4. Inverses

Definition 5.3.1 (p. 103)

Let D be a digraph with vertices $1, 2, \ldots, n$. Let i and j be vertices of D. A *1-connection* of vertex i to vertex j is a spanning subdigraph $D[i \to j]$ of D with the following properties:

1° If $i \neq j$, then

(i) exactly one edge leaves, but no edge enters, vertex i;

(ii) exactly one edge enters, but no edge leaves, vertex j;

(iii) for each vertex $k \neq i, j$, exactly one edge enters, and exactly one edge leaves, vertex k;

Definition 5.3.1 (p. 103)

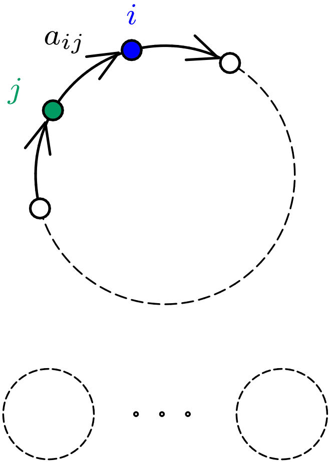
Let D be a digraph with vertices $1, 2, \ldots, n$. Let i and j be vertices of D. A *1-connection* of vertex i to vertex j is a spanning subdigraph $D[i \to j]$ of D with the following properties:

2° If $i = j$, then

(i) no edges enter or leave vertex i;

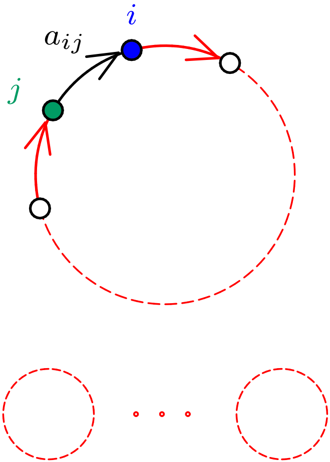
(ii) for each vertex $k \neq i$, exactly one edge enters, and exactly one edge leaves, vertex k;

$\mathbf{1^\circ}$ If $i \neq j$



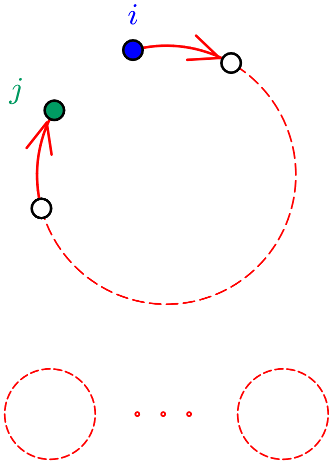
linear subdigraph L

$\mathbf{1}^\circ$ If $i \neq j$



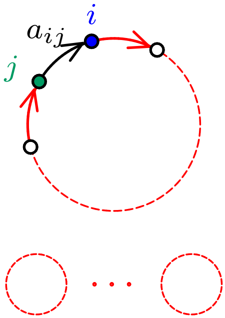
linear subdigraph L

$\mathbf{1}^\circ$ If $i \neq j$



1-connection $D^*[i \to j]$

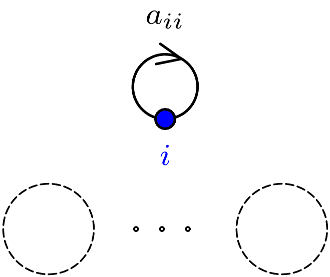
$\mathbf{1}^\circ$ If $i \neq j$



1-connection $D^*[i \to j]$

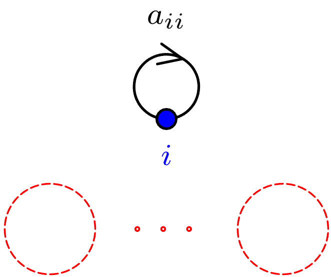
$$c(L) = c(D^*[i \to j]) + 1$$
$$w(L) = a_{ij} \cdot w(D^*[i \to j])$$

$\mathbf{2}^\circ$ If $i = j$



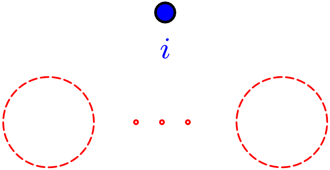
linear subdigraph L

2° If $i = j$



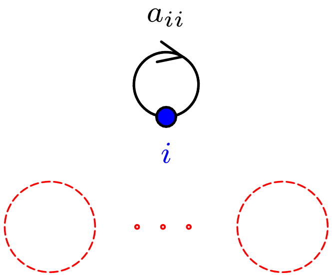
linear subdigraph L

$\mathbf{2}^\circ$ If $i = j$



1-connection $D^*[i \to i]$

2° If $i = j$



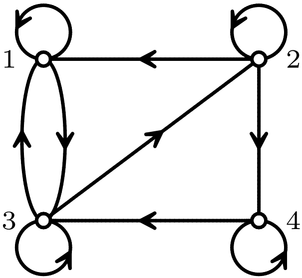
1-connection $D^*[i \to i]$

$$c(L) = c(D^*[i \to i]) + 1$$

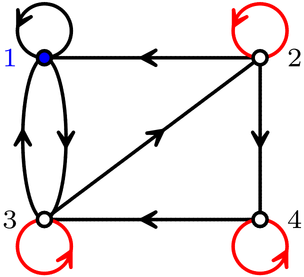
$$w(L) = a_{ii} \cdot w(D^*[i \to i])$$

Example (p. 104)

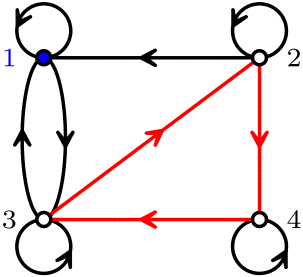
Find all 1-connections of digraph D.



D

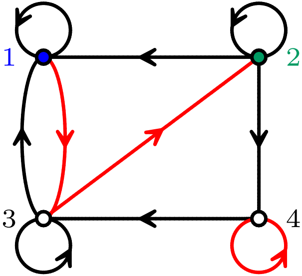


1.



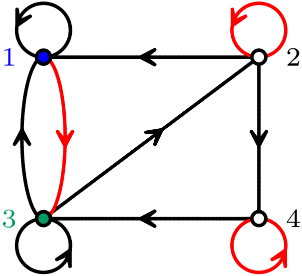
2.

$$D[1 \to 1]$$



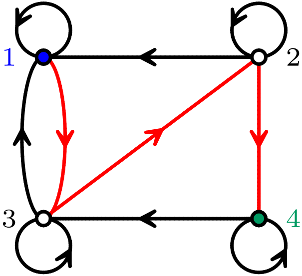
3.

$$D[1 \to 2]$$



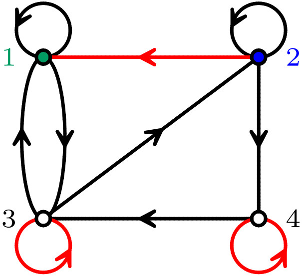
4.

$$D[1 \to 3]$$

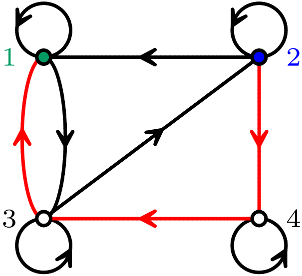


5.

$$D[1 \to 4]$$

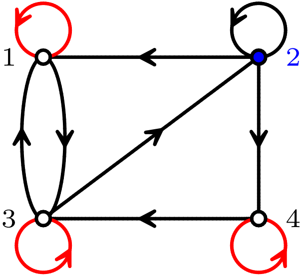


6.

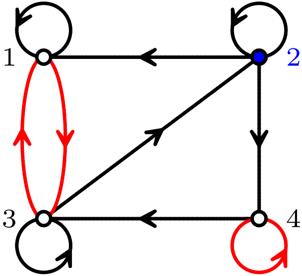


7.

$$D[2 \to 1]$$

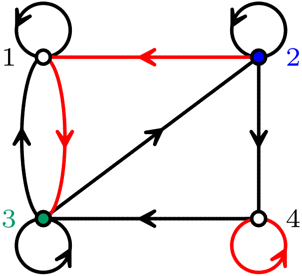


8.

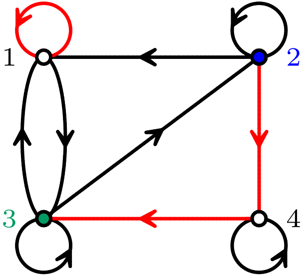


9.

$$D[2 \to 2]$$

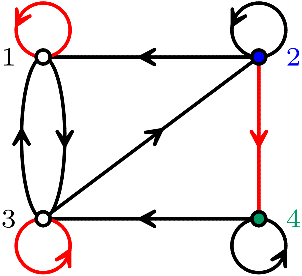


10.

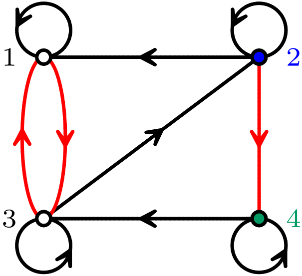


11.

$$D[2 \to 3]$$

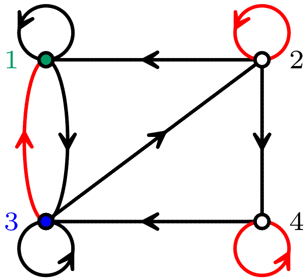


12.

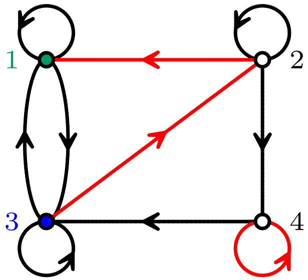


13.

$$D[2 \to 4]$$

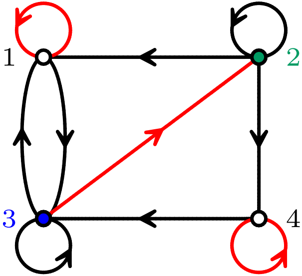


14.



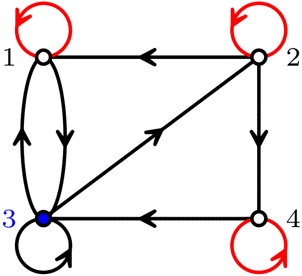
15.

$$D[3 \to 1]$$



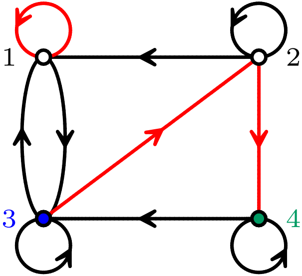
16.

$$D[3 \to 2]$$



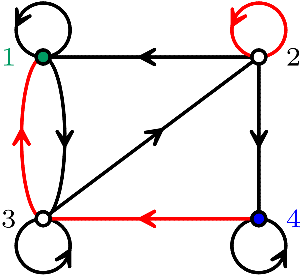
17.

$$D[3 \to 3]$$

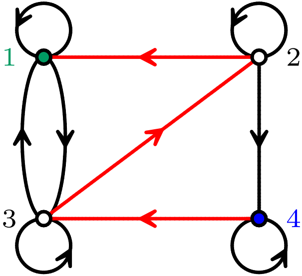


18.

$$D[3 \to 4]$$

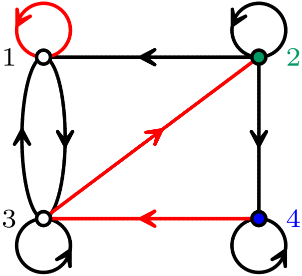


19.



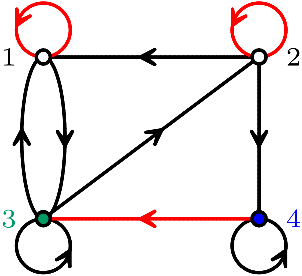
20.

$$D[4 \to 1]$$



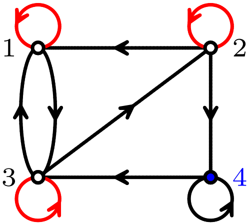
21.

$$D[4 \to 2]$$

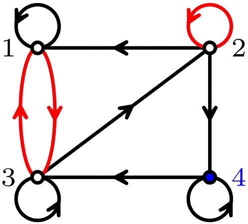


22.

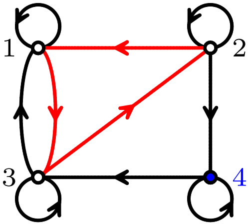
$$D[4 \to 3]$$



23.



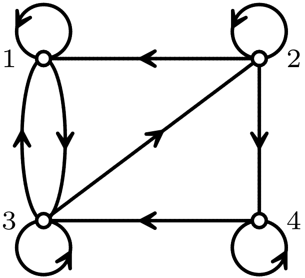
24.



25.

$$D[4 \to 4]$$

There are 25 1-connections of digraph D.



D

Theorem 5.3.2. (p. 105)

Let $A = [a_{ij}]$ be an invertible matrix of order n, and let $A^{-1} = [a'_{ji}]$. Then

$$a_{ji} = \frac{\sum_{D^*[i \to j]} (-1)^{c(D^*[i \to j])+1} w(D^*[i \to j])}{\sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L)},$$

$(1 \leq i, j \leq n)$.

Example 5.3.3. (p. 106)

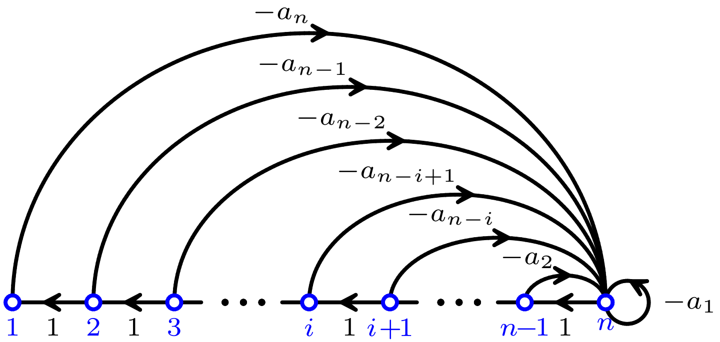
Let

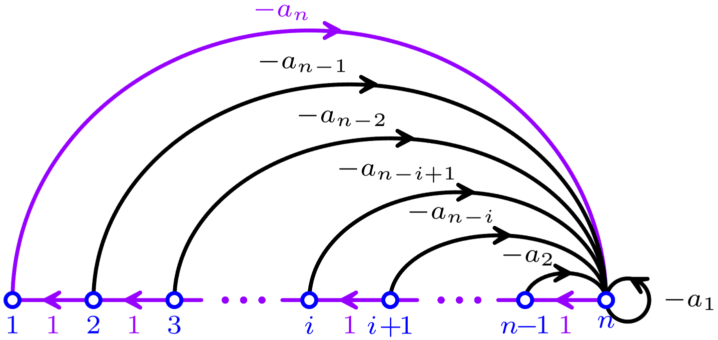
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}.$$

Find A^{-1}.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}$$

Coates' digraph $D^*(A)$

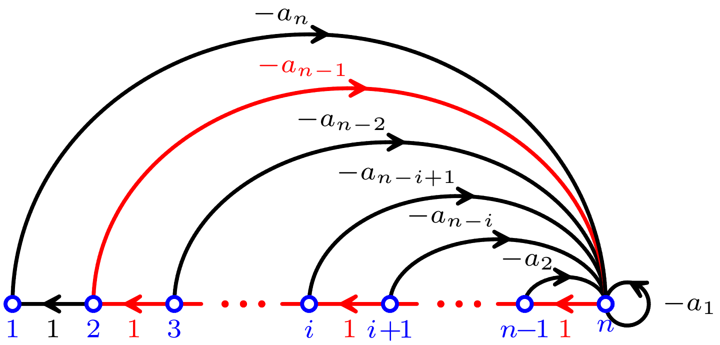




The Digraph $D^*(A)$ has only 1 linear subdigraph with 1 cycle, and we get

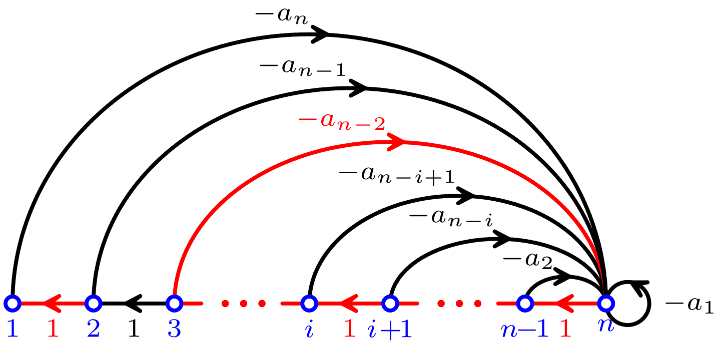
$$\det(A) = (-1)^n(-1)^1(-a_n) = (-1)^n a_n.$$

All nominators in formula in Theorem 5.3.2. are equal to $(-1)^1(-a_n) = a_n$.



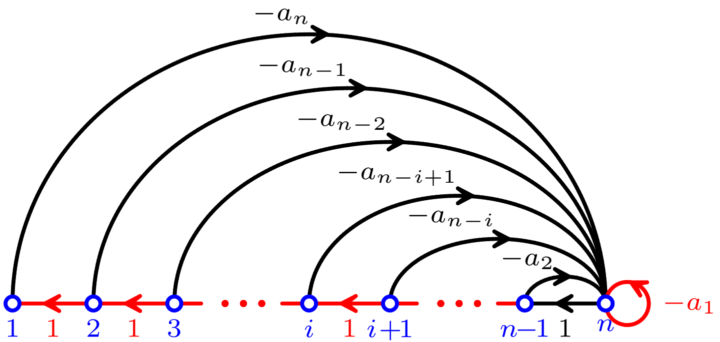
$$1\text{-connection } D^*(A)[1 \to 1] \text{ with } (-1)^{c+1}w = -a_{n-1}$$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



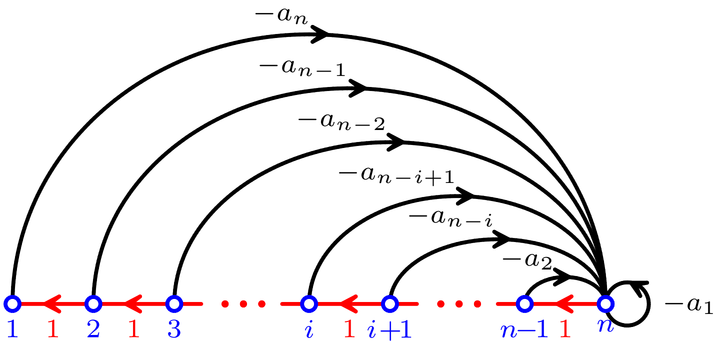
1-connection $D^*(A)[2 \to 1]$ with $(-1)^{c+1}w = -a_{n-2}$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



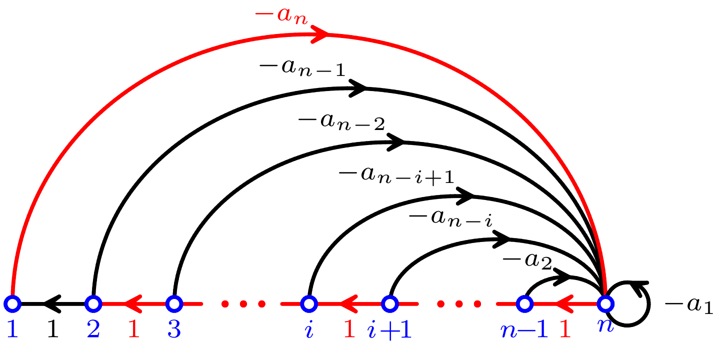
1-connection $D^*(A)[n-1 \to 1]$ with $(-1)^{c+1}w = -a_1$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



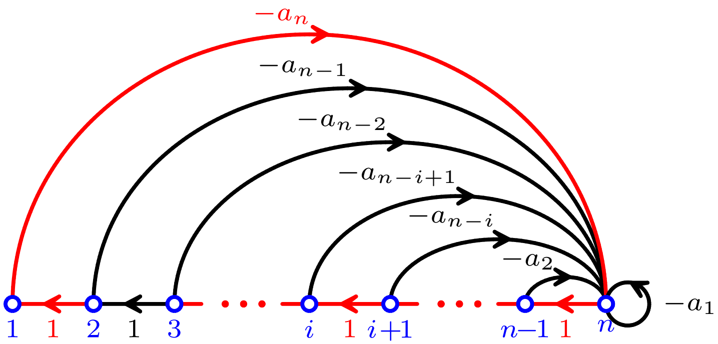
1-connection $D^*(A)[n \to 1]$ with $(-1)^{c+1}w = -1$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



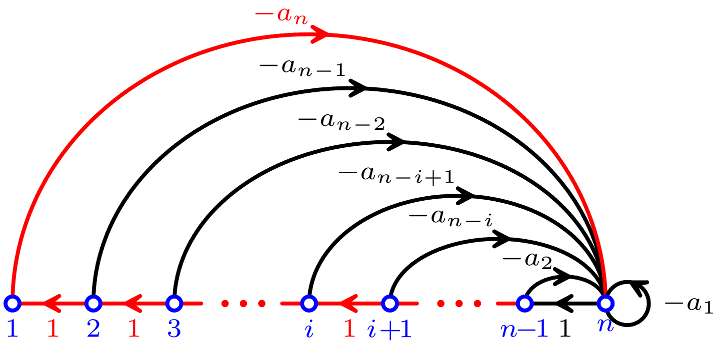
1-connection $D^*(A)[1 \to 2]$ with $(-1)^{c+1}w = a_n$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ \frac{a_n}{a_n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



1-connection $D^*(A)[2 \to 3]$ with $(-1)^{c+1}w = a_n$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{a_n}{a_n} & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



1-connection $D^*(A)[n-1 \to n]$ with $(-1)^{c+1}w = a_n$

$$A^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \frac{-a_{n-3}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{a_n}{a_n} & 0 \end{bmatrix}$$