A generalization of Fiedler's lemma and some applications

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Aim

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In this work, having a result presented by Fiedler as motivation, we obtain, for graphs with a special type of adjacency matrix, bounds for its energy and we do some applications to graph spectra.

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- $\lambda_1, \ldots, \lambda_n$ - eigenvalues of $A(G)$ (are considered as the eigenvalues of G).
- For a square matrix A, its eigenvalue λ and a corresponding eigenvector $\mathbf{u}$ then, $(\lambda, \mathbf{u})$ is an eigenpair of A.

Fiedler's Lemma

Lemma 1

Let A, B symmetric matrices of orders m and n, respectively, with corresponding eigenpairs $(\alpha_i, \mathbf{u}_i)$ $i = 1, \ldots, m$ and $(\beta_i, \mathbf{v}_i)$ $i = 1, \ldots, n$, respectively. Suppose that $\|\mathbf{u}_1\| = 1 = \|\mathbf{v}_1\|$. Then, for any ρ, the matrix

$$C = \begin{pmatrix} A & \rho \mathbf{u}_1 \mathbf{v}_1^T \\ \rho \mathbf{v}_1 \mathbf{u}_1^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2$, where γ_1, γ_2 are the eigenvalues of

$$\widehat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

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- B be the adjacency matrix of a q-regular graph G_2.
- In the eigenpairs $(\alpha_1, \mathbf{u}_1)$ and $(\beta_1, \mathbf{v}_1)$, $\alpha_1 = p$, $\beta_1 = q$.
- the vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are the normalized all one vectors with m and n components, respectively.

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$$(p - \gamma)(q - \gamma) - mn = \gamma^2 - (p + q)\gamma + pq - mn .$$

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$$[p + q \pm \sqrt{(p-q)^2 + 4(pq - mn)}]/2.$$

Generalization of the Fiedler's lemma

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- with corresponding eigenpairs $(\alpha_{ij}, \mathbf{u}_{ij})$, $i = 1, \ldots, n_j$.
- Moreover, for $p = 1, 2, \ldots, k-1$, let ρ_p be arbitrary constants.

Then, we consider the matrices

$$C = \begin{pmatrix} A_1 & \rho_1 \mathbf{u}_{11}\mathbf{u}_{12}^T & & & \\ \rho_1 \mathbf{u}_{12}\mathbf{u}_{11}^T & A_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & & A_{k-1} & \rho_{k-1}\mathbf{u}_{1k-1}\mathbf{u}_{1k}^T \\ & & & \rho_{k-1}\mathbf{u}_{1k}\mathbf{u}_{1k-1}^T & A_k \end{pmatrix}$$

$$\widehat{C} = \begin{pmatrix} \alpha_{11} & \rho_1 & & & \\ \rho_1 & \alpha_{12} & \ddots & & \\ & \ddots & \ddots & & \\ & & & \alpha_{1k-1} & \rho_{k-1} \\ & & & \rho_{k-1} & \alpha_{1k} \end{pmatrix}.$$

Lemma 2

For $j = 1, 2, \ldots, k$, let A_j be $n_j \times n_j$ symmetric matrices, with corresponding eigenpairs $(\alpha_{ij}, \mathbf{u}_{ij})$, $i = 1, \ldots, n_j$. Furthermore, suppose that for each j the system of eigenvectors $\mathbf{u}_{ij}$, $i = 1, \ldots, n_j$, is orthonormal. Then, for any $\rho_1, \rho_2, \ldots, \rho_{k-1}$, the matrix C has eigenvalues

$$\alpha_{21}, \alpha_{31}, \ldots, \alpha_{n_1 1}, \alpha_{22}, \ldots, \alpha_{n_2 2}, \ldots, \alpha_{2k}, \ldots, \alpha_{n_k k}, \gamma_1, \gamma_2, \ldots, \gamma_k$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k$ are eigenvalues of the matrix $\widehat{C}$.

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  - $\alpha_{1j} = p_j$
  - $\mathbf{u}_{1j} = \frac{1}{\sqrt{n_j}}\, \hat{\mathrm{e}}_{(n_j)}$, where $\hat{\mathrm{e}}_{(n_j)}$ is the all-one vector with n_j components, for $j = 1, \ldots, k$.

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- Define the operation: $\bigoplus_{j=1}^{k} G_j$, as the graph G obtained from $G_1, \dots, G_k$, by connecting each vertex of G_j to all vertices of G_{j+1} for $j = 1, \dots, k-1$.

Let $\rho_j = \sqrt{n_j\, n_{j+1}}$ for $j = 1, \dots, k-1$. Applying previous lemma,

- C is the adjacency matrix of the graph $G = \bigoplus_{j=1}^{k} G_j$.

Let $\rho_j = \sqrt{n_j\, n_{j+1}}$ for $j = 1, \ldots, k-1$. Applying previous lemma,

- C is the adjacency matrix of the graph $G = \bigoplus_{j=1}^{k} G_j$.
- Thus, the eigenvalues of G are the eigenvalues of G_j for $j = 1, \ldots, k$ considered altogether except for

$$\alpha_{1j}\ ,\ j = 1, \ldots, k$$

which are replaced by the eigenvalues of the matrix $\widehat{C}$,

$$\gamma_j\ ,\ j = 1, \ldots, k.$$

.

Then we arrive at the following result.

Theorem 3

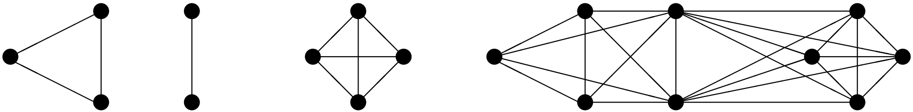
Let for $j = 1, \ldots, k$, G_j be a p_j-regular graph of order n_j, with spectrum $\sigma(G_j)$. If $\widetilde{G} = \bigoplus_{j=1}^{k} G_j$, $\rho_j = \sqrt{n_j\, n_{j+1}}$ and $\widehat{C}$ is the above specified matrix, then the spectrum of $\widetilde{G}$ is

$$\sigma(\widetilde{G}) = \left(\bigcup_{j=1}^{k} \left(\sigma(G_j) \setminus \{p_j\}\right) \right) \cup \sigma(\widehat{C})$$

where $\sigma(G_j)$, $j = 1, \ldots, k$, are multisets and their union is to be considered with possible repetition of eigenvalues.

An Example

For the regular graphs $G_1 = K_3$, $G_2 = K_2$, and $G_3 = K_4$, the graph $\widetilde{G} = \bigoplus_{j=1}^{3} G_j$ is depicted in Fig. 1.



G_1 $\quad$ G_2 $\quad$ G_3 $\quad$ $\widetilde{G} = \bigoplus_{j=1}^{3} G_j$

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Application to graph energy

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Taking into account this definition, as a direct consequence of previous theorem, we have:

Corollary 4

Let for $j = 1, \ldots, k$, G_j be a p_j-regular graph of order n_j, with spectrum $\sigma(G_j)$. If $\widetilde{G} = \bigoplus_{j=1}^{k} G_j$ and

$$\widehat{C} = \begin{pmatrix} p_1 & \sqrt{n_1 n_2} & & \\ \sqrt{n_1 n_2} & p_2 & & \\ & & \ddots & \sqrt{n_{k-1} n_k} \\ & & \sqrt{n_{k-1} n_k} & p_k \end{pmatrix}$$

then

$$E(\widetilde{G}) = \sum_{j=1}^{k} (E(G_j) - p_j) + E(\widehat{C}).$$

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Let B be an $m \times n$ complex matrix. The energy of B is defined as

$$E(B) = \sum_j s_j(B)$$

where $s_1(B), s_2(B), \ldots$ are the singular values of B .

This extends the concept of graph energy and therefore, if the matrix $B \in \mathbb{R}^{n \times n}$ is symmetric with eigenvalues $\gamma_1(B), \ldots, \gamma_n(B)$, then its energy is given by

$$E(B) = \sum_{i=1}^{n} |\gamma_i(B)| \ .$$

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$$A(G) = \begin{pmatrix} A_1 & X_1 & & & \\ X_1^T & A_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & & A_{k-1} & X_{k-1} \\ & & & X_{k-1}^T & A_k \end{pmatrix} \quad (1)$$

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where, for each $j \in \{1, 2, \ldots, k\}$, the matrix A_j is the adjacency matrix of the subgraph G_j induced by the vertices with indices i such that $\sum_{p=0}^{j-1} n_p < i \leq \sum_{p=0}^{j} n_p$ (setting $n_0 = 0$).

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$$C = \begin{pmatrix} A_1 & \rho_1 \mathbf{u}_{11} \mathbf{u}_{12}^T & & & \\ \rho_1 \mathbf{u}_{12} \mathbf{u}_{11}^T & A_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & & A_{k-1} & \rho_{k-1} \mathbf{u}_{1k-1} \mathbf{u}_{1k}^T \\ & & & \rho_{k-1} \mathbf{u}_{1k} \mathbf{u}_{1k-1}^T & A_k \end{pmatrix}$$

$$
T = \begin{pmatrix}
0 & \rho_1 \mathbf{u}_{11}\mathbf{u}_{12}^T & & & \\
\rho_1 \mathbf{u}_{12}\mathbf{u}_{11}^T & 0 & \ddots & & \\
 & \ddots & \ddots & & \\
 & & & 0 & \rho_{k-1}\mathbf{u}_{1k-1}\mathbf{u}_{1k}^T \\
 & & & \rho_{k-1}\mathbf{u}_{1k}\mathbf{u}_{1k-1}^T & 0
\end{pmatrix}
$$

where $\rho_1, \ldots, \rho_{k-1}$ are positive. Then

$$
E(G) \geq E(C) - E(T) \; .
$$

Based on previous theorem we obtain the following:

Corollary 6

Let G be a graph, with adjacency matrix as defined in (1), such that

- each of the induced subgraphs G_j , $j = 1, \ldots, k$, is p_j-regular
- $\overline{G} = \bigoplus_{j=1}^{k} \overline{G}_j$, where each of the graphs $\overline{G}_j$, $j = 1, \ldots, k$, is 0-regular of order n_j
- $\widetilde{G} = \bigoplus_{j=1}^{k} G_j$,

then

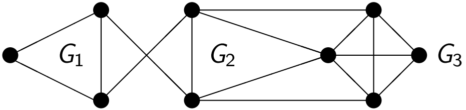
$$E(G) \geq E(\widetilde{G}) - E(\overline{G}) .$$

Proof

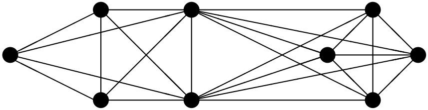
Since each of the induced subgraphs G_j , $j = 1, \ldots, k$, is p_j-regular, by setting $\rho_j = \sqrt{n_j \, n_{j+1}}$, $j = 1, \ldots, k-1$,

- C of previous theorem becomes the adjacency matrix of the graph $\widetilde{G} = \bigoplus_{j=1}^{k} G_j$
- T is the adjacency matrix of the graph $\overline{G} = \bigoplus_{j=1}^{k} \overline{G}_j$.

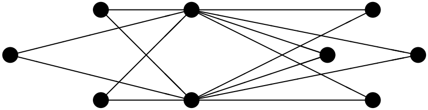
The graphs G , $\widetilde{G} = \bigoplus_{j=1}^{3} G_j$, and $\overline{G} = \bigoplus_{j=1}^{3} \overline{G}_j$. For the induced subgraphs G_1 , G_2 , and G_3.



G



$\widetilde{G} = \bigoplus_{j=1}^{3} G_j$



$\overline{G} = \bigoplus_{j=1}^{3} \overline{G}_j$

Another application

Let $a_j > 0$ for $j = 1, 2, \ldots, n-1$, and $\varepsilon \geq 0$.

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Let $a_j > 0$ for $j = 1, 2, \dots, n-1$, and $\varepsilon \geq 0$. Consider the tridiagonal $n \times n$ symmetric matrix

$$T_{n,\varepsilon} = \begin{pmatrix} 0 & a_1 & & & \\ a_1 & 0 & \ddots & & \\ & \ddots & \ddots & & \\ & & & 0 & a_{n-1}+\varepsilon \\ & & & a_{n-1}+\varepsilon & 0 \end{pmatrix}$$

Theorem 7

Let $T_{n,\varepsilon}$ be the matrix specified above with $\varepsilon > 0$ and let $T_{n,0}$ be the matrix resulting from it by replacing ε by zero. Then

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Let $T_{n,\varepsilon}$ be the matrix specified above with $\varepsilon > 0$ and let $T_{n,0}$ be the matrix resulting from it by replacing ε by zero. Then

$$E(T_{n,\varepsilon}) > E(T_{n,0}) \ .$$

Corollary 8

If any of the entries a_j , $j = 1, \ldots, n-1$, of the tridiagonal matrix $T_{n,\varepsilon}$ increases, then its energy also increases. In particular, if all entries a_j , $j = 1, \ldots, n-1$, increase, then the energy of $T_{n,\varepsilon}$ increases.

Theorem 9

Let $\overline{G} = \bigoplus_{j=1}^{k} \overline{G}_j$, where each $\overline{G}_j$ is 0-regular of order n_j, $j = 1, \ldots, k$, and consider

- $n^* = \max_{1 \leq j \leq k} n_j$ and
- $n_* = \min_{1 \leq j \leq k} n_j$.

If $n^* > 1$, then

$$n_* E(P_k) < E(\overline{G}) < n^* E(P_k),$$

where P_k is the k-vertex path.

Proof

From Corollary 4, $E(\overline{G}) = \sum_{j=1}^{k} E(\overline{G}_j) + E(\widehat{C})$, where

$$\widehat{C} = \begin{pmatrix} 0 & \sqrt{n_1 n_2} & & & \\ \sqrt{n_1 n_2} & 0 & & & \\ & & \ddots & & \\ & & & & \sqrt{n_{k-1} n_k} \\ & & & \sqrt{n_{k-1} n_k} & 0 \end{pmatrix}$$

Considering the k-vertex path P_k, using Corollary 8, since $n^* > 1$, we may conclude the strict inequalities

$$n_* E(P_k) < E(\widehat{C}) < n^* E(P_k).$$

Therefore, since $E(\overline{G}_j) = 0, j = 1, \ldots, k$, the result follows.

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