7. Cayley-Hamilton Theorem

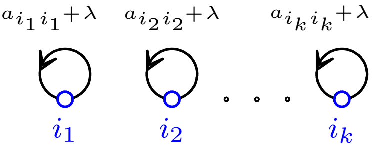
Coefficients of the characteristic polynomial

Theorem 4.3.1 (p. 86)

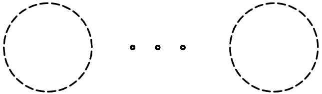
Let A be a matrix of order n. Then

$$\det(A + \lambda I) = \sum_{p=0}^{n} \lambda^p c_{n-p},$$

where c_{n-p} equals the sum of the principal minors of order $n - p$ of A.



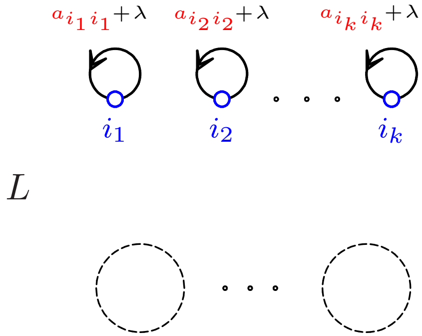
L



L is linear subdigraph with k loops.

There are 2^k terms of the form

$$\lambda^p a_{j_1 j_1} a_{j_2 j_2} \cdots a_{j_{k-p} j_{k-p}}.$$



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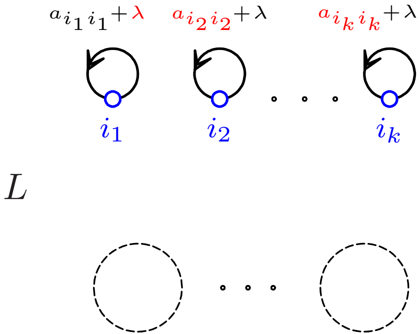

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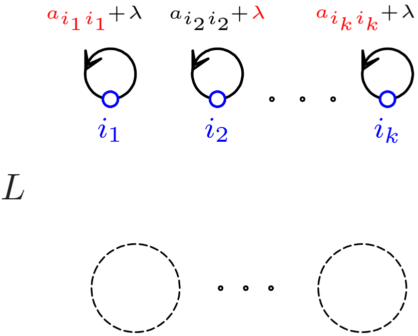

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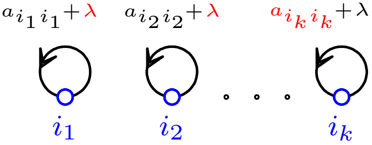

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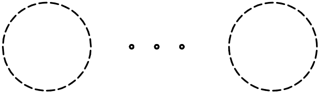

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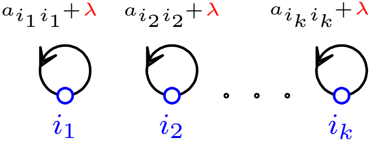

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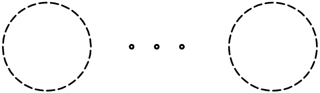

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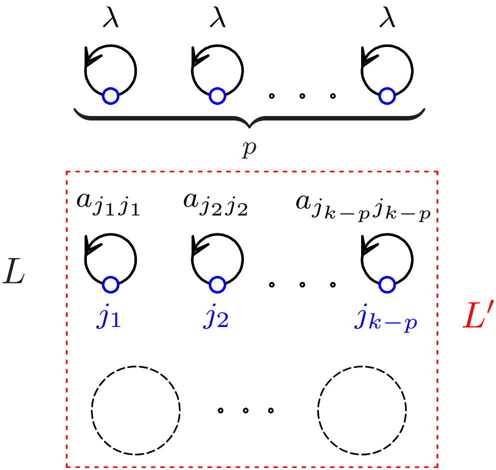
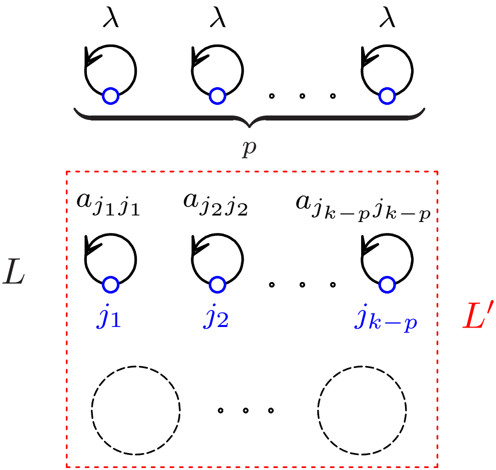


L' is linear subdigraph of digraph $D \backslash$the set of p vertices.



$$w(L) = \sum \lambda^p w(L'),$$

where sum goes over all choices of p vertices.

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Hence,

$$\det(A + \lambda I) = \sum_{p=0}^{n} \lambda^p c_{n-p},$$

where c_{n-p} equals the sum of the principal minors of order $n - p$ of A.

By replacing λ with $-\lambda$ in Theorem 4.3.1 we obtain the following corollary.

Corollary 4.3.2 (p. 86)

Let A be a matrix of order n. Then

$$\det(A - \lambda I) = \sum_{p=0}^{n} (-1)^p \lambda^p c_{n-p},$$

where c_{n-p} equals the sum of the principal minors of order $n-p$ of A.

Definition 5.3.1 (p. 103)

Let D be a digraph with vertices $1, 2, \ldots, n$. Let i and j be vertices of D.

We recall that a 1-connection $D[i \to j]$ of vertex i to vertex j is a spanning subdigraph of D consisting of a path from i to j and a possibly empty collection of pairwise vertex disjoint cycles having no vertex in common with the path.

Definition 5.3.1 (p. 103)

Let D be a digraph with vertices $1, 2, \ldots, n$. Let i and j be vertices of D.

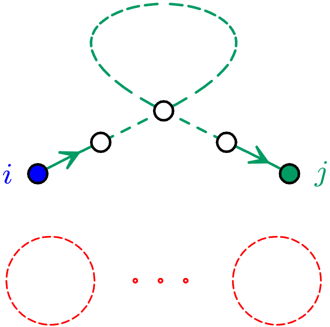
A *quasi-1-connection* $D[i \to j]^*$ of vertex i to vertex j is a spanning subdigraph of D consisting of a walk from i to j of a length at most n and a possibly empty collection of pairwise vertex disjoint cycles, where the walk may intersect the cycles and where the total number of edges in the walk and cycles equals n.

Thus a quasi-1-connection $D[i \to j]^*$ is a pair

$$D[i \to j]^* = (\gamma, C),$$

γ – walk from i to j

C – collection of disjoint cycles.



Theorem 7.2.2 (p. 147–149)

(Cayley–Hamilton theorem)

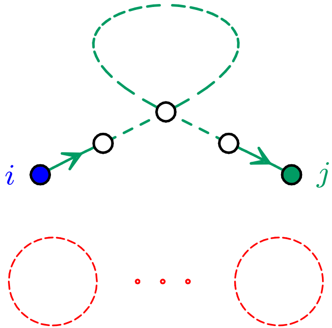
Let $A = [a_{ij}]$ be a matrix of order n and let

$$p_A(\lambda) = \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - \cdots + (-1)^k c_{n-k}\lambda^k + \cdots + (-1)^n c_n$$

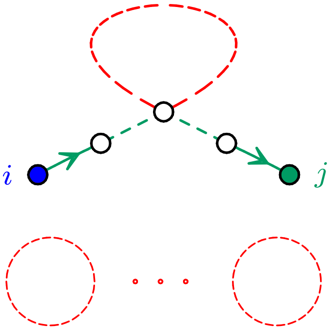
be the characteristic polynomial of A.

Then $p(A) = O$, that is,

$$A^n - c_1A^{n-1} + c_2A^{n-2} - \cdots + (-1)^k c_{n-k}A^k + \cdots + (-1)^n c_n I_n = O.$$



quasi-1-connection $D[i \to j]^*$



quasi-1-connection $D[i \to j]^*$

The coefficient c_{n-k} of λ^k in the characteristic polynomial equals the sum of all the determinants of the principal submatrices of A of order $n-k$:

$$c_{n-k} = (-1)^{n-k} \sum_{L} (-1)^{c(L)} w(L)$$

(summation extends over all linear subdigraphs of the $D^*(A)$ having $n-k$ vertices).

The entry in position (i, j) of A^k equals the sum of the weights of all walks of length k from vertex i to vertex j.

The entry in position (i, j) of $(-1)^{n-k} c_{n-k} A^k$ equals

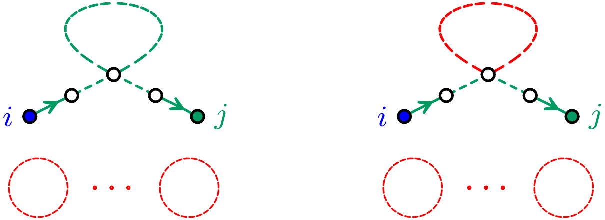
$$\sum_{D[i \to j]_k^*} (-1)^{c(D[i \to j]_k^*)} w(D[i \to j]_k^*)$$

(summation extends over all quasi-1-connections whose walk γ from i to j has length k).

The entry in position (i, j) of $p_A(A)$ equals

$$\sum_{D[i \to j]^*} (-1)^{c(D[i \to j]^*)} w(D[i \to j]^*)$$

(summation extends over all quasi-1-connections from i to j).



$$w(D[i \to j]^{*\prime}) = w(D[i \to j]^{*\prime\prime})$$

$$c(D[i \to j]^{*\prime}) - 1 = c(D[i \to j]^{*\prime\prime})$$

$$-(-1)^{c(D[i \to j]^{*\prime})} = (-1)^{c(D[i \to j]^{*\prime\prime})}$$

$$(-1)^{c(D[i \to j]^{*\prime})} w(D[i \to j]^{*\prime}) + (-1)^{c(D[i \to j]^{*\prime\prime})} w(D[i \to j]^{*\prime\prime}) = 0$$

$$\sum_{D[i \to j]^*} (-1)^{c(D[i \to j]^*)} w(D[i \to j]^*) = 0$$

$$P_A(A) = 0.$$