Graph Eigenvalues in Combinatorial Optimization

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16th ILAS Conference - Minisymposia on Spectral Graph Theory, June 21-25, 2010, Pisa, Italy

Outline

Graph eigenvalues and the chromatic number

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  are the earliest graph spectra applications in combinatorial optimization.
- The left inequality was obtained by Hoffman in **1970** and the right by Wilf in **1967**.

A lower bound on the stability number

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Theorem [Wilf, 1967]

Let G be a graph with at least one edge. Then

$$\alpha(G) \geq \frac{s^2}{s^2 - \lambda_1(\overline{G})},$$

where s is the sum of the entries of the normalized eigenvector corresponding to $\lambda_1(\overline{G})$ (the index of the complement of G).

Another lower bound on the chromatic number

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$$\chi(G) \geq \omega(G) = \alpha(\overline{G}) \geq \frac{s^2}{s^2 - \lambda_1(G)},$$

$s^2 \leq n$ and $\frac{s^2}{s^2-\lambda_1(G)}$ is a decreasing function on s^2, the Cvetković's bound on chromatic number can be deduced.

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Theorem [Cvetković, 1972]

If $G \neq K_n$ is a graph with n vertices, then

$$\chi(G) \geq \frac{n}{n - \lambda_1(G)}.$$

Upper bounds on the stability number

Theorem [Ryacek and Sciermeyer, 1995]

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Theorem [Cvetković, 1973]

Let $|\sigma^-(G)|, |\sigma^0(G)|, |\sigma^+(G)|$ the number of eigenvalues of the graph G smaller than, equal to, and greater than zero, respectively. Then

$$\alpha(G) \leq |\sigma^0(G)| + \min\{|\sigma^-(G)|, |\sigma^+(G)|\}.$$

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This problem can be formulated as the following integer quadratic program:

$$\begin{aligned} mc(G,w) \quad = \quad & \text{Maximize } \frac{1}{2} \sum_{ij \in E(G)} w_{ij}(1 - x_i x_j) \\ & \text{subject to } x_i \in \{1,-1\} \qquad \forall i \in V(G). \end{aligned}$$

Upper bound on the maximum cut problem

Theorem [Mohar and Poljak, 1990]

Let $L_{(G,w)}$ be the Laplacian matrix of the weighted graph (G, w). Then

$$mc(G, w) \leq \frac{n}{4}\lambda_1(L_{(G,w)}).$$

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Theorem [Delorme and Poljak, 1990]

Let $L_{(G,w)}$ be the Laplacian matrix of the weighted graph alert(G, w). Then

$$mc(G, w) \leq \min_u \frac{n}{4}\lambda_1(L_{(G,w)} + \mathrm{diag}(u)),$$

where the minimum is obtained among overall $u \in \mathbb{R}^n$ such that$\sum_i u_i = 0$.

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- The **quadratic assignment problem** (QAP) can be defined as follows: given the set $\mathbf{N} = \{1, \ldots, n\}$ and three $n \times n$ matrices $\mathbf{A} = (a_{ik})$, $\mathbf{B} = (b_{jl})$ and $\mathbf{C} = (c_{ij})$, find a permutation π of the set $\mathbf{N}$ which minimizes

$$\sum_{i=1}^{n} c_{i\pi(i)} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\pi(i)\pi(j)}.$$

The quadratic assignment problem (alternative formulation)

- Equivalently, QAP consists on findind a $n \times n$ permutation matrix P which minimizes the trace

$$\min_{P \in \Pi} \operatorname{tr}(C + APB^t)P^t,$$

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- **Theorem [Finke, Burkard and Rendl, 1987]**
Let A and B be symmetric $n \times n$ matrices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Then
$$\min_\pi \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\pi(i)\pi(j)} \geq \sum_{i=1}^n \lambda_i \mu_{n-i+1},$$
where the minimum is taken over all permutations π of $\{1, \ldots, n\}$.

Upper bound on the size of a vertex-set inducing subgraph with mean degree k

- Using a quadratic programming technique jointly with the main angles of G, as proved in (C and Rowlinson, 2010), the size of a vertex subset $S \subset V(G)$ inducing a subgraph with mean degree k has the following upper bound:

$$|S| \leq \inf\{h_k^G(t) : t > -\lambda_n(G)\},$$

where $h_k^G(t) = (k+t)\left(1 - \frac{P_{\overline{G}}(t-1)}{(-1)^n P_G(-t)}\right)$.

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- Considering $H_G(t) = \sum_{k=0}^{\infty} N_k t^k$, where N_k is the number of walks of length k in G (the walk-generating function of G), we may write $h_k^G(t) = \left(1 + \frac{k}{t}\right) H_G(-\frac{1}{t})$.

Induced matchings and dominating induced matchings

Lemma

Let G be a graph with an induced matching $M \subseteq E(G)$. Then

- $|M| \leq \min\{|\sigma^{+}(G)|, |\sigma^{-}(G)|\}$;

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Theorem [C, Cerdeira, Delorme and Silva, 2008]

Let G be a graph. If $M \subseteq E(G)$ is a dominating induced matching, then $|M| \geq \frac{1}{2}\max\{|\sigma^+(G)|, |\sigma^-(G)|\}$.

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Corollary

If M is a dominating induced matching of a graph G, then

$$\frac{1}{2}\max\{|\sigma^+(G)|, |\sigma^-(G)|\} \leq |M| \leq \min\{|\sigma^+(G)|, |\sigma^-(G)|\}.$$

Dominating induced matchings and maximum induced matchings

- It is clear that if M is and induced matching of G, then

$$|M| \leq \frac{|E(G)|}{2\delta(G) - 1},$$

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- If G is p-regular and M is a dominating induced matching, then the above upper bound is attained.
- Combining the above upper bound with the previous corollary, we may conclude that some graphs have no dominating induced matchings.

Graph eigenvalues and matchings

- **Theorem[Ming and Wang, 2000]**
  Let T be a tree with n vertices. If T has a perfect matching then $\lambda_{n/2}(T) = 2$.

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  Let T be a tree with n vertices. If T has a perfect matching then $\lambda_{n/2}(T) = 2$.
- **Theorem[Brouwer and Haemers, 2005]**
  Let G be a connected k-regular graph with $2p$ vertices and with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{2p}$. If

$$\lambda_3 \leq \begin{cases} k - 1 + \frac{3}{k+1}, & \text{if k is even} \\ k - 1 + \frac{3}{k+2}, & \text{otherwise,} \end{cases}$$

then G has a perfect matching.

Hamiltonian paths and Hamiltonian cycles

Considering a graph G of order n, in [Fiedler and Nikiforov, 2010] the following results are deduced.

- If $\lambda_1(G) \geq n - 2$ then G has a Hamiltonian path, unless $G = K_{n-1} + v$ (a complete graph plus an isolated vertex v).

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- If $\lambda_1(G) > n - 2$ then G has a Hamiltonian cycle, unless $G = K_{n-1} + e$ (a complete graph plus a pendent edge e).

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Definition of (k, τ)-regular set

A vertex subset $S \subseteq V(G)$ is (k, τ)**-regular** if induces a k-regular subgraph and

$$\forall v \notin S \;\; |N_G(v) \cap S| = \tau.$$

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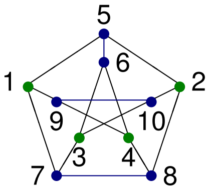

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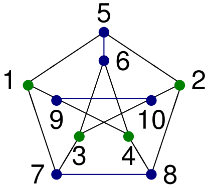

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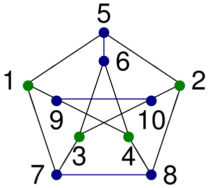

$S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular.
$S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular.

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- If G has a (k,τ)-regular set S then in its complement, $\bar{G}$, S is $(|S|-k-1, |S|-\tau)$-regular.

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- By convention, if G is k-regular, then we say that $V(G)$ is $(k, 0)$-regular.
- If a p-regular graph G has a (k, τ)-regular set S, then $V(G) \setminus S$ is $(p - \tau, p - k)$-regular.

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Theorem[Barbosa and C, 2004]

A graph G has a maximum stable set which is $(0, 1)$-regular if and only if each vertex belongs to exactly one simplex.

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Theorem[C, Cerdeira, Delorme and Silva, 2008]

If $M \subset E(G)$ is a dominated induced matching of G, then M is a maximum induced matching.

A connected graph G with more than one edge has a dominating induced matching $D \subset E(G)$ if and only if $L(D)$ is $(0, 1)$-regular in the line graph $L(G)$.

Independent sets

Theorem

Let G be a graph and $S \subset V(G)$. If S is $(0, \tau)$-regular with $\tau = -\lambda_n(G)$, then S is a maximum stable set and every maximum stable set is $(0, \tau)$-regular.

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Theorem

Let G be a graph and $S \subset V(G)$. If S is $(0, \tau)$-regular with $\tau = -\lambda_n(G)$, then S is a maximum stable set and every maximum stable set is $(0, \tau)$-regular.

- The Petersen graph P has a $(0, 2)$-regular set with cardinality 4 and $\lambda_n(P) = -2$. Then $\alpha(P) = 4$.
- The Hoffman-Singleton graph HS has a $(0, 3)$-regular set with cardinality 15 and $\lambda_n(HS) = -3$. Then $\alpha(HS) = 15$.

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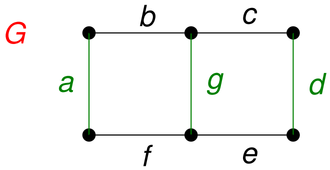
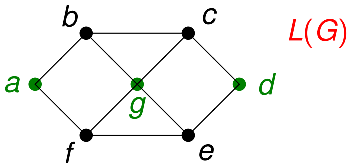


$M = \{a, d, g\}$



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Strongly regular graphs

Theorem [C, Sciriha and Zerafa, 2009]

A k-regular graph G of order n is strongly regular with parameters (n, k, a, c) if and only if $\forall v \in V(G)$, $S = N_G(v)$ is (a, c)-regular in $H = G - \{v\}$.

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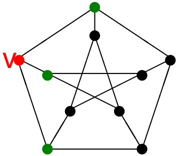
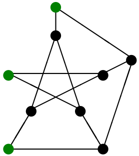


The Petersen graph P



The graph $P - \{v\}$

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Lemma

A graph G has a Hamiltonian cycle C if and only if $L(C)$ is $(2, 4)$-regular inducing a connected subgraph in the line graph $L(G)$.

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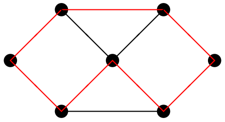

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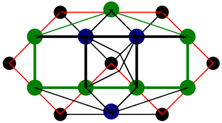

Eigenvalues and (k, τ)-regular sets

Theorem[Thompson, 1981]

Let G be a p-regular graph and $\mathbf{x}$ the characteristic vector of $S \subseteq V(G)$. Then S is a (k, τ)-regular set, with $\tau > 0$, if and only if $k - \tau \in \sigma(G)$ with corresponding eigenvector $(p - k + \tau)\mathbf{x} - \tau j$, where j is the all-one vector.

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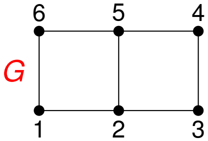
Theorem[C and Rama, 2007]

Let $\lambda \in \mathbb{Z}$ and G be a graph with a (k_1, τ_1)-regular set S_1 ($\tau_1 > 0$) and a (k_2, τ_2)-regular set S_2, such that $S_1 \neq S_2$ and $k_1 - \tau_1 = k_2 - \tau_2 = \lambda$. Then $\lambda \in \sigma(G)$ with corresponding eigenvector $\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the characteristic vectors of S_1 and S_2, respectively.

Eigenvalues and eigenvectors



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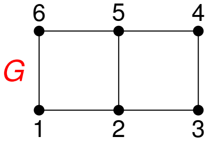

$S_1 = \{1, 2, 5, 6\}$ and $S_2 = \{2, 3, 4, 5\}$ are $(2, 1)$-regular.

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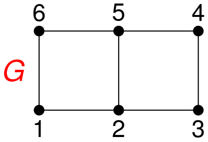

$S_1 = \{1, 2, 5, 6\}$ and $S_2 = \{2, 3, 4, 5\}$ are $(2, 1)$-regular.
$T_1 = \{1, 4\}$ and $T_2 = \{3, 6\}$ are $(0, 1)$-regular.

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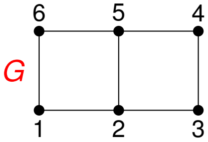

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$T_1 = \{1, 4\}$ and $T_2 = \{3, 6\}$ are $(0, 1)$-regular.
Then $\{-1, 1\} \subset \sigma(G) = \{-2.41, -1, -0.41, 0.41, 1, 2.41\}$ and

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in \mathcal{E}_G(1) \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathcal{E}_G(-1).$$

Maximum induced k-regular subgraphs

Theorem[C, Kamiński and Lozin, 2007]

Let G be a graph and $\tau = -\lambda_n(G)$. If $S \subseteq V(G)$ is $(k, k+\tau)$-regular, then S is a maximum cardinality set inducing a k-regular subgraph.

Maximum induced k-regular subgraphs

Theorem[C, Kamiński and Lozin, 2007]

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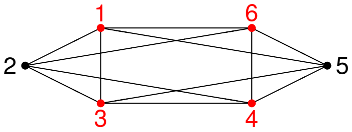

Graph G with $\sigma(G) = \{-2, 0, 4\}$ and the $(2,4)$-regular set $\{1, 3, 4, 6\}$.

The main characteristic polynomial

Definition

Each distinct eigenvalue $\mu_1, \ldots, \mu_p$, $1 \leq p \leq n$, of a graph G such that $\mathcal{E}_G(\mu_i)$ is *not* orthogonal to the all-one vector $\mathbf{j}$ is said to be **main.** The remaining distinct eigenvalues are **non-main.**

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Lemma [D. Cvetković and M. Petrić, 1984]

If G is a graph with p distinct main eigenvalues $\mu_1, \ldots, \mu_p$, then the **main characteristic polynomial** of G

$$m_G(\lambda) \quad = \quad \lambda^p - c_0\lambda^{p-1} - c_1\lambda^{p-2} - \cdots - c_{p-2}\lambda - c_{p-1}$$

has integer coefficients.

The walk matrix

Definition

Given a graph G of order n, the $n \times k$ walk matrix of G is the matrix $\mathbf{W'} = \left(\mathbf{j}, \mathbf{A_G}\mathbf{j}, \mathbf{A}_G^2\mathbf{j}, \ldots, \mathbf{A}_G^{k-1}\mathbf{j}\right)$.

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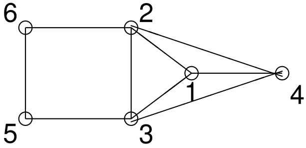

$$\begin{aligned} \mathbf{W}' &= (j, A_G j, A_G^2 j) \\ &= \begin{pmatrix} 1 & 3 & 11 \\ 1 & 4 & 12 \\ 1 & 4 & 12 \\ 1 & 3 & 11 \\ 1 & 2 & 6 \\ 1 & 2 & 6 \end{pmatrix} \end{aligned}$$

The column space of the walking matrix

Theorem (Hagos, 2002)

Let G be a graph of order n with p distinct main eigenvalues. The rank of its $n \times k$ walk matrix $\mathbf{W}'$, is p for $k \geq p$.

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$$p = \text{min}\{k : \{j, Aj, A^2j, \dots, A^kj\} \text{ is linear dependent}\}.$$

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The pth column of $\mathbf{AW}$ is $\mathbf{A}^p\mathbf{j} = \mathbf{W}\begin{pmatrix} c_{p-1} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix}$, where c_j, for $j = 0, \ldots, p-1$, are the coefficients of $m_G(\lambda)$.

Main (non-main) eigenvalues of graphs with (κ, τ)-regular sets

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The main eigenspace of a graph G, $Main(G)$, is the subspace spanned by the eigenvectors not orthogonal to the all one vector j.

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Theorem[C, Sciriha and Zerafa, 2008]

Let G be a graph with a (k, τ)-regular set S, where $\tau > 0$, and $\lambda \in \sigma(G)$.

1. The eigenvalue λ is non-main if and only if
$$\lambda = k - \tau \qquad \text{or} \qquad \mathbf{x}_S \in (\mathcal{E}_G(\lambda))^\perp .$$
2. If λ is main with associated eigenvector $\mathbf{u} \in Main(G)$, then
$$\mathbf{u}^t\mathbf{x}_S \neq 0 \text{ and } \lambda = \tau\frac{\mathbf{u}^t\mathbf{x}_{\bar{S}}}{\mathbf{u}^t\mathbf{x}_S} + k.$$

(κ, τ)-parametric vectors

Definition

Let G be a graph with p distinct main eigenvalues $\mu_1, \ldots, \mu_p$ and $\{x_1, \ldots, x_p\}$ an orthonormal basis of $Main(G)$. Considering the nonnegative integer κ and the positive integer τ such that $(\kappa - \tau) \notin \{\mu_1, \ldots, \mu_p\}$, the vector

$$g = \sum_{i=1}^{p} \tau \frac{j^t x_i}{\mu_i - (\kappa - \tau)} x_i$$

is referred as the (κ, τ)-parametric vector of G.

Graphs with (κ, τ)-regular sets

Theorem[C, Sciriha and Zerafa, 2008]

Let G be a graph with p distinct main eigenvalues $\mu_1, \ldots, \mu_p$ and $\{x_1, \ldots, x_p\}$ an orthonormal basis of $Main(G)$. Consider a (κ, τ)-regular set $S \subseteq V(G)$, with $\tau > 0$. Then the characteristic vector of S, x_S, is such that

$$x_S = g + q,$$

where $q \in (Main(G))^{\perp}$ and the following holds.

1. If $k - \tau \notin \sigma(G)$, then $q = 0$ and $x_S = \tau (A - (k - \tau) I)^{-1} j$;
2. If $\kappa - \tau \in \sigma(G)$ then $q \in \varepsilon_G(\kappa - \tau)$.

Graphs with (κ, τ)-regular sets

Corollary[C, Sciriha and Zerafa, 2008]

If a graph G with p distinct main eigenvalues $\mu_1, \ldots, \mu_p$ and $\{x_1, \ldots, x_p\}$ an orthonormal basis of $Main(G)$ has a (κ, τ)-regular set $S \subset V(G)$ and $(\kappa - \tau) \in \sigma(G)$, then

$$|S| = \sum_{i=1}^{p} \tau \frac{(j^t x_i)^2}{\mu_i - (\kappa - \tau)}.$$

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