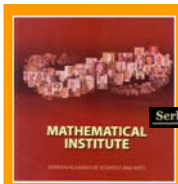

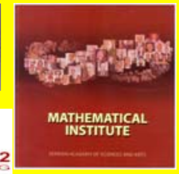

APPROXIMATIONS OF NONLINEAR DIFFERENTIAL EQUATION SOLUTIONS

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Abstract: A list of approximations of nonlinear functions of one and two arguments is done. The linearizations of nonlinear differential equations around stationary points correspond to equilibrium positions or relative equilibrium positions of mechanical system dynamics with trigger of coupled singularities are obtained. By using known analytical solutions of linearized nonlinear differential equations around stationary point, as the starting solutions, by application Krilov-Bogolyubov-Mitropolyski asymptotic methods and method of variation constants and averaging, different expressions of first approximations of nonlinear differential equation solution are obtained. First approximations of a nonlinear differential equation obtained by different methods and around different known analytical solutions were compared and corresponding conclusions are presented. As special examples are used nonlinear differential equations describing nonlinear dynamics of the mechanical system with coupled rotations in damping field. A list of approximations of nonlinear functions of one and two arguments is done. The linearizations of nonlinear differential equations around stationary points correspond to equilibrium positions or relative equilibrium positions of mechanical system dynamics with trigger of coupled singularities are obtained. By using known analytical solutions of linearized nonlinear differential equations around stationary point, as the starting solutions, by application Krilov-Bogolyubov-Mitropolyski asymptotic methods and method of variation constants and averaging, different expressions of first approximations of nonlinear differential equation solution are obtained. First approximations of a nonlinear differential equation obtained by different methods and around different known analytical solutions were compared and corresponding conclusions are presented. As special examples are used nonlinear differential equations describing nonlinear dynamics of the mechanical system with coupled rotations in

$$\frac{d\varphi}{dt}=f_1(\varphi,v)\qquad \frac{dv}{dt}=f_2(\varphi,v)$$

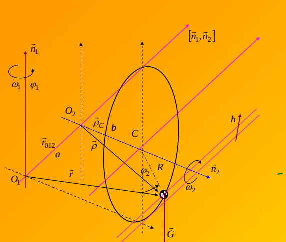
damping field.

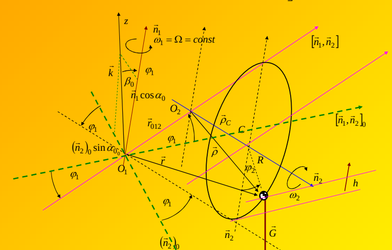
$$\frac{d\varphi}{dt}=\left(\frac{\partial f_1(\varphi,v)}{\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi+\left(\frac{\partial f_1(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v+\frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial\varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2+\frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v^2+2\frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v+$$

$$+\frac{1}{3!}\left[\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^3+3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 v+3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v^2+\left(\frac{\partial^3 f_1(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v3\right]+....$$

$$\frac{dv}{dt}=\left(\frac{\partial f_2(\varphi,v)}{\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi+\left(\frac{\partial f_2(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v+\left(\frac{\partial^2 f_2(\varphi,v)}{\partial\varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2+\left(\frac{\partial^2 f_2(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v^2+2\frac{1}{2!}\left(\frac{\partial^2 f_2(\varphi,v)}{\partial v\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v+$$

$$+\frac{1}{3!}\left[\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^3+3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 v+3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v^2+\left(\frac{\partial^3 f_2(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}v3\right]+....$$





$$\ddot{\varphi}_2+\Omega^2\left\langle\frac{g}{R\Omega^2}-\cos\varphi_2\right\rangle\sin\varphi_2-\Omega^2\frac{r_{012}}{R}\cos\varphi_2=0$$

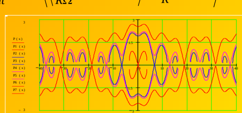
$$\ddot{\varphi}_2+\Omega^2\left(\frac{g}{\Omega^2 R}\cos\beta_0-\cos\varphi_2\right)\sin\varphi_2-\Omega^2\frac{r_{012}}{R}\cos\varphi_2=\frac{g}{R}\cos\varphi_2\sin\beta_0\sin\Omega t_1$$

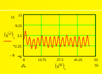
$$\frac{d^2\varphi}{dt^2}+\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle\varphi=0$$

$$\ddot{\varphi}_2+\varphi_2\left[\omega_0^2\cos^2\alpha_0+\Omega^2\sin 2\varphi_{s2}-\Omega^2\frac{r_{012}}{R}\sin\varphi_{s2}+\frac{g}{R}\sin\alpha_0\sin\varphi_{s2}\cos\Omega t\right]+$$

$$+\omega_0^2\cos\alpha_0\sin\varphi_{s2}-\frac{\Omega^2}{2}\cos 2\varphi_{s2}-\Omega^2\frac{r_{012}}{R}\cos\varphi_{s2}=\frac{g}{R}\sin\alpha_0\cos\varphi_{s2}\cos\Omega t$$

$$\ddot{\varphi}_2+\varphi_2\left[\lambda+\gamma\sin\varphi_{s2}\cos\Omega t\right]=h_s\cos\Omega t$$





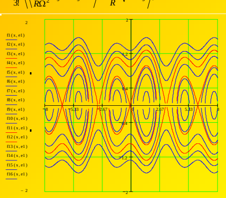
$$\frac{d^2\varphi}{dt^2}+\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle\varphi-\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2}-4\lambda_s\sqrt{1-\lambda_s^2}\right\rangle-\frac{r_{012}}{R}\lambda_s\right\rangle\varphi^2-$$

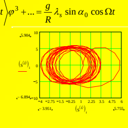
$$-\frac{\Omega^2}{3!}\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle\varphi^3+....=0$$

$$\frac{d^2\varphi}{dt^2}+\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}+\frac{g}{R\Omega^2}\sin\alpha_0\sqrt{1-\lambda_s^2}\cos\Omega t\right\rangle\varphi-$$

$$-\frac{\Omega^2}{2!}\left\langle\left\langle\frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2}-4\lambda_s\sqrt{1-\lambda_s^2}\right\rangle-\frac{r_{012}}{R}\lambda_s-\frac{g}{R\Omega^2}\lambda_s\sin\alpha_0\cos\Omega t\right\rangle\varphi^2-$$

$$-\frac{\Omega^2}{3!}\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}+\frac{g}{R\Omega^2}\sin\alpha_0\sqrt{1-\lambda_s^2}\cos\Omega t\right\rangle\varphi^3+...=\frac{g}{R}\lambda_s\sin\alpha_0\cos\Omega t$$





$$\chi=\frac{g}{R}\sin\alpha_0$$

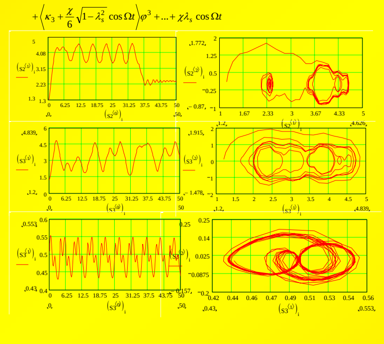
$$\frac{d^2\varphi}{dt^2}+\left\langle\omega_{0,lin}^2+\chi\sqrt{1-\lambda_s^2}\cos\Omega t\right\rangle\varphi=\left\langle\kappa_2-\frac{\chi}{2}\lambda_s\cos\Omega t\right\rangle\varphi^2+$$

$$+\left\langle\kappa_3+\frac{\chi}{6}\sqrt{1-\lambda_s^2}\cos\Omega t\right\rangle\varphi^3+...+\chi\lambda_s\cos\Omega t$$

$$\omega_{0,lin}^2=\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle$$

$$\kappa_2=\Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2}-4\lambda_s\sqrt{1-\lambda_s^2}\right\rangle-\frac{r_{012}}{R}\lambda_s\right\rangle$$

$$\kappa_3=\frac{\Omega^2}{3!}\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle$$



$$\ddot{\varphi}+\omega_{0,lin}^2\varphi=\kappa_2\varphi^2+\kappa_3\varphi^3\qquad \dot{a}(t)=0$$

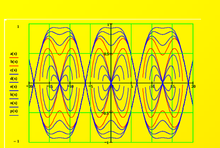
$$\varphi(t)=a(t)\cos\Phi(t)\qquad \dot{\phi}(t)=\frac{3\kappa_3}{8\omega_0}[a(t)]^2$$

$$a(t)=a_0=const$$

$$\phi(t)=\frac{\Omega\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\sqrt{\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}}[a(t)]^2 t+\phi(0)$$

$$\Phi(t)=\left\langle\Omega\sqrt{\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}+\frac{\Omega\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\sqrt{\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}}[a(t)]^2\right\rangle t+\phi(0)$$

$$\omega_{nel}=\Omega\sqrt{\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}\left\{1+\frac{\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-8\lambda_s^2+4\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s-2\lambda_s^2+1\right\rangle+\frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}a_0^2\right\}$$



- For nonlinear differential equation with small cubic nonlinearity

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=\mp\tilde{\omega}_{N1}^2 x_1^3(t)$$

- from numerous world known monographs and books it is known the following first aproximation:

$$x_1(t)=a_o e^{-\delta_1 t}\cos\left[\omega_1 t-\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\psi_o\right].$$

- Starting analytical solution is in the form:
- $x_1(t)=R(t)\cos(\omega_1 t+\psi(t))$ of liner differential equation in the form: $\ddot{x}_1(t)+\omega_1^2 x_1(t)=0$
- For the case, that $\omega_{N1}^2\to 0$ nonlinearuty in equation is equal to zero linearized differential equation is:

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=0$$

- and from first approximation, we obtain the following solution of previous equation:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos(\omega_1 t+\psi_0)\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon=0\qquad \tilde{\omega}_{N1}^2=o$$

- which is not correct for this limit case. The correct solution of previous linear differential equation is:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos(p_1 t+\alpha_{01})\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon=0\qquad \tilde{\omega}_{N1}^2=o\qquad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

By use as a starting known analytical solution in previous form with amplitude and phase as a function in the following form:

$$x_1(t)=R(t)e^{-\delta_1 t}\cos(p_1 t+\phi(t))\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon\neq 0\qquad \tilde{\omega}_{N1}^2\neq o$$

For first asymptotic approximation of the nonlinear differential equation solution, we obtain the following expression:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t\mp\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t}+\alpha_{01}\right)\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon\neq 0\qquad \tilde{\omega}_{N1}^2\neq o\qquad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

or for known initial condition

$$R_1(0)=R_{01}\qquad \phi_1(0)=\phi_{01}=-\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2+\alpha_{01}\qquad \alpha_{01}=\phi_{01}\pm\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2$$

First asymptotic aproximation of the nonlinear differential equation solution is in the following form:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t\mp\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\phi_{01}\right)\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon\neq 0\qquad \tilde{\omega}_{N1}^2\neq o\qquad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

The previous obtained first aproximation of the starting nonlinear differential equation solution is possible to obtain by different methods: 1* Combination of the methods: Variation constants of the known analytical solution of the corresponding linear to the nonlinear differential equation with small cubic nonlinearity and applied averaging along full phase as proposed by Hedrih; 2* Asymptotic method Krilov-Bogolyubov Mitropolyski adopted by Mitropolyski for obtaining asymptotic approximation of the solutions of the nonlinear differential equation with small nonlinearity expressed by nonlinear function depending on the slowchanging time.

$$x_1(t)=R_{01}\cos\left\langle\left(\omega_1\pm\frac{3}{8\omega_1}\tilde{\omega}_{N1}^2 R_{01}^2\right)t+\phi_{01}\right\rangle\qquad \delta_1=0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon\neq 0\qquad \tilde{\omega}_{N1}^2\neq 0$$

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos(p_1 t+\alpha_{01})\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\qquad \varepsilon=0\qquad \tilde{\omega}_{N1}^2=0\qquad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

By comparison, border cases for solutions obtained from the first approximation:
1* obtained by Hedrih, which start from known analytical solution of the linear differential equation

CONCLUDING REMARKS

Let we made a general review of the obtained results for approximately solving of the nonlinear differential equation with small cubic nonlinearity and linear damping in the form:

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=\mp\tilde{\omega}_{N1}^2 x_1^3(t)\qquad (7)$$

in which hard or soft, refers to sign $\mp$, $\delta_1=\varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2=\varepsilon\tilde{\tilde{\omega}}_{N1}^2$ and ε and ε_1 are small parameters. By use two methods [9] and [8] starting known analytical solutions, $x_1(t)=R_1(t)e^{-\delta_1 t}\cos(p_1 t+\phi(t))$, $p_1=\sqrt{\omega_1^2-\delta_1^2}$ and $x(t)=a(t)\cos[\omega_1 t+\phi(t)]$
and we obtained same first approximation of the solution in the following forms [9]:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t\mp\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\phi_{01}\right)\ \text{for}\ \delta_1\neq 0\quad \omega_1^2>\delta_1^2\quad \varepsilon\neq 0\quad \tilde{\omega}_{N1}^2\neq o\quad p_1=\sqrt{\omega_1^2-\delta_1^2}\qquad (8)$$

$$x_1(t)=a_0 e^{-\delta_1 t}\cos\left(\omega_1 t\mp\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\phi_0\right)\ \text{for}\ \delta_1\neq 0\quad \omega_1^2>\delta_1^2\quad \varepsilon\neq 0\quad \tilde{\omega}_{N1}^2\neq o\qquad (9)$$

For the case that damping coefficient tends to zero, from both first approximations (8) and (9), we obtain same analytical approximation of the solution for conservative nonlinear system dynamics. For the case that coefficient of the cubic nonlinearity tends to zero, from first approximation (8), we obtain known analytical solution of the linear no conservative system dynamics in the following form:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos(p_1 t+\alpha_{01})\quad \text{for}\quad \delta_1\neq 0\quad \omega_1^2>\delta_1^2\quad \varepsilon=0\quad \tilde{\omega}_{N1}^2=o\quad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

but from the second form (9) obtained solution

$$x_1(t)=a_0 e^{-\delta_1 t}\cos(\omega_1 t+\alpha_{01})\quad \text{for}\quad \delta_1\neq 0\quad \omega_1^2>\delta_1^2\quad \varepsilon\neq 0\quad \tilde{\omega}_{N1}^2=o$$

is not correct. Because is not solution of the differential equation:

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=0\qquad \delta_1\neq 0\qquad \omega_1^2>\delta_1^2\quad \varepsilon=0\quad \tilde{\omega}_{N1}^2=o\quad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

Then we can conclude that, starting different known analytical solutions, for obtaining first approximations are acceptable, but limited by corresponding conditions. Approximation of the solution of nonlinear differential George Duffing differential equations (7) in the form (8) is better them (9) known from numerous literatures. Presentation of full original results is limited by length of the paper.

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