Algebraic Connectivity and Vertex-Deleted Subgraphs

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Basics

Given a graph G on vertices labeled $1, \ldots, n$, the corresponding *Laplacian matrix* is the $n \times n$ matrix L such that for each $i, j = 1, \ldots, n$, we have: $L_{ij} = -1$ if $i \sim j$, $L_{ij} = 0$ if $i \neq j$ and vertices i and j are not adjacent, and $L_{ii} =$ degree of vertex i.

The corresponding Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

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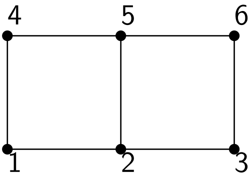

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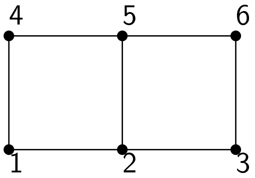

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Properties of the Laplacian matrix

For any graph G on n vertices, its Laplacian matrix L is
symmetric;
positive semi-definite (i.e. $x^T Lx \geq 0$ for any real vector x); and
singular, since the all ones vector, $\mathbf{1}$, is a null vector.

Label the eigenvalues of L as $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

It turns out that the nullity of L (i.e. the multiplicity of 0 as an eigenvalue of L) coincides with the number of connected components of G. In particular, $\lambda_2 > 0$ if and only if G is connected.

It seems that the algebraic properties of the Laplacian matrix for a graph carry information about the graph's combinatorial structure.

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Definition and basics on algebraic connectivity

The *algebraic connectivity of* G, $\alpha(G)$, is the second smallest eigenvalue of the corresponding Laplacian matrix. (Fiedler, 1973)

Sample facts: Let G be a graph on n vertices with Laplacian matrix L.

Then $\alpha(G) = min\{x^T Lx | x^T x = 1, x^T \mathbf{1} = 0\} = min\{\sum_{i<j, i\sim j}(x_i - x_j)^2 | \sum_{1\le i\le n} {x_i}^2 = 1, \sum_{1\le i\le n} x_i = 0\}$. (Fiedler, 1973)

Suppose that $\hat{G}$ is formed from G by adding an edge not already present in G. Then $\alpha(G) \le \alpha(\hat{G})$. (Fiedler, 1973)

We have $\alpha(P_n) = 2(1 - cos(\frac{\pi}{n})) \le \alpha(G) \le n = \alpha(K_n)$

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Fiedler vectors

For a connected graph G there is also interest in the eigenvectors of L associated with $\alpha(G)$; these are known as *Fiedler vectors* for G.

The interest stems from the following phenomenon (Fiedler, 1975): Label the vertices of G with the integers $1, \ldots, n$, and let y be a corresponding Fiedler vector. Then y finds a 'middle of the graph' in the sense that there is a collection of vertices S whose corresponding $|y_i|$'s are small, and such that as we move along paths away from S, the corresponding entries in y exhibit a monotonic behaviour (positive/increasing, negative/decreasing, identically zero).

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Yet another result of Fiedler's

Given a graph G and a vertex v of G, we let $G \setminus v$ denote the graph formed from G by deleting v and all edges incident with v.

Theorem

(Fiedler, 1973) Let G be a graph, and suppose that v is a vertex of G. Then $\alpha(G) \leq \alpha(G \setminus v) + 1$.

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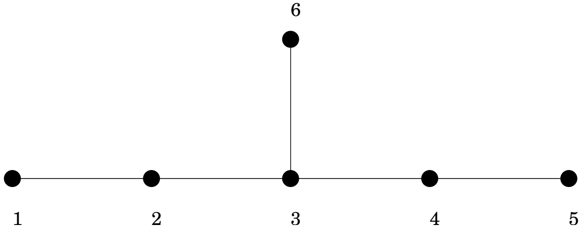

For the graph G above, we have $\alpha(G) > \alpha(G \setminus v)$ for $v = 2, 3, 4$, $\alpha(G) = \alpha(G \setminus 6)$, and $\alpha(G) < \alpha(G \setminus v)$ for $v = 1, 5$.

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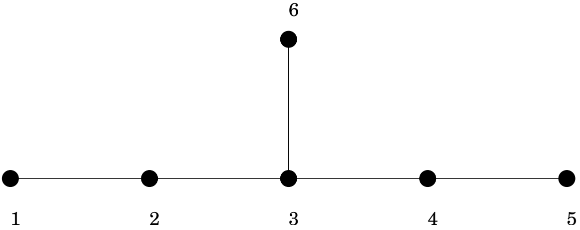

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Two functions

Given a connected graph G, we define the following two functions on its set of vertices:
$\phi(v) = \alpha(G) - \alpha(G \setminus v)$, and $\kappa(v) = \frac{\alpha(G \setminus v)}{\alpha(G)}$.

Evidently $\phi(v) < 0, \phi(v) = 0, \phi(v) > 0$ according as $\kappa(v) > 1, \kappa(v) = 1, \kappa(v) < 1$.

Interpretation: If $\kappa(v) < 1$, then $\alpha(G \setminus v) < \alpha(G)$, so the presence of v and its incident edges serves to *increase* the algebraic connectivity, while if $\kappa(v) > 1$, then v and its incident edges *decrease* the algebraic connectivity. The functions ϕ and κ measure that increase/decrease in absolute and relative terms, respectively.

The plan: Investigate the functions ϕ and κ.

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Technique

The approach to generating many of the results relies on the following idea:

Suppose that G has n vertices, and let w be a Fielder vector for G; form $\tilde{w}$ from w by deleting the entry w_v of w that corresponds to vertex v;
define the vector u of order $n-1$ via $u = \tilde{w} + \frac{w_v}{n-1}\mathbf{1}$;
consider the quadratic form $u^T\tilde{L}u$, where $\tilde{L}$ is the Laplacian matrix for $G \setminus v$;
that quadratic form generates an upper bound on $\alpha(G \setminus v)$ in terms of $\alpha(G)$ and the entries in w.

A consequence: If v is a vertex of G corresponding to an entry in w of smallest absolute value, then $\phi(v) \geq 0$ (equivalently, $\kappa(v) \leq 1$). This reinforces the notion that entries in a Fiedler vector of small absolute value are in the 'middle' of the graph.

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that quadratic form generates an upper bound on $\alpha(G \setminus v)$ in terms of $\alpha(G)$ and the entries in w.

A consequence: If v is a vertex of G corresponding to an entry in w of smallest absolute value, then $\phi(v) \geq 0$ (equivalently, $\kappa(v) \leq 1$). This reinforces the notion that entries in a Fiedler vector of small absolute value are in the 'middle' of the graph.

Technique

The approach to generating many of the results relies on the following idea:

Suppose that G has n vertices, and let w be a Fielder vector for G;

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Bounds on ϕ

We have already seen that for any graph G and any vertex v of G, $\phi(v) \le 1$. Here is a companion result.

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Let G be a connected graph on $n \ge 3$ vertices. Then for any vertex v of $G, \phi(v) \ge -(n-2)$. Equality holds in the lower bound if and only if G is constructed from K_{n-1} by adding in the vertex v and a single edge joining v to one vertex of K_{n-1}.

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Bounds on κ

If G is a graph and v is one of its vertices, we have $\kappa(v) \geq 0$, with $\kappa(v) = 0$ if and only if $G \setminus v$ is not connected (such a vertex is called a *cutpoint* of G).

Theorem

Let G be a connected graph G on $n \geq 3$ vertices, and suppose that v is a vertex of G that is not a cutpoint. Then $\kappa(v) \geq \frac{2-2\cos(\frac{\pi}{n-1})}{3-2\cos(\frac{\pi}{n-1})}$.
Equality holds in the lower bound if and only if either
a) G is constructed from P_{n-1} by adding the vertex v, along with edges from v to all vertices of P_{n-1}, or
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κ bounds, cont'd

Theorem

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Theorem

Let G be a connected graph on $n \geq 3$ vertices. Then $\min\{\kappa(v) | v \in G\} \leq \frac{n-1}{n}$. Equality holds in the inequality if and only if $G = K_n$.

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We saw before that if $\phi(v) > 0$, then we may take the interpretation that v and its incident edges increase the algebraic connectivity. How many such vertices are there?

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Let G be a connected graph on $n \geq 4$ vertices. Then there are least $\lfloor \frac{\alpha(G)(n-2)}{n-1} \rfloor + 1$ vertices u for which $\phi(u) \geq 0$.

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An illustration using food webs

A *food web* is a graph that is used to represent certain relationships in an ecosystem. The vertices of the graph correspond to the various species in an ecosystem; for each pair of vertices in a predator/prey relationship, the graph contains an edge between the predator vertex and the prey vertex.

There is an obvious lack of symmetry in the relationship between predator and prey. However, since effects due to changes to predator and prey species can propagate throughout a food web, it has been argued in the ecology literature that there is merit in considering a food web as an undirected graph.

One might try to use the κ function to determine which vertices (species) are important in terms of the connectivity properties of the graph (food web).

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Northeast US shelf ecosystem

Example: Based on data for the Northeast US shelf ecosystem, we have a graph G on 81 vertices, primarily representing fishes, other vertebrates, invertebrates, and basal groups.

We have $\alpha(G) = 7.5421$.

Note that for any v, $\kappa(v) \geq 1 - \frac{1}{\alpha(G)} = 0.8674$.

The eight smallest values of κ are: 0.8755 (cancer crabs); 0.8755 (other crabs); 0.8763 (clams, mussels); 0.8791 (scallops); 0.8793 (phytoplankton); 0.8812 (lobsters); 0.8851 (detritus); and 0.8852 (urchins).

The maximum value of κ is 1.2516 (snails).

All remaining vertices yield values of κ lie between 0.9928 and 1.0023.

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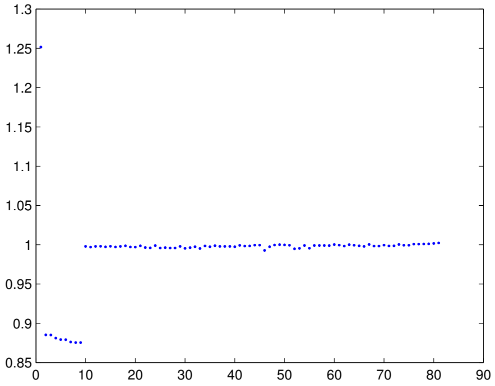
κ values sorted according to the Fiedler vector for the shelf ecosystem



For further details . . .

S. Kirkland, Algebraic Connectivity for Vertex-Deleted Subgraphs, and a Notion of Vertex Centrality, *Discrete Mathematics* 310 (2010), 911-921.