Decompositions of Graphs and Hypergraphs

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Bicliques and Graham-Pollak Theorem

A complete bipartite graph is called a **biclique**.

Theorem (Graham-Pollak 1971)

If the edge-set of the complete graph K_n on n vertices are partitioned into k bicliques, then

$$k \geq n - 1.$$

Remarks

All the known proofs of this theorem use methods from linear algebra. There are many partitions using $n - 1$ bicliques.

The **biclique partition number** $bp(G)$ of a graph G is the minimum number of bicliques that partition the edge-set of G.

THEOREM (WITSENHAUSEN 1971)

If G is a graph, then

$$bp(G) \geq \max(n_+(G), n_-(G))$$

where $n_+(G)$ is the number of positive eigenvalues and $n_-(G)$ is the number of negative eigenvalues of the adjacency matrix of G.

Applications

$K_{a,b}$ has eigenvalues $\sqrt{ab}^{(1)}, 0^{(a+b-2)}, -\sqrt{ab}^{(1)}$ so $bp(K_{a,b}) \geq \max(n_+(K_{a,b}), n_-(K_{a,b})) = 1$.

K_n has eigenvalues $(n-1)^{(1)}, (-1)^{(n-1)}$ so $bp(K_n) \geq \max(n_+(K_n), n_-(K_n)) = n-1$.

The Kneser Graph $K(n,k)$

The vertices of the Kneser graph $K(n,k)$ are the k-subsets of $[n] := \{1, \ldots, n\}$ and $A \sim B$ iff $A \cap B = \emptyset$.

$$bp(K(n,k)) \geq \max(n_+(K(n,k)), n_-(K(n,k))) = \binom{n-1}{k}.$$

Hypergraph Decomposition into HyperBicliques

Complete r-uniform hypergraphs

Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices. Its vertex set is $[n] = \{1, \ldots, n\}$ and its **hyper**edges are the r-subsets of $[n]$.

Complete r-partite r-uniform hypergraph

If $A_1, \ldots, A_r \subset [n]$ are pairwise disjoint, then the r-uniform hypergraph whose edge set is $A_1 \times \cdots \times A_r$ is called a **complete r-partite r-uniform hypergraph**.

Example

The 3-uniform hypergraph whose edges are $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}$ is complete 3-partite 3-uniform hypergraph with edge set $\{1\} \times \{2\} \times \{3,4,5\}$.

Graham-Pollak Theorm for Hypergraphs

Let $f_r(n)$ denote the minimum number of complete r-partite r-uniform hypergraphs that partition the edge-set of $K_n^{(r)}$.

Graham-Pollak Theorem states that $f_2(n) = n - 1$.

Question (Aharoni-Linial 1980s)

For fixed $r \geq 3$, is $f_r(n)$ linear in n ?

Theorem (Alon 1986)

$$f_3(n) = n - 2$$

and for fixed $r \geq 4$

$$f_r(n) = \Theta\left(n^{\lfloor \frac{r}{2} \rfloor}\right).$$

Our Results

Theorem (Cioabă-Kündgen-Verstraëte 2009)

For $2k \leq n$,

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leq f_{2k}(n) \leq f_{2k+1}(n+1) \leq \binom{n-k}{k}.$$

Proof of the lower bound $f_{2k}(n) \geq \frac{2\binom{n-1}{k}}{\binom{2k}{k}}$

Proof Idea 1

From a complete $2k$-partite $2k$-uniform hypergraph with edge set $A_1 \times \cdots \times A_{2k}$ construct $\frac{\binom{2k}{k}}{2}$ bicliques in the Kneser graph $K(n,k)$: for each partition $I \cup J = [2k], |I| = |J| = k$, take the biclique whose color classes are $A_I := \times_{i \in I} A_i$ and $A_J := \times_{j \in J} A_j$.

Proof Idea 2

From an optimal decomposition of $K_n^{(2k)}$ with $f_{2k}(n)$ hypergraphs, we obtain $f_{2k}(n) \cdot \frac{\binom{2k}{k}}{2}$ bicliques that partition the edge-set of the Kneser graph $K(n,k)$:

$$f_{2k}(n) \cdot \frac{\binom{2k}{k}}{2} \geq bp(K(n,k)) = \binom{n-1}{k}.$$

Proof of the upper bound $f_{2k}(n) \le f_{2k+1}(n+1) \le \binom{n-k}{k}$

Construction

For $1 < i_1 < \cdots < i_k < n+1$ with $i_{j+1} - i_j \ge 2$ for each j, let $H_{i_1,\ldots,i_k}$ be the complete $(2k+1)$-partite $(2k+1)$-uniform hypergraph whose edge-set is

$$\{1,\ldots,i_1-1\} \times \{i_1\} \times \{i_1+1,\ldots,i_2-1\} \times \cdots \times \{i_k\} \times \{i_k+1,\ldots,n+1\}.$$

The $\binom{n-k}{k}$ hypergraphs $H_{i_1,\ldots,i_k}$ partition the edge-set of $K_{n+1}^{(2k+1)}$.

Open Problems

Determining $f_r(n)$ when $r \geq 4$

$$\frac{n^2-3n+2}{6} = \frac{2\binom{n-1}{2}}{\binom{4}{2}} \leq f_4(n) \leq \binom{n-2}{2} = \frac{n^2-5n+6}{2}$$

Question (Brualdi 2010)

For fixed $k \geq 1$, do any of the following limits exist ?

$$\lim_{n\to\infty} \frac{f_{2k+1}(n+1)}{f_{2k}(n)}$$

$$\lim_{n\to\infty} \frac{f_{2k}(n)}{n^k}$$

Open Problems

De Caen Conjecture

In any partition of K_n with $n-1$ **colored** bicliques, there exists a **multicolored path** with $n-1$ edges.

Theorem (Alon, Brualdi and Shader 1991)

In any partition of K_n with $n-1$ **colored** *bicliques, there exists a* **multicolored tree** *with $n-1$ edges.*

De Caen-Gregory-Pritikin Conjecture

Given $\lambda \geq 2$, there exists $n(\lambda)$ such that $bp(\lambda K_n) = n-1$ for any $n \geq n(\lambda)$.

Open Problems

Covering each edge of K_n at most twice

What is the minimum number of bicliques that cover the edges of K_n such that each edge of K_n belongs to precisely one or two bicliques ?

Huang-Sudakov 2010: this number is between $\sqrt{n}$ and $2\sqrt{n}$.

Hoffman's Honeymoon Hotel

The Honeymoon Hotel $HH(m)$ is obtained from K_{2m} by adding a perfect matching.

$$m + \lfloor \sqrt{2m} \rfloor - 1 \leq bp(HH(m)) \leq \begin{cases} \frac{3m-2}{2} & \text{m even} \\ \frac{3m-1}{2} & \text{m odd} \end{cases}$$

Conjecture (Hoffman 2001)

$bp(HH(m)) = RHS$. True for $m \leq 10$. (Mike Tait, UDel Undergrad).