Constructing infinite families of ALQ-integral graphs

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Topics

Basic concepts and results;

A new operation and ALQ-integral graphs;

ALQ-integral graphs and regular graphs;

ALQ-integral graphs and split-like graphs.

Basic definitions

- $G = (V, E)$ is a simple and connected graph on n vertices and m edges;
- $A(G)$ is the adjacency matrix of G and $\lambda_1 \geq \ldots \geq \lambda_n$ its eigenvalues;
- $D(G)$ is the diagonal matrix where the elements on the main diagonal are the degrees of the vertices of G;
- $L(G) = D(G) - A(G)$ is the Laplacian matrix of G and $\mu_1 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ its eigenvalues;
- $Q(G) = D(G) + A(G)$ is the signless Laplacian matrix of G and $q_1 \geq \ldots \geq q_n$ its eigenvalues.

M-integral graphs

Definition: *Let M be a matrix associated to a graph G. A graph is M-integral if and only if the eigenvalues of M are all integer numbers.*

- If $M = A$ is the adjacency matrix of G, G is called an A-*integral graph*;
- If $M = L$ is the Laplacian matrix of G, G is called an L-*integral graph*;
- If $M = Q$ is the Signless Laplacian matrix of G, G is called a Q-*integral graph*.

Regular graphs and integrality

Theorem: If G is an r-regular graph, then

$$P_A(G,x) = P_Q(G,x+r) \text{ and } P_L(G,x) = (-1)^n P_Q(G,2r-x).$$

Consequently,

$$P_L(G,x) = (-1)^n P_A(G,r-x).$$

Corollary: Let G be a regular graph. G is an A-integral graph $\Leftrightarrow$ G is an L-integral graph $\Leftrightarrow$ G is a Q-integral graph.

Bipartite graphs and integrality

Theorem: Let G be a bipartite graph. G is an L-integral graph $\Leftrightarrow$ G is a Q-integral graph.

ALQ-integral graphs

Definition: *A graph is called an ALQ-integral when it is simultaneously A, L and Q-integral graph.*

- In 2007, Z. Stanić enumerated all connected Q-integral graphs up to 10 vertices.
- There are 172 such graphs.
- Among them, there are 42 ALQ-integral graphs.
- It is interesting to note that only one of them is neither regular nor bipartite.

The smallest connected neither bipartite nor regular ALQ-integral graph

There are no connected neither regular nor bipartite ALQ-integral graph up to 9 vertices.

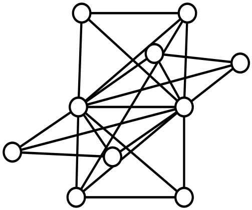


Figure: The smallest connected ALQ-integral graph (it has 10 vertices) which is neither bipartite nor regular.

Objective

- **Our aim:** To construct infinite families of ALQ-integral graphs which necessarily contain neither regular nor bipartite graphs.

- **Remark:** If we consider the computational experiments of Stanić(2007), we can hope that the ALQ-integral graphs, neither bipartite nor regular, have huge orders. So, we are looking for them through the operation of graphs.

Union and Cartesian product of graphs

Let G_1 and G_2 be graphs such that, for $i = 1, 2$, V_i and E_i are, respectively, the sets of vertices and edges of G_i.

Union of graphs: *The union $G_1 \cup G_2$ of G_1 and G_2 is the graph whose vertex set is $V = V_1 \cup V_2$ and whose edge set is $E = E_1 \cup E_2$.*

Cartesian Product: *The Cartesian Product $G_1 \times G_2$ of G_1 and G_2 is the graph whose vertex set is $V = V_1 \times V_2$ and where*

$$(u_1, u_2) \sim (v_1, v_2) \Leftrightarrow \left\{ \begin{array}{l} u_1 \sim v_1 \text{ in } G_1 \text{ and } u_2 = v_2, \text{ or} \\ u_1 = v_1 \text{ and } u_2 \sim v_2 \text{ in } G_2. \end{array} \right\}$$

Spectra of resultant graphs from operations

- G_1 and G_2 are A-integral graphs $\Rightarrow G_1 \cup G_2$ and $G_1 \times G_2$ are A-integral graphs;
- G_1 and G_2 are L-integral graphs $\Rightarrow G_1 \cup G_2$ and $G_1 \times G_2$ are L-integral graphs;
- G_1 and G_2 are Q-integral graphs $\Rightarrow G_1 \cup G_2$ and $G_1 \times G_2$ are Q-integral graphs.

Definition of k-pla operation: *Let $G = (V, E)$ be a graph on n vertices and, for $k \in \mathbb{N}$, let $K = \{1, \dots, k\} \subset \mathbb{N}$. We call a k-pla graph $G^{(k)} = (V_k, E_k)$ of G that one which set of vertices is $V_k = V \times K$ and the set of edges is $E_k = \{\{(x_1, y_1), (x_2, y_2)\}; \{x_1, x_2\} \in E; y_1, y_2 \in K\}$.*

Remark:

- This definition generalizes the definition of a double graph of G, given by Munarini *et al* (2008). In fact, the double graph is a 2-pla graph of G.
- It is a special case of NEPS (see Cvetković *et al.*)

An example of a 3-pla graph

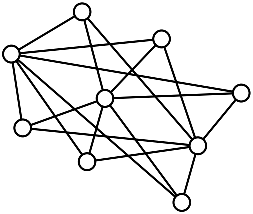


Figure: $3-$pla graph of P_3

Matrices of a k-pla graph

- Note that $G^{(1)} \simeq G$.
- The adjacency matrix of $G^{(k)}$ can be represented as
$$A(G^{(k)}) = \mathbb{J}_k \otimes A(G),$$
where $\mathbb{J}_k$ denotes the all ones matrix of order k and $\otimes$ is the Kronecker product of matrices.
- Consequently,
$$L(G^{(k)}) = k\,\mathbb{I}_k \otimes D(G) \,-\, \mathbb{J}_k \otimes A(G)$$
and
$$Q(G^{(k)}) = k\,\mathbb{I}_k \otimes D(G) \,+\, \mathbb{J}_k \otimes A(G).$$

The spectra of a k-pla graph

Theorem: Let G be a graph on n vertices and $D(G) = diag(d_1, \ldots, d_n)$, $d_i, 1 \leq i, n$, is degree of the vertex v_i in G. Let $k \in \mathbb{N}$, $k \geq 2$.

- The A-eigenvalues of $G^{(k)}$ are $0^{(k-1)n}, k\lambda_1, \ldots, k\lambda_n$, where $Sp_A(G) = (\lambda_1, \ldots, \lambda_n)$;
- The L-eigenvalues of $G^{(k)}$ are $nd_1^{k-1}, \ldots, nd_n^{k-1}, k\mu_1, \ldots, k\mu_n$, where $Sp_L(G) = (\mu_1, \ldots, \mu_n)$;
- The Q-eigenvalues of $G^{(k)}$ are $nd_1^{k-1}, \ldots, kd_n^{k-1}, kq_1, \ldots, kq_n$, where $Sp_Q(G) = (q_1, \ldots, q_n)$.

Observation: The exponents denote multiplicities.

Sketch of the Proof

Sketch of the Proof: We have,

- $A(G)\mathbf{v} = \lambda\mathbf{v} \Longrightarrow A(G^{(k)})\mathbf{w} = (\mathbb{J}_k \otimes A(G))\, \mathbf{1}_k \otimes \mathbf{v} = k\lambda\mathbf{w}$, where $\mathbf{w} = \mathbf{1}_k \otimes \mathbf{v}$.
- $\mathbf{u} \in \mathbb{R}^k$ is orthogonal to $\mathbf{1}_k \Longrightarrow A(G^{(k)})\mathbf{w}_i = \mathbf{0}$, where $\mathbf{w}_i = \mathbf{u} \otimes \mathbf{e}_i$, for $1 \le i \le n$.

Consequently, the A-eigenvalues of $G^{(k)}$ are

$$0^{(k-1)n}, k\lambda_1, \ldots, k\lambda_n,$$

where $Sp_A(G) = (\lambda_1, \ldots, \lambda_n)$. ■

Observation: The proofs of the others items are similar.

Corollary: *The graph $G^{(k)}$, $k \in \mathbb{N}$, $k \geq 2$, is ALQ-integral if and only if G is an ALQ-integral graph.*

Corollary: *For each ALQ-integral graph G, $\{G^{(k)}, k \in \mathbb{N}\}$ is an infinite family of ALQ-integral graphs.*

Remark: Note that if G is neither regular nor bipartite, the same occurs with $G^{(k)}$, for each $k \in \mathbb{N}$.

Join of graphs

Definition of Join Operation: *Let G_1 and G_2 be graphs such that, for $i = 1, 2$, V_i and E_i are, respectively, the sets of vertices and edges of G_i. The Join $G_1 \vee G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2.*

The L-characteristic polynomial of a join of graphs

Theorem: *For $i = 1, 2$, let G_i be a graph on n_i vertices. Then, the L-characteristic polynomial of $G_1 \vee G_2$ is*

$$P_L(G_1 \vee G_2, x) = x(x - (n_1 + n_2)) \frac{P_L(G_1,\, x - n_2) P_L(G_2,\, x - n_1)}{(x - n_2)(x - n_1)}. \quad (1)$$

Corollary: *$G_1 \vee G_2$ is L-integral $\Leftrightarrow$ G_1 and G_2 are L-integral.*

The A-characteristic polynomial of a join of regular graphs

Theorem: *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices. Then, the A-characteristic polynomial of $G_1 \vee G_2$ is*

$$P_A(G_1 \vee G_2, x) = \frac{P_A(G_1, x)P_A(G_2, x)}{(x - r_1)(x - r_2)} f(x), \qquad (2)$$

where $f(x) = x^2 - (r_1 + r_2)x + r_1 r_2 - n_1 n_2$.

The Q-polynomial of a join of regular graphs

Theorem(Freitas et al., 2010) *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices. Then, the Q-characteristic polynomial of $G_1 \vee G_2$ is*

$$P_Q(G_1 \vee G_2, x) = \frac{P_Q(G_1, x - n_2) P_Q(G_2, x - n_1)}{(x - 2r_1 - n_2)(x - 2r_2 - n_1)} f(x), \quad (3)$$

where
$f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1 r_2 + r_1 n_1 + r_2 n_2)$.

ALQ-integrality condition for a join of regular graphs

Theorem: *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i. vertices. The graph $G_1 \vee G_2$ is ALQ-integral if and only if G_1 and G_2 are ALQ-integral and $(r_1 - r_2)^2 + 4n_1n_2$ and $((2r_1 - n_1) - (2r_2 - n_2))^2 + 4n_1n_2$ are perfect squares.*

Note that if $r_1 - r_2 \neq 0$ and $r_1 - r_2 \neq n_1 - n_2$ then $G_1 \vee G_2$ is neither regular nor bipartite.

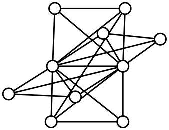
From this result, we obtain some corollaries which give us more infinite families of ALQ-integral graphs.

$K_2 \vee 4K_2$ satisfies the last theorem

The smallest connected neither bipartite nor regular ALQ-integral graph is isomorphic to $K_2 \vee 4K_2$ and it satisfies the conditions of the last theorem.

- $G_1 = K_2$ has 2 vertices and $G_2 = 4K_2$ has 8 vertices and both graphs are 1-regular;
- $(r_1 - r_2)^2 + 4n_1n_2 = 64$ is a perfect square;
- $((2r_1 - n_1) - (2r_2 - n_2))^2 + 4n_1n_2 = 36$;

Also, we have $r_1 - r_2 = 0 \neq n_2 - n_1 = 6$.



The first corollary

Corollary I: *Let G_1 and G_2 be r-regular ALQ-integral graphs. Then, $G_1 \vee G_2$ is ALQ-integral if and only if $|G_1||G_2|$ is a perfect square.*

Sketch of the Proof: Let $|G_i| = n_i$, $i = 1, 2$, both regular graphs with the same degree r. From last proposition, we have:

$G_1 \vee G_2$ is ALQ-integral $\Leftrightarrow$ $((2r - n_1) - (2r - n_2))^2 + 4n_1n_2 = (n_1 + n_2)^2$ and $4n_1n_2$ are perfect squares.

So, $G_1 \vee G_2$ is ALQ-integral $\Longleftrightarrow$ $|G_1||G_2|$ is a perfect square. $\blacksquare$

The first corollary generalizes a result of Stanić(2007)

Lemma: A complete bipartite graph $K_{t,s}$ is ALQ-integral if and only if ts is a perfect square.

The second corollary

Corollary II: *For all $j, k \in \mathbb{N}$, the graph $K_j \vee kK_j$ is ALQ-integral if and only if k is a perfect square.*

The second corollary also generalizes the other result of Stanić for $j = 2$. His result was recently published in Ars Combinatoria (2009).

The third corollary

Corollary III: *For all* $j \in \mathbb{N}$ *and* $k = \dfrac{j(j+1)}{2}$*, the graph* $K_{j,j} \vee kK_{j+1}$ *is* ALQ*-integral.*

An example which satisfies the third corollary

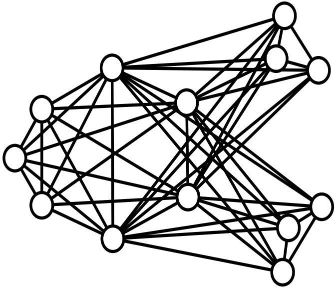


Figure: $K_{2,2} \vee 3K_3$

In 2002, Hansen *et al* characterized A-integral graphs and, in 2010, Freitas *et al* characterized Q-integral graphs in the classes constituted by joins of complete graphs with the complements of complete graphs.

Let $a, b, k \in \mathbb{N}$. They are the following classes:

Complete split graphs, $CS_b^a \cong \overline{K_a} \vee K_b$;

Multiple complete split-like graphs, $MCS_{b,k}^a \cong \overline{K_a} \vee (kK_b)$;

Multiple extended complete split-like graphs, $MECS_{b,k}^a \cong \overline{K_a} \vee (k(K_b \times K_2))$.

ALQ-integrality condition for split like graphs

Corollary 4: For $a, b \in \mathbb{N}$ and $k \in \mathbb{N}$,

the complete split graph CS_b^a is ALQ-integral if and only if $(b-1)^2 + 4ab$ and $(a+b-2)^2 + 4ab$ are perfect squares;

the multiple complete split-like graph $MCS_{b,k}^a$ is ALQ-integral if and only if $(a + 2(b-1) - kb)^2 + 4abk$ and $(b-1)^2 + 4abk$ are perfect squares;

the multiple extended complete split-like graph $MECS_{b,k}^a$ is ALQ-integral if and only if $(a + 2b(k-1))^2 + 8ab$ and $b^2 + 8abk$ are perfect squares.

Sketch of the Proof:

Sketch of the Proof of the first item: $\overline{K_a}$ and K_b are regular ALQ-integral graphs of orders a and b and degrees 0 and $b-1$, respectively.

From the last corollary, CS_b^a is ALQ-integral $\Leftrightarrow (b-1)^2+4ab$ and $(a+b-2)^2+4ab$ are perfect squares. ■

Observation: The proof of the others items are similar.

Can we determine some graphs which satisfy the conditions of the last theorem?

Some examples: If one of the conditions below holds, CS_b^a is Q-integral:

$a = 3j_t$ and $b = 2j_t - 1$, where $j_0 = 3$, $m_0 = 7$ and there is $m_t \in \mathbb{Z}, t \geq 1$ such that

$$j_{t+1} = 127j_t + 484m_t - 45$$
$$m_{t+1} = 336j_t + 127m_t - 120.$$

$a = 10j_t - 4$, $b = 3j_t$, where $j_0 = 4$, $m_0 = 43$ and there is $m_t \in \mathbb{Z}$ such that

$$j_{t+1} = -16855j_t - 1484m_t + 3528$$
$$m_{t+1} = -191436j_t - 16855m_t + 40068;$$

$a = 3j_t - 2$ and $b = 2j_t$, where $j_0 = 10$, $m_0 = -51$ and there is $m_t \in \mathbb{Z}$ such that

$$j_{t+1} = 127j_t + 24m_t - 45$$
$$m_{t+1} = 672j_t + 127m_t - 240;$$

Observation: As we built the families of ALQ-integral graphs for the complete split graphs, we can construct the infinite families for the multiple complete split graphs and the multiple extended complete split graphs in the similar way.

A complete split graph and its spectra

Figure 1 shows the complete split graph CS_5^9 whose spectra relative to A, L and Q respectively are

$$Sp_A(CS_5^9) = (9, 0^8, -1^4, -5),$$
$$Sp_L(CS_5^9) = (14^5, 5^8, 0) \text{ and}$$
$$Sp_Q(CS_5^9) = (20, 12^4, 5^8, 2).$$

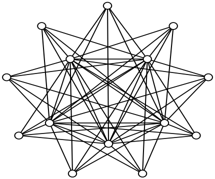


Figure: CS_5^9

A multiple complete split graph and its spectra

Figure below displays the multiple complete split graph, $MCS^7_{3,3}$, whose spectra relative to A, L and Q respectively are:

$Sp_A(MCS^7_{3,3}) = (9, 2^2, 0^6, -1^6, -7)$,
$Sp_L(MCS^7_{3,3}) = (16, 10^6, 9^6, 7^2, 0)$ and
$Sp_Q(MCS^7_{3,3}) = (18, 11^2, 9^6, 8^6, 2)$.

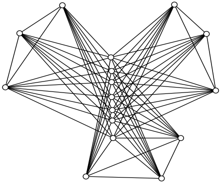


Figure: The graph $MCS^7_{3,3}$ is ALQ-integral

A multiple extended complete split graph and its spectra

Example: Figure below shows $G = MECS^{6}_{2,2}$ whose spectra relative to A, L and Q are:

$$Sp_{\,A}(G) = (8, 2, 0^9, -2^2, -6),$$

$$Sp_{\,L}(G) = (14, 10^2, 8^9, 6, 0)$$

$$Sp_{\,Q}(G) = (16, 10, 8^9, 6^2, 2).$$

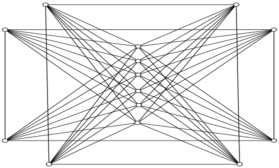


Figure: The graph $MECS^{6}_{2,2}$ is ALQ-integral

Bibliography

F. Harary, A.J. Schwenk, *Which graphs have integral spectra?*, em Bari, R., Harary, F.(Eds), Graphs and Combinatorics, Springer, Berlim, 1974, p. 45-51.

D. Cvetković, P. Rowlinson, S. Simić, *Signless Laplacian of finite graphs*, Linear Algebra and its Applications **423** (2007) 155-171.

S. Simić, Z. Stanić, *Q-integral graphs with edge-degrees at most five*, Discrete Math. **308** (2008) 4625-4634.

Z. Stanić, *There are exactly 172 connected Q-integral graphs up to 10 vertices*, Novi Sad J. Math. **37** (2) (2007) 193-205.

Stanić, Z., *Some results on Q-integral graphs*, Ars Combinatoria **90** (2009), 321-335.

M.A.A. de Freitas, N.M.M. de Abreu, R.R. Del-Vecchio, S.Jurkiewicz, *Infinite families of Q-integral graphs,* Linear Algebra Appl. **432** (2010), 2353-2360

D. Cvetković, M. Doob, H. Sachs: *Spectra of graphs-theory and application.* Deutscher Verlag der Wissenschaften-Academic Press, Berlin-New York, 1980; second edition 1982; third edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

P. Hansen, H. Melot, D, Stevanović, *Integral complete split graphs*, Univ. Beograd, *Publ. Elektrotehn. Fak. Ser. Mat.* **13**, 2002, 89-95.