Graphs of given order and size and minimal algebraic connectivity

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joint work with

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Algebraic Connectivity

- $G = (V, E)$ undirected simple graph, V vertices, E edges
- Laplacian matrix of G $L(G) = D(G) - A(G)$
- first eigenvalue λ_1 of $L(G)$ is always zero
- second smallest eigenvalue λ_2 algebraic connectivity
- eigenvector of λ_2 Fiedler vector
- (Fiedler 1974) $\lambda_2 \leq v(G) \leq e(G)$
  connectivity and edge connectivity $v(G), e(G)$, respectively

Minimum Algebraic Connectivity

Problem (Surprisingly very recently)

Let $H(n, m)$ be the set of all connected graphs with n vertices and m edges.
Which graph $G \in H(n, m)$ has the minimum algebraic connectivity?

Short History of Minimum Algebraic Connectivity

- Godsil and Royle conjecture (Algebraic Graph Theory Book 2001)
  Graphs with small algebraic connectivity have a large diameter with bridges.

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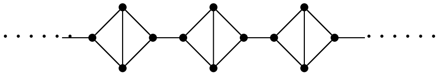
- AGX-system path-complete conjecture
  (Belheaiza, de Abreu, Hansen and Olivieira 2005)
  Path-completes graph have the minimum algebraic connectivity

Short History of Minimum Algebraic Connectivity

- (Brand, Guiduli and Imrich 2007)
  Cubic (3-regular) graphs with minimum algebraic connectivity look like a "string of a perl"



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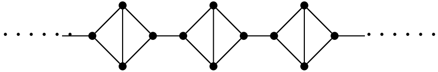


- (Fallat and Kirkland 1998)
  A path with a star in both ends (almost equal size) has the minimum algebraic connectivity from all trees with a fixed diameter.

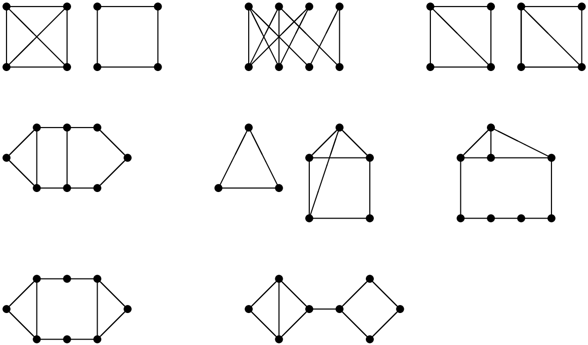
Degree Sequence

degree of a vertex v is the number of its neighbors

- Let $(d_1, \cdots, d_n)$ be a sequence of nonnegative integers
- $(d_1, \cdots, d_n)$ is called degree sequence, if there exists a graph with n vertices and the degrees of the vertices $d_1, \cdots, d_n$.

- T_d is the set of all trees with degree sequence $d = (d_1, \ldots, d_n)$
- (Folklore) $d = (d_1, \ldots, d_n)$ is a tree sequence iff $\sum_i d_i = 2(n-1)$ and $1 \leq d_i \leq n-1$.

Example: $(3, 3, 3, 3, 2, 2, 2, 2)$



Tree Sequences and Algebraic Connectivity

T_d is the set of all trees with degree sequence $d = (d_1, \dots, d_n)$

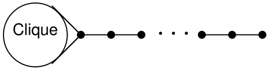
Theorem (T.B. and Leydold 2009)

Caterpillar tree has the minimum algebraic connectivity in T_d. The degrees and eigenvector of this tree in T_d has the spiral form.

The structure of the Caterpillar is an open problem
The problem is suddenly very complicated

AGX-system Conjecture

path-complete graph (Šoltés 1991)



AGX-system Path-complete Conjecture:
(Belheaiza, de Abreu, Hansen and Oliveira 2005)
Path-complete graph has minimum algebraic connectivity among all connected graphs with n vertices and m edges.

Theorem (T.B. and Leydold News From the Kitchen)

The path-clique graph has minimum algebraic connectivity graph with n vertices and m edges.

path-clique (see blackboard)

Still working on that...

Proof Methods and Techniques

- Eigenvector structure of the extremal graph
- Geometric Nodal Domains
- Graph Operations compatible with Rayleih Quotient
  - switching
  - shifting
- Properties of Perron vector and Fiedler vector

see Josef's talk

Graph with a Boundary and Discrete Dirichlet Matrix

Graph with a boundary (Joel Friedman 1993)

- $G = (V_o \cup \partial V, E_o \cup \partial E)$ graph with a boundary
- ∂V boundary vertices
- V_o interior vertices
- ∂E boundary edges
- E_o interior edges
- there is no edges between boundary vertices
- $L(G) = D(G) - A(G)$, G Laplacian of G
- we delete rows and columns of boundary vertices of $L(G)$
- So we get Dirichlet Matrix of G

Discrete Faber-Krahn Theorem

Faber-Krahn type results:

Problem

Which graph has the minimum first Dirichlet eigenvalue under all graphs with the same volume?

In discrete problem, we can talk about more than one volume

T_d is the set of all trees with degree sequence $d = (d_1, \ldots, d_n)$

Theorem (Discrete Faber-Krahn Theorem II T.B. and Leydold 2007)

Spiral Tree has the minimum first Dirichlet eigenvalue in T_d.

Discrete Faber-Krahn Theorem

T_d is the set of all trees with degree sequence $d = (d_1, \ldots, d_n)$

Theorem (Discrete Faber-Krahn Theorem II T.B. and Leydold 2007)

Spiral Tree has the minimum first Dirichlet eigenvalue in T_d.

Theorem (Discrete Faber-Krahn Theorem I T.B. and Leydold 2007)

Comet has the minimum first Dirichlet eigenvalue, under all connected graphs (trees) with n vertices and k boundary vertices.

Analogous results for the Laplace-Beltrami operators on manifolds with non-constant curvature are rare