# APPROXIMATIONS OF NONLINEAR DIFFERENTIAL EQUATION SOLUTIONS

**Hedrih R. (Stevanović) Katica**
Mathematical Institute SANU Belgrade,
and Faulty of Mechanical Engineering University of Niš, Serbia

Paper ID: SM16-053
Poster: P-477

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=\mp\tilde{\omega}_{N1}^2 x_1^3(t)$$

## First asymptotic approximations starting from different analytical solutions

$$x_1(t)=a_o e^{-\delta_1 t}\cos\left[\omega_1 t-\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\psi_o\right].$$

## Better first asymptotic approximations starting from analytical solution of damping linear vibrations

$$p_1=\sqrt{\omega_1^2-\delta_1^2}$$

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t\mp\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\phi_{01}\right)$$


XXIII ICTAM, 19-24 August 2012, Beijing, China

$$\ddot{x}_1(t) + 2\delta_1 \dot{x}_1(t) + \omega_1^2 x_1(t) = \mp \tilde{\omega}_{N1}^2 x_1^3(t)$$

**Starting analytical solution for aproximation**

$$\ddot{x}_1(t) + \omega_1^2 x_1(t) = 0 \qquad x_1(t) = R(t)\cos(\omega_1 t + \psi(t))$$

**First asymptotic Approximation**

$$x_1(t) = a_o e^{-\delta_1 t} \cos\left[\omega_1 t - \frac{3}{16\delta_1 \omega_1} \omega_{N1}^2 a_o^2 \left(e^{-2\delta_1 t} - 1\right) + \psi_o\right].$$

**For limit case when nonlinearyti is zero**

$\delta_1 \neq 0$ $\quad \varepsilon = 0$

$\omega_1^{\mathbf{2}} > \delta_1^2$ $\quad \tilde{\omega}_{N1}^2 = o$

$\delta_1 \neq 0$ $\quad \omega_1^{\mathbf{2}} > \delta_1^2$

$$\ddot{x}_1(t) + 2\delta_1 \dot{x}_1(t) + \omega_1^2 x_1(t) = 0$$

**Error solution**

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos(\omega_1 t + \psi_0)$$

**Analytical solution**

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos(p_1 t + \alpha_{01})$$

$$p_1 = \sqrt{\omega_1^{\mathbf{2}} - \delta_1^2}$$

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=\mp\tilde{\omega}_{N1}^2 x_1^3(t)$$

**Starting analytical solution for aproximation**

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=0 \qquad x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t+\alpha_{01}\right)$$

**First asymptotic approximation**

$$x_1(t)=R(t)e^{-\delta_1 t}\cos\left(p_1 t+\phi(t)\right) \qquad p_1=\sqrt{\omega_1^2-\delta_1^2}$$

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t\mp\frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t}-1\right)+\phi_{01}\right)$$

**For limit case when nonlinearyti is zero**

$$\delta_1\neq 0 \qquad \varepsilon=0$$

$$\omega_1^2>\delta_1^2 \qquad \tilde{\omega}_{N1}^2=o$$

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=0$$

$$\delta_1\neq 0 \qquad \omega_1^2>\delta_1^2$$

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1 t+\alpha_{01}\right)$$

**Analytical solution**

$$p_1=\sqrt{\omega_1^2-\delta_1^2}$$

# Nonlinear differential equations nonlinear approximations around stationary points

$$\frac{d\varphi}{dt} = f_1(\varphi, v) \qquad \frac{dv}{dt} = f_2(\varphi, v)$$

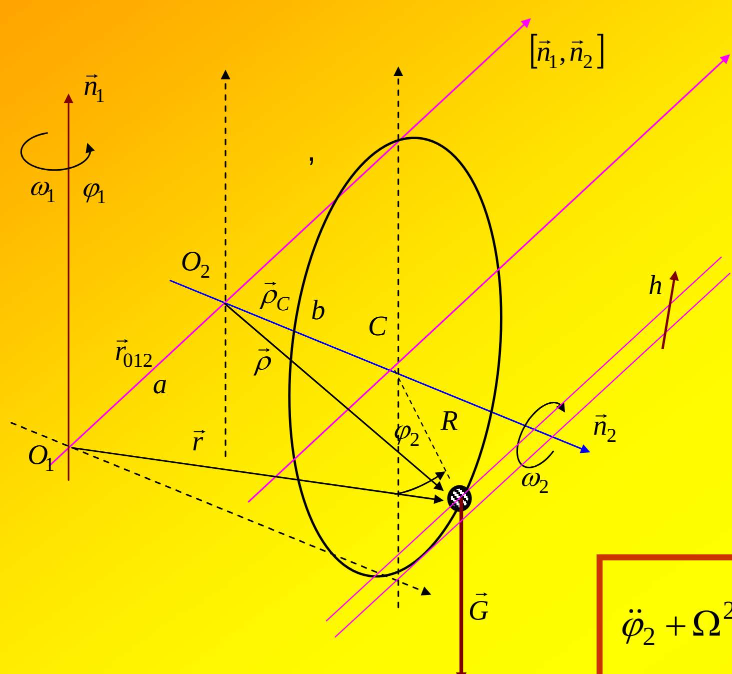
$$\frac{d\varphi}{dt} = \left(\frac{\partial f_1(\varphi,v)}{\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi + \left(\frac{\partial f_1(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v + \frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial\varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^2 + \frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v^2 + 2\frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi v +$$

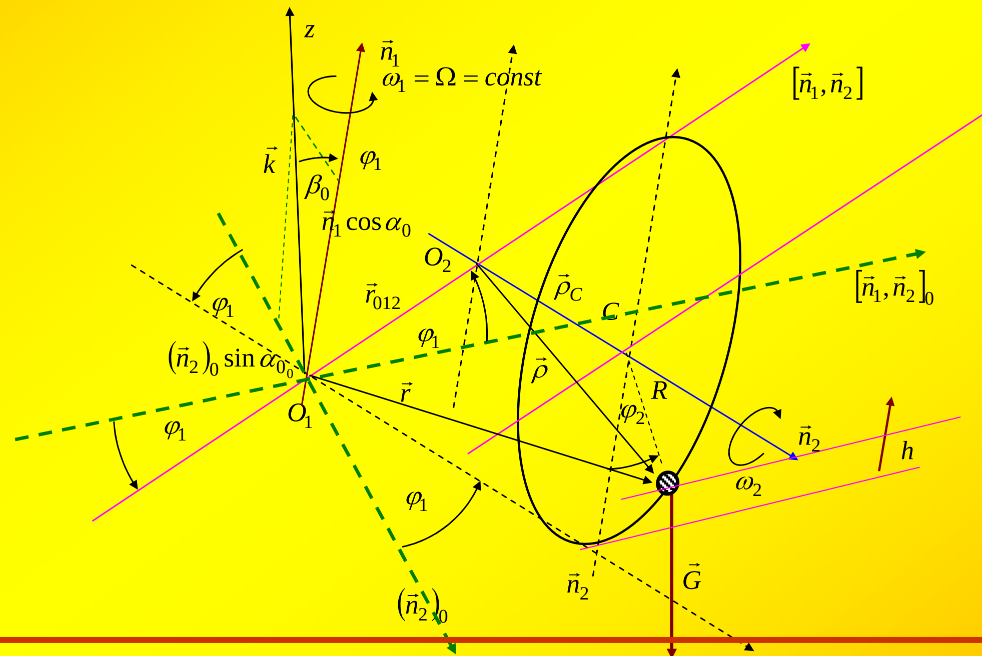
$$+\frac{1}{3!}\left[\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^3 + 3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^2 v + 3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial\varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi v^2 + \left(\frac{\partial^3 f_1(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v3\right] + \ldots.$$

$$\frac{dv}{dt} = \left(\frac{\partial f_2(\varphi,v)}{\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi + \left(\frac{\partial f_2(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v + \left(\frac{\partial^2 f_2(\varphi,v)}{\partial\varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^2 + \left(\frac{\partial^2 f_2(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v^2 + 2\frac{1}{2!}\left(\frac{\partial^2 f_2(\varphi,v)}{\partial v\partial\varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi v +$$

$$+\frac{1}{3!}\left[\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^3 + 3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi^2 v + 3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial\varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} \varphi v^2 + \left(\frac{\partial^3 f_2(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v3\right] + \ldots.$$

# Examples of nonlinear dynamics of heavy mass particle dynamics with coupled rotations about no intersecing axes- Linearrization- Linear aproximation




$$\ddot{\varphi}_2 + \Omega^2\left(\frac{g}{\Omega^2 R}\cos\beta_0 - \cos\varphi_2\right)\sin\varphi_2 - \Omega^2\frac{r_{012}}{R}\cos\varphi_2 = \frac{g}{R}\cos\varphi_2\sin\beta_0\sin\Omega t_1$$

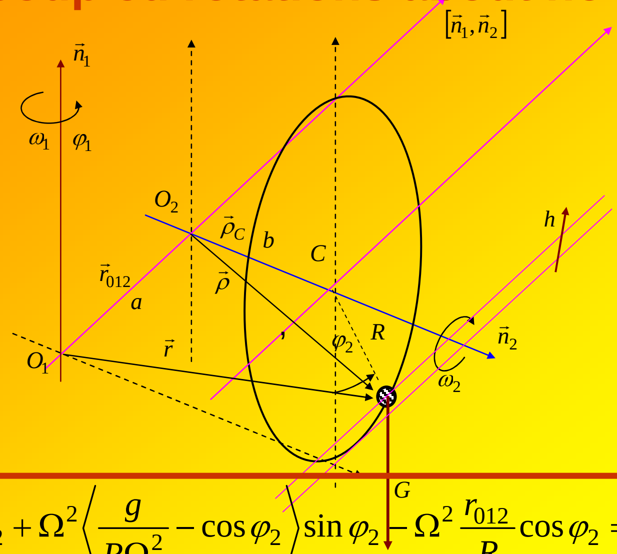
$$\ddot{\varphi}_2 + \Omega^2\left\langle\frac{g}{R\Omega^2} - \cos\varphi_2\right\rangle\sin\varphi_2 - \Omega^2\frac{r_{012}}{R}\cos\varphi_2 = 0$$

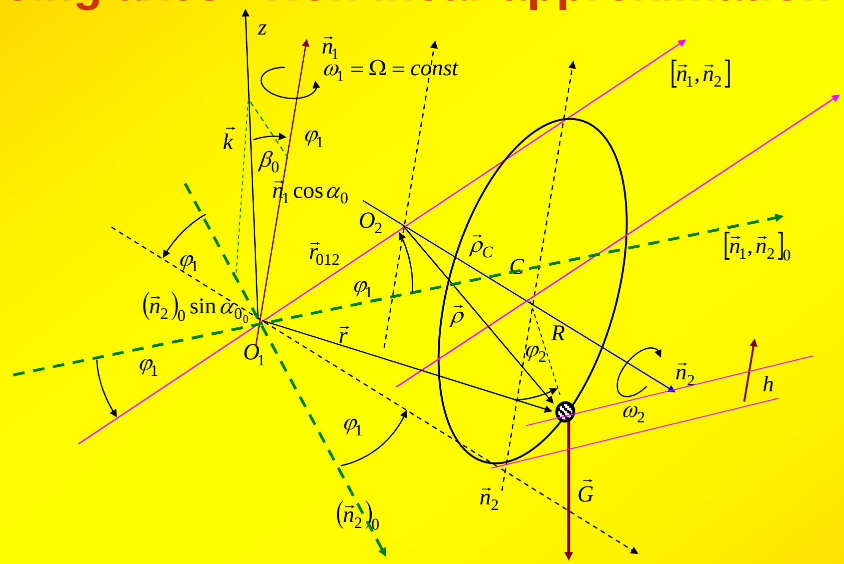
$$\frac{d^2\varphi}{dt^2} + \Omega^2\left\langle\left\langle\frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle\varphi = 0$$

$$\ddot{\varphi}_2 + \varphi_2\left[\omega_0^2\cos^2\alpha_0 + \Omega^2\sin 2\varphi_{s2} \_ \Omega^2\frac{r_{012}}{R}\sin\varphi_{s2} + \frac{g}{R}\sin\alpha_0\sin\varphi_{s2}\cos\Omega t\right] +$$
$$+\omega_0^2\cos\alpha_0\sin\varphi_{s2} - \frac{\Omega^2}{2}\cos 2\varphi_{s2} - \Omega^2\frac{r_{012}}{R}\cos\varphi_{s2} = \frac{g}{R}\sin\alpha_0\cos\varphi_{s2}\cos\Omega t$$

$$\ddot{\varphi}_2 + \varphi_2\left[\lambda + \gamma\sin\varphi_{s2}\cos\Omega t\right] = h_s\cos\Omega t$$

# Examples of nonlinear dynamics of heavy mass particle dynamics with coupled rotations about no intersecing axes - Nonlinear approximation




$$\ddot{\varphi}_2 + \Omega^2 \left\langle \frac{g}{R\Omega^2} - \cos\varphi_2 \right\rangle \sin\varphi_2 - \Omega^2 \frac{r_{012}}{R} \cos\varphi_2 = 0$$

$$\ddot{\varphi}_2 + \Omega^2 \left( \frac{g}{\Omega^2 R} \cos\beta_0 - \cos\varphi_2 \right) \sin\varphi_2 - \Omega^2 \frac{r_{012}}{R} \cos\varphi_2 = \frac{g}{R} \cos\varphi_2 \sin\beta_0 \sin\Omega t_1$$

$$\frac{d^2\varphi}{dt^2} + \Omega^2 \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle \varphi - \Omega^2 \left\langle \left\langle \frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2} - 4\lambda_s\sqrt{1-\lambda_s^2} \right\rangle - \frac{r_{012}}{R}\lambda_s \right\rangle \varphi^2 -$$
$$-\frac{\Omega^2}{3!} \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle \varphi^3 + .... = 0$$

$$\frac{d^2\varphi}{dt^2} + \Omega^2 \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} + \frac{g}{R\Omega^2}\sin\alpha_0\sqrt{1-\lambda_s^2}\cos\Omega t \right\rangle \varphi -$$
$$-\frac{\Omega^2}{2!} \left\langle \left\langle \frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2} - 4\lambda_s\sqrt{1-\lambda_s^2} \right\rangle - \frac{r_{012}}{R}\lambda_s - \frac{g}{R\Omega^2}\lambda_s\sin\alpha_0\cos\Omega t \right\rangle \varphi^2 -$$
$$-\frac{\Omega^2}{3!} \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} + \frac{g}{R\Omega^2}\sin\alpha_0\sqrt{1-\lambda_s^2}\cos\Omega t \right\rangle \varphi^3 + ... = \frac{g}{R}\lambda_s \sin\alpha_0 \cos\Omega t$$

$$\omega_{0,lin}^2 = \Omega^2 \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle \qquad \kappa_2 = \Omega^2 \left\langle \left\langle \frac{g}{R\Omega^2}\sqrt{1-\lambda_s^2} - 4\lambda_s\sqrt{1-\lambda_s^2} \right\rangle - \frac{r_{012}}{R}\lambda_s \right\rangle$$

$$\kappa_3 = \frac{\Omega^2}{3!} \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle \qquad \chi = \frac{g}{R}\sin\alpha_0$$

$$\ddot{\varphi} + \omega_{0,lin}^2\varphi = \kappa_2\varphi^2 + \kappa_3\varphi^3$$

$$\varphi(t) = a(t)\cos\Phi(t)$$

$$\dot{a}(t) = 0$$

$$\dot{\phi}(t) = \frac{3\kappa_3}{8\omega_0}\left[a(t)\right]^2$$

$$a(t) = a_0 = const$$

$$\frac{d^2\varphi}{dt^2} + \left\langle \omega_{0,lin}^2 + \chi\sqrt{1-\lambda_s^2}\cos\Omega t \right\rangle \varphi = \left\langle \kappa_2 - \frac{\chi}{2}\lambda_s\cos\Omega t \right\rangle \varphi^2 +$$

$$+ \left\langle \kappa_3 + \frac{\chi}{6}\sqrt{1-\lambda_s^2}\cos\Omega t \right\rangle \varphi^3 + \ldots + \chi\lambda_s\cos\Omega t$$

$$\phi(t) = \frac{\Omega \left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle}{16\sqrt{\left\langle \left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1 \right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2} \right\rangle}}\left[a(t)\right]^2 t + \phi(0)$$

$$\varphi(t) = a(t)\cos\Phi(t)$$

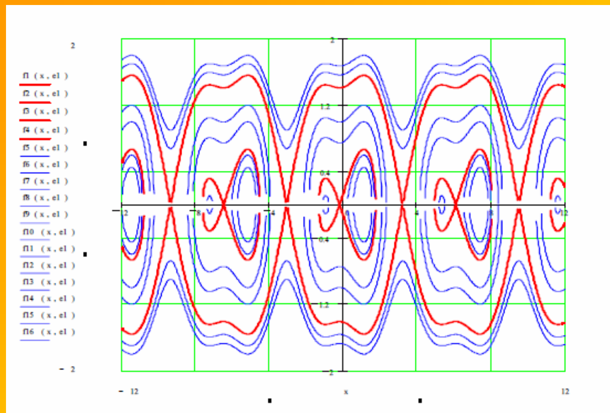
$$a(t) = a_0 = const$$

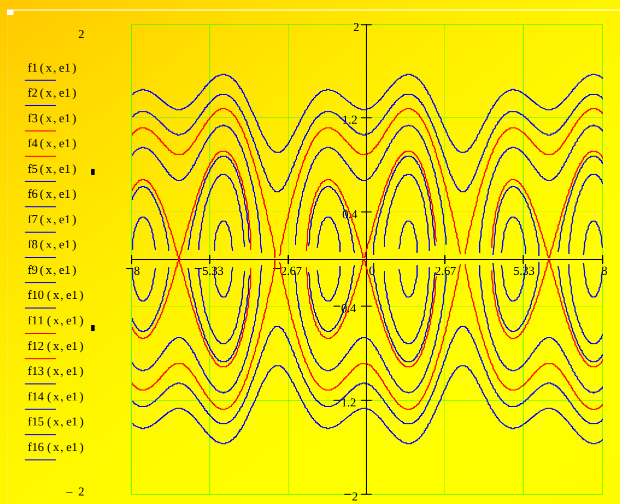
$$\dot{\phi}(t) = \frac{3\kappa_3}{8\omega_0}\left[a(t)\right]^2$$

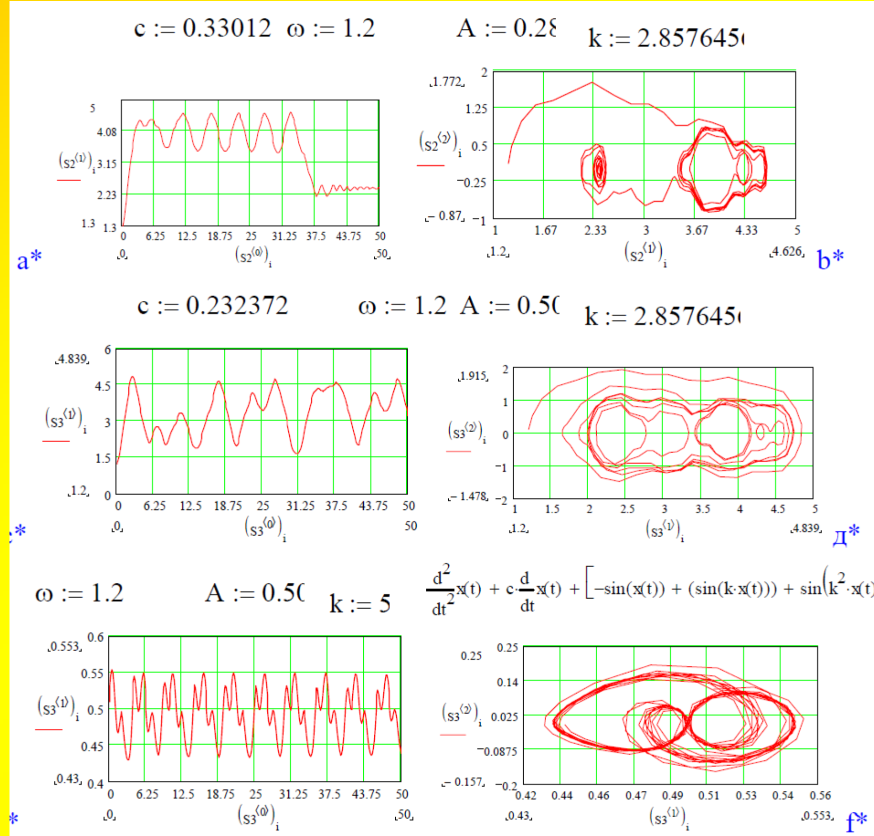
$$\phi(t) = \frac{\Omega\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\sqrt{\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}}\left[a(t)\right]^2 t + \phi(0)$$

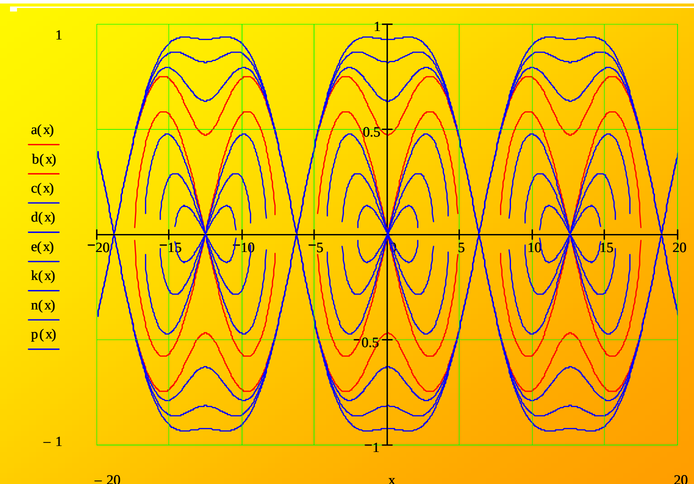
$$\Phi(t) = \left\langle \Omega\sqrt{\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle} + \frac{\Omega\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\sqrt{\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}}\left[a(t)\right]^2 \right\rangle t + \phi(0)$$

$$\omega_{nel} = \Omega\sqrt{\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}\left\{1 + \frac{\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 8\lambda_s^2 + 4\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle}{16\left\langle\left\langle \frac{g}{R\Omega^2}\lambda_s - 2\lambda_s^2 + 1\right\rangle + \frac{r_{012}}{R}\sqrt{1-\lambda_s^2}\right\rangle} a_0^2\right\}$$








**Paper ID: SM16-053 in the seminar session of SM16 Vibrations and control of structures**

**Poster board number: P-477**

# Thank You for Attention!


Thank You for Attention!
ВОЈНОТЕХНИЧКИ ИНСТИТУТ
MILANKOVITCH ANNIVERSARY SYMPOSIUM 2004
Milutin Milanković
Milanković Crater, Mars
Welcome to Serbia