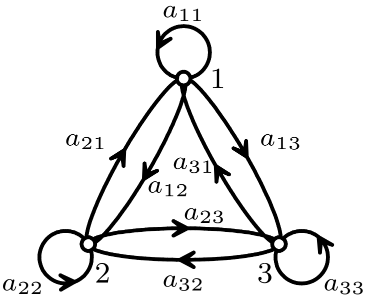
2. Powers of Matrices

matrix A

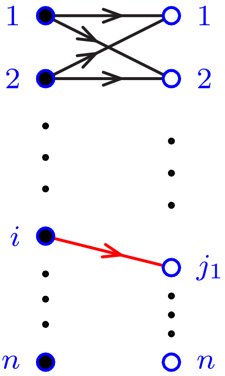
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

weighted digraph $D(A)$

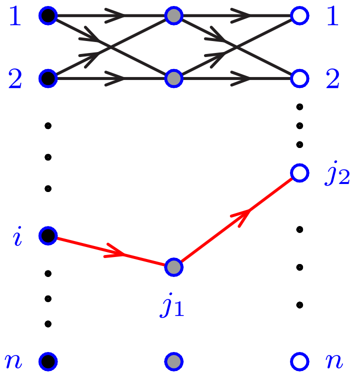


Theorem 3.1.2 (pp. 51-52)

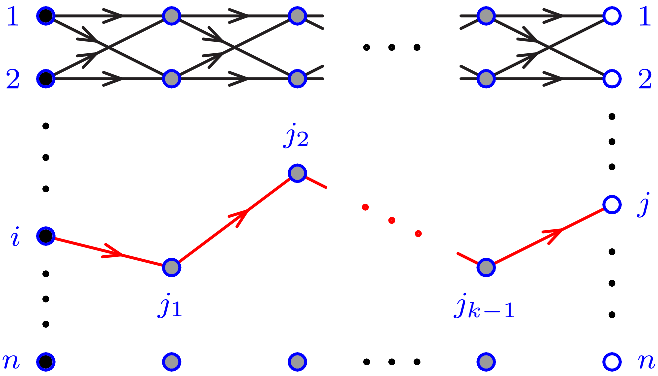
Let $A = [a_{ij}]$ be a matrix of order n. For each positive integer k, the entry $a_{ij}^{(k)}$ of A^k in the ith row and jth column equals the sum of the weights of all walks in $D(A)$ of length k from vertex i to vertex j.



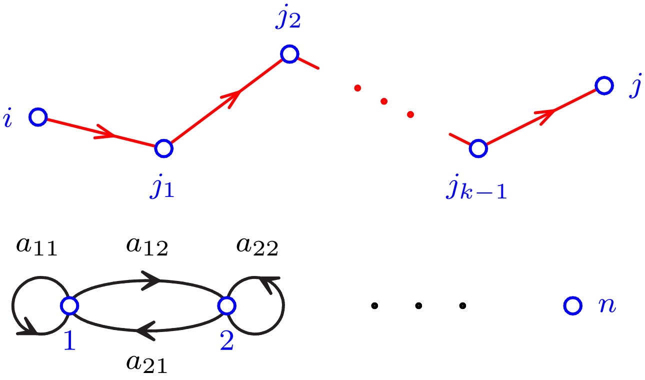
$G(A)$



$$G(A) \quad * \quad G(A)$$



$$G(A) \quad * \quad G(A) \quad * \quad \cdots \quad * \quad G(A)$$



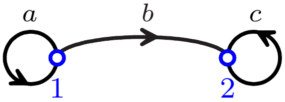
$D(A)$

Example 3.1.3 (p. 52)

Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Use the digraph $D(A)$ to compute A^k.

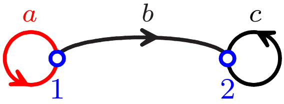
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$D(A)$



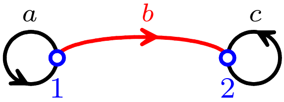
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$D(A)$



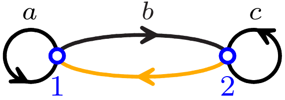
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$D(A)$



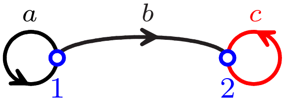
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$D(A)$



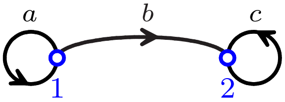
$$A = \begin{bmatrix} a & b \\ 0 & \color{red}{c} \end{bmatrix}$$

$D(A)$



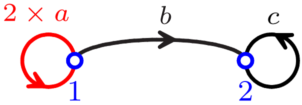
$$A^2 = \begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix}$$

$D(A)$



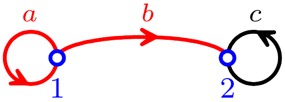
$$A^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix}$$

$D(A)$



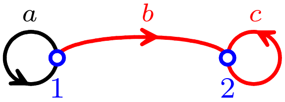
$$A^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix}$$

$D(A)$



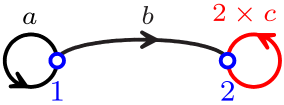
$$A^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix}$$

$D(A)$



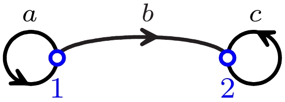
$$A^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix}$$

$D(A)$



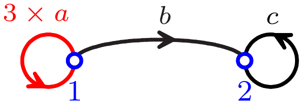
$$A^3 = \begin{bmatrix} a^3 & a^2b + abc + bc^2 \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



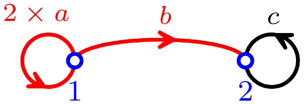
$$A^3 = \begin{bmatrix} a^3 & a^2b + abc + bc^2 \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



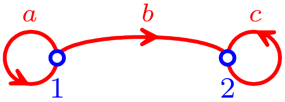
$$A^3 = \begin{bmatrix} a^3 & a^2b + abc + bc^2 \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



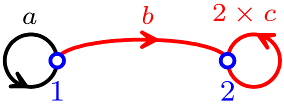
$$A^3 = \begin{bmatrix} a^3 & a^2b + {\color{red}abc} + bc^2 \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



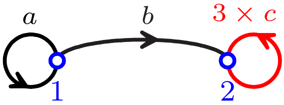
$$A^3 = \begin{bmatrix} a^3 & a^2b + abc + \color{red}{bc^2} \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



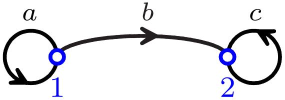
$$A^3 = \begin{bmatrix} a^3 & a^2b + abc + bc^2 \\ 0 & c^3 \end{bmatrix}$$

$D(A)$



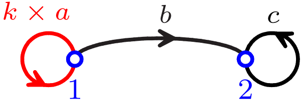
$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



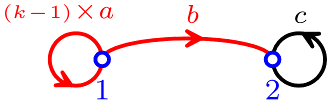
$$A^k = \begin{bmatrix} \color{red}{a^k} & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



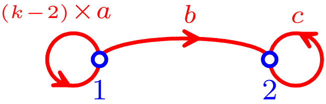
$$A^k = \begin{bmatrix} a^k & {\color{red}a^{k-1}b} + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



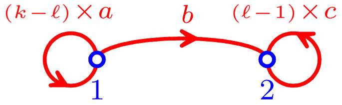
$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



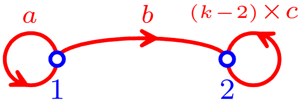
$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



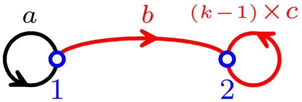
$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + {\color{red} abc^{k-2}} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



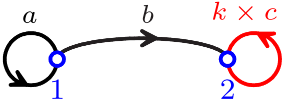
$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



$$A^k = \begin{bmatrix} a^k & a^{k-1}b + a^{k-2}bc + \cdots + abc^{k-2} + bc^{k-1} \\ 0 & c^k \end{bmatrix}$$

$D(A)$



Theorem 3.1.4 (p. 53)

Let A be a square matrix of order n. Then A is nilpotent if the corresponding digraph $D(A)$ does not have any cycles; in this case, $A^n = O$.

Exercise 3.4.2 (p. 60)

Hint (p. 246)

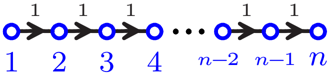
Let $A = [a_{ij}]$ be the matrix of order n defined by

$$a_{ij} = \delta_{i,j-1} = \begin{cases} 1, & i = j-1 \\ 0, & i \neq j-1 \end{cases} \qquad (i, j = 1, 2, \ldots, n).$$

For k a positive integer use the digraph $D(A)$ to compute A^k.

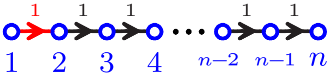
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



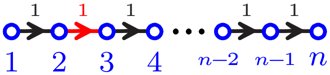
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



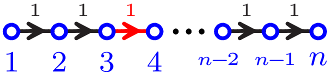
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



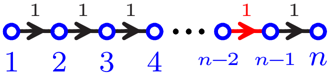
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \color{red}{1} & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



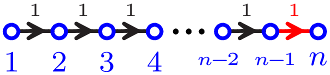
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



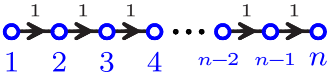
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



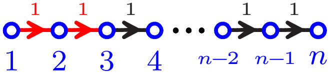
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



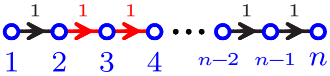
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



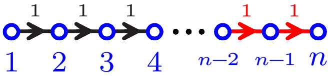
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \color{red}{1} & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



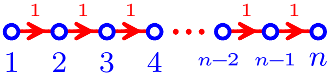
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



$$A^{n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \color{red}{1} \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$D(A)$



$$A^n = O$$

by Theorem 3.1.4 (because $D(A)$ does not have any cycles)

or

by Theorem 3.1.2 (because there are no walks of length n in $D(A)$).

Solution:

$$A^k = \begin{cases} O, & k \geq n \\ B, & k < n \end{cases},$$

where matrix $B = [b_{ij}]$ is defined by

$$b_{ij} = \delta_{i,j-k} = \begin{cases} 1, & i = j - k \\ 0, & i \neq j - k \end{cases} \qquad (i, j = 1, 2, \ldots, n).$$