Forbidden subgraphs for some classes of treelike reflexive graphs

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Introduction

- **Graph** = simple graph
  (finite, nonoriented, without loops and/or multiple edges)

- **Spectrum** = adjacency spectrum
  (spectrum of (0,1) adjacency matrix)

- Graphs are **connected**
  (the spectrum of a disconnected graph is the union of the spectra of its components)

- A graph G is **reflexive** if its second largest eigenvalue does not exceed 2
- Being reflexive is a **hereditary property**
- Presentation of all reflexive graphs inside given set:
  via maximal graphs/ **via minimal forbidden graphs**
- A graph is **treelike** or a **cactus** if any pair of its cycles has at most one common vertex
- A cycle (as a subgraph of a connected graph) is **free** if it has only one vertex whose degree exceeds 2.

Theorem Let A be a symmetric matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ and B one of its principal submatrices with eigenvalues $\mu_1,\dots,\mu_m$. Then the inequalities $\lambda_{n\text{-}m+i} \le \mu_i \le \lambda_i \ (i=1,\dots,m)$ hold.

Lemma Given a graph G, let $C(v)$ ($C(uv)$) denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then

$$P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u' \in Adj(v)} P_{G-v-u'}(\lambda) - 2\sum_{C\in\varphi(v)} P_{G-V(C)}(\lambda),$$

$$P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-u-v}(\lambda) - 2\sum_{C\in\varphi(uv)} P_{G-V(C)}(\lambda),$$

where $Adj(v)$ denotes the set of neighbours of v, while $G-V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

Corollary Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' (G_2') be the subgraph of G_1 (G_2) obtained by deleting the vertex v_1(v_2) from G_1 (resp. G_2). Then

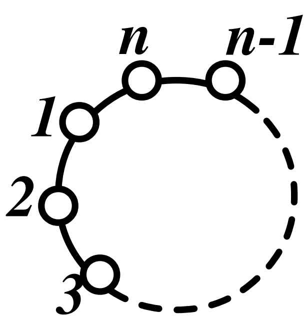
$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary Let G be a graph with a pendant edge v_1v_2, v_1 being of degree 1. Then

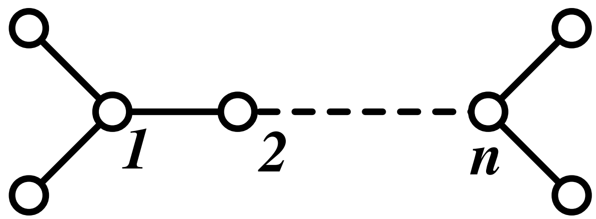
$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where G_1 (G_2) is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

Smith graphs - all connected graphs whose index is equal to 2



Cn



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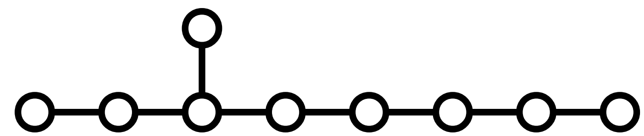
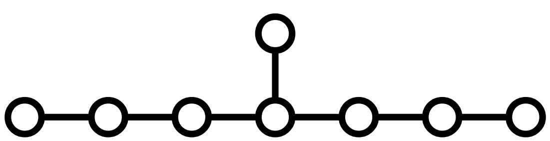
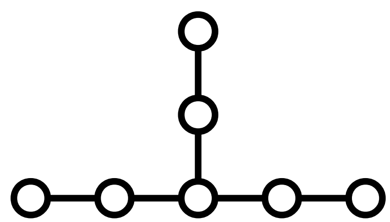
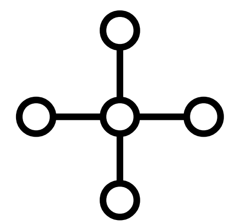
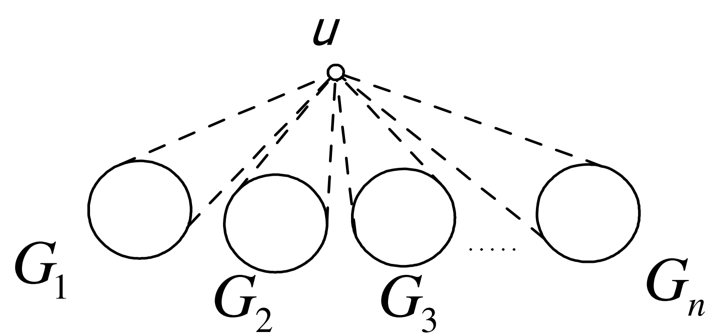










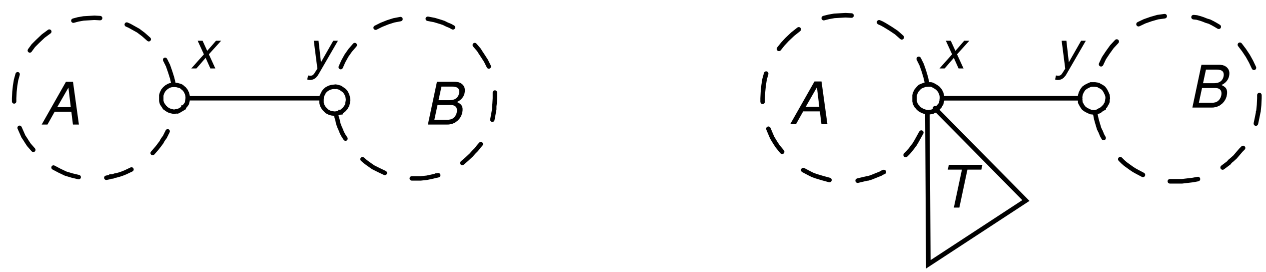
- ***Theorem 2 (RS)*** Let *G* be a graph with a cut-vertex *u*.

1) If at least two components of *G-u* are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.
2) If at least two components of *G-u* are Smith graphs and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$.
3) If at most one component of *G-u* is a Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.

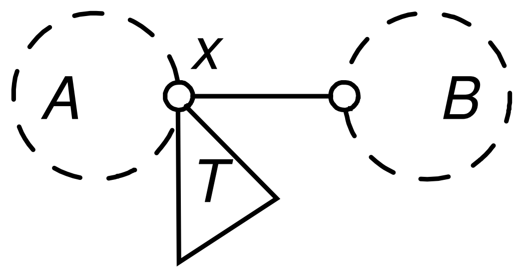
Two conditions:

- 1. RS theorem can not be applied
- 2. cycles of the graph do not make a bundle

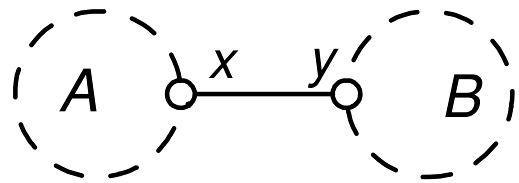
Bicyclic graphs with the bridge



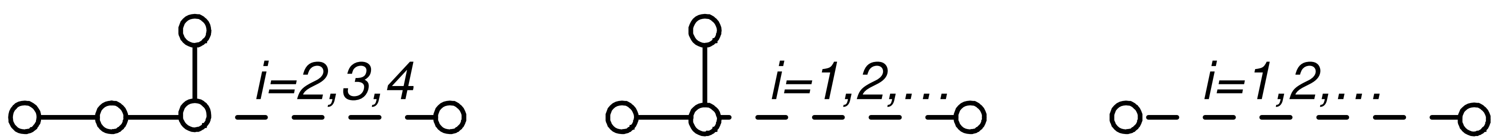
- Let graph *A* be a cycle of an arbitrary length, additionally loaded with some edges and let *B* be also a cycle of an arbitrary length. Let *T* be a tree.



- If graph A-x (or T-x) is supergraph of one of Smith graphs, then $\lambda_2(G) > 2$.
- If graph A-x is one of Smith graphs and T-x is subgraph of one of Smith graphs (or vice versa) then $\lambda_2(G) = 2$.
- If graphs A-x and T-x are subgraphs of some of Smith graphs, then $\lambda_2(G) < 2$.

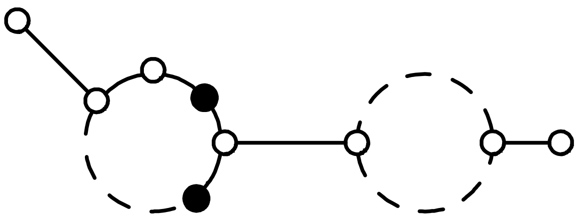


- Let A-x and B-y be subgraphs of some of Smith graphs. Then each of them has to be one of the following graphs:

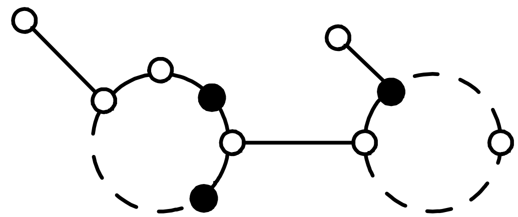


Let the vertices of the two cycles which are adjacent with the vertices x and y be the black vertices, and let all other vertices, different from x and y, be the white vertices. If there is a white vertex of degree at least 3, its distance from the nearest black vertex is 1 or 2.

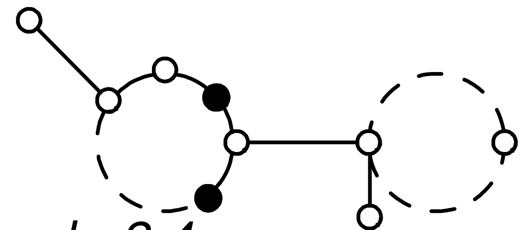
Cases where distance is 2:



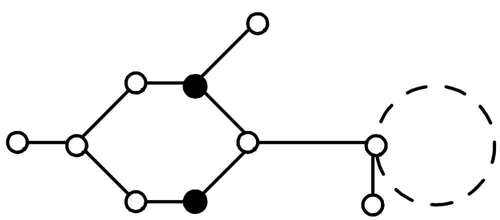
k=2,3,4

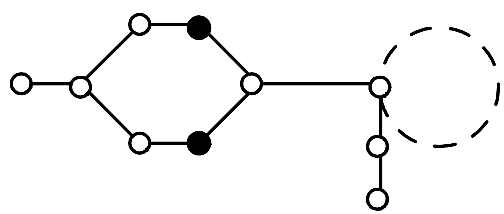


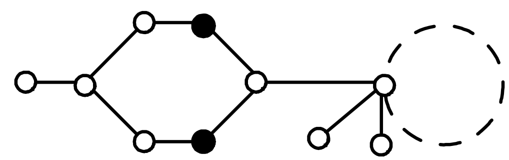
k=2,3,4



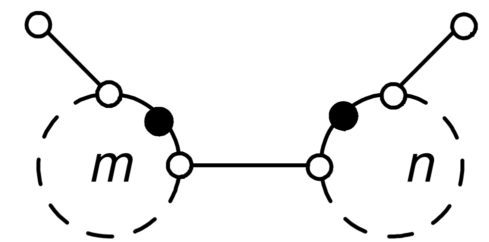
k=3,4



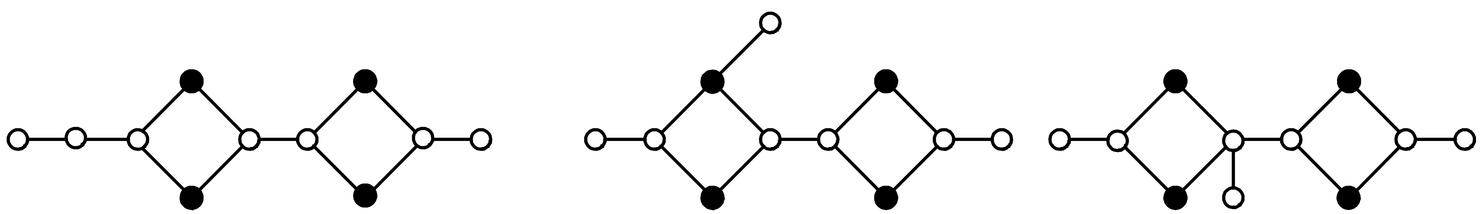




Cases where distance is 1 (2 white vertices loaded):

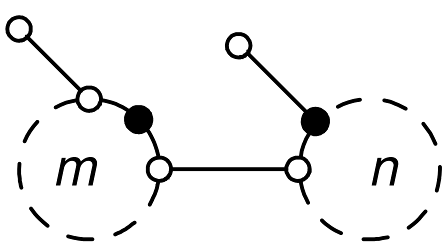


m,n > 4

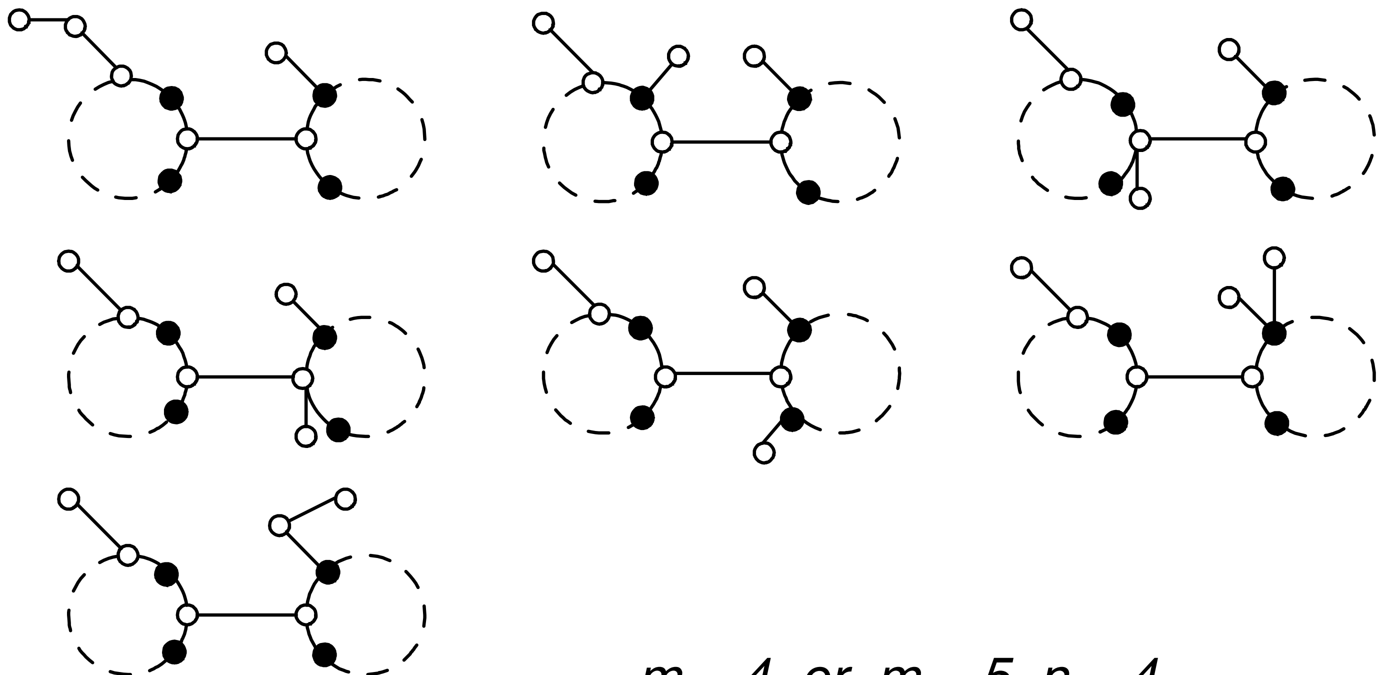


m = n = 4

Cases where distance is 1 (1 white and 1 black vertex loaded):

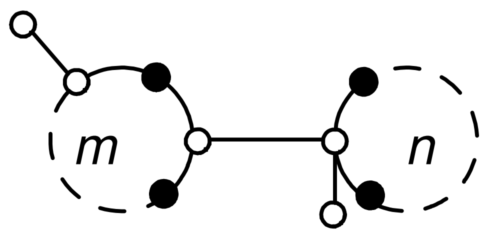


m > 4 or m = 5, n > 4



m = 4 or *m* = 5, *n* = 4

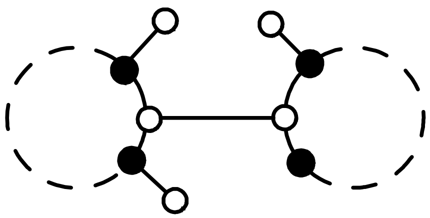
Cases where distance is 1 (*B*-*y* is a path):



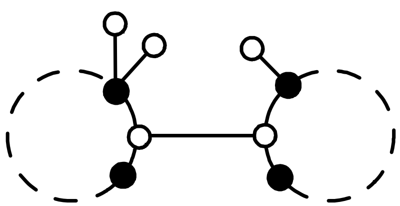
$m > 8$

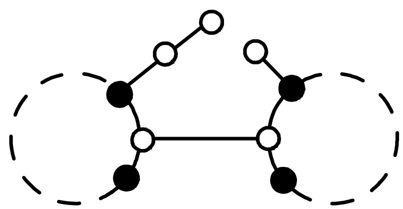
- for $m = 8$, there are 4 minimal forbidden graphs
- for $m = 7$, there are 6 minimal forbidden graphs
- for $m = 6$, there are 9 minimal forbidden graphs
- for $m = 5$, there are 13 minimal forbidden graphs
- for $m = 4$, there are 21 minimal forbidden graphs

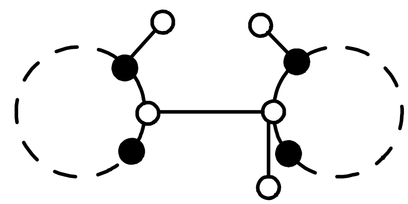
Cases where only black vertices are loaded:



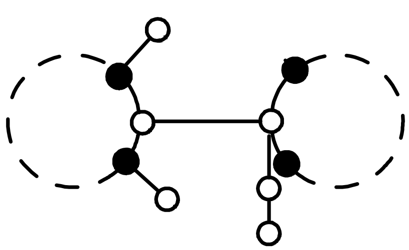
3 black vertices loaded

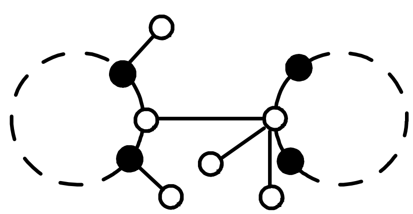


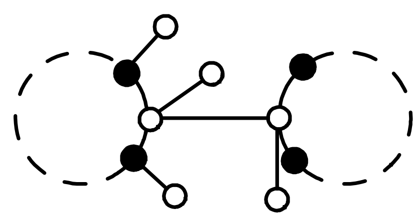




2 black vertices from different cycles loaded







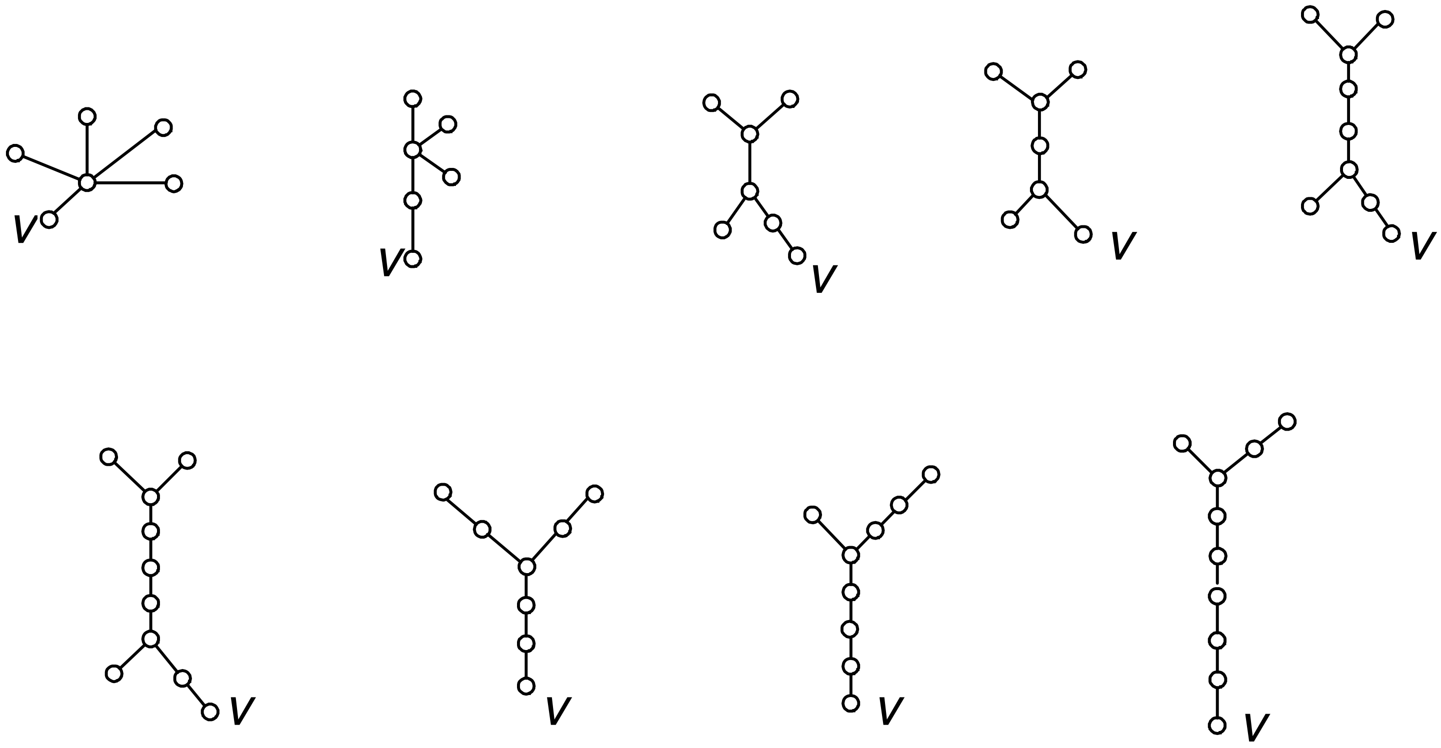
2 black vertices from the same cycle loaded

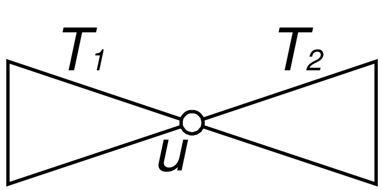
One black vertex is loaded:

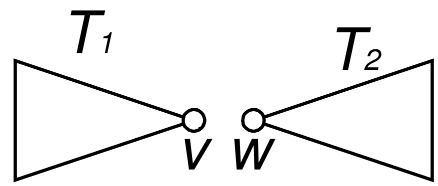
there are some special families of graphs that may be described together using pouring etc.

Cases where only vertices *x* and *y* are loaded:

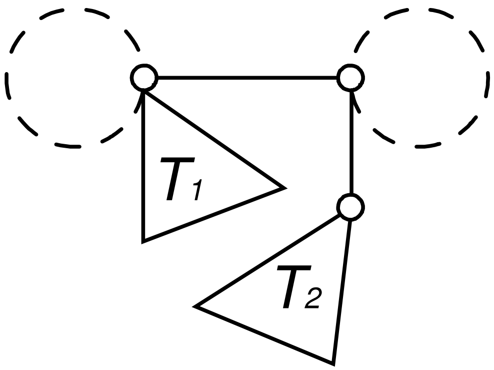
All minimal forbidden trees for the property $\lambda_1 = 2$



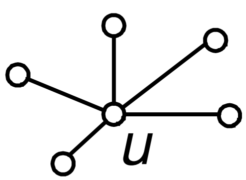


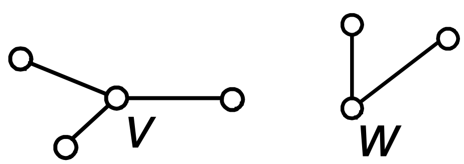


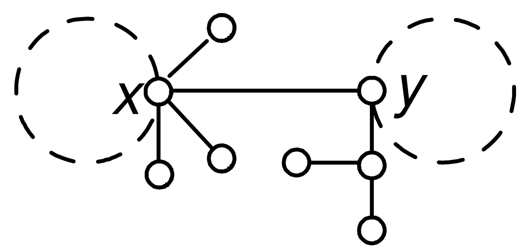
splitting

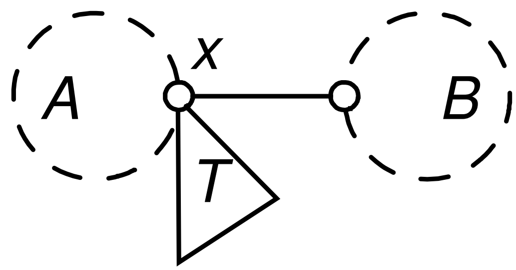


pouring

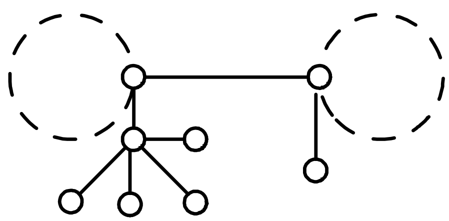


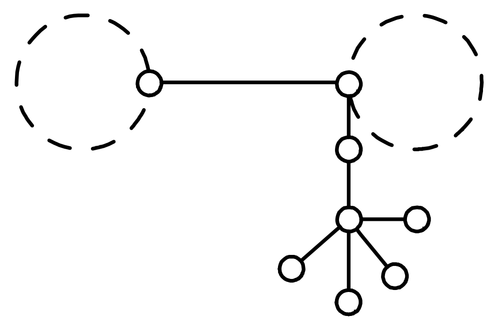






- If graph *A*-*x* (or *T*-*x*) is supergraph of one of Smith graphs, then $\lambda_2(G) > 2$.





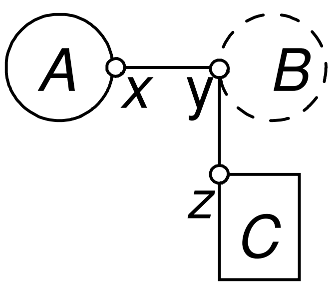
One condition:

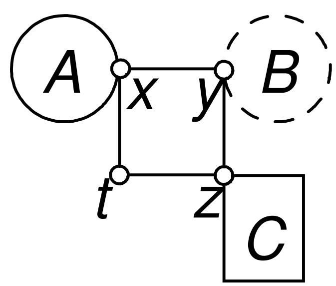
- 1. RS theorem can not be applied
- **2. cycles of the graph do not make a bundle**

New classes

- Let A and B be cycles of an arbitrary length. Let B be a free cycle, while vertices of A and vertex z may be additionally loaded.

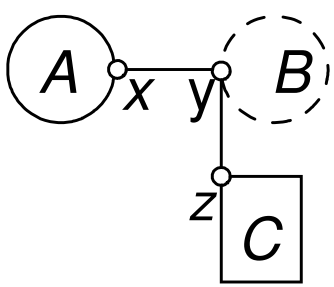
  Let K_1 and K_2 be the correspondent classes of bicyclic and tricyclic graphs. A graph G_1 is minimal forbidden in the class K_1 if and only if the correspondent graph G_2 is minimal forbidden in the class K_2.

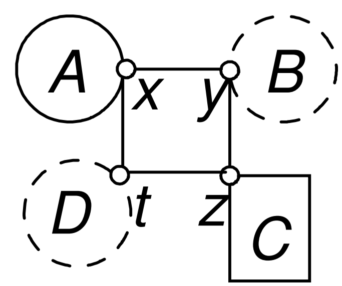




- Let *A* and *B* be cycles of an arbitrary length. Let *B* and *D* be free cycles, while vertices of *A* and vertex *z* may be additionally loaded.

  Let K_1 and K_3 be the correspondent classes of bicyclic and graphs four cycles. If a graph G_1 is minimal forbidden in the class K_1 then the correspondent graph G_3 is minimal forbidden in the class K_3.





- Let A and B be cycles of an arbitrary length. Let B be a free cycle, while vertices of A and vertex z may be additionally loaded and let S be a Smith tree.

  Let K_1 and K_4 be the correspondent classes of bicyclic and unicyclic graphs. If a graph G_1 is minimal forbidden in the class K_1 then the correspondent graph G_4 is minimal forbidden in the class K_4.