SOME TOPICS ON INTEGRAL GRAPHS

Dragoš Cvetković

Faculty of Electrical Engineering, University of Belgrade,
and
Mathematical Institute SANU, Belgrade,
11000 Belgrade, Serbia

e-mail: ecvetkod@etf.rs

The M-spectrum of a graph is the spectrum of a graph matrix M (adjacency matrix A, Laplacian L, signless Laplacian Q, etc.). A graph is called *M-integral* if its M-spectrum consists entirely of integers. If the matrix M is fixed, we say, for short, *integral* instead of M-integral. A graph which is A-, L- and Q-integral is called *ALQ-integral*. A survey on integral graphs can be found in the paper

K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, A survey on integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat., **13**(2002), 42-65.

For the product we have the following interesting formula

$$Q_{G\times K_2}(x) = Q_G(x)L_G(x) = L_{G\times K_2}(x).$$

The formula is easily obtained by elementary determinantal transformations. Therefore it follows that

$$q_1, q_2, \ldots, q_n \text{ and } \mu_1, \mu_2, \ldots, \mu_n$$

are the Q–eigenvalues (and as well the L–eigenvalues) of the graph $G\times K_2$. In particular, we have that the Q–indices of G and $G\times K_2$ are equal (as is the case for A–indices of these graphs).

Proposition. *If G is an ALQ-integral graph, then the product $G \times K_2$ is a bipartite ALQ-integral graph.*

A-integral graphs are very rare. Other kinds of integral graphs could be more frequent. For example, out of 112 connected integral graphs on 6 vertices there are only 6 A-integral graphs while the number of L-integral graphs is 37. According to a table of Q-eigenvalues of the 112 connected graphs on six vertices from there are 13 Q-integral graphs.

The reason for high number of L-integral graphs is, among other things, the fact that the complement of an L-integral graph is also L-integral. There are no corresponding formulas for the A-polynomial and for the Q-polynomial which would preserve the property of being integral and this is reflected in statistics of integral graphs.

There are exactly 150 connected A-integral graphs up to 10 vertices.

It is established by a computer search that there are exactly 172 connected Q-integral graphs up to 10 vertices . Among them there exists exactly one graph which is ALQ-integral but not regular and not bipartite. It has 10 vertices. There is another ALQ-integral (on 10 vertices) which is bipartite (and not regular).

The problem of determining all connected, non-regular ALQ-integral graphs was posed at the Aveiro Workshop on Graph Spectra, 2006. More desirable problem would appear if we require, in addition, that the graphs are non-bipartite.

Cvetković D., Simić S.K., Towards a spectral theory of graphs based on signless Laplacian, II, *Linear Algebra Appl.*, 432(2010), 2257-2272.

Integral graphs have recently found some applications in quantum computing, multiprocessor systems and chemistry.

Let G be a graph with the largest A-eigenvalue λ_1 and the diameter D. The quantity $(D+1)\lambda_1$ is called the *tightness* of G and is denoted by $t(G)$. There are exactly 69 non-trivial connected graphs G with $t(G) \leq 9$ and among them 14 graphs are A-integral.

D. Cvetković, T. Davidović, Multiprocessor interconnection networks with small tightness, Internat. J. Foundations Computer Sci., 20(2009), No. 5, 941-963.

Load balancing and multiprocessor interconnection networks

The job which has to be executed by a multiprocessor system is divided into parts (elementary jobs or items) that are given to particular processors. Elementary jobs distribution among processors can be represented by a vector x whose coordinates are non-negative integers associated to graph vertices and indicate how many elementary jobs are given to corresponding processors.

The load balancing problem requires creation of algorithms for moving elementary jobs among processors in order to achieve the uniform distribution, i.e., that the vector x is an integer multiple of the vector j whose all coordinates are equal to 1.

Let G be a connected graph on n vertices. Eigenvalues and corresponding orthonormal eigenvectors of the Laplacian $L = D - A$ of G are denoted by $\nu_1, \nu_2, \ldots, \nu_n = 0$ and $u_1, u_2, \ldots, u_n$, respectively. Any vector x from R^n can be represented as a linear combination of the form $x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$.

Suppose now that G has distinct Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_m = 0$ with multiplicities $k_1, k_2, \ldots, k_m = 1$, respectively. Vector x can now be represented in the form $x = y_1 + y_2 + \cdots + y_m$ where y_i belongs to the eigenspace of μ_i for $i = 1, 2, \ldots, m$. We also have $y_m = \beta j$ for some β.

Since $Lx = L(y_1+y_2+\cdots+y_m) = \mu_1 y_1+\mu_2 y_2+\cdots+\mu_m y_m$, we have $x^{(1)} = x-\frac{1}{\mu_1}Lx = (I-\frac{1}{\mu_1}L)x = (1-\frac{\mu_2}{\mu_1})y_2+\cdots+\beta j$. We see that the component of x in the eigenspace of μ_1 has been cancelled by the transformation by the matrix $I-\frac{1}{\mu_1}L$ while the component in the eigenspace of μ_m remains unchanged. The transformation $I-\frac{1}{\mu_2}L$ will cause that the component of $x^{(2)} = (I-\frac{1}{\mu_2}L)x^{(1)}$ in the eigenspace of μ_2 disappears. Continuing in this way

$$x^{(k)} = (I-\frac{1}{\mu_k}L)x^{(k-1)}, \quad k = 1,2,\ldots,m-1 \tag{1}$$

we shall obtain $x^{(m-1)} = \beta j$.

We have seen how a vector x can be transformed to a multiple of j using the iteration process (1) which involves the Laplacian matrix of the multiprocessor graph G.

Let vector $x^{(k)}$ have coordinates $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$. Relations (1) can be rewritten in the form

$$x_i^{(k)} = x_i^{(k-1)} - \frac{1}{\mu_k} \sum_{i * j} (d_i x_i^{(k-1)} - x_j^{(k-1)}) \qquad (2)$$

where d_i is the degree of vertex i. This means that the current load at vertex i is changed in such a way that vertex (processor) i sends $\frac{1}{\mu_k}$-th part of its load to each of its d_i neighbors and, because this holds for every vertex, also receives $\frac{1}{\mu_k}$-th part of the load from each of its d_i neighbors.

We have a load flow on the edge set of G. If $x_i^{(k-1)}$ is negative, then vertex i, in fact, receives the corresponding amount. For each edge ij we have two parts of the flow: the part which is sent (or received) by i and the part which is sent (or received) by j. Adding algebraically we get final value of the flow through edge ij. This flow at the end has a non-negative value which is sent either from i to j or vice versa.

The number of iterations in (1) is equal to the number m of non-zero distinct Laplacian eigenvalues of the underlying graph. The maximum vertex degree Δ of G also affects computation of the balancing flow. The complexity of the balancing flow calculations essentially depends on the product $m\Delta$ and that is why this quantity was proposed in

R. Elsässer, R. Královič, B. Monien, *Sparse topologies with small spectrum size*, Theor. Comput. Sci. 307:549–565, 2003.

as a parameter relevant for the choice and the design of multiprocessor interconnection networks.

The following definitions of four kinds of graph *tightness* have been introduced and used in Cvetković D., Davidović D., 2008, 2009.

First type mixed tightness $t_1(G)$ of a graph G is defined as the product of the number of distinct eigenvalues m and the maximum vertex degree Δ of G, i.e., $t_1(G) = m\Delta$.

Structural tightness $stt(G)$ is the product $(D+1)\Delta$ where D is diameter and Δ is the maximum vertex degree of a graph G.

Spectral tightness $spt(G)$ is the product of the number of distinct eigenvalues m and the largest eigenvalue λ_1 of a graph G.

Second type mixed tightness $t_2(G)$ is defined as a function of the diameter D of G and the largest eigenvalue λ_1, i.e., $t_2(G) = (D+1)\lambda_1$.

Several arguments were given which support the claim that graphs with small tightness t_2 are well suited for multiprocessor interconnection networks.

It was proved that the number of connected graphs with a bounded tightness is finite and graphs with tightness values not exceeding 9 are determined explicitly. There are 69 such graphs and they contain up to 10 vertices. In addition, graphs with minimal tightness values when the number of vertices is $n = 2, \ldots, 10$ are identified.

In integral graphs on n vertices there exist sets of n independent integral eigenvectors. Such sets can be constructed using star partitions of graphs and can be useful in treating the load balancing problems in multiprocessor systems and some problems in combinatorial optimization.

Integral graphs in load balancing

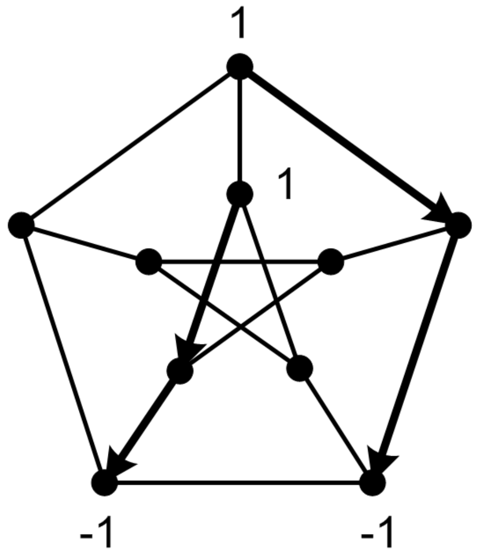
As defined, a graph is called *integral* if its spectrum consists entirely of integers. Each eigenvalue has integral eigenvectors and each eigenspace has a basis consisting of such eigenvectors.

In integral graphs load balancing algorithms, which use eigenvalues and eigenvectors, can be executed in integer arithmetics as noted in the paper

Cvetković D., Davidović T., *Multiprocessor interconnection networks with small tightness*, Internat. J. Foundations Computer Sci., 20(2009), No. 5, 941-963.

The further study of integral graphs in connection to multiprocessor topologies seems to be a promising subject for future research.

Recall that $3, 1^5, (-2)^4$ is the spectrum of the Petersen graph.



An eigenvector for eigenvalue 1 and a load balancing flow

A basis of integral eigenvectors of an integral graph can be found using the theory of star partitions of a graph (see the book

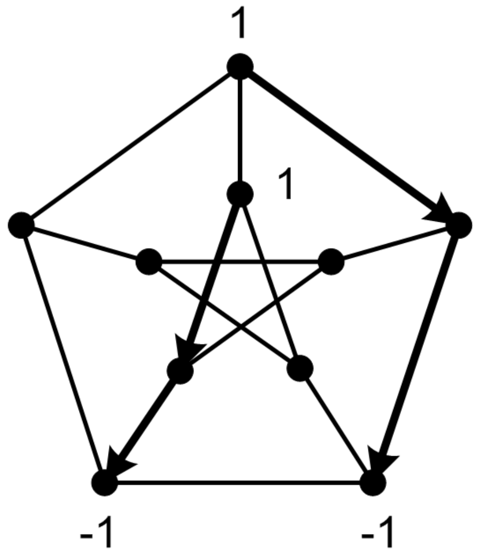
Cvetković D., Rowlinson P., Simić S., *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2009, Chapter 5).

Let G be a graph an n vertices with distinct eigenvalues $\mu_1, \ldots, \mu_m$. A partition $X_1 \dot{\cup} \cdots \dot{\cup} X_m$ of the vertex set $V(G)$ of G is a *star partition* of G if for each $i \in \{1, \ldots, m\}$, μ_i is not an eigenvalue of $G - X_i$.

For any $x \in X_i$ the subgraph of G induced by $V(G) - X_i + x$ has a simple eigenvalue μ_i. If G is an integral graph, then, of course, μ_i is an integer and the corresponding eigenvector can be chosen to be integral. Extending it with zeros for coordinates corresponding to vertices from $X_i - x$, we obtain an n-dimensional integral vector which is an eigenvector of G for μ_i. In this way n independent integral eigenvectors can be found.

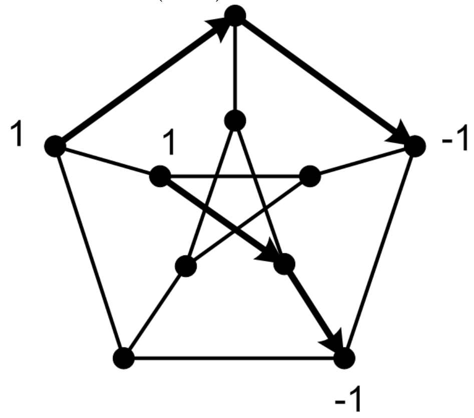
They are not necessarily mutually orthogonal but the Gram-Schmidt orthogonalization procedure can be applied afterward.

spectrum: $\quad 3, 1^5, (-2)^4$



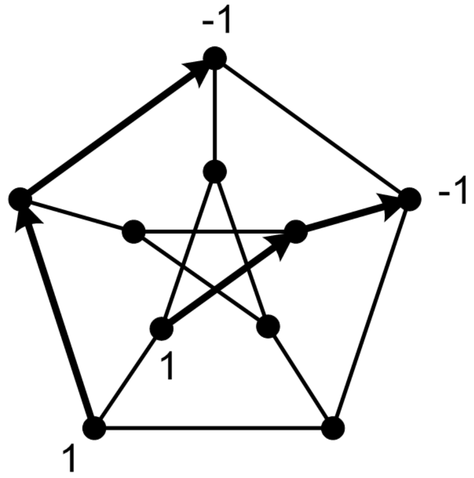
An eigenvector for eigenvalue 1 and a load balancing flow

spectrum: $3, 1^5, (-2)^4$



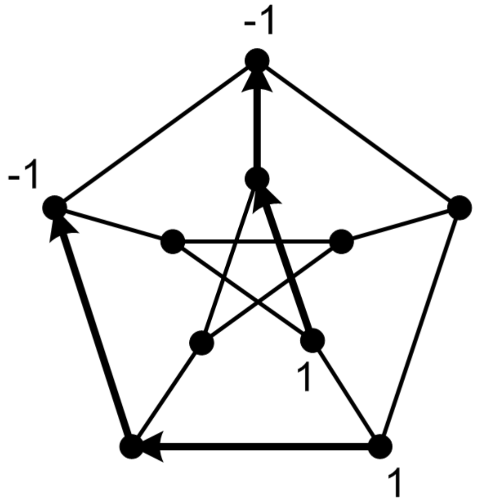
An eigenvector for eigenvalue 1 and a load balancing flow

spectrum: $3, 1^5, (-2)^4$



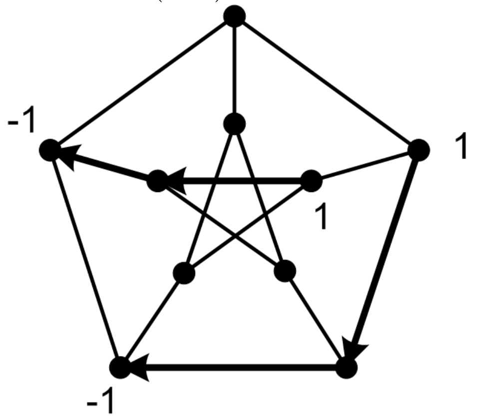
An eigenvector for eigenvalue 1 and a load balancing flow

spectrum: $3, 1^5, (-2)^4$



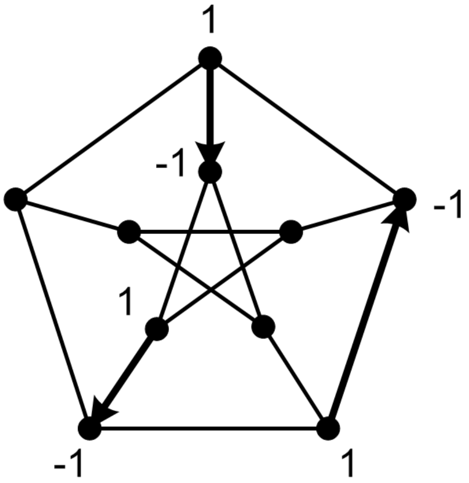
An eigenvector for eigenvalue 1 and a load balancing flow

spectrum: $3, 1^5, (-2)^4$

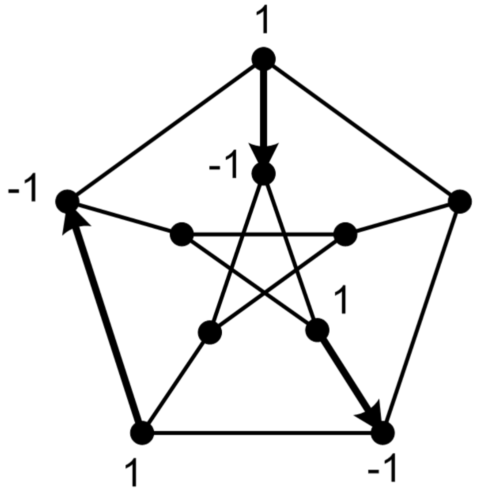


An eigenvector for eigenvalue 1 and a load balancing flow

spectrum: $3, 1^5, (-2)^4$

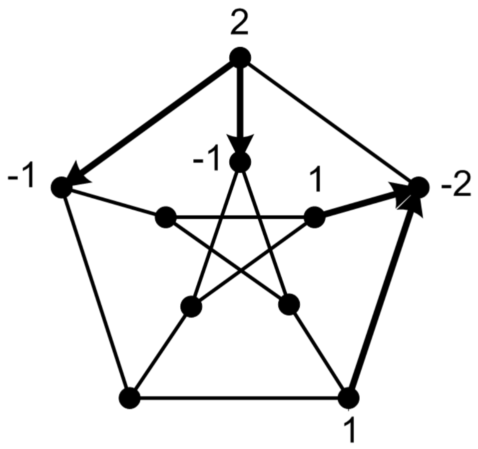


An eigenvector for eigenvalue -2 and a load balancing flow (hexagon as a subgraph) spectrum: $\quad 3, 1^5, (-2)^4$



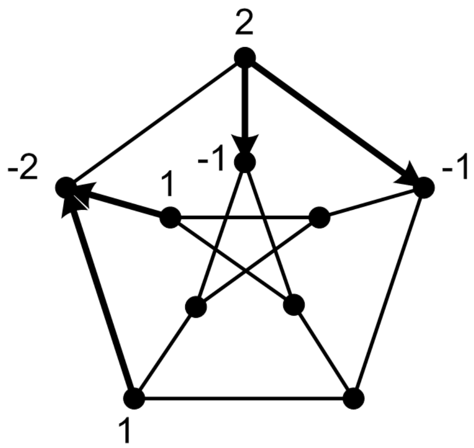
An eigenvector for eigenvalue -2 and a load balancing flow (hexagon as a subgraph)

spectrum: $3, 1^5, (-2)^4$



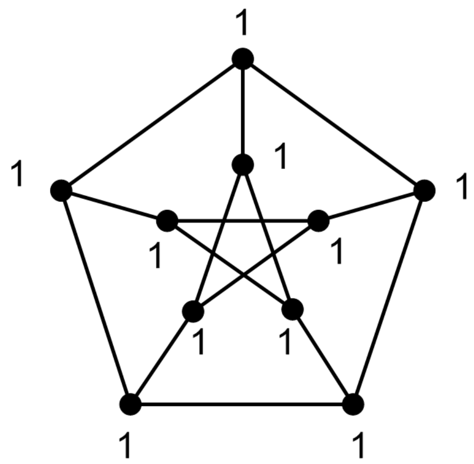
An eigenvector for eigenvalue -2 and a load balancing flow (Smith graph W_6 as a subgraph)

spectrum: $3, 1^5, (-2)^4$



An eigenvector for eigenvalue -2 and a load balancing flow (Smith graph W_6 as a subgraph)

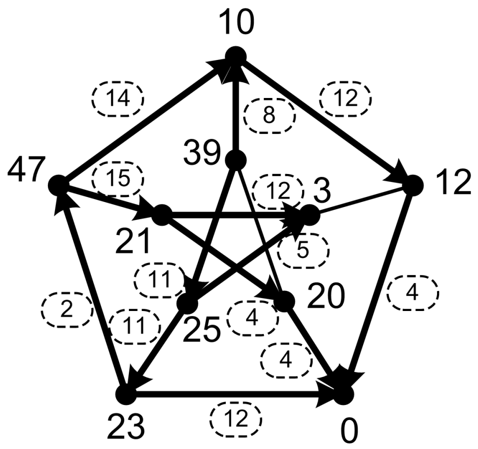
spectrum: $3, 1^5, (-2)^4$



An eigenvector for eigenvalue 3

(all-1 vector, i.e. a balanced load distribution)

spectrum: $3, 1^5, (-2)^4$



A load distribution and a load balancing flow

The last copy of the Petersen graph represents a load distribution among processors given by weights on vertices. This load vector can be represented as linear combination

$$20v_0 + 11v_1 + 4v_2 + 5v_3 - 12v_5 - 5v_8 - 3v_9$$

and the resulting balancing flow is given as flows along oriented edges. This global balancing flow is obtained by the same linear combination of balancing flows of eigenvectors $v_1, \ldots, v_9$. When the flow is realized each vertex has a load equal to 20.

Integral graphs with small tightness

In this section we survey known integral graphs with small tightness indicating the open enumeration problems.

$t_2(G) = (D+1)\lambda_1$

We have $t_2(K_n) = 2n - 2$.

$t_2(G) = (D+1)\lambda_1$

Proposition 1. *If $p, p > 2$ is a prime, there are no graphs G such that $t_2(G) = p$.*

Proof. As known, a rational eigenvalue of a graph is an integer. Therefore, if $t_2(G) = p$, then either $\lambda_1 = p$ and $D = 0$ or $\lambda_1 = 1$ and $D = p - 1$. In both cases G does not exist. ∎

An important role play the graphs with $\lambda_1 = 2$, known as Smith graphs. There are 6 types of Smith graphs. Four of them are concrete graphs, while the remaining two types (cycles C_n and *double-head snakes* W_n on n vertices) can have an arbitrary number of vertices. On Fig. 1 we reproduce some of them. The remaining Smith graphs are cycles C_n and the star $S_5 = K_{1,4}$.

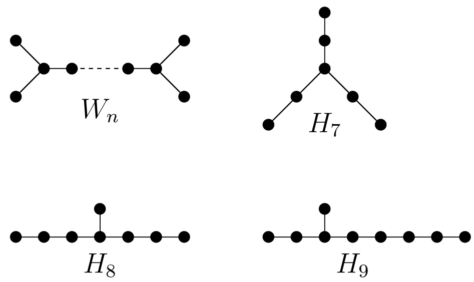


Figure 1: Some Smith graphs

In our study we need also graphs with $\lambda_1 < 2$. By removing vertices out of Smith graphs, we obtain paths P_n, $n = 2, 3, \ldots$; *single-head snakes* Z_n, $n = 4, 5, \ldots$, given in the upper row of Fig. 2 up to $n = 7$; and the three other graphs given in the second row of Fig. 2 and denoted by E_6, E_7 and E_8.

Eigenvalues of Smith graphs and of their connected subgraphs are explicitly calculated in Cvetković, Gutman 1975.

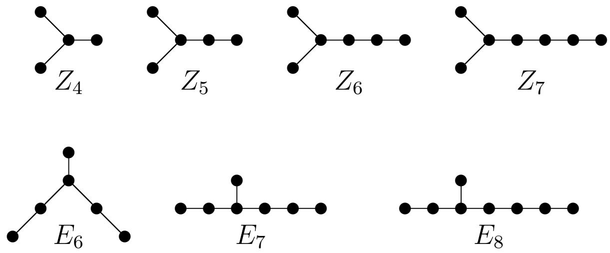


Figure 2: Subgraphs of some Smith graphs

Proposition 2. *Integral Smith graphs are the following graphs:* $K_2,\ K_3,\ C_4,\ K_{1,4},\ C_6, W_6\ ,\ H_7$.

Proof. By inspecting spectra of Smith graphs we easily get the graphs quoted. ∎

$t_2(G) = (D+1)\lambda_1$

Proposition 3. *The only graph G with $t_2(G) = 2k$, k being a prime greater than 5, is the graph K_{k+1}.*

Proof. If we put $D+1 = 2$ and $\lambda_1 = k$, then clearly $G = K_{k+1}$. In the case $D+1 = k$ and $\lambda_1 = 2$ we have to find integral Smith graphs. By Proposition 2 they do not exist. ■

We present a classification of A-integral graphs G with $t(G) < 24$.

By Proposition 1 there are no graphs with tightness values 3, 5, 7, 11, 13, 17, 19, 23.

$t_2(G) = 2$. We have $G = K_2$.

$t_2(G) = 4$. We have $G = K_3$.

$t_2(G) = 6$. We have $G = K_4, C_4, K_{1,4}$ by Proposition 2.

$t_2(G) = 8$. We have $G = K_5, C_6, W_6$ by Proposition 2.

$t_2(G) = 9$. All connected graphs G with $t_2(G) \leq 9$ have been determined in

D. Cvetković, T. Davidović, Multiprocessor interconnection networks with small tightness, Internat. J. Foundations Computer Sci., 20(2009), No. 5, 941-963.

There are 69 such graphs and among them exactly 14 are integral. Those with $t_2(G) = 9$ are the following graphs: regular graphs $K_{3,3}$, the three side prism and the Petersen graph and three non-regular graphs $K_{1,9}$ and two others.

$t_2(G) = 10$. We have $G = K_6, H_7$ by Propositions 2 and 3.

$t_2(G) = 12$. Here we have K_7 for $\lambda_1 = 6$ and by Proposition 2 there are no integral graphs for $\lambda_1 = 2$. The following two cases remain.

$\lambda_1 = 3$ and $D = 3$. All integral cubic graphs are well known

Bussemaker F. C., Cvetković D., *There are exactly 13 connected, cubic, integral graphs*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz., No. 544-No. 576(1976), 43-48.

Those with $D = 3$ are graphs denoted there by $G_4, G_7, G_8, G_{11}$and they have 8, 10, 12, 10 vertices respectively. G_4 is the cube graph.

$\lambda_1 = 4$ and $D = 2$. Integral regular graphs of degree 4 up to 24 vertices are listed in

Stevanović D., Abreu, N.M.M. de, Freitas, M.A.A. de, Del-Veccio R., *Walks and regular integral graphs*, Linear Algebra Appl., 423(2007), 119-135.

Those which fulfill the requirements are

$$B_1 = K_{4,4}, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_{11}$$

(graph names as in the cited paper).

$t_2(G) = 14$. We have $G = K_8$ by Proposition 3.

One can go on up to $t_2(G) = 23$.

Most of the presented results are included into the paper

Cvetković D., Davidović T., Ilić A., Simić S.K., *Graphs for small multiprocessor interconnection networks*, to appear.

Thank you for your attention