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Guest Editors: Katica R. (Stevanović) Hedrih
Dragoslav Šumarac

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Jubilee 135 years anniversary of birthday

ANTON DIMITRIJA BILIMOVIČ

(July 20, 1879 - September 17, 1970)

Dear Readers,

Before 135 years Anton Dimitrija Bilimovič was born on July 20, 1879 in Zhitomir, Ukraine. Professor Bilimović died on September 17, 1970, at the age of 91 in Belgrade, Serbia (and Yugoslavia).

He worked in Russia (and Ukraine) from November 25, 1903 until January 1920, when he left Russia (and Ukraine). After that, he worked in Serbia (and Yugoslavia) from April 20, 1920, to February 15, 1955 when he was retired.

Anton Bilimovič defended his doctoral dissertation, entitled: "Contact motion of rigid body, first part: motion with one degree of freedom" in Odessa, in 1907.

In 1907 he became associate professor of the Kiev University, and in 1915 full professor for the subject mechanics, at the Novorossisky University in Odessa.

On April 20, 1920 Anton Bilimovič was elected professor under contract at the Faculty of Philosophy, Belgrade University. On November 3,1926 he was elected full professor of same Faculty for the subjects rational mechanics and applied mathematics.

On February 18, 1925 he was elected a corresponding member of the Serbian Academy of Sciences. On February 17,1936 he was elected full member of the Serbian Academy of Sciences, in which he held the position of Secretary General of the Department of Natural Sciences and Mathematics, from 1939 to 1940.

Professor Bilimovič was one of the founders of the Belgrade school of mechanics. His first doctorate students at the Belgrade University were Vječeslav Žardecki, Demčenko, Konstantin Voronjec and Tatomir Anđelić. Bilimovic's scientific activity deserves recognition for dissemination of the Russian school of mechanics in Serbia.

Professor Bilimovč is founder of the journal Publications de l'Institut mathematique de l'Academie Serbe des Sciences. In 2012 this Journal celebrates 80 years of subsistence. This Journal is founded as journal for mechanics and mathematics, but nowday by strong aspration of mathematicians, this journal is only journal for pure mathematics.

Professor Bilimovič initiated foundation of Institute for Mathematics of the Serbian Academy of Sciences and Arts which was founded in 1946. Due to his engagement the first post-second world-war journal Publications de l'Institute mathematique de l'Academie Serbe des Sciences, was published in 1949.

Academicians Konstantin Voronyec, Anton Bilimovič, Tatomir Andjelić (from right to left) and Professor Danilo Rašković – Members of Commission for doctoral dissertation evaluation and defence at Natural-Mathematica Faculty University of Belgrade (Aprl 1961).

Together with the Serbian scientists Milutin Milanković, Jakov Hlitčijev, Konstantin Voronjec and Tatomir Anđelić Professor Bilimovič founded study-group for mechanics Mathematical Institute. On the suggestion of professor Hlitčijev the same group was founded at the Faculty of Natural Sciences and Mathematics of the Belgrade University, in 1952. The first professors were above mentioned founders, and its first Head of the group, until his retirement, was professor Bilimovič. But nowday, by strong aspiration of mathematicians same as to the journal, and working mechanicians without corresponding personality in area of theoretical and applied mechanics, this study group of mechanics, after working of 50 years, dissapears from Faculty of mathematics.

Professor Bilimovič published his scientific results in the most acknowledged world journals. He published 138 research scientific works, 28 scientific papers, 35 books and university text books, many of them with several editions, 9 texts for popularization of science, 15 reviews and 15 reports. Papers and works of Bilimovič numerous times was cited by world and Yougoslav scientists. His first Ph.D students at the Belgrade University was Vječeslav Žardecki, Basile Demčenko, and Konstantin Voronjec and Tatomir Anđelić, late also academician. Professor Bilimovuč is laurete of important state prize the Labor Medal of the Firest Class, in 1855.

Professor Bilimovič in the Reference titled: "***A. M. Lyapunov in Odesse***" presented and showed his meetings, acquaintance with Lyapunov and his impressions of the famous Russian and world scientist Alexand Mihailovich Lyapunov. He described the tragic death of Lyapunov. Before he met Lyapunov, it was familiar with scientific work and character and its important personality , because it's heard from his teacher and Professor Gavril Konstantinovich Suslov. Lyapunov and Suslv was students of Dimitriya Konstantinovich Bobilyev at Saint Petersburg University. Both, as the students in the same generation, wrote on the same topic works titled: "On rigid body equilibrium in fluid". And, both were awarded a Gold medal for work. Lyapunov's large and more complex addressed this issue and received the additional award, so that his work published in Saint Petersburg University scientific publicatin. How are at Sain Petersburg Univetsity all professorship was busy, Lyapunov was sent to Kharkov, and Suslov in Kiev, at corresponding chair of Mechanics. Lyapunov founded the Kharkiv scientific school of mechanics, up to nowdays with world very inportant scientific results and scientists. First student of Lyapunov at Kharkov was Vladimir Andreevich Steklov. In Kiev scientific school of mechanics, founded by Suslov , first student was Petar Vasilyevich Voronec.

Bilimovič writes that Suslov described Lyapunov, as a man who is still in his youth thinking about scientific ideas, issues, completely no interested for the environment, ignoring the fact and living conditions. Bilimovuič visited Sain Petersburg University, but does he not had the opportunity to meet Lyapunov. At that time Lyapunov works were not observed in Russia and also in aboard, because a small number of scientists involved in these area of science. Since Bilimovič was in Paris, hi listened to lectures by P. Appe;l on *the form of rotating fluids*, why in a popular way represent the scientific results of H. Poncare, G.H. Darwin and A.M. Lyapinov. In one of his lectures Appell recommended works Lyapunov and advised mathematicians to study scientific results of Lyaounov. Next, Bilimovič says that to his knowledge, the research task of the form of rotating fluid, Pafnutiy Levovich Chebyshev give to Lyapunov in recommendation with following words: "You, Alexander Mihailovich, need to deal with only serious problems of mathematical tasks." As reported by Bili, the Lyapunov adhered this recommendation during all his life.

Bilimovič met Lyapunov in Saint Petersburg, and their relationship was beginning to formal acquaintance, which lagged turn into spiritual scientific closeness.

1918 Lypunov was in Odessa with his brothers who worked at the History-Philosophical Faculty in Odessa. At this period Bilimovič was Rector of Odessa University.

While in Odessa had difficult living conditions, Lyapunov continued to work on problems of form of rotating fluids. He had finished his fundamental work, which in 1925 published in the Saint Petersbug's Academy under the title: "Sur certaines séries de figures d'équilibred'un liquide hétérogène en rotation".

Next Bilimovič weote, that in 1918 Lyapunov's wife dead and that in this time was in Kieve.

Although taking care of Lyapunov, which is difficult submitted his wife death at a time when he was alone in room with his wife, he fired a bullet in his head and so he ended at tragic way his life. This happened ar November 3, 1918 in Odessa.

In closing his article on Lyapunov in Odessa, Blimovič wrote the following: In considering the life work and tragic dead of Lyapunov in Odessa, believed in a need to write about it to your knowledge, because much more about Lyapunov in Odessa was written false. Such inaccuracy of Lyapunov appear in the publications of the French Academy of Sciences.

Bilimović this work finished by saying that he wanted to clarify this period of life and death of important world scientist A.M. Lyapunov in Odessa.

REFERENCES

1. Đorđe Đukić., ANTON DIMITRIJA BILIMOVIČ (1879 - 1970), WEB site of Mathematical Institute SANU, http://www.mi.sanu.ac.rs/History/bilimovic.htm
2. Đorđe Đukić., ANTON DIMITRIJA BILIMOVIČ, Publication de L'Institut Mathematique,Nouvele Serie, Tome 91(105) (2012), 3-17. , DOI: 10.2298/ PIM1205003D, Mathematical Institute SANU. http://elib.mi.sanu.ac.rs/files/journals/publ/111/n105p003.pdf, http://elib.mi.sanu.ac.rs/pages/browse_issue.php?db=publ&rbr=111
3. Archive of the Journal Publication de L'Institut Mathematique, Nouvele Serie, Mathematical Institute SANU written by Anton Bilimovič.,
http://publications.mi.sanu.ac.rs/home
4. Anton D, Bilimovič, A. M. Lyapunov in Odesse (in Russian- А. М. Ляпунов В Одессе), , Publication de L'Institut Mathematique, Tome (S.S.) IX (09) / 1956, pp. 1-7.
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ANTON DMITRIEVICH BILIMOVICH, AN OUTSTANDING SCIENTIST OF MECHANICS

UDC 501: 531

I.E. Rikun, chief bibliographer of Odessa National Scientific Library named after M. Gorkiy

Anton Dmitrievich Bilimovich was born on the 20thof July, 1879, in Zhitomir. In 1903 he graduated from the Mechanical Division of Kiev University Physics and Mathematics Faculty with the first-degree diploma and gold medal for the research work “Application of geometric derivable in the theory of curves and surfaces”. He was the follower of G.K. Suslov and P.V. Voronets. On the same year he became the member of Kiev Society in Physics and Mathematics and took up the position of a secretary. He has been working as an assistant of Theoretical and Applied Mathematics Department. Since 1909 he became a privat-docent of this Department. He gave lectures in analytical mechanics. In 1910–1911 Bilimovich went on a business trip abroad to prepare for the professorial degree. In 1912 he defended his Master’s thesis “The motion equitation for conservative systems and its application” in Kiev University. In 1912–1914 the scientist visited Paris and Gottingen during his academic trip.

In April 1915 he occupied the position of a full professor of Mechanics Department in Novorossiysk University. He lectured the basic course in theoretical mechanics, additional sections of rigid body dynamics, the theory of elasticity, special courses for the integration of mechanics equitation and the theory of airplane. During that time he was in charge of University mechanics shop and in 1915–1917 he became the Head of Mechanics cabinet. A.D. Bilimovich also ran the mathematical circle for young students.

On February 26th (March 11th) 1918 he was elected for the position of University Chancellor. On July 19th, 1918 he was approved for the position of a Chancellor by Higher Education Board of Ministry of Public Education and Art under hetman P.P. Skoropadskiy’s government.

He has been occupying this position till April 1919 and afterwards during the period from August to November. This pause was explained by the presence of Soviet Government in Odessa. In November 1919 by the order of A.I. Denikin, the commander-in-chief of the armed forces at the south of Russia, he was appointed at the position of a trustee in Odessa Educational District.

He attached A.M. Lyapunov to the cooperation in the University. After the death of scientist, A.D. Bilimovich was a member of the Commission for his manuscripts preservation. In 1918 he became a member of Odessa Polytechnic Institute Organizational Committee and up to 1920 the scientist was acting as a professor and the Head of Mechanics Department. In 1919 he was lecturing the Theoretical Mechanics at Advanced Female Courses in Agricultural Institute. He submitted his research work with the title "The osculating motion of a rigid body", as a doctoral thesis, for the University Board consideration, but he didn't manage to defend it in Odessa.

He has been living at the following address: Uyutnaya st.9.

In 1920 he moved to Yugoslavia and in April the scientist became a Head of Theoretical Mechanics Department in Belgrade University. In1925 he was elected as an associate member and, thereon, in 1936 – as an active member of the Academy of Science in Serbia. In 1939-1940 the scientist held the position of a secretary at the Division of Mathematical and Natural Sciences.

A.D. Bilimovich was one of the founders of Mathematical Institute in Serbian Academy of Science, Yugoslav Society of Mechanics; in 1964 he was elected for the position of an Honorary Chairman. This scientist also established the journal "Publications Mathématiques de l'Université de Belgrade".

During the World War II he abandoned the cooperation with occupants and was retired at his own free will. After Belgrade has been set free, he became a professor of Belgrade University Mathematical and Natural Sciences Faculty. He has been working there till his retirement in 1955.

The scientist died on September 15th, 1970 in Belgrade.

A.D. Bilimovich is the author of above 200 scientific works. His first works were devoted to differential geometry, which he also continued to examine later. His essential achievements are related to the development of the different sections in Analytical Mechanics. He paid special attention to the fundamentals of Mechanics (general principle of Pfaff and its application in perturbation theory). In 1958 he discovered the new differential and variation concept in mechanics, which is equal to d'Alembert-Euler principle. He realized the fundamental researches in mechanics of a rigid body (dynamics of a rigid body with a fixed point and others). He studied canonical equitation of rigid body and system motion (Master's thesis and other research works during 1910–1932), material system motion through the construction and examination of the relevant model movement. His distinguished research works refer to nonholonomic mechanics, in which he continued and developed G.K. Suslov's ideas. This is the description of several unsteady nonholonomic mechanisms structures, which broke the ice for the further elaboration of nonholonomic mechanisms application in machinery, computing instruments theory and other branches of techniques: research works about the integration of canonical nonholonomic mechanics equations, development of the equations of curves of nonholonomic system paths, doctoral thesis, elaborated in Odessa, and other research works. He also made his great contribution in the area of gravitational mechanics: in 1911 he found the new integratable case in the *n*-bodies problem and also studied the issue of three bodies. The scientist received the important results in the theory of Earth's rotation around its axis (particularly, about seasonals).

He also developed the issues about vector analysis and methods of mathematics teaching in school.

G.K. Suslov's ideas elaboration and propagation in Yugoslavia is considered to be the most important achievement of this scientist. He founded a big school of Analytical Mechanics, raised a lot of gifted scientists and fully participated in publishing activities of Russian Emigrants.

His brother – O.D. Bilimovich, a well-known economist.

REFERENCES

1. Geometric differentials in the theory of surfaces. – K., 1905. – 13 p.
2. APPLICATION of geometric derivatives in the theory of curves and surfaces. – K., 1906. – 108 p., 2 sh. of draw.
3. REFORM of mathematics teaching in Germany. – K., 1907. – 28p.
4. MOTION EQUITATION for the conservative systems with line integrals// Izv. of Kiev Univ. – 1910. – Vol. 50. No. 10. – P. 23–43.
5. MOTION EQUITATION for the conservative systems and its applications. – K., – 1912. – VIII, 160p., 9sh. of draw.
6. TO A THEORY about nonholonomic system path / /ZINU. Physics and Mathematics Faculty. – 1916. – ed. 10. – P. 1–57.
7. To an issue about mechanics fundamentals // Rus. mysl. – 1917. – Nov-Dec P.1–83.
8. Osculating movement of a rigid body. Vol.1 part 1. Unidirectional motion. – Odessa:Technik, 1919. – XI, 335 p.
9. Prrodne jednachine kretana chvstog tela // Glas Serpske AS. – 1922. – T.XCIX vol. 42. – P.45-67.
10. Geometricheskie osnovy rachuna sa diadama. – Belgrade., 1930. – vol.1. Belgrade University: (speech, delivered at the General Meeting as of September 19, 1929) // Tr.VI SRAOZ. – Belgrade, 1929. – Part 1. – P.6. (Publication of Russian Scientific Institute, Belgrade).
11. Application of absolute geometry to classical mechanics // Tr.VI SRAOZ. – Belgrade, 1929. – Part 2. – P. 4 (Publication of Russian Scientific Institute, Belgrade).
12. About the equations of mechanics related to the major axes // ZRNIB. – Belgrade, 1930. Ed.2. – P. 6.
13. Modern volume of knowledge in mathematics, applied in technique // Ingener. – 1931. – No. 2. – P. 8.
14. About the rotation of some material system as an integral. // ZRNIB. – Belgrade, 1932. ed.6. – P. 7.
15. Elementary theory of correlation.// ZRNIB. - Belgrade, 1935. ed.12. –P.16.
16. Über einige nichtholonome Mechanismen.// Publ. math. Univ. Belgrad. – 1936. – T.V. –P.169-176.
17. Über einige nichtholonome Pendel.// Publ. math. Univ. Belgrad. – 1938. – T.VI-VII. –P.262–272.

18. A.M.Lyapunov in Odessa.// Publications de l'institute mathématique (Belgrade Academie Serbe des sciences). – 1956. – T. IX. – P.1–7.
19. BILIMOVIC ANTON – matematicar // Enciklopedije Jugoclavije. – Zegreb. 1955. – Enj.I. – S.565-566. – Reference list of the basic research works (21 titles).
20. Angelich T.P. The review of mechanics developments in Serbia// Researches in mechanics history: set of art. – M., 1981. – P.71–91:ill.
    About Bilimovich. – P.82 (port.), 83, 88.
21. Bilimovich Anton Dmitrievich // Russkoe Zarubezh'ye. Zolotaya kniga emigratsii. M.,1997. – P.88–90.
22. Hausmann G. Universität und Städtische Gesellschaft in Odesa, 1865–1917. – Stuttgart, 1998. – S. 582.
23. Professors of Odessa (Novorossyisk) University:biogr.slov. – Odessa, 2000. – T.1. Rectory. – P. 65–68.
24. The same publication: the 2nd enlarged edition. – 2005. – P.66–69. port.
25. Yovanovich M. Russian Emigration in Balkans: 1920–40/tr. from serb. by A.Yu. Timofeeva; sc.ed.: A.V. Gromov-Kolly, Ye.V. Mikhaylov. – M.: Library fund "Russkoye zarubezh'ye"; Rus.put, 2005. – 448 p.
    About Bilimovich. – P. 333.
26. Tanin S.Yu. Russian Belgrade. – M.: Veche, 2009. – 304 p. – (Russians abroad).
    About Bilimovich. – P. 213–214.

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ON THE LAGRANGIAN AND HAMILTONIAN MECHANICAL SYSTEMS

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Dedicated to Memory of Anton Dimitrija Blimovici

Abstract

One introduces and studies the Lagrangian and Hamiltonian mechanical systems by means of geometrical methods. One established the fundamental equations: the Lagrange and Hamilton equations, respectively. Some examples and applications are pointed out.

Preface

Recently has been published the authors's monograph *Lagrangian and Hamiltonian geometries. Application to Analytical Mechanics* (Ed. Academiei Române, 2011). A half of the book is devoted to introducing and investigating new analytical Mechanics: Finslerian, Lagrangian and Hamiltonian.

One knows (R. Abraham [1], J. Klein [11], R. Miron [20] et al.) that the geometrical theory of nonconservative mechanical systems can not be rigourously constructed without the use of the geometry of the tangent bundle of the configuration space.

The solution of this problem is based on the Lagrangian and Hamiltonian geometries, [3], [5], [18], [19], [20], [21]. In fact, the construction of these geometries relies on the mechanical principles and on the notion of Legendre transformation.

The whole edifice has as support the sequence of inclusions:

$$\{\mathscr{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}$$

formed by Riemannian, Finslerian, Lagrangian, and generalized Lagrangian spaces. The $\mathscr{L}-$duality transforms this sequence into a similar one formed by Hamiltonian spaces, [8], [20].

Of course, these sequences suggest the introduction of the correspondent Mechanics: Riemannian, Finslerian, Lagrangian, Hamiltonian etc.

The fundamental equations (or evolution equations) of these Mechanics are derived from the variational calculus applied to the integral of action and these can be studied by using the methods of Lagrangian or Hamiltonian geometries.

For short, in the monograph are presented the solutions of the problems:

1° A solution of the problem of geometrization of the classical nonconservative mechanical systems, whose external forces depend on velocities, based on the differential geometry of velocity space.

2° The introduction of the notion of Finslerian mechanical system.

3° The definition of Cartan mechanical system.

4° The study of theory of Lagrangian and Hamiltonian mechanical systems by means of the geometry of tangent and cotangent bundles.

5° The geometrization of the higher order Lagrangian and Hamiltonian mechanical systems.

6° The determination of the fundamental equations of the Riemannian mechanical systems whose external forces depend on the higher order accelerations.

In the present paper we introduce the notions of Lagrangian and Hamiltonian mechanical systems and present only their fundamental equations.

1. Riemannian mechanical systems

Let $g_{ij}(x)$ be Riemannian tensor field on the configuration space M. So its kinetic energy is

$$T = \frac{1}{2} g_{ij}(x) y^i y^j, \;\; y^i = \frac{dx^i}{dt} = \dot{x}^i. \tag{1.1}$$

Following J. Klein [11], we can give:

Definition 1.1 A Riemannian Mechanical System (shortly RMS) is a triple $\Sigma_{\mathscr{R}} = (M, T, Fe)$, where

1° M is an $n-$dimensional, real, differentiable manifold (called configuration space).

2° $T = \dfrac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j$ is the kinetic energy of an a priori given Riemannian space $\mathscr{R}^n = (M, g_{ij}(x))$.

3° $Fe(x,y) = F^i(x,y)\dfrac{\partial}{\partial y^i}$ is a vertical vector field on the velocity space TM (Fe are called external forces).

Of course, $\Sigma_{\mathscr{R}}$ is a scleronomic mechanical system. The covariant components of Fe are:

$$F_i(x,y) = g_{ij}(x)F^j(x,y). \tag{1.2}$$

Examples:

1. RMS - for which $Fe(x,y) = a(x,y)\mathbb{C}$, $a \neq 0$. Thus $F^i = a(x,y)y^i$ and $\Sigma_{\mathscr{R}}$ is called a Liouville RMS.
2. The RMS $\Sigma_{\mathscr{R}}$, where $Fe(x,y) = F^i(x)\frac{\partial}{\partial y^i}$, and $F_i(x) = grad_i f(x)$, called conservative systems.
3. The RMS $\Sigma_{\mathscr{R}}$, where $Fe(x,y) = F^i(x)\frac{\partial}{\partial y^i}$, but $F_i(x) \neq grad_i f(x)$, called non-conservative.

Remarks 1.1. A conservative system $\Sigma_{\mathscr{R}}$ is called by J. Klein [11] *a Lagrangian system*. One should pay attention to not make confusion of this kind of mechanical systems with the “Lagrangian mechanical systems” $\Sigma_L = (M, L(x,y), Fe(x,y))$ introduced by R. Miron [20], where $L : TM \to \mathbb{R}$ is a regular Lagrangian.

Starting from Definition 1, in a very similar manner as in the geometrical theory of mechanical systems, one introduces

Postulate. The evolution equations of a RSM $\Sigma_{\mathscr{R}}$ are the *Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i(x,y), \quad y^i = \frac{dx^i}{dt}, \; L = 2T. \tag{1.3}$$

This postulate will be geometrically justified by the existence of a semispray S on TM whose integral curves are given by the equations (1.3). Therefore, the integral curves of Lagrange equations will be called the *evolution curves* of the RSM $\Sigma_{\mathscr{R}}$.

The Lagrangian $L = 2T$ has the fundamental tensor $g_{ij}(x)$.

Remark 1.2. In classical Analytical Mechanics, the coordinates (x^i) of a material point $x \in M$ are denoted by (q^i), and the velocities $y^i = \dfrac{dx^i}{dt}$ by $\dot{q}^i = \dfrac{dq^i}{dt}$. However, we prefer to use the notations (x^i) and (y^i) which are often used in the geometry of the tangent manifold TM.

2. Lagrangian Mechanical systems

A natural extension of the notion of the Riemannian mechanical system is that of the Lagrangian mechanical system. It is defined as a triple $\Sigma_L = (M, L(x,y), Fe(x,y))$ where: M is a real $n-$dimensional C^∞ manifold called the configuration space; $L(x,y)$ is a regular Lagrangian, $Fe(x,y)$ is an a priori given vertical vector field on the velocity space TM called the external forces. The number n is the number of freedom degree of Σ_L.

The Lagrangian $L(x,y)$ is regular if its fundamental tensor

$$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L(x,y) \quad (i,j = 1,2,...,n)$$

is nonsingular, i.e. $\det(g_{ij}(x,y)) \neq 0$ (with notations $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$, $\partial_i = \dfrac{\partial}{\partial x^i}$). In this case the pair $L^n = (M, L(x,y))$ is called Lagrange space associated to the Lagrangian mechanical system Σ_L. For instance, we can consider the systems Σ_L where $L(x,y)$ is the Lagrangian from electrodynamics:

$$L(x,y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathscr{U}(x).$$

γ_{ij} for $(i,j = 1,...,n)$ are the gravity potentials ($\gamma_{ij}(x)$ is a pseudo-riemannian metric), $A_i(x)$, $(i = 1,...,n)$ are the electromagnetic potentials, $\mathscr{U}(x)$ is the potential function and m, c, e are the known physical constants.

The fundamental tensor of $L(x,y)$ is $g_{ij} = mc\gamma_{ij}$. The external force $F_e(x,y)$ of a system Σ_L can be given in the form $F_e = F^i(x,y)\dot{\partial}_i$. Thus $F^i(x,y)$, $(i = 1,...,n)$ is a contravariant $d-$vector field on the velocity manifold TM.

The covariant components $F_i(x,y)$ of F_e are

$$F_i(x,y) = g_{ij}(x,y)F^j(x,y).$$

Exactly, as in the Riemannian mechanical systems $\Sigma_{\mathscr{R}}$, [23], [24], we introduce the following Postulate:

Postulate. *The evolution equations of the Lagrangian mechanical system $\Sigma_L = (M, L, Fe)$ are the Lagrange equations:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y),\ y^i = \frac{dx^i}{dt},\ \ (i = 1,2,...,n). \tag{2.1}$$

But the both members of the Lagrange equations (2.1) are $d-$covectors on the velocity spaces TM. Consequently, we have:

Theorem 2.1. *The Lagrange equations* (2.1) *of a Lagrangian mechanical system $\Sigma_L = (M, L, Fe)$ have a geometrical meaning.*

Theorem 2.2. *The trajectories without external forces of the Lagrangian mechanical system $\Sigma_L = (M, L, Fe)$ are the geodesics of the Lagrange space $L^n = (M, L)$.*

Indeed, $F_i(x, y) \equiv 0$ implies the previous affirmations.

Since $L(x, y)$ is a regular, the Lagrange equations (2.1) are equivalent to the second order differential equations (SODE):

$$\frac{d^2x^i}{dt^2} + 2\overset{\circ}{G}{}^i\left(x, \frac{dx}{dt}\right) = \frac{1}{2}F^i\left(x, \frac{dx}{dt}\right), \tag{2.2}$$

where the functions $2\overset{\circ}{G}{}^i = \frac{1}{2}g^{is}\{(\dot{\partial}_s\partial_h L)y^h - \partial_s L\}$ are the local coefficients of canonical semispray $\overset{\circ}{S}$ of Lagrange space $L^n = (M, L(x, y))$, $\overset{\circ}{S} = y^i\partial_i - 2\overset{\circ}{G}{}^i(x, y)\dot{\partial}_i$.

The equations (2.2) are called fundamental equations of Σ_L, too.

Remark. The Lagrangian $L(x, y)$ and the external forces $Fe(x, y)$ do not explicitly depend on the time t. Therefore Σ_L is a scleronomic Lagrangian mechanical system, [23], [24].

The evolution semispray of Σ_L

The Lagrange equations determine a semispray S which depend on the Lagrangian mechanical system Σ_L, only.

Indeed, the vector field on TM:

$$S = y^i\partial_i - 2G^i(x, y)\dot{\partial}_i \tag{2.3}$$

with

$$2G^i(x, y) = 2\overset{\circ}{G}{}^i(x, y) - \frac{1}{2}F^i(x, y) \tag{2.4}$$

has the property $JS = \mathbb{C}$. So it is a semispray depending only on Σ_L.

Theorem 2.3. [Miron] *For a Lagrangian mechanical system Σ_L the following properties hold:*

1° *The semispray S is given by*

$$S = \overset{\circ}{S} + \frac{1}{2}Fe \tag{2.5}$$

2° *S is a dynamical system on the velocity space TM.*

3° *The integral curves of S are the evolution curves of Σ_L.*

In fact, 1° derives from the formulas (2.3) and (2.4).
2° S being a vector field on TM, compatible with the geometric structure of TM, (i.e. $JS = \mathbb{C}$), it is a dynamical system on the manifold TM.
3° The integral curves of S are determined by the system of differential equations

$$\frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} + 2G^i(x,y) = 0. \tag{2.6}$$

By means of the expression (2.4) of the coefficients $G^i(x,y)$ of the semispray S, the system (2.6) is coincident to (2.2).

The vector field S is called the *evolution* (or *canonical*) semispray of the Lagrangian mechanical system Σ_L. Exactly as in the case of Riemannian or Finslerian mechanical systems, one can prove:

Theorem 2.4. *The evolution semispray S is the unique vector field, on the velocity space TM, solution of the equation*

$$i_S\overset{\circ}{\theta} = -d\mathscr{E}_L + \sigma \tag{2.7}$$

where $\mathscr{E}_L = \dfrac{\partial L}{\partial y^i} - L$ is the energy of Lagragian L, $\overset{\circ}{\theta}$ is the symplectic structure of the Lagrange space L^n, [5], [20] and $\sigma = F_i(x,y)dx^i$.

But S being a solution of the previous equations, we get:

$$d\mathscr{E}_L(S) = \sigma(S) = F_iy^i = g_{ij}F^iy^j.$$

So, we have:

Theorem 2.5. *The variation of energy $\mathscr{E}_L$ along the evolution curves of mechanical system Σ_L is given by*

$$\frac{d\mathscr{E}_L}{dt} = F_i\left(x, \frac{dx}{dt}\right)\frac{dx^i}{dt}. \tag{2.8}$$

The external forces field Fe is called dissipative if $g(\mathbb{C}, Fe) = g_{ij}F^iy^j \leq 0$. Thus, the previous theorem implies:

Theorem 2.6. *The energy of Lagrange space $L^n = (M,L)$ is decreasing along the evolution curves of the mechanical system Σ_L if and only if the external forces field Fe is dissipative.*

Evidently, the semispray S being a dynamical system on the velocity space TM it can be used for study the important problems, as the stability of evolution curves of Σ_L, the equilibrium points etc.

3. The Hamiltonian mechanical systems

Following the ideas from the section 3, we can introduce the next definition:

Definition 3.1. A Hamiltonian mechanical system is a triple:

$$\Sigma_H = (M, H(x,p), Fe(x,p)), \tag{3.1}$$

where $H^n = (M, H(x,p))$ is a Hamilton space and

$$Fe(x,p) = F_i(x,p)\dot{\partial}^i \tag{3.2}$$

is a given vertical vector field on the momenta space T^*M and $\dot{\partial}^i = \dfrac{\partial}{\partial p_i}$, $(i = 1,\dots,n)$.

Fe is called the external forces field.

The evolution equations of Σ_H can be defined by means of the following Postulate:

Postulate 3.1. *The evolution equations of the Hamiltonian mechanical system* Σ_H *are the following* **Hamilton equations***:*

$$\frac{dx^i}{dt} - \dot{\partial}^i \mathscr{H} = 0, \ \frac{dp_i}{dt} + \partial_i \mathscr{H} = \frac{1}{2}F_i(x,p), \ \ \mathscr{H} = \frac{1}{2}H. \tag{3.3}$$

Evidently, for $Fe = 0$, the equations (3.3) give us the geodesics of the Hamilton space H^n.

Using the canonical nonlinear connection $\overset{\circ}{N}$ we can write in an invariant form the Hamilton equations, which allow to prove the geometrical meaning of these equations, [21].

Examples. 1° Consider $H^n = (M, H(x,p))$ the Hamilton spaces of electrodynamics, [8], [19], [21]:

$$H(x,p) = \frac{1}{mc}\gamma^{ij}(x)p_ip_j - \frac{2e}{mc^2}A^i(x)p_i + \frac{e^3}{mc^3}A_i(x)A^i(x)$$

and $Fe = p_i\dot{\partial}^i$. Then Σ_H is a Hamiltonian mechanical system determined only by H^n.

2° $H^n = (M, K^2(x,p))$ is a Cartan space and $Fe = p_i\dot{\partial}^i$.

3° $H^n = (M, H(x,p))$ with $H^2(x,p) = \gamma^{ij}(x)p_ip_j$ and $Fe = a(x)p_i\dot{\partial}^i$.

Returning to the general theory, we can prove:

Theorem 3.1. *The following properties hold:*

1° ξ *given by*

$$\xi = \frac{1}{2}[\dot{\partial}^i H \partial_i - (\partial_i H - F_i)\dot{\partial}^i] \tag{3.4}$$

is a vector field on $\widetilde{T^*M}$.

2° ξ *is determined only by the Hamiltonian mechanical system* Σ_H.

3° *The integral curves of* ξ *are given by the Hamilton equation* (3.3).

The previous Theorem is not difficult to prove if we remark the following expression of ξ:

$$\xi = \overset{\circ}{\xi} + \frac{1}{2}Fe. \tag{3.5}$$

Also we have:

Proposition 3.1. *The variation of the Hamiltonian* $H(x,p)$ *along the evolution curves of* Σ_H *is given by:*

$$\frac{dH}{dt} = F_i \frac{dx^i}{dt}. \tag{3.6}$$

As we know, the external forces Fe are dissipative if $\langle Fe, \mathbb{C}\rangle \geq 0$.

Looking at the formula (3.6) one can say:

Proposition 3.2. *The fundamental function* $H(x,p)$ *of the Hamiltonian mechanical system* Σ_H *is decreasing on the evolution curves of* Σ_H*, if and only if, the external forces* Fe *are dissipative.*

The vector field ξ on $\widetilde{T^*M}$ is called the canonical dynamical system of the Hamilton mechanical system Σ_H.

Therefore we can say: The geometry of Σ_H is the geometry of pair (H^n, ξ).

For other details see [3], [8], [12], [16], [19], [21].

References

[1] Abraham, R., Marsden, J., *Foundations of Mechanics*, Benjamin, New-York, (1978).

[2] Anastasiei, M., *Symplectic structures and Lagrange geometry*, in M. Anastasiei, P.L. Antonelli (eds.), **Finsler and Lagrange Geometry**, Kluwer Academic Publisher, pp. 1–6, 2003.

[3] Bucataru, I., Miron, R., *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Romanian Academy Publ. House, (2007).

[4] Comic, I., *Curvature Theory of Generalized Mirons d-Connection*, Diff. Geom. and Appl. Proc. Conf. Brno, 1989, 17–26.

[5] Djukic, D., Vujanovic, B., *On Some Geometrical Aspects of Classical Nonconservative Mechanics*, Journal of Mathematical Physics, vol. 16, no. 10, p. 2099–2102, (1975).

[6] Djukic, D., Vujanovic, B., *Noether's Theory in Classical Nonconservative Mechanics*, Acta Mathematica, no.23, p.17-27, (1975).

[7] Grigore, D.R., *The derivation of Einstein equations from invariance principles*, 1992, Class. Quantum Grav. 9.

[8] Hrimiuc, D., Shimada, H., *On the $\mathscr{L}-$duality between Lagrange and Hamilton manifolds*, Nonlinear World 3 (1996), 613–641.

[9] Ikeda, S., *Advanced Study in Applied Geometry*, Seizansha, Kanagawa, 1995.

[10] Kern, J., *Lagrange Geometry*, Arch. Math., 25, 1974, 438–443.

[11] Klein, J., *Espaces Variationnels et Mécanique*, Ann. Inst. Fourier, Grenoble, 12, p. 1–124, (1962).

[12] Krupková, O., *Variational Analysis on Fibered Manifolds over One-dimensional Bases*, PhD Thesis, Silesian University, Opava & Charles University, Prague, 1992, 67 pp.

[13] de Leon, M., Rodrigues, P. R., *Almost tangent geometry and higher order mechanical systems*, Proceedings of the Conference on Differential Geometry and Its Applications, August, 24-30, 1986, Brno, Reidel and J. E. Purkyne, Brno (1987) 179–195.

[14] Miron, R., *Lagrange Geometry*, Math. Comput. Modelling 20 (4/5), 1994.

[15] Miron, R., *General Randers Spaces, Lagrange and Finsler Geometry*, in P.L. Antonelli and R. Miron, Editors, FTPH no. 59, Kluwer Acad. Publ., Otsu (1996).

[16] Miron, R., *Hamilton geometry*, Univ. Timisoara Sem. Mecanica **3**(1987).

[17] Miron, R., *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto Univ. 23 (1983), pp. 219–224.

[18] Miron, R., *A Lagrangian theory of relativity*, I, II, An. Şt. Univ. "Al. I. Cuza" Iaşi, 32(1986), 37–62, 7–16.

[19] Miron, R., *Dynamical Sytems of Lagrangian and Hamiltonian Mechanical Systems*, Advance Studies in Pure Mathematics 48, p.309–340, (2007).

[20] Miron, R., *Lagrangian and Hamiltonian geometries. Applications to Analytical Mechanics*, Ed. Academiei & Ed. Fair Partners, Bucuresti, 2011.

[21] Miron, R., Hrimiuc, D., Shimada, H., Sabău, S.V., *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publ., FTPH nr. 118, 2001.

[22] Puta, M., *Hamiltonian mechanical systems and geometric cuantization*, Kluwer Acad. Publ. 260, 1993.

[23] Santilli, R. M., *Foundations of theoretical mechanics. I. The inverse problem in Newtonian mechanics*, Texts and Monographs in Physics. Springer-Verlag New York, New York-Heidelberg, 1978, xix+266 pp.

[24] Santilli, R. M., *Foundations of theoretical mechanics. II. Birkhoffian generalization of Hamiltonian mechanics*, Texts and Monographs in Physics. Springer-Verlag, New York-Berlin, 1983, xix+370 pp.

[25] Vujici, V.A., Hedrih (Stevanovic), K., *The Rheonomic constraints change force*, Facta Universitatis, 1, p.313-322, (1991).

О МЕХАНИЧКИМ СИСТЕМИМА ЛАГРАНЖИАНА И ХАМИЛТОНИАНА

Раду Мирон

Геометријским методама су уведени и изучавани механички системи који одговарају Лагранжијану и Хамилтониану. Постављене ду фундаменталне једначине: Лагранжеове и Хамилтонове једначине, редом. Неки примери и примене су представљене.

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INVARIANT RELATIONS METHOD DEVELOPMENT IN THE PROBLEMS OF RIGID BODY DYNAMICS

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Abstract. *In this paper the problems of invariant manifolds inclusion into the integral manifolds' set and the first integrals construction of differential equations based on the invariant relations, including the equations of rigid body dynamics, and also the integration of these equations by means of K. Jacobi theorems at the invariant relations of T. Levi-Civita and S.A. Chaplygin have been solved. The first integral existence of Euler–Poisson equations under Hess conditions has been proved and its first approximation has been obtained.*

Keywords: *invariant manifolds, integral manifolds, invariant relations, Euler–Poisson equations.*

Introduction. The issue of the first integrals existence along with Cauchy problem holds a central position in the theory of ordinary differential equations (ODE). Generally, for the equations, considered in mechanics, Cauchy problem has a solution, which depends on the properties of phase-space points. All phase-space points are divided into ordinary and singular, in the neighborhood of these points not only the solutions properties, but also the integral manifold analytical structure are different. The information, concerning the fact that in the neighborhood of a an ordinary point, a full set of the first integrals is existed and the availability of the first integrals, which leads to the boundedness of solution, makes it possible to get the solution at the infinite horizon. The classical problems of mechanics (the problem of N bodies and the problem about heavy rigid body motion with a fixed point) are common; a special attention is drawn to them not only due to the importance of the application results, but also because the equations of motion assume the first integrals. In these problems the cases of a full set of the first integrals existence are rendered. For instance, in the problem about the heavy rigid body motion the integrals of Euler [1], Lagrange [2], Kovalevskaya [3] are specified. Hence, the results, received in this area, are not sufficient for getting the complete solution of the issue for the rigid body motion general properties determination in a classic problem. This circumstance is explained by the fact, that due to the research works of Husson [4], Burgatti [5], Poincare [6], Lyapunov [7], Kozlov [8] and other authors the facts of non-existence of the new integrals of algebraic and analytic structure are only established. Therefore, with the examination of the integrals, the development of the research trend, based on specific solutions construction, has been initiated practically at the same time. The most famous solutions of the following authors may be distinguished: Hess [9], Bobylev [10], Goryachev [11, 12], Steklov [13, 14], Chaplygin [15, 16], N. Kovalevskiy [17], Grioli [18] and other authors (see the reviews [19–22]).

Specific solutions construction methods are closely related to the invariant manifolds and invariant relations theorems, which determine these manifolds. Poincare [6] was the first to give the definition for the invariant relations system. The equations for the functions determination, which characterize the invariant manifolds, were received by T. Levi-Civita [23, 24] and S.A. Chaplygin [25]. The equations of Levi-Civita represent the system of linear inhomogeneous differential equations of first order, which contain the undetermined coefficients. Evidently, the availability of these undetermined coefficients caused the fact, that these equations were not entirely known and examined. However, the approaches of Barbashin–Krasovskiy in the theory of stability and Kalman's approaches in the control theory require the detailed examination of the invariant manifold properties [26] and, particularly, the possibility of its replacement by their integral manifolds. In dynamic systems theory the last issue may be formulated as a problem about the possibility of the stated invariant manifold inclusion in a set of integrals. The first four sections of this paper are devoted to the solution of this problem [27].

In the first section of this research the task is given and necessary definitions and equations have been provided. The issues about the invariant manifold inclusion in the set of integrals have been considered at the second section. The role of the special manifold in the dynamic system phase portrait and the first equations integrals formation were mentioned herein. The equations of rigid body motion with a fixed point in the gravity field were brought in the third section; also there is a description of the fourth integrals in cases of Euler, Lagrange, Kovalevskaya. The fourth section gives us the proof that under Hess condition the fourth integral is existed, the special case of which is the integrals of Euler and Lagrange and the solutions of Hess and Dokshevich.

The notion about the invariant relations only always gives the opportunity to reduce the solution of the integrating of the motion equation problem to quadrature (for instance, in Hess solution). So in line with the above-mentioned task, the dynamics equations theory extension, offered by Jacobi, in case of n–2 first integrals, at the invariant relation of different nature remains relevant [28]. In section 5 of this paper the main formula of Jacobi is set, which is regarded as a reduced, on the basis of the integrals of differential equations system. The sufficient conditions of dynamic equations, integrating in quadratures on the basis of n–3 first integrals and an invariant relation as per Levi-Civita case and on the basis of n–4 first integrals and two invariant relations as per Levi-Civita case, provided in Sections 6 and 7. The connection with the Chaplygin's results about integrating of the equations on the first and particular integrals (Hamilton dictionary) considered in Section 8.

1. STATEMENT OF A PROBLEM

Let us consider the dynamic system

$$\dot{x} = f(x), \tag{1}$$

where $x = (x_1, ..., x_n)^T$ – phase vector: $x \in D \subseteq R^n$, $\dot{x} = \frac{dx}{dt}$, $t \in [0, \infty)$; $f(x)$ is a sufficient number of times continuously differentiable function for $x \in D$.

Let $x(t; x^0)$ is a solution of the system (1) with initial condition $x(0; x^0) = x^0 \in D$. If $x(t; x^0) = x^0$ for all $t \geq 0$, therefore the point x^0 is called singular and we have $f(x^0) = 0$. If the function $f(x^0) \neq 0$, then the point x^0 is called ordinary one.

Definition 1. *The function $F(x)$ is called the first integral of the system (1), if it preserves the constant value at any solution of the system (1), viz. $F(x(t; x^0)) = c(x^0)$ for all values $t \geq 0$ and arbitrary value $x^0 \in D$.*

The integrals of the equation (1) are considered as the solutions of a homogeneous partial differential equation

$$L_f F(x) = 0 \qquad \left(L_f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \right). \tag{2}$$

So, it is assumed, that $F(x)$ is a continuously differentiable function to a certain order.

It has been known [29], that in the neighborhood of an ordinary point of equation (1) there are exist $n-1$ independent first integrals

$$F_i(x) = c_i \qquad (i = \overline{1, n-1}), \tag{3}$$

viz. the equation (2) has $n-1$ solution.

Definition 2. *The integral manifold of the equation (1) is called the manifold S_t of the phase space, filled with the integral curves, defined for all $t \in R$, which can be presented as the equation*

$$x = g(t; c), \tag{4}$$

where $g(t; c)$ is defined for all t of R, and the point $c = (c_1, ..., c_k)$ refers to some area G. This function possesses the certain smoothness; the vectors $\frac{\partial g}{\partial c_i}$ $(i = \overline{1, k})$ are linearly independent for each t. The integral manifold (4) is called k-dimensional integral manifold with the same smoothness as the function (4).

It should be noted, that if $k = n$, then formula (4) gives the general solution of the equation (1).

In some cases, instead of the Definition 2, it is convenient to use the definition, based on the first integrals of the vector equation (1).

Definition 3. *The manifold $N = \{x : F_i(x) = c_i, \quad (i = (\overline{1, k})\}$, where $F_i(x) \in C^{m_1}$ (m_1 is natural number) – the first integrals of the equation (1); $c = (c_1, ..., c_k) \in G$, is called k-dimensional integral manifold.*

The main role in the dynamics of differential equations integrating theory plays invariant manifolds.

Definition 4. *The set* $M \in D$ *is called the invariant manifold of the equation (1), if* $x(t;x^0) \in M$ *as* $t \geq 0$ *and some* $x^0 \in M$. *If all* $x^0 \in M$ *are singular points of the system (1), then the manifold* M *is called the singular invariant manifold.*

Sometimes the invariant manifold of the dynamic system is defined as the integral submanifold M *of the phase space of the dynamic system*, the invariant one with regard to "time shifts": $\psi_t : X \to X$ – phase flow conversion (ψ_t – "the shift for time t"), then invariant manifold is set in by the inclusion $\psi_t(M) \subseteq M$ for all admissible time points t.

Comments. While using the Definition 3 we will exclude the cases, which are indicated in research work [30], because manifold N formation may be referred to the main issue in the researching of the equation properties (1), and excluding the extraneous solutions (the solution, which does not comply with the equation (1)) is not considered as the problematic.

For the further description and identification of the first integrals of the invariant relations of the equation (1) it is necessary to have the constructive researching method, based on the functional approach. Therefore, let us give the following definition.

Definition 5. *Let the function* $\varphi(x)$ *is a continuously differentiable up to some order and* $\overline{\mathrm{grad}}\,\varphi(x_1,...,x_n) \neq \mathbf{0}$ *in* D. *The relation*

$$\varphi(x_1,...,x_n) = 0 \tag{5}$$

is called the invariant relation of the equation (1), if it defines the set σ, *which contains the invariant manifold* $M \subset \sigma$.

In definition 5 the terms of T. Levi-Civita [23, 24] and P.V. Kharlamov [31] are regarded as the particular cases. Actually, according to the term of T. Levi-Civita, the relation (5) is called the invariant relation (IR) relative to the equation (1), if from the condition $\varphi(x(0;x^0)) = 0$ follows the equality $\varphi(x(t;x^0)) = 0$ for $t > 0$. So, this case we receive from the definition 5 if $\sigma = M$. But it is very important for the investigation of the IR issue problem, because we can get the equation of T. Levi-Civita for it [23]

$$\frac{d\varphi(x_1,...,x_n)}{dt} = \varphi(x_1,...,x_n)\,\lambda(x_1,...,x_n) \ . \tag{6}$$

According to the P.V. Kharlamov's definition [31], the relation (5) is called invariant and relative to the equation (1), if the manifold is not void

$$\begin{gathered} \varphi(x_1,...,x_n) = 0\,, \\ L_f\varphi(x_1,...,x_n) = \varphi_1(x_1,...,x_n) = 0\,, \\ L_f\varphi_1(x_1,...,x_n) = \varphi_2(x_1,...,x_n) = 0\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ L_f\varphi_k(x_1,...,x_n) = \varphi_{k+1}(x_1,...,x_n) = 0\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\ . \end{gathered} \tag{7}$$

Therefore, his definition is closely associated with the approach, which helps to get the indicated manifold. Whilst as in Definition 5 the way of invariant manifold σ formation may be different from the above-mentioned approach (see the relation (7)).

The definition 5 may be generalized in case of IR system. As this generalization can't be regarded as problematic, we will not analyze it. It should be only noted, that it will be different from the terms of A. Poincare [5], S.A. Chaplygin [25], T. Levi-Civita [24], stated in different periods of time. Indeed, these terms allow formulating the general definition [6]: the system from $m < n$ finite representations

$$\varphi_j(x_1,...,x_n) = 0 \quad (j = \overline{1,l}), \tag{8}$$

where $\varphi_j(x) \in C^1$, the rank of matrix $\left(\dfrac{\partial \varphi_j(x_1,...,x_n)}{\partial x_i}\right)$ is equal to l, is called the invariant, if it will comply with any solution of the system (1) $x = x(t;x^0)$, the initial value of $x(0;x^0)$ will comply with it.

Special attention should be drawn to the equations at the functions (8). The most common equations are indicated by S.A. Chaplygin [25]

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= \varphi_1^{m_{11}}(x_1,...,x_n)\lambda_{11}(x_1,...,x_n) + ... + \varphi_l^{m_{1l}}(x_1,...,x_n)\lambda_{1l}(x_1,...,x_n)\,, \\
\frac{d\varphi_2}{dt} &= \varphi_1^{m_{21}}(x_1,...,x_n)\lambda_{21}(x_1,...,x_n) + ... + \varphi_l^{m_{2l}}(x_1,...,x_n)\lambda_{2l}(x_1,...,x_n)\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{d\varphi_l}{dt} &= \varphi_1^{m_{l1}}(x_1,...,x_n)\lambda_{l1}(x_1,...,x_n) + ... + \varphi_l^{m_{ll}}(x_1,...,x_n)\lambda_{ll}(x_1,...,x_n)\,,
\end{aligned} \tag{9}$$

where $m_{ij} > 0$. It should be observed, that IR has been indicated as particular integrals by S.A. Chaplygin.

We can get the equations of T. Levi-Civita from (9) if $m_{ij} = 1 \quad \forall i, j = \overline{1,l}$. Let us put in the reduced form [23]

$$\frac{d\varphi_i(x_1,...,x_n)}{dt} = \sum_{j=1}^{l} \varphi_j(x_1,...,x_n)\lambda_{ij}(x_1,...,x_n) \qquad (i = \overline{1,l}). \tag{10}$$

In this paper we consider two problems.

Problem 1. The invariant manifold $M = \{x : V_i(x) = 0, i = 1,...,n-m; \quad m = \dim M\}$ of the system (1) is given. It is required to include it in some set of the integral manifolds N, in which in particular values of constants we get the given invariant manifold (IM).

Problem 2. Given: either one IR according to Levi-Civita or IR system according to Levi-Civita or S.A. Chaplygin. It requires on these relations to integrate the differential equations system in quadrature that results from (1).

2. INCLUSION OF THE INVARIANT MANIFOLD IN A SET OF INTEGRAL MANIFOLDS

Let the invariant manifold is to be described by the relations of class (8), i.e. indicating redifinitions [27] by the equations

$$V_i(x) = 0 \qquad (i = \overline{1, n-m}) . \tag{11}$$

We will use parametric approach, i.e. from the equalities (11) let us find

$$x = \varphi(\tau_1, ..., \tau_{n-m}) . \tag{12}$$

Where as $\tau_1, ..., \tau_{n-m}$ we can tale local coordinates.

Among various parameterizations we would like to highlight the one, in which the time t is determined as one of the characteristics. Let us call such parameterization as natural. It can be presented in the following way

$$x = x(t; x^0(\tau_1, ..., \tau_{n-m-1})) ,$$

where $x^0(\tau_1, ..., \tau_{n-m-1})$ is the parametrization of the initial state, which can assume the arbitrary rule, as well as function φ. Ordinary parameterization always exists in the neighborhood of the ordinary point; besides, it comprises the following property (a): for the values under consideration $\tau_1, ..., \tau_{n-m-1}$ vectors $f(x^0), \partial x^0 / \partial \tau_1, ..., \partial x^0 / \partial \tau_{n-m-1}$ are linearly independent. The usage of the ordinary parameterization for the manifolds, comprising the singular points, needs the additional review. Finally, special invariant manifold doesn't assume any ordinary parameterization. In particular it is explained by the fact, that the singular points are not transmitted with the time-change, so it is not possible to choose it as local coordinate. The issue about the invariant manifolds inclusion in the set of integrals is decided in different ways for special and non-special invariant manifolds.

Theorem 1. *The invariant manifold of $n-k$ dimension in the neighborhood of ordinary point is included in k-parametric set of the integral manifolds.*

Proof. Let us define the invariant manifold for the $n-k$ dimension in the neighborhood of a simple point by means of ordinary parameterization: $x = x(t; x^0(\tau_1, ..., \tau_{n-k-1}))$. We will put comprehensive $n-1$-dimensional manifolds of the initial states $X^0(\tau_1, ..., \tau_{n-1})$, which include, if $\tau_{n-k} = 0, ..., \tau_{n-1} = 0$, the initial first manifold: $X^0(\tau_1, ..., \tau_{n-k-1}, 0, ..., 0) = x^0(\tau_1, ..., \tau_{n-k-1})$ and satisfy the characteristic (a): vectors $f(X^0), \partial X^0 / \partial \tau_1, ..., \partial X^0 / \partial \tau_{n-1}$ are linearly independent. The set of solutions $x(t; X^0(\tau_1, ..., \tau_{n-1}))$ of the system (1) comprises the invariant manifold under consideration $x(t; x^0(\tau_1, ..., \tau_{n-k-1}))$ and coincides with it, if $\tau_{n-k} = 0, ..., \tau_{n-1} = 0$. If we solve the equations $x = x(t; X^0(\tau_1, ..., \tau_{n-1}))$ relatively to $t, \tau_1, ..., \tau_{n-1}$, we will get $n-1$ of independent integrals $\tau_i(x),\ i = 1, ..., n-1$. Under the construction k-dimensional set of integral manifolds $\tau_i(x) = c_i,\quad i = n-k, ..., n-1$ includes the invariant manifold under consideration, which can be singled out from it, if $c_i = 0,\quad i = n-k, ..., n-1$.

Theorem 2. *The singular invariant manifold of* $n-k$ *dimension is included in* k*-1-parametric set of the integral manifolds.*

Proof. Let us define the special invariant manifold of $n-k$ dimension by means of parameterization $x = x^0(\tau_1,...,\tau_{n-k})$. We shall put the comprehensive $(n-1)$-dimensional manifold of the initial states $X^0(\tau_1,...,\tau_{n-1})$, comprising as $\tau_{n-k+1}=0,...,\tau_{n-1}=0$ the special manifold $X^0(\tau_1,...,\tau_{n-k},0,...,0) = x^0(\tau_1,...,\tau_{n-k})$ and complying with the characteristic (*a*) if $\tau_{n-k+1}^2+...+\tau_{n-1}^2 \neq 0$, vectors $f(X^0)$, $\partial X^0/\partial\tau_{n-k+1},...,\partial X^0/\partial\tau_{n-1}$ are linearly independent. The set of solutions $x(t;X^0(\tau_1,...,\tau_{n-1}))$ of the system (1) complies with the condition $x(t;X^0(\tau_1,...,\tau_{n-k},0,...,0)) = x^0(\tau_1,...,\tau_{n-k})$. The points $X^0(\tau_1,...,\tau_{n-1})$ are ordinary points in the system (1) as $\tau_{n-k+1}^2+...+\tau_{n-1}^2 \neq 0$, therefore, if in the neighborhood of these points we solve the equations $x = x(t;X^0(\tau_1,...,\tau_{n-1}))$ relatively to $t,\tau_1,...,\tau_{n-1}$, we will get $(n-1)$ of independent integrals $\tau_i(x), i=1,...,n-1$. Under the construction $k-1$-dimensional set of integral manifolds $\tau_i(x)=c_i, i=n-k+1,...,n-1$, includes the invariant manifold. The integral manifold $\tau_i(x)=0 \quad (i=n-k+1,...,n-1)$, either coincides with the special invariant manifolds under consideration, or it is $(n-k+1)$-dimensional invariant manifold which contains the stated special manifold.

From the theorem 2 for $k=1$ we get the following statement.

Corollary 1. *The singular invariant manifold of* $n-1$ *dimension is included in the set of integral manifolds only in the cases, when the multiplier of Levi-Civita is identically equal to zero.*

The received conclusion about the inclusion of the invariant manifold inclusion in the set of the integrals indicates the important role of the invariant relations and particular solutions of the differential equations, namely, the possibility of its usage for integrals construction and the description of their structure. The solution of this problem for the invariant manifolds of the arbitrary dimensions is sufficiently difficult; however, for the manifolds of $n-1$ dimension the following property may be received.

Theorem 3. *Let for the system (1)* k*-values for the invariant manifolds of* $n-1$ *dimension are known, which are described by the equations* $V_i(x)=0,\ i=1,...,k$, *for which the multipliers of Levi-Civita in the equations*

$$\frac{dV_i}{dt} = \lambda_i V_i \tag{13}$$

complies with the condition $\alpha_1\lambda_1+...+\alpha_k\lambda_k=0$, α_i *– some numbers. Then the system (1) assumes the integral* $V_1^{\alpha_1}V_2^{\alpha_2}...V_k^{\alpha_k}=c$.

The theorem is proved by the direct checking by means of equations (2), (13).

It should be noted, that the statement of the theorem doesn't depend on the fact, whether the equation $V_i(x)=0$ has the solution or not, i.e. for the integral construction it is possible to use so-called "imaginary" solutions, which are known, but, for instance, they don't attach any importance in rigid body dynamics and physic sense. The consequence of the theorem 3 is a known theorem about integral construction as per two integrating multipliers. Indeed, the integrating multiplier $\mu(x)$ of the system (1) satisfies the equation $L_f\mu = \mu \operatorname{div} f$. So, for two solutions μ_1, μ_2 the conditions of the theorem 3 are realized if $\alpha_1 = -\alpha_2 = 1$, that will result to the known integral $\mu_1 \mu_2^{-1} = c$. Let us additionally indicate the property of the integrating multiplier: if the equation $\mu(x) = 0$ has the solution, then it determines the invariant manifold. Unfortunately, for the Euler–Poisson equations, the known integrating multiplier $\mu(x) = 1$ doesn't determine the invariant manifold.

The theorem 3 can be generalized in different ways and used the integrals structure description. So, if $k = 2$ and invariant manifold is of $n-2$ dimension, the classification of the singular points of the differential equations can be used at the plane.

Theorem 4. *Let the invariant manifolds are stated by the equations* $V_1 = 0,\ V_2 = 0$. *Depending on the type of the equations of Levi-Civita (12) it is possible to get the following types of the integrals:*

1. *hyperbolic integral:* $V_1 V_2 = c$: $L_f V_1 = \lambda V_1$, $L_f V_2 = -\lambda V_2$;
2. *dicritical integral* $V_1 / V_2 = c$: $L_f V_i = \lambda V_i \ (i = 1,2)$;
3. *elliptic integral:* $V_1^2 + V_2^2 = c$: $L_f V_1 = -\lambda V_2$, $L_f V_2 = \lambda V_1$;
4. *spiral integral:* $\ln(V_1^2 + V_2^2) + 2\operatorname{arctg}(V_2 / V_1) = c$:

$$L_f V_1 = -V_1 - V_2; \qquad L_f V_2 = V_1 - V_2.$$

The theorem is proved by the direct checking the equations (2) and (13).

The theorems 3, 4 and similar statements may be used for integrals construction from the known particular solutions and, vice versa, to receive the invariant manifolds from the known integrals. As an example, let us analyze the invariant manifolds and the integrals of one system, which demonstrates the complexity of invariant relations method.

$$\dot{x} = x, \qquad \dot{y} = x^2 + z^2 - a^2, \qquad \dot{z} = -\frac{1}{z}(2x^2 + z^2 - a^2). \tag{14}$$

The system (14) is considered in the domain $D = R^3 \setminus \{(x, y, z) : z = 0\}$. The special invariant manifolds are determined by the equations

$$x = 0, \qquad x^2 + z^2 - a^2 = 0, \qquad 2x^2 + z^2 - a^2 = 0 \tag{15}$$

and presented in the form of straight lines. The first two equations from (15) are independent and may be used to select of the special invariant manifold. To determine its functions $V_1 = x$, $V_2 = x^2 + z^2 - a^2$ let us put the equations of Levi-Civita

$$L_f V_1 = V_1; \qquad L_f V_2 = -2V_2.$$

Therefore, two-dimensional manifolds, defined by the equations $V_1 = 0$ and $V_2 = 0$, are invariant. For the multipliers of Levi-Civita the following condition $2\lambda_1 + \lambda_2 = 0$ is preserved, according to theorem 3 the system (14) has the integral

$I_1 = V_1^2 V_2 = x^2(x^2 + z^2 - a^2) = c_1$. Using the integral from the first two equations (14) we define $dy/dx = c_1/x^3$, i.e. this system possesses additional integral $I_2 = 2y + x^2 + z^2 = c_2$. So, for the system (14) the full set of integrals is indicated

$$I_1 = x^2(x^2 + z^2 - a^2) = c_1, \qquad I_2 = 2y + x^2 + z^2 = c_2 .$$

The phase space structure can be described in the following way. In the space there is a special invariant manifold, consisting of two straight lines $x = 0$, $z = \pm a$. This one-dimensional manifold is included in two-dimensional integral manifold $I_1 = 0$, consisting of two connected components: a plane $x = 0$ and a circular cylinder $x^2 + z^2 = a^2$, which are crossed at special manifold. Any other invariant manifold is integral and defined by the equations: $I_1 = c_1$, $I_2 = c_2$ – the path – one-dimensional invariant manifold; $F(I_1, I_2) = c$ – two-dimensional invariant manifold. The first integral I_1 gives us sufficiently clear picture of the phase space stratification on the integral planes: it is a set of cylindrical surfaces: $x^2(x^2 + z^2 - a^2) = c_1$.

3. APPLICATION OF THE RESULTS FOR THE SOLUTIONS ANALYSIS OF EULER–POISSON EQUATIONS

Let us consider the classical problem of a heavy rigid body motion with a fixed point. We shall put down the equations in the principal coordinate system [19, 20]

$$A_1\dot{\omega}_1 = (A_2 - A_3)\omega_2\omega_3 + \Gamma(e_2\nu_3 - e_3\nu_2), \qquad \dot{\nu}_1 = \omega_2\nu_3 - \omega_3\nu_2 \qquad (123). \qquad (16)$$

In formulas (16) two equations are indicated, four others we shall get from the data after the cyclic interchange of the numbers 1, 2, 3, that shows the symbol (123).

It should be observed, that in (16) ω_i – angular velocity components; ν_i – vertical vector components; A_i – principal inertia moments of rigid bodies; e_i – unit vector components, directed to a vector $\bar{r}_c = \overline{OC}$ (C – center of solid masses); $\Gamma = mg\,|\,\bar{r}_c\,|$, m – solid mass; g – acceleration of gravity.

The equations (16) have first integrals

$$A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 - 2\Gamma(e_1\nu_1 + e_2\nu_2 + e_3\nu_3) = 2E,$$

$$A_1\omega_1\nu_1 + A_2\omega_2\nu_2 + A_3\omega_3\nu_3 = k, \qquad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1. \qquad (17)$$

As it was mentioned above, the equations (16) have three general cases of the additional first integral existence. Let us define the conditions for the parameters and additional integral.

Euler's case ($\Gamma = 0$):

$$A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 = g_0^2. \qquad (18)$$

Lagrange's case: $e_1 = e_2 = 0$, $A_1 = A_2$:

$$\omega_3 = \omega_3^{(0)}. \qquad (19)$$

Kovalevskaya's case: $e_2 = e_3 = 0$, $A_1 = A_2 = 2A_3$:

$$(\omega_1^2-\omega_2^2+c\nu_1)^2+(2\omega_1\omega_2+c\nu_2)^2=k^2, \qquad (20)$$

where $c=\Gamma/A_3$.

Let us consider Euler's case and assume that $A_2>A_1>A_3$. It is easy to prove, that the following equations can be formed

$$L_fV_1=-\lambda V_1; \qquad L_fV_2=\lambda V_2, \qquad (21)$$

where

$$\lambda=\omega_1\sqrt{\frac{(A_2-A_1)(A_1-A_3)}{A_2A_3}}, \qquad (22)$$

$$V_1=\omega_2\sqrt{A_2(A_2-A_1)}+\omega_3\sqrt{A_3(A_1-A_3)}, \qquad V_2=\omega_2\sqrt{A_2(A_2-A_1)}-\omega_3\sqrt{A_3(A_1-A_3)}. \qquad (23)$$

Due to the theorem3 from (21)–(23) we have the first integral

$$I=V_1V_2=\mathrm{const}, \qquad (24)$$

in which owing to the equality $I=g_0^2-2A_1E$, received from (17), (23), (24), we shall find the integral of Euler (18). The integral (24) with zero value of the constant gives two invariant manifolds $V_1=0,\ V_2=0$.

Lagrange's case will be further considered, as the method of integral formation (19) directly from the equations of Euler–Poisson can't indicate any difficulties.

In case of S.V. Kovalevskaya from (16) we shall get the following equations

$$L_fV_1=\omega_3V_2, \quad L_fV_2=-\omega_3V_1, \quad V_1=\omega_1^2-\omega_2^2+c\nu_1, \quad V_2=2\omega_1\omega_2+c\nu_2, \qquad (25)$$

which form the integral of Kovalevskaya (20). Zero level of the integral (20) splits into two manifolds: $V_1=0,\quad V_2=0$, where V_i indicated in the system (25).

4. THE INTEGRAL UNDER HESS CONDITION

For generality the following item, after the above-mentioned equations, goes V. Hess solution. Hess conditions for the rigid bodies masses arrangement are used, as follows [9]

$$e_2=0, \qquad e_1\sqrt{A_1(A_2-A_3)}+e_3\sqrt{A_2(A_1-A_2)}=0. \qquad (26)$$

While preserving the equality (26) the equations (16) allow the invariant manifold

$$V=A_1e_1\omega_1+A_3e_3\omega_3=0, \qquad (27)$$

because for the function V, Levi-Civita equation takes place.

$$L_fV=\lambda V, \qquad \lambda=\frac{e_1(A_2-A_1)}{e_3A_3}\omega_2. \qquad (28)$$

Euler–Poisson equations, under Hess conditions, and the equations (27), (28) are easy to review in a special coordinate system, in which the first axis is directed to barycentric line. Let us use the components of angular momentum x,y,z and vertical components ν_1,ν_2,ν_3 as the variables. If we set by $\tilde{a}$ the gyro tenser, then under Hess conditions it can be presented as follows [27]

$$\tilde{a} = \begin{pmatrix} a & b_1 & 0 \\ b_1 & a_* & 0 \\ 0 & 0 & a_* \end{pmatrix} . \tag{29}$$

Euler–Poisson equations due to (29) are the following [31, 32]

$$\dot{x} = -b_1 zx , \quad \dot{y} = (a - a_*)zx + b_1 yz - \nu_3 \Gamma , \quad \dot{z} = -(a - a_*)yx + b_1(x^2 - y^2) + \nu_2 \Gamma , \tag{30}$$
$$\dot{\nu}_1 = a_* z \nu_2 - (a_* y + b_1 x)\nu_3 , \quad \dot{\nu}_2 = (a\ x + b_1 y)\nu_3 - a_* z \nu_1 , \quad \dot{\nu}_3 = (a_* y + b_1 x)\nu_1 - (ax + b_1 y)\nu_2 .$$

Let us put down the first integrals of the equations (30)

$$ax^2 + a_*(y^2 + z^2) + 2b_1 yx \ - 2\nu_1 \Gamma = 2E , \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1 , \quad x\nu_1 + y\nu_2 + z\nu_3 = k . \tag{31}$$

From the first equation of the system (30) comes IR of Hess

$$V \ = x = 0 \tag{32}$$

and Levi-Civita equation

$$L_f V \ = -b_1 z V \ . \tag{33}$$

In Euler's case from the system (30) in addition to the equation (33) we get

$$L_f V_1 = b_1 z V_1 , \qquad V_1 = (a - a_*)x + 2b_1 y \tag{34}$$

and thus on the basis of the theorem 3 we shall get the first integral

$$I = VV_1 = x[(a - a_*)x + 2b_1 y] = \text{const} . \tag{35}$$

Euler's integral $x^2 + y^2 + z^2 = g_0^2$ will be received from the formula $I = -a_* g_0^2 + 2E$, where I is defined by the relation (35), and $2E$ – by the first equality from the system (31).

In Lagrange's case $b_1 = 0$, so Lagrange integral

$$x = \text{const} \tag{36}$$

may be received from the formula (35).

Let us turn into consideration of the equations (30) with integrals (31) in general case. Based on the theorem 3 due to the equation (33) and the first equation of the system (34)

$$L_f V_1 = b_1 z V_1 , \tag{37}$$

the initial system (30) accepts the first integral

$$I = xV_1 \ . \tag{38}$$

It is only necessary to indicate, that the integral (38) is not a combination of the integrals (31). For this, we shall use the properties of a special invariant manifold of Hess gyroscope, which is represented as a manifold (including the integral $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$), consisting of six curves $S_1,...,S_6$ [27]

S_1, S_2:

$$x = 0 ,$$
$$\Gamma^2 a_*^2 = b_1^2 y^2 (z^2 a_*^2 + (a_*^2 + b_1^2) y^2) ,$$
$$\Gamma \nu_1 = b_1^2 y^2 / a_* , \qquad \Gamma \nu_2 = b_1\, y^2 , \qquad \Gamma \nu_3 = b_1\, y\ z ;$$

$S_3,...,S_6$:

$$z = 0 ,$$

$$[(a-a_*)yx - b_1(x^2-y^2)]^2[(ax+b_1y)^2+(a_*y+b_1x)^2] = \Gamma^2(a_*y+b_1x)^2\,, \qquad (39)$$

$$\Gamma\nu_1 = \frac{ax+b_1y}{a_*y+b_1x}[(a-a_*)yx - b_1(x^2-y^2)]\,,$$

$$\Gamma\nu_2 = (a-a_*)yx - b_1(x^2-y^2)\,, \qquad \nu_3 = 0\ .$$

The integral $I = xV_1 = c$ if $c = 0$ defines the integral manifold, which includes the curves S_1, S_2 of a special manifold. We shall verify that any combination of the integrals (31), which defines the integral manifold, including the curves S_1, S_2, cannot have the same structure.

Lemma. *Any combination of the integrals (31), which defines the integral manifold, including the curves* S_1, S_2 *of a special invariant manifold does not contain x, as a multiplier.*

Proof. By means of formulas (39) we shall exclude the variables ν_i from the integrals (31) of energies and surfaces

$$\Gamma k = b_1 y(y^2+z^2)\,, \qquad (40)$$

$$2a_*E = a_*^2(y^2+z^2) - 2b_1^2y^2\,. \qquad (41)$$

Excluding from the equation (41) the expression y^2+z^2 by means of formula (40); and the variable z by the second equation (39), we get the following equations

$$2b_1y(b_1^2y^2+a_*E) - a_*^2\Gamma k = 0\,, \qquad (42)$$

$$3b_1^2y^2 - \Gamma^2a_*^2(b_1^2y^2)^{-1} + 2a_*E = 0\,. \qquad (43)$$

Excluding from the equations (42), (43) the variable y, we receive the desired expression for the combination of integrals

$$-E^4 + 2\kappa_*E^3 + 2\Gamma^2E^2 - 18\Gamma^2\kappa_*E + 27\Gamma^2\kappa_*^2 - \Gamma^4 = 0\,, \qquad (44)$$

where $4\kappa_* = a_*k^2$. Its explicit expression in the initial variables, received by means of computer program for the analytical calculations, shows, that the equation (44) does not contain x as a multiplier.

Before the formulation of the final conclusion we shall indicate that for Hess gyroscope, except Hess solution, Dokshevich solution also exists. Let us enter his note in special axes, following the treatise [21]:

$$\Gamma\nu_1 = \alpha_0x^2+\alpha_2\,, \qquad \Gamma\nu_2 = \beta_0x^2+\beta\ , \qquad \Gamma\nu_3 = \gamma_0xz\,, \qquad b_1y = (p_0-a)x + p_2/x\,, \qquad (45)$$

$$(\alpha_0x^2+\alpha_2)^2 + (\beta_0x^2+\beta)^2 + \gamma_0^2x^2z^2 = \Gamma^2\,.$$

The explicit expressions for the coefficients, contained in formulas (45), may be found in the research work [21].

The above-mentioned arguments allow making the following conclusion.

Statement. *Euler–Poisson equations under Hess conditions (29) have the additional integral of* $I = xV$ *, where* V *is the solution for the equation* $L_fV = b_1zV$ *. The special case of this integral are Euler and Lagrange integrals. Hess and Dokshevich solutions define invariant manifolds of the integral manifold* $I = xV = c$ *.*

With a view of integral structure description, under Hess conditions, let us consider the integrals of the system (30), linearized near the curve (39). Let us introduce the disturbances as per formulas

$$x = x_1\,,\quad y = y_0 + x_2\,,\quad z = z_0 + x_3\,,\quad \nu_1 = \nu_{10} + x_4\,,\quad \nu_2 = \nu_{20} + x_5\,,\quad \nu_3 = \nu_{30} + x_6\,.$$

The values y_0, z_0 comply with the second equation from (39), and $\nu_{10}, \nu_{20}, \nu_{30}$ are expressed by y_0, z_0 as per three last formulas from (39).

The equations of linear approximation are the following

$$\dot{x}_1 = -b_1 z_0 x_1\,,\quad \dot{x}_2 = (a - a_*) z_0 x_1 + b_1 z_0 x_2 + b_1 y_0 x_3 - \Gamma x_6\,,$$

$$\dot{x}_3 = (a_* - a) y_0 x_1 - 2b_1 y_0 x_2 + \Gamma x_5\,,$$

$$\dot{x}_4 = -\frac{b_1 y_0 z_0}{\Gamma}(b_1 x_1 + a_* x_2) + \frac{a_* b_1 y_0^2}{\Gamma} x_3 + a_* z_0 x_5 - a_* - y_0 x_6\,,$$

$$\dot{x}_5 = \frac{b_1 y_0 z_0}{\Gamma}(a x_1 + b_1 x_2) - \frac{b_1^2 y_0^2}{\Gamma} x_3 - a_* z_0 x_4 + b_1 y_0 x_6\,,\ \dot{x}_6 = \frac{b_1 y_0^2}{\Gamma a_*}(b_1^2 - a a_*) x_1 + a_* y_0 x_4 - b_1 y_0 x_5$$

To the eigenvalues $\lambda_1 = -b_1 z_0$, $\lambda_2 = b_1 z_0$ of the characteristic equation of this system the following integral is corresponded

$$x_1[(a_*(a_*^{\,2} + b_1^2)(a_* - a)(z_0^2 + y_0^2) + b_1^2 y_0^2 (b_1^2 - 2a_*^{\,2} - 3aa_*))x_1 -$$

$$- 2a_* b_1 (a_*^{\,2} + b_1^2)(z_0^2 + 2y_0^2)x_2 - 2a_* b_1\, y_0 z_0 (a_*^{\,2} + b_1^2)x_3 +$$

$$+ 6a_*^{\,2} + b_1\, y_0 \Gamma x_4 + 2a_* y_0 (a_*^{\,2} - 2b_1^2)\Gamma x_5 + 2a_* z_0 (a_*^{\,2} + b_1^2)\Gamma x_6] = \text{const}\,.$$

This integral is the first member of the Hess integral decomposition near the curve (39).

Finally, it should be noted that Dokshevich solution belongs to integral manifold with a non-zero constant of Hess integral, which is testified by the availability of x^{-1} member in formulas (45).

Comment. As a result, the local way of invariant manifold inclusion in the set of integral manifolds has been indicated. It was demonstrated on the proof of the statement about the first integral existence of Euler–Poisson equations under Hess conditions. The first integrals construction, which states the invariant manifold on function by means of Levi-Civita equations, may be realized in a global way (theorem 3).

5. THE INTEGRATION OF DYNAMICS DIFFERENTIAL EQUATIONS WITH $n-2$ FIRST INTEGRALS

Rigid body dynamics equations are autonomous and comply with the following conditions: the right sides of the differential equations do not depend on that variable, in reference to which its left side has been written (see the equations (16))

$$\dot{x}_i = X_i(x_1,\dots,x_n)\,,\qquad \frac{\partial X_i(x_1,\dots,x_n)}{\partial x_i} = 0\,. \tag{46}$$

Here right side satisfies the conditions $\dfrac{\partial X_i}{\partial x_j} \in C^{k_1}$ in the domain $E_n \subset R_n$ $(i = \overline{1,n})$.

The equations (46) have a range of the first integrals (for better understanding let us change the identifications comparing with (21))

$$\varphi_i(x_1,...,x_n) = c_i \qquad (i = \overline{1,k}), \tag{47}$$

where functions $\varphi_i(x_1,...,x_n) \in C^1$ and rank of matrix $\left(\dfrac{\partial \varphi_i(x_1,...,x_n)}{\partial x_j} \right)$ is equal to k .

Let us put the new variables $y_1,..., y_n$ in the point $x^0 = (x_1^0,...,x_n^0) \in E_n$ and its neighborhood

$$x_i = g_i(y_1,..., y_n) \qquad (i = \overline{1,n}), \tag{48}$$

where $y = (y_1,..., y_n)^T \in E_n^* \in R_n$. Jacobi replacement (48) in the domain E_*

$$D(y_1,..., y_n) = \left| \frac{\partial g_i(y_1,..., y_n)}{\partial y_j} \right| \neq 0 .$$

So the replacement (48) is inversible

$$y_i = G_i(x_1,...,x_n), \tag{49}$$

and $y_i^0 = G_i(x_1^0,...,x_n^0)$. Due to $D \neq 0$ the condition is realized

$$D^*(x_1,...,x_n) = \left| \frac{\partial G_i(x_1,...,x_n)}{\partial x_j} \right| = \frac{1}{\langle D(y_1,..., y_n) \rangle} \neq 0, \tag{50}$$

where the identification is taken into account

$$\langle D(y_1,..., y_n) \rangle = D(G_1(x_1,...,x_n),...,G_n(x_1,...,x_n)). \tag{51}$$

Obviously, due to $D \neq 0$ the value D^* from (50) is not transformed in zero.

With the replacement (48) the equations (46) are transformed to the system

$$\dot{y}_i = Y_i(y_1,..., y_n) \qquad (i = \overline{1,n}). \tag{52}$$

For the equations (46), (52) Jacobi identity takes place

$$\sum_{i=1}^{n} \frac{\partial X_i(x_1,...,x_n)}{\partial x_i} = \left\langle \frac{1}{D(y_1,..., y_n)} \sum_{j=1}^{n} \frac{\partial D(y_1,..., y_n) Y_j(y_1,..., y_n)}{\partial y_j} \right\rangle. \tag{53}$$

Jacobi theorem is known: *Let the system (46) has* $n-2$ *first integrals* $\varphi_i(x_1,...,x_n) = c_i \quad (i = \overline{1,n-2})$. *So the system (52) allows the additional first integral and it is integrated in quadrature.*

This theorem proof is based on the usage of the first integrals (47) in the replacement (49):

$$y_k = \varphi_k(x_1,...,x_n) = c_k \qquad (k = \overline{1,n-2}), \quad y_{n-1} = x_{n-1}, \quad y_n = x_n . \tag{54}$$

So due to (54) the system (52) is presented in the following way

$$\dot{y}_k = 0 \ \ (k = \overline{1,n-2}), \quad \dot{y}_{n-1} = Y_{n-1}(c_1,...,c_{n-2}, y_{n-1}, y_n), \quad \dot{y}_n = Y_n(c_1,...,c_{n-2}, y_{n-1}, y_n). \tag{55}$$

This system due to Jacobi condition (53) has the integrating multiplier

$$M(c_1,...,c_{n-2}, y_{n-1}, y_n) = D(c_1,...,c_{n-2}, y_{n-1}, y_n) . \tag{56}$$

So it is integrated in quadrature, as two last equations from (55) allow the existence of additional integral

$$\varphi_{n-1}(c_1,...,c_{n-2},y_{n-1},y_n)=c_{n-1}\,. \qquad (57)$$

So the relation (57) allows getting the solution for the equations (55)

$$y_k=c_k\,,\quad (k=\overline{1.n-2})\,,\qquad y_{n-1}=y_{n-1}(c_1,...,c_{n-1},t)\,,\qquad y_n=y_n(c_1,...,c_{n-1},t)\,.$$

The equations (46) solutions (46) we can take from (48).

In classic problem three cases of the additional (forth) integral existence in Euler–Poisson equation are known. These are the cases of Euler, Lagrange and Kovalevskaya.

There are no any other additional algebraic and analytical integrals [4-8]. Therefore the task to integrate the dynamic equations at invariant relations is challenging.

6. THE INTEGRATION OF THE EQUATIONS (46) BASED ON $n-3$ FIRST INTEGRALS AND ONE IR ACCORDING TO LEVI-CHIVITA

Let the equations (46) have $n-3$ first integrals:

$$\varphi_k(x_1,...,x_n)=c_k \qquad (k=\overline{1,n-3}) \qquad (58)$$

and one invariant relation (5). Let us introduce the new variables

$$y_i=\varphi_i(x_1,...,x_n) \qquad (i=\overline{1,n-3})\,,$$
$$y_{n-2}=\varphi(x_1,...,x_n)\,,\qquad y_{n-1}=x_{n-1}\,,\qquad y_n=x_n\,. \qquad (59)$$

So based on (6), (58), (59) the system (46) is reduced to

$$\dot y_i=0\quad (i=\overline{1,n-3})\,,\qquad \dot y_{n-2}=y_{n-2}Y_{n-2}(y_1,...,y_n)\,,$$
$$\dot y_{n-1}=Y_{n-1}(y_1,...,y_n)\,,\qquad \dot y_n=Y_n(y_1,...,y_n)\,. \qquad (60)$$

We shall consider the system (60) integration at invariant relation $\varphi(x_1,...,x_n)=0$, viz. assume in (60) $y_{n-2}=0$ and bear in mind the first integrals (58). Then from (60) we shall get [28]

$$y_i=c_i\quad (i=\overline{1,n-3})\,,\qquad y_{n-2}=0\,,$$
$$\dot y_{n-1}=Y_{n-1}(c_1,...,c_{n-3},0,y_{n-1},y_n)\,,\qquad \dot y_n=Y_n(c_1,...,c_{n-3},0,y_{n-1},y_n)\,. \qquad (61)$$

The equations (61) have been received in the assumption, that

$$\Delta^*(x_1,...,x_n)=\left|\frac{\partial y_k}{\partial x_i}\right|\neq 0 \qquad (i=\overline{1,n};\ k=\overline{1,n})\,, \qquad (62)$$

where $x=(x_1,...,x_n)^T\in E_n$, a y_i $(i=\overline{1,n})$ complies with the relations (59). Due to (62) the condition takes place

$$\Delta(y_1,...,y_n)=\left|\frac{\partial x_i}{\partial y_j}\right|=\frac{1}{\left\langle\Delta^*(x_1,...,x_n)\right\rangle}\neq 0 \qquad (i=\overline{1,n};\ j=\overline{1,n})\,. \qquad (63)$$

Here $x_i(y_1,...,y_n)$ are received from the equality (59). Taking into account the equations (60), (61), out of condition (53) we get

$$\Delta(c_1,...,c_{n-3},0,y_{n-1},y_n)Y_{n-2}(c_1,...,c_{n-3},0,y_{n-1},y_n)+$$

$$+\frac{\partial\Delta(c_1,...,c_{n-3},0,y_{n-1},y_n)Y_{n-1}(c_1,...,c_{n-3},0,y_{n-1},y_n)}{\partial y_{n-1}}+ \qquad (64)$$
$$+\frac{\partial\Delta(c_1,...,c_{n-3},0,y_{n-1},y_n)Y_n(c_1,...,c_{n-3},0,y_{n-1},y_n)}{\partial y_n}=0 .$$

As the condition (63) is to be realized, so the following item is resulted out of the condition (64), that due to the availability of the equality

$$Y_{n-2}(c_1,...,c_{n-3},0,y_{n-1},y_n)=0 . \qquad (65)$$

The function $\Delta(c_1,...,c_{n-3},0,y_{n-1},y_n)$ will be the integrating multiplier of the two last equations of the system (61). Therefore, the sufficient condition (65) for the system (46) integration at the invariant relation $\varphi(x_1,...,x_n)=0$ has been defined. The further system (61) integration is realized similar to the case of $n-2$ first integrals existence in the system (64).

7. THE EQUATIONS (46) INTEGRATION, INVOLVING $n-4$ FIRST INTEGRALS AND TWO INVARIANT RELATIONS ACCORDING TO LEVI-CIVITA

Let us consider the case, when the system (46) has $n-4$ first integrals

$$\varphi_i(x_1,...,x_n)=c_i \qquad (i=\overline{1,n-4}) \qquad (66)$$

and two IR according to Levi-Civita

$$\varphi(x_1,...,x_n)=0 , \qquad g(x_1,...,x_n)=0 . \qquad (67)$$

If we consider, that the derivative coefficients from IR (67) are the analytical functions of the variables $\varphi=\varphi(x_1,...,x_n)$, $g=g(x_1,...,x_n)$, then IR (67) complies with the equation of a class (10) [24]

$$\begin{aligned}\dot\varphi&=\lambda_1(x_1,...,x_n)\varphi(x_1,...,x_n)+\lambda_2(x_1,...,x_n)g(x_1,...,x_n) ,\\ \dot g&=\lambda_3(x_1,...,x_n)\varphi(x_1,...,x_n)+\lambda_4(x_1,...,x_n)g(x_1,...,x_n) .\end{aligned} \qquad (68)$$

Adding in (46) the new variables $y_1,...,y_n$ as per formulas

$$\begin{aligned}&y_i=\varphi_i(x_1,...,x_n)=c_i \quad (i=\overline{1,n-4}) , \qquad y_{n-3}=\varphi(x_1,...,x_n) ,\\ &y_{n-2}=g(x_1,...,x_n) , \qquad y_{n-1}=x_{n-1} , \qquad y_n=x_n ,\end{aligned} \qquad (69)$$

where c_i – constants of the first integrals, we shall put down the reduced equations, using (66), (68):

$$\begin{aligned}&\dot y_i=0 \quad (i=\overline{1,n-4}) ,\\ &\dot y_{n-3}=y_{n-3}Y_{n-3}^{(1)}(y_1,...,y_n)+y_{n-2}Y_{n-3}^{(2)}(y_1,...,y_n) ,\\ &\dot y_{n-2}=y_{n-3}Y_{n-2}^{(1)}(y_1,...,y_n)+y_{n-2}Y_{n-2}^{(2)}(y_1,...,y_n) ,\\ &\dot y_{n-1}=Y_{n-1}(y_1,...,y_n) , \qquad \dot y_n=Y_n(y_1,...,y_n) .\end{aligned} \qquad (70)$$

Let us consider the system (70) integration at IR (67). Assuming in Jacobi formula (53) that $y_i=c_i \quad (i=\overline{1,n-4})$, $y_{n-3}=0$, $y_{n-2}=0$, we shall get

$$\Delta(c_1,...,c_{n-4},0,0,y_{n-1},y_n)[Y^{(1)}_{n-3}(c_1,...,c_{n-4},0,0,y_{n-1},y_n)+$$
$$+Y^{(2)}_{n-2}(c_1,...,c_{n-4},0,0,y_{n-1},y_n)]+$$
$$+\frac{\partial\Delta(c_1,...,c_{n-4},0,y_{n-1},y_n)Y_{n-1}(c_1,...,c_{n-4},0,y_{n-1},y_n)}{\partial y_{n-1}}+ \tag{71}$$
$$+\frac{\partial\Delta(c_1,...,c_{n-4},0,0,y_{n-1},y_n)Y_n(c_1,...,c_{n-4},0,y_{n-1},y_n)}{\partial y_n}=0.$$

In formula (71) $\Delta=\left|\frac{\partial x_i}{\partial y_j}\right|$ $(i,j=\overline{1,n})$ and

$$Y^{(1)}_{n-3}(y_1,...,y_n)=\langle\lambda_1(x_1,...,x_n)\rangle,$$
$$Y^{(2)}_{n-2}(y_1,...,y_n)=\langle\lambda_4(x_1,...,x_n)\rangle.$$

If in equality (71) we set

$$Y^{(1)}_{n-3}(c_1,...,c_{n-4},0,0,y_{n-1},y_n)+Y^{(2)}_{n-2}(c_1,...,c_{n-4},0,0,y_{n-1},y_n)=0, \tag{72}$$

then from two last equations of the system (70) and the equality (71) it is implied that the integrating multiplier of the equations (70) at IR (67) may be taken as follows

$$M^*(c_1,...,c_{n-4},0,0,y_{n-1},y_n)=\Delta(c_1,...,c_{n-4},0,0,y_{n-1},y_n). \tag{73}$$

Therefore, we received the sufficient condition (72) of the solutions existence for the equations (46) at two IR (in quadrature) [28], based at formula (53). The other cases of the equations (46) integration at IR are analyzed in a similar way.

Examples.

For the demonstration of the received results, first of all, let us make an example of the three-dimensional system, which allows having one invariant relation.

Assume that for the variables x_1,x_2,x_3 the following system has been given

$$\dot{x}_1=\alpha_0^2+\alpha_0(x_2+x_3)+2x_2x_3+\frac{1}{2}(1-a)x_2^2,$$
$$\dot{x}_2=\alpha_0(x_1+x_3)+2x_1x_3+\frac{1}{2}(1-b)x_1^2+x_3^2, \tag{74}$$
$$\dot{x}_3=\alpha_0(x_1+x_2)+2x_1x_2+\frac{1}{2}(1+b)x_1^2+\frac{1}{2}(1+a)x_2^2,$$

where α_0,a,b – fixed characteristics. Obviously, the right-hand sides of the equations (74) comply with the conditions $\frac{\partial X_i}{\partial x_i}=0$. This system accepts the invariant relation

$$\alpha_0+x_1+x_2+x_3=\varphi(x_1,x_2,x_3)=0, \tag{75}$$

so for the function from (75) due to (74) we have the following equation

$$\dot{\varphi}(x_1,x_2,x_3)=\varphi^2(x_1,x_2,x_3). \tag{76}$$

Let us enter in system (74) the new following variables $y_1=x_1$, $y_2=x_2$, $y_3=\varphi(x_1,x_2,x_3)$. Then on the basis of (74), (76) we have

$$\dot{y}_1 = -\alpha_0 y_1 - 2\alpha_0 y_2 + \alpha_0 y_3 - 2y_1 y_2 - \frac{(a+3)}{2} y_2^2 + 2y_2 y_3,$$

$$\dot{y}_2 = \alpha_0 y_2 - \alpha_0 y_3 - \frac{(1+b)}{2} y_1^2 - 2y_2 y_3 + y_2^2 + y_3^2, \qquad \dot{y}_3 = y_3^2. \tag{77}$$

It is obvious, that for the system (74) the condition (65) has been maintained. For the system

$$\dot{y}_1 = -\alpha_0 y_1 - 2\alpha_0 y_2 - 2y_1 y_2 - \frac{(a+3)}{2} y_2^2, \qquad \dot{y}_2 = \alpha_0 y_2 - \frac{(1+b)}{2} y_1^2 + y_2^2, \tag{78}$$

which comes from a system (77) as $y_3 = 0$, the integrating multiplier is 1. The first integral of the system (78) has the following form

$$\alpha_0 y_1 y_2 + y_1 y_2^2 - \frac{(1+b)}{6} y_1^3 + \alpha_0 y_2^2 + \frac{(a+3)}{6} y_2^3 = c, \tag{79}$$

where c – arbitrary constant. By means of the integral (79) the system (78) is integrated in quadratures.

By means of the system (74) it is easy to make an example, when the system has a first integral and one invariant relation. Let us consider the system of the 4-th rank, consisting of three equations (74) and one equation [28]

$$\dot{x}_4 = \alpha_0^2 + 2\alpha_0(x_1 + x_2 + x_3) - 2(x_1 x_2 + x_1 x_3 + x_2 x_3) + x_1^2 + x_2^2 + x_3^2. \tag{80}$$

The equations system (74), (80) allows the first integral

$$x_4 - x_1 - x_2 - x_3 = c,$$

where c – arbitrary constant. While replacing $y_1 = x_1$, $y_2 = x_2$, $y_3 = \alpha_0 + x_1 + x_2 + x_3$, $y_4 = x_4 - x_1 - x_2 - x_3$ the system (74), (80) is modified to the system (77) and $\dot{y}_4 = 0$. As $y_3 = 0$ it is integrated and assumes two integrals: the integral (79) and $y_4 = c_1$. The result, received in these examples, is stipulated by the condition (72).

8. EQUATIONS (46) INTEGRATING AT S.A. CHAPLYGIN INVARIANT RELATIONS

S.A. Chaplygin reviewed the issue about the integrating multiplier of the equations (46), which allow l invariant relations [25]

$$\varphi_1(x_1,...,x_n) = 0, ..., \varphi_l(x_1,...,x_n) = 0 \tag{81}$$

and $n-2-l$ first integrals

$$\varphi_{l+1}(x_1,...,x_n) = c_{l+1}, ..., \varphi_{n-2}(x_1,...,x_n) = c_{n-2}. \tag{82}$$

It is expected that the functions, which are placed on the left sides of the equalities (81), (82), are continuously differentiable and $\operatorname{rank}\left(\frac{\partial \varphi_j(x_1,...,x_n)}{\partial x_i}\right)$ $(j = \overline{1, n-2};\ i = \overline{1, n})$ is equal to $n-2$. Herewith, the relations (81) were called as particular integrals by S.A. Chaplygin and he put down the equations (9), in which the values $m_{ij} > 0$. As mentioned above, the system (9) in S.A. Chaplygin's case is more generalized, than the system (10). However, this exception may make difficult the application of the results, as in general case the right sides of the equations (9) are not considered as analytic functions of the variables $\varphi_1,...,\varphi_l$, which set the IR (81).

If, following [25], we are to arrange the transformation of the variables $x_1,...,x_n$ according to the formulas

$$y_1 = \varphi_1(x_1,...,x_n),\ ...,\ y_l = \varphi_l(x_1,...,x_n), \qquad y_{l+1} = \varphi_{l+1}(x_1,...,x_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \tag{83}$$
$$y_{n-2} = \varphi_{n-2}(x_1,...,x_n), \qquad y_{n-1} = x_{n-1}, \qquad y_n = x_n,$$

then the system (64) due to (9), (82), (83)will be reduced to

$$\dot{y}_i = \sum_{j=1}^{l} y_i^{m_{ij}} \Lambda_{ij}(y_1,...,y_l,c_{l+1},...,c_{n-2},y_{n-1},y_n) \qquad (i=\overline{1,l}),$$
$$y_{l+1} = c_{l+1},\ \ldots,\ y_{n-2} = c_{n-2}, \tag{84}$$
$$\dot{y}_{n-1} = Y_{n-1}(y_1,...,y_l,c_{l+1},...,c_{n-2},y_{n-1},y_n),$$
$$\dot{y}_n = Y_n(y_1,...,y_l,c_{l+1},...,c_{n-2},y_{n-1},y_n).$$

Having transformed the system (46) by means of the replacement (83) and indicated by $D = \left|\frac{\partial y_k}{\partial x_i}\right|$ $(i,k=\overline{1,n})$ Jacobi form of a backward transformation, for the application of Jacobi form (53) S.A. Chaplygin put down the following expressions [25]

$$\frac{\partial DY_i}{\partial y_i} = m_{ii} y_i^{m_{ii}-1} D\Lambda_{ii} + y_i^{m_{ii}} \frac{\partial D\Lambda_{ii}}{\partial y_i}. \tag{85}$$

By means of (85) the formula (53) may be reduced to

$$\sum_{i=1}^{l}\left(m_{ii} y_i^{m_{ii}-1} D\Lambda_{ii} + y_i^{m_{ii}} \frac{\partial D\Lambda_{ii}}{\partial y_i}\right) + \frac{\partial DY_{n-1}}{\partial y_{n-1}} + \frac{\partial DY_n}{\partial y_n} = 0. \tag{86}$$

Considering the system (84) integration at IR (81), S.A. Chaplygin assumes that $m_{ii} > 1$ in the equation (9). That is similar to the approach, which was applied during the integration of the dynamics equations, allowing $n-4$ first integrals and two IR, we can verify the two last equations from the system (84) allow the integrating multiplier $D(0,...,0,c_{l+1},...,c_{n-2},y_{n-1},y_n)$, i.e. the system (84) is integrated in quadratures at IR (81) and at first integrals (82). Thereby, S.A. Chaplygin condition is formed on the basis of the statement, that m_{ii} dimensions of φ_i functions of the diagonal system elements (9) are above one.

As stated in the research work by A.V. Maznev [33], the result of S.A. Chaplygin [25] may be generalized, by assuming that some $m_{ii} > 1$ $(i = n_1, n_2, ..., n_m,\ m < l)$, and for others $m_{\sigma\sigma} = 1$ and

$$\sum_{(\sigma \neq n_1,...,\sigma \neq n_l)} \Lambda_{\sigma\sigma}(0,0,...,0,c_{l+1},...,c_{n-2},y_{n-1},y_n) = 0.$$

Then again as an integrating multiplier of the system (84) the function $D(0,...,0,c_{l+1},...,c_{n-2},y_{n-1},y_n)$ can be held. So the supplement to the results of S.A. Chaplygin lies herein.

9. CONCLUSIONS

In this paper two problems, concerning the invariant relations method development in the rigid bodies dynamics equations, have been reviewed. There problems are closely associated with the issues of ordinary differential equations integration, which assume the first integrals existence and invariant relations. The applicability of this research in the area of mechanics was highlighted in the articles of the following outstanding scientists: A. Poincare, E. Husson, P. Burgatti, A.M. Lyapunov, S.A. Chaplygin and others. Due to the approval of nonexistence of algebraic and analytical integrals of the Euler–Poisson equations in general case (A. Poincare, V.V. Kozlov and others) the problems, which were solved here, allow using the new approaches for the further investigation of integral manifolds properties in the dynamic equations and thereby establish the general patterns in the motion of mechanic systems. For rigid body dynamics it is very important to attract the scientists attention to the investigation of the new fourth integral (A.M. Kovalev [27, 34]) which is obtained under the Hess conditions.

REFERENCES

1. Euler L. *Découverte d'un nouveau principe de mécanique,* Histoire de l'Académie Royale des Science et Belles Lettres, Berlin, vol. 6 (1750, 1752), p. 185–217.
2. Lagrange G. *Analytical mechanics*, Gostekhizdat, Moscow; Leningrad 1950, vol. 2, 440 p.
3. Kovalevskaya S.V. *The problem about solid rotation in a neighborhood of a fixed point.* – in book: Kovalevskaya S.V. Scientific works, USSR Academy of Science edition, Moscow 1948, p. 153–220.
4. Hüsson E. *Recherche des integrales algebriques dans le mouvement d'un solide pesant autour d'un point fixe*, Ann. Fac. sci. Univ. Toulouse 1906, ser. 2, vol. 9, p. 73–152.
5. Burgatti P. *Dimostrazione della non esistenza d'integrali algebrici (oltre i noti) nel Problema del moto d'un corpo pesante intorno a un punto fisso*, Rend. Circ. mat. Palermo 1910, vol. 29, p. 369–377.
6. Poincare A. *New methods of celestial mechanics*. Collected work. In 3 volumes.vol.1. Moscow 1971, 771 p.
7. Lyapunov A.M. *About one property of differential equations of heavy rigid body motion, having a fixed point.* col.edition vol.1. USSR Academy of Science edition, Moscow 1954, p.402–417.
8. Kozlov V.V. *Integrability and non-integrability in Hamilton mechanics,* Uspekhi matematichskikh nauk, vol.38, ed. 1 (1983), p. 3–67.
9. Hess W. *Über die Euler'schen Bewegungsgleichungen und über eine neue partikülare Lösung des Problems der Bewegung eines starren schweren Körpers um einen festen Punkt*, Math. Ann. 37. H. 2 (1890), s. 153–181.
10. Bobylev D.K. *About one particular solution of differential equations for a heavy rigid body rotation around a fixed point*, The department of Physics in the Society of Natural Science Amateurs, vol. 8, No. 2 (1896), p. 21–25.
11. Goryachev D.N. *New particular solution of the problem about a heavy rigid body motion around a fixed point*, The department of Physics and Mathematics in the Society of Natural Science Amateurs, vol. 10, ed. 1 (1899), p. 23–24.

12. Goryachev D.N. *About a heavy rigid body motion around a fixed point in case of $A = B = 4C$*, M. Mathematics Club Publications, vol. 21, ed. 3 (1900), p. 431–438.
13. Steklov V.A. *One case of a heavy rigid body motion, having a fixed point*, The department of Physics in the Society of Natural Science Amateurs, vol. 8, ed. 2 (1896), p. 19–21.
14. Steklov V.A. *New particular solution of the problem about a heavy rigid body motion having a fixed point*, The department of Physics in the Society of Natural Science Amateurs, vol. 10, No. 1 (1899), p. 1–3.
15. Chaplygin S.A. *New case of heavy rigid body rotation, braced on one point*, the department of Physics in the Society of Natural Science Amateurs, vol. 10, ed. 2 (1901), p. 32–34.
16. Chaplygin S.A. *New particular solution of the problem about a heavy rigid body motion around a fixed point*, the department of Physics in the Society of Natural Science Amateurs, vol. 12, ed. 1 (1904),p.1–4.
17. Kowalewsky N. *Eine neue partikulare Lösung der differentialgleichungen der Bewegung eines schweren starren Körpers um einen festen Punkt*, Math. Ann. vol. 65 (1908), s. 528–537.
18. Grioli G. *Esistenza e determinazione delle prezessioni regolari dinamicamente possibili per un solido pesanto asimmetrico*, Ann. Mat. pura ed Appl., vol. 26, fasc. 3–4 (1947), p. 271–281.
19. Gorr G.V., Kudryashova L.V., Stepanova L.A. *Classical problems of rigid body dynamics*, Naukova dumka, Kiev 1978, 296 p.
20. Kharlamov P.V. *Lectures in rigid body dynamics*, publishing office of Novosibirsk State University, Novosibirsk 1965, 221 p.
21. Dokshevich A.I. *Solutions in the final variant of Euler–Poisson equations*, Naukova dumka, Kiev 1992, 168 p.
22. Gorr G.V., Maznev A.V. *Gyrostat dynamics, having a fixed point*, Donetsk National University, Donetsk 2010, 364 p.
23. Levi-Civita T. Amaldi U. *Lezioni di Meccanica Razionale*, Nicola Zanichelli, Bologna 1927, vol. 2, 671 p.
24. Levi-Civita T. Amaldi U. *Course of theoretical mechanics*. in2 volumes. Publishing office of foreign literature, Moscow 1951, vol. 2, part 2, 555 p.
25. Chaplygin S.A. *About final multiplier principle.* Collected edition. vol.I, M.–L. 1948, p. 4–9.
26. Kovalev A.M. *The equations of invariant and oriented manifolds of dynamic systems*, Report of National Academy of Science of Ukraine, No. 9 (1998), p. 21–25.
27. Kovalev A.M. *Inclusion of the invariant manifolds in a set of integral manifolds and Hess solution analysis*, Mekhanica tverdogo tela, ed. 32 (2002), p. 16–31.
28. Gorr G.V., Schetinina Ye.K. *About integrating multiplier of dynamic equations of rigid body at invariant manifolds*, Report of National Academy of Science of Ukraine, No. 1 (2007), p. 60–66.
29. Pontryagin L.S. *Simple differential equations*, Nauka, Moscow 1965, 331 p.
30. Gorr G.V., Ilyukhin A.A., Kharlamova Ye.I. *About particular solutions of one equations form for rigid body motion, having a fixed point,* Mekhanica tverdogo tela, ed. 6 (1974), p. 3–9.
31. Kharlamov P.V. *About invariant relations of differential equations system*, Mekhanica tverdogo tela, ed. 6 (1974), p. 15–24.
32. Kovalev A.M., Kirichenko V.V. *The equations and integrals of Hess gyroscope motion near the proportional rotations*, Mekhanica tverdogo tela, ed. 30 (2000), p. 87–93.

33. Maznev A.V. *About one case of* an *integrating multiplier existence of the rigid body dynamic equations at invariant relations of Chaplygin's class*, Research works of Institute of Applied Mechanics and Mathematics, vol. 19 (2009), p. 155–161.
34. Kovalev A.M. *Invariant and integral manifolds of dynamical systems and the problem of integration of the Euler–Poisson equations*, Regular and chaotic dynamics, vol. 9, No 1 (2004), p. 59–72.

РАЗВОЈ МЕТОДЕ ИНВАРИЈАНТНИХ РЕЛАЦИЈА У ПРОБЛЕМИМА ДИНАМИКЕ КРУТИХ ТЕЛА

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У раду се представљају решења проблема укључења инваријантних многструкости у скуп многострујости интеграла и конструкције првих интеграла диференцијалних једначина постављене на основу инваријантних релација укључујући и једначине динамике крутих тела. Такође је представљена интеграција тих диференцијалних једначина помоћу K. Jacobi-јевих теорема на инваријантним релацијама T. Levi-Civita и S.A. Chaplygin. Егсистенција првих интеграла Euler–Poisson-овох једначина уз задовољење Hess-ових услова је доказано и њихова прва апроксимација је добијена.

Кључне речи: инваријантне многострукости, многострукости интеграла, инваријантне релације, Euler–Poisson-ове једначине.

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MULTI MEMBRANE FRACTIONAL ORDER SYSTEM VIBRATIONS

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Abstract. *A model of multi membrane fractional order oscillations is presented and corresponding partial fractional order differential equations are solved. A hybrid fractional order element with translator and rotator inertia properties is introduced by corresponding constitutive relations. Generalized function of fractional order energy dissipation is introduced. Generalized forces of two membrane and fractional order layer as well as of its constitutive element are expressed by energies and generalized function of fractional order energy dissipation.*
For obtaining solution of system of partial fractional order differential equations, it is used Euler-Bernoulli method of particular integral and transformation of the system of ordinary fractional order differential equations along eigen time functions introducing eigen main coordinates of fractional order system. In result it is obtained a system of independent ordinary fractional order differential equations each along one eigen fractional order main coordinates. Eigen fractional order main modes of an eigen time function in each of infinite number of eigen amplitude shapes are defined.

Key words: *Hybrid system, fractional order element, translator and rotator inertia, discrete continuum fractional order layer, membrane, fractional order energy dissipation, eigen time functions, eigen main fractional order mode, trigonometric method.*

1. INTRODUCTION

Aim of this paper is to investigate properties of the vibrations of a hybrid system vibrations consisting of finite number of membranes coupled by discrete continuum fractional order layers with translator and rotator inertia properties. Idea of discrete continuum layer consisting by standard light hereditary elements [1-3] or standard light fractional order elements [4-8] appear in the papers [9-13] written by (Stevanović) Hedrih. Standard light hereditary element is defined in the monograph [1] authored by Goroshko and (Stevanović) Hedrih. Standard light fractional order element is defined in the numerous References [8-10] by (Stevanović) Hedrih, and both elements was used as coupling elements between rigid or deformable bodies for obtaining model of different kind hybrid system dynamics. Discrete continuum fractional order layer is defined on the basis of discrete continuum method, containing standard light fractional order elements distributed homogeneously along line or surface for coupling deformable beams or plates, or deformable bodies with same boundary conditions.

Idea that coupling element can be with translator and rotator inertia properties appears in the papers [14-15] by (Stevanović) Hedrih and schematically presented by a rolling element between two surfaces. Late a standard visco-elastic element with translator and rotator inertia properties is used in a discrete continuum layer between two circular plate system in the first submitted dissertation [16] authored by Simonović, and also in the paper [17] by (Stevanović) Hedrih and Simonović, and also in the papers [18] by (Stevanović) Hedrih and [19] by Simonović.

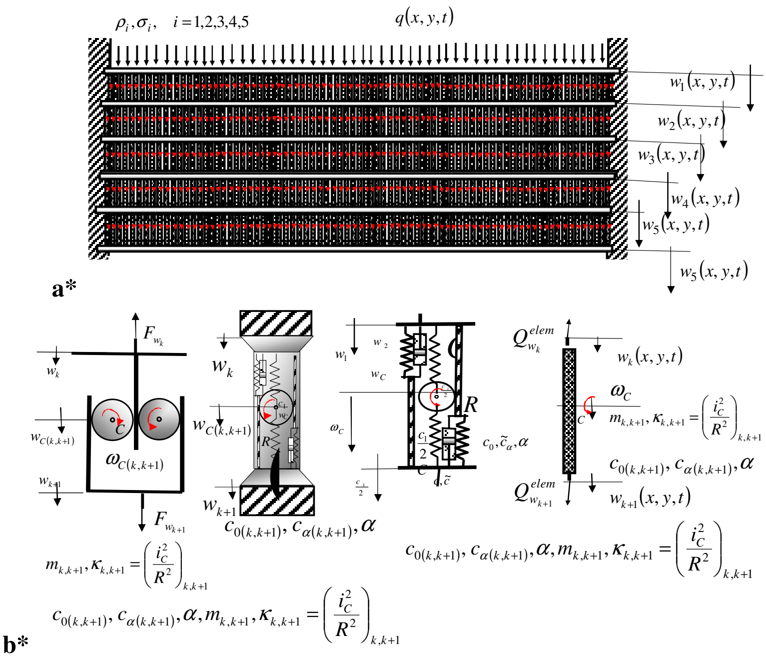

Figure 1. Discrete continuum fractional order layers with translator and inertia properties: Five membranes with same contours and boundary conditions, coupled by discrete continuum fractional order layers (a*); and Standard fractional order element with translator and inertia properties – schematically presentation (b*).

Next chapter of this paper is focused to the constitutive relation of standard hybrid fractional order element with translator and inertia properties as subelements of discrete continuum fractional order layer with translator and rotator inertia properties. Third chapter is related to partial fractional order differential equations of transversal vibrations of multi membrane system with fractional order layers of translator and inertia properties. Next two chapters are focussed to solution of system of partial fractional order differential equations.

2. DISCRETE CONTINUUM FRACTIONAL ORDER LAYER WITH TRANSLATOR AND ROTATOR INERTIA PROPERTIES

In this chapter we describe discrete continuum layer consisting of numerous standard hybrid fractional order elements, each containing parallel coupled standard ligh ideal elastic element, standard light fractional order element [20-21] and standard translator and rotator inertia element. These elements are presented in Figure 1. b* schematically as a hybrid element with the following coefficients: of rigidity $c_{0(k,k+1)}$, fractional order disipative properties $c_{\alpha(k,k+1)}, \alpha$ and coefficient of translator and rotator inertia properties: $m_{k,k+1}$ mass of rolling sub-element and its ratio $\kappa_{k,k+1} = \left(\frac{i_C^2}{R^2}\right)_{k,k+1}$ between square of inertia radius and square of radius of its contour.

2.1. Constitutive relation of standard light fractional order element.

Taking into account, that each standard light fractional order element is between two deformable membranes, with membrane transversal displacements $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$, respectivelly in same direction, then extension of this element is: $\Delta w_{k+1,k}(x,y,t) = w_{k+1}(x,y,t) - w_k(x,y,t)$, we can use expression for generalized forces for generalized coordinates $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$ in the following constitutive relations:

$$Q_{\alpha,k}(t) = -\left\{c_{0(k,k+1)}\left[w_{k+1}(x,y,t) - w_k(x,y,t)\right] + c_{\alpha(k,k+1)}\mathfrak{D}_t^{\alpha}\left[w_{k+1}(x,y,t) - w_k(x,y,t)\right]\right\} = -Q_{\alpha,k+1}(t)$$
$$k = 1,2,3,4,....M \tag{1}$$

where $\mathfrak{D}_t^{\alpha}[\bullet]$ is a fractional order differential operator of the α^{th} derivative with respect to time t in the following form [20, 21,22]:

$$\mathfrak{D}_t^{\alpha}\left[w_{k+1}(x,y,t) - w_k(x,y,t)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\left[w_{k+1}(x,y,\tau) - w_k(x,y,\tau)\right]}{(t-\tau)^{\alpha}}d\tau \tag{2}$$

where c, c_{α} are rigidity coefficients – momentary and prolonged one, and α is rational number between 0 and 1, $0 < \alpha < 1$.

2.2. Standard inertia element with translator and rotator inertia properties

By use primary idea from 2007 and published References by Hedrih (Stevanović) in Refs. [14], [15] and [18], and late in [16] and [19] by Simonović J. in her doctoral dissertation used in a number of investigated examples, in this paper, we extended number of basic elements by a *standard hybrid fractional order element* with translator and inertia properties. Taking into account mass and mass inertia moments and realized by a rolling disk or sphere a *standard inertia element with translator and inertia properties* is presented in Figure 1. In same Figure 1, two different realization of the standard element with translator and rotator inertia properties by two rolling disks or two spheres and with transversal forces compensations are presented. This structure of element is only schematically represented for simulation inertia properties at macro level, but this element must to accept at micro-level.

Basic constitutive relation for standard inertia element is possible to obtain by use kinetic energy and corresponding expression for inertia forces. Taking into account that standard inertia element have only kinetic energy $\mathbf{E}_{K(k,k+1)}^{elem-layer}$, we can use expression for generalized forces of inertia for generalized coordinates $w_k(x,y,t)$ expressed by kinetic energy in the following form:

$$Q_{inert,w_k}^{elem-layer} = F_{j,w_k} = -\left\langle \frac{d}{dt}\frac{\partial \mathbf{E}_{K(k,k+1)}^{elem-layer}}{\partial\left(\frac{\partial w_k}{\partial t}\right)} - \frac{\partial \mathbf{E}_{K(k,k+1)}^{elem-layer}}{\partial w_k}\right\rangle \tag{3}$$

and corresponding constitutive relations between corresponding generalized force and generalized coordinates, $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$ are expressed by following double relation:

$$Q_{inert,w_k/w_{k+1}}^{elem-layer} = F_{j,w_k/w_{k+1}} = -\frac{1}{4}m_{k,k+1}\left[\left(\frac{\partial^2 w_{k+1}}{\partial t^2}+\frac{\partial^2 w_k}{\partial^2 t}\right)\mp\kappa_{k,k+1}\left(\frac{\partial^2 w_{k+1}}{\partial t^2}-\frac{\partial^2 w_k}{\partial^2 t}\right)\right] \tag{4}$$
$$k=1,2,3,4,....M$$

where $m_{k,k+1}$ is specific mass of discrete inertial element, $\kappa_{k,k+1}=\left(\frac{i_C^2}{R^2}\right)_{k,k+1}$ coefficient depending of rolling inertia properties-of inertia constitutive sub-element, for disc $\kappa_{k,k+1}=1/2$, or for sphere $\kappa_{k,k+1}=3/5$, or for hollow sphere with thin value $\kappa_{k,k+1}\approx 2/3$. We can see that this coefficient $\kappa_{k,k+1}$ is constant, and no depending of dimension, but only of mass distribution around axis of rolling and of ratio between radius of axial mass inertia and radius of rolling surface.

From previous expressions, we can see that generalized forces in axial direction are with different values. Then difference between previous generalized forces is

$$Q_{inert,w_{k+1}}^{elem-layer} - Q_{inert,w_k}^{elem-layer} = -\frac{1}{2}m_{k,k+1}\kappa_{k,k+1}\left(\frac{\partial^2 w_{k+1}}{\partial t^2}-\frac{\partial^2 w_{1k}}{\partial^2 t}\right) \tag{5}$$

and represent influence of rolling element inertia rotation in the form of kinetic angular momentum in the form:

$$\mathbf{J}_{\mathrm{C}(k,k+1)}\frac{d\omega_{C(k,k+1)}}{dt}=-\left\{-\left\langle\frac{d}{dt}\frac{\partial\mathbf{E}^{elem-layer}_{K(k,k+1)}}{\partial\left(\frac{\partial w_{k+1}}{\partial t}\right)}-\frac{\partial\mathbf{E}^{elem-layer}_{K(k,k+1)}}{\partial w_{k+1}}\right\rangle-\left\{-\left\langle\frac{d}{dt}\frac{\partial\mathbf{E}^{elem-layer}_{K(k,k+1)}}{\partial\left(\frac{\partial w_{k}}{\partial t}\right)}-\frac{\partial\mathbf{E}^{elem-layer}_{K(k,k+1)}}{\partial w_{k}}\right\rangle\right\}\right\}R_{k,k+1} \tag{6}$$

or in form

$$\mathbf{J}_{\mathrm{C}(k,k+1)}\frac{d\omega_{C(k,k+1)}}{dt}=\left(Q^{elem-layer}_{inert,w_{k+1}}-Q^{elem-layer}_{inert,w_{k}}\right)R_{k,k+1}=-\frac{1}{2}m_{k,k+1}\kappa_{k,k+1}R_{k,k+1}\left(\frac{\partial^2 w_{k+1}}{\partial t^2}-\frac{\partial^2 w_{1k}}{\partial^2 t}\right) \tag{7}$$

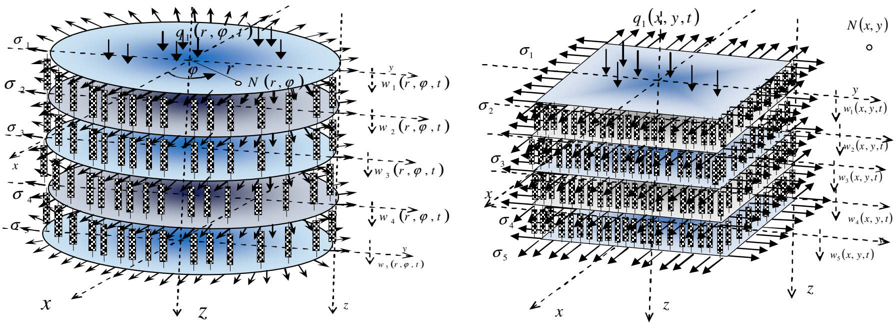

Figure 2. Models of multi membrane hybrid system: Five membranes, same contours and boundary conditions, coupled by discrete continuum fractional order layers with translator and rotator inertia properties: (a*) circular membranes; and (b*) rectangular membranes.

2.3. Standard hybrid fractional order element with translator and rolling inertia properties

In the beginning of this chapter we describe structure of a standard hybrid fractional order element with translator and inertia properties, each containing paralel coupled a standard ligh ideal elastic element, a standard light fractional order element and standard inertia element with translator and rotator inertia properties. This definition give us formula fo obtaining constitutive relation between generalized forces and extenssion of the element. Expressions of the generalized forces of interactions between deformable bodies- membranes and standard hybrid fractional order element with inertia properties in the discrete continuum model, in function of independent generalized coordinates $w_k(x,y,t)$ are possible to obtain by different principles. In this paper, generalized forces $Q^{elem-layer}_{w_k}$ between standard hybrid fractional order element and membranes with transversal displacements $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$ which are independent genarlized coordinates of a hybrid fractional order multi membrane system dynamics, (presented in figures 1.a*, and 2.a* and b*, are in the form:

1* If it is known expressions for kinetic energy $\mathbf{E}_{K(k,k+1)}^{elem-layer}$, potential energy $\mathbf{E}_{P(k,k+1)}^{elem-layer}$ and generalized function $\Phi_{\alpha(k,k+1)}^{elem-layer}$ of fractional order dissipation of mechanical energy of hybrid fractional order element with inertia properties, the generalized forces are in the following forms:

$$Q_{w_k}^{elem-layer} = -\left\langle \frac{d}{dt}\frac{\partial \mathbf{E}_{K(k,k+1)}^{elem-layer}}{\partial\left(\frac{\partial w_k}{\partial t}\right)} - \frac{\partial \mathbf{E}_{K(k,k+1)}^{elem-layer}}{\partial w_k} \right\rangle - \frac{\partial \mathbf{E}_{P(k,k+1)}^{elem-layer}}{\partial w_k} - \frac{\partial \Phi_{\alpha(k,k+1)}^{elem-layer}}{\partial\left(\mathfrak{D}_t^{\alpha}[w_k]\right)} = Q_{w_k}^{elem-plate} \tag{8}$$

or

2* in developed form in function of generalized coordinates $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$:

$$\begin{aligned} Q_{inert,w_k/w_{k+1}}^{elem-plate} &= \mp\left\{c_{0(k,k+1)}\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]+c_{\alpha(k,k+1)}\mathfrak{D}_t^{\alpha}\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]\right\}- \\ &-\frac{1}{4}m_{k,k+1}\left[\left(\frac{\partial^2 w_{k+1}}{\partial t^2}+\frac{\partial^2 w_k}{\partial^2 t}\right)\mp\kappa_{k,k+1}\left(\frac{\partial^2 w_{k+1}}{\partial t^2}-\frac{\partial^2 w_k}{\partial^2 t}\right)\right] \end{aligned} \tag{9}$$

Last expressions (8) and (9) present generalized force interactions between two bodies-membranes coupled by standard hybrid fractional order element with inertia properties.

3. Governing Partial Fractional Order Differential Equations of Transversal Oscillations of a Hybrid Multi Membrane System with Fractional Order Layers with Translator and Rotator Inertia Properties

Let's consider transversal oscillations of a hybrid multi deformable membrane system presented in Figure 1. b* or in Figure 2. a* and b*. Hybrid multi membrane system contains M ideal elastic membranes coupled by $M-1$ discrete continuum fractional order layers with translator and rotator inertia properties. Constitutive elements of these layers are standard hybrid fractional order elements with translator and rotator inertia properties of which constitutive relation between generalized forces and generalized coordinates are defined by expressions (8) or (9). These generalized forces $Q_{inert,w_k/w_{k+1}}^{elem}$ are defined as forces distributed along unit area of contact membrane surface between membrane and discrete continuum layer. We propose that all M membranes are same contour line in parallel planes, and that membranes are loaded by distributed force along contour producing stress intensities σ_k, $k=1,2,3,4,....M$ and that membranes are with mass density distribution along surface of membranes ρ_k. Then, taking into account that $c_k^2=\frac{\sigma_k}{\rho_k}, k=1,2,3,4,....M$ are surface velocities of corresponding membrane transversal waves propagations in the hybrid multi-membrane system dynamics, the system of partial differential equations describing transversal oscillations of a hybrid multi deformable membrane fractional order system (presented in Figure 1, b* or in Figure 2. a* and b*) is in the following form (see References by Rašković [23, 24] and by Hedrih (Stevanović) [9, 10]):

$$\begin{gathered}
\rho_1 \frac{\partial^2 w_1(x,y,t)}{\partial t^2} = \rho_1 c_1^2 \Delta w_1(x,y,t) + c_{0(1.2)}\left[w_2(x,y,t) - w_1(x,y,t)\right] + \\
+ c_{\alpha(1,2)} \mathfrak{D}_t^\alpha \left[w_2(x,y,t) - w_1(x,y,t)\right] - \\
-\frac{1}{4} m_1 \left[\left(\frac{\partial^2 w_2}{\partial t^2} + \frac{\partial^2 w_1}{\partial^2 t}\right) - \kappa_{1,2}\left(\frac{\partial^2 w_2}{\partial t^2} - \frac{\partial^2 w_1}{\partial^2 t}\right)\right] + q_1(x,y,t) \\
\rho_k \frac{\partial^2 w_k(x,y,t)}{\partial t^2} = \rho_k c_k^2 \Delta w_k(x,y,t) - \\
-\frac{1}{4} m_{k-1,k} \left[\left(\frac{\partial^2 w_k}{\partial t^2} + \frac{\partial^2 w_{k-1}}{\partial^2 t}\right) + \kappa_{k-1,k}\left(\frac{\partial^2 w_k}{\partial t^2} - \frac{\partial^2 w_{k-1}}{\partial^2 t}\right)\right] - \\
-\frac{1}{4} m_{k,k+1} \left[\left(\frac{\partial^2 w_{k+1}}{\partial t^2} + \frac{\partial^2 w_k}{\partial^2 t}\right) - \kappa_{k,k+1}\left(\frac{\partial^2 w_{k+1}}{\partial t^2} - \frac{\partial^2 w_k}{\partial^2 t}\right)\right] - \\
- c_{0(k-1.k)}\left[w_k(x,y,t) - w_{k-1}(x,y,t)\right] - c_{\alpha(k-1.k)} \mathfrak{D}_t^\alpha \left[w_k(x,y,t) - w_{k-1}(x,y,t)\right] + \\
+ c_{0(k.k+1)}\left[w_{k+1}(x,y,,t) - w_k(x,y,t)\right] + c_{\alpha(k.k+1)} \mathfrak{D}_t^\alpha \left[w_{k+1}(x,y,t) - w_k(x,y,t)\right] + \\
+ q_k(x,y,t) \\
k = 2,3,4,\ldots M-2, M-1 \\
\rho_M \frac{\partial^2 w_M(x,y,t)}{\partial t^2} = \rho_M c_M^2 \Delta w_M(x,y,t) - c_{0(M-1,M)}\left[w_M(x,y,t) - w_{M-1}(x,y,t)\right] - \\
- c_{\alpha(M-1,M)} \mathfrak{D}_t^\alpha \left[w_M(x,y,t) - w_{M-1}(x,y,t)\right] - \\
-\frac{1}{4} m_{M-1,M} \left[\left(\frac{\partial^2 w_M}{\partial t^2} + \frac{\partial^2 w_{M-1}}{\partial^2 t}\right) + \kappa_{M-1,M}\left(\frac{\partial^2 w_M}{\partial t^2} - \frac{\partial^2 w_{M-1}}{\partial^2 t}\right)\right] + q_M(x,y,t)
\end{gathered} \tag{10}$$

where Laplace differential operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in Descartes coordinates (x,y), suitable for the case that membranes are with rectangular contour line, and $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}$ in polar coordinate system (r,φ), suitable for the case that membranes are with circular contour line.

To previous system of partial fractional order differential equations a system of the boundary conditions is add in the following form:

1* for the system containing M membranes with rectangular contours:

$$\begin{array}{llll}
x = 0 & 0 \le y \le b & w_k(0,y,t) = 0 & k = 1,2,3,4,\ldots M \\
x = a & 0 \le y \le b & w_k(a,y,t) = 0 & k = 1,2,3,4,\ldots M \\
0 \le x \le a & y = 0 & w_k(x,0,t) = 0 & k = 1,2,3,4,\ldots M \\
0 \le x \le a & y = b & w_k(x,b,t) = 0 & k = 1,2,3,4,\ldots M
\end{array} \tag{11}$$

2* for the system containing M membranes with circular contours:

$$r = R \quad 0 \le y \le 2\pi \quad w_k(R,\varphi,t) = 0 \quad k = 1,2,3,4,\ldots M \tag{12}$$

To previous system of partial fractional order differential equations and a system of the boundary conditions for complete definition of the vibrations of the system is necessary to add corresponding initial conditions in the following forms:

$$1^* \text{for } t=0, \quad w_k(x,y,t)\big|_{t=0} = f(x,y) \text{ and } \left.\frac{\partial w_k(x,y,t)}{\partial t}\right|_{t=0} = F(x,y), k = 1,2,3,4,....M$$

$$\mathfrak{D}_t^\alpha\left[w_k(x,y,t)\right]\Big|_{\substack{t=0\\ \alpha=0}} = f(x,y) \text{ and } \mathfrak{D}_t^\alpha\left[w_k(x,y,t)\right]\Big|_{\substack{t=0\\ \alpha=1}} = \left.\frac{\partial w_k(x,y,t)}{\partial t}\right|_{t=0} = F(x,y), \; k = 1,2,3,4,....M$$

for the system containing M membranes with rectangular contours;
and

$$2^* \text{ for } t=0, \quad w_k(r,\varphi,t)\big|_{t=0} = f(r,\varphi) \text{ and } \left.\frac{\partial w_k(r,\varphi,t)}{\partial t}\right|_{t=0} = F(r,\varphi), k = 1,2,3,4,....M$$

$$\mathfrak{D}_t^\alpha\left[w_k(r,\varphi,t)\right]\Big|_{\substack{t=0\\ \alpha=0}} = f(r,\varphi) \text{ and } \mathfrak{D}_t^\alpha\left[w_k(r,\varphi,t)\right]\Big|_{\substack{t=0\\ \alpha=1}} = \left.\frac{\partial w_k(r,\varphi,t)}{\partial t}\right|_{t=0} = F(r,\varphi), k = 1,2,3,4,....M$$

for the system containing membranes with circular contours.

In the both cases functions $f(x,y)$ and $F(x,y)$ as well as $f(r,\varphi)$ and $F(r,\varphi)$ must to satisfy boundary conditions (11) or (12) respectively.

4. Solution of governing partial fractional order differential equations of transversal oscillations of a hybrid multi-membrane system

As it is suposed all M membranes are same contour-lines and same boundary conditions (11) or (12). This fact permit us to suppose that eigen amplitude functions $W_{nm}(x,y)$ or $W_{nm}(r,\varphi)$, for all M membranes in the considered multi membrane system are possible to take in the same form $W_{nm}(x,y)$ or $W_{nm}(r,\varphi)$ $n,m = 1,2,3,4...\infty$ as in the case of one single membrane with same baoundary conditions (11) or (12) depending of form contour line of membrane in rectangular form or circular form. Taking, this fact into account, solution of system (15), the posible solution suppose in the following forms (see Reference [23] by Rašković and [25] by Janković, Potić and Hedrih (Stevanović)):

$$w_k(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{nm}(x,y)T_{k(nm)}(t), \; k = 1,2,3,4,....M \tag{16}$$

and that distributed excitation along membrane surfaces are :

$$\frac{q_k(x,y,t)}{\rho} = \sum_{m=1}^{M}\sum_{n=1}^{N} h_{0k,nm} W_{nm}(x,y)\sin(\Omega_{knm}t + \vartheta_{k,nm}), k = 1,2,3,4,....M \tag{17}$$

Suposed solutions of previous system (15) of partial fractional order differential equations are in the forms of series along eigen amplitude functions, $W_{nm}(x,y)$, $n,m = 1,2,3,4,....,\infty$ satisfaying boudary conditions and with eigen time functiones $T_{k(nm)}(t)$, $k = 1,2,3,4,...M$, $n,m = 1,2,3,4,....,\infty$ and expresed by (16).

4.1. Ordinary differential equations along eigen time functions of a hybrid multi membrane system fractional order transversal oscillations

For beginning, let's introduce the following denotations:

$$\mu_k = \frac{m_{k,k+1}}{\rho_k}, \quad a^2_{0(k,k+1)} = \frac{c_{0(k.k+1)}}{\rho_k}, \quad a^2_{\alpha(k,k+1)} = \frac{c_{\alpha(k.k+1)}}{\rho_k}, \quad \mu_{k,k+1} = \frac{m_{k,k+1}}{\rho_k},$$
$$\tilde{\mu}_{k-1,k} = \frac{m_{k-1,k}}{\rho_k}, \quad \tilde{a}^2_{0(k-1,k)} = \frac{c_{0(k-1,k)}}{\rho_k}, \quad \tilde{a}^2_{\alpha(k-1,k)} = \frac{c_{\alpha(k-1,k)}}{\rho_k}, \quad \tilde{q}_k(x,y,t) = \frac{q_k(x,y,t)}{\rho_k} \tag{18}$$

For first, we introduce supposed solutions (16) and expressions (17) into partial fractional order differential equations of the system (15). Then, we multiply all M previous obtained fractional order differential equations with sr-th eigen amplitude functions $W_{sr}(x,y)dxdy$, $s,r = 1,2,3,4...,\infty$ and after integrating by following way $\iint_A * W_{sr}(x,y)dxdy$ all terms in all M equations along surface of the membranes, taking into account orthogonally conditions of the eigen amplitude functions expressed by formula $\iint_A W_{mn}(x,y)W_{sr}(x,y)dxdy = 0$, for $mn \neq sr$ and $m,n,s,r = 1,2,3,4...,\infty$ and taking denotations (18), we obtain system of the coupled ordinary fractional order differential equations along eigen time functions $T_{k(nm)}(t)$, $k = 1,2,...,M$, $n,m = 1,2,3,4,....,\infty$, in the following forms:

$$\begin{gathered}
\left[1+\frac{1}{4}\mu_{1,2}(1+\kappa)\right]\ddot{T}_{1(nm)}(t) + c_1^2 k_{nm}^2 T_{1(nm)}(t) + a^2_{0(1,2)}T_{1(nm)}(t) - \\
+ a^2_{\alpha(1,2)}\mathfrak{D}_t^{\alpha}\left[T_{1(nm)}(t)\right] - a^2_{0(1,2)}T_{2(nm)}(t) - a^2_{\alpha(1,2)}\mathfrak{D}_t^{\alpha}\left[T_{2(nm)}(t)\right] - \\
-\frac{1}{4}\mu_{1,2}(1-\kappa)\ddot{T}_{2(nm)}(t) = h_{01,nm}\sin\left(\Omega_{1,nm}t + \vartheta_{1,nm}\right) \\
n,m = 1,2,3,4,....,\infty \\
\left[1+\frac{1}{4}\tilde{\mu}_{k-1,k}(1+\kappa)+\frac{1}{4}\mu_{k,k+1}(1+\kappa)\right]\ddot{T}_{k(nm)}(t) + c_k^2 k_{nm}^2 T_{k(nm)}(t) + \left[\tilde{a}^2_{0(k-1,k)} + a^2_{0(k,k1)}\right]T_{k(nm)}(t) + \\
+\left[\tilde{a}^2_{\alpha(k-1,k)} + a^2_{\alpha(k,k+1)}\right]\mathfrak{D}_t^{\alpha}\left[T_{k(nm)}(t)\right] - \tilde{a}^2_{0(k-1,k)}T_{k-1(nm)}(t) - \tilde{a}^2_{\alpha(k-1,k)}\mathfrak{D}_t^{\alpha}\left[T_{k-1(nm)}(t)\right] - \\
- a^2_{0(k,k+1)}T_{k+1(nm)}(t) - a^2_{\alpha(k,k+1)}\mathfrak{D}_t^{\alpha}\left[T_{k+1(nm)}(t)\right] + \frac{1}{4}\tilde{\mu}_{k-1,k}(1-\kappa)\ddot{T}_{k-1(nm)}(t) + \\
+\frac{1}{4}\mu_{k,k+1}(1-\kappa)\ddot{T}_{k+1(nm)}(t) = h_{0k,nm}\sin\left(\Omega_{knm}t + \vartheta_{k,nm}\right) \\
n,m = 1,2,3,4,....,\infty, \quad k = 2,3,4,...M-2, M-1 \\
\left[1+\frac{1}{4}\tilde{\mu}_{M-1,M}(1+\kappa)\right]\ddot{T}_{M(nm)}(t) + c_k^2 k_{nm}^2 T_{M(nm)}(t) + \tilde{a}^2_{0(M-1,M)}T_{M(nm)}(t) + \\
+\frac{1}{4}\tilde{\mu}_{M-1,M}(1-\kappa)\ddot{T}_{M-1(nm)}(t) - \tilde{a}^2_{0(M-1,M)}T_{k-1(nm)}(t) + \tilde{a}^2_{\alpha(M-1,M)}\mathfrak{D}_t^{\alpha}\left[T_{M(nm)}(t)\right] - \\
-\tilde{a}^2_{\alpha(M-1,M)}\mathfrak{D}_t^{\alpha}\left[T_{M-1(nm)}(t)\right] = h_{M,nm}\sin\left(\Omega_{Mnm}t + \vartheta_{M,nm}\right)
\end{gathered} \tag{19}$$

$$n,m = 1,2,3,4,....,\infty$$

4.2 Transformation of ordinary differential equations along eigen time functions of a hybrid multi membrane system fractional order transversal oscillations to the eigen main coordinates and eigen main fractional order modes

System (19) contains infinite number of independent sub-systems corresponding to each of eigen amplitude function from a set with $W_{mn}(x,y)$, $n,m = 1,2,3,4,....,\infty$. Each of these sub-systems contains M coupled ordinary fractional order differential equations along eigen time functions $T_{k(nm)}(t), k = 1,2,...,M$, $n,m = 1,2,3,4,....,\infty$,. There are sets with the ordinary fractional order differential equations (19) of the same type, but with different coefficients. We are going to analyze one of the subsystem, for $h_{0k,nm} = 0$, corresponding to free fractional order vibrations of membranes, that we can write in matrix form (see References [23], [20] and [21)]:

$$\mathbf{A}_{nm}\left\{\ddot{T}_{k(nm)}\right\}^{\downarrow\, k=1,2,3,...,M} + C_{nm}\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,...,M} + C_{\alpha,nm}\mathfrak{D}_{t}^{\alpha}\left[\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,...,M}\right] = \{0\} \quad (20)$$

$$n,m = 1,2,3,4,....,\infty$$

where $\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,...,M}$ is a matrix column with M elements which are eigen time functions $T_{k(nm)}(t), k = 1,2,...,M$ for free vibrations, corresponding to one eigen amplitude function from the set of infinite number of $W_{mn}(x,y)$, $n,m = 1,2,3,4,....,\infty$;

$\mathbf{A}_{nm}$ is the matrix, in rank $M \times M$,of coefficients of system (19) mass inertia properties in the form:

$$\mathbf{A}_{nm} = \left(a_{nm,kj}\right)^{\downarrow\, k=1,2,3,...,M}_{\rightarrow\, j=1,2,3,...,M} \quad , \quad n,m = 1,2,3,4,....,\infty \quad (21)$$

$\mathbf{C}_{nm}$ is the matrix, in rank $M \times M$, of coefficients of system (19) rigidity properties in the form:

$$\mathbf{C}_{nm} = \left(c_{nm,kj}\right)^{\downarrow\, k=1,2,3,...,M}_{\rightarrow\, j=1,2,3,...,M} \quad , \quad n,m = 1,2,3,4,....,\infty \quad (22)$$

and $C_{\alpha,nm}$ is the matrix, in rank $M \times M$, of coefficients of system (19) visco-elastic creep fractional order properties, in the form:

$$C_{\alpha,nm} = \left(c_{\alpha,nm,kj}\right)^{\downarrow\, k=1,2,3,...,M}_{\rightarrow\, j=1,2,3,...,M} , \quad n,m = 1,2,3,4,....,\infty . \quad (23)$$

The modal matrix, in rank $M \times M$, that of linear system correspond to fractional order system defined by (19) or (20) is :

$$\mathbf{R}_{nm} = \left(\left\{K^{s}_{nm,Mk}\right\}\right) = \left(K^{s}_{nm,Mk}\right)^{\downarrow\, k=1,2,3,...,M}_{\rightarrow\, s=1,2,3,...,M} , \quad n,m = 1,2,3,4,....,\infty \quad (24)$$

where $K^{s}_{nm,Mk}$ are corresponding cofactors of determinate, in rank $M \times M$, of defined characteristic equation of corresponding linear system to matrix equation (20), and in the following form:

$$f\left(\omega^2_{nm}\right) = \left|C_{nm} - \omega^2_{nm}\mathbf{A}_{nm}\right| = 0 \quad , \quad n,m = 1,2,3,4,....,\infty . \quad (25)$$

with M roots $\omega^2_{nm,s}$, $s=1,2,3,...,M$, for each combination from $n,m=1,2,3,4,....,\infty$. $\omega_{nm,s}$ are eigen circular frequencies of linear (harmonic) vibrations.

If we apply modal matrix $\mathbf{R}_{nm}$ to the fractional order system matrices $\mathbf{A}_{nm}$, $\mathbf{C}_{nm}$ and $\mathbf{C}_{\alpha,nm}$ in results of their transformations from generalized coordinates - eigen time functions $T_{k(nm)}(t)$, $k=1,2,...,M$ along eigen main coordinates of corresponding linear system, we obtain M diagonal matrices (see References [23], [20] and [21]):

$$\mathfrak{A}_{nm}=\mathbf{R}'_{nm}\mathbf{A}_{nm}\mathbf{R}_{nm}=diag\left(a_{nm,ss}\right),\quad \mathfrak{C}_{nm}=\mathbf{R}'_{nm}C_{nm}\mathbf{R}_{nm}=diag\left(c_{nm,ss}\right),$$
$$\mathfrak{C}_{\alpha,nm,s}=\mathbf{R}'_{nm}\mathbf{C}_{\alpha,nm}\mathbf{R}_{nm}=diag\left(c_{(\alpha)nm,sS}\right),\quad n,m=1,2,3,4,....,\infty \tag{26}$$

where $\mathfrak{A}_{nm}$, $\mathfrak{C}_{nm}$ and $\mathfrak{C}_{\alpha,nm,s}$ are diagonal matrices, also each in rank $M\times M$.

According to previous obtained results in transformations of matrices of the system of ordinary fractional order differential equations and theorem presented in References [23], [20], [21] or [26], we can take the following formula of coordinate transformation:

$$\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,...,M}=\mathbf{R}_{nm}\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,...,M}=\left(K^{s}_{nm,Mk}\right)^{\downarrow\, k=1,2,3,...,M}_{\to\, s=1,2,3,...,M}\left\{\xi_{m\backslash nm,s}\right\}^{\downarrow\, s=1,2,3,...,M} \tag{27}$$

from the generalized coordinates $T_{k(nm)}(t)$ of linear system to corresponding eigen main coordinates $\xi_{nm,s}(t)$, $s=1,2,3,...,M$ and eigen main modes of free oscillations:

$$\xi_{nm,s}(t)=C_{nm,s}\cos\left(\omega_{nm,s}t+\vartheta_{nm,s}\right). \tag{28}$$

where $C_{nm,s}$ and $\vartheta_{nm,s}$ are integral constant for linear system determined by initial conditions, and $\omega_{nm,s}$ eigen circular frequencies of linear system obtained as roots from characteristic equations (25).

Taking into account results (26) of the transformations of matrices of fractional order system in corresponding diagonal form and formula of coordinate transformation (27) of eigen time functions $T_{k(nm)}(t)$ applied to the system of ordinary fractional order differential equations (19) or in matrix form (20), after transformation of system of fractional order differential equation (19), or in matrix form (20), we have series of subsystems with all independent fractional order differential equations along one eigen main coordinate $\xi_{nm,s}(t)$, $s=1,2,3,...,M$ in the following form:

$$\mathfrak{A}_{nm}\left\{\ddot{\xi}_{nm,s}\right\}^{\downarrow\, s=1,2,3,...,M}+\mathfrak{C}_{nm}\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,...,M}+C_{\alpha,nm}\mathfrak{D}^{\alpha}_{t}\left[\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,...,M}\right]=\{0\} \tag{29}$$

where $\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,...,M}$ matrix column with M elements which are eigen main coordinates $\xi_{nm,s}$ free fractional order vibrations of system (20) of fractional order differential equations along of each time functions $T_{k(nm)}(t), k=1,2,...,M$ correspond to one eigen amplitude function from the set of infinite number of $W_{mn}(x,y)$, $n,m=1,2,3,4,....,\infty$. From obtained subsystems of the fractional order differential equations in matrix form (29) is easier to separate the corresponding subsystems of fractional order differential equations only along one eigen main coordinate $\xi_{nm,s}$, $n,m=1,2,3,4,....,\infty$, $s=1,2,3,...,M$ in the following form:

$$\ddot{\xi}_{nm,s}+\omega^2_{nm,s}\xi_{nm,s}+\omega^2_{(\alpha)nm,s}\mathfrak{D}^{\alpha}_{t}\left[\xi_{nm,s}\right]=0\,,\ n,m=1,2,3,4,....,\infty\,,\ s=1,2,3,...,M \qquad (30)$$

where $\omega^2_{nm,s}$ and $\omega^2_{(\alpha)nm,s}$ are characteristic numbers of fractional order vibrations of eigen time functions $T_{k(nm)}(t)$, $k=1,2,...,M$ for free fractional order vibrations, correspond to one of the eigen amplitude functions ,$W_{mn}(x,y)$ $n,m=1,2,3,4,....,\infty$ from the infinite set of these functions determined by boundary conditions of all membranes.

First set of M characteristic numbers $\omega^2_{nm,s}$, $s=1,2,3,...,M$ for each combination of $n,m=1,2,3,4,....,\infty$, contains elements which are roots of characteristic frequency equation (25) which correspond to linear system. These characteristic numbers $\omega^2_{nm,_s}$ are square of the eigen circular frequencies of linear (harmonic) vibrations. Also these characteristic numbers satisfy the following relations $\omega^2_{nm,s}=\dfrac{c_{nm,ss}}{a_{nm,ss}}$. Second set of M characteristic numbers $\omega^2_{(\alpha)nm,s}$, $s=1,2,3,...,M$, for each combination of $n,m=1,2,3,4,....,\infty$, present characteristic numbers which express fractional order properties of the system and corresponding characteristic coefficients of fractional order dissipation of the multi membrane system energy. There characteristic numbers is possible to determine by $\omega^2_{(\alpha)nm,_s}=\dfrac{c_{(\alpha)nm,ss}}{a_{nm,ss}}$ for $0<\alpha<1$.

1* For $\alpha=0$ all differential equations are in the form $\ddot{\xi}_{nm,s}+\left(\omega^2_{nm,s}+\omega^2_{(\alpha=0)nm,s}\right)\xi_{nm,s}=0$ and characteristic numbers are $\omega^2_{nm,_s}\Big|_{\alpha=0}=\dfrac{c_{nm,ss}+c_{(\alpha=0)nm,ss}}{a_{nm,ss}}=\omega^2_{nm,_s}+\omega^2_{(\alpha=0)nm,s}$ and eigen main modes are pure periodic with eigen circular frequency $\omega_{nm,s}\Big|_{\alpha=0}=\sqrt{\dfrac{c_{nm,ss}+c_{(\alpha=0)nm,ss}}{a_{nm,ss}}}=\sqrt{\omega^2_{nm,s}+\omega^2_{(\alpha=0)nm,s}}$ and main coordinates are

$$\xi_{nm,s}\Big|_{\alpha=0}=C_{nm,s}\cos\left(t\sqrt{\omega^2_{nm,s}+\omega^2_{(\alpha=0)nm,s}}+\vartheta_{nm,s}\right). \qquad (31)$$

2* For $\alpha=1$ all differential equations are in the form $\ddot{\xi}_{nm,s}+\omega^2_{nm,s}\xi_{nm,s}+\omega^2_{(\alpha=1)nm,s}\dot{\xi}_{nm,s}=0$ and characteristic numbers are complese and conjugate $\lambda_{nm,s}\Big|_{\alpha=1}=-\dfrac{\omega^2_{(\alpha=1)nm,s}}{2}\mp\mathbf{i}\sqrt{\omega^2_{nm,s}-\dfrac{\omega^4_{(\alpha=1)nm,s}}{2}}$, for $\omega^2_{nm,s}>\dfrac{\omega^4_{(\alpha=1)nm,s}}{2}$ and eigen main modes are pure aperiodic with eigen circular frequency $\omega_{nm,s}\big|_{\alpha=1} = \sqrt{\omega_{nm,s}^2 - \frac{\omega_{(\alpha=1)nm,s}^4}{2}}$ and coefficient of dissipation $\delta_{nm,s}\big|_{\alpha=1} = -\frac{\omega_{(\alpha=1)nm,s}^2}{2}$ and main coordinate are :

$$\xi_{nm,s}\big|_{\alpha=1} = C_{nm,s} e^{-\frac{\omega_{(\alpha=1)nm,s}^2}{2}t} \cos\left(t\sqrt{\omega_{nm,s}^2 - \frac{\omega_{(\alpha=1)nm,s}^4}{4}} + \vartheta_{nm,s}\right). \tag{32}$$

3* For $\alpha = 1$ all differential equations are in the form $\ddot{\xi}_{nm,s} + \omega_{nm,s}^2 \xi_{nm,s} + \omega_{(\alpha=1)nm,s}^2 \dot{\xi}_{nm,s} = 0$ and characteristic numbers are real $\lambda_{nm,s}\big|_{\alpha=1} = -\frac{\omega_{(\alpha=1)nm,s}^2}{2} \mp \sqrt{\frac{\omega_{(\alpha=1)nm,s}^4}{2} - \omega_{nm,s}^2}$, for $\omega_{nm,s}^2 < \frac{\omega_{(\alpha=1)nm,s}^4}{2}$ and eigen main modes are aperiodic with eigen $q_{nm,s}\big|_{\alpha=1} = \sqrt{\frac{\omega_{(\alpha=1)nm,s}^4}{2} - \omega_{nm,s}^2}$ and eigen main coordinate are linear combination of two particular integrals:

$$\xi_{nm,s}\big|_{\alpha=1} = C_{nm,s} e^{-\frac{\omega_{(\alpha=1)nm,s}^2}{2}t} ch\left(t\sqrt{\omega_{nm,s}^2 - \frac{\omega_{(\alpha=1)nm,s}^4}{4}}\right) \quad \text{and}$$

$$\xi_{nm,s}\big|_{\alpha=1} = D_{nm,s} e^{-\frac{\omega_{(\alpha=1)nm,s}^2}{2}t} sh\left(t\sqrt{\omega_{nm,s}^2 - \frac{\omega_{(\alpha=1)nm,s}^4}{4}}\right) \tag{33}$$

Type of fractional order differential equation (30) is known from literature (see References [22] and [27]) and particular integrals $\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \cos}}$ like cosine and $\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \sin}}$ like sine are in the following forms:

$$\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \cos}} = \sum_{k=0}^{\infty}(-1)^k \omega_{(\alpha)nm,s}^{2k} t^{2k} \sum_{j=0}^{k}\binom{k}{j} \frac{\omega_{(\alpha),nm,s}^{-2j} t^{-\alpha j}}{\omega_{nm,s}^{2j}\Gamma(2k+1-\alpha j)},\ 0 \le \alpha \le 1 \tag{34}$$

$s = 1,2,3,...,M$, $n,m = 1,2,3,4,....,\infty$

$$\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \sin s}} = \sum_{k=0}^{\infty}(-1)^k \omega_{(\alpha)nm,s}^{2k} t^{2k+1} \sum_{j=0}^{k}\binom{k}{j} \frac{\omega_{(\alpha)nm,s}^{-2j} t^{-\alpha j}}{\omega_{nm,s}^{2j}\Gamma(2k+2-\alpha j)},\ 0 \le \alpha \le 1 \tag{35}$$

$s = 1,2,3,...,M$, $n,m = 1,2,3,4,....,\infty$

Eigen time functions $T_{k(nm)}(t)$, $k = 1,2,...,M$ corresponding to one eigen amplitude function from the set of infinite numbers of $W_{mn}(x,y)$, $n,m = 1,2,3,4,....,\infty$ are expressed by particular integrals $\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \cos}}$ like cosine and $\xi_{nm,s}(t,\alpha)\big|_{\substack{0\le\alpha\le1\\ \sin}}$ like sine, in the form

$$T_{k(nm)}(t)=\sum_{s=1}^{s=M}K_{nm,Mk}^{s}\left\langle \xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}+\xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\right\rangle$$

$$k=1,2,...,M\ ,\ n,m=1,2,3,4,....,\infty \tag{36}$$

Then, it is possible to obtain expressions for transversal displacements of all M membranes coupled by $M-1$ discrete continuum fractional order layers, in the following form

1* for rectangular membranes:

$$w_k(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(x,y)\sum_{s=1}^{s=M}K_{nm,Mk}^{s}\xi_{nm,s}(t),\ k=1,2,3,4,....M \tag{37}$$

or in the following form developed form:

$$\{w_k(x,y,t)\}^{\downarrow\ l=1,2,3,...,M}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(x,y)\mathbf{R}_{nm}\left\{\xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}\right\}^{\downarrow\ s=1,2,3,...,M}+$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(x,y)\mathbf{R}_{nm}\left\{\xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\right\}^{\downarrow\ s=1,2,3,...,M} \tag{38}$$

where for boundary conditions (11) eigen amplitude functions are in the forms:

$$W_{nm}(x,y)=\sin\frac{n\pi}{a}x\sin\frac{m\pi}{a}y\ ,\ n,m=1,2,3,4,....,\infty \tag{39}$$

2* for circular membranes:

$$w_k(r,\varphi,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(r,\varphi)\sum_{s=1}^{s=M}K_{nm,Mk}^{s}\xi_{nm,s}(t),\ k=1,2,3,4,....M \tag{40}$$

or in the following developed form:

$$\{w_k(r,\varphi,t)\}^{\downarrow\ l=1,2,3,...,M}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(r,\varphi)\mathbf{R}_{nm}\left\{\xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \cos}}\right\}^{\downarrow\ s=1,2,3,...,M}+$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}(r,\varphi)\mathbf{R}_{nm}\left\{\xi_{nm,s}(0,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\xi_{nm,s}(t,\alpha)\Big|_{\substack{0\le\alpha\le1\\ \sin}}\right\}^{\downarrow\ s=1,2,3,...,M} \tag{41}$$

where for boundary conditions (12) eigen amplitude functions are in the forms [23]:

$$W_{nm}(r,\varphi)=R_n(r)\phi_m(\varphi)=\mathbf{J}_m(k_{nm}r)\sin m\varphi \tag{42}$$

where $\mathbf{J}_m(kr)$ Bessel's or cylindrical function of first order n-th order in one of the form [23]:

$$\mathbf{J}_m(kr)=\left(\frac{kr}{2}\right)^m\sum_{n=0}^{\infty}\frac{\left(\mathbf{i}\frac{kr}{2}\right)^{2n}}{(m+n)!n!},\ \text{or in the form}\ \mathbf{J}_m(kr)=\frac{(-\mathbf{i})^m}{2\pi}\int_{-\pi}^{\pi}e^{\mathbf{i}kr\cos\tau}\cos m\tau \tag{43}$$

where $\mathbf{i}=\sqrt{-1}$ imaginary unit. (44)

5. CONCLUDING REMARKS

Theoretically task for solving system of coupled partial fractional order differential equations describing transversal fractional order oscillatory displacements of finite number of M membranes coupled by $M-1$ discrete continuum fractional order layers is solved. For particular solutions it is necessary to obtain eigen characteristic numbers and modal matrix of linear system corresponding to fractional order system for free linear vibrations.

For solving problem for forced vibrations of M membranes coupled by $M-1$ discrete continuum fractional order layers, when each of membrane is loaded by single frequency external excitation distributed along each membrane, system of fractional order differential equations (19) is possible to write in the form:

$$\mathbf{A}_{nm}\left\{\ddot{T}_{k(nm)}\right\}^{\downarrow\, k=1,2,3,\dots,M}+C_{nm}\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,\dots,M}+C_{\alpha,nm}\mathfrak{D}_{\mathbf{t}}^{\alpha}\left[\left\{T_{k(nm)}\right\}^{\downarrow\, k=1,2,3,\dots,M}\right]= \\ =\left\{h_{0k,nm}\sin\left(\Omega_{knm}t+\vartheta_{k,nm}\right)\right\}^{\downarrow\, k=1,2,3,\dots,M} \tag{44}$$

After that, using same formula transformation (27) and taking into account previous results, transformation of matrix fractional order differential equation (45) give the following result:

$$\mathfrak{A}_{nm}\left\{\ddot{\xi}_{nm,s}\right\}^{\downarrow\, s=1,2,3,\dots,M}+\mathfrak{C}_{nm}\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,\dots,M}+C_{\alpha,nm}\mathfrak{D}_{\mathbf{t}}^{\alpha}\left[\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,\dots,M}\right]= \\ =\mathbf{R}'_{nm}\left\{h_{0k,nm}\sin\left(\Omega_{knm}t+\vartheta_{k,nm}\right)\right\}^{\downarrow\, k=1,2,3,\dots,M} \tag{45}$$

where $\left\{\xi_{nm,s}\right\}^{\downarrow\, s=1,2,3,\dots,M}$ is matrix column with M elements which are eigen main coordinates $\xi_{nm,s}$ for forced fractional order vibrations, of system (20) of fractional order differential equations of each eigen time function $T_{k(nm)}(t)$, $k=1,2,\dots,M$ correspond to one eigen amplitude function from the set of infinite number of $W_{mn}(x,y)$, $n,m=1,2,3,4,\dots,\infty$:

$$\ddot{\xi}_{nm,s}+\omega_{nm,s}^{2}\xi_{nm,s}+\omega_{(\alpha)nm,s}^{2}\mathfrak{D}_{t}^{\alpha}\left[\xi_{nm,s}\right]=\sum_{k=1}^{k=M}K_{nm,Mk}^{s}h_{0k,nm}\sin\left(\Omega_{knm}t+\vartheta_{k,nm}\right), \\ n,m=1,2,3,4,\dots,\infty,\ s=1,2,3,\dots,M \tag{46}$$

Each of obtained independent ordinary fractional order differential equation (46) describes a fractional order oscillator loaded by M frequency external excitation, which is also solvable by Laplace direct and inverse transformations or by generalized Laplace method of variation constant in the solution (36) or (38) (for detail see References [23] and [28].

At the end it is necessary point out series of the Review References [29]. [30] and [31] containing presentation of all the world selected publications concerning applications of fractional calculus for dynamic problem of solid mechanics with novel trends and resent results, and also applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. In Reference [31] a reflections on two parallel ways in the progress of fractional calculus in mechanics of solids are pointed out, a s results of the closed groups of researchers at large world west and east. These trends appear, also, in the very small countries as it is in Serbia in present time.

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REFERENCES

1. Oleg Aleksandrovič Goroško (Ukrajina) i Katica (Stevanović) Hedrih (Jugoslavija): Analitička dinamika (mehanika) diskretnih naslednih sistema, (Analytical Dynamics (Mechanics) of Discrete Hereditary Systems), University of Niš, 2001, Monograph, p. 426 (in Serbian), YU ISBN 86-7181-054-2.
2. Hedrih (Stevanović K., Energy analysis in a nonlinear hybrid system containing linear and nonlinear subsystems coupled by hereditary element (Article), NONLINEAR DYNAMICS , (2008) vol.51 br.1-2 str. 127 -140. DOI 10.1007/s11071-007-9197-2
3. Hedrih (Stevanović), K., *Mathematical Analogy and Phenomenological Mapping: Vibrations of Multi Plate and Multi Beam Homogeneous Systems,* Scientific Buletletin of the "Politehnica" University of Timisoara, Romania, Transaction on Mechanics, Edited by L. Bereteu, T. Cioara, M. Toth-Tascau, C. Vigaru, Editura Politehnica, Tom 50 (64), Special Issue, 2005, 11-18, ISSN 1224-6077, 2005.
4. Hedrih (Stevanović) Katica, (2004), *Creep Vibrations of a Fractional Derivative Order Constitutive Relation Deformable Bodies*, Proceedings Eighth American Congress of Applied Mechanics PACAM VIII. La Habana, Cuba, 5-9 January 2004, Series ***Applied Mechanics in Americas***, Vol. 10. pp. 548-551, ISBN 959-7056-20-8.
5. Hedrih (Stevanović) K., (2004), *Discrete Continuum Method,* COMPUTATIONAL MECHANICS, **WCCM** VI in conjunction with **APCOM**'04, *Sept. 5-10, 2004, Beijing, China,* © 2004 Tsinghua University Press & Springer-Verlag, pp. 1-11, CD. IACAM International Association for Computational Mechanics – www. iacm.info
6. Hedtrih (Stevanović) K., (2008), Dynamics of coupled systems, Journal Nonlinear Analysis: Hybrid Systems, Volume 2, Issue 2, June 2008, Pages 310-334. doi:10.1016/j.nahs.2006.06.003. http://www.sciencedirect.com/science/journal/1751570X
7. Hedtrih (Stevanović) K., (2008), *Dynamics of multipendulum systems with fractional order creep elements, Special Issue Vibration and Chaos,* Journal of Theoretical and Applied Mechanics, Quaterly, (Warsaw, Poland) No.3 Vol. 46, pp. 483-509. http://nainfo.nbs.bg.ac.yu.nainfo.nbs.bg.ac.yu:2048/Kobson/service/jcr.aspx?ISSN=1429-2955
8. Hedrih (Stevanović) K., (2011), Analytical mechanics of fractional order discrete system vibrations. Chap in Monograph. Advances in nonlinear sciences, Вол. 3, JANN, Belgrade, pp. 101-148, 2011. ISSN: 978-86-905633-3-3.
9. Hedrih (Stevanović) K., (2004), *Partial fractional order differential equations of transversal vibrations of creep connected double plates systems*, Workshop Preprints/Proceedings No 2004-1 IFAC workshop on Fractional Differentiation and its Applications FDA 04, ENSEIRB, Bordeaux France, July 19-21, 2004., pp. 299-304.
10. Hedrih (Stevanović K., Vibration Modes of a axially moving double belt system with creep layer ,Journal of Vibration and Control, (2008), 14(10-Sep): 1333-1347. http://nainfo.nbs.bg.ac.yu.nainfo.nbs.bg.ac.yu:2048/Kobson/service/jcr.aspx?ISSN=1077-5463
11. Hedrih (Stevanović) K., (2005), Partial Fractional Differential Equations of Creeping and Vibrations of Plate and their Solutions (First Part), Journal of the Mechanical Behavior of Materials, Freund Publishing House Ltd. 2005, pp. 305-314. ISSN 0334-8938. http://www.freundpublishing.com/JOURNALS/materials_science_and_engineering.htm

12. Hedrih (Stevanović) K., (2008), The fractional order hybrid system vibrations, Monograph, Chap in Monograph. Advances in Nonlinear Sciences, ANN, 2008, Vol. 2, pp. 226-326.
13. Hedrih (Stevanović) K., Energy and Nonlinear Dynamics of Hybrid System, Chapter in Book Dynamical Systems and Methods, Edited by Albert Luo. 2011, Chapter oko 50 strana, Springer 2012, Part 1, 29-83, DOI: 10.1007/978-1-4614-0454-5_2, (in press)
14. Hedrih (Stevanović) K., (2003), Frequency equations of small oscillations mixed systems of the coupled discrete and continuous subsystems, Mehanika tverdogo tela (Rigid Body Mechanics), Donetsk, ISSN 0321-1975, vip. 33, pp. 174-189. Ukraine.
15. Hedrih (Stevanović) R. Katica, (2010), Energy analysis in the hybrid system forced regimes, Zbornik praci Institutu Matematiki NANU Ukeaini (Proceeding of Institute of Mathematics NANU Ikraine) , 2010, Vol. 7, No. 3, pp. 90-107.
16. Simonović J., Dinamika i stabilnost hibridnih dinamičkih sistema (Dynamics and Stability of Dynamics Hybrid Systems), [in Serbian], Doctor's Degree Thesis, Faculty of Mechanical Engineering in Niš,Febr. 2011. Supervisor K. Hedrih (Stevanović).
17. Hedrih (Stevanović) K., Simonović J., (2010), "*Models of Hybrid Multi-Plates Systems Dynamics",* The International Conference-Mechanical Engineering in XXI Century, Niš, Serbia, 25-26 September 2010, Proceedings, pp.17-20.
18. Hedrih (Stevanović K., (2012), Models of dynamical dislocation in continuum, Proceedings of First International Conference on Damage Mechanics – ICDM 1, Edited by Chi L. Chow, J. Woody and D. Sumarac, Belgrade, Serbia, 25 – 27 June 2012, pp. 141-144. ISBN 978-86-86115-09-6 (SCE) ISBN 978-86-7518-153-8 (FCE). tp://www.icdm.rs/icdm/?id=pocetna
19. Simonovic J., (2012), Non-Linear Dynamics of a Double-Plate System Coupled by a Layer with Viscoelastic and Inertia Properties, Scientific Technical Review, 2012,Vol.62, No.1, pp.40-54. ISSN 1820-0206. http://www.vti.mod.gov.rs/ntp/entp.htm.
20. Hedrih (Stevanović) K., (2013), Generalized function of fractional order dissipation of system energy and extended Lagrange differential equation in matrix form, *Dedicated to 86th Anniversary of Radu MIRON'S Birth. 30 minutes* Plenary Lecture, Abstracts of THE 13TH INTERNATIONAL CONFERENCE OF TENSOR SOCIETY ON DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS, AND INFORMATICS BESIDES., ***The 86th Anniversary of Radu MIRON'S Birth.*** September 3rd (Tuesday) to September 7th (Saturday) in 2013. Faculty of Mathematics, Alexandru Ioan Cuza University and Mathematical Institute "O.Mayer" in Iaşi Romania And Tensor Society, Japan, 2013, p.3. (Full paper paper submitted for publishing in the journal Tensor of Tensor Society, Japan.). http://www.math.uaic.ro/~tensorconference2013/.
21. Hedrih (Stevanović) K., J. Tenreiro Machado, (2013), Discrete fractional order system vibrations, (*Preprint submitted for review-January 6, 2014).*
22. Gorenflo, R., Mainardi, F., (2000) Fractional Calculus, Integral and Differential Equations of Fractional Order, CISM Lecture Notes, Udine, Italy, Preprint 54 pages, pp. 223-276.
23. Rašković D., Teorija oscilacija (Theory of Oscillations), Naučna knjiga, 1952.
24. Rašković D,, Analitička mehanika (Analytical Mechanics), Mašinski fakultet Kragujevac, 1974.
25. Janković, V. S., Potić, V. P. and Hedrih (Stevanović), K., (1999), Parcijalne diferencijalne jednačine i integralne jednačine sa primenama u inženjerstvu (Partial differential equations and integro-differential equations with examples in engineering), Univerzitet u Nišu, 1999, str. 347. (in Serbian).
26. Katica R. (Stevanović) Hedrih, (201`3), Fractional order hybrid system dynamics, PAMM, Proc. Appl. Math. Mech. 13, 25 – 26 (2013) / DOI 10.1002/pamm.201310008.

http://onlinelibrary.wiley.com/doi/10.1002/pamm.v13.1/issuetoc

27. Bačlić, B.S., Atanacković, T., (2000), M., *Stability and creep of a fractional derivative order viscoelastic Rod*, Bulletin T, CXXI de L'Academie Serbe des Sciences st de Arts - 2000, Class des Sciences mathematques et naturelles Sciences mathematiques, No. 25, 115-131.
28. Hedrih (Stevanović) K., (2013), A generalization of Lagrange method of variation constants, SCIENTIFIC REVIEW, Special Issue Nonlinear Dynamics, Dedicated to Milutin Milanković (1879-1958), Guest Editors: Katica R. (Stevanović) Hedrih and Žarko Mijajlović, S2 (2013) pp. 37-66, SERBIAN SCIENTIFIC SOCIENTY, YU ISSN 0350-2910.
http://afrodita.rcub.bg.ac.rs/~nds/ ,http://afrodita.rcub.bg.ac.rs/~nds/indexe.html .
29. Rosikin Yuri and Maria Shitikova: (2010), Application of Fractional Calculus for Dynamic Problem of Solid Mechanics, Novel Trends and Resent Results, Applied Mechanics Reviews, American Society of Mechanical Enginewers –ASME, JANUAR 2010, Vol. 63/ pp. 010801-1 - 010801-52.
30. Rosikin Yuri and Maria Shitikova:, (2002), Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, Transmitted by Associate Editor Isaac Elishakoff, ASME Reprint No AMR202 $42, Appl Mech Rev vol 50, no 1, January 1997 15 © 1997 , American Society of Mechanical Engineers Downloaded**,** pp. 15-67.
31. Rossikhin Yuriy A., (2010), Reflections on Two Parallel Ways in the Progress of Fractional Calculus in Mechanics of Solids, Applied Mechanics Reviews, Copyright © 2010 by ASME JANUARY 2010, Vol. 63 / 010701-1, pp. / (010701-1)-/(010701-12).

OSCILACIJE SISTEMA VIŠE MEMBRANA POVEZANIH SLOJEVIMA SA FRAKCIONIM I INERCIONIM SVOJSTVIMA

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Apstrakt. Jedan model sistema konačnog broja membrana spregnutih diskretno kontinualnim slojevima sa frakcionim i inercionim svojstvima je definisan. Uveden je standardni hibridni element frakcionih i inercionih svojstava i za isti generalisana funkcija disipacije energije frakcionih svojstava. Dati su izrazi za određivanje generalisanih sila interakcije elementa i membrane pomoću energija i disipacije energije frakcionog reda. Napisan je odgovarajući sistem parcijalnih diferencijalnih jednačina sa izvodima necelog (racionalnog) reda. Rešenje dobijenog sistema parcijalnih diferencijalnih jednačina je dobijeno pomoću Euler-Bernoulli-jeve metode partikularnih integrala i transformacijom dobijenih običnih diferencijanih jednačina sa necelim izvidima po sopstvenim vremenskim funkcijama koje odgovaraju jednom od beskonačnog broja sopstvenih amplitudnih funckija. Određene su glavne koordinate sistema frakcionih svojstava i modovi parcijalnig oscilatora sistema frakcionih svojstava za slučajeve sopstvenih i prinudnih modova fracionih svojstava.

Ključne reči: *Hibridni sistem, elementi frakcionog redda, inercija translacije i rotacije, discretno kontinualni sloj frakcionih svojstava, membrane, disipacija frakcionih svojstava, generalisana funkcija disipacije energije frakcionih svojstava, sopstvene vremenske funkcije, sopstveni modovi frakcionih svojstava, sopstveni karakteristični brojevi sistema frakcionih svojstava..*

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STABILITY ANALYSIS OF STATIONARY OSCILLATORY REGIMES OF COUPLED DEFORMABLE BODIES SYSTEM

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Abstract. *The paper is addressed at analytical and numerical analysis on local stability of oscillatory regimes in systems of coupled deformable bodies. Systems consists of coupled deformable bodies like plates, beams, belts or membranes that are connected through visco-elastic non-linear layer, modeled by continuously distributed elements of Kelvin-Voigt type with nonlinearity of third order. Using the mathematical analogies the similarities of structural models in systems of plates, beams, belts or membranes is obvious. The structural models consists by a set of two coupled non-homogenous partial differential equations. The problems to solve are divided into space and time domains by classical Bernoulli-Fourier method. In the time domains the systems of nonlinear coupled ordinary differential equations are completely analog for different systems of deformable bodies and are solved by using the Krilov-Bogolyubov-Mitropolskiy asymptotic method. The first asymptotic approximation of the solutions describing stationary behavior, in the regions around the resonances, consists of four differential equations on amplitudes and phases of two nonlinear coupled modes. This paper presents the beauty of mathematical analytical calculus which could be the same even for physically different systems.*
The stability was investigated applying the Lyapunov's method and for stationary regimes stability we used the theorems of stability by linearized the obtained systems of solutions for amplitudes and phases of component harmonics in the vicinity of stationary solutions. The solutions of characteristic equation of linearized systems are numerically treated for any stationary values from resonant regimes and the conclusions of local stability have been obtained. The mathematical numerical calculus is powerful and useful tool for making the final conclusions between to many input and output values.

Key words: *phenomenological and mathematical analogy, multi-bodies system, non-linear dynamics, local stability, multi-frequency stationary regimes, mode interactions.*

1. Introduction

This paper presents systems of coupled deformable bodies: plates, beams, belts or membranes, with non-linear interconnecting layer. The mathematical non-linear descriptions are treated in a sense of making the qualitative analysis of the local stability behavior of such systems.

In engineering systems with non-linearity, high frequency excitations are the sources of multi frequency resonant regimes, appearance of high as well as low frequency coupled modes. The many experimental and theoretical research results ([2-5]) demonstrate that. List of the valuable research results in connected area of the objects of author's research is large; some of them may be finding in the reference lists of referred papers.

The study of transversal vibrations of a double, or multi, deformable bodies systems with elastic, visco-elastic or creep connections is important for both theoretical and pragmatic reason. Many important structures may be modeled from coupled systems and possess a big importance in many appliances. For example, in civil engineering for roofs, floors, walls, in thermo and acoustics isolation systems of walls and floors constructions, orthotropic bridge decks or for building any structural application in which the traditional method of construction uses stiffened steel.

An analysis of system energy of the excited modes depending on amplitudes, phases and frequencies of different non-linear modes are obtained with averaging and asymptotic methods for obtaining system of ordinary differential equations of amplitudes and phases in first approximations in [2, 5]. The energy analysis of mode interaction in the multi frequency free and forced vibration regimes of non-linear elastic systems (beams, plates, and belts) excited by initial conditions of system deformation and velocities, was made and a series of resonant jumps as well as energy transfer features were identified. Meaning that excitation was done by perturbation of equilibrium state of the double plate system at initial moment, defined by initial conditions for displacements and velocities of both plate middle surface points. Besides, two or more resonant energy jumps at the non-linear modes are present for the case of an external excitation in the resonant frequency range near one of the natural eigen frequency of the basic linear system.

Based on power of mathematical analytical and numerical analyses and noted analogies this paper gives an access to investigation of stability of stationary regimes in different systems of coupled deformable bodies. By applying the Lyapunov's method and the theorems of stability to the obtained systems of solutions for amplitudes and phases of component time harmonics are linearized. The features of roots of characteristic equations of linearized system for any values from the resonant regions give us information of stability of particular stationary regimes. It also clarifies the non-linear phenomena like: passing through resonant range with discrete values changes of external excitation frequencies and appearance of one or several resonant jumps in the amplitude-frequency and phase-frequency curves, like as the multi-non-linear mode mutual interactions between amplitudes and phases of different non-linear modes of stationary regimes.

2. Mathematical model of transversal vibrations of deformable bodies systems

When presenting physical models of deformable bodies systems shown in the Fig.1.a, b, c. then it is clear that the mathematical model of such a system may be expressed by the system of two coupled partial differential equations (1), [2, 3 and 5]. The partial differential equations in this case are formulated in terms of two unknowns: the transversal displacement $w_i(\wp,t)$, $i=1,2$ in direction of the z axis, of the upper body middle surface and of the lower body middle surface. We present the interconnecting layer as a model of one standard light visco-elastic element with started spring's length l_0 and non-linearity ot third order in the elastic part of the layer as shown in Fig.1d.

The system of partial differential equations (1) is derived by using the Principle of dynamic equilibrium in the following forms (see Refs. [2, 3 and 5]):

$$\frac{\partial^2 w_i(\wp,t)}{\partial t^2}+c_{(i)}^4\Delta\Delta w_i(\wp,t)-2\delta_{(i)}\left[\frac{\partial w_{i+1}(\wp,t)}{\partial t}-\frac{\partial w_i(\wp,t)}{\partial t}\right]- \tag{1}$$

$$-a_{(i)}^2\left[w_{i+1}(\wp,t)-w_i(\wp,t)\right]=\pm\varepsilon\beta_{(i)}\left[w_{i+1}(\wp,t)-w_i(\wp,t)\right]^3+\tilde{q}_{(i)}(\wp,t)\qquad for\qquad i=1,2$$

where are : $\mathfrak{D}_i=\dfrac{E_ih_i^3}{12\left(1-\mu_i^2\right)}$, $\varepsilon\beta_{(i)}=\dfrac{\beta}{\rho_ih_i}$, $a_{(i)}^2=\dfrac{c}{\rho_ih_i}$, $c_{(i)}^4=\dfrac{\mathfrak{D}_i}{\rho_ih_i}$, $2\delta_i=\dfrac{b}{\rho_ih_i}$, $\wp\equiv r,\varphi$ for circular plates; $\mathrm{B}_i=E_iI_x$, $c_{(i)}^4=\dfrac{\mathrm{B}_i}{\rho_iA_i}$, $a_{(i)}^2=\dfrac{c_e}{\rho_iA_i}$, $\varepsilon\beta_{(i)}=\dfrac{\beta}{\rho_iA_i}$, $2\delta_i=\dfrac{b}{\rho_iA_i}$, $\wp\equiv z$ for beams; or and the sign $\pm$ on the right hand side corresponds to the soft (sign +) or hard (sign -) properties of the non-linear elastic layer. We suppose that the functions of external excitation at nm-mode of oscillations are the two-frequency process in the form: $\tilde{q}_{(i)nm}(t)=h_{01nm}\cos[\Omega_{1nm}t+\phi_{1nm}]+h_{02nm}\cos[\Omega_{2nm}t+\phi_{2nm}]$. The solution, in one eigen mod of oscillation, for the system (1) is taken in the form of the eigen amplitude functions $W_{i(nm)}(r,\varphi)$, $n,m=1,2,3\ldots\infty$ for plates or membranes, or $W_{i(n)}(z)$, $n=1,2,3\ldots\infty$ for beams or belts, satisfying the same boundary conditions and orthogonally conditions, expansion with time coefficients in the form of unknown time functions $T_i(t)$, and describing their time evolution (see Refs. [2, 3 and 5]):

$$w_{i(eigen)}(r,\varphi,t)=\mathrm{W}_i(\wp)T_i(t)=\mathrm{W}_i(\wp)\left[K_i^{(1)}e^{-\hat{\delta}_1t}R_1(t)\cos\Phi_1(t)+K_i^{(2)}e^{-\hat{\delta}_2t}R_2(t)\cos\Phi_2(t)\right] \tag{2}$$

where are: K_{ij}^s co-factors of determinant corresponding to basic homegenous coupled system , $-\hat{\delta}_i$ and $\hat{p}_i$ are real and imaginary parts of the appropriate pair of the roots of the characteristic equation (see Refs. [2, 5]).

Amplitudes $R_i(t)$ and phases $\Phi_i(t)=\Omega_it+\phi_i(t)$ are unknown time functions which were obtained by using the Krilov-Bogolyubov-Mitropolyskiy asymptotic method. It is taken into account that defined task satisfies all necessary conditions for applying asymptotic method Krilov-Bogolyubov-Mitropolskiy concerning small parameter. The external excitation frequencies $\Omega_1\approx\hat{p}_1$ and $\Omega_2\approx\hat{p}_2$ are in the resonant range of the corresponding eigen frequencies of unperturbed linear system solutions. By applying the asymptotic method as well as the method of averaging to the right-hand sides of that system with respect to the full phases $\Phi_1^{(i)}(t)$ and $\Phi_2^{(i)}(t)$, we obtain the system of the first order differential equations according to unknown amplitudes and phases in the first asymptotic averaged approximation [2-4] as follow:

$$\begin{aligned}
\dot{a}_1(t) &= -\delta_1 a_1(t) - \frac{\varepsilon \mathrm{P}_1}{(\Omega_1+\hat{\mathrm{p}}_1)}\cos\phi_1 = \sigma_1\left(a_1(t), a_2(t), \phi_1(t), \phi_2(t), \Omega_{1s}, \Omega_{2s}\right)\\
\dot{\phi}_1(t) &= (\hat{\mathrm{p}}_1-\Omega_1) - \frac{3}{8}\frac{\alpha_1}{\hat{\mathrm{p}}_1}a_1^2(t) - \frac{1}{4}\frac{\beta_1}{\hat{\mathrm{p}}_1}a_2^2(t) + \frac{\varepsilon \mathrm{P}_1}{(\Omega_1+\hat{\mathrm{p}}_1)a_1(t)}\sin\phi_1 = \tau_1\left(a_1(t), a_2(t), \phi_1(t), \phi_2(t), \Omega_{1s}, \Omega_{2s}\right)\\
\dot{a}_2(t) &= -\delta_2 a_2(t) - \frac{\varepsilon \mathrm{P}_2}{(\Omega_2+\hat{\mathrm{p}}_2)}\cos\phi_2 = \sigma_2\left(a_1(t), a_2(t), \phi_1(t), \phi_2(t), \Omega_{1s}, \Omega_{2s}\right)\\
\dot{\phi}_2(t) &= (\hat{\mathrm{p}}_2-\Omega_2) - \frac{3}{8}\frac{\alpha_2}{\hat{\mathrm{p}}_2}a_2^2(t) - \frac{1}{4}\frac{\beta_2}{\hat{\mathrm{p}}_2}a_1^2(t) + \frac{\varepsilon \mathrm{P}_2}{(\Omega_2+\hat{\mathrm{p}}_2)a_2(t)}\sin\phi_2 = \tau_2\left(a_1(t), a_2(t), \phi_1(t), \phi_2(t), \Omega_{1s}, \Omega_{2s}\right)
\end{aligned} \tag{3}$$

where $a_i(t) = R_i(t)e^{-\hat{\delta}_i t}$ is the change of variables hence $\dot{a}_i(t) = \left(\dot{R}_i(t) - \hat{\delta}_i R_i(t)\right)e^{-\hat{\delta}_i t}$. The full forms of constants δ_i, α_i, β_i and p_i were presented in [5]. Here it was underlined that these constants all rely on coefficients of coupling properties via cofactors $\mathrm{K}_{2i}^{(s)}$, that δ_i depends of damping coefficients of visco-elastic layer $\tilde{\delta}_{(i)}$, $\varepsilon \mathrm{P}_i$ depend of excited amplitudes, and α_i, β_i of non-linearity layer properties. Coefficients β_i are coefficients of mode mutual interactions.

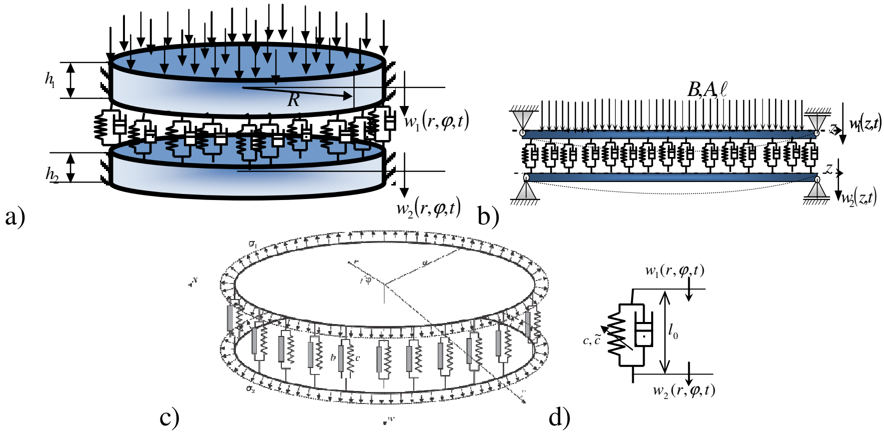

Fig. 1. A visco-elastically connected: a) double circular plate system; b) double beams system; c) membranes system; d) model of discrete element at visco-elastic non-linear interconnected layer.

It was observed the case when external distributed two-frequencies force acts at normal direction and along middle plain of upper body with frequencies near eigen circular frequencies of coupled linearized plate systems $\Omega_i \approx \hat{p}_i$. In this case the lower body is free of load. This means that we were observed the passing thought main resonant states by discrete changing the values of the forced frequencies. By using the first asymptotic approximation of the amplitudes and phases of multi frequency particular solutions of the non-linear system dynamics (3), we are in position to make analytical analysis of the stability of nonlinear modes in stationary regimes and to present results of theirs numerical solutions, for particular eigen mod of oscillations, $n, m = 1,2,3\ldots\infty$ for plates (membranes) or $n = 1,2,3\ldots\infty$ for beams (belts).

3. STABILITY OF THE STATIONARY REGIMES OF TRANSVERSAL VIBRATIONS OF COUPLED SYSTEMS

For the analysis of the stationary regime of oscillations, we equal the right hand sides of differential equations (3) with null. Eliminating the phases ϕ_1 and ϕ_2 we obtained system of two non-linear algebraic equations by unknown amplitudes a_1 and a_2 (for detail see Refs. [2, 5]). Also, with elimination of amplitudes a_1 and a_2, we obtained the algebraic equations for phases ϕ_1 and ϕ_2 in the case of two-frequencies forced oscillations in stationary regime of one eigen (nm for plates or mode n for beams) mode of double bodies system oscillations. Solving these algebraic systems by numerical Newton-Kantorovic's method in computer program Mathematica, we obtained stationary amplitudes and phases curves of two-frequencies regime of one eigen amplitude mode oscillations in double bodies system coupling with visco-elastic nonlinear layer depending on frequencies of external excitation force. If we fixed the value of on external excitation frequency of two possible, we obtained amlitude- and phase-frequency curves of stationary resonant vibration regime in the following forms:

1* for $\Omega_2 = const$ proper amplirude-frequency and phase-frequency curves are:

$a_1 = f_1(\Omega_1)$, $a_2 = f_2(\Omega_1)$, $\phi_1 = f_3(\Omega_1)$ and $\phi_2 = f_4(\Omega_1)$ and

2* for $\Omega_1 = const$ proper amplirude-frequency and phase-frequency curves:

$a_1 = f_5(\Omega_2)$, $a_2 = f_6(\Omega_2)$, $\phi_1 = f_7(\Omega_2)$ and $\phi_2 = f_8(\Omega_2)$.

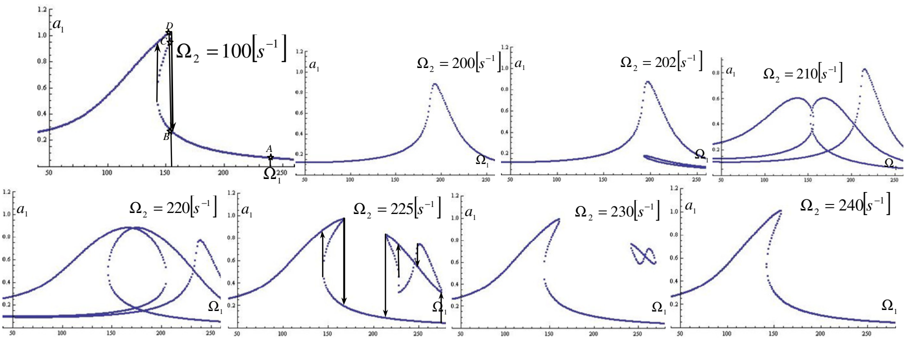

Fig. 2. Amplitude-frequency characteristic curves for the amplitudes of the first time harmonics $a_1 = f_1(\Omega_1)$ for hard characteristics of interconnected layer and for the different discrete values of excited frequency $\Omega_2 = const$ with noted proper one or more resonant jumps. Arrows means directions of the resonant jumps.

For any different discrete value of external force frequencies, we get characteristic diagram of that amplitide-frequency and proper phase-frequency curves. The Fig. 2 illustrates the series of that diagrams representing the passing through discrete stationary states alog resonant frequency intervals. We will follow the changes at that characteristics for the frequencies of external force in the range of eigen frequencies of coupling in one eigen amplitude mode of proper linearized system oscillations.

The phenomena of the resonant transition for stationary regime are evident from diagrams. Those are the distinctive jumps of the amplitude and phase response in the vicinity of the resonant values $\Omega_i \approx \hat{p}_i$, appearance of the new stable and unstable branches causing the more value-system responses and the emergence of two stable solutions of the system in the area of those new branches, the mutual interaction of the time harmonics and the jumps of the system energies.

Also, amplitude jumps are followed with new instability branches appearances, so there is more than one instability branch in the proper amplitude-frequency curves. It is visible that in the listed discrete values of the external excitation frequency from the proper resonant intervals two pairs plus one or three pairs with one more resonant jump appear together with proper instable branch presented by dot line in the listed diagrams.

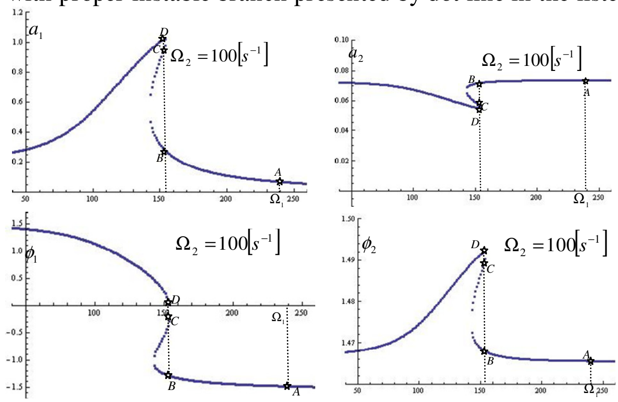

Fig. 3. Frequency characteristic curves for the amplitude of the first time harmonic $a_1 = f_1(\Omega_1)$, for the amplitude of the second time harmonic $a_2 = f_2(\Omega_1)$, for the phase of the first time harmonic $\phi_1 = f_3(\Omega_1)$ and for the phase of the second time harmonic $\phi_2 = f_4(\Omega_1)$ on discrete value of excited frequency $\Omega_2 = 100[s^{-1}]$

The stability was investigated applying the Lyapunov's method and for stationary regimes stability. We've used the theorems of stability by linearized the obtained systems of solutions for amplitudes and phases of component harmonics in the vicinity of stationary solutions. The data of stability or instability of the stationary amplitude and phase were obtained by using the linearization of the system of first approximation equations in each discrete stationary vibration state. The matrix notation of the stationary equations of system (3) has the form:

$$\begin{Bmatrix} \dot{a}_1 \\ \dot{\phi}_1 \\ \dot{a}_2 \\ \dot{\phi}_2 \end{Bmatrix} = \begin{bmatrix} \frac{\partial \sigma_1}{\partial a_1} & \frac{\partial \sigma_1}{\partial \phi_1} & \frac{\partial \sigma_1}{\partial a_2} & \frac{\partial \sigma_1}{\partial \phi_2} \\ \frac{\partial \tau_1}{\partial a_1} & \frac{\partial \tau_1}{\partial \phi_1} & \frac{\partial \tau_1}{\partial a_2} & \frac{\partial \tau_1}{\partial \phi_2} \\ \frac{\partial \sigma_2}{\partial a_1} & \frac{\partial \sigma_2}{\partial \phi_1} & \frac{\partial \sigma_2}{\partial a_2} & \frac{\partial \sigma_2}{\partial \phi_2} \\ \frac{\partial \tau_2}{\partial a_1} & \frac{\partial \tau_2}{\partial \phi_1} & \frac{\partial \tau_2}{\partial a_2} & \frac{\partial \tau_2}{\partial \phi_2} \end{bmatrix}_{a_{is},\phi_{is},\Omega_s} \begin{Bmatrix} a_1 \\ \phi_1 \\ a_2 \\ \phi_2 \end{Bmatrix} = \mathbf{J}_{a_{is},\phi_{is},\Omega_s} \begin{Bmatrix} a_1 \\ \phi_1 \\ a_2 \\ \phi_2 \end{Bmatrix} = \mathbf{0} \quad , \ i=1,2 \tag{4}$$

where the $\mathbf{J}_{a_{is},\phi_{is},\Omega_s}$ is Jacobi-an matrix of system (3). The eigen values of that matrix need to be known, consequently corresponding characteristic equation was composed in the form:

$$\begin{vmatrix} -\delta_1-\lambda & \frac{\varepsilon P_1 \sin\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)} & 0 & 0 \\ -\frac{3}{4}\frac{\alpha_1}{\hat{p}_1}a_{1s} - \frac{\varepsilon P_1 \sin\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)a_{1s}^2} & \frac{\varepsilon P_1 \cos\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)a_{1s}} - \lambda & -\frac{1}{2}\frac{\beta_1}{\hat{p}_1}a_{2s} & 0 \\ 0 & 0 & -\delta_2-\lambda & \frac{\varepsilon P_2 \sin\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)} \\ -\frac{1}{2}\frac{\beta_2}{\hat{p}_2}a_{1s} & 0 & -\frac{3}{4}\frac{\alpha_2}{\hat{p}_2}a_{2s} - \frac{\varepsilon P_2 \sin\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)a_{2s}^2} & \frac{\varepsilon P_2 \cos\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)a_{2s}} - \lambda \end{vmatrix}_{a_{is},\phi_{is},\Omega_s} = 0 \quad , i=1,2$$

or in the extend form:

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \tag{5}$$

where the coefficients have the forms:

$$A = \delta_1 + \delta_2 - \varepsilon\left(\frac{P_1\cos\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)a_{1s}} + \frac{P_2\cos\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)a_{2s}}\right),$$

$$B = \frac{3}{4}\frac{\alpha_1}{\hat{p}_1}a_{1s}\frac{\varepsilon P_1\sin\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)} + \frac{\varepsilon^2 P_1^2\sin^2\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)^2 a_{1s}^2} - \frac{\varepsilon P_1\cos\phi_{1s}(\delta_2+\delta_1)}{(\Omega_{1s}+\hat{p}_1)a_{1s}} + \frac{3}{4}\frac{\alpha_2}{\hat{p}_2}a_{2s}\frac{\varepsilon P_2\sin\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)} + \frac{\varepsilon^2 P_2^2\sin^2\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)^2 a_{2s}^2} -,$$
$$-\frac{\varepsilon P_2\cos\phi_{2s}(\delta_2+\delta_1)}{(\Omega_{2s}+\hat{p}_2)a_{2s}} + \frac{\varepsilon P_2\cos\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)a_{2s}}\frac{\varepsilon P_1\cos\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)a_{1s}} + \delta_2\delta_1$$

$$C = \left(\delta_2 - \frac{\varepsilon P_2\cos\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)a_{2s}}\right)\left(\frac{3}{4}\frac{\alpha_1}{\hat{p}_1}a_{1s}\frac{\varepsilon P_1\sin\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)} + \frac{\varepsilon^2 P_1^2\sin^2\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)^2 a_{1s}^2} - \frac{\varepsilon P_1\cos\phi_{1s}\delta_1}{(\Omega_{1s}+\hat{p}_1)a_{1s}}\right) + \quad \text{and}$$
$$+\left(\delta_1 - \frac{\varepsilon P_1\cos\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)a_{1s}}\right)\left(\frac{3}{4}\frac{\alpha_2}{\hat{p}_2}a_{2s}\frac{\varepsilon P_2\sin\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)} + \frac{\varepsilon^2 P_2^2\sin^2\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)^2 a_{2s}^2} - \frac{\varepsilon P_2\cos\phi_{2s}\delta_2}{(\Omega_{2s}+\hat{p}_2)a_{2s}}\right)$$

$$D = \frac{1}{4}\frac{\beta_1\beta_2}{\hat{p}_1\hat{p}_2}a_{2s}a_{1s}\frac{\varepsilon^2 P_2 P_1\sin\phi_{2s}\sin\phi_{1s}}{(\Omega_{2s}+\hat{p}_2)(\Omega_{1s}+\hat{p}_1)} +$$
$$+\left(\frac{3}{4}\frac{\alpha_1}{\hat{p}_1}a_{1s}\frac{\varepsilon P_1\sin\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)} + \frac{\varepsilon^2 P_1^2\sin^2\phi_{1s}}{(\Omega_{1s}+\hat{p}_1)^2 a_{1s}^2} - \frac{\varepsilon P_1\cos\phi_{1s}\delta_1}{(\Omega_{1s}+\hat{p}_1)a_{1s}}\right)\left(\frac{3}{4}\frac{\alpha_2}{\hat{p}_2}a_{2s}\frac{\varepsilon P_2\sin\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)} + \frac{\varepsilon^2 P_2^2\sin^2\phi_{2s}}{(\Omega_{2s}+\hat{p}_2)^2 a_{2s}^2} - \frac{\varepsilon P_2\cos\phi_{2s}\delta_2}{(\Omega_{2s}+\hat{p}_2)a_{2s}}\right).$$

The values of these coefficients need to be valued for any value of Ω_{is} and determined values of a_{is} for $i=1,2$ from the above diagrams and of ϕ_{is} from proper diagrams of phase-frequencies curves. Than, the corresponding roots of the equation (5) are obtained numerically.

If all real parts of the all roots of the characteristic equation are negative, then stationary resonant regime is stable. For example, the noted star point A, on a diagrams at Fig. 3, with coordinate $(a_1;\phi_1;a_2;\phi_2;\Omega_1) = (0.0654;-1.485;0.0734;1.465;240)$, is stable because the roots of equation (5) are all complex with negative real parts: $(\lambda_{1/2}, \lambda_{3/4}) = (-11.121 \pm 125.196i; -8.347 \pm 29.577i)$.

If only one is positive, then stationary resonant regime is unstable, per example star point C. It has coordinate $(a_1;\phi_1;a_2;\phi_2;\Omega_1) = (0.9785;-0.2881;0.0561;1.4904;154)$ and gives the complex roots of equation (5): $(\lambda_{1/2}, \lambda_{3/4}) = (9.084 \pm 15.35i; -24.313 \pm 12.686i)$, two with positive real parts. The start points D with coordinate

$(a_1;\phi_1;a_2;\phi_2;\Omega_1) = (1.0204;0.0183;0.0549;1.492;154)$ and roots :

$(\lambda_{1/2}, \lambda_3, \lambda_4)_D = (-6{,}355 \pm 13.531i; -0.324; -17.398)$, and point B with coordinate:

$(a_1;\phi_1;a_2;\phi_2;\Omega_1) = (0.2751;-1.2978;0.0717;1.467;154)$ and roots :

$(\lambda_{1/2}, \lambda_3, \lambda_4)_B = (-3.063 \pm 13.069i; -15.541; -40.209)$ are stable on the same external frequency, $\Omega_1 = 154 s^{-1}$, like as point C. These three star points present trigger of the coupled three singularities- two stable stationary values and one unstable saddle type value [1]. A series of the amplitude-frequency curves of the two-frequency like vibration regimes are obtained numerically and presented with noted branches corresponding to unstable stationary vibration resonant regimes. In the listed diagrams of Fig. 2 branches presented with dot line corresponds to unstable stationary vibration regimes.

4. Concluding Remarks

By using asymptotic method Krilov-Bogolyubov-Mitropoksiy, we solved system of PDE`s (1) semi analytically in averaged asymptotic first approximation to analyzed the stationary regimes of forced resonant non-linear oscillations for presented model of double bodies system. This paper presents the beauty of mathematical analytical calculus which could be the same even for physically different systems. The mathematical numerical calculus is powerful and useful tool for making the final conclusions between to many output values. One step (part) of solutions were obtained numerically and presented at a series of the amplitudes-frequency characteristics. We could conclude that there exist complexities in the system forced non-linear response, depending of initial conditions and of proper relation between the system kinetic parameters.

Non-linearity is source for appearing two resonant jumps at the amplitude-frequency and phase-frequency curves in the resonant frequency interval. Between two jumps there appear three or five, or seven or more singular values of the stationary amplitudes and phases with alternatively stable and unstable values which build coupled singularities and trigger of coupled singularities, two stable around one unstable amplitudes and corresponding phases. Passing through resonant ranges of the external excitation frequencies unique values of the stationary amplitudes and phases lose its stability and split into trigger of the coupled three singularities- two stable stationary values and one unstable saddle type of the amplitudes (or phases) for simple case without non-linear interactions between time modes. But, in the case when there are resonant interactions between modes, more than one pair of the resonant jumps appears, and there are possibilities for appearance of the coupled triggers of the coupled singularities consisting of odd number of the alternating coupled stable and unstable singularities, [1]. The mentioned instabilities of the stationary vibration regimes are associated with Hopf bifurcations in mathematical descriptions of the first asymptotic approximations of solutions.

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REFERENCES

1. Hedrih (Stevanović) K., A Trigger of Coupled Singularities, *MECCANICA*, Vol.39, No. 3, 2004., pp. 295-314. International Journal of the Italian Association of Theoretical and Applied Mechanics, Springer
2. Hedrih K, Simonović J (2012) Energies of the Dynamics in a Double Plate Nonlinear System, *International Journal of Bifurcation and Chaos*, Vol. 21, No. 10, pp. 2993-3011.
3. Hedrih K, Simonović J (2010) Non-linear dynamics of the sandwich double circular plate system, *International Journal of Non-Linear Mechanics* 45 (9) 902-918.
4. Nayfeh A.H. (2000) *Nonlinear Interactions: Analytical, Computational and Experimental Methods*, Wiley, New York
5. Simonović J (2012) Dynamics and Stability of Dynamics Hybrid Systems, *doctoral dissertation,* Faculty of Mechanical Engineering, University of Niš, p. 341.

АНАЛИЗА СТАБИЛНОСТИ СТАЦИОНАРНИХ РЕЖИМА ОСЦИЛОВАЊА СИСТЕМА СПРЕГНУТИХ ДЕФОРМАБИЛНИХ ТЕЛА

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Апстракт. *Рад је посвећен аналитичкој и нумеричкој анализи локалне стабилности осцилаторних режима у системима спрегнутих деформабилних тела. Системи су сачињени од ситема деформабилних тела: како што су плоче, греде, појасеви или мемебране, која су повезана виско-еластичним нелинеарним слојем. Слој је моделираним као систем континуално раподељених дискретних елемената типа Келвин-Воигт-а са нелинеарношћу трећег реда. Познавајући математичке аналогије јасно је да постоји сличност структурних модела система плоча, греда, појасева или мембране. Структурни модели се састоје од по две спрегнуте нехомогене парцијалне диференцијалне једначине. Проблем је најпре раздвојен на временски и просторни домен применом класичне Берноулли-Фоуриер-ове методе. У временском домену систем две спрегнуте нехомогене обичне диференцијалне једначине потпуно је аналоган за различите системе деформабилних тела и решаван је Крилов-Боголуубов-Митрополскиу асимптотском методом. Прва асимптотска апроксимација решења која*

описује стационарна понашања система у резонантној области састоји се од система четири диференцијалне једначине по амплитудама и фазама два нелинеарна спрегнута мода. У овом смислу рад представља лепоту математичког аналитичког рачуна који може бити исти за физички различите системе.
Стабилност је проучавана применом Луапунов-љеве методе и теореме стабилности за стационарне режиме по којој се линеаризује изведени систем апроксимација решења за амплитуде и фазе компонентних хармоника у околини стационарних решења. Решења карактеристичне једначине линеаризованог система добијају се нумерички за све стационарне вредности из резонантних области и добијају се закључци о локалној стабилоности. У том смислу показано је да је нумерички прорачун моћан и користан алат за крајње закључке при великом броју улазних и излазних вредности.

Кључне речи: *феноменолошка и математичка аналогија, системи више тела, нелинеарна динамика, локална стабилност, више-фреквентни резонантни режими, интеракција модова.*

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ANALYSIS OF HAMILTON'S PRINCIPLE FOR NONLINEAR NONHOLONOMIC SYSTEMS WITH REGARD TO TRANSPOSITIONAL RELATIONS IN QUASICOORDINATES

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Abstract. *The paper analyzes Hamilton's principle for nonlinear nonholonomic systems with regard to different viewpoints on transpositional relations in the mechanics of nonholonomic systems, for both real coordinates and quasicoordinates. In the first part it is shown why the application of variation instead of mechanical variation results in the correct form of Hamilton's principle, wherefrom there follow the differential equations of motion. Thereafter, a detailed analysis proves why the approach presented in [5] is the only one applicable, i.e., the operator of variation and the operator of differentiation are commutative for all quasicoordinates.*

Key words: *nonholonomic, nonlinear, principle, transposition, quasicoordinate*

1. FORMS OF HAMILLTON'S PRINCIPLE FOR NONLINEAR NONHOLONOMIC SYSTEMS IN QUASICOORDINATES

In literature, there are two different viewpoints on the formation of Hamilton's principle for the case of nonholonomic systems. According to the first, maintained by Hertz, Appel and Suslov, the operators of variation and differentiation are not commutative for the case of kinematically dependent coordinates. However, according to the second, advocated by Hölder, the operators of variation and differentiation are commutative for all real coordinates. As for quasicoordinates, in literature (for example, [1], [2], [3], [4]) there is an undivided attitude that the operators of variation and differentiation are non-commutative for all quasicoordinates. In this paper, the problem is solved only for the case when quasicoordinates are introduced and the initial approach is that the operators of variation and differentiation are always commutative [5]. The same approach, i.e., of Hölder's, is also applicable only for the case of real coordinates, as shown in [3], [6].

Let us consider a mechanical system subject to conservative forces, whose position is determined at an arbitrary instant of time by Lagrange coordinates q^i, but its motion is limited by nonlinear nonholonomic constraints of the form

$$\varphi^{\nu}\left(q^i,\dot{q}^i\right)=0.^{*} \tag{1.1}$$

The equations that determine the relations between the quasivelocities (derivatives with respect to time of quasicoordinates π^i) and generalized velocities have the form

$$\dot{\pi}^i=\theta^i\left(q^j,\dot{q}^j\right). \tag{1.2}$$

Eqs (1.2) can be also written in the form (where $\det\left[\frac{\partial\theta^i}{\partial\dot{q}^j}\right]\neq 0,\ \frac{\partial\theta^i}{\partial\dot{q}^j}\frac{\partial\psi^j}{\partial\dot{\pi}^k}=\delta^i_k$)

$$\dot{q}^i=\psi^i\left(q^j,\dot{\pi}^j\right). \tag{1.3}$$

Now, let nonlinear nonholonomic constraints (1.1) be determined by equations

$$\dot{\pi}^{\nu}=\theta^{\nu}\left(q^j,\dot{q}^j\right)=\varphi^{\nu}\left(q^j,\dot{q}^j\right)=0\,. \tag{1.4}$$

Based on Eqs (1.2), we find

$$\delta\dot{\pi}^i=\delta\theta^i=\frac{\partial\theta^i}{\partial q^j}\delta q^j+\frac{\partial\theta^i}{\partial\dot{q}^j}\delta\dot{q}^j \tag{1.5}$$

however, based on the constraints between virtual variations

$$\delta\pi^i=\frac{\partial\theta^i}{\partial\dot{q}^j}\delta q^j \tag{1.6}$$

for the derivatives with respect to time $\frac{d}{dt}\delta\pi^i$ we obtain

$$\frac{d}{dt}\delta\pi^i=\frac{d}{dt}\left(\frac{\partial\theta^i}{\partial\dot{q}^j}\delta q^j\right). \tag{1.7}$$

From (1.5) and (1.7) there follow

$$\frac{d}{dt}\delta\pi^i-\delta\dot{\pi}^i=\frac{\partial\theta^i}{\partial\dot{q}^j}\left[\frac{d}{dt}\left(\delta q^j\right)-\delta\dot{q}^j\right]+\varepsilon^i_k\delta\pi^k \tag{1.8}$$

where the coefficients ε^i_k are determined by expressions

$$\varepsilon^i_k=\left(\frac{\partial^2\theta^i}{\partial\dot{q}^j\partial q^r}\dot{q}^r+\frac{\partial^2\theta^i}{\partial\dot{q}^j\partial\dot{q}^r}\ddot{q}^r-\frac{\partial\theta^i}{\partial q^j}\right)\frac{\partial\psi^j}{\partial\dot{\pi}^k}\,. \tag{1.9}$$

*Einstein summation convention is used in the paper. Indices take the following values: $i, j, k, r, s = 1, \ldots, n$; $\alpha = 1, \ldots, m$; $\rho = m+1, \ldots, m+l = n$

Assuming Hölder's viewpoint on commutability properties of the operators of variation and differentiation for all real coordinates ([3], [6]), Eqs (1.8) read

$$\frac{d}{dt}\delta\pi^i - \delta\dot{\pi}^i = \varepsilon^i_k \delta\pi^k \tag{1.8'}$$

and, based on above, an approach to non-commutability of the operators of variation and differentiation for all quasicoordinates ([1], [2], [3], [4]) is derived.

Prior to performing the analysis of the relations (1.8') that are inconsistent with the rules of variational calculus, let us derive the form of Hamilton's principle in quasicoordinates. Starting from

$$\int_{t_0}^{t_1} \delta L dt = 0 \tag{1.10}$$

and taking into account that the application of mechanical principles requires that the constraints between virtual variations hold

$$\delta\pi^i = \frac{\partial\theta^i}{\partial\dot{q}^j}\delta q^j \tag{1.6}$$

that is

$$\frac{d}{dt}\delta\pi^i = \frac{d}{dt}\left(\frac{\partial\theta^i}{\partial\dot{q}^j}\delta q^j\right) \tag{1.7}$$

let us first calculate the variation of the Lagrange function $\tilde{L} = L_{(\dot{q}^i = \psi^i)} = L(q^i, \dot{\pi}^i)$. It reads

$$\delta\tilde{L} = \frac{\partial\tilde{L}}{\partial q^k}\delta q^k + \frac{\partial\tilde{L}}{\partial\dot{\pi}^k}\delta\dot{\pi}^k = \left(\frac{\partial L}{\partial q^k}\delta q^k + \frac{\partial L}{\partial\dot{q}^r}\frac{\partial\psi^r}{\partial q^k}\delta q^k\right) + \frac{\partial L}{\partial\dot{q}^r}\frac{\partial\psi^r}{\partial\dot{\pi}^k}\delta\dot{\pi}^k \tag{1.11}$$

where from it follows that

$$\frac{\partial L}{\partial q^k}\delta q^k = \delta\tilde{L} - \frac{\partial L}{\partial\dot{q}^r}\left(\frac{\partial\psi^r}{\partial q^k}\delta q^k + \frac{\partial\psi^r}{\partial\dot{\pi}^k}\delta\dot{\pi}^k\right)$$

the variation of the Lagrange function δL now reads

$$\delta L = \frac{\partial L}{\partial q^k}\delta q^k + \frac{\partial L}{\partial\dot{q}^k}\delta\dot{q}^k = \delta\tilde{L} - \frac{\partial L}{\partial\dot{q}^r}\left(\frac{\partial\psi^r}{\partial q^k}\delta q^k + \frac{\partial\psi^r}{\partial\dot{\pi}^k}\delta\dot{\pi}^k\right) + \frac{\partial L}{\partial\dot{q}^k}\delta\dot{q}^k. \tag{1.12}$$

In further calculations we obtain

$$\delta L = \delta\tilde{L} + \frac{\partial L}{\partial \dot{q}^r}\left(\delta\dot{q}^r - \delta\psi^r\right) = \delta\tilde{L} + \frac{\partial L}{\partial \dot{q}^r}\left(\frac{d}{dt}\delta q^r - \delta\psi^r\right)$$

$$\delta L = \delta\tilde{L} + \frac{\partial \tilde{L}}{\partial \dot{\pi}^i}\gamma_k^i\delta\pi^k \tag{1.13}$$

where the coefficients γ_k^i are determined by expressions

$$\gamma_k^i = -\frac{\partial\theta^i}{\partial \dot{q}^r}\left(\frac{\partial^2\psi^r}{\partial\dot{\pi}^k\partial q^s}\dot{q}^s + \frac{\partial^2\psi^r}{\partial\dot{\pi}^k\partial\dot{\pi}^s}\ddot{\pi}^s - \frac{\partial\psi^r}{\partial q^s}\frac{\partial\psi^s}{\partial\dot{\pi}^k}\right). \tag{1.14}$$

Based on Eq. (1.13), the expression (1.10) reads

$$\int_{t_0}^{t_1}\delta L dt = \int_{t_0}^{t_1}\left(\delta\tilde{L} - \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\gamma_k^i\delta\pi^k\right)dt = 0\,. \tag{1.15}$$

For nonlinear nonholonomic systems, taking into account that the expression for

$$\dot{\pi}^\nu = \theta^\nu = \varphi^\nu = 0 \rightarrow \delta\pi^\nu = 0$$

$\delta\tilde{L}$ has the form

$$\delta\tilde{L} = \frac{\partial\tilde{L}}{\partial q^k}\delta q^k + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\nu}\delta\dot{\pi}^\nu = \frac{\partial\tilde{L}}{\partial q^k}\delta q^k + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\nu}\frac{d}{dt}\delta\pi^\nu \tag{1.16}$$

where due to the constraints $\dot{\pi}^\nu = 0$

$$\left[\delta\tilde{L}\right]_{\dot{\pi}^\nu=0} = \left[\frac{\partial\tilde{L}}{\partial q^k}\right]_{\dot{\pi}^\nu=0}\delta q^k + \left[\frac{\partial\tilde{L}}{\partial\dot{\pi}^\alpha}\right]_{\dot{\pi}^\nu=0}\frac{d}{dt}\delta\pi^\alpha$$

the variation of the function $\tilde{L}^* = \tilde{L}_{(\dot{\pi}^\nu=0)} = L\left(q^i, \dot{\pi}^\alpha\right)$ has the form

$$\delta L^* = \frac{\partial\tilde{L}^*}{\partial q^k}\delta q^k + \frac{\partial\tilde{L}^*}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha = \left(\frac{\partial\tilde{L}}{\partial q^k} + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\nu}\frac{\partial\dot{\pi}^\nu}{\partial q^k}\right)\delta q^k + \left(\frac{\partial\tilde{L}}{\partial\dot{\pi}^\alpha} + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\nu}\frac{\partial\dot{\pi}^\nu}{\partial\dot{\pi}^\alpha}\right)\delta\dot{\pi}^\alpha$$

$$\partial\tilde{L}^* = \frac{\partial\tilde{L}}{\partial q^k}\delta q^k + \frac{\partial\tilde{L}}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha = \partial\tilde{L}\,. \tag{1.17}$$

So, the final form of Hamilton's principle for nonlinear nonholonomic mechanical systems in quasicoordinates is

$$\int_{t_0}^{t_1}\left(\delta\tilde{L} + \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\gamma_k^i\delta\pi^k\right)dt = \int_{t_0}^{t_1}\left(\delta\tilde{L}^* + \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha + \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\gamma_\nu^i\delta\pi^\nu\right)dt\,.$$

Respectively

$$\int_{t_0}^{t_1}\left(\delta\tilde{L}^{*}-\frac{\partial\tilde{L}}{\partial\dot{\pi}^{i}}\gamma_{\alpha}^{i}\delta\pi^{\alpha}\right)dt\,. \tag{1.18}$$

It can be shown that such form of Hamilton's principle (1.18) can be also obtained when using the variations

$$\delta\pi^{i}=\frac{\partial\theta^{i}}{\partial\dot{q}^{j}}\delta q^{j} \tag{1.6}$$

and the variations

$$\delta\dot{\pi}^{i}=\delta\theta^{i}=\frac{\partial\theta^{i}}{\partial q^{j}}\delta q^{j}+\frac{\partial\theta^{i}}{\partial\dot{q}^{j}}\delta\dot{q}^{j}\,. \tag{1.5}$$

G.K. Suslov considered that for the case of nonholonomic systems D'Alembert's principle should be transformed in the following way, [7]. Starting from D'Alembert's principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}^{k}}\right)\delta q^{k}-\frac{\partial L}{\partial\dot{q}^{k}}\delta q^{k}=0 \tag{1.19}$$

or from the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}^{k}}\delta q^{k}\right)-\left(\frac{\partial L}{\partial q^{k}}\delta q^{k}+\frac{\partial L}{\partial\dot{q}^{k}}\delta\dot{q}^{k}\right)-\frac{\partial L}{\partial\dot{q}^{k}}\left(\frac{d}{dt}\delta q^{k}-\delta\dot{q}^{k}\right)=0 \tag{1.20}$$

taking into account that

$$\delta q^{k}=\frac{\partial\psi^{k}}{\partial\dot{\pi}^{i}}\delta\pi^{i}\,.$$

Eq. (1.20) in quasicoordinates has the form as follows

$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\pi}^{i}}\delta\pi^{i}\right)-\left[\frac{\partial\tilde{L}}{\partial q^{k}}\frac{\partial\psi^{k}}{\partial\dot{\pi}^{i}}\delta\pi^{i}+\frac{\partial\tilde{L}}{\partial\dot{\pi}^{s}}\frac{\partial\theta^{s}}{\partial\dot{q}^{k}}\left(\frac{d}{dt}\frac{\partial\psi^{k}}{\partial\dot{\pi}^{i}}\right)\delta\pi^{i}-\frac{\partial\tilde{L}}{\partial\dot{\pi}^{k}}\frac{\partial\psi^{k}}{\partial q^{i}}\delta\pi^{i}+\frac{\partial\tilde{L}}{\partial\dot{\pi}^{i}}\delta\dot{\pi}^{i}\right]-$$

$$-\frac{\partial\tilde{L}}{\partial\dot{\pi}^{i}}\left[\frac{d}{dt}\delta\pi^{i}-\delta\dot{\pi}^{i}\right]=0\,. \tag{1.21}$$

Taking into account that concurrently (1.5) and (1.6) hold, we would arrive at the proper form of Hamilton's principle, wherefrom there follow correct differential equations of motion of nonholonomic systems

$$\int_{t_0}^{t_1}\left(\delta\tilde{L}^* - \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\gamma^i_\alpha\delta\pi^\alpha\right)dt = 0\,. \tag{1.18}$$

A question can be raised in what is a contradiction when using variations (1.5) instead of (1.6) (as required by the principles of mechanics) one arrives at correct form of Hamilton's principle (1.18).

Note that D'Alembert's principle (1.19) can be written in the form (1.21) in quasicoordinates, i.e., in the form

$$\begin{aligned}\int_{t_0}^{t_1}\Bigg(\Bigg[-\frac{\partial\tilde{L}}{\partial q^k}\frac{\partial\psi^k}{\partial\dot{\pi}}\delta\pi^i - \frac{\partial\tilde{L}}{\partial\dot{\pi}^s}\frac{\partial\theta^s}{\partial\dot{q}^k}\left(\frac{d}{dt}\frac{\partial\psi^k}{\partial\dot{\pi}^i}\right)\delta\pi^i + \frac{\partial\tilde{L}}{\partial\dot{\pi}^k}\frac{\partial\psi^k}{\partial q^i}\delta\pi^i - \\ - \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\delta\dot{\pi}^i - \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\frac{d}{dt}\delta\pi^i + \frac{\partial\tilde{L}}{\partial\dot{\pi}^i}\delta\dot{\pi}^i\Bigg]\Bigg)dt = 0\end{aligned} \tag{1.22}$$

using the conditions

$$\delta\pi^i_{(t_0)} = \delta\pi^i_{(t_1)} = 0\,. \tag{1.23}$$

Based on Eq. (1.22), it is evident that any assumption on the values of $\delta\dot{\pi}^i$ does not affect the final result of transformation (1.22). Accordingly, to obtain the correct form of Hamilton's principle, the assumption for variations $\delta\pi^i$ (but not for $\delta\dot{\pi}^i$) is of relevance grounded on a physical model of the corresponding constraints and presented in the form of a basic assumption.

2. APPROACH CORRECTNES ANALYSIS OF TRANSPOSITIONAL RELATIONS IN QUASICOORDINATES

Hamilton's principle is applicable for nonholonomic systems, because it is derived from D'Alembert's principle that holds for both holonomic and nonholonomic systems, [3]. However, Hamilton's principle for the case of nonholonomic systems does not have identical meaning as it does for the case of holonomic systems, [7], [8]. Namely, it does not differ significantly from the transformed D'Alembert's principle and does not have the stationary property of action in a general case, [7].

However, if it is taken into account that concurrently hold

$$\delta\dot{\pi}^i = \delta\theta^i \tag{1.5}$$

and

$$\delta\pi^i = \frac{\partial\theta^i}{\partial\dot{q}^j}\delta q^j \tag{1.6}$$

there occurs contradiction, because simultaneous fulfillment of conditions (1.5) and (1.6) means either the variation of the trajectory into itself or the integrability of constraints (in a general case, in nonholonomic systems the constraints are not maintained on the varied trajectory). In holonomic systems the variations (1.5) and (1.6) can be used, however, not simultaneously and in the corresponding variational problems, [9]. Variations (1.5) differ from variations (1.6) in that that the system with variations (1.5) does not perform virtual displacements.

$$\delta\int_{t_0}^{t_1}\left[\tilde{L} + \lambda_i\left(\dot{\pi}^i - \theta^i\right) + \nu_\rho \dot{\pi}^\rho\right]dt = 0 \tag{2.1}$$

$$\int_{t_0}^{t_1}\left\{\delta\left[\tilde{L} + \lambda_\alpha\left(\dot{\pi}^\alpha - \theta^\alpha\right)\right] + \frac{\partial \tilde{L}}{\partial \dot{\pi}^\rho}\gamma_i^\rho \delta q^i + \nu_\rho \delta\pi^\rho + \mu_\rho\left(\delta\pi^\rho - \frac{\partial\theta^\rho}{\partial \dot{q}^i}\delta q^i\right)\right\}dt = 0 \tag{2.2}$$

that is, two different problems.

The equation (2.1) represents a conditional variational problem and expresses the condition of stationarity of the functional. The problem (2.2) does not express the condition of stationarity of the functional and because of that the Hamilton principle for nonholonomic system differs from the Hamilton principle for holonomic systems. Hence, if the conditions (1.5) and (1.6) are used simultaneously, we have a contradiction, as these conditions belong to different variational problems. The variations set up by the expression (1.5) differ from the variations set up by the expression (1.6), because the particles of the system with the variations (1.5) do not perform virtual displacements.

Let us form, according to (1.3), a system of n differential equations that can be used to determine variations δq^k We obtain

$$\delta\dot{q}^k = \delta\psi^k\left(q^r, \dot{\pi}^\alpha\right) \to \frac{d}{dt}\delta q^k = \frac{\partial\psi^k}{\partial q^r}\delta q^r + \frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha \tag{2.3}$$

$$\frac{d}{dt}\delta q^k - \left(\frac{\partial\psi^k}{\partial q^r}\right)_{(t)}\delta q^r = \left(\frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\right)_{(t)}\frac{d}{dt}\delta\pi^\alpha \tag{2.3'}$$

whose solution from the condition $\delta q_0^k = 0$ (condition $\delta q_0^k = 0$ does not disturb the general solution) read

$$\delta q^k = \frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\delta\pi^\alpha + \phi_r^k\int_{t_0}^{t}\Gamma_s^r\gamma_{\alpha(t)}^i\left(\frac{\partial\psi^s}{\partial\dot{\pi}^i}\right)_{(t)}\delta\pi^\alpha dt\,. \tag{2.4}$$

Where

$$\phi_r^k \Gamma_s^r = \delta_s^k$$

ϕ_r^k is a fundamental matrix of a corresponding homogeneous system of differential equations (2.3'). Based on this, evidently, there is a difference between variations determined by (1.5) and variations determined by (1.6). Manners of varying (1.5) and (1.6) will coincide only for the case when

$$\int_{t_0}^{t} \Gamma_s^r \gamma_{\alpha(t)}^i \left(\frac{\partial \psi^s}{\partial \dot{\pi}^i} \right)_{(t)} \delta\pi^\alpha dt = 0 \qquad (2.5)$$

respectively, when $\gamma_{\alpha(t)}^i = 0$, i.e., when the expressions (1.2) and (1.3) are integrable, [5], [6].

Now, it is shown that using variations (2.4) leads to another variational problem. As shown from (1.3) there follows (from the condition $\delta q_0^k = 0$)

$$\delta q^k = \frac{\partial \psi^k}{\partial \dot{\pi}^\alpha} \delta\pi^\alpha + \phi_r^k \int_{t_0}^{t} \Gamma_s^r \frac{\partial \psi^s}{\partial \dot{\pi}^i} \gamma_\alpha^i \delta\pi^\alpha dt .$$

For the case of variations

$$\delta \dot{q}^r = \delta \psi^r$$

from Eq. (1.12) for the variation of Lagrange's function it is obtained

$$\delta L = \delta \tilde{L} + \frac{\partial L}{\partial \dot{q}^r} \delta\left(\dot{q}^r - \psi^r\right) = \delta \tilde{L}$$

and as shown by Eq. (1.17) there follows

$$\delta \tilde{L} = \delta \tilde{L}^* .$$

(If Hamilton's principle is applied now, for example, to nonlinear nonholonomic system of the Chaplygin type, we obtain

$$\int_{t_0}^{t_1} \delta L dt = \int_{t_0}^{t_1} \delta \tilde{L}^* dt = \int_{t_0}^{t_1} \delta \tilde{L}^* \left(q^\alpha, \dot{\pi}^\alpha\right) dt = 0 \rightarrow \frac{d}{dt} \frac{\partial \tilde{L}^*}{\partial \dot{\pi}^\alpha} - \frac{\partial \tilde{L}^*}{\partial q^\alpha} = 0$$

i.e. equations that do not correspond to the motion of a mechanical system. This means that the relation $\delta \tilde{L} = \delta \tilde{L}^*$ is inconsistent with Hamilton's principle.)

Accordingly, for this case, Hamilton's principle reads

$$\int_{t_0}^{t_1} \delta L dt = \int_{t_0}^{t_1} \delta \tilde{L} dt = \int_{t_0}^{t_1} \delta \tilde{L}^* dt = 0 \qquad (2.6)$$

that is

$$\int_{t_0}^{t_1}\left[\frac{\partial\widetilde{L}^*}{\partial q^k}\left(\frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\delta\pi^\alpha + \phi_r^k\int_{t_0}^{t_1}\Gamma_s^r\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha dt\right) + \frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha\right]dt = 0 \tag{2.6'}$$

where

$$\int_{t_0}^{t_1}\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha}\frac{d}{dt}\delta\pi^\alpha dt = \left(\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha}\delta\pi^\alpha\right)_{(t_1)} - \int_{t_0}^{t_1}\left(\frac{d}{dt}\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha}\right)\delta\pi^\alpha .$$

Then, we perform calculations of the integral

$$\begin{aligned} J_1 &= \int_{t_0}^{t_1}\frac{\partial\widetilde{L}^*}{\partial q^k}\phi_r^k\int_{t_0}^{t}\Gamma_s^r\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha dt = \\ &= \int_{t_0}^{t_1}\frac{\partial\widetilde{L}^*}{\partial q^k}\phi_r^k dt\int_{t_0}^{t}\Gamma_s^r\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha dt - \int_{t_0}^{t_1}\left[\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha\Gamma_s^r\int_{t_0}^{t}\phi_r^k\frac{\partial\widetilde{L}^*}{\partial q^k}dt\right]dt. \end{aligned} \tag{2.7}$$

Taking into account that

$$\lambda_k = \Gamma_k^s\left[\int_{t_0}^{t_1}\phi_s^j\frac{\partial\widetilde{L}^*}{\partial q^j}dt + \lambda_{s(t_0)}\right] \tag{2.8}$$

is the solution of the system of differential equations

$$\frac{d}{dt}\lambda_k + \frac{\partial\psi^r}{\partial q^k}\lambda_r = \frac{\partial\widetilde{L}^*}{\partial q^k}. \tag{2.9}$$

Now, integral (2.7) reads

$$J_1 = -\left(\frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\delta\pi^\alpha\lambda_k\right)_{(t_1)} - \int_{t_0}^{t_1}\lambda_s\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha dt . \tag{2.7'}$$

Based on (2.7'), Eq. (2.6') has the form

$$\begin{aligned} \int_{t_0}^{t_1}\delta\widetilde{L}^* dt &= \int_{t_0}^{t_1}\left(\frac{\partial\widetilde{L}^*}{\partial q^k}\frac{\partial\psi^k}{\partial\dot{\pi}^\alpha}\delta\pi^\alpha - \frac{d}{dt}\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha}\delta\dot{\pi}^\alpha - \lambda_s\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i\delta\pi^\alpha\right)dt + \\ &+ \left(\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha} - \lambda_s\frac{\partial\psi^s}{\partial\dot{\pi}^\alpha}\right)_{(t_1)}\delta\pi_{(t_1)}^\alpha = 0 \end{aligned} \tag{2.6''}$$

wherefrom for the case $\delta\pi_{(t_1)}^\alpha = 0$ there follow differential equations

$$\frac{d}{dt}\frac{\partial\widetilde{L}^*}{\partial\dot{\pi}^\alpha} - \frac{\partial\widetilde{L}^*}{\partial q^k}\frac{\partial\psi^k}{\partial\dot{\pi}^\alpha} + \lambda_s\frac{\partial\psi^s}{\partial\dot{\pi}^i}\gamma_\alpha^i = 0 . \tag{2.10}$$

If the boundary condition for $\delta\pi^{\alpha}_{(t_1)}$ is eliminated, then

$$\left(\frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}} - \lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\right)_{(t_1)} = 0,\ \delta\pi^{\alpha}_{(t_1)} \neq 0. \tag{2.11}$$

Note that the system of differential equations (2.9) can be written in the following form

$$\frac{d}{dt}\left(\lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\right) + \frac{\partial\psi^k}{\partial\dot{\pi}^{i}}\gamma^i_{\alpha}\lambda_k - \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\frac{\partial\tilde{L}^*}{\partial q^k} = 0 \tag{2.12}$$

because when developed that expression has the form

$$\left(\frac{d}{dt}\lambda_k\right)\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} + \lambda_k \frac{d}{dt}\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} + \left(\frac{\partial\psi^k}{\partial q^s}\frac{\partial\psi^s}{\partial\dot{\pi}^{\alpha}} - \frac{d}{dt}\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\right)\lambda_k - \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\frac{\partial\tilde{L}^*}{\partial q^k} = 0$$

that is

$$\left(\frac{d}{dt}\lambda_k + \frac{\partial\psi^s}{\partial q^k}\lambda_s - \frac{\partial\tilde{L}^*}{\partial q^k}\right)\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} = 0. \tag{2.12'}$$

Based on Eqs (2.10) and (2.12) it is obtained

$$\left.\begin{aligned} &\frac{d}{dt}\left(\lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\right) + \frac{\partial\psi^k}{\partial\dot{\pi}^{i}}\gamma^i_{\alpha}\lambda_k - \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\frac{\partial\tilde{L}^*}{\partial q^k} = 0 \\ &-\frac{d}{dt}\frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}} - \frac{\partial\psi^s}{\partial\dot{\pi}^{i}}\gamma^i_{\alpha}\lambda_s + \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\frac{\partial\tilde{L}^*}{\partial q^k} = 0 \end{aligned}\right\} \frac{d}{dt}\left(\lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} - \frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}}\right) = 0 \tag{2.13}$$

that is

$$\lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} - \frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}} = C_{\alpha}$$

respectively, since according to (2.11)

$$\left(\frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}} - \lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}}\right)_{(t_1)} = 0 \rightarrow C_{\alpha} = 0. \tag{2.14}$$

Based on (2.14) there follows

$$\lambda_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} - \frac{\partial L}{\partial\dot{q}^k}\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} = \left(\lambda_k - \frac{\partial L}{\partial\dot{q}^k}\right)\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} = \mu_k \frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} = 0.$$

Accordingly, Eqs (2.10) and (2.12) finally read

$$\frac{d}{dt}\frac{\partial\tilde{L}^*}{\partial\dot{\pi}^{\alpha}} - \frac{\partial\tilde{L}^*}{\partial q^k}\frac{\partial\psi^k}{\partial\dot{\pi}^{\alpha}} + \left(\mu_s + \frac{\partial L}{\partial\dot{q}^s}\right)\frac{\partial\psi^s}{\partial\dot{\pi}^{i}}\gamma^i_{\alpha} = 0 \tag{2.10'}$$

$$\frac{d}{dt}\left[\left(\mu_k + \frac{\partial L}{\partial \dot{q}^k}\right)\frac{\partial \psi^k}{\partial \dot{\pi}^\alpha}\right] + \left(\mu_k + \frac{\partial L}{\partial \dot{q}^k}\right)\frac{\partial \psi^k}{\partial \dot{\pi}^i}\gamma^i_\alpha - \frac{\partial \psi^k}{\partial \dot{\pi}^\alpha}\frac{\partial \tilde{L}^*}{\partial q^k} = 0 . \qquad (2.12'')$$

The variational problem corresponds to the principles of mechanics

$$\int_{t_0}^{t_1} \delta L dt = 0 .$$

Where

$$\delta q^k = \frac{\partial \psi^k}{\partial \dot{\pi}^\alpha}\delta\pi^\alpha$$

wherefrom there follow the differential equations of motion

$$\frac{d}{dt}\frac{\partial \tilde{L}^*}{\partial \dot{\pi}^\alpha} - \frac{\partial \tilde{L}^*}{\partial q^k}\frac{\partial \psi^k}{\partial \dot{\pi}^\alpha} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \psi^i}{\partial \dot{\pi}^k}\gamma^k_\alpha = \frac{d}{dt}\frac{\partial \tilde{L}^*}{\partial \dot{\pi}^\alpha} - \frac{\partial \tilde{L}^*}{\partial q^k}\frac{\partial \psi^k}{\partial \dot{\pi}^\alpha} + \frac{\partial \tilde{L}}{\partial \dot{\pi}^s}\frac{\partial \theta^s}{\partial \dot{q}^i}\frac{\partial \psi^i}{\partial \dot{\pi}^k}\gamma^k_\alpha = 0 \qquad (2.15)$$

$$\frac{d}{dt}\frac{\partial \tilde{L}^*}{\partial \dot{\pi}^\alpha} - \frac{\partial \tilde{L}^*}{\partial q^k}\frac{\partial \psi^k}{\partial \dot{\pi}^\alpha} + \frac{\partial \tilde{L}}{\partial \dot{\pi}^s}\gamma^s_\alpha = 0 \qquad (2.15')$$

whereas the variational problem

$$\int_{t_0}^{t_1} \delta L dt = 0$$

with variations

$$\delta q^k = \frac{\partial \psi^k}{\partial \dot{\pi}^\alpha}\delta\pi^\alpha + \phi^k_r \int_{t_0}^{t_1} \Gamma^r_s \frac{\partial \psi^s}{\partial \dot{\pi}^i}\gamma^i_\alpha \delta\pi^\alpha dt$$

leads to Eqs (2.10), the difference being obvious.

Concluding Remarks

On the basis of the previous analysis we can conclude the following: in the mechanics of nonlinear nonholonomic systems the operators for the variation and the differentiation are commutative for all quasicoordinates.

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REFERENCES

1. Neimark Y.I., Fufayev N.A., (1967), *The Dynamics of Non-holonomic Systems*, Nauka, Moscow. [In Russian]
2. Hamel G. I., (1949) , *Teoretische Mechanik*, Springer-Verlag, Berlin, Göttingen, Heidelberg.
3. Hölder O., (1959), *On the Principles of Hamilton's and Maupertuis'*, Annals of Variational Principles of Mechanics, FIZMATGIZ, Moscow. [In Russian]
4. Lurje A.I., (1961), *Analytical Mechanics*, Gosud. Izd. F.M. literat., Moscow. [In Russian]
5. Čović V., Lukačević M., (1986), Contribution to Analytical Mechanics of Nonholonomic Systems, Glas CCCXLVI SANU, Vol. 50, Belgrade. [In Serbian]
6. Čović V., (1983), On Commutativity of the Operators of Variation and Differentiation in the Mechanics of Nonholonomic Systems, Vol. 4(2), Annals of Mathematics Institute, New series, Belgrade. [In Serbian]
7. Rumyantsev V.V. (1978), Hamilton's Principle for Nonholonomic Systems, PMM, Vol. 42(3). [In Russian]
8. Suslov G.K., (1944), *Theoretical Mechanics*, OGIZ, Moscow-Leningrad. [In Russian]
9. Lukačević M., Čović V., (1984) On the Variation of the Quasicoordinates, Annals of Mathematics Institute, Vol. 4(12), New series, Belgrade, 1984. [In Serbian]

АНАЛИЗА ХАМИЛТОНОВОГ ПРИНЦИПА ЗА НЕЛИНЕАРНЕ НЕХОЛОНОМНЕ СИСТЕМЕ С ОБЗИРОМ НА ТРАНСПОЗИЦИОНЕ ОДНОСЕ У КВАЗИКООРДИНАТАМА

Драгомир Н. Зековић

Apstrakt. У погледу формирања Хамилтоновог принципа за случај нехолономних система, у литератури постоје два различита схватања што се тиче стварних координата. Међутим, што се тиче квазикоордината, у литератури постоји јединствено гледиште да су оператори варирања и диференцирања некомутативни за све квазикоординате. У овом раду решава се проблем само за случај квазикоордината и полази се од става по коме су оператори варирања и диференцирања комутативни [5].

Key words (in Serbian): *нехолономни, нелинеарни, принцип, транспозиција, квазикоординате*

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ON THE MOTION OF A TWO-BLADE SYSTEM WITH A NON-LINEAR SPRING

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Abstract: *A system consisting of two particles that move in a horizontal plane and are connected with a non-linear spring is considered. The particles are placed on two blades causing their velocities to be perpendicular mutually and yielding two non-holonomic constraints. Two differential equations of motion are solved approximately and are used then together with the non-holonomic constraints to obtain the motion of one point analytically as well as to determine its trajectory. Analytical results are verified numerically. In addition, the condition when this trajectory becomes unbounded is found. The case when the spring non-linearity is beneficial in this respect is determined, too.*

Key words: *non-holonomic constraint, non-linear spring, frequency, trajectory*

1. INTRODUCTION

Systems with blades and skates have been recognized for decades as the exemplary non-holonomic constraints [1]. This type of constraint is such that a fixed point on the object can only move relative to the plane below in a direction that is also fixed in the object [2]. The constraint is realized by an ice skate which glides easily in one direction without allowing sideways motion. One of the most famous systems that uses this skate constraint is the Chaplygin sleigh [3]. This system has various disguises, such as no hands tricycle (Tenessee racer) [2], fins on an underwater missile [4] and locked wheels on a skidding car [5].

An interesting model of a system with such constraints consists of two particles whose velocities can have certain directions realizable through the use of blades attached to the particles, while they are connected by a pitchfork of negligible mass which permits the distance between them to vary [6]. The case when two particles are connected by a linear spring is considered in [7]. The system is shown to be integrable and general theoretical considerations carried out for one of the particles imply that its trajectory stays bounded under certain conditions, which is confirmed numerically. The following study is seen to generalise the analysis performed in [7] for the case when the particles are connected by a non-linear spring, which has not been considered so far. In addition, a deeper analytical analysis is conducted to obtain analytical solutions for motion as well as the conditions for (un)bounded response.

The paper is organised as follows. Mechanical and mathematical descriptions of the system considered are given in Section 2. Section 3 provides approximate approaches developed to obtain motion of the system analytically and includes numerical confirmations of these new results. The final section contains conclusions, which point out all new findings as well as the case when the use of certain non-linear spring is advantageous.

2. On the Model

The system under consideration consists of two particles, Point 1 and Point 2 (Fig. 1). Their masses are equal m_1=m_2=m, and each of them is placed on a blade that makes the velocities $\mathbf{v}_1$ perpendicular to the direction 12 and the velocity $\mathbf{v}_2$ passing through these points, as shown in Fig. 1. The masses, which move in a horizontal plane, are connected by a non-linear spring. The restoring force F_k is assumed to be of a power form of the deflection Δl, i.e. $F_k = k\,\mathrm{sgn}(\Delta l)\cdot|\Delta l|^{\alpha}$, where α and k are real and positive. In order to assure that this force is an odd function of the deflection, and, thus yields oscillatory motion, the sign and absolute value functions are used. Four generalised coordinates are assigned to the system: x_1 and y_1 - two Descartes' coordinates defining the position of Point 1, the coordinate ρ - defining the position of the second point with respect to the first one and the angle φ between the direction passing through these two points and the horizontal, all as defined in Fig. 1. Two unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ define respectively the direction perpendicular and passing through the points considered.

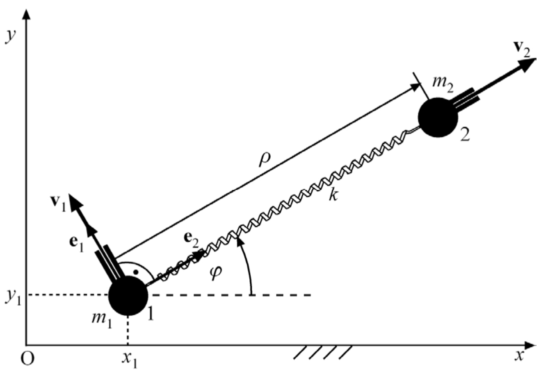

Fig. 1 System under consideration

The fact that the velocities are perpendicular mutually implies $\mathbf{v_1} \cdot \mathbf{v_2} = 0$, i.e. $\mathbf{v_1} \cdot \mathbf{e_2} = 0$, which yields the following non-holonomic constraint

$$\dot{x}_1 \cos\varphi + \dot{y}_1 \cos\varphi = 0, \tag{1}$$

where a dot stands for differentiation with respect to time *t*.
Further, as the velocity of Point 2 is given by

$$\mathbf{v}_2 = \mathbf{v}_1 + \dot{\rho}\mathbf{e}_2 + \rho\dot{\varphi}\mathbf{e}_1, \tag{2}$$

the second non-holonomic constraint is obtained by projecting Eq. (2) to the direction of $\mathbf{e_1}$:

$$-\dot{x}_1 \sin\varphi + \dot{y}_1 \cos\varphi + \rho\dot{\varphi} = 0. \tag{3}$$

By using Eq. (3), the kinetic energy of the system can be written down in the form

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\dot{\rho}^2. \tag{4}$$

The potential energy corresponds to the potential energy of the spring and is given by

$$V = \frac{k}{\alpha+1}\left|\rho - l_0\right|^{\alpha+1}, \tag{5}$$

where l_0 stands for the length of the underformed spring.
By expressing the non-holonomic constraints (1) and (3) as

$$f_1 \equiv \dot{x}_1 - \rho\dot{\varphi}\sin\varphi = 0, \tag{6}$$

$$f_2 \equiv \dot{y}_1 + \rho\dot{\varphi}\cos\varphi = 0, \tag{7}$$

Lagrange's equations with unknown multipliers λ_i (*i*=1, 2)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q_q^* + \sum_{i=1}^{2}\lambda_i \frac{\partial f_i}{\partial \dot{q}}, \tag{8}$$

where $q \in \{x_1, y_1, \rho, \varphi\}$ and Q_q^* are non-conservative generalised forces, have the form

$$m\ddot{x}_1 = \lambda_1, \tag{9}$$

$$m\ddot{y}_1 = \lambda_2, \tag{10}$$

$$m\ddot{\rho} + k\,\mathrm{sgn}(\rho - l_0)\cdot\left|\rho - l_0\right|^{\alpha} = 0, \tag{11}$$

$$0 = \lambda_1\rho\sin\varphi - \lambda_2\rho\cos\varphi. \tag{12}$$

Equation (11) yields the following first integral

$$\frac{1}{2}m\dot{\rho}^2+\frac{k}{\alpha+1}\cdot\left|\rho-l_0\right|^{\alpha+1}=c_1, \tag{13}$$

where c_1 is a constant.

Equations (12), (9), (10), (6) and (7) lead another first integral

$$\rho\dot{\varphi}=c_2, \tag{14}$$

where c_2 is a constant.

Since the position of the system is defined by four generalised coordinates and there are two non-holonomic constraints, the system has two degrees of freedom. On the other hand, six initial conditions need to be specified and they are chosen to be:

$$x_1(0)=0,\quad y_1(0)=0,\quad \rho(0)=l_0+A,\quad \dot{\rho}(0)=0,\quad \varphi(0)=0,\quad \dot{\varphi}(0)=\dot{\varphi}_0, \tag{15a-f}$$

where A is a real non-zero constant ($A\neq 0$). Due to the physical limitation related to the compressed spring, the initial amplitude has to be smaller than l_0. It is also assumed that the initial extension is not longer than this length, which, all in all, yields $|A|<l_0$. Besides this, φ_0 and $\dot{\varphi}_0$ are also constants, where the latter one in conjunction with Eq. (14) yields $\dot{\varphi}_0=c_2/(l_0+A)$.

By introducing the following non-dimensional variables

$$\tilde{x}_1=\frac{x_1}{l_0},\quad \tilde{y}_1=\frac{y_1}{l_0},\quad \tilde{\rho}=\frac{\rho}{l_0},\quad \tau=\frac{t}{\sqrt{\frac{m}{k}}\cdot(l_0)^{\frac{1-\alpha}{2}}}, \tag{16a-f}$$

Eqs. (11), (14), (6) and (7) can be written down as

$$\tilde{\rho}''+\mathrm{sgn}(\tilde{\rho}-1)\cdot|\tilde{\rho}-1|^{\alpha}=0, \tag{17}$$

$$\tilde{\rho}\varphi'=\tilde{c}_2, \tag{18}$$

$$x_1'=\tilde{c}_2\sin\varphi, \tag{19}$$

$$y_1'=-\tilde{c}_2\cos\varphi, \tag{20}$$

where primes denote differentiation with respect to the non-dimensional time τ and

$$\tilde{c}_2=c_2\sqrt{\frac{m}{k}}\cdot(l_0)^{-\frac{\alpha+1}{2}}=\dot{\varphi}_0(l_0+A)\sqrt{\frac{m}{k}}\cdot(l_0)^{-\frac{\alpha+1}{2}}. \tag{21}$$

It is of interest here to find the motion of Point 1, i.e. to determine $x_1(\tau)$ and $y_1(\tau)$. To that end, Eq. (17) needs to be solved first and then Eq. (18) integrated to find $\varphi(\tau)$. This can be used further to obtain the equation of motion of Point 1 and its trajectory. The procedure yielding these results is presented in the following section.

3. ANALYTICAL AND NUMERICAL ANALYSIS

By introducing the substitution $\tilde{u} = \tilde{\rho} - 1$, Eq. (17) can be represented in the form

$$\tilde{u}'' + \operatorname{sgn}(\tilde{u}) \cdot |\tilde{u}|^{\alpha} = 0, \tag{22}$$

with the initial conditions

$$u(0) = \frac{A}{l_0} = \tilde{A}, \quad u'(0) = 0. \tag{23}$$

For α=1, the model (22) corresponds to a harmonic oscillator and for $\alpha \neq 1$ governs truly/purely non-linear oscillators [8]. Despite the fact that this equation has an exact solution in the form of the special Ateb(h) function [9], this function is not suitable for further analytical considerations. A more suitable form can be written down by using the fact that the exact period of oscillations for the oscillator (22) can be calculated [10], [11] as follows

$$T_{\text{ex}} = 4\int_0^{\tilde{A}} \frac{\mathrm{d}u}{|\tilde{u}'|} = 4\sqrt{\frac{\alpha+1}{2}}\int_0^{\tilde{A}} \frac{\mathrm{d}u}{\sqrt{|\tilde{A}|^{\alpha+1} - |\tilde{u}|^{\alpha+1}}} = \sqrt{\frac{8\pi}{(\alpha+1)}}\,\frac{\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}|\tilde{A}|^{\frac{1-\alpha}{2}}, \tag{24}$$

where Γ is the Euler Gamma function [12]. Assuming that the response corresponds to harmonic oscillations with the period 2π, the angular frequency is related to this period by $\omega = 2\pi/T_{\text{ex}}$. Then, the approximate solution for motion can be expressed as [11]

$$\tilde{u} = \tilde{A}\cos\omega\tau, \quad \omega = c(\alpha)|\tilde{A}|^{\frac{\alpha-1}{2}}, \quad c(\alpha) = \sqrt{\frac{\pi(\alpha+1)}{2}}\,\frac{\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}{\Gamma\left(\frac{1}{\alpha+1}\right)}. \tag{25a-c}$$

Figure 2 shows how the angular frequency defined by Eqs. (25b,c) changes with the power of non-linearity α.

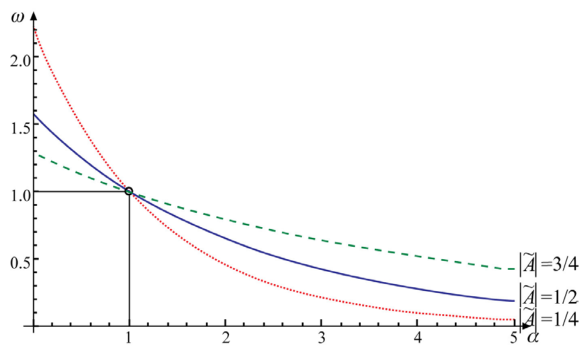

Fig. 2 Change of the angular frequency, Eqs. (25b,c) with the power α for several values of the parameter $|\tilde{A}|$

Given the fact that $|\tilde{A}|<1$, the angular frequency ω decreases as the power α increases. For under-linear oscillators ($\alpha<1$), the angular frequency ω is higher than the angular frequency of the linear oscillator and is smaller for higher values of the parameter $|\tilde{A}|$. For over-linear oscillators ($\alpha>1$), the opposite is true.

Now, Eq. (18) can be considered to obtain how the angle φ changes with time. To that end, the integrand $I=1/(1+\tilde{A}\cos\omega\tau)$ is approximated by the following Fourier series expansion

$$I_{\text{app}}=a_0+a_1\cos\omega\tau+a_2\cos 2\omega\tau+a_3\cos 3\omega\tau, \tag{26}$$

where the Fourier coefficients are

$$\begin{gathered}a_0=\frac{1}{\sqrt{1-\tilde{A}^2}},\quad a_1=\frac{2}{\tilde{A}}\left(1-\frac{1}{\sqrt{1-\tilde{A}^2}}\right),\quad a_2=-\frac{4}{\tilde{A}^2}+\frac{4-2\tilde{A}^2}{\tilde{A}^2\sqrt{1-\tilde{A}^2}},\\ a_3=-\frac{4}{\tilde{A}^2(1-\tilde{A})}+\frac{4}{\tilde{A}(1-\tilde{A})}-\frac{2}{\sqrt{1-\tilde{A}^2}}+\frac{4}{\tilde{A}^2\sqrt{1-\tilde{A}^2}},\ldots.\end{gathered} \tag{27a-d}$$

Figure 3 displays how the values of these coefficients change with $\tilde{A}$. It is seen that their values and the way how they compare with each other, and especially with the free term a_0, point out the importance of the first and the second harmonic as well as the fact that they all diverge as $|\tilde{A}|$ approaches unity from below.

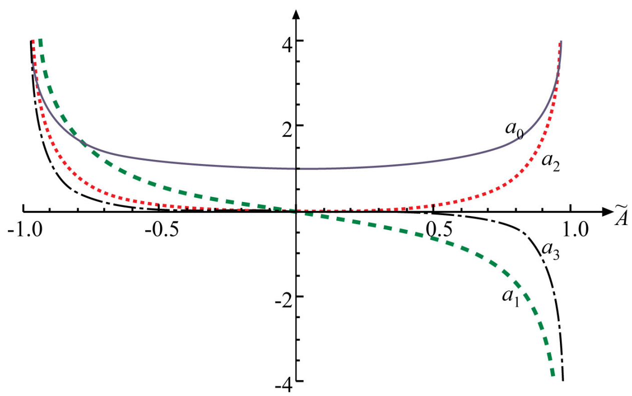

Fig. 3 Change of the Fourier coefficients given by Eqs. (27a-d) with the parameter $\tilde{A}$

To confirm the validity of the overall approximation, the comparisons of the graphs representing the integrand I and its approximation I_{app} are shown in Fig. 4 for the under-linear spring $(\alpha = 1/3)$ and the over-linear spring $(\alpha = 3)$. It is confirmed that the approximate values I_{app} agree well with the values of I.

This leads to the solution for φ

$$\varphi = \tilde{c}_2 \left(a_0 \tau + \frac{a_1}{\omega} \sin \omega\tau + \frac{a_2}{2\omega} \sin 2\omega\tau + \frac{a_3}{3\omega} \sin 3\omega\tau \right). \tag{28}$$

In Fig. 5, the corresponding solution $\varphi / \tilde{c}_2$ (thicker red dashed line) is plotted together with the numerical solution of Eq. (18) with Eq. (25a-c) (blue solid line) for the same power of non-linearity as in Fig. 4. These solutions coincide and indicate that $\varphi / \tilde{c}_2$ oscillates around the straight line $\varphi / \tilde{c}_2 = a_0 \tau$, shown as a green dotted line in Fig. 5.

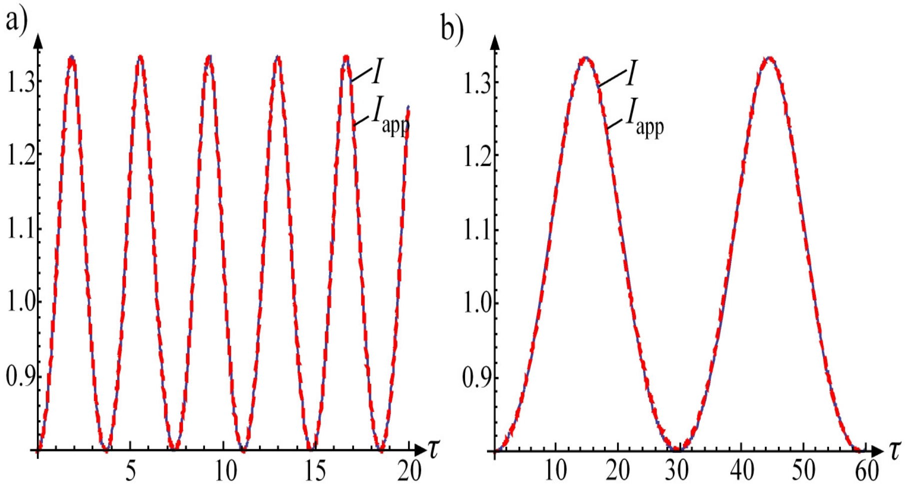

Fig. 4 Change of the integrand I (blue solid line) and its approximation I_{app} (thicker red dashed line) given by Eq. (26) for $\tilde{A} = 1/4$ and: a) $\alpha = 1/3$; b) $\alpha = 3$

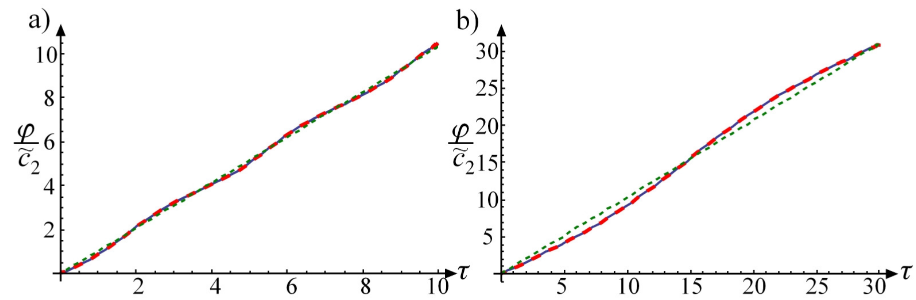

Fig. 5 The approximate solution for $\varphi/\tilde{c}_2$ (thicker red dashed line) given by Eq. (28), the numerical solution of Eq. (18) with Eq. (25a-c) (blue solid line) and the solution $\varphi/\tilde{c}_2 = a_0\tau$ (green dotted line) for $\varphi(0)=0$, $\tilde{A}=1/4$ and: a) $\alpha=1/3$; b) $\alpha=3$

By introducing the complex variable $z_1 = -y_1 + i\,x_1$ [13] and by using Eqs. (19) and (20), the velocity of Point 1 is found to be

$$\dot{z}_1 = \tilde{c}_2\, e^{ib_0\tau} \prod_{n=1}^{3} e^{ib_n \sin n\omega\tau}, \tag{29}$$

where $b_0 = \tilde{c}_2\, a_0$ and $b_n = \tilde{c}_2\, a_n /(n\omega)$. Further, the use can be made of the following Fourier series approximation

$$e^{ib_n \sin n\omega\tau} \approx J_0(b_n) + J_1(b_n)\cdot\left(e^{in\omega\tau} - e^{-in\omega\tau}\right) + J_2(b_n)\cdot\left(e^{i2n\omega\tau} + e^{-i2n\omega\tau}\right) + J_3(b_n)\cdot\left(e^{i3n\omega\tau} - e^{-i3n\omega\tau}\right), \tag{30}$$

where the coefficients are represented by means of the Bessel function of the first kind $J_j(b_n)$, $j=0,1,2,3$ [12]. For the sake of brevity, the notation $J_j(b_n) \equiv J_{jn}$ will be used in the following text. After keeping only the approximations with the first, second and third harmonics included, Eq. (29) is integrated, taking into account the initial conditions (15a,b). The following solution is derived:

$$z_1 = z_1^*(\tau) - z_1^*(0), \tag{31}$$

where

$$z_1^*(\tau) = \tilde{c}_2\left[K_0 \frac{ie^{ib_0\tau}}{b_0} + K_{1m}\frac{ie^{i(b_0-\omega)\tau}}{b_0-\omega} + K_{1p}\frac{ie^{i(b_0+\omega)\tau}}{b_0+\omega} + K_{2m}\frac{ie^{i(b_0-2\omega)\tau}}{b_0-2\omega} + K_{2p}\frac{ie^{i(b_0+2\omega)\tau}}{b_0+2\omega} + K_{3m}\frac{ie^{i(b_0-3\omega)\tau}}{b_0-3\omega} + K_{3p}\frac{ie^{i(b_0+3\omega)\tau}}{b_0+3\omega}\right], \tag{32}$$

and:

$$
\begin{aligned}
K_0 &= -J_{02}\left(J_{01}J_{03} - 2J_{13}J_{31}\right),\\
K_{1m} &= J_{02}J_{03}J_{11} + J_{03}J_{11}J_{12} + J_{01}J_{12}J_{13} + J_{02}J_{13}J_{21} + J_{12}J_{13}J_{21} + J_{03}J_{12}J_{31},\\
K_{1p} &= -J_{02}J_{03}J_{11} + J_{03}J_{11}J_{12} + J_{01}J_{12}J_{13} - J_{02}J_{13}J_{21} + J_{12}J_{13}J_{21} + J_{03}J_{12}J_{31}\\
K_{2m} &= J_{01}J_{03}J_{12} + J_{02}J_{11}J_{13} - J_{11}J_{12}J_{13} - J_{02}J_{03}J_{21} - 2J_{12}J_{13}J_{31},\\
K_{2p} &= -J_{01}J_{03}J_{12} + J_{02}J_{11}J_{13} + J_{11}J_{12}J_{13} - J_{02}J_{03}J_{21} + 2J_{12}J_{13}J_{31},\\
K_{3m} &= -J_{03}J_{11}J_{12} + J_{01}J_{02}J_{13} + J_{02}J_{03}J_{31},\\
K_{3p} &= -J_{03}J_{11}J_{12} - J_{01}J_{02}J_{13} - J_{02}J_{03}J_{31}.
\end{aligned}
\tag{33a-h}
$$

Given the definition for the variable z_1, the following approximate parametric equations are derived for the motion of Point 1:

$$
\begin{aligned}
x_1(\tau) = \tilde{c}_2\Bigg[& \frac{K_0}{b_0}\cos(b_0\tau) + \frac{K_{1m}}{b_0-\omega}\cos((b_0-\omega)\tau) + \frac{K_{1p}}{b_0+\omega}\cos((b_0+\omega)\tau) +\\
& \frac{K_{2m}}{b_0-2\omega}\cos((b_0-2\omega)\tau) + \frac{K_{2p}}{b_0+2\omega}\cos((b_0+2\omega)\tau) +\\
& \frac{K_{3m}}{b_0-3\omega}\cos((b_0-3\omega)\tau) + \frac{K_{3p}}{b_0+3\omega}\cos((b_0+3\omega)\tau) -\\
& \left(\frac{K_0}{b_0} + \frac{K_{1m}}{b_0-\omega} + \frac{K_{1p}}{b_0+\omega} + \frac{K_{2m}}{b_0-2\omega} + \frac{K_{2p}}{b_0+2\omega} + \frac{K_{3m}}{b_0-3\omega} + \frac{K_{3p}}{b_0+3\omega}\right)\Bigg],
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
y_1(\tau) = \tilde{c}_2\Bigg[& \frac{K_0}{b_0}\sin(b_0\tau) + \frac{K_{1m}}{b_0-\omega}\sin((b_0-\omega)\tau) + \frac{K_{1p}}{b_0+\omega}\sin((b_0+\omega)\tau) +\\
& \frac{K_{2m}}{b_0-2\omega}\sin((b_0-2\omega)\tau) + \frac{K_{2p}}{b_0+2\omega}\sin((b_0+2\omega)\tau) +\\
& \frac{K_{3m}}{b_0-3\omega}\sin((b_0-3\omega)\tau) + \frac{K_{3p}}{b_0+3\omega}\sin((b_0+3\omega)\tau)\Bigg].
\end{aligned}
\tag{35}
$$

At this point, one can recognize the problem of 'small denominators' [14]: it is seen that the values of the coordinates x_1 and y_1 can become unbounded if the denominators that depend on b_0 and ω are equal to zero. This yields the corresponding resonant values of the constant $\tilde{c}_2^*$:

$$
\tilde{c}_2^* = \pm N\sqrt{\frac{\pi(\alpha+1)}{2}\,\frac{\Gamma\left(\dfrac{\alpha+3}{2(\alpha+1)}\right)}{\Gamma\left(\dfrac{1}{\alpha+1}\right)}}\sqrt{\left|\tilde{A}\right|^{\alpha-1} - \left|\tilde{A}\right|^{\alpha+1}}, \quad N = 1, 2, 3\ldots \; . \tag{36}
$$

One should note that although Eqs. (34) and (35) imply the existence of an unbounded solution for $N=1$, 2 and 3 only, this value actually corresponds to any natural number, as indicated in Eq. (36). This stems from the fact that the series expansion in Eq. (26) contains an infinite number of harmonics, as a result of which the part of the final solution (31) can be presented as

$$z_1^*(\tau)=\tilde{c}_2\left[\frac{d_0}{b_0}e^{i b_0\tau}+\sum_{N=1}^{\infty}\frac{d_n}{(b_0-N\omega)}e^{i(b_0-N\omega)\tau}+\sum_{N=1}^{\infty}\frac{\tilde{d}_n}{(b_0+N\omega)}e^{i(b_0+N\omega)\tau}\right], \qquad (37)$$

where d_0, d_n and $\tilde{d}_n$ are constants.

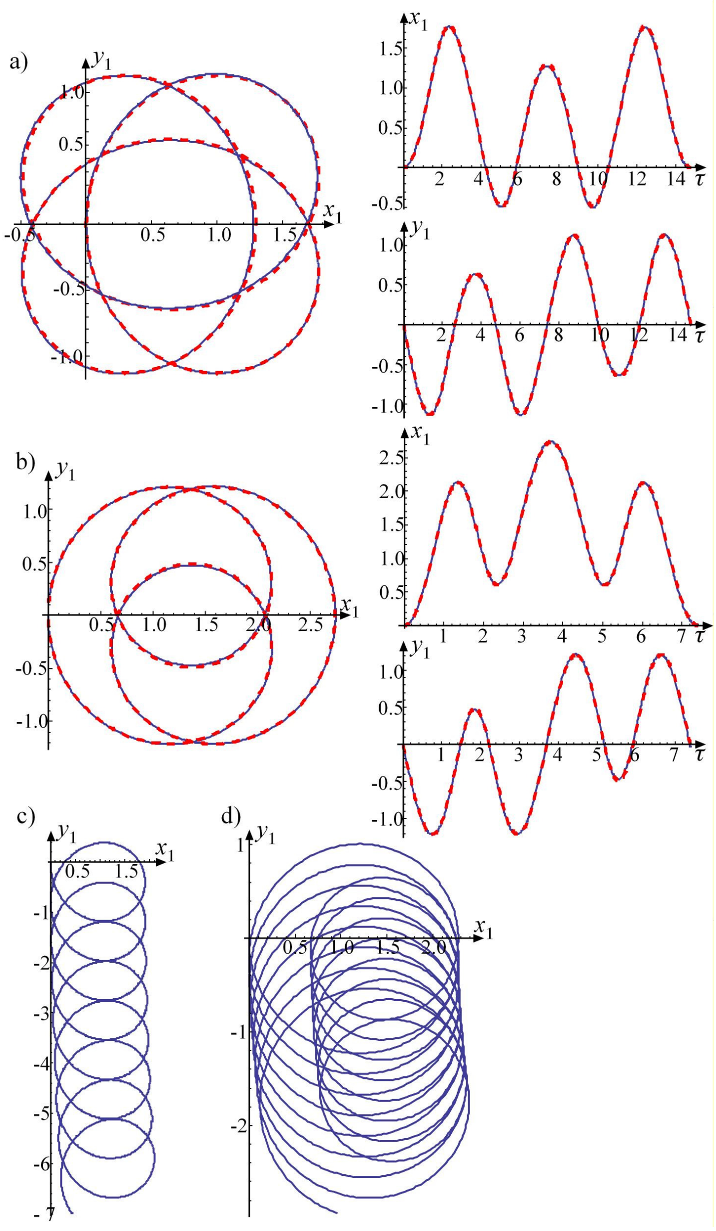

Fig. 6 Trajectory of Point 1 for $\alpha = 1/3$, $\tilde{A} = 1/4$ and: a) $\tilde{c}_2 = 0.75\tilde{c}_2^*$; b) $\tilde{c}_2 = 1.5\tilde{c}_2^*$; c) $\tilde{c}_2^* = 1$; d) $\tilde{c}_2^* = 2$. Numerical solutions (blue solid line), approximate analytical solutions (thicker red dashed line), Eqs. (34) and (35)

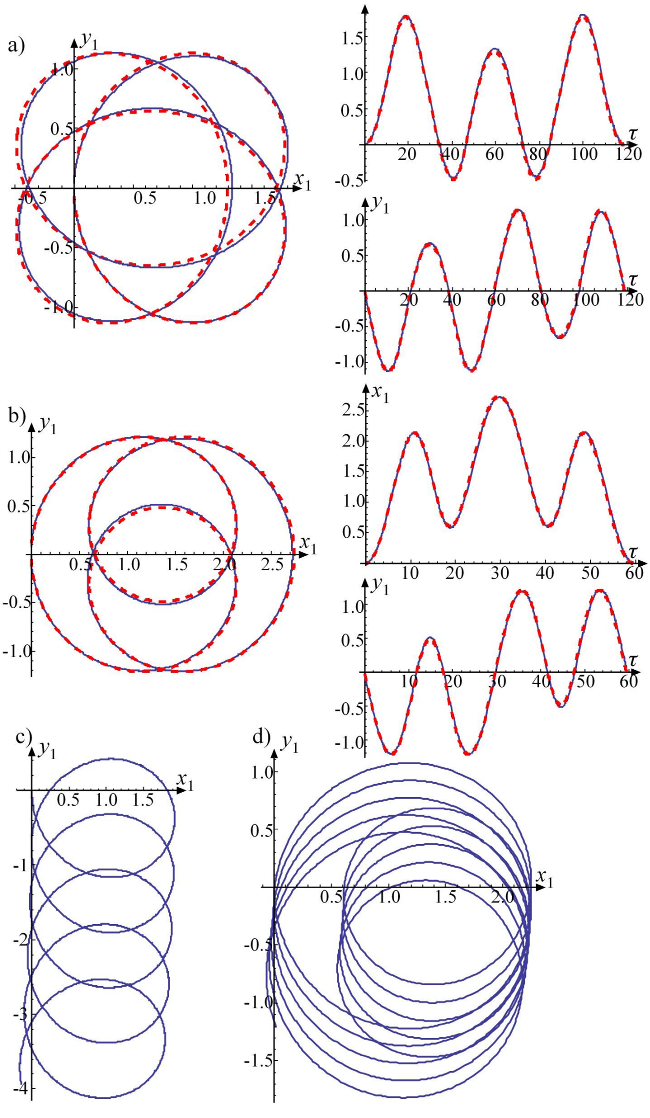

Fig. 7 Trajectory of Point 1 for $\alpha = 3$, $\tilde{A} = 1/4$ and: a) $\tilde{c}_2 = 0.75\tilde{c}_2^*$; b) $\tilde{c}_2 = 1.5\tilde{c}_2^*$; c) $\tilde{c}_2^* = 1$; d) $\tilde{c}_2^* = 2$. Numerical solutions (blue solid line), approximate analytical solutions (thicker red dashed line), Eqs. (34) and (35)

To check the accuracy of the approximate solutions for motion of Point 1 given by Eqs. (34) and (35), this solution is compared with the numerical solutions of Eqs. (17)-(20) with the corresponding initial conditions. The time interval of interest is the one from the initial moment until the moment when Point 1 returns to the initial position. These solutions are plotted in Fig. 6a, b for $\alpha = 1/3$ and in Fig. 7a, b for $\alpha = 3$ in terms of time-history diagrams and the corresponding trajectories for two non-resonant values of the constant $\tilde{c}_2$.

A reasonably good match between the solutions is seen. In addition, Fig. 6c, d and Fig. 7c, d show numerically obtained time responses of these oscillators for two resonant values of the constant $\tilde{c}_2^*$, which confirm the conclusion about the unbounded response.

The resonant value of the parameter $\tilde{c}_2^*$ given by Eq. (36) is plotted in Fig. 8 as a function of $|\tilde{A}|$. Given the initial values, one has $|\tilde{A}| \in (0,1)$. The excluded values $|\tilde{A}| = 0$ and $|\tilde{A}| = 1$ are shown in this figure as black dotes. The cases corresponding to $N = \pm 1$ and three different values of the power of the spring corresponding to under-linear, linear and over-linear springs are presented. It can be concluded that the use of under-linear springs can be beneficial as it increases the resonant value of the constant $\tilde{c}_2^*$, which is important as this value puts the limitations on the system parameter values, as seen from Eq. (21).

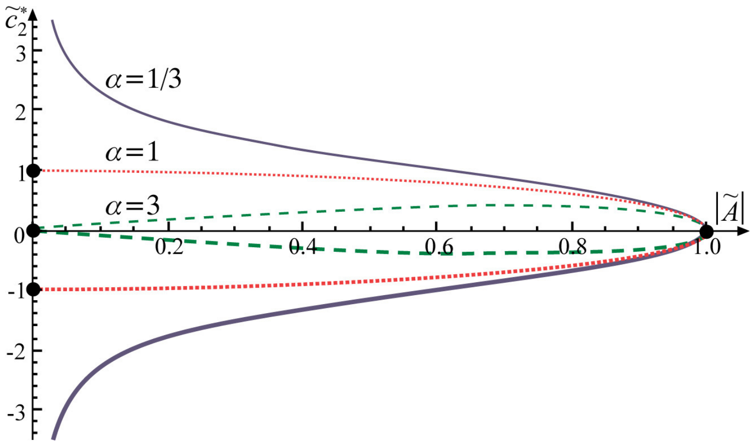

Fig. 8 Resonant value of $\tilde{c}_2^*$, Eq. (36) a) as a function of the parameter $|\tilde{A}|$ for different values of the power α and $N = -1$ (thicker lines), when $\tilde{c}_2 < 0$, as well as for $N = 1$ when $\tilde{c}_2 > 0$

4. CONCLUSIONS

In this paper, a system that comprises two particles of equal masses moving in a horizontal plane, connected with a purely non-linear power-form spring, has been studied. Each of the particles is placed on a blade, so that their velocities are perpendicular mutually. This has also yielded the existence of two non-holonomic constraints. Four generalised coordinates have been assigned to the system, which due to the number of non-holonomic constraints, has two degrees of freedom. Two differential equations of motion have been solved approximately to obtain first how the distance between the points, i.e. the length of the spring, changes with time, and then to derive analytically the angle between the direction passing through these two points and the horizontal as a function of time. It has been demonstrated that the length of the spring oscillates with the angular frequency ω that decreases as the power of the non-linear spring increases. The influence of the initial deflection on this angular frequency has been shown to be different for under-linear and over-linear springs. Further, it has been derived that the angle between the direction passing through the points and the horizontal oscillates around the value defined by the corresponding constant angular velocity, which depends on the system parameter values, the initial angular velocity and the initial deflection of the spring. These oscillations are multi-frequency oscillations whose angular frequencies are positive whole-number multiplications of the afore-mentioned angular frequency ω. These results have been used then together with the non-holonomic constraints to obtain approximations for motion of one of the points considered. The corresponding oscillatory response has been derived in terms of harmonics whose coefficients have been expressed by means of the Bessel function of the first kind and are characterized by the existence of the denominators that can have zero values. The condition when this happens and when the corresponding trajectory is consequently unbounded has been found. All of the analytical results obtained have been verified numerically. Finally, it has been shown that the use of under-linear springs can be beneficial as it increases the value of the parameter that yields an unbounded trajectory.

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REFERENCES

[1] J.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems*, Nauka, Moscow, 1967 (in Russian).

[2] A. Ruina, Nonholonomic stability aspects of piecewise holonomic systems, *Reports on Mathematical Physics*, 42 (1998), 91-100.

[3] S.A. Chaplygin, Theory of motion of non-holonomic systems. Reducing Factor Theorem. In: collected works, Vol. 1. Gostekhizdat, Moscow-Leningrad, 1948, 15-25.

[4] R.H. Rand and D.V. Ramani, Relaxing nonholonomic constraints, Proceedings of the First International Symposium on Impact and Friction of Solids, Structures and Machines, Ottawa, Canada, June 26-30, 1998, pp 113-116

[5] F.C. Moon, *Applied Dynamics: with Applications to Multibody and Mechatronics Systems*, John Wiley & Sons, New York, 1998.

[6] D.N. Zekovic, Dynamics of mechanical systems with nonlinear nonholonomic constraints – I The history of solving the problem of a material realization of a nonlinear nonholonomic constraint, *ZAMM Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewante mathematic und Mechanik* 91 (2011), 883 – 898.

[7] D. Zekovic, On the motion of a nonholonomically constrained system in the nonresonance case, *Mechanics Research Communications* 38 (2011) 330–333.

[8] R.E. Mickens, *Truly Nonlinear Oscillations: Harmonic Balance, Parametric Expansions, Iteration, and Averaging Methods*, World Scientific, Singapore, 2010.

[9] R.M. Rosenberg, The Ateb(h)-functions and their properties, *Quarterly Applied Mathematics* 21 (1963), 37–47.

[10] H.P.W. Gottlieb, Frequencies of oscillators with fractional-power nonlinearities, *Journal of Sound and Vibration* 261 (2003) 557-566.

[11] L. Cveticanin, Oscillator with fraction order restoring force, *Journal of Sound and Vibration* 320 (2009) 1064-1077.

[12] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 2000.

[13] V.V. Kozlov and N.N. Kolesnikov, About the theorems in dynamics, *Applied Mathematics and Mechanics* 42 (1978), 28–33 (in Russian).

[14] J. Kevorkian and J.D. Cole, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York, 1981.

O KRETANJU SISTEMA SA DVA SEČIVA I NELINEARNOM OPRUGOM

Ivana Kovačić

U ovom radu je razmatran sistem koji se sastoji od dve materijalne tačke koje se kreću u horizontalnoj ravni i povezane su nelinearnom oprugom. Svaka od materijalnih tačaka je postavljena na sečivo usled čega su njihove brzine međusobno upravnih pravaca i postoje dve jednačine neholonomnih veza. Dve diferencijalne jednačine kretanja su rešene približnim metodama, a zatim je korišćenjem jednačina neholonomnih veza dobijeno približno analitičko rešenje za kretanje jedne tačke, i određena njena trajektorija. Analitički rezultati su potvrđeni numeričkim putem. Osim toga, određen je i uslov pod kojim trajektorija ove tačke nije ograničena. Utvrđen je slučaj kada je u ovom smislu od koristi upotreba određenog tipa nelinearne opruge.

Ključne reči: *neholonomna veza, nelinearna opruga, frekvencija, trajektorija.*

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ANALYSIS OF CYCLIC PLASTICITY OF TRUSSES USING THE PREISACH MODEL OF HYSTERESIS

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Abstract *In the present paper, the Preisach model of hysteresis, which was already successfully implemented for solving problems of cyclic plasticity of axially loaded bar and cyclic bending of elastoplastic beam, is extended to structural analysis of trusses subjected to cyclic loading. It is also shown that damage effects can be included in presented analysis by introducing basic concepts of continuum damage mechanics. Using finite element method, equilibrium equations are obtained and algorithm for numerical solution is defined. Some advantages of this approach are underlined, compared with already existing procedures and shown on various numerical examples.*

Key words: *cyclic plasticity, Preisach model, trusses, damage*

1. INTRODUCTION

Although there are numerous well known models of cyclic plasticity defined, in this paper, it is shown that for uniaxial stress state, hysteresis can be defined, based on experimental data, in one particular rigorously mathematical form and implemented in finite element equations for trusses.

The hysteresis operator is a mathematical concept and it is not directly related to the intrinsic physical causes of hysteresis. Since there are numerous examples of hysteresis phenomena occurring in physical processes (hysteresis in continuum mechanics, in ferromagnetism, in filtration through porous media etc.), appropriate modeling of hysteresis is of great interest for engineers and physicists. One of the most powerful scalar model of hysteresis, among those that are known so far, was proposed by the physicist F. Preisach in 1935 [1] to represent scalar ferromagnetism. There are numerous mathematical models that describe hysteretic behavior and some of them were used to model hysteresis in solid mechanics (Prandt-Ishlikii, Bouc, Wen, Baber-Noori). Application of the Preisach model to cyclic behavior of elasto-plastic material was introduced in 1993 by ([Lubarda, Sumarac and Krajcinovic [2],[3]). One of the most important properties of the Preisach operator is the so-called memory map [16], but in addition it is shown in [2] that suggested (Preisach) model also possesses congruency and wiping out property, which makes this model [2],[3] appropriate to describe hysteretic behavior of elasto-plastic material. It was also shown that Preisach model can be defined in purely geometric terms[8], without any reference to analytical definition, which is less attractive approach for engineers.

Using finite element method, equilibrium equations for structural analysis of trusses are obtained and algorithm for numerical solution is defined in C++ code. Several numerical examples will be presented and results obtained by suggested model are compared with the already existing in the literature. The second and the third part of this paper contains basic outline of the Preisach model and its application of modeling ductile materials subjected to cyclic loading, as explained in [3][4] and [5]. In the fourth part, finite element equations for static nonlinear analysis of trusses subjected to cyclic loading are shown. It is also shown that damage effects can be included in elastoplastic analysis by taking into account basic concepts of continuum damage mechanics. In the fifth part, numerical examples are presented and results, obtained by this model, are analyzed and compared with the results obtained by the Bouc-Wen model of hysteresis [13], [14], [15], applied in SAP2000 [18], and results obtained by GP (Generalized Plasticity) model explained in [20].

2. The Preisach model of hysteresis

According to Mayergoyz [8], the Preisach model implies the mapping of an input $u(t)$ on the output $f(t)$ in the integral form:

$$f(t)=\iint P(\alpha,\beta)\,G_{\alpha,\beta}u(t)\,d\alpha d\beta, \tag{1}$$

where $G_{\alpha,\beta}$ is an elementary hysteresis operator given in Figure 1.a. Parameters α and β are up and down switching values of the input, while $P(\alpha, \beta)$ is the Preisach function. i.e.a weight (Green′s) function of the hysteresis nonlinearity to be represented by the Preisach model. The domain of integration of integral (1) is right triangle in the α, β plane, with $\alpha=\beta$ being the hypotenuse and ($\alpha_0, \beta_0= - \alpha_0$) being the triangular vertex (Fig.1.b). History of loading corresponds to staircase line $L(t)$ which divides triangle into two parts (Lubarda et al.[2]).

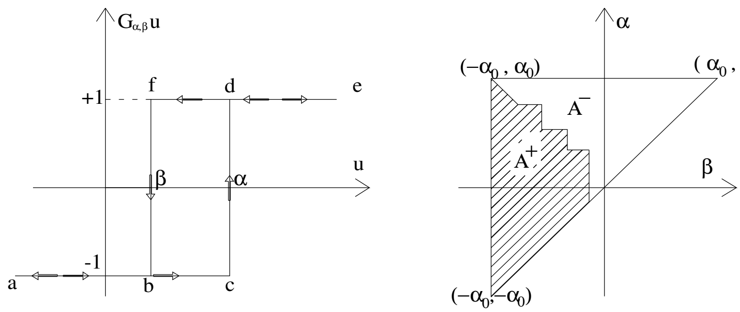

Fig.1 (a) Elementary hysteresis operator; (b) Staircase line in the Preisach triangle

Maxima or minima of loading history are represented by the vertices with coordinates *(α,β)* on staircase line *L(t)* in such a way if the input at a previous instant of time is increased, the final link of *L(t)* is horizontal, and vice versa if it is decreased it is vertical. Therefore, the triangle is divided into two parts with the positive and negative values of $G_{\alpha,\beta}$ by the interface staircase line *L(t).* From formula (1) it is obtained:

$$f(t) = \iint_{A^+(t)} P(\alpha,\beta) G_{\alpha,\beta} u(t)\, d\alpha d\beta - \iint_{A^-(t)} P(\alpha,\beta) G_{\alpha,\beta} u(t)\, d\alpha d\beta \,. \tag{2}$$

Denoting the output value at $u=\beta$ by $f_{\alpha,\beta}$ from the limiting triangle, it follows that

$$f_{\alpha,\beta} - f_\alpha = -2\int_\beta^\alpha \left(\int_{\beta'}^{\alpha} P(\alpha',\beta')\, d\alpha' \right) d\beta' \,. \tag{3}$$

By differentiating expression (3) twice, with respect to α and β, the Preisach weight function is derived in the form

$$P(\alpha,\beta) = \frac{1}{2}\frac{\partial^2 f_{\alpha,\beta}}{\partial\alpha\partial\beta} \,. \tag{4}$$

The Preisach model explained above possesses two properties: wiping out and congruency properties. Those properties and much more about Preisach model is explained in the Lubarda et al. [2] and [3].

3 THE PREISACH MODEL FOR CYCLIC BEHAVIOR OF DUCTILE MATERIALS

One dimensional hysteretic behavior of elasto-plastic material can be successfully described by the Preisach model. Ductile material is represented in various ways by a series or parallel connections of elastic (spring) and plastic (slip) elements Lubarda, at al. [2]. These results have advantage in comparison with classically obtained Iwan, [7], Asaro, [10] because of simplicity and strict mathematical rigorous procedure. Parallel connection of elastic and slip elements, Series connection of elastic and slip elements are discussed elsewhere Sumarac and Stosic, [4], Lubarda, at al. [2]. Here we will consider a three element unit.

Elastic-linearly hardening material behavior, characterized by the stress-strain curve shown in Fig. 2a. (E and E_h are elastic and hardening moduli respectively), can be modeled by a three-element unit shown in Fig. 2b.

Elastic element of length l and modules E_0 is connected in a series with a parallel connection of elastic and slip element, of length L modulus h_0 and yield strength Y. It then follows that $E = E_0(l_0+L_0)/l_0$ and $E_h = Eh(E+h)$, where $h=h_0(l_0+L_0)/L_0$. Since in this paper, displacement-based finite element method is used where displacement (strain) is unknown variable, only three-element units connected in parallel will be used to model material.

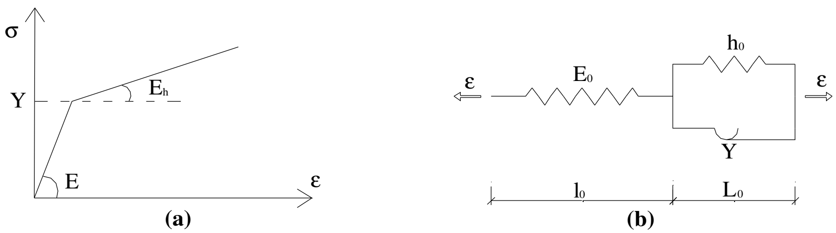

Figure 2 (a) Elastic-linearly hardening stress-strain behavior with elastic modulus E, initial yield stress Y and hardening modulus E_h;(b) Three-element unit reproducing the stress-strain behavior in (a)

3.1. A Three-Element Unit Connected in Parallel

In this case the Preisach function can be determined from the hysteresis nonlinearity shown in Fig.2a again, taking into consideration that strain is input and stress is output. The Preisach function in this case has support along the lines $\alpha-\beta=0$ and $\alpha-\beta=2Y/E$, i.e.it is given by

$$P(\alpha,\beta)=\frac{E}{2}\left[\delta(\alpha-\beta)+\frac{E-E_h}{2}\delta(\alpha-\beta-2Y/E)\right]. \tag{5}$$

The expression for stress as a function of applied strain is, consequently,

$$\sigma(t)=\frac{E}{2}\left[\int_{-\varepsilon_0}^{\varepsilon_0} G_{\alpha,\alpha}\varepsilon(t)\,d\alpha-\frac{(E-E_h)}{E}\int_{2Y/E-\varepsilon_0}^{\varepsilon_0} G_{\alpha,\alpha-2Y/E}\varepsilon(t)\,d\alpha\right]. \tag{6}$$

The first and second term on the right-hand side of (6) are elastic and plastic stress, respectively. For a system consisting of infinitely many of three-element units, connected in a parallel and with uniform yield strength distribution within the range $Y_{min}\le Y\le Y_{max}$, the total stress is

$$\sigma(t)=\frac{E}{2}\left[\int_{-\varepsilon_0}^{\varepsilon_0} G_{\alpha,\alpha}\varepsilon(t)\,d\alpha-\frac{E-E_h}{2}\frac{1}{Y_{\max}-Y_{\min}}\iint_A G_{\alpha,\beta}\varepsilon(t)\,d\alpha d\beta\right]. \tag{7}$$

In (7) the integration domain A is the area of the band contained between the lines $\alpha-\beta=2Y_{min}/E$ and $\alpha-\beta=2Y_{max}/E$ in the limiting triangle, shown in Fig.3.b.

3.2. Comparison with experimental results for the strain control uniaxial loading

Stress-strain behavior of material model in plastic domain presented in Fig.3.a is considered to be consisted of two parts, linear and nonlinear.

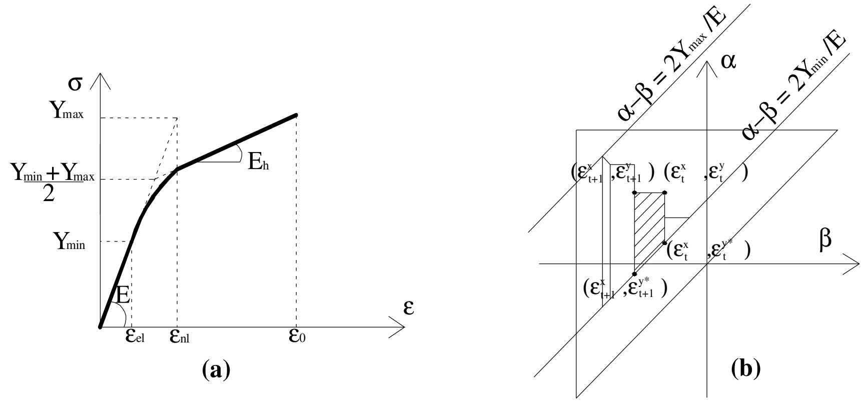

Fig.3 (a)Stress-strain behavior of material modeled by parallel connection of infinite number of three element units; (b) Set $A^{+}(t)$ of Preisach triangle subdivided into N trapezes.

Nonlinear part of this curve is determined by appropriately adopting stress limits Y_{min} and Y_{max} in addition to elastic modulus E and hardening modulus E_h that defines slope of linear part of strain hardening. By analyzing experimental curve, these parameters could be obtained.

For results obtained in experiment of cyclic loading of material in stable cycle loop, published in the paper [9], analytical solution was determined based on model, presented in this paper, of parallel connection of infinitely many elements given in Fig.2b. In this experiment, sample of Titanium alloy was subjected to strain controlled cyclic loadings ε=± 1.2% and stable hysteretic curves were obtained. By analyzing shape of this hysteresis, parameters for material behavior defined in (7) could be determined by considering geometry of experimental curve in Fig.4. Slope of the curve in elastic loading and reloading segments defines modulus of elasticity E=114GPa and linear part of strain hardening gives hardening modulus E_h=17.2MPa.

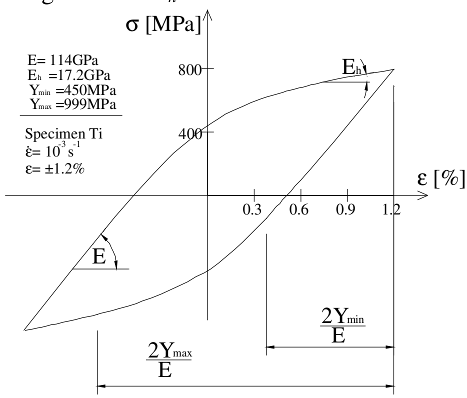

Fig. 4 Results of experiment of cyclic loading of alloy of Tittanium published in [9] and determination of parameters for the Preisach model

If Preisach triangle (Fig.3.b) is analyzed, it can be seen that elastic part of curve's reloading segment always defines constant strain value of $2Y_{min}/E$, while the elastic and nonlinear plastic part of curve's reloading segment give constant strain value of $2Y_{max}/E$. Hence, stress limits Y_{min}=450MPa and Y_{max}=999MPa are defined. Experimentally obtained stable cycle loop was in excellent agreement with one obtained using model of parallel connection of infinitely many elements as shown on Fig.5.a.

In all numerical examples, presented in this paper, the same material of truss structure is taken and compared with the results obtained by the Bouc-Wen model of hysteresis, one of the most recognizable and probably the model that had the largest application in structural analysis. Therefore parameters for Bouc-Wen hysteresis operator are also determined based on the same experimental results.

Detailed formulation and definition of the Bouc –Wen model and its parameters can be found in [13] [14] and [15]. In presented examples, uniaxial case, with no additional possibilities such as degradation and pinching

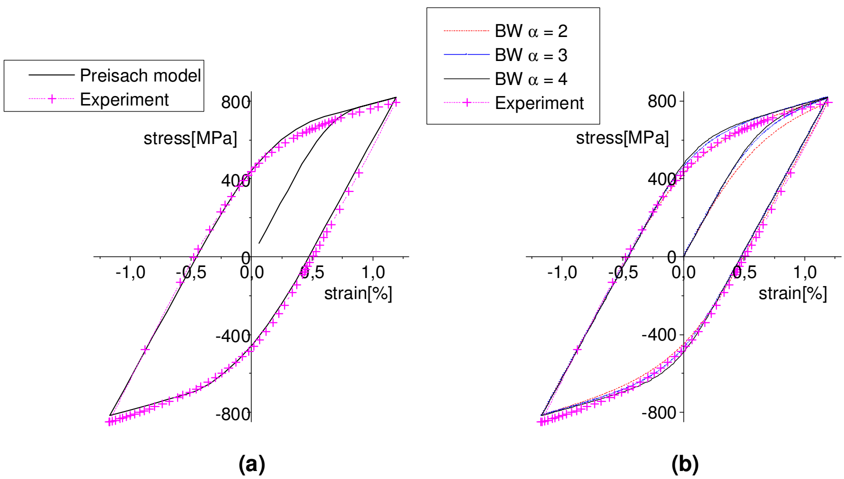

Fig. 5 (a) Comparison of experimentally obtained loop ([9]) with the loop obtained by suggested model of hysteresis; (b) Determination of parameters for the Bouc-Wen model of hysteresis for the best fitting of experimentally obtained loop [9]

effects modeling, is used. All parameters for this model could be determined from experimental results, but the exponent for transition zone was varied to achieve the best approximation of experiment (α=3 gave the best fitting), and results were presented in Fig.5.b.

In addition, results from numerical examples are compared with results obtained by GP finite element [19] based on generalized plasticity material model developed by [20]. GP model has both yield function and limit function and therefore smooth transition between elastic and plastic states is achieved. Material parameters for GP model are also defined from experimental results [9].

4. FINITE ELEMENT EQUATIONS FOR TRUSSES SUBJECTED TO CYCLIC LOADING IN PLASTIC DOMAIN

Using principle of virtual displacements, equations for finite element procedures can be obtained. If only truss elements are considered, (body forces and surface forces are zero), only concentrated loads at nodes, as externally applied load are possible. In the finite element analysis we approximate the structure (in this case truss) as the assemblage of the discrete finite elements interconnected at the nodal points on the element boundaries. The expression for principle of virtual displacements then becomes:

$$\sum_m \int_{V^{(m)}} \overline{\varepsilon}^{(m)T} \sigma^{(m)} dV^{(m)} = \sum_i \overline{u}^{iT} R_C^i \quad , \tag{8}$$

where σ represents stresses in equilibrium with applied loads, R_C^i denotes concentrated forces on point i of applied loads, $\overline{u}^i$ denotes virtual displacements, $\overline{\varepsilon}$ corresponding virtual strains and $m= 1,2...k$, where k is the number of elements (bars). If only one element m of structure is analyzed, substituting equation (7) into (8), it is obtained:

$$\int_V \overline{\varepsilon}^T \frac{E}{2}\left[\int_{-\varepsilon_0}^{\varepsilon_0} G_{\alpha,\alpha}\varepsilon(t)\,d\alpha\right]dV - \int_V \overline{\varepsilon}^T \left[\frac{E(E-E_h)}{4}\frac{1}{Y_{\max}-Y_{\min}}\iint_A G_{\alpha,\beta}\varepsilon(t)\,d\alpha\, d\beta\right]dV = \overline{u}^T R_C \ . \tag{9}$$

It was demonstrated in [3] and [4] that for corresponding strain limits $\varepsilon_{el}=Y_{min}/E$and $\varepsilon_{nl}=Y_{max}/E$, Preisach triangle is formed as presented in Fig.3.(b). It is shown in [2] and [3] that the first part of the expression in Eq.(7) represents elastic stress of axially loaded bars. The second part of the expression in Eq.(7) defines plastic stress, when strain in material exceeds elastic limit $(\varepsilon > \varepsilon_{el})$ and by geometric interpretation it is shown in [2] and [3], that it actually represents difference of integrals over positive and negative area $A^+(t)$ and $A^-(t)$ in the Preisach triangle. It is obvious that area $A^+(t)$ is consisted of sum of N trapezes whose vertices have coordinates equal to past input extrema [2], and therefore it represents function of predominant input strain data values $(\varepsilon_t^x,\ \varepsilon_t^y)$(Fig.3.b):

$$A^+(t) = \sum_{t=1}^{N}\left[\left(\varepsilon_{t+1}^x - \varepsilon_t^x\right)\left(\varepsilon_{t+1}^y - \varepsilon_{t+1}^{y*} + \varepsilon_t^y - \varepsilon_t^{y*}\right)/2\right]. \tag{10}$$

Considering that displacement-based finite element method is used, it is necessary to exchange strain variable ε with bar length change Δu, and because of Eq.(10), $A^+(t)$ will therefore represent function of predominant input bar length change data values Δu. Second part of the equation (10) then becomes:

$$\iint_A G_{\alpha,\beta}\times\varepsilon(t)\times d\alpha d\beta = \iint_{\overline{A}} G_{\overline{\alpha},\overline{\beta}}\times\Delta u(t)\times d\overline{\alpha}d\overline{\beta} = A^+(t) - A^-(t) = \frac{1}{L^2}\times u_{pl} = \varepsilon_{pl}\,, \tag{11}$$

where ε_{pl} and u_{pl} represent differences of positive and negative sets in the Preisach triangles where strain and bar length change are used as input functions, respectively. L represents bar length. Substituting Eq.(11) in (9), (12) is obtained:

$$\overline{u}^{(m)T}\left[\int_{V^{(m)}} B^{(m)T} E B^{(m)} dV^{(m)}\right]\overline{u}^{(m)} - \overline{u}^{(m)T}\left[\int_{V^{(m)}} B^{(m)T}\frac{E(E-E_h)}{4(Y_{\max}-Y_{\min})}\frac{1}{\left(L^{(m)}\right)^2}dV^{(m)}\right]\cdot u_{pl}^{(m)} = \overline{u}^{(m)T} R_C^i \ . \tag{12}$$

It is considered that this problem would not require large displacement and large strain analysis, and if strain displacement matrix B is introduced, expressions in brackets of first and second part of (12) are actually defining elastic stiffness matrix and plastic stiffness matrix respectively:

$$K_{el}^{(m)}\overline{u}^{(m)} - K_{pl}^{(m)} \cdot u_{pl}^{(m)} = R_C^i \,. \tag{13}$$

For the finite element assemblage, expression in Eq.(11) becomes

$$K_{el}U - K_{pl} \cdot U_{pl} = R \,. \tag{14}$$

It is important to emphasize that elements of vectors U represent nodal displacements of the global system while elements of vector U_{pl} represent differences of positive and negative sets $A^+(t)$ and $A^-(t)$ in corresponding Preisach triangle, transformed in global system. For solving problem of nonlinear static analysis, iterative procedure using Newton-Raphson initial stress method can be applied:

$$\begin{aligned} &K_{el}\Delta U^{(i)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} \\ &{}^{t+\Delta t}U^{(i)} = {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)} \\ &{}^{t+\Delta t}F^{(i)} = K_{el}\,{}^{t+\Delta t}U^{(i)} + K_{pl}\,{}^{t+\Delta t}U_{pl}^{(i)} \,. \end{aligned} \tag{15}$$

Procedure for iteration i in Eqs.(15) is repeated until convergence is achieved. According to defined procedures for numerical analysis from Eqs.(13) to (15), algorithm for elastoplastic analysis of trusses subjected to cyclic loading was defined in C++ code. During every step and iteration in expressions (13) to (15), in every bar of truss structure, plastic part from Eq.(15) is being calculated according to current state of corresponding bar and then assembled in global matrix in Eq.(14). For assigned material properties, corresponding stress-strain behavior obtained by the Preisach model can be presented as shown in Fig.3.a, where ε_{el} is elastic strain limit of material, ε_{nl} is plastic strain limit at the onset of linear hardening and ε_0 is optional maximum strain limit of material. In static analysis, if material has very small or no strain hardening ($E_h \approx 0$), in order to provide some indication of when both the displacements and the forces are near their equilibrium values, it is recommended [6] that convergence criteria should be based on energy tolerance condition as shown in Eq.(16). In every iteration increment of internal energy is compared to initial internal energy increment:

$$\Delta U^{(i)T}\left({}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}\right) \leq \epsilon_E \left(\Delta U^{(1)T}\left({}^{t+\Delta t}R - {}^{t}F\right)\right). \tag{16}$$

4.1. Modeling of damage

In presented analysis basic concepts of macroscopic damage is introduced. Simple isotropic damage theory is implemented by introducing scalar damage measure in form of scalar variable ω that evolves from 0 (undamaged material) to 1 (fully damaged material):

$$\sigma = (1-\omega)\hat{\sigma}, \tag{17}$$

where $\hat{\sigma}$ represents effective stress of undamaged body (in case of elastic or elastoplastic analysis) and σ represents actual stress caused by damage. Effective strain of undamaged body $\hat{\varepsilon}$ is considered to be equal to effective strain of damaged body ε. Since in presented paper, uniaxial stress state is analyzed, with homogenous behavior of each element (bar) of structure (truss), local damage definition is sufficient and well suited for implementation since it is considered that damage is constant throughout each element of truss structure. Algorithm for elastoplastic analysis including damage can be defined as explained in [23]:

$$d\hat{\sigma} = D^{ep} \cdot d\hat{\varepsilon} \tag{18}$$

$$d\sigma = (1-\omega) D^{ep} \cdot d\varepsilon - d\omega \cdot \hat{\sigma} \tag{19}$$

$$\sigma^{(i+1)} = \sigma^{(i)} + \left(1-\omega^{(i)}\right) D^{ep(i+1)} \cdot d\varepsilon - d\omega \cdot \hat{\sigma}^{(i)} \tag{20}$$

$$D^{ep(i+1)} \cdot d\varepsilon = \left(\hat{\sigma}^{(i+1)} - \hat{\sigma}^{(i)}\right), \tag{21}$$

where D^{ep} represents elastoplastic matrix of material in multiaxial stress state or tangent modulus in presented unixial case. Hence $D^{ep} d\varepsilon$ represents elastoplastic stress increment. Note that $\hat{\sigma}^{(i+1)}$ and $\hat{\sigma}^{(i)}$ represents effective stress of undamaged body in elastoplastic analysis at time increments $i+1$ and i respectively, and they can be determined as presented in paragraphs 3.1. and 4.

Ductile damage variable ω can be defined as function of damage history parameter κ^d and and it grows from zero to one as the parameter κ^d grows from threshold κ_o to its ultimate value κ_u. Damage evolution can be defined as function that limits elastoplastic behavior in stress space and determines initiation of damage:

$$f^d = \bar{\varepsilon} - \kappa^d , \tag{22}$$

where measure $\bar{\varepsilon}$ can be adopted as equivalent plastic strain. The damage growth function governs damage variable evolution and it can be determined experimentally [24] in linear, power law or exponential form. In following numerical analysis, modified power law form of damage variable evolution is used:

$$\omega\left(\kappa^d\right) = 1 - \left(\frac{\kappa^o}{\kappa^d}\right)^{\beta} \left(\frac{\kappa^u - \kappa^d}{\kappa^u - \kappa^o}\right)^{\gamma}, \tag{23}$$

where γ, β represent material parameters.

5. NUMERICAL EXAMPLES

In all numerical examples, material properties for all truss bars are taken from experimental results [9], shown in paragraph 3.2. In order to outline the advantages of the Preisach hysteresis operator, results from presented model were compared to the results of the Bouc-Wen and GP model.

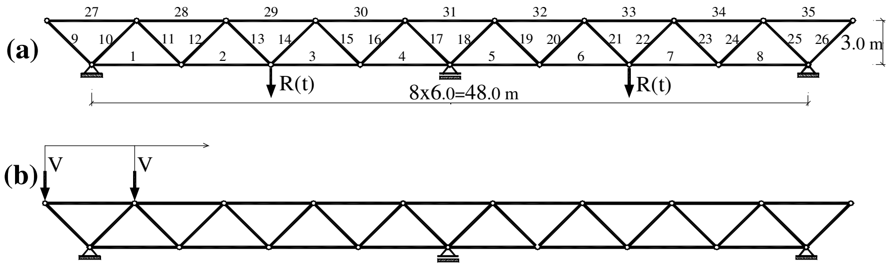

Fig. 6 (a) Geometry and loading $R(t)$ of truss structure for the first numerical example
(b) Moving load V in the second numerical example

5.1. Quasi-static analysis of two-span truss without damage

In the first example, truss structure shown in Fig.6.a is analyzed under two types of load (Fig.6.a and Fig.6.b). The first load case ($R(t)$) has cyclic character and its input function is shown in Fig.11.b, while in the second case truss was subjected to moving load pattern of two concentrated forces $2xV$ ($V=8000kN$). Structure consists of two types of bars. Horizontal bars with length of $6m$, and cross section areas $A_{hor}= 0.02m^2$ and diagonal bars with cross section areas $A_{dia}= 0.015m^2$.

In the first case of loading, all bars of structure, which undergo plastic deformation, have stable hysteretic loops and resulting stress-strain diagram for some of characteristic bars (29, 31) are presented in Fig.7.

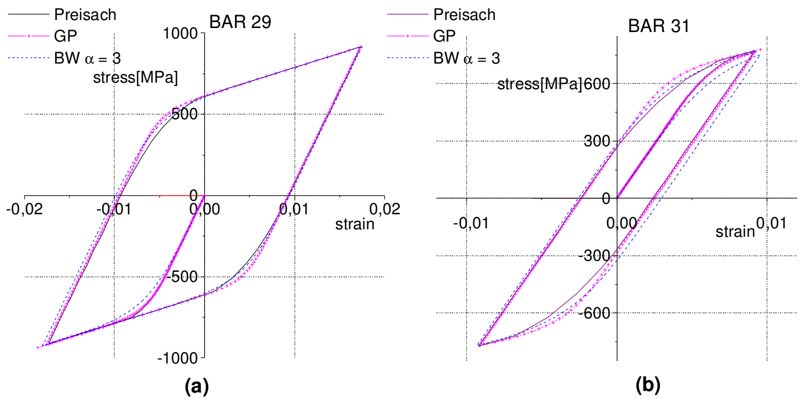

Fig. 7 Stress strain hysteresis curves for the first numerical example: Comparison of the Preisach model, GP model and the Bouc-Wen model (α=3) for bar 29 (a) and bar 31 (b) respectively

Although in the second case applied moving load pattern $2xV$ doesn't have cyclic character, bars will be subjected to load reversals, since these concentrate forces move across two span of continuous truss structure. Structure is subjected to five consecutive cycles of moving load pattern (Fig.6.b). Note that bars whose position is symmetrical around midpoint of structure should have identical response, according to position of load, in case of elastic analysis. However, all bars of the structure are being plastically deformed gradually and therefore symmetrical bars have different deformed state at same instant of time of loading and therefore different strain history. Results according to different models are presented in Fig.8, where maximum absolute vertical displacement during each cycle and residual maximum vertical displacement after each cycle are presented and compared. When results obtained using the Preisach model are analyzed, it can be seen that loading after first cycle doesn't increase deformation significantly. In summary, after the second and all subsequent cycles, identical and stable deformation is observed and there is no further significant increase of plastic strain as it is shown in Fig.9.

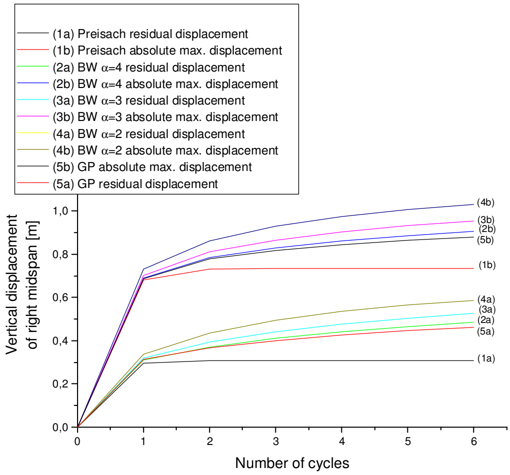

Fig. 8 Maximum absolute vertical displacement of the right midspan of truss during each cycle and residual maximum vertical displacement after each cycle in the second numerical example obtained using different models

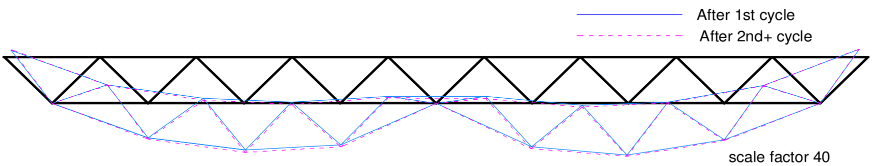

Figure 9 Deformed shape of structure for the second numerical example, obtained by suggested model, (scale factor = 40) after first cycle and after 2[nd] and all subsequent cycles

The most obvious difference between Preisach model and the Bouc-Wen model, can be seen when form of resulting hysteresis is analyzed. While resulting loops obtained by the Preisach model are decreased significantly after first cycle, providing elastic shakedown behaviour, that is not the case with corresponding results of the Bouc Wen and GP model. Hence, when results of the Bouc-Wen and GP model of hysteresis are compared, it can be seen that stabilization of plastic deformation in second numerical example occurred in higher number of cycles and higher strain values as shown in Fig.9.

5.2. Quasi-static analysis of two-span truss with damage

Results of elastoplastic analysis obtained using Preisach model without damage in first and second numerical example are compared to results of the same model that includes damage. Evolution law for damage variable is adopted in modified power law form (23) and parameters for damage variable ω are adopted as follows: κ_o = 0.01, κ_u = 0.1, β = 0.2 while coefficient γ is varied. Results obtained in the first numerical example are presented on the Figure 10.

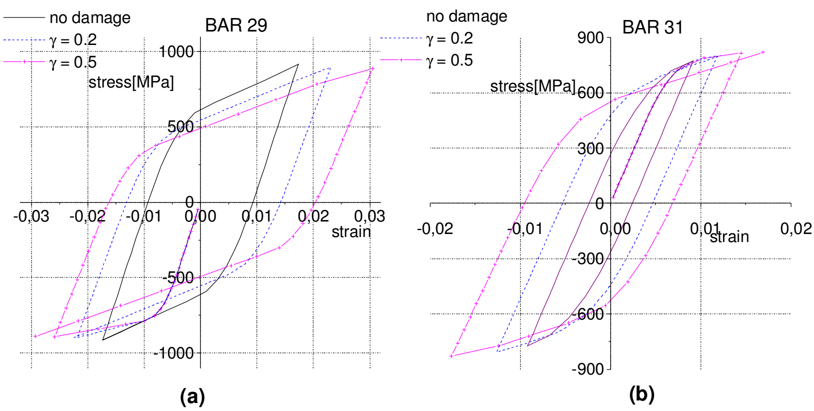

Fig. 10 Stress - strain hysteresis curves for the first numerical example: Comparison of the Preisach model without damage , Preisach model with damage (γ=0.2), Preisach model with damage (γ=0.5) for bar 29 (a) and bar 31 (b) respectively

By analyzing change of the tangent modulus on stress-strain curves, degradation of elastic and hardening modulus can be observed. In second numerical example, there is also stabilization of deformation occurred in models that included damage of material. It can be seen that resulting behavior is dependent form parameter γ and appropriate attention should be made for determination of damage parameters. Function of vertical displacement of right midspan, obtained using different models, is analyzed throughout all cycles and presented on Fig.11.

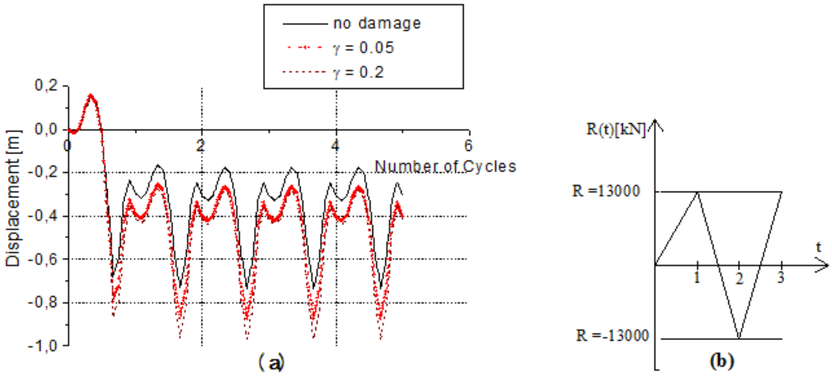

Fig.11 (a) Vertical displacement of right midspan for the second numerical example: Comparison of Preisach model without damage , Preisach model with damage (γ=0.05), Preisach model with damage (γ=0.2) ; (b) Time history of loading *R(t)* for the first numerical example

6 Conclusions

In the present paper it is shown that the Preisach model of hysteresis can be successfully applied in structural analysis of trusses, besides previously shown advanced application in the case of uniaxial cyclic loading of bar [2],[3] and cyclic bending of beam [4],[5]. Program in C++ code using finite element procedure is made for analyzing static and dynamic problems of trusses subjected to cyclic loading in the plastic range. The procedure leads to Newton-Raphson initial stress method. Damage can be included in presented algorithm by introducing scalar damage variable and basic concepts of continuum damage mechanics. It is also shown that the Preisach model can be defined in purely geometric terms, without any reference to analytical definition.

Results obtained by this model were compared with the result from the Bouc-Wen and the GP (Generalized Plasticity) models for several numerical examples. Even the agreement of obtained results is excellent, this model has several advantages. Analytical solution in closed form provides mathematical rigor of the Preisach model, while its absolute equivalent geometric interpretation enables numerical effective solution and less computational cost. Considering all possibilities that Preisach model poses, this type of analysis in finite element procedures is yet to be applied.

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References

1. Preisach, F., Über die magnetische Nachwirkung, *Z. Phys.*, 94, 277- 302 (1935)

2. Lubarda, A.V., Sumarac, D., Krajcinovic, D.: Hysteretic response of ductile materials subjected to cyclic loads. In: Ju, J.W. (ed.) *Recent Advances in Damage Mechanics and Plasticity, ASME Publication*, AMD, 123, 145-157 (1992)

3. Lubarda, A.V., Sumarac, D., Krajcinovic, D.: Preisach model and hysteretic behavior of ductile materials. *Eur. J. Mech.*, A/Solids, **12**, (4), 445-470 (1993)

4. Sumarac, D., Stosic, S.: The Preisach model for the cyclic bending of elasto-plastic beams, *Eur. J. Mech.*, A/Solids **15** (1), 155-172 (1996)

5. Šumarac, D. Petrašković, Z: Hysteretic behavior of rectangular tube (box) sections based on Preisach model, *Archive of Applied Mechanics*, 82 (10), 1663-1673(2012)

6. Bathe, K J.: *Finite Element Procedures*, Prentice Hall (1996)

7. Iwan, W.D.: On a class of models for the yielding behavior of continuous and composite systems, *J. Appl. Mech.*, 34, 612-617 (1967)

8. Mayergoyz, I. D.: *Mathematical Models of Hysteresis*, Springer-Verlag, New York (1991)

9. Kujawski, D., Krempl, E.: The Rate (Time)-Dependent Behaviour of Ti-7Al-2Cb-1Ta Titanium Alloy at Room Temperature Under Quasi Static Monotonic and Cyclic Loading, *J. Appl. Mech.*, 48, 55-63 (1981)

10. Asaro, R.J.: Elastic-plastic memory and kinematic type hardening, *Acta Metall.* **23**, 1255-1265 (1975)

11. Rabotnov,Yu.N.:*Elements of Hereditary Solid Mechanics*,Mir Publishers, (1980)

12. Erlicher, S., Point, N.:Endochronic theory, nonlinear kinematic hardening rule and generalized plasticity: A new interpretation based on generalized normality assumption, *International Journal of Solids and Structures*,43,4175-4200, (2006)

13. Bouc, R.:Modèle Mathématique d'Hystérésis, *Acustica*, 24, 16—25, (1971) (in French)

14. Wen, Y.K.: Method for Random Vibration of Hysteretic Systems,Journal of the Engineering Mechanics Division, Proc. ASCE, 102, 249—263, (1976)

15. Ismail, M., Ikbouane, F., Rodellar, J.: The Hysteresis Bouc-Wen model, a Survey, *Arch Comut Methods Eng*,16,161-188, (2009)

16. Erlicher, S.: Hysteretic degrading models for the low-cycle fatigue behavior of structural elements, Theory, Numerical aspects and Applications, Dottorato di ricerca in Modellazione, Conservazione e Controllo dei Materiali e delle Strutture, *Universita Degli studi di Trento*, (2003)

17. Visintin, A.: Mathematical Models of Hysteresis, *Dipartimento di Matematica dell'Universita degli Studi di Trento*, Italia, (2005)

18. CSI Analysis Reference Manual for SAP2000, ETABS, SAFE and CSiBridge, Computers and Structures Inc., (2009)
19. Lubliner, J., Taylor, RL, Auricchio F.: A new model of generalized plasticity and its numerical implementation , *International Journal of Solids and Stuctures* 30(22):3171-3184, (1993)
20. Kostic, S., Filippou, F., Lee, C.L.: An efficient beam-column element for inelastic 3D frame analysis, *Computational Methods in Applied Sciences*, 30, 49-67, (2013).
21. Sumarac, D. and Krajcinovic, D.: "A Self-consistent Model for Microcrack-weakened Solids", *Mechanics of Materials*, 6, pp. 39-52, 1987.
22. Sumarac,D. and Krajcinovic, D.: "A Mesomechanical Model for Brittle Deformation Processes", Part II, *Journal of Applied Mechanics*, 56, pp. 57-62, 1989.
23. de Borst, R., Pamin, J., Geers, M.G.D.: On coupled gradient – dependent plasticity and damage theories with view to localization analysis, *Eur. J.Mech.* A/Solids,18, 939-962, (1999)
24. Panis, M.W., van der Slius, O., Peerlings, R.H.J.: Experimental identification of damage evolution law for steel, Eindhoven, (2004).

ANALIZA CIKLIČNE PLASTIČNOSTI REŠETKASTIH NOSAČA KORISTEĆI PRAJZAKOV MODEL HISTEREZISA

Z. Perović, D. Šumarac

Abstract *U ovom radu, Prajzakov model histerezisa, koji je već uspešno primenjen za rešavanje problema ciklične plastičnosti aksijalno napregnutih štapova i cikličnog savijanja elastoplastične grede, je primenjen u strukturnoj analizi rešetkastih nosača koji su izloženi cikličnom opetrećenju. Takođe je prikazano da se efekti oštećenja mogu uključiti u prikazanu analizu uvođenjem osnovnih principa mehanike oštećenja u kontinuumu. Koristeći metodu konačnih elemenata, jednačine ravnoteže i algoritam za numeričko rešavanje je definisan. Neke prednosti ovakvog pristupa su naglašene, upoređene sa već postojećim postupcima za rešavanje, i prikazane na različitim numeričkim primerima.*

Ključne reči: *ciklična plastičnost, Prajzakov model, rešetkasti nosači, oštećenje*

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THE IN-PLANE STRESS INFLUENCE ON CRACK KINKING OUT FROM THE INTERFACE

UDC 551.324.85:532.61.042:537.226.86:621.833

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ABSTRACT. *Crack that lies between the two elastic isotropic materials can continue to propagate along the interface or it can kink out from it and continue to propagate in one of the two materials. The "competition" between the crack kinking out from the interface and propagation along it can be estimated by comparing the energy release rate needed for the crack kinking from the interface G_s and energy release rate needed for the crack propagation along the interface G. The in-plane stress parallel to the interface affects the energy release rate for the crack that is kinking out from the interface and can significantly change the conditions under which the crack would kink out. In this paper is presented dependence of the energy release rates ratio G_s/G on in-plane stresses. Results presented here enable comparison of the interface toughness to toughness of material without interface, in order to determine whether the crack would kink out from the interface or would it continue to propagate along it. This is significant in design of joints between the two materials, for instance in composite materials between fibers and substrate and for coatings over the substrates.*

Key words: *Interfacial crack, Crack kinking, In-plane stress*

1. INTRODUCTION

The crack most frequently appears at the interface between the two materials, since the interface toughness is smaller than toughness of materials that are forming the interface. However, in some cases, the crack can appear in one of the materials, in such a way that it can be parallel to the interface (the so-called subinterfacial crack) or it can attack the interface at a certain angle. Also, in some other cases a crack that propagates along the interface can kink out from it and it can G and G_s, respectively. Those "driving forces" depend on intensity of the applied load, ratio of opening mode load and the shear mode load and on elastic constants. The "competition" between crack kinking out and propagation along the interface also includes fracture toughness of materials Γ and of interface Γ_i. Roughly speaking, the tough interface would prevent the crack to propagate along it. The crack kinking out from the interface is possible if the following holds, He and Hutchinson (1989):

$$\frac{G_s}{G} > \frac{\Gamma}{\Gamma_i}. \tag{1}$$

There exist several papers where the problems related to crack kinking out from the interface were analyzed. The crack kinking out from the interface between the two elastic isotropic materials was analyzed by He and Hutchinson (1989) and Veljkovic (2005). Wang et al. (1991) and Nikolić et al. (2010) were considering crack kinking out from the interface between the two anisotropic materials. Behavior of an interfacial crack in conditions of the in-plane stresses parallel to the interface is the subject of this work.

2. PROBLEM FORMULATION

The problem considered is presented in Figure 1.the main crack lies at the interface between the two different semi-infinite elastic bodies, in the plane strain conditions. The straight-line crack, of length a kinks out at angle ω into the material below the interface, material 2. It is assumed that the length of the crack that kinks out from the interface, a, is small in comparison to length of the main crack and that the asymptotic problem of the semi-infinite main crack is being analyzed.

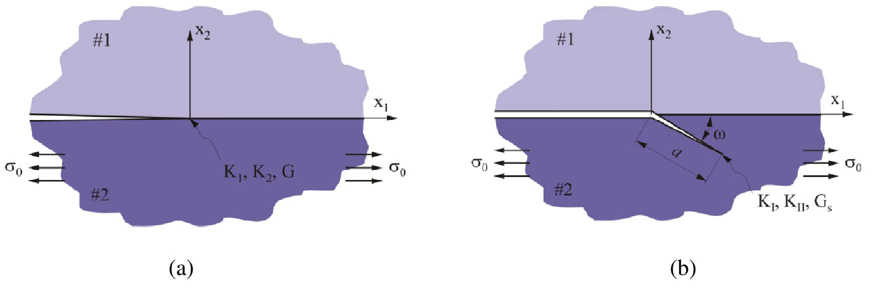

Fig. 1. Geometry of the considered problem:
(a) crack at the interface; (b) kinked crack.

The stress field prior to crack kinking is a singular field of an interfacial crack, which is characterized by the complex stress intensity factor, $K = K_1 + iK_2$ with addition of the stress σ_0 parallel to the interface, which exists within the material into which the crack is kinking, Figure 1. (a). The stress field at the tip of a crack that has kinked from the interface is characterized by the combination of the standard intensity factors for Mode I and Mode II of crack propagation, K_I and K_{II}, respectively, Figure 1. (b). Analysis should establish the relation between stress intensity factors K_I and K_{II} for the crack that has kinked from the interface and stress intensity factors K_1 and K_2 for the crack at the interface in terms of the angle ω and elastic moduli of the two materials. The energy release rate for the crack that has kinked from the interface is compared to the energy release rate for the interfacial crack.

The complex stress intensity factor represents the characteristic of the interface. This factor has the general form, Rice (1988):

$$K = YT\sqrt{L}\,L^{-i\varepsilon}e^{-i\psi}\,, \tag{2}$$

where T represents intensity of stress due to the load applied to a sample, L is the characteristic length (crack length or layer thickness, etc.), Y is the dimensionless real positive variable, ψ is the phase angle of $KL^{i\varepsilon}$, but it is often called "the phase angle of the complex stress intensity factor" or "the phase angle of applied load". Variables, Y and ψ both depend on applied load and generally on ratio of the elasticity moduli and the characteristic sizes of the cracked body.

When the stress σ_0 exists in the material below the interface, which acts parallel to the interface, Figure 1. (a), regardless whether the matter is residual stress or applied load, the additional dimensionless parameter is being introduced:

$$\eta = \frac{\sigma_0\sqrt{a}}{\cosh \pi\varepsilon\sqrt{E_* G}}, \tag{3}$$

where E_* is determined by the expression:

$$\frac{2}{E_*} = \frac{1}{\bar{E}_1} + \frac{1}{\bar{E}_2}\,, \tag{4}$$

where: $\bar{E}_i = E_i/(1-\nu_i^2)$ for the plane strain conditions, E_i (i = 1, 2) is the Young's modulus of elasticity, ν_i is the Poisson's ratio, while indices 1 and 2 refer to materials above and below the interface, respectively.

Parameter ε is called the bielastic constant or the oscillatory index; it is the characteristics of the interfacial crack, and it is determined as, Rice (1988):

$$\varepsilon = \frac{1}{2\pi}\ln\left(\frac{1-\beta}{1+\beta}\right), \tag{5}$$

where β is one of the two Dundurs parameters, defined with:

$$\alpha = \frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}\,,\ \beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}\,, \tag{6}$$

where: μ_i is the shear modulus, while $\kappa_i = 3 - 4\,\nu_i$ for the plane strain conditions.

Energy release rate for the interfacial crack is being determined based on expression, Hutchinson and Suo (1992):

$$G = \left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2} \right) \frac{K\,\bar{K}}{16ch^2(\varepsilon\pi)}\,. \tag{7}$$

The stress field for the semi-infinite interfacial crack (a = 0) has the form determined by the following equation:

$$\sigma_{ij} = \frac{1}{\sqrt{2\pi\, r}} \left[\mathrm{Re}\,(Kr^{i\varepsilon})\sigma^{I}_{\alpha\beta}(\theta,\varepsilon) + \mathrm{Im}\,(Kr^{i\varepsilon})\sigma^{II}_{\alpha\beta}(\theta,\varepsilon) \right], \tag{8}$$

where $\sigma^{I,II}_{\alpha\beta}(\theta,\varepsilon)$ are the angular functions which correspond to tensile tractions and in-plane shear tractions across the interface, Nikolic and Djokovic (2011). Thus, tractions n the interface, ahead of the crack tip, at a distance r have the form:

$$(\sigma_{22} + i\sigma_{12})_{\theta=0} = \frac{(K_1 + iK_2)r^{i\varepsilon}}{\sqrt{2\pi\, r}}\,. \tag{9}$$

The stress field at the tip of a crack, which has kinked from the interface into the material 2, is the common stress field with the usual stress intensity factors K_{I} and K_{II}, determined by equations:

$$\sigma_{22} = K_I\,(2\pi r)^{-1/2}, \qquad \sigma_{12} = K_{II}\,(2\pi r)^{-1/2}\,. \tag{10}$$

The relation between stress intensity factors for the crack that has kinked and the main crack at the interface can be written as:

$$K^s_I + iK^s_{II} = cKa^{i\varepsilon} + \bar{d}\,\bar{K}a^{-i\varepsilon} + b\sigma_0\sqrt{a}\,, \tag{11}$$

where $(\bar{\ })$ denotes the complex conjugate value, c, d and b are the complex functions of ω, α and β.

Coefficient $b = b_1 + ib_2$ is being determined based on procedure defined by He and Hutchinson (1988) for coefficients c and d. For approximate determination of coefficients c and d one can use the approximate expressions proposed in Veljkovic (2005):

$$\begin{aligned} c(\omega) &= \frac{1}{2}\sqrt{\frac{1-\beta}{1+\alpha}}\left(e^{-\frac{i\omega}{2}} + e^{-\frac{3i\omega}{2}} \right) \\ d(\omega) &= \frac{1}{4}\sqrt{\frac{1-\beta}{1-\alpha}}\left(e^{-\frac{i\omega}{2}} - e^{-\frac{3i\omega}{2}} \right) \end{aligned}. \tag{12}$$

For approximate determination of coefficients b_1 and b_2 one can use the approximation defined by Cotterell and Rice (1980):

$$b_1 = 2\sqrt{\frac{2}{\pi}}\sin^2\omega, \qquad b_2 = \sqrt{\frac{2}{\pi}}\sin 2\omega. \tag{13}$$

In the case that $\beta = 0$ equation (11) can be written as:

$$\begin{aligned} K_I^s &= (c_R + d_R)K_1 - (c_I + d_I)K_2 + b_1\sigma_0\sqrt{a} \\ K_{II}^s &= (c_I - d_I)K_1 + (c_R + d_R)K_2 + b_2\sigma_0\sqrt{a}. \end{aligned} \tag{14}$$

The relation between the energy release rate G for the interfacial crack, which propagates along the interface, and the stress intensity factor K is determined by expression (7).

Energy release rate G_s, for the crack that is kinking out from the interface ($a > 0$), can be determined based on Irwin's relation between the energy release rate for the straight-line quasi-static crack growth and the stress intensity factor, i.e.

$$G = \frac{K_I^2 + K_{II}^2}{\bar{E}}. \tag{15}$$

Making use of equation (11) yields:
where is defined by equation:

$$G_s = (G_s)_{\eta=0} + \frac{2\sigma_0\sqrt{a}}{\bar{E}_2}\mathrm{Re}[\bar{b}(cKa^{i\varepsilon} + \bar{d}\bar{K}a^{-i\varepsilon})] + \frac{\sigma_0^2 a}{\bar{E}_2}(b_1^2 + b_2^2), \tag{16}$$

where $(G_s)_{\eta=0}$ is defined by equation:

$$(G_s)_{\eta=0} = \frac{\kappa_2 + 1}{8\mu_2}[(|c|^2 + |d|^2)K\bar{K} + 2\,\mathrm{Re}(c\,d\,K^2 a^{2i\varepsilon})] \tag{17}$$

and it represents the energy release rate for $\sigma_0 = 0$.

Finally, based on equations (7), (16) and (2), one obtains relation between the energy release rates for the kinked and the main crack as:

$$G_s / G = q^{-2}\left\{[(|c|^2 + |d|^2) + 2\,\mathrm{Re}(c\,d\,e^{2i\bar{\psi}})] + 2\eta\,\mathrm{Re}[\bar{b}(ce^{i\bar{\psi}} + \bar{d}e^{-i\bar{\psi}})] + \eta^2(b_1^2 + b_2^2)\right\}, \tag{18}$$

where $q = \sqrt{(1-\beta^2)/(1+\alpha)}$, $\bar{\psi} = \psi + \varepsilon\ln(a/L)$ and L is the characteristic length for the interfacial crack problem when ($a > 0$).

3. RESULTS AND DISCUSSION

The role of the σ_0 stress in competition whether the interfacial crack would continue to propagate along the interface or would it kink out from it, is presented in Figures 2 and 3.

The ratio G_s/G, calculated based on equation (18), is presented in Figure 2, in terms of kink angle ω for the case $\alpha = \beta = 0$. The main crack is in the mixed mode conditions, with equal shares of Mode I and Mode II, i.e., for $\psi = 45°$. The significant influence of the σ_0 stress on the energy release rate begins to appear when $\eta = 0.1$. For values $\eta = \pm 0.25$, ratio G_s/G either increases or decreases for about 50 %, depending on the sign of σ_0.

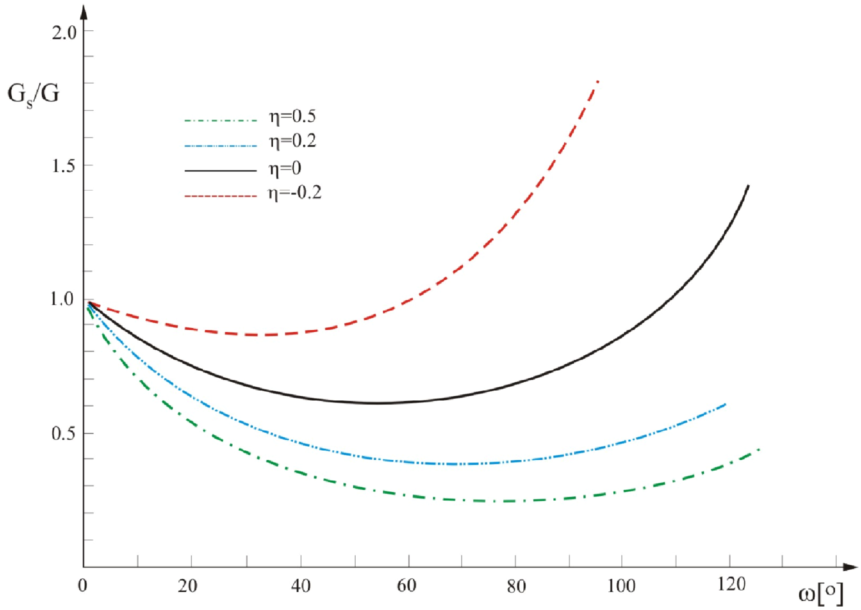

Fig. 2. Ratio G_s/G as a function of the crack kinking angle ω for different values of parameter η and values of the load phase angle $\psi = arctg(K_{II} / K_I) = 45°$ and $\alpha = \beta = 0$.

Ratio G_s/G as a function of the load phase angle ψ is presented in Figure 3. The role of in-plane tension ($\eta > 0$) can be explained based on Figure 3. The ratio G_s/G is determined based on the initial value of the crack length a and σ_0. The kinking out of a crack from the interface is possible if inequality (1) is satisfied. If that was not the case, the crack would continue to propagate along the interface, since the applied load is too small to cause the crack kinking. For the case of the in-plane tension, if the crack has kinked out from the interface, as it propagates so will η increase thus causing increase of the driving force at the crack tip and the kinking becomes unstable. For the case of the in-plane compression ($\eta < 0$), the behavior in the material below the interface is completely different. The energy release rate G_s decreases with increase of the kinked crack length a, consequently the kinked crack will tend to close.

Influence of different characteristics of materials on the either sides of the interface on ratio G_s/G is in Figure 3 shown via parameters α and β. One can notice from Figure 3 that with increase of the relative compliance of a material into which the crack is kinking, the energy release rate for the crack kinking out also increases, thus enhancing the tendency of a crack to kink into the material below the interface.

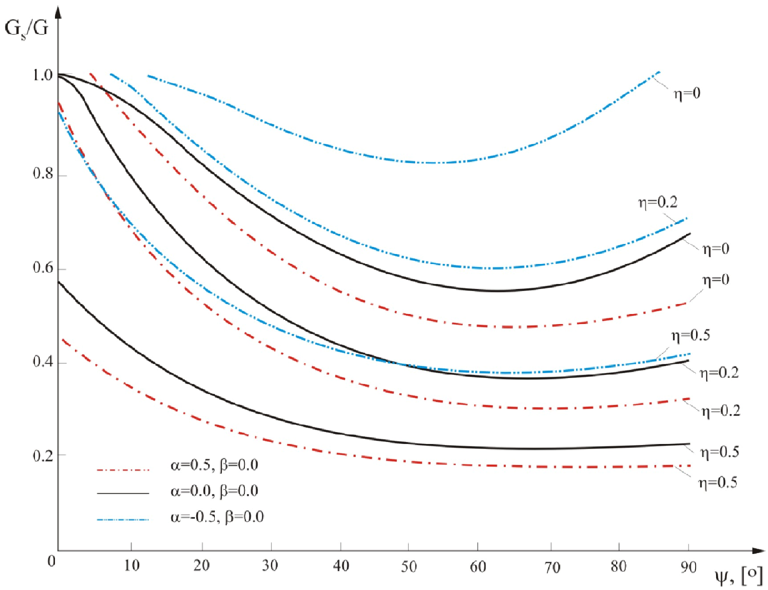

Fig. 3. Ratio G_s/G as a function of the load phase angle ψ for different values of parameter η and parameters α and β.

4. CONCLUSION

The in-plane stresses impose a strong influence on the interfacial crack behavior, especially the tensile ones. If the crack kinking out from the interface existed, the tensile in-plane stresses would cause destabilizing of the interfacial crack and its further kinking into the material below the interface and distancing from it. On the other hand, the compressive in-plane stresses would stabilize the interfacial crack and would lead to closing of the kinked crack.

Such a result is important for design of joints where either the reactive layer or the coating are exposed to different stresses, as well as for fracture of fibers in composite materials. Imposing compressive in-plane stresses would prevent propagation of the kinked portion of interfacial cracks and thus eliminate possibility for coating delamination from the substrate in the former case or fiber separation from the substrate in the latter case.

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REFERENCES

1. He M. -Y. and J. W. Hutchinson, (1989), Kinking of a crack out of an interface, *Journal of Applied Mechanics,* Vol. **56**, pp. 270-278.
2. Veljkovic J.M., (2005), The crack kinking out of an interface, Theoretical *and Applied Mechanics*, Vol. **32** (3), pp. 208-216.
3. Wang T.C. , Shih C.F., Suo Z., (1992), Crack extension and kinking in laminates and bicrystals, *International Journal of Solid and Structures*, Vol. **29**, pp. 327-344.
4. Nikolic R. R., Djokovic J. M., Mićunović M.V., (2010), The Competition Between the Crack Kinking Away From the Interface and Crack Propagation Along the Interface in Elastic Bicrystals, *International Journal of Fracture,* Vol. **164**, No.1, pp. 73-82.
5. Rice J.R., (1988), Elastic fracture mechanics concepts for interfacial cracks, *Journal of Applied Mechanics,* Vol. **55**, pp. 98-103.
6. Hutchinson, J.W. and Z. Suo (1992), Mixed mode cracking in layered materials, *Advances in Applied Mechanics,* Vol. **29**, pp. 63-191.
7. Nikolic, R.R. and J.M. Djokovic, Interfacial Cracks in Bicrystals and Bimaterials, In "Crack Growth: Rates, Prediction and Prevention", Edited by D. Kubair, Nova Publishers, Inc., New York, (2011), pp. 101-126. ISBN: 978-1-61470-799-8.
8. Cotterell B. and Rice J.R., (1980), Slightly curved or kinked cracks, *International Journal of Fracture*, Vol. **16**, No. 2, pp. 155-169.

UTICAJ NAPONA U RAVNI NA SKRETANJE PRSLINE SA INTERFEJSA

Jelena M. Djoković, Ružica R. Nikolić

Prslina koja leži izmedju dva elastična izotropna materijala može da nastavi da se širi duž interfejsa ili može da skrene sa njega i da nastavi da se širi u jednom od dva materijala. "Takmičenje" izmedju skretanja prsline sa interfejsa i širenja duž njega može da se oceni poredjenjem odnosa brzine oslobadjanja energije za skretanje prsline sa interfejsa G_s i brzine oslobadjanja energije za prslinu na interfejsu G. Napon u ravni paralelan sa interfejsom utiče na brzinu oslobadjanja energije za prslinu koja skreće sa interfejsa i može značajno promeniti uslove pod kojima će prslina skrenuti sa interfejsa. U ovom radu je prikazana zavisnost odnosa brzina oslobadjanja energije G_s/G od napona u ravni. Rezultati prikazani u ovom radu daju mogućnost uporedjivanja žilavosti interfejsa sa žilavošću materijala van interfejsa, a u cilju odredjivanja da li će prslina da skrene sa interfejsa, ili će da nastavi dalje da se širi duž njega. Ovo je značajno za primenu u projektovanju spojeva izmedju dva materijala, na primer u kompozitnim materijalima izmedju vlakana i matrice ili kod prevlaka na osnovnim materijalima.

Key words: *Interfejsna prslina, Skretanje prsline, Napon u ravni.*

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UNSTEADY MAGNETOHYDRODYNAMIC HEAT AND MASS TRANSFER PAST THE BODY WITH TIME VARYING WALL TEMPERATURE

UDC 532.5: 536.24:537.6

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Abstract. *The paper discusses the unsteady two-dimensional laminar magnetohydrodynamic (MHD) thermal boundary layer of incompressible fluid. Outer electric filed is neglected, magnetic Reynolds number is significantly lower then one i.e. considered problem is in induction-less approximation, characteristic properties of fluid are constant and body surface temperature varies with time. The boundary-layer equations are generalized such that the equations and the boundary conditions are independent of the particular conditions of the problem, and this form is considered as universal. Obtained universal equations are numerically solved using progonka method. Numerical results for the dimensionless velocity, temperature and dimensionless friction factor in function of introduced sets of parameters are obtained, displayed graphically and used to carry out general conclusions about development of temperature MHD boundary layer..*

Key words: *MHD, magnetic field, similarity parameters, universal equations*

1. INTRODUCTION

The problem of boundary-layer separation and control has attracted considerable attention over several decades because of the fundamental flow physics and technological applications. Some of the essential ideas related to boundary-layer separation and the need to prevent the same from occurring have been addressed by Prandtl [1]. The possibility to act on a fluid flow in a contactless way, offered by magnetohydrodynamics (MHD), stimulated the imagination of aerodynamists and naval engineers relatively early [2]. In the 1950s, a multitude of aerospace applications of MHD flow control techniques has been envisioned using the fact that at high enough speeds air gets ionised by the action of shock waves and frictional heating, and thus becomes a conductor. Such high-speed conditions are typical for re-entry problems. Resler and Sears [3] and Busemann [4] proposed, among others, to use magnetic fields to control heat transfer, to decelerate or to accelerate vehicles, and to prevent flow separation. Although enthusiasm for the practical application of these ideas waned later on, the topic is now again under investigation in connection with scramjets [5], heat transfer mitigation [6] and electromagnetic braking [7]. Flow of an incompressible viscous fluid over a surface [8] has an important influence on several technological applications in the field of metallurgy and chemical engineering. For example, extrusion of plastics in the manufacture of rayon and nylon, the purification of crude oils, the textile industry, etc. In many process industries the cooling of threads or sheets of some polymer materials is important in the production line. The rate of the cooling can be controlled effectively to achieve final products of desired characteristics by drawing threads, etc., in the presence of an electrically conducting fluid subjected to magnetic field. Many MHD flow problems have important applications in the extrusion process. The extrudate from the die is generally drawn and simultaneously stretched into a sheet which is then solidified through quenching or gradual cooling by direct contact with cooling fluid [9]. In these cases the properties of the final product depend to a great extent on the rate of cooling which is governed by the conditions in the boundary-layer.

In this paper, for the sake of richness of mentioned research, mathematical model of unsteady temperature two-dimensional laminar MHD boundary-layer of incompressible fluid is studied, which is directly related with previously mentioned physical models. The system of partial differential equations that describe the considered problem can be solved for each particular case using modern numerical methods and computer. In this paper, quite different approach is used based on ideas of generalized similarity method given in papers [10-13], which is extended in papers [14-16]. The boundary-layer equations are generalized such that the equations and the boundary conditions are independent of the particular conditions of the problem, and this form is considered as universal.

1.1. Mathematical model

Unsteady two-dimensional temperature laminar MHD boundary-layer of incompressible fluid is considered. Magnetic field is function of longitudinal coordinate and perpendicular to the surface, considered problem is in induction-less approximation (magnetic Reynolds number is significantly lower then one) and electric field is neglected. Characteristic properties of fluid are constant, and surface temperature is time function. Described problem is mathematically presented with following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0; \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}(u-U); \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B^2}{\rho c_p}(u-U)^2+\frac{1}{c_p}\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)(U-u); \tag{3}$$

and corresponding boundary and initial conditions:

$$u = 0, v = 0, T = T_w(t) \text{ for } y = 0; \quad u \to U(x,t), T \to T_\infty \text{ for } y \to \infty;$$

$$u = u_0(x, y), T = T_0(x, y) \text{ for } t = t_0; \quad u = u_1(t, y), T = T_1(t, y) \text{ for } x = x_0. \tag{4}$$

In the given equations and in the boundary conditions the notations common in the boundary-layer theory are used for different physical values. Here, x, y is longitudinal and transversal coordinate respectively; t-time; u, v-longitudinal and transversal velocity component respectively; $U(x,t)$- free stream velocity; ν-kinematic viscosity of fluid; σ-fluid electrical conductivity; ρ-fluid density; B-magnetic induction; T-fluid temperature; λ-thermal conductivity, c_p-specific heat capacity; μ-dynamic viscosity; $T_w(t)$-body surface temperature; T_∞-free stream temperature; $u_0(x, y)$ and $T_0(x, y)$-distribution of longitudinal velocity and fluid temperature in time moment $t = t_0$ respectively; $u_1(x, y)$ and $T_1(x, y)$-distribution of longitudinal velocity and fluid temperature in cross section $x = x_0$.

For further consideration stream function, $\Psi(x, y, t)$ is introduced with following relations:

$$\frac{\partial \Psi}{\partial x} = -v, \quad \frac{\partial \Psi}{\partial y} = u; \tag{5}$$

which satisfies equation (1) identically and transform momentum equation (2) into equation:

$$\frac{\partial^2 \Psi}{\partial t \partial y} + \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^3 \Psi}{\partial y^3} - \frac{\sigma B^2}{\rho}\left(\frac{\partial \Psi}{\partial y} - U\right); \tag{6}$$

and energy equation (3) into equation:

$$\begin{gathered}\frac{\partial T}{\partial t} + \frac{\partial \Psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial T}{\partial y} = \frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2 + \\ + \frac{\sigma B^2}{\rho c_p}\left(\frac{\partial \Psi}{\partial y} - U\right)^2 + \frac{1}{c_p}\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial t}\right)\left(U - \frac{\partial \Psi}{\partial y}\right).\end{gathered} \tag{7}$$

Boundary and initial conditions (4) are transformed into conditions:

$$\Psi = 0, \ \partial\Psi/\partial y = 0; \ T = T_w(t) \text{ for } y = 0; \quad \partial\Psi/\partial y \to U(x,t); T \to T_\infty \text{ for } y \to \infty;$$

$$\partial\Psi/\partial y = u_0(x, y), \ T = T_0(x, y) \text{ for } t = t_0; \quad \partial\Psi/\partial y = u_1(t, y), \ T = T_1(t, y) \text{ for } x = x_0. \tag{8}$$

Equation (9) does not depend on the equation (10) and it can be solved independently. Solution of equation (9) is used for solving of equation (10).

1.2 Universal equations

In order to analyze described flow problem following new variables are introduced:

$$x = x,\ t = t,\ \eta = \frac{Dy}{h(x,t)};\ \Phi(x,t,\eta) = \frac{D\Psi(x,y,t)}{U(x,t)h(x,t)};\ \Theta(x,t,\eta) = \frac{T_w - T}{T_w - T_\infty}; \tag{9}$$

where D is normalizing constant, η-dimensionless transversal coordinate, $h(x,t)$ is characteristic linear scale of transversal coordinate in boundary-layer, $\Phi(x,t,\eta)$-dimensionless stream function and $\Theta(x,t,\eta)$-dimensionless temperature difference. According to introduced variables, equations (6) and (7) are transformed in following system:

$$\begin{gathered} D^2\frac{\partial^3\Phi}{\partial\eta^3} + f_{1,0}\left(\Phi\frac{\partial^2\Phi}{\partial\eta^2} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2 + 1\right) + \left(f_{0,1} + g_{1,0}\right)\left(1 - \frac{\partial\Phi}{\partial\eta}\right) + \\ + \frac{1}{2}\left(F\Phi + \eta g\right)\frac{\partial^2\Phi}{\partial\eta^2} = z\frac{\partial^2\Phi}{\partial t\partial\eta} + UzX(\eta;x); \end{gathered} \tag{10}$$

$$\begin{gathered} \frac{D^2}{P_r}\frac{\partial^2\Theta}{\partial\eta^2} - D^2E_c\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2 - E_c g_{1,0}\left(1 - \frac{\partial\Phi}{\partial\eta}\right)^2 + (1-\Theta)l_1\frac{\partial\Phi}{\partial\eta} + \frac{1}{2}\eta g\frac{\partial\Theta}{\partial\eta} - \\ -E_c\left(f_{0,1} + f_{1,0}\right)\left(1 - \frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{2}\left(F + 2f_{1,0}\right)\Phi\frac{\partial\Theta}{\partial\eta} = z\frac{\partial\Theta}{\partial t} - UzY(x;\eta); \end{gathered} \tag{11}$$

where for the sake of shorter expression, the notations are introduced:

$$z = \frac{h^2}{\nu},\ g = \frac{\partial z}{\partial t},\ N = \frac{\sigma B^2}{\rho};\ g_{1,0} = Nz,\ F = U\frac{\partial z}{\partial x};$$

$$f_{1,0} = z\frac{\partial U}{\partial x},\ f_{0,1} = \frac{z}{U}\frac{\partial U}{\partial t};\ l_1 = \frac{z}{T_w - T_\infty}\frac{dT_w}{dt};$$

$$X(x_1;x_2) = \frac{\partial\Phi}{\partial x_1}\frac{\partial^2\Phi}{\partial\eta\partial x_2} - \frac{\partial\Phi}{\partial x_2}\frac{\partial^2\Phi}{\partial x_1\partial\eta};\ Y(x_1;x_2) = \frac{\partial\Phi}{\partial x_1}\frac{\partial\Theta}{\partial x_2} - \frac{\partial\Phi}{\partial x_2}\frac{\partial\Theta}{\partial x_1};$$

$$P_r = \frac{\nu\rho c_p}{\lambda}\ \text{- Prandtl number};\ E_c = \frac{U^2}{c_p\left(T_w - T_\infty\right)}\ \text{- Eckert number}. \tag{12}$$

Now we introduce sets of parameters:

$$f_{k,n} = U^{k-1}\frac{\partial^{k+n}U}{\partial x^k \partial t^n}z^{k+n} \quad (k,n = 0,1,2,...;k \vee n \neq 0); \tag{13}$$

$$g_{k,n} = U^{k-1}\frac{\partial^{k-1+n}N}{\partial x^{k-1} \partial t^n}z^{k+n} \quad (k,n = 0,1,2,...,;k \neq 0); \tag{14}$$

$$l_k = \frac{1}{q}\frac{d^k q}{dt^k}z^k, \ (k = 1,2,...); \ \text{ where } q = T_w - T_\infty; \tag{15}$$

$$g = \frac{\partial z}{\partial t} = const. \tag{16}$$

In previous equations $f_{k,n}$ is dynamic parameter, $g_{k,n}$-magnetic parameter, l_k-temperature parameter and g-cosntant parameter. It can be noticed that the first parameters are already given in the equations (12). Introduced sets of parameters reflect the nature of the change of free stream velocity, alteration characteristic of variable N and the change of surface temperature, and a part from that, in the integral form (by means of z and $\partial z / \partial t$) pre-history of flow in boundary layer.

Using, introduced sets of parameters (13-16) like new independent variables instead of x and t, and following differentiation operators for x and t:

$$\frac{\partial}{\partial \varphi} = \sum_{\substack{k,n=0 \\ k\vee n\neq 0}}^{\infty} \frac{\partial f_{k,n}}{\partial \varphi}\frac{\partial}{\partial f_{k,n}} + \sum_{\substack{k=1 \\ n=0}}^{\infty} \frac{\partial g_{k,n}}{\partial \varphi}\frac{\partial}{\partial g_{k,n}} + \begin{Bmatrix} 0 \ \text{ for } \Phi \\ \sum\limits_{k=1}^{\infty} \frac{\partial l_k}{\partial \varphi}\frac{\partial}{\partial l_k} \ \text{ for } \Theta \end{Bmatrix}; \quad \varphi = x \ \text{ or } \ t \tag{17}$$

parameter derivates along coordinate x and time t are obtained by differentiation of equations (13-16):

$$\frac{\partial f_{k,n}}{\partial x} = \frac{1}{Uz}\left\{(k-1)f_{1,0}f_{k,n} + (k+n)Ff_{k,n} + f_{k+1,n}\right\} = \frac{1}{Uz}Q_{k,n};$$

$$\frac{\partial f_{k,n}}{\partial t} = \frac{1}{z}\left\{(k-1)f_{0,1}f_{k,n} + (k+n)gf_{k,n} + f_{k,n+1}\right\} = \frac{1}{z}E_{k,n};$$

$$\frac{\partial g_{k,n}}{\partial x} = \frac{1}{Uz}\left\{(k-1)f_{1,0}g_{k,n} + (k+n)Fg_{k,n} + g_{k+1,n}\right\} = \frac{1}{Uz}K_{k,n};$$

$$\frac{\partial g_{k,n}}{\partial t} = \frac{1}{z}\left\{(k-1)f_{0,1}\, g_{k,n} + (k+n)gg_{k,n} + g_{k,n+1}\right\} = \frac{1}{z}L_{k,n};$$

$$\frac{\partial l_k}{\partial x} = \frac{1}{Uz}kFl_k = \frac{1}{Uz}M_k;$$

$$\frac{\partial l_k}{\partial t} = \frac{1}{z}\left\{(kg - l_1)l_k + l_{k+1}\right\} = \frac{1}{z}N_k; \tag{18}$$

where $Q_{k,n};E_{k,n};K_{k,n};L_{k,n};M_k;N_k;$ are terms in curly brackets in obtained equations. It is important to notice that $Q_{k,n};K_{k,n};M_k;$ beside the parameters depend on value $U\partial z/\partial x = F$. Using parameters (13-16), operator (17) and terms (18) system of equations (10) and (11) is transformed into system:

$$\Im_1 = \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}\left[E_{k,n}\frac{\partial^2\Phi}{\partial\eta\partial f_{k,n}} + Q_{k,n}X\left(\eta;f_{k,n}\right)\right] + \sum_{\substack{k=1\\ n=0}}^{\infty}\left[L_{k,n}\frac{\partial^2\Phi}{\partial\eta\partial g_{k,n}} + K_{k,n}X\left(\eta;g_{k,n}\right)\right];$$

$$\begin{gathered}\Im_2 = \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}\left[E_{k,n}\frac{\partial\Theta}{\partial f_{k,n}} + Q_{k,n}Y\left(\eta;f_{k,n}\right)\right] + \sum_{\substack{k=1\\ n=0}}^{\infty}\left[L_{k,n}\frac{\partial\Theta}{\partial g_{k,n}} + K_{k,n}Y\left(\eta;g_{k,n}\right)\right] + \\ +\sum_{k=1}^{\infty}\left[N_k\frac{\partial\Theta}{\partial l_k} + M_kY\left(\eta;l_k\right)\right];\end{gathered} \tag{19}$$

where the following markings have been used for shorter statement: $\Im_1$ -left side of equation (10), $\Im_2$ - left side of equation (11).

In order to make system (19) universal it is necessary to show that value which appears in terms for $Q_{k,n};K_{k,n};M_k;$ can be expressed by means of introduced parameters. In order to prove mentioned we start from impulse equation of described problem:

$$\frac{\partial}{\partial t}\left(U\delta^*\right) + \frac{\partial}{\partial x}\left(U^2\delta^{**}\right) + U\left(\frac{\partial U}{\partial x} + N\right)\delta^* - \frac{\tau_w}{\rho} = 0; \tag{20}$$

where:

$$\delta^*\left(x,t\right) = \int_0^\infty\left(1-\frac{u}{U}\right)dy \text{ -displacement thickness;} \tag{21}$$

$$\delta^{**}\left(x,t\right) = \int_0^\infty\frac{u}{U}\left(1-\frac{u}{U}\right)dy \text{ -momentum thickness;} \tag{22}$$

$$\tau_w\left(x,t\right) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \text{ -friction stress on the body.} \tag{23}$$

Now we introduce dimensionless characteristic functions:

$$H^*\left(x,t\right) = \frac{\delta^*}{h};\ \ H^{**}\left(x,t\right) = \frac{\delta^{**}}{h};\ \ \xi\left(x,t\right) = \frac{\tau_w h}{\mu U}; \tag{24}$$

which, according to Eqs. (9) and (21-23) , can be expressed in the following form:

$$H^*(x,t)=\frac{1}{D}\int_0^\infty\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta;\quad H^{**}(x,t)=\frac{1}{D}\int_0^\infty\frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta;$$
$$\xi(x,t)=D^2\frac{\partial^2\Phi}{\partial\eta^2}\Big|_{\eta=0}. \tag{25}$$

After transition to new independent variables (introduced parameters) in terms (25) values H^*, H^{**} and ξ become functions of parameters $f_{k,n}$, $g_{k,n}$ l_k and g. Now, using parameters from impulse Eq. (20) after simple transformation next equation is obtained:

$$F=\frac{P}{Q}; \tag{26}$$

where, for the sake of shorter expression following marks are used:

$$P=\xi-f_{1,0}\left(2H^{**}+H^*\right)-\left(f_{0,1}+g_{1,0}+\frac{1}{2}g\right)H^*-$$
$$-\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}\left\{E_{k,n}\frac{\partial H^*}{\partial f_{k,n}}+(k-1)f_{1,0}f_{k,n}\frac{\partial H^{**}}{\partial f_{k,n}}+f_{k+1,n}\frac{\partial H^{**}}{\partial f_{k,n}}\right\}$$
$$-\sum_{k=1}^{\infty}\left\{L_{k,n}\frac{\partial H^*}{\partial g_{k,n}}+\left[(k-1)f_{1,0}g_{k,n}+g_{k+1,n}\right]\frac{\partial H^{**}}{\partial g_{k,n}}\right\};$$
$$Q=\frac{1}{2}H^{**}+\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}(k+n)f_{k,n}\frac{\partial H^{**}}{\partial f_{k,n}}+\sum_{\substack{k=1\\ n=0}}^{\infty}(k+n)g_{k,n}\frac{\partial H^{**}}{\partial g_{k,n}}. \tag{27}$$

Last two equations define a value F as a function of introduced parameters. Equation system (19) is now universal system of equations of described problem. Boundary conditions, also universal, are given with terms:

$$\Phi=0,\frac{\partial\Phi}{\partial\eta}=0,\Theta=0\text{ for }\eta=0;\quad \Phi\to 1,\Theta\to 1\text{ for }\eta\to\infty;$$
$$\Phi=\Phi_0(\eta),\Theta=\Theta_0(\eta)\ \text{for}\begin{Bmatrix} f_{k,n}=0\ (k,n=0,1,2,...,k\vee n\neq 0)\\ g_{k,n}=0\ (k,n=0,1,2,...,k\neq 0)\\ l_k=0\ (k=0,1,2,...)\\ g=0\end{Bmatrix}; \tag{28}$$

where $\Phi_0(\eta)$-Blasius solution for stationary boundary layer on the flat plate and $\Theta_0(\eta)$ is solution of following equation:

$$\frac{D^2}{P_r}\frac{d^2\Theta_0}{d\eta^2}-D^2E_c\left(\frac{d^2\Phi_0}{d\eta^2}\right)^2+\frac{\xi_0}{H^{**}}\Phi_0\frac{d\Theta_0}{d\eta}=0. \tag{29}$$

A universal system of equations (19) with boundary conditions (28) are accurate for wide class of problems where $z = At + C(x)$ (A is arbitrary constant and $C(x)$ some function of longitudinal coordinate). For other problems this equations are approximated universal equations.

System of equations (19) is integrated in m-parametric approximation once for good and all. Obtained characteristic functions can be used to yield general conclusions about heat and mass transfer in boundary layer and to solve any particular problem.

Before integration for scale of transversal coordinate in boundary layer $h(x,t)$ some characteristic value is chosen. In this case $h=\delta^{**}$ and accordingly to Eq. (24) $H^{**}=1$, $H^*=\delta^*/\delta^{**}=H$, and equality (26) now have form:

$$F=2\left[\xi-f_{1,0}\left(2+H\right)-\left(f_{0,1}+g_{1,0}+\frac{1}{2}g\right)H-\sum_{\substack{k,n=0\\k\vee n\neq 0}}^{\infty}E_{k,n}\frac{\partial H}{\partial f_{k,n}}-\sum_{\substack{k=1\\n=0}}^{\infty}L_{k,n}\frac{\partial H}{\partial g_{k,n}}\right]. \tag{30}$$

Taking parameters $f_{k,n}=0$, $g_{k,n}=0$, $g=0$ first equation of system (19) is simplified into form:

$$\frac{d^3\Phi_0}{d\eta^3}+\frac{\xi_0}{D^2}\Phi_0\frac{d^2\Phi_0}{d\eta^2}=0; \tag{31}$$

and if $D^2=\xi_0$ then previous equation became well-known Blasius equation. According to previous statement for normalizing constant D value 0,47 must be chosen. For selected value h equation (29) for determining variable Θ_0 became:

$$\frac{1}{P_r}\frac{d^2\Theta_0}{d\eta^2}+\Phi_0\frac{d\Theta_0}{d\eta}-E_c\left(\frac{d^2\Phi_0}{d\eta^2}\right)^2=0. \tag{32}$$

In this paper approximated system of equations (19) is solved in which influence of parameters $f_{1,0}$, $f_{0,1}$, $g_{1,0}$, l_1, and g are detained and influence of parameters $f_{0,1}$, l_1 derivatives are disregarded. In this way, Eqs. (19) is simplified into following form:

$$\Im_1=Ff_{1,0}X\left(\eta;f_{1,0}\right)+gf_{1,0}\frac{\partial^2\Phi}{\partial\eta\partial f_{1,0}}+Fg_{1,0}X\left(\eta;g_{1,0}\right)+gg_{1,0}\frac{\partial^2\Phi}{\partial\eta\partial g_{1,0}};$$

$$\Im_2=Ff_{1,0}Y\left(\eta;f_{1,0}\right)+gf_{1,0}\frac{\partial\Theta}{\partial f_{1,0}}+Fg_{1,0}Y\left(\eta;g_{1,0}\right)+gg_{1,0}\frac{\partial\Theta}{\partial g_{1,0}}; \tag{33}$$

where function F in same approximation have form:

$$F = 2\left[\xi - f_{1,0}\left(2+H\right) - \left(f_{0,1} + g_{1,0} + \frac{1}{2}g \right)H - gf_{1,0}\frac{\partial H}{\partial f_{1,0}} - gg_{1,0}\frac{\partial H}{\partial g_{1,0}} \right]. \quad (34)$$

Boundary conditions, which coincide to previous system of equations are:

$$\Phi = 0, \frac{\partial \Phi}{\partial \eta} = 0, \Theta = 0 \text{ for } \eta = 0; \quad \Phi \to 1, \Theta \to 1 \text{ for } \eta \to \infty;$$

$$\Phi = \Phi_0\left(\eta\right), \Theta = \Theta_0\left(\eta\right) \text{ for } \begin{Bmatrix} f_{1,0} = 0, f_{0,1} = 0, g_{1,0} = 0 \\ l_1 = 0, g = 0 \end{Bmatrix}; \quad (35)$$

which is obtained from conditions (28), using same simplifications like for system of equations. First equation of system (32) is four-parametric once localized approximation and second is five-parametric twice-localized approximation of second equation of system (19).

2 Results and discusion

In this section, part of results obtained with numerical integration (using tridiagonal algorithm) of equation system (32) with boundary conditions (34) is given. All results are given for $P_r = 1.0$, $E_c = 0.3$ and $g = -0.013$. Figure 1 presents the variations of quantities F and ξ in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$. It may be noted that with increase of magnetic parameter value ξ (dimensionless friction) also increase. This remark lead to conclusion that distance between front stagnation point and boundary layer separation point increase with increase of magnetic field intensity.

It is interesting to note decreasing of function F with increase of magnetic parameter. This result confirms the delay of boundary layer separation and greater postponement is achieved with increasing of magnetic parameter $g_{1,0}$. Figure 1 is given for the case of accelerated free stream $(f_{0,1} = 0.01)$, however the same conclusion is obtained for the case of free stream deceleration $(f_{0,1} < 0)$.

The effect of magnetic parameter $g_{1,0}$ on ratio of boundary layer displacement thickness and momentum thickness H in function of dynamic parameter is shown in Figure 2. The Figure presents the case of accelerated outer flow $(f_{0,1} > 0)$. It is interesting to note decreasing of function H with increase of magnetic parameter and also with increase of dynamic parameter. This result confirms the delay of the boundary-layer separation and greater postponement is achieved with increasing of magnetic parameter. Figures also show that increasing the magnetic field decreases the velocity boundary layer thickness due to its damping effect.

Figure 3 presents the variations of F and ξ in function of dynamic parameter for different values of unsteadiness parameter. It may be noted that function F decrease with increase of dynamic parameter. Function F have higher values for the case of decelerated free stream outer flow $(f_{0,1} < 0)$, and lower for the acceleration case in relation to stationary outer flow.

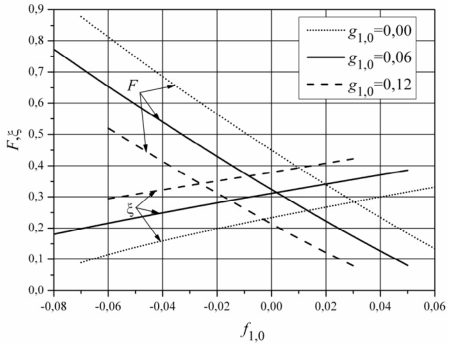

Figure 1. Variations of F and ξ in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$

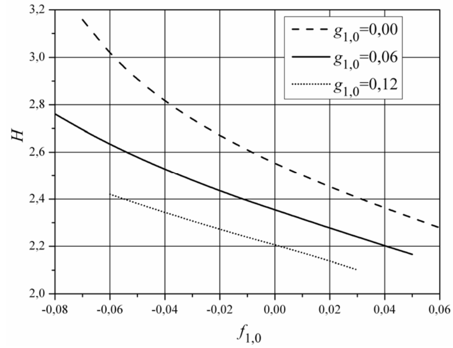

Figure 2. Variations of ∗ in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$

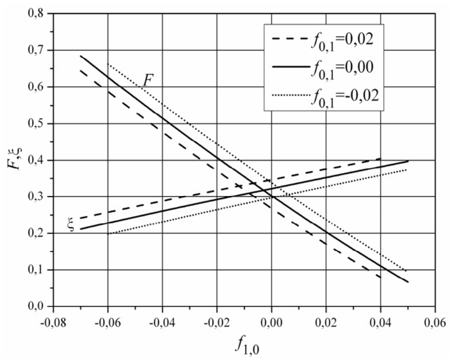

Figure 3. Variations of F and ξ in function of dynamic parameter $f_{1,0}$ for different values of unsteadiness parameter $f_{0,1}$

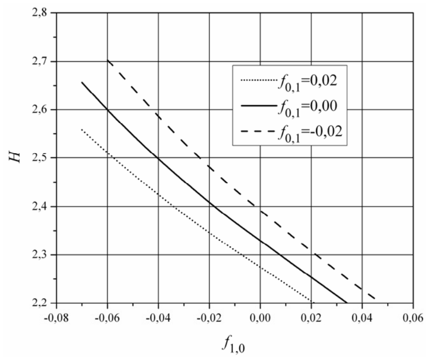

Figure 4. Variations of H in function of dynamic parameter $f_{1,0}$ for different values of unsteadiness parameter $f_{0,1}$

Figure 4 shows the ratio of boundary layer displacement thickness and momentum thickness H in function of dynamic parameter for different values of unsteadiness parameter, while magnetic parameter is set to $g_{1,0} = 0.08$. This ratio decrease with increase of dynamic parameter. It may be noted also that for the same value of dynamic parameter ratio is higher for the case of decelerated free stream and lower for the case of acceleration. According to derived conclusions, it may be observed that free stream acceleration have positive influence on boundary layer developmentFree stream function Φ (dimensionless velocity) is shown in the Figures 5 and 6 in function of dimensionless transversal coordinate η for different values of magnetic and unsteadiness parameter. From Figure 5, we observe that with increase of magnetic parameter this ratio also increase and the minimal value is obtained for the case of non-conducting fluid or for the case of magnetic field absence.

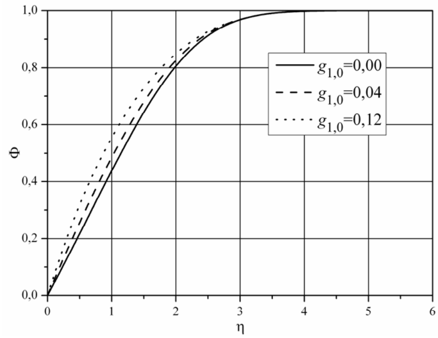

Figure 5. Effect of magnetic parameter $g_{1,0}$ on dimensionless velocity

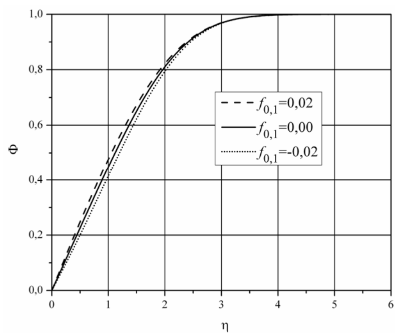

Figure 6. Effect of unsteadiness parameter $f_{0,1}$ on dimensionless velocity

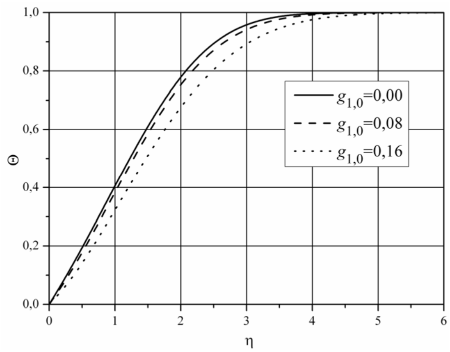

Figure 7. Temperature profiles for different values of magnetic parameter $g_{1,0}$

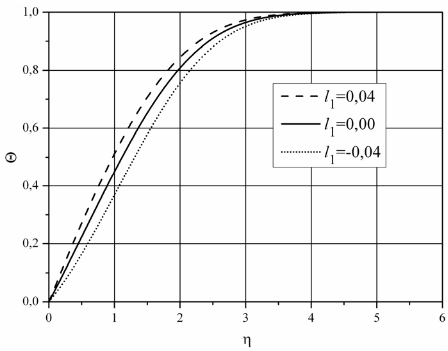

Figure 8. Temperature function for different values of temperature parameter l_1

This analysis indicates the significant influence of magnetic field on increasing velocity in boundary-layer. The results clearly show that the magnetic field tends to delay or prevent separation. Velocity distribution is given on figure 6 for cross-section, which coincides to dynamic parameter $f_{1,0} = -0.02$ and decelerated, stationary and accelerated free stream. It may be noted that velocity in boundary layer faster tends to the free stream velocity for the case of accelerated free stream and slower for the case of decelerated flow compared with stationary outer flow. Same conclusion is valid for other cross-sections of boundary-layer and for all values of dynamic and magnetic parameter.

In Figure 7 the variation of dimensionless temperature in function of value η for different values of magnetic parameter is given. Figure presents the results obtained for cross-section which coincide to value of dynamic parameter $f_{1,0} = 0.01$. It may be noted that the highest temperature value is obtained for the case of non-conducting fluid or for the case of outer magnetic field absence, and increase of magnetic parameter results in temperature decreasing.

Figure 8 describe temperature distribution in function of dimensionless transversal coordinate η for different values of temperature parameter. Solid line presents the case of constant body surface temperature. With increasing of temperature parameter ($l_1 > 0$) dimensionless temperature also increase and in the case for body surface temperature decreasing dimensionless temperature also decrease. Figures 9 and 10 presents the temperature distribution in function of dimensionless transversal coordinate for different values of unsteadiness parameter and dynamic parameter. It may me noted that temperature faster tends to value on outer edge of boundary layer for the case of decelerated free stream.

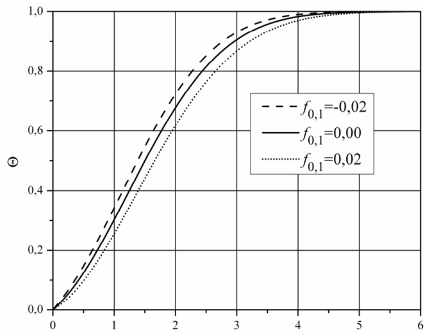

Figure 9. Temperature profiles for different values of unsteadiness parameter $f_{0,1}$

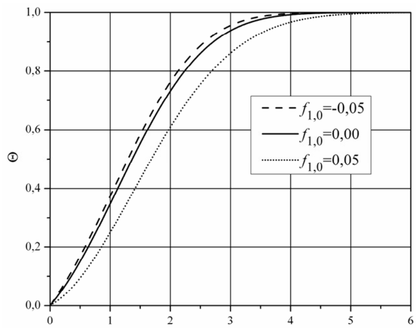

Figure 10. Temperature profiles for different values of dynamic parameter $f_{1,0}$

CONCLUSION

In this paper unsteady two-dimensional MHD boundary-layer on the body whose temperature varies with time is considered. This problem can be analyzed for each particular case, i.e. for given free stream velocity. Here is used quite different approach in order to use benefits of generalized similarity method and universal equations of observed problem are derived. These equations are solved numerically in some approximation and integration results are given in the form of diagrams and conclusions. The obtained results can be used in drawing about general conclusions of boundary-layer development and in calculation of particular problems.

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REFERENCES

1. H. Schlichting, (1982), *Grenzschicht-Theorie*, Braun-Verlag, Karlsruhe.
2. E.J. Blum, J. A. Mihailov, (1962), *Heat transfer in electroconductive fluid in presence of transversal magnetic field* (in Russian), Magnetohydrodynamics, 1966, Vol. 5, pp. 2-18.
3. E. J. Resler, W. R. Sears, (1958), *The prospects for magneto aerodynamics,* Journal of Aerospace Science and Technologies, Vol. 25, pp.235.245
4. A. Busemann, (1961), *Is aerodynamics breaking an ionic barrier?*, NASA-TM-X-56147.
5. H. Y. Lu, C. H. Lee, (2009), *Simulation of three-dimensional nonideal MHD flow at low magnetic Reynolds number*, Science China Series E-Technological Sciences, Vol. 52(12), pp. 3690- 3697.
6. J. Poggie, D. Gaitonde, (2002), *Magnetic control of flow past a blunt body: Numerical validation and exploration*, Physics of Fluids, Vol. 14(5), pp. 1720.1731.
7. K. Cukierski, B. G. Thomas, (2008), *Flow Control with Local Electromagnetic Braking in Continuous Casting of Steel Slabs*, Metallurgical and Materials Transactions B, Vol. 39(1), pp. 94-107.
8. I. A. Abbas, G. Palani, (2010), *Effects of magnetohydrodynamic .ow past a vertical plate with variable surface temperature*, Applied Mathematics and Mechanics (English Edition), Vol. 31(3), pp. 329.338.
9. J. Zhu, L. Zheng, Z Zhang, (2010), *Effects of slip condition on MHD stagnation-point .ow over a power-law stretching sheet*, Applied Mathematics and Mechanics (English Edition), Vol. 31(4), pp. 439.448.
10. L.G. Loicijanski, (1965), *Universal equations and parametric approximations in theory of laminar boundary-layer* (in Russian), AN SSSR Applied mathematics and mechanics, Vol. 29, pp. 70-87.
11. Yu. V. Saraev, (1975), *Application of the parametric method to the solution of problems of the unsteady thermal boundary-layer*, Journal of Engineering Physics and Thermophysics, Vol. 28 (2), pp. 200-206.
12. V.N. Saljnikov, (1978), *A Contribution to Universal Solutions of the boundary-layer theory*, Theoretical and Applied Mechanics, Vol. 4, pp.139-163.
13. O.N, Busmarin, J.V. Saraev, (1974), *Parametric method in theory of unsteady boundary-layer* (in Russian), Inz.-fiz. zhurnal,, Vol. XXVII (1), pp. 110-118.
14. Z. Boricic, D. Nikodijevic, D. Milenkovic, Z. Stamenkovic, (2005), *A form of MHD universal equations of unsteady incompressible fluid flow with variable electroconductivity on heated moving plate*, Theoretical and Applied Mechanics, Vol. 32(1), pp. 65-77.
15. B. Obrovic, D. Nikodijevic, S. Savic, (2009), *Boundary-layer of dissociated gas on bodies of revolution of a porous contour,* Strojniski vestnik - Journal of Mechanical Engineering,, Vol. 55(4), pp.244-253.
16. Z. Boricic, D. Nikodijevic, B. Blagojevic, Z. Stamenkovic, (2009), *Universal solutions of unsteady two-dimensional MHD boundary layer on the body with temperature gradient along surface*, WSEAS Transactions on Fluid Mechanics, Vol. 4, No. 3, pp. 97-106.
17. B. V. Alexeev,(2004), *Generalized Boltzmann Physical Kinetics,* Elsevier, pp.347-355.

18. Iu. A. Poveshchenko, Iu. P. Popov, E.A. Samarskaia, (1982), *Iterative methods for solving difference schemes for equations of gas dynamics with heat conduction* (in Russian), Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, Vol. 22, pp. 903-912.

NESTACIONARNI MAGNETNO HIDRODINAMIČKI PRENOS MASE I TOPLOTE NA TELU ČIJA SE TEMPERATURA MENJA TOKOM VREMENA

Dragiša Nikodijević, Živojin Stamenković

U ovom radu se razmatra nestacionarni dvodimenzionalni temperaturski magneno hidrodinamički granični sloj nestišljivog fluida. Spoljašnje električno polje je zanemareno, magnetni Reynolds-ov broj je znatno manji od jedinice tj. problem se razmatra u bezindukcionoj aproksimaciji. Karaktersitična fizička svojstva fluida su konstantna dok se temperature tela menja tokom vremena. Jednačine i granični uslovi koji opisuju problem prenosa mase i toplote u graničnom sloju uopšteni su tako da ne zavise od partikularnih uslova razmatranog problema, i u tom smislu one se smatraju univerzalnim. Dobijene univerzalne jednačine rešene su primenommetode progonka. Numerički rezultati za bezdimenionu brzinu, temperaturu i trenje u funkciji od uvedenih parametara predstavljeni su grafički i iskorišćeni za donošenje generalnih zaključaka o razvoju posmatranog temperaturskog MHD graničnog sloja.

Key words: *MHD, magnetno polje, parametric sličnosti, univerzalne jednačine*

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INFLUENCE OF THE REYNOLDS NUMBER ON THE STATISTICAL AND CORRELATION-SPECTRAL PROPERTIES OF TURBULENT SWIRL FLOW

UDC 531: 534

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Abstract. *Average velocity distributions of the turbulent swirl flow for four Reynolds numbers, obtained with one-component laser Doppler anemometry (LDA) and stereo particle image velocimetry (SPIV) behind axial fan in the pipe, has been reported in this paper. The highest levels of turbulence are observed in the vortex core region as well the non-Gaussian values for skewness and flatness factors. Additional information on turbulent structure physics were obtained on the basis of experimentally determined autocorrelation functions and time integral scale for circumferential velocity.*

Key words: *swirl flow, turbulence, SPIV, LDA.*

1. INTRODUCTION

Turbulent swirl flow investigation attracts attention of numerous researchers, but behind the axial fan in the circular pipe, by use of modern measuring techniques, is thoroughly studied in Lečić (2003), Oro et al. (2008) with hot-wire anemometry (HWA), in Protić et al. (2010) with one-component LDA and stereo PIV, while in Čantrak (2012) with one- and two-component laser Doppler anemometry (LDA) and stereo particle image velocimetry (SPIV) with low and high speed cameras and lasers.

Investigation of the turbulent swirl flow behind axial fan in the circular pipe, which generates Rankine swirl, has been reported in this paper. Average total velocity fields for various Reynolds numbers are given, as well average velocity fields for all three components. Reynolds numbers were generated by changing the fan rotation speed. Four flow regions, characteristic for Rankine swirl, are identified. Influence of the Reynolds number on these average characteristics is obvious in the intensity, but not in the velocity distribution characters. Level of turbulence is the highest, for all cases, in the vortex core region. Skewness and flatness factors with time integral scale reveal some of the turbulent mechanisms.

2. Experimental Test Rig and Measuring Methods

2.1. Experimental Test Rig

The test rig is 27.74·D long, where D=0.4m is inner average pipe diameter (Fig. 1.). Measuring section for both measuring techniques is in the position x/D=3.35, where x is the axial coordinate along a pipe axis and measured from the test rig inlet.

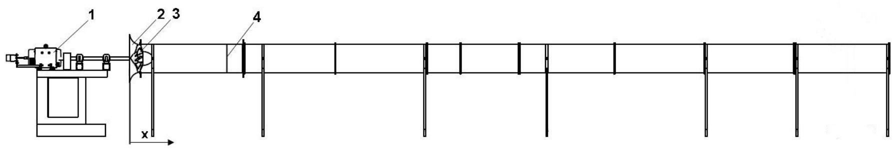


Fig. 1 Experimental test rig: 1-DC motor, 2-profiled inlet nozzle, 3-axial fan and 4-LDA and SPIV measuring section.

The fan rotation speed was precisely regulated. Each obtained Reynolds number correspond to the regime defined with rotation number, i.e. n=1000, 1500, 2000 and 2500rpm correspond, respectively, to the Re=182602, 277018, 369612 and 469612, calculated on the basis of average axial velocity U_m (Table 1.).

Axial fan with nine blades, impeller diameter D_a=0.399 m and the dimensionless ratio D_i/D_a=0.5, where D_i is the hub ratio, designed by Protić[†], is the Rankine swirl generator. The blade angle at the impeller diameter is β_a=30º.

2.2. Laser Doppler Anemometry

One-component LDA system was used for measuring all three velocity components subsequently in points on 10mm distance along the vertical diameter in position x/D=3.35 (Fig. 1., pos. 4). Used LDA system is the Flow Explorer Mini LDA, Dantec with the BSA F30 signal processor and adequate software for data acquisition and processing. Transit time was used as the weighting factor. Recording time was 10 s for all measurements. Sampling frequency and data validation varied along the diameter. Flow is seeded by the Antari Z3000II thermal fog generator which produced quality seeding for LDA and PIV at the same time. Corrections of the measuring volume position are discussed in Ristić et al. (2012).

2.3. Stereo Particle Image Velocimetry

SPIV measurements have been performed in the vertical cross-section in defined position (*x/D*=3.35, Fig. 1, pos. 4), in target plane *X-Y* of approximate size 200x90 mm. The target origin is on the pipe axis and target coordinate system for software nomenclature is given (Fig. 2., pos. 4).

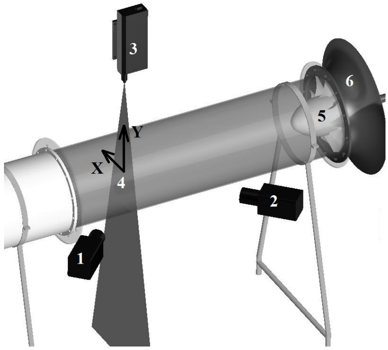


Fig. 2 SPIV measuring arrangement in the measuring section: 1-left camera, 2-right camera, 3-Nd:Yag laser, 4-illuminated pipe cross-section, 5-swirl generator and 6-profiled inlet nozzle.

Dual head Nd:Yag laser (max power: 30mJ/pulse, wavelength 532 nm, 15 Hz), was used for flow illumination, while two 12-bit CCD cameras with the resolution of 1660x1200 pixels and 32fps were in Scheimpflug setup. The INSIGHT 3G TSI software was used for data acquisition and processing. Results are obtained by averaging 400 pictures obtained with laser frequency of 2Hz. Image processing was performed using the central difference image correction (CDIC) deformation algorithm combined with the FFT correlator (Čantrak et al., 2012).

3. EXPERIMENTAL RESULTS AND DISCUSSION

3.1. Integral Flow Characteristics

Distribution of all three time-average velocities is presented in Fig. 3. Angle φ=90º denotes the upper part (above-pipe axis) of the vertical diameter, while φ=270º the lower part (under-pipe axis). Circumferential velocity was measured in both planes from both sides due to the LDA laser optics technical specification and there is points overlapping in the vortex core region.

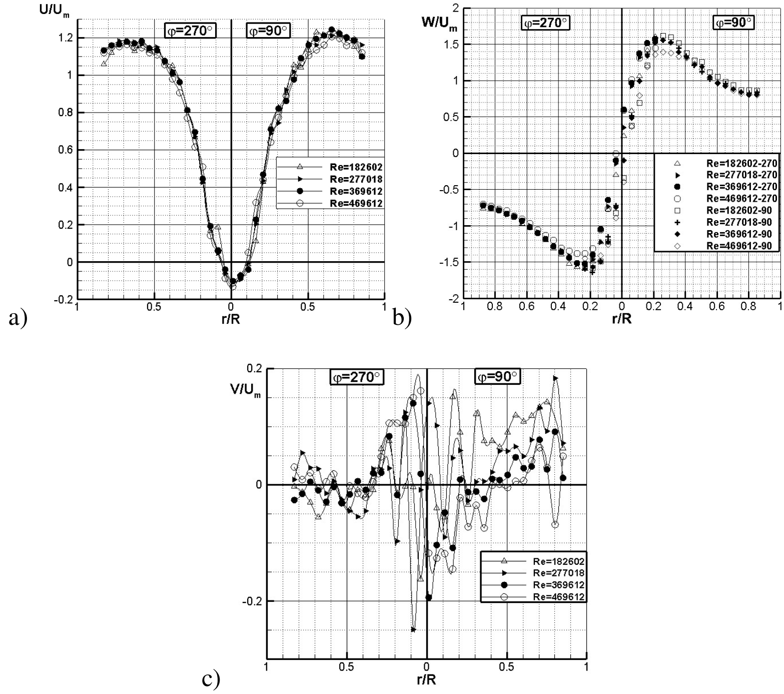


Fig. 3 Distribution of the time averaged velocities in measuring section 1: a) axial, b) circumferential and c) radial.

Similarity of axial and circumferential velocity profiles for all Reynolds numbers is obvious in Fig. 3.a,b. The minimum of axial velocity is in position r/R=0, where r is radial distance form pipe axis and $R=D/2$. The reverse flow is evident in the vortex core region up to r/R=0.1. Maximum of non-dimensional axial velocity is $(U/U_m)_{max}\approx 1.2$ for all regimes. Circumferential velocity maximum, approximately for all cases $(W/U_m)_{max}\approx 1.6$, is reached in the point r/R=0. The distribution of circumferential and axial velocity is almost symmetric with respect to the pipe axis. Linear increase of circumferential velocity is characteristic for solid body flow region, what is obvious in the vortex core region. Additional characteristic flow region for generated Rankine swirl is in the sound flow region rW=const (Fig. 3b). In Fig. 3c is also evident finite value of radial velocities, with minimum value V=-2.53 m/s for n_3=1500rpm.

Volume flow rate is calculated on the basis of axial velocity distribution, while average circulation and swirl number on the basis of axial and circumferential velocities as follows

$$Q = 2\pi R^2 \int_0^1 kU dk,\ U = \frac{Q}{R^2\pi},\ \ \Gamma = \frac{4\pi^2 R^3}{Q}\int_0^1 k^2 UW dk,\ \Omega = \frac{Q}{R\Gamma} \qquad (1)$$

where $k=r/R$ is the dimensionless radius. Obtained data are presented in Table 1. Calculation of these integral values involved assumption of flow axis-symmetry, what doesn't generate significant error, what is proved with calculation in both meridian planes φ=90° and φ=270°.

Table 1 Calculated integral flow values.

Fan rot. speed n [rpm]	U_m [m/s]	Q [m^3/s]	Re	Γ [m^2/s]	Ω
1000	6.68	0.86	182602	5.41	0.79
1500	10.13	1.305	277018	7.91	0.81
2000	13.52	1.741	369612	10.51	0.82
2500	17.17	2.212	469612	13.12	0.83

Average circulation increases with the increase of axial and radial velocities, while swirl number defined as in eq. (1) remains almost constant for all Reynolds numbers.

Total velocity vectors are presented in Fig. 4 for all four regimes. Vortex core region is visible in all directions, while the maximum position is r/R=0.4, what belongs to the sound flow region. It is obvious that minimum of the total velocity average field is not on the pipe axis what is in accordance with diagrams in Fig. 3, where distributions are slightly off the pipe axis.

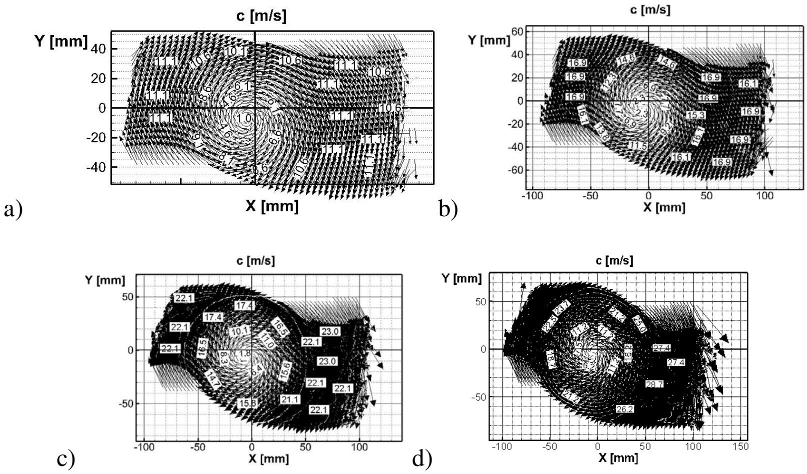
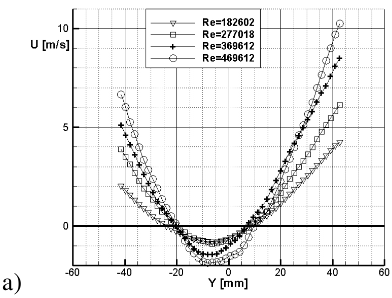


Fig. 4 Velocity vectors in measuring pipe cross-section for all four regimes from n=1000 rpm till 2500 rpm, respectively.

Velocity components are recalculated to the polar-cylindrical coordinate system and presented in Fig. 5. They are obtained by section of velocity profile with the plane $X=X_{VC}$. Reverse flow is more intensive as the Reynolds number is higher (Fig. 5a). In Fig. 5b occur negative circumferential velocities due to the recalculation on the target with center on the pipe axis, while real center of the vortex core is not on the pipe axis.



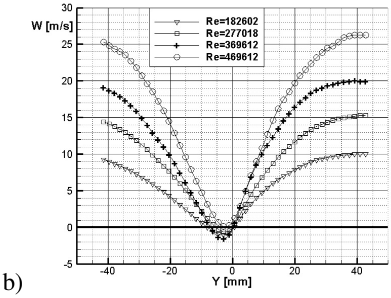


Fig. 5 Velocity components for all four regimes: a) axial and b) circumferential.

3.2. Turbulence Statistics

Reynolds normal stresses and level of turbulence are calculated as follows

$$\overline{u_i^2} = \sum_{j=0}^{N-1} \eta_j \left(u_i^2\right)_j, \quad \eta_j = \frac{t_j}{\sum_{k=0}^{N-1} t_k} \quad \text{and} \quad \frac{\sigma_i}{U_m} = \frac{\sqrt{\overline{u_i^2}}}{U_m} \tag{2}$$

where t_j is transit time of the j-th particle crossing the measuring volume and u_i=u,v,w are fluctuating velocities in axial (x), radial (r) and circumferential (φ) directions respectively. Here will be discussed turbulence statistics for circumferential velocity, very important velocity component, and also due to the obtained highest data sampling rates and validations with accent on the plane φ=270º till position r/R=0.88. Wall region is omitted due to technical reasons in the wall vicinity. Character of distribution of turbulence levels is similar for all Reynolds numbers, except in the case for the highest Reynolds number in the shear layer (Fig. 6).

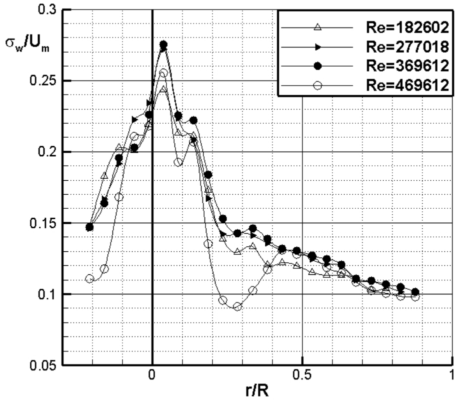


Fig. 6 Turbulence levels for circumferential velocity in the plane φ=270º.

Normalized central moments for all three velocity components of the third S_i (skewness), and the fourth order F_i (flatness), where $u_i = u,v,w$, are calculated by introducing the weighting factor (η_j):

$$S_i = \overline{u_i^3} / \sigma_i^3, \ F_i = \overline{u_i^4} / \sigma_i^4 \tag{3}$$

Skewness and flatness factors for circumferential velocity in the plane φ=270º are presented in Fig. 7.

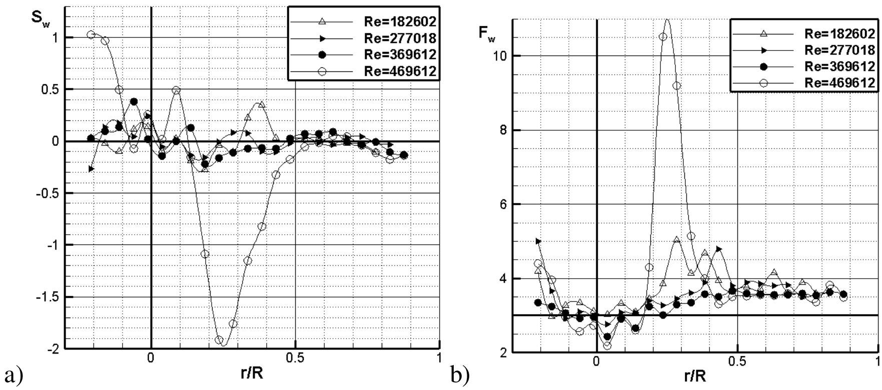


Fig. 7 Normalized central moments for circumferential velocity in plane φ=270º: a) skewness and b) flatness factors.

All skewness and flatness factors differ from the values for normal distribution S_i=0 and F_i=3. Negative skewness factors indicate that large velocity fluctuations are negative. All extremes are obtained for the highest Reynolds number. Positive and negative values of the coefficient S_w reveal various processes of turbulent diffusion, as well complex interactions of fluctuating and deformation fields. The minimum skewness value is $S_{w,min}$=-1.92 in position *r/R*=0.23 which belongs to the shear layer. In the same position is the maximum flatness factor $F_{w,max}$=10.52. Flatness factors for F_i>3 indicate great probability of small fluctuations. All this reveals the swirl flow nature (Čantrak, 1981).

3.3. Correlation and Spectral Analysis

Coefficients of the time autocorrelation function for circumferential fluctuating velocity ($R_{ww}(\tau)$) are presented in Fig. 8. Points are representative for three various flow regions in the plane φ=270°, i.e. *r/R*=0.1 in vortex core region, *r/R*=0.23 in shear layer and *r/R*=0.63 in the main flow region. They are calculated till 100ms, but presented here till 40ms.

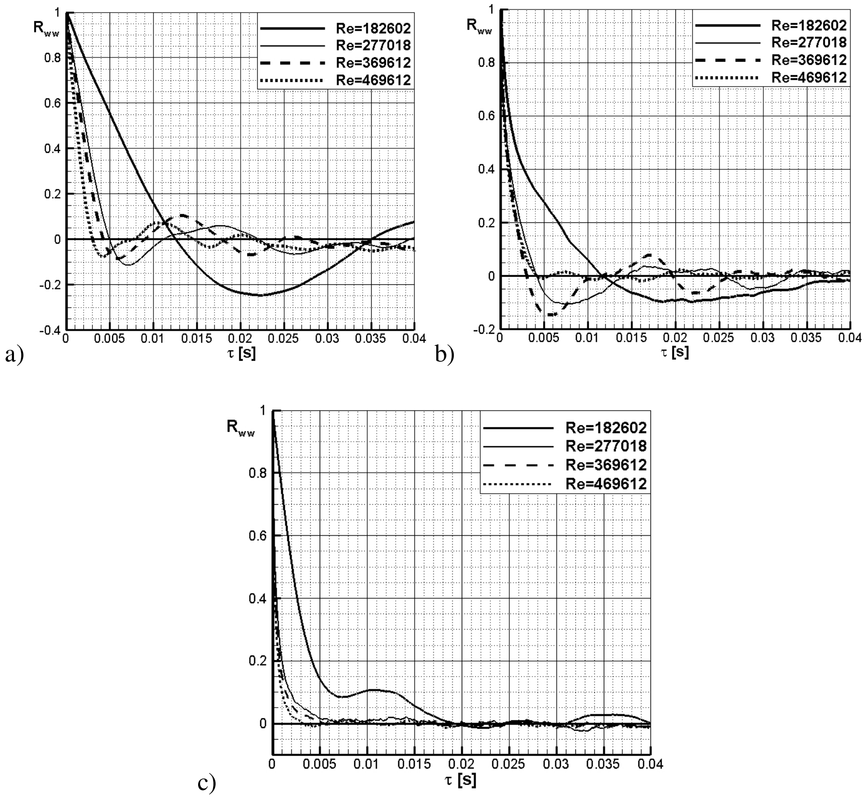


Fig. 8 Coefficients of the time autocorrelation function for circumferential fluctuating velocity ($R_{ww}(\tau)$) in various positions in plane φ=270° for all four Reynolds numbers: a) r/R=0.1, b) r/R=0.23 and c) r/R=0.63.

Characters of the experimentally determined correlation curves point out various dynamics of the circumferential fluctuating velocity field in various points of the pipe cross-section. Changes of the correlation coefficients are significant, i.e. significantly high with time increase for higher Reynolds numbers in all represented points. It can be observed the regime with Re=277018. It is obvious that the steepest correlation curve is in the point r/R=0.63 in main flow region. This means that in this point high frequency components play the most important role. Character of the correlation curves in the vortex core region for this regime point out the dominant role of the low frequency fluctuations in the wide spectral density of circumferential fluctuating velocities. The situation is similar for other Reynolds numbers.

In Fig. 9 are presented correlation curves for each second point in the plane φ=270° for Reynolds number Re=277018 in the time intervals till 100ms and afterwards 20ms.

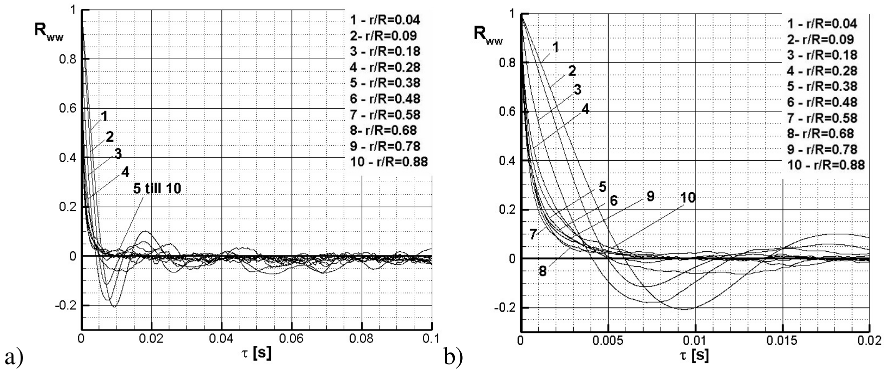


Fig. 9 Coefficients of the time autocorrelation function for circumferential fluctuating velocity ($R_{ww}(\tau)$) in various positions in plane φ=270° for all Reynolds number *Re*=277018 in time interval: a) 100 ms and b) 20 ms.

In Fig. 9 is obvious that correlation curves $R_{ww}(\tau)$ change sign numerous times, asymptotically approaching zero value. Hierarchical distribution of the curves till τ=3ms, from the pipe axis till position *r/R*=0.58, is obvious in Fig. 9b. Afterwards there is opposite process till final position *r/R*=0.88. It is now proved, on higher number of points, that high frequency components play the most important role in the main flow region, while in the vortex core low frequency fluctuations have a dominant role. Time τ_0 denotes time interval till the first $R_{ww}(\tau_0)=0$. It is related to the dominant frequency of circumferential fluctuating velocities, when the maxima of the spectral density occur.

Significant structural parameter is time integral scale T_{Ew} defined as follows

$$T_{Ew} = \frac{1}{2}\int_{-\infty}^{\infty} R_{ww}(\tau)d\tau = \int_{0}^{\infty} R_{ww}(\tau)d\tau \tag{4}$$

It denotes average mean time duration of correlated turbulent disorders, i.e. mean duration of existence of turbulent eddies. Therefore, the calculated values of integral time scale of the turbulent eddies T_{Ew} characterizes various eddy scales and structural properties in the regions of vortex core, shear layer and the main flow (Čantrak, 2012). Radial distributions of the time integral scale in plane φ=270° for four Reynolds numbers and all points are presented in Fig. 10. Here is time integral scale calculated in the interval [0, τ_0], i.e. on the basis of the equation (4) where τ_0 is the upper integral limit and correlation curves.

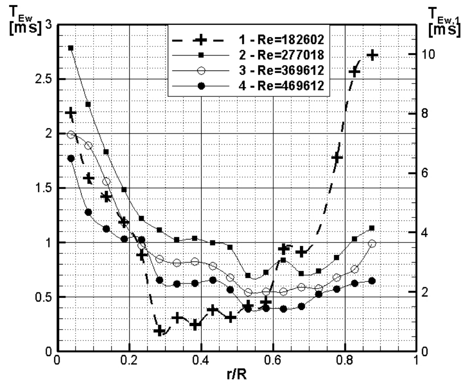


Fig. 10 Distribution of the time integral scale T_{Ew} in plane φ=270° for various Reynolds numbers.

In the point r/R=0.04, closest to the pipe axis and which belongs to the vortex core region, time integral scale T_{Ew} has maximum values for three Reynolds numbers. Hierarchy occurs for the whole domain and for three Reynolds numbers Re=277018, 369612 and 469612 has following maximum values: $T_{Ew,max}$=2.78, 1.99 and 1.77 ms. This indicates the presence of large eddy structures in the core zone. Similar situation is for the lowest Reynolds number in the vortex core region and shear layer. On the contrary to these distributions, there is a significant increase afterwards in the main flow region starting from the r/R=0.53. This curve reaches its maximum $T_{Ew,max}$=9.95 in the point r/R=0.88.

Time integral scale physically represents time of turbulence disturbance existence, what leads to the correlation of T_{Ew} with distribution of statistical moments. The highest values T_{Ew} correspond to the lowest values of flatness factor F_w (Fig. 7b). These results show that the probability of the small velocity fluctuations in the circumferential direction in point r/R=0.04 is very small, as well that in this zone exist eddies of various scales, even the biggest, what is the reason why T_{Ew} reaches its maximum values. Time integral scale for all three Reynolds numbers have extremely negative gradient in the vortex core region, while it is kept almost constant in the main flow region. There is an increase in the wall vicinity for all Reynolds numbers which is the most distinguished in the case of the lowest Reynolds number.

4. CONCLUSIONS

Turbulent swirl flow behind axial fan has been investigated in this paper by use of one-component LDA and SPIV measurements. The Rankine swirl structure is detected for all Reynolds numbers at the axial fan outlet on the basis of circumferential velocity profile. It is shown that velocity profiles of axial and circumferential velocities are almost identical in the non-dimensional form for all Reynolds numbers. Average circulation is increased by increasing the Reynolds number, while swirl number stays almost constant for all Reynolds numbers.

SPIV measurements have reveled total velocity structure in the domain of approximate size 200x90mm. SPIV measurements proved non-axisymmetry and the Rankine vortex. The whole vortex core region is captured with linear distribution of the circumferential velocity. Also radial velocity existence has been revealed with both measuring techniques.

Level of turbulence is great and reaches its maximum in the vortex core region for all Reynolds numbers. Skewness and flatness factors for circumferential velocity differ from normal distribution values. Correlation analysis has revealed various vortex structures along the pipe radius and the highest amount of the energy is distributed for lower frequencies.

Autocorrelation function for circumferential fluctuating velocities, based on the obtained experimental results, point out that in various domains of cross-section exist different scales vortex structures. Important relations between time integral scale and correlation moments of the higher order have been established. Time integral scale T_{Ew} for circumferential velocity has maximum values for all Reynolds numbers on the pipe axis, i.e. in the vortex core region, with exception for the lowest Reynolds number where the maximum value is reached in the position r/R=0.88. It was shown that the highest values $T_{Ew,max}$ correspond to the lowest values of $F_{w,min}$, what implies that the probability of the small velocity fluctuations in the circumferential direction on pipe axis is very small what indicates the biggest eddies. It was also shown that the time integral scale for all three Reynolds numbers have extremely negative gradient in the vortex core region, while it is kept almost constant in the main flow region and with increase in the wall vicinity. All these conclusions allow a closer look into the physics of turbulent swirl flow.

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REFERENCES

1. Čantrak Đ., (2012), *Analysis of the Vortex Core and Turbulence Structure behind Axial Fans in a Straight Pipe using PIV, LDA and HWA Methods*, University of Belgrade, 2012, Doctoral Dissertation, p. 375. UDC 621.634:532.517.4(043.3) (in Serbian)
2. Čantrak Đ., Nedeljković M. and Janković N., (2012), Turbulent swirl flow characteristics and vortex core dynamics behind axial fan in a circular pipe, *Proceedings* CMFF'12, Budapest University of Technology and Economy, Budapest Hungary, September 4-7, 2012, Vol. II, pp. 749-756.
3. Čantrak S., (1981), *Experimental Investigation of the Statistical Properties of the Turbulent Swirl Flow in Pipe and Diffuser*, University Karlsruhe, 1981, Doctoral Dissertation, p. 44. (in German)
4. Lečić M., (2003), *Theoretical and Experimental Investigation of Turbulent Swirling Flows*, University of Belgrade, 2003, Doctoral Dissertation, p. 200. (in Serbian)
5. Oro J.M.F., Ballesteros-Tajadura R., Marigorta E.B., Diaz K.M.A., Morros C.S., (2008), Turbulence and secondary flows in an axial flow fan with variable pitch blades, *Journal of Fluids Engineering*, Vol. 130, No 4, 2008, pp. 041101-1-11

6. Protić Z.D., Nedeljković M.S., Čantrak Đ., Janković N., (2010), Novel methods for axial fan impeller geometry analysis and experimental investigations of the generated swirl turbulent flow, *Thermal Science*, Vol. 14, Suppl, 2010 pp. S125-S139
7. Ristić S.S., Ilić J.T., Čantrak D.S., Ristić O.R., Janković N.Z., (2012), Estimation of laser-Doppler anemometry measuring volume displacement in cylindrical pipe flow, *Thermal Science*, Vol. 16, No 4, 2012 pp. 1127-1142

УТИЦАЈ РЕЈНОЛДСОВОГ БРОЈА НА СТАТИСТИЧКЕ И КОРЕЛАЦИОНО-СПЕКТРАЛНЕ КАРАКТЕРИСТИКЕ ТУРБУЛЕНТНОГ ВИХОРНОГ СТРУЈАЊА

Ђорђе С. Чантрак и Новица З. Јанковић

У оквиру овог рада су приказана осредњена поља брзине турбулентног вихорног струјања за четири вредности Рејнолдсовог броја, добијених са једнокомпонентним ЛДА (ласер Доплер анемометрија) системом и стерео ПИВ (particle image velocimetry (PIV)) мерном техником на потису аксијалног вентилатора у цеви. Највећи нивои турбуленције су уочени у вртложном језгру. На основу експерименталних резултата утврђене су велике вредности статистичких момената трећег и четвртог реда и њихова значајна одступања од Гаусове расподеле у мерним пресецима. Додатне информације, у вези са физиком турбуленције, су добијене на бази експериментално одређених аутокорелационих функција и интегралне временске размере турбуленције за обимску брзину.

Кључне речи: *вихорно струјање, турбуленција, SPIV, ЛДА.*

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FINITE-TIME BOUNDEDNESS OF UNCERTAIN DISCRETE-TIME DESCRIPTOR SYSTEMS WITH TIME-VARYING EXOGENOUS DISTURBANCE

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Abstract. *In this paper, the class of linear uncertain discrete-time descriptor systems with time-varying exogenous disturbance are considered. By combining Lyapunov-like approach and matrix inequality technique, some sufficient conditions of the finite-time boundedness for this kind of systems is presented by a set of feasible problems involving linear matrix inequalities (LMIs). The effectiveness of the approach proposed in this paper is presented by a numerical example.*

Key words: *finite-time boundedness, uncertain discrete-time systems, descriptor systems, exogenous disturbance*

1. INTRODUCTION

In many practical applications, it could be required that the state of a system does not exceed a certain bound during a specified time interval for given bound on the initial state. For this purpose, the concepts of the finite-time stability (FTS) and practical stability are used. While the concept of the Lyapunov stability is used for the analysis of the system behavior on infinite time interval, the concept of FTS is defined on a finite (or very short) time interval. A system is said to be finite-time stable if, once a time interval is fixed, its state does not exceed some bounds during this time interval. Some early results on FTS can be found in the 1960s [1,2]. In [3–7], the authors studied the FTS problem for continuous-time or discrete-time linear systems and proposed some criteria for the design of a controller such that the resulting closed-loop system is finite-time stable. The concept of FTS is also extended to finite-time boundedness (FTB) by introducing an exogenous input and sufficient conditions for FTB are also given in [4, 8]. Sufficient conditions for the existence of state feedback laws, which guarantee FTB of a closed-loop system, are given for linear continuous-time systems [3,4,8] and for linear discrete-time systems [4].

Singular systems, which are also referred to as descriptor systems, generalized state-space systems or semi-state systems, have been extensively studied in the last few decades because this class of systems is more appropriate to describe the behavior of some practical systems like electrical systems, mechanical systems, and chemical systems [9-10]. In general, the singular representation consists of differential and algebraic equations, and therefore is a generalized representation of the state-space system. Numerical solution of singular systems is more difficult compared to regular models, due to the existence of linear and non-linear algebraic equations and due to discontinuities in the algebraic variables over the independent variable space. The large number of fundamental concepts and results based on the theory of the normal state-space systems has been extended to singular systems [11-12]. In recent years, the stability problems for singular systems have been investigated by many researchers, for example [13-16].

It is noted that all the above-mentioned works about the stability of singular systems are focused on Lyapunov asymptotic stability. A small number of articles have appeared on the subject of the FTS and FTB analysis of singular systems. Some results on FTS and FTB can be found in [11, 17-27]. In [22-23], the authors introduced the concept of FTS into linear time-varying singular systems. Robust finite-time stabilization problem was studied for linear singular systems with parametric uncertainties and exogenous disturbances in [24], and a sufficient condition in terms of LMIs was obtained for robust finite-time stabilization via state feedback. It should be noticed that all the FTS and FTB-related literatures for singular systems mentioned above were developed in the context of continuous singular systems while very little attention has been paid to the discrete case.

In this paper, we extend the concepts of FTB to the uncertain discrete-time descriptor systems. Using LMI approach, novel sufficient conditions for the FTB are derived. A numerical example has been provided to show the advantage of developed results.

The following notations will be used throughout the paper. Superscript "T" stands for matrix transposition. $\Re^n$ denotes the n-dimensional Euclidean space and $\Re^{n\times m}$ is the set of all real matrices of dimension $n\times m$. $X>0$ means that X is real symmetric and positive definite, and $X>Y$ means that the matrix $X-Y$ is positive definite. In symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem formulation

Consider the uncertain linear discrete-time descriptor systems with time-varying exogenous disturbance described by:

$$\begin{aligned} &E\,x(k+1) = \left(A + \Delta A(k)\right)x(k) + \left(G + \Delta G(k)\right)z(k), \\ &x(0) = x_0 \end{aligned} \tag{1}$$

where $x(k) \in \Re^n$ is a state vector, $x(0)$ is vector of the initial conditions and $z(k) \in \Re^m$ is a time-varying exogenous disturbance which satisfied

$$z^T(k)\,z(k) < \varepsilon, \quad \forall k \in \{1, \cdots, N\}. \tag{2}$$

E, A and G are known real constant matrices of appropriate dimensions, where E may be singular and we assume that $rank\,E = r < n$. $\Delta A(k)$ and $\Delta G(k)$ are unknown and possibly time-varying matrices representing norm-bounded parameter uncertainties and are assumed to be of the following form

$$\Delta A(t) = MF(k)N_A, \quad \Delta G(t) = MF(k)N_G \tag{3}$$

where M, N_A and N_G are known constant matrices, and $G(k)$ is an unknown matrix function satisfying:

$$F(k)F^T(k) \le I \tag{4}$$

The parameter uncertainties $\Delta A(k)$ and $\Delta G(k)$ are said to be admissible if both (3) and (4) hold.

The nominal discrete-time descriptor systems of (1) can be written as:

$$\begin{aligned} &E\,x(k+1) = Ax(k) + G\,z(k), \\ &x(0) = x_0 \end{aligned} \tag{5}$$

For the linear singular discrete systems (1) and (5) we present the following definition that will be used in the proof of the main results.

Definition 1. [12] The matrix pair (E, A) is said to be regular if $\det(zE - A)$ is not identically zero.

Definition 2. [12] The matrix pair (E, A) is said to be causal if $\deg(\det(zE - A)) = rank(E) = r$.

The regularity and causality of the matrix pair (E, A) ensure the existence, uniqueness and causality of solution of the system (5) for $k = 1, 2, 3, \cdots$.

Lemma 1. [28] Suppose the pair (E, A) is regular and impulse free, then the solution to (5) exists and is impulse free and unique on $(0, \infty)$.

Definition 3. The linear discrete descriptor systems (1) and (5) with $z(k) \equiv 0$ is finite-time stable (FTS) with respect to $\{\alpha, \beta, N\}$, $\alpha < \beta$ if:

$$x^T(0)\,E^T E\,x(0) < \alpha \tag{6}$$

implies:

$$x^T(k)E^T E\,x(k) < \beta,\quad \forall k \in \{1,\cdots,N\} \tag{7}$$

Definition 4. The linear discrete descriptor systems (1) and (5), subject to exogenous disturbances, is finite-time bounded (FTB) with respect to $\{\alpha,\ \beta,\ N\}$, $\alpha < \beta$, if :

$$x^{\mathrm{T}}(0)E^{\mathrm{T}}Ex(0) \le \alpha \tag{8}$$

implies:

$$x^{\mathrm{T}}(k)E^{\mathrm{T}}Ex(k) < \beta,\quad \forall k \in \{1,\cdots,N\} \tag{9}$$

when:

$$z^T(k)z(k) < \varepsilon,\quad \forall k \in \{1,\cdots,N\} \tag{10}$$

The following lemma is employed to handle the time-varying structured uncertainties in the system.

Lemma 2. Given matrices $\Delta = \Delta^T$, H, E and $R = R^T$ of appropriate dimensions,

$$\Delta + HFE + E^T F^T H^T < 0$$

for all F satisfying $F^T F \le R$, if and only if there exists some $\lambda > 0$ such that

$$\Delta + \lambda HH^T + \lambda^{-1}E^T RE < 0$$

3. Main Result

In this section, some finite-time boundedness criteria for nomimal and uncertain discrete-time descriptor systems (5) and (1) are presented.

Theorem 1. The nominal linear discrete descriptor system (5) is regular, causal and FTB with respect to $\{\alpha,\ \beta,\ N\}$, $\alpha < \beta$, if there exist a scalar $\gamma > 1$, positive scalars λ_1, λ_2 and λ_3, two symmetric positive definite matrices $P \in \Re^{n\times n}$, $Q \in \Re^{m\times m}$ and matrix $S \in \Re^{r\times(n-r)}$ such that:

$$\Omega = \begin{bmatrix} -\gamma E^T PE + A^T RS^T + SR^T A & SR^T G & A^T P \\ * & -Q & G^T P \\ * & * & -P \end{bmatrix} < 0 \tag{11}$$

$$\lambda_1 I < P < \lambda_2 I,\quad 0 < Q < \lambda_3 I \tag{12}$$

$$-\beta\lambda_1 + \alpha\gamma^N\lambda_2 + \varepsilon\rho_N\lambda_3 < 0 \tag{13}$$

where:

$$\rho_N = \frac{\gamma^N - 1}{\gamma - 1} \tag{14}$$

and $R \in \Re^{n\times(n-r)}$ is any full-column rank matrix satisfying:

$$E^T R = 0 \tag{15}$$

Proof. First, we prove the regularity and causality of the system. By Schur complement, (11) is equivalent to

$$\Gamma = \begin{bmatrix} -\gamma E^T PE + A^T RS^T + SR^T A & SR^T G \\ * & -Q \end{bmatrix} < 0 \tag{16}$$

i.e.

$$-\gamma E^T PE + A^T RS^T + SR^T A < 0 \tag{17}$$

Whereas $\gamma \geq 1$, we have

$$\Sigma = -E^T PE + A^T RS^T + SR^T A < 0 \tag{18}$$

Now, we choose two nonsingular matrices M and N such that:

$$\hat{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{A} = MAN = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{19}$$

$$\hat{P} = M^{-T} PM^{-1} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^{\ T} & \hat{P}_{22} \end{bmatrix} \tag{20}$$

$$\hat{S} = N^T S = \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} \tag{21}$$

$$\hat{R} = M^{-T} R = \begin{bmatrix} 0 \\ \hat{R}_2 \end{bmatrix}, \quad \hat{R}_2 \in \Re^{(n-r)\times(n-r)}, \quad R \neq 0 \tag{22}$$

Then:

$$\hat{\Sigma} = N^T \Sigma N = -\hat{E}^T \hat{P}\hat{E} + \hat{A}^T \hat{R}\hat{S}^T + \hat{S}\hat{R}^T \hat{A} < 0 \tag{23}$$

From (23) we get

$$\hat{E}^T \hat{P}\hat{E} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^{\ T} & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{24}$$

$$\hat{S}\hat{R}^T \hat{A} = \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} \begin{bmatrix} 0 & \hat{R}_2^T \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} \hat{S}_1 \hat{R}_2^T \hat{A}_{21} & \hat{S}_1 \hat{R}_{22}^T \hat{A}_{22} \\ \hat{S}_2 \hat{R}_2^T \hat{A}_{21} & \hat{S}_2 \hat{R}_2^T \hat{A}_{22} \end{bmatrix} \tag{25}$$

$$\hat{A}^T \hat{R}\hat{S}^T = \left(\hat{S}\hat{R}^T \hat{A}\right)^T = \begin{bmatrix} \hat{S}_1 \hat{R}_2^T \hat{A}_{21} & \hat{S}_1 \hat{R}_{22}^T \hat{A}_{22} \\ \hat{S}_2 \hat{R}_2^T \hat{A}_{21} & \hat{S}_2 \hat{R}_2^T \hat{A}_{22} \end{bmatrix}^T = \begin{bmatrix} \left(\hat{S}_1 \hat{R}_2^T \hat{A}_{21}\right)^T & \left(\hat{S}_2 \hat{R}_2^T \hat{A}_{21}\right)^T \\ \left(\hat{S}_1 \hat{R}_{22}^T \hat{A}_{22}\right)^T & \left(\hat{S}_2 \hat{R}_2^T \hat{A}_{22}\right)^T \end{bmatrix} \tag{26}$$

$$\hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_{11} & \hat{S}_1 \hat{R}^T \hat{A}_{22} + \left(\hat{S}_2 \hat{R}_2^T \hat{A}_{21}\right)^T \\ \hat{\Sigma}_{21} & \hat{S}_2 \hat{R}_2^T \hat{A}_{22} + \left(\hat{S}_2 \hat{R}_2^T \hat{A}_{22}\right)^T \end{bmatrix} < 0, \quad \begin{matrix} \hat{\Sigma}_{11} = \hat{P}_{11} + \hat{S}_1 \hat{R}_2^T \hat{A}_{21} + \left(\hat{S}_1 \hat{R}_2^T \hat{A}_{21}\right)^T \\ \hat{\Sigma}_{21} = \hat{S}_2 \hat{R}_2^T \hat{A}_{21} + \left(\hat{S}_1 \hat{R}_{22}^T \hat{A}_{22}\right)^T \end{matrix} \tag{27}$$

i.e.

$$\hat{S}_2\hat{R}_2^T\hat{A}_{22} + \hat{A}_{22}^T\hat{R}_2\hat{S}_2^T < 0 \tag{28}$$

From (28), we obtain that $\hat{A}_{22}$ is nonsingular and from

$$\begin{aligned}\det(zE - A) &= \det(M^{-1})\det(z\hat{E} - \hat{A})\det(N^{-1}) \\ &= \det(M^{-1})\det(N^{-1})\times\det\left(\begin{bmatrix} zI_r - \hat{A}_{11} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} & -\hat{A}_{12} \\ 0 & -\hat{A}_{22}\end{bmatrix}\right) \\ &= \det(M^{-1})\det(N^{-1})\det(-\hat{A}_{22})\times\det\left(zI_r - (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\right)\end{aligned} \tag{29}$$

we get

$$\det(zE - A) \not\equiv 0, \quad \deg\det(zE - A) \ = r \tag{30}$$

which implies that (E, A) is regular and causal.

Next, we will show that the nominal linear discrete descriptor system (5) with exogenous disturbances is FTB. Choose a Lyapunov-like function as follows:

$$V\left(x(k)\right) = x^T\left(k\right)E^T PE\,x\left(k\right) \tag{31}$$

A difference of $V\left(x(k)\right)$ along the trajectories of the system (5) is:

$$\begin{aligned}\Delta V\left(x(k)\right) &= V\left(x(k+1)\right) - V\left(x(k)\right) \\ &= \left[Ax(k) + G\,z(k)\right]^T P\left[Ax(k) + G\,z(k)\right] \\ &\quad - x^T(k)E^T PEx(k) + 2x^T(k+1)E^T RS^T x(k) \\ &= \xi^T(k)\hat{\Gamma}\xi(k)\end{aligned} \tag{32}$$

where

$$\xi(k) = \begin{bmatrix} x(k) \\ z(k)\end{bmatrix}, \quad \hat{\Gamma} = \begin{bmatrix} A^T PA - E^T PE + A^T RS^T + SR^T A & A^T PG + SR^T G \\ * & G^T PG\end{bmatrix} \tag{33}$$

From (16) and (32) we have:

$$\Delta V\left(x(k)\right) = \xi^T(k)\hat{\Gamma}\xi(k) = \xi^T(k)\left(\Gamma + \begin{bmatrix}(\gamma-1)E^T PE & 0 \\ 0 & Q\end{bmatrix}\right)\xi(k) \tag{34}$$

If (11) (i.e. (16)) is satisfied, then (34) reduces to:

$$\begin{aligned}\Delta V\left(x(k)\right) &\le (\gamma-1)\,x^T(k)E^T PE\,x(k) + z^T(k)Qz(k) \\ &= (\gamma-1)V\left(x(k)\right) + z^T(k)Qz(k)\end{aligned} \tag{35}$$

so, we obtain:

$$V\left(x(k+1)\right) \le \gamma V\left(x(k)\right) + z^T(k)Qz(k) \tag{36}$$

Applying iterative procedure on (36), we obtain:

$$V(x(k)) \le \gamma^k V(x(0)) + \sum_{j=0}^{k-1} \gamma^{k-1-j} z^T(j) Q z(j) \tag{37}$$

Furthermore:

$$\begin{aligned} V(x(k)) &< \gamma^k x^T(0) E^T P E\, x(0) + \varepsilon \lambda_{\max}(Q)\, \rho_k \\ &\le \gamma^k \lambda_{\max}(P) x^T(0) E^T E\, x(0) + \varepsilon \lambda_{\max}(Q)\, \rho_k \\ &< \gamma^N \alpha \lambda_{\max}(P) + \varepsilon \lambda_{\max}(Q)\, \rho_N \end{aligned} \tag{38}$$

On the other hand:

$$\lambda_{\min}(P) x^T(k) E^T E\, x(k) \le V(x(k)) \tag{39}$$

Combining (38) and (39), yields:

$$x^T(k) E^T E\, x(k) < \left[\gamma^N \alpha \lambda_{\max}(P) + \varepsilon \lambda_{\max}(Q) \rho_N\right] / \lambda_{\min}(P) \tag{40}$$

If the following condition is satisfied:

$$\gamma^N \alpha\, \lambda_{\max}(P) + \rho_N \varepsilon \lambda_{\max}(Q) < \beta \lambda_{\min}(P) \tag{41}$$

then from (40) follows:

$$x^T(k) E^T E\, x(k) < \beta, \quad \forall k \in \{1, \cdots, N\} \tag{42}$$

Accordingly, from Definition 4, we conclude that the system is FTB with respect to $\{\alpha, \beta, N\}$, $\alpha < \beta$.

Let

$$0 < \lambda_1 < \lambda_{\min}(P), \quad \lambda_2 > \lambda_{\max}(P), \quad \lambda_3 > \lambda_{\max}(Q)$$

Then

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_1 > 0, \quad 0 < Q < \lambda_3 I \tag{43}$$

and from (41) we get (13). The proof is completed.

Theorem 2. The uncertain linear discrete descriptor system (1) is regular, causal and FTB with respect to $\{\alpha, \beta, N\}$, $\alpha < \beta$, if there exist a scalar $\gamma > 1$, positive scalars λ_1, λ_2 λ_3, η, δ, μ and σ, two symmetric positive definite matrices $P \in \Re^{n \times n}$, $Q \in \Re^{m \times m}$ and matrix $S \in \Re^{r \times (n-r)}$ such that:

$$\Omega=\begin{bmatrix}\Omega_{11} & SR^TG & A^TP & SR^TM & SR^TM & 0 & 0\\ * & \Omega_{22} & G^TP & 0 & 0 & 0 & 0\\ * & * & -P & 0 & 0 & PM & PM\\ * & * & * & -\eta I & 0 & 0 & 0\\ * & * & * & * & -\delta I & 0 & 0\\ * & * & * & * & * & -\mu I & 0\\ * & * & * & * & * & * & -\sigma I\end{bmatrix}<0 \tag{44}$$

$$\Omega_{11}=-\gamma E^TPE+A^TRS^T+SR^TA+(\eta+\mu)N_A^TN_A,\quad \Omega_{22}=-Q+(\delta+\sigma)N_G^TN_G$$

$$\lambda_1 I<P<\lambda_2 I,\quad 0<Q<\lambda_3 I \tag{45}$$

$$-\beta\lambda_1+\alpha\gamma^N\lambda_2+\varepsilon\rho_N\lambda_3<0 \tag{46}$$

where:

$$\rho_N=\frac{\gamma^N-1}{\gamma-1} \tag{47}$$

and $R\in\Re^{n\times(n-r)}$ is any full-column rank matrix satisfying $E^TR=0$.

Proof. Replacing A and G in (11) with $A+MF(k)N_A$ and $G+MF(k)N_G$, respectively, we get:

$$\begin{aligned}\Omega=&\begin{bmatrix}-\gamma E^TPE+A^TRS^T+SR^TA & SR^T(G+MF(k)N_G) & (A+MF(k)N_A)^TP\\ * & -Q & (G+MF(k)N_G)^TP\\ * & * & -P\end{bmatrix}\\ &+\begin{bmatrix}N_A^T\\0\\0\end{bmatrix}F^T(k)\begin{bmatrix}M^TRS^T & 0 & 0\end{bmatrix}+\begin{bmatrix}SR^TM\\0\\0\end{bmatrix}F(k)\begin{bmatrix}N_A & 0 & 0\end{bmatrix}<0\end{aligned} \tag{48}$$

Using Lemma 1 and Schur complement we find that (48) is equivalent to:

$$\begin{aligned}&\begin{bmatrix}-\gamma E^TPE+A^TRS^T+SR^TA+\eta N_A^TN_A & SR^TG & (A+MF(k)N_A)^TP & SR^TM\\ * & -Q & (G+MF(k)N_G)^TP & 0\\ * & * & -P & 0\\ * & * & * & -\eta I\end{bmatrix}\\ &+\begin{bmatrix}0\\N_A^T\\0\\0\end{bmatrix}F^T(k)\begin{bmatrix}M^TRS^T & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}SR^TM\\0\\0\\0\end{bmatrix}F(k)\begin{bmatrix}0 & N_G & 0 & 0\end{bmatrix}<0\end{aligned} \tag{49}$$

$$\begin{bmatrix} -\gamma E^T PE + A^T RS^T + SR^T A + \eta N_A^T N_A & SR^T G & A^T P & SR^T M & SR^T M \\ * & -Q + \delta N_G^T N_G & (G + MF(k)N_G)^T P & 0 & 0 \\ * & * & -P & 0 & 0 \\ * & * & * & -\eta I & 0 \\ * & * & * & * & -\delta I \end{bmatrix} +$$

$$+ \begin{bmatrix} N_A^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F^T(k) \begin{bmatrix} 0 & 0 & M^T P & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ PM \\ 0 \\ 0 \end{bmatrix} F(k) \begin{bmatrix} N_A & 0 & 0 & 0 & 0 \end{bmatrix} < 0 \qquad (50)$$

$$\begin{bmatrix} -\gamma E^T PE + A^T RS^T + SR^T A + (\eta + \mu) N_A^T N_A & SR^T G & A^T P & SR^T M & SR^T M & 0 \\ * & -Q + \delta N_G^T N_G & G^T P & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & PM \\ * & * & * & -\eta I & 0 & 0 \\ * & * & * & * & -\delta I & 0 \\ * & * & * & * & * & -\mu I \end{bmatrix} \qquad (51)$$

$$+ \begin{bmatrix} 0 \\ N_A^T \\ 0 \\ 0 \\ 0 \end{bmatrix} F^T(k) \begin{bmatrix} 0 & 0 & M^T P & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ PM \\ 0 \\ 0 \end{bmatrix} F(k) \begin{bmatrix} 0 & N_G & 0 & 0 & 0 & 0 \end{bmatrix} < 0$$

Finaly, from (51) follows (44).

4. Numerical Example and Simulation

Example 1. Consider the following linear discrete-time descriptor systems with exogenous disturbance $z(k)$:

$$
\begin{aligned}
&Ex(k+1) = \left(A + \Delta A(k)\right)x(k) + \left(G + \Delta G(k)\right)z(k) \\
&\Delta A(k) = MF(k)N_A, \quad \Delta G(k) = MF(k)N_G \\
&E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.5 & 1.1 & 0.4 \\ 0.2 & 0 & 0.8 \end{bmatrix}, \quad G = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix} \\
&M = N_A = \begin{bmatrix} 0.1 & 0.02 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.02 & 0.1 & 0.1 \end{bmatrix}, \quad N_G = 0.1 \\
&x(0) = \begin{bmatrix} 2 & 1 & -0.5 \end{bmatrix}, \\
&z(k) = 5, \quad \forall k > 0, \quad z(0) = 0
\end{aligned}
\tag{52}
$$

We will give two examples to demonstrate the effectiveness of the approach proposed in this paper.

First, we consider the nominal system with $\Delta A(k) = 0$ and $\Delta G(k) = 0$. Fig. 1 shows time histories of the state trajectories of the nominal system (52) with the initial condition $x(0) = [2 \ 1 - 0.5]^T$ and $z(0) = 0$. Time dependent of norm of the state trajectories is illustrated on Fig. 2. Obviously, the observed system is not asymptotically stable. However, in the rest of the paper we will show that this system is FTB, for certain values of the parameters α, β and N. In this sense, it is necessary to investigate FTB with respect to $(6, 469, 10)$. For given matrix E we choose $R = [0 \, 0 \, 1]^T$ such that (15) is satisfied. Solving LMIs (11)-(13) for fixed $\gamma = 1.42$, we obtain the following feasible solution:

$$
P = \begin{bmatrix} 226.12 & 20.43 & -8.65 \\ 20.43 & 168.43 & -2.76 \\ -8.65 & -2.76 & 163.18 \end{bmatrix}, \quad Q = 10.43, \quad S = \begin{bmatrix} -157.59 \\ -112.31 \\ -185.66 \end{bmatrix},
$$

$$
\lambda_1 = 161.91, \quad \lambda_2 = 233.82, \quad \lambda_3 = 10.439
$$

This demonstrates that the system (52) is causal, regular and FTB with respect to $(6, 469, 10)$. For given numerical values $\alpha = 6$ and $N = 10$, the maximum allowed value for parameter β is $\beta = \beta_m = 469$ (see Fig. 2). From Fig 2, the actual values of parameter β, β_a, is estimated from the norm of the state vector and its value is $\beta_a = 224$.

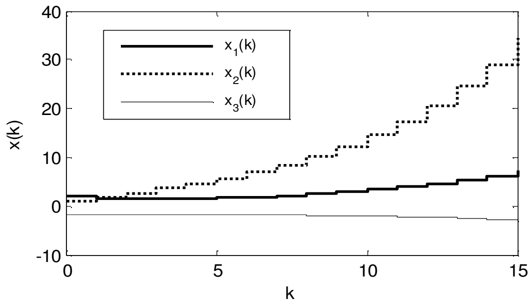

Fig. 1. The initial response of the system (52) with the initial condition $x(0) = [2 \;\; 1 - 0.5]^T$.

Finally, we consider the uncertain system with given matrices M, N_A and N_B. Based on Theorem 2, we obtain the following feasible solution for $\alpha = 6$, $\beta = 120$, $N = 5$ and fixed $\gamma = 1.66$:

$$P = \begin{bmatrix} 1367.6 & 5.85 & -5.67 \\ 5.85 & 1353.2 & -2.24 \\ -5.67 & -2.24 & 1353.5 \end{bmatrix}, \; Q = 132.62 \;, \; S = \begin{bmatrix} -1165.3 \\ -892.8 \\ -1567.4 \end{bmatrix},$$

$$\lambda_1 = 1350.9, \quad \lambda_2 = 1371.9, \quad \lambda_3 = 1327.1$$

$$\eta = 1046.3, \quad \delta = 1424.0, \quad \mu = 2402.7, \quad \sigma = 3268.3$$

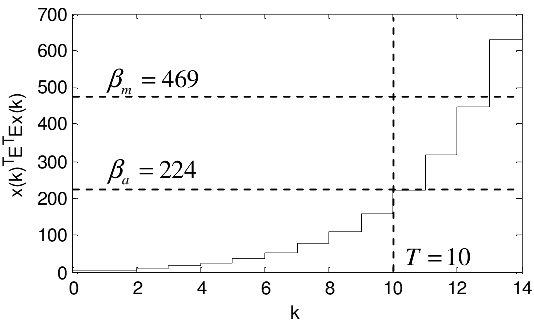

Fig. 2. The norm $x^T(t)E^T Ex(t)$ of the state trajectories.

5. CONCLUSION

In this paper, the finite-time boundedness for linear uncertain discrete-time descriptor systems with time-varying exogenous disturbance is discussed. New stability criteria are derived in terms of LMIs using Lyapunov-like approach and matrix inequality technique.

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REFERENCES

1. Dorato P., Short Time Stability in Linear Time-Varying Systems, *Proceedings of the IRE International Convention Record*, Part 4, New York, 1961, pp. 83–87.
2. Weiss L., Infante, E., Finite Time Stability Under Perturbing Forces and on Product Spaces, *IEEE Trans. Autom. Control*, Vol. 12, No. 1, 1967, pp. 54–59.
3. Amato F., Ariola M., Dorato P., Finite-Time Control of Linear Systems Subject to Parametric Uncertainties and Disturbances, *Automatica*, Vol. 37 N0. 9, 2001, pp. 1459–1463.
4. Amato F., Ariola M., Finite-Time Control of Discrete-Time Linear Systems, *IEEE Trans. Autom. Control*, Vol. 50, No. 5, 2005, pp. 724–729.
5. Amato F., Ariola M., Cosentino C., Finite-Time Stabilization via Dynamic Output Feedback, *Automatica*, Vol. 42, No. 2, 2006, pp. 337–342.
6. Garcia G., Tarbouriech S., Bernussou J., Finite-Time Stabilization of Linear Time-Varying Continuous Systems, *IEEE Trans. Autom. Control*, Vol. 54, No. 2, 2009, pp. 364–369.
7. Amato F., Ariola M., Cosentino C., Finite-Time Control of Discrete-Time Linear Systems: Analysis and Design Conditions, *Automatica*, Vol. 46, No. 5, 2010, pp. 919–924.
8. Amato F., Ariola M., Carbone M., Cosentino C., Finite-time control of linear systems: A survey in current trends in nonlinear systems and control: In honor of Petar Kokotovic and Turi Nicosia, 2006, pp. 195–213.
9. Campbell S.L., *Singular Systems of Differential Equations*, Pitman, London, 1980.
10. Muller P.C., Stability of Linear Mechanical Systems with Holonomic Constraints, *Appl. Mech. Rev*., Vol. 46, No. 11, 1993, pp. 60–164.
11. Dai L., Singular Control Systems: Lecture Notes in Control and Information Sciences, New York, Springer, Vol. 118, 1989.
12. Xu S., Lam J., Robust Control and Filtering of Singular Systems, Springer, Berlin, 2006.
13. Xu S., Lam J., Robust Stability and Stabilization of Discrete Singular Systems: An Equivalent Characterization, *IEEE Trans. Autom. Control*, Vol. 49, No. 4, 2004, pp. 568–574.
14. Ma S., Cheng Z., Zhang C., Delay-Dependent Robust Stability and Stabilisation for Uncertain Discrete Singular Systems With Time-Varying Delays, *IET Control Theory Appl.*, Vol. 1, No. 4, 2007, pp. 1086–1095.
15. Wu Z., Su H., Chu J., Robust Stabilization for Uncertain Discrete Singular Systems With State Delay, *Int. J. Robust Nonlinear Control*, Vol. 18, No. 16, 2008, pp. 1532–1550.
16. Xia Y., Boukas E., Shi P., Zhang J., Stability and Stabilization of Continuous-Time Singular Hybrid Systems, *Automatica*, Vol. 45, No. 6, 2009, pp. 1504–1509.
17. Debeljkovic D.Lj, Lazarevic M.P., Koruga Dj., Tomasevic S., Finite time stability of singular systems operating under perturbing forces: Matrix measure approach, *Proc. AMSE Conference*, Melbourne Australia, October 29 – 31, 1997, pp. 447–450.
18. Debeljkovic D.Lj., Kablar N.A., Finite time stability of linear singular systems: Bellman-Gronwall approach, *Proc. IEEE ACC 99*, San Diego, USA, June 2-4, 1999, pp. 1803-1806.
19. Yangy C., Zhang Y.Q., Linz Y., Zhouy L., Practical stability of closed-loop descriptor systems, *Int. J. Syst. Sci.*, Vol. 37, No. 14, 2006, pp. 1059–1067.
20. Jun-E F., Zhen W., Jia-Bing S., Finite-time control of linear singular systems subject to parametric uncertain and disturbances, *Acta Autom. Sin*., Vol. 31, No. 4, 2005, pp. 634-637.
21. Yang C.Y., Jing X., Zhang Q.L., Zhou L.N., Practical stability analysis and synthesis of linear descriptor systems with disturbances, *Int. J. Autom. Comput*., Vol. 5, No. 2, 2008 pp. 138–144.
22. Zhao S., Sun J., Liu L., Finite-Time Stability of Linear Time- Varying Singular Systems With Impulsive Effects, *Int. J. Control*, Vol. 81, No. 11, 2008, pp. 1824–1829.
23. Xu J., Sun J., Finite-Time Stability of Linear Time-Varying Singular Impulsive Systems, *IET Control Theory Appl.*, Vol 4, No. 10, 2010, pp. 2239–2244.
24. Feng J., Wu Z., Sun J., Finite-Time Control of Linear Singular Systems With Parametric Uncertainties and Disturbances, *Acta Automatica Sin*., Vol. 31, No. 4, 2005, pp. 634–637.
25. Hou L.L., Zong G.D., Wu Y.Q., Finite-time Control for Discrete-Time Switched Systems with Time Delay, *Int J Control Autom*, Vol. 10, No. 4, 2012, pp. 855-860.
26. Xue W., Mao W., Admissible Finite-Time Stability and Stabilization of Uncertain Discrete Singular Systems, *J. Dyn. Sys., Meas., Control*, Vol. 135, 2013, doi:10.1115/1.4023213.

27. Lin Y., An F., Finite-Time Control of Linear Discrete Singular Systems with Disturbances, *D. Jin and S. Lin* (Eds.): Advances in MSEC, Vol. 1, 2011, pp. 569-573.
28. Xu S., Van Dooren P., Stefan R, Lam, J., Robust stability and stabilization for singular systems with state delay and parameter uncertainty, IEEE Transactions on Automatic Control, Vol. 47, No. 7, pp. 1122,1128, 2002.

OGRANIČENOST DISKRETNIH DESKRIPTIVNIH SISTEMA SA PRISUTNIM NEODREĐENOSTIMA I VREMENSKI PROMENLJIVIM SPOLJAŠNJIM POREMEĆAJEM NA KONAČNOM VREMENSKOM INTERVALU

Sreten B. Stojanovic, Dragutin Lj. Debeljkovic, Goran Simeunovic, Nebojša Dimitrijevic

U radu je proučavana klasa linearnih, diskretnih, deskriptivnih sistema sa prisutnim neodređenostima i vremenski promenljivim spoljašnjim poremećajem. Kombinovanjem metode slične Ljapunovoj metodi i tehnike linearnih matričnih nejednačina, izvedeni su dovoljni uslovi ograničenosti sistema na konačnom vremenskom intervalu u obliku sistema linearnih matričnih nejednakosti. Efikasnost predloženog pristupa prezentovana je jednim numeričkim primerom.

Ključne reči: *ograničenost na konačnom vremenskom intervalu, diskretni sistemi sa neodređenostima, descriptor sistemi, spoljašnji poremećaji*

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ACADEMICIAN BOŽIDAR D. VUJANOVIĆ (1930 - 2014)

Academician BOŽIDAR D. VUJANOVIĆ
(September 8, 1930 - March 11, 2014)

THE BIOGRAPHY OF BOŽIDAR D. VUJANOVIĆ

Božidar D.Vujanović was born in the city of Smederevo (Serbia) in 1930. After finishing the elementary school, music school and Gyimnazium, he graduated at the Department of Mechanics at the University of Belgrade 1956., and his doctorate was conferred to him at the same University at 1963. The title of the doctorate is :"The Geometrization of Motion and Disturbances of Nonconservative Dynamical Systems" from the area of mechanics. He was employed at the Mechanical Engineering Faculty as an Assistent of Mechanics at the University of Belgrade from 1957 to 1963. Since then he has been at the University of Novi Sad – Department of Theoretical and applied Mechanics at the Faculty of Technical Sciences and spent there all the time until he retired as a Full Professor at 1995. Between 1967 to 1969 he visited the USA as a Research Associate at the University of Kentucky, Lexington KY at the Department of Theoretical and Applied Mechanics. From 1977 to 1978 he has been a Visiting Professor at the Institute of Electronics and Information Sciences at the University of Tsukuba, Japan. At 1984 he spent six months at the Institute of Engineering and Material Sciences as a Visiting Professor at the Vanderbilt University in Nashville, Tennessee, USA. The scientific interest of Professor Vujanović is Theoreticaland Applied Mechanics, Variational Principles and their applications to conservative and nonconservative dynamical systems, the heat conduction theory, optimal control theory, nonlinear oscillations with dissipative elements etc.

In 1990 he was elected as a Corresponding Member of the Academy of Sciences and Arts of Vojvodina in Novi Sad, and after the fusion of this Institution with the Serbioan Academy of Sciences and Arts in Belgrade he has been adopted as a Member of the same rank. At 2000 he has been elected as a Full Member of the Sebian Academy of Sciences and Arts in Belgrade. At 2009 he has been elected as a Foreign Member of the Academy of Sciences in Turin, Italy (Accademia delle Scienze di Torino , Classe di Scienze fisiche, matematiche e naturali).

Professor Vujanović scientific and university activities have been recognized by a numeruos awards. To mention just a few, he obtained „The October's Prize of the City of Novi Sad for Science" 1970, The University of Novi Sad "Golden Memorial Award"1996, The Golden Placard of the Faculty of Technical Sciences 1990, for the distinguished contribution to the field of Mechanics, the Golden Placard of the Society of the University Professors of Serbia, 1996 etc. He was Commissioned A "Kentucky Colonel" from the Governor of the State Kentucky W.G. Wilkinson, 1990.The Silver Placard "Antico Segillio della Cita di Torino" 1984. For more than forthy years he is Reviewer of the Journals: Zentralblatt fur Mathematik und ihre Grenzgebite (Berlin, Germany) and Mathematical Reviews (Ann Arbor, USA). From 1986-1988 and 1988-2000 he was the President of the Yugoslav Society of the Theoretical and Applied Mechanics. He is the Member of the American Mathematical Society, The Tensor Society (Japan), he is the member of the American Scientific Society "Sigma Ksi" etc.

A BRIEF DESCRIPTION OF THE SCIENTIFIC WORKS

The scientific activity of Professor Vujanović, as mentioned above, is concentrated to the several area of theoretical and applied mechanics, frequently called Analytical Mechanics. This areas, roughly speaking are: Geometrization of motion nonconservative dynamical systems, Variational Principles of Mechanics which can be used in the study of Irreversibile dynamical systems with the finite and infinite degrees of freedom. Generalization of the Hamilton-Jacobi theory and its applications to nonconservative systems. Variational description of the heat transfer theory through the solids including the change of phase, the Study of the Conservation Laws of the conservative and nonconservative dynamical systems with the finite number of the degrees of freedom etc. Generally speaking the most important works of Professor Vujanović can be devided into following three groups:

1. Many years, Professor Vujanović devoted to the study of Conservation Laws of conservative and nonconservative dynamical systems (linear and nonlinear) with the finite number of degrees of freedom. It is well known that the Conservation laws play a very important role in physics and engineering from both theoretical and practical

standpoint. One or more conservation laws can considerably simplifly the integration of the differential equations of motion. It is also beloved that the conservation laws, in some specific way reflect the physical mechanism acting in the dynamical system. Probably the best-known and most popular modern method for finding conservation laws is based upon the study of the invariant properties of the Hamilton action integral with respect to infinitesimal transformations of the generalized coordinates, described the position of the system and time. This traditional approach is based on the famous Emmy Noether theorem which states*: For every given infinitesimal transformation of the generalized coordinates and time, which lives the Hamilton action integral absolute or gauge-invariant, there exists a conservation law of the dynamical system.* Since the Noether theorem does not offer any suggestions as to how to find the infinitesimal transformations that leave the Hamilton action integral unchanged. Professor Vujanović studied the question of finding the aforementioned infinitesimal transformations and corresponding conservation laws. It is shown that the solution of this problem leads to to a system of first-order partial differential equations which he named the generalized Killing's equations. It is obvious that the classical Noether theorem is valid only to the Lagrangian- type dynamical systems. In order to generalize the theory to the purely nonconservative dynamical systems. Vujanović started to study the transformation properties of the D'Alambert differential variational principle which is equally valid for conservative (Langrangian) and nonconservative dynamical systems, and succeeded to enlarge the finding of conservation laws to the purely nonconservative systems. In addition, the further study showed that as the starting point for finding conservation laws can be based also to the Gauss and Jourdain differential variational principles.

2. The next part of his research interest, Prof. Vujanović devoted to the variational principles suitable for the study of irreversible phenomena whose physical manifestations are described by means of partial differential equations and appropriate initial and boundary conditions. The effort to find an appropriate variational principle suitable for a nonconservative physical field is entirely pragmatical. In fact, the merit and efficiency of each variational formulation is tasted for the possibility of obtaining information about the behaviour of the physical systems in questions by applying the direct methods of variational calculus. As a matter of fact for almost all of the important processes of irreversible physics, the exact Lagrangian function of the problem in the sense of classical mechanics does not exist. For example, the transient parabolic differential equation of heat conduction in solids, even in the linear one-dimensional case, does not have any classical Langrangian function. Thus in order to give irreversible phenomena some variational characteristics, especially in the sense of Hamilton's variational principle, Vujanović has been compelled to modify some of the basic rules of the classical variational calculus, whose structure has an exclusively potential character. The main characteristic of this new variational approach is based upon the so called the "Variational principle with a vanishing parameter". The essence of this approach is that the Hamilton principle generates more complex field than the relevant differential equations of the physical process. The differential equations thus obtained, contain a parameter that is let tends to zero after the finishing the process of variation. Doing this, one arrives at the correct differential equations of the process. It is

important to note, that in the realm of heat transfer theory the "vanishing parameter" has a clearly specified physical interpretation that is related to the finite velocity of propagation of the thermal disturbance which is based upon the Cataneo hyperbolic heat conduction theory. From this point of view, the principle with a vanishing parameter represents a transition from the generalized (hyperbolic) heat transfer theory to the classical (parabolic) heat transfer mechanism of the Fourier type, which has an infinite velocity for the propagation of the thermal signal. The variational principle with a vanishing parameter is employed as a starting point for obtaining approximate solutions in two physical area that have a remarquable nonconservative nature: linear and nonlinear transient heat transfer in slids and the theory of laminar boundary layer of the fluid flow.It should be noted that the variational principle with a vanishing parameter is profoundly different than the variational formulations of Glansdorff-Prigogine, Bateman and Biot. Another variational approach introduced by Vujanović is called the "variational principle with uncommutative rules of variations". In this variational principle of Hamilton type are intoroduced the special rules for variation of velocity and velocity of variations, which are not equal, as it is the case in the classical variational calculus. These new rules represent the mausure of nonconservativity of a dynamical systems and are equally applied to the dynamical systems with a finite and infinite number of degrees of freedom. It is also shown that the applications of the Gauss differential variational principle can be of use in obtaining the approximate solutions of various irreversible processes by applying the direct methods of variational calculus.

3. It is well known that the famous Hamilton Jacobi method can be advantageously used in many practical situations as an exact method for solving the canonical differntial equations of motion. In addition, a variety of approximate solution can be built up, based upon this method, for solving nonlinear problems for which an exact, complete solution of the Hamilton-Jacobi nonlinear partial differential equation is not avaliable. However, the method of Hamilton and Jacobi can be employed only with those dynamical systems described by the Langrangian or Hamiltonian function, and *purely nonconservative (non-Hamiltonian) systems* remain outside of the areas treated by this method.

The author introduced a field method suitable for finding the motion of conservative or *purely nonconservative dynamical systems*, which is conceptually different than the method of Hamilton and Jacobi. The basic supposition in this field method is that one of the state variables (a generalized coordinate or generalized momentum) figuring in the dynamical system can be interpreted as a field depending on time and the rest of the state variables of the dynamical system. The resulting field equation, which the author calls *the basic equation*, is a single quasy-linear partial differential equation of the first order. If one is able to find a complete solution of this equation, the motion of the dynamical system can be obtained without any additional integration. It is to be noted that this field method can be used in solving conservative (Hamiltonian) dynamical systems (for which the Hamilton-Jacobi method can be also applied) but in this case the field method is not identical to the Hamilton-Jacobi method. An important advantage of this field method is that one has to solve the quasy linear partial differential equation which is much more menageable than to find a complete solution of the nonlinear Hamilton-Jacobi equation. It is to be noted that one of the dynamical variables is interpreted as the basic field. Thus, the corresponding field equation is more intimately connected with the dynamical problem in question than the Hamilton principal function, which is not by itself a constituent of the dynamical problem. The field method introduced by the author is succesefully employed in the linear and nonlinear boundary value problems and also to the study of nonlinear vibration theory in which an approximate method is introduced.

INVITED LECTURES

The scientific results of the author have attracted the interest of many universities, scientific institutions, and interested researches and the author has been invited to present his scientific results to some places of which, we list a few.
Carnegie-Mellon University, Pitsburg, 1968. Department of physics, Czechoslovakian Academy of Sciences, Prague, 1971. Kings College, London 1972. Department of Mechanics, Thechnical University Budapest, 1975. Instituto di Matemathica, Universita di Torino, 1977. Instituto Matemathico "J.L.Lagrange" Universita di Torino, 1977. Instituto Mathematico, "Ulise Dini", Universita di Firenza, 1977. Department of Mathematics and Physics, Tokio University, 1978. Deprtment of Theoretical and Applied Mechanics, University of Kyoto, Japan 1978. Summer School of Theoretical and applied mechanics, Hiroshima, Japan 1978. Summer school of mechanics, Institute of electonics and information sciences, Tsukuba University, Japan 1978. "Colloquia on Mechanics" Department of Theoretical and Applied Mechanics, University of Kentucky, Lexington, Ky 1984. Department of Mechanical and Material Engineering, Vanderbuilt University, Neshville, TN 1984. Faculdad de Ciencias Fisicas, Departmento de Fisica teoretica, Universidad Complutense de Madrid, 1988.

REFERENCES

The papers published in domestic journals

1. B. Vujanović, Knoblauch's formulae and their application to the rolling of sphere without sliding on the unmovable surface, Publikacije Mašinskog fakulteta u Beogradu, 1962, No. 5. str.51-58.
2. B.Vujanović, Prilog geometrizaciji poremećenog kretanja holonomnih, skleronomnih dinamičkih sistema, Tehnika, avgust 1963, br. 8, str. 1411-1412.
3. B.Vujanović, Jedan matrično-divektorski način za odredjivanje dinamičkih sila u epicikličnim mehanizmima, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 1, Novi Sad, 1965, str.56-74
4. Б. Вујановић, Геометризација кретања и поремећаја неконзервативних динамичких система, Докторска дисертација, Посебна издања, Научно дело, Београд 1964. 32 стр.
5. B.Vujanović, Prilog teoriji oscilacija visko-elastičnih konstrukcija, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 2, Novi Sad, 1964. str.145-163

6. B.Vujanović, Singove poremećajne jednačine kao varijacioni problem i njihovi prvi integrali, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 1, Novi Sad, 1965, str. 93-104.
7. B.Vujanović, Hamiltonov varijacioni princip i njegova primena na provođenje toplote, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 5, Novi Sad, 1970, str.112 - 129.
8. B.Vujanović, Provođenje toplote kroz polu-beskonačna tela sa varijabilnom temperaturom na čeonoj površini, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 6, Novi Sad, 1971. str. 151-171.
9. B.Vujanović, Prilog varijacionom proučavanju konvektivnog provođenja toplote nestišljive tečnosti, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 6, Novi Sad, 1971. str. 161-172.
10. B.Vujanović, O nekim varijacionim aspektima optimalnog upravljanja u smislu Pontrjagina, Zbornik radova Mašinskog fakulteta u Novom Sadu, br. 6, Novi Sad, 1971. str. 173-188.
11. B.Vujanović, Đ.Đukić, A variational principle of Hamilton's type for the laminar boundary layers in incompressible fluids, Publications of Mathematical Institute - Belgrade, Vol. 11 (25), 1971, str. 73-83.
12. B.Vujanović, Koncept polja u nekonzervativnoj mehanici i njegove primene u teoriji nelinearnih oscilacija, Zbornik radova Matematičkog instituta u Beogradu, Nova serija, tom 4 (12), 1984, str. 223-231.
13. B.Vujanović, Conservation laws of nonconservative dynamical systems via Hamel's variational principle, Bulletin T.CXI de L'Academie Serbe des Sciences et des Arts, Classe des Sciences Mathematiques et naturelles, 21, 31-46, 1996.
14. B.D.Vujanović: The bilinear canonical forms of variational principles in heat conduction theory, Bulletin, T.CXVIII, Classe des sciences mathematiques et naturelles, sciences mathematiques No 24, Beograd 1999, str. 1-18.
15. Bozidar D. Vujanovic, A Variational Principle with Uncommutative Rules, Inaugural Lectures of Newly-Elected Full Members of the Academy, Extraid du Bulletin T. CXXIV de l Academie serbe des sciences et des arts,Classe des sciences mathematiques et naturelles, No. 40, Beograd 2003, str.108-124,.

The papers presented at the meetings

16. B.Vujanovic, Dj.Djukic, A.M.Strauss, A new varitaional principle for irreversible phenomena, XIII International Congress of Theoretical and Applied Mechanics (IUTAM), Moskva, 1972. Book of abstracts, Moscow, SSSR, Nauka, p.108.
17. B.Vujanović, On some variational principles for the conduction of heat, First Heat Transfer Conference, Vol. I, str. 262-272, 1973, Iasi, Romania.
18. B.Vujanović, O jednom varijacionom principu mehanike, XII Jugoslovenski kongres teorijske i primenjene mehanike, Ohrid, 1974. Пленарно саопштење, Књига радова са Конгреса, Секција А, стр.1-10.(Југословенско друштво за теоријску и примењену механику)
19. B.Vujanović, B.Bačlić, O direktnim metodama zasnovanim na Gausovom principu najmanje prinude, XIII Jugoslovenski kongres teorijske i primenjene mehanike, Sarajevo, 1976. 07.-11.06.1976. A opšta mehanika, A1-9, (10 strana), Izdanje: Jugoslovensko društvo za mehaniku, 11 000 Beograd Kneza Miloša 9/1
20. B.Vujanović, T.M.Atanacković, O primeni Žurdenovog principa u nelinearnoj mehanici, XIII Jugoslovenski kongres teorijske i primenjene mehanike, Sarajevo, 07.-

11.06.1976. A opšta mehanika, A1-8, (10 strana), Izdanje: Jugoslovensko društvo za mehaniku, 11 000 Beograd Kneza Miloša 9/1

21. B.Vujanović, Savremeni razvoj varijacionih principa u nekonzervativnoj mehanici - Plenarno saopstenje, XIII Jugoslovenski kongres teorijske i primenjene mehanike, Sarajevo, 7-11.6.1976., K-Konferencije, K-4, str. 1-15, it Jugoslovensko drustvo za mehaniku, 11000 Beograd, Kneza Milosa 9/I, 1976.
22. B.Vujanovic, Dj.Djukic, A new variational principle with uncommutative rules and its applications to the nonconservative phenomena, XIV International Congress of Theoretical and Applied Mechanics (IUTAM), 1976, Delft, the Netherlands. Book of abstracts p. 301.
23. B.Vujanovic: On a gradient method in nonconservative mechanics, GAMM, Gesellschaft fur Angewandte Mathematik und Mechanik, Mitteilungen, Heft 1, Marz 1979, Wiesbaden, p.129.
24. B.Vujanovic, Dj.Djukic, On the first integrals and adiabatic invariants of nonconservative dynamical systems, XV International Congress of Theoretical and Applied Mechanics (IUTAM) 1980, Toronto, Canada. Book of abstracts, p. 66.
25. B.Vujanović, Metod polja generalisanog impulsa u nekonzervativnoj mehanici, VII Kongres matematičara, fizičara i astronoma Jugoslavije, 1980, Bečići-Budva.
26. B.Vujanović, O zakonima održanja nekonzervativnih dinamičkih sistema, XV Jugoslovenski kongres teorijske i primenjene mehanike, Kupari, 1981.Књига радова са Конгреса, Секција А-7; стр. 215-222.
27. B.Vujanović, O zakonima održanja nekonzervativnih dinamičkih sistema i njihova primena, Plenarno saopstenje, XVII Jugoslovenski kongres teorijske i primenjene mehanike, Zadar, 2-6. juni 1986., Uvodni referati, str. 97-110, it Jugoslovensko drustvo za mehaniku, 11000 Beograd, Kneza Milosa 9/I, 1986.
28. B.D.Vujanović, B.S.Bačlić, A bilinear form of of the variational principle with vanishing parameter (in English), XXII Jugoslovenski kongres teorijske i primenjene mehanike, June 2-7 1997, Vrnjacka Banja, A – General Mechanics, Aa 04, str. 18-23, Faculty of Mechanical Engineering, University of Belgrade, Mathematical intitute SANU.
29. B.D.Vujanović, B.S.Bačlić, O kanonskim bilinearnim formama varijacionih principa u provođenju toplote, skup "300 godina optimizacije", Odeljenje Tehničkih nauka SANU - Matematički institut SANU, Decembar 1997.
30. B.S. Bačlić, B.D. Vujanović, The Field Method for Arbitrary Rheo-Linear Dynamical Systems, 25-th Yugoslav Congress on Theoretical and Applied Mechanics, Novi Sad, Serbia and Montenegro, June 1-3, 2005. Book of abstracts, p. 24.
31. B. D. Vujanovic, B. S. Baclic, The action conservation laws for the second and fourth order systems, 3RD Serbian-Greek Symposium, Recent Advances in Mechanics,Organized by Serbian Academy of Sciences and Arts, Branch in Novi Sad and Office of Theoretical and Applied mechanics, Academy of Athens and Hellenic Society of Theoretical and Applied Mechanics, Novi Sad, September 15-17, 2008.

The papers printed in foreign journals

32. B.Vujanovic, Synge's distributed equations as a variational problem and their first integrals, Academie Royal de Belgique, Bulletin de la classe des sciences 5, tom LI-1965, pp. 263-269, 1965, Bruxelles.

33. B.Vujanovic, A contribution to the geometrization of nonconservative dynamical systems, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), Akademie-Verlag, Berlin, Band 47, Heft 1, pp. 63-64, 1967.
34. B.Vujanovic, Contribution to geometrization of perturbances of nonconservative dynamical systems, Tensor (N.S.), vol. 18, pp. 328-338, 1967, Japan.
35. B.Vujanovic, The Hamilton's canonical equations as a problem in nonholonomic coordinates, Transactions of the West Virginia Academy of Sciences, Vol. 41, pp. 324-328, 1969.
36. B.Vujanovic, A group variational method for finding first integrals of dynamical systems, International Journal of Non-Linear Mechanics, Pergamon-Oxford, Vol. 5, pp. 269-278, 1970.
37. B.Vujanovic, On one first vector integral of planetary motion, Astronautica Acta, Pergamon, Oxford, Vol. 15, pp. 231-233, 1970.
38. B.Vujanovic, On nonholonomic parameterization, Journal of Applied Mechanics, Transactions of the American Society of Mechanical Engineers (ASME), Vol. 26, pp. 128-132, 1970, New York.
39. B.Vujanovic, On certain aspects of motion in the presence of dissipative forces, Tensor (N.S.), vol. 21, pp. 48-50, 1970, Japan.
40. Б.Вуянович, О некаторих своиствах движения при действйй рассеяваюших сыл, Механика, МИР, Том 1, 35-38, 1971, Москва.
41. B.Vujanovic, An approach to linear and nonlinear heat transfer using a Lagrangian, American Institute of Aeronautics and Astronautics (AIAA-Journal), Vol. 9, No. 1, pp. 131-134, 1971, New York.
42. B.Vujanovic, A.M.Strauss, Heat transfer with nonlinear boundary conditions via a variational principle, American Institute of Aeronautics and Astronautics (AIAA-Journal), Vol. 9, No. 2, pp. 327-330, 1971, New York.
43. B.Vujanovic, A.M.Strauss, A new variational principle for linear viscoelasticity, Tensor (N.S.), Vol. 24, pp. 231-238, 1971, Japan.
44. Dj.Djukic, B.Vujanovic, On one variational principle of Hamiltons type for classical field theory, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), Akademie-Verlag, Berlin, Band 54, Heft 8, pp. 611-617, 1971.
45. Dj.Djukic, B.Vujanovic, Canonical transformations and the Hamilton-Jacobi theorem in the optimal control theory, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), Akademie-Verlag, Berlin, Vol. 57, pp. 651-657, 1971.
46. B.Vujanovic, Dj.Djukic, On one variational principle of Hamilton's type for nonlinear heat transfer problem, International Journal of Heat and Mass Transfer, Pergamon-Oxford, Vol. 15, pp. 1111-1123, 1972.
47. B.Vujanovic, Dj.Djukic, A.M.Strauss, A variational solution of the Raleigh problem for a power-law non-Newtonian fluid, Ingenieur-Archiv, Springer-Berlin, Vol. 41, pp. 381-386, 1972.
48. B.Vujanovic, Dj.Djukic, M.Pavlovic, A variational principle for the laminar boundary layer theory, Bollettino Unione Matematica Italiana, Roma, vol. 4 (7), pp. 377-391, 1973.
49. Dj.Djukic, B.Vujanovic, N.Tatic, A.M.Strauss, On two variational methods for obtaining solutions to transport problems, The Chemical Engineering Journal, The Netherlands, Vol. 5, pp. 145-152, 1973.
50. B.Vujanovic, Application of the optimal linearization method to the heat transfer problem, International Journal of Heat and Mass Transfer, Pergamon-Oxford, Vol. 16, pp. 1111-1117, 1973.

51. B.Vujanovic, On one variational principle for irreversible phenomena, Acta Mechanica, Springer, Wien-New York, vol. 19, pp. 259-275, 1974.52.Vujanovic, A variational principle for nonconservative dynamic systems, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), Akademie-Verlag, Berlin, Vol. 55, pp. 321-331, 1975.
52. Dj.Djukic, B.Vujanovic, A variational principle for the two-dimensional boundary layer of non-Newtonian power-law fluids, Rheologica Acta, The Netherlands, Vol. 14, pp. 881-890, 1975.
53. Dj.Djukic, B.Vujanovic, Noether's theory in classical nonconservative mechanics, Acta Mechanica, Springer, Wien-New York, Vol. 23, pp. 17-27, 1975.
54. Dj.Djukic, B.Vujanovic, On some geometrical aspects of classical nonconservative mechanics, Journal of Mathematical Physics, New York, Vol. 16, p. 2099-2102, 1975.
55. B.Vujanovic, The practical use of Gauss's principle of least constraint, Transactions of the American Society of Mechanical Engineers (ASME), New York, Vol. 32, pp. 1-6, 1976.
56. B.Vujanovic, B.Baclic, Application of Gauss's principle of least constraint to the nonlinear heat transfer problem, International Journal of Heat and Mass Transfer, Pergamon-Oxford, Vol. 19, pp. 721-730, 1976.
57. B.Vujanovic, Applications of analytical mechanics to nonconservative field theory, Rendiconti Seminario Matematico, Universita di Torino, Torino, Vol, 35, pp. 89-95, 1976-77.
58. B.Vujanovic, T.M.Atanackovic, On application of Jourdain's variational principle in nonlinear mechanics, Acta Mechanica, Springer, Wien-New York, Vol. 29, pp. 221-238, 1978.
59. B.Vujanovic, Conservation laws of dynamical systems via D'Alambert's principle, International Journal of Non-Linear Mechanics, Pergamon-Oxford, Vol, 13, pp. 185-197, 1978.
60. B.Vujanovic, A geometrical approach to the conservation laws of nonconservative dynamical systems, Tensor (N.S.), Vol. 32, pp. 357-365, 1978, Japan.
61. B.Vujanovic, On the field-momenta method in mechanics, Tensor (N.S.), vol. 33, No. 1, pp. 117-122, 1979, Japan.
62. B.Vujanovic, On a gradient method in nonconservative mechanics, Acta Mechanica, Springer, Wien-New York, Vol. 34, pp. 167-179, 1979.
63. B.Vujanovic, A.M.Strauss, Linear and quadratic first integrals of a forced linearly dumped oscillator with a single degree of freedom, Journal of Acoustic Society of America, Vol. 69 (4), pp. 1213-1214, 1981, New York.
64. B.Vujanovic, On the integration of the nonconservative Hamilton's dynamical equations, International Journal of Engineering Sciences, Pergamon-Oxford, Vol. 19, No. 12, pp. 1739-1747, 1981.
65. Tytti Sutela, B.Vujanovic, Motion of a nonconservative dynamical system via a complete integral of a partial differential equation, Tensor (N.S.), Vol. 38, pp. 303-310, 1982, Japan.
66. B.Vujanovic, A.M.Strauss, Study of the motion and conservation laws of nonconservative dynamical systems, Hadronic Journal, Vol. 7, pp. 163-185, 1984, USA.
67. B.Vujanovic, A field method and its applications to the theory of vibrations, International Journal of Non-Linear Mechanics, Pergamon-Oxford, vol. 19, No. 4, pp. 383-396, 1984.
68. B.Vujanovic, A.M.Strauss, A new method for solving linear system of algebraic equations, Tensor (N.S.), vol. 41, pp. 173-180, 1984, Japan.

69. B.Vujanovic, A.M.Strauss and S.E.Jones, On some conservation laws of conservative and nonconservative dynamical systems, International Journal of Nonlinear Mechanics, Pergamon-Oxford, Vol. 21, No. 6, pp. 489-499, 1986.
70. B.Vujanovic, A study of conservation laws of dynamical systems by means of the differential variational principles of Jourdain and Gauss, Acta Mechanica, Springer, Wien-New York, Vol. 65, pp. 63-80, 1986.
71. B.Vujanovic, A.M.Strauss, Applications of a field method to the theory of vibrations, Journal of Sound and Vibration, England, Vol. 114, No. 2, pp. 375-387, 1987.
72. B.Vujanovic, A.M.Strauss, Applications of the Hamilton-Jacobi method to linear and nonlinear vibration theory, Journal of Mathematical Physics, New York, Vol. 29, No. 7, pp. 1604-1609, 1988.
73. S.E.Jones, B.Vujanovic, On the inverse Lagrangian problem, Acta Mechanica, Springer, Wien-New York, Vol. 73, pp. 245-251, 1988.
74. B.Vujanovic, S.E.Jones, T.Kawaguchi, On certain Gaussian aspects of Biot's canonical equations in heat transfer theory, Tensor (N.S.), vol. 47, pp. 286-298, 1988, Japan.
75. B.Vujanovic, A variational approach to the problem of solidification of a metal semi-infinite thermally nonlinear body, Acta Mechanica, Springer, Wien-New York, Vol. 77, pp. 231-240, 1989.
76. S.E.Jones, P.P.Gillis, B.Vujanovic, The motion of projectiles launched from elevated positions, International Journal of Applied Engineering Education, vol. 6, No. 3, pp. 365-369. 1990.
77. B.Vujanovic, S.E.Jones, Approximate solutions of canonical heat conducting equations, Journal of Heat Transfer, Transactions of the American Society of Mechanical Engineers (ASME), New York, Vol. 112, November, pp. 836-842, 1990.
78. B.D.Vujanovic, Application of the Hamilton-Jacobi method to the study of rheo-linear oscillators, Acta Mechanica, 93, 179-190, Sringer,Wien-New York, 1992.
79. B.D.Vujanovic, Conservation laws of rheo-linear dynamical systems with one and two-degrees of freedom, International Journal of Nonlinear Mechanics, Vol. 27, pp. 309-322, Pergamon-Oxford, 1992.
80. B.Vujanovic, Application of the Field-Momentum Method to Rheonomic Dynamics, International Journal of Non-Linear Mechanics, Vol. 29, No. 44, 515-528, Pergamon-Oxford, 1994.
81. B.Vujanovic, Conservation Laws and Reduction to Quadratures of the Generalized Time-Dependent Duffing Equation, International Journal of Non-Linear Mechanics, Vol. 30, No. 6, 783-792, Pergamon-Oxford, 1995.
82. B.Vujanovic, T.Kawaguchi, S.Simic, A Class of Conservation Laws of Linear Time-Dependent Dynamical Systems, Tensor, Vol. 58, No. 3, 1997, Japan.
83. B.Vujanovic, A.M.Strauss, S.E.Jones, P.P.Gillis, Polynomial conservation laws of the generalized Emden-Fowler equation, International Journal of Non-Linear Mechanics, Vol. 33, No. 2, Pergamon-Oxford, 377-384, 1998.
84. B.Vujanovic, B.S.Baclic, A bilinear form of the variational principle with vanishing parameter, Archive of Applied Mechanics, 68, (1998), 122-127, Springer, 1998.
85. T.M. Atanackovic, B.D. Vujanovic and B.S. Baclic, A variational principle motivated by the optimal rod theory, Acta Mechanica, Springer, Wien, New York, 139, 57-71, 2000.
86. B.S.Baclic, B.D.Vujanovic, The Hamilton-Jacobi method for arbitrary rheo-linear dynamical systems, Acta Mechanica 163, 51-79, Springer, Wien-New York, 2003.
87. B. D. Vujanovic, B. S. Baclic, A. M. Strauss, Physical Consequences of Action Conservation Laws and Their Applications. Physical Review E 78, 056604(11) (2008),2008 The Americam Physical Societ

Monographs and papers printed in monographs

88. B.Vujanovic, S.E.Jones, Variational Methods in Irreversible Phenomena, Academic Press, New York-London, 1989, štampano u ediciji: Mathematics in Science and Engineering, Vol. 182, Boston, New York, San Diego, Berkeley, London, Sidney, Tokyo, Toronto. 371 pages.
89. B.D.Vujanovic and B.S.Baclic, Variational solutions of transient heat conduction through the bodies of finite length, Advances in Heat Transfer, J.P. Hartnett and T.F. Irvine, Jr., Editors, Vol. 24, 39-99, Academic Press, 1994. Boston.
90. B. Vujanovic,Conservation laws and a Hamilton-Jacobi-like method in nonconservative mechanics, stampano u monografiji, Dynamical Systems and Microphysics, Geometry and Mechanics, A. Avez, A. Blaquiere, A Marzollo editors, Academic Press, New York, pp.293-301,1982.
91. B. D. Vujanovic, A variational approach to transient heat transfer theory, Stampano u monografiji, Advances in Thermodynamics, Vol.6, Flow, Diffusion and Rate Processes, Edited by S. Sieninutycz and P. Salamon, Taylor and Frances, New York,1992, pp.200-217.
92. B. D. Vujanovic, T.M. Atanackovic, An Introduction to Modern Variational Techniques in Mechanics and Engineering, Birkhauser, Boston, Basel, Berlin, 2004,346 pages.

Универзитетски уџбеници
(University Textbooks)

1. B.Vujanović, Odabrana poglavlja iz dinamike, Mašinski fakultet u Novom Sadu, 1964.
2. B.Vujanović, Kratak kurs tenzorskog računa sa primenama u fizici, Mašinski fakultet u Novom Sadu, 1966.
3. B. Vujanovic, Metodi Optimizacije, (Varijacioni racun, Analiticka mehanika, Optimalno upravljanje). Univerzitet u Novom Sadu, Novi Sad, 1980.
4. B.Vujanović, Teorija oscilacija, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, 1977, 1987.
5. B.Vujanović, Teorija oscilacija, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, 1995.
6. B.Vujanović, Dinamika, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, Naučna knjiga, Beograd, 1977, 1988. i 1992.
7. B.Vujanović, D.Spasić, Metodi optimizacije, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, Novi Sad, 1997. (Drugo, dopunjeno I preradjeno izdanje udzbenika navedenog pod br. 3)
8. B. Vujanović, Teorija oscilacija, Treće izdanje, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, FTN Izdavaštvo, Novi Sad, 2007.

International Projects

1. On the application of variational principles, conservation laws and the field-momentum method in nonconservative phenomena. Serbian Academy of Sciences and Arts, (1982. -1986.).

2. Utilization of variational methods in nonlinear and irreversible engineering problems. Bilaterar Yugoslavia – U.S.A. (1988. - 1991.). Finacialy supported by National Science Foundation, Washington D.C.

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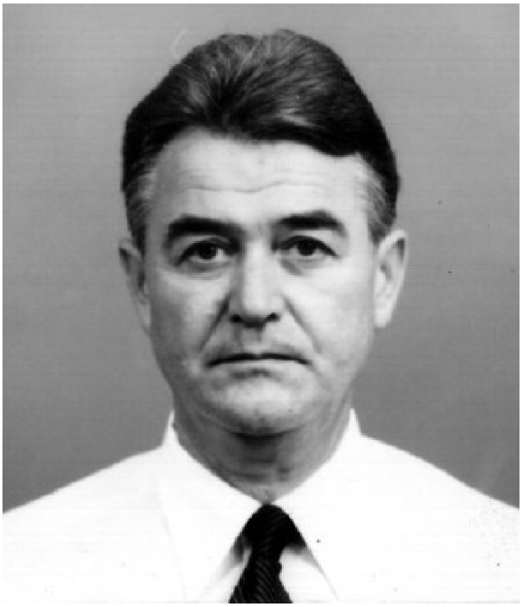
АКАДЕМИК БОЖИДАР Д. ВУЈАНОВИЋ (1930 - 2014)



Академик БОЖИДАР Д. ВУЈАНОВИЋ
(Септембар 8, 1930 - Март 11, 2014

ОСНОВНИ БИОГРФАСКИ ПОДАЦИ

Божидар Д. Вујановић рођен је 8. септембра 1930. године у Смедереву од оца Драгутина и мајке Косаре рођене Матић. У Смедереву је завршио основну школу, музичку школу, гимназију. Дипломирао је 1955. године на Групи за механику

Природно математичког факултета у Београду где је и докторирао 1963 године. Од 1958. до 1963. године био је асистент на предметима механике на Машинском факултету у Београду. Од 1963. године запослио се на Машинском факултету у Новом Саду као доцент за механику. На овом факултету је прошао сва научна звања и изабран је за редовног професора 1972. године. На Факултету (који је касније променио име у Факултет техничких наука, у даљем тексту ФТН) радио је до пензионисања 1995. године. У периоду од 1967. до 1969. године радио је као истраживач (Reserch Associate) на Департману за механику Универзитета у Кентакију, САД (Lexington, KY) у трајању од две године, где је осим истраживања у области Варијационих принципа механике предавао курсеве из динамике и теорије осцилација. Гостовао је као професор по позиву (Visiting Professor) на Универзитету Тсукуба у Јапану 1978.-1979. године у трајању од једне године на Институту за електонику и информационе науке. На овом Институту радио је на уопштењу Хамилтон – Јакобијеве методе у неконзервативној теорији поља, која је од фундаменталног значаја у информационим наукама и предавао је на последипломским студијама Теорију оптималног управљања (Optimal Control Theory). Такође је гостовао као Visiting Professor на Вандербилт Универзитету (Vanderbilt University, Nashville, Tennessee) на Департману за машинску технику и материјале током 1984. године, где је предавао на последипломским курсевима и поставио курс из Оптималног управљања динамичким системима са концентрисаним и распоређеним параметрима.

Добио је Октобарску награду града Новог Сад за науку 1970. године, Награду за „Животно дело“ 1997. године од Удружења универзитетских професора и научника Србије, Златну плакету Универзитета у Новом Саду 1996. године и Златну плакету ФТН-а 1990. године. Добитник је Повеље Универзитета у Новом Саду и ФТН-а за „Изизетан допринос развоју Факултета 2005. године“. Добитник је Светосавске повеље града Смедерева за 2009. годину. Добио је такође златну плакету „Kentucky Collonel“ 1990. године, издату од Гувернера Савазне државе Кентаки, као признање за успешну сарадњу између Универзитета у Новом Саду и Универзитета у Кентакију. Добитник је плакете „Antico Sigillio della Cita di Torino del Sec. XVIII“ града Торина за успешну вишегодишњу срадњу Катедре за механику ФТН-а и „Istituto di Fisica Matematica „J.-Louis Lagrange“ Universita di Torino“ 1983. године. Дугогодишњи је рецензент Референтних журнала: Zentralblatt fur Mathematik und ihre Grenzgebite (Berlin) и Mathematical Reviews (USA) за област механике. У периоду од 1986.-1988. и 1988.-1990. био је Председник Југословенског друштва за Теоријску и примењену механику. Члан је америчког научног друштва „Sigma Ksi“ (1970-), члан је европског друштва за механику Euromech и америчког математичког друштва. Члан је јапанског друштва за примењену геометрију Tensor Society. Члан је редакционог одбора часописа Theoretical and Applied Mechanics који издаје Југословенско (сада Српско) друштво за теоријску и примењену механику и часописа Facta Universitatis који издаје Универзитет у Нишу.

Дописни члан Војвођанске академије наука и уметности постао је 1990. године, а за дописног члана САНУ примљен је 1991. године. За редовног члана САНУ изабран је 2000. године. За иностраног члана торинске Академија наука изабран је

једногласно 13.маја 2009. године (Accademia delle Scienze di Torino, Classe di Scienze Fisiche, Matematiche e Naturali).
Био је ментор 11 докторских дисертација и бројних магистарских теза које су брањене на више универзитета у Југославији и две у иностранству (Хелсинки и Хајдерабад).

НАУЧНА ДЕЛАТНОСТ

Научна Делатност Божидара Д. Вујановића (у даљем тексту Б.Д.В.), односи се на неколико области савремене теоријске и примењене механике (Аналитичке механике). Те области су: Геометризација кретања неконзервативних динамичких система. Варијациони принципи механике погодни за проучавање неповратних динамичких процеса са коначним и бесконачним бројем степени слободе, Уопштење Хамилтон-Јакобијеве методе у неконзервативној механици, Варијационо описивање нелинеарних, нестационарних термичких процеса, Студија закона конзервације динамичких система са коначним бројем степени слободе и др. Грубо говорећи, важнији резултати Б.Д.В. могу се разврстаи у следеће три групе:

1. У својој дугогодишњој активности у проучавању закона конзервације конзервативних и неконзервативних динамичких система са коначним бројем степени слободе Б.Д.В. је радио на проширењу Теореме Emmy Noether. Ова теорема (кључна теорема аналитичке механике) тврди да за сваку <u>задату</u> инфинитезималну трансформацију просторних и временске промењљиве која оставља Хамилтоново дејство апсолутно или градијентно инваријантним, следи неки закон конзевације задатог динамичког система. Први пут је показано <u>како да се нађу</u> инфинитезималне трансформације које остављају инваријантним (апслоутно или градијентно) Хамилтоново дејство, о чему Теорема Noether не говори. Показано је да се налажење ових инфинитезималних трансформација проблем своди на налажење решења система парцијалних диференцијалних једначина, које у случају простог кретања по инерцији постају Килингове једначине познате у диференцијалној геометрији. Уз то, Б.Д.В. је као полазну основу проучавао инваријантност Даламберовог диференцијалног варијационог принципа, а касније и Гаусовог и Журденовог варијационог принципа, па је тако успешно проширен метод налажења закона конзервације неконзервативних динамичких система о чему Теорема Noether, такође не говори.

2. Знатан део истарживања Б.Д.В. је посветио варијационом описивању дисипативних и неповратних процеса, а посебно варијационом принципу Хамилтоновог типа у линеарном и нелинеарном провођењу топлоте у чврстим телима, укључујући промену фазе, теорији граничног слоја у хидродинамици и сл. Добро је познато да нестационарни линеарни проблеми провођења топлоте, описани параболичним једначинама Фуријеовог типа не допуштају варијационо описивање у смислу налажења одговарајуће Лагранжеве функције. Б.Д.В. је први показао да се тачан акциони интеграл може наћи за случај генералисане теорије провођења топлоте када је брзина термичког сигнала коначна. Варирањем овог интеграла и пуштањем да време термичке релаксације тежи нули, долази се посредно до класичне Фуријеове теорије нестационарног провођења. Овако формулисан варијациони принцип може се са великом предношћу користити за добијање решења линеарних и нелинеарних проблема провођења, комбиновањем разних директних метода варијационог рачуна као на пример метод Рица или метод парцијалне интеграције. Ова проучавања довела су до формулације принципа са ишчезавајућим параметром. Коришћењем структурне аналогије овај варијациони принцип примењен је на бројне проблеме течења флуида како њутновског тако и нењутновских флуида. Други варијациони прилаз у неповратним проблемима механике постигнут је формулацијом тзв. Варијационог принципа са некомутативним правилом варирања. Карактеристика овог варијационог принципа је заснована на увођењу посебне класе допустивих варијација код којих „ Варијација извода по времену није једнака изводу по врему вариране функције“ у случају да је динамички систем строго неповратан (неконзервативан). Овај варијациони принцип показао се не само успешним у дисипативним системима типа провођења топлоте, већ и у добијању решења динамичких система са коначним бројем степена слободе, проналажењу адијабатских инваријаната, теорији нелинеарних осцилација и др. Варијационим принципима билинеарног типа посвећена је такође велика пажња.

3. Доборо је познато да је Хамилтон-Јакобојева метода најопштији и најуспешнији метод интеграције Лагранжевих или Хамилтонових динамичких система. Ова моћна и чувена метода не може се применити у динамици неконзервативних система. У свом раду, Б.Д.В. је развио један паралелан метод који је еквивалентан Хамилтон-Јакобојевом методу али се од њега сушаствено разликује чак и у случају Хамилтонових динамичких система. У поменутој методи, претпоставља се да се један од генералисаних импулса (момената)може представити као поље које зависи од времена, генералисаних координата и осталих генералисаних импулса. На тај начин, долази се до једне квази-линеарне парцијалне диференцијалне једначине. Показано је, да уколико смо у стању да нађемо један потпуни интеграл ове једначине, тада без додатне интеграције, долазимо до општег решења неконзервативног динамичког система. Овај метод, који је аутор назвао Метод поља генералисаног импулса успешно је примењен у бројним проблемима интеграције неконзервативиних динамичких система, теорији нелинеарних осцилација и теорији оптималног управљања.

Резултатати научног рада Б.Д.В. и његових сарадника привукли су пажњу бројних истраживачких центара који се баве сличном проблематиком, па је Б.Д.В. одржао бројна предавања по позиву , од којих помињемо само нека:

Carnegie-Mellon University, Pitsburgh, 1968. Одељење за физику чврстог стања Чехословачке академије наука, Праг, 1971. Департман за механику, Kings college, London, 1972. Катедра за механику Техничког Универзитета у Будимпешти, 1975. Istituto di Matematica Universita di Torino, 1977. Istituto Matematico „J.L.Lagrange, Universita di Torino, 1977. Istituto Matematico „Ulise Dini“, Universita di Firenza, 1977. Департман за математику и физику, Универзитет у Токију, 1978. Департман за механику Унивезитет у Кјото-у, Јапан, 1978. Summer School of Theoretical and Applied Mechanics, Хирошима, Јапан 1978. Institute of Information Sciences and Electronics Summer School, University of Tsukuba, Japan, 1978. „Colloquia on Mechanics“ Department of Engineering Mechanics, University of Kentacky, Lexington, Ky, 1984. Department of Mechanical and Material Engineering, Vanderbilt University, Neshville, Tennessee, 1984. Faculdad de Ciencias Fisicas, Departmento de Fisica Teoretica, Universidad Complutense de Madrid, 1988.

YUGOSLAV CONGRESS ON THEORETICAL AND APPLIED MECHANICS NIS'1995

Left photo: Chairmen of the Yugoslav Congress of Theoretical and Applied Mechanics Nis 1995: Members of Serbian Academy of Science and Arts, academician Božidar Vujanović and academician Vladan Djordjević in Nis

Right photo: At Congress breaks: Member of Serbian Academy of Science and Arts Božidar Vujanović and Professors Dusan Stokic and Katica (Stevanovic) Hedrih

Note: Main body of this text and list of references are based on the autobibliography written by academician BOŽIDAR D. VUJANOVIĆ.

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THE SCIENTIFIC AND SOCIAL ACTIVITY OF PROFESSOR N. N. SALTYKOV IN RUSSIA IN 1894–1919

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Abstract. *The scientific and social activity of Professor N. N. Saltykov in Russia in period 1894–1919 is presented.*

Key words: *Nikola Saltikov, scientific activity, university, Serbian academy of Science and Art, Russia, Serbia.*

Fig.1 Nikola N.Saltykov (May 24, 1872-2961)

The name of N. N. Saltykov, who was the Academician of the Serbian Academy of Sciences and Arts, is widely known in Serbia. He was one of the founders of the Serbian

Mathematics. His scientific activity in Serbia is reflected in the sixth volume of the ten-volume edition, dedicated to Serbian scientists [1, p. 43–71]. However, the information about his pre-revolutionary activity in Russia is almost absent. The objective of this article is to fill this gap and provide a Serbian reader with the information about the life and scientific research of Academician Nikolai Saltykov. The sources of our research: few publications about N. N. Saltykov, mostly from the Kharkov National University [2–4], and the National Technical University "Kharkov Polytechnic Institute" [5; 6], where he worked, as well as the materials of regional archives of Kharkov (Ukraine) and Tomsk (Russia) [7; 8].

N. N. Saltykov was born in May 24, 1872 in Vishny Volochyok of Tver province in a family of an engineer-technologist. In 1891 he graduated from the Kharkov high school and entered the Mathematics Department of Physics and Mathematics Faculty of Kharkov University. It was one of the first universities of Russian Empire that later became famous for its high level of teaching of Mathematics and Mechanics. It got the most success in Mathematics and Mechanics at the end of XIX and early XX centuries. The world-famous academicians of Kharkov University are: V. G Imshenetsky, A. M. Lyapunov, V. A. Steklov, K. A. Andreev, S. N. Bernstein, D. M. Sintsov, professors I. D. Sokolov, D. M. Delarue, V. P. Alekseev, M. A. Tihomandritsky, A. B. P. Psheborsky and many other scientists [2, p. 280–282; 6, p. 38–56]. They were the teachers and colleagues of N. N. Saltykov at the university. Two outstanding scientists influenced him greatly: Aleksandr Mikhailovich Lyapunov (1857–1918) – a graduate of St. Petersburg University, a student of P. L. Chebyshev. He was a great mathematician and engineer, the founder of the Stability of Motion theory, the Academician of the Petersburg Academy of Sciences (since 1902). The Lyapunov's sense of life was his devotion to science. He was the first who outlined a specification for Mechanics tasks in their mathematical formulation to be solved with accuracy, or the evaluation of the accuracy of approximate solutions to be defined constantly. Lyapunov delivered Mechanics courses at the Kharkov University (1885–1902) and the Technological Institute (1887–1893).

Vladimir Andreevich Steklov (1863–1926) – a Kharkov University graduate, a student of Lyapunov. He delivered Mechanics at the Kharkov University (1891–1906) and at the Institute of Technology (1893–1906). In 1912 he was elected as an Academician of the Petersburg Academy of Sciences (corresponding member since 1902). In 1919–1926 – he is a Vice-President of the Academy of Sciences of the USSR. In 1921, Steklov organized and managed the Institute of Physics and Mathematics, on the base of which the Mathematical Institute was created in 1934. Now the Mathematical Institute of the Russian Academy of Sciences got the name of Steklov.

While a student, Nikolai Saltykov published his first scientific paper on the integration of the Lauserbracht's equation in Paris in 1894 («Integration de l'equation de Lauserbracht», L'intermedieur de mathematicians. – Paris, 1894, t. II) [1, p. 50, 60]. This gifted young man attracted the attention of teachers and was invited to work at the university to accomplish professorship. Since January 1896 Saltykov was awarded with the scholarship of the Ministry of Education in the Department of Abstract Mathematics of the Kharkov University, and the training period for him was elongated to three years. Nicholai Saltykov successfully defended the dissertation "On the Integration of Equation with Partial First Derivatives of One Unknown Function" at the Academic Council of the Kharkov University in December 5, 1899 and was awarded with a Master's degree in Abstract Mathematics.

In January 1900 the Ministry of Education assigned Saltykov with a scientific mission to study the methods of teaching of Theoretical Mechanics at the Western Europe famous universities. He interned in France and Germany. After return he was invited by the Council of the Tomsk Technological Institute (TTI) to occupy a position of the chief of the Theoretical Mechanics Department. In August 1, 1901 Nicholai Saltykov was appointed as an extraordinary professor of that department. In Tomsk, he delivered lectures in Theoretical Mechanics to the 1st year students of all departments (2 hours per week) and students of the 2nd year of Mechanical and Civil Engineering departments (3 hours per week), conducted practical skill classes. In the first half of 1903/04, during the Professor Nekrasov's assignment, Saltykov also delivered a course of lectures and practical classes in infinitesimal calculus to the students of the 1st course of the Mining and Chemistry Departments. In 1901–1902 he occupied a position of a Secretary (Deputy Dean) of the TTI Mining Department.

Despite the hard academic assignment, N. N. Saltykov constantly improved upon his scientific level. So during the summer vacation from May 15 to September 1, 1902, when he was abroad, he aimed to continue his research in the field of the theory of integration of canonical differential equations as well as equations in partial derivatives. During this trip he worked at the libraries of Paris (Sorbonne) and Leipzig universities and the Paris Academy of Sciences. The results of the study were performed at the Paris Mathematical Society as well as at the Paris Academy of Sciences (partially). At the University of Paris a young scientist attended lectures of professors G. Hadamard and E. Picard, studied their works on the theory of differential equations, function theory, number theory and astronomy. He performed scientific reports about the current development of the celestial mechanics at the Kassel Congress in Germany. He also attended Leipzig and Göttingen Universities, perceived the courses of lectures in Theoretical Mechanics of F. Klein as well as the research of the Norwegian mathematician Sophus Lie, established professional contacts with professors A. Mayer and F. Engel. He took part in the congress of German naturalists and physicians [7, p. 126, Op. 2. u Mts. 1921, 1553, 1842, p. 194, Op. 6a, ed. Mts. 11, 113].

In March, 1904 N. N. Saltykov was transferred to Kiev Polytechnic Institute (KPI) as extraordinary professor. There he continued his scientific research with differential equations with partial derivatives. The professor of the Kiev University and the KPI, corresponding member of the St. Peterburg Academy of Sciences V. P. Ermakov (1845–1922) influenced his scientific research greatly. He mainly focused on the differential equations with partial derivatives. His both dissertations were devoted to the above mentioned subject [9]. In 1906, Nikolai Saltykov defended his doctoral thesis for the

Doctor of Abstract Mathematics Degree: "The Research on the Theory of Partial Derivative Equations of the First Order of an Unknown Function." After the defense, Saltykov became a professor in ordinary of the Applied Mathematics Department of the KPI.

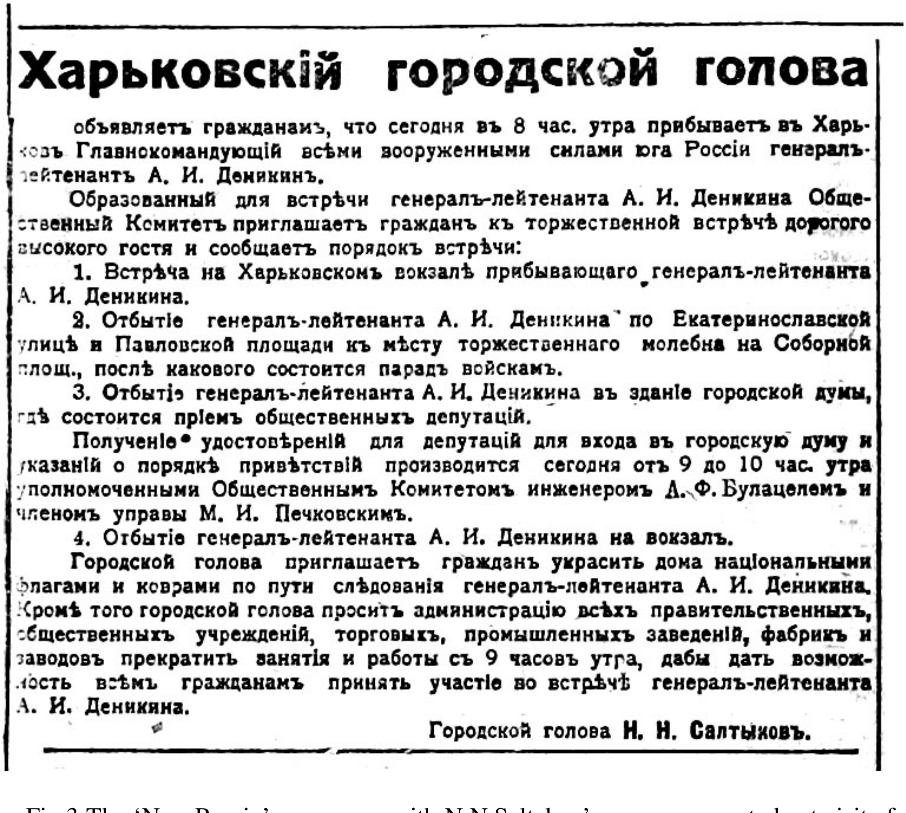
Saltykov hadn't to work long in Kiev. In 1906, his teacher Steklov moved to St. Petersburg, and Saltykov returned to his alma mater. It was in February 5, 1907, when he was appointed as a professor in ordinary of Mechanics at the Kharkov University. To maintain a high level of teaching Mechanics at the university, that had been developed by Lyapunov and Steklov before, Nikolai Saltykov payed much attention to methodological issues. In addition to obligatory courses, he delivered an optional course "Mechanical Principles of Airplane Flight", conducted the scientific seminars on differential equations of Mechanics and the History of Mechanics. Within the study of Theoretical Mechanics, students were offered more complicated problems on the dynamics of gyroscopes and systems of points, three-body problem, variational problems, the problem of small oscillations, problems of the stability of the elastic rod systems and stability of the motion [2, p. 281]. Among the issues Saltykov was interested in at that period was the question of mathematical education in secondary schools. He conducted seminars at the University in order to harmonize teaching in secondary and high schools. Understanding the importance of the Theoretical Mechanics course for secondary school teachers, N. N. Saltykov did his best to perform it at the University.

In 1906–1908 Saltykov also delivered lectures at the Kharkov Practical Institute of Technology (now the National Technical University "Kharkov Polytechnic Institute"), where the courses in Theoretical and Analytical Mechanics were delivered by Lyapunov and Steklov before. Here he and P. V. Shepelev have developed new courses on these subjects on the basis of the generalized experience of A. M Lyapunov and V. A. Steklov. These courses have been agreed with the teaching of other disciplines of mechanical course and Mathematics. Statics and basis of Kinematics and Dynamics were delivered to the first course. The main attention was focused on the elucidation of the mechanical and geometrical issues of the concepts, phenomena and laws introduced. This course comprised three hours a week during a year. It was developed as a complete object and fully satisfied the requirements of the Chemical Department. After sufficient knowledge in Mathematical Analysis on the second course (two hours per week) were obtained, the Dynamics of points system and the basis of Analytical Mechanics were delivered. This program was focused on the students of Mechanical Department [6, p. 48–49]. At the beginning of XX century, the name Saltykov acquired fame. 53 of his works were published before 1918, 27 were published in Paris, 14 in Kharkov, four in Moscow and Kiev, two in St. Petersburg and one in Rome and Cambridge. He was a member of the Mathematical Societies of Kharkov, Kiev, Moscow, Paris, Berlin and Palermo [1, p. 60–63].

In politics Nikolai Saltykov was a liberal. In 1905–1906, in Kiev he was a member of the "Union of Professors." From 1917 he was a member of the Constitutional Democratic Party (Party of National Freedom), and joined the department of the Russian national center when it was organized in July, 1919 in Kharkov. Moreover he was elected to the board of this department. He got in the city council on the list of the national-democratic union, headed by cadets during the local elections in Kharkov in October 1919, and later he was elected as a mayor. Heading the municipal government in such a critical moment, Saltykov demonstrated remarkable personal qualities – optimism, responsibility, commitment to a difficult and thankless work for the good of the city. During the Revolution and the Civil War N. N. Saltykov didn't keep out of the events. When in summer 1919 Volunteer Army came to Kharkov, he was elected as a mayor of the city. As the head of the city government, Saltykov demonstrated his personal qualities – optimism, responsibility, did his best for the city. In this difficult time Nikolai represented the sample of honesty and integrity. In particular, he didn't obey the decision of the National Center about the alliance with the Black Hundreds, the Russian nationalist organization [4]. At the end of June 1919 the Chief of the Armed Forces of South Russia, Lieutenant-General Anton Denikin visited Kharkov. The reception ceremony was organized by Mayor of Kharkov, N. N. Saltykov [10]. Denikin attended the special prayer service dedicated to the city liberation which took place in the square in front of St. Nicholas Cathedral. He was presented with bread and salt on a special dish. Later this dish was captured (looted) by the Red Army during the Denikin's army retreat, and now it is kept in the Central Museum of the Armed Forces of Russia [11].

Fig.2 The meeting of general A.I.Denikin in Kharkov

12.02.1934, N. N. Saltykov was elected as a corresponding member of the Serbian Royal Academy of Natural Sciences department. In accord with the Ministerial Council decision of 22.11.1941, N. N. Saltykov was sent into retirement. During the war, the septuagenarian scientist was imprisoned in a concentration camp Banjica near Belgrade. After the camp liberation he was reinstated as a full-time professor of the Philosophical Department of the Belgrade University by decision of 22. 11. 1945. 02.03.1946 N. N. Saltykov was elected as an active member on the Natural-Mathematical department by the Serbian Academy of Sciences and Arts. In April 1946 the Mathematical Institute of the Serbian Academy of Sciences in Belgrade was founded and here Nikolai Saltykov became a research assistant. In 1955, being already a retiree, he continued his work there as an honorary researcher. The scientific research to which N. N. Saltykov devoted almost 70 years, (taking into account his first publication in 1894), became the main devotion of his life. His bibliography includes 181 scientific papers, comprising several monographs. A complete list of publications is given in [1, p. 60–71].



Харьковскій городской голова

объявляетъ гражданамъ, что сегодня въ 8 час. утра прибываетъ въ Харьковъ Главнокомандующій всѣми вооруженными силами юга Россіи генералъ-лейтенантъ А. И. Деникинъ.

Образованный для встрѣчи генералъ-лейтенанта А. И. Деникина Общественный Комитетъ приглашаетъ гражданъ къ торжественной встрѣчѣ дорогого высокого гостя и сообщаетъ порядокъ встрѣчи:

1. Встрѣча на Харьковскомъ вокзалѣ прибывающаго генералъ-лейтенанта А. И. Деникина.

2. Отбытіе генералъ-лейтенанта А. И. Деникина по Екатеринославской улицѣ и Павловской площади къ мѣсту торжественнаго молебна на Соборной площ., послѣ каковаго состоится парадъ войскамъ.

3. Отбытіе генералъ-лейтенанта А. И. Деникина въ зданіе городской думы, гдѣ состоится пріемъ общественныхъ депутацій.

Полученіе удостовѣреній для депутацій для входа въ городскую думу и указаній о порядкѣ привѣтствій производится сегодня отъ 9 до 10 час. утра уполномоченными Общественнымъ Комитетомъ инженеромъ Д. Ф. Булацелемъ и членомъ управы М. И. Печковскимъ.

4. Отбытіе генералъ-лейтенанта А. И. Деникина на вокзалъ.

Городской голова приглашаетъ гражданъ украсить дома національными флагами и коврами по пути слѣдованія генералъ-лейтенанта А. И. Деникина. Кромѣ того городской голова проситъ администрацію всѣхъ правительственныхъ, общественныхъ учрежденій, торговыхъ, промышленныхъ заведеній, фабрикъ и заводовъ прекратить занятія и работы съ 9 часовъ утра, дабы дать возможность всѣмъ гражданамъ принять участіе во встрѣчѣ генералъ-лейтенанта А. И. Деникина.

Городской голова **Н. Н. Салтыковъ.**

Fig.3 The ‘New Russia’ newspaper with N.N.Saltykov’s announcement about visit of general Denikin

The fundamental direction of Professor Saltykov's scientific activity was the study of partial differential equations of the first order. Lagrange suggested the beginning of general studies of the mentioned above equations. Later, they were developed by I. Pfaff, C. Jacobi, A. Cauchy, J. Bertrand, J. Liouville, A. N. Korkin and many others. In 1870s the scientific papers of Adolf Meyer and Sophus Lie appeared. These studies became the starting point for further research of V. P. Ermakov in this field [9, p. 35–45]. Undoubtedly, the Ermakov's research influenced greatly the N. N. Saltykov's work. The basic tenets of the theory of partial differential equations of the first order and Lee's simplified summary of this study were considered in Saltykov's master thesis from a classical point of view. He critically analyzes and develops the theory of Lee in his doctoral thesis. His official opponent D. M. Sintsov in a review denotes the connection between Liouville's and Lee's scientific papers that was proved by Saltykov. The correlation of the Lee's theory with the classical research of Liouville and Jacobi and other mathematicians is considered to be one of the major Saltykov's merits. More details about the content of the theses of Nikolai Saltykov can be found in the essay of I. A. Naumov "Mechanics in Kharkov" [2, p. 280–282]. Saltykov was dealing with the problem of partial differential equations of the first order during his entire life. The most important of his works, that were published between the World Wars were the works published in Paris [1, p. 64, 65] – "On the theory of partial differential equations of the first order with one unknown function» («Sur la theore des equations aux derivees partielles du premier ordre d'une seule fonction inconnue», Bui. Des Sc. Math. 2 ser. T. XLIX , Juillet – Paris, 1925);

– "Classical methods for the integration of equations in partial derivatives of the first order» («Methodes classique d'integration des equations aux derivees partielles du premier ordre», Memoriales des Sciences Mathematiques, 1931, fasc. L. – Paris);

– "Modern methods for the integration of differential equations with partial derivatives of the first order for one unknown function" («Methodes modernes d'integration des equations aux derivees partielles du premier ordre a une fonction inconnue», Memoriales des Sciences Mathematiques, 1933, fasc. LXX. – Paris).

The last two papers were published in one of the most prestigious collections of monographs, where the works of the most famous French and foreign mathematicians of the time were published. Apart from Saltykov, among Yugoslav mathematicians, only professor of the Belgrad University, Academician Mikhail Petrovich (1868–1943) was awarded with this honor [1, 50–51]. A detailed analysis of N. N. Saltykov's contribution to the theory of differential equations development was performed in monograph [1, 50–57]. The result of his research activities was a monograph "Methods of integrating the differential equations of the first order with one unknown function", published in Belgrade by Serbian Academy of Sciences and Arts in Serbian [12]. It is an encyclopedia of this branch of Mathematics.

Working in Belgrade, Saltykov devoted part of his publications to the reform of the mathematics education in high school and wrote a textbook on Analytical Geometry [13]. A special place in the scientific work of N. N. Saltykov belongs to the history of mathematics. Historical research goes through all his work that distinguishes his works from of other scientists. But apart from this he has many works of historical direction. First of all, the works dedicated to the study of differential equations of Jacobi, Jean d'Alembert, and other mathematicians of the past. Saltykov studied unpublished memoirs on differential equations of the mathematician of XVIII century Charpy as well as revealed its scientific value. He wrote essays about life and activities of the French mathematicians Poincaré and Cartan, Yugoslav mathematicians M. Petrovic and M. Getaldi, Russian mathematician, expat D. F. Selivanov, as well as articles about Archimedes and Descartes as creators of mathematical methods. Among his achievements is the Russian mathematics history to Western auditory. His last papers were published in 1962–1963, after the author's death.

Being a person with active citizenship, N. N. Saltykov took an active part in the Russian academic team in Belgrade and in the Russian Scientific Institute, which brought together Russian scientists – emigrants of different scientific fields. He published his works in the "Notes" of the Institute. Saltykov participated in various activities related to the Russian emigration: 4th Congress of the Russian academic organizations abroad (Belgrade, 1929), the International Congress of Mathematicians (Zurich, 1932), as a delegate from the Russian academic group in Yugoslavia, during the 1st Congress of Mathematicians of the Slavic countries he represented the Russian Scientific Institute in Belgrade. He delivered lectures on inter-Balkan Mathematical Congress (Athens, 1934). His activity to promote the achievements of Russian scientists was of great importance. During 15 years N. N. Saltykov was invited to deliver a series of lectures on various areas of the theory of differential equations in a number of French (Paris and Strasbourg) and Belgium (Brussels, Liege, Leuven and Ghent) universities before World War II. During the postwar period, he delivered lectures at the University of Brussels and the Poincaré Institute in Paris [1, p. 47]. N. N. Saltykov was awarded with a medal by the University of Brussels. Nikolai Saltykov was an active member of the Society of mathematicians, physicists and astronomers of People's Republic of Serbia, as well as the Union of mathematicians, physicists and astronomers of Yugoslavia. During his scientific career N. N. Saltykov participated in international conferences in Rome, Cambridge, Amsterdam, Nancy, Beche and Belgrade.

Nikolai Saltykov created the scientific school of the theory of differential equations in Yugoslavia. Among his students are Professor D. Mihnevich, L. Shchedrin, K. Orlov and M. Stojadinovic. The name of Saltykov is connected with the establishment and progress of Yugoslavia and, in particular, the Serbian mathematics. His merits were estimated pro vita – by the Decree of the Chairman of the Federal People's Republic of Yugoslavia, Josip Broz Tito on 03.30.1956, N. N. Saltykov was awarded with the Order of the I degree. Nicholai lived a long life. He died in September 28, 1961 in Belgrade. Despite the fact that since that time it's been 50 years, the memory about him is still alive, not only in Serbia but also in Ukraine, and in particular in Kharkov – the town which Nikolai Saltykov devoted his best years to.

REFERENCES

1. Život i delo srpskih naučnika [Biografije i bibliografije, urednik akademik Miloje Sarić] Beograd: Srpska Akademija Nauka i Umetnosti, knjiga VI. – 2000. – 423 str.
2. The History of Mechanics in Russia. – Kyev: Naukova Dumka, 1987. –392 p. (In Russian).
3. V. A. Zolotarev. On the formation and development of mathematics education at the University of Kharkov / V. A. Zolotarev // Universitates. Science and Education, 2004. – № 1. – P. 68–72 (In Russian)
4. Pavlova T. G. Kharkov Professor emigration 20–30-s of XX cent / T. G. Pavlova // Universitates. Science and education, 2010. – № 1. – P. 30–51. (In Russian)
5. Larin A. A. Life and career of academician N. N. Saltykov/ A. A. Larin // Proc. of Dnipropetrovsk University (Visnik Dnipropetrovskogo universitetu), 2011. – № 1/2. – T. 19. – P. 108–114 (In Russian)
6. Academician Alexander Aleksandr Mikhailovich Lyapunov: the 150th anniversary of the birth: Monograph / [L. L. Tovazhnyansky, D. V. Breslavsky, A. A. Larin and others]. – Kharkov: NTU "KPI", 2007. – 288 p.
7. State Archive of Tomsk region (Russia)
8. State Archives of Kharkov region (Ukraine)
9. Dobrovolsky V. A. Vasiliy Ermakov, 1845–1922 / B. A. Dobrovolsky. – Moscow: Nauka. – 1981. – 88 p. (In Russian)
10. The newspaper "New Russia", Kharkov, June 22, 1919
11. http://ru.wikipedia.org/wiki/Добровольческая_армия_в_Харькове (Volunteer Army in Kharkov, In Russian)
12. 12. Салтиков Н. Методе интегральена парциальних једначина првог реда са једном непознатом функцијом / Н. Салтиков. – Београд.: Српска академја наука и уметности, 1947. – 748 с.
13. Салтиков Н. Аналитичка геометрја, ч. I / Н. Салтиков. – Београд.: Просвета, 1947. – 271 с.

NAUČNA I AKADEMSKA AKTIVNOST PROFESORA NIKOLE N. SALTIKOVA U RUSIJI I PERIODU 1894-1919

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Apstrakt. *Prikazana je naučna i akademska aktivnost Profesora Nikole N. Saltykov u Rusiji u period 1894–1919.*

Ključne reči: *Nikola Saltikov, naučna aktivnost, univerzitetski rad, Srpska akademija nauka i umetbđnsti,Rusija, Srbija.*

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Lagrangian and Hamiltonian Geometries. Applications to Analytical Mechanics

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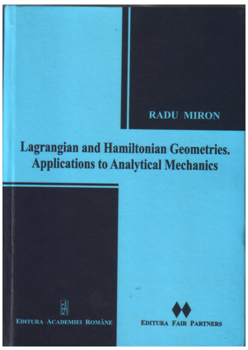
București, 2011

Radu Miron: Lagrangian and Hamiltonian Geometries. Applications in Analytical Mechanics.

Editura Academiei Romane and Editura Fair Partners, Bucuresti, 2011. ISBN 978-973-27-2129-2. ISBN 978-973-1877-63-1.



Monograph titled: "**Lagrangian and Hamiltonian Geometries. Applications in Analytical Mechanics**" authored by Radu Miron**, academician of Romanian Academy of sciences contain new approach to Analytical Mechanics on the basis of new knowledge in **Lagrangian and Hamiltonian Geometries.**

Monograph contain the following Parts:

Part I. Lagrange and Hamilton Spaces (pp. 1-102) includes:

* *The geometry of tangent manifold* (the manifold TM , semisprays on the manifold TM , Nonlinear connections, N -Linear connections, Parallelisms, Structure equations) ;

* *Lagrange spaces* (Variation problem, Euler-Lagrange equations, Canonical semispray, Hamilton-Jacobi equations, Metrical N -linear connections, The

electromagnetic and gravitational fields, The almost Kählerian model of a Lagrange space, Generalized Lagrange spaces);

* *Finsler spaces* (Finsler metrics, Geodesics, Cartan nonlinear connection, Cartan metrical connection);

* *The geometry of cotangent manifold* (congruent bundle, Variational problem, Hamilton-Jacobi equations, Nonlinear connections, N-Linear connections, Parallelisms, paths and structure equations);

* Hamilton spaces (…Generalized Hamilton space GH^n, The almost Kählerian model of a Hamilton space, …);

* Cartan spaces (…., Cartan metrical connection of C^n, The duality between Lagrange and Hamiltion spaces).

Part II. Lagrangian and Hamiltonian Spaces of higher-order (pp. 103-148) includes: * The geometry of manifolds, *Lagrange space of higher-order, * Higher-order Finsler space, * The geometry of k-cotangent bundle).

Part III. Analytical Mecjanics of Lagrangian and Hamiltonian Mechanical Systems (pp. 149-244) includes:

* *Riemannan mechanical systems* (Reimannian mechanical systems, Examples of Reimannian mechanical systems, The evolution semispray of the mechanical system, The nonlinear connection, The canonical metrical connection, The electromagnetism in the theory of the Reimannian mechanical systems, The almost Hermitian model of the Reimannian mechanical systems);

* *Finslerian mechanical systems* (Semidefinite Finsler spaces, The notion of Finsled Mechanical system, The evolution semispray of the Finsler spaces, The canonical nonlinear connection of the Finsled Mechanical system, The dynamical derivative determined by the evolution nonlinear connection N, Metrical N-linear connection of Finsler spaces, The electromagnetism in the theory of the Finsler mechanical systems, The almost Hermitian model of the tangent manifold of the Finslerian mechanical systems);

* *Lagrangian mechanical systems* (Lagrange spaces, Lagrangian mechanical systems, The evolution semispray of Lagrangial space, The evolution nonlinear connection of Lagrangial space, Canonical N-metrical connection of Lagrangial space, Structural equations, Electromagnetic field, The almost Hermitian model of the Lagrangian mechanical systems, Generalized Lagrangian mechanical systems.);

* *Hamiltonian and Cartanian mechanical systems* (Hamilton spaces, Hamiltonian mechanical systems. Cartanian mechanical systems);

* *Lagrangian, Finslerian and Hamiltonian mechanical systems* of order $k \geq 1$ (Lagrangian mechanical systems of order $k \geq 1$, Canonical k-semispray of mechanical system, Canonical nonlinear connection of mechanical system, Canonical metrical N connection, The Reimanian $(k-1)n$ almost contact model of the Lagrangian mechanical systems of order k, Classical Reimanian mechanical systems with external

forces damping on the higher-order accelerations, Finsler mechanical systems of order k , Finslerian mechanical systems of order $k \geq 1$).

In opinion of author, academician Rdu Miron, the aim of the present monograph is twofold:

1* to provide a Compendium of Lagrangian and Hamiltonian geometries;

2* to introduce and investigate new analytical Mechanics: Finslerian, Lagrangian and Hamiltonian.

Some sequences in monograph suggest, also Rimanian analytical mechanics and etc.

The fundamental equations (or evolution equations) on the Mechanics are derived from variational calculus applied to the integral of action and these can be studied by using methods of Lagrangian and Hamiltonian geometries.

More general, the notions of higher-order Lagrange or Hamiltonian spaces have been introduced by Radu Miron, the present author, and developed by means of two sequences of inclusions similarly with those of the geometry of the order 1. The applications leads to the notion of Lagrangian or Hamiltonian Analytical Mechanics of order k .

In short presentation, in this monograph author radu Miron aim to solve some difficult problems:

* the problem of geometrization of classical nonconservative mechanical systems;

* the foundations of geometrical theory of new mechanics: Finslerian, Lagrangian and Hamiltonian;

* to determine the evolution equations of the classical mechanical systems for whose external forces depend on the higher-order accelerations.

The Monograph is based on the theory and and important remarks taken from numerous cited books and references prese4nted in the List of the 258 References.

. Clarify; the classical Lagrange equations are not valid for all mechanical systems with different types of constraints. In the last part of the monograph Radu Miron present the solution of the problem for considered class of the mechanical systems.

Finally, answers to question: What is new in this presented monograph? New are:

* A solution of the problem of geometrization of the classical nonconservative mechanical systems, whose external forces depend on velocities, based on the differential geometry of velocity space;

* The introduction of the notion of Finslerian mechanical system;

* The definition of the Cartanian mechanical system;

* The study of the Lagrangian and Hamiltonian mechanical systems by means of the geometry of tangent and cotangent bundles;

* The geometrization of the higher-order Lagrangian and Hamiltonian mechanical systems;

* The determination of the fundamental equations of the Reimaninan mechanical systems whose external forces depend on the higher-order accelerations.

Monograph contains Index on the three pages.
Monograph is presented on the 266 pages.
With pleasure, I recommend to read this unique and original monograph as very useful for scientists, researchers and Ph. D student in area of mechanics as well as in mathematics

Katica R. (Stevanović) Hedrih

**Professor Radu Miron, doctor of sciences (born on Sctober 3, 1927) is a Member of Romanian Academy of Sciences and of the Academy of Sciences of Moldava Republic.He is Doctor Honoris Causa of Universities from Craiova, Constanta, Oradea, Galati, Bacau and Brasov, Romania and Kishinev. Moldava. He is awarded by the Romanian Academy with "CH.Titeica" Prize and by Romanian Ministry of Education with the "Opera Omnia" Prize.

Professor Radu Miron discovered the Geometries of Lagrange and Hamilton spaces, as well as the Lagrange and Hamilton geometries of higher-order. He difined and studied for the forst time the notions of Lagrangian and Hamiltonian Mechanical Systems, which are the basic in Analytical Mechanics.

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