9. Non-negative Matrices

Definition 8.1.1 (p. 172)

A square matrix A of order n is *irreducible* provided that its digraph $D(A)$ is strongly connected; otherwise, A is *reducible*.

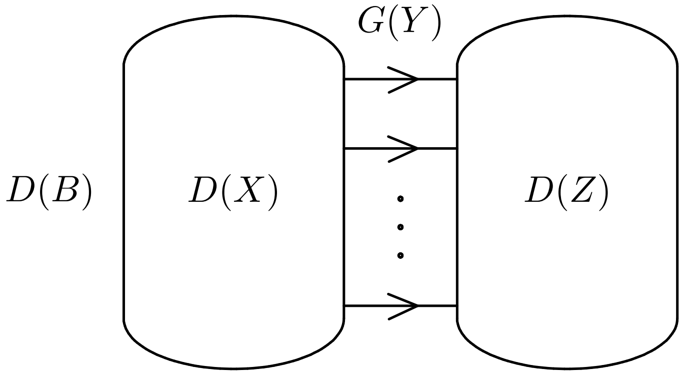
A digraph is strongly connected if for each pair u and v of distinct vertices there is a path from u to v and a path from v to u.

A digraph is strongly connected if and only if there does not exist a partition of its vertex set into two nonempty sets U and W such that each edge between U and W has its initial vertex in U and its terminal vertex in W. Thus if we simultaneously permute the rows and columns of A so that the first rows correspond to U, we obtain that A is reducible if and only if there is a permutation matrix P such that

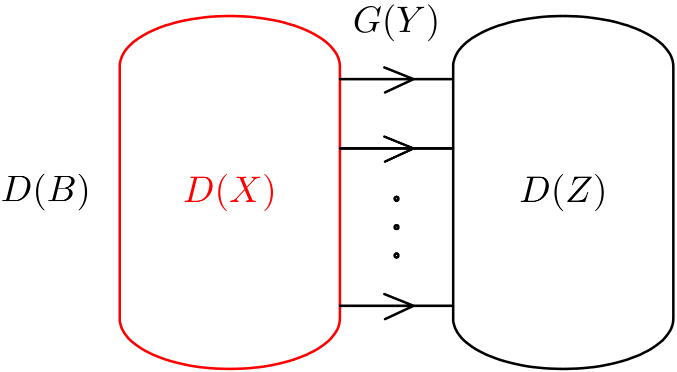
$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} = B,$$

where X and Z are square matrices of order at least 1.

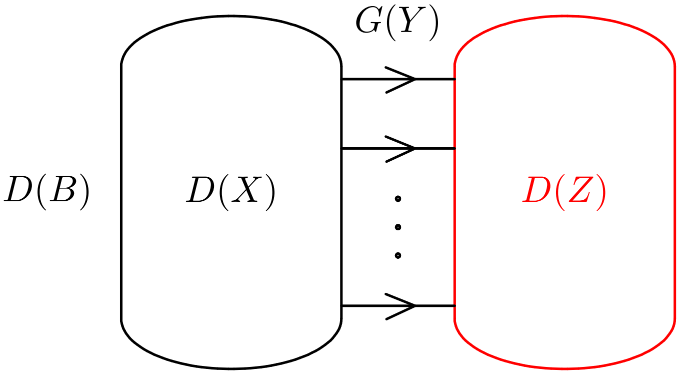
$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} = B$$



$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} = B$$

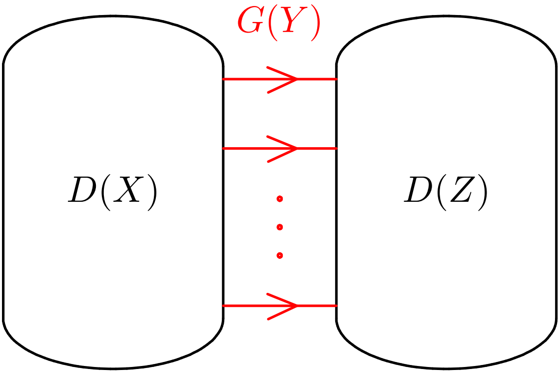


$$PAP^T = \begin{bmatrix} X & Y \\ O & {\color{red}Z} \end{bmatrix} = B$$

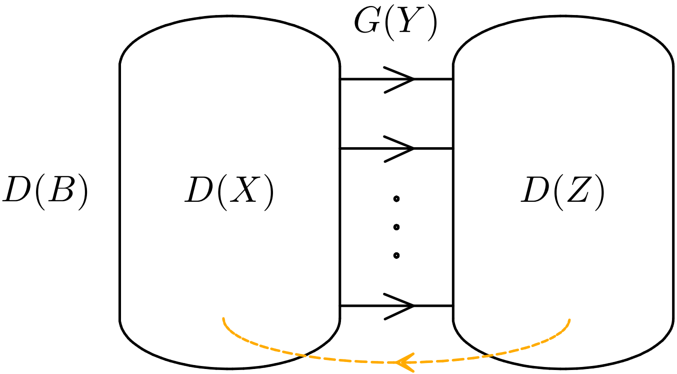


$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} = B$$

$D(B)$



$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} = B$$



$\boxed{D(B) \text{ not strongly connected}}$

Theorem 4.2.13 (p. 79)

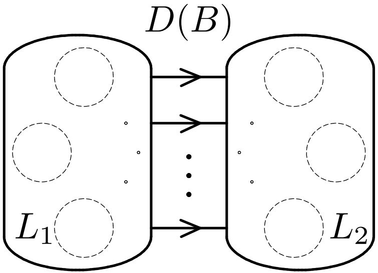
Let

$$B = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix},$$

where X and Z are square submatrices of B. Then

$$\det B = \det X \cdot \det Z.$$

In particular, the determinant of B does not depend on Y.



$$L = L_1 \cup L_2 \qquad n = n_1 + n_2$$

$$c(L) = c(L_1) + c(L_2) \qquad w(L) = w(L_1) \cdot w(L_2)$$

$$\det B = (-1)^n \sum_L (-1)^{c(L)} w(L) =$$

$$(-1)^{n_1} \sum_{L_1} (-1)^{c(L_1)} w(L_1) \cdot (-1)^{n_2} \sum_{L_2} (-1)^{c(L_2)} w(L_2)$$

$$= \det X \cdot \det Z.$$