the symmetry group pmg in Paleolithic art: (a) Mezin, USSR, around 12000–10000 B.C.; (b) the Paleolithic of Europe; (c) Pernak, Estonia, around 10000 B.C.; (d) Shtetin, Magdalenian period.



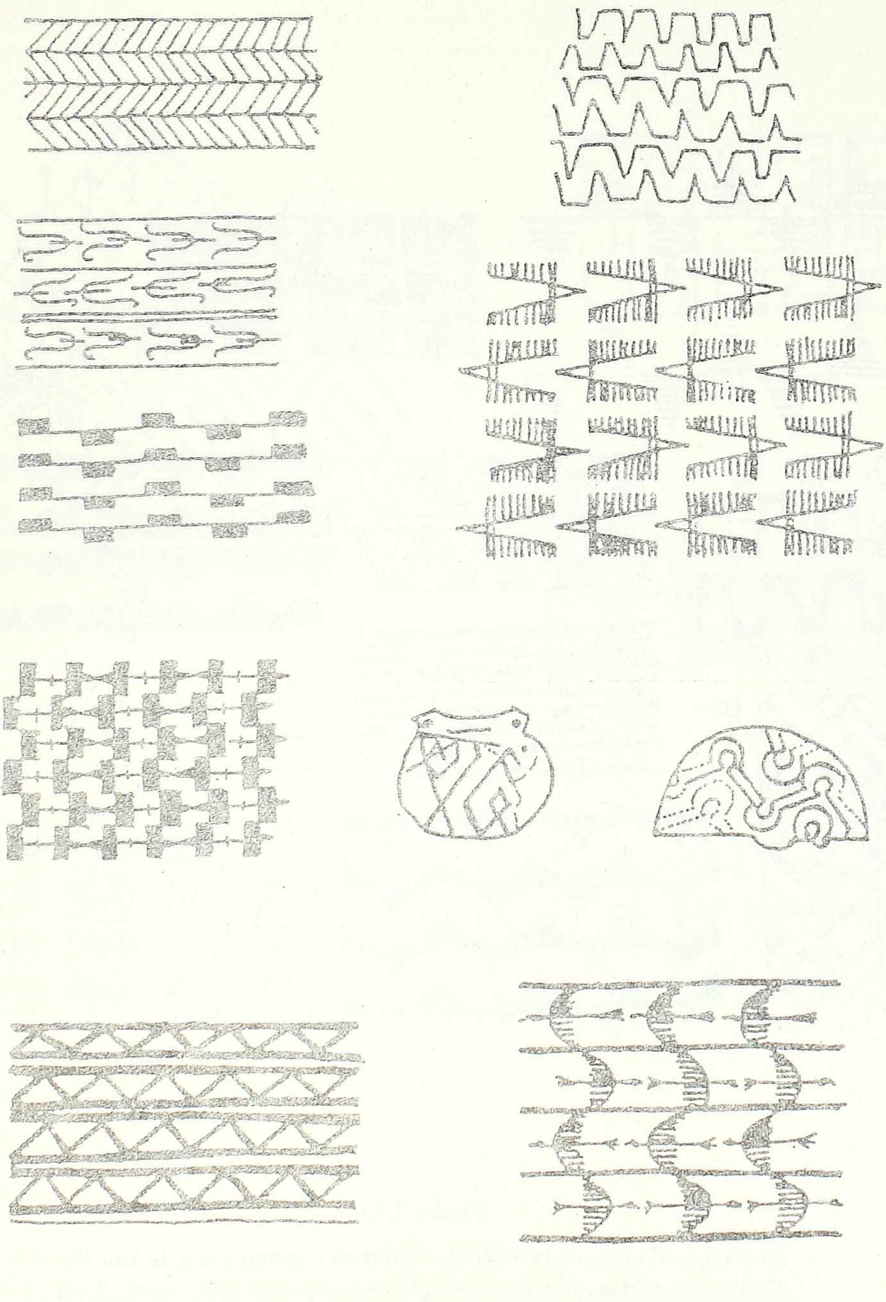


Figure 2.71

Examples of ornaments with the symmetry group pmg in Neolithic art (Susa, Harrap, Butmir, Danilo, Vincha A, Lobositz).

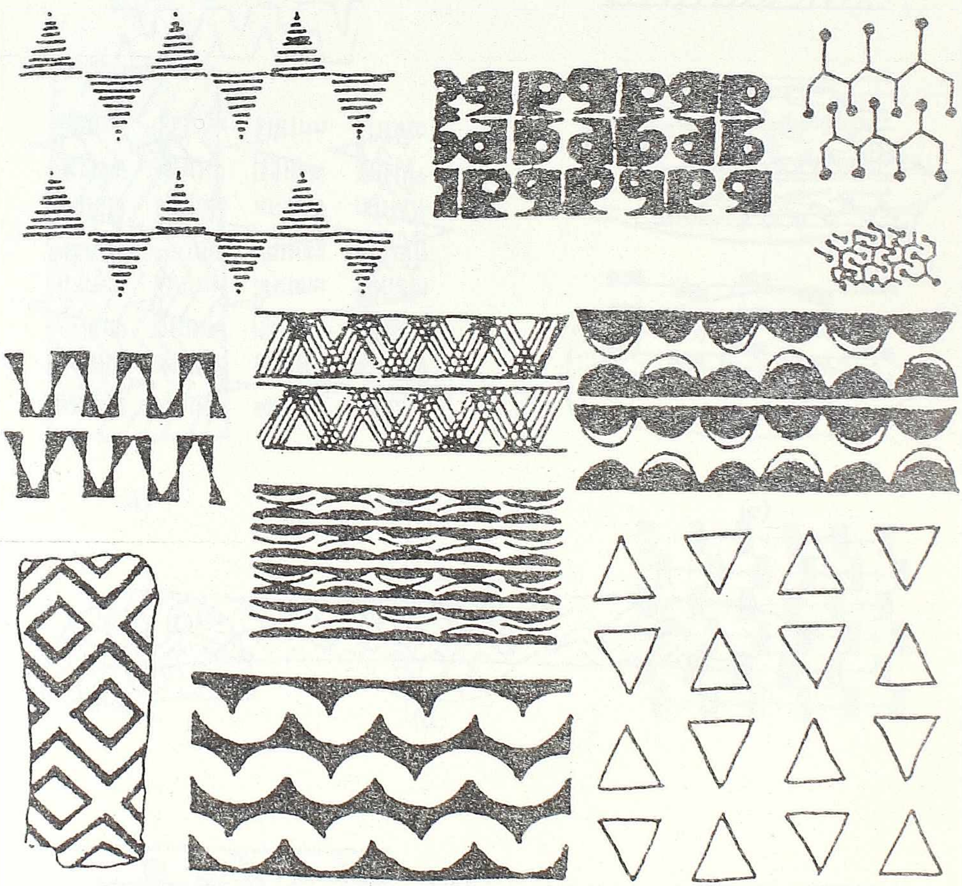


Figure 2.72

Examples of ornaments with the symmetry group pmg in the Neolithic art of the Middle East (Persia, around 4000–3000 B.C.; Samara, around 6000 B.C.; Siyalk, around 4000 B.C.; Susa 5500–5000 B.C.) and in the art of the pre-dynastic and early dynastic period of Egypt (Dendereh, Abydos).

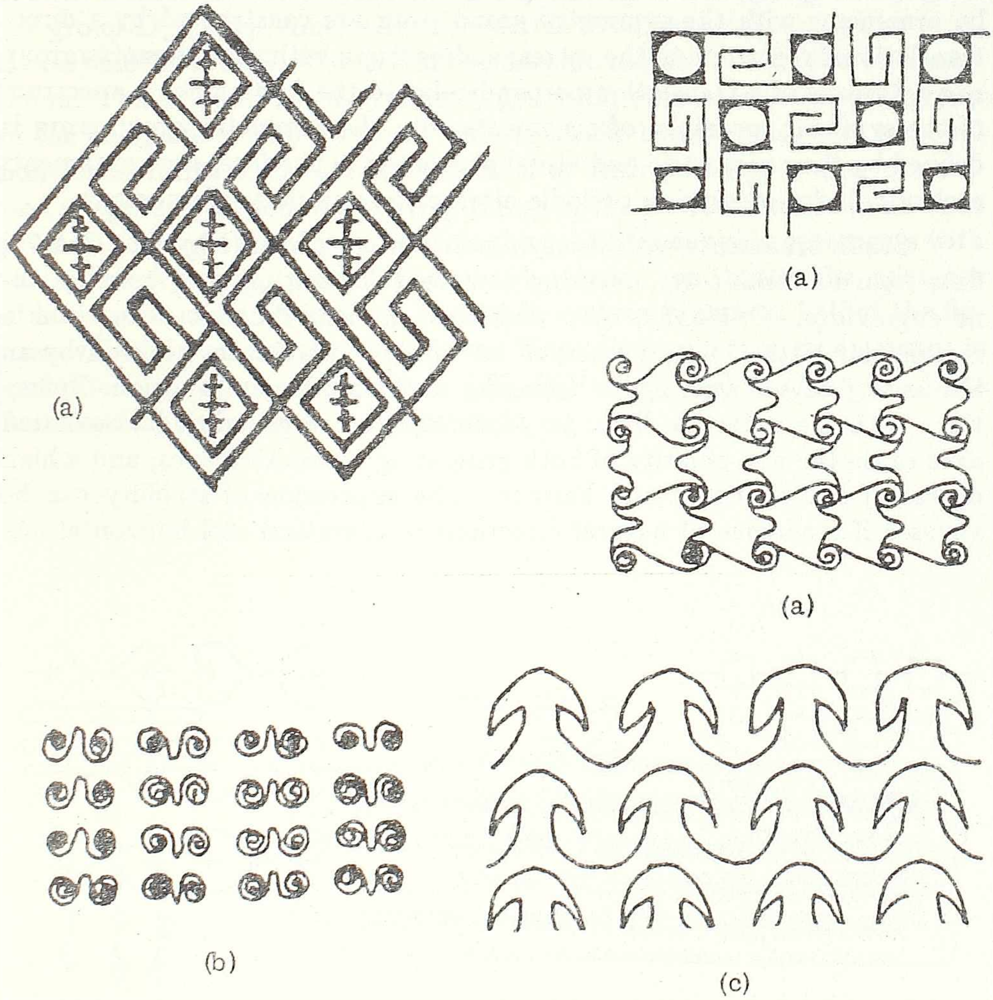


Figure 2.73

Ornaments with the symmetry group pmg: (a) Egypt, dynastic period; (b) Troy, around 1500 B.C.; (c) the ornamental art of the primitive peoples of Oceania.

an initial figure — a rosette with the symmetry group D_1 (m) — is multiplied by means of a glide reflection, perpendicular to the reflection line of the rosette. Afterward, the constructed frieze with the symmetry group mg is repeated by a translation perpendicular to the frieze axis. Most frequently, ornaments with the symmetry group pmg are constructed by a direct translational repetition of the corresponding frieze with the symmetry group mg by means of a translation perpendicular to the frieze axis. A spectrum of the symbolic meanings of ornaments with the symmetry group pmg is defined by their geometric and visual properties. Therefore, such ornaments are suitable for presenting periodic alternating non-polar phenomena.

Static ornaments with the symmetry group pmm (Figure 2.74–2.76) date from Paleolithic art, occuring as cave wall paintings and bone or stone engravings. Originally, they were used in their simplest form — as a rectangular lattice — and later, as its various paraphrases realized by an elementary asymmetric figure belonging to the fundamental region. Reflections with the reflection lines perpendicular to the generating translation axes cause the non-polarity of both generating translation axes, and a high degree of stationariness and balance. The impression of stability can be stressed if fundamental natural directions — a vertical and horizontal line

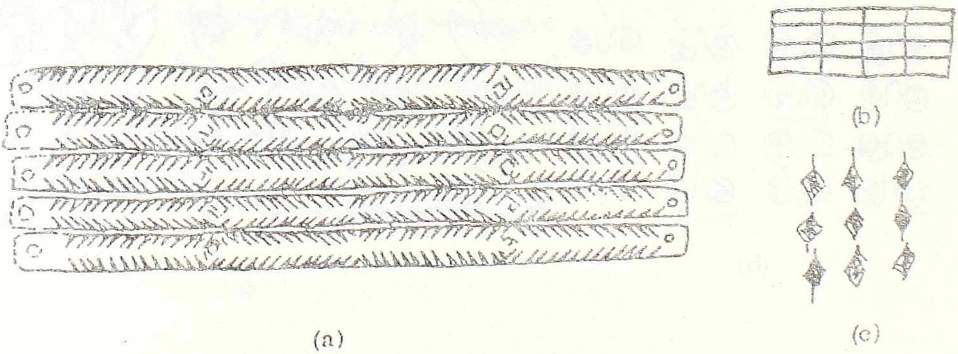
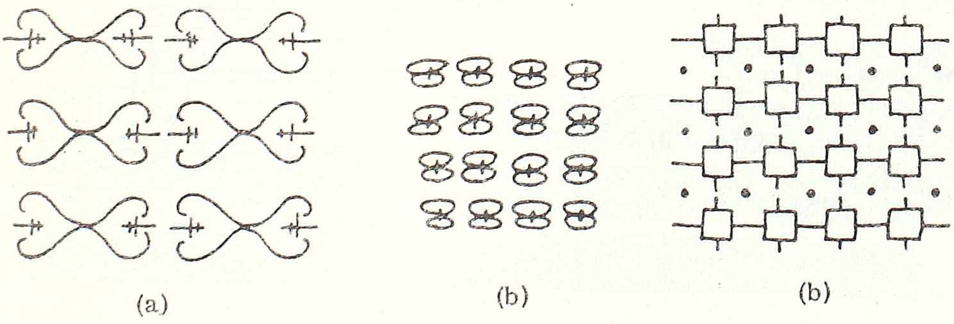


Figure 2.74

Examples of ornaments with the symmetry group pmm in Paleolithic art: (a) Mezin, USSR, around 12000 B.C.; (b) example of a rectangular Bravais lattice, the Lasco cave; (c) Laugerie Haute.

— coincide with the generating translation axes. In ornamental art, ornaments with the symmetry group **pmm** usually are in that position. Also, the existence of horizontal reflections restricts the area of suitable ornamental motifs to rosettes with a symmetry group having as a subgroup the symmetry group $\mathbf{D_2}$ (**2m**). Among them, rosettes with geometric and plant (e.g., flower) motifs prevail. The use of zoomorphic motifs is restricted to a minimum. This is because the symmetry group $\mathbf{D_2}$ (**2m**) occurs as a subgroup of the complete symmetry group only in sessile living organisms, while all non-sessile forms of life are characterized by their polarity — their orientation in space. The polarity of living things in the vertical direction and their upward orientation makes impossible the use of zoomorphic ornaments with the symmetry group **pmm**. The stated polarity contradicts the existence of horizontal mirror symmetry, so that in such an ornament half of the figures will be in an unnatural position. Since the symmetry group **pmm** is generated by reflections (**pmm** $= \{R_1, R_3\} \times \{R_2, R_4\} = D_\infty \times D_\infty$), its fundamental region must be a rectangle.



(a) (b) (b)

Figure 2.75

Ornaments with the symmetry group **pmm** (a) Middle Empire, Egypt; (b) the primitive art of Africa.

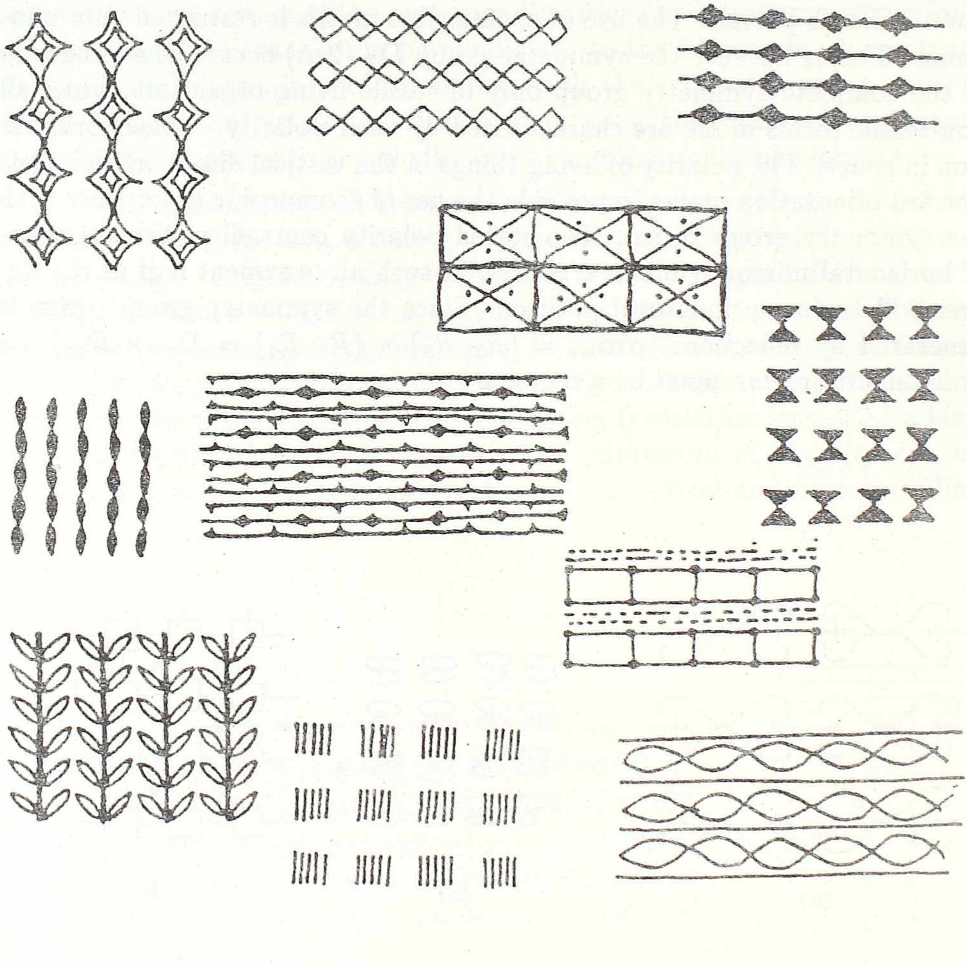


Figure 2.76

Examples of ornaments with the symmetry group pmm in the Neolithic art of the Middle East (Eridu culture, Hallaf, Catal Hüjük) and the pre-dynastic and early dynastic period of Egypt (Dashasheh, Abydos).

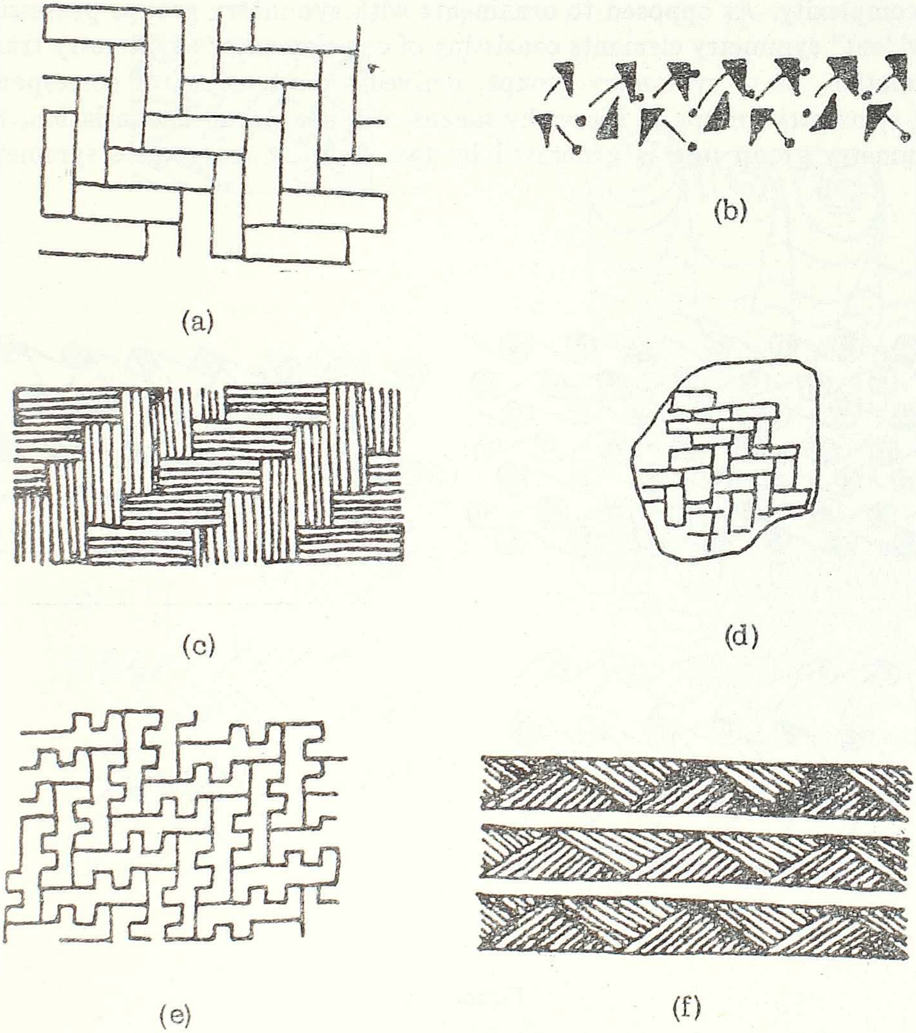


Figure 2.77

Ornaments with the symmetry group pgg in Neolithic art: (a) Jarmo culture, around 5300 B.C.; (b) Catal Hüjük, around 6380–5790 B.C.; (c) Djeblet el Beda, around 6000 B.C.; (d) Tripolian culture, USSR, around 4000–3000 B.C.; (e) Siyalk, Iran, around 4000 B.C.; (f) Susa and Butmir, around 5000–4000 B.C.

The reasons for a relatively rare use of the symmetry group **pgg** in ornamental art (Figure 2.77–2.79) are connected with the difficulty in recognizing the regularities these ornaments are based on and their constructional complexity. As opposed to ornaments with symmetry groups possessing "evident" symmetry elements consisting of one elementary symmetry transformation, or to symmetry groups, derived as extensions of corresponding symmetry groups of friezes by means of a non-parallel translation, the symmetry group **pgg** is generated by two complex, composite symmetry

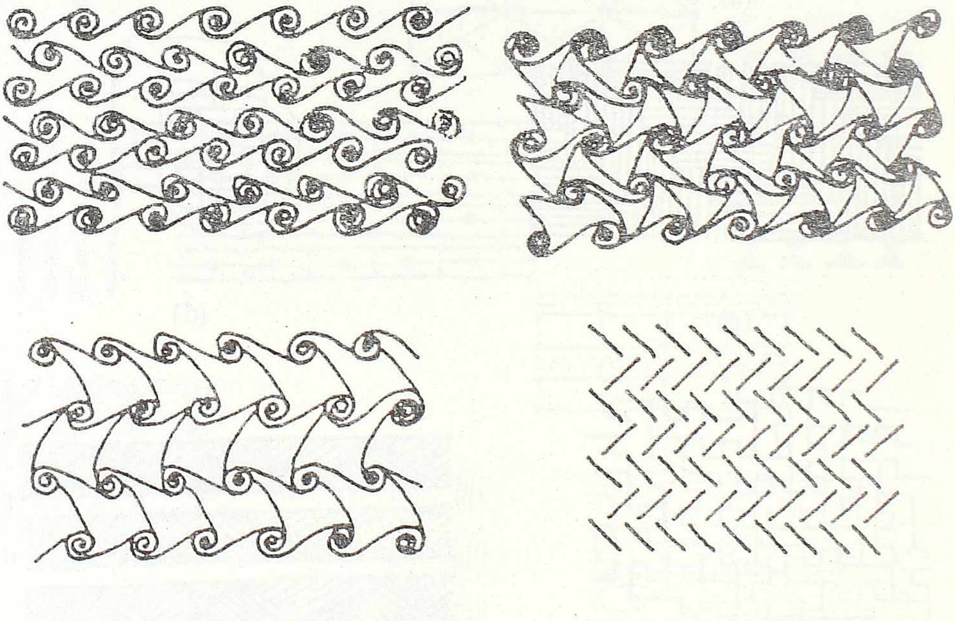
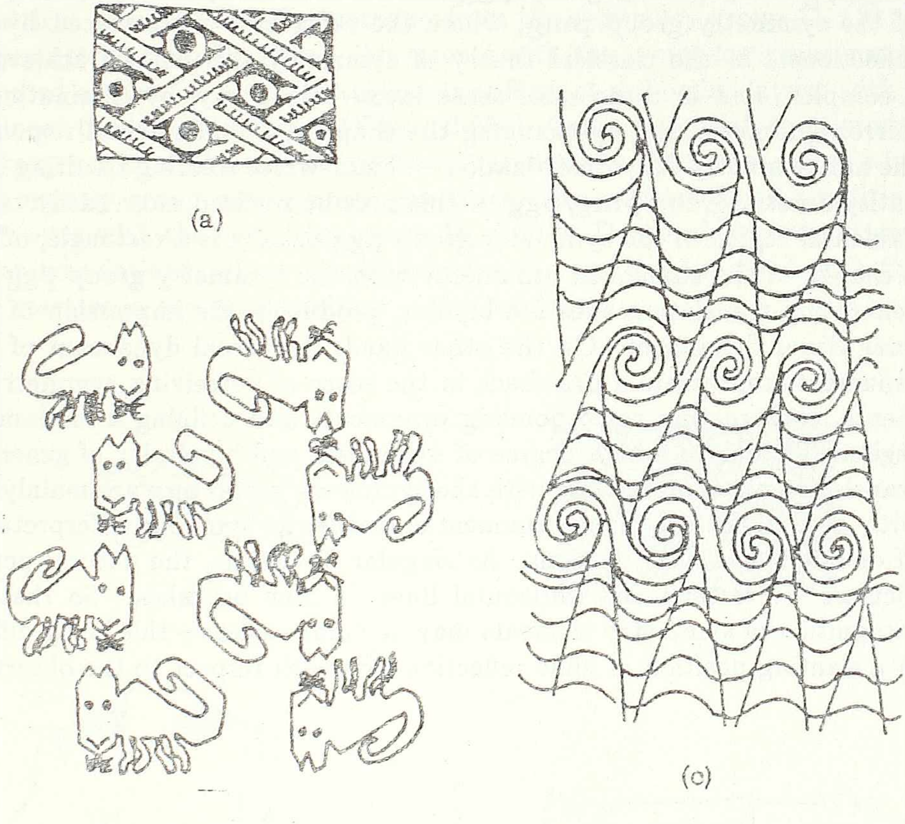


Figure 2.78

Examples of ornaments with the symmetry group **pgg** in the art of Egypt and the Aegean cultures.

transformations — perpendicular glide reflections. Just how difficult it is to perceive these "hidden symmetries", recognize them, construct the corresponding ornaments and discover the symmetry group **pgg**, is proved by the fact that it is the only symmetry group of ornaments omitted by C. Jordan (1868/69). Ornaments with the symmetry group **pgg** are usually realized by the multiplication of a frieze with the symmetry group **1g** by a glide



(a)

(b)

(c)

Figure 2.79

Ornaments with the symmetry group pgg: (a) Columbia, around 800–1500; (b) pre-Columbian art, Peru; (c) the art of the primitive peoples of Oceania.

reflection perpendicular to the frieze axis. Since in Paleolithic and Neolithic ornamental art friezes with the symmetry group 1g occur seldom, this especially refers to ornaments with the symmetry group pgg. The oldest examples of these ornaments can be found in the Neolithic (Figure 2.77) and in the ornamental art of ancient civilizations (Figure 2.78). Besides layered patterns with the resulting symmetry group **pgg**, antisymmetry ornaments with the antisymmetry group **pmg/pgg** (Figure 2.102l) are treated by the classical theory of symmetry as ornaments with the symmetry group **pgg**. Namely, the symmetry group **pgg** can be realized by a desymmetrization of the symmetry group **pmg**, where the reflections are replaced by glide reflections. In the classical theory of symmetry this can be achieved by a complex, and in a technical sense inconvenient way, by eliminating the mirror symmetry and by changing the shape of a fundamental region. By the antisymmetry desymmetrization — black-white coloring resulting in the antisymmetry group **pmg/pgg** — this may be realized more easily. A fundamental region of the symmetry group **pgg** usually is a rectangle, offering a change of the shape. In ornaments with the symmetry group **pgg** both generating translation axes are bipolar, producing the impression of maximal visual dynamism. On the other hand, the visual dynamism of these ornaments represents a drawback in the sense of perceiving symmetry elements, constructing corresponding ornaments and defining a fundamental region. Thanks to a high degree of dynamism and bipolarity of generating translation axes, ornaments with the symmetry group **pgg** are mainly used with geometric and plant ornamental motifs, as symbolic interpretations of double-alternating motions. As singular directions, the natural perpendiculars — vertical and horizontal lines — may be taken. So that, the recognition of symmetry elements may be made easier — this being difficult in a slanting position of glide reflection axis with respect to the observer.

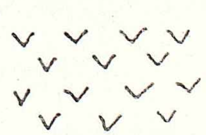
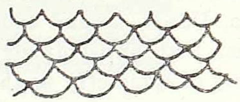


Figure 2.80

The example of the ornament with the symmetry group **cm** in Paleolithic art.

Ornaments with the symmetry group **cm** or **cmm** (Figure 2.80–2.88) are based on a rhombic lattice, the lattice with equal sides, appearing in Paleolithic ornamental art on bone engravings. The origin of a rhombic lattice and its use may be interpreted as a manifestation of the principle of visual entropy — maximal symmetry and maximal simplicity in the visual and constructional sense. The replacement of the unequal sides of a lattice of parallelograms with equal sides results in the symmetrization of the lattice, changing its cell symmetry from the symmetry group C_2 (2) to the symmetry group D_2 (2m). Ornaments with the symmetry group **cmm** originated earlier, in their most elementary form as a rhombic lattice with a rectilinear fundamental region (Figure 2.85c). Ornaments with the symmetry group **cm** or **cmm** maybe originate, respectively, from ornaments with the symmetry group **pm** or **pmm** constructed by a translational repetition of a rosette with the symmetry group D_1 (m) or D_2 (2m) by two perpendicular translations, where "gaps" between the rosettes are filled with the same rosettes in the same position — by *centering*.


(a)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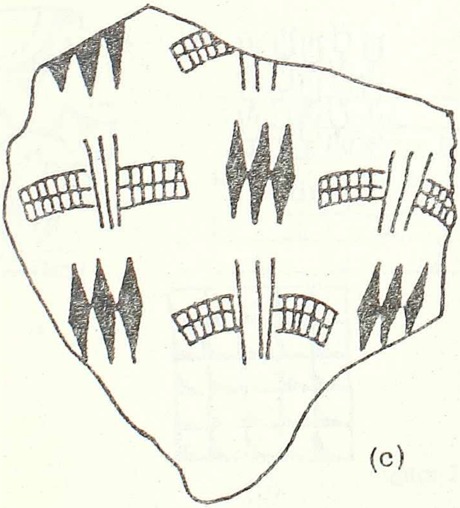
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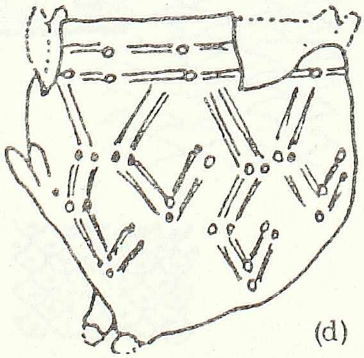
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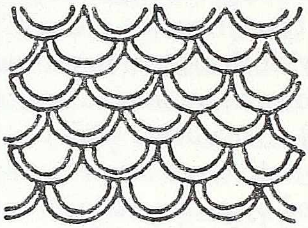

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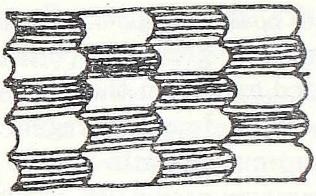
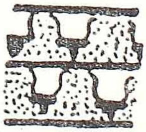


(f)

Figure 2.81

Examples of ornaments with the symmetry group cm in Neolithic art: (a) Susa, around 6000 B.C.; (b) Hallaf, around 6000 B.C.; (c) Haldea, around 5000 B.C.; (d) Nezvisko, USSR, around 5000 B.C.; (e) Hallaf; (f) antisymmetry ornament with antisymmetry group pm/cm, treated by the classical theory of symmetry as the symmetry group cm, Hallaf.



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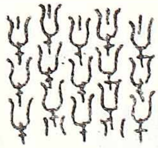

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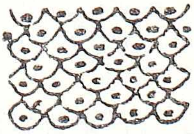

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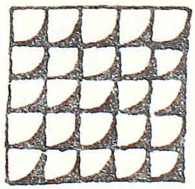

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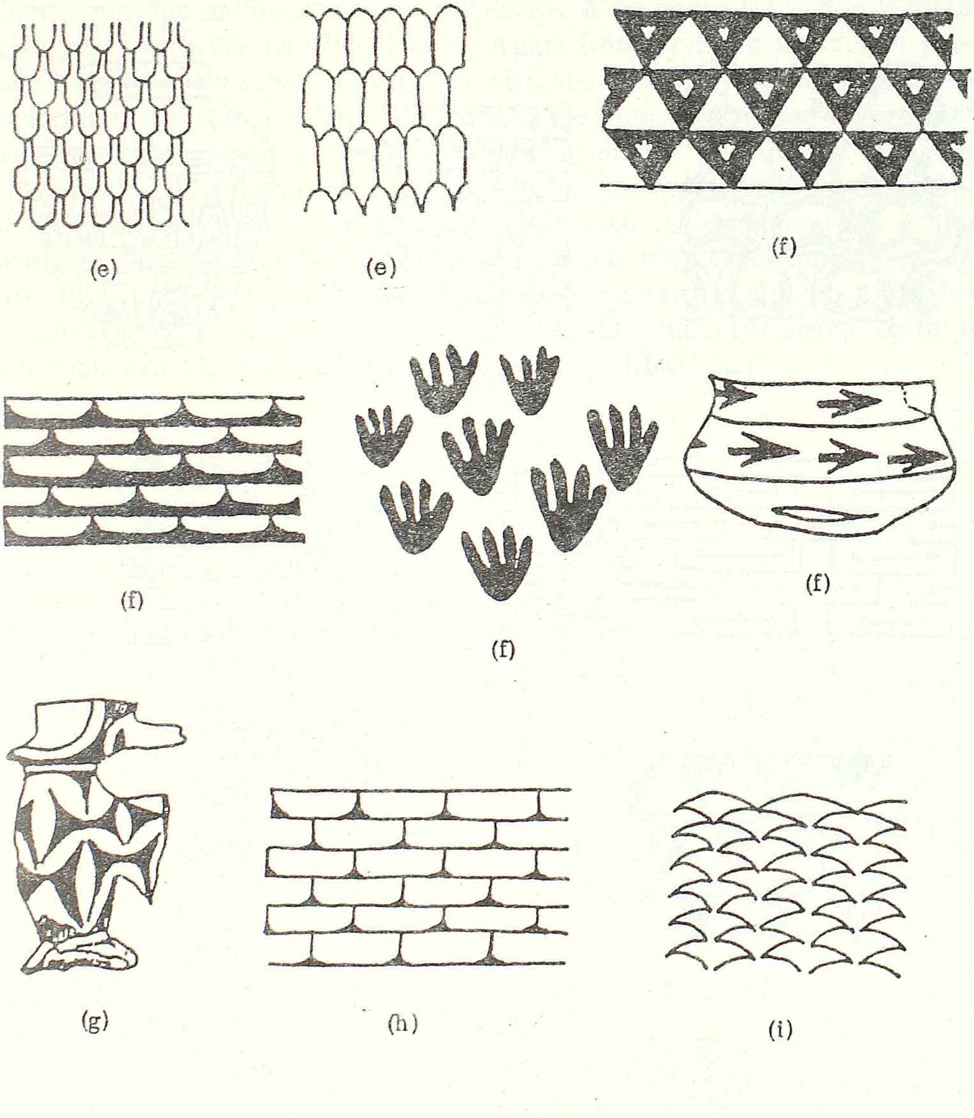


Figure 2.82

Ornaments with the symmetry group cm in Neolithic art: (a) Tell Tschagar Bazaar; (b) Tell Arpachiyah; (c) Hallaf; (d) Eridu culture; (e) Susa; (f) Hacilar; (g) Tripolian culture; (h) Namazga; (i) Starchevo. These ornaments belong to the period 6500–3500 B.C.

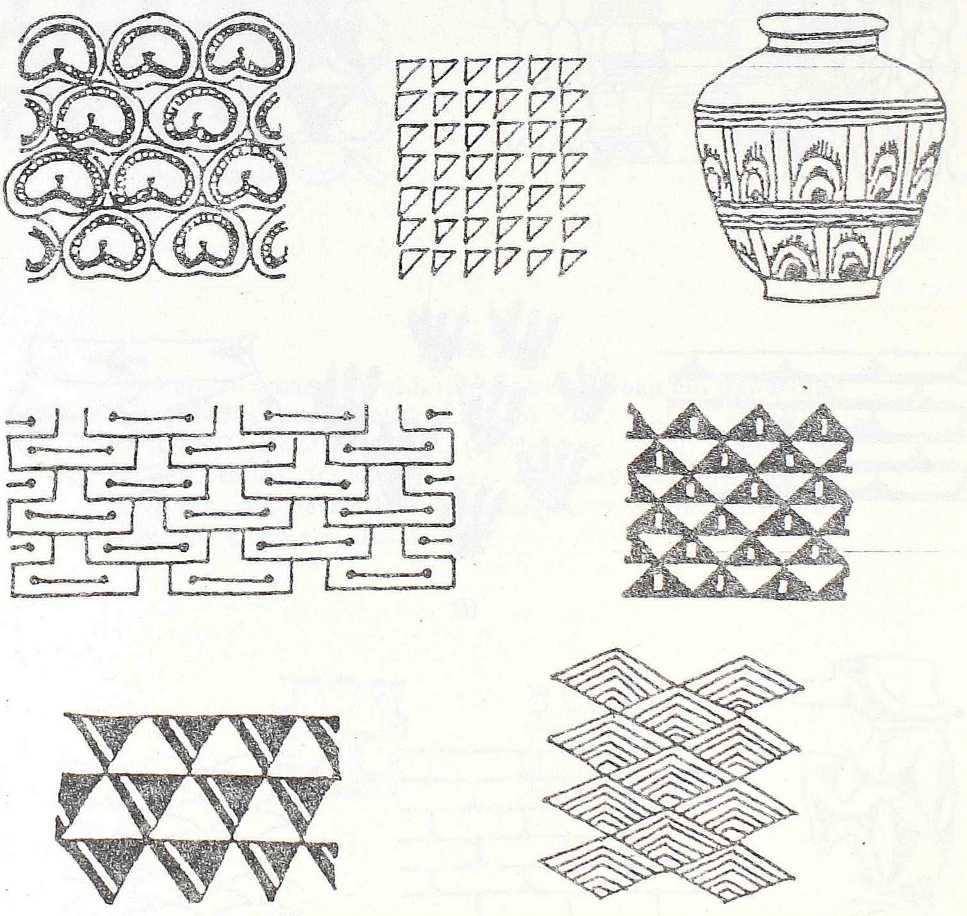
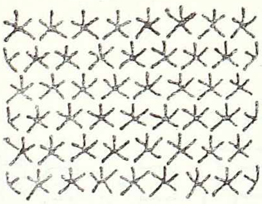


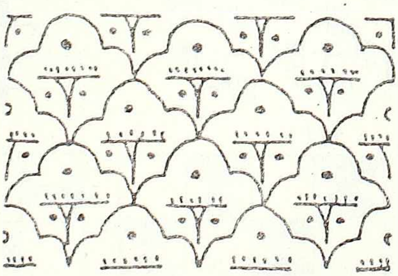
Figure 2.83

Ornaments with the symmetry group cm in Neolithic art (Mohenjo Daro, Niniva, Tepe Aly, Hacilar, Tell i Jari, around 6000–3500 B.C.).

Ornaments with the symmetry group cm (Figure 2.80–2.84) possess one polar diagonal generating translation. They are convenient for suggesting directed motion. Owing to the indirect isometries — reflections — enantiomorphic modifications do not occur. A fundamental region is often triangular, with one rectilinear side. Apart from by the construction proposed above — by means of rosettes with the symmetry group D_1 (m) — such ornaments may be also obtained by a desymmetrization of the symmetry group cmm, where one reflection must be eliminated either by changing the shape of the fundamental region or by a coloring. In the classical theory of symmetry, antisymmetry ornaments with the antisymmetry group cmm/cm are included in the class of ornaments with the symmetry group cm (Figure 2.102g). Ornaments with the symmetry group cm may also be constructed by multiplying a frieze with the symmetry group 1g by a reflection with the reflection line parallel to the frieze axis.



(a)



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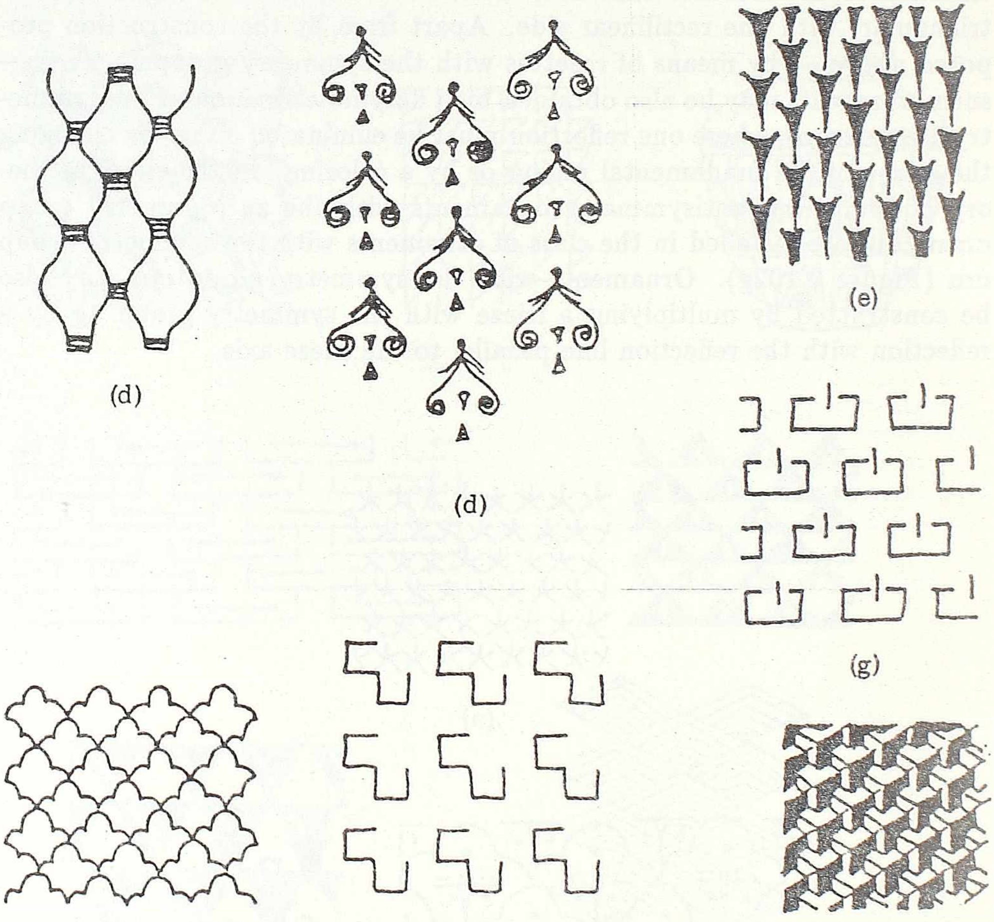


Figure 2.84

Examples of ornaments with the symmetry group cm: (a) Egypt, 2310 B.C.; (b) the Knossos palace; (c) Egypt; (d) Pseira; (e) Egypt, 1830 B.C.; (f) Arabian ornament; (g) the Mittla palace, the pre-Columbian period of America; (h) Gothic ornament.

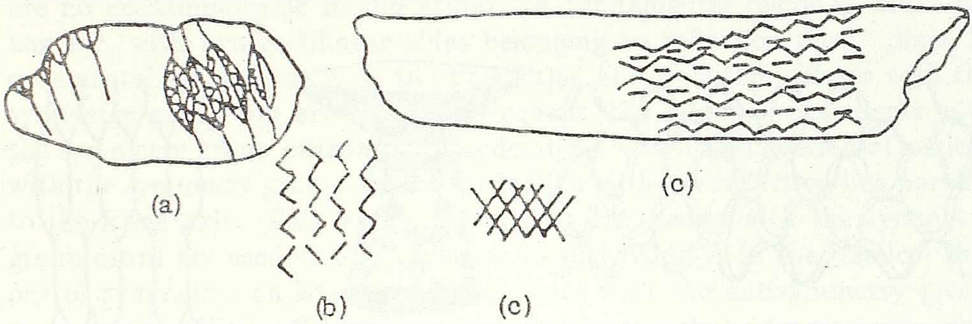


Figure 2.85

Ornaments with the symmetry group cmm in Paleolithic art: (a) Polesini cave; (b) Laugerie Haute; (c) Pindel, Vogelherd.

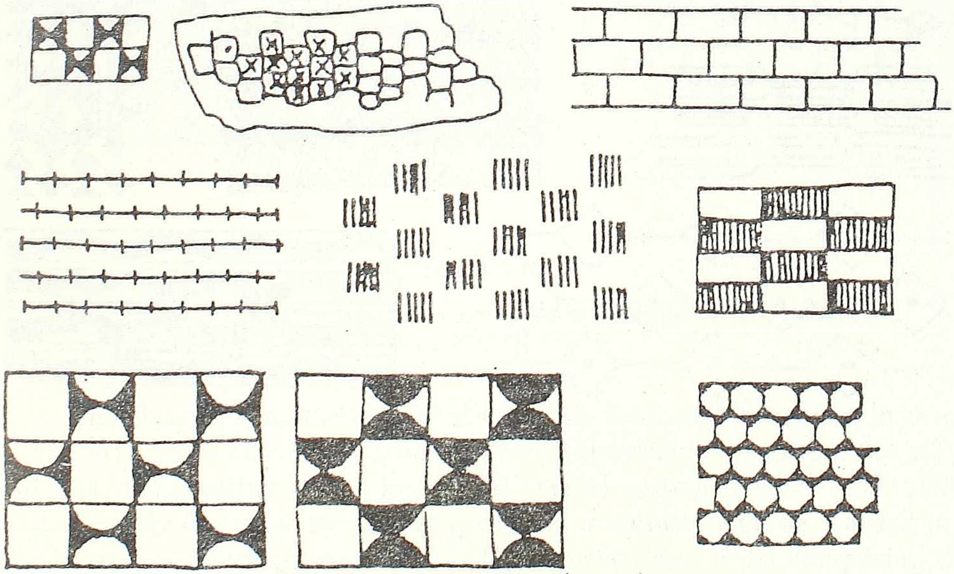


Figure 2.86

Examples of ornaments with the symmetry group cmm in the Neolithic art of the Middle East (Hallaf, Catal Hüjük).

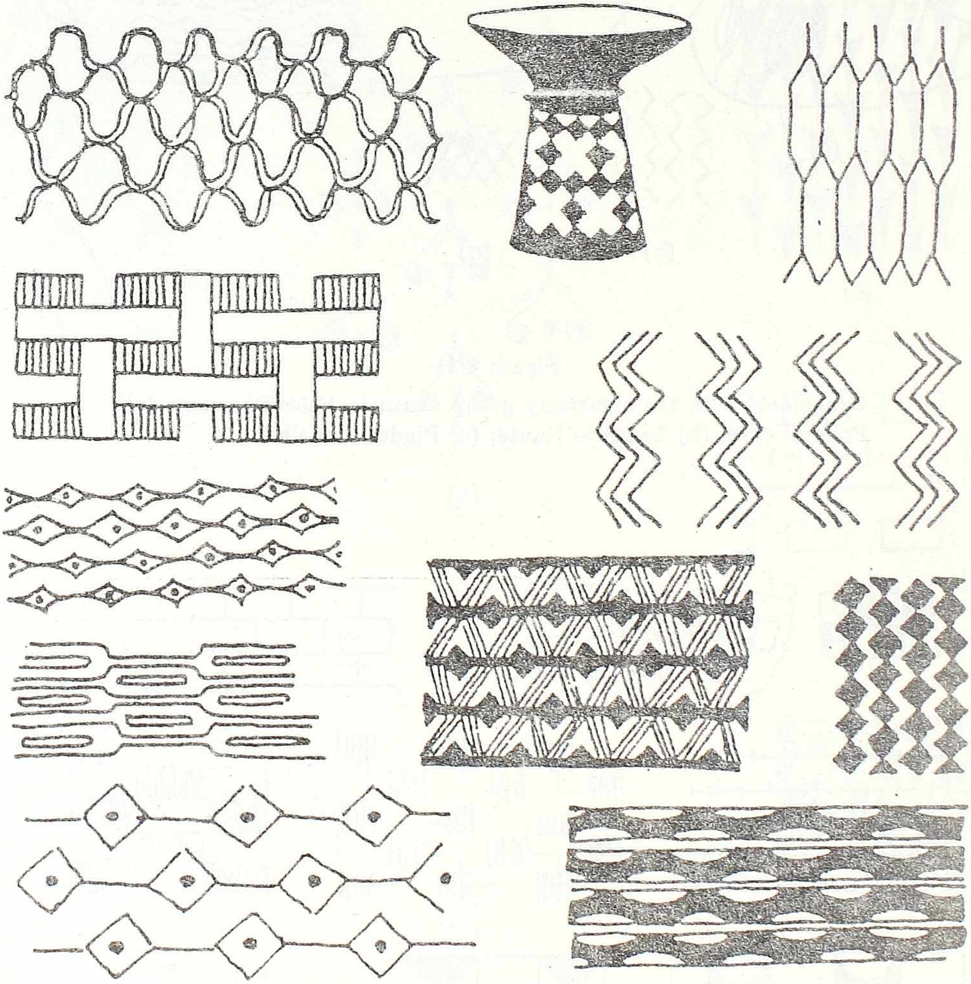


Figure 2.87

Examples of ornaments with the symmetry group cmm in Neolithic art (Susa, Hajji Mohammed, Siyalk, Hacilar, Lendel culture, around 6000–3000 B.C.).

Ornaments with the symmetry group **cmm** (Figure 2.85–2.88) possess one non-polar diagonal and one bipolar generating translation axis. There are no enantiomorphic modifications. A fundamental region is often triangular, with two rectilinear sides belonging to reflection lines. Since in ornamental art, especially in the Paleolithic and Neolithic, friezes with the symmetry group **mg** are the most frequent, the origin of ornaments with the symmetry group **cmm** can be understood as a multiplication of a frieze with the symmetry group **mg** by a reflection with the reflection line parallel to the frieze axis. This is the reason that ornaments with the symmetry group **cmm** are used so much and are in such variety. In the classical theory of symmetry, antisymmetry ornaments with the antisymmetry group **pmm/cmm** (Figure 2.102b) are included in the class of ornaments with the symmetry group **cmm**, thus enriching it considerably.

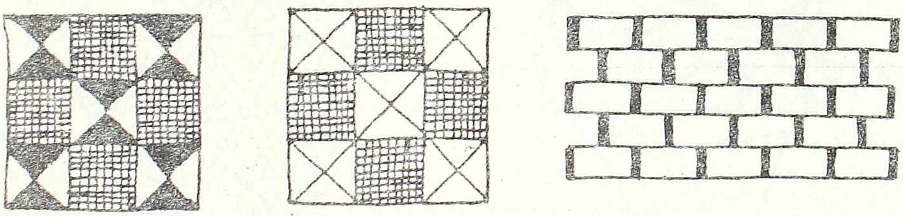


Figure 2.88

Ornaments with the symmetry group cmm in Neolithic art, Hallaf, around 6000 B.C. (7600–6900 B.C.?).

The class of ornaments appearing much later and occurring in ornamental art seldom consists of ornaments with the symmetry groups **p3**, **p3m** and **p31m**, belonging to the hexagonal crystal system (Figure 2.89, 2.90). All previously discussed symmetry groups of ornaments can be realized in a simple way, as primary plane lattices (Bravais lattices) or by composing the symmetry groups of friezes with the simplest symmetry groups of rosettes D_1 (m) or C_2 (2), generated by a reflection or by a central reflection.

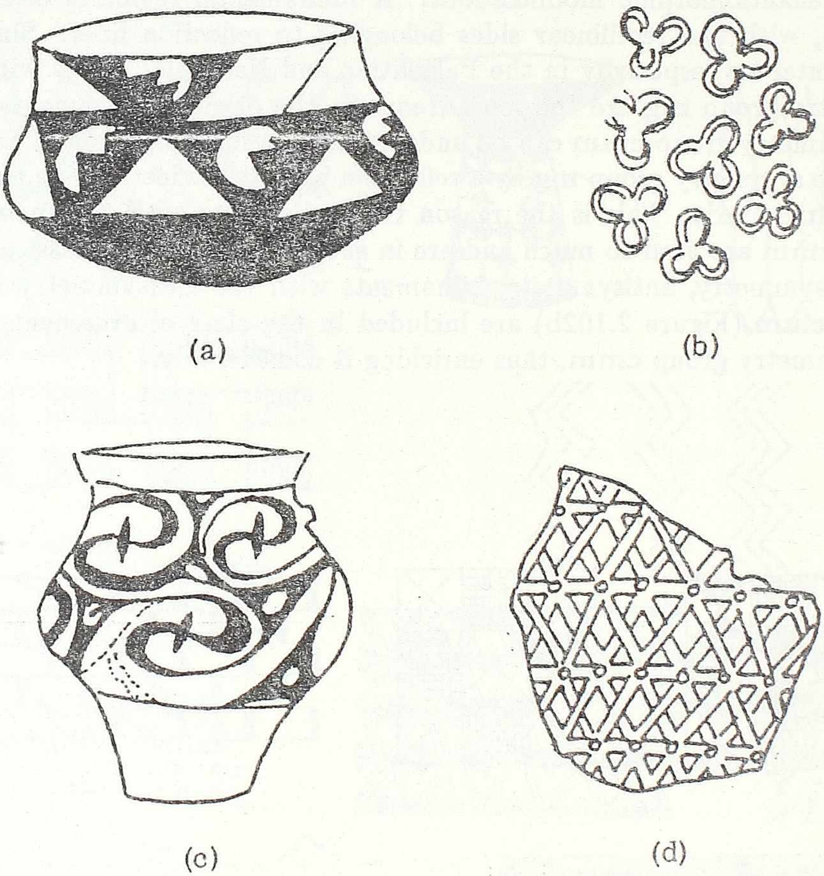


Figure 2.89

Ornamental motifs that suggest the symmetry of plane ornaments with three-fold rotations: (a) Hallaf, around 6000 B.C. (7600–6900 B.C.?); (b) Mohenjo Daro, around 4500–4000 B.C.; (c) Tripolian culture, around 4000–3500 B.C.; (d) the interlaced motif, Egypt, early dynastic period.

Ornaments with the symmetry group **p3** may be constructed by multiplying a rosette with the symmetry group C_3 (**3**) by two non-parallel translations with translation vectors of the same intensity, constructing a $60°$-angle or by multiplying a frieze with the symmetry group **11** by a trigonal rotation. Since natural forms with the symmetry group C_3 (**3**) occur seldom, the absence of these natural models accounted for the absence of ornaments with the symmetry group **p3** in the earliest periods of ornamental art. A few older examples of ornaments containing rosettes with the symmetry group C_3 (3) deviate from the exact rules of symmetry (Figure 2.89) and only suggest the **p3**-symmetry, which would be used consequently in the ornamental art of the ancient civilizations (of Egypt, China, etc.) (Figure 2.90). The usual shape of a fundamental region of the symmetry group **p3** is a rhombus with a 60°-angle, offering great possibilities for variations and for curvilinear boundaries. Ornaments with the symmetry group **p3** possess enantiomorphic modifications. Since they contain two dynamic polar components — polar translation axes and trigonal polar oriented rotations, such ornaments are dynamic ones. Their visual dynamism can be stressed by using a suitable shape of the fundamental region or of an elementary asymmetric figure belonging to the fundamental region. Representing the result of a superposition of the symmetry groups **11** and C_3 (3), the symmetry group **p3** possesses three singular polar directions, so that the corresponding ornaments produce an impression of a three-way directed motion. As a subgroup of the index 4 of the symmetry group **p6**, the symmetry group **p3** may be derived by a desymmetrization of the symmetry group **p6m**, or even by a desymmetrization of the symmetry group **p6**. The symmetry group **p6** is the subgroup of the index 2 of the symmetry group **p6m**, and the symmetry group **p3** is the subgroup of the index 2 of the symmetry group **p6**. This offers many possibilities for classical-symmetry or antisymmetry desymmetrizations, in particular, for the antisymmetry desymmetrization resulting in the antisymmetry group **p6/p3**, treated by the classical theory of symmetry as the symmetry group **p3** (Figure 2.90a, 2.102d).

Ornaments with the symmetry groups **p31m** and **p3m1** differ among themselves by the position of reflection lines. In ornaments with the symmetry group **p31m** reflection lines are parallel to the generating translation axis, while in ornaments with the symmetry group **p3m1** they are perpendicular to it. Regarding their origin, both groups can be understood as a result of superposition of the symmetry group of rosettes D_3 (3m) and the symmetry group of friezes **11**, where in ornaments with the symmetry group **p31m** the frieze axis is parallel to the reflection line of the rosette, while in ornaments with the symmetry group **p3m1** it is perpendicular to the reflection line. Also, we may consider the symmetry group **p31m** as the superposition of the symmetry group of rosettes C_3 (3) and the symmetry group of friezes **1m**, and the symmetry group **p3m1** as the superposition of the same symmetry group C_3 (3) and the symmetry group of friezes **m1**.

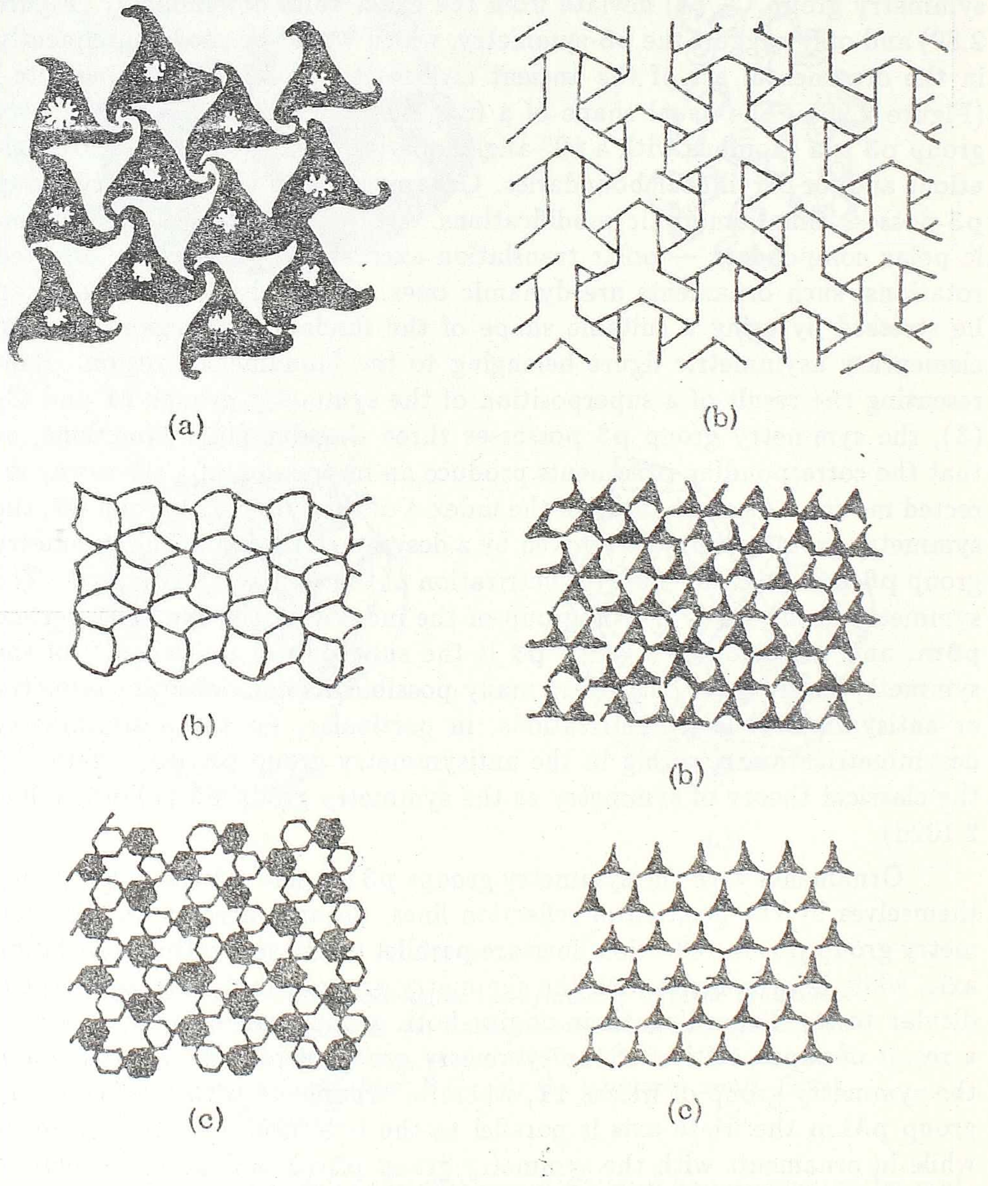


Figure 2.90

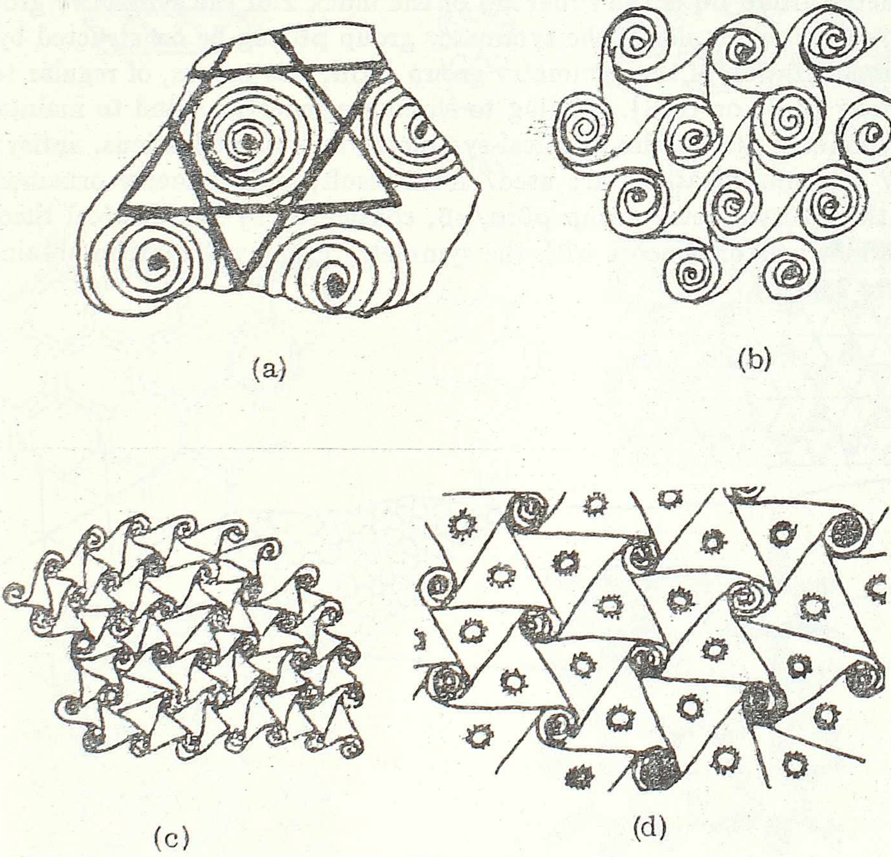
Ornaments with three-fold rotations: (a) p3; (b) p31m; (c) p3m1.

From there result the following common properties of these two groups: the symmetry groups **p31m** and **p3m1** are the subgroups of the index 2 of the symmetry group **p6m**, so they can be derived by desymmetrizations of the symmetry group **p6m**, in particular, by antisymmetry desymmetrizations. Owing to reflections with reflection lines incident with trigonal rotation centers, the symmetry group **p3 m1** is characterized by the non-polarity of all trigonal rotations. In ornaments with the symmetry group **p31m** half of the trigonal rotation centers are incident to reflection lines, and the others are not, so the symmetry group **p31m** contains the polar and non-polar trigonal rotations. In both symmetry groups, the indirect transformations — reflections — cause the absence of the enantiomorphism. In ornaments with the symmetry groups **p31m** and **p3m1** glide reflections parallel with reflection lines appear as secondary elements of symmetry. The different position of reflection lines with respect to the generating translation axes causes certain relevant constructional and visual-symbolic differences between the corresponding ornaments.

Same as their generating friezes with the symmetry group **1m**, ornaments with the symmetry group **p31m**, that contain a reflection parallel to the generating translation axis, have a polar, oriented generating translation axis. From this the visual dynamism of ornaments with the symmetry group **p31m** results. They produce the impression of three-way directed motion. A fundamental region of the symmetry group **p31m** is usually defined by a longer diagonal and by two sides of the rhombic fundamental region already mentioned in the case of the symmetry group **p3**, where these two sides can be replaced with adequate curvilinear contours. This makes possible different variations and the emphasis or alleviation of the dynamic visual impression produced by the corresponding ornaments. Despite the existing conditions for a variety of ornaments with the symmetry group **p31m**, in old ornamental art they occur relatively seldom, mainly because of their constructional complexity and the absence of natural models. Their full affirmation will come in the ornamental art of the developed ancient civilizations (Figure 2.90). Examples of the symmetry group **p31m** may be obtained by a desymmetrization of the symmetry group **p6m** in which the symmetry group **p31m** is the subgroup of the index 2. Owing to the constructional complexity of the corresponding antisymmetry mosaics, the antisymmetry group **p6m/p31m**, treated by the classical theory of symmetry as the symmetry group **p31m** , is poorly represented in ornamental art (A.V. Shubnikov, N.V. Belov et al., 1964, pp. 220).

Similar to the corresponding symmetry group of friezes **m1**, the symmetry group of ornaments **p3m1** contains reflections with reflection lines perpendicular to the generating translation axis, that cause the non-polarity of the axis. Therefore, in a geometric and visual-symbolic sense, the symmetry group **p3m1** can be considered as a static equivalent of the symmetry group **p31m**. The fundamental region of the symmetry group **p3m1** is an equilateral triangle. The symmetry group **p3m1** (△) is generated by reflections, that require a rectilinear triangular fundamental region and restrict the area of the corresponding classical-symmetry ornaments to the use of some elementary asymmetric figure belonging to a fundamental region, resulting in a *marked isohedral tiling* (B. Grünbaum, G.C. Shephard, 1987). No desymmetrization of a fundamental region of the symmetry group D_3 (3m) can be achieved by changing its shape. Therefore, an internal desymmetrization of the fundamental region becomes indispensable, making impossible a visual interpretation of the symmetry group **p3m1** accompanied by an isohedral tiling, without a previous internal desymmetrization of the fundamental region. A fundamental region must be rectilinear in all the groups generated by reflections: **pmm**, **p4m**, **p6m** and **p3m1**. Except the symmetry group **p3m1**, this fact has no influence on the possibility to construct an isohedral tiling with a tile serving as the fundamental region of the symmetry group discussed. Only in the symmetry group **p3m1**, where this tiling is a regular tessellation $\{3, 6\}$, there comes the symmetrization by composing six equilateral triangles with a common vertex and each with the symmetry group D_3 (3m). The result is the isohedral tiling with the symmetry group **p6m**. Therefore, in ornamental art, the symmetry group **p3m1** occurs seldom, especially in the earlier period (Figure 2.90c). Distinct from classical-symmetry ornaments with the symmetry group **p3m1**, antisymmetry ornaments with the antisymmetry group **p6m/p3m1**, in the classical theory of symmetry treated as ornaments with the symmetry group **p3m1**, are some most frequent antisymmetry ornaments. The antisymmetry group **p6m/p3m1** can be derived by the antisymmetry desymmetrization of the symmetry group **p6m**, where the adjacent equilateral triangles of a regular tessellation $\{3, 6\}$ are colored oppositely (Figure 2.102n). Their generating symmetry group **p6m** is one of the most frequent symmetry groups in the whole of ornamental art, while their constructional and visual simplicity caused the frequent occurrence of ornaments with the antisymmetry group **p6m/p3m1**.

The symmetry group **p6** belongs to the hexagonal crystal system. The corresponding ornaments (Figure 2.91) may be constructed by multiplying a rosette with the symmetry group C_6 (6) by means of two non-parallel translations with translation vectors of the same intensity, constructing a 60^o-angle. The symmetry group **p6** can be obtained by superposing the symmetry group of friezes **11** and the symmetry group of rosettes $\mathbf{C_6}$ (**6**), or by composing the symmetry group of friezes **12** and the symmetry group of rosettes $\mathbf{C_3}$ (**3**). The oldest examples of ornaments with the symmetry group **p6** date from the ancient civilizations (Figure 2.91c,d). Owing to

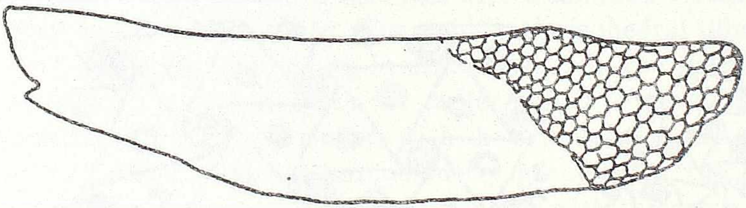


(a) (b) (c) (d)

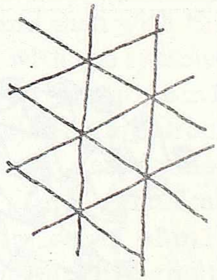
Figure 2.91

Ornaments with the symmetry group **p6**: (a) Butmir, around 4000 B.C.; (b) Pseira; (c) Cyclades, around 2500 B.C.; (d) Crete and Egypt.

their constructional complexity and the lack of models in nature, in ornamental art they are rather rare. A fundamental region of the symmetry group **p6** may by an equilateral triangle with sides that can be replaced by curved lines. This results in the variety of ornaments with the symmetry group **p6**. The enantiomorphism occurs. Since central reflections are elements of the symmetry group **p6**, the generating translation axes will be bipolar. Although their dynamic visual effect is alleviated by central reflections and by the bipolarity of the generating translation axes, ornaments with the symmetry group **p6** belong to a family of dynamic ornaments. In them, it is possible to recognize three singular bipolar directions. The symmetry group **p6** is the subgroup of the index 2 of the symmetry group **p6m**, so that examples of the symmetry group **p6** can be constructed by a desymmetrization of the symmetry group **p6m**, this means, of regular tessellations $\{3,6\}$ or $\{6,3\}$. Aiming to eliminate reflections and to maintain the symmetry **p6**, besides classical-symmetry desymmetrizations, antisymmetry desymmetrizations are used. As a result, antisymmetry ornaments with the antisymmetry group **p6m/p6**, considered by the classical theory of symmetry as ornaments with the symmetry group **p6**, may be obtained (Figure 2.102o).



(a)



(b)

Figure 2.92

Ornaments with the symmetry group **p6m** in Paleolithic art: (a) regular tessellation $\{6,3\}$, the motif of honeycomb, Yeliseevichi, USSR, around 10000 B.C.; (b) the example of regular tessellation $\{3,6\}$ in the Paleolithic art of Europe (Magdalenian).

The symmetry group **p6m** (Figure 2.92, 2.93) is the maximal symmetry group of the hexagonal crystal system. The oldest and most frequent ornaments with the symmetry group **p6m** are regular tessellations $\{3,6\}$ and $\{6,3\}$. The tessellation $\{6,3\}$ dates from the Paleolithic (Figure 2.92a),

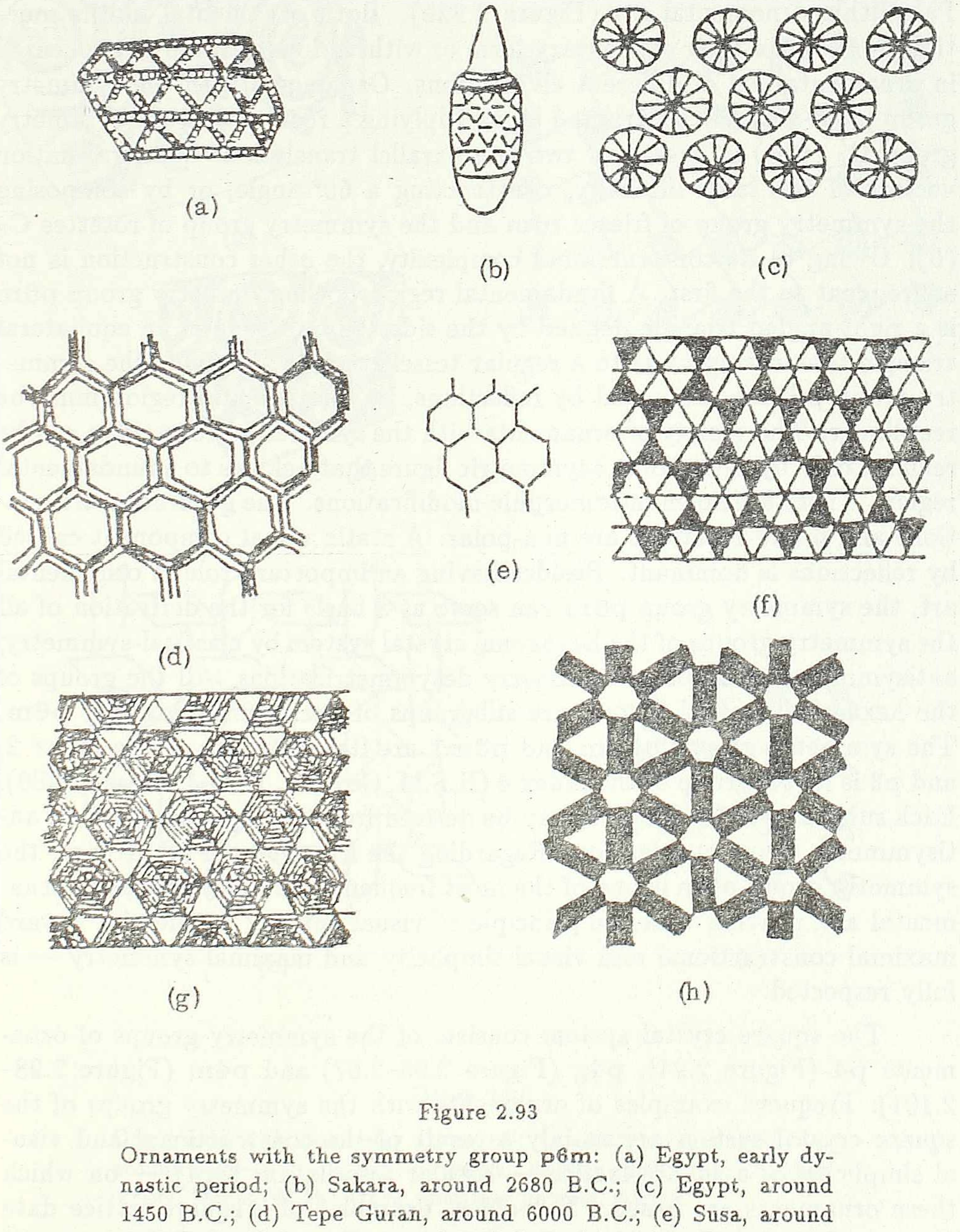


Figure 2.93

Ornaments with the symmetry group p6m: (a) Egypt, early dynastic period; (b) Sakara, around 2680 B.C.; (c) Egypt, around 1450 B.C.; (d) Tepe Guran, around 6000 B.C.; (e) Susa, around 6000 B.C.; (f) Samara, around 5000 B.C.; (g) Middle East, around 5000–4000 B.C.; (h) Greco-Roman mosaic.

occurring on a bone engraving. This tessellation and its corresponding symmetry group **p6m** probably originated from natural models — honeycombs — the regularity and symmetry of which has always attracted the attention of artists and mathematicians. Its dual tessellation $\{3,6\}$ also dates from Paleolithic ornamental art (Figure 2.92b). Both ornamental motifs mentioned, either in their elementary form or with various paraphrases, occured in ornamental art of different civilizations. Ornaments with the symmetry group **p6m** may be constructed by multiplying a rosette with the symmetry group D_6 (**6m**) by means of two non-parallel translations with translation vectors of the same intensity, constructing a 60°-angle, or by composing the symmetry group of friezes **mm** and the symmetry group of rosettes C_6 (**6**). Owing to its constructional complexity, the other construction is not as frequent as the first. A fundamental region of the symmetry group **p6m** is a right-angled triangle defined by the sides and altitude of an equilateral triangle that corresponds to a regular tessellation $\{3,6\}$. Since the symmetry group **p6m** is generated by reflections, its fundamental region must be rectilinear. The variety of ornaments with the symmetry group **p6m** can be realized only by using some asymmetric figure that belongs to a fundamental region. There are no enantiomorphic modifications. The generating translations and 6-fold rotations are non-polar. A static visual component caused by reflections is dominant. Besides having an important role in ornamental art, the symmetry group **p6m** can serve as a basis for the derivation of all the symmetry groups of the hexagonal crystal system by classical-symmetry, antisymmetry and colored-symmetry desymmetrizations. All the groups of the hexagonal crystal system are subgroups of the symmetry group **p6m**. The symmetry groups **p31m** and **p3m1** are its subgroups of the index 2, and **p3** is its subgroup of the index 4 (H.S.M. Coxeter, W.O.J. Moser, 1980). Each subgroup of the index 2 may be derived from its supergroup by an antisymmetry desymmetrization. Regarding the frequency of occurrence, the symmetry group **p6m** is one of the most frequent symmetry groups in ornamental art, proving that the principle of visual entropy — the aim toward maximal constructional and visual simplicity and maximal symmetry — is fully respected.

The square crystal system consists of the symmetry groups of ornaments **p4** (Figure 2.94), **p4g** (Figure 2.95–2.97) and **p4m** (Figure 2.98–2.101). Frequent examples of ornaments with the symmetry groups of the square crystal system are mainly a result of the constructional and visual simplicity of a square lattice — regular tessellation $\{4,4\}$ — on which these ornaments are based. The oldest examples of a square lattice date from the Paleolithic stone or bone engravings. By uniting two fundamental

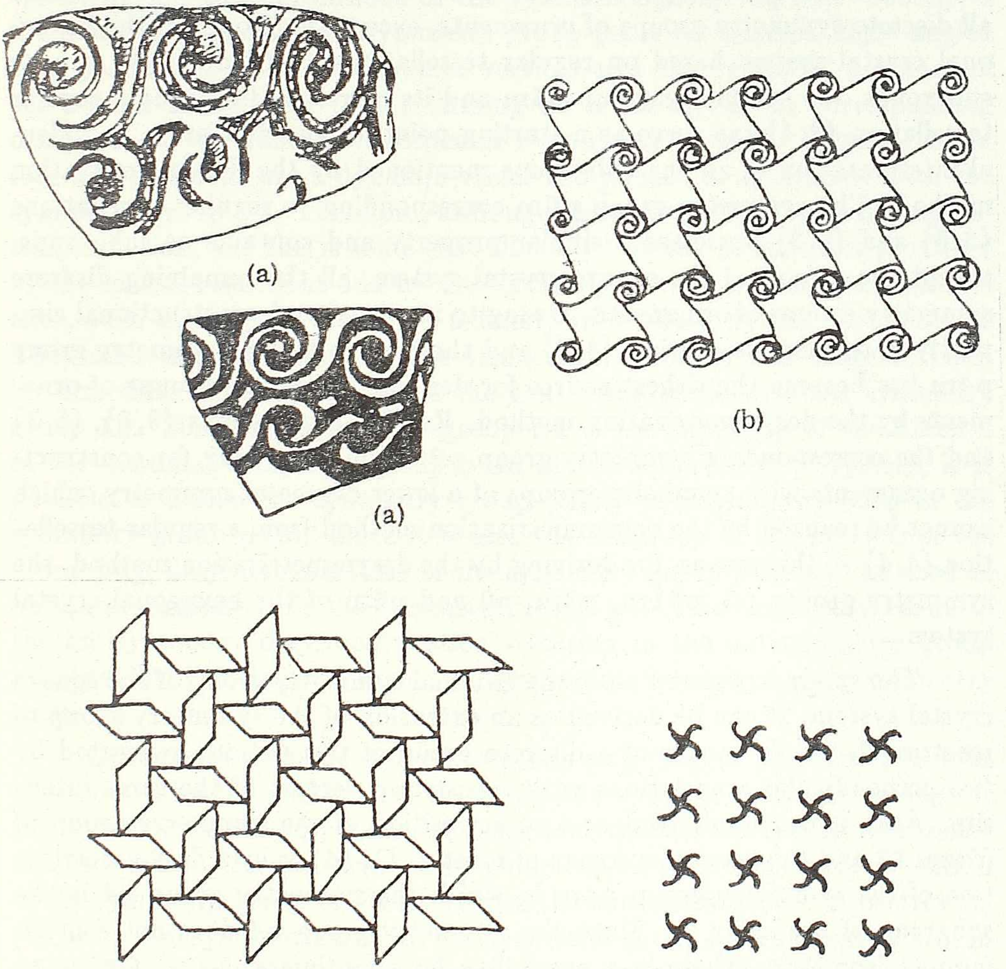


Figure 2.94

Ornaments with symmetry group **p4**: (a) Starchevo, Yugoslavia, around 5500–5000 B.C.; (b) Egypt and Aegean cultures; (c) Akhbar school, India; (d) the art of primitive peoples, Africa.

properties — division of a plane into squares, and perpendicularity — a regular tessellation $\{4,4\}$ possesses the maximal symmetry group **p4m** of the square crystal system and serves as a basis for the construction of all ornaments with the symmetry groups belonging to this crystal system. Because all discrete symmetry groups of ornaments, except the groups of the hexagonal crystal system based on regular tessellations $\{3,6\}$ or $\{6,3\}$, are its subgroups, the symmetry group **p4m** and its corresponding square regular tessellation $\{4,4\}$ can serve as a starting point for the derivation and visual interpretation of all the subgroups mentioned by the desymmetrization method. The symmetry group **p6m** corresponding to regular tessellations $\{3,6\}$ and $\{6,3\}$ possesses a similar property and contains as subgroups, except the groups of the square crystal system, all the remaining discrete symmetry groups of ornaments. Owing to its visual and constructional simplicity, a regular tessellation $\{4,4\}$ and the corresponding symmetry group **p4m** has become the richest source for deriving symmetry groups of ornaments by the desymmetrization method. Regular tessellations $\{3,6\}$, $\{6,3\}$ and the corresponding symmetry group **p6m** will serve only for constructing ornaments with symmetry groups of a lower degree of symmetry, which cannot be realized by the desymmetrization method from a regular tessellation $\{4,4\}$ — this means, for deriving by the desymmetrization method, the symmetry groups **p3**, **p31m**, **p3m**, **p6** and **p6m** of the hexagonal crystal system.

The symmetry group **p4** is the minimal symmetry group of the square crystal system. It can be derived as an extension of the symmetry group of rosettes C_4 (4) by means of a discrete group of translations generated by two perpendicular translations with translation vectors of the same intensity. Also, it can be derived as a superposition of the symmetry group of friezes **12** and the symmetry group of rosettes C_4 (4) or by a desymmetrization of the symmetry group **p4m** in which the symmetry group **p4** is the subgroup of the index 2 . Since the symmetry group **p4** does not contain indirect isometries, there is a possibility for enantiomorphism. Owing to the visual dynamism caused by the absence of reflections, the possibility of enantiomorphism, and the bipolarity of generating translation axes and polarity of 4-rotations, the visual and constructional simplicity of a regular tessellation $\{4,4\}$ is not so expressed in ornaments with the symmetry group **p4**. In constructing ornaments with the symmetry group **p4**, all construction methods will have almost equal importance. Owing to their low degree of symmetry and complex construction, ornaments with the symmetry group **p4** are not as frequent as ornaments with the symmetry groups **p4g** and **p4m** belonging to the same crystal system. The oldest examples of ornaments with the symmetry group **p4** originate from Neolithic ceramics (Figure 2.94). These ornaments are constructed by multiplying a frieze with the symmetry group **12**, with the "wave" motif based on a double spiral, by the transformations of the symmetry group C_4 (4). Usually, a fundamental region of the symmetry group **p4** is an isosceles right-angled triangle defined by the immediate vertices and the center of a square of the regular tessellation $\{4,4\}$. Aiming for a variety of the corresponding ornaments, a curvilinear fundamental region may be used. Polar, oriented four-fold rotations are a dynamic visual component of ornaments with the symmetry group **p4**. Therefore, although they do produce a visual suggestion of motion, and this is somewhat alleviated by the visual effect produced by the central reflections and by the bipolarity of the generating translation axes, such ornaments belong to a family of visually dynamic ornaments. They offer the possibility for the visual distinction of four singular bipolar directions that correspond to the generating frieze with the symmetry group **12**. Since the symmetry group **p4** is the subgroup of the index 2 of the maximal symmetry group **p4m** of the square crystal system, it is possible to derive the symmetry group **p4** by a desymmetrization of the symmetry group **p4m**. Since it is also the subgroup of the index 2 of the group **p4g**, desymmetrizations of the symmetry group **p4g** may be used to obtain ornaments with the symmetry group **p4**. That especially refers to the antisymmetry desymmetrization resulting in the antisymmetry group **p4g/p4**, by the classical theory of symmetry discussed as the symmetry group **p4**.

Ornaments with the symmetry group **p4g** (Figure 2.95–2.97) may be constructed by the multiplication of a rosette with the symmetry group C_4 (4) by means of two perpendicular reflections that do not contain the center of this rosette. The same symmetry group can be derived as a superposition of the symmetry group of friezes **1g** and the symmetry group of rosettes C_4 (4) or by a desymmetrization of the symmetry group of ornaments **p4m**, in which the group **p4g** is the subgroup of the index 2. The symmetry group **p4g** corresponds to a uniform tessellation $s\{4,4\}$, i.e. (4.8^2) (H.S.M. Coxeter, W.O.J. Moser, 1980; B. Grünbaum, G.C. Shephard, 1987). According to the criterion of maximal constructional simplicity, a desymmetrization of the symmetry group **p4m** or the multiplication of a rosette with the symmetry group C_4 (4) by means of two perpendicular reflections, non incident with the rosette center, will be the prevailing methods for constructing ornaments with the symmetry group **p4g**. The oldest ornaments with the symmetry group **p4g** appear in Neolithic ceramics (Figure 2.95). In ornamental art of the ancient civilizations such ornaments, with somewhat changed Neolithic motifs, are very frequent. This especially refers to the ornaments with the symmetry group p4g using "swastika" motifs, which appeared as early as the Neolithic (Figure 2.95b).

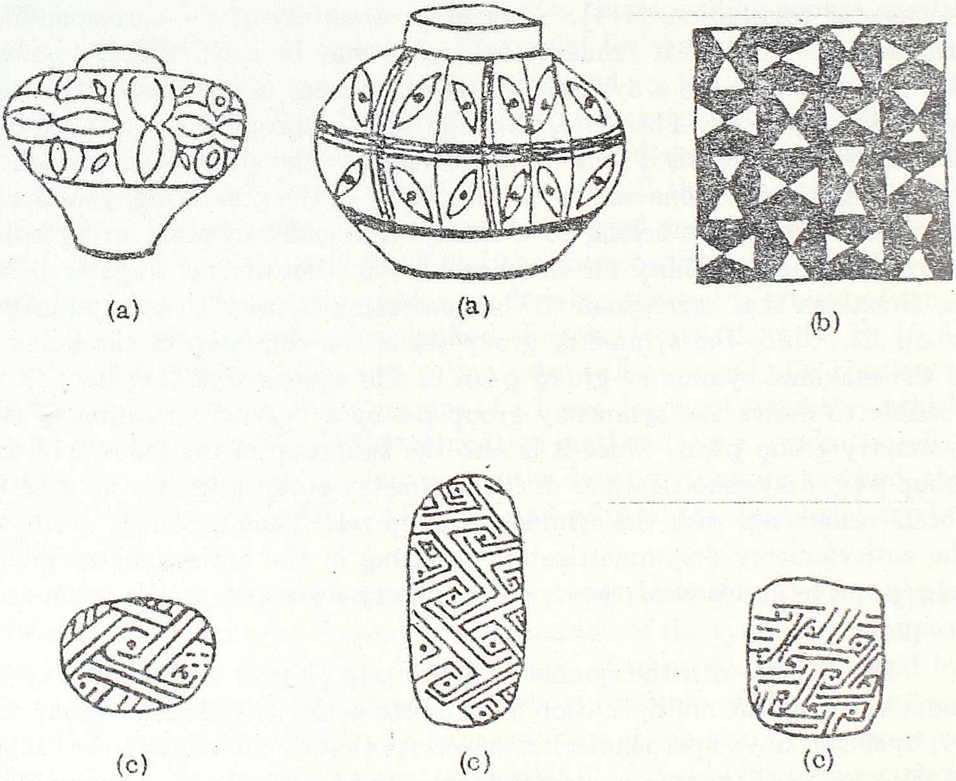


Figure 2.95

Examples of ornaments with the symmetry group p4g in Neolithic art: (a) Tripolian culture, USSR, around 4000 B.C.; (b) Hallaf, around 6000 B.C.; (c) Catal Hüjük, around 6400–5800 B.C.

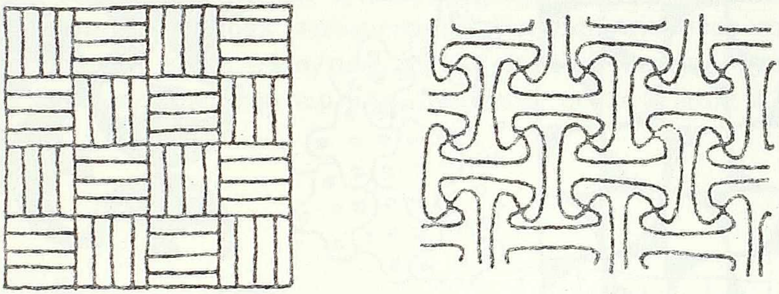


Figure 2.96

Examples of ornaments with the symmetry group **p4g** in the art of the dynastic period of Egypt.

Therefore, a detailed comparative analysis of the repetition of that and some other ornamental motifs can be useful in the study of the inter-cultural relations and influences that occurred in the Neolithic period and at the beginning of ancient civilizations. Usually, a fundamental region of the symmetry group **p4g** is an isosceles right-angled triangle defined by the centers of adjacent sides and by the corresponding vertex of a square of the regular tessellation $\{4,4\}$. Aiming to increase the variety of ornaments with the symmetry group **p4g** and to emphasize or decrease the intensity of the dynamic visual effect produced by them, its edges can be replaced by curved lines. The generating translation axes are bipolar. Despite the secondary reflections and the absence of the enantiomorphism, the four-fold rotations are polar, since these reflections do not contain the four-fold rotation centers. Owing to the four-fold polar, oriented rotations and glide reflections, ornaments with the symmetry group **p4g** belong to a family of extremely dynamic ornaments in the visual sense. They suggest a motion by their parts and by the whole, which is only partly offset by the static visual effect produced by the secondary reflections. Ornaments with the symmetry group **p4g** offer the possibility for the visual distinction of generating rosettes with the symmetry group C_4 (4) and generating friezes with the symmetry group **mg**. According to the principle of visual entropy, a visual recognition of their symmetry substructures will be hindered by the visual dynamism of these ornaments and by the visual dominance of glide reflections as the elements

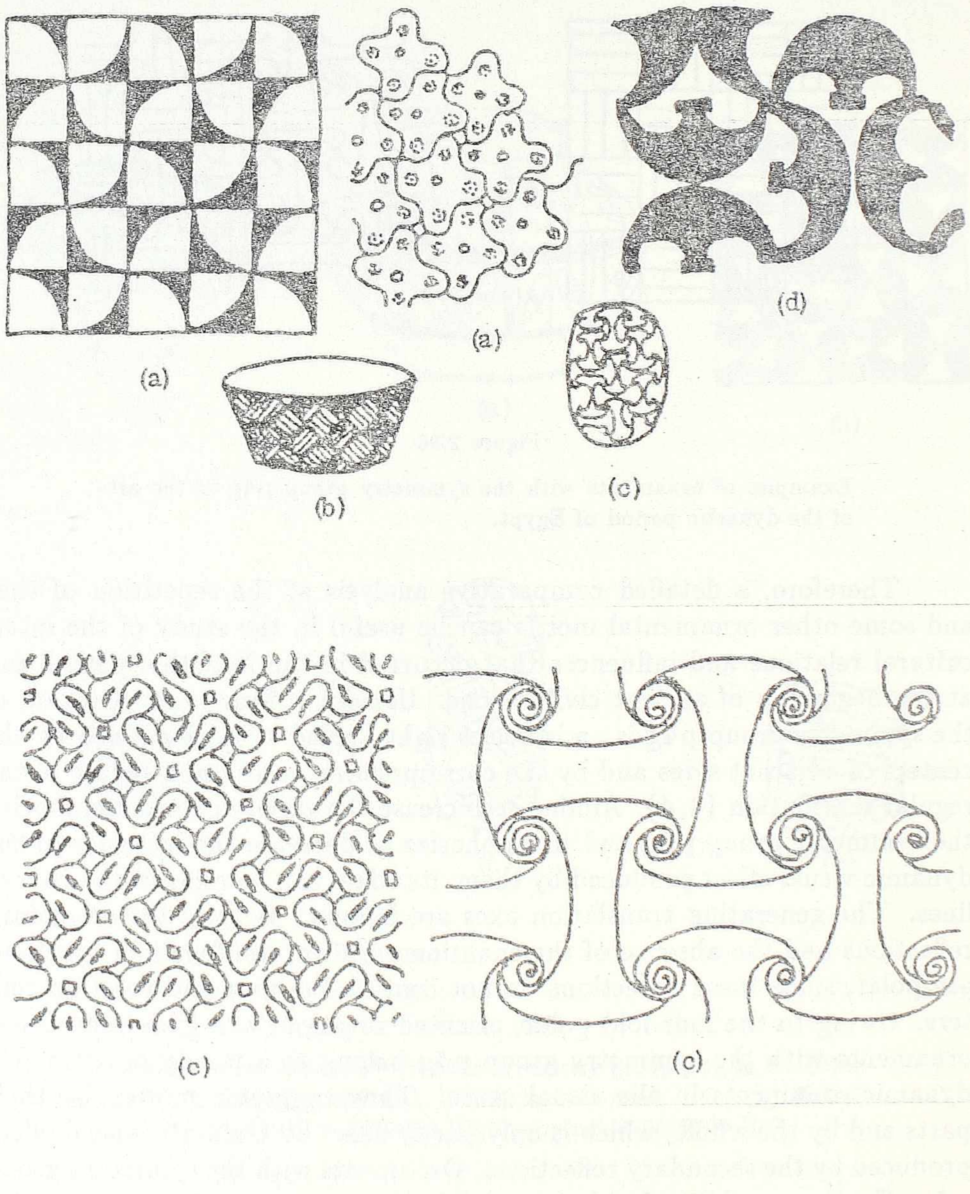
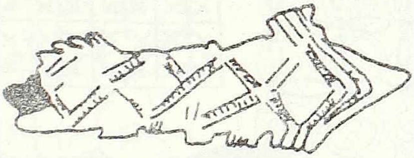


Figure 2.97

Ornaments with the symmetry group p4g: (a) Crete; (b) Aegina, around 5000 B.C.; (c) Thebes, Egypt, around 1500 B.C.; (d) Greco-Roman mosaic.

of symmetry. The symmetry group p4g is the subgroup of the index 2 of the symmetry group p4m, so that desymmetrizations of the symmetry group p4m and of a regular tessellation {4,4} can be efficiently used for constructing ornaments with the symmetry group p4g. Besides classical-symmetry desymmetrizations, antisymmetry desymmetrizations resulting in the antisymmetry group p4m/p4g, by the classical theory of symmetry considered as the symmetry group p4g, frequently occur (Figure 2.102h).



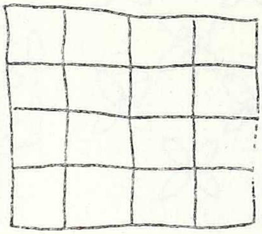


Figure 2.98

Examples of ornaments with the symmetry group p4m in Paleolithic art; the regular tessellation {4,4}.

The most frequent symmetry group of ornaments is the maximal symmetry group p4m of the square crystal system (Figure 2.98–2.101), which corresponds to a regular tessellation {4,4}. Besides their large independent application, the symmetry group p4m plays an important role in the construction of all ornaments, except those with the symmetry groups of the hexagonal crystal system, by the desymmetrization method. Besides the property of regularity, a regular tessellation {4,4} possesses another fundamental property — the existence of two perpendicular generating translation axes incident to reflection lines. Since all the discrete symmetry groups of ornaments, except the groups of the hexagonal crystal system, are subgroups of the symmetry group p4m, it can serve as a basis for the derivation of all the discrete symmetry groups of ornaments by the desymmetrization method, in the first place for those symmetry groups with perpendicular generating translation axes: pm , pg, pmm, pmg, pgg, cm, cmm, p4, p4g. Besides classical-symmetry desymmetrizations, antisymmetry desymmetrizations resulting in all the subgroups of the index 2 of the symmetry group **p4m** — **pmm**, **cmm**, **p4**, **p4g**, **p4m** and color-symmetry desymmetrizations, may also be used. Ornaments with the symmetry group **p4m** can be constructed by multiplying a frieze with the maximal symmetry group of friezes **mm** by means of a four-fold rotation, by multiplying a rosette with the symmetry group $\mathbf{D_4}$ (**4m**) by means of two perpendicular

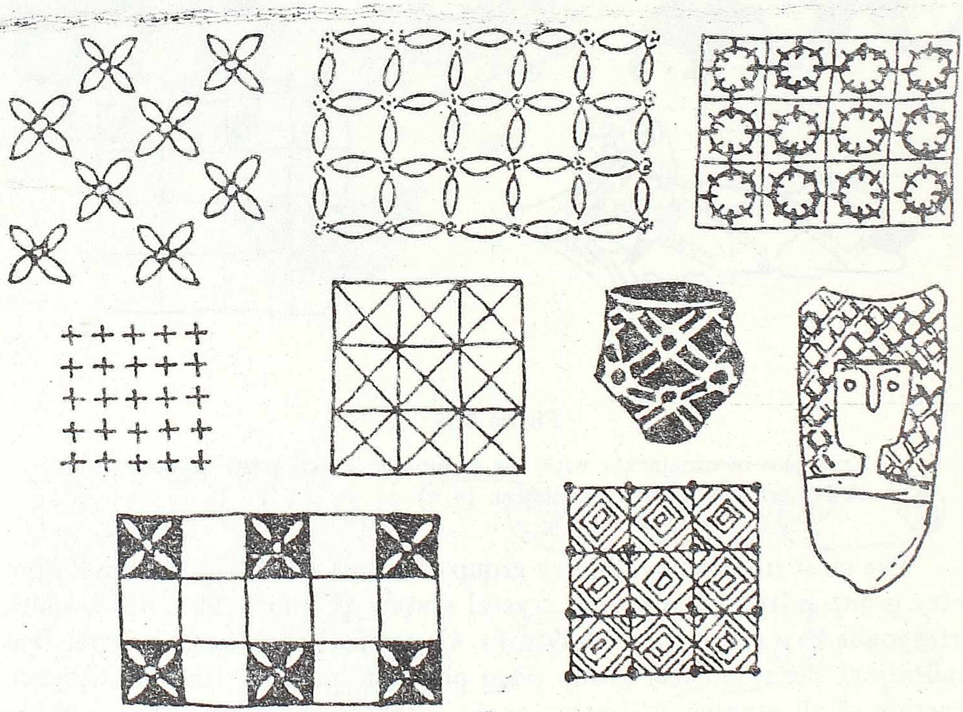


Figure 2.99

Ornaments with the symmetry group p4m in Neolithic art (Catal Hüjük, around 6400–5800 B.C.; Tell Brak, around 6000 B.C.; Tell Arpachiyah, around 6000 B.C.; Hacilar, around 5700–5000 B.C.).

translations with translation vectors of the same intensity, or by means of a regular tessellation $\{4,4\}$. According to the principle of maximal constructional simplicity, the dominant methods for constructing ornaments with the symmetry group **p4m** will be by a square regular tessellation $\{4,4\}$ or the translational multiplication of a rosette with the symmetry group $\mathbf{D_4}$

(4m). Early ornaments with the symmetry group p4m date from the Paleolithic, occurring as bone engravings representing an elementary square lattice — regular tessellation {4,4} (Figure 2.98). The further development of ornamental art tended toward the enrichment of ornamental motifs (Figure 2.99, 2.100). A fundamental region of the symmetry group p4m is an isosceles right-angled triangle defined by the center of a side, its belonging

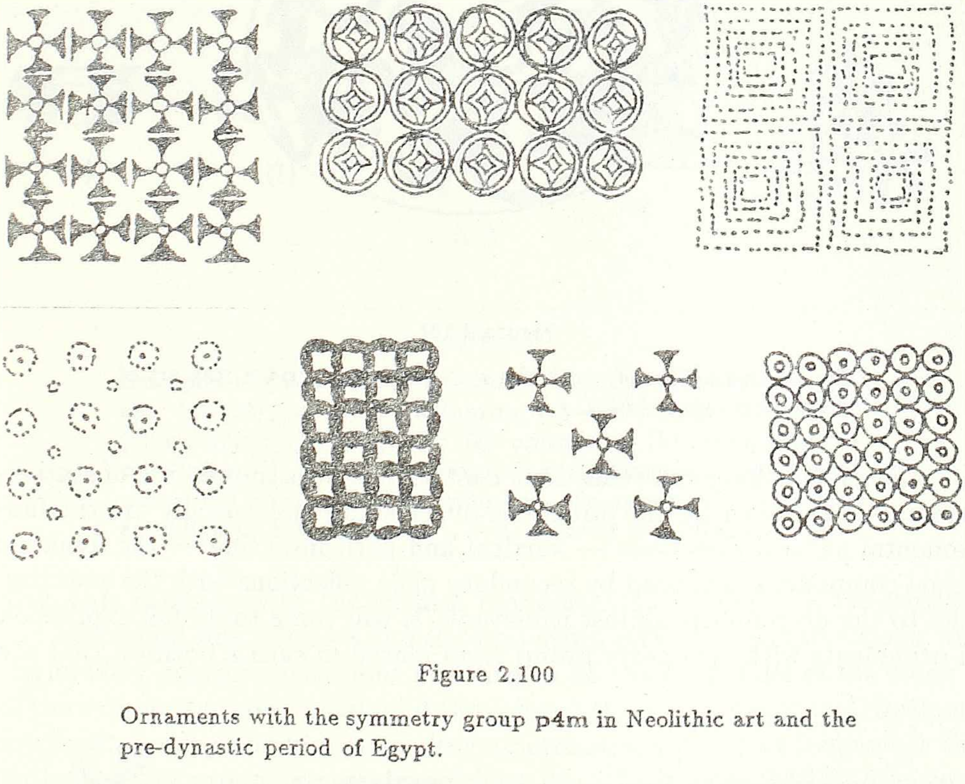


Figure 2.100

Ornaments with the symmetry group p4m in Neolithic art and the pre-dynastic period of Egypt.

vertex, and the center of a square that corresponds to the regular tessellation {4,4}. A fundamental region of all the groups generated by reflections must be rectilinear, so that the variety of ornaments with the symmetry group p4m is reduced to the use of an elementary asymmetric figure belonging to a fundamental region. Four-fold rotations and generating translations are non-polar. Enantiomorphic modifications do not occur. Therefore, ornaments with the symmetry group p4m produce an impression of stationariness and balance, caused by perpendicular reflections incident to the generating translation axes.

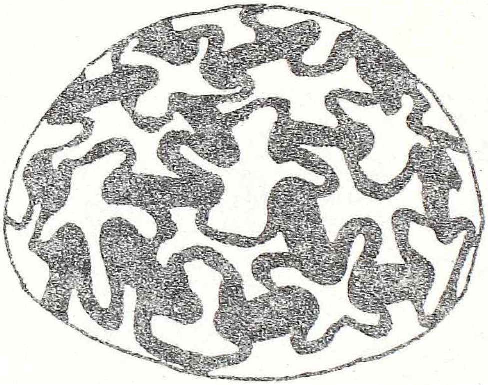
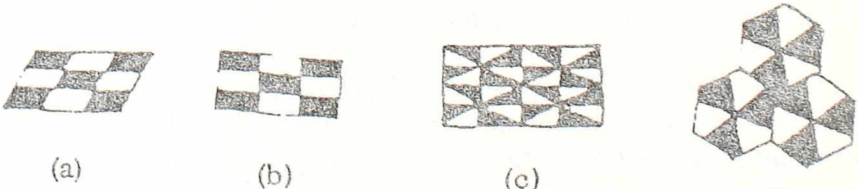


Figure 2.101

The ornament "Frogs" with the symmetry group p4m, the art of primitive peoples, Africa.

Secondary diagonal reflections contribute to the impression of stationariness. Since taken for the directions of reflection lines usually are the fundamental natural directions — vertical and horizontal line — the dynamic visual component produced by secondary glide reflections with the axes parallel to the diagonals is almost irrelevant. It will come to its full expression in ornaments with symmetry group p4m placed in such a position that the



(a) (b) (c) (d)

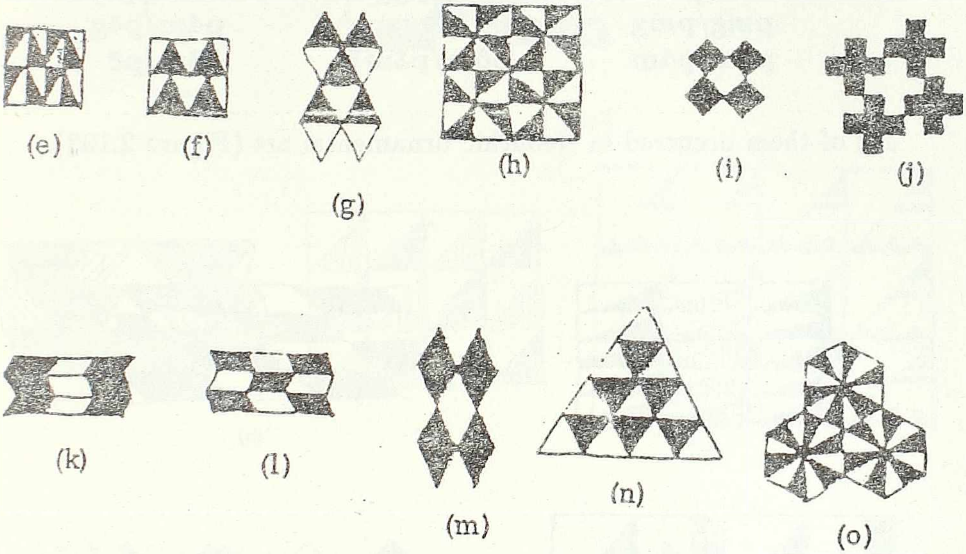


Figure 2.102

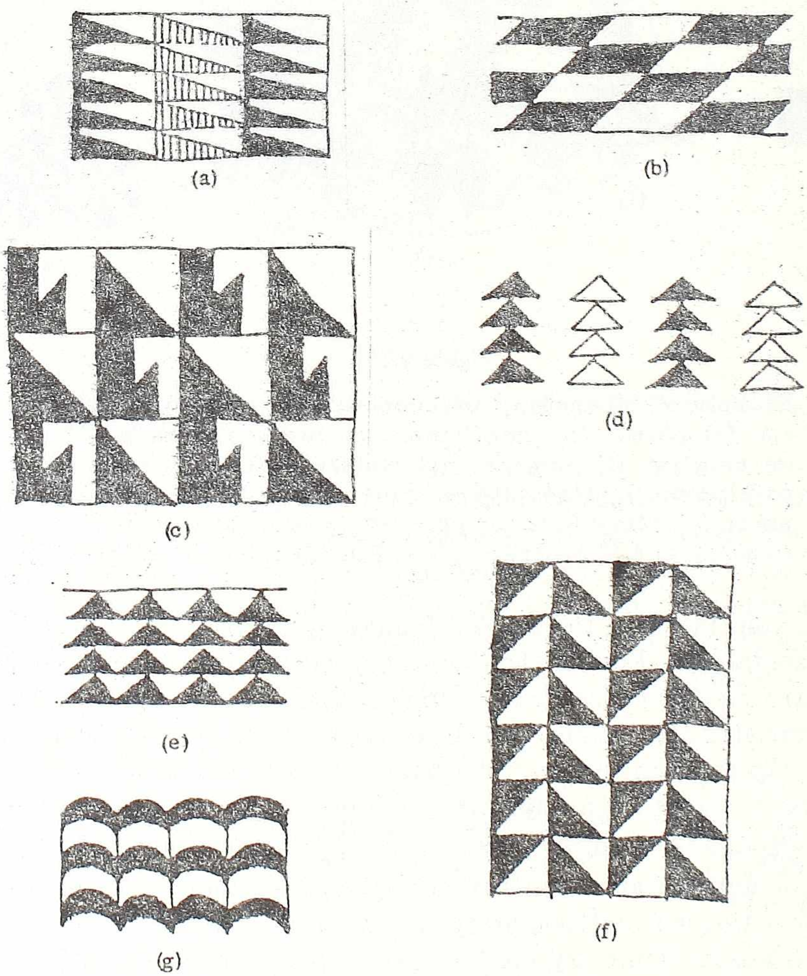
Examples of antisymmetry ornaments with the antisymmetry group: (a) p2/p2; (b) pmm/cmm; (c) cmm/pgg; (d) p6/p3; (e) pmg/pg; (f) pmg/pm; (g) cmm/cm; (h) p4m/p4g; (i) p4m/p4m; (j) p4/p4; (k) pmg/pmg; (l) pmg/pgg; (m) cmm/pmm; (n) p6m/p3m1; (o) p6m/p6, the antisymmetry mosaics according to A.V. Shubnikov , N.V. Belov et al. (1964, pp. 220).

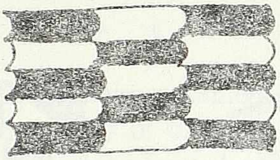
diagonals coincide with the natural fundamental directions — the vertical and horizontal line. Besides the classical-symmetry desymmetrizations, antisymmetry desymmetrizations resulting in all the subgroups of the index 2 of the symmetry group p4m, and the colored-symmetry desymmetrizations, are used. Among antisymmetry desymmetrizations the most frequent is the antisymmetry group p4m/p4m ("chess board") (Figure 2.102j) discussed in the classical theory of symmetry as the symmetry group p4m.

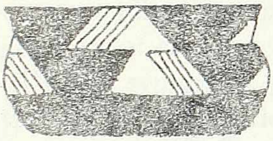
Basic data on antisymmetry desymmetrizations is given in the table of the most frequent antisymmetry groups of ornaments G'_2, i.e. the most frequent classical-symmetry groups that may be derived by antisymmetry desymmetrizations:

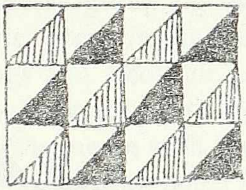
p2/p2	pmm/cmm	pmg/pm
cmm/cm	pmg/pgg	cmm/pmm
pmg/pmg	p4/p4	p4m/p4g
p4m/p4m	p6m/p3m1	p6m/p6

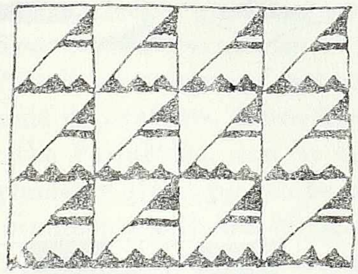
All of them occurred in Neolithic ornamental art (Figure 2.103).

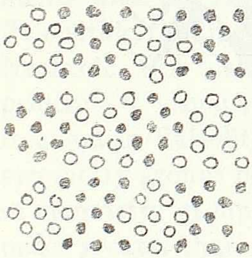



(h)


(i)


(j)


(k)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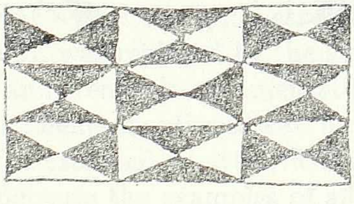
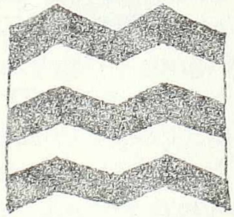
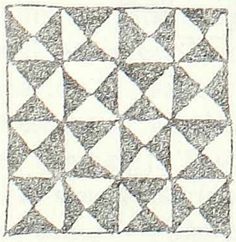

(m)


(n)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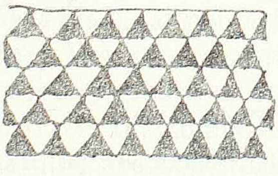
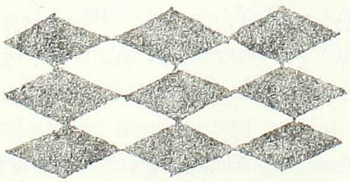

(p)


(q)

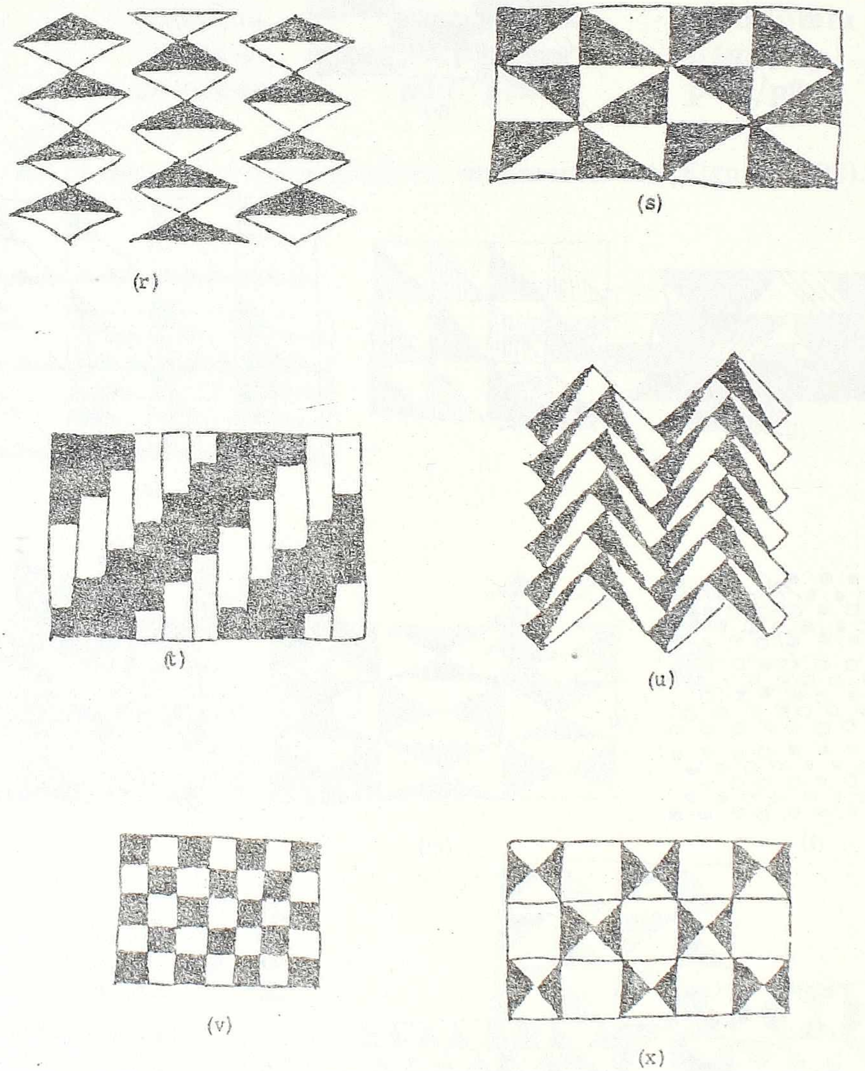


Figure 2.103

Antisymmetry ornaments in Neolithic art: (a) p1/p1; (b) p2/p2; (c) p2/p1; (d) pm/p1m; (e) pmg/pm; (f) pmg/pg; (g) pm/ p-m1; (h) pm/cm; (i) pg/p1; (j) cm/pm; (k) cm/p1; (l) pmm/pmm; (m) pmm/cmm; (n) pmg/pmg; (o) p4m/p4g; (p) p6m/p3m1; (q) cmm/pmm; (r) cmm/cm; (s) cmm/pgg; (t) pgg/pgg; (u) pgg/pg; (v) p4m/p4m; (x) p4m/cmm.

* * *

Analyzing connections between ornamental art and the theory of symmetry, we can use the chronology of ornaments, considering as the main characteristics the time, the methods and the origin of ornaments. The oldest examples of ornaments date from the Paleolithic and Neolithic and belong to ornaments with the symmetry groups **p1**, **p2**, (**pg**), **pm**, **pmg**, **pmm**, **cm**, **cmm**, **p4m** and **p6m**. For all the symmetry groups of ornaments, except the symmetry group **pg**, we have concrete examples of the corresponding ornaments, occurring in the Paleolithic stone and bone engravings or cave drawings and Neolithic ceramic decorations. Since friezes with the symmetry group **1g** originate from the Paleolithic, and by their translational repetition ornaments with the symmetry group **pg** can be constructed, probably the same dating holds for ornaments with the symmetry group **pg**. Another argument in favor of their early appearance is the existence of models in nature — the arrangements of leaves in some plants. On the other hand, the visual dynamism and low degree of symmetry of ornaments with the symmetry group **pg** could be account for their absence in Paleolithic ornamental art and for their somewhat later appearance in the Neolithic. It is also possible that the symmetry group **pg**, according to the principle of visual entropy, was replaced by the symmetry group **pmg**, very frequent in Paleolithic and Neolithic ornamental art. The first eight of the symmetry groups of ornaments mentioned can be derived as superpositions of symmetry groups of friezes and rosettes, without using rotations of the order greater than 2. Because the examples of all seven discrete symmetry groups of friezes **11**, **1g**, **12**, **m1**, **1m**, **mg** and **mm** occur in Paleolithic, this is the simplest method for constructing ornaments. Five of these symmetry groups of ornaments appear in the Paleolithic as the Bravais lattices (a lattice of parallelograms with the symmetry group **p2**, rectangular (**pmm**), rhombic (**cmm**), square (**p4m**) and hexagonal (**p6m**)) — the simplest visual interpretations of the maximal symmetry groups of the crystal systems bearing the same names. The importance of reflections is proved by their presence in all the symmetry groups of ornaments mentioned, except **p1**, **p2** (and **pg**). This illustrates the dominance of static ornaments in the visual sense, expressed by the absence of almost all the dynamic symmetry elements — polar rotations, glide reflections and by a relative dominance of non-polar and bipolar generating translations over the polar ones. As generating rosettes there are those with the most frequent symmetry groups of rosettes: D_1 (**m**), C_2 (**2**), D_2 (**2m**), D_4 (**4m**) and D_6 (**6m**). The symmetry group of rosettes D_2 (**2m**), besides the presence of reflections also possesses the other fundamental property — perpendicularity of reflection lines that coincide to the fundamental natural directions — the vertical and horizontal line. This symmetry group is a subgroup of four of the symmetry groups of ornaments mentioned: **pmm**, **cmm**, **p4m**, **p6m**. The two stated ornaments, **p4m** and **p6m**, correspond to regular tessellations $\{4,4\}$ and $\{3,6\}$ or $\{6,3\}$, i.e. to the perfect ornamental forms. With isogons, isohedrons, etc., regular tessellations belong together to the general tiling theory, discussed by E.S. Fedorov (1916), B.N. Delone (1959), H. Heesch, O. Kienzle (1963), L. Fejes Tóth (1964), H. Heesch (1968), A.V. Shubnikov, V.A. Koptsik (1974), B. Grünbaum, D. Lockenhoff, G.C. Shephard, A.H. Temesvari (1985), in many works by B. Grünbaum and G.C. Shephard (e.g., 1977c, 1978, 1983) and in their monograph *Tilings and Patterns* (1987).

Generating rosettes and friezes serve as the basis for constructing ornaments. They caused the time of appearance and the frequency of occurrence of particular symmetry groups of ornaments. Such constructions are extensions from the "local symmetry" of rosettes and friezes to the "global symmetry" of ornaments. As a common denominator for all the characteristics of Paleolithic ornaments we can use the principle of visual entropy — maximal constructional and visual simplicity and maximal symmetry.

Models existing in nature are prerequisites for the early appearance of some symmetry groups of ornaments (**p2** — waves, **pmg** — water, **p6m** — a honeycomb). On the other hand, they impose certain restrictions in ornamental motifs. Since almost all animals and many plants are mirror-symmetrical, the same holds for ornaments inspired by those natural models. Even in geometric ornaments, where such restrictions have no influence, the importance of mirror symmetry in nature served as the implicit model and caused the dominance of ornaments containing reflections. The causes of this phenomenon can be found also in the constructional simplicity of ornaments with reflections, in human mirror symmetry and binocularity.

In the lather periods — in the Neolithic and in the period of the ancient civilizations — ornaments with the symmetry groups **pg**, **pgg**, **p3**, **p31m**, **p3m1**, **p4**, **p4g**, **p6**, appeared. Ornaments with the symmetry groups **p3**, **p31m**, **p3m1**, **p4**, **p4g**, **p6** contain rotations of the higher order — 3, 4, 6 — occurring, until till then, only in regular tessellations $\{4,4\}$, $\{3,6\}$ and $\{6,3\}$, i.e. in the symmetry groups **p4m**, **p6m**. The new symmetry groups caused new construction problems, in that the construction of ornaments required a very complicated multiplication of friezes. In these symmetry groups of ornaments, dynamic symmetry elements — polar rotations, polar generating translations and glide reflections — prevail. Therefore, with the constructional complexity of the corresponding ornaments, there is the problem of their visual complexity — i.e. difficulties in perceiving the regularities and symmetry principles they are based on. Also, these symmetry groups of ornaments presented the most problems to mathematicians.

The exception in this class is the symmetry group **p3m1** (△), belonging to the family of the symmetry groups generated by reflections, and containing non-polar three-fold rotations, non-polar translations and a high degree of symmetry — favourable conditions for the early appearance in ornamental art. Its fundamental region is an equilateral triangle of a regular tessellation $\{3,6\}$, which must be rectilinear because the group **p3m1** (△) is generated by reflections. The symmetry group of a regular tessellation $\{3,6\}$ is the group **p6m**, so that ornaments with the symmetry group **p3m1** can be constructed by a translational multiplication of rosettes with the symmetry group D_3 (**3m**) or by means of an internal classical-symmetry desymmetrization of the fundamental region. The use of such an elementary asymmetric figure belonging to the fundamental region results in a marked isohedral tiling with the symmetry group **p3m1**. However, both methods were not used in the Paleolithic.

Distinct from classical-symmetry ornaments with the symmetry group **p3m1** occurring relatively seldom in ornamental art, antisymmetry ornaments with the antisymmetry group **p6m/p3m1**, considered in the classical theory of symmetry as the symmetry group **p3m1**, are some oldest and most frequent antisymmetry ornaments, because they can be obtained by the antisymmetry desymmetrization of the symmetry group **p6m**, in which the symmetry group **p3m1** is the subgroup of the index 2, this means, from a regular tessellation $\{3,6\}$.

The stated chronological analysis offers an insight into all construction problems, the methods of forming ornaments and their origin. It points out the parallelism between the mathematical approach to ornaments from the theory of symmetry, and their origins and construction by methods developed in ornamental art. As basic common construction methods we can distinguish a frieze multiplication, a rosette multiplication and different aspects of the desymmetrization method — classical-symmetry, antisymmetry and color-symmetry desymmetrizations.

Owing to its simplicity, the construction of ornaments by using friezes (A.V. Shubnikov, V.A. Koptsik, 1974) is, most probably, the oldest method of constructing ornaments, since the conditions for so-doing — the existence of examples of all the symmetry groups of friezes — were fulfilled already in Paleolithic ornamental art. Very successful in the construction of Paleolithic ornaments, this method shows its deficiency in the constructional complexity for the ornaments with rotations of the higher order ($n=3, 4, 6$).

Somewhat more complex is the method of rosette multiplication in which we must consider, besides rosettes with a definite symmetry, directions and intensities of the vectors of generating discrete translations multiplying these rosettes. Therefore, such a construction method implicitly introduces problems of plane Bravais lattices, crystal systems with the same names, and tessellations.

Occurring in Paleolithic ornamental art are examples of all the five Bravais lattices corresponding to the symmetry groups **p2**, **pmm**, **cmm**, **p4m** and **p6m**, i.e. to the maximal symmetry groups of the crystal systems bearing the same names, including regular tessellations $\{4,4\}$ (**p4m**), $\{3,6\}$ and $\{6,3\}$ (**p6m**). Till the Neolithic, the method of rosette multiplication remained on the level of elementary forms — Bravais lattices. Later, it was used for the construction of all kinds of ornaments. In the mathematical theory of symmetry this approach is discussed by different authors, e.g., by A.V. Shubnikov, N.V. Belov et al. (1964), and A.V. Shubnikov, V.A. Koptsik (1974).

The desymmetrization method is based on the elimination of adequate symmetry elements of a given symmetry group, aiming to obtain one of its subgroups. All the discrete symmetry groups of ornaments are subgroups of the groups **p4m** and **p6m** generated by reflections (H.S.M. Coxeter, W.O.J. Moser, 1980). Knowledge of group-subgroup relations between the symmetry groups of ornaments makes possible the consequent application of desymmetrizations. In ornamental art, this method appears in the Neolithic, with the wider application of colors and decorative elements, to become in time, along with the method of rosette and frieze multiplication, one of the dominant construction methods.

Antisymmetry desymmetrizations originate from the use of colors in ornamental art, beginning with Neolithic ceramics. Although colors, as an artistic means, were used even in the Paleolithic, they were mainly used for figurative contour drawings. Paleolithic ornaments mostly occur as stone and bone engravings or drawings, so that antisymmetry desymmetrizations came to their full expression with Neolithic colored, dichromatic ceramics. They can serve as the basis for the derivation of all subgroups of the index 2 of a given symmetry group. In cases when classical-symmetry desymmetrizations are complex, this is the most fruitful method for the derivation of certain less frequent symmetry groups of ornaments (**cmm/pgg**, **pmg/pg**, **p4m/p4g**, **pmg/pgg**, **p6m/p3m1**, **p6m/p6**), because the resulting symmetry groups (**pgg**, **pg**, **p4g**, **pgg**, **p3m1**, **p6**) are derived by antisymmetry desymmetrizations of the frequently used generating symmetry groups (**cmm**, **pmg**, **p4m**, **pmg**, **p6m**, **p6m**).

The list of antisymmetry desymmetrizations of the symmetry groups of ornaments, i.e. the list of antisymmetry groups of ornaments G'_2, can serve as the basis for antisymmetry desymmetrizations. A survey of the minimal indexes of subgroups in the symmetry groups of ornaments (H.S.M. Coxeter, W.O.J. Moser, 1980, pp. 136), mosaics for dichromatic plane groups (A.V. Shubnikov, N.V. Belov et al., 1964, pp. 220) and numerous works on the theory of the antisymmetry of ornaments (H.J. Woods, 1935; A.M. Zamorzaev, A.F. Palistrant, 1960, 1961; A.V. Shubnikov, N.V. Belov et al., 1964; A.M. Zamorzaev, 1976; S.V. Jablan, 1986a) are some of the possible sources for future consideration of antisymmetry desymmetrizations.

In the table of antisymmetry desymmetrizations the antisymmetry groups of ornaments G'_2 are given in the group/subgroup notation G/H giving information on the generating symmetry group G and its subgroup H of the index 2 obtained by the antisymmetry desymmetrization. Aiming to differentiate between two antisymmetry groups possessing the same group/subgroup symbol and to denote two different possible positions of reflections in the symmetry group pm, the symbols **pm1** and **p1m**, are used.

The table of antisymmetry desymmetrizations of symmetry groups of ornaments G_2:

p1/p1	pmg/pmg	p4g/p4
	pmg/pgg	p4g/cmm
p2/p2	pmg/pm	p4g/pgg
p2/p1	pmg/pg	
	pmg/p2	p4m/p4m
pg/pg		p4m/p4g
pg/p1	pmm/pmm	p4m/p4
	pmm/cmm	p4m/cmm
pm/cm	pmm/pmg	p4m/pmm
pm/pm1	pmm/pm	
pm/p1m	pmm/p2	p3m1/p3
pm/pg		
pm/p1	cmm/pmm	p31m/p3
	cmm/pmg	
cm/pm	cmm/pgg	p6/p3

cm/pg	cmm/cm	
cm/p1	cmm/p2	p6m/p6
		p6m/p31m
pgg/pg	p4/p4	p6m/p3m1
pgg/p2	p4/p2	

Antisymmetry and color-symmetry desymmetrizations are the very efficient tools for finding all subgroups of the index N of a given symmetry group. The complete list of group-subgroup relations between 17 discrete symmetry groups of ornaments G_2 and the minimal indexes of subgroups in groups are given in the monograph by H.S.M. Coxeter and W.O.J. Moser (1980, pp. 136). From the table of color-symmetry desymmetrizations we can conclude that [cm:cm]= 3 (pp. 182, the color-symmetry desymmetrization cm/cm, N = 3) and [cmm:cmm]=*3* (pp. 182, the color-symmetry desymmetrization cmm/cmm/cm, N = 3). After these corrections and the corrections: [pmg:pmg]= 2, [p6:p6]=*3*, already given by the authors in the reprint of their monograph, this table reads as follows.

	p1	p2	pg	pm	cm	pgg	pmg	pmm	cmm	p4	p4g	p4m	p3	p31m	p3m1	p6	p6m
p1	2																
p2	2	2															
pg	2		2														
pm	2		2	2	2												
cm	2		2	2	3												
pgg	4	2	2			*3*											
pmg	4	2	2	2	*4*	2	2										
pmm	4	2	4	2	4	4	2	2	2								
cmm	4	2	4	4	2	2	2	2	*3*								
p4	4	2								2							
p4g	8	4	*4*	*8*	*4*	2	*4*	4	2	2	*9*						
p4m	8	4	*8*	*4*	*4*	4	*4*	2	2	2	2	2					
p3	3												3				
p31m	6		*6*	*6*	*3*								2	*4*	3		
p3m1	6		*6*	*6*	*3*								2	*3*	*4*		
p6	6	3											2			*3*	
p6m	12	6	*12*	*12*	6	6	6	6	*3*				4	2	2	2	*3*

Because the construction of antisymmetry ornaments and the application of antisymmetry desymmetrizations is followed by various construction problems, their use in ornamental art is usually restricted to the antisymmetry groups stated in the table listing the most frequent antisymmetry desymmetrizations (pp. 172).

The use of color-symmetry desymmetrizations in ornamental art began with Neolithic polychromatic ceramics, to reach its peak in the ornamental art of the ancient civilizations (Egypt), in Roman floor mosaics and in Moorish ornaments. When considering color-symmetry desymmetrizations one should distinguish, as in antisymmetry desymmetrizations, two different possibilities. In the first case the use of colors results in a desymmetrization of the symmetry group of the ornament, but neither an antisymmetry nor color-symmetry group is obtained. Although the color is a means of such desymmetrization, this is, in fact, a classical-symmetry desymmetrization, as opposed to real antisymmetry or color-symmetry desymmetrizations resulting in antisymmetry and colored symmetry groups of ornaments. Only those desymmetrizations resulting in antisymmetry or color-symmetry groups of ornaments will be accepted and analyzed under the term of antisymmetry or color-symmetry desymmetrizations, while all the other desymmetrizations using colors only as a technical means, will be discussed as classical-symmetry desymmetrizations.

In line with the principle of visual entropy — maximal constructional and visual simplicity and maximal symmetry — from the point of view of ornamental art, especially important will be color-symmetry desymmetrizations of the maximal symmetry groups of the crystal systems — desymmetrizations of Bravais lattices.

The discussion on color-symmetry desymmetrizations of the symmetry groups of ornaments relied on the works of A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant (1978), M. Senechal (1979), A.F. Palistrant (1980a), J.D. Jarratt, R.L.E. Schwarzenberger (1980), R.L.E. Schwarzenberger (1984), A.M. Zamorzaev, Yu.S. Karpova, A.P. Lungu, A.F. Palistrant (1986) and the monograph by T.W. Wieting *The Mathematical Theory of Chromatic Plane Ornaments* (1982). In this monograph the numbers of the color-symmetry groups of ornaments for $N \leq 60$ colors, the catalogue of the color-symmetry groups of ornaments for $N \leq 8$ colors and the colored mosaics for $N = 4$ colors, are given.

Because of the large number of the color-symmetry groups of ornaments, in the table of the color-symmetry desymmetrizations of the symmetry groups of ornaments, the restriction $N \leq 8$ is accepted. In that way, the usual practical needs for the construction and analysis of colored ornaments, where the number of colors N does not exceed 8, are also satisfied. According to the definition of color-symmetry groups given by R.L.E.Schwarzenberger (1984), all the color-symmetry groups discussed require the even use of colors. The uneven use of colors or the use of colors in a given ratio, e.g., 2:1:1, 4:2:1:1, 6:2:1, 6:3:1:1:1 (B. Grünbaum, Z. Grünbaum, G.C. Shephard, 1986), which occur in ornamental art and demand a special mathematical approach, represents an open research field.

In the table of the color-symmetry desymmetrizations of the symmetry groups of ornaments, by analogy to the corresponding tables of the color-symmetry desymmetrizations of the symmetry groups of rosettes and friezes, every color-symmetry group is denoted, for a fixed N, by a symbol $G/H/H_1$. For $H = H_1$, i.e. iff H is a normal subgroup of the group G, such a symbol is reduced to the symbol G/H. If the first or the second symbol does not uniquely determine the color-symmetry group, i.e. the corresponding color-symmetry desymmetrization, such symbols supplemented by the symbol of the translational group that corresponds to the subgroup H, are used (D. Harker, 1981). In the supplementary symbols $-a$ is denoted by $\underline{a}$, and the symbol of the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $[ab, cd]$. The incidence of the symmetry element S of the group G to the symmetry element S_1 of the subgroup H is denoted by the supplementary symbol $(S \equiv S_1)$. To differentiate between the symmetry groups pg with the glide reflections g_x, g_y, the symbols pg1, p1g are used respectively. The symbols cm1, c1m, and pmg, pgm, are used analogously for denoting the two different possible positions of the reflection m.

The table of color-symmetry desymmetrizations of symmetry groups of ornaments G_2:

$N = 3$

p1/p1	pm/pm/p1	pgg/pgg/pg
	pm/pm	
p2/p2/p1		p31m/cm/p1
	pg/pg/p1	p31m/p3m1
p3/p1	pg/pg	
p3/p3		p3m1/cm/p1
	cmm/cmm/cm	p3m1/p31m/p3 p6/p2
p6/p6/p3	pmm/pmm/pm	p6m/cmm/p2
		p6m/p6m/p3m1
cm/cm/p1	pmg/pmg/pm	
cm/cm	pmg/pmg/pg	

$N = 4$

p1/p1[10, 04]
p1/p1[20, 02]

p2/p1
p2/p2/p1
p2/p2

p3/p3/p1

p4/p1
p4/p2/p2
p4/p2(4≡2)
p4/p2(2≡2)
p4/p4/p2

p6/p6/p2

cm/p1/p1
cm/p1
cm/cm/p1
cm/pm/p1
cm/pg/p1
cm/pm
cm/pg

pm/p1[10, 02]
pm/p1[20, 01]
pm/p1[11, $\underline{1}$1]
pm/pm/p1
pm/pm[20, 02]
pm/pg[20, 02]
pm/cm/p1
pm/pm[40, 01]
pm/pg[40, 01]
pm/cm

pg/p1[10, 02]
pg/p1[20, 01]

cmm/p1
cmm/p2/p2
cmm/p2 (mm≡2)
cmm/p2(2≡2)
cmm/pm
cmm/pg
cmm/cmm/p2
cmm/pmg/pg
cmm/pmm/pm
cmm/pgg/pg
cmm/pmg/pm

pmm/p1
pmm/p2[10, 02]
pmm/p2[11, $\underline{1}$1]
pmm/pm[10, 02]
pmm/pm[20, 01]
pmm/pg
pmm/cm
pmm/pmm
pmm/pgg
pmm/pmg
pmm/pmg/pg
pmm/pmm/pm
pmm/cmm/cm

pmg/p1
pmg/p2
pmg/p2[20, 01]/p1
pmg/p2[11, $\underline{1}$1]/p1
pmg/pm
pmg/pg1
pmg/pm/p1
pmg/cm/p1
pmg/p1g
pmg/pgg/pg
pmg/pmg/pm

pgg/p1
pgg/p2/p1
pgg/p2
pgg/pg/p1

p31m/p31m/p1

p3m1/p3m1/p1

p4m/p2
p4m/p4
p4m/cm/p1
p4m/pm/p1
p4m/pmm
p4m/pgg
p4m/pmg[11, $\underline{1}$1]/p2
p4m/pmg[10, 02]/p2
p4m/pmm/pmm
p4m/cmm
p4m/cmm/p2
p4m/p4m/pmm
p4m/p4g/pgg

p4g/p2
p4g/p4/p2
p4g/cm/p1
p4g/pg/p1
p4g/pmg/p2
p4g/pgg
p4g/pmm

p6m/p3
p6m/p6m/p2

pg/p1[11, $\underline{1}$1]
pg/pg/p1

$N = 5$

p1/p1	pm/pm/p1	pmm/pmm/pm
	pm/pm	
p2/p2/p1		pmg/pmg/pm
	pg/pg/p1	pmg/pmg/pg
p4/p4/p1	pg/pg	
		pgg/pgg/pg
cm/cm/p1	cmm/cmm/cm	
cm/cm		

$N = 6$

p1/p1	pg/p1[10, 03]	pgg/p2[10, 03]/p1
	pg/p1[30, 01]	pgg/p2[2$\underline{1}$, 11]/p1
p2/p1	pg/p1/p1	pgg/pg/p1
p2/p2/p1	pg/pg/p1	pgg/pg
	pg/pg	
p3/p1/p1		p31m/p1
	cmm/p2[10, 03]/p1	p31m/p3
p4/p2[10, 03]/p1	cmm/p2[2$\underline{1}$, 11]/p1	p31m/p3/p1
p4/p2[2$\underline{1}$, 11]/p1	cmm/cm/p1	p31m/pm/p1
	cmm/cm	p31m/pg/p1
p6/p1	cmm/pmm/pm	
p6/p2/p2	cmm/pgg/pg	p3m1/p1
p6/p2/p1	cmm/pmg/pg	p3m1/p3
p6/p3	cmm/pmg/pm	p3m1/pm/p1
p6/p3/p1		p3m1/pg/p1
	pmm/p2[10, 03]/p1	
cm/p1/p1	pmm/p2[2$\underline{1}$, 11]/p1	p4m/cmm/p1
cm/p1[2$\underline{1}$, 11]	pmm/pm/p1	p4m/pmm/p1
cm/p1[21, $\underline{1}$1]	pmm/pm	
cm/pm/p1	pmm/pmm[20, 03]/pm	p4g/cmm/p1
cm/pg/p1	pmm/pmg/pm	p4g/pmm/p1
cm/pm	pmm/pmg/pg	
cm/pg	pmm/pmm[10, 06]/pm	p6m/p2
	pmm/cmm/cm	p6m/p6/p3
pm/p1[10, 03]		p6m/cm1/p1
pm/p1[30, 01]	pmg/p2[10, 03]/p1	p6m/c1m/p1

pm/p1/p1	pmg/p2[30,01]/p1	p6m/pmm/p2
pm/pm[10,06]/p1	pmg/p2[2$\underline{1}$,11]/p1	p6m/pgg/p2
pm/pm[20,03]/p1	pmg/pm	p6m/pmg/p1
pm/pg/p1	pmg/pm/p1	p6m/pgm/p1
pm/cm/p1	pmg/pg/p1	p6m/p31m/p3
pm/pm[30,02]	pmg/pg	p6m/p31m/p1
pm/pm[60,01]	pmg/pgg[10,06]/p1	p6m/p3m1
pm/pg	pmg/pmg/pm	
pm/cm	pmg/pgg[30,02]/pg	
	pmg/pmg/pg	

$N = 7$

p1/p1	cm/cm/p1	cmm/cmm/cm
	cm/cm	
p2/p2/p1		pmm/pmm/pm
	pm/pm/p1	
p3/p3/p1	pm/pm	pmg/pmg/pm
		pmg/pmg/pg
p6/p6/p1	pg/pg/p1	
	pg/pg	pgg/pgg/pg

$N = 8$

p1/p1[10,08]	cmm/p1/p1	pgg/p1[10,02]
p1/p1[20,04]	cmm/p1	pgg/p1[11,$\underline{1}$1]
	cmm/p2(2≡2)	pgg/p2[20,02]/p1
p2/p1[10,04]	cmm/p2(mm≡2)	pgg/p2[10,04]/p1
p2/p1[20,02]	cmm/p2[10,04]/p1	pgg/p2[2$\underline{2}$,11]/p1
p2/p2[10,08]/p1	cmm/p2[20,12]/p1	pgg/pg/p1
p2/p2[20,04]/p1	cmm/p2[2$\underline{2}$,11]/p1(mm≡2)	
	cmm/p2[2$\underline{2}$,11]/p1(2≡2)	p31m/p3/p1
p4/p1/p1	cmm/cm/p1	
p4/p1	cmm/pm/p1	p3m1/p3/p1
p4/p2[20,02]/p1	cmm/pg/p1	
p4/p2	cmm/pm	p4m/p1
p4/p2[10,04]/p1	cmm/pg	p4m/p2/p2
p4/p2[20,12]/p1	cmm/pmg/p1	p4m/p2(4m≡2)
p4/p2[2$\underline{2}$,11]/p1(4≡2)	cmm/pgg/p1	p4m/p2(2m≡2)
p4/p2[2$\underline{2}$,11]/p1(2≡2)	cmm/pmm/p1	p4m/p4/p2
p4/p4/p2	cmm/pmm/pm	p4m/pm[11,$\underline{1}$1]/p1

p6/p3/p1

cm/p1[20, 02]
cm/p1[40, 01]/p1
cm/p1[2$\underline{2}$, 11]
cm/p1[20, 12]/p1
cm/p1[22, $\underline{1}$1]
cm/pg[22, $\underline{2}$2]/p1
cm/pm[22, $\underline{2}$2]/p1
cm/pm[4$\underline{4}$, 11]/p1
cm/pg[4$\underline{4}$, 11]/p1
cm/cm[31, $\underline{2}$2]/p1
cm/cm[3$\underline{1}$, 22]/p1
cm/pm
cm/pg

pm/p1[10, 04]
pm/p1[20, 02]
pm/p1[20, 12]
pm/p1[40, 01]
pm/p1[21, 02]
pm/p1[2$\underline{2}$, 11]
pm/pm[10, 08]/p1
pm/pm[20, 04]/p1
pm/pg/p1
pm/cm[20, 14]/p1
pm/pm[40, 02]
pm/pg[40, 02]
pm/cm[22, $\underline{2}$2]/p1
pm/pm[80, 01]
pm/pg[80, 01]
pm/cm

pg/p1[10, 04]
pg/p1[20, 02]
pg/p1[20, 12]
pg/p1[40, 01]

cmm/pmg/pg
cmm/pmg/pm
cmm/pgg/pg

pmm/p1[10, 02]
pmm/p1[11, $\underline{1}$1]
pmm/p2
pmm/p2[10, 04]/p1
pmm/p2[20, 12]/p1
pmm/p2[2$\underline{2}$, 11]/p1
pmm/pm/p1
pmm/pm[20, 02]
pmm/pg[20, 02]
pmm/cm/p1
pmm/pm[40, 01]
pmm/pg[40, 01]
pmm/cm
pmm/pmg[20, 04]/pg
pmm/pmm[20, 04]/pm
pmm/pgg/pg
pmm/pmg/pm
pmm/cmm/p1
pmm/pmg[10, 08]/pg
pmm/pmm[10, 08]/pm
pmm/cmm/cm

pmg/p1[10, 02]
pmg/p1[20, 01]
pmg/p1[11, $\underline{1}$1]
pmg/p2[10, 04]/p1
pmg/p2[11, $\underline{1}$1]/p1
pmg/p2[20, 12]/p1
pmg/p2[40, 01]/p1
pmg/p2[21, 02]/p1
pmg/p2[2$\underline{2}$, 11]/p1
pmg/pm
pmg/pg
pmg/pm[20, 02]/p1

p4m/pg[11, $\underline{1}$1]/p1
p4m/pm[10, 02]/p1
p4m/pm[20, 01]/p1
p4m/pg[20, 01]/p1
p4m/cm/p1
p4m/cmm/p2
p4m/pmg[2$\underline{2}$, 11]/p1
p4m/pmm[2$\underline{2}$, 11]/p1
p4m/pgg[2$\underline{2}$, 11]/p1
p4m/pmg[2$\underline{2}$, 11]/p1
p4m/pmm[20, 02]/p1
p4m/pmg[20, 02]/p1
p4m/pgg[20, 02]/p1
p4m/pmm
p4m/pgg
p4m/pmg[20, 02]/p2
p4m/pmg[10, 04]/pg
p4m/pmm[10, 04]/p1
p4m/cmm/p1(2m≡2)
p4m/cmm/p1(4m≡2)
p4m/p4m/p1
p4m/p4g/p1

p4g/p1
p4g/p2/p1
p4g/p2(4≡2)
p4g/p2(2≡2)
p4g/p4/p1
p4g/pg[11, $\underline{1}$1]/p1
p4g/pm/p1
p4g/pg[10, 02]/p1
p4g/cmm/p1
p4g/pgg/p1
p4g/pmg/p1(4≡2)
p4g/pmg/p1(2≡2)

p6m/p6/p2
p6m/p3m1/p1

pg/p1[21, 02]	pmg/pg[11, $\underline{1}$1]/p1	p6m/p31m/p1
pg/p1[2$\underline{2}$, 11]	pmg/cm[20, 12]/p1	
pg/pg/p1	pmg/pm[40, 01]/p1	
	pmg/cm[21, 02]/p1	
	pmg/pg[10, 04]/p1	
	pmg/pgg/pg	
	pmg/pmg/pm	

A homogeneous and isotropic plane possesses the maximal continuous symmetry group of ornaments $p_{00}\infty m$ ($s^{\infty}\infty$), while all the other symmetry groups of ornaments are its subgroups. In ornamental art, it may be identified as a ground representing the environment where the remaining symmetry groups of ornaments exist. Among the symmetry groups of semicontinua, only the symmetry groups $p_{10}1m$ (s1m) and $p_{10}mm$ (smm) possess their adequate visual interpretations. The condition for the visual presentability of the continuous symmetry groups of ornaments is the nonpolarity of their continuous translations and continuous rotations. For the symmetry groups of semicontinua this condition is equivalent to the existence of the corresponding visually presentable continuous friezes, by the translational multiplication of which the corresponding semicontinua can be derived.

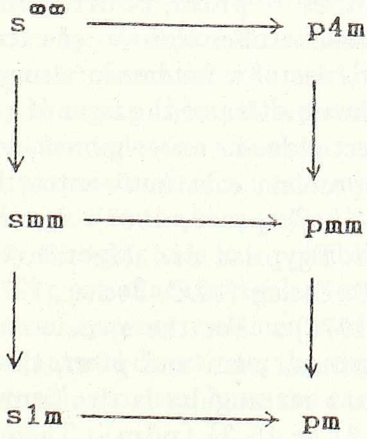


Figure 2.104

Between the visually presentable continuous symmetry groups of ornaments, the following group-subgroup relations hold (Figure 2.104). These relations point out desymmetrizations suitable to derive continuous symmetry groups of a lower degree of symmetry and to recognize symmetry substructures of visually presentable plane continua and semicontinua. Since all discrete symmetry groups of ornaments G_2 are subgroups of the symmetry groups **p4m** and **p6m**, for further work on group-subgroup relations between the continuous and discrete symmetry groups of ornaments, it is possible to use this data.

The geometric-algebraic properties of the symmetry groups of ornaments G_2 — their presentations, data on their structure, properties of generators, polarity, non-polarity and bipolarity, enantiomorphism, form of the fundamental region, tables of group-subgroup relations, Cayley diagrams, etc. — offer the possibility to plan the visual properties of ornaments before their construction. Besides the usual approach to ornamental art, where visual structures serve as the objects for analyses from the point of view of the theory of symmetry, there is also an opposite approach — from abstract geometric-algebraic structures to the anticipation of their visual properties. Then ornaments may be understood as visual models of the corresponding symmetry groups.

Generators define a possible form of a fundamental region, so that the incidence of a reflection line with a segment of the boundary of the fundamental region means that such a part of the boundary must be rectilinear. A fundamental region must be rectilinear in the symmetry groups of ornaments generated by reflections — **pmm**, **p3m1**, **p4m**, **p6m**. All the other symmetry groups of ornaments offer the use of curvilinear boundaries or curvilinear parts of boundaries of a fundamental region, which do not belong to reflection lines. The question of the form of a fundamental region is directly linked to the perfect plane forms — *monohedral tilings*, plane tilings by congruent tiles. This problem constantly attracts mathematicians and artists (B. Grünbaum, G.C. Shephard, 1987). In ornamental art it came to its fullest expression in Egyptian and Moorish ornamental art and in the graphic works by M.C. Escher (M.C. Escher, 1971a, b, 1986; B. Ernst, 1976; C.H. Macgillavry, 1976) . For the symmetry groups of ornaments generated by reflections **pmm**, **p4m** and **p6m**, the problem of isohedral tilings is simply reduced to a rectangular lattice (**pmm**) and regular tessellations $\{4,4\}$ (**p4m**), $\{3,6\}$ or $\{6,3\}$ (**p6m**). There is a certain anomaly in the symmetry group **p3m1** ($\triangle$). Owing to a symmetrization, its corresponding tessellation $\{3,6\}$ possesses the symmetry group **p6m**. Therefore, this is the only symmetry group of ornaments not offering any possibility for the corresponding isohedral unmarked tilings and requiring the use of an asymmetric figure within a fundamental region, this means, a marked tiling. This contradiction can be solved by taking rotation centers always inside the tiles, so that the symmetry group **p3m1** will correspond to a regular tessellation $\{3,6\}$, and the symmetry group **p6m** to a regular tessellation $\{6,3\}$. Under such conditions the symmetry group **p3m1** belongs to the family of the symmetry groups of ornaments appearing in Paleolithic ornamental art, since the regular tessellation $\{3,6\}$ dates from the Paleolithic (Figure 2.92b).

A generating translation axis of an ornament is non-polar iff there exists a reflection perpendicular to it. If a central reflection center belongs to a translation axis, this axis will be bipolar, while in the other cases that axis will be polar. There will be no enantiomorphic modifications of an ornament iff its symmetry group contains indirect isometries. A rotation of an ornament will be non-polar iff there exists a reflection incident to the center of this rotation n (n=2, 3, 4, 6), i.e. iff the corresponding dihedral group D_n (nm) is a subgroup of the symmetry group of the ornament.

The visual stationariness of an ornament is conditioned by the non-polarity of its axes and rotations, by the presence of reflections, especially perpendicular ones, by the absence of glide reflections suggesting alternating motions and by the absence of the enantiomorphism. On the other hand, the factors that determine whether an ornament will be a dynamic one in the visual sense, will be the polarity of translation axes suggesting one-way directed motions, the polarity of rotations, the presence of glide reflections, the absence of reflections and the existence of the enantiomorphism.

Besides the stated objective geometric-visual elements defining visual stationariness or dynamism of certain ornament, the "subjective" elements are also important. By changing them, it is possible to influence the visual impression that the ornament gives to the observer. Under these changeable factors — the "visual parameters" of an ornament — we can include the form of a fundamental region or the shape of an asymmetric figure within a fundamental region. By changing this it is possible to decrease or emphasize the visual dynamism. The use of acuteangular forms in dynamic ornaments, the change of a position of symmetry elements with respect to the observer and to the fundamental natural directions, the visual role of the "ascending" and "descending" diagonal, etc., can have the same function.

Real ornaments occurring in ornamental art are bounded parts of "ideal" unbounded geometric ornaments, i.e. their finite factor groups, derived by the identifications $p_x^m = E$, $p_y^n = E$ $(m,n \in N)$. Such a bounded part of an ornament is defined by the intensities of the generating translations

p_x, p_y, their mutual position and by the parameters m,n. In ornaments with the p_x- and p_y-translation vectors of the same intensity, especially interesting is the case of $m = n$.

The visual effect given to the observer by a real ornament is the result of the interaction between the symmetry group of the ornament, the plane symmetry of the observer and binocularity, the symmetry of a bounded part of the "ideal" ornament, symmetry of the environment, etc. It is also formed under the visual-physiological influence of the fundamental natural directions (the vertical and horizontal line, gravitation, etc.). All these "subjective" factors participate in the formation of the primary visual impression, as desymmetrization or symmetrization factors. Some of them (e.g., the vertical and horizontal line), may occur as a specific implicit coordinate system having a relevant positive or negative influence on the recognition of the symmetry of ornament. In the further course of the process of visual perception, the observer tends to recognize the symmetry of the ornament itself. Even after the elimination of disturbing elements, the eye often perceives a visually dominant structure, reducing it to the simplest one possible. Sometimes, according to the principle of visual entropy, the visual introduction of new symmetry elements — symmetrization of the structure — and visual substitution of dynamic elements of symmetry by their static equivalents takes its place. This is often manifested in experiments to recognize or reproduce certain motifs.

One comes across similar problems during the visual perception of symmetry substructures — rosettes, friezes or ornaments — belonging to certain ornament. Their symmetry groups appear as the subgroups of the symmetry group of the ornament. The primary factors relevant for the visual perception of substructures will be the visual simplicity of substructures themselves (their visual stationariness or dynamism), the degree of visual simplicity of the ornament and the position of substructures with respect to the fundamental directions of the ornament, to the fundamental natural directions, etc.

Visually simpler, static non-polar forms with a high degree of symmetry, occurring as substructures, will be more easily perceived and visually recognized. The possibility for recognizing its symmetry substructures will depend also on the visual simplicity of the ornament itself. A high degree of symmetry of the ornament in such a case is the aggravating factor for registering subentities, so that the same symmetry subgroup will be easily recognized in ornaments with a low degree of symmetry. For recognizing the symmetry of ornaments and their substructures, it is very important to visually recognize and discern a fundamental region or an elementary asymmetric figure belonging to the fundamental region. Otherwise, a slow recognition of symmetry elements and the symmetry group of the ornament, is unavoidable. Problems with the recognition of symmetry substructures of one ornament can be efficiently solved by using the table of subgroups of a given symmetry group of ornaments, and especially, by using their visual interpretations — tables of the graphic symbols of symmetry elements (A.V. Shubnikov, V.A. Koptsik, 1974; B. Grünbaum, G.C. Shephard, 1987) and Cayley diagrams of ornaments (H.S.M. Coxeter, W.O.J. Moser, 1980).

Visualization was an important element in the development of the theory of symmetry of ornaments, so that the visual characteristics of ornaments caused the occurrence of similar ideas, construction methods and even a chronological parallelism between these fields. In ornamental art and in the theory of symmetry the oldest methods for the construction of ornaments, used the frieze and rosette multiplication, including the problem of Bravais lattices, crystal systems and tessellations. The criteria of maximal constructional and visual simplicity and maximal symmetry, united by the principle of visual entropy, played the same important role in both fields, where the symmetry structures with emphasized visual simplicity, in which prevail static elements, were dominant. Considering dynamic structures, mathematicians and artists were faced with the problem of perceiving the regularities on which they were based, when defining the elements of symmetry, generating and other symmetry substructures.

Throughout history, visuality was usually the cause and the basis for geometric discussions. Only lately, with non-Euclidean geometries, have the roles been partly replaced, so that visualization is becoming more and more the way of modeling already existing theories. From a methodological aspect, this is the evolution from the empirical-inductive to the deductive approach. Similarly, we can note the way ornaments progressed from their origin linked to concrete meanings — models in nature — or the symbolic meanings of ornamental motifs. After their meaning and form were harmonized, ornaments became a means of communication. In the final phase, after having complete insight into the geometric and formal properties of ornaments, the question of the meaning of ornaments is almost solved, but new possibilities for artistic investigations of the variety and decorativeness of ornaments, are opened.

This analysis of the visual properties of ornaments and their connection with the symmetry of ornaments, points out the inseparability of these two fields. Besides the possibility for the exact analysis of ornaments, "the theory of symmetry approach" to ornamental art provides the groundwork for planning, constructing and considering ornaments as visual creations with desired visual properties. These visual properties can be anticipated from the corresponding symmetry groups of ornaments, their presentations, properties of generators, structures, etc. Ornaments, treated as visual models of different symmetry structures, can be the subject of scientific studies, and can be used in all scientific fields requiring a visualization of such symmetry structures.

Chapter 3

SIMILARITY SYMMETRY IN E^2

A similarity symmetry group is any group of similarity transformations (H.S.M. Coxeter, 1969, pp. 72), at least one of which is not an isometry. According to the theorem on the existence of an invariant point of every discrete similarity transformation that is not an isometry (E.I. Galyarski, A.M. Zamorzaev, 1963; H.S.M. Coxeter, 1969), all the similarity symmetry groups of the plane E^2 belong to the similarity symmetry groups with an invariant point. From the relationships: $S_2 = S_{20}$, $S_{21} = S_{210}$, $S_{210} \subset S_2$, we can conclude that for a full understanding of the similarity symmetry groups of the plane E^2, it is enough to analyze the similarity symmetry groups of the category S_{20}. Owing to the existence of an invariant point, the similarity symmetry groups of the category S_{20} are also called the similarity symmetry groups of rosettes S_{20}, and the corresponding figures possessing such a symmetry group are called similarity symmetry rosettes.

3.1. Similarity Symmetry Groups of Rosettes S_{20}

The idea of similarity symmetry and the possibility for its exact mathematical treatment was introduced in the monograph by H. Weyl (1952), who defines two similarity transformations of the plane E^2: a central dilatation (or simply, dilatation) and dilative rotation, with the restriction for the dilatation coefficient $k > 0$, and he establishes the connection between the transformations mentioned and the corresponding space isometries — a translation and twist, respectively. His analysis is based on natural forms satisfying similarity symmetry (e.g., the Nautilus shell, Figure 3.1.; the sunflower *Heliantus maximus*, etc.). In considering a spiral tendency in nature Weyl quotes certain older authors (e.g., Leonardo and Goethe), who also studied these problems and also that of a *phyllotaxis*, the connection between the way of growth of certain plants and the *Fibonnaci* *sequence*, linked to a *golden section* (H.S.M. Coxeter, 1953, 1969). The sequence $1, 1, 2, 3, 5, 8, 13\ldots$ defined by the recursion formula: $f_1 = 1$, $f_2 = 1$, $f_n + f_{n+1} = f_{n+2}$, $n \in N$, is called the Fibonnaci sequence. A golden section ("*aurea sectio*" or "*de divina proportione*", according to L. Paccioli) is the division of a line segment so that the ratio of the larger part to the smaller is equal to the ratio of the whole segment to the larger part, i.e. its division in the ratio $\tau : 1$, where τ is the positive root of the quadratic equation $\tau^2 + \tau + 1 = 0$, $\tau = (\sqrt{5} + 1)/2 \approx 1,618033989\ldots$

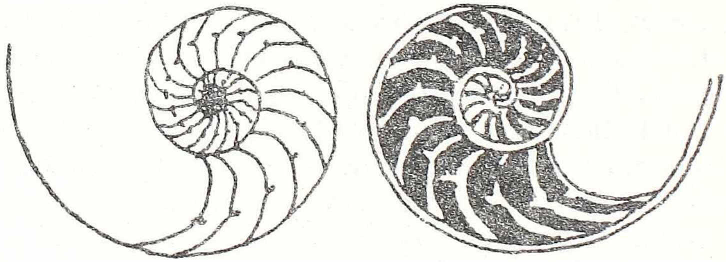


Figure 3.1
Cross-section of a Nautilus shell.

The next step in the development of the theory of similarity symmetry in the plane E^2 was a contribution by A.V. Shubnikov (1960). He described all the similarity transformations of the plane E^2: central dilatation K, dilative rotation L and dilative reflection M and the symmetry groups derived by one of the transformations mentioned and by isometries having the same invariant point — rotations and reflections. Shubnikov derived six types of discrete similarity symmetry groups of rosettes S_{20}: $\mathrm{C}_n\mathrm{K}$, $\mathrm{C}_n\mathrm{L}$, $\mathrm{C}_n\mathrm{M}$, $\mathrm{D}_n\mathrm{K}$, $\mathrm{D}_n\mathrm{L}$, $\mathrm{D}_n\mathrm{M}$, denoted by Shubnikov nK, nL, nM, nmK, nmL, nmM respectively. Since the types $\mathrm{D}_n\mathrm{M}$ (nmM) and $\mathrm{D}_n\mathrm{L}$ (nmL) coincide, there are, in fact, five types of the discrete similarity symmetry groups of rosettes S_{20}: $\mathrm{C}_n\mathrm{K}$ (nK), $\mathrm{C}_n\mathrm{M}$ (nM), $\mathrm{C}_n\mathrm{L}$ (nL), $\mathrm{D}_n\mathrm{K}$ (nmK), D_n L (nmL) and two types of the visually presentable continuous similarity symmetry groups of rosettes S_{20}: $\mathrm{D}_\infty\mathrm{K}$ (∞K) and $\mathrm{C}_n\mathrm{L}_1$ (nL_1). The term "type of similarity symmetry groups of rosettes" and the corresponding type symbol denote all the similarity symmetry groups defined by this symbol, that can be obtained by different combinations of parameters defining them. For example, by the symbol C_nK (nK) are denoted all the corresponding similarity symmetry groups which can be obtained for different values of n ($n \in N$) and k (where $K = K(k)$).

Presentations and structures:

$C_nK(nK)$ $\{S,K\}$ $S^n = E$ $SK = KS$ $C_n \times C_\infty$

$C_nL(nL)$ $\{S,L\}$ $S^n = E$ $SL = LS$ $C_n \times C_\infty$

$C_nM(nM)$ $\{S,M\}$ $S^n = E$ $SMS = M$

$D_nK(nmK)$ $\{S,R,K\}$ $S^n = R^2 = (SR)^2 = E$ $KR = RK$ $KS = SK$ $D_n \times C_\infty$

$\{R,R_1,K\}$ $R^2 = R_1^2 = (RR_1)^n = E$ $KR = RK$ $KR_1 = R_1K$

$D_nL(nmL)$ $\{S,R,L\}$ $S^n = R^2 = (SR)^2 = E$ $LS = SL$ $LRLR = RLRL$ $RLR = LS$

$\{R,R_1,L\}$ $R^2 = R_1^2 = (RR_1)^n = E$

$LRLR = RLRL$ $LR_1LR_1 = R_1LR_1L$ $R_1L = LR$

$(L = L_{2n} = L(k,\pi/n))$

Form of the fundamental region: bounded, allows changes of the shape of boundaries that do not belong to reflection lines, so symmetry groups of the types C_nK (nK), C_nL (nL), C_nM (nM) allow changes of the shape of all the boundaries, while symmetry groups of the types D_nK (nmK), D_nL (nmL) allow only changes of the shape of boundaries that do not belong to reflection lines.

Number of edges of the fundamental region: D_nK (nmK) — 4; C_nK (nK), C_nL (nL), C_nM (nM) — 4,6; D_nL (nmL) — 3,4,5,6.

Enantiomorphism: symmetry groups of the types C_nK (nK), C_nL (nL), C_nL_1 (nL_1) give the possibility for the enantiomorphism. In all other cases the enantiomorphism does not occur.

Polarity of rotations: coincides with the polarity of rotations of the generating symmetry groups of rosettes C_n (n), D_n (nm).

Polarity of radial rays: if they exist, radial rays are polar.

The table of group-subgroup relations between discrete similarity symmetry groups of rosettes S_{20}:

	C_nK	C_nM	D_nK	D_nL
C_nK	2			
C_nM	2	2		
D_nK	2	2	2	
D_nL	2	2	2	3

$[K(k):K(k^m)] = m,$

$[M(k):M(k^m)] = m,$

$[L(k,\theta):L(k^m, m\theta)] = m.$

If $\theta = p\pi/q$, $(p,q) = 1$, then:

$[L(k,\theta):K((-1)^p k^q)] = q,$

$[D_nL(k,\theta):C_nK((-1)^p k^q)] = q,$

$[D_nL:D_nK(k^2)] = 2,\ [D_nK(k^2):C_nM(k^2)] = 2,\ [C_nM(k^2):C_nK(k^4)] = 2.$

Further analysis on similarity symmetry groups was undertaken by E.I. Galyarski and A.M. Zamorzaev (1963). Besides giving the precise definitions of the similarity transformations K, L, M, they used the adequate names for these transformations, comparing them, respectively, with the corresponding isometries of the space E^3 — translation, twist and glide reflection. They also successfully established the isomorphism between the similarity symmetry groups of rosettes S_{20} and the corresponding symmetry groups of oriented, polar rods G_{31}. In this way, consideration of the similarity symmetry groups of rosettes S_{20} and their generalizations is reduced to the consideration of the corresponding, far better known symmetry, antisymmetry and color-symmetry groups of polar, oriented rods G_{31}. The principle of crystallographic restriction (n=1,2,3,4,6) is followed by E.I. Galyarski and A.M. Zamorzaev.

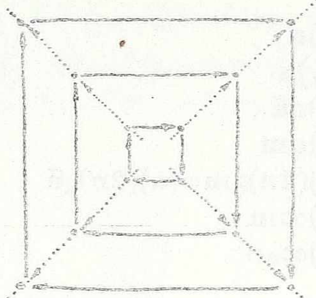
Isomorphism between similarity symmetry groups of rosettes S_{20} and symmetry groups of polar rods G_{31} is, according to A.V. Shubnikov and V.A. Koptsik (1974):

C_nK (nK)	(a)n
C_nL (nL)	(a_t)n
C_nM (nM)	(a)nã
D_nK (nmK)	(a)nm
D_nL (nmL)	$(a)(2n)_n m=(a)(2n)_n \tilde{a}$
$D_\infty K$ (∞mK)	(a)∞m
C_nL_1 (nL_1)	$(a)\infty_0 n$

In the work by E.I. Galyarski and A.M. Zamorzaev (1963), there is no the restriction for the dilatation coefficient $k > 0$, used by H. Weyl (1952). This restriction does not result in any loss of generality, but only in the somewhat different classification of the similarity symmetry groups of rosettes S_{20}.

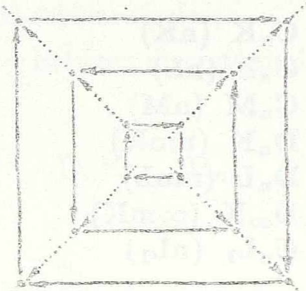
There is also the problem that for every particular similarity symmetry group of rosettes S_{20}, its corresponding type is not always uniquely defined. Namely, under certain conditions, the same symmetry group can be included in two different types. Such a case is, e.g., that symmetry groups of the type C_nK (nK), because of the relationship $K(k) = L(k,0)$, also belong to the type C_nL (nL). If we accept the condition $K = K(k) = L(k,0) = L_0$, then there also exists the subtype D_nL_0 (nmL_0), but symmetry groups of the subtype mentioned are not included in the type D_nL_{2n} (nmL_{2n}). If we accept the criterion of subordination, which means, if we consider symmetry groups existing in two different types within the larger type, certain types would not exist at all. For example, all the symmetry groups of the type C_nK (nK) would be included in the type C_nL (nL), so that the type C_nK (nK) would not exist at all, and so on. A similar problem may occur with the same similarity symmetry group that can be defined by different sets of parameters n, k, θ... To consequently solve that problem, it is necessary to accept the common criterion of maximal symmetry. Such an overlapping of different types of the similarity symmetry groups of rosettes S_{20} is possible to avoid by accepting Weyl's condition $k > 0$ for all the similarity symmetry groups of rosettes S_{20} and the condition $0 <|\theta|< \pi/n$ for symmetry groups of the type C_nL (nL).

Cayley diagrams (Figure 3.2):



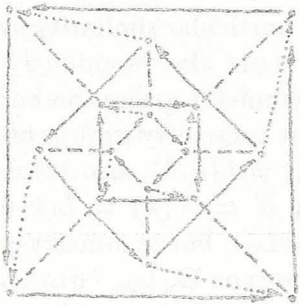
C_nK (nK), C_nL (nL)

S
K, *L*

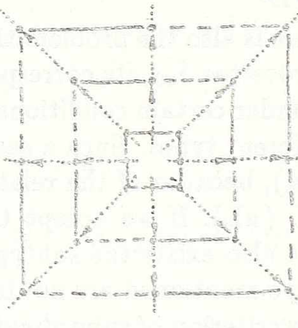


C_nM (nM)

S
M

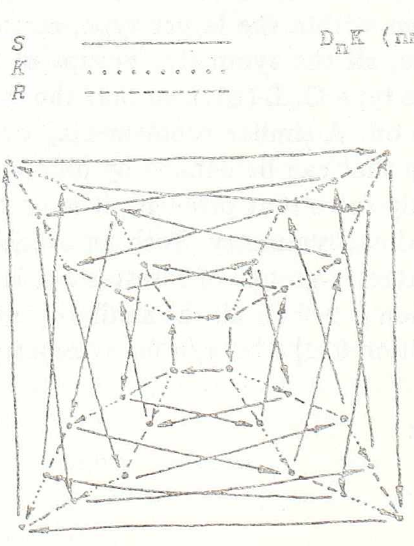


S
K
R



D_nK (nmK)

K
R
R_1



D_nL (nmL)

S
L
R

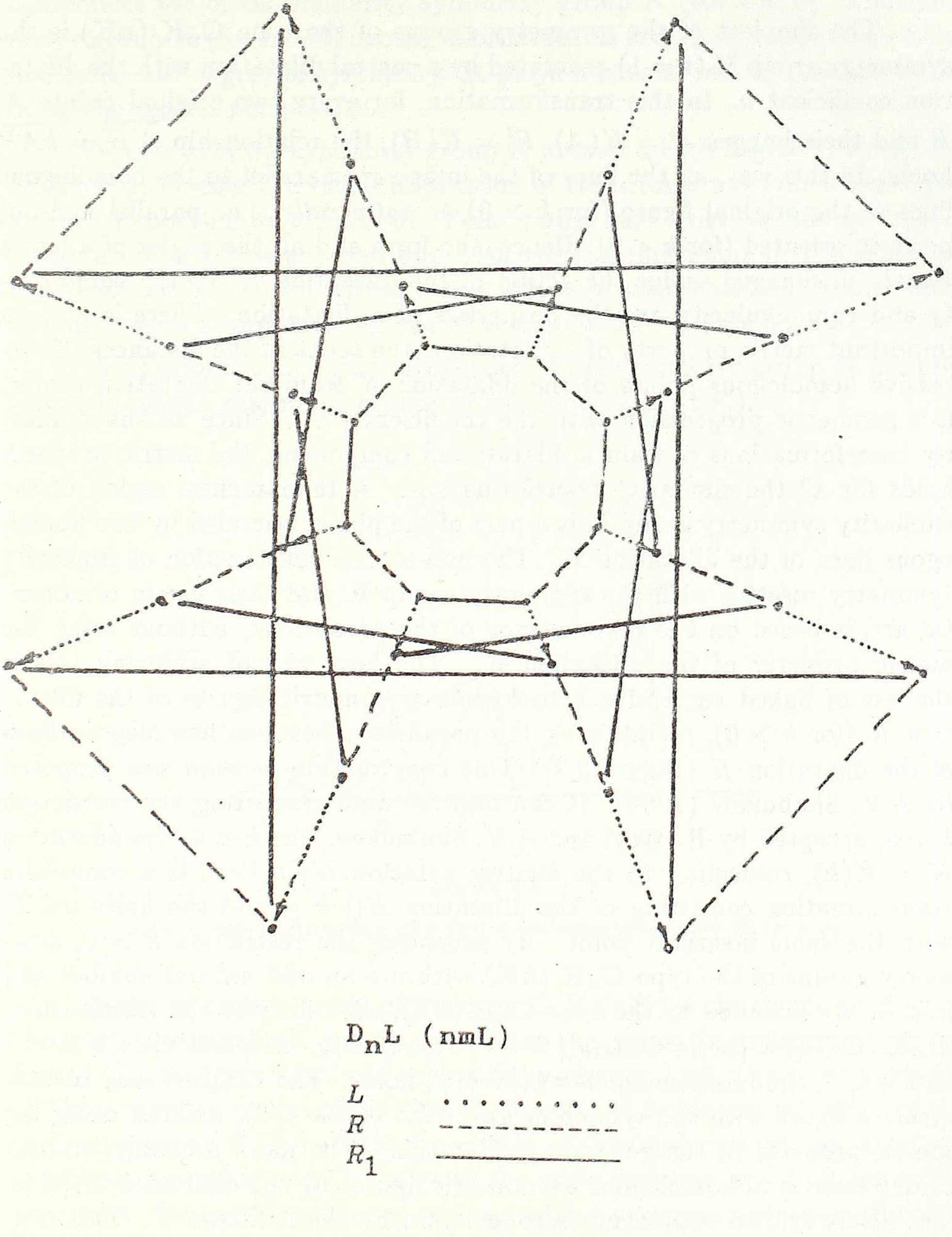


Figure 3.2

3.2. Similarity Symmetry Rosettes and Ornamental Art

The simplest of the symmetry groups of the type C_nK (nK) is the symmetry group K (n= 1) generated by a central dilatation with the dilatation coefficient k. In this transformation, for every two original points A, B and their images $A' = K(A)$, $B' = K(B)$, the relationship $\overrightarrow{A'B'} = k\overrightarrow{AB}$ holds. In this way, all the lines of the image are parallel to the homologous lines of the original figure (for $k > 0$) or *antiparallel*, i.e. parallel and oppositely oriented (for $k < 0$). Hence, the form and all the angles of a figure remain unchanged under the action of the dilatation K, so the equiformity and equiangularity are the properties of a dilatation. There is also an important metric property of a dilatation: the series of the distances of successive homologous points of the dilatation K from the dilatation center, is a geometric progression with the coefficient $| k |$. Since all the similarity transformations contain a dilatational component, the metric property holds for all the similarity transformations. A fundamental region of the similarity symmetry group K is a part of the plane, bounded by two homologous lines of the dilatation K. The non-metric construction of similarity symmetry rosettes with the symmetry group K and their use in ornamental art, is based on the maintenance of the parallelism, without using the metric property of the dilatation K. The best way of achieving this is the use of linked successive homologous asymmetric figures of the dilatation K (for $k > 0$), maintaining the parallelism between homologous lines of the dilatation K (Figure 3.3). This construction method was proposed by A.V. Shubnikov (1960). It can only be used respecting the restriction $k > 0$, accepted by H. Weyl and A.V. Shubnikov. For $k < 0$, the dilatation $K = K(k)$, coinciding to the dilative rotation $L(| k |, \pi)$, is a composite transformation consisting of the dilatation $K(| k |)$ and the half-turn T, with the same invariant point. By accepting the restriction $k > 0$, symmetry groups of the type C_nK (nK) with n - an odd natural number and $k < 0$, are included in the type C_nL (nL), according to the relationship: $C_nK = C_nL(| k |, \pi) = C_nL_{2n}(| k |, \pi/n) = C_nL_{2n}$. In particular, if $n = 1$ and $k < 0$, the relationship $K = L(| k |, \pi)$ holds. The simplest way to construct a figure with the symmetry group K (for $k < 0$), without using the metric property of the generating dilatation K is, most probably, to construct a series of homologous asymmetric figures by the dilatation $K(| k |)$, and afterward, to copy every second figure by the half-turn T. This construction is the same as the construction of a figure with the symmetry group $L(| k |, \pi)$. Owing to the complexity of the construction itself, in the earlier phases of ornamental art, it is very difficult to find examples of the consequent use of the similarity symmetry group K (for $k < 0$). The symmetry group K (for $k > 0$) occurs in ornamental art, though not frequently, due to its low degree of symmetry. It plays a special role in fine art works using the central perspective.

Figures with the symmetry group K possess a high degree of visual dynamism, producing the visual impression of centrifugal motion. They occur as enantiomorphic modifications. Polar radial rays exist. In the geometric sense, a radial ray is any half-line with the starting point in the center of the dilatation K. In the visual sense, this is a basic half-line of any series of homologous asymmetric figures of the dilatation K (Figure 3.3).

The characteristic visual properties of the symmetry group K are preserved in all the symmetry groups of the type C_nK (nK).

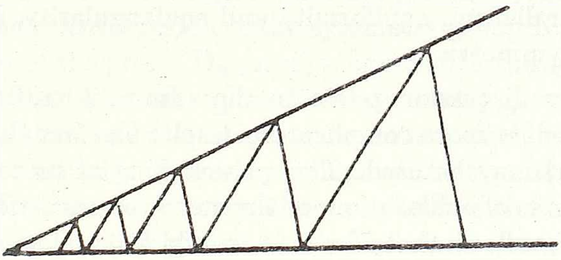


Figure 3.3

Non-metric construction of a figure with the symmetry K ($k > 0$).

Every symmetry group of the type C_nK (nK) is the direct product of the symmetry groups K and C_n (n). Since the generating symmetry groups K and C_n (n) occur relatively seldom in ornamental art, the same holds for all the symmetry groups of the type C_nK (nK). A fundamental region of the symmetry group C_nK (nK) is the section of the fundamental regions of the symmetry groups K and C_n (n) with the same invariant point. Polar radial rays exist. Similarity symmetry rosettes with a symmetry group of the type C_nK (nK) may be obtained from rosettes with the symmetry group C_n (n), multiplying them by the dilatation K with the same center. The same result can be obtained multiplying by means of the n-fold rotation, a figure with the symmetry group K, belonging to a fundamental region of the symmetry group C_n (n) with the same rotation and dilatation center. Besides the construction mentioned, based on the maintenance of the parallelism, applicable when successive homologous asymmetric figures of the dilatation K (for $k > 0$) are linked to each other, constructions based on the metric property of the dilatation are also possible, and also the combinations of these two methods.

Owing to its complexity, the metric construction of similarity symmetry rosettes with the symmetry group C_nK (nK) was very seldom used in ornamental art. Where such attempts exist, they are usually followed by deviations from the regularity, so that similarity symmetry rosettes obtained only suggest the symmetry C_nK (nK), without satisfying it in the strict sense. Aiming for maximal constructional simplicity, usually triumphant is the simplest as possible metric regularity, where the geometric progression formed by distances of successive homologous points of the dilatation K from the dilatation center, is replaced by an arithmetic progression. That disturbs the parallelism, equiformity and equiangularity, and consequently, the similarity symmetry.

The above discussion refers to the case of $k > 0$. For $k < 0$ the construction itself is more complicated, despite the fact that the parallelism (antiparallelism) may be used. Then, the linking of successive homologous asymmetric figures of a dilatation, or the metric construction method cannot be exclusively used, so that, for n – an odd natural number and $k < 0$, adequate examples of similarity symmetry rosettes with the symmetry group C_nK (nK) are rarely found in early ornamental art. For n – a fixed even natural number, there is no difference between symmetry groups of the type C_nK (nK), depending on the sign of the dilatation coefficient k, so the relationship $C_nK(k)=C_nK(-k)$ holds. That is because, for n – an even natural number, the half-turn already exists in the symmetry group C_n (n), so it is included in the symmetry group C_nK (nK) derived from it. The difference between the symmetry groups $C_nK(k)$ and $C_nK(-k)$ occurs only for n – a fixed odd natural number. In the symmetry group with $k < 0$ and n – an odd natural number, a half-turn does not exist as an independent symmetry transformation of the symmetry group C_nK (nK), but only as a part of the composite transformation $K = TK(|\,k\,|)$. For n – an odd natural number and $k < 0$, the relationship $C_nK=C_nL(|\,k\,|,\pi/n)=C_nL_{2n}$ holds.

Apart from by the construction methods mentioned above, similarity symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{K}$ (nK) may be constructed by a desymmetrization of the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK), where by a coloring or some other desymmetrization procedure, reflections are eliminated. The similarity symmetry rosettes derived often belong to similarity antisymmetry rosettes with the antisymmetry group $\mathbf{D}_n\mathbf{K}/\mathbf{C}_n\mathbf{K}$ (nmK/nK), treated by the classical theory of symmetry as similarity symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{K}$ (nK).

The causes of the origin of similarity symmetry groups of the type $\mathbf{C}_n\mathbf{K}$ (nK) in ornamental art, can be found in their visual-symbolic meaning, where two dynamic components — the centrifugal component produced by the dilatation K, and the suggestion of a rotational motion produced by the subgroup $\mathbf{C}_n$ (n) — come to their full expression.

Every similarity symmetry group of the type $\mathbf{D}_n\mathbf{K}$ (nmK) is the direct product of the symmetry groups $\mathbf{D}_n$ (nm) and $\mathbf{K}$. Similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK) can be constructed multiplying by the dilatation K a figure with the symmetry group $\mathbf{D}_n$ (nm), belonging to a fundamental region of the symmetry group $\mathbf{K}$, or multiplying by the symmetries of the group $\mathbf{D}_n$ (nm) a figure with the symmetry group $\mathbf{K}$, belonging to a fundamental region of the symmetry group $\mathbf{D}_n$ (nm). In both cases, the rotation and dilatation center must coincide. Existing models in nature with similarity symmetry groups of the type $\mathbf{D}_n\mathbf{K}$ (nmK) are the cause of the frequent occurrence of corresponding similarity symmetry rosettes in ornamental art (Figure 3.4). A fundamental region of the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK) is the section of the fundamental regions of the symmetry groups $\mathbf{D}_n$ (nm) and $\mathbf{K}$ with the same invariant point. Due to the existence of at least one reflection in every symmetry group of the type $\mathbf{D}_n\mathbf{K}$ (nmK), there is no enantiomorphism. Polar radial rays exist.

For constructing a figure with the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK), it is possible to use the parallelism of homologous lines of the dilatation K, by using the linking of successive homologous asymmetric figures of the dilatation K, analogously to the construction of figures with the similarity symmetry group $\mathbf{C}_n\mathbf{K}$ (nK). The metric construction method also can be used. Like for similarity symmetry groups of the type $\mathbf{C}_n\mathbf{K}$ (nK) and their visual interpretations, the same deviations from the regularity dictated by the dilatation K frequently occur — the use of the equidistance and disturbance of the similarity symmetry. In the visual sense, all similarity symmetry rosettes with symmetry groups of the type $\mathbf{D}_n\mathbf{K}$ (nmK) possess a static visual component resulting from the existence of the subgroup D_n (nm), and a dynamic, centrifugal component resulting from the visual function of the dilatation K. For n – a fixed even natural number, for the same reason like the symmetry groups $C_nK(k)$ and $C_nK(-k)$, the symmetry groups $D_nK(k)$ and $D_nK(-k)$ will not differ. Like for the symmetry groups of the type C_nK (nK), for n – an odd natural number and $k < 0$, constructions of similarity symmetry rosettes of the type D_nK (nmK) are more complicated. Therefore, in ornamental art, adequate examples of those similarity symmetry groups are considerably less frequent. Accepting the restriction $k > 0$, symmetry groups of the type D_nK (nmK), for n – an odd natural number and $k < 0$, can be discussed within the type D_nL (nmL), where, for $k < 0$, the relationships $K=L(|k|,\pi)$ and $D_nK=D_nL(|k|,\pi)=D_nL$, hold.

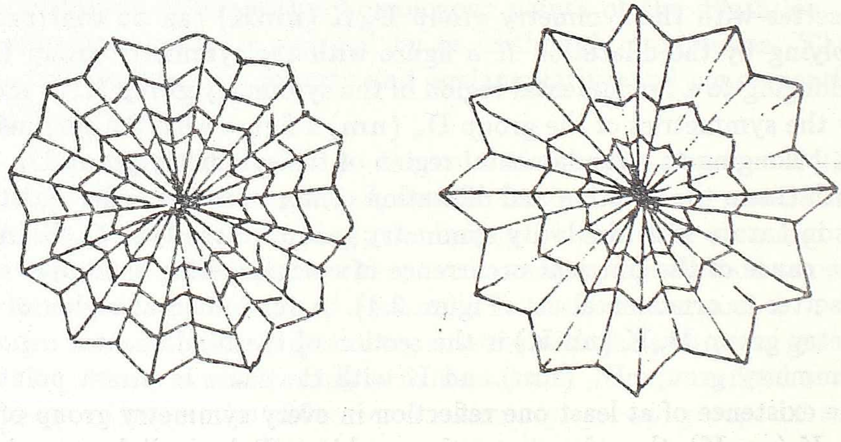


Figure 3.4

Rosettes with the similarity symmetry group $D_{12}K$ (12mK) (the monastery of Dechani, Yugoslavia).

A special place among similarity symmetry groups of the type D_nK (nmK) is taken by the symmetry group D_1K (mK). Apart from its use in ornamental art, it frequently occurs in painting works with the application of the central perspective, where the motif (e.g., a street, architectural objects, the road, the tree-lined path, etc.) possesses plane symmetry.

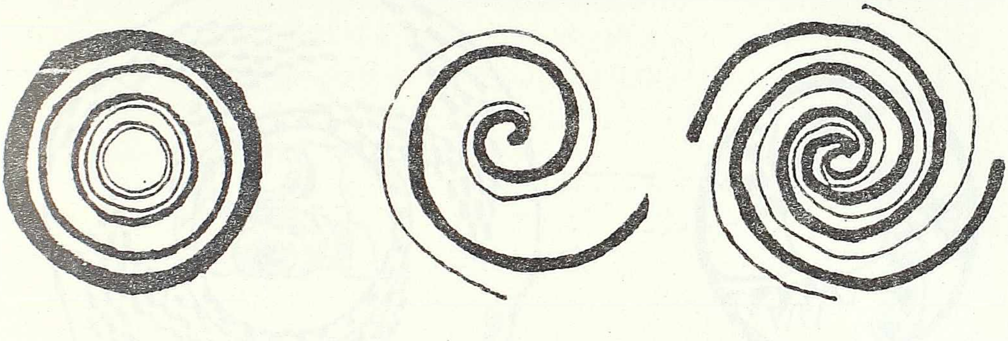
The appearance of similarity symmetry rosettes with symmetry groups of the type $\mathbf{D}_n\mathbf{K}$ (nmK) in ornamental art, can be explained, in the first place, as the imitation of natural forms usually possessing or suggesting that kind of symmetry. The dihedral symmetry group $\mathbf{D}_n$ (nm) is present in many living beings, as the symmetry group of the entity or its parts, while the similarity symmetry group K is the result of the growth of living beings. In ornamental art, an important role in the formation of similarity symmetry rosettes with a symmetry group of the type $\mathbf{D}_n\mathbf{K}$ (nmK), was their visual-symbolic meaning and the possibility for the visual suggestion of a radial expansion from the center. This makes possible the use of similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK) as dynamic symbols, possessing also a certain degree of visual stationariness and balance, resulting from the existence of the subgroup $\mathbf{D}_n$ (nm).

Regarding the frequency of occurrence, according to the principle of visual entropy, similarity symmetry rosettes with similarity symmetry groups of the type $\mathbf{D}_n\mathbf{K}$ (nmK), especially for n=1,2,4,8,12..., dominate in ornamental art.

If we accept the criterion of subordination, previously considered similarity symmetry groups of the types $\mathbf{C}_n\mathbf{K}$ (nK) and $\mathbf{D}_n\mathbf{K}$ (nmK) can be discussed, respectively, within the types $\mathbf{C}_n\mathbf{L}$ (nL) and $\mathbf{D}_n\mathbf{L}$ (nmL). Symmetry groups of the type $\mathbf{C}_n\mathbf{K}$ (nK) can be discussed within the type $\mathbf{C}_n\mathbf{L}$ (nL), where the dilatation $K(k)$ can be understood as the dilative rotation $L_0(k,0)$, so the relationship $\mathbf{C}_n\mathbf{K}=\mathbf{C}_n\mathbf{L}_0$ holds. For n - an odd natural number and $k < 0$, the relationships $\mathbf{C}_n\mathbf{K}=\mathbf{C}_n\mathbf{L}(|\ k\ |,\pi)$ and $\mathbf{D}_n\mathbf{K}=\mathbf{D}_n\mathbf{L}(|\ k\ |,\pi)=\mathbf{D}_n\mathbf{L}$ hold.

The simplest symmetry group of the type $\mathbf{C}_n\mathbf{L}$ (nL) is the symmetry group L (n=1) generated by the dilative rotation $L = L(k,\theta)$ - a composite transformation representing the commutative product of a dilatation K and the rotation S with the rotation angle θ. Under the action of the transformation L, every vector $\overrightarrow{AB}$ defined by two original points A, B, is transformed onto the vector $\overrightarrow{A'B'}$ of the intensity $|\ k\overrightarrow{AB}\ |$, defined by the image points $A' = L(A)$, $B' = L(B)$, which forms with the vector $\overrightarrow{AB}$ the oriented angle θ (for $k > 0$) or $\pi - \theta$ (for $k < 0$). The following vector equalities: $|\ \overrightarrow{A'B'}\ |= k\ |\ \overrightarrow{AB}\ |$, $\overrightarrow{AB} \circ \overrightarrow{A'B'}/k\overrightarrow{AB}^2 = \cos\theta$ hold. Due to the maintenance of the angle between two original vectors, all the angles and the form of an original figure remain unchanged under the action the transformation L. Hence, the equiangularity and equiformity are the properties of a dilative rotation. Where the angle θ of the dilative rotation L is a rational one ($\theta = p\pi/q, (p,q) = 1, p,q \in Z$), among the symmetry transformations of the symmetry group L, there will be the dilatation $K((-1)^p k^q)$. In a visual sense, it makes possible a division of the figure with the symmetry group L into the sectors of the dilatation $K((-1)^p k^q)$ (A.V. Shubnikov, 1960) resulting in the appearance of polar radial rays. A *sector of dilatation* is any sector between two successive radial rays. A fundamental region of the symmetry group L is a part of the plane defined by two homologous lines of the transformation L.

There exist several ways of constructing figures with the similarity symmetry group L. Like the constructions of similarity symmetry groups of the type C_nK (nK), they can be divided into two basic kinds: constructions by using or not the metric property of a dilatation, which is a constituent part of the composite transformation L. Combinations of these methods are also possible. In ornamental art, the most frequent are non-metric constructions. The easiest method for construction of visual interpretations of the symmetry group L is the non-metric construction of a series of linked homologous asymmetric figures of the dilatation $K(|k|)$, and afterward the rotation of every n-th figure ($n \in N$) through the angle $n\theta$ (for $k > 0$) or $n(\pi - \theta)$ (for $k < 0$). The metric construction is based on the construction of a series of homologous points of the dilative rotation L, which satisfy, simultaneously, both the rotation and the geometric progression of their distances from the invariant point. Besides the construction methods mentioned, for obtaining examples of the symmetry group L, we can use a desymmetrization of the continuous visually presentable similarity symmetry group D_∞K (∞mK), a visual interpretation of which consists of a series of concentric circles satisfying the dilatation K (Figure 3.5a). A construction of a series of consecutive θ-segments (for $k > 0$) or ($\pi - \theta$)-segments (for $k < 0$) followed by the elimination of reflections and maintenance of the equiformity of homologous asymmetric figures of the dilatative rotation L, which belong to the consecutive segments and consecutive circular rings, is very efficient for obtaining visual interpretations of the symmetry group L. Such a construction, in a certain sense, remains in some technical procedures (e.g., weaving baskets, etc.), but, because of the technical characteristics of the work itself, there usually occur deviations from the regularity dictated by the dilatation, the use of the simplest metric regularity — equidistance, and the disturbance of the similarity symmetry (Figure 3.6).



(a) (b) (c)

Figure 3.5

Visual interpretations of the continuous similarity symmetry groups: (a) $D_\infty K$ (∞mK); (b) L_1; (c) C_2L_1 ($2L_1$).

Since every symmetry group $L(k,\theta)$ for $k < 0$ can be reduced to the symmetry group $L(|k|, \pi - \theta)$, a particular analysis of the symmetry groups L, depending on the sign of the coefficient k, is not necessary. Nevertheless, in ornamental art, examples of the symmetry groups L with an acute minimal angle of the dilative rotation L and $k > 0$, are more frequent. In the symmetry group L, there is the possibility of the enantiomorphism. If the dilative rotation angle is a rational one, i.e. $\theta = \pi p/q, (p,q) = 1, p, q \in Z$, there exist polar radial rays.

In nature, there are frequent and various examples of spiral forms. Most of them are the results of different rotational motions. The symmetry group L occurs as the symmetry group of complete natural forms or their parts, in non-living natural forms (e.g., as galaxies, whirlpools at turbulent motion of fluids, etc.) as well in at living creatures (the spiral tendency of shell growth in some mollusks, the spiral tendency of growth in certain plants or plant products, etc.).

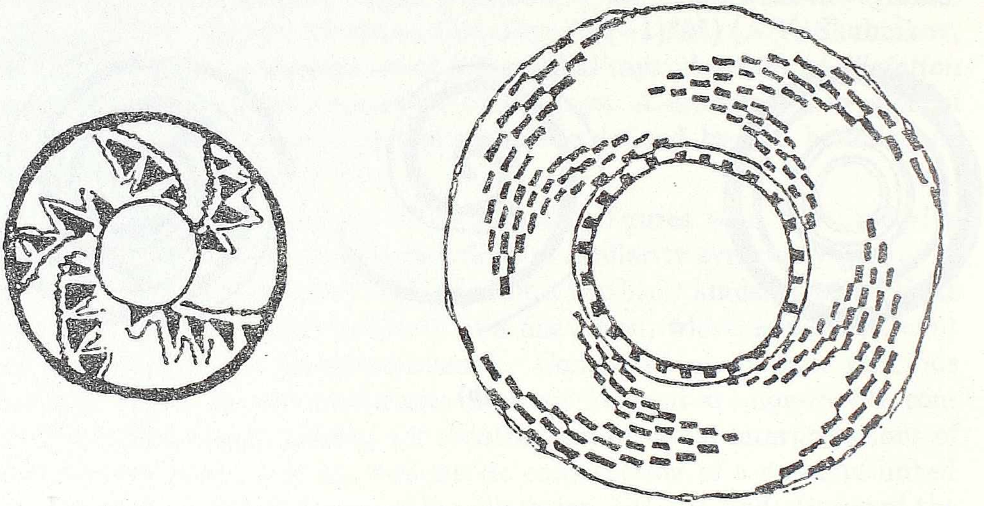


Figure 3.6

Woven baskets of the American Indians, which suggest the similarity symmetry of rosettes C_5L (5L) and C_4L (4L).

As a fundamental geometric figure, and by using models in nature, the spiral became one of the most frequent dynamic symbols in the whole of art. Regarding its visual-symbolic meaning and its frequency of occurrence in ornamental art, it can be considered as the dynamic equivalent of the circle. The oldest examples of spirals date to the Paleolithic (Figure 3.7). In the further development of ornamental art, the spiral appears, most probably independently, in all cultures, distant both in place and time, as one of the basic ornamental motifs (J. Purce, 1975).

All the orbit points $L(P)$ of a point P in general position with respect to the symmetry group L belong to a logarithmic, equiangular spiral. A *logarithmic spiral* is the orbit of a point of the plane $E^2 \setminus \{O\}$, with respect to the continuous visually presentable conformal symmetry group L_1Z_I. A logarithmic spiral satisfies the condition of "uniform motion", according to

H. Weyl (1952). It is the only plane curve with the property of equiangularity. This means it intersects all its radius-vectors under a constant angle. When the angle between a radius-vector and the corresponding tangent line is 90°, the logarithmic spiral is reduced to a circle. The fact that every linear

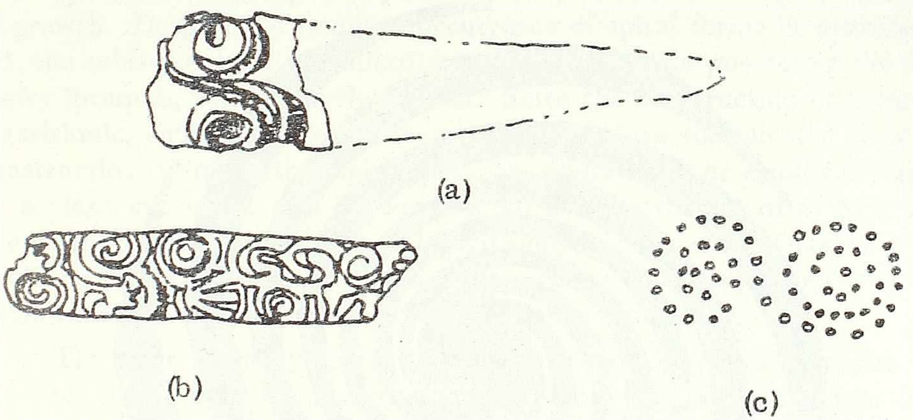


Figure 3.7

Paleolithic spiral ornaments: (a) Arudy; (b) Isturiz; (c) Mal'ta, USSR (Magdalenian, around 10000 B.C.).

transformation of the plane transforms a logarithmic spiral onto the logarithmic spiral congruent to it, led J. Bernulli to name it "spira mirabilis" (A.A. Savelov, 1960). From the point of view of the theory of similarity symmetry, of special interest is the invariance of a logarithmic spiral with respect to certain similarity transformations. In a visual sense, Weyl's condition of the "uniform motion" is expressed by the fact that by the uniform rotation of a logarithmic spiral around its center, it is possible to realize the visual impression of the change in dimensions of the logarithmic spiral — its increase or decrease. This visual phenomenon shows the equivalence of the action of a dilatation and rotation with the same center. This property is used in applied art to decorate rotating elements (e.g., wheels) (Figure 3.8).

A bipolar, non-oriented logarithmic spiral corresponds to the visually presentable continuous conformal symmetry group $L_1 Z_I$. An oriented logarithmic spiral corresponds to the visually presentable continuous similarity symmetry group L_1 (Figure 3.5b), the subgroup of the index 2 of the symmetry group $L_1 Z_I$. An equiangular, logarithmic spiral is the invariant

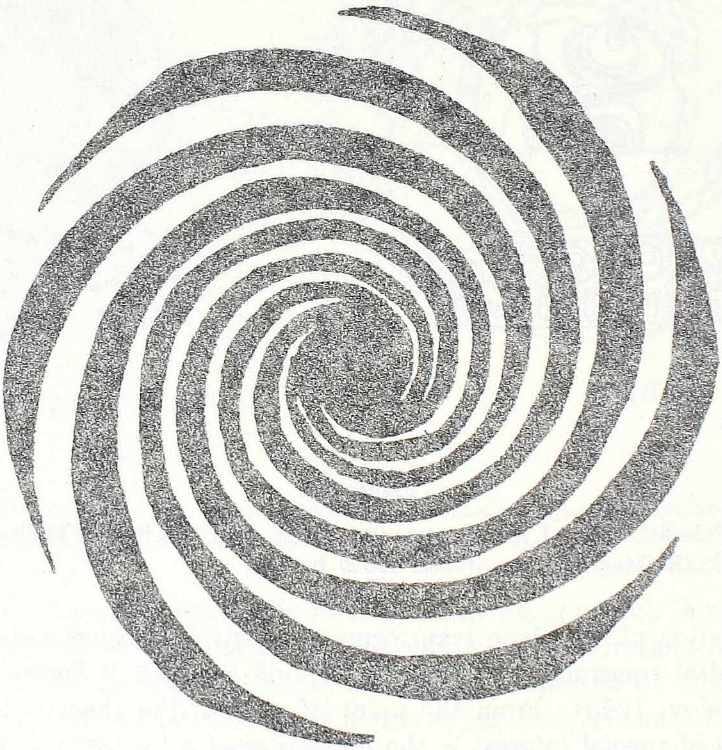


Figure 3.8

Example of the equiangular, logarithmic spiral occurring as an ornamental motif. The impression of the changes in dimensions of the logarithmic spiral can be achieved by the rotation of this rosette around the singular point. This indicates the equivalence of the visual effect resulting from the dilatation and such a rotation of equiangular spiral.

figure of this continuous conformal symmetry group and of its subgroups. By using that property, constructions based on the maintenance of the equiangularity and equiformity may be simplified. So that, it can serve as a basis for the construction of all similarity symmetry rosettes with the symmetry group L, by applying the desymmetrization method to the symmetry group $L_1 Z_I$ with the same dilative rotation angle (A.V. Shubnikov, 1960).

In nature, there are forms that almost perfectly satisfy the regularity of a logarithmic, equiangular spiral. This is, so called, the spiral tendency of growth. Despite the frequent occurrence of spiral forms in ornamental art, the subsequent use of similarity symmetry rosettes possessing the symmetry group L, was relatively seldom. Since the construction of a certain logarithmic, equiangular spiral is considerably more complicated than the construction of some other spiral (e.g., an *Archimede's* or *equidistant spiral* — a plane curve that can be constructed as the evolvent of a circle by a simple mechanical procedure), a logarithmic spiral is often replaced by an equidistant spiral. That results in a disturbance of the equiangularity and equiformity, and consequently, of the similarity symmetry.

The basic visual-symbolic characteristic of the symmetry group L is a double visual dynamism, caused by the visual suggestion of a rotational motion and centrifugal expansion, resulting from the rotational and dilatational component. Polar radial rays exist under the condition that $\theta = \pi p/q$, $(p,q) = 1$, $p,q \in Z$. A degree of the visual dynamism produced by corresponding similarity symmetry rosettes depends on the coefficient k and on the dilative rotation angle θ. Changes in these parameters result in different visual impressions.

The symmetry group L is applied in painting works having the central perspective as the element, or even as a basis of the complete central dynamic composition of the work (e.g., in the baroque, in Tintoretto's works), creating thus the visual impression of an expanding rotational motion.

Similarity symmetry groups the type C_nL (nL) are formed by composing the symmetry groups L and C_n (n) with the same invariant point. Corresponding similarity symmetry groups can be constructed multiplying by the dilative rotation L a figure with the symmetry group C_n (n), belonging to a fundamental region of the symmetry group L, or multiplying by the n-fold rotation a figure with the symmetry group L, belonging to a fundamental region of the symmetry group C_n (n) (Figure 3.9a, 3.10). In both cases the rotation center and dilative rotation center coincide. Constructional methods used are analogous to that considered with the symmetry

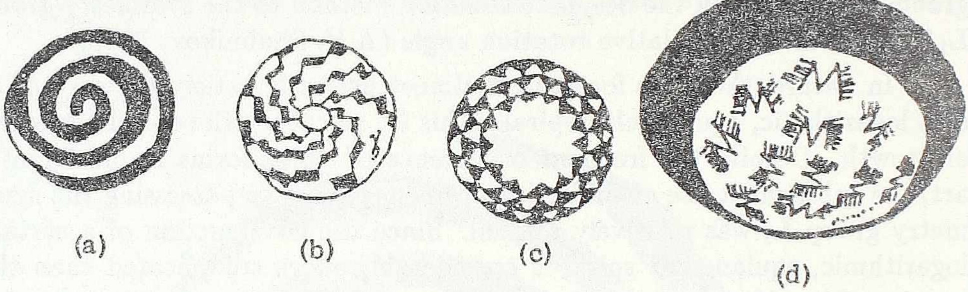


Figure 3.9

Examples of similarity symmetry rosettes in the art of Neolithic and pre-dynastic period of the ancient civilizations, around 4500–3500 B.C.: (a) Egypt; (b), (c) examples of rosettes with the similarity symmetry group of the type C_nL (nL) and D_nL (nmL), Egypt and Iran; (d) example of the rosette with the similarity symmetry group of the type D_nL (nmL), Susa ceramics.

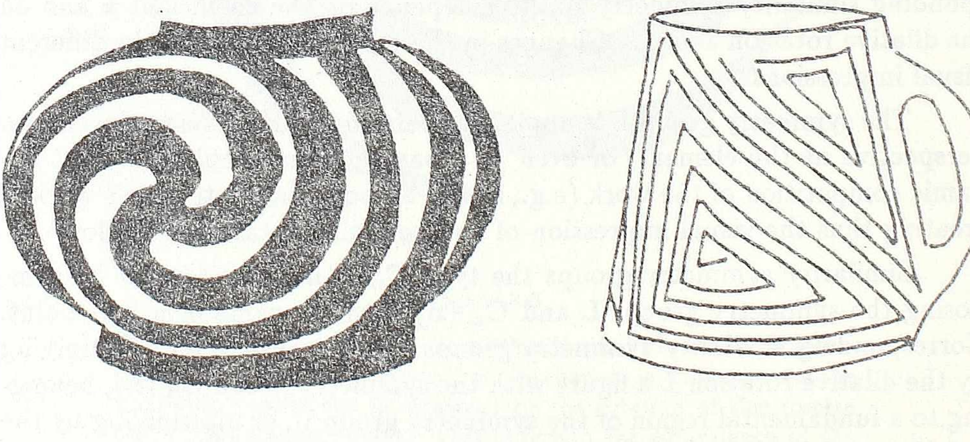


Figure 3.10

Examples of ornamental motifs in the ceramics of the American Indians, that suggest similarity symmetry.

group L. In ornamental art, especially when the metric construction method is applied, deviations from the regularity dictated by the metric property of the dilative rotation L, frequently occur. A tendency toward maximal constructional simplicity results in the appearance of the equidistance, disturbance of the equiangularity and equiformity, desymmetrization of the similarity symmetry group C_nL (nL) and its reduction to the symmetry group C_n (n).

By applying the criterion of maximal symmetry, it is possible to eliminate certain repetitions and overlappings of symmetry groups, otherwise occurring within the type C_nL (nL). The existence of the n-fold rotation with the rotation angle $2\pi/n$ and the dilative rotation L with the dilative rotation angle θ (for $k > 0$) or $(\pi - \theta)$ (for $k < 0$), within the symmetry group C_nL (nL) results in the appearance of the new dilative rotation L' with the minimal dilative rotation angle θ', which is less than the dilative rotation angle θ. According to the criterion of maximal symmetry, every symmetry group C_nL (nL) can be considered as the symmetry group C_nL' (nL'). If we accept the condition $C_nL_0{=}C_nL(k,0) = C_nK$, the type C_nK (nK) is the subtype of the type C_nL (nL). For n - an odd natural number and $k < 0$, the relationship $C_nL_{2n}{=}C_nL(k,\pi/n) = C_nK(|\ k\ |) = C_nK$ holds. For n - an odd natural number, according to the above relationship and the relationship $C_nK{=}C_nL_{2n}$, holding for n - an odd natural number and $k < 0$, we can conclude that the types C_nK (nK) and C_nL_{2n} (nL_{2n}), are dual with respect to the change of the sign of the coefficient k.

A fundamental region of the symmetry group C_nL (nL) is the section of the fundamental regions of its generating symmetry groups C_n (n) and L with the same invariant point. Between symmetry groups of the type C_nL (nL) there will be no essential difference depending on the sign of the coefficient k. Since examples of rosettes with the symmetry group C_n (n) are relatively rare in ornamental art, the same refers to similarity symmetry rosettes with the symmetry group C_nL (nL). Such similarity symmetry rosettes occur in enantiomorphic modifications. If the angle of the dilative rotation L is a rational one, polar radial rays exist. Then, the existence of a dilatation as the element of the symmetry group C_nL (nL) makes it possible to divide the corresponding similarity symmetry rosette into the sectors of the dilatation.

Similarity symmetry rosettes with the symmetry group C_nL (nL) (Figure 3.9b, 3.11c, e) possessing a very high degree of visual dynamism, caused by the polarity of both the relevant components — n-fold rotation and dilative rotation L — produce a visual impression of centrifugal rotational expansion. The existence of models in nature, the dynamic visual impression that suggest, their expressiveness and visual-symbolic function resulted in the appearance and use of similarity symmetry rosettes with the symmetry group C_nL (nL) in ornamental art.

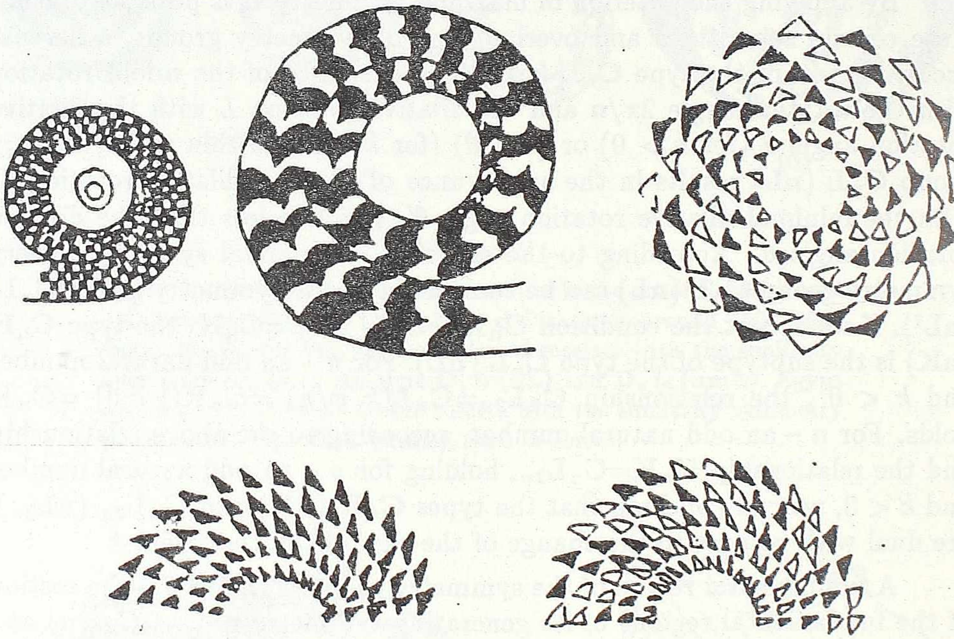


Figure 3.11

Examples of similarity symmetry rosettes in Greek and Byzantine ornamental art.

Similarity symmetry groups of the subtype $C_nL_{2n}(k, \pi/n)$ ($nL_{2n}(k, \pi/n)$) can be derived by desymmetrizations of similarity symmetry groups of the type D_nL (nmL), which are more frequent in ornamental art. By choosing an appropriate desymmetrization and eliminating reflections of the symmetry group D_nL (nmL), the symmetry group C_nL_{2n} (nL_{2n}) can be obtained. By the antisymmetry desymmetrization of the symmetry group D_nL (nmL), the antisymmetry group D_nL/C_nL_{2n} (nmL/nL_{2n}), treated by the classical theory of symmetry as the symmetry group C_nL_{2n} (nL_{2n}) belonging to the type C_nL (nL), can be derived (Figure 3.12b).

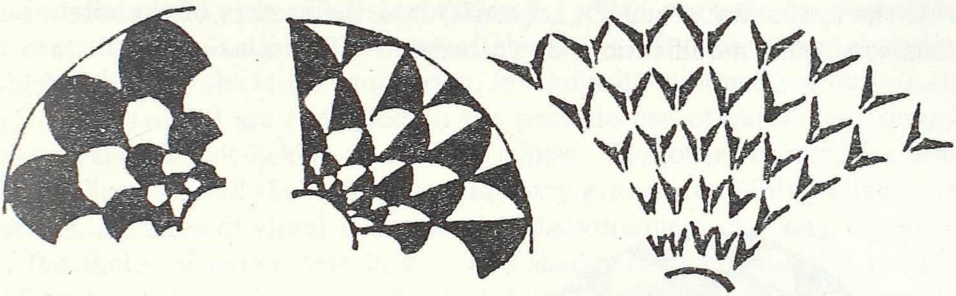


Figure 3.12

Examples of similarity symmetry rosettes in Roman ornaments.

Every symmetry group of the type $\mathbf{D}_n\mathbf{L}$ (nmL) is the composition of the symmetry groups $\mathbf{L}(k, \pi/n)$ and $\mathbf{D}_n$ (nm) with the same invariant point. A fundamental region of the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) is the section of fundamental regions of these two symmetry groups. Similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) (Figure 3.9c, d, 3.12c, 3.13) can be constructed multiplying by the dilative rotation L a rosette with the symmetry group $\mathbf{D}_n$ (nm), belonging to a fundamental region of the symmetry group $\mathbf{L}$, where the rosette center and the dilative rotation center coincide. Construction methods used for obtaining similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) are analogous to the construction methods previously discussed, used with similarity symmetry groups of the type $\mathbf{C}_n\mathbf{L}$ (nL). Owing to a very high degree of symmetry, the existence of models in nature (e.g., flowers and the fruits of certain plants) and frequent applications of the symmetry group of rosettes $\mathbf{D}_n$ (nm), the type $\mathbf{D}_n\mathbf{L}$ (nmL), regarded from the point of view of ornamental art, is one of the largest and most heterogeneous types of the similarity symmetry groups of rosettes S_{20}. For n – an even natural number, there is no difference between individual symmetry groups of the type $\mathbf{D}_n\mathbf{L}$ (nmL), depending on the sign of the dilatation coefficient k, but for n – an odd natural number and $k < 0$, the relationship $\mathbf{D}_n\mathbf{L}{=}\mathbf{D}_n\mathbf{L}(k, \pi/n) = \mathbf{D}_n\mathbf{K}(|\,k\,|) = \mathbf{D}_n\mathbf{K}$ holds. According to this relationship and the relationship $\mathbf{D}_n\mathbf{K}{=}\mathbf{D}_n\mathbf{L}$, holding for

n - an odd natural number and $k < 0$, the types $\mathbf{D}_n\mathbf{K}$ (nmK) and $\mathbf{D}_n\mathbf{L}$ (nmL) are dual with respect to the change of the sign of the coefficient k. Owing to a rational angle of dilative rotation L, $\theta = \pi/n$, there are the polar radial rays — the axes of the dilatation $K(k^2)$, incident to the reflection lines. Therefore, it is possible to divide a similarity symmetry rosette with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) into the sectors of the dilatation. Enantiomorphic modifications do not exist.

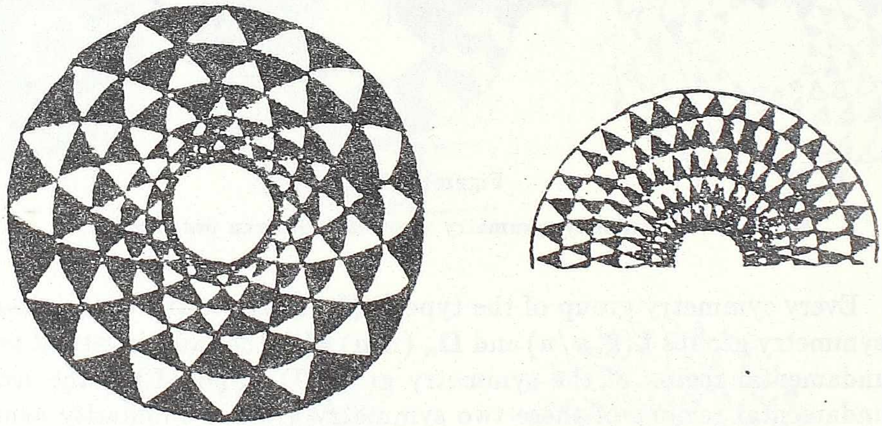


Figure 3.13

Examples of Roman floor mosaics with the similarity symmetry groups of the type $\mathbf{D}_n\mathbf{L}$ (nmL).

A similarity symmetry rosette with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) can be simply derived from a similarity symmetry rosette with the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK) by its "centering" — by rotating every second set of its fundamental regions, homologous regarding transformations of its subgroup $\mathbf{D}_n$ (nm), through the angle $\theta = \pi/n$. The symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) can be derived also by a desymmetrization of the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK). Since the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) is the subgroup of the index 2 of the symmetry group $\mathbf{D}_n\mathbf{K}$ (nmK), by using the antisymmetry desymmetrization, the antisymmetry group $\mathbf{D}_n\mathbf{K}/\mathbf{D}_n\mathbf{L}$ (nmK/nmL), treated by the classical theory of symmetry as the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL), can be obtained.

Similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL) possess a specific unity of visual dynamism and stationariness, produced, respectively, by the dynamic component — dilative rotation L — and by the static component — subgroup $\mathbf{D}_n$ (nm). The reflections of this subgroup cause the non-polarity of rotations and alleviate the dynamic visual effect produced by the dilative rotation L, which suggests an impression of centrifugal expansion. Changes of the shape of a fundamental region, which influences the visual impression, in similarity symmetry groups of the type $\mathbf{D}_n\mathbf{L}$ (nmL) are restricted to the possible use of curvilinear boundaries, which do not belong to reflection lines. The other boundaries must be rectilinear. In all the similarity symmetry groups containing dilative rotations, a degree of visual dynamism or stationariness can vary according to the choice of parameters k, θ... The spectrum of possibilities includes different varieties. This range from visually dynamic similarity symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{L}$ (nL), with an irrational angle of the dilative rotation L, to similarity symmetry rosettes with a rational angle, which offer a perception of the sectors of dilatation, through to static similarity symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{L}$ (nmL), with the coefficient of dilative rotation $k \approx 1$, which are, by their visual properties, similar to rosettes with the symmetry group $\mathbf{D}_n$ (nm).

The simplest among similarity symmetry groups of the type $\mathbf{C}_n\mathbf{M}$ (nM) (Figure 3.14, 3.15) is the symmetry group $\mathbf{M}(n = 1)$ generated by the dilative reflection M — a composite transformation representing the commutative product of a dilatation and reflection. A fundamental region of the symmetry group $\mathbf{M}$ is a part of the plane defined by two homologous lines of the dilative reflection M. The polar radial rays exist. Due to the presence of the indirect transformation — dilative reflection M — there is no the possibility of the enantiomorphism.

There are several ways to construct figures with the similarity symmetry group $\mathbf{M}$. They can be divided into non-metric constructions, based on the use of the non-metric properties of the dilatation K — parallelism or antiparallelism of homologous vectors of the dilatation K, equiformity, equiangularity and linking of its successive homologous asymmetric figures — and metric constructions, based on the use of the metric property of the dilatation K that is a constitutive part of the composite transformation $M(k,\mathrm{m})$. Such a construction always begins with the metric construction of a series of homologous asymmetric figures of the dilatation K. After that, it is necessary to copy by the reflection in the reflection line m, every second homologous figure mentioned. Combinations of these methods are also possible.

For the needs of ornamental art, probably the most efficient is the non-metric construction, consisting of the construction of a series of asymmetric figures that satisfy the dilatation $K(|k|)$, by applying the linking of successive homologous asymmetric figures of the dilatation K. After that, every second figure must be copied by the reflection with the reflection line m for $k > 0$, or by the reflection with the reflection line m' perpendicular to the reflection line m in the invariant point for $k < 0$. In line with this, when analyzing the similarity symmetry group M, it is not necessary to discern the cases of $k > 0$, $k < 0$.

By applying the metric construction method, aiming for maximal constructional simplicity, there frequently occur deviations from the requirements of similarity symmetry. In such a case, the geometric progression mentioned above, is replaced by an analogous arithmetic progression.

Since a dilative reflection is present in nature (e.g., in the arrangement and growth of leaves in certain plants), natural models are imitated by ornamental art. Therefore, the similarity symmetry group M appears even in Paleolithic ornamental art, although followed by deviations with respect to the geometric consistency. The other reason for the origin and the use of the similarity symmetry group M can be found in the visual effect and symbolic meanings which corresponding similarity symmetry rosettes possess. Owing to the polarity of the radial ray incident to the reflection line m and due to the dynamic visual properties of the dilative reflection, similar to that of a glide reflection, figures with the symmetry group M can serve as the visual symbols of oriented, polar alternating phenomena of a growing intensity. It is, probably, the origin and reason for the frequent occurrence of similarity symmetry rosettes with the symmetry group M in primitive art. They occurr independently, or within more complex similarity symmetry rosettes with a symmetry group of the type C_nM (nM) (Figure 3.14, 3.15). By varying the dilatation coefficient k and the angle between the reflection line m and the radial ray of the dilatation $K(k^2)$ which belongs to the symmetry group M and generates its subgroup of the index 2, it is possible to emphasize or alleviate the dynamic visual effect produced by the polar radial ray, which goes from suggesting an impression of dynamism, similar to that produced by a glide reflection, to an impression of stationariness similar to that produced by a reflection.

The use of the similarity symmetry group M in painting, comes to its full expression when presenting objects with the symmetry group 1g by applying the central perspective.

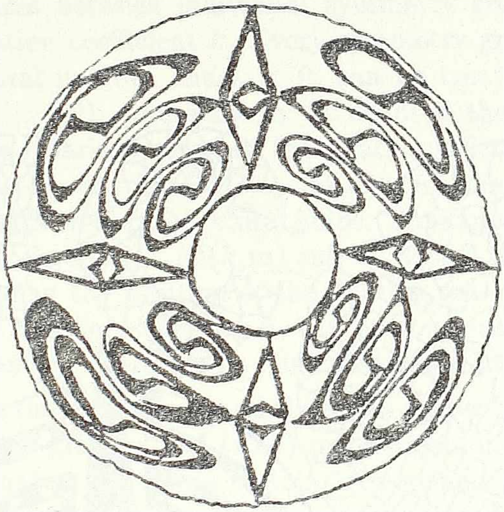


Figure 3.14

The rosette with the similarity symmetry group C_4M (4M) in the ornamental art of Oceania, Bali.

Similar characteristics of all the similarity symmetry groups of the type C_nM (nM) are conditioned by the essential properties of the similarity symmetry group M. Similarity symmetry groups of the type C_nM (nM) are the result obtained when composing the symmetry groups M and C_n (n) with the same invariant point. Similarity symmetry rosettes with the symmetry group C_nM (nM) can be constructed by multiplying by the n-fold rotation a figure with the similarity symmetry group M, belonging to a fundamental region of the symmetry group C_n (n), or multiplying by the dilative reflection M a figure with the symmetry group C_n (n), belonging to a fundamental region of the symmetry group C_n (n). In both cases the rosette center and the dilative reflection center must coincide. The application of the non-metric construction method, combined with the use of the linking of asymmetric homologous figures of the dilative reflection M, is also possible. With the use of the metric construction method there often occur deviations from the regularities of the similarity symmetry group M — the replacement of the geometric progression mentioned above with a corresponding arithmetic progression, the disturbance of equiformity and equiangularity, and, consequently, of the similarity symmetry. These deviations are the result aiming for maximal constructional simplicity.

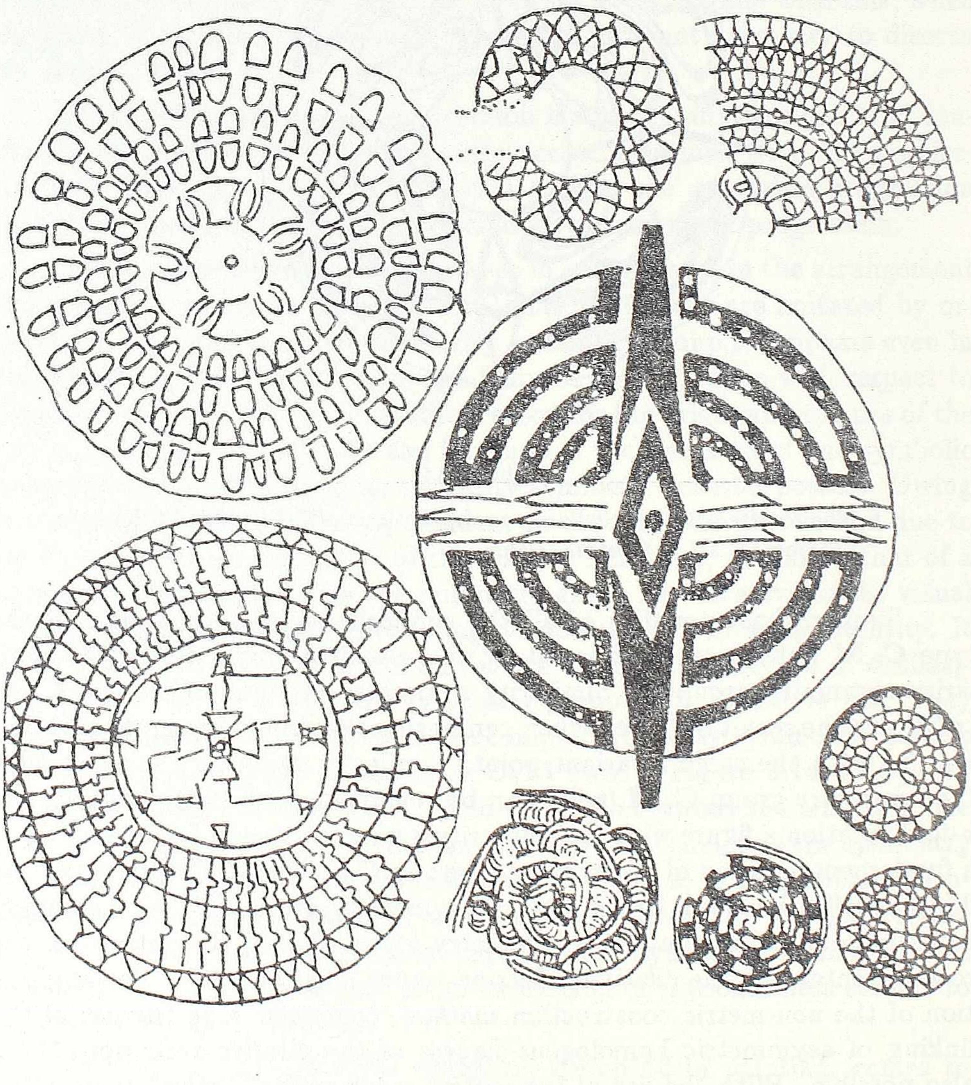


Figure 3.15

Examples of similarity symmetry rosettes in the ornamental art of Oceania (New Zealand, New Guinea, Solomon Islands).

A fundamental region of the symmetry group C_nM (nM) is the section of fundamental regions of the symmetry groups C_n (n) and M with the same invariant point. Within the type C_nM (nM), there will be no essential differences between individual symmetry groups, caused by the sign of the dilatation coefficient k. Every symmetry group $C_nM(k,m)$, for n - an odd natural number and $k < 0$, can be treated as the symmetry group $C_nM(|k|, m')$, where by m' is denoted the reflection line perpendicular in the invariant point of the dilative reflection $M(k,m)$ to the reflection line m. Hence, for n - an odd natural number and $k < 0$, the relationship $C_nM(k,m)=C_nM(k,m')$ holds. Similarity symmetry rosettes with the symmetry groups $C_nM(k,m)$ and $C_nM(-k,m)$ will differ between themselves regarding the position of the dilative reflection axis, only for n - an odd natural number, while for n - an even natural number, there will be no such difference. There are no enantiomorphic modifications.

Reasons for the appearance and the use of similarity symmetry rosettes with the symmetry group C_nM (nM) in ornamental art, can be found in the imitation of natural forms, in certain arrangements of leaves and in the growth of some plants, combined with a decorative effect of rosettes with the symmetry group C_n (n). Among all similarity symmetry rosettes, those with similarity symmetry groups of the type C_nM (nM) possess the maximal degree of visual dynamism, conditioned by two dynamic components — the n-fold rotation and dilative reflection M, which combines by itself the visual dynamism of alternating motion and that of centrifugal expansion, caused by its dilative component. The intensity of the dynamic visual effect can be influenced by choosing the parameter k and the position of the reflection line m.

The symmetry groups C_n (n) and M are relatively rare in ornamental art. The same refers to the similarity symmetry groups of the type C_nM (nM), formed as their compositions. A similarity symmetry group C_nM (nM) can be obtained also by a desymmetrization of the symmetry group D_nK (nmK) or D_nL (nmL), examples of which are, due to their higher degree of symmetry, visual and constructional simplicity, more frequent in ornamental art.

Desymmetrizations achieved by a dichromatic coloring often result in antisymmetry groups of the type D_nK/C_nM (nmK/nM) or D_nL/C_nM (nmL/nM), which in the classical theory of symmetry are considered within the type C_nM (nM). The same can be realized by suitable classical-symmetry desymmetrizations.

Similarity symmetry groups of the type D_nM (nmM), the existence of which was proposed by A.V. Shubnikov (1960), coincide to the similarity symmetry groups of the type D_nL (nm).

Among the continuous similarity symmetry groups of rosettes S_{20}, the symmetry groups $D_\infty K$ (∞mK) and C_nL_1 (nL_1) will have adequate visual interpretations, without using textures (Figure 3.5). As a visual model of the symmetry group $D_\infty K$ (∞mK), a series of concentric circles can be used — this model being obtained multiplying by the dilatation K two different concentric circles with the center incident with the dilatation center.

Adequate visual interpretations of all the other continuous similarity symmetry groups can be obtained only by using textures — the average even density of some elementary asymmetric figure, arranged according to the given continuous symmetry group. Concerning physical interpretations, all the continuous similarity symmetry groups have adequate interpretations, which can be realized by using physical factors (e.g., motion, rotation, the effect of a physical field, etc.).

* * *

The central problem with the similarity symmetry groups of rosettes S_{20} and their examples in ornamental art is the question of the construction of corresponding similarity symmetry rosettes. As basic construction methods, it is possible to distinguish, first, the non-metric method, based, directly or indirectly, on the parallelism of homologous asymmetric figures of a dilatation, their equiangularity and equiformity; second, the metric method, founded on the fact that the distances of homologous points of the similarity transformations K, L, M from the invariant point form a geometric progression; and, third, combinations of these methods.

The non-metric construction method gives the best results, guaranteeing that equiformity and equiangularity will be respected, and may serve for the direct construction of similarity symmetry rosettes of the type C_nK (nK) or D_nK (nmK) with the dilatation coefficient $k > 0$, by applying the linking of successive homologous asymmetric figures of the dilatation K or dilative reflection M (Figure 3.3, according to A.V. Shubnikov, 1960). In the other cases, when similarity symmetry transformations are composite transformations, this means, in symmetry groups of the types C_nK (nK), D_nK (nmK) with $k < 0$, and all the symmetry groups of the types C_nL (nL), D_nL (nmL), C_nM (nM), it is not possible to use exclusively linking and parallelism. In these cases, after the first part of the construction, the copying of successive homologous asymmetric figures of the dilatation K ($k > 0$) by means of corresponding rotations and reflections becomes indispensable, so the construction becomes more complicated. In those cases where is not possible to link homologous asymmetric figures of a dilatation, because of the complexity of construction, the non-metric method has a relatively limited application.

The metric construction method shows its superiority, in the sense of constructional simplicity, in those situations when the non-metric construction method is difficult to apply — with composite similarity transformations or with unlinked homologous asymmetric figures obtained by dilatation. A negative aspect of the metric construction method, coming to its expression in ornamental art, is the possibility for replacing the geometric progression mentioned above by a corresponding arithmetic progression, aiming for maximal constructional simplicity. Such an inconsistent application of the metric construction method unavoidably disturbs the equiangularity, equiformity, and consequently, the similarity symmetry.

In ornamental art we can find many similarity symmetry rosettes, formed by the inconsistent use of the construction methods mentioned. Such rosettes do not satisfy the similarity symmetry but only suggest it. In early ornamental art, this is not the exception but the rule.

Both construction methods mentioned are used to construct similarity symmetry rosettes, formed by applying the similarity transformations K, L, M on a rosette with the symmetry group C_n (n) or D_n (nm) belonging to a fundamental region of the corresponding similarity symmetry group K, L or M. An opposite approach — the multiplication of a figure with the similarity symmetry group K, L, M, by the symmetries of the symmetry group C_n (n) or D_n (nm) — is not so frequent. Such a construction requires a better understanding of the similarity symmetry, especially concerning the fundamental regions of the generating groups, to avoid the possible overlapping of figures. In all those cases, the generating symmetry group of rosettes and the similarity symmetry group possess the same invariant point.

The desymmetrization method is not an independent construction method. It can be used exclusively if we know similarity symmetry groups of rosettes with a higher degree of symmetry, which can be reduced to a lower degree of symmetry by the elimination of certain symmetry elements, to derive their similarity symmetry subgroups. However, since similarity symmetry groups of a higher degree of symmetry, due to the principle of visual entropy, are more frequent and much older, this construction method has been abundantly used in ornamental art, with the classical-symmetry, antisymmetry and color-symmetry desymmetrizations.

Like the antisymmetry and colored symmetry groups of rosettes, friezes and ornaments, such desymmetrizations of similarity symmetry groups are of a somewhat later date, appearing in ornamental art with dichromatic and polychromatic ceramics (in the Neolithic and in the period of the ancient civilizations). Classical-symmetry desymmetrizations can be used to derive similarity symmetry subgroups of the arbitrary index of the given similarity symmetry group. Desymmetrizations of the continuous visually presentable similarity symmetry groups of the type $D_{\infty}K$ (∞mK), which can be visually interpreted by a system of concentric circles, obtained from two different concentric circles multiplied by the dilatation K with the same center, frequently occur (Figure 3.5a). This continuous similarity symmetry group is a perfect basis on which to apply the desymmetrization method. Continuous similarity symmetry groups of the type C_nL_1 (nL_1) (Figure 3.5b, c) are based on the continuous visually presentable conformal symmetry group L_1Z_I, which can be visually interpreted by the corresponding logarithmic spiral. Therefore, they make possible a very simple transition from the visually presentable continuous, to the corresponding discrete similarity symmetry groups.

The classical-symmetry desymmetrization method can be very successfully applied on the similarity symmetry groups generated by the symmetry group D_n (nm), to obtain their subgroups, generated by the symmetry group C_n (n). Since the symmetry group C_n (n) is the subgroup of the index 2 of the symmetry group D_n (nm), there is a possibility for antisymmetry desymmetrizations.

More detailed information on possible desymmetrizations of similarity symmetry groups can be found in the table of the group-subgroup relations existing between different types of the similarity symmetry groups of rosettes, and in the tables of antisymmetry and color-symmetry desymmetrizations.

Since the continuous similarity symmetry groups $D_{\infty}K$ (∞mK) and C_nL_1 (nL_1) are visually presentable, very important are the group-subgroup relations between the continuous and discrete similarity symmetry groups of rosettes S_{20}: $D_{\infty}K \to DK$, DL, $CL_1 \to CL$. Between the different types of discrete similarity symmetry groups, the following relations hold: DL $\to$ DK $\to$ CM $\to$ CK, using the symbols D, C instead of the symbols D_n, C_n, for denoting the group-subgroup relations between the types, and not between the individual symmetry groups.

When establishing the group-subgroup relations between the individual similarity symmetry groups of rosettes S_{20} and their subgroups, we can use the group-subgroup relations existing between the symmetry groups

$\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm) and the group-subgroup relations between the symmetry groups K, L, M, since all the similarity symmetry groups of rosettes S_{20} are derived as the superpositions of the symmetry groups mentioned, i.e. as the extensions of the symmetry groups of rosettes G_{20}: $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm) by the similarity transformations K, L, M. For the discrete similarity symmetry groups K, L, M, the following relationships hold: $[\mathrm{K}:\mathrm{K}(k^m)] = m$, $[\mathrm{M}:\mathrm{M}(k^m)] = m$, $[\mathrm{L}:\mathrm{L}(k^m, k\theta)] = m$ $(m \in N)$. For a rational angle of the dilative rotation $\theta = p\pi/q, (p,q) = 1, p, q \in Z$, the following relationships hold: $[\mathrm{L}:\mathrm{L}(k^q, q\theta)] = q$ and $\mathrm{L}(k^q, q\theta) = \mathrm{L}((-1)^p k^q, 0) = \mathrm{K}((-1)^p k^q)$, showing that every symmetry group L with a rational angle of dilative rotation θ contains the subgroup generated by the dilatation $K((-1)^p k^q)$. The relationship $[\mathrm{M}:\mathrm{K}(k^2)] = 2$ highlights the existence of the subgroup of the index 2 generated by the dilatation $K(k^2)$ in every symmetry group M, while the relationship $\mathbf{C}_n\mathrm{K} = \mathbf{C}_n\mathrm{L}(k, 0) = \mathbf{C}_n\mathrm{L}_0$ highlights the different type possibilities for the symmetry groups $\mathbf{C}_n\mathrm{K}$ (nK). This means that they can be discussed within the type $\mathbf{C}_n\mathrm{L}$ (nL), as the subtype $\mathbf{C}_n\mathrm{L}_0$ (nL$_0$).

By accepting the criterion of subordination, by treating the symmetry group K within the type $\mathbf{C}_n\mathrm{L}$ (in accordance with the relationship $\mathrm{K} = \mathrm{L}_0$), and the type $\mathbf{C}_n\mathrm{K}$ (nK) as the subtype of the type $\mathbf{C}_n\mathrm{L}$ (nL) (in accordance with the relationship $\mathbf{C}_n\mathrm{K} = \mathbf{C}_n\mathrm{L}_0$), the whole discussion on the discrete similarity symmetry groups of rosettes can be reduced to the analysis of the symmetry groups of the types $\mathbf{C}_n\mathrm{L}$ (nL), $\mathbf{C}_n\mathrm{M}$ (nM), $\mathbf{D}_n\mathrm{K}$ (nmK) and $\mathbf{D}_n\mathrm{L}$ (nmL). The criterion of the maximal symmetry can be introduced even between individual symmetry groups of the type $\mathbf{C}_n\mathrm{L}$ (nL), where the symmetrization caused by a superposition of the n-fold rotation and the rotational component of the dilative rotation L results in the change of the minimal angle of the dilative rotation, and in the appearance of the new dilative rotation L', i.e. in the new symmetry group $\mathbf{C}_n\mathrm{L}'$ (nL').

In the table of antisymmetry desymmetrizations of discrete similarity symmetry groups of rosettes, the symbols of antisymmetry groups, i.e. the corresponding antisymmetry desymmetrizations, are given in the group/subgroup notation G/H. The symbol θ' corresponds to a newly derived minimal angle of the dilative rotation L'.

The table of antisymmetry desymmetrizations of similarity symmetry groups of rosettes S_{20}:

C_nK/C_nK	C_nM/C_nK
$C_{2n}K/C_nK$	$C_{2n}M/C_nM$
$C_{2n}K/C_nL_{2n}$	
	$C_nL/C_nL(k^2, 2\theta)$
D_nK/D_nK	$C_{2n}L/C_nL$
D_nK/C_nM	$C_nL/C_nL'(k, \theta')$
D_nK/C_nK	
$D_{2n}K/D_nK$	D_nL/D_nK
$D_{2n}K/C_nL_{2n}$	D_nL/C_nL_{2n}
	D_nL/C_nM

Besides the possibilities to apply the antisymmetry desymmetrization method, this table gives evidence for all the subgroups of the index 2 of any given discrete similarity symmetry group of rosettes. By using data given by A.M. Zamorzaev (1976), it is possible to compare similarity antisymmetry groups with the corresponding crystallographic antisymmetry groups of polar, oriented rods G_{31}. A complete catalogue of the similarity antisymmetry groups of rosettes S'_{20} is given by S.V. Jablan (1985).

The color-symmetry desymmetrizations of the discrete crystallographic similarity symmetry groups of rosettes can be partially considered by using the work of E.I. Galyarski (1970, 1974b), A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant (1978), and A.F. Palistrant (1980c).

Different problems of tiling theory (B. Grünbaum, G.C. Shephard, 1987) are extended to the similarity symmetry groups of rosettes S_{20} by E.A. Zamorzaeva (1979, 1984). In the works mentioned, a link is established between the similarity symmetry groups of rosettes S_{20}, the symmetry groups of polar oriented rods G_{31} and corresponding symmetry groups of ornaments G_2, resulting in the following relationships: C_nK (nK), C_nL (nL) $\simeq$ **p1**, C_nM (nM) $\simeq$ **pg**, D_nK (nmK) $\simeq$ **pm**, D_nL (nmL) $\simeq$ **cm**. In this way, different problems of similarity symmetry plane tilings are reduced to the much better known problems of tilings that correspond to the symmetry groups of ornaments **p1**, **pg**, **pm**, **cm**. By using such an approach, the problems of isohedral and 2-homeohedral similarity symmetry plane tilings are solved by E.A. Zamorzaeva.

The chronology of similarity symmetry rosettes in ornamental art is connected with the problem of their construction. The oldest examples of rosettes suggesting similarity symmetry date to the Paleolithic and Neolithic, beginning with the appearance of the first spiral forms in art (Figure 3.7), series of concentric circles or concentric squares with parallel sides, and motifs based on natural models with the similarity symmetry group

M or $\mathbf{D}_1\mathbf{K}$ (mK), etc. In the Neolithic we come across more diverse and complex examples of rosettes with similarity symmetry groups of the type $\mathbf{C}_n\mathbf{L}$ (nL) or $\mathbf{D}_n\mathbf{L}$ (nmL) (Figure 3.9). Already in the Neolithic and in the ornamental art of ancient civilizations, there are examples of all the types of similarity symmetry groups of rosettes. Though, almost unavoidably, there are deviations from geometric regularity, these being due to the approximate constructions used in ornamental art. Ornamental motifs with the application of similarity symmetry reached their peak in the ornamental art of Rome and Byzantium (Figure 3.11–3.13), mainly in floor mosaics. Here we find examples of all the types of the similarity symmetry groups of rosettes, without any deviations from strict geometric regularity.

One of the conditions necessary for the appearance of corresponding similarity symmetry rosettes in ornamental art is the existence of models in nature, i.e. a spiral tendency in nature, expressed through the way of growth of certain living beings or as a result of rotational motions (e.g., whirlpools in a turbulent fluid motion, etc.). In the earlier periods of ornamental art, it is possible to note the imitation of models in nature that possess similarity symmetry. In the further development of ornamental art, a visual-symbolic component based on a suggestion of the impression of centrifugal expansion, produced by similarity symmetry rosettes, became the main reason for the use of similarity symmetry. After empirically solving the construction problems and discovering all the symmetry possibilities, i.e. all the types of the similarity symmetry groups of rosettes, primary symbolic meanings retreated into a concern for decorativeness. That opened new possibilities for the enrichment and variety of similarity symmetry rosettes in ornamental art.

Like with the symmetry groups of rosettes G_{20}, where rosettes with the symmetry groups of the type $\mathbf{D}_n$ (nm) are more frequent than rosettes with the symmetry groups of the type $\mathbf{C}_n$ (n), the principle of visual entropy and numerous models in nature caused the dominance of rosettes with similarity symmetry groups of the type $\mathbf{D}_n\mathbf{K}$ (nmK), $\mathbf{D}_n\mathbf{L}$ (nmL), over those with similarity symmetry groups of the types $\mathbf{C}_n\mathbf{K}$ (nK), $\mathbf{C}_n\mathbf{M}$ (nM), $\mathbf{C}_n\mathbf{L}$ (nL). As generating symmetry groups of the type $\mathbf{D}_n$ (nm), most frequently are used symmetry groups of rosettes $\mathbf{D}_1$ (m), $\mathbf{D}_2$ (2m), $\mathbf{D}_4$ (4m), $\mathbf{D}_6$ (6m), etc., mainly with n – an even natural number. In such rosettes the incidence of reflection lines to the fundamental natural directions — vertical and horizontal line — is possible.

A fundamental region of similarity symmetry groups offers the variation and the use of curvilinear boundaries. Rectilinear must be only those parts of the boundaries of the fundamental region that coincide with reflection lines. By changing the form of a fundamental region we can influence the intensity of static or dynamic visual impression produced by the given similarity symmetry rosette and intensify desired visual impression. In all similarity symmetry rosettes, it is possible to realize the corresponding (unmarked) isohedral plane tilings.

A basic visual property of similarity symmetry rosettes is the impression of centrifugal expansion, which these rosettes render to the observer. The intensity of that impression will depend primarily on the value of the coefficient k, on the form of a fundamental region or an elementary asymmetric figure belonging to the fundamental region, where the adequate use of acuteangular forms may stress a dynamic effect of a dilatation, occurring as the independent or dependent symmetry transformation. Polar, oriented rotations existing in subgroups of the type C_n (n) play the role of visual dynamic symmetry elements. Dilative reflections have a double, contradictory role, since they cause the absence of the enantiomorphism in groups of the type C_nM (nM). On the other hand, they increase visual dynamism, by suggesting the impression of a centrifugal alternating expansion. By varying the parameter k and the position of the reflection line m, we can stress the visual static or dynamic function of the dilative reflection $M(k,m)$.

Enantiomorphic modifications do not exist in similarity symmetry groups of the types C_nM (nM), D_nK (nmK), D_nL (nmL), $D_\infty K$ (∞mK), i.e. in groups containing at least one indirect symmetry transformation. The presence of the dilatation K or $K(k^2)$ is obligatory in all the similarity symmetry groups of rosettes, except groups of the type C_nL (nL), which contain a dilatation only when the angle of the dilative rotation L is rational. Then is possible to perceive sectors of dilatation. Since the presence of a dilatation within the symmetry group C_nL (nL) increases the number of different symmetry transformations and simplifies the construction of corresponding rosettes, in line with the principle of visual entropy, similarity symmetry groups of the type C_nL (nL), offering a division of the corresponding similarity symmetry rosettes into sectors of dilatation, will be more frequent in ornamental art than groups of the type C_nL (nL) with an irrational angle of the dilative rotation L.

Because of a high degree of symmetry and the possibility for the simple construction of their corresponding visual interpretations by desymmetrizations of groups of the type $D_\infty K$ (∞mK), of special interest will be groups of the types D_nK (nmK) or D_nL (nmL). According to the principle of visual entropy, similarity symmetry groups generated by the symmetry groups of rosettes of the type D_n (nm), for n=1,2,3,4,6,8,12,..., are the oldest and most frequent in ornamental art. In visual interpretations of the derived similarity symmetry groups of rosettes a dynamic visual component — the suggestion of a centrifugal expansion conditioned by dilatation — is in visual balance with the static component produced by reflections. The result is non-polarity of rotations and absence of the enantiomorphism. On the other hand, in the older ornamental art and that of primitive people, visually dynamic rosettes with similarity symmetry groups of the types $\mathbf{C}_n\mathbf{K}$ (nK), $\mathbf{C}_n\mathbf{L}$ (nL), $\mathbf{C}_n\mathbf{M}$ (nM), $\mathbf{C}_n\mathbf{L}_1$ (nL$_1$), with polar rotations and dilative reflections, are very frequent. Their abundant use in ornamental art, is due to their symbolic function.

Besides serving as a basis for the application of the desymmetrization method, the tables of the group-subgroup relations between the types of similarity symmetry groups or between the individual groups are, at the same time, an indicator of symmetry substructures of a given similarity symmetry group. They represent the groundwork for the exact registering of the subentities mentioned, which with an empirical visual-perceptive approach is sometimes very difficult. The surveys given consist of a series of inclusion relations beginning with the maximal visually presentable continuous similarity symmetry groups of the types $\mathbf{D}_\infty\mathbf{K}$ (∞mK) and $\mathbf{C}_n\mathbf{L}_1$ (nL$_1$), including all discrete similarity symmetry groups and ending with the symmetry groups of rosettes $\mathbf{D}_n$ (nm) and $\mathbf{C}_n$ (n) and their subgroups. When discussing continuous similarity symmetry groups, only the visually presentable groups are considered, since ornamental art imposes this restriction. Visually non-presentable similarity symmetry groups will have their physical interpretations, owing to the possibility of including physical desymmetrization factors (e.g., a uniform rotation of a rosette with the similarity symmetry group $\mathbf{D}_\infty\mathbf{K}$ (∞mK) around the invariant point, when its symmetry group is reduced to the symmetry group $\mathbf{C}_\infty\mathbf{K}$ (∞K), or by using similar methods). In ornamental art, visual presentations of such continuous similarity symmetry groups can be obtained by using textures. As physical interpretations of these groups, we may consider different similarity symmetry structures realized by means of a physical field with a singular point, the intensity of which depends on distance from the singular point, according to the requirements of the similarity symmetry.

In analyzing the visual properties of similarity symmetry groups we can use, very efficiently, their visual interpretations: similarity symmetry rosettes, tables of the graphic symbols of symmetry elements and Cayley diagrams. Owing to the existence of the isomorphism between the similarity symmetry groups of rosettes S_{20} and the symmetry groups of polar, oriented rods G_{31}, the properties of the similarity symmetry groups of rosettes S_{20}, the characteristics of similarity transformations and relations which are included in their presentations, will be, sometimes, more evident in the symmetry groups of polar, oriented rods G_{31}. The symmetry groups of rods G_{31}, that in the isomorphism mentioned correspond to the similarity symmetry groups of rosettes S_{20}, possess the same presentations and geometric characteristics. By analyzing the symmetry groups of rods G_{31}, the conclusion on the absence of the type $\mathbf{D}_n\mathbf{M}$ (nmM) of the similarity symmetry groups of rosettes and its reduction to the type $\mathbf{D}_n\mathbf{L}$ (nmL), becomes absolutely clear. The same is proved by the table of the symmetry groups of rods G_{31} (A.V.Shubnikov, V.A.Koptsik, 1974) in which, because of the justification already given, there is no individual type $(\mathbf{a})(2\mathbf{n})_n\tilde{\mathbf{a}}$, consisting of groups isomorphic to similarity symmetry groups of the type $\mathbf{D}_n\mathbf{M}$ (nmM). These symmetry groups of rods are included in the type $(\mathbf{a})(2\mathbf{n})_n\mathbf{m}$, consisting of groups isomorphic to the similarity symmetry groups of the type $\mathbf{D}_n\mathbf{L}$ (nmL).

The problem of plane symmetry groups isomorphic to the symmetry groups of non-polar rods G_{31} is solved in the theory of conformal symmetry introduced by A.M. Zamorzaev, E.I. Galyarski and A.F. Palistrant (1978), in the Euclidean plane with a singular point O removed, i.e. in the plane $E^2 \setminus \{O\}$.

All the other problems in the field of visual interpretations of the similarity symmetry groups of rosettes S_{20} — "objective" and "subjective" symmetry, problems of perceiving the objective symmetry and eliminating other visual symmetry factors, desymmetrizations or symmetrizations caused by physiological-physical reasons, the effect of the principle of visual entropy, problems of visual perception of substructures, treatment of symmetry groups of "real" similarity symmetry rosettes as finite factor groups of "ideal", infinite similarity symmetry groups of rosettes, etc. — can be discussed analogously to the similar problems of visual perception previously analyzed with the symmetry groups of rosettes G_{20}, friezes G_{21} and ornaments G_2.

The chronological parallelism and the use of similar construction methods in ornamental art and the theory of similarity symmetry, the more profound connection between the similarity symmetry groups used in ornamental art and the theory of symmetry, the possibility of a different approach to ornaments treating them as models of geometric-algebraic structures and many other similar questions, are some of the problems raised in this work that demand a more detailed study.

Chapter 4

CONFORMAL SYMMETRY IN $E^2 \setminus \{O\}$

A group of conformal symmetry in $E^2 \setminus \{O\}$ is a group of conformal transformations of the plane $E^2 \setminus \{O\}$ (H.S.M. Coxeter, 1969; A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant, 1978) at least one of which is not a similarity transformation. According to the relationships: $C_{20} \subset C_{21}$, $C_{20} = C_{210}$, any discussion on the conformal symmetry groups in $E^2 \setminus \{O\}$ can be reduced to the analysis of the two categories of conformal symmetry groups — C_{21} and C_2.

4.1. Conformal Symmetry Groups in $E^2 \setminus \{O\}$

The idea of conformal symmetry was given in the monograph *Colored Symmetry, its Generalizations and Applications* by A.M. Zamorzaev, E.I. Galyarski and A.F. Palistrant (1978) as a generalization of similarity symmetry. Along with it, there was established the isomorphism between the symmetry groups of non-polar rods G_{31} and the conformal symmetry groups of the category C_2. The isomorphism between the similarity symmetry groups S_{20} and the symmetry groups of polar rods G_{31} is pointed out within the analysis of the similarity symmetry groups in E^2 (Chapter 3). According to the same isomorphism, the category C_2 consists of the conformal symmetry groups isomorphic to the symmetry groups of non-polar rods G_{31}.

In conformal symmetry groups of the category C_2, besides the isometries and similarity transformations, which are the elements of the similarity symmetry groups S_{20}, there will be, as basic transformations, three more conformal symmetry transformations: inversion R_I in a circle with the center in the singular point O of the plane E^2, inversional reflection Z_I and inversional rotation S_I. An inversion circle can be denoted by $\mathbf{m}_I$, inversional reflection center by $\mathbf{2}_I$ and inversional rotation center by $\mathbf{n}_I$.

According to the isomorphism existing between the types of the discrete symmetry groups of non-polar rods G_{31} (A.V. Shubnikov, V.A. Koptsik, 1974) and the corresponding types of discrete conformal symmetry groups of the category C_2, it is possible to conclude that there will be ten types of discrete conformal symmetry groups of the category C_2.

Isomorphism between discrete symmetry groups of non-polar rods G_{31} and discrete conformal symmetry groups of category C_2:

$\mathbf{KN}_I$	$(a)(\widetilde{2n})$	$\mathbf{MN}_I$	$(a)(\widetilde{2n})\tilde{a}$
$\mathbf{KC}_n\mathbf{R}_I$	(a)n:m	$\mathbf{KD}_n\mathbf{R}_I$	(a)mn:m
$\mathbf{KC}_n\mathbf{Z}_I$	(a)n:2	$\mathbf{MC}_n\mathbf{R}_I$	$(a)\tilde{a}n$:m
$\mathbf{LC}_n\mathbf{Z}_I$	(a_t)n:2	$\mathbf{LC}_n\mathbf{R}_I$	$(a)(2n)_n$:m
$\mathbf{KD}_1\mathbf{N}_I$	$(a)(\widetilde{2n})$m	$\mathbf{LD}_n\mathbf{R}_I$	$(a)m(2n)_n$:m

Besides conformal symmetry groups of the category C_2, there exist five types of discrete conformal symmetry groups of the category C_{21}, isomorphic to the five existing types of the discrete symmetry groups of tablets G_{320} (A.V. Shubnikov, V.A. Koptsik, 1974): $\mathbf{C}_n\mathbf{Z}_I \simeq$n:2, $\mathbf{C}_n\mathbf{R}_I \simeq$n:m, $\mathbf{N}_I \simeq (\widetilde{2n})$, $\mathbf{D}_1\mathbf{N}_I \simeq (\widetilde{2n})$m and $\mathbf{D}_n\mathbf{R}_I \simeq$mn:m.

Those isomorphisms make possible the defining of the continuous conformal symmetry groups of the categories C_2 and C_{21}, which in these isomorphisms correspond, respectively, to the continuous symmetry groups of non-polar rods G_{31} and tablets G_{320}.

Owing to the existence of a singular point, any figure the symmetry group of which is a conformal symmetry group is called a conformal symmetry rosette. All the conformal symmetry groups of the category C_2 can be derived as the extensions of the conformal symmetry groups of the category C_{21} by the similarity transformations K, L, M. Therefore, every conformal symmetry rosette with a conformal symmetry group of the category C_2 can be constructed, multiplying by those similarity transformations a conformal symmetry rosette with the corresponding conformal symmetry group of the category C_{21}. Developing our analysis further we will consider first the conformal symmetry groups of the category C_{21}. They will be used to generate conformal symmetry groups of the category C_2. An analogous approach to the symmetry groups of rods G_{31} treated as extensions of the symmetry groups of tablets G_{320}, is given by A.V. Shubnikov and V.A. Koptsik (1974).

According to the criterion of subordination, for the conformal symmetry groups of the category C_2, it is possible to consider a certain type as the subtype of the more general type (e.g., the type $\mathrm{KC}_n\mathrm{Z}_I$ as the subtype of the type $\mathrm{LC}_n\mathrm{Z}_I$, according to the relationship $K = L(k, 0) = L_0$). That and similar problems can be solved in the same way as with the similarity symmetry groups S_{20}.

Presentations and structures of conformal symmetry groups of category C_{21}:

N_I $\quad \{S_I\} \quad S_I^{2n} = E \quad C_{2n}$

$\mathrm{D}_1\mathrm{N}_I$ $\quad \{S_I, R_I\} \quad S_I^{2n} = E \quad R_I^2 = E \quad R^2 = E \quad RS_I = S_I R \quad C_{2n} \times D_1$

$\mathrm{C}_n\mathrm{R}_I$ $\quad \{S, R_I\} \quad S^n = E \quad SR_I = R_I S \quad C_n \times D_1$

$\mathrm{C}_n\mathrm{Z}_I$ $\quad \{S, Z_I\} \quad S^n = Z_I^2 = (SZ_I)^2 = E \quad D_n$

$\{Z_I, Z_I'\} \quad Z_I^2 = Z_I'^2 = (Z_I Z_I')^n = E$

$\mathrm{D}_n\mathrm{R}_I$ $\quad \{S, R, R_I\} \quad S^n = R^2 = (SR)^2 = E \quad R_I^2 = E$

$SR_I = R_I S \quad RR_I = R_I R \quad D_n \times D_1$

$\{R, R_1, R_I\} \quad R^2 = R_1^2 = (RR_1)^n = E \quad R_I^2 = E$

$RR_I = R_I R \quad R_1 R_I = R_I R_1$

Presentations of conformal symmetry groups of category C_2:

KN_I $\quad \{K, S_I\} \quad S_I^{2n} = E \quad KS_I K = S_I$

MN_I $\quad \{M, S_I\} \quad S_I^{2n} = E \quad (MS_I)^2 = E$

$\mathrm{KD}_1\mathrm{N}_I$ $\quad \{K, S_I, R\} \quad S_I^{2n} = R^2 = (RS_I)^2 = E \quad KR = RK \quad (KS_I)^{2n} = E$

$\mathrm{KC}_n\mathrm{R}_I$ $\quad \{K, S, R_I\} \quad S^n = R_I^2 = E \quad SR_I = R_I S \quad KS = SK \quad (KR_I)^2 = E$

$\mathrm{KC}_n\mathrm{Z}_I$ $\quad \{K, S, Z_I\} \quad S^n = Z_I^2 = (SZ_I)^2 = E \quad KS = SK \quad (KZ_I)^2 = E$

$\mathrm{LC}_n\mathrm{R}_I$ $\quad \{L, S, R_I\} \quad S^n = R_I^2 = E \quad SR_I = R_I S \quad SL = LS$

$LR_I LR_I = R_I LR_I L = S \quad (L = L_{2n} = L(k, \pi/n))$

$\mathrm{LC}_n\mathrm{Z}_I$ $\quad \{L, S, Z_I\} \quad S^n = Z_I^2 = (SZ_I)^2 = E \quad SL = LS \quad (LZ_I)^2 = E$

$\mathrm{MC}_n\mathrm{R}_I$ $\quad \{M, S, R_I\} \quad S^n = R_I^2 = E \quad SR_I = R_I S \quad SMS = M \quad (MR_I)^2 = E$

$\mathrm{KD}_n\mathrm{R}_I$ $\quad \{K, S, R, R_I\} \quad S^n = R^2 = (SR)^2 = E \quad R_I^2 = E \quad SR_I = R_I S$

$RR_I = R_I R \quad KS = SK \quad KR = RK \quad (KR_I)^2 = E$

$\mathrm{LD}_n\mathrm{R}_I$ $\quad \{L, S, R, R_I\} \quad S^n = R^2 = (SR)^2 = E \quad R_I^2 = E \quad SR_I = R_I S$

$RR_I = R_I R \quad LS = LS \quad LRLR = RLRL \quad LR_I LR_I = R_I LR_I L = S \quad RLR = LS$

$(L = L_{2n} = L(k, \pi/n))$

Besides these presentations of conformal symmetry C_2, it is possible to have presentations with a different choice of generators. For example, groups of the types $\mathbf{KC}_n\mathbf{R}_I$, $\mathbf{KC}_n\mathbf{Z}_I$, $\mathbf{KD}_n\mathbf{R}_I$ offer the following possibilities:

$\mathbf{KC}_n\mathbf{R}_I$ $\{S, R_I, R_I'\}$ $S^n = R_I^2 = R_I'^2 = E$ $SR_I = R_IS$ $SR_I' = R_I'S$ $(R_I' = R_IK)$

$\mathbf{KC}_n\mathbf{Z}_I$ $\{S, Z_I, Z_I'\}$ $S^n = Z_I^2 = Z_I'^2 = E$ $(SZ_I)^2 = (SZ_I')^2 = E$ $(Z_I' = Z_IK)$

$\mathbf{KD}_n\mathbf{R}_I$ $\{S, R, R_I, R_I'\}$ $S^n = R^2 = (SR)^2 = E$ $SR_I = R_IS$ $SR_I' = R_I'S$

$R_I^2 = R_I'^2 = E$ $RR_I = R_IR$ $RR_I' = R_I'R$ $(R_I' = R_IK)$

Similar possibilities exist in all the other conformal symmetry groups. Besides indicating the different presentation possibilities, certain choices of generators and corresponding presentations make possible a direct recognition of the structure of a group considered (e.g., the structure of groups of the type $\mathbf{KC}_n\mathbf{R}_I$ is $D_\infty \times C_n$, of groups of the type $\mathbf{KD}_n\mathbf{R}_I$ is $D_\infty \times D_n$, etc.).

Enantiomorphism: the enantiomorphism exists in the conformal symmetry groups C_{21} of the type $\mathbf{C}_n\mathbf{Z}_I$ type and conformal symmetry groups of the category C_2 of the types $\mathbf{KC}_n\mathbf{Z}_I$ and $\mathbf{LC}_n\mathbf{R}_I$.

Polarity of rotations: polar rotations — $\mathbf{N}_I$, $\mathbf{BC}_n\mathbf{R}_I$, $\mathbf{KN}_I$, $\mathbf{KC}_n\mathbf{R}_I$, $\mathbf{LC}_n\mathbf{R}_I$;
bipolar rotations — $\mathbf{C}_n\mathbf{Z}_I$, $\mathbf{MN}_I$, $\mathbf{MC}_n\mathbf{R}_I$, $\mathbf{KC}_n\mathbf{Z}_I$, $\mathbf{LC}_n\mathbf{Z}_I$;
non-polar rotations — $\mathbf{D}_1\mathbf{N}_I$, $\mathbf{D}_n\mathbf{R}_I$, $\mathbf{KD}_1\mathbf{N}_I$, $\mathbf{KD}_n\mathbf{R}_I$, $\mathbf{L}_n\mathbf{R}_I$.

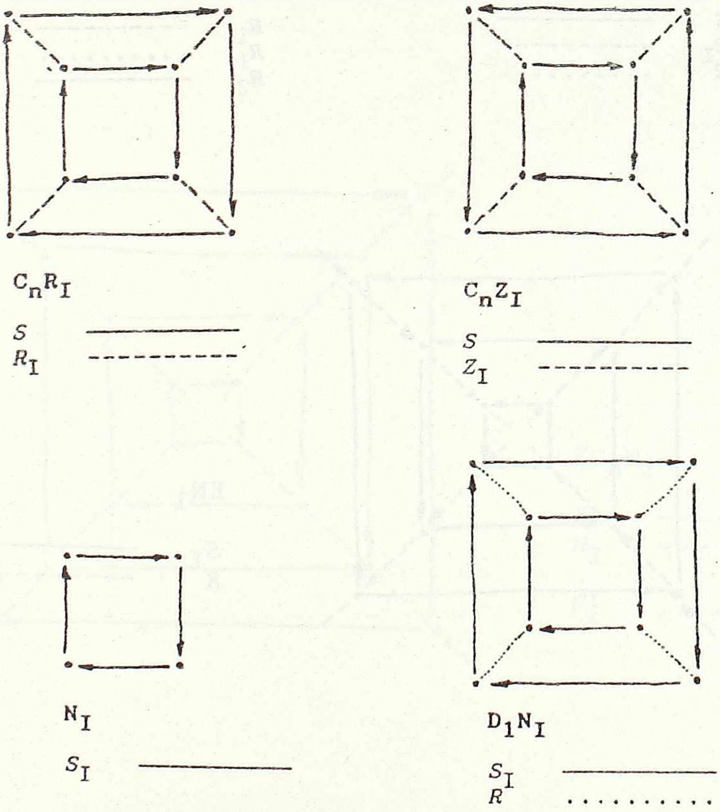
Polarity of radial rays: polar — $\mathbf{KN}_I$;
bipolar — $\mathbf{MN}_I$, $\mathbf{KD}_1\mathbf{N}_I$, $\mathbf{KC}_n\mathbf{Z}_I$, $\mathbf{LC}_n\mathbf{Z}_I$ (if they exist, i.e. if the dilative rotation angle is a rational one);
non-polar — $\mathbf{KC}_n\mathbf{R}_I$, $\mathbf{KD}_n\mathbf{R}_I$, $\mathbf{MC}_n\mathbf{R}_I$, $\mathbf{LC}_n\mathbf{R}_I$, $\mathbf{LD}_n\mathbf{R}_I$.

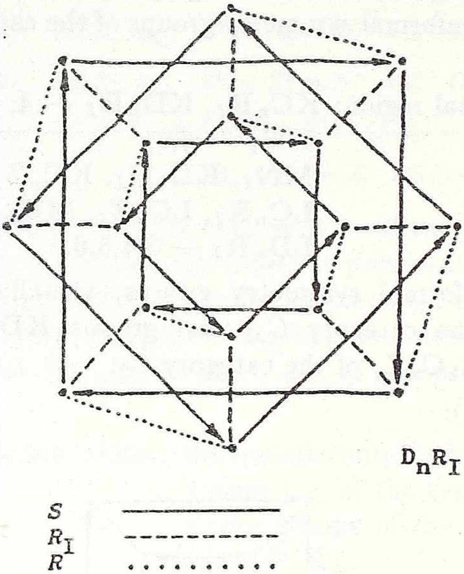
Form of the fundamental region: unbounded in the conformal symmetry groups C_{21}; bounded in conformal symmetry groups of the category C_2. There exists the possibility of varying the shape of boundaries that do not belong to reflection lines or inversion circles. In conformal symmetry groups of the category C_{21}, variation of all the boundaries is possible in groups of the types $\mathbf{N}_I$, $\mathbf{C}_n\mathbf{Z}_I$, of non-reflectional boundaries in groups of the type $\mathbf{D}_1\mathbf{N}_I$, of non-inversional boundaries in groups of the type $\mathbf{C}_n\mathbf{R}_I$, while in groups of the type $\mathbf{D}_n\mathbf{R}_I$, it is not possible at all. Regaring the changes of the form of a fundamental region, conformal symmetry groups of the category C_2 offer similar possibilities as generating conformal symmetry groups of the category C_{21}.

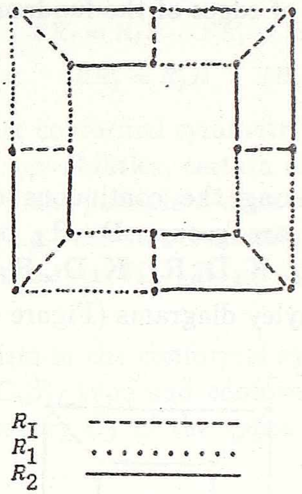
Number of edges of the fundamental region: $\mathbf{KC}_n\mathbf{R}_I$, $\mathbf{KD}_n\mathbf{R}_I$ — 4;
$\mathbf{KN}_I$ — 4,6;
$\mathbf{MN}_I$, $\mathbf{KD}_1\mathbf{N}_I$, $\mathbf{KC}_n\mathbf{Z}_I$,
$\mathbf{LC}_n\mathbf{R}_I$, $\mathbf{LC}_n\mathbf{Z}_I$, $\mathbf{MC}_n\mathbf{R}_I$,
$\mathbf{LD}_n\mathbf{R}_I$ — 3,4,5,6.

Among the continuous conformal symmetry groups, visually presentable are groups $\mathbf{D}_\infty\mathbf{R}_I$ of the category C_{21} and groups $\mathbf{KD}_\infty\mathbf{R}_I$, $\mathbf{K}_1\mathbf{C}_n\mathbf{R}_I$, $\mathbf{K}_1\mathbf{D}_n\mathbf{R}_I$, $\mathbf{K}_1\mathbf{D}_\infty\mathbf{R}_I$, $\mathbf{L}_1\mathbf{C}_n\mathbf{Z}_I$ of the category C_2.

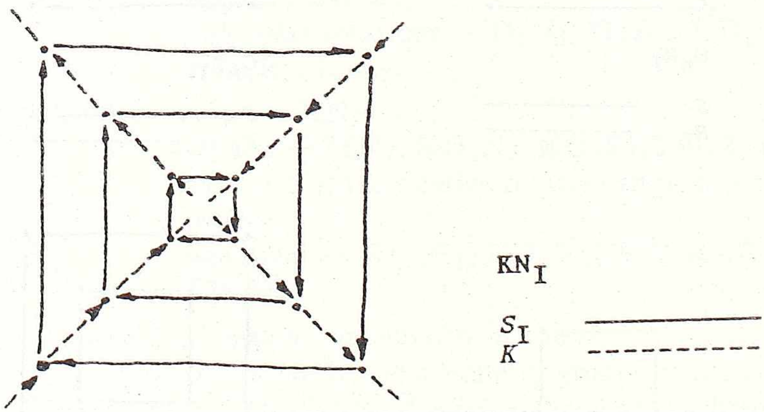
Cayley diagrams (Figure 4.1):



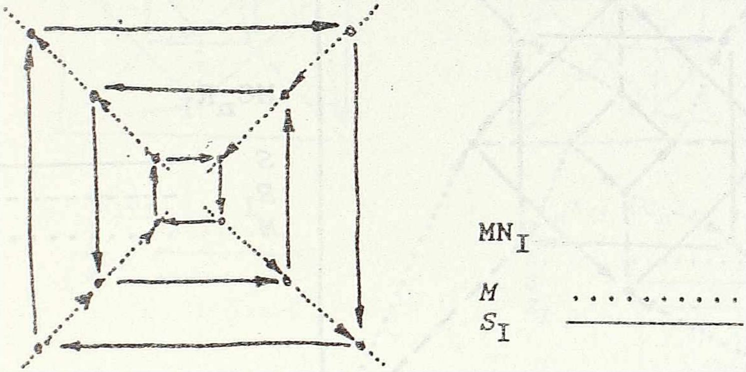


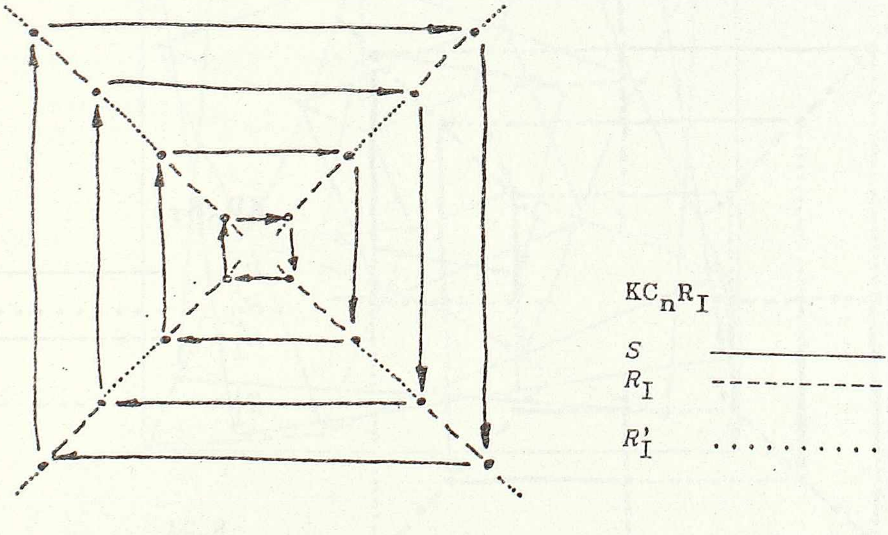


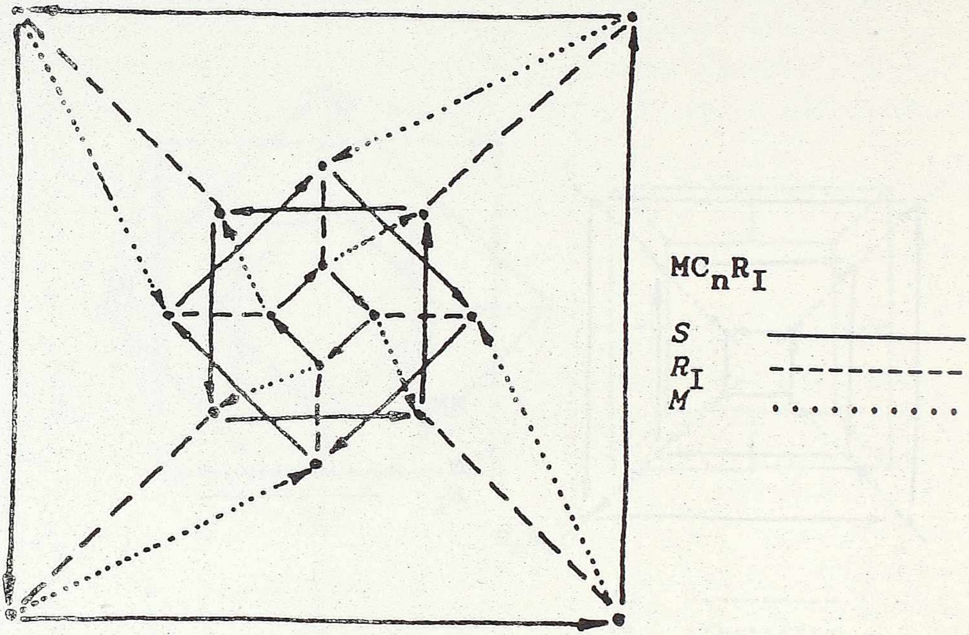
D_nR_I

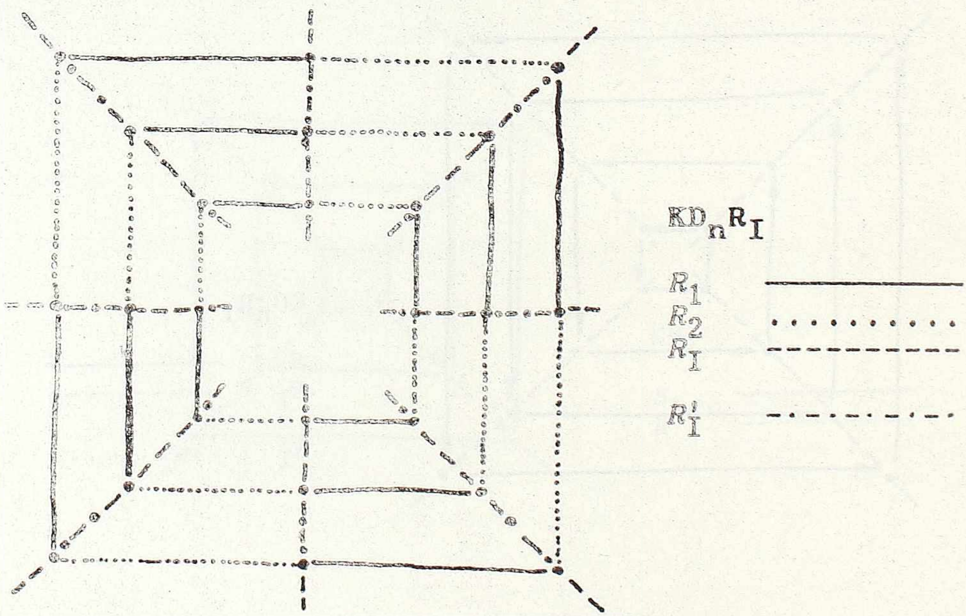


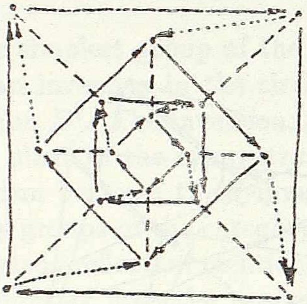
KN_I


MN_I
M
S_I


KC_nR_I
S
R_I
R'_I


MC_nR_I
S
R_I
M


KD_nR_I
R_1
R_2
R_I
R_I'

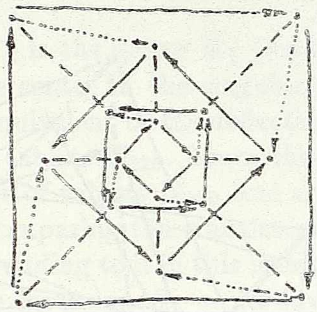


KD_1N_I

K

S_I

R

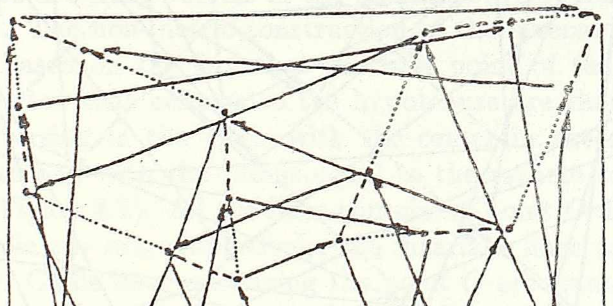


KC_nZ_I, LC_nZ_I

K, L

S

Z_I



LC_nR_I

S

R_I

L

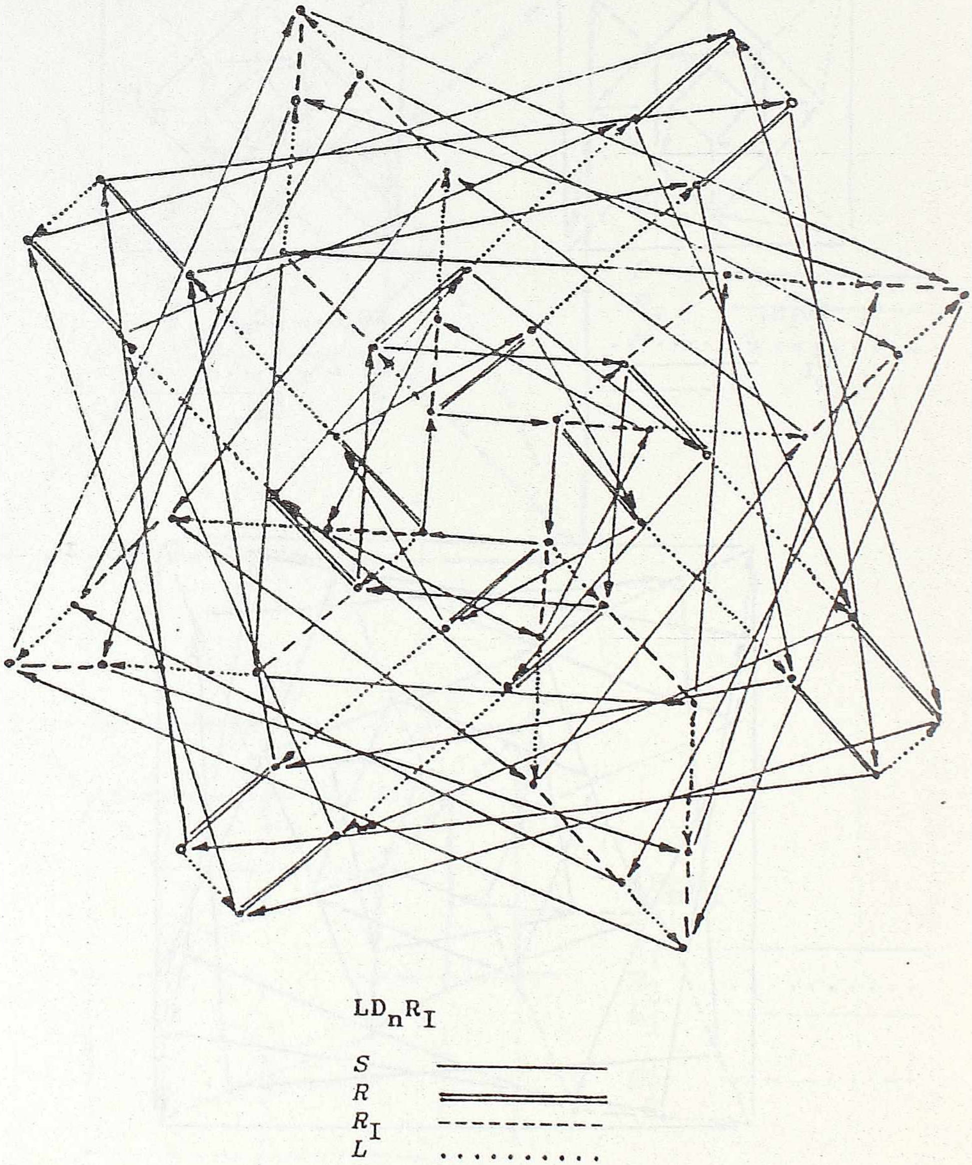


Figure 4.1

4.2. Conformal Symmetry Rosettes and Ornamental Art

The simplest group of the type $\mathbf{C}_n\mathbf{R}_I$ is the group $\mathbf{R}_I$ (n=1) generated by an inversion in the circle with the center in the singular point O of the plane E^2. The inversion R_I is the equivalent of the reflection in the invariant plane of the symmetry groups of tablets G_{320}. From the existing isomorphism between the symmetry groups of tablets G_{320} and conformal symmetry groups of the category C_{21}, by comparing the transformation R_I with the plane reflection mentioned, corresponding to it in this isomorphism, one can directly note the properties of the inversion R_I — its involutionality and the relations between the inversion R_I and the other conformal symmetry transformations.

An inversion transforms a point A of the plane $E^2\setminus\{O\}$ onto the point A' of the same plane, where the relationship $\overrightarrow{OA} \circ \overrightarrow{OA'} = r^2$ holds, and r is the length of a radius of the inversion circle. An important characteristic of the inversion R_I is the property of equiangularity — the maintenance of the angle between two arbitrary vectors in the plane $E^2 \setminus \{O\}$, transformed by the inversion R_I. The non-metric construction of the inverse point $R_I(A)$ of a point A is based on the fact that the base point of the hypotenuse altitude and a vertex that belongs to the hypotenuse are the homologous points of the inversion in the circle with the center in the other vertex of the hypotenuse and with the radius equal to the cathete to which this vertex belongs (Figure 4.2). All the lines containing point O and points of the inversion circle $\mathbf{m}_I$ are, respectively, the invariant lines and points of the inversion R_I. Circle lines containing the point O are transformed onto the lines that do not contain the point O, circle lines that do not contain the point O are transformed onto the circle lines that do not contain the point O, while to all the lines not containing the point O correspond the circle lines containing the point O. Hence, the following relationships hold: $R_I(A) = A \iff A \in \mathbf{m}_I$, $R_I(l) = l \iff O \in l$, $R_I(l) = c \iff O \notin l$ and $O \in c$, $R_I(c) = c_1 \iff O \notin c$ and $O \notin c_1$.

Every circle perpendicular to the inversion circle $\mathbf{m}_I$ is transformed by the inversion R_I onto itself and represents its invariant, so the relationship $R_I(c) = c \iff c \perp \mathbf{m}_I$ holds. All the constructions in which an inversion takes part can be considerably simplified by using those invariance relations — the invariance of all the points of the inversion circle $\mathbf{m}_I$, of lines containing the singular point O and of circles perpendicular to the inversion circle $\mathbf{m}_I$. Very important for the simplification of constructions is the fact that every circle line containing the point O and touching the inversion circle $\mathbf{m}_I$, is transformed onto the tangent line of the circle $\mathbf{m}_I$ in the touch point and vice versa, and also every secant of the inversion circle m_I is transformed onto the circle line containing the singular point O and the intersection points of the secant and inversion circle. The reverse transformations also hold. Owing to those characteristics of the non-metric construction of inverse figures, according to the criterion of maximal constructional simplicity, the metric construction was rarely used.

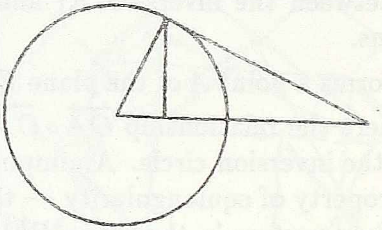


Figure 4.2
The construction of homologous points of the inversion R_I.

Every conformal symmetry group of the type C_nR_I is the direct product of the symmetry groups C_n (n) and R_I. Hence, visual interpretations of conformal symmetry groups of the type C_nR_I (Figure 4.3, 4.4) can be constructed multiplying by the inversion R_I a rosette with the symmetry group C_n (n), belonging to a fundamental region of the group R_I, or multiplying by the n-fold rotation a figure with the symmetry group R_I, belonging to the fundamental region of the group C_n (n). A fundamental region of the group C_nR_I is the section of the fundamental regions of the groups C_n (n) and R_I. Owing to the presence of the indirect symmetry transformation — inversion R_I — in groups of the type C_nR_I enantiomorphic modifications do not occur. The visual effect and degree of visual dynamism in rosettes with the conformal symmetry group C_nR_I depend exclusively on the choice of the form of a fundamental region, or on the position and form of an elementary asymmetric figure within a fundamental region. A degree of visual dynamism goes from conformal symmetry rosettes alike to rosettes with the symmetry group C_n (n) (Figure 4.3), to the conformal symmetry rosettes with a fully expressed stationary visual component resulting from

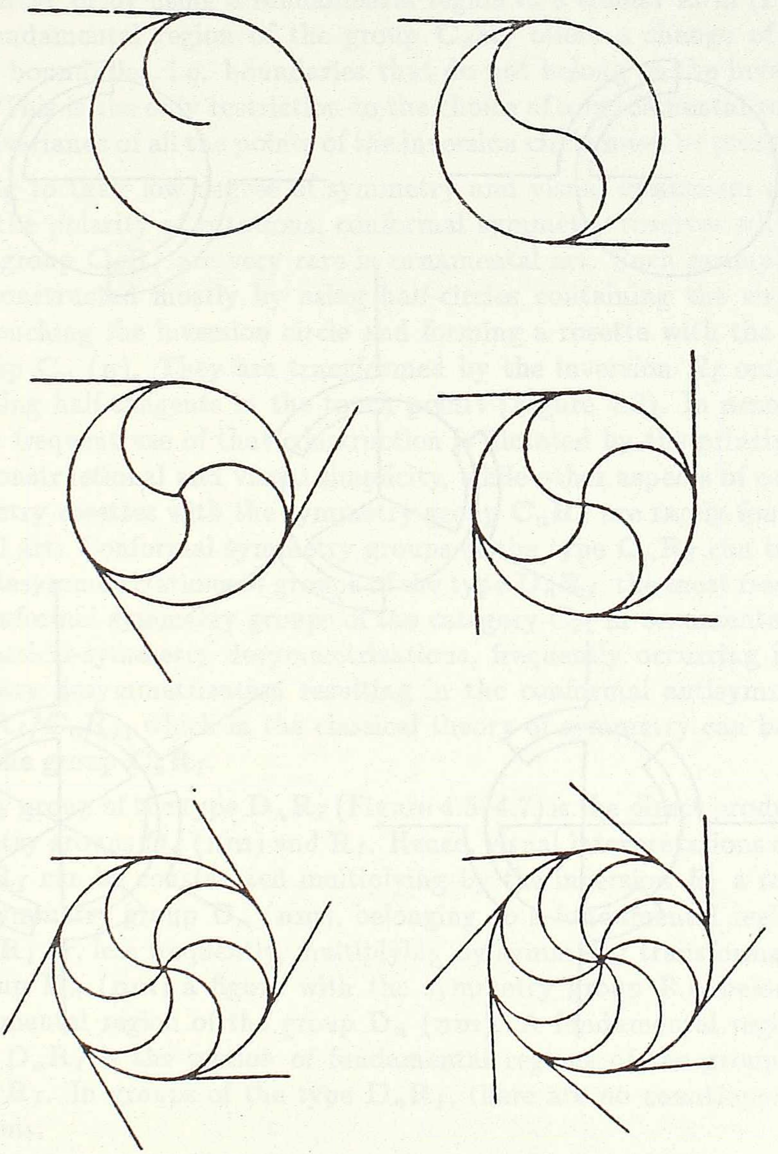


Figure 4.3

Conformal symmetry rosettes with the symmetry groups of the type $C_n R_I$, which satisfy the principle of maximal constructional simplicity.

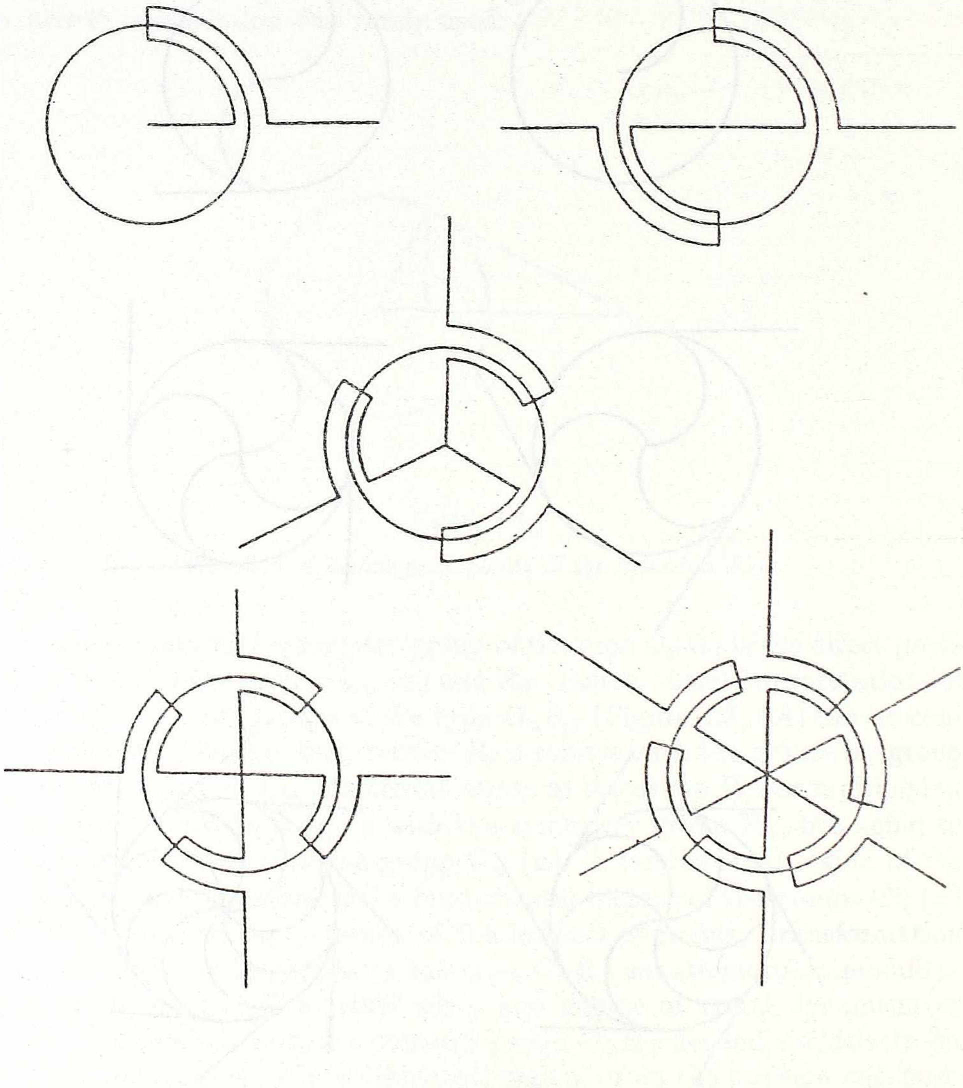


Figure 4.4

Examples of conformal symmetry rosettes with the symmetry groups of the type $\mathbf{C}_n\mathbf{R}_I$, with dominant static visual component produced by the inversion R_I.

the visual effect of the inversion R_I, similar to the visual effect of a reflection. Those "static" conformal symmetry rosettes with the group $\mathbf{C}_n\mathbf{R}_I$ can be constructed by using an asymmetric figure, with its shape very close to the inversion circle, or by using a fundamental region of a similar form (Figure 4.4). A fundamental region of the group $\mathbf{C}_n\mathbf{R}_I$ offers a change of non-inversional boundaries, i.e. boundaries that do not belong to the inversion circle $\mathbf{m}_I$. This is the only restriction to the choice of a fundamental region, since the invariance of all the points of the inversion circle must be preserved.

Owing to their low degree of symmetry and visual dynamism conditioned by the polarity of rotations, conformal symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{R}_I$ are very rare in ornamental art. Such examples as exist are constructed mostly by using half-circles containing the singular point O, touching the inversion circle and forming a rosette with the symmetry group $\mathbf{C}_n$ (n). They are transformed by the inversion R_I onto the corresponding half-tangents in the touch points (Figure 4.3). In ornamental art, the frequent use of that construction is dictated by the principle of maximal constructional and visual simplicity, while other aspects of conformal symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{R}_I$ are rarely found in ornamental art. Conformal symmetry groups of the type $\mathbf{C}_n\mathbf{R}_I$ can be obtained by desymmetrizations of groups of the type $\mathbf{D}_n\mathbf{R}_I$, the most frequent discrete conformal symmetry groups of the category C_{21} in ornamental art. Besides classical-symmetry desymmetrizations, frequently occurring is the antisymmetry desymmetrization resulting in the conformal antisymmetry group $\mathbf{D}_n\mathbf{R}_I/\mathbf{C}_n\mathbf{R}_I$, which in the classical theory of symmetry can be discussed as the group $\mathbf{C}_n\mathbf{R}_I$.

Every group of the type $\mathbf{D}_n\mathbf{R}_I$ (Figure 4.5–4.7) is the direct product of the symmetry groups $\mathbf{D}_n$ (nm) and $\mathbf{R}_I$. Hence, visual interpretations of the group $\mathbf{D}_n\mathbf{R}_I$ can be constructed multiplying by the inversion R_I a rosette with the symmetry group $\mathbf{D}_n$ (nm), belonging to a fundamental region of the group $\mathbf{R}_I$ or, less frequently, multiplying by symmetry transformations of the group $\mathbf{D}_n$ (nm) a figure with the symmetry group $\mathbf{R}_I$, belonging to a fundamental region of the group $\mathbf{D}_n$ (nm). A fundamental region of the group $\mathbf{D}_n\mathbf{R}_I$ is the section of fundamental regions of the groups $\mathbf{D}_n$ (nm) and $\mathbf{R}_I$. In groups of the type $\mathbf{D}_n\mathbf{R}_I$, there are no enantiomorphic modifications.

Conformal symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{R}_I$ have visual characteristics similar to that of generating rosettes with the symmetry group $\mathbf{D}_n$ (nm). Owing to the presence of reflections and inversions, these conformal symmetry groups belong to the family of visually static symmetry groups with non-polar rotations.

Since groups of the type $\mathbf{D}_n\mathbf{R}_I$ are generated by reflections (reflections and inversions), there is no possibility for changing boundaries of a fundamental region. Owing to the fixed shape of a fundamental region, tilings corresponding to the group $\mathbf{D}_n\mathbf{R}_I$, for fixed n, are reduced to only one figure (Figure 4.5). In ornamental art, the variety, richness and visual interest of conformal symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{R}_I$, is achieved by applying different elementary asymmetric figures within a fundamental region. The visual effect of the inversion R_I, within the group $\mathbf{D}_n\mathbf{R}_I$ depends on the shape of that elementary asymmetric figure and its position within a fundamental region. The static visual function of the inversion R_I comes to its full expression for an elementary asymmetric figure, by the shape and position being very close to the inversion circle.

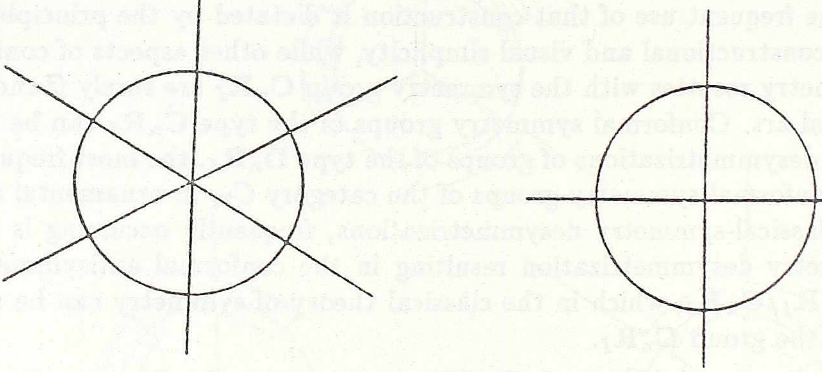


Figure 4.5
Examples of conformal symmetry rosettes with the symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$, constructed according to the principle of maximal constructional simplicity.

In ornamental art, there are many examples of conformal symmetry rosettes with symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$. The frequency of occurrence of a particular group depends on the frequency of occurrence of its generating group $\mathbf{D}_n$ (nm). Therefore, the groups $\mathbf{D}_n\mathbf{R}_I$, for n – an even natural number, especially for n=2,4,6,8,12..., occur most often. These groups satisfy the principle of visual entropy and offer the possibility of choosing the position of a corresponding conformal symmetry rosette, such that its reflection lines coincide with the fundamental natural directions — the vertical and horizontal line.

Owing to maximal constructional simplicity, groups of the type $\mathbf{D}_n\mathbf{R}_I$ have a special role in ornamental art. Very interesting visual interpretations of these groups are obtained, reproducing by the inversion R_I a rosette with the symmetry group $\mathbf{D}_n$ (nm), constructed by circles (or their arcs) containing the singular point O and touching the inversion circle $\mathbf{m}_I$. The inversion R_I transforms these circles (arcs) onto the tangent lines (parts of tangent lines) of the inversion circle in the touch points (Figure 4.6, 4.7). These conformal symmetry rosettes are used in ornamental art by almost all cultures. They have a special place in Romanesque and Gothic art, within rosettes used in architecture.

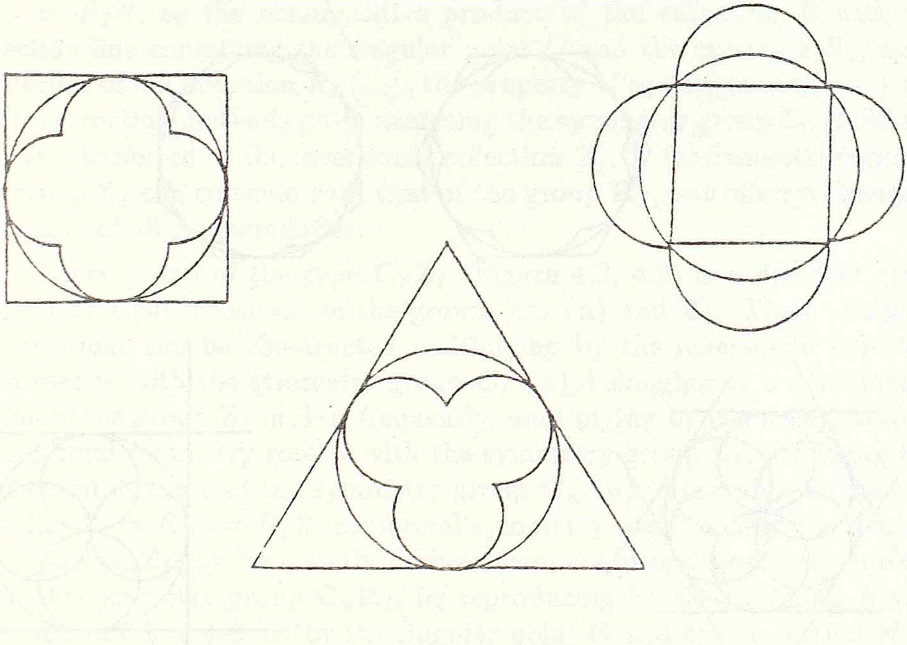


Figure 4.6

Examples of conformal symmetry rosettes with the symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$, which are used in ornamental art.

The continuous conformal symmetry group $\mathbf{D}_\infty\mathbf{R}_I$ possesses adequate visual interpretations. One of them is a circle. Regarded from the point of view of the isometric theory of symmetry, a circle possesses the continuous symmetry group of rosettes $\mathbf{D}_\infty$, but for conformal symmetry, its symmetry

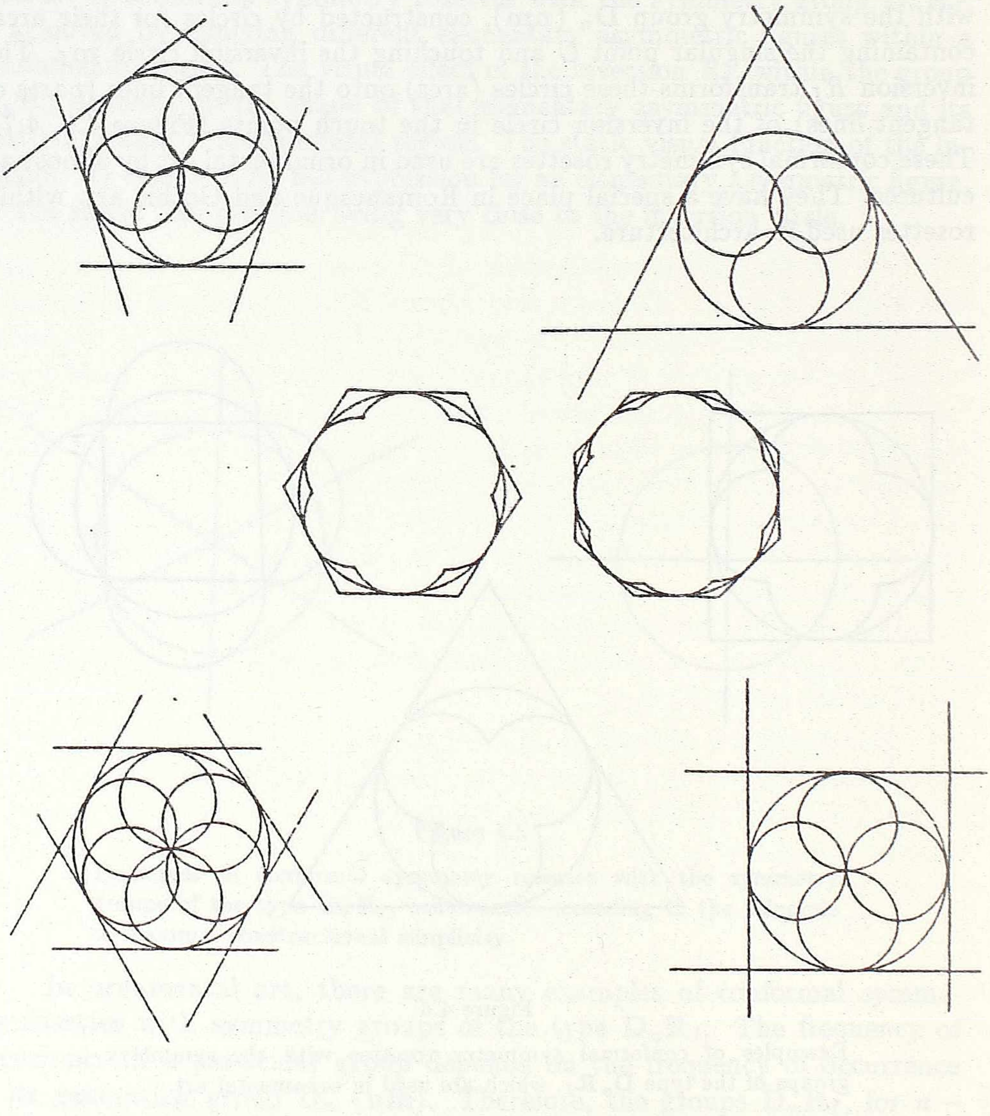


Figure 4.7

Examples of conformal symmetry rosettes with the symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$, which are used in ornamental art.

group is $\mathbf{D}_\infty \mathbf{R}_I$. Such a possibility for different symmetry treatments of the same figure occurs in all situations when a certain theory (e.g., the isometric theory of symmetry) is extended to a larger, more general theory (e.g., the theory of conformal symmetry).

The simplest group of the type $\mathbf{C}_n \mathbf{Z}_I$ is the group $\mathbf{Z}_I$ ($n = 1$), generated by the inversional reflection Z_I, the commutative composition of a reflection and an inversion. The inversional reflection Z_I is an equivalent of a two-fold rotation with the axis belonging to the invariant plane of the symmetry groups of tablets G_{320}. Those isomorphism between conformal symmetry groups of the category C_{21} and the symmetry groups of tablets G_{320}, in which the inversional reflection Z_I corresponds to this two-fold rotation, indicates the properties of the inversional reflection Z_I — its involutionality and relations to the other conformal symmetry transformations.

Since the inversional reflection Z_I can be represented in the form $Z_I = RR_I = R_I R$, as the commutative product of the reflection R with the reflection line containing the singular point O and the inversion R_I, many properties of the inversion R_I (e.g., the property of equiangularity, etc.) and the construction methods given analyzing the symmetry group $\mathbf{R}_I$, hold and can be transferred to the inversional reflection Z_I. A fundamental region of the group $\mathbf{Z}_I$ can coincide with that of the group $\mathbf{R}_I$, but offers a change of the shape of all its boundaries.

Every group of the type $\mathbf{C}_n \mathbf{Z}_I$ (Figure 4.8, 4.9) is a dihedral group derived as a superposition of the groups $\mathbf{C}_n$ (n) and $\mathbf{Z}_I$. Their visual interpretations can be constructed multiplying by the inversional reflection Z_I a rosette with the symmetry group $\mathbf{C}_n$ (n), belonging to a fundamental region of the group $\mathbf{Z}_I$ or, less frequently, multiplying by the n-fold rotation a conformal symmetry rosette with the symmetry group $\mathbf{Z}_I$, belonging to a fundamental region of the symmetry group $\mathbf{C}_n$ (n). According to the relationship $Z_I = RR_I = R_I R$, conformal symmetry rosettes with the symmetry group $\mathbf{C}_n \mathbf{Z}_I$ can be directly derived from conformal symmetry rosettes with the symmetry group $\mathbf{C}_n \mathbf{R}_I$, by reproducing by the reflection R with the reflection line defined by the singular point O and by the section of the boundaries of the fundamental region of the group $\mathbf{C}_n \mathbf{R}_I$ with the inversion circle $\mathbf{m}_I$, one class of fundamental regions of the group $\mathbf{C}_n \mathbf{R}_I$ (the internal or external fundamental regions) (Figure 4.8). Conformal symmetry groups of the type $\mathbf{C}_n \mathbf{Z}_I$ offer the possibility for the enantiomorphism. All the other properties of groups of the type $\mathbf{C}_n \mathbf{R}_I$ can be attributed to groups of the type $\mathbf{C}_n \mathbf{Z}_I$.

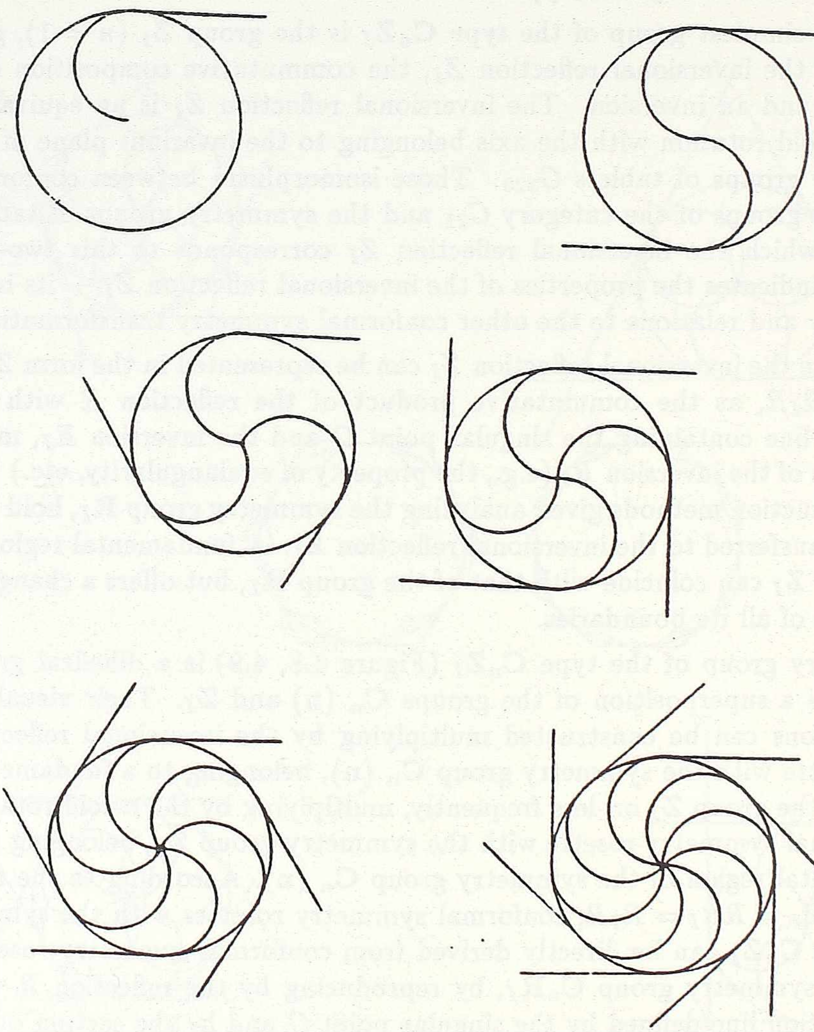


Figure 4.8

Examples of conformal symmetry rosettes with the symmetry groups of the type $C_n Z_I$, constructed according to the principle of maximal constructional simplicity.

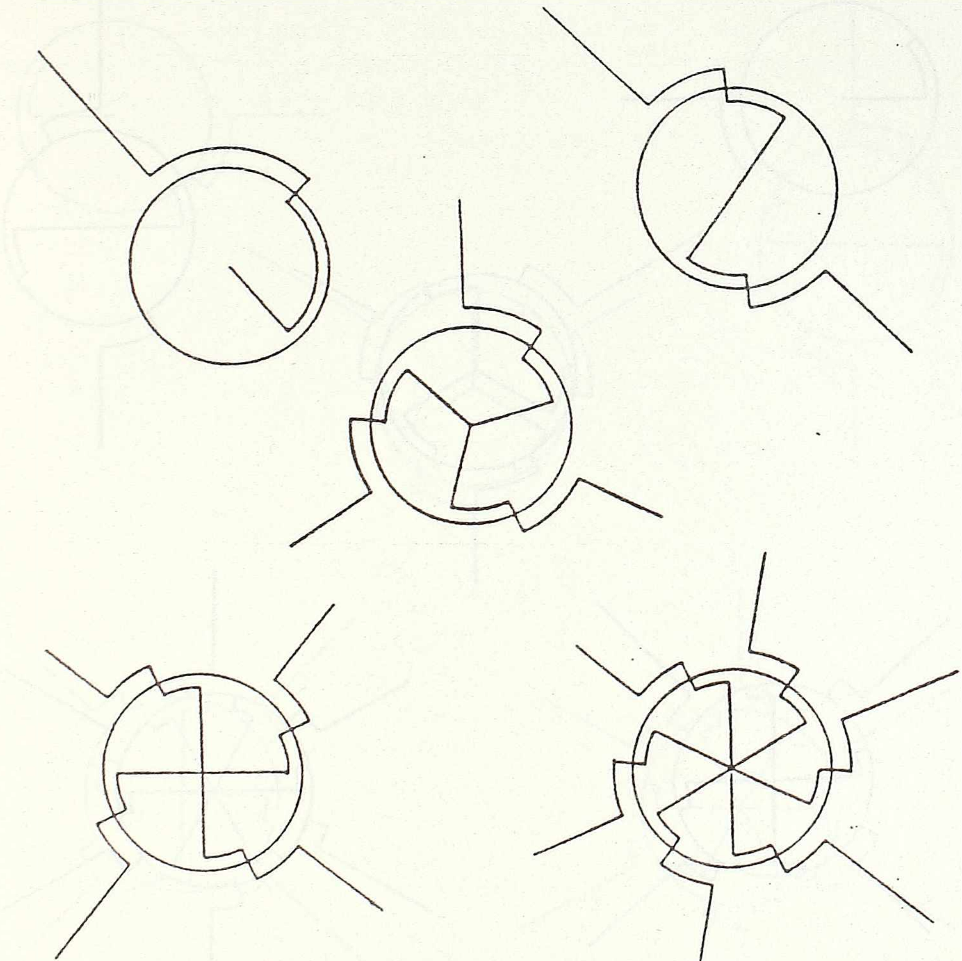


Figure 4.9

Examples of conformal symmetry rosettes with the symmetry groups of the type $\mathbf{C}_n \mathbf{Z}_I$.

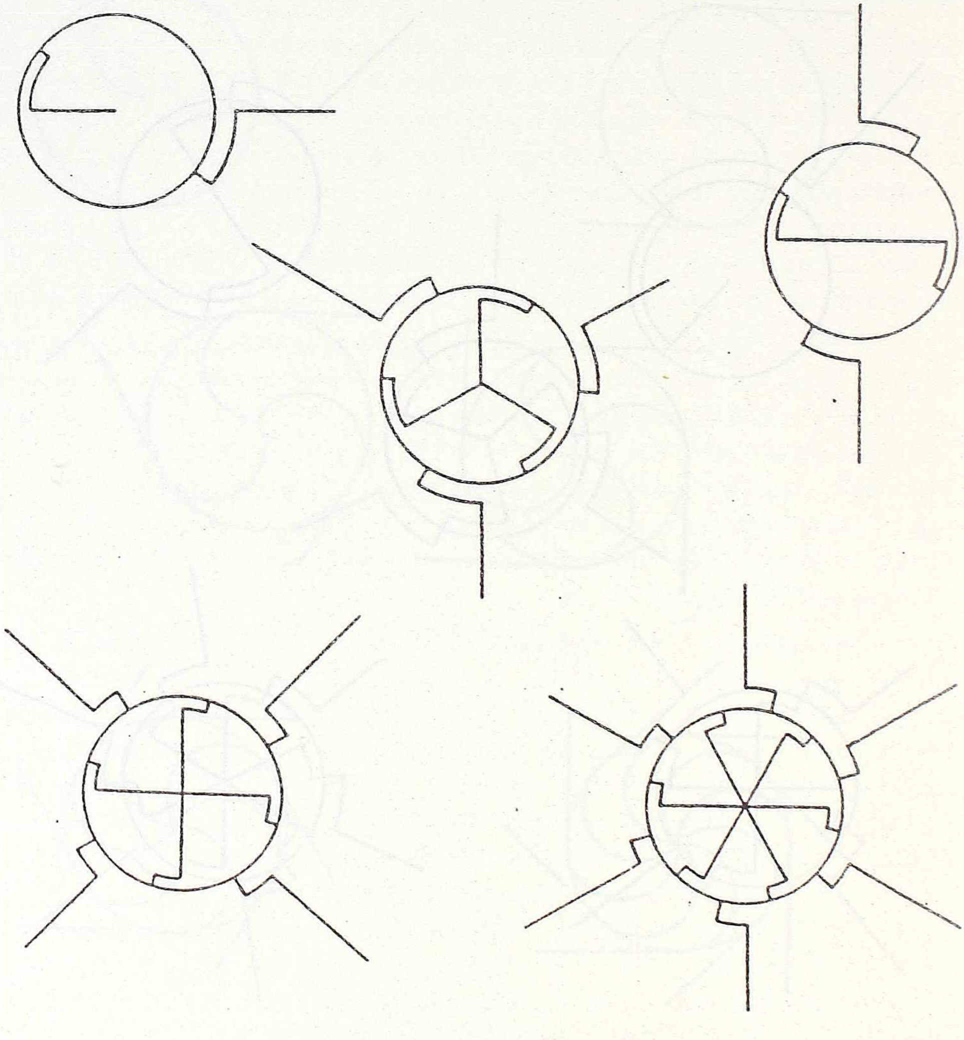


Figure 4.10

Examples of conformal symmetry rosettes with the symmetry groups of the type N_I, which satisfy the principle of maximal constructional simplicity.

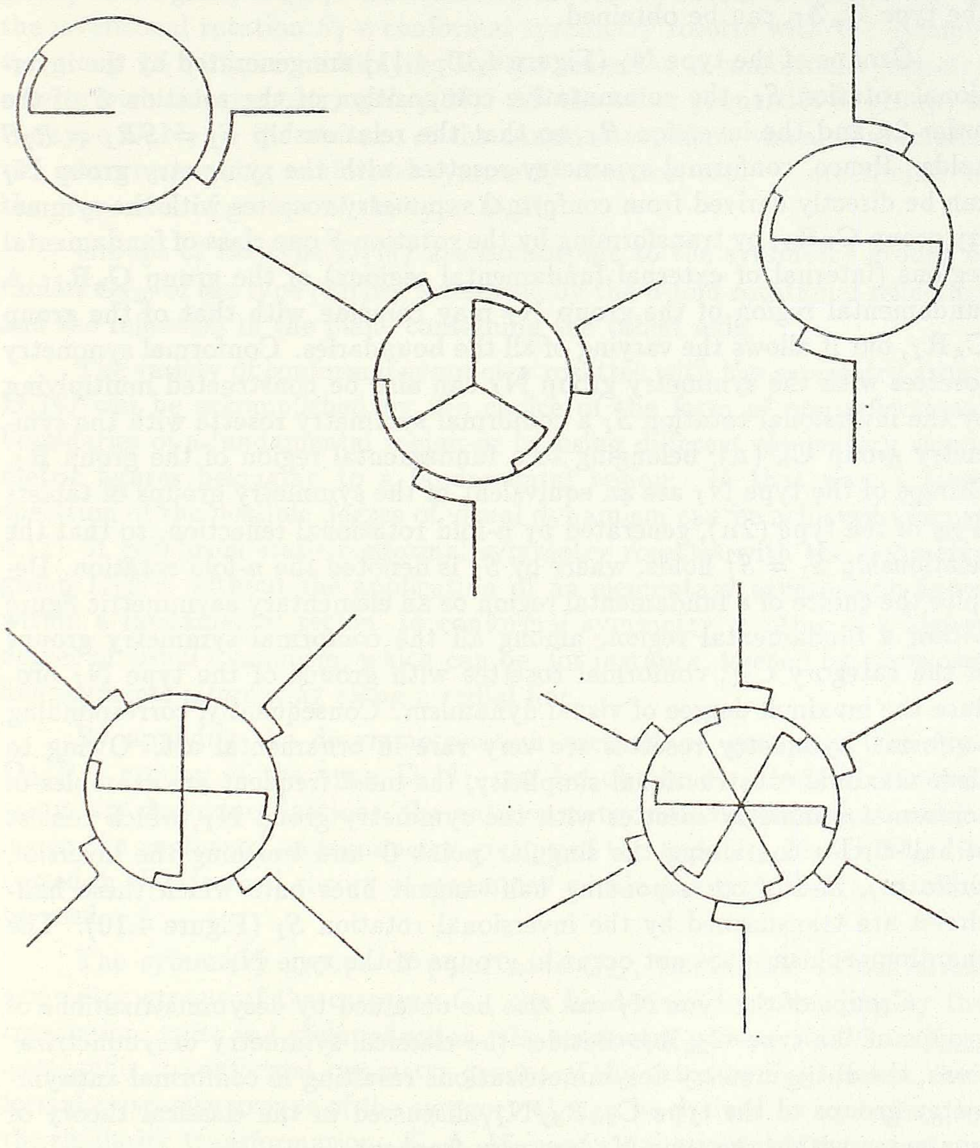


Figure 4.11

Examples of conformal symmetry rosettes with the symmetry groups of the type N_I.

Like the symmetry groups of the type $\mathbf{C}_n\mathbf{R}_I$, the groups of the type $\mathbf{C}_n\mathbf{Z}_I$ can be derived by a desymmetrization of the symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$. Besides the classical-symmetry desymmetrizations, antisymmetry desymmetrizations resulting in conformal antisymmetry groups of the type $\mathbf{D}_n\mathbf{R}_I/\mathbf{C}_n\mathbf{Z}_I$, discussed in the classical theory of symmetry within the type $\mathbf{C}_n\mathbf{Z}_I$, can be obtained.

Groups of the type $\mathbf{N}_I$ (Figure 4.10, 4.11) are generated by the inversional rotation S_I, the commutative composition of the rotation S of the order $2n$ and the inversion R_I, so that the relationship $S_I = SR_I = R_IS$ holds. Hence, conformal symmetry rosettes with the symmetry group $\mathbf{N}_I$ can be directly derived from conformal symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{R}_I$, by transforming by the rotation S one class of fundamental regions (internal or external fundamental regions) of the group $\mathbf{C}_n\mathbf{R}_I$. A fundamental region of the group $\mathbf{N}_I$ may coincide with that of the group $\mathbf{C}_n\mathbf{R}_I$, but it allows the varying of all the boundaries. Conformal symmetry rosettes with the symmetry group $\mathbf{N}_I$ can also be constructed multiplying by the inversional rotation S_I a conformal symmetry rosette with the symmetry group $\mathbf{C}_n$ (n), belonging to a fundamental region of the group $\mathbf{R}_I$. Groups of the type $\mathbf{N}_I$ are an equivalent of the symmetry groups of tablets G_{320} of the type $(\widetilde{2\mathrm{n}})$, generated by n-fold rotational reflection, so that the relationship $S_1 = S_I^2$ holds, where by S_1 is denoted the n-fold rotation. Despite the choice of a fundamental region or an elementary asymmetric figure within a fundamental region, among all the conformal symmetry groups of the category C_{21}, conformal rosettes with groups of the type $\mathbf{N}_I$ produce the maximal degree of visual dynamism. Consequently, corresponding conformal symmetry rosettes are very rare in ornamental art. Owing to their maximal constructional simplicity, the most frequent are examples of conformal symmetry rosettes with the symmetry group $\mathbf{N}_I$, which consist of half-circles containing the singular point O and touching the inversion circle $\mathbf{m}_I$, and of corresponding half-tangent lines onto which these half-circles are transformed by the inversional rotation S_I (Figure 4.10). The enantiomorphism does not occur in groups of the type $\mathbf{N}_I$.

Groups of the type $\mathbf{N}_I$ can also be obtained by desymmetrizations of groups of the type $\mathbf{C}_{2n}\mathbf{R}_I$. Besides the classical-symmetry desymmetrizations, the antisymmetry desymmetrizations resulting in conformal antisymmetry groups of the type $\mathbf{C}_{2n}\mathbf{R}_I/\mathbf{N}_I$, discussed in the classical theory of symmetry within the type $\mathbf{N}_I$, are very frequent.

According to the relationship $S_I = SR_I = R_IS$, where by S is denoted the rotation of the order $2n$, a connection, analogous to that existing between the types $\mathbf{C}_n\mathbf{R}_I$ and $\mathbf{N}_I$, exists between the types $\mathbf{D}_n\mathbf{R}_I$ and $\mathbf{D}_1\mathbf{N}_I$.

Hence, conformal symmetry rosettes with the symmetry group $\mathbf{D}_1\mathbf{N}_I$ (Figure 4.12, 4.13) can be directly derived from conformal symmetry rosettes with the symmetry group $\mathbf{D}_n\mathbf{R}_I$, by reproducing by the rotation S one class of the fundamental regions (the internal or external fundamental regions) of the group $\mathbf{D}_n\mathbf{R}_I$. The same result can be obtained multiplying by the inversional rotation S_I a conformal symmetry rosette with the symmetry group $\mathbf{D}_1$ (m), or multiplying by the reflection a conformal symmetry rosette with the symmetry group $\mathbf{N}_I$. All the other properties of groups of the type $\mathbf{D}_1\mathbf{N}_I$ — the absence of the enantiomorphism, visual characteristics of corresponding conformal symmetry rosettes, etc. — are similar to the properties of groups of the type $\mathbf{D}_n\mathbf{R}_I$.

Groups of the type $\mathbf{D}_1\mathbf{N}_I$ are isomorphic to the symmetry groups of tablets G_{320} of the type $(\widetilde{2n})$m, generated by the n-fold rotational reflection and the reflection in the plane containing the tablet axis.

The variety of conformal symmetry rosettes with the symmetry group $\mathbf{D}_1\mathbf{N}_I$ can be accomplished by the choice of the form of non-reflectional boundaries of a fundamental region or by using different elementary asymmetric figures belonging to a fundamental region. In that way, a large spectrum of the possible degree of visual dynamism can be achieved (Figure 4.12). It goes from static conformal symmetry rosettes with the symmetry group $\mathbf{D}_1\mathbf{N}_I$, without the application of an elementary asymmetric figure within a fundamental region, to conformal symmetry rosettes of a higher degree of visual dynamism, which can be, for instance, formed by circle and line segments alternating along a radial line.

By applying the desymmetrization method on groups of the type $\mathbf{D}_{2n}\mathbf{R}_I$, groups of the type $\mathbf{D}_1\mathbf{N}_I$ can be obtained. Besides classical-symmetry desymmetrizations, the antisymmetry desymmetrizations resulting in the conformal antisymmetry groups of the type $\mathbf{D}_{2n}\mathbf{R}_I/\mathbf{D}_1\mathbf{N}_I$, discussed in the classical theory of symmetry within the type $\mathbf{D}_1\mathbf{N}_I$, can also be used.

The symmetry groups of polar rods G_{31}, isomorphic to conformal symmetry groups of the category C_2, can be derived by extending by the translation, twist and glide reflection, the symmetry groups of tablets G_{320}, isomorphic to conformal symmetry groups of the category C_{21}. Hence, conformal symmetry groups of the category C_2 can be derived by extending by the similarity transformations K, L, M, conformal symmetry groups of the category C_{21}.

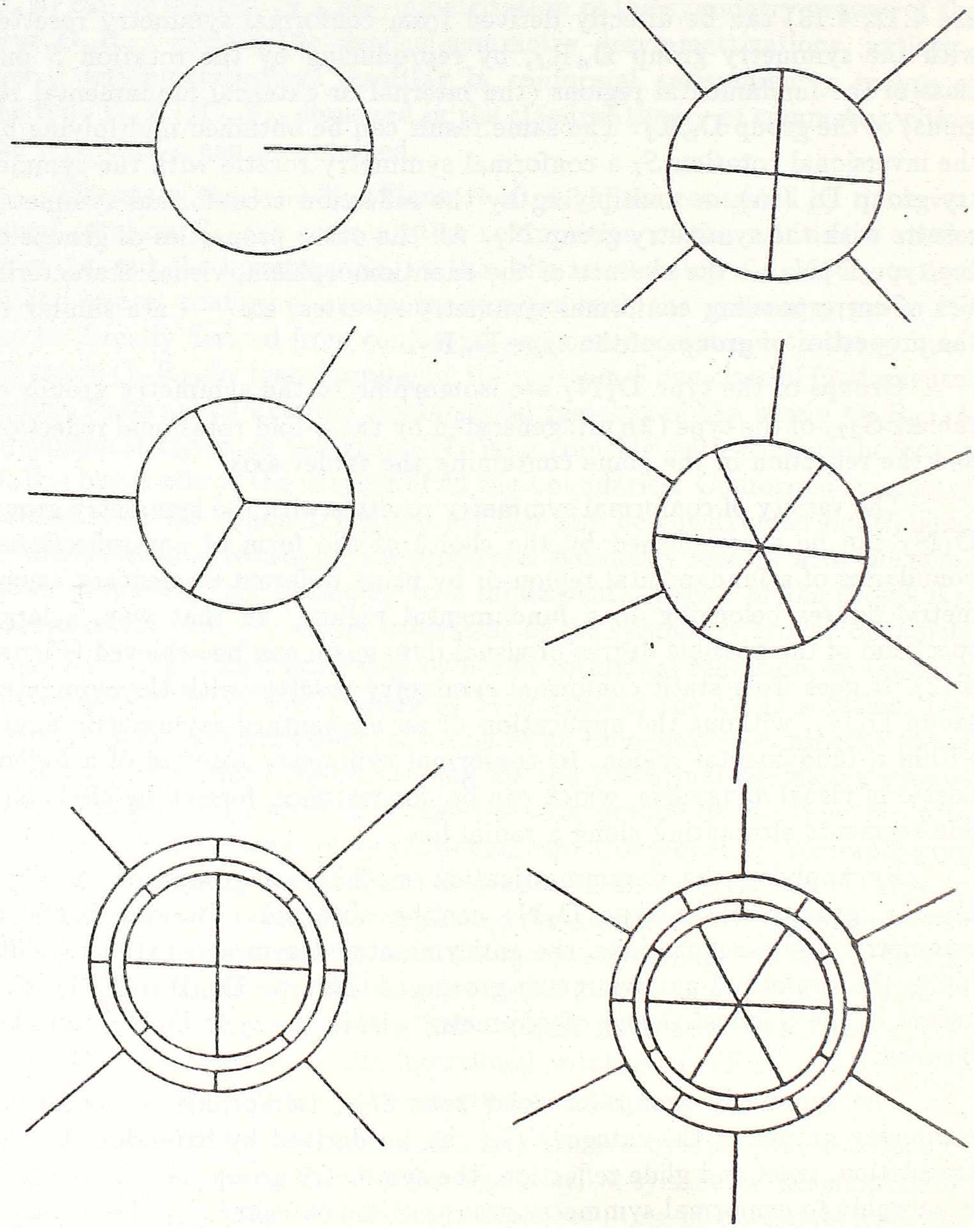


Figure 4.12

Examples of conformal symmetry rosettes with the symmetry groups of the type D_1N_I.

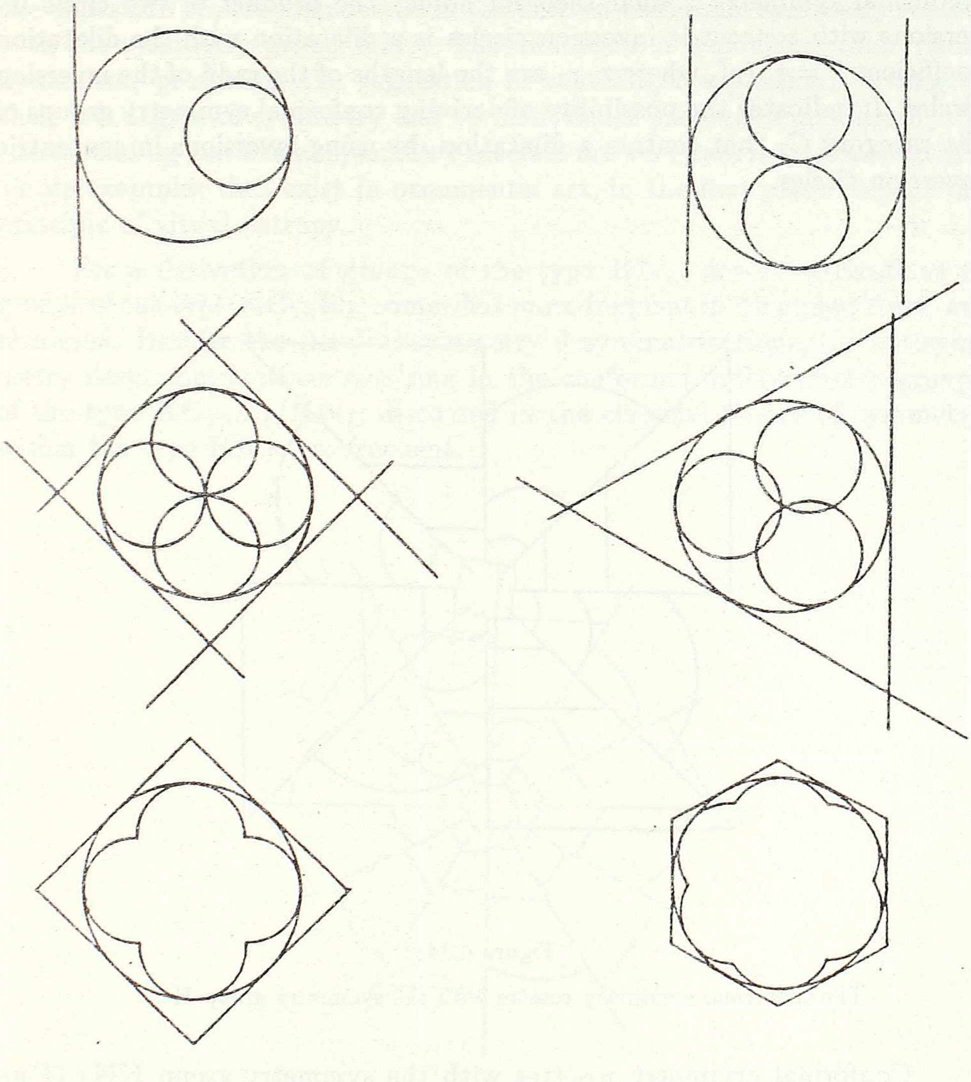


Figure 4.13

Examples of conformal symmetry rosettes with the symmetry groups of the type D_1N_I, which are used in ornamental art.

According to the theorem that the product of two reflections with parallel reflection lines is a translation, the modulus of the translation vector of which is twice the distance between the reflection lines, in the field of conformal symmetry a dual theorem holds: the product of two circle inversions with concentric inversion circles is a dilatation with the dilatation coefficient $k = r^2/r_1^2$, where r, r_1 are the lengths of the radii of the inversion circles. It indicates the possibility of deriving conformal symmetry groups of the category C_2 that contain a dilatation, by using inversions in concentric inversion circles.

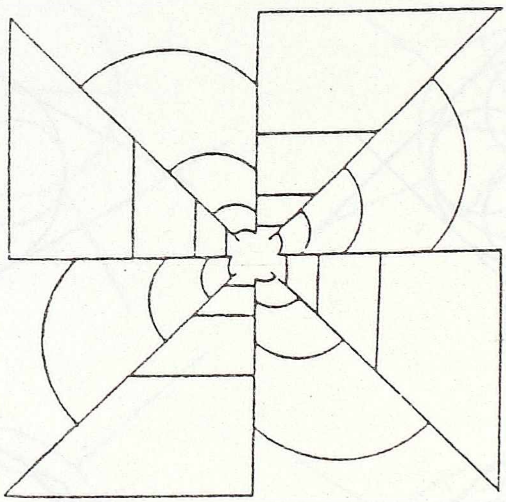


Figure 4.14

The conformal symmetry rosette with the symmetry group $\mathrm{K4}_I$.

Conformal symmetry rosettes with the symmetry group KN_I (Figure 4.14) can be constructed multiplying by the dilatation K (with $k > 0$) a conformal symmetry rosette with the symmetry group N_I, belonging to a fundamental region of the group $\mathbf{K}$. A fundamental region of the group KN_I is the section of fundamental regions of the groups $\mathbf{K}$ and N_I, and it allows a varying of the form of all the boundaries. Since the dilatation K is the element of every group of the type KN_I, to efficiently construct the corresponding visual interpretations one applies two inversions in the concentric inversion circles. All the visual properties of a generating conformal symmetry rosette with the symmetry group N_I, after the introduction of the dilatation K, are maintained in the derived conformal symmetry rosette with the symmetry group KN_I. The dilatation K stimulates their visual dynamism, producing the suggestion of centrifugal expansion. Owing to their low degree of symmetry and to their visual properties mentioned, the corresponding conformal symmetry rosettes are very rare in ornamental art. Those examples that exist in ornamental art, in the first place respect the principle of visual entropy.

For a derivation of groups of the type KN_I, desymmetrizations of groups of the type $KC_{2n}R_I$, somewhat more frequent in ornamental art, are also used. Besides the classical-symmetry desymmetrizations, the antisymmetry desymmetrizations resulting in the conformal antisymmetry groups of the type $KC_{2n}R_I/KN_I$, discussed in the classical theory of symmetry within the type KN_I, are frequent.

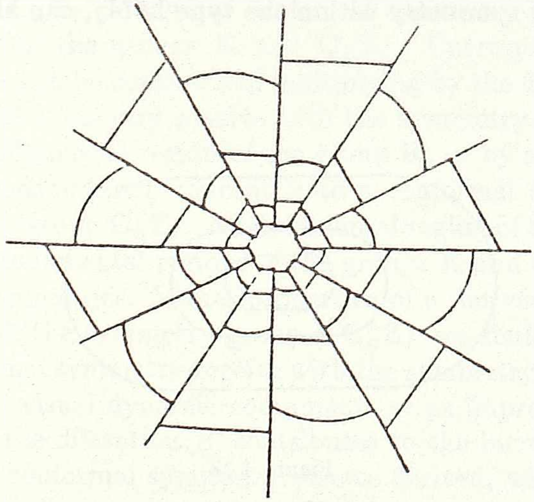


Figure 4.15
The conformal symmetry rosette with the symmetry group $M6_I$.

The type MN_I consists of the groups derived by the superposition of the groups M and N_I. Hence, corresponding conformal symmetry rosettes with the group MN_I can be constructed multiplying by the dilative reflection M (with $k > 0$) a conformal symmetry rosette with the group N_I, belonging to a fundamental region of the group M (Figure 4.15). A fundamental region of the group MN_I is the section of fundamental regions of the groups M and N_I, and allows a varying of all the boundaries.

After the introduction of the dilative reflection M, the visual properties of a generating conformal symmetry rosette with the symmetry group N_I, remain unchanged. In the conformal symmetry rosette with the symmetry group MN_I obtained, the presence of the dilative reflection M increases the visual dynamism, suggesting alternating centrifugal expansion. Since conformal symmetry rosettes with the symmetry group MN_I belong to a family of dynamic, complicated conformal symmetry rosettes, the construction and symmetry of which is not comprehensible by empirical methods, they are very rare in ornamental art.

In aiming to obtain groups of the type MN_I, the desymmetrization method can be applied on groups of the type KD_1N_I. Besides the classical-symmetry desymmetrizations, also the antisymmetry desymmetrizations resulting in antisymmetry groups of the type KD_1N_I/MN_I, discussed in the classical theory of symmetry within the type MN_I, can also be derived.

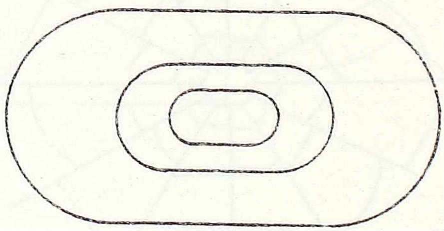


Figure 4.16

The conformal symmetry rosette with the symmetry group KD_12_I.

Conformal symmetry rosettes with the symmetry group KD_1N_I (Figure 4.16) can be constructed multiplying by the dilatation K (with $k > 0$) a conformal symmetry rosette with the symmetry group D_1N_I, belonging to a fundamental region of the symmetry group K. A fundamental region of the group KD_1N_I is the section of fundamental regions of the groups K and D_1N_I, and it allows a varying of non-reflectional boundaries.

The visual properties of generating conformal symmetry rosettes with the symmetry group $\mathbf{D}_1\mathbf{N}_I$ are maintained in derived conformal symmetry rosettes with the symmetry group $\mathbf{KD}_1\mathbf{N}_I$. The introduction of the dilatation K results in the appearance of a new dynamic visual component — the suggestion of centrifugal expansion. As in all the other cases of conformal symmetry groups of the category C_2 containing a dilatation, for the construction of conformal symmetry rosettes with the symmetry group $\mathbf{KD}_1\mathbf{N}_I$, it is very efficient to make the use of two circle inversions with the concentric inversion circles. Owing to the static visual component produced by reflections and the non-polarity of rotations, examples of conformal symmetry rosettes with the symmetry group $\mathbf{KD}_1\mathbf{N}_I$ are more frequent in ornamental art, than conformal symmetry rosettes with the symmetry group $\mathbf{KN}_I$ or $\mathbf{MN}_I$.

By using the desymmetrization method, besides classical-symmetry desymmetrizations, the antisymmetry desymmetrizations of groups of the type $\mathbf{KD}_{2n}\mathbf{R}_I$, resulting in conformal antisymmetry groups of the type $\mathbf{KD}_{2n}\mathbf{R}_I/\mathbf{KD}_1\mathbf{N}_I$, discussed in the classical theory of symmetry within the type $\mathbf{KD}_1\mathbf{N}_I$, can be obtained.

The type $\mathbf{KC}_n\mathbf{Z}_I$ consists of conformal symmetry groups formed by the superposition of the groups $\mathbf{K}$ and $\mathbf{C}_n\mathbf{Z}_I$. Corresponding conformal symmetry rosettes can be constructed multiplying by the dilatation K (with $k > 0$) a conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{Z}_I$, belonging to the fundamental region of the group $\mathbf{K}$, or by applying an inversion with the inversion circle concentric to a conformal symmetry rosette with thé symmetry group $\mathbf{C}_n\mathbf{Z}_I$. A fundamental region of the group $\mathbf{KC}_n\mathbf{Z}_I$ is the section of fundamental regions of the groups $\mathbf{K}$ and $\mathbf{C}_n\mathbf{Z}_I$, and allows a varying of all boundaries. Visual properties of a derived conformal symmetry rosette with the symmetry group $\mathbf{KC}_n\mathbf{Z}_I$ are similar to that of the generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{Z}_I$. By introducing a new visual dynamic component — an impression of centrifugal expansion — the dilatation K contributes to the increase in the visual dynamism of the conformal symmetry rosette derived, with respect to the generating conformal symmetry rosette. Due to their visual dynamism conditioned by the bipolarity of rotations, enantiomorphism, visual function of the dilatation K, etc., examples of conformal symmetry rosettes with the symmetry group $\mathbf{KC}_n\mathbf{Z}_I$, are very rare in ornamental art (Figure 4.17).

By desymmetrizations of groups of the type $\mathbf{KD}_n\mathbf{R}_I$ — the most frequent conformal symmetry groups of the category C_2 in ornamental art — it is possible to derive groups of the type $\mathbf{KC}_n\mathbf{Z}_I$. Besides classical-symmetry desymmetrizations, the antisymmetry desymmetrizations resulting in conformal antisymmetry groups of the type $\mathbf{KD}_n\mathbf{R}_I/\mathbf{KC}_n\mathbf{Z}_I$, discussed in the classical theory of symmetry within the type $\mathbf{KC}_n\mathbf{Z}_I$, can be obtained.

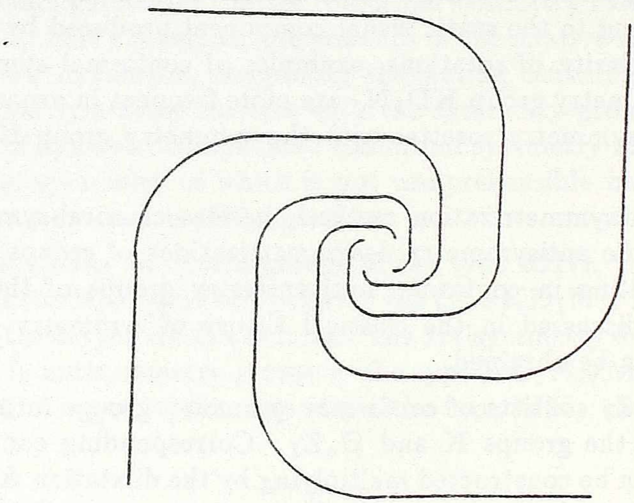


Figure 4.17

The conformal symmetry rosette with the symmetry group $\mathbf{KC}_4\mathbf{Z}_I$.

The type $\mathbf{LC}_n\mathbf{Z}_I$ consists of conformal symmetry groups that are the result of the superposition of the groups $\mathbf{L}$ and $\mathbf{C}_n\mathbf{Z}_I$. Visual examples of conformal symmetry groups of the type $\mathbf{LC}_n\mathbf{Z}_I$ can be constructed multiplying by the dilative rotation L (with $k > 0$) a conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{Z}_I$, belonging to a fundamental region of the group $\mathbf{L}$ (Figure 4.18). The visual effect of the dilative rotation L is the appearance of a dynamic spiral-motion component. For a rational angle θ of the dilative rotation L, $\theta = p\pi/q$, $(p,q) = 1$, $p, q \in Z$, it is possible to divide a conformal symmetry rosette with the symmetry group $\mathbf{LC}_n\mathbf{Z}_I$ into sectors of the dilatation $K((-1)^p k^q)$. A fundamental region of the group $\mathbf{LC}_n\mathbf{Z}_I$ is the section of fundamental regions of the groups $\mathbf{L}$ and $\mathbf{C}_n\mathbf{Z}_I$. Hence, a varying of the shape of all the boundaries of a fundamental region is allowed. Since the dilative rotation L is a composite transformation, the relationship $L = KS = SK$ holds, where a rotation with the rotation angle θ is denoted by S. Conformal symmetry rosettes with the symmetry group $\mathbf{KC}_n\mathbf{Z}_I$ can be constructed by using a generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{Z}_I$, but such a construction is, in a certain degree, complicated. The conformal rosette mentioned, must be first transformed by an inversion with the inversion circle $\mathbf{m}_I$ concentric to this rosette. After that, the image obtained must be transformed by the reflection R with the reflection line containing the singular point O, and finally, by the rotation S.

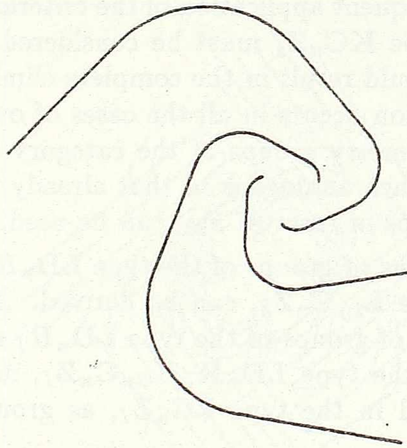


Figure 4.18

The conformal symmetry rosette with the symmetry group $\mathbf{LC}_3\mathbf{Z}_I$.

Groups of the type $\mathbf{LC}_n\mathbf{Z}_I$ belong to a family of visually dynamic conformal symmetry groups with bipolar rotations, and with the possibility for the enantiomorphism. The impression of visual dynamism, suggested by the corresponding conformal symmetry rosettes, is greater than that suggested by the generating conformal symmetry rosettes with a symmetry group of the type $\mathbf{C}_n\mathbf{Z}_I$. It is the result of the presence of the dilative rotation L, producing the visual impression of spiral-motion rotational expansion, and representing by itself a visual interpretation of a twist within the plane. The desired intensity of a visual dynamic impression can be achieved by varying the form of a fundamental region, applying different elementary asymmetric figures within a fundamental region and choosing the parameters k, θ. In ornamental art, the variety of conformal symmetry rosettes with symmetry groups of the type $\mathbf{LC}_n\mathbf{Z}_I$ is restricted by the principle of visual entropy. Therefore, the most frequent conformal symmetry rosettes with the symmetry group $\mathbf{LC}_n\mathbf{Z}_I$ are constructed multiplying by a dilative rotation, the simplest conformal symmetry rosettes with the symmetry group $\mathbf{C}_n\mathbf{Z}_I$ (Figure 4.18). For the same reason, more frequent are conformal symmetry rosettes with symmetry groups of the type $\mathbf{LC}_n\mathbf{Z}_I$, with a rational angle of the dilative rotation L. Very important in ornamental art are conformal symmetry rosettes with a symmetry group of the subtype $\mathbf{L}_{2n}\mathbf{C}_n\mathbf{Z}_I$ ($L_{2n} = L(k,\pi/n)$). The symmetry group mentioned is the subgroup of the index 2 of the group $\mathbf{LD}_n\mathbf{R}_I$. According to the relationship $K = L(k,0) = L_0$, a consequent application of the criterion of subordination requires also that the type $\mathbf{KC}_n\mathbf{Z}_I$ must be considered as the subtype of the type $\mathbf{LC}_n\mathbf{Z}_I$. This would result in the complete elimination of the type $\mathbf{KC}_n\mathbf{Z}_I$. A similar situation occurs in all the cases of overlapping types or individual conformal symmetry groups of the category C_2. When solving such a problem, an approach analogous to that already discussed with the similarity symmetry groups of rosettes S_{20}, can be used.

By desymmetrizations of groups of the type $\mathbf{LD}_n\mathbf{R}_I$, the corresponding groups of the subtype $\mathbf{L}_{2n}\mathbf{C}_n\mathbf{Z}_I$, can be derived. Adequate antisymmetry desymmetrizations of groups of the type $\mathbf{LD}_n\mathbf{R}_I$ result in conformal antisymmetry groups of the type $\mathbf{LD}_n\mathbf{R}_I/\mathbf{L}_{2n}\mathbf{C}_n\mathbf{Z}_I$, in the classical theory of symmetry included in the type $\mathbf{LC}_n\mathbf{Z}_I$, as groups of the subtype $\mathbf{L}_{2n}\mathbf{C}_n\mathbf{Z}_I$.

Conformal symmetry rosettes with the symmetry group $\mathbf{KC}_n\mathbf{R}_I$ can be constructed multiplying by the dilatation K (with $k > 0$) a conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$, belonging to a fundamental region of the group $\mathbf{K}$, or multiplying the same conformal symmetry rosette by an inversion with the inversion circle concentric with it (Figure 4.19). A fundamental region of the group $\mathbf{KC}_n\mathbf{R}_I$ is the section of fundamental regions of the groups $\mathbf{K}$ and $\mathbf{C}_n\mathbf{R}_I$, and allows a varying of boundaries that do not belong to the inversion circles, i.e. a varying of radial boundaries. The visual effect of the conformal symmetry rosettes derived is very similar to that produced by the generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$. The introduction of the dilatation K representing the new dynamic visual component — the suggestion of centrifugal expansion — results in an increase in the visual dynamism. Owing to their dynamic visual qualities, we would expect that conformal symmetry rosettes with symmetry groups of the type $\mathbf{KC}_n\mathbf{R}_I$ are not so frequent in ornamental art. However, the possibility of deriving conformal symmetry groups of the type $\mathbf{KC}_n\mathbf{R}_I$ by desymmetrizations of groups of the type $\mathbf{KD}_n\mathbf{R}_I$, the most frequently used conformal symmetry groups of the category C_2 in ornamental art, caused their more frequent occurrence. Besides classical-symmetry desymmetrizations, the antisymmetry desymmetrizations, resulting in the conformal antisymmetry groups of the type $\mathbf{KD}_n\mathbf{R}_I/\mathbf{KC}_n\mathbf{R}_I$, discussed in the classical theory of symmetry within the type $\mathbf{KC}_n\mathbf{R}_I$, can be obtained.

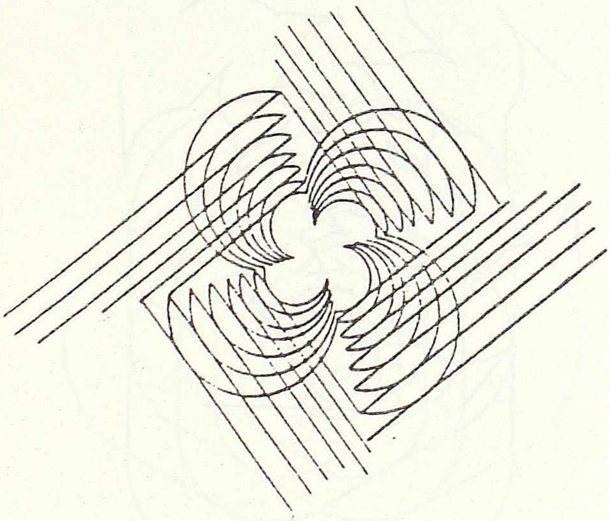


Figure 4.19
The conformal symmetry rosette with the symmetry group $\mathbf{KC}_4\mathbf{R}_I$.

The type $\mathbf{MC}_n\mathbf{R}_I$ consists of conformal symmetry groups derived by extending by the dilative reflection M (with $k > 0$) conformal symmetry groups of the type $\mathbf{C}_n\mathbf{R}_I$. Corresponding conformal symmetry rosettes can be constructed multiplying by the dilative reflection M a conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$, belonging to a fundamental region of the group $\mathbf{M}$. The same conformal symmetry rosettes can be constructed transforming by an inversion with the inversion circle $\mathbf{m}_I$ concentric to it, a generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$. Afterward, that image obtained we must to copy by a reflection in the reflection line containing the singular point O (Figure 4.20). The extension of the symmetry group $\mathbf{C}_n\mathbf{R}_I$ by the dilative reflection M will result, in the visual sense, in the appearance of a new dynamic visual component — centrifugal alternating expansion. The dominance of dynamic components caused the relatively rare occurrence of conformal symmetry groups of the type $\mathbf{MC}_n\mathbf{R}_I$ in ornamental art.

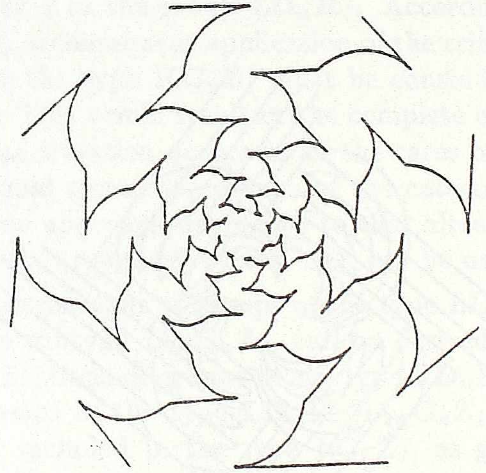


Figure 4.20
The conformal symmetry rosette with the symmetry group $\mathbf{MC}_8\mathbf{R}_I$.

By desymmetrizations of groups of the type $\mathbf{KD}_n\mathbf{R}_I$, conformal symmetry groups of the type $\mathbf{MC}_n\mathbf{R}_I$ can be derived. In particular, antisymmetry desymmetrizations, resulting in antisymmetry groups of the type $\mathbf{KD}_n\mathbf{R}_I/\mathbf{MC}_n\mathbf{R}_I$, discussed by the classical theory of symmetry within the type $\mathbf{MC}_n\mathbf{R}_I$, can be obtained.

The group $\mathbf{KD}_n\mathbf{R}_I$ can be derived extending by the dilatation K the conformal symmetry group $\mathbf{D}_n\mathbf{R}_I$. Corresponding conformal symmetry rosettes can be constructed multiplying by the dilatation K a conformal symmetry rosette with the symmetry group $\mathbf{D}_n\mathbf{R}_I$, belonging to a fundamental region of the group $\mathbf{K}$ (with $k > 0$), or multiplying the same conformal symmetry rosette by an inversion with the inversion circle $\mathbf{m}_I$ concentric to it (Figure 4.21). A fundamental region of the group $\mathbf{KD}_n\mathbf{R}_I$ is the section of fundamental regions of the groups $\mathbf{K}$ and $\mathbf{D}_n\mathbf{R}_I$. Since the group $\mathbf{KD}_n\mathbf{R}_I$ is generated by reflections (reflections and inversions), its fundamental region is fixed. Therefore, a fundamental region of the group $\mathbf{KD}_n\mathbf{R}_I$ is defined by two successive reflection lines and two successive inversion circles, corresponding to this conformal symmetry group. The varying of conformal symmetry rosettes with the symmetry group $\mathbf{KD}_n\mathbf{R}_I$ is reduced to the use of different elementary asymmetric figures belonging to a fundamental region and to a change in the value of the parameter k.

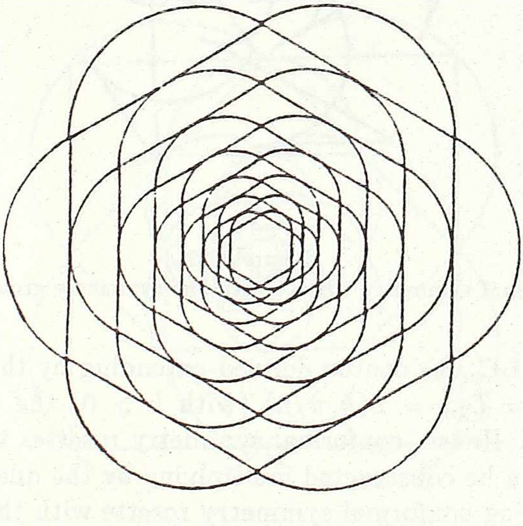


Figure 4.21
The conformal symmetry rosette with the symmetry group KD_6R_I.

The effect of the dilatation K on a generating conformal symmetry rosette with the symmetry group $\mathbf{D}_n\mathbf{R}_I$ is reduced, in the visual sense, to the increase in the visual dynamism and the suggestion of centrifugal expansion. Since there are a large number of models in nature with the symmetry group of rosettes $\mathbf{D}_n$, with a high degree of constructional and visual simplicity and symmetry, and with the dominance of the static visual impression, conformal symmetry groups of the type $\mathbf{KD}_n\mathbf{R}_I$ are the most frequent discrete conformal symmetry groups of the category C_2, in ornamental art. Besides their individual use, groups of the type $\mathbf{KD}_n\mathbf{R}_I$ form the basis for applying the desymmetrization method, aiming to derive the other types of conformal symmetry groups of the category C_{21}.

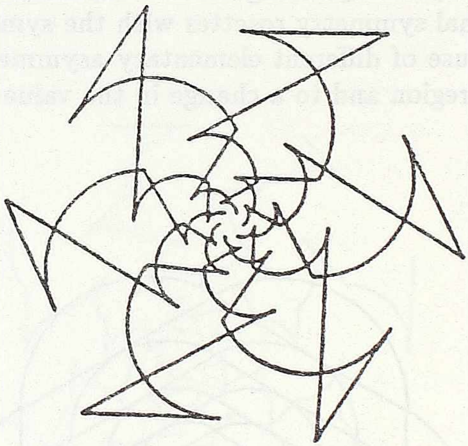


Figure 4.22
The conformal symmetry rosette with the symmetry group $\mathbf{L}_{12}\mathbf{C}_6\mathbf{R}_I$.

The group $\mathbf{LC}_n\mathbf{R}_I$ can be derived extending by the "centering" dilative rotation $L = L_{2n} = L(k, \pi/n)$ (with $k > 0$) the conformal symmetry group $\mathbf{C}_n\mathbf{R}_I$. Hence, conformal symmetry rosettes with the symmetry group $\mathbf{LC}_n\mathbf{R}_I$ can be constructed multiplying by the dilative rotation mentioned, a generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$, belonging to a fundamental region of the group $\mathbf{L}$ (Figure 4.22). The same conformal symmetry rosettes can be constructed by using an inversion with the inversion circle concentric to the generating rosette. In that case, after transforming it by the inversion, the image obtained must be rotated through the angle $\theta = \pi/n$. A fundamental region of the group $\mathbf{LC}_n\mathbf{R}_I$ is the section of fundamental regions of the groups $\mathbf{L}_{2n}$ and $\mathbf{C}_n\mathbf{R}_I$, and allows a varying of the form of non-inversional boundaries, while the remaining boundaries are defined by the concentric inversion circles (their corresponding arcs).

The influence of the dilative rotation $L = L_{2n}$ on a generating conformal symmetry rosette with the symmetry group $\mathbf{C}_n\mathbf{R}_I$ results in the formation of the visual impression of a spiral motion. All the other visual properties of the generating conformal symmetry rosette remain unchanged. Because of the rational dilative rotation angle $\theta = \pi/n$, there are sectors of the dilatation $K(k^n)$.

In ornamental art, apart from by those construction methods, groups of the type $\mathbf{LC}_n\mathbf{R}_I$ can be obtained by desymmetrizations of groups of the type $\mathbf{LD}_n\mathbf{R}_I$. Since the group $\mathbf{LC}_n\mathbf{R}_I$ is the subgroup of the index 2 of the group $\mathbf{LD}_n\mathbf{R}_I$, besides classical-symmetry desymmetrizations, very frequent are antisymmetry desymmetrizations, resulting in antisymmetry groups of the type $\mathbf{LD}_n\mathbf{R}_I/\mathbf{LC}_n\mathbf{R}_I$, discussed in the classical theory of symmetry within the type $\mathbf{LC}_n\mathbf{R}_I$.

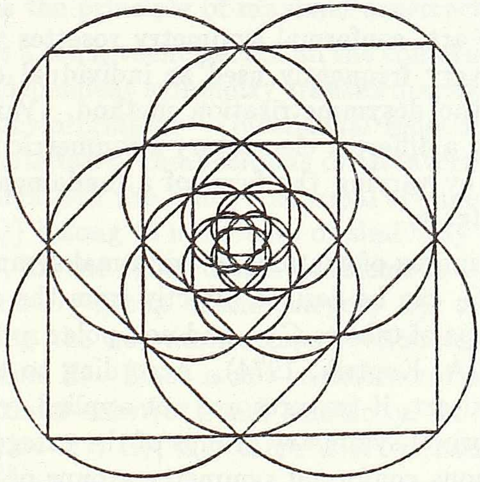


Figure 4.23

The conformal symmetry rosette with the symmetry group $\mathbf{LD}_4\mathbf{R}_I$.

The group $\mathbf{LD}_n\mathbf{R}_I$ can be derived extending the group $\mathbf{D}_n\mathbf{R}_I$ by the "centering" dilative rotation $L = L_{2n} = L(k, \pi/n)$ (with $k > 0$). In ornamental art, conformal symmetry rosettes with the symmetry group $\mathbf{LD}_n\mathbf{R}_I$ are very frequent, since they can be derived multiplying by the dilative rotation $L = L_{2n}$ a conformal symmetry rosette with a symmetry group of the type $\mathbf{D}_n\mathbf{R}_I$, the most frequent conformal symmetry group of the category C_2, which belongs to a fundamental region of the group $\mathbf{L}$ (Figure 4.23). The same conformal symmetry rosettes can be constructed transforming by a circle inversion with the concentric inversion circle $\mathbf{m}_I$, the generating rosette mentioned. Afterward, the image obtained must be rotated through the angle $\theta = \pi/n$. A fundamental region of the group $\mathbf{LD}_n\mathbf{R}_I$ is the section of fundamental regions of the groups $\mathbf{D}_n\mathbf{R}_I$ and $\mathbf{L}$. Hence, a varying of the form of a fundamental region is restricted to a change in the shape of boundaries that do not belong to the reflection lines or inversion circles.

The dilative rotation L produces, in the visual sense, a dynamic effect and gives the impression of a spiral motion. Although, since the "centered dihedral" similarity symmetry group $\mathbf{LD}_n$ is the subgroup of the index 2 of the group $\mathbf{LD}_n\mathbf{R}_I$, there exists a specific balance between static and dynamic visual components in conformal symmetry rosettes with the symmetry group $\mathbf{LD}_n\mathbf{R}_I$, and even a dominance of the static ones. Because of the rational dilative rotation angle $\theta = \pi/n$, there are sectors of the dilatation $K(k^n)$.

In ornamental art, conformal symmetry rosettes with the symmetry group $\mathbf{LD}_n\mathbf{R}_I$ are very frequently used as individual ones, or as a basis for applications of the desymmetrization method. Various examples are obtained by applying a different elementary asymmetric figure within a fundamental region or by varying the form of a fundamental region and the value of the parameter k.

The complete survey of continuous conformal symmetry groups of the categories C_{21} and C_2 can be derived directly from the data on the continuous symmetry groups of tablets G_{320} and non-polar rods G_{31}, respectively (A.V. Shubnikov, V.A. Koptsik, 1974). According to the restrictions imposed by ornamental art, if textures are not applied, visually presentable are continuous conformal symmetry groups of the category C_{21} of the type $\mathbf{D}_\infty\mathbf{R}_I$, and continuous conformal symmetry groups of the category C_2 of the types $\mathbf{KD}_\infty\mathbf{R}_I$, $\mathbf{K}_1\mathbf{C}_n\mathbf{R}_I$, $\mathbf{K}_1\mathbf{D}_n\mathbf{R}_I$, $\mathbf{L}_1\mathbf{D}_\infty\mathbf{R}_I$ and $\mathbf{L}_1\mathbf{C}_n\mathbf{Z}_I$, where a continuous dilatation group and continuous dilative rotation group is denoted by $\mathbf{K}_1$, $\mathbf{L}_1$, respectively. In terms of ornamental art, the most interesting are continuous conformal symmetry rosettes with the symmetry group $\mathbf{L}_1\mathbf{C}_n\mathbf{Z}_I$, which can be constructed multiplying by the n-fold rotation a logarithmic spiral — the invariant line of the continuous conformal symmetry group $\mathbf{L}_1\mathbf{Z}_I$. All the other visually non-presentable continuous conformal symmetry groups can be visually interpreted by using textures. Regarding the physical interpretations, all the continuous conformal symmetry groups of the categories C_{21} and C_2 can be modeled in the plane $E^2\setminus\{O\}$, by means of physical desymmetrization factors (e.g., by the uniform rotation around the singular point O, by adequate physical fields, etc.), as was done with the continuous symmetry groups of tablets G_{320} and rods G_{31}, isomorphic to them (A.V. Shubnikov, V.A. Koptsik, 1974).

* * *

As with all the previously discussed symmetry groups occurring in ornamental art, a significant prerequisite for their early appearance and frequent use in ornamental art is that they satisfy the principle of visual entropy — maximal constructional and visual simplicity and maximal symmetry. For many ornamental art motifs, their origin and use is not directly a function of the existence of models in nature with the corresponding symmetry. This especially refers to conformal symmetry rosettes. Hence, probably the most significant of the three mentioned criteria conditioning the time of origin and frequency of occurrence of different conformal symmetry groups in ornamental art, is the principle of maximal constructional simplicity.

An inversion is a constituent part of all the conformal symmetry transformations, as an independent symmetry transformation, or as a component of the composite transformations — inversional reflection Z_I or inversional rotation S_I, the commutative compositions of an inversion with a reflection or with n-fold rotation. All the other conformal symmetry transformations of the plane $E^2 \setminus \{O\}$ belong to isometries or similarity symmetry transformations. Therefore, all the construction problems in conformal symmetry rosettes with a symmetry group of the category C_{21} or C_2, can be solved, in principle, by using the non-metric construction to obtain homologous points of the inversion R_I. Since a non-metric construction fully satisfies the criterion of maximal constructional simplicity, for the conformal symmetry transformations R_I, Z_I, S_I, there is no reason to use the metric construction method.

The property of equiangularity is satisfied by every inversion and by all the other isometries and similarity symmetry transformations consisting of conformal symmetry groups. Hence, this property is an invariant of all the conformal symmetry groups. When constructing conformal symmetry rosettes with an elementary asymmetric figure of an arbitrary form, belonging to a fundamental region, as with the similarity symmetry groups of rosettes S_2, a construction of the type "point by point" is unavoidable. Since such a construction is very complicated, the invariance of the points of the inversion circle $\mathbf{m}_I$ and the fact that circles and lines are homologous figures of an inversion, expressed by the relationships:

$R_I(A) = A \iff A \in m_I$, $R_I(l) = l \iff O \in l$, $R_I(l) = c \iff O \notin l$ and $O \in c$, $R_I(c) = c_1 \iff O \notin c$ and $O \notin c_1$, $R_I(c) = c \iff c \perp m_I$ are a basis upon which we can simplify constructions of all conformal symmetry rosettes.

For conformal symmetry groups of the category C_{21}, this is sufficient for the construction of their visual interpretations. Besides the fact that they can be constructed multiplying by similarity transformations K, L, M a generating conformal symmetry rosette with a symmetry group of the category C_{21}, for a construction of the conformal symmetry groups of the category C_2 it is possible to use an inversion with the inversion circle concentric to the generating rosette mentioned. Due to the simplicity of constructions by circles and lines, in many cases this is the most suitable construction method. Whatever the approach is, the construction of visual interpretations of conformal symmetry rosettes with a symmetry group of the category C_2 is reduced to a multiplication of a conformal symmetry rosette with the conformal symmetry group of the category C_{21}. Since conformal symmetry groups of the category C_{21} are the extensions of the symmetry groups of rosettes $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm), every construction of conformal symmetry rosettes can be reduced to the following procedure: the transformation of a generating rosette with the symmetry group $\mathbf{C}_n$ (n) or $\mathbf{D}_n$ (nm) by two circle inversions with the concentric inversion circles, by rotations with the singular point O and by reflections with the reflection line containing the point O.

For constructions of conformal symmetry rosettes with the desired symmetry, the desymmetrization method is also used. This indirect construction method is mainly applied to the conformal symmetry rosettes of the types $\mathbf{D}_n\mathbf{R}_I$, $\mathbf{KD}_n\mathbf{R}_I$, $\mathbf{LD}_n\mathbf{R}_I$. Such conformal symmetry rosettes, possessing a high degree of symmetry and representing the most frequent conformal symmetry groups in ornamental art, are a suitable medium for constructions of other conformal symmetry rosettes of a lower degree of symmetry.

Using the desymmetrization method, besides classical-symmetry desymmetrizations, antisymmetry desymmetrizations can be used in all the cases when the desired symmetry group is the subgroup of the index 2 of a certain larger group. With conformal symmetry groups of the category C_{21}, a complete survey of them is given in the table of the antisymmetry desymmetrizations, i.e. of the corresponding conformal antisymmetry groups of the category C'_{21}. In this table, symbols of antisymmetry groups are given in the group/subgroup notation G/H.

The table of antisymmetry desymmetrizations of conformal symmetry groups of category C_{21}:

$\mathrm{N}_I/\mathrm{C}_n$	$\mathrm{C}_n\mathrm{R}_I/\mathrm{C}_n$	$\mathrm{D}_n\mathrm{R}_I/\mathrm{D}_n$
	$\mathrm{C}_{2n}\mathrm{R}_I/\mathrm{C}_n\mathrm{R}_I$	$\mathrm{D}_{2n}\mathrm{R}_I/\mathrm{D}_n\mathrm{R}_I$
$\mathrm{D}_1\mathrm{N}_I/\mathrm{D}_n$	$\mathrm{C}_{2n}\mathrm{R}_I/\mathrm{C}_n\mathrm{Z}_I$	$\mathrm{D}_n\mathrm{R}_I/\mathrm{C}_n\mathrm{R}_I$
$\mathrm{D}_1\mathrm{N}_I/\mathrm{N}_I$		$\mathrm{D}_{2n}\mathrm{R}_I/\mathrm{D}_1\mathrm{N}_I$
$\mathrm{D}_1\mathrm{N}_I/\mathrm{C}_n\mathrm{Z}_I$	$\mathrm{C}_n\mathrm{Z}_I/\mathrm{C}_n$	$\mathrm{D}_n\mathrm{R}_I/\mathrm{C}_n\mathrm{Z}_I$
	$\mathrm{C}_{2n}\mathrm{Z}_I/\mathrm{C}_n\mathrm{Z}_I$	

Besides being the basis for applications of the antisymmetry desymmetrization method, this table is an indicator of all the subgroups of the index 2 of conformal symmetry groups of the category C_{21}.

The complete derivation and catalogue of conformal antisymmetry groups of the categories C'_{21} and C'_2 is given by S.V. Jablan (1985).

Information on some possible color-symmetry desymmetrizations of crystallographic conformal symmetry groups of the categories C_{21} and C_2, can be obtained from the work of A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant (1978), A.F. Palistrant (1980c), and E.I. Galyarski (1986), who discuss the color-symmetry groups of tablets G_{320} and non-polar rods G_{31}.

With conformal symmetry groups of the category C_{21}, it is possible to establish a connection between these and the corresponding symmetry groups of friezes G_{21}. The following relationships hold: $\mathbf{C}_n\mathbf{R}_I \simeq$ **m1**, $\mathbf{D}_n\mathbf{R}_I \simeq$ **mm**, $\mathbf{C}_n\mathbf{Z}_I \simeq$ **12**, $\mathbf{D}_1\mathbf{N}_I \simeq$ **mg**, $\mathbf{N}_I \simeq$ **1g**. In this way, the problem of color-symmetry groups derived from conformal symmetry groups of the category C_{21} can be reduced to the color-symmetry groups of friezes (J.D. Jarratt, R.L.E. Schwarzenberger, 1981), i.e. to the use of the table of color-symmetry desymmetrizations of the corresponding symmetry groups of friezes. In doing so, it is necessary to be aware on the identification $\mathrm{p}_x^n = E$.

Conformal symmetry tilings of the plane $E^2 \setminus \{O\}$, are discussed by E.A. Zamorzaeva (1985). In this work, a connection is established between the different types of conformal symmetry groups of the category C_2, the symmetry groups of non-polar rods G_{31} and the symmetry groups of ornaments G_2. The following relationships hold : $\mathbf{KC}_n\mathbf{Z}_I$, $\mathbf{LC}_n\mathbf{Z}_I \simeq$ **p2**, $\mathbf{KN}_I \simeq$ **pg**, $\mathbf{KC}_n\mathbf{R}_I \simeq$ **pm**, $\mathbf{LC}_n\mathbf{R}_I \simeq$ **cm**, $\mathbf{MN}_I \simeq$ **pgg**, $\mathbf{KD}_1\mathbf{N}_I$, $\mathbf{MC}_n\mathbf{R}_I \simeq$ **pmg**, $\mathbf{LD}_n\mathbf{R}_I \simeq$ **cmm**, $\mathbf{KD}_n\mathbf{R}_I \simeq$ **pmm**. Using such an approach, the problem of isogonal tilings, which correspond to conformal symmetry groups of the category C_2, is solved.

The discussion on the visual properties of conformal symmetry groups of the category C_{21}, can be reduced to an analysis of the effects of the conformal transformations R_I, Z_I, S_I on generating rosettes with the symmetry groups $\mathbf{C}_n$ (n), $\mathbf{D}_n$ (nm).

In groups of the type $\mathbf{C}_n\mathbf{R}_I$, the inversion R_I causes the absence of the enantiomorphism, existing in the symmetry group $\mathbf{C}_n$ (n). The intensity of the static visual impression produced by an inversion depends on the position and form of an elementary asymmetric figure belonging to a fundamental region of the conformal symmetry group containing this inversion. It comes to its full expression only for figures that are, by their shape, close to the inversion circle. In the geometric sense, the inversion R_I causes the constancy of the shape of the boundary of a fundamental region, which coincides with the inversion circle $\mathbf{m}_I$ (its arc), and the non-polarity of radial rays.

The inversional rotation S_I mainly keeps the properties of generating rosettes with symmetry groups of the category G_{20} and somewhat intensifies their dynamic visual properties. The inversional reflection Z_I causes the bipolarity of rotations and of radial rays, and preserves the property of the enantiomorphism. The dynamic or static visual properties of conformal symmetry rosettes with symmetry groups of the category C_{21} will depend on the analogous properties of generating rosettes with symmetry groups of the category G_{20}.

Conformal symmetry groups of the category C_2 are derived extending by the similarity transformations K, L, M, conformal symmetry groups of the category C_{21}. The dilatation K and dilative rotation L maintain all the geometric-visual properties of the generating conformal symmetry groups of the category C_{21} and introduce a new dynamic visual component — a suggestion of centrifugal expansion, or of rotational centrifugal expansion. The dilative reflection M, in the visual sense, produces the impression of centrifugal alternating expansion. In the geometric sense, it eliminates the possibility for the enantiomorphism.

The form of a fundamental region of conformal symmetry groups is defined by the invariance of all the points of inversion circles and reflection lines. In this way, the conformal symmetry groups of the types $\mathbf{N}_I$, $\mathbf{C}_n\mathbf{Z}_I$, $\mathbf{KN}_I$, $\mathbf{KC}_n\mathbf{Z}_I$, $\mathbf{LC}_n\mathbf{Z}_I$, $\mathbf{MN}_I$, offer the possibility to change the shape of all the boundaries of a fundamental region; groups of the types $\mathbf{C}_n\mathbf{R}_I$, $\mathbf{MC}_n\mathbf{R}_I$, $\mathbf{LC}_n\mathbf{R}_I$ offer the possibility to change the shape of non-inversional boundaries; groups of the types $\mathbf{D}_1\mathbf{N}_I$, $\mathbf{KD}_1\mathbf{N}_I$ offer the possibility to change the shape of non-reflectional boundaries; groups of the type $\mathbf{LD}_n\mathbf{R}_I$ type offer the possibility to change the shape of non-reflectional and non-inversional boundaries, while groups of the types $\mathbf{D}_n\mathbf{R}_I$, $\mathbf{KD}_n\mathbf{R}_I$, generated by reflections and inversions, do not offer the possibility to change the shape of boundaries of a fundamental region.

In conformal symmetry groups that do not require the constancy of the form of a fundamental region, a variety of corresponding conformal symmetry rosettes in ornamental art is achieved by varying the boundaries of a fundamental region or the form of an elementary asymmetric figure belonging to a fundamental region. In the remaining conformal rosettes the variety is achieved exclusively by the second of these possibilities.

In the geometric-visual sense, the inversion R_I represents an adequate interpretation of "two-sideness" in the "one-sided" plane, i.e. the interpretation of the symmetry transformation of the space E^3 — the plane reflection in the invariant plane of the symmetry groups of tablets G_{320}, in the plane $E^2 \setminus \{O\}$. In the same way, because of the isomorphism between the symmetry groups of tablets G_{320} and conformal symmetry groups of the category C_{21}, and the isomorphism between the symmetry groups of non-polar rods G_{31} and conformal symmetry groups of the category C_2, apart from the schematic visual interpretations — Cayley diagrams and tables of the graphic symbols of symmetry elements — conformal symmetry rosettes represent a completely adequate visual model of the symmetry groups of tablets G_{320} and non-polar rods G_{31}. The symmetry groups of polar rods G_{31} possess a similar visual interpretation in the plane E^2 — similarity symmetry rosettes . On the basis of those isomorphisms, the presentations, the geometric and visual properties of conformal symmetry groups of the categories C_{21}, C_2 can be fully transferred, respectively, to the symmetry groups of tablets G_{320} and non-polar rods G_{31}.

In the table of the group-subgroup relations (Figure 4.24), a survey is given of all the group-subgroup relations between the visually-presentable continuous conformal symmetry groups, the discrete conformal symmetry groups of the categories C_{21}, C_2 and group-subgroup relations between conformal symmetry groups of the category C_2 and the similarity symmetry groups of the category S_{20}. Although incomplete, as they do not include all the group-subgroup relations but only the most important ones, the tables can serve as a basis on which to apply the desymmetrization method for obtaining conformal symmetry groups or similarity symmetry groups and also for the geometric-visual evidence of symmetry substructures of conformal symmetry groups. Aiming for a more complete consideration of those problems, the given tables can be used with the analogous tables corresponding to similarity symmetry groups of the category S_{20}.

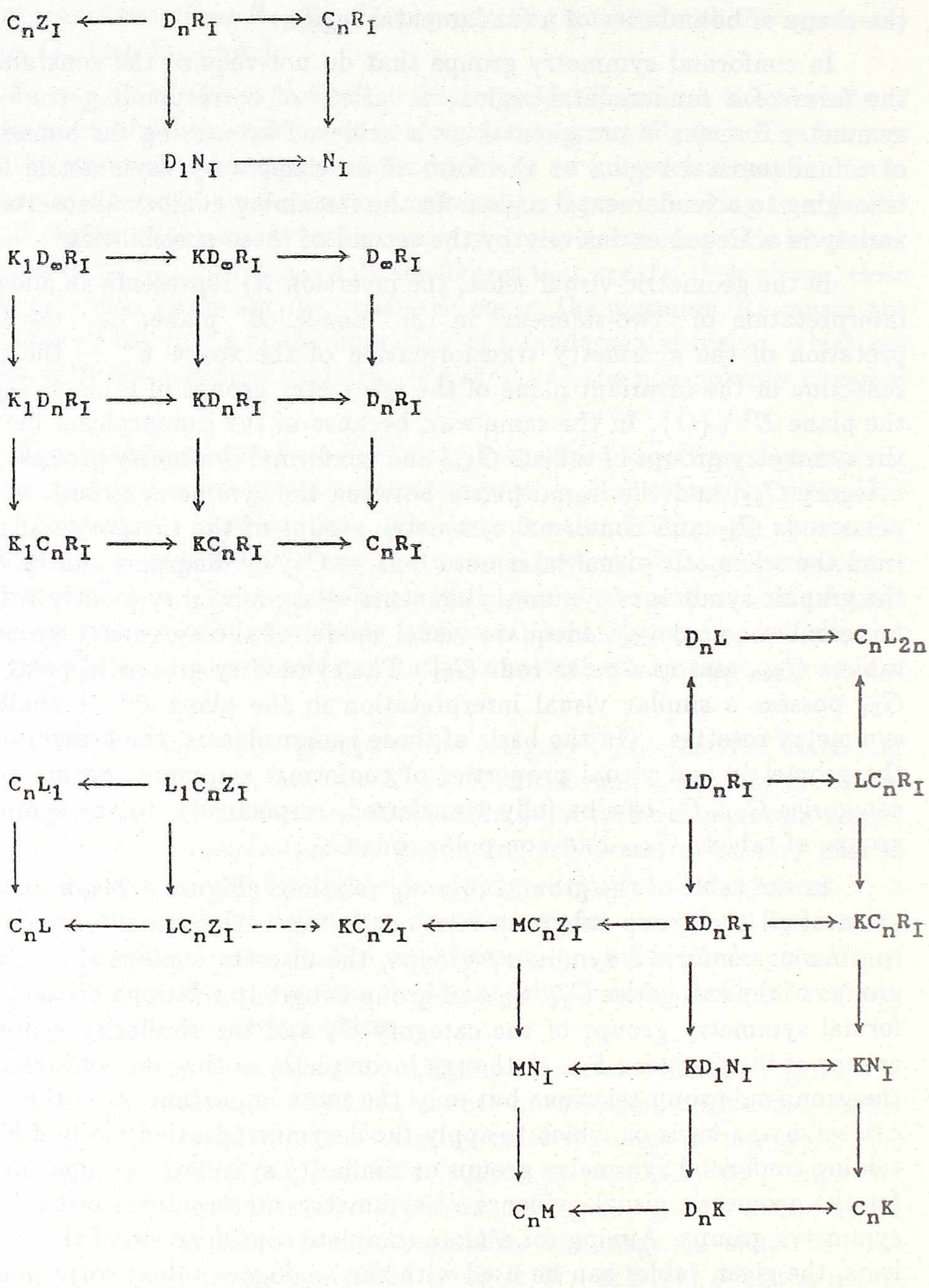

C_nZ_I
D_nR_I
C_nR_I
D_1N_I
N_I
$K_1D_\infty R_I$
$KD_\infty R_I$
$D_\infty R_I$
$K_1D_nR_I$
KD_nR_I
D_nR_I
$K_1C_nR_I$
KC_nR_I
C_nR_I
D_nL
C_nL_{2n}
C_nL_1
$L_1C_nZ_I$
LD_nR_I
LC_nR_I
C_nL
LC_nZ_I
KC_nZ_I
MC_nZ_I
KD_nR_I
KC_nR_I
MN_I
KD_1N_I
KN_I
C_nM
D_nK
C_nK

Figure 4.24

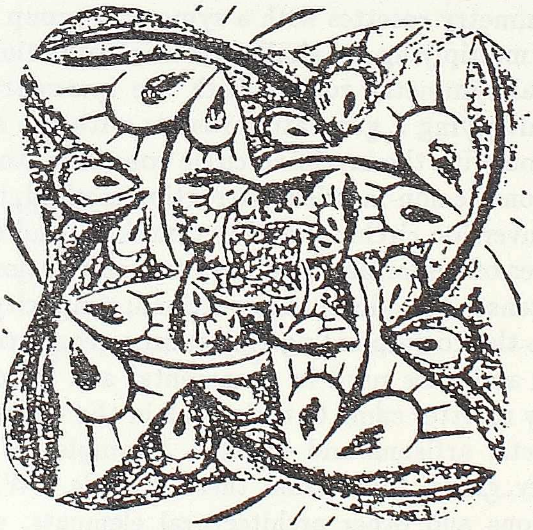
The time and frequency of occurrence of different conformal symmetry groups in ornamental art are related to the periods when various constructional problems were solved. According to the criterion of maximal constructional and visual simplicity, constructions of conformal symmetry rosettes are mostly based on the use of circles and lines as homologous elements of conformal symmetry transformations. Combinations of elementary geometric figures (regular polygons, circles) with a common singular point, found in the earliest periods of ornamental art, gave as a result the first examples of conformal symmetry rosettes with symmetry groups of the category C_{21}, mostly of the type $\mathbf{D}_n\mathbf{R}_I$ (n=1,2,3,4,6,8,...). In the further development of ornamental art, examples of all other conformal symmetry groups of the category C_{21} appeared. The dominance of visually static conformal symmetry rosettes with a higher degree of symmetry, is expressed throughout the history of ornamental art.

Very important in the formation of conformal symmetry rosettes was the existence of certain models in nature — the flowers of different plants, forms of growth, etc., possessing or suggesting different kinds of conformal symmetry.

Conformal symmetry rosettes with a symmetry group of the category C_2 are constructed multiplying by similarity transformations K, L, M, a generating conformal symmetry rosette with the symmetry group of the category C_{21}, or multiplying a generating rosette with the same symmetry group, by an inversion with the inversion circle concentric to it. The second construction, based on the non-metric construction method, using invariance of all the points of inversion circles and reflection lines, and circles and lines as homologous figures of conformal symmetry transformations, offers better possibilities, in the sense of maximal constructional simplicity. It came to its fullest expression in the corresponding elementary geometric constructions by means of circles and lines used in ornamental art. In ornamental art, conformal symmetry rosettes came to their peak in the work of Romanesque and Gothic architects, artisans and artists. Examples of almost all the conformal symmetry groups date from these periods. When calculating a building proportions and other architectural elements, especially when drawing-up the plans of the decorative architectural elements — window and floor rosettes — Medieval architects used those constructions.

Directly linked to these problems, and covered by the theory of similarity symmetry and conformal symmetry, are the questions of the theory of proportions, the roots of which date from Greek geometry. It held a special place in Medieval and Renaissance architectural planning and it reached its fullest expression in applications of the "*aurea sectio*" (or the "golden section") and musical harmonies used in architecture and in the visual arts. In more recent periods, examples of conformal symmetry rosettes with symmetry groups of the category C_2 can be found in the work of M.C. Escher (1971a, b; 1986) (Figure 4.25), who, besides classical-symmetry, often used conformal antisymmetry and color-symmetry rosettes, and greatly contributed to the analysis of different conformal color-symmetry groups and conformal tilings.

The problems of visual perception, referring here to conformal symmetry rosettes, can be solved analogously to the same problems previously discussed in the other categories of symmetry groups, through the analysis of the symmetrization and desymmetrization factors caused by the visual effects of the physiological-psychological elements of visual perception.



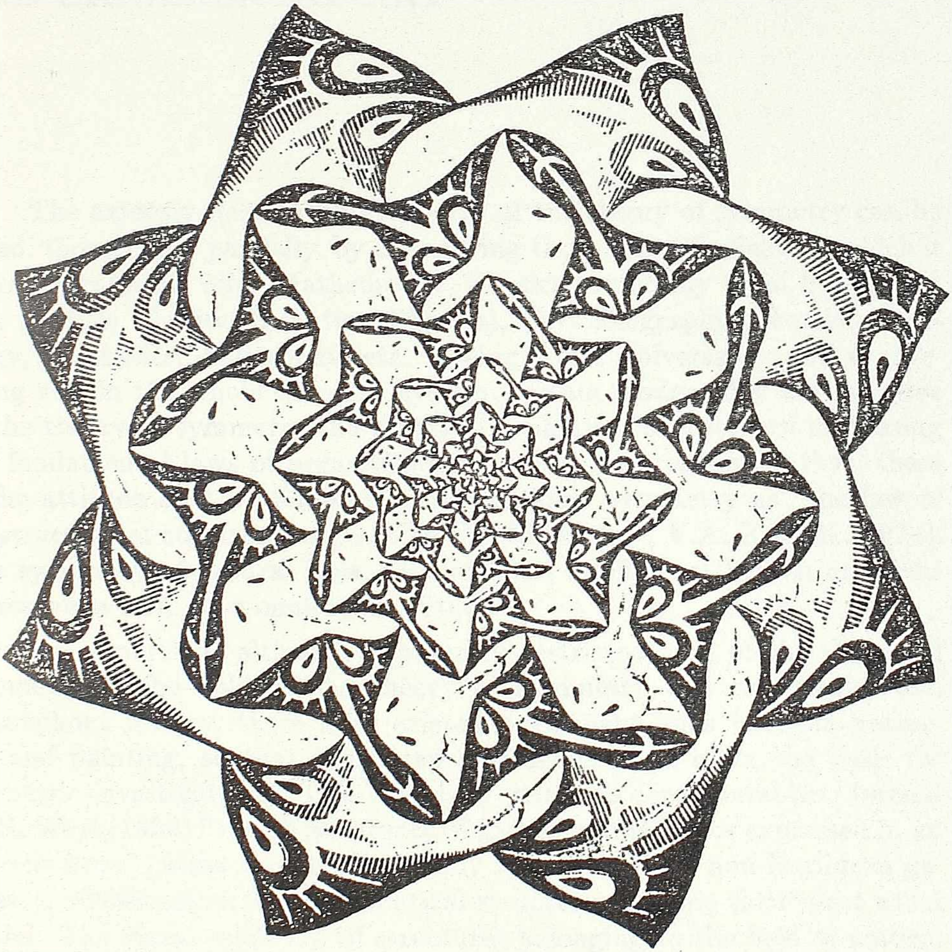


Figure 4.25

Examples of conformal symmetry rosettes from the work of M.C. Escher.

The approach to ornamental art from the theory of symmetry, makes possible the recognition, classification and the exact analysis of all the various kinds of conformal symmetry rosettes occurring in ornamental art, and also highlights the different possibilities for constructing these conformal symmetry rosettes possessing the symmetry and geometric-visual properties already anticipated.

Chapter 5

THE THEORY OF SYMMETRY AND ORNAMENTAL ART

The extensiveness and universality of the theory of symmetry can be noted, though only partially, by considering those scientific fields in which it plays a significant role: Mathematics, Physics (especially Solid State Physics, Particle Physics, Quantum Physics), Crystallography, Chemistry, Biology, Aesthetics, Philosophy, etc. Owing to its universality and synthesizing role in the whole scientific system, certain modern-day authors give to the theory of symmetry the status of a philosophic category expressing the fundamental laws of organization in nature. According to that, there is the attitude of A.V. Shubnikov, who defined symmetry as "the law of construction of structural objects" (A.V. Shubnikov, V.A. Koptsik, 1974). The symmetry of natural laws, material and intellectual human creations represents a form of symmetry in nature.

An important, although apparently restricted area of the theory of symmetry is the field of the theory of symmetry and ornamental art. Throughout history, there have existed permanent links between geometry and painting, so that visual representations were often the basis for geometric investigations. This especially refers to ornamental art, termed by H. Weyl (1952) "the oldest aspect of higher mathematics expressed in an implicit form". More recently, especially for the needs of non-Euclidean geometry, visualizations of mathematical structures became their most usual model. The visual modeling of structures belonging to the field of natural sciences (Physics, Crystallography, Biology, Chemistry, etc.) by means of different visual representations — diagrams, graphics, graphic symbols of symmetry elements, Cayley diagrams, graphs, etc. — brought into being a complete visual language for the expression and representation of symmetry structures. The long lasting interaction between geometry and painting, especially present in the periods of scientific and art synthesis (e.g., in Egyptian, Greece, Renaissance science and art) is reflected in the simultaneity of the most significant epochs, tendencies and ideas in geometry and painting. In the modern period, this connection is expressed as a prolific exchange of experience between mathematicians, scientists and artists, especially in the period of the formation and domination of the geometric abstraction (e.g., in work of P. Mondrian, K. Malevich, V. Vasarely). There are also examples of direct cooperation between artists and scientists (e.g., the long standing contacts between H.S.M. Coxeter and M.C. Escher, and the retrospective exhibition of M.C. Escher during the International Congress of the World Crystallographic Union in Cambridge, 1960) and their joint projects (M.C. Escher, 1971a, b, 1986; C.H. Macgillavry, 1976). Progress in the field of visual communications (press, photography, film, TV) and its further development open new possibilities for the visual representation of different symmetry structures — the subjects of scientific studies — thus making new scientific knowledge more accessible and expressible in a more comprehensible form.

The very roots of the theory of symmetry (in Greece) are inseparably linked to the establishment of the aesthetic principles — the canons and theory of proportions. The links between the theory of symmetry and aesthetics developed and were strengthened throughout history, where works of ornamental art represented the common ground between the theory of symmetry and painting. A new motive for the analysis and revision of aesthetic criteria was the appearance of abstract painting, especially involving geometric abstraction. Since figurative painting works are usually based on reality — on models found in nature — in the field of the aesthetics of painting there existed criteria not connected to visuality: e.g., contents of myths, degree of realism, etc. Non-figurative painting, abstract and ornamental, pointed out the inconsistency and incompleteness of aesthetics founded on the classical basis and the necessity to construct new visual-aesthetic criteria, formed according to the theory of symmetry (asymmetry, dissymmetry, antisymmetry, colored symmetry, curved symmetry, etc.).

The fundamental role of symmetry in the art is not exhausted by its connection with ornament or geometric abstraction. Art historians often used symmetry to characterize the formal qualities of a work of art, distinguishing symmetry as a basic principle of all artistic rules — the canons, laws of composition, criteria of well-balanced form... As the most significant property of harmony and regularity, symmetry is one of the main organizational principles in every art: painting, sculpture, architecture, music, dance, poetry... Even in the most extreme modern art — conceptualism or minimalism, it lays in their intellectual background.

This work is restricted to the area of ornamental art. This is so because of the fact that examples of visual interpretations of symmetry groups occur most frequently in their explicit form in this very area. They occur, also, in other fields of painting, but mostly in an implicit form. The possibilities of the theory of symmetry and ornamental art combined are, due to the extensiveness of their application (covering science, aesthetics, visual communications, etc.) far greater than those of ornamental art itself.

In this work, discrete symmetry groups formed by isometries and similarity symmetries in the plane E^2 and also by conformal symmetries in the plane $E^2 \setminus \{O\}$, are discussed. The condition of discreteness directly causes the existence of a bounded or unbounded fundamental region of a discrete symmetry group.

In the plane E^2 and $E^2 \setminus \{O\}$, the following categories of the symmetry groups of isometries were discussed: the symmetry groups of rosettes G_{20}, friezes G_{21}, ornaments G_2 , similarity symmetry groups S_{20} and conformal symmetry groups C_{21} and C_2. The maintenance of the metric properties of figures, holding for symmetry groups of isometries, for similarity symmetry groups is replaced by the similarity condition, so that the properties of equiformity and equiangularity remain preserved. In conformal symmetry groups, only the property of equiangularity holds. Similarity symmetry groups S_{20} and conformal symmetry groups C_{21} and C_2 are characterized, respectively, by their isomorphism to symmetry groups of polar rods G_{31}, tablets G_{320} and non-polar rods G_{31}, making possible an adequate visual interpretation of these symmetry groups of the space E^3 in the plane E^2 ($E^2 \setminus \{O\}$). Further extensions of the theory of symmetry leading toward elliptic and hyperbolic groups of symmetry and their interpretations in the inversive, conformal plane can be achieved without eliminating the condition of equiangularity, preserved by all the conformal symmetry transformations. A final consequences of such extensions are homology and curved symmetry groups belonging to a family of discrete affine and topological groups. That approach is in accord with the concept expressed in the Erlangen program by F. Klein (1872), who proposed a derivation of sequences of symmetry group extensions and their generalizations.

From the standpoint of the theory of symmetry and ornamental art, it is also considerable extension of the classical theory of symmetry to antisymmetry and colored symmetry. They are not the main topic of this work, so we have discussed only those cases where antisymmetry or colored symmetry is used for deriving classical-symmetry groups by desymmetrizations. Antisymmetry and colored symmetry have been discussed in detail by A.V. Shubnikov, N.V. Belov et al. (1964), A.L. Loeb (1971), A.V. Shubnikov, V.A. Koptsik (1974), A.M. Zamorzaev (1976), B. Grünbaum, G.C. Shephard (1977b, 1983, 1987), A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant (1978), M. Senechal (1979), J.D. Jarratt, R.L.E. Schwarzenberger (1984), T.W. Wieting (1981), R.L.E. Schwarzenberger (1984), S.V. Jablan (1984a, b, c, 1985, 1986a, b), H.S.M. Coxeter (1985, 1987), A.M. Zamorzaev, Yu.S. Karpova, A.P. Lungu, A.F. Palistrant (1987), etc.

When analyzing the visual characteristics of ornaments, the most fundamental appears to be the principle of visual entropy — maximal constructional and visual simplicity and maximal symmetry. From the examples and the accompanying arguments given in this work we can conclude that the appearance and frequency of occurrence of different symmetry groups in ornamental art are conditioned by the degree of agreement of corresponding ornaments with this principle. The principle of visual entropy is an affirmation of the universal natural principle of economy — its affirmation in the field of visuality. The individual components of this principle — the criterion of maximal visual and constructional simplicity and maximal symmetry and their role have been analyzed regarding all the categories of plane symmetry structures discussed. The maximal degree of symmetry is inseparably connected to constructional and visual simplicity. This is proved in ornamental art by the chronological priority and domination of examples of visually presentable continuous symmetry groups and maximal symmetry groups generated by reflections or their equivalents, containing all the other symmetry groups of the same category as subgroups. The principle of maximal symmetry represents the basis of the desymmetrization method, thus making possible the derivation of symmetry groups of a lower degree of symmetry by a desymmetrization of groups of a higher degree of symmetry, examples of which appear in ornamental art much earlier and occur more often. This can be achieved by classical-symmetry, antisymmetry and color-symmetry desymmetrizations. A consequent application of the desymmetrization method makes possible the realization of all symmetry groups as subgroups of the maximal symmetry groups of certain categories, where these subgroups can be derived directly from them or by sequences of successive desymmetrizations. In such a complementary approach to the theory of symmetry, asymmetry could be understood as a positive-defined property, in the sense of desymmetrization transformations, and not only as a negative-defined property — the absence of symmetry (I.D. Akopyan, 1980).

The principle of visual entropy is in accord with the standpoint of Gestalt psychology in considering the theory of visual perception (R. Arnheim, 1975, 1979). This point of view underlines the primary importance of the perception of a whole (Gestalt) and its essential structural organization laws, among which symmetry occupies an important position. That means the priority of the synthetic part of the visual perception. The position of Gestalt psychology has been greatly strengthened by recent research work on the physiological and psychological basis of visual perception. One of the arguments proving this is, for example, the study of chronology and frequency of occurrence of certain aspects of symmetry existing in ornamental art, outlined in this book.

By the term "visual simplicity", the stationariness or dynamism of symmetry structures is considered, where the dynamism of a visual object in many cases coincides with the complexity of its form and structure, resulting in difficulties when registering its symmetry. The enantiomorphism is the possibility of existence of the "left" or "right" form of discussed symmetry structure — their enantiomorphic modifications. The absence of enantiomorphism is the result of the existence of at least one indirect symmetry transformation within the symmetry group discussed. Somewhat more restricted, the term "polarity" introduces questions of polarity, non-polarity and bipolarity of symmetry elements and their corresponding invariant lines. The non-polarity of a certain symmetry element is conditioned by the existence of adequate reflections (inversions) commuting with it, while bipolarity is caused by the presence of adequate central reflections or their equivalents.

The polarity of generators of a symmetry group, especially of generators of the infinite order, introduces into ornamental art the time component — a suggestion of movement — that can be seen in polar rosettes, friezes and ornaments, or similarity symmetry rosettes producing the visual impression of centrifugal expansion. The polarity considerably affects the degree of visual dynamism. A dynamic visual effect produced by polar generators can be stressed or lessened by the right choice of the relevant visual parameters (e.g., by using acute-angular forms oriented toward the orientation of a polar element of symmetry, by changing the form of a fundamental region or the form of an elementary asymmetric figure belonging to it, or by choosing the coefficient of dilatation at the similarity symmetry groups S_{20}, etc.).

The chronology and frequency of occurrence of certain visual examples of symmetry groups are caused also by the principle of maximal constructional simplicity. Because visually presentable continuous symmetry groups and maximal discrete symmetry groups generated by reflections or their equivalents are, also, the simplest ones in a constructional and in a visual sense, constructional simplicity is directly connected to the degree of symmetry.

In the oldest phases of ornamental art, after the intuitive and empirical perception of construction problems, the solutions to these problems and the forming of adequate construction methods came. According to the criterion of maximal constructional simplicity, direct, non-metric construction methods prevail in ornamental art in all the cases where their application is possible.

The problem of the exact construction of rosettes with symmetry groups of the category G_{20} is reduced to the question of the construction of regular polygons, which is possible for the polygons with $2^m p_1 p_2 \ldots p_n$ sides, where $p_1, p_2, \ldots, p_n$ are the prime Fermat numbers and $n \in N$, $m \in N \cup \{0\}$, while in other cases only approximate constructions are possible. Although approximate constructions are often used in ornamental art, in the history of rosettal ornaments there is an apparent domination of rosettes with rotations of the order $n = 1, 2, 3, 4, 6, 8, 12, \ldots$, while, for instance, rosettes with rotations of the order $n = 7, 9, \ldots$ are extremely rare. A more detailed analysis of the causes of this, (e.g., the existence of natural models) is given in Chapter 2.

Symmetry groups of friezes G_{21} possess a high degree of constructional simplicity. Among other reasons, this caused the appearance of all seven symmetry groups of friezes in Paleolithic ornamental art. Usually, friezes are constructed by the method of rosette multiplication, where discrete friezes are derived multiplying by means of a discrete translation or glide reflection a certain rosette with the symmetry group C_1 (1), C_2 (2), D_1 (m), D_2 (2m) — that means, by an extension from the "local symmetry" of the symmetry groups of finite friezes G_{210} to the "global symmetry" of the symmetry groups of friezes G_{21}.

Apart from their independent use, friezes are used for constructing ornaments with symmetry groups of the category G_2. In Paleolithic ornamental art, there occurred a multiplication of friezes by some other isometry — by a translation, reflection, half-turn or, maybe, by a glide reflection. The results obtained were superpositions of friezes — plane ornaments with the absence of a rotation of an order greater than 2. From the Paleolithic date the five plane Bravais lattices corresponding to the maximal symmetry groups of relevant syngonies, which are, at the same time, the basis for applying the method of rosette multiplication for constructing plane ornaments — the multiplication of a rosette with the symmetry group $\mathbf{C}_n$ $(\mathbf{n})$, $\mathbf{D}_n$ $(\mathbf{nm})$ $(\mathbf{n} = 1, 2, 3, 4, 6)$ by the discrete translational symmetry group of ornaments **p1**. This method for constructing ornaments, which demands a knowledge of plane Bravais lattices, was much more used in Neolithic ornamental art. Interesting empirical results obtained by its use can be traced in Moorish ornamental art, where experiments with rotations of the order 5 were made, representing an empirical analysis of the principle of crystallographic restriction (W. Barlow, 1894).

Constructions of similarity symmetry rosettes are based on the multiplication of a rosette with the symmetry group $\mathbf{C}_n$ $(\mathbf{n})$, $\mathbf{D}_n$ $(\mathbf{nm})$ by a similarity symmetry transformation K, L or M, with a common invariant point. Constructional difficulties occurring with the non-metric method of construction, artists try to solve by using also the metric method, but this often results in deviations from the similarity symmetry, caused by the inconsequent application of the metric construction method.

Since the non-metric construction method fully satisfies the criterion of maximal constructional simplicity, it prevails when constructing conformal symmetry rosettes with the symmetry groups of the category C_{21} or C_2. By applying this method, visual examples of finite C_{21} and infinite C_2 groups of conformal symmetry in $E^2 \setminus \{O\}$ can be derived by multiplying some rosette with the symmetry group $\mathbf{C}_n$ $(\mathbf{n})$, $\mathbf{D}_n$ $(\mathbf{nm})$, by means of conformal symmetry transformations R_I, Z_I, S_I and similarity symmetry transformations K, L, M. When doing this, aiming to simplify the construction, we can use the correspondence between circles and lines as homologous figures of conformal symmetry transformations, the invariance of all the points of reflection lines and inversion circles, and the invariance of circles perpendicular to inversion circles.

The desymmetrization method is a universal, dependent construction method, consisting of classical-symmetry, antisymmetry and color-symmetry desymmetrizations, making it possible to obtain visual examples of all the subgroups of a given symmetry group. It is applied mostly on continuous visually presentable symmetry groups or on discrete groups of a high degree of symmetry — the symmetry groups of rosettes $\mathbf{D}_n$ (nm), friezes mm, ornaments pmm, p4m, p6m, similarity symmetry groups $\mathbf{D}_n\mathbf{K}$ (nmK) and finite and infinite conformal symmetry groups $\mathbf{D}_n\mathbf{R}_I$ and $\mathbf{KD}_n\mathbf{R}_I$. By antisymmetry desymmetrizations we can derive all the subgroups of the index 2 of a given group. Therefore, they can be used to find all its subgroups of the index 2. As for the period of origin, antisymmetry and color-symmetry desymmetrizations are somewhat younger than classic-symmetry desymmetrizations, and they first appear in the Neolithic, with the beginning of dichromatic and polychromatic ceramics. The consequent use of the desymmetrization method requires knowledge of the tables of group-subgroup relations.

Owing to their high degree of symmetry, and maximal visual and constructional simplicity, continuous symmetry groups belong to the oldest and most frequent symmetry groups used in art. Without using textures, only continuous symmetry groups with non-polar continuous elements of symmetry are visually presentable. Because of that, their application in ornamental art is strongly restricted. The first examples of visually presentable continuous rosettes, friezes, semicontinua and continua date from the Paleolithic and belong to the family of elementary geometric figures — circles, lines, parallels, spirals — representing the oldest expression of human geometric perception and knowledge. The possibilities for the visual representation of continuous symmetry groups can be extended with textures. Continuous visually presentable symmetry groups fully satisfy the criterion of visual entropy, so besides having a special role as independent symmetry groups, they serve as the most important source for deriving symmetry groups of a lower degree of symmetry by a desymmetrization. The link between the approaches existing in ornamental art and in the theory of symmetry is apparent regarding their similar origins and the use of similar construction methods.

In studying symmetry structures, besides objective elements conditioning a visual impression, subjective factors — e.g., physical, physiological, or psychological — also play their part. Regarding a symmetry structure itself, they can represent symmetrization or desymmetrization factors. The most important of them are — the influence of human plane symmetry and binocularity, the position of the symmetry structure regarding the fundamental natural directions, the effect of orientation ("left" or "right"), the influence of the symmetry of surrounding structures, etc. The fact that realistic ornaments represent finite parts of "ideal" ornaments — their factor groups — can also have a great influence on the visual impression produced by a certain ornament. Therefore, aiming for a more thorough analysis from the visual point of view, realistic plane symmetry structures should be considered as the result of the interaction between all the objective and subjective factors mentioned.

Similar problems appear in attempting to perceive all the symmetry substructures of a given symmetry structure. In this case, elements of symmetry of the larger structure represent, with respect to substructures, the secondary visual symmetrization or desymmetrization factors. Visual perception of substructures and of their symmetry can be immensely simplified by using tables of group-subgroup relations.

The theory of symmetry is one of the most efficient means for studying the principles of balance and harmony in art. Since symmetry is one of the basic structure-organization laws in nature, the existence of natural models was one of the origins of ornaments and an inexhaustible source of ideas during all the history of ornamental art. In the field of rosettal ornaments, frequently used basic models were objects with the mirror symmetry D_1 (m), shapes with the symmetry group D_2 (2m) expressing the relation between a vertical and horizontal line, and rosettes with the symmetry groups C_n (n), D_n (nm), $n = 1, 2, 3, 4, 5, 6 \ldots$ corresponding to the symmetry of certain plants, flowers and some other forms of life (e.g., a starfish, jellyfish). The basis of friezes are the models found in nature, the distribution of leaves on plants, the shape of waves, the periodic character of many natural phenomena (the turn of day and night, the phases of the Moon, the seasons of the year, etc.). Resulting from there we have the primary calendar role of friezes, witnessed by the names of many friezes preserved in the art of primitive peoples (R. Smeets, 1975).

As complex plane symmetry structures, besides imitating natural models — honeycombs or net structures — many ornaments are the result of a human longing to express regularity and to construct perfect visual forms — discrete regular plane tilings. The similarity symmetry rosettes and infinite conformal symmetry rosettes produce, in the visual sense, the impression of centrifugal expansion, giving an adequate visual interpretation of the basic natural tendency of living matter — growth, directly connected with spiral structures. The spiral itself is one of oldest archetypical dynamic symbols used in art, occurring in nature in different types of snails, flowers and plants.

This study of the origin and development of ornamental art has been based mainly on examples of the oldest ornaments from the Paleolithic, Neolithic, the art of the ancient civilizations, and the native art of primitive peoples. After an intuitive-empirical perception of symmetry by the use of natural models (in the Paleolithic) and by solving elementary construction problems, the first symbolic meanings of ornaments were formed. In time, the visual-symbolic language of ornaments became a specific form of visual communication. After having solved constructional problems, in ornamental art there, then came the phase of an empirical analysis of the visual form of ornaments: by bringing into accord the visual form and the symbolic meaning, by solving the problems of visual stationariness or dynamism and orientation, and by studying tessellations and perfect ornamental forms. These were the first attempts to achieve a desired visual effect through the choice of relevant visual parameters.

The methods for obtaining different ornaments with the same symmetry group, by a change of the shape of the fundamental region or of the shape of an elementary asymmetric figure belonging to it, can be traced in the ornamental art of Neolithic, the ancient civilizations and primitive peoples. In this way, a special contribution to ornamental art is represented by Islamic ornaments (D. Hill, O. Grabar, 1964; K. Critchlow, 1976) and Moorish ornaments, and recently by the work of M.C. Escher (1971a, b, 1986).

Regarding the form of the fundamental region, we can distinguish visually static groups in the strictest sense — groups generated by reflections and their equivalents (circle inversions) — not allowing any kind of change of the boundary of a fundamental region, from the other symmetry groups allowing a change of boundaries not belonging to the reflection lines or inversion circles. Since the form of the fundamental region is fixed in the symmetry groups of rosettes $\mathbf{D}_n$ (nm), friezes **m1**, **mm**, ornaments **pmm**, **p3m1**, **p4m**, **p6m** and conformal symmetry groups $\mathbf{D}_n\mathbf{R}_I$, $\mathbf{KD}_n\mathbf{R}_I$ generated by reflections (and inversions), the abundance and variety of such ornamental motifs can be exclusively achieved by changing the metric parameters — the dimensions and shape of an elementary asymmetric figure belonging to the fundamental region. At least one isohedral plane tiling corresponds to each of these groups, except to the symmetry group of ornaments **p3m1** (△), whose corresponding regular tessellation {3, 6} has the symmetry group **p6m**. This one and similar problems are an important part of the theory of plane tilings (B. Grünbaum, G.C. Shephard, 1987).

By solving the construction problems and by investigating the possibilities for obtaining different ornamental motifs, generations of artist and artisans have opened vast possibilities for decorativeness in ornamental art. In time, the symbolic meanings of ornaments were lost, and the role of ornaments was gradually reduced to pure decorativeness, almost without any symbolic meaning.

In ornamental art, by using an empirical approach, probably all the different possibilities for plane symmetry structures are exhausted. Symmetry groups present in ornamental art, their existence and uniqueness (completeness), have been recently scientifically verified by the theory of symmetry. So, we can divide all the discrete symmetry groups of the plane E^2 and $E^2 \setminus \{O\}$ into the following categories: two types of symmetry groups of rosettes G_{20}; seven symmetry groups of friezes G_{21}; 17 symmetry groups of ornaments G_2 ; five types of similarity symmetry groups S_{20}; five types of finite C_{21} and ten types of infinite C_2 conformal symmetry groups. Throughout history, the role of ornaments was mostly symbolic or decorative, but in our time, thanks to their link with science, especially with the natural sciences, ornaments have gained new meaning. Since they can be understood as models of structures that are the subject of scientific studies, ornaments today have outgrown the restricted area of ornamental art.

The history of ornamental art began in the period of the middle and late Paleolithic, around the tenth millennium B.C., when we have the first examples of discrete symmetry groups of rosettes G_{20} among which prevail visually static rosettes with the symmetry group $\mathbf{D}_n$ (**nm**) ($\mathbf{n} = 1, 2, 3, 4, 6$), examples of all seven discrete symmetry groups of friezes, all five plane Bravais lattices and examples of ornaments derived by elementary superpositions of friezes. From Paleolithic art we can date examples of almost all visually presentable continuous symmetry groups of the categories mentioned and the first intuitive premonitions of similarity symmetry — spirals, radial structures, series of concentric circles or squares and also the oldest examples of finite conformal symmetry groups of the type $\mathbf{D}_n\mathbf{R}_I$.

From the ornamental art of the Neolithic and the first ancient civilizations originate examples of all the 17 symmetry groups of ornaments

G_2 . After solving construction problems and creating ornaments that possessed a higher degree of constructional and visual complexity, they began the artistic experiments that opened the way to decorativeness, e.g., by changing the form of a fundamental region or the shape of an elementary asymmetric figure belonging to it. From the Neolithic date the oldest examples of antisymmetry and colored symmetry groups, being most completely realized in the ornamental art of Egypt. The ornamental art of Greece, Rome and Byzantium gave new results in similarity symmetry, antisymmetry, colored symmetry and conformal symmetry, while Gothic art almost completely exhausted the possibilities of conformal symmetry of rosettes, because dominating in this period were architectural constructions with rosettes constructed by circles and lines. Islamic and Moorish were the peaks of ornamental art. Renaissance and post-Renaissance ornamental art in Europe was almost completely reduced to decorativeness, which automatically reduced its status to that of a "second rate art". Lately, with geometric styles in painting, ornamental art gained a new affirmation with the work of V. Vasarely, M.C. Escher, by op-art (C. Barrett, 1970) and computer art (M.L. Prueitt, 1984). With the development of visual communications and accompanying visual design, ornamental art found a new place within the applied arts.

Interesting and not sufficiently investigated fields, referring to the chronology of ornamental art, appear to be the complete dating of the first appearance of all the symmetry groups in ornamental art, the evidence of the most important archaeological locations and civilizations that have achieved the most as for the completeness and variety of ornamental motifs, etc. Very important are comparative analyses aiming to find connections between civilizations, where ornaments can be the relevant indicators of these relations; either regarding repetition of details, elementary figures and the same forms of fundamental regions in ornamental art of different civilizations; or, regarding the use of the same symmetry groups. According to Gestalt theory, when visually perceiving an ornament the observer records and recognizes it as a whole, often abstracting details, but trying to understand and remember its law of organization — symmetry. Therefore, the use of the same symmetry groups by different civilizations can be a relevant indicator of their connections. Problems of the sense and symbolic meanings of ornaments, questions about relations between cosmological theories of different civilizations and their ornaments, etc., are only parts of this large field of investigation.

Even the elementary symmetry structures: rosettes C_n or D_n and the corresponding geometrical figures: square, circle, cross, have a symbolic meaning:

"*Point*: the primeval element, beginning, and kernel; symbol of the number. It is the symbol of the beginning (grain of seed) and of the end (grain of dust); it represents the smallest substance (atom, nucleus). The point is in fact imaginary: it occupies no space...

Vertical line: the sign of life, health, activity, certainty, effective stability, manliness. It is the symbol of spirit directed upward, of grandeur and loftiness, and of man running erect; it is the sign of right and might...

Horizontal line: the polar opposite of the vertical, and symbol of the earth, the passive, woman, death and rest; the material and the earthbound...

Cross: one of the oldest and most universal signs, uniting the polar contrasts of vertical and horizontal, of God and the world, of the spiritual and the material, of life and death, of man and woman. It indicates the four points of the compass and the point of intersection. After the Crucifixion it became a holy symbol in Christianity and was used in many variations...

Circle: together with the square and triangle, the primeval signs. Alike on all sides and the only geometric figure formed with one line with no beginning or end, it is a sign of infinity, eternity, perfection, and God. As a round form it is likewise a symbol of the Sun, the Cosmos, the Earth and the planets. As a pure form it is a sign of purity; as an embracing sign, a symbol of community...

Yang-yin: the symbol of perfect antithesis, ideal balance of opposites. Yin signifies womanly, dark, bound to the earth, cool, reticent, oppressed; yang manly, light, heavenly, aggressive, warm, governing. The white dot in the dark yin and the dark dot in the light yang signifies that each is always a part of the other...

Spiral: indicates that all life develops from one point and, still spinning from that one point, grows to adulthood. Also a symbol of the rising sun and the year. Much loved as an ornamental sign in many variations and combinations in all times and by all peoples...

Double spiral: the magnificent sign for perfection — in fact, a completed S-line. Symbol of the day between the rising and the setting sun and leap year (since a single spiral represents one year)...

Figure eight: the loop without end and therefore symbolic of endless, eternal time, which has no beginning or end...

Square: one of the three basic signs. Symbol of massiveness, sturdy peace, and stability: it stands fast and firm on the ground. It is the same on all the sides and the token for the number four, therefore, it symbolizes the four seasons, the four points of the compass, the four elements, the four rivers of Paradise, the four Evangelists...

Triangle: the third of the basic, primary signs. It is an aggressive sign with its points directed outwards. It is the symbol of the Trinity and, with a point in the middle, a sign of the all-seeing eye of God. A triangle standing with its base firmly planted on the ground and its point striving upwards has a womanly character, as opposed to a triangle balanced on its point with broad "shoulders" above, which has a more manly character...

Hexagram: two triangles passing through each other create a new sign, a beautiful symmetrical star. It is a magic sign of preservation and protection against destruction; also a very old Jewish sign, the Star of David, that crowns the synagogue and decorates the Torah rolls, as well as an emblem of the cosmos, the divine Creator, and His work...

Pentacle: another very old sign, known as the druid's foot for its magical meaning. Pointing upwards, it is a symbol of white magic; downwards, of black magic. The sign shows the five senses and indicates the powers and forms in nature. The lines intersect one another in golden-section proportions..." (R. Smeets, 1975, pp. 54–56).

Symbols move the deep, secret recesses of the human soul. They carry the mind over the borders of the finite into the realm of infinite: they are signs of unspeakable. Even only one of them — the spiral, may be the subject of the monograph (J. Purce, 1975). Some of them, e.g. Islamic patterns, are the artistic vision of cosmology (K. Critchlow, 1976), but many of them, e.g. the signs discovered by prehistoric archaeology, are still not decoded. Their symmetry and its accord with the message may be the keys for deciphering their symbolic meaning.

The word "symmetry" has its roots in Greek philosophy and aesthetics, where it was used to express balance, proportion, and to point out a whole spectrum of the philosophic-aesthetic synonym terms: harmony, accord, completeness, which were used in history. This term entered science in the 1830-ies, with the beginning of the study of crystal classes and their analysis based on the theory of groups, introduced by E. Galois (1831) in work published in 1848. The essence of the theory of symmetry, based on the theory of groups, is expressed in the Erlangen program of F. Klein (1872), distinguishing the theory of symmetry as a universal approach to different geometries by registering manifolds, their groups of transformations and invariants of these groups. Further development of the theory of symmetry cannot be separated from crystallography and the theory of groups. Of central interest to the problems covered by this work are the results of the theory of symmetry in the planes E^2 and $E^2 \setminus \{O\}$.

The answer to the question about the existence and completeness of the classification of symmetry groups of rosettes G_{20}, H. Weyl (1952, pp. 119) attributed to Leonardo. The derivation of the 32 crystal classes (J.F.Ch. Hessel, 1930) and 14 space Bravais lattices (O. Bravais, 1848) laid the basis for the complete derivation of the 230 G_3 crystallographic space symmetry groups (E.S. Fedorov, 1891a; A. Schönflies, 1891), while W. Barlow (1894) proved the crystallographic restriction, showing that rotations of the symmetry group of a lattice can only have periods $\mathbf{n} = 1,2,3,4,6$.

The derivation of 17 discrete symmetry groups of ornaments G_2 , given incompletely by C. Jordan (1868/69), where the symmetry group **pgg** is omitted, and by L. Sohncke (1874), is completely realized as a partial result of the derivation of the 230 space groups G_3 (E.S. Fedorov, 1891b). The 7 discrete symmetry groups of friezes G_{21} are derived by G. Pólya (1924), P. Niggli (1924) and A. Speiser (1927). The first two of them derived independently also 17 symmetry groups of ornaments (G. Pólya, 1924; P. Niggli, 1924).

Antisymmetry introduced by H. Heesch (1929), linked to the question of the visual interpretation of subperiodic symmetry groups of the space E^3 — symmetry groups of bands G_{321} and layers G_{32} in the plane E^2 by Weber black-white diagrams, was further developed by H.J. Woods (1935) and A.V. Shubnikov (1951). Recent development of the theory of antisymmetry, multiple antisymmetry, colored symmetry and its extensions has been seen in the contributions of many authors (e.g. A.V. Shubnikov, V.A. Koptsik, N.V. Belov, A.M. Zamorzaev, A.F. Palistrant, E.I. Galyarski, M. Senechal, A. Loeb, R.L.E. Schwarzenberger, T.W. Wieting, H.S.M. Coxeter, etc.).

The idea of similarity symmetry, put forward by H. Weyl (1952) was developed by A.V. Shubnikov (1960), E.I. Galyarski and A.M. Zamorzaev (1963). The discovery of the isomorphism between similarity symmetry groups S_{20}, finite C_{21} and infinite C_2 conformal symmetry groups and symmetry groups of polar rods G_{31}, tablets G_{320} and non-polar rods G_{31}, respectively, incited the development of the theory of similarity symmetry in E^2 and conformal symmetry in $E^2 \setminus \{O\}$.

Besides ornaments, as the most obvious visual models of symmetry groups in the plane E^2 and $E^2 \setminus \{O\}$, Cayley diagrams (A. Cayley, 1878; M. Dehn, 1910) and tables of graphic symbols of symmetry elements elaborated in crystallography, are used.

The generalized Niggli's categorization of symmetry groups, resulting in Bohm symbols of symmetry group categories (J. Bohm, K. Dornberger-Schiff, 1966) is consequently used in this work. Different systems for denoting symmetry groups, introduced by many authors (e.g., A. Schönflies, A.V. Shubnikov, M. Senechal, etc.) are unified for classical-symmetry groups by using a simplified version of the International symbols of symmetry groups of ornaments G_2 (H.S.M. Coxeter, W.O.J. Moser, 1980) and by the symbols of symmetry groups of friezes G_{21} introduced by M. Senechal (1975). For denoting symmetry groups of rosettes G_{20}, similarity symmetry groups S_{20} and conformal symmetry groups C_{21}, C_2, symbols derived according to those introduced by A. Schönflies and A.V. Shubnikov, are used.

Aiming for a more complete knowledge of the theory of symmetry, very inspiring might be the books by H. Weyl (1952), L. Fejes Tóth (1964), A.V. Shubnikov, N.V. Belov et al. (1964), A.V. Shubnikov, V.A. Koptsik (1974), A.M. Zamorzaev (1976), A.M. Zamorzaev, E.I. Galyarski, A.F. Palistrant (1978), E.H. Lockwood, R.H. Macmillan (1978), T.W. Wieting (1982), B. Grünbaum, G.C. Shephard (1987), A.M. Zamorzaev, Yu.S. Karpova, A.P. Lungu, A.F. Palistrant (1987) and the monograph "*Generators and Relations for Discrete Groups*" by H.S.M. Coxeter and W.O.J. Moser (1980).

The survey of ornamental art and the theory of symmetry given jointly in this book makes possible their comparison in the sense of common constructional approaches, methods and their final results — symmetry groups and their visual interpretations. In both fields, more sophisticated results are obtained by a similar approach — by extending symmetry groups of isometries in the plane E^2 to similarity symmetry groups S_{20}, conformal symmetry groups C_{21} and C_2, antisymmetry and colored symmetry groups. Regarding the chronology, the history of ornamental art fully satisfies the inductive series of extensions leading from the symmetry groups of rosettes G_{20}, over symmetry groups of friezes G_{21}, ornaments G_2 and similarity symmetry groups S_{20} in the plane E^2, to conformal symmetry groups C_{21}

and C_2 in the plane $E^2 \setminus \{O\}$. In the development of the theory of symmetry there are only a few exceptions from this sequence, mostly conditioned by the practical interests of crystallographers (e.g., the derivation of the symmetry groups of ornaments G_2 before the symmetry groups of friezes G_{21}, or the very early derivation of the 230 space symmetry groups G_3). In this way, the connection between ornamental art and the theory of symmetry represents a component of the universal, eternal link between art and science.

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NOTATION INDEX

30	$\tilde{a}$	glide reflection
30	m	reflection
30	g	glide reflection
32	T	two-fold rotation
41	$(k, m) = 1$	k, m are coprime numbers
63	p_0	continuous line group of translations
89	$[p,q]$	symmetry group of regular tessellation $\{p, q\}$
94	p_{00}	continuous plane group of translations
185	m_I	inversion circle

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