On Hosoya's dormants and sprouts

Dragan Stevanović

Mathematical Institute of the Serbian Academy of Sciences and Arts

(Joint work with Salem Al-Yakoob and Ali Kanso)

The beginnings of cospectral graphs

Günthard and Primas in 1956 first expressed the belief that the multiset of eigenvalues of adjacency matrix characterizes graphs.

The first counterexample of *cospectral* graphs was constructed by Collatz and Sinogowitz in their seminal paper in 1957.

In 1966, Kac modeled the shape of a drum in a continuous fashion, while Fischer modeled it in a discrete manner by a graph, and then posed the famous question:

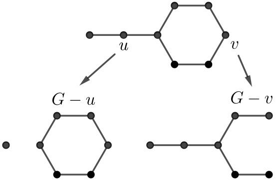
Can one hear the shape of a drum?,

or can a graph be characterized by the multiset of its eigenvalues? Fischer then found additional examples of cospectral graphs. . .

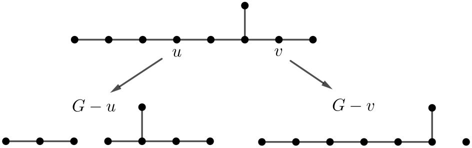
The 1970s and the 1980s

Development of theoretical methods for constructing arbitrary numbers of new pairs of cospectral graphs.

Herndon and Ellzey (1975, 1986) used isospectral vertices: $G - u$ and $G - v$ are cospectral, but not isomorphic.



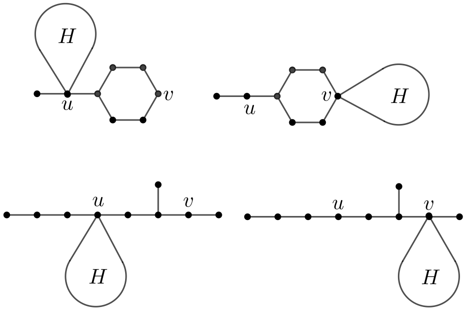
$$P(G-u) = P(G-v) = x^7 - 6x^5 + 9x^3 - 4x$$



$$P(G-u) = P(G-v) = x^8 - 6x^6 + 10x^4 - 4x^2$$

Coalescences at isospectral vertices

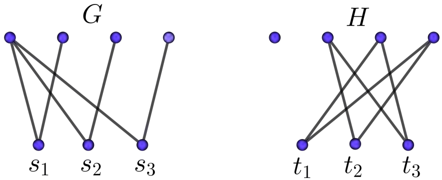
Cospectral graphs are then obtained by identifying isospectral vertices with the root of an arbitrary rooted graph H:



Schwenk's celebrated 1973 result that almost every tree is cospectral to another tree relies on the fact that the proportion of trees with either of these forms tends to 1 as $n \to \infty$.

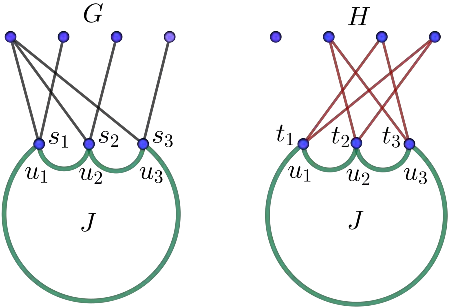
Removal-cospectral sets

Schwenk further generalized this approach in 1979:
for a graph G with a subset of vertices $S = \{s_1, \ldots, s_k\}$ and a graph H with a subset of vertices $T = \{t_1, \ldots, t_k\}$, S and T are *removal-cospectral* if $G - A$ and $H - \theta(A)$ are cospectral for each $A \subset S$, where $\theta(s_i) = t_i$.



Removal-cospectral sets

Schwenk further generalized this approach in 1979:
for a graph G with a subset of vertices $S = \{s_1, \ldots, s_k\}$
and a graph H with a subset of vertices $T = \{t_1, \ldots, t_k\}$,
S and T are *removal-cospectral* if $G - A$ and $H - \theta(A)$ are cospectral for each $A \subset S$, where $\theta(s_i) = t_i$.



Then the coalescences of G and H with arbitrary J are cospectral.

Godsil-McKay switching

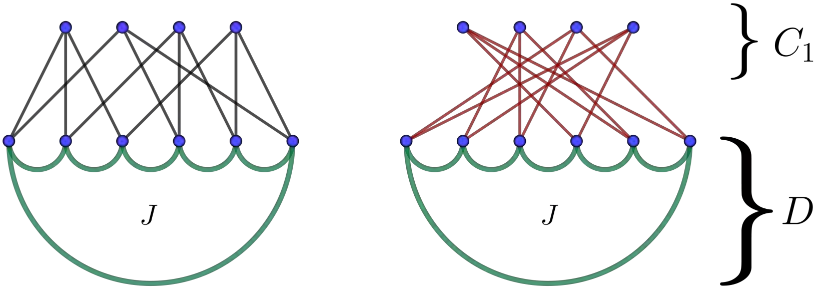
A celebrated 1982 approach for constructing cospectral graphs.

Let $\pi = (C_1, \ldots, C_k, D)$ be a partition of the vertex set of G s.t.:
a) any two vertices in C_i have equally many neighbours in C_j,
b) any $v \in D$ has either 0, $\frac{|C_i|}{2}$ or $|C_i|$ neighbours in C_i.

Construct $G^{(\pi)}$ in the following way:
for each $v \in D$ which has $\frac{|C_i|}{2}$ neighbours in C_i,
delete the existing edges between v and C_i, and
add the edges between v and the remaining vertices in C_i.

Then G and $G^{(\pi)}$ are cospectral, with cospectral complements.
(Actually, D is a removal-cospectral set in both G and $G^{(\pi)}$.)

Godsil-McKay switching, example 2



Enumerations of cospectral graphs

- Godsil-McKay, 1976: connected graphs up to 9 vertices
- Lepović, 1998: connected graphs on 10 vertices
- Haemers-Spence, 2004: connected graphs on 11 vertices
- Brouwer-Spence, 2009: connected graphs on 12 vertices

Findings:

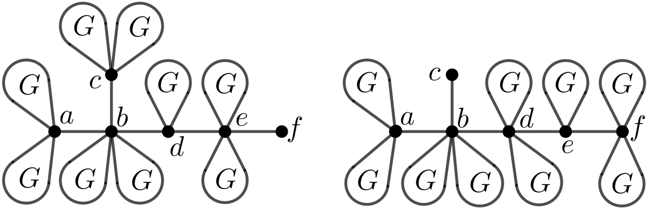
- 21.3% of cospectral graphs on 10 vertices
- 21.1% of cospectral graphs on 11 vertices
- 18.8% of cospectral graphs on 12 vertices

van Dam and Haemers conjectured that (unlike the case for trees) *almost all graphs ought to be determined by their spectrum*.

This revived the interest for cospectrality of graphs:
the paper of van Dam and Haemers is already cited $>$ 400 times...

Hosoya's coalescences

Hosoya drew attention more recently (2016–2019) that cospectral graphs may be constructed in another way using coalescences with a number of copies of the same rooted graph:



- The underlying vertex sets are not removal-cospectral;
- Different number of copies are attached at different vertices.

Our goal: *explain theoretical background for this cospectrality*.

Notation for coalescences

Let T and $G_1, \ldots, G_k$ be vertex-disjoint graphs. Let $u_1, \ldots, u_k$ be distinct vertices of T, and let v_i be a vertex of G_i for $i = 1, \ldots, k$.

$$T(u_1 = v_1)G_1(u_2 = v_2)G_2 \cdots (u_k = v_k)G_k$$

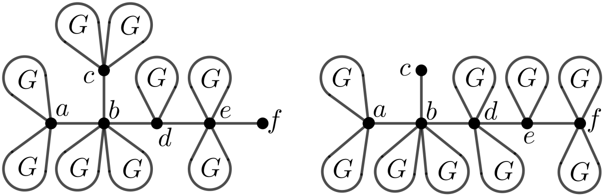
denotes the multiple coalescence obtained from $T \cup G_1 \cup \cdots \cup G_k$ by identifying the vertices u_i and v_i for $i = 1, \ldots, k$.

For a rooted graph G with the root r and $a \geq 1$, $G^{(a)}$ is short for

$$G(r = r)G(r = r) \cdots (r = r)G,$$

in which the roots of a copies of G are mutually identified.

Denoting Hosoya's coalescences



T is the underlying tree with vertices $\{a, b, c, d, e, f\}$.
The left coalescence:

$$T(a = r)G^{(2)}(b = r)G^{(2)}(c = r)G^{(2)}(e = r)G^{(2)}(d = r)G.$$

The right coalescence:

$$T(a = r)G^{(2)}(b = r)G^{(2)}(d = r)G^{(2)}(f = r)G^{(2)}(e = r)G.$$

The sequence of "exponents" $(2, 2, 2, 2, 1)$ is the *signature*.

Characteristic polynomial of multiple coalescences

Main theorem

Let T be a graph with distinct vertices $u_1, \ldots, u_k$ for some $k \geq 1$. For $i = 1, \ldots, k$, let G_i be a rooted graph with the root r_i and let

$$Q_i = PG_i - xP(G_i - r_i) \quad \text{and} \quad R_i = P(G_i - r_i).$$

Then for any signature $(a_1, \ldots, a_k)$, we have

$$PT(u_1 = r_1)G_1^{(a_1)} \cdots (u_k = r_k)G_k^{(a_k)} =$$

$$\sum_{J \subseteq \{1,\ldots,k\}} P(T - \sum_{i \in J} u_i) \prod_{j \in J} a_j \prod_{l \in J} Q_l \prod_{m=1}^{k} R_m^{a_m - |\{m\} \cap J|}.$$

Proof by induction on k, relying on the Schwenk's formula

$$PG(u=v)H = PGP(H-v) + P(G-u)PH - xP(G-u)P(H-v).$$

Corollaries of the main theorem

When all rooted graphs $G_1, \ldots, G_k$ are the same:

Corollary

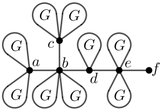
Let T be a graph with distinct vertices $u_1, \ldots, u_k$ for some $k \geq 1$. For a rooted graph G with the root r let

$$Q = PG - xP(G-r) \quad \text{and} \quad R = P(G-r).$$

Then for any signature $(a_1, \ldots, a_k)$, we have

$$PT(u_1\!=\!r)G^{(a_1)} \cdots (u_k\!=\!r)G^{(a_k)} = \sum_{J \subseteq \{1,\ldots,k\}} P(T - \sum_{i \in J} u_i) \prod_{j \in J} a_j \, Q^{|J|} R^{\sum_{m=1}^{k} a_m - |J|}.$$

Example



$$PT(a=r)G^{(2)}(b=r)G^{(2)}(c=r)G^{(2)}(e=r)G^{(2)}(d=r)G^{(1)}$$

$$= PT\,R^9$$

$$+ [2P(T-a)+2P(T-b)+2P(T-c)+2P(T-e)+P(T-d)]QR^8$$

$$+ [4P(T-a-b)+4P(T-a-c)+4P(T-a-e)+2P(T-a-d)+4P(T-b-c)$$

$$+4P(T-b-e)+2P(T-b-d)+4P(T-c-e)+2P(T-c-d)+2P(T-e-d)]Q^2R^7$$

$$+ [8P(T-a-b-c)+8P(T-a-b-e)+4P(T-a-b-d)+8P(T-a-c-e)$$

$$+4P(T-a-c-d)+4P(T-a-e-d)+8P(T-b-c-e)+4P(T-b-c-d)$$

$$+4P(T-b-e-d)+4P(T-c-e-d)]Q^3R^6$$

$$+ [16P(T-a-b-c-e)+8P(T-a-b-c-d)+8P(T-a-b-e-d)$$

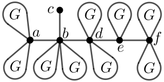
$$+8P(T-a-c-e-d)+8P(T-b-c-e-d)]Q^4R^5$$

$$+ 16P(T-a-b-c-e-d)Q^5R^4$$

$$= PT\,R^9+(9x^5-29x^3+14x)QR^8+(32x^4-58x^2+8)Q^2R^7$$

$$+(56x^3-44x)Q^3R^6+(48x^2-8)Q^4R^5+16xQ^5R^4.$$

Example



$$
\begin{aligned}
&PT(a{=}r)G^{(2)}(b{=}r)G^{(2)}(d{=}r)G^{(2)}(f{=}r)G^{(2)}(e{=}r)G^{(1)}\\
&= PT\,R^9\\
&+[2P(T-a)+2P(T-b)+2P(T-d)+2P(T-f)+P(T-e)]QR^8\\
&+[4P(T-a-b)+4P(T-a-d)+4P(T-a-f)+2P(T-a-e)+4P(T-b-d)\\
&\quad+4P(T-b-f)+2P(T-b-e)+4P(T-d-f)+2P(T-d-e)+2P(T-f-e)]Q^2R^7\\
&+[8P(T-a-b-d)+8P(T-a-b-f)+4P(T-a-b-e)+8P(T-a-d-f)\\
&\quad+4P(T-a-d-e)+4P(T-a-f-e)+8P(T-b-d-f)+4P(T-b-d-e)\\
&\quad+4P(T-b-f-e)+4P(T-d-f-e)]Q^3R^6\\
&+[16P(T-a-b-d-f)+8P(T-a-b-d-e)+8P(T-a-b-f-e)\\
&\quad+8P(T-a-d-f-e)+8P(T-b-d-f-e)]Q^4R^5\\
&+16P(T-a-b-d-f-e)Q^5R^4\\
&= PT\,R^9+(9x^5-29x^3+14x)QR^8+(32x^4-58x^2+8)Q^2R^7\\
&\quad+(56x^3-44x)Q^3R^6+(48x^2-8)Q^4R^5+16xQ^5R^4.
\end{aligned}
$$

Corollaries of the main theorem (2)

Corollary

Let T_1 be a graph with distinct vertices $u_1, \ldots, u_k$, and T_2 a graph with distinct vertices $v_1, \ldots, v_k$ for some $k \geq 1$. For a fixed signature $(a_1, \ldots, a_k)$, the coalescences

$$T_1(u_1 = r_1)G_1^{(a_1)} \cdots (u_k = r_k)G_k^{(a_k)}$$

and

$$T_2(v_1 = r_1)G_1^{(a_1)} \cdots (v_k = r_k)G_k^{(a_k)}$$

are cospectral **for all choices of $G_1, \ldots, G_k$ and their roots** *$r_1, \ldots, r_k$ if and only if T_1 and T_2 are cospectral graphs with the removal-cospectral sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$.*

Corollaries of the main theorem (3)

Corollary

Let T_1 be a graph with distinct vertices $u_1, \ldots, u_k$, and T_2 a graph with distinct vertices $v_1, \ldots, v_k$ for some $k \geq 1$.
For fixed signatures $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ with $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$, the coalescences

$$T_1(u_1 = r)G^{(a_1)} \cdots (u_k = r)G^{(a_k)}$$

and

$$T_2(v_1 = r)G^{(b_1)} \cdots (v_k = r)G^{(b_k)}$$

are cospectral **for all choices of** G **and its root** r *if and only if* $(a_1, \ldots, a_k) = (b_1, \ldots, b_k)$ *and for each* $0 \leq l \leq k$

$$(*) \quad \sum_{J \subseteq \{1,\ldots,k\}, |J|=l} P(T_1 - \sum_{i \in J} u_i) \prod_{j \in J} a_j = \sum_{J \subseteq \{1,\ldots,k\}, |J|=l} P(T_2 - \sum_{i \in J} v_i) \prod_{j \in J} a_j.$$

Software for the Hosoya-type coalescences

The Java code, available at `zenodo.org/record/4896776`, was written to find examples of Hosoya-type coalescences.

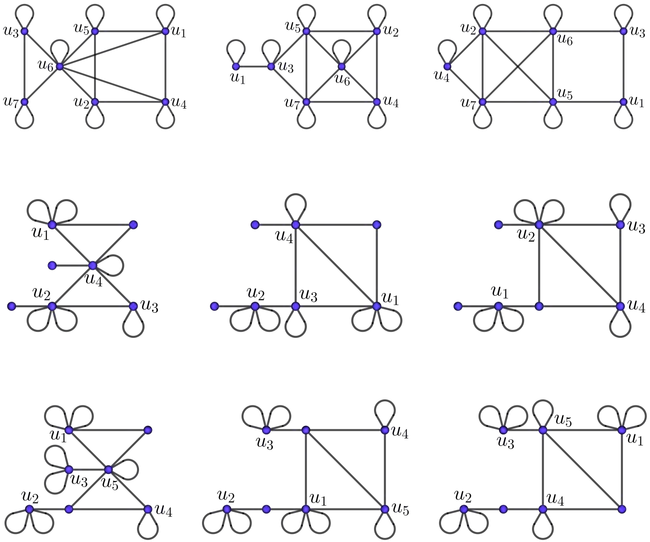
For a given set of underlying graphs,
sorted by the characteristic polynomials,
it processes all groups of cospectral graphs,
all signatures with entries between 1 and a given max value,
and all variations of vertices from each cospectral graph
to identify the examples that satisfy conditions (*).
(the removal-cospectral sets are skipped).

It succumbs to combinatorial explosion
when underlying graphs have more than 10–12 vertices.

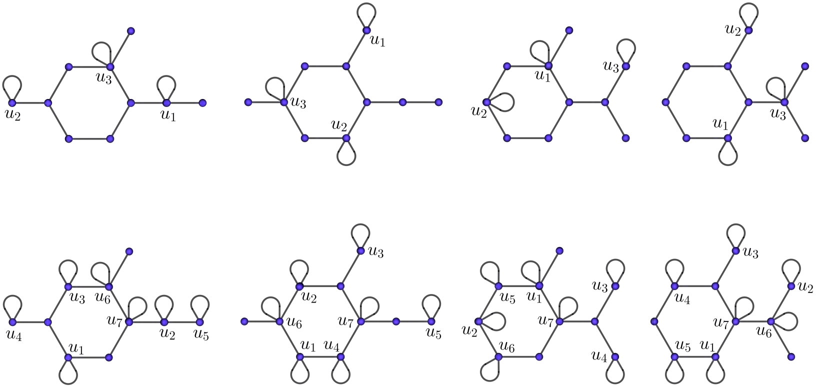
Computational results

Set of graphs	MSE	# Pairs of CMC	# Triplets of CMC	# Quadruplets of CMC
Connected graphs, 5 vertices	4	20		
Connected graphs, 6 vertices	4	277		
Connected graphs, 7 vertices	2	1215	3	
Path, 8 vertices	10	2788		
Path, 11 vertices	1	4		
Path, 14 vertices	1	10		
Path, 15 vertices	1	3		
Path, 17 vertices	1	11		
Path, 19 vertices	1	10		
Path, 20 vertices	1	9		
Trees with perfect matchings, 6 vertices	3	14		
Trees with perfect matchings, 8 vertices	2	89		
Trees with perfect matchings, 10 vertices	1	105		
Unicyclic graphs, girth 6, 6 vertices	2	1		
Unicyclic graphs, girth 6, 7 vertices	2	2		
Unicyclic graphs, girth 6, 8 vertices	2	52		
Unicyclic graphs, girth 6, 9 vertices	2	745	4	
Unicyclic graphs, girth 6, 10 vertices	1	429	6	2
Benzenoid, 2 hexagons	2	4		

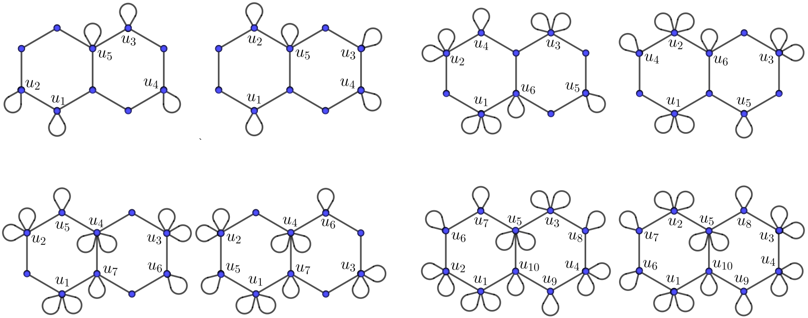
Triplets with underlying graphs on 7 vertices



Quadruplets with underlying unicyclic graphs on 10 vertices

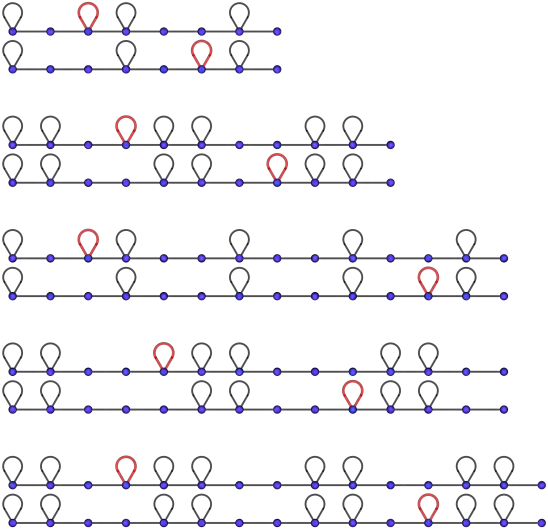


Pairs with underlying benzenoids with 2 hexagons



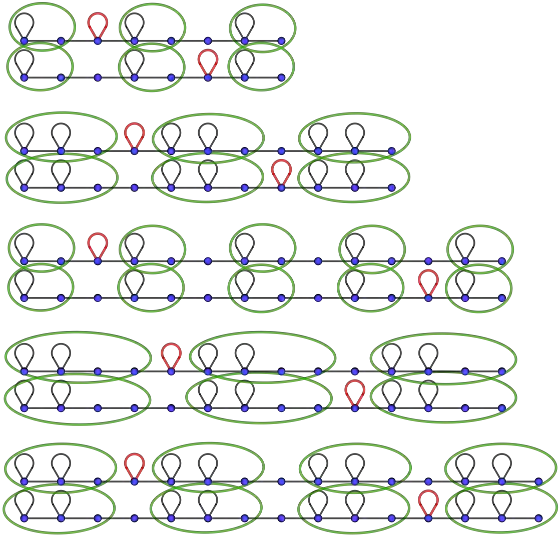
Pairs with paths as underlying graphs

These were deemed rare by Hosoya...

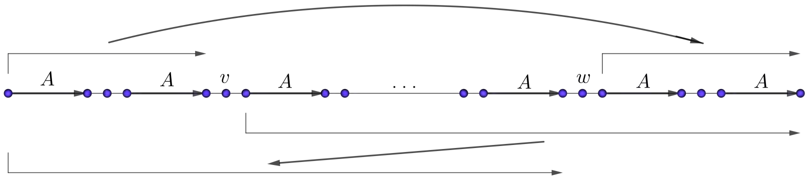


Pairs with paths as underlying graphs

These were deemed rare by Hosoya. . .



General structure of these pairs



Here A is some subset of initial path vertices,
that then gets translated by multiples of d along the path,
while v and w are symmetrically placed between copies of A.

An infinite family with paths as underlying graphs

Theorem

For arbitrary integers $k \geq 3$, $m < k/2$, $d \geq 2$, *and the integers* $(a_1, \ldots, a_p)$ *such that* $0 \leq a_1 < \cdots < a_p \leq d-2$, *let* $n = kd - 1$, $v = md - 1$, $w = (k-m)d - 1$, *and let* $u_{(i,j)} = (i-1)d + a_j$ *for* $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, p\}$. *Then for any rooted graph* G *with the root* r *the coalescences*

$$P_n(v = r)G(u_{(1,1)} = r)G(u_{(1,2)} = r)G \cdots (u_{(k,p)} = r)G$$

and

$$P_n(w = r)G(u_{(1,1)} = r)G(u_{(1,2)} = r)G \cdots (u_{(k,p)} = r)G$$

are cospectral.

Instead of a conclusion

From: hosoya.haruo@ocha.ac.jp
To: dragance106@yahoo.com
Date: Sat, Sep 4 at 6:32 AM
Subject: Re: On construction of cospectral graphs using multiple coalescences

Dear Dragon Stevanovic:
This is an astonishing information which you could send me. Congratulation and a lot of thanks to your effort in realizing my wish and dream. As I am not good at programming, it is rather difficult to follow your algorithm precisely. However, I am convinced of its correctness, because some parts of the results in your paper support my results. To say the truth I was astonished at the results in Table 1 showing Path, 8 vertices, MSE=10, # Pairs of CMC=2788. Could you please, show me only some part of it other than for (1, 1, 1, 1)? At this moment I cannot write more, but I am deeply expressing my thanks and astonishment to your group.

Best regards,

Haruo Hosoya

Thanks!

Preprint:
S. Al-Yakoob, A. Kanso, D. Stevanović,
On Hosoya's dormants and sprouts,
`arxiv.org/abs/2109.09369`