KEY SLIDES FOR A COURSE IN COMBINATORIAL MATRIX THEORY

Richard Brualdi,
University of Wisconsin at Madison, Madison, Wisconsin, USA
Dragoš Cvetković,
Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

INTRODUCTION

The purpose of these slides is to support teaching matrix theory with a combinatorial approach, in particular, in the way described in the book

1. Brualdi R.A., Cvetković D., *A Combinatorial Approach to Matrix Theory and Its Application*, CRC Press, Boca Raton, 2008.

Slides are contained in ten PDF-files. Each file is devoted to a theme described in the book [1]. Selected themes do not cover the whole book; we have taken only those subjects which require extensive graph drawings in teaching. The idea is that a teacher of a course in matrix theory can select some of our slides and combine them with his/her own slides or other tools (board, oral explanations, etc.).

The list of themes and description and explanation of the corresponding slides is given below with references to book [1].

The slides are produced in Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia, within the scientific project "Graph Theory and Mathematical Programming with Applications to Chemistry and Engineering". The slides are distributed free of charge and can be downloaded from web sites of the book publisher and of the Mathematical Institute, Belgrade.

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Themes

1. Product of matrices, 2. Powers of matrices, 3. Determinants, 4. Inverses, 5. Coates formula, 6. Signal flow graphs, 7. Cayley-Hamilton theorem, 8. Jordan canonical form, 9. Non-negative matrices, 10. Control theory.

DESCRIPTION OF SLIDES

For each of the themes the corresponding PDF-file contains several slides.

1. Product of matrices

(related to Chapter 2 of the book [1])

Slide No. 1.1. *The König digraph of a matrix and its transpose*
(related to p. 36 and p. 40 of [1])

The König digraph $G(A)$ of a 2×3 matrix A is given. The digraph of the transpose A^T is obtained by reversing orientation to all edges and by exchanging colors (black and white) in all vertices. Since for the original matrix black vertices are placed on the left hand side of the drawing, one can do the same for the transpose as shown.

Slide No. 1.2. *Matrix multiplication via König digraphs*
(related to pp. 36-39 of [1])

König digraphs of two matrices are given. The number of white vertices in the first digraph equals the number of black vertices in the second digraph so that the multiplication of the corresponding matrices is possible. Only entries of the first row of the first matrix and of the first column of the second matrix are given in matrices as well as in the digraphs. Next picture shows the composition of these two digraphs which is obtained by identifying each white vertex of the first digraph with the correspondingly labeled black vertex of the second digraph. The new vertices become gray. Further, all paths from the black vertex 1 to the white vertex 1 are colored red. The sum of weights of these paths determines the entry at position (1,1) of the matrix product and becomes the weight of the edge between the black vertex 1 and the white vertex 1 in the corresponding digraph.

Slide No. 1.3. *Transpose of a matrix product*
(related to p. 41 of [1])

Matrices A and B are given by their König digraphs $G(A)$ and $G(B)$. The slide gradually gives the composition $G(A) * G(B)$ and digraphs $G(AB)$ and $G((AB)^T)$. The last digraph is then transformed into the composition $G(B^T) * G(A^T)$ which should convince the student that $(AB)^T$ equals $B^T A^T$.

Slide No. 1.4. *Associative law for matrix multiplication*
(related to p. 39 of [1])

Three matrices (A, B and C) are given by their König digraphs. The point is that both double products $(AB)C$ and $A(BC)$ can be obtained by inspecting paths of length 3 between black and white vertices in the double composition $G(A) * G(B) * G(C)$. Concatenation of paths in such multiple compositions of König digraphs is obviously associative implying that composition $*$ as well as the matrix multiplication are associative operations.

Slide No. 1.5. *An example of finding the k-th power of a square matrix*

The task of Example 3.1.3 is treated here by the König digraph. The edge with a zero weight is omitted when drawing the digraph since paths of zero weight do not contribute to the value of entries in the product of matrices. Gradually, matrix powers A^2, A^3 and then A^k are calculated using compositions of the corresponding number of copies of the König digraph $G(A)$ of the matrix A. Relevant paths in these compositions and corresponding terms of resulting matrix entries are indicated red.

2. Powers of matrices

(related to Chapter 3 of the book [1])

Slide No. 2.1. *Graph theoretical interpretation of matrix powers*
(related to pp. 51 - 52 of [1])

First, the student is reminded, by a square matrix A of order 3, how the digraph $D(A)$ is constructed. For Theorem 3.1.2, p. 51, the second proof, given on p. 52, is illustrated. Starting from the König digraph $G(A)$ of a square matrix A, the slide gives gradually the composition of $G(A)$ with copies of itself, up to the composition of k copies. A particular path of length k from the black vertex i to the white vertex j is indicated by red color. The same path with the same weight exists in the digraph $D(A)$. Since this holds for any path of length k from the black vertex i to the white vertex j, we conclude that the theorem is true.

Slide No. 2.2. *Examples of finding powers of a matrix*
(related to p. 52, p. 60 and p. 246 of [1])

Now the task of Example 3.1.3 is treated here by the digraph $D(A)$. In addition, the matrix A from Exercise 2 on p. 60 is treated by Theorem 3.1.2 and also by Theorem 3.1.4.

3. Determinants

(related to Chapter 4 of the book [1])

Slide No. 3.1. *The Coates digraph of a matrix*

(related to p. 65 of [1])

This slide illustrates the representation of a matrix in the form of a square scheme and in the form of a weighted digraph. Both representations are useful; the first one for recording a matrix, the second one in proofs of many theorems. In addition, one should point out that both digraphs $D(A)$ and $D^*(A)$ are useful. In particular, in the theory of determinants both are equally good in view of the fact that a square matrix and its transpose have the same determinant. A less natural choice, to use $D^*(A)$ for the definition of a determinant, can be justified by the more natural form of the expression for cofactors (slide 4.1) and of the Coates formula for solving the system of linear algebraic equations (slide 5.1).

Slide No. 3.2. *Evaluation of determinants of order 2*

(related to p. 66 of [1])

Slide No. 3.3. *Evaluation of determinants of order 3*

(related to p. 66-68 of [1])

The six linear subdigraphs are colored red within the digraph but also they are extracted. This example should make students familiar with the structure of a linear subdigraph (collection of disjoint cycles which cover all vertices of the digraph). In this way the student will be able to realize how linear subdigraphs look in the general case.

Slide No. 3.4. *Determinant of a transpose*

(related to p. 72 of [1])

The slide helps the student to accept the proof that the matrix and its transpose have the same determinant. First, it is made clear that the transposition of a matrix changes the orientation of all edges in the corresponding digraph. (One should draw the attention that the orientation of a loop is not important so that the change of orientation of a loop is actually not performed on slides.) For each linear subdigraph from the previous slide, it is shown that it goes over into a linear subdigraph of the digraph associated to the transpose. The weight and the number of cycles are preserved which proves the theorem. The teacher should point out that the argument does not depend on the example shown in slides, i.e. it holds for matrices of any size.

Slide No. 3.5. *Effect of zero entries*

(related to p. 69 of [1])

Example with a matrix of order 3 is further exploited to illustrate the effect of a zero entry on the calculation of the value of a determinant via the associated digraph. By considering all the six linear subdigraphs, it is suggested that linear subdigraphs containing an edge with zero weight do not contribute to the value of the determinant, and that they can be eliminated by deleting the edge with zero weight from the graph drawing. In this case only four linear subdigraphs remain but the example suggests that in the general case the deletion of such edges is allowed since such deletion eliminates exactly those linear subdigraphs that contain at least one such edge and that such linear subdigraphs are unnecessary since their weights are equal to 0 and do not contribute to the value of the determinant. The teacher should tell the students that matrices with a lot of zero entries (sparse matrices) are very frequent in theoretical considerations as well as in applications. On the other hand, the proofs of matrix theorems based on associated digraphs do hold in general case irrespectively of whether the matrix is sparse or not.

Slide No. 3.6. *Multiplying a row by a number*

(related to p. 72 of [1])

The slide illustrates the theorem on multiplying entries of a row by a number α. Entries of the i-the row are weights of the edges going into vertex i. Each linear subdigraph contains exactly one such edge and its weight is multiplied by α.

Slide No. 3.7. *Interchanging two rows*

(related to pp. 73-74 of [1])

This is an illustration of the proof that the determinant changes its sign if two rows exchange their positions. If i-the and j-th row change positions, the associated digraph is modified in such a way that each edge going into vertex i becomes an edge going into vertex j and vice versa. The effect of this transformation on a linear subdigraph depends on whether vertices i and j belong to the same or to different cycles. In both cases the parity of the number of cycles contained in the linear subdigraph is changed while the weight remains unchanged. Hence, all terms in the determinant defining expression change their sign and so does the determinant.

Slide No. 3.8. *Examples of evaluation of determinants*
(related to pp. 69-71, p. 95 and p.248 of [1])

There are three examples: a diagonal matrix of order n, a tridiagonal matrix of order 4 and a matrix of order $n+1$.

4. Inverses
(related to Chapter 5 of the book [1])

Slide No. 4.1. *Cofactors and 1-connections*
(related to pp. 103-105 of [1])
The definition of a 1-connection $D[i \to j]$ from vertex i to vertex j in a digraph D, p. 103, is reproduced. The slide illustrates how a 1-connection is obtained from a linear subdigraph by deleting an edge. If we apply this procedure to linear subdigraphs of the digraph $D^*(A)$, we get expressions for cofactors. Relations between the numbers of cycles and weights are given, separately for $i \neq j$ and $i = j$. Finally, as an illustration, all 1-connections of a digraph are indicated.

Slide No. 4.2. *Example of finding the inverse of a matrix*
(related to p. 106 of [1])
Example 5.3.3. is reproduced.

5. Coates formula
(related to Section 6.3 of the book [1])

Slide No. 5.1. *Deriving the Coates formula*
(related to pp. 123-124 of [1])
The slide illustrates a detail in deriving the Coates formula (6.14) for solving a system of linear algebraic equations. It is shown how a 1-connection $D^*[j \to i]$ is extended to a 1-connection $D^*[0 \to i]$ and corresponding relations concerning the number of cycles and weights are given.

Slide No. 5.2. *Example of solving a system of equations*
(related to pp. 122-126 of [1])
Examples 6.3.2 and 6.3.3 are illustrated. Linear subdigraphs and 1-connections together with corresponding terms in the solution are colored red.

Slide No. 5.3. *Example of solving a system of equations*
(related to p. 138 and pp. 249-250 of [1])

The solution of the system of Exercise 8 on p.138 is given on pp. 249-250 and this slide reproduces both the system and the solution for x_1. Linear subdigraphs and 1-connections together with corresponding terms in the solution are colored red.

6. Signal flow graphs

(related to Section 6.4 of the book [1])

Slide No. 6.1. *Example of solving a system of equations*
(related to pp. 128-131 of [1])

The slide displays the system of linear algebraic equations of Example 6.4.4 and the corresponding signal flow graph. Then the solutions for all three unknowns are obtained using Mason's formula. Relevant subdigraphs together with corresponding terms in the solution are appropriately colored.

7. Cayley-Hamilton theorem

(related to Section 7.2 of the book [1])

Slide No. 7.1. *Coefficients of the characteristic polynomial*
(related to pp. 85-87 of [1])

The slide first reproduces Theorem 4.3.1, p. 86, which essentially gives coefficients of the characteristic polynomial of a matrix. In the proof we have to evaluate the determinant $\det(A + \lambda I)$. Here linear subdigraphs of the corresponding digraph are relevant and we consider a linear subdigraph with exactly k loops. Its contribution to the value of the determinant is replaced by 2^k terms of the form λ^p times the sum of weights of linear subdigraphs of the digraph corresponding to a principal submatrix of order $n - p$. It turns out that the coefficient of λ^p is equal to the sum of principal minors of order $n - p$ of the matrix A.

Slide No. 7.2. *A proof of the Cayley-Hamilton theorem*
(related to pp. 148-149 of [1])

The slide first reproduces Definition 5.3.1, p. 103, of a quasi-1-connection. It consists of a walk and a collection of disjoint cycles with a total number of n edges. The crucial step in the proof is the conclusion that the entry in position (i, j) of $p_A(A)$ equals the sum of quantities $(-1)^{c(Q)}w(Q)$ where Q runs over all quasi-1-connections from i to j. Then the set of all quasi-1-connections is partitioned into pairs such that in each pair we have quasi-1-connections with opposite quantities $(-1)^{c(Q)}w(Q)$.

In a quasi-1-connection its walk is either a path which touches one of the cycles or has a self-crossing. On the slide the edges of the walk are colored green and those in cycles are colored red. By switching between green and red color, the slide suggests how quasi-1-connections are paired so that they have opposite quantities $(-1)^{c(Q)}w(Q)$.

8. Jordan canonical form

(related to Section 7.3 of the book [1])

Slide No. 8.1. *Digraphs of Jordan matrices*

(related to p. 160 of [1])

Jordan blocks of order 1, 2 and 3 together with the corresponding digraphs are given. diagonal and off-diagonal entries are distinguished by colorings. The Jordan matrix of Example 7.3.10, p. 161, with the corresponding digraph is also given. Red color is used to specify particular blocks. Algebraic and geometric multiplicities of eigenvalues are given.

9. Non-negative matrices

(related to Chapter 8 of the book [1])

Slide No. 9.1. *Reducibility and connectedness of digraphs*

(related to p. 172 and pp. 79-80 of [1])

By Definition 8.1.1, p. 172, a square matrix A is irreducible if its digraph $D(A)$ is strongly connected. The slide illustrates the situation when the matrix A is reducible. Then by simultaneous permutation of rows and columns (the same permutation is applied in both cases) the matrix can be transformed into the matrix B whose digraph $D(B)$ is presented. The student can see that there is no path connecting a vertex from $D(Z)$ with a vertex from $D(X)$. In addition it is shown that $\det B = \det X \det Z$, what is the content, with another notation, of Theorem 4.2.13.

10. Control theory

(related to Section 10.1 of the book [1])

Slide No. 10.1. *Solving an electrical circuit*

(related to pp. 219-221 of [1])

Example 10.1.1. is reproduced.

Slide No. 10.2. *Finding transfer function of a system*
(related to pp. 222-223 of [1])

The block diagram of the system of control theory of Figure 10.5, p.222, is reproduced in the slide. Equations describing the flow of signals throughout the system are gradually introduced following particular vertices of the block diagram. The corresponding signal flow digraph has identical structure as the block diagram. In practice, the equations need not to be written, and even the signal flow digraph has not to be drawn, since the Mason formula can be applied directly to the block diagram. In this example, due to its simplicity, only paths from the input to the output vertex are relevant for terms of the numerator and only the cycles are relevant for terms of the denominator. Red color is again used to emphasize current subdigraphs and corresponding terms.