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PROJECTIVE POLYGON RECUTTING

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Abstract. In this paper, we propose a variant of Adler's recutting for projective polygons. We discuss its properties and overview its connection with the cross-ratio relation.

1. Introduction

In his celebrated papers [1,2], V. Adler presented a family of elementary polygon transformations, called *polygon recutting*, representing the action of the affine symmetric group on the space of polygons. Polygon recutting of the j-th vertex of a polygon $\mathbf{P} = \{P_1, \dots, P_n\}$ is defined as the polygon $P_1 \dots \tilde{P}_j \dots P_n$, where $\tilde{P}_j$ is the reflection of P_j in the perpendicular bisector to the short diagonal $P_{j-1}P_{j+1}$. The complete *recutting* of the polygon is obtained by the composition of elementary recuttings in vertices from 1 to n. Recutting is completely integrable transformation (in the sense of Arnold-Liouville), on the set of polygons. In [7], elementary recuttings were interpreted as cluster transformations thus providing the set of conserved quantities, together with the invariant Poisson structure, preserved by any composition of elementary recuttings.

It has been shown in [8] that Adler's recutting is closely related to the *bicycle correspondence* on polygons. For instance, both recutting and the bicycle correspondence share a family of conserved quantities.

The centroaffine analog of the above correspondence and recutting was recently studied in [5], where it was shown that the conserved quantities carry over to the centroaffine setting.

Finally, in [4] it was proven that the *cross-ratio correspondence* on projective polygons is integrable in the Arnold-Liouville sense. Two projective polygons are in α-correspondence if all the cross-ratios formed by pairs of respective sides of these polygons are equal to α. This correspondence can be thought of as a degeneration of the bicycle correspondence in the centroaffine setting. However, no projectively natural recutting transformation was presented as a naive degeneration results in the identical mapping.

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In this paper, we present a projectively natural analog of Adler's recutting. We present two conserved quantities in the spirit of the cross-ratio correspondence and deduce integrable behavior for the first two non-trivial cases.

2. Notations and definitions

We will start with the description of the analog of the perpendicular bisector. Throughout this paper, we will identify the real projective line $\mathbb{RP}^1$ with the absolute of the hyperbolic plane, which is in turn, identified with the unit circle on the plane. We parametrize points on this circle as

$$P = \Big(\frac{1-p^2}{1+p^2}, \frac{2p}{1+p^2}\Big). \tag{2.1}$$

Throughout the text, we denote points on the plane by upper case letters, while the corresponding lower case letters stand for the parameters in the above parametrization. Whenever it won't lead to the confusion, we won't distinguish the points on the absolute and their parameters. If not stated otherwise, the indices are understood (mod n). By a *projective polygon*, we will understand an ordered collection $\mathbf{P} = (p_1, \dots, p_n)$ of points in $\mathbb{RP}^1$ considered up to the action of the group of projective transformations. For $n \geqslant 4$ the cross-ratios $c_j = [p_j, p_{j+1}, p_{j+2}, p_{j+3}]$ of quadruples of consecutive vertices of a polygon provide a coordinate chart for a Zariski open subset of the set of projective polygons with no coinciding consecutive vertices or sides, supplying this set with the structure of $(n-3)$-dimensional manifold. Throughout the paper, for the cross-ratio we use the convention

$$[a, b, c, d] = \frac{(b-a)(d-c)}{(c-a)(d-b)}.$$

For geometric constructions we use Beltrami-Cayley-Klein model of the hyperbolic plane: geodesics correspond to straight lines; note, that the Euclidean angles

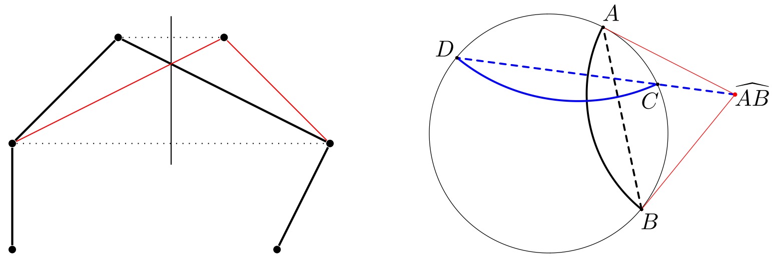

FIGURE 1. Left: Elementary Adler's recutting. Right: Hyperbolic perpendicular. The pole to the line through ideal points A and B is the center $\widehat{AB}$ of the circular arc AB in Poincaré model of the hyperbolic plane. Equivalently it is the common point of tangents to the absolute at the end-points. Every line through $\widehat{AB}$ is hyperbolically orthogonal AB. Dashed lines are geodesics in Beltrami–Klein model. Solid arcs stand for Poincaré gedosecis.

between these lines do not coincide with the hyperbolic angles. To construct a perpendicular to the line through the ideal points A and B one has to find a *pole* to this line, i.e. the common point $\widehat{AB}$ of tangents to the absolute through points A and B. Every geodesic through the pole is hyperbolically perpendicular to the side AB (see the right panel of Figure 1).

While projective polygons are thought of as ideal hyperbolic polygons, i.e. hyperbolic polygons with all their vertices lying on the absolute, the side lengths are undefined, and so one needs some aside considerations to pick a particular perpendicular out of the pencil of lines passing through the pole $\widehat{p_{j-1}p_{j+1}}$. For the present note, we chose the perpendicular passing through the intersection of the diagonals $p_{j-1}p_{j+2}$ and $p_{j-2}p_{j+1}$.

Elementary projective recutting. It is a well-known fact (see e.g. [3]) that

LEMMA 2.1. *Common perpendicular to the sides AB and CD of the ideal quadrilateral $ABCD$ passes through the intersection of diagonals AC and BD.*

Any line through the pole $\widehat{AB}$ is perpendicular to the side AB. Thus, the line through $\widehat{AB}$ and $\widehat{CD}$ is the common perpendicular to the sides AB and CD (see left panel of Figure 2). The fact that the common point of the lines AC and BD belongs to this perpendicular is the limit case of the Brianchon theorem.

DEFINITION 2.1. *The elementary projective recutting* of an n-gon $\mathbf{P} = (p_1, \dots, p_n)$ in j-th vertex is the polygon $F_j(\mathbf{P}) = (p_1, \dots, p_{j-1}, \tilde{p}_j, p_{j+1}, \dots, p_n)$,where $\tilde{p}_j$ is the hyperbolic reflection of p_j in the common perpendicular to the diagonals $p_{j-1}p_{j+1}$ and $p_{j-2}p_{j+2}$ (as in the right panel of Figure 2). The polygon

$$F(\mathbf{P}) := F_n \circ \cdots \circ F_1(\mathbf{P})$$

is called *the complete projective recutting* [1] of the polygon $\mathbf{P}$.

Notice that the above construction is purely projective, as $\tilde{P}_j$ is the hyperbolic reflection of P_j in the line AB if and only if the pole to $P_j\tilde{P}_j$ belongs to AB. Equivalently, one can observe that all three lines $P_{j-1}P_{j+1}$, $P_{j-2}P_{j+2}$ and $P_j\tilde{P}_j$ are orthogonal to AB, hence have to pass through the pole of AB. This remark leads to yet another construction of $\tilde{P}_j$. Let Q be the common point of $P_{j-1}P_{j+1}$ and $P_{j-2}P_{j+2}$. That is, Q is the pole of AB (see Figure 2). Then $\tilde{P}_j$ is the second intersection point of the line QP_j with the absolute.

LEMMA 2.2. *Let $\tilde{p}_j$ be the flip of p_j. Then*

$$\tilde{p}_j = \frac{p_{j+2}p_{j-2}(p_{j-1}+p_{j+1}-p_j) - p_{j+1}p_{j-1}(p_{j+2}+p_{j-2}-p_j)}{p_{j+2}p_{j-2} - p_{j+1}p_{j-1} + p_j(p_{j+1}-p_{j+2}+p_{j-1}-p_{j-2})}. \tag{2.2}$$

PROOF. First we find the coordinates of the pole of the line through ideal points a and b. We obtain

$$\widehat{ab} = \Big(\frac{1-ab}{1+ab}, \frac{a+b}{1+ab}\Big). \tag{2.3}$$

[1]Complete recutting depends on the order of the vertices, not only on the cyclic order.

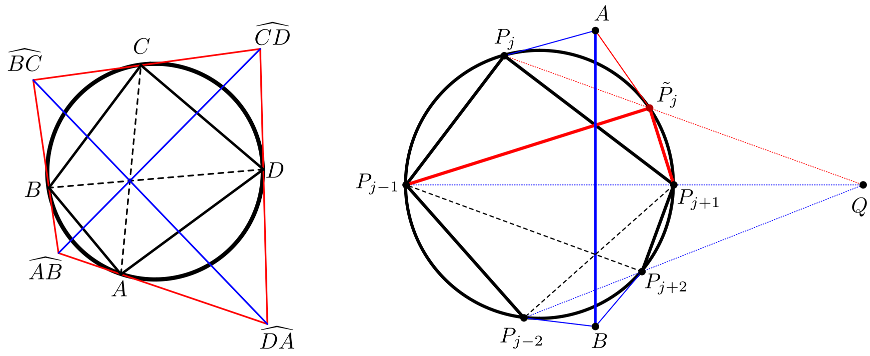

Figure 2. Left: Lemma 2.1. Right: Elementary projective recuting. Line AB is the common perpendicular to the diagonals $P_{j-1}P_{j+1}$ and $P_{j-2}P_{j+2}$. According to Lemma 2.1 it also passes via the common point of diagonals $P_{j-1}P_{j+2}$ and $P_{j-2}P_{j+1}$. Point $\tilde{P}_j$ is the hyperbolic reflection of the point P_j in the line AB. Lines $P_{j-1}P_{j+1}$, $P_{j-2}P_{j+2}$ and $P_j\tilde{P}_j$ are concurrent at the point Q – the pole to the line AB.

The common perpendicular to the diagonals $p_{j-1}p_{j+1}$ and $p_{j-2}p_{j+2}$ is the line $\ell(t) = t\widehat{p_{j-1}p_{j+1}} + (1-t)\widehat{p_{j-2}p_{j+2}}$.

For $\tilde{p}_j$ to be the hyperbolic reflection of p_j in the line ℓ, the pole $\widehat{p_j\tilde{p}_j}$ has to belong to ℓ.

Collinearity of three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) in the plane is equivalent to coplanarity of the three lines spanned by the vectors $(x_k, y_k, 1)$ in $\mathbb{R}^3$. Thus, the collinearity condition of the three poles, thanks to (2.3), can be written as

$$\det \begin{pmatrix} 1 - p_{j+1}p_{j-1} & 1 - p_{j+2}p_{j-2} & 1 - p_j\tilde{p}_j \\ p_{j+1} + p_{j-1} & p_{j+2} + p_{j-2} & p_j + \tilde{p}_j \\ 1 + p_{j+1}p_{j-1} & 1 + p_{j+2}p_{j-2} & 1 + p_j\tilde{p}_j \end{pmatrix} = 0.$$

Solving the above equation for $\tilde{p}_j$ yields (2.2). □

Since elementary recutting at j-th vertex (2.2) is a reflection in the line depending only on the vertices with the indices j, $j \pm 1$, and $j \pm 2$ we have the following statement.

Lemma 2.3. *The following relations hold for the elementary recuttings:*

$$F_j^2 = \mathrm{Id}, \quad F_jF_k = F_kF_j \quad \text{for } |j-k| \geqslant 3.$$

where indices are understood cyclically.

Coxeter incidence calculus. In what follows, we will adhere to the ideology that many incidence theorems of classical geometry can be interpreted as algebraic identities due to the following two configuration theorems by Coxeter (see [6]). We will use the following notations: points O, E and N stand for the stereographic projections (2.1) of the points 0, 1 and ∞, and other points denote the same projection of respective expressions in variables x and y.

THEOREM 2.1 (addition). *Let X and Y be the stereographic projections of the points x, $y \in \mathbb{R}$ onto the circle (see left panel of Figure 3) Let A be the point of intersection of the line OX with the line NA, tangent to the circle at point N. Then, the line YA intersects the circle in the stereographic projection of the point $(x - y)$.*

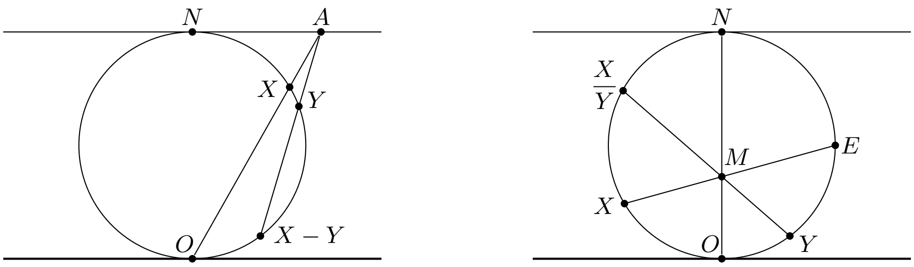

FIGURE 3. Left: Coxeter's addition theorem. Right: Coxeter's multiplication theorem.

THEOREM 2.2 (multiplication). *Let X and Y be the stereographic projections of the points x, $y \in \mathbb{R}$ onto the circle. (see right panel of Figure 3). Let M be the point of intersection of the lines ON and EX. Then, the line MY intersects the circle in the stereographic projection of the point x/y.*

For the sake of completeness, we reproduce the proofs of the above-mentioned theorems.

PROOF OF THEOREM 2.1. Rotate Figure 3 by $\pi/2$ counterclockwise. The parametrization (2.1) is then the usual parametrization of the unit circle $X = (\cos\alpha, \sin\alpha)$ provided that $x = \tan\frac{\alpha}{2}$. From the triangle ONA, we have $NA = 2\cot\frac{\alpha}{2}$. Let two points of intersection of the line through A with the circle have coordinates $(\cos\beta, \sin\beta)$ and $(\cos\gamma, \sin\gamma)$. Then, from the collinearity one gets

$$\Big(2\cot\frac{\alpha}{2} + \sin\beta\Big)(1+\cos\gamma) = (1+\cos\beta)\Big(2\cot\frac{\alpha}{2} + \sin\gamma\Big),$$

or, after simplification,

$$2\cot\frac{\alpha}{2}(\cos\gamma - \cos\beta) = \sin\gamma - \sin\beta + \sin(\gamma - \beta).$$

Expressing the differences of sines and cosines yields

$$\cot\frac{\alpha}{2}\Big(\sin\frac{\beta+\gamma}{2}\Big) = \cos\frac{\gamma}{2}\cos\frac{\beta}{2},$$

or

$$\cot\frac{\alpha}{2}\Big(\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\Big) = 1$$

and Theorem 2.1 follows. □

Proof of Theorem 2.2. Similarly, for $X = (\cos\alpha, \sin\alpha)$ and $E = (0, 1)$ one can find the coordinates of the point $M = (m, 0)$. One gets

$$m = \frac{\cos\alpha}{1 - \sin\alpha}.$$

Let β and γ be the angular coordinates for the points of intersection of the line through M with the unit circle. Then, from the colinearity condition one has:

$$(m - \cos\beta)\sin\gamma = (m - \cos\gamma)\sin\beta.$$

After simplification, using the expressions for the sine of the difference of angles and for the difference of sines, one has

$$m\Big(1 - \tan\frac{\beta}{2}\tan\frac{\gamma}{2}\Big) = 1 + \tan\frac{\beta}{2}\tan\frac{\gamma}{2}.$$

Introducing the expression for m yields

$$\Big(2\cos^2\frac{\alpha}{2} - 1\Big)\Big(1 - \tan\frac{\beta}{2}\tan\frac{\gamma}{2}\Big) = \Big(1 - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\Big)\Big(1 + \tan\frac{\beta}{2}\tan\frac{\gamma}{2}\Big),$$

or, after some simplification

$$\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\Big(1 - \tan\frac{\alpha}{2}\Big) = \tan^2\frac{\alpha}{2} - \tan\frac{\alpha}{2},$$

and Theorem 2.2 follows. □

Now, we can interpret the elementary projective recutting in a more conceptual way. To fix the conventions, consider a Möbius transformation

$$\mathcal{M}(p) = \frac{(p - p_{j-1})(p_j - p_{j+1})}{(p - p_{j+1})(p_j - p_{j-1})},$$

mapping the vertices (p_{j-1}, p_j, p_{j+1}) to the points $(0, 1, \infty)$ and denote $\mathcal{M}(p_{j-2}) = x$ and $\mathcal{M}(p_{j+2}) = y$.

Every line through the common point Q of the lines $\mathcal{M}(p_{j-1})\mathcal{M}(p_{j+1})$ and $\mathcal{M}(p_{j-2})\mathcal{M}(p_{j+2})$, thanks to Theorem 2.2, intersect the absolute in two points whose parameters multiply to xy. Hence, if $\mathcal{M}(p_j) = 1$, it follows that $\mathcal{M}(\tilde{p}_j) = xy$. Taking the inverse $\mathcal{M}^{-1}(xy)$ we thus obtain another derivation of the formula (2.2).

3. Conserved quantities

From now on we will use the following modification of the coordinates from the paper [4] for the moduli space of projective equivalence classes of ideal polygons[2]

$$c_j = -[p_{j-3}, p_{j-2}, p_{j-1}, p_j].$$

Lemma 3.1. *Let $\tilde{p}_j$ be the flip of p_j. Denote by c_j and $\tilde{c}_j$ the corresponding cross-ratios with p_j replaced by $\tilde{p}_j$. Then, for the elementary flip one has*

$$\tilde{c}_j = \frac{c_j(1 + c_{j+2})}{1 + c_{j+1}}, \qquad \tilde{c}_{j+1} = c_{j+2},$$
$$\tilde{c}_{j+2} = c_{j+1}, \qquad \tilde{c}_{j+3} = \frac{c_{j+3}(1 + c_{j+1})}{1 + c_{j+2}}. \tag{3.1}$$

[2] Our coordinates have opposite sign to the coordinates from [4].

PROOF. Thanks to the discussion in section 2, it is sufficient to check the above formulae for the case when (p_{j-1}, p_j, p_{j+1}) equals to $(0, 1, \infty)$ and therefore $\tilde{p}_j = p_{j-2}p_{j+2}$. Then (3.1) follow from the straightforward computation of the cross-ratios. □

In this section, we provide several conserved quantities for the complete projective recutting of the polygon. For instance, the cross-ratios of the quadruple from the second to the fifth vertex of $\mathbf{P}$ and of $F(\mathbf{P})$ coincide. Indeed, the flip F_1 shifts c_2 to the third position, next flip F_2 shifts the third coordinate of the projective class to the fourth position, etc. Hence, after n elementary flips c_2 will be moved to its initial position.

On the other hand, one can apply elementary flips in different order, thus it is natural to investigate other quantities, preserved by all the elementary flips.

THEOREM 3.1. *$I = \prod_{j=1}^{n} c_j$ is preserved by any elementary flip.*

PROOF. Indeed, only four cross-ratios are affected by an elementary flip F_j. c_{j+1} and c_{j+2} are swapped, and the factors for $\tilde{c}_j$ and $\tilde{c}_{j+3}$ are reciprocals of each other. □

THEOREM 3.2. *The k-multi-index $[i_1, \ldots, i_k]$ is called sparse if $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $|i_j - i_{j+1}| > 1$ for all $j = 1, \ldots, k-1$. If, in addition $(n + i_1 - i_k) > 1$, the multi-index is called cyclically sparse. Let $I_k = \sum_{[i_1,\ldots,i_k]} \prod_{\ell=1}^{k} c_{i_\ell}$, where the sum is taken over all cyclically sparse k-multi-indices. Then, $G = \sum_{k=1}^{\lfloor n/2 \rfloor} I_k$ is preserved by any elementary flip.*

PROOF. Let $\tilde{c}_i$ be the cross-ratios of the polygon with the vertex p_1 flipped. Then $\tilde{c}_i = c_i$ for all $i > 4$. Thus $G_k := I_k - \tilde{I}_k$ contains only terms with $i_1 \in \{1, 2, 3, 4\}$. Note that since the multi-index is cyclically sparse, no more than two first indices may belong to the set $\{1, 2, 3, 4\}$. Thus, for any k we can represent

$$G_k = G_k^{(1)} + G_k^{(2)} + G_k^{(3)} + G_k^{(4)} + G_k^{(1,4)} + G_k^{(1,3)} + G_k^{(2,4)},$$

where

$$G_k^{(j)} = \sum_{j \subseteq [i_1,\ldots,i_k]} \Big(\prod_{\ell=1}^{k} c_{i_\ell} - \prod_{\ell=1}^{k} \tilde{c}_{i_\ell} \Big),$$

with summation taken over cyclically sparse multiindices, containing the subset j. Hence $G_k^{(l)} = (c_l - \tilde{c}_l) I_{k-1}$ for $l = 1, 2, 3, 4$. Since $c_1 c_4 = \tilde{c}_1 \tilde{c}_4$ it follows that $G_k^{(1,4)} = 0$. For the remaining terms one has $G_k^{(1,3)} = (c_1 c_3 - \tilde{c}_1 \tilde{c}_3) I_{k-2}$ and $G_k^{(2,4)} = (c_2 c_4 - \tilde{c}_2 \tilde{c}_4) I_{k-2}$. Therefore,

$$G_{k-1}^{(4)} + G_k^{(2,4)} = \big((c_2 + 1)c_4 - (\tilde{c}_2 + 1)\tilde{c}_4 \big) I_{k-2}.$$

But from the expressions (3.1) it follows that $(c_2 + 1)c_4 - (\tilde{c}_2 + 1)\tilde{c}_4 = 0$ and so the above sum vanishes. Similarly we get $G_{k-1}^{(1)} + G_k^{(1,3)} = 0$.

The sum $G_k^{(2)} + G_k^{(3)}$ is identically zero since it is invariant with respect to the involution $c_2 \leftrightarrow c_3$ and at the same time changes sign under its action. □

4. Case studies

In this section, we will describe the dynamics of projective recutting for the case of small-gons.

Pentagons. For closed pentagons coordinates c_j satisfy the relation

$$1 + c_j + c_{j+1} + c_{j+2} + c_j c_{j+2} = 0.$$

Hence, for the elementary recutting in the first vertex, thanks to (3.1), one gets $\tilde{c}_1 = c_4$ and $\tilde{c}_4 = c_1$. Thus, application of the elementary recuttings in five consecutive vertices results in the following sequence of transforms:

$$(c_1, c_2, c_3, c_4, c_5) \mapsto (c_4, c_3, c_2, c_1, c_5) \mapsto (c_4, c_5, c_1, c_2, c_3)$$
$$\mapsto (c_1, c_5, c_4, c_3, c_2) \mapsto (c_2, c_3, c_4, c_5, c_1) \mapsto (c_3, c_2, c_1, c_5, c_4).$$

In other words, the complete projective recutting of the pentagon corresponds to the permutation of the coordinates and so is an involution on the projective equivalence class.

Hexagons. For closed hexagons we have the relations

$$1 + c_j + c_{j+1} + c_{j+2} + c_{j+3} + c_j c_{j+2} + c_{j+1} c_{j+3} + c_j c_{j+3} = 0.$$

Therefore, the moduli space of projective equivalence classes of hexagons is three-dimensional. The conserved quantities from Theorems 3.1 and 3.2 constrain the dynamics to one-dimensional ovals. While the experiments show that the dynamics on these ovals is topologically conjugated to irrational rotation, we won't dwell on this here as we haven't investigated its smoothness.

Interestingly, the dynamics for $n = 7$ is also restricted to a one-dimensional torus, while the dynamics for $n = 8$ seems to represent the quasi-periodic motion on two-dimensional tori in 5-dimensional space (see Figure 4).

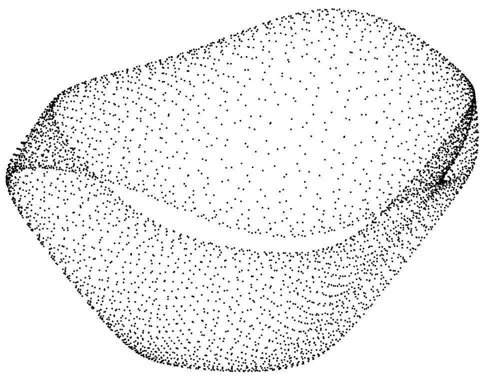

FIGURE 4. 5000 iterations of the complete projective recutting of an octagon, projected onto the three-dimensional subspace, spanned by (c_1, c_3, c_5).

5. Discussion

The aim of this paper is to initiate the study of seemingly natural projective analog of Adler's recutting. Below we collect a few open questions in this regard.

- Currently we have observed only two non-trivial conserved quantities for the described dynamics. However, numerical experiments suggest integrable behavior for any composition $F_{s_1} \cdots F_{s_n}$ of elementary flips, corresponding to any given permutation $(s_1, \ldots, s_n) \in \Sigma_n$.
- While we were able to construct only first integrals, the integrable behavior suggests the existence of a suitable Poisson structure. Expressions (3.1) resemble the underlying Y-mutation on a bipartite graph, similar to one in [7].

  Recall the definition of the Y-pattern. By the quiver we will understand the directed simply-laced graph with the vertices labeled by the *cluster* variables y_j. The mutation at the j-th vertex is the transformation of the graph according to the following rules:

  – Update the labels: $y_k \mapsto \begin{cases} y_j^{-1}, & k = j \\ y_k(1+y_j), & y_k \leftarrow y_j \\ y_k(1+y_j^{-1})^{-1}, & y_k \to y_j \\ y_k, & \text{else} \end{cases}.$
  – Add the edge $y_i \leftarrow y_k$ for each 2-chain $y_i \leftarrow y_j \leftarrow y_k$.
  – Delete all appearing 2-cycles.
  – Reverse the orientation of all edges incident to y_j.

  Consider the graph in Figure 5 with the labels $c_j^{(-1)^j}$, following the black zig-zag path and $a_j^{(-1)^j}$, following gray zig-zag path.

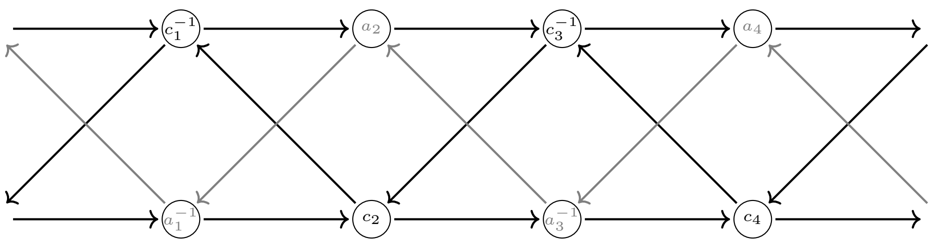

Figure 5. Bi-partite Y-quiver for the formulae (3.1)

  Mutations of the given graph in the vertices a_2 and a_3^{-1}, followed by the mutations in c_2 and in c_3^{-1} provide the expressions (3.1) in the limit $a_j \to 0$. For the moment we do not have appropriate geometric interpretation for this phenomena.
- Numerical experiments show similar behavior for deeper projective bisectors, i.e., common perpendiculars to $P_{j-1}P_{j+1}$ and $P_{j-k}P_{j+k}$. It would be interesting to investigate the dynamics there as well.

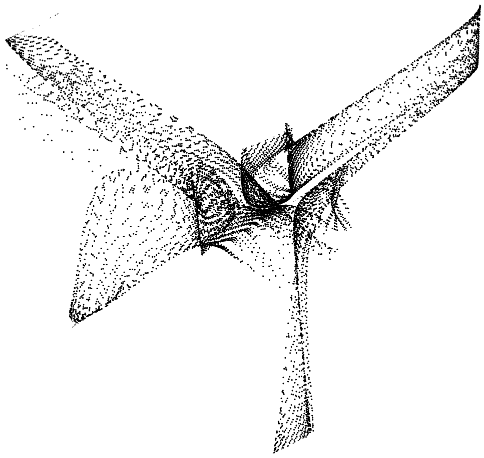

Figure 6. Complete recutting in the common perpendiculars to (P_{j-1}, P_{j+1}) and $(P_{j-3}P_{j+3})$ of an octagon. Arctangents of the first, third and fifth cross-ratios are plotted.

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ПРОЈЕКТИВНО ПОНОВНО РАСЕЦАЊЕ ПОЛИГОНА

Резиме. У овом раду предлажемо варијанту Адлеровог поновног расецања за пројективне полигоне. Разматрамо својства дате трансформације, као и њену везу са дворазмером.

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