3. Determinants

empty scheme

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

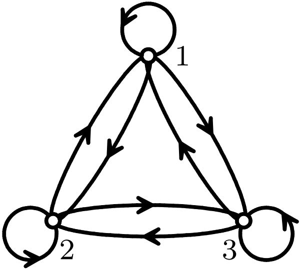
$$\text{empty scheme} \qquad\qquad\qquad \text{matrix}$$

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \xrightarrow{\;f\;} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

empty scheme | matrix

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \xrightarrow{\;f\;} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

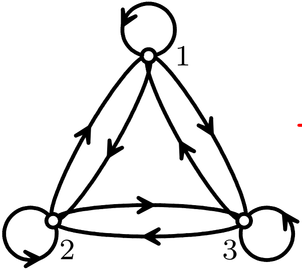
digraph



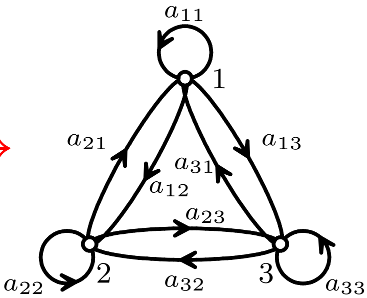
empty scheme $\xrightarrow{f}$ matrix

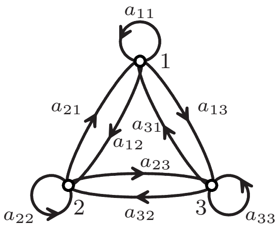
$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \xrightarrow{f} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

digraph $\xrightarrow{f}$ weighted digraph

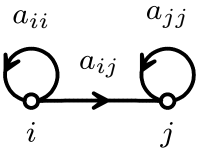


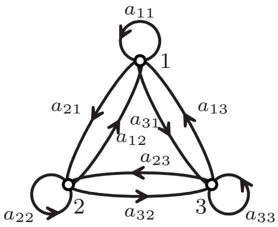
$\xrightarrow{f}$



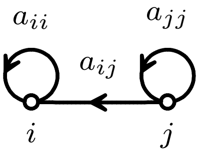

a_{11}
1
a_{21}
a_{13}
a_{31}
a_{12}
a_{23}
a_{22}
2
a_{32}
3
a_{33}

$$D(A)$$


a_{ii}
a_{jj}
a_{ij}
i
j


a_{11}
1
a_{21}
a_{13}
a_{31}
a_{12}
a_{23}
a_{22}
2
a_{32}
3
a_{33}

$$D^*(A)$$


a_{ii}
a_{jj}
a_{ij}
i
j

Definition 4.1.1 (p. 65)

Let $A = [a_{ij}]$ be a square matrix of order n. The *determinant* of A is the number $\det A$ defined by the sum

$$\det A = (-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L)$$

where the summation extends over all linear subdigraphs L of the digraph $D^*(A)$.

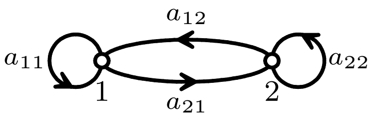
Example 4.1.2 (p. 66)

Calculate the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

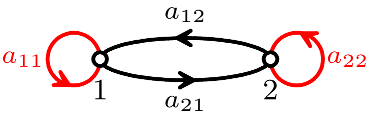
$$\begin{aligned} \det A &= (-1)^{2+2} a_{11}a_{22} + (-1)^{2+1} a_{12}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$



$D^*(A)$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{aligned} \det A &= (-1)^{2+2} a_{11} a_{22} + (-1)^{2+1} a_{12} a_{21} \\ &= a_{11} a_{22} - a_{12} a_{21} \end{aligned}$$

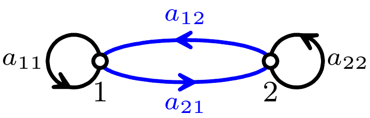


$D^*(A)$

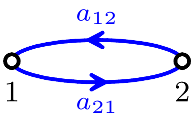
L_1

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad |A| = \begin{vmatrix} a_{11} & {\color{blue}a_{12}} \\ {\color{blue}a_{21}} & a_{22} \end{vmatrix}$$

$$\begin{aligned} \det A &= (-1)^{2+2} a_{11} a_{22} + {\color{blue}(-1)^{2+1} a_{12} a_{21}} \\ &= a_{11} a_{22} {\color{blue}- a_{12} a_{21}} \end{aligned}$$



$D^*(A)$

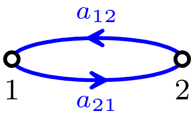


L_2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{aligned} \det A &= (-1)^{2+2} a_{11} a_{22} + (-1)^{2+1} a_{12} a_{21} \\ &= a_{11} a_{22} - a_{12} a_{21} \end{aligned}$$

L_1



L_2

Example 4.1.2 (p. 66)

Calculate the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

using

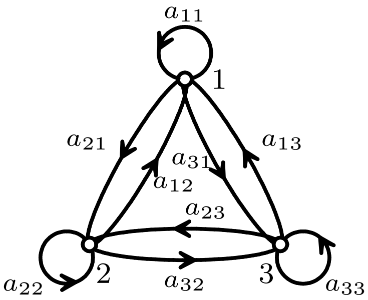
$$\det A = (-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L).$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

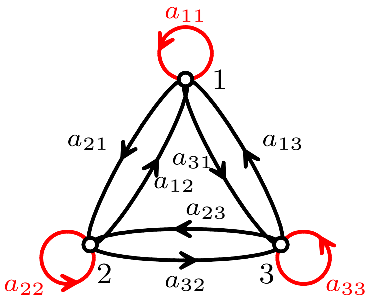
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23} + \\ & (-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32} + \\ & (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21} \end{aligned}$$

$$\begin{aligned} \det A \quad = & \quad\;\; a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{21}a_{32}a_{13} \\ = & \; -a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21} \end{aligned}$$

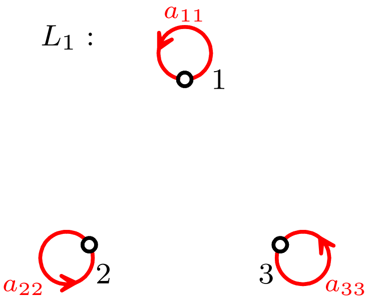
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



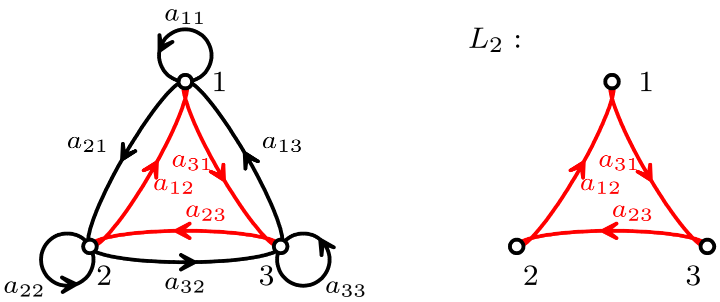
$$\det A = (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}$$



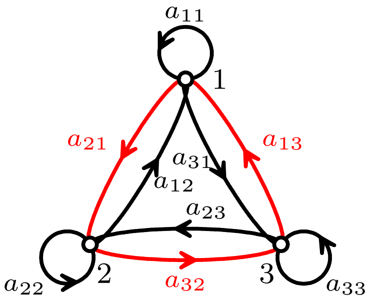
L_1 :



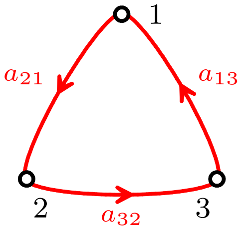
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



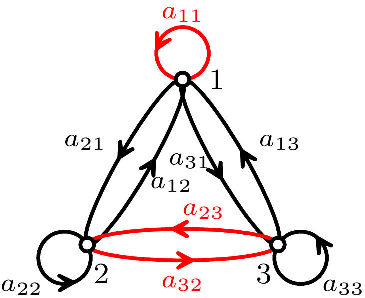
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



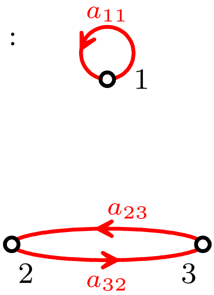
L_3 :



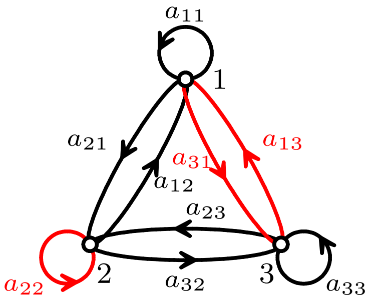
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + {\color{red}(-1)^{3+2}a_{11}a_{23}a_{32}} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



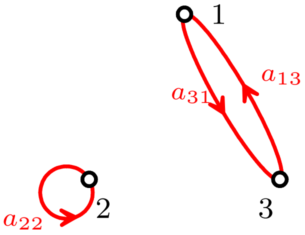
L_4 :



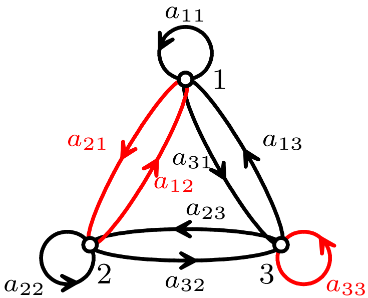
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



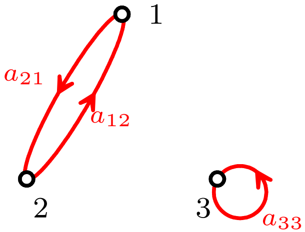
L_5 :



$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + {\color{red}(-1)^{3+2}a_{33}a_{12}a_{21}}\end{aligned}$$



L_6 :



Theorem 4.2.1 (p. 72)

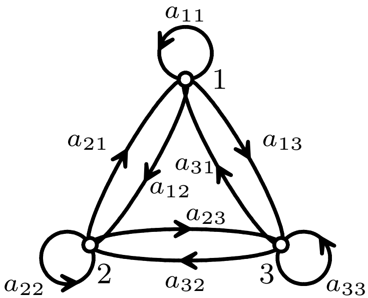
$$\det A^T = \det A.$$

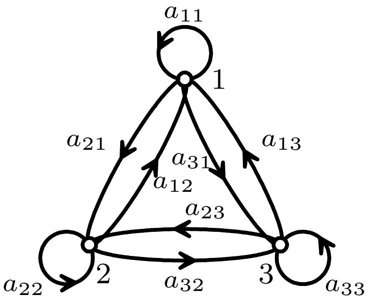
Theorem 4.2.1 (p. 72)

$$\det A^T = \det A.$$

Theorem 4.2.1 implies that every statement that holds for the rows of a matrix also holds for the columns.

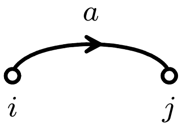
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$





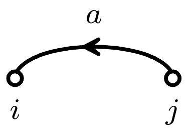
Effect of transposition

entry a at position (i, j)



Effect of transposition

entry a at position (j, i)

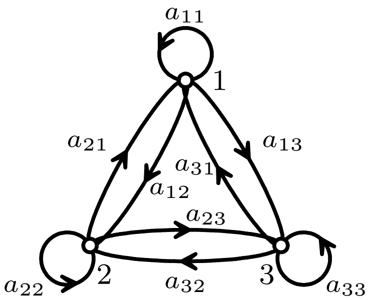


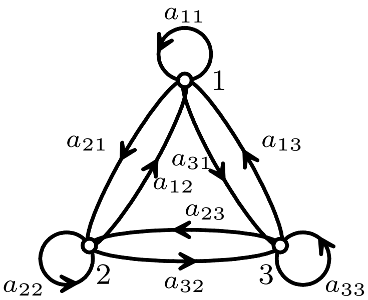
$$\det A = \qquad\qquad \det A^T =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



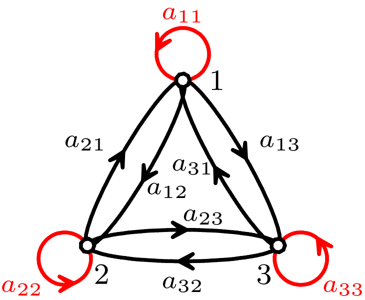


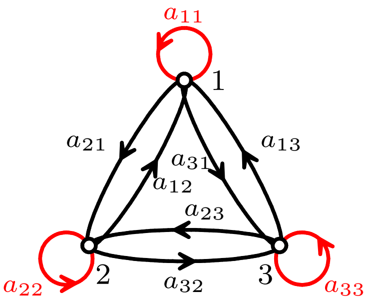
$$\det A = \qquad\qquad \det A^T =$$

$$= {\color{red}(-1)^{3+3} a_{11} a_{22} a_{33}} + (-1)^{3+1} a_{12} a_{31} a_{23}$$

$$+(-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32}$$

$$+(-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}$$



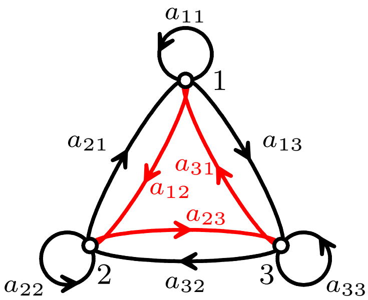


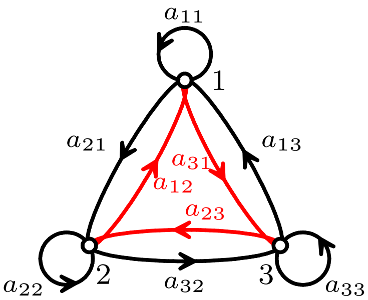
$$\det A = \qquad\qquad \det A^T =$$

$$= (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23}$$

$$+ (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32}$$

$$+ (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}$$



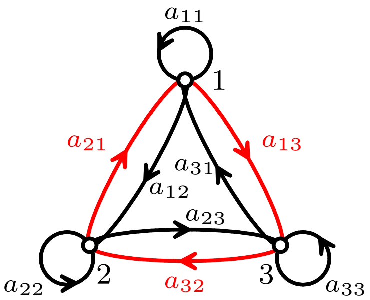


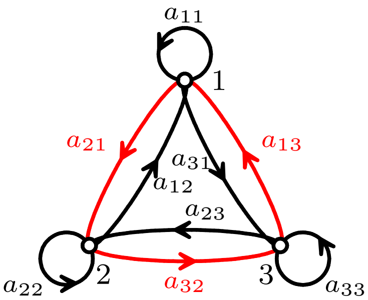
$$\det A = \qquad\qquad \det A^T =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



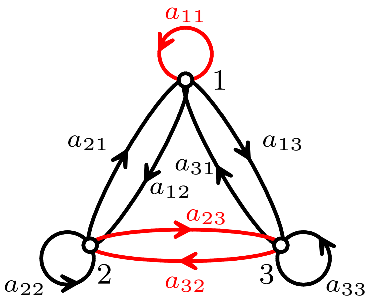


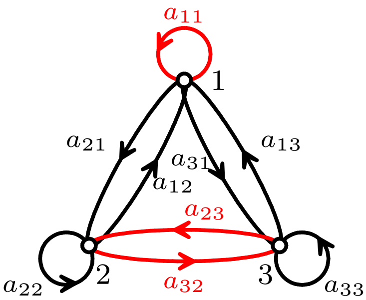
$$\det A = \qquad\qquad \det A^T =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



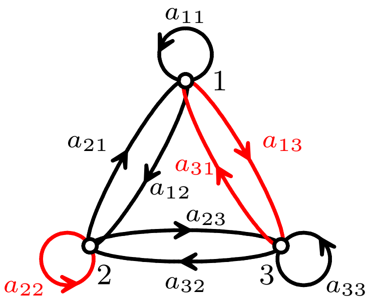


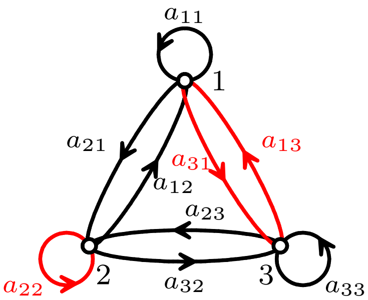
$$\det A = \qquad\qquad \det A^T =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



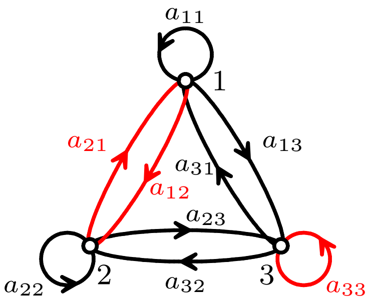


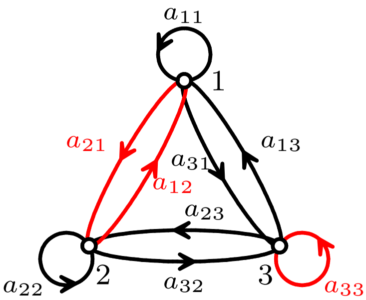
$$\det A = \qquad \det A^T =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + \color{red}{(-1)^{3+2} a_{33}a_{12}a_{21}}$$

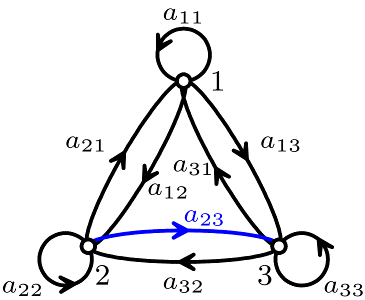




The effect of a zero entry

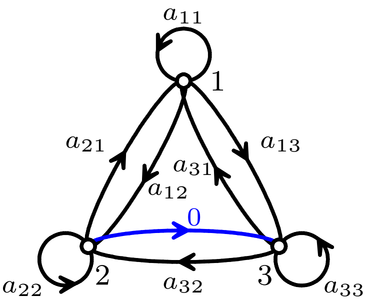
(related to remark on p. 69)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{blue}{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



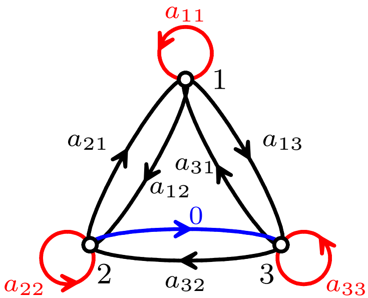
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}\color{blue}{a_{23}} \color{black}{} + \\ & (-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}\color{blue}{a_{23}}\color{black}{a_{32}} + \\ & (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21} \end{aligned}$$

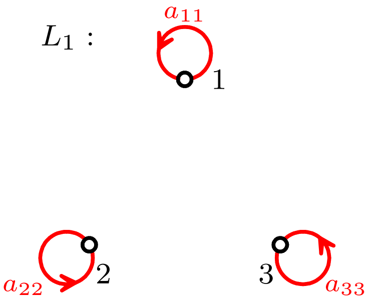
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{blue}{0} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



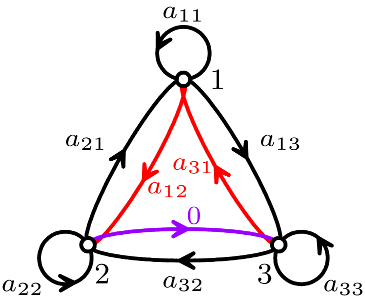
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} \cdot \color{blue}{0} \color{black} + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} \cdot \color{blue}{0} \color{black} \cdot a_{32} + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$

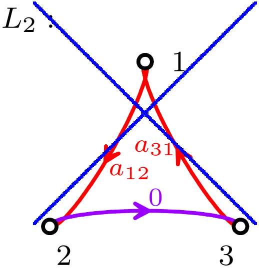
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



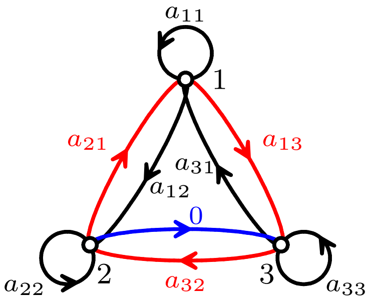


$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \cancel{(-1)^{3+1}a_{12}a_{31}\cdot 0} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$

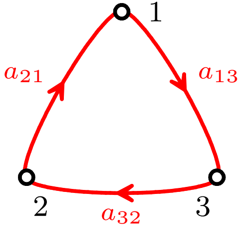




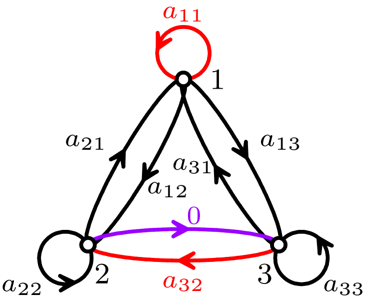
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\ & {\color{red}(-1)^{3+1}a_{21}a_{32}a_{13}} + \qquad 0 \qquad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$

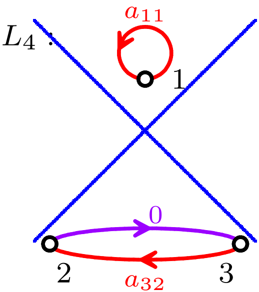


L_3 :

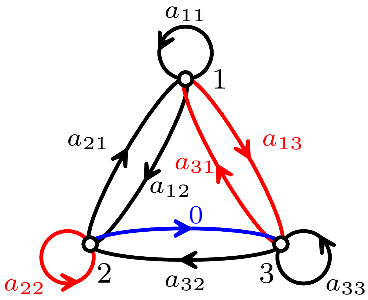


$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21}a_{32}a_{13} + \cancel{(-1)^{3+2} a_{11}\cdot 0\cdot a_{32}} + \\ & (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}\end{aligned}$$

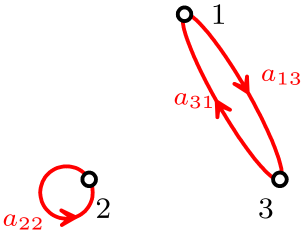




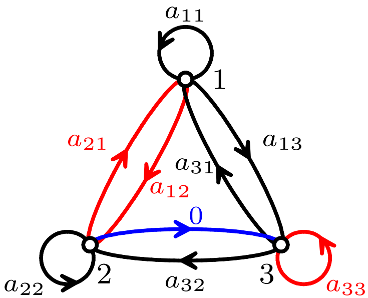
$$\det A = (-1)^{3+3} a_{11}a_{22}a_{33} + \quad 0 \quad + \\ (-1)^{3+1} a_{21}a_{32}a_{13} + \quad 0 \quad + \\ \color{red}{(-1)^{3+2} a_{22}a_{13}a_{31}} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



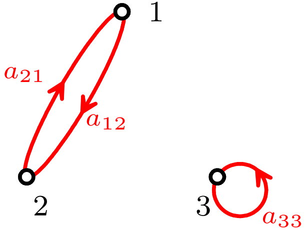
L_5 :



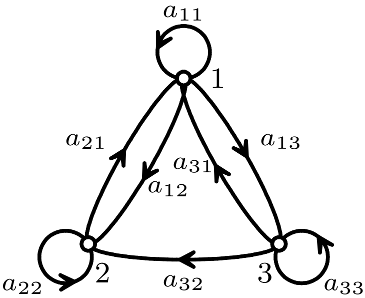
$$\begin{aligned}\det A \;=\; & (-1)^{3+3}a_{11}a_{22}a_{33} + \quad 0 \quad + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + \quad 0 \quad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + {\color{red}(-1)^{3+2}a_{33}a_{12}a_{21}}\end{aligned}$$



L_6 :

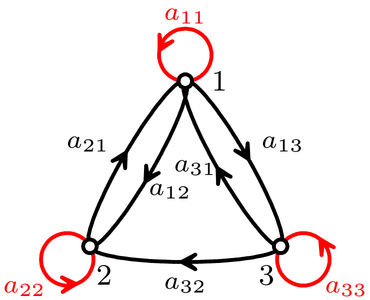


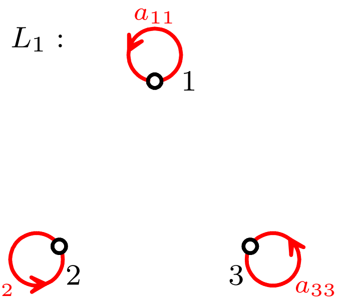
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{blue}{0} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



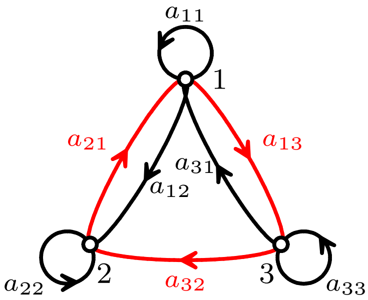
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad \color{blue}{0} \qquad + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad \color{blue}{0} \qquad + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$

$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}\end{aligned}$$

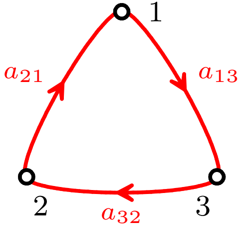




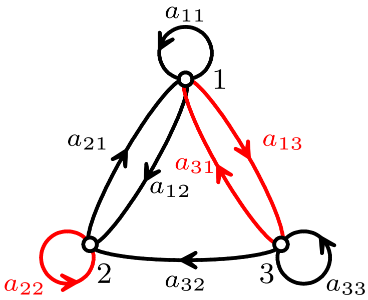
$$\begin{aligned}\det A = \; & (-1)^{3+3}a_{11}a_{22}a_{33} + \quad 0 \quad + \\ & {\color{red}(-1)^{3+1}a_{21}a_{32}a_{13}} + \quad 0 \quad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



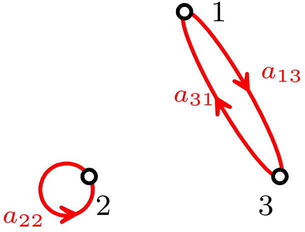
L_3 :



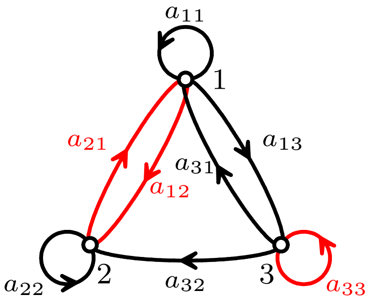
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\ & {\color{red}(-1)^{3+2} a_{22} a_{13} a_{31}} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$



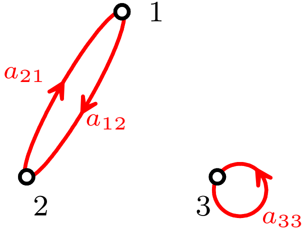
L_5 :



$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2} a_{22}a_{13}a_{31} + {\color{red}(-1)^{3+2} a_{33}a_{12}a_{21}}\end{aligned}$$



L_6 :



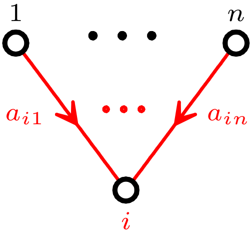
Theorem 4.2.2 (p. 72)

Let each element of some row (row i) of the matrix A be multiplied by c, resulting in matrix B. Then

$$\det B = c \cdot \det A.$$

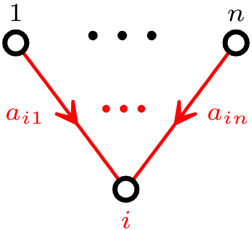
$$A = \begin{array}{cl} \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} & \begin{array}{l} \\ i\text{-th row} \\ \\ \end{array} \end{array}$$

$$A = \begin{array}{cl} \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} & i\text{-th row} \end{array}$$



$D^*(A)$

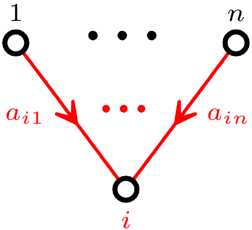
$$A = \begin{array}{cc} \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} & i\text{-th row} \end{array}$$



$D^*(A)$

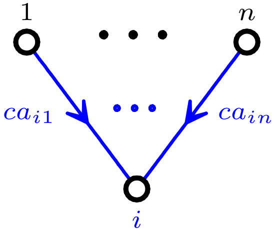
$$B = \begin{array}{cc} \begin{bmatrix} & \text{unchanged} & & \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ & \text{unchanged} & & \end{bmatrix} & i\text{-th row} \end{array}$$

$$A = \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \quad i\text{-th row}$$



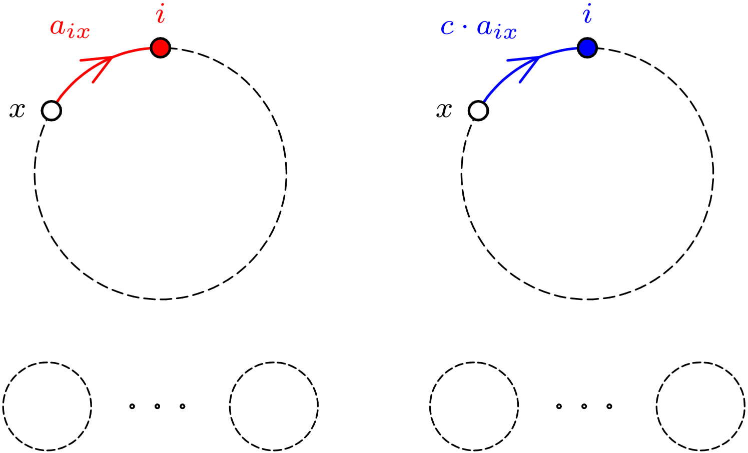
$D^*(A)$

$$B = \begin{bmatrix} & \text{unchanged} & & \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ & \text{unchanged} & & \end{bmatrix} \quad i\text{-th row}$$

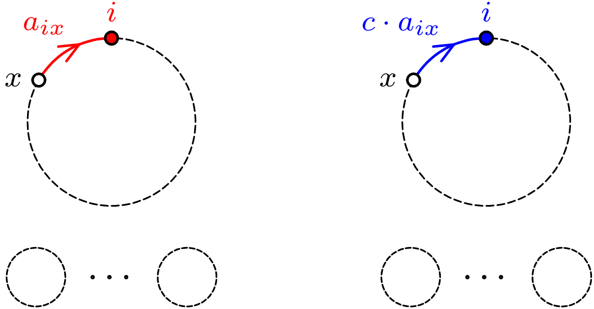


$D^*(B)$

weights of other edges unchanged



$L_A \in \mathcal{L}(A)$ $\qquad$ $L_B \in \mathcal{L}(B)$



$$c \cdot w(L_A) = w(L_B)$$

$$c(L_A) = c(L_B)$$

$$(-1)^{c(L_A)} = (-1)^{c(L_B)}$$

$$(-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} c \cdot w(L) = (-1)^n \sum_{L \in \mathcal{L}(B)} (-1)^{c(L)} w(L)$$

$$c \cdot \det A = \det B$$

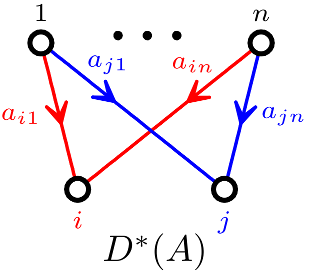
Theorem 4.2.3 (p. 73)

Let two rows (rows i and j) of the matrix A be interchanged, resulting in matrix B. Then

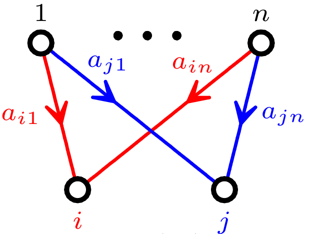
$$\det B = -\det A.$$

$$A = \begin{array}{cc} \begin{bmatrix} \color{red}{a_{i1}} & \color{red}{a_{i2}} & \color{red}{\cdots} & \color{red}{a_{in}} \\ \color{blue}{a_{j1}} & \color{blue}{a_{j2}} & \color{blue}{\cdots} & \color{blue}{a_{jn}} \end{bmatrix} & \begin{array}{l} i\text{-th row} \\ j\text{-th row} \end{array} \end{array}$$

$$A = \begin{array}{c} \left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{array}\right] \end{array} \begin{array}{l} i\text{-th row} \\ j\text{-th row} \end{array}$$



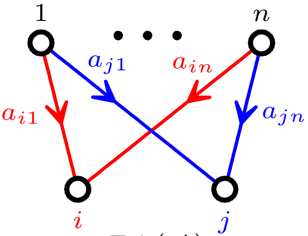
$$A = \begin{array}{cl} \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} & \begin{array}{l} i\text{-th row} \\ j\text{-th row} \end{array} \end{array}$$



$D^*(A)$

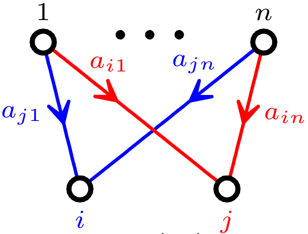
$$B = \begin{array}{cl} \begin{bmatrix} & \text{unchanged} & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \text{unchanged} & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \text{unchanged} & & \end{bmatrix} & \begin{array}{l} \\ i\text{-th row} \\ \\ j\text{-th row} \\ \\ \end{array} \end{array}$$

$$A = \begin{array}{cl} \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} & \begin{array}{l} i\text{-th row} \\ j\text{-th row} \end{array} \end{array}$$



$D^*(A)$

$$B = \begin{array}{cl} \begin{bmatrix} & \text{unchanged} & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \text{unchanged} & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \text{unchanged} & & \end{bmatrix} & \begin{array}{l} \\ i\text{-th row} \\ \\ j\text{-th row} \\ \\ \end{array} \end{array}$$

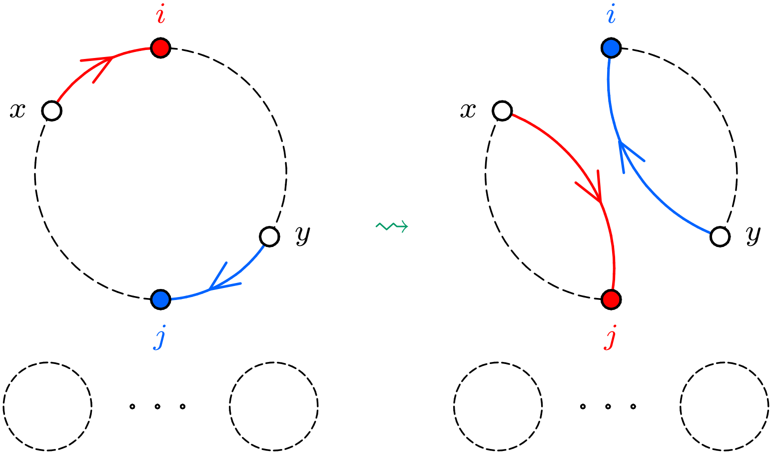


$D^*(B)$

weights of other edges unchanged

i and j belong to the same cycle

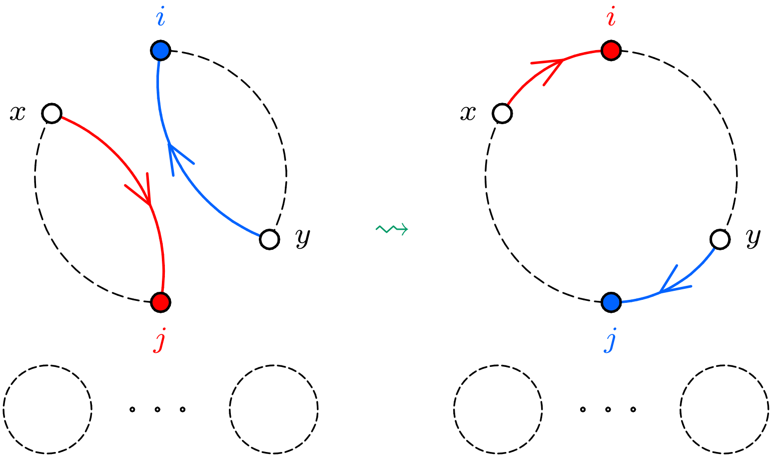
i and j are in different cycles



$L_A \in \mathcal{L}(A)$ $\rightsquigarrow$ $L_B \in \mathcal{L}(B)$

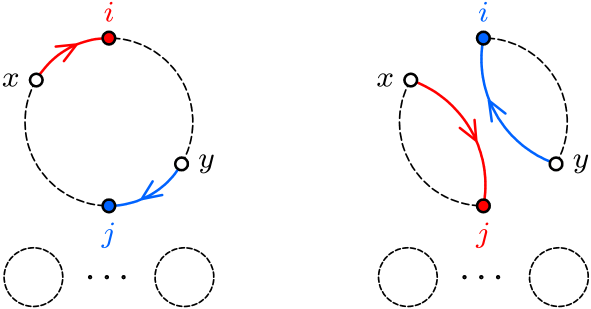
i and j are in different cycles

i and j belong to the same cycle



$L_A \in \mathcal{L}(A)$

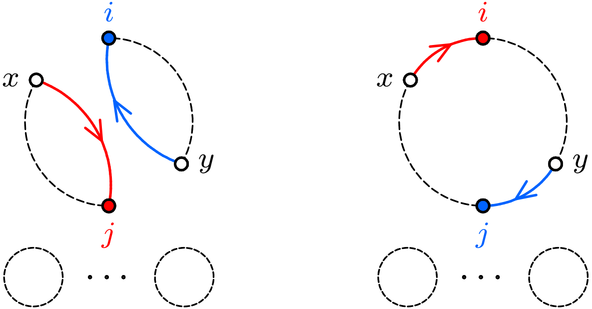
$L_B \in \mathcal{L}(B)$



$$w(L_A) = w(L_B)$$

$$c(L_A) + 1 = c(L_B)$$

$$-(-1)^{c(L_A)} = (-1)^{c(L_B)}$$



$$w(L_A) = w(L_B)$$

$$c(L_A) - 1 = c(L_B)$$

$$-(-1)^{c(L_A)} = (-1)^{c(L_B)}$$

$$-(-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L) = (-1)^n \sum_{L \in \mathcal{L}(B)} (-1)^{c(L)} w(L)$$

$$-\det A = \det B$$

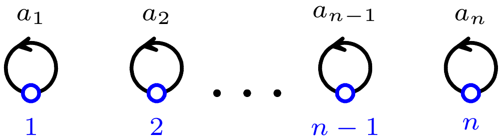
Example 4.1.3 (p. 69)

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Calculate the determinant $\det A$.

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

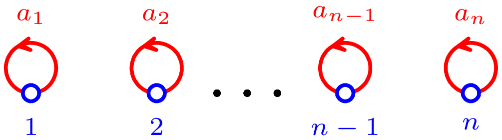
digraph $D^*(A)$



$$\begin{aligned} \det A &= {\color{blue}(-1)^n}\Big((-1)^n a_1 a_2 \cdots a_n\Big) \\ &= a_1 a_2 \cdots a_n \end{aligned}$$

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^n\left((-1)^n a_1 a_2 \cdots a_n\right) \\ &= a_1 a_2 \cdots a_n \end{aligned}$$

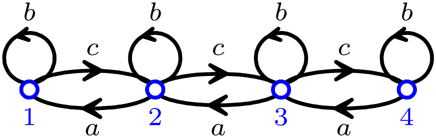
Example 4.1.4 (p. 70)

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

Calculate the determinant $\det A$.

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

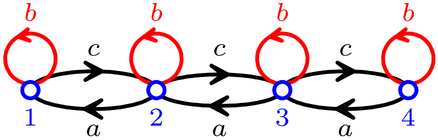
digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

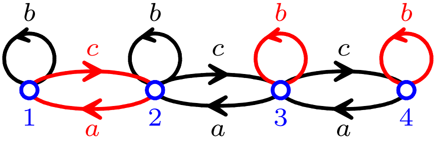
digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

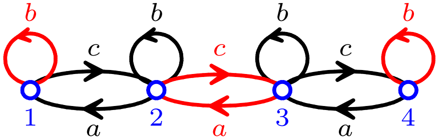
digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

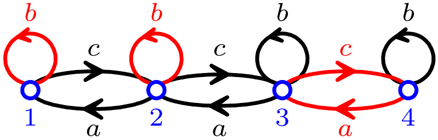
digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

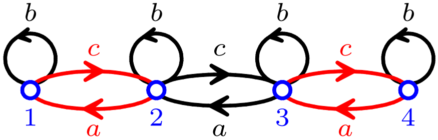
digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

$$A = \begin{bmatrix} b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \\ 0 & 0 & c & b \end{bmatrix}$$

digraph $D^*(A)$



$$\begin{aligned} \det A &= (-1)^4\Big((-1)^4 b^4 + 3(-1)^3 ab^2c + (-1)^2 a^2c^2\Big) \\ &= b^4 - ab^2c - ab^2c - ab^2c + a^2c^2 \\ &= b^4 - 3ab^2c + a^2c^2 \end{aligned}$$

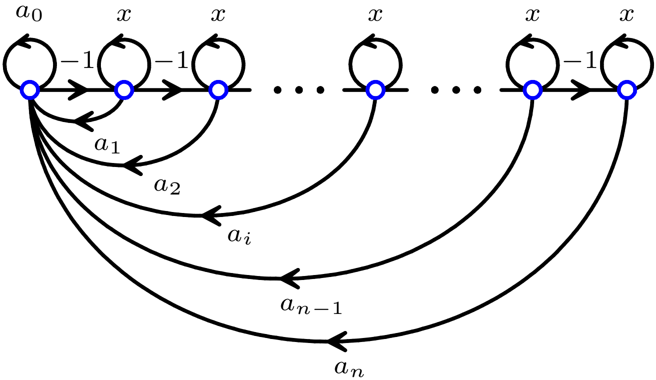
Exercise 4. (p. 95)

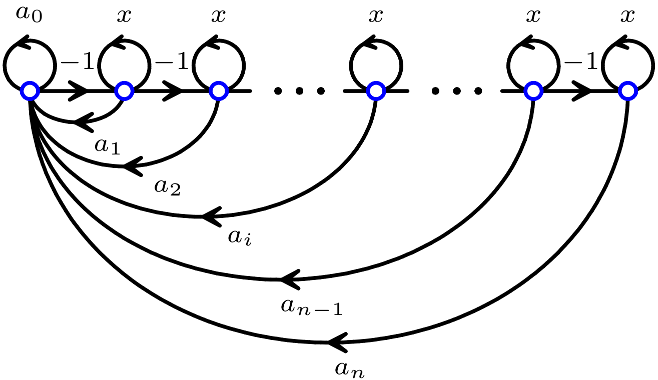
Solution (p. 248)

Prove that

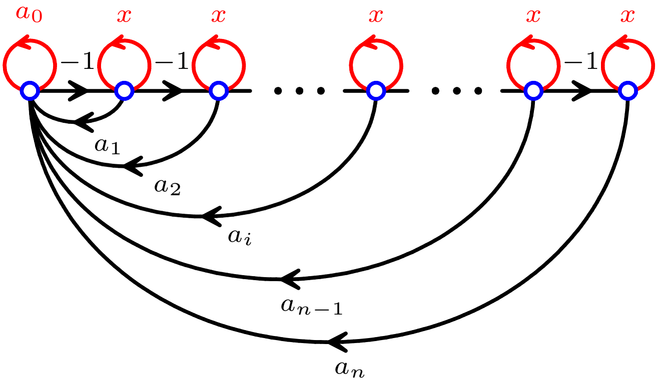
$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & & -1 & x \end{vmatrix} = \sum_{i=0}^{n} a_i x^{n-i}.$$

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & & -1 & x \end{bmatrix}$$

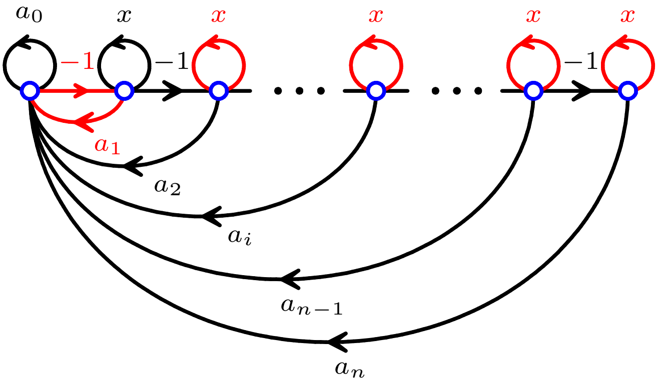




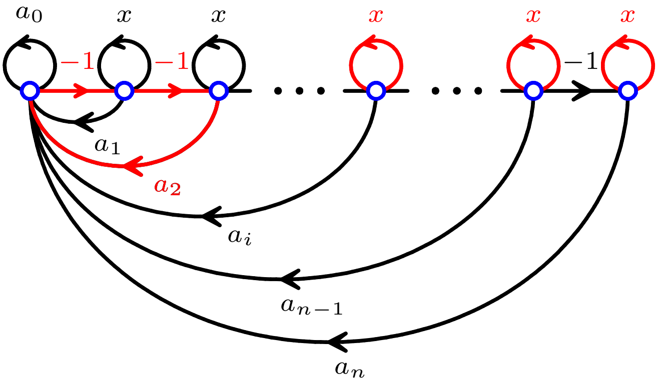
$$\det A = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$



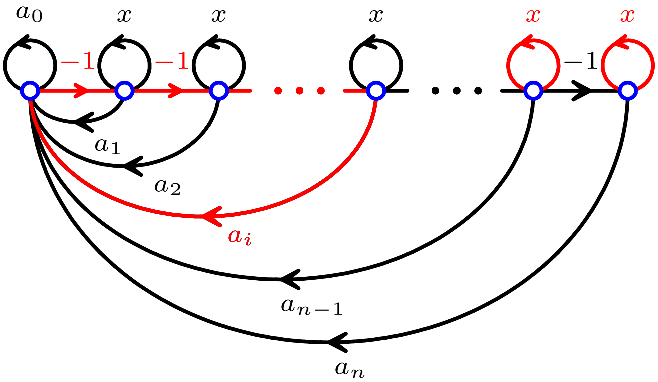
$$\det A = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$



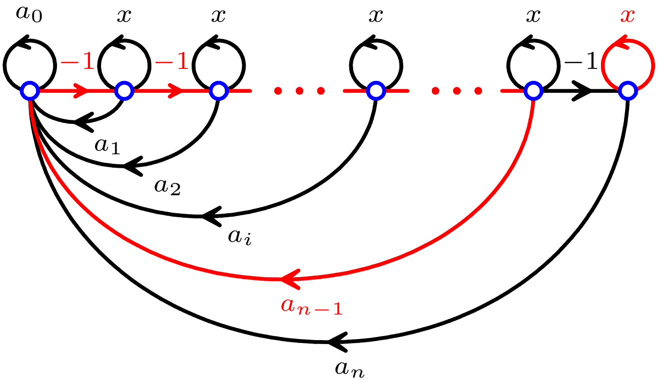
$$\det A = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$



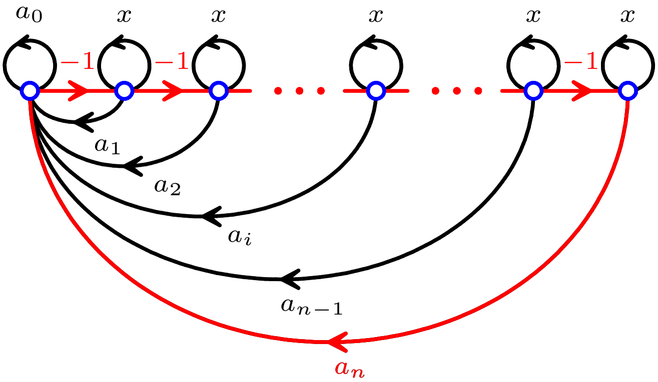
$$\begin{aligned}\det A \quad &= \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} \\ &\qquad + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}\end{aligned}$$



$$\det A \quad = \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i}$$
$$+ \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$



$$\det A = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$



$$\det A = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_ix^{n-i} + \cdots + a_{n-1}x + a_n = \sum_{i=0}^{n} a_ix^{n-i}$$