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ROLLING GEODESICS ON SYMMETRIC SEMI-RIEMANNIAN SPACES

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Abstract. This paper is an outgrowth of the results in the domain of rolling obtained in our recent paper written with F. Silva Leite and I. Markina, and the earlier papers on the rollings of spheres produced with J. Zimmerman. We show that the rolling equations associated with a symmetric semi-Riemannian manifold rolling on its tangent space at a fixed point on the manifold essentially have the same structure as the rolling equations for the n-dimensional sphere rolling on the horizontal hyperplane; that is, we show that the rolling equations are described by a left-invariant distribution $\mathcal{D}$ on a Lie group $\mathbf{G}$ with the Lie bracket growth

$$\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = T\mathbf{G},$$

reminiscent of the growth $(2, 3, 5)$ for the two spheres rolling on the horizontal plane. We then define rolling geodesics on semi-Riemannian spaces as extensions of sub-Riemannian geodesics in the Riemannian symmetric spaces, and show that the rolling geodesics are the projections of the extremal curves, which, remarkably, are the solution curves of a completely integrable Hamiltonian system in the cotangent bundle of the configuration space. Finally, we illustrate the theory with a few noteworthy examples.

1. Introduction

This paper is a continuation of my long-standing interest in the role of Lie groups and Lie algebras in the theory of integrable systems and the equations of mathematical physics. The interest in this topic originated from two seemingly unrelated phenomena, the presence of elastica in the theory of rolling spheres [1, 2]), and the presence of the heavy top in the equations describing the equilibrium configurations of an elastic rod [3–5]. My interest in these phenomena was further renewed by the results of our recent paper [6] that showed that the rollings of the semi-Riemannian symmetric manifolds on their tangent planes admitted a conceptually simple description analogous to the equations for the rolling spheres [7].

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Briefly, this paper outlines the relevant theory of symmetric semi-Riemannian spaces for the results in [6] and defines a new class of curves called rolling geodesics. Then, the paper demonstrates that the rolling geodesics are the projections of the extremal curves generated by a completely integrable Hamiltonian system on the cotangent bundle of the configuration space. The passage to Hamiltonian systems reveals a geometric significance of rolling for the problems of geometric mechanics.

Of course, mathematical considerations of objects rolling on one another have a long and diverse history. The interest in the rolling phenomena very likely began with the wheel rolling without slipping on a flat surface and the discovery of the cycloid, which, in time, got to be known as the crown jewel of all curves because of two remarkable properties: the tautochrone property, whereby a particle, sliding without friction under uniform gravity, reaches the lowest point of the curve in time independent of the initial position (C. Huygens, (1659)), and also the brachistochrone property, as the curve along which a particle slides under gravity in shortest time (discovered by the Bernoulli brothers around 1696).

More generally, rollings of curves along other curves became a standard topic in classical mechanics in which the cardioid and the astroid stood out as particularly beautiful examples. In these early studies, rolling was regarded as a part of the mechanical world in which rolling objects were seen as rigid bodies moved by the forces that preserved their geometric properties and kept them in contact with each other. The passage to geometric rolling, principally surfaces rolling on their affine tangent planes, was initiated by the geometers in the early part of the 20^{th} century. In particular, T. Levi-Civita, in his famous treatise on intrinsic calculus [8], motivates the notion of parallelism on a surface in $\mathbb{R}^3$ by appealing to the rollings of a surface on its tangent plane. In the case of a developable surface, which Levi-Civita describes as flexible and inextensible and which can be made to coincide with a region of the plane, like the cylinder and the cone, tangential directions u_1 and u_2 situated at the points p_1 and p_2 on the surface are parallel whenever these directions are parallel in the ordinary sense when the surface is unfolded on the plane. In the case of a more general surface, he states that a surface S is rolled along a curve $\alpha(t)$ on the tangent plane P at a point p_1 on S if during the rolling the point of contact with the stationary plane P traces a curve $\hat{\alpha}(t)$ in P, called the development of $\alpha(t)$, if the tangential directions u_1 and u_2 at the points p_1 and p_2 are parallel along $\alpha(t)$ if they are parallel, in the ordinary sense, along the developed curve $\hat{\alpha}(t)$ [8, p. 102].

Since then, rolling and parallel transport took somewhat divergent paths: parallel transport found its natural definition within the realm of the affine connection and the covariant derivative, while rolling continued to be of interest to mechanics as a prototype of phenomena encountered in systems with non-holonomic constraints [1, 9]. In this interim period, the paper of E. Cartan [10] on the rollings of spheres on each other and their relation to the 14-dimensional Lie group G_2 stands out as the most fascinating contribution to the subject of rolling.

It was only recently that rolling got its first axiomatic treatment by R. Sharpe for Riemannian manifolds M and $\hat{M}$ that are embedded isometrically in a Euclidean space $\mathbb{E}^n$ [11]. For this class of manifolds, Sharpe showed that each curve $\alpha(t)$ in M

can be rolled on a curve $\hat{\alpha}(t)$ in $\hat{M}$, called the development of $\alpha(t)$, by an isometry $g(t)$ that satisfies the no slipping and no twisting condition. Moreover, he showed that the isometry curve that does this rolling is unique. The axiomatic notion of rolling made this subject matter more accessible and led a way to several studies [12, 13] with a particular interest on the rollings of symmetric Riemannian spaces on their affine tangent spaces [14]. Sharpe's definition of rolling, however, made no explicit connection to the parallel transport.

Independently of the studies based on Sharpe's definition of rolling, R. Bryant and L. Hsu [15] introduced another definition of rolling according to which a curve $\alpha(t)$ in a Riemannian manifold M rolls on a curve $\hat{\alpha}(t)$ in another Riemannian manifold $\hat{M}$ if there is an isometry $A(t)$ such that

$$A(t)\colon T_{\alpha(t)}M \to T_{\hat{\alpha}(t)}\hat{M}, \quad \frac{d\hat{\alpha}}{dt}(t) = A(t)\frac{d\alpha}{dt}(t), \tag{1.1}$$

and $\hat{v}(t) = A(t)v(t)$ is a parallel transport along $\hat{\alpha}(t)$ for each parallel transport $v(t)$ along $\alpha(t)$. New studies emerged based on this notion of rolling [16–20] with a philosophical outlook sufficiently different from the papers based on Sharpe's formulation, suggesting conceptual differences between the two definitions. As a result, these two notions of rollings became known as the extrinsic rolling (Sharpe) and intrinsic rolling (Bryant and Hsu). However, it was shown recently [6] that a curve $\alpha(t)$ rolls on a curve $\hat{\alpha}(t)$ independently of the definition used, but the isometry curve $A(t)$ that does the rolling may be different. Sharpe's definition imposes an additional condition, dependent on the embedding, under which $A(t)$ becomes a unique isometry that rolls $\alpha(t)$ on $\hat{\alpha}(t)$.

This paper provides a comprehensive overview of semi-Riemannian symmetric spaces rolling on their tangent spaces and then focuses on the extremal equations associated with the rolling curves. The material is presented in five distinctive parts. The first part establishes the essential properties of symmetric semi-Riemannian spaces, largely a synthesis of the theory presented in the seminal work of B. O'Neill [21]. The second part derives rolling equations and sets the stage for the associated variational problems (a complement to the results presented in [6]). The third part discusses some noteworthy examples that give the reader a glimpse into the large variety of systems covered in the first section. This section also includes an original treatment of the rolling spheres and the rolling hyperboloids that play an important role in the latter sections. The fourth section outlines the passage to the Hamiltonian systems and the associated extremal equations. This section also shows the connections between the Poisson systems associated with the affine-quadratic systems, which figure prominently in the equations of geometric mechanics, and the rolling systems. Finally, section five returns to the space forms, hyperboloids and spheres, with an original rendition of the results obtained in [7] linking rolling to the elastic curves.

2. Semi-Riemannian spaces

Because of the relative novelty of rollings in the semi-Riemannian environment, it may be advantageous to present a self-contained account of the relevant theory, in order to make the paper more accessible to a wider audience. For that reason some of the exposition presented below is a re-hash of the material presented in [21].

A semi-Riemannian manifold M is a smooth manifold endowed with a non-degenerate symmetric tensor $g\langle .,.\rangle$. We write $n=\dim M$, and denote by p the number of positive eigenvalues of the tensor g, so that $n-p$ is the number of negative eigenvalues of g. Any vector space V equipped with a symmetric and non-degenerate form (,) is semi-Riemannian. Any such form can be diagonalized, that is, there exists a basis $e_1,\dots,e_n$ in V such that $(e_i,e_j)=\pm\delta_{ij}$. Then, there exist integers $p\leqslant n$ and $n-p$ such that $(e_i,e_i)=1$, for $i\leqslant p$ and $(e_i,e_i)=-1$ for $i>p$. It then follows that

$$(x,y)=\sum_{k=1}^{p}x_ky_k-\sum_{k=p+1}^{n}x_ky_k,\quad x,y\in\mathbb{R}^{p,n-p}.$$

where $x_1,\dots,x_n$ and $y_1,\dots,y_n$ are the coordinates of x and y in the above basis. Following the terminology in [21], we will refer to the above form as a scalar product.

On any semi-Riemannian manifold M, there is a unique affine connection D_XY that satisfies

- $[V,W]=D_VW-D_WV$
- $X\langle V,W\rangle=\langle D_XV,W\rangle+\langle V,D_XW\rangle$ for all smooth vector fields X,V,W on M.

Such a connection is called the Levi-Civita connection. The Levi-Civita connection is characterized by the Koszul formula

$$\begin{aligned}2\langle D_VW,X\rangle=&\,V\langle W,X\rangle+W\langle X,V\rangle-X\langle V,W\rangle\\&-\langle V,[W,X]\rangle+\langle W,[X,V]\rangle+\langle X,[V,W]\rangle.\end{aligned}$$

The Levi-Civita connection induces the covariant derivative $\frac{DV}{dt}$ of a vector field $V(t)$ along a differentiable curve $c(t)$ in M according to the usual formula $\frac{DV}{dt}=D_{\frac{dc}{dt}}\tilde{V}$ where $\tilde{V}$ is a vector field on M such that $V(t)=\tilde{V}(c(t))$. A vector filed $V(t)$ along a curve $c(t)$ is called parallel if $\frac{DV}{dt}=0$. Parallel vector fields exist along any curve $c(t)$ in M. In fact, if w is a vector in $T_{c(t_0)}M$, then there exist a parallel vector field $V(t)$ along $c(t)$ such that $V(t_0)=w$.

A curve $\gamma(t)$ in a semi-Riemannian manifold M is called a geodesic if the tangent vector field $\frac{d\gamma}{dt}$ is parallel along $\gamma(t)$. It then follows that for any $v\in T_pM$ there exists an open interval I around the origin and a unique geodesic $\gamma_v(t)$ such that $\frac{d\gamma_v}{dt}(0)=v$ (of course,$\gamma_v(0)=p$). Any semi-Riemannian manifold in which the geodesics are defined for all $t\in\mathbb{R}$ is said to be complete. In general, there exists a maximal open interval I on which a geodesic $\gamma_v(t)$ is defined. In such a case $\gamma_v(t)$ is unique. If the domain of γ_v includes the interval $[0,1]$ then the end point map $v\to\gamma_v(1)$ is one to one and it is easy to show that $\gamma_{tv}(1)=\gamma_v(t)$.

As in the Riemannian case, the exponential map $\exp_o(tv)=\gamma_v(t)$ is a diffeomorphism from a star shaped open neighbourhood $\mathcal{U}=\{tv : t\in[0,1],v\in U\}$ of the origin in T_oM onto an open neighbourhood $\bar{\mathcal{U}}$ of the point o in M. This neighbourhood is called normal. This implies that any two points p and q in a connected manifold M can be connected by a broken geodesic, a continuous curve $c(t)$ defined on a finite union of closed intervals $I_j = [t_j, t_{j+1}]$, $j = 1, \ldots, n$, such that on each interval I_j $c(t)$ is a geodesic $\gamma_{v_j}(t)$ that satisfies $\gamma_{v_j}(t_j) = c(t_j)$ and $\frac{d\gamma_{v_j}}{dt}(t_j) = v_j$.

A diffeomorphism ϕ on M that leaves its semi-Riemannian metric invariant is called an isometry. The set of all isometries on M is a Lie group $I(M)$ that naturally acts on M by the left action $g, p \to gp$. A semi-Riemannian manifold is said to be homogeneous if $I(M)$ acts transitively on M. One can easily show using the Koszul's formula that $D_{d\phi X} d\phi Y = d\phi(D_X Y)$ is the Levi-Civita connection on M for any isometry ϕ. But then $d\phi(D_X Y) = D_X Y$ by the uniqueness of the Levi-Civita connection. Therefore, isometries preserve the geometric quantities of M. In particular, they preserve geodesics. We also have

PROPOSITION 2.1. *Suppose that g and h are elements in $I(M)$ such that $go = ho$ and $d_o g = d_o h$ for some point o in M. Then $g = h$ whenever M is connected.*

PROOF. Suppose that $go = ho$ then $h^{-1}go = o$. So, it suffices to show that $go = o$ and $d_o g = \mathrm{Id}$ implies $g = \mathrm{Id}$. Let $U = \{p \in M : gp = p, d_o g = \mathrm{Id}\}$. By continuity U is a closed set in M. It remains to show that U is open. Let p be an arbitrary point in U and let $\alpha(t)$ be any geodesic in M such that $\alpha(0) = p$. Since $d_p g(\dot\alpha(0) = \dot\alpha(0)$, $g\alpha(t) = \alpha(t)$. Therefore, U contains a normal neighbourhood at p, and hence is open. But then $U = M$, since M is connected. Thus $gp = p$ for all $p \in M$ $\square$

2.1. Semi-Riemannian symmetric spaces. A smooth manifold M together with a transitive group action of a Lie group G on M is called homogeneous. If the action of G is represented by the diffeomorphisms $\{\phi_g : g \in G\}$, then M can be written as the quotient $M = G/K$ where $K = \{g \in G : \phi_g(o) = o\}$ for some fixed point $o \in M$. In this representation, the natural projection $\pi \colon g \to gK$ is identified with $\phi_g(o)$, in which case we have $\pi(L_g) = \phi_g \pi$, where L_g is the left action $L_g(h) = gh$ of G on itself.

Any homogeneous manifold admits a finite dimensional family of complete vector fields $\mathcal{F}$ that span each tangent space $T_p M$. They are the infinitesimal generators of the flows $\{\phi_{\exp tX} : t \in \mathbb{R}, X \in \mathfrak{g}\}$, where ϕ_g denotes the action of G on M. When G is the isometry group of M, these vector fields are known as the Killing vector fields (a vector field X on a semi-Riemannian manifold M is called Killing if its one-parameter group of diffeomorphisms $\{\exp tX : t \in \mathbb{R}\}$ acts on M as a group of isometries [21, 23]). The correspondence

$$X \in \mathfrak{g} \to \vec{X}, \vec{X}(p) = \frac{d}{dt}\phi_{\exp tX}(p)|_{t=0}$$

is an anti-isomorphism since $[X, Y] \to -[\vec{X}, \vec{Y}]$.

A homogeneous space G/K is called reductive if the Lie algebra $\mathfrak{g}$ admits a vector space $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and $Ad_K(\mathfrak{p}) \subseteq \mathfrak{p}$. This invariance implies that $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$. On reductive homogeneous spaces $T_p M$ can be identified with $\mathfrak{p}$ via the correspondence $X \in \mathfrak{p} \to \vec{X}(p) \in T_pM$. If, in addition, a reductive space G/K is equipped with a G-invariant metric $\langle\,,\,\rangle_p$, then $\langle d_o\phi_g\vec{X}(o), d_o\phi_g\vec{Y}(o)\rangle_p = \langle\vec{X}(o),\vec{Y}(o)\rangle_o$ for any $g \in G$ and any $\vec{X}(o)$ and $\vec{Y}(o)$ in T_oM by the invariance of the metric. In particular, $\langle d_o\phi_h\vec{X}(o), d_o\phi_h\vec{Y}(o)\rangle_p = \langle\vec{X}(o),\vec{Y}(o)\rangle_o$ when $h \in K$. Since $\phi_h(o) = o$, $d_o\phi_h\vec{X}(o) \in T_oM$, and therefore $d_o\phi_h\vec{X}(o) = \vec{\tilde{X}}(o)$ for some $\tilde{X}$ in $\mathfrak{p}$. We now have

$$d_o\phi_h\vec{X}(o) = \frac{d}{dt}\phi_{h\exp tX}(o)|_{t=0} = \frac{d}{dt}\phi_{h\exp tXh^{-1}}(o)|_{t=0} = \frac{d}{dt}\phi_{\exp tAd_h(X)}(o)|_{t=0}$$

Therefore, $d_o\phi_h\vec{X}(o) = \vec{\tilde{X}}(o)$ where $\tilde{X} = Ad_h(X)$. If we now define an isometric scalar product $\langle\langle\,,\,\rangle\rangle$ on $\mathfrak{p}$ via the formula

$$\langle\langle X, Y\rangle\rangle = \langle\vec{X}(o),\vec{Y}(o)\rangle_o \tag{2.1}$$

then $\langle\langle Ad_hX, Ad_hY\rangle\rangle = \langle\langle X, Y\rangle\rangle$, that is, $\langle\langle\,,\,\rangle\rangle$ is an Ad_K invariant scalar product on $\mathfrak{p}$. Let now $\langle\langle\,,\,\rangle\rangle$ denote any scalar product on $\mathfrak{g}$ which agrees with $\langle\langle\,,\,\rangle\rangle$ on $\mathfrak{p}$. Then G together with the left-invariant metric induced by $\langle\langle\,,\,\rangle\rangle$ becomes a semi-Riemannian manifold.

Conversely, suppose that G/K is a reductive manifold with G equipped with a left-invariant metric defined by a scalar product $\langle\langle\,,\,\rangle\rangle$ that is Ad_K invariant on $\mathfrak{p}$. Then, formula (2.1) can be used to define a scalar product $\langle\,,\,\rangle_o$ on T_oM which can be extended to a G-invariant metric on M via

$$\langle d_o\phi_g\vec{X}(o), d_o\phi_g\vec{Y}(o)\rangle_p = \langle\vec{X}(o),\vec{Y}(o)\rangle_o.$$

Definition 2.1. A connected semi-Riemannian manifold (M, g) is called symmetric if for each $o \in M$ there exists an isometry $\zeta_o\colon M \to M$ called the global symmetry of M at o, such that $\zeta_o o = o$ and $d_o\zeta_o = -I$ on T_oM.

Proposition 2.2. *Any symmetric semi-Riemannian connected manifold is complete and homogeneous.*

Proof. Since ζ_o is an isometry, it acts on the geodesics. So if $\gamma(t)$ is a geodesic that satisfies $\gamma(0) = o$, then $c(t) = \zeta_0\gamma(t) = \gamma(-t)$ because $\dot{c}(0) = -\dot{\gamma}(0)$. More generally,

$$\zeta_{\gamma(s)}\gamma(s-t) = \gamma(s+t) = c(t) \tag{2.2}$$

for any geodesic curve $\gamma(t)$. So, if $\gamma(t)$ is defined on an interval $[-s, s]$ then $c(t)$ is a geodesic defined on $[0, 2s]$ that satisfies $c(0) = \gamma(s)$ and $\dot{c}(0) = \dot{\gamma}(s)$. Hence, the domain of $\gamma(t)$ can be extended to the interval $[0, 2s]$. The same argument with $-s$ shows that the domain of γ can be extended to the interval $[-2s, 2s]$. Hence, each geodesic is defined on $(-\infty, \infty)$.

To prove homogeneity, we will use the fact that any two points p and q in M can be connected by a broken geodesic curve. So, let $\alpha(t)$ be a continuous curve in M on an interval $[0, T]$, partitioned by n points $0 < t_1 < t_2 < \cdots < t_{n+1} = T$, such that on each interval $[t_j, t_{j+1}]$, $j \leqslant n$, $\alpha(t) = \gamma_j(t)$ for some geodesic $\gamma_j(t)$.

If $\alpha(0) = p$ and $\alpha(T) = q$ then $\zeta_{\gamma_1(\frac{t_1}{2})}\alpha(0) = \alpha(t_1) = \gamma_2(t_1)$ by (2.2). Then $\zeta_{\gamma_2(\frac{t_2-t_1}{2})}\alpha(t_1) = \alpha(t_2) = \gamma_3(t_2)$. Continuing we get

$$q = \alpha(t_{n+1}) = \zeta_{\gamma_n(\frac{t_{n+1}-t_n}{2})} \cdot \zeta_{\gamma_{n-1}(\frac{t_n-t_{n-1}}{2})} \cdots \zeta_{\gamma_2(\frac{t_2-t_1}{2})} \cdot \zeta_{\gamma_1(\frac{t_1}{2})} p$$

Hence, $I(M)$ acts transitively on M. □

Let $G = I_0(M)$ be the connected component of $I(M)$ that contains the group identity. Since $I(M)$ acts transitively on M so does G. Hence, M is diffeomorphic to the quotient manifold G/K where K is the isotropy group of a point o in M [**22**]. In this representation $\pi \circ L_g = \phi_g \circ \pi$, and therefore the left actions $L_g, g \in G$, become isometries.

The existence of global symmetry on M induces an involutive automorphism $\sigma\colon G \to G$ defined by $\phi_{\sigma(g)} = \zeta_o \phi_g \zeta_o$. One easily shows using Proposition 2.1 that $\zeta_o^2 = \mathrm{Id}$, hence $\zeta_o^{-1} = \zeta_o$. Therefore, σ satisfies

$$\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2), \sigma^2 = I.$$

PROPOSITION 2.3. *Let $H = \{g \in G : \sigma(g) = g\}$. Then $H_o \subseteq K \subseteq H$, where H_0 is the connected component of H through the group identity.*

PROOF. Let $g \in G$ satisfy $\phi_g o = o$, i.e., $g \in K$. Then $\phi_{\sigma(g)} o = \zeta_o \phi_g \zeta_o(o) = o$, and

$$d_o\phi_{\sigma(g)} = d_o\zeta_o d_o\phi_g d_o\zeta_o = (-\mathrm{Id}) d_o\phi_g (-\mathrm{Id}) = d_o\phi_g$$

It then follows from Proposition 2.1 that $\sigma(g) = g$, that is, $g \in H$. This shows that $K \subseteq H$. To show that $H_0 \subseteq K$, let $\exp tX$ denote the one-parameter group generated by an element X in the Lie algebra of H. Then $\exp tX$ belongs to H_0 for all t , hence satisfies $\sigma(\exp tX) = \exp tX$. It follows that $\zeta_o \exp tX(o) = \exp tX(o)$. Since o the only fixed point by ζ_o in a neighbourhood of o in M, $\exp tX(o) = o$ for small t. But then $\exp tX(o) = o$ for all t by analyticity of $\exp tX$. This shows that $\{\exp tX : t \in \mathbb{R}\} \subseteq K$ for each X in the Lie algebra of H. But, H_0 is generated by the exponentials in the Lie algebra of H, hence $H_0 \subseteq K$. □

COROLLARY 2.1. *The Lie algebra of K is the same as the Lie algebra of H.*

The automorphism σ induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{p} = \{X \in \mathfrak{g} : d\sigma_e(X) = -X\}$, and $\mathfrak{k} = \{X \in \mathfrak{g} : d\sigma_e(X) = X\}$, where e denotes the group identity in G. Evidently $\mathfrak{k}$ is the Lie algebra of K, and therefore $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$. In addition,

$$[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

COROLLARY 2.2. *$Ad_K(\mathfrak{p}) \subset \mathfrak{p}$.*

PROOF. Since σ is an autotomorphism, $d\sigma_e Ad_h = Ad_{\sigma(h)} d\sigma_e$. When $X \in \mathfrak{p}$, and $h \in K$,

$$d\sigma_e Ad_h X = Ad_{\sigma(h)} d\sigma_e X = -Ad_{\sigma(h)} X = -Ad_h(X).$$ □

Remarkably, the existence of an involutive automorphisms on G implies the existence of global symmetries according to the following proposition.

Proposition 2.4. *Let K be a closed subgroup of a connected Lie group G. Let σ be an involutive automorphism of G such that $H_0 \subseteq K \subseteq H$, where $H = \{g \in G : \sigma(g) = g\}$. Then, any G-invariant metric tensor on $M = G/K$ makes M a semi-Riemannian symmetric space with ζ, the global symmetry of M at o, defined by $\zeta \circ \pi = \pi \circ \sigma$.*

Remark 2.1. The existence of global symmetry at a single point o implies the existence of global symmetry at every point $p \in M$. For if ζ_o is the symmetry at o, then $\zeta_p = g\zeta_o g^{-1}$ is the symmetry at $p = g(o)$.

It follows that a symmetric semi-Riemannian space M is a reductive space and satisfies an extra property $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. In this setting, every curve $m(t)$ in M can be lifted to a curve $g(t) \in G$ via the formula $\phi_{g(t)}(o) = m(t)$. Then, $\dot{m}(t) = d_o\phi_{g(t)}\vec{X}(t)(o)$ where $X(t)$ is a curve in $\mathfrak{g}$ defined by $\frac{dg}{dt} = g(t)X(t)$. It follows that $X(t) = U(t) + V(t)$ for $U(t) \in \mathfrak{p}$ and $V(t) \in \mathfrak{k}$, and, therefore, $\dot{m}(t) = d_o\phi_{g(t)}(\vec{U}(t)(o) + \vec{V}(t)(o)) = d_o\phi_{g(t)}\vec{U}(t)(o)$ because $\vec{V}(o) = 0$. An absolutely continuous curve $g(t) \in G$ is called horizontal if $g(t)$ is a solution of $\frac{dg}{dt} = g(t)U(t)$ for some bounded measurable curve $U(t)$ in $\mathfrak{p}$. It follows from above that every curve $m(t)$ in M can be lifted to a horizontal curve $g(t)$. Any two horizontal lifts $g_1(t)$ and $g_2(t)$ satisfy $g_2(t) = g_1(t)h$ for some fixed element $h \in K$. Then $\frac{dg_2}{dt} = g_2(t)U_2(t) = g_2(t)Ad_{h^{-1}}U_1(t)$ where $U_1(t) = g^{-1}(t)\frac{dg_1}{dt}$. Hence

$$\frac{dm}{dt} = d_o\phi_{g_2(t)}\vec{U_2}(t)(o) = d_o d\phi_{g_1(t)}\vec{U}_1(t)(o).$$

To summarize, every curve $m(t)$ in M is the projection of a horizontal curve $g(t)$ a solution of $\frac{dg}{dt} = g(t)U(t)$ for some absolutely continuous curve $U(t) \in \mathfrak{p}$, in which case

$$\frac{dm}{dt} = d_o\phi_{g(t)}\vec{U}(t)(o). \tag{2.3}$$

2.2. The Levi-Civita connection and the parallel transport. Suppose now that $M = G/K$ is a symmetric semi-Riemannian manifold, and suppose that its isometry group $G = I_0(M)$ is equipped with a left-invariant semi-Riemannian metric induced by an (arbitrary) orthogonal extension to $\mathfrak{g}$ of an Ad_K invariant scalar product $\langle\langle\, ,\, \rangle\rangle$ on $\mathfrak{p}$. Then,

Lemma 2.1.
$$\langle\langle V, [W, U]\rangle\rangle = \langle\langle [V, W], U\rangle\rangle$$
for any elements U,V in $\mathfrak{p}$ and W in $\mathfrak{k}$.

Proof. $\langle\langle Ad_hU, Ad_hV\rangle\rangle = \langle\langle U, V\rangle\rangle$ for any $h \in K$. If $h(t) = \exp tW$, then

$$0 = \frac{d}{dt}\langle\langle Ad_hU, Ad_hV\rangle\rangle|_{t=0} = \langle\langle [W, U], V\rangle\rangle + \langle\langle U, [W, V]\rangle\rangle. \qquad \square$$

Let $\nabla_U V$ denote the corresponding Levi-Civita connection on TG corresponding to the above metric.

PROPOSITION 2.5. *Suppose that $\bar{U}(g) = gU$, and $\bar{V}(g) = gV$ are left-invariant vector fields defined by U and V in $\mathfrak{p}$. Then ,*

$$\bigtriangledown_{\bar{U}} \bar{V}(g) = \frac{1}{2} g[U, V],$$

PROOF. We will use Koszul's formula

$$2\langle\langle \bigtriangledown_{\bar{U}} \bar{V}, \bar{X} \rangle\rangle = \bar{U} \langle\langle \bar{V}, \bar{X} \rangle\rangle + \bar{V} \langle\langle \bar{X}, \bar{U} \rangle\rangle - \bar{X} \langle\langle \bar{U}, \bar{V} \rangle\rangle - \langle\langle \bar{U}, [\bar{V}, \bar{X}] \rangle\rangle + \langle\langle \bar{V}, [\bar{X}, \bar{U}] \rangle\rangle + \langle\langle X, [\bar{U}, \bar{V}] \rangle\rangle.$$

Let now $\bigtriangledown_{\bar{U}} \bar{V} = g(D_U V)$ for some element $D_U V$ in $\mathfrak{g}$ and let $\bar{X} = gX$ for $X \in \mathfrak{g}$. By the invariance of the metric $\langle\langle gX, gY \rangle\rangle_g = \langle\langle X, Y \rangle\rangle_e$, and hence is constant. Therefore, $\bar{Z} \langle\langle \bar{X}, \bar{Y} \rangle\rangle = 0$ for any left-nvariant vector field $\bar{Z}$. Koszul's formula then simplifies to

$$2\langle\langle D_U V, X \rangle\rangle_e = -\langle\langle U, [V, X] \rangle\rangle_e + \langle\langle V, [X, U] \rangle\rangle_e + \langle\langle X, [U, V] \rangle\rangle_e$$

If $X \in \mathfrak{k}$ the first two terms cancel due to Lemma 2.1, and if $X \in \mathfrak{p}$, then both $[V, X]$ and $[U, X]$ are in $\mathfrak{k}$, and, then, by the orthogonality, the fist two terms are both equal to zero. Hence $2D_U V = [U, V]$. □

PROPOSITION 2.6. *Suppose that $W(t) = g(t)V(t), V(t) \in \mathfrak{p}$ is a curve of tangent vector fields along a horizontal curve $g(t)$, a solution of $\frac{dg}{dt} = gU(t), U(t) \in \mathfrak{p}$. Then, the covariant derivative $\bigtriangledown_{g(t)U(t)} W(t)$ of $W(t)$ along $g(t)$ is given by*

$$\bigtriangledown_{g(t)U(t)} W(t) = g(t) \Big(\frac{dV}{dt} + \frac{1}{2} [U(t), V(t)] \Big).$$

PROOF. Let $A_1, \ldots, A_n$ denote an orthonormal basis in $\mathfrak{p}$. Then $U(t) = \sum_{i=1}^n u_i(t) A_i$ and $V(t) = \sum_{i=1}^n v_i(t) A_i$for some functions $u_1(t), \ldots, u_n(t)$ and $v_1(t), \ldots, v_n(t)$. If $X_i(g) = gA_i$, then $\frac{dg}{dt} = \sum_{i=1}^n u_i(t) X_i(g(t))$ and $V(t) = \sum_{i=1}^n v_i(t) X_i(g(t))$. It follows that

$$\begin{aligned} \frac{D_{g(t)} V(t)}{dt} &= \sum_{i=1}^n \frac{dv_i}{dt} X_i(g(t)) + v_i(t) \frac{D_{g(t)} X_i}{dt} \\ &= \sum_{i=1}^n \frac{dv_i}{dt} X_i(g(t)) + v_i \bigtriangledown_{\sum_{j=1}^n u_j(t) X_j} X_i(g) \\ &= \sum_{i=1}^n \frac{dv_i}{dt} X_i(g(t)) + \sum_{i,j=1}^n v_i u_j \bigtriangledown_{X_j} X_i \\ &= \sum_{i=1}^n \frac{dv_i}{dt} X_i(g(t)) + \frac{1}{2} \sum_{i,j=1}^n v_i u_j [X_i, X_j] = g(t) \Big(\frac{dV}{dt} + \frac{1}{2} [U(t) V(t)] \Big). \end{aligned}$$ □

PROPOSITION 2.7. *Let $\vec{U}$ and $\vec{V}$ be any Killing vector fields defined by U and V in $\mathfrak{p}$. Let $\bigtriangledown_{\vec{U}} \vec{V}$ denote the Levi Civita connection on M. Then $\bigtriangledown_{\vec{U}} \vec{V} = 0$. The covariant derivative $\frac{D_{m(t)}}{dt} v(t)$ of a vector field $v(t) = d_o \phi_{g(t)} \vec{V}(t)$ along a curve $m(t) = \phi_{g(t)}(o)$ in M is given by $\frac{D_{m(t)}}{dt} v(t) = d_o \phi_{g(t)} \frac{d\vec{V}(t)}{dt}(o)$.*

PROOF. It is easy to show using Koszul's formula that $\bigtriangledown_{\vec{U}}\vec{V}$ is equal to the orthogonal projection of $\bigtriangledown_{\bar{U}}\bar{V}$ on the horizontal distribution $\mathcal{H}(g)=\{gX : X\in\mathfrak{p}\}$. As in the Proposition 2.5, $\bar{U}$ and $\bar{V}$ are the left invariant vector fields defined by U and V. The same proposition gives $\bigtriangledown_{\bar{U}}\bar{V}=\frac{1}{2}[\bar{U},\bar{V}]$. Since $[U,V]\in\mathfrak{k}$, the projection of $\bigtriangledown_{\bar{U}}\bar{V}$ on $\mathcal{H}$ is zero.

Suppose now that $v(t)$ is a vector field along $m(t)=\phi_{g(t)}(o)$ for $g(t)$ a solution of $\frac{dg}{dt}=g(t)U(t), U(t)\in\mathfrak{p}$. It follows from above that there is a curve $V(t)\in\mathfrak{p}$ such that $v(t)=d_o\phi_{g(t)}\vec{V}(t)(o)$. Then $\frac{D_{m(t)}}{dt}v(t)=d_o\phi_{g(t)}\Big(\frac{d\vec{V}}{dt}+\bigtriangledown_{\vec{U}(t)}\vec{V}(t)\Big)(o)=d_o\phi_{g(t)}\frac{d\vec{V}}{dt}(o)$. □

3. Rollings of symmetric semi-Riemannian manifolds

Let us now return to the intrinsic equations of rolling (1.1), whereby a curve $\alpha(t)$ in a Riemannian manifold M rolls on a curve $\hat{\alpha}(t)$ in another Riemannian manifold $\hat{M}$ of the same dimension if there is an isometry $A(t)$ such that

$$A(t)\colon T_{\alpha(t)}M\to T_{\hat{\alpha}(t)}\hat{M},\frac{d\hat{\alpha}}{dt}(t)=A(t)\frac{d\alpha}{dt}(t),$$

and $\hat{v}(t)=A(t)v(t)$ is a parallel transport along $\hat{\alpha}(t)$ for each parallel transport $v(t)$ along $\alpha(t)$. Note that the following properties follow from the above definition.

- *Reflexive property:* if $A(t)$ rolls $\alpha(t)$ in M onto a curve $\hat{\alpha}(t)$ in $\hat{M}$ then $A^{-1}(t)$ rolls $\hat{\alpha}(t)$ onto $\alpha(t)$.
- *Transitive property:* if $A(t)$ rolls $\alpha(t)$ on $\hat{\alpha}(t)$, and if $B(t)$ is an isometry that rolls $\hat{\alpha}(t)$ on $\beta(t)$, then $B(t)A(t)$ is the isometry that rolls $\alpha(t)$ on $\beta(t)$.
- *no need for the initial contact between M and $\hat{M}$.*

On symmetric semi-Riemannian spaces rolling on their tangent spaces the equations of rolling equations follow almost immediately from the definition according to the following proposition.

PROPOSITION 3.1. *Suppose that $\hat{M}$ is the tangent space T_oM at o in a symmetric semi-Riemannian space M. A curve $\alpha(t)$ in M rolls on a curve $\hat{\alpha}(t)$ in $\hat{M}$, a solution of $\dot{\hat{\alpha}}(t)=\vec{U}(t)(o)$, by the isometry $A(t)=(d_g\phi_{g(t)})^{-1}$ defined by a horizontal lift $g(t)$ of $\alpha(t)$ and a solution of $\frac{dg}{dt}=g(t)U(t), U(t)\in\mathfrak{p}$.*

PROOF. Let $\alpha(t)=\phi_{g(t)}(o)$ with $g(t)$ a solution of $\frac{dg}{dt}=g(t)U(t)$ for some curve $U(t)\in\mathfrak{p}$. It follows (equation (2.3)) that

$$\frac{d\alpha}{dt}=d_o\phi_{g(t)}\vec{U}(t)(o).$$

If we now let $\hat{\alpha}(t)$ denote a solution of $\frac{d\hat{\alpha}}{dt}=\vec{U}(t)(o)$, then the preceding equation can be written as

$$\frac{d\alpha}{dt}=d_o\phi_{g(t)}\frac{d\hat{\alpha}}{dt}=A^{-1}(t)\frac{d\hat{\alpha}}{dt},$$

where $A^{-1}(t)=d_o\phi_{g(t)}$. So we have

$$\frac{d\hat{\alpha}(t)}{dt}=A(t)\frac{d\alpha(t)}{dt}. \tag{3.1}$$

We need to show that $A(t)$ maps parallel transport $v(t)$ along $\alpha(t)$ onto a parallel transport $\hat{v}(t)$ along $\hat{\alpha}(t)$. Every vector field $v(t)$ along $\alpha(t)$ can be represented as $v(t) = d_o\phi_{g(t)}\vec{V}(t)(o)$ for some $V(t) \in \mathfrak{p}$. According to Proposition 2.7, $\frac{D_{\alpha(t)}}{dt}v(t) = d_o\phi_{g(t)}\frac{d\vec{V}}{dt}$. Then $v(t)$ is parallel whenever $\frac{D_{\alpha(t)}}{dt}v(t) = 0 = d_o\phi_{g(t)}\frac{d\vec{V}}{dt}$, that is, whenever $\frac{d\vec{V}(t)}{dt} = 0$. Therefore $\hat{v}(t) = A(t)v(t) = d_o\phi_{g(t)^{-1}}v(t) = \vec{V}(t)$, and $\vec{V}(t)$ is constant, hence parallel to $\hat{\alpha}(t)$. □

The above shows that each horizontal curve $g(t)$ in G, a solution of $\frac{dg}{dt} = g(t)U(t), U(t) \in \mathfrak{p}$, defines a family of curves $\hat{\alpha}(t)$ in T_oM, each a solution of $\frac{d\hat{\alpha}}{dt} = \vec{U}(t)(o)$, that roll on $\alpha(t) = \phi_{g(t)}(o)$. Conversely, every solution $(g(t), \hat{\alpha}(t))$ of the differential system

$$\frac{dg}{dt} = g(t)U(t), \frac{d\hat{\alpha}(t)}{dt} = \vec{U}(t)(o), U(t) \in \mathfrak{p} \tag{3.2}$$

singles out a curve $\alpha(t) = \phi_{g(t)}(o)$ in M that is rolled on $\hat{\alpha}(t)$ in T_oM by $d_{g^{-1}}\phi_{g^{-1}(t)}$. The triple $(\hat{\alpha}(t), d_g\phi_{g(t)}, \alpha(t))$ is known as a rolling curve in the existing literature on rolling [**17**, **18**].

We shall refer to (3.2) as the rolling distribution and we will denote it by $\mathcal{H}(g,p)$. Solutions of (3.2) generated by bounded and measurable curves $U(t), t \in [0,T]$ will be referred to as the rolling motions. Rolling motions take place in the configuration space

$$\mathbf{G} = G \times \hat{M}, \hat{M} = T_aM$$

which we regard as a Lie group, with the group operation $\mathbf{gh} = (g,p)(h,q) = (gh, p+q)$, $\mathbf{g} = (g,p)$ and $\mathbf{h} = (h,q)$. Then the Lie algebra $\mathcal{G}$ of $\mathbf{G}$ is naturally identified with $\mathfrak{g} \times \hat{M}$ and its canonical Lie bracket $[(X,p),(Y,q)] = ([X,Y],0)$. It then follows that $\mathcal{H}(g,p)$ is a left-invariant distribution given by

$$\mathcal{H}(\mathbf{g}) = \mathcal{H}(g,p) = \{(gU, \vec{U}(o)) : U \in \mathfrak{p}\}, \mathbf{g} = (g,p) \in \mathbf{G}.$$

Let $\Gamma = \{(U, \vec{U}(o)) : U \in \mathfrak{p}\}$, so that $\mathcal{H}(g,p) = \mathbf{g}\Gamma$. Since Γ is a vector subspace in $\mathcal{G}$ that satisfies

$$\Gamma + [\Gamma,\Gamma] + [\Gamma,[\Gamma,\Gamma]] = \mathcal{G}, \tag{3.3}$$

the Lie algebra generated by the left-invariant vector fields tangent to $\mathcal{H}$ is equal to $\mathbf{G}$ and, therefore, any two points in $\mathbf{G}$ can be connected by a rolling motion (by the Chow-Rashevsky theorem [**27**]).

In the Riemannian case rolling motions inherit natural length $\int_0^T \sqrt{\langle\langle U(t), U(t)\rangle\rangle}dt$ from G. Since vector fields in $\mathcal{H}$ are complete, any pair of points in $\mathbf{G}$ can be connected by an integral curve of $\mathcal{H}$ of minimal length [**16**]. An integral curve $\gamma(t)$ of $\mathcal{H}$ is called a *rolling geodesic* if for any t_0 and t_1, sufficiently close to each other, the length of $\gamma(t)$ in the interval $[t_0, t_1]$ is minimal among all other integral curves of $\mathcal{H}$ that connect $\gamma(t_0)$ to $\gamma(t_1)$.

To put the matter in a control theoretic context, let $A_1, \dots, A_m$ be an orthonormal basis in $\mathfrak{p}$ so that $(A_i, \vec{A}_i(o))$ is an orthonormal basis in Γ. Then, an absolutely continuous curve $\mathbf{g}(t) = (g(t), p(t))$ is a rolling motion if and only if

$$\frac{dg}{dt} = g(t)(\sum_{i=1}^{m} u_i(t)A_i), \frac{dp}{dt} = \sum_{i=1}^{m} u_i(t)\vec{A}_i(o), \tag{3.4}$$

for some bounded and measurable control functions $u_1(t), \ldots, u_m(t)$. Then, the length of $\mathbf{g}(t)$ is given by $\int_0^T \sqrt{u_1^2(t) + \cdots + u_m^2(t)}dt$. In the semi-Riemannian case $\langle\langle U(t), U(t)\rangle\rangle$ is of indefinite sign and does not lend itself to any length interpretations. This potential obstacle is bypassed when the length functional is replaced by the energy functional $E = \frac{1}{2}\int_0^T \langle\langle U(t), U(t)\rangle\rangle dt$, in which case the rolling geodesics are redefined as the rolling motions along which the energy of transfer is critical (rather than optimal) relative to the given boundary conditions $\mathbf{g}(0) = \mathbf{g}_0$ and $\mathbf{g}(T) = \mathbf{g}_1$. Our ultimate aim is to find the differential equations that the rolling geodesics must satisfy. But first let us look at some particular cases.

4. Some noteworthy cases

4.1. Rolling spheres and rolling hyperboloids. According to the Killing-Hopf theorem, the universal cover of a complete Riemannian manifold of constant curvature is isometric to the hyperboloid $\mathbb{H}^n$, the sphere S^n, or the Euclidean space $\mathbb{E}^n$ depending whether the curvature is negative, positive or zero [24]. These three prototypes are known as the space forms. Let us now consider rollings of non-Euclidean space forms on their tangent spaces under the action of the isometry groups. For that purpose let

$$(x, y)_\epsilon = x_0y_0 + \epsilon\sum_{i=1}^{n} x_iy_i, \epsilon = \pm 1, \tag{4.1}$$

for x and y in $\mathbb{R}^{n+1}$. For $\epsilon = 1$ this scalar product is Euclidean, and for $\epsilon = -1$ it is Lorentzian. Let now $\|x\|_\epsilon^2 = (x, x)_\epsilon$ and then define

$$S_\epsilon^n(\rho) = \{x \in \mathbb{R}^{n+1} : \|x\|_\epsilon^2 = \rho^2, x_0 > 0 \text{ when } \epsilon = -1\}.$$

We will refer to $S_\epsilon^n(\rho)$ as the sphere of radius ρ (Euclidean when $\epsilon = 1$ and hyperbolic when $\epsilon = -1$). Indeed for $\epsilon = -1$, the hyperbolic sphere is the connected component of the hyperboloid $x_0^2 = \rho^2 + \sum_{i=1}^n x_i^2$ through $x_0 = \rho$. Below we will show that each sphere $S_\epsilon^n(\rho)$ is a Riemannian symmetric manifold with its metric defined by $\langle\dot{x}, \dot{y}\rangle_\epsilon = \epsilon(\dot{x}, \dot{y})_\epsilon$ for each pair of tangent vectors $\dot{x}$ and $\dot{y}$ in $T_pS_\epsilon^n(\rho)$.

Let now G_ϵ be $SO(n+1)$ when $\epsilon = 1$, and be the connected component of $SO(1, n)$ that contains the group identity when $\epsilon = -1$. Each group G_ϵ acts on the points of $\mathbb{R}^{n+1}$ by the matrix multiplication and preserves the bilinear form (4.1). We will regard G_ϵ as a semi-Riemannian manifold with its semi-Riemannian left-invariant metric given by

$$\langle\langle RX, RY\rangle\rangle_\epsilon = \langle\langle X, Y\rangle\rangle_\epsilon = -\frac{\epsilon\rho^2}{2}Tr(XY),$$

for any X and Y in the Lie algebra $\mathfrak{g}_\epsilon$ of G_ϵ, where RX and RY are the abreviates for $d_eL_R(X)$ and d_eL_RY. This metric is G_ϵ invariant because the trace form is Ad_{G_ϵ} invariant.

Each group G_ϵ acts transitively on the sphere $S^n_\epsilon(\rho)$ via the action $\phi_R(x) = Rx$ and identifies the sphere $S^n_\epsilon(\rho)$ as the quotient $S^n_\epsilon(\rho) = SO_\epsilon(n+1)/K_\epsilon$, where K_ϵ is the isotropy group of a given point $o \in S^n_\epsilon(\rho)$. That is, $\{Ro : R \in SO_\epsilon(n+1)\}$ is identified with $\{RK_\epsilon : R \in G_\epsilon\}$. If π denotes the natural projection from G_ϵ onto the quotient space G_ϵ/K_ϵ then $\pi(R) = \phi_R(o) = Ro$. We then have

$$\pi(gR) = \phi_{gR}o = (gR)o = g(Ro) = \phi_g(\pi(R)), g, R, \in G_\epsilon.$$

The realization of $S^n_\epsilon(\rho)$ as the orbit $\{Ro : R \in G_\epsilon\}$ induces a decomposition $\mathfrak{g}_\epsilon = \mathfrak{p}_\epsilon \oplus \mathfrak{k}_\epsilon$ with $\mathfrak{k}_\epsilon$ the Lie algebra of the isotropy group K_ϵ and $\mathfrak{p}_\epsilon$ its orthogonal complement in $\mathfrak{g}_\epsilon$. Since G_ϵ preserves the scalar product $(\ ,\ )_\epsilon$, the elements of $\mathfrak{g}_\epsilon$ are skew-symmetric, that is, satisfy $(Xu, v)_\epsilon = -(u, Xv)_\epsilon$ for each $X \in \mathfrak{g}_\epsilon$.

Elements in $\mathfrak{g}_\epsilon$ can be represented in terms of linear operators $(u \wedge_\epsilon v) = u \otimes_\epsilon v - v \otimes_\epsilon u$, $u \in R^{n+1}, v \in R^{n+1}$, where $(u \otimes_\epsilon v)x = (v, x)_\epsilon u, x \in \mathbb{R}^{n+1}$. One can easily verify that

$$((u \wedge_\epsilon v)x, y)_\epsilon + (x, (u \wedge_\epsilon v)y)_\epsilon = 0,$$

hence, $(u \wedge_\epsilon v) \in \mathfrak{g}_\epsilon$. Then one can show that $\mathfrak{g}_\epsilon$ is the linear span of $\{u \wedge_\epsilon v : u, v \in \mathbb{R}^{n+1}\}$ by an easy dimensionality argument. The preceding operators satisfy

$$\begin{aligned} [u \wedge_\epsilon v, w \wedge_\epsilon z] = {} & (u, w)_\epsilon (v \wedge_\epsilon z) + (v, z)_\epsilon (u \wedge_\epsilon w) \\ & - (v, w)_\epsilon (u \wedge_\epsilon z) - (u, z)_\epsilon (v \wedge_\epsilon w). \end{aligned} \tag{4.2}$$

In addition, they conform to the following inner product in $\mathfrak{g}_\epsilon$:

$$\begin{aligned} \langle\langle u \wedge_\epsilon v, w \wedge_\epsilon z \rangle\rangle_\epsilon &= -\frac{\epsilon\rho^2}{2} Tr((u \wedge_\epsilon v)(w \wedge_\epsilon z)) \\ &= \epsilon\rho^2((v, z)_\epsilon (u, w)_\epsilon - (w, v)_\epsilon (u, z)_\epsilon) \end{aligned} \tag{4.3}$$

The following formula will be needed later in the paper.

Proposition 4.1. *Let* $U = u \wedge_\epsilon o, V = v \wedge_\epsilon o, W = w \wedge_\epsilon o$, *where* $\langle\langle V, W \rangle\rangle_\epsilon = 0$. *Then*

$$\|[U, [V.W]]\|^2_\epsilon = \frac{1}{\rho^4}(\langle\langle V, W \rangle\rangle^2_\epsilon \|V\|^2_\epsilon + \langle\langle U, V \rangle\rangle^2_\epsilon \|W\|^2_\epsilon. \tag{4.4}$$

Proof. It follows from above that $[V, W] = \rho^2(v \wedge_\epsilon w)$, and

$$\begin{aligned} [U, [V, W]] &= \rho^2 [u \wedge_\epsilon o, v \wedge_\epsilon w] \\ &= \rho^2((u, w)_\epsilon V - (u, v)_\epsilon W). \end{aligned}$$

Hence,

$$\|[U, [V, W]]\|^2_\epsilon = \rho^4 (u, w)^2_\epsilon \|V\|^2_\epsilon + (u, v)^2_\epsilon \|W\|^2_\epsilon.$$

But then,

$$\langle\langle U, V \rangle\rangle = \epsilon\rho^4 (u, v)\epsilon, \langle\langle U, W \rangle\rangle_\epsilon = \epsilon\rho^4(u, w)\epsilon$$

gives the desired formula. $\square$

In this notation,

$$\mathfrak{k}_\epsilon = \{u \wedge_\epsilon v, (u,o)_\epsilon = (v,o)_\epsilon = 0\}, \mathfrak{p}_\epsilon = \{u \wedge_\epsilon o, u \in \mathbb{R}^{n+1}, (u,o)_\epsilon = 0\}.$$

The reader can readily verify using (4.2) that these factors satisfy Cartan's Lie algebraic relations

$$[\mathfrak{p}_\epsilon, \mathfrak{p}_\epsilon] = \mathfrak{k}_\epsilon, [\mathfrak{p}_\epsilon, \mathfrak{k}_\epsilon] = \mathfrak{p}_\epsilon, [\mathfrak{k}_\epsilon, \mathfrak{k}_\epsilon] \subseteq \mathfrak{k}_\epsilon,$$

Then each $U = u \wedge_\epsilon o$ gives rise to a Killing vector field $\vec{U}$ through $\vec{U}(o) = \frac{d}{dt}\exp(tU(o)|_{t=0} = Uo = \rho^2 u$. Then for each U, V in $\mathfrak{p}_\epsilon$ we have

$$\langle\langle U, V\rangle\rangle_\epsilon = \epsilon\rho^4(u,v)_\epsilon = \langle \vec{U}o, \vec{V}o\rangle_\epsilon \tag{4.5}$$

hence the mapping $U \to \vec{U}$ is an isometry. According to the lemma below, $\langle\langle\, ,\, \rangle\rangle_\epsilon$ is positive on $\mathfrak{p}_\epsilon$, so each sphere $S^n_\epsilon(\rho)$ is a Riemannian symmetric space.

Lemma 4.1. *The bilinear form $\langle\langle\, ,\, \rangle\rangle_\epsilon$ is positive on $\mathfrak{p}_\epsilon$, and negative on $\mathfrak{k}_\epsilon$ when $\epsilon = -1$.*

Proof. It follows from (4.3) that $\langle\langle u \wedge_\epsilon o, v \wedge_\epsilon o\rangle\rangle_\epsilon = \epsilon\rho^4(u,v)_\epsilon$. When $o = \rho e_0$ then $\|o\|^2_\epsilon = \rho^2$ and $(u,v)_\epsilon = \epsilon\sum_{i=1}^n u_i v_i$. Therefore,

$$\langle\langle u \wedge_\epsilon \rho e_0, v \wedge_\epsilon \rho e_0\rangle\rangle_\epsilon = \rho^4 \sum_{i=1}^n u_i v_i.$$

Evidently, this form is positive independently of ϵ, but on $\mathfrak{k}_\epsilon$ it is negative for $\epsilon = -1$.

Suppose now that o is any point on the sphere $S^n_\epsilon(\rho)$. Since SO_ϵ acts transitively on each sphere $S^n_\epsilon(\rho)$ there exists an element R of SO_ϵ such $Re_0 = o$. Then $R(u \wedge_\epsilon e_0)R^{-1} = Ru \wedge_\epsilon o$ and Ru is orthogonal to o. Since the quadratic form $\langle\langle\, ,\, \rangle\rangle_\epsilon$ is invariant under the conjugations by elements in G_ϵ we have

$$\langle u \wedge_\epsilon e_0, v \wedge_\epsilon e_0\rangle = \langle R(u \wedge_\epsilon e_0)R^{-1}, R(v \wedge_\epsilon e_0)R^{-1})\rangle = \langle Ru \wedge_\epsilon o, Rv \wedge_\epsilon o\rangle.$$

Likewise, one shows that $\langle\, ,\, \rangle_{-1}$ is negative on $\mathfrak{k}_o$ when $\epsilon = -1$. □

Let us now return to the rolling on tangent spaces. The following proposition is a direct corollary of Proposition 3.1.

Proposition 4.2. *A curve $\alpha(t) = R(t)o$ in $S^n_\epsilon(\rho)$ is rolled on $\hat{\alpha}(t)$ in $T_o S^n_\epsilon(\rho)$ by an isometry $R^{-1}(t)$ whenever the following equations hold*

$$\alpha(t) = R(t)o, \frac{dR}{dt} = R(t)(u(t) \wedge_\epsilon o), \frac{d\hat{\alpha}}{dt} = \vec{U}(o) = \rho^2 u(t). \tag{4.6}$$

Rolling equations (4.6) can be modified to include the rollings of spheres centred at any point c in R^{n+1}. If $S^n_\epsilon(\rho, c)$ is such a sphere then G_ϵ acts on it by the action $\phi_{R(t)}p = c + R(t)(p - c)$. If o is a point on $S^n_\epsilon(\rho, c)$, and if $\alpha(t)$ based at o then $\alpha(t) = \phi_{R(t)}(o) = c + R(t)(o - c)$. Relative to this action the Killing vector fields are given by $\vec{U}(p) = U(p - c)$ for each $U \in \mathfrak{p}_\epsilon$. It follows that $\alpha(t)$ is rolled on $\hat{\alpha}(t)$ in the tangent space $T_o S^n_\epsilon(\rho, c)$ by a curve $R(t)$ in G_ϵ whenever the following modified equations hold

$$\frac{dR}{dt} = R(t)(u(t) \wedge_\epsilon (o - c), \frac{d\hat{\alpha}}{dt}(t) = \vec{U}(t)(o) = U(t)(o - c), \tag{4.7}$$

For instance, when $S^n_\epsilon(\rho, c)$ rolls on the hyperplane $P = \{q \in \mathbb{R}^{n+1} : q_{n+1} = 0\}$ its centre is confined to $\{q \in \mathbb{R}^{n+1} : q_{n+1} = \rho\}$. Hence $o = 0$ is on the sphere when $c = \rho e_{n+1}$ in which case $o - c = -\rho e_{n+1}$. Then $\alpha(t) = c + R(t)(-\rho e_{n+1})$ is rolled on $\dot{\hat{\alpha}}(t) = \vec{U}(t)(o) = -\rho U(t)(e_{n+1})$ by the isometry $R^{-1}(t)$, a solution of $\frac{dR}{dt} = R(t)U(t)$, with $U(t) = -\rho u(t) \wedge_\epsilon (e_{n+1})$. It follows that $U(t) = \rho\begin{pmatrix} 0 & u(t) \\ -\epsilon u^T(t) & 0 \end{pmatrix}$, and

$$\dot{\hat{\alpha}}(t) = -\rho u(t) \wedge_\epsilon e_{n+1}(-\rho e_{n+1}) = \rho^2 \begin{pmatrix} u(t) \\ 0 \end{pmatrix}.$$

On the other hand, when the sphere $S^n_\epsilon(\rho)$ is rolled on the hyperplane $P = \{q \in \mathbb{R}^{n+1} : q_1 = 0\}$ then $o - c = -\rho e_1$. Hence $U(t) = -u(t) \wedge_\epsilon (\rho e_1) = \rho\begin{pmatrix} 0 & \epsilon u^T(t) \\ -u(t) & 0 \end{pmatrix}$ and $\dot{\hat{\alpha}}(t) = \rho^2\begin{pmatrix} u(t) \\ o \end{pmatrix}$ which agrees with the results obtained in [7].

4.1.1. *Spheres rolling on spheres.* Equations (4.6) and (4.7) can be used to obtain the rolling equations for the sphere of radius ρ rolling on the sphere of radius σ by employing a brilliant observation of F. Silva Leite and F. Louro [25] that the rollings of the spheres can be done through the intermediate rolling on the common tangent plane. In particular, we will consider the case where the sphere $S^n_\epsilon(\rho, (\rho + \sigma)e_0)$ is rolled on the stationary sphere $S^n_\epsilon(\sigma, 0)$. Then $x_0 = \sigma e_0$ is the "north pole" for the stationary sphere centred at $c = 0$ as well as the "south pole" for the rolling sphere centred at $c = (\rho + \sigma)e_0$.

Let $M_1(\rho, \epsilon)$ denote the sphere of radius ρ centred at $c = (\rho + \sigma)e_0$ and let $M_2(\sigma, \epsilon)$ denote the stationary sphere of radius σ centred at the origin of R^{n+1}. Then $o = \sigma e_0$ is the common point for the two spheres in both the hyperbolic and the Euclidean case. If $\alpha(t), \alpha(0) = o$ is a curve on $M_1(\rho, \epsilon)$ then $\alpha(t)$ is rolled onto $\beta_1(t)$ in the tangent plane $T_o M_1(\rho, \epsilon)$ by a horizontal curve $g_1(t)$ that is a solution of $\frac{dg_1}{dt} = g_1(t)U_1(t), U_1(t) = u_1(t) \wedge_\epsilon (o - c)$ with

$$\dot{\beta}_1(t) = \vec{U}_1(t)(o) = U_1(t)(o - c) = U_1(t)(-\rho e_0).$$

Likewise, a curve $\alpha_2(t)$ in $M_2(\sigma, \epsilon)$, $\alpha_2(0) = \sigma e_0$, is rolled on a curve $\beta_2(t)$ in $T_{\sigma e_0} M_2(\sigma, \epsilon)$ by a curve $g_2(t)$ that projects onto $\alpha_2(t)$, that is, $\alpha_2(t) = g_2(t)(\sigma e_0)$, and is a solution of

$$\frac{dg_2}{dt} = g_2(t)U_2(t), U_2(t) = u_2(t) \wedge_\epsilon \sigma e_0$$

with $\dot{\beta}_2(t) = \vec{U}_2(t)(\sigma e_0) = U_2(t)(\sigma e_0)$.

If we now impose the condition that $\beta_1(t) = \beta_2(t)$ then by the transitivity property of rollings, $\alpha_1(t)$ is rolled on $\alpha_2(t)$ by $g(t) = g_2(t)g_1^{-1}(t)$ since $g_2^{-1}\dot{\alpha}_2(t) = \dot{\beta}_2 = \dot{\beta}_1 = g_1^{-1}\dot{\alpha}_1(t)$. Then

$$U_1(-\rho e_0) = \dot{\beta}_1(t) = \dot{\beta}_2(t) = U_2(\sigma e_0) \Rightarrow u_1 = \frac{\sigma^2}{\rho^2} u_2,$$

therefore, $U_1 = \frac{\sigma^2}{\rho^2} u_2 \wedge_\epsilon (-\rho e_0) = -\frac{\sigma}{\rho} U_2$.

$$\Omega_\epsilon(t)g(t) = \frac{d}{dt} g_2(t)g_1^{-1}(t) = g_2 U_2 g_1^{-1} - g_2 U_1 g_1^{-1} = \frac{\rho + \sigma}{\rho} g_2 U_2 g_2^{-1}(t)g(t).$$

Thus,

$$\Omega_\epsilon(t) = \frac{\rho+\sigma}{\rho} g_2(t)U_2(t)g_2^{-1}(t).$$

Hence

$$\dot{\alpha}_2(t) = \frac{dg_2}{dt}(\sigma e_0) = \sigma g_2(t)U_2(t)e_0 = (g_2U_2g_2^{-1})\alpha_2(t) = \frac{\rho}{\rho+\sigma}\Omega_\epsilon(t)\alpha_2(t).$$

If we now write $U_2(t) = \frac{\rho}{\rho+\sigma}U_\epsilon(t)$ where $\mathrm{U}_\epsilon(t) = u(t)\wedge_\epsilon e_0$, and rename the variables $g_2 = S$, $g_1 = \hat{S}$ and $g(t) = R(t)$, then we get:

$$\Omega_\epsilon(t) = S(t)U_\epsilon(t)S^{-1}(t) = S(t)u(t)\wedge_\epsilon S(t)e_0, \frac{dR}{dt} = \Omega_\epsilon(t)R(t),$$
$$\alpha_1(t) = \hat{S}(o) = c + \hat{S}(o-c) = -\rho\hat{S}(t)e_0 + (\rho+\sigma)e_0, \alpha_2(t) = \sigma S(t)e_0,$$
$$\frac{dS}{dt} = \frac{\rho}{\rho+\sigma}SU_\epsilon(t), \frac{d\hat{S}}{dt} = -\frac{\sigma}{\rho+\sigma}\hat{S}U_\epsilon(t).$$

These equations agree with the rolling equations obtained in [7].

4.2. Semi-simple Lie groups. Every semi-simple Lie algebra of a connected Lie group G admits an Ad_G invariant and non-degenerate scalar product $\langle X, Y\rangle$ equal to a constant multiple of the Killing form $Kl(X,Y) = Trace(adX \circ adY)$. The scalar product induced a left-invariant metric $\langle\langle gX, gY\rangle\rangle_g = \langle X, Y\rangle$ where gX and gY stand respectively for $d_e\phi_g(X)$ and $d_e\phi_g Y$. This left-invariant metric is also bi-invariant because $\langle\,,\,\rangle$ is Ad_G invariant. It follows that the group of left translations $\{\phi_g : g \in G\}$ is the connected component $I_0(G)$ of the isometry group $I(G)$. Therefore, G together with the metric $\langle\langle\,,\,\rangle\rangle$ is a semi-Riemannian manifold.

However, G is also a symmetric semi-Riemannian manifold because $\zeta_e(g) = g^{-1}$ is a global symmetry at the group identity e, with $\zeta_g = \phi_g\zeta_e\phi_{g^{-1}}$ a global symmetry at any other point g in G.

Alternatively, this fact could be verified through the action of $G\times G$ on G given by

$$\phi_{(g_1,g_2)}h = g_1hg_2^{-1}, (g_1,g_2) \in G\times G.$$

For then, $G = G\times G/K$, where $K = \{(g,g) : g \in G\}$ is the isotropy group of this action through the group identity e. The group K is also the group of fixed points of the involutive automorphism $\sigma((g_1,g_2)) = (g_2,g_1)$. Hence $\mathfrak{p} = \{(X,-X) : X \in \mathfrak{h}\}$ and $\mathfrak{k} = \{X,X) : X \in \mathfrak{h}\}$ are the Cartan factors induced by σ.

When $G\times G$ is endowed with a left-invariant metric induced by the scalar product $\langle\langle (X_1,X_2),(Y_1,Y_2)\rangle\rangle = 2(\langle X_1,Y_1\rangle + \langle X_2,Y_2\rangle)$ then any horizontal curve $(g_1(t),g_2(t))$, a solution of $\frac{dg_1}{dt} = g_1(t)U(t)$, $\frac{dg_2}{dt} = -g_2(t)U(t)$ satisfies

$$\begin{aligned} 4\langle\langle U(t),U(t)\rangle\rangle &= \Big\langle\Big\langle \frac{d}{dt}(g_1(t),g_2(t)), \frac{d}{dt}(g_1(t),g_2(t))\Big\rangle\Big\rangle \\ &= \Big\langle \frac{d}{dt}(g_1(t)g_2^{-1}(t)), \frac{d}{dt}(g_1(t)g_2^{-1}(t))\Big\rangle \\ &= 4\langle g_1U(t)g_2^{-1}, g_1U(t)g_2^{-1}\rangle = 4\langle U(t),U(t)\rangle \end{aligned}$$

Thus G, with it metric induced by the Killing form, becomes a symmetric semi-Riemannian space (it is a symmetric Riemannian space only when G is compact [29]). Then Proposition 2.4 can be used to obtain the global symmetry. We have $\zeta_e \circ \pi = \pi \circ \sigma$ which yields

$$\zeta_e \pi(g_1, g_2) = \zeta_2(g_1 g_2^{-1}) = g_2 g_1^{-1} = (g_1 g_2^{-1})^{-1},$$

which agrees with the global symmetry quoted above.

In the category of matrix groups, $SL(n,\mathbb{R})$, the group of non-singular matrices with determinant one, is a particularly important semi-simple case. On $SL(n,\mathbb{R})$, $Kl(X,Y) = 2nTr(XY)$ [29]. For computational purposes it is more convenient to endow G with the left-invariant metric defined by $\langle\langle X, Y\rangle\rangle = \frac{1}{2}Tr(XY)$. Then, $\langle\langle X, X\rangle\rangle = \pm \sum_{i,j=1}^{n} x_{ij}^2$, depending whether X is symmetric or skew-symmetric.

4.3. Self adjoint subgroups of $SL(n,\mathbb{R})$. A closed subgroup H of $SL(n,\mathbb{R})$ is said to be self-adjoint if g^T, the transpose of g, is in H for each $g \in H$ [26]. Then $SL(n,\mathbb{R})$ as well as any self-adjoint subgroup H, admits an involutive automorphism $\sigma(g) = (g^T)^{-1}$. It follows that $K = \{\sigma(g) = g, g \in H\} = H \cap SO(n,\mathbb{R})$. In each case, H/K is a symmetric Riemannian space. In particular, $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is equal to the space of positive definite matrices with determinant one, and $SO_0(1,n)/SO(n,\mathbb{R})$ is the hyperboloid $\mathbb{H}^n$. For other cases, see [26, 27].

Let us consider $M = SL(n,\mathbb{R})/SO(n,\mathbb{R})$ in some detail. Then $K = SO(n,\mathbb{R})$ is the Lie group of points fixed by the automorphism σ and $\mathfrak{k} = so(n,\mathbb{R})$ is its Lie algebra. It follows that $d_e\sigma(X) = \frac{d}{dt}\exp(-tX)^T|_{t=0} = -X^T$. Hence $d_e\sigma(X) = -X$ if and only if X symmetric. Thus $\mathfrak{p}$ is equal to the space of symmetric matrices in $\mathfrak{g}$. It is easy to verify that $\langle\, ,\, \rangle$ is positive on $\mathfrak{p}$ and negative on $\mathfrak{k} = so(n)$. Therefore, G with its left-invariant metric induced by $\langle\, ,\, \rangle$ is a semi-Riemannian manifold.

Then, the quotient space $M = G/K$ will be identified with the space of positive-definite matrices of determinant one, denoted by $\mathcal{P}_n$, through the action

$$\tau_g(P) = gPg^T, \quad g \in SL(n,\mathbb{R}), \quad P \in \mathcal{P}_n.$$

Since any positive definite matrix P with $Det(P) = 1$ can be written as $P = SS^T$ for some $S \in SL(n,\mathbb{R})$ the action is transitive, and $\mathcal{P}_n$ can be identified with the orbit through the identity I. Horizontal curves are the solutions of $\frac{dg}{dt} = g(t)U(t)$, with $U(t) \in sl(n,\mathbb{R})$ symmetric. Any curve $\alpha(t)$ in $\mathcal{P}_n$ is the projection of a horizontal curve $g(t)$ and the length of $\alpha(t)$ is given by $\int_0^T \sqrt{\langle U(t), U(t)\rangle}\, dt$. Killing vector fields are given by $\vec{U}(P) = UP + PU^T$, $U \in sl(n,\mathbb{R})$ and $P \in M$. The rolling distribution is given by

$$\frac{dg}{dt} = g(t)U(t), \quad U(t) \in sl(n,\mathbb{R}), \quad U(t) = U^T(t), \quad \frac{dp}{dt} = \vec{U}(t)(o) = 2U(t).$$

The case $n = 2$ is somewhat special, for then $\mathcal{P}_2$ is isometrically diffeomorphic to the Poincaré upper half plane $\mathcal{P} = \{z = x + iy : y > 0\}$ with its metric $\frac{1}{y}\sqrt{\dot{x}^2 + \dot{y}^2}$. To elaborate, note that every $g \in SL(2)$ can be written as $g = PR$

where P is an upper triangular matrix and R a rotation matrix in $SO(2)$. In fact, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $SL(2)$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} 1 & ac+bd \\ 0 & c^2+d^2 \end{pmatrix} \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

Let now

$$P = \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} 1 & ac+bd \\ 0 & c^2+d^2 \end{pmatrix} = \begin{pmatrix} \frac{y}{\sqrt{y}} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix},$$

where $x = \frac{ac+bd}{c^2+d^2}, y = \frac{1}{c^2+d^2}$, and then define a mapping F from the Poincaré upper half plane to the space of positive definite matrices with determinant one by

$$F(x+iy) = gg^T = PP^T = \frac{1}{y} \begin{pmatrix} x^2+y^2 & x \\ x & 1 \end{pmatrix},$$

We will now show that F is an isometry from $\mathcal{P}$ with its hyperbolic metric onto $\mathcal{P}_2$ with its H-invariant metric. If $\tilde{\alpha}(t) = F(\alpha(t)$ then

$$\dot{\tilde{\alpha}}(t) = \dot{P}P^T + P\dot{P}^T = P(P^{-1}\dot{P} + \dot{P}^T(P^{-1})^T)P^T,$$

and therefore, $\|\dot{\tilde{\alpha}}(t)\| = \|P^{-1}\dot{P} + \dot{P}^T(P^{-1})^T\|$. If $Y = \frac{y}{\sqrt{y}}$ and $X = \frac{x}{\sqrt{y}}$, then an easy calculation shows that

$$P^{-1}\dot{P} + \dot{P}^T(P^T)^{-1} = \begin{pmatrix} 2\frac{\dot{Y}}{Y} & \frac{X\dot{Y}+\dot{X}Y}{Y^2} \\ \frac{X\dot{Y}+\dot{X}Y}{Y^2} & -2\frac{\dot{Y}}{Y} \end{pmatrix} = \frac{1}{y} \begin{pmatrix} \dot{y} & \dot{x} \\ \dot{x} & -\dot{y} \end{pmatrix},$$

and hence $\|\dot{\tilde{\alpha}}(t)\| = \frac{1}{y}\sqrt{\dot{x}^2+\dot{y}^2}$. It follows that $\|\dot{\alpha}(t)\| = \|\dot{\tilde{\alpha}}(t)\|$ and, therefore, F is an isometry.

It then follows that the rolling distribution has its isometric analogue on $\mathcal{P}$ rolling on the tangent space at i. In this scenario $SL(2,\mathbb{R})$ acts on $\mathcal{P}$ via the Moebius transformations $\tau_g(z) = \frac{az+b}{cz+d}, g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $\mathcal{P}$ is represented by the orbit $\{\tau_g(i) : g \in SL_(2,\mathbb{R})\}$. Horizontal curves are the solutions of $\frac{dg}{dt} = g(t)\left(\begin{smallmatrix} u_1(t) & u_2(t) \\ u_2(t) & -u_1(t) \end{smallmatrix}\right)$ and their projections on $\mathcal{P}$ are given by $z(t) = g(t)(i)$.

Suppose that $\alpha(t)$ is a curve such that $\alpha(0) = i$. Then $\alpha(t) = g(t)i$ for a horizontal curve $g(t)$ such that $g(0) = I$. Then

$$\frac{d\alpha(t)}{dt}|_{t=0} = \frac{d}{dt}g(t)(i)|_{t=0} = \frac{d}{dt}\frac{1}{c^2+d^2}(bd+ac+i)|_{t=0} = 2i(u_1 - iu_2).$$

Therefore rolling motions are the solutions of

$$\frac{dg}{dt} = g(t) \begin{pmatrix} u_1(t) & u_2(t) \\ u_2(t) & -u_1(t) \end{pmatrix}, \frac{dp}{dt} = 2i(u_1(t) - iu_2(t)).$$

5. Hamiltonian and Poisson systems: Extremal curves

We now come to the central part of the paper, the Hamiltonian systems associated with rolling problems. We will apply the Maximum Principle to obtain the extremal equations.

REMARK 5.1. Even though the Maximum Principle is invented for problems of optimal control for which it provides the necessary conditions that an optimal control must satisfy, it can be also used to provide the necessary conditions that a critical control (a control along which the variational equation is equal to zero) must satisfy.

5.1. Rolling Hamiltonians. Let us recall the rolling equations (4.6) for symmetric spaces M rolling on $\hat{M} = T_oM$

$$\frac{dg}{dt} = g(t)\left(\sum_{i=1}^{m} u_i(t)A_i\right), \quad \frac{dp}{dt} = \sum_{i=1}^{m} u_i(t)\vec{A}_i(o),$$

and the variational problem associated with the energy function

$$E = \frac{1}{2}\int_0^T \sum_{i=1}^{m} u_i^2(t)dt.$$

Our immediate aim is to use the Maximum Principle to obtain the equations for the extremal curves in the cotangent bundle $T^*\mathbf{G}$ of the configuration space $\mathbf{G} = G\times\hat{M}$. To emphasize the structure of the problem we will rewrite (3.4) as

$$\frac{d\mathbf{g}}{dt} = \sum_{i=1}^{m} u_i(t)X_i(\mathbf{g}),$$

where each X_i a left-invariant vector field $X_i(\mathbf{g}) = (gA_i, \vec{A}_i(o))$, $\mathbf{g} = (g, p)$ and we assume that $\mathbf{g}(t)$ is a critical trajectory generated by a control $\mathbf{u}(t)$ that satisfies the given boundary conditions $\mathbf{g}(0) = \mathbf{g}_0$ and $\mathbf{g}(T) = \mathbf{g}_1$. Then according to the Maximum Principle, $\mathbf{g}(t)$ is either the projection of a normal extremal curve $\xi(t)$ in $T^*\mathbf{G}$ along which the extended Hamiltonian

$$\mathcal{H}(\xi(t), u_1(t), \dots, u_m(t)) = -\frac{1}{2}\sum_{i=1}^{m} u_i^2(t) + \sum_{i=1}^{m} u_i(t)H_i(\xi(t)),$$

satisfies $\frac{\partial \mathcal{H}}{\partial u_j} = 0, j = 1, \dots, m$ at $u(t) = \mathbf{u}(t)$, or $\mathbf{g}(t)$ is the projection of an abnormal extremal curve $\xi(t), \xi(t) \neq 0$, that satisfies the constraints

$$H_i(\xi(t)) = 0, \quad i = 1, \dots, m.$$

In this notation, each $H_i(\xi(t)$ is the Hamiltonian $H_i(\xi(t)) = \xi(t)(X_i(\mathbf{g}(t)))$.

In what follows we will confine our attention to the normal extremal curves. It then follows that the normal extremal curves are generated by the controls $u_i(t) = H_i(\xi(t)), i = 1, \dots, m$, that is, the normal extremal curves are the solution curves of a single Hamiltonian vector field $\vec{H}$ corresponding to the rolling Hamiltonian

$$H(\xi) = \frac{1}{2}\sum_{i=1}^{m} H_i^2(\xi).$$

REMARK 5.2. On Riemannian symmetric spaces of constant curvature every optimal trajectory is the projection of a normal extremal curve, so the abnormal extremal curves can be ignored [7]. I suspect that the same is true for arbitrary symmetric Riemannian spaces. It is not clear if the same could be said about general symmetric semi-Riemannian spaces.

In the left-invariant trivialization $T^*\mathbf{G} = \mathbf{G} \times \mathcal{G}^*$ the above Hamiltonian is $\mathbf{G}$ invariant, hence it can be viewed as a function on $\mathcal{G}^*$. Thus, H is both a Hamiltonian relative to the symplectic structure of $T^*\mathbf{G}$ as well as a Hamiltonian relative to the Poisson structure of $\mathcal{G}^*$. Recall that the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson space relative to the Poisson bracket $\{f,h\}(\ell) = \ell([dh, df])$, $\ell \in \mathfrak{g}^*$, for any smooth functions f and h on $\mathfrak{g}^*$. Then $\vec{f}$ defined by $\vec{f}(h) = \{f,h\}$, $h \in C^\infty(\mathfrak{g}^*)$ is the Poisson vector field generated by f. As a consequence, the Hamiltonian system generated by H is given in the quadrature form:

$$\frac{d\ell}{dt} = -ad^*dH(\ell(t))(\ell(t)), \quad \frac{d\mathbf{g}}{dt} = \sum_{i=1}^m H_i(\ell(t))X_i(\mathbf{g}(t)), \ell \in \mathcal{G}^*,$$

where $ad^*dH(\ell)(\ell)X = -\ell[dH, X]$, $X \in \mathcal{G}$. It follows that

$$\frac{d\ell}{dt} = -ad^*dH(\ell(t))(\ell(t)) \tag{5.1}$$

is the Poisson equation on $\mathcal{G}^*$ associated with H. Most of this theory is taken from my earlier publications [**3**, **27**]. We will now examine the solutions of the above equations in more detail.

Let us also recall several fundamental facts about the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. First, $\mathfrak{g}^*$ is foliated by the coadjoint orbits of $\mathbf{G}$. Second, each coadjoint orbit is a symplectic submanifold in $\mathfrak{g}^*$, and, third, each coadjoint orbit is an invariant set for each Poisson vector field $\vec{f}$ [**27**]. We will now concentrate on the solutions of the associated Poisson equation.

Let us first comment on the structure of the coadjoint orbits in this situation. Since $\hat{M}$ is a vector space, its tangent space at 0 can be identified with $\hat{M}$. Then, the Lie algebra $\mathcal{G}$ can be identified with $\mathfrak{g} \oplus \hat{M}$, and its dual can be identified with $\mathcal{G}^* = \mathfrak{g}^* \oplus \hat{M}^*$, where

$$\mathfrak{g}^* = \{\ell \in \mathcal{G}^* : \ell(\dot{p}) = 0, \dot{p} \in \hat{M}\}, \quad \hat{M}^* = \{\ell \in \mathcal{G}^* : \ell(\mathfrak{g}) = 0\}.$$

It then follows that every $\ell \in \mathcal{G}^*$ can be written as $\ell = \ell_1 + \ell_2$ with $\ell_1 \in \mathfrak{g}^*$ and $\ell_2 \in \hat{M}^*$. Since $\hat{M}$, as a vector space, is an abelian algebra, the projection ℓ_2 on $\hat{M}^*$ is constant on each coadjoint orbit of $\mathbf{G}$. The argument is straightforward: if $\mathbf{g} = (g, p)$, and if $(X, \dot{p}) \in \mathcal{G}$ then

$$Ad^*_{\mathbf{g}}(\ell)(X + \dot{p}) = \ell(Ad_{\mathbf{g}^{-1}}(X + \dot{p})) = \ell(Ad_{g^{-1}}(X) + \dot{p}) = \ell_1(Ad_{g^{-1}}(X)) + \ell_2(\dot{p}),$$

It follows that the coadjoint orbits in $\mathcal{G}^*$ are of the form

$$\{Ad^*_g(\ell_1) : g \in G\} + \ell_2, \quad \text{for any} \quad \ell = \ell_1 + \ell_2.$$

This fact can be also verified directly from equation (5.1): we have

$$\frac{d\ell}{dt}V = -\ell[dH, V], \quad \text{for any any} \quad V = X + \dot{p} \ \text{ in } \ \mathcal{G},$$

where $dH = \sum_{i=1}^m H_i(\ell)(A_i + \vec{A}_i(o))$ and $H_i(\ell) = \ell_1(A_i) + \ell_2(\vec{A}_i(o))$. Therefore,

$$\frac{d\ell_1}{dt}(X) + \frac{d\ell_2}{dt}(\dot{p}) = -(\ell_1 + \ell_2)([dH, X + \dot{p}]) = -\sum_{i=1}^m H_i(\ell_1)[A_i, X].$$

from which follows that

$$\frac{d\ell_1}{dt}(X) = -\sum_{i=1}^m H_i \ell_1 [A_i, X], \quad X \in \mathfrak{g}, \frac{d\ell_2}{dt}(\dot{p}) = 0.$$

Since $\dot{p}$ is arbitrary $\frac{d\ell_2}{dt} = 0$.

To uncover other constants of motion, identify $\mathcal{G}^*$ with $\mathcal{G}$ via the natural quadratic forms on each of the factors, and then recast the preceding equations on $\mathcal{G}$. More precisely, identify each ℓ_2 in $\hat{M}^*$ with a tangent vector $l = \sum_{i=1}^m l_i \vec{A}_i(o)$ via the formula $\ell_2(\dot{p}) = \langle l, \dot{p} \rangle \dot{p} \in \hat{M}$. Similarly, identify $\ell_1 \in \mathfrak{g}^*$ with $L \in \mathfrak{g}$ via the formula $\ell_1(X) = \langle\langle L, X \rangle\rangle, X \in \mathfrak{g}$. Note that in this identification of the Lie algebras with their duals, coadjoint orbits $\{Ad_g^*(\ell_1) + \ell_2 : g \in G\}$ are identified with the affine sets $\{Ad_g(L) + l : g \in G\}$. Also, $L \in \mathfrak{g}$ is identified with the sum $L = L_\mathfrak{p} + L_\mathfrak{k}$, $L_\mathfrak{p} \in \mathfrak{p}$ and $L_\mathfrak{k} \in \mathfrak{k}$. Relative to the orthonormal basis $A_1, \ldots, A_m$ in $\mathfrak{p}$, $L_\mathfrak{p} = \sum_{i=1}^m P_i A_i$ where $P_i = \ell_1(A_i) = \langle\langle L, A_i \rangle\rangle$. It follows that

$$H_i(\xi) = \ell(A_i) + \vec{A}_i(o) = \ell_1(A_i) + \ell_2(\vec{A}_i(o)) = P_i + l_i,$$

and

$$\frac{d\ell_1}{dt}(X) = \left\langle\left\langle \frac{dL}{dt}, X \right\rangle\right\rangle = -\left\langle\left\langle L, \left[\sum_{i=1}^m (l_i + P_i)A_i, X\right]\right\rangle\right\rangle$$
$$= -\left\langle\left\langle \left[L, \sum_{i=1}^m (l_i + P_i)A_i\right], X \right\rangle\right\rangle, \quad \left\langle \frac{dl}{dt}, \dot{p} \right\rangle = \frac{d\ell_2}{dt}(t)(\dot{p}) = 0$$

Since X and $\dot{p}$ are arbitrary,

$$\frac{dL}{dt} = \left[\sum_{i=1}^m (l_i + P_i)A_i, L\right] = [A + L_\mathfrak{p}, L], \quad A = \sum_{i=1}^m l_i A_i, \quad \frac{dl}{dt} = 0. \tag{5.2}$$

Equations (5.2) constitute the Poisson equations on $\mathcal{G}$ generated by the rolling Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^m H_i^2 = \frac{1}{2}\sum_{i=1}^m (l_i + P_i)^2 = \frac{1}{2}\|A + L_\mathfrak{p}\|^2.$$

They may be also written as

$$\frac{dL_\mathfrak{p}}{dt} = [A + L_\mathfrak{p}, L_\mathfrak{k}], \frac{dL_\mathfrak{k}}{dt} = [A, L_\mathfrak{p}], \quad A = \sum_{i=1}^m l_i A_i, \quad \frac{dl}{dt} = 0. \tag{5.3}$$

We will refer to the above equations as *the rolling extremals*. Its solutions project onto the rolling geodesics via the differential equations

$$\frac{dg}{dt} = g(t)(A + L_\mathfrak{p}(t)), \quad \frac{dp}{dt} = \sum_{i=1}^m (l_i + P_i(t))\vec{A}_i(o).$$

Each extremal curve $(L_{\mathfrak{p}}(t), L_{\mathfrak{k}}(t), l)$ projects onto a geodesic $\mathbf{g}(t) = (g(t), p(t))$, and each geodesic further projects onto the pair of curves $\alpha(t) = \phi_{g(t)}(o)$ in M and $\hat{\alpha}(t) = p(t)$ in $\hat{M}$ that are rolled upon each other by $g(t)$.

5.2. Rolling and affine-quadratic Hamiltonians. The isometry groups of symmetric Riemannian spaces harbour another class of Hamiltonian systems that show remarkable connections with the equations of mechanical tops. These Hamiltonians are induced by a class of optimal control problems associated with an affine control system $\frac{dg}{dt} = g(t)(A + U(t))$ where A is an element in $\mathfrak{p}$ and where $U(t)$ is an arbitrary bounded and measurable control curve in $\mathfrak{k}$. In the case that $M = G/K$ is a Riemannian symmetric space, K is compact and the Killing form is negative definite on $\mathfrak{k}$. Hence $\langle\,,\,\rangle$, a negative scalar multiple of the Killing form, can be used to define an energy function $E = \frac{1}{2}\int_0^T \langle U(t), U(t)\rangle dt$. More generally, any positive linear operator $\mathcal{P}$ on $\mathfrak{k}$ could be used to define a modified energy function $E = \frac{1}{2}\int_0^T \langle \mathcal{P}(U(t)), U(t)\rangle dt$. This energy functional is bounded below, and assuming that the preceding control system is controllable, admits trajectories that satisfy the given boundary conditions along which the energy is minimal. Then every such optimal trajectory is the projection of an integral curve of the Hamiltonian system induced by $\mathcal{H} = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle$ (assuming that there are no abnormal optimal extremals).The case $\mathcal{P} = Id$ is called canonical. In this case the associated Hamiltonian system is given by

$$\frac{dL_{\mathfrak{k}}}{dt} = [A, L_{\mathfrak{p}}], \quad \frac{dL_{\mathfrak{p}}}{dt} = [L_{\mathfrak{k}}, L_{\mathfrak{p}}] + [A, L_{\mathfrak{k}}], \quad \frac{dg}{dt} = g(t)(A + L_{\mathfrak{k}}), \tag{5.4}$$

where $L \in \mathfrak{g}$ and $L = L_{\mathfrak{p}} + L_{\mathfrak{k}}$, $L_{\mathfrak{p}} \in \mathfrak{p}, L_{\mathfrak{k}} \in \mathfrak{k}$ [27].

This Hamiltonian system remains valid for semi-Riemannian spaces, in the sense that the solution curves project onto the rolling motions along which the energy is critical (rather than optimal, as in the Riemannian case).

The proposition below shows that the Poisson system generated by the affine-quadratic Hamiltonian $\mathcal{H}$ is an invariant subsystem of the Poisson system generated by the rolling Hamiltonian H (see also [28]).

PROPOSITION 5.1. *Let $\mathbf{g}(t) = (g(t), p(t)), \mathbf{L}_{\mathfrak{p}}(t), \mathbf{L}_{\mathfrak{k}}(t)$ be any integral curve of the rolling Hamiltonian $H = \frac{1}{2}\|\mathbf{A} + \mathbf{L}_{\mathfrak{p}}\|^2$, that is,*

$$\frac{dg}{dt} = g(t)(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t)), \qquad \frac{dp}{dt} = \sum_{i=1}^m (l_i + P_i)\vec{A}_i(o),$$

$$\frac{d\mathbf{L}_{\mathfrak{k}}}{dt} = [\mathbf{A}, \mathbf{L}_{\mathfrak{p}}], \qquad \frac{d\mathbf{L}_{\mathfrak{p}}}{dt} = [\mathbf{A} + \mathbf{L}_{\mathfrak{p}}, \mathbf{L}_{\mathfrak{k}}], \qquad \mathbf{A} = \sum_{i=1}^m l_i A_i$$

Then

$$\tilde{g}(t) = g(t)h(t), \quad L_{\mathfrak{p}}(t) = Ad_{h^{-1}(t)}(\mathbf{L}_{\mathfrak{p}}(t)), \quad L_{\mathfrak{k}} = Ad_{h^{-1}(t)}(\mathbf{L}_{\mathfrak{k}}(t))$$

is an integral curve of the affine-quadratic Hamiltonian $\mathcal{H} = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle$, where $A = Ad_{h^{-1}(t)}(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t))$, and $h(t)$ is the solution of $\frac{dh}{dt} = \mathbf{L}_{\mathfrak{k}}(t)h(t)$ with $h(0) = I$.

PROOF. If A is any element in $\mathfrak{p}$ then $\frac{d}{dt}Ad_{h(t)}(A) = [Ad_{h(t)}(A), \mathbf{L}_\mathfrak{k}]$. Since $\frac{d}{dt}(\mathbf{A} + \mathbf{L}_\mathfrak{p}(t)) = [\mathbf{A} + \mathbf{L}_\mathfrak{p}(t), \mathbf{L}_\mathfrak{k}(t)]$, $Ad_{h(t)}(A)$ and $\mathbf{A} + \mathbf{L}_\mathfrak{p}(t)$ are the solutions of the same differential equation, hence, they will be equal to each other whenever $Ad_{h(0)}(A) = \mathbf{A} + \mathbf{L}_\mathfrak{p}(0)$, that is, when $A = \mathbf{A} + L_\mathfrak{p}(0)$.

Assume that $Ad_{h(t)}(A) = \mathbf{A} + \mathbf{L}_\mathfrak{p}(t)$. Then,

$$\begin{aligned}\frac{d\tilde{g}}{dt} &= g(t)(\mathbf{A} + \mathbf{L}_\mathfrak{p}(t))h(t) + g(t)\mathbf{L}_\mathfrak{k}(t)h(t)\\ &= \tilde{g}(t)(Ad_{h^{-1}(t)}(\mathbf{A} + \mathbf{L}_\mathfrak{p}(t)) + Ad_{h^{-1}(t)}\mathbf{L}_\mathfrak{k}(t)) = \tilde{g}(t)(A + L_\mathfrak{k}(t)).\end{aligned}$$

Additionally,

$$\begin{aligned}\frac{dL_\mathfrak{p}}{dt} &= \frac{d}{dt}Ad_{h^{-1}(t)}(\mathbf{L}_\mathfrak{p}(t)) = Ad_{h^{-1}(t)}([\mathbf{L}_\mathfrak{k}, \mathbf{L}_\mathfrak{p}]) + Ad_{h^{-1}(t)}([\mathbf{A} + \mathbf{L}_\mathfrak{p}(t), \mathbf{L}_\mathfrak{k}(t)])\\ &= Ad_{h^{-1}(t)}[\mathbf{A}, \mathbf{L}_\mathfrak{k}(t)] = [Ad_{h^{-1}(t)}\mathbf{A}, Ad_{h^{-1}(t)}\mathbf{L}_\mathfrak{k}(t)]\\ &= [A - Ad_{h^{-1}(t)}\mathbf{L}_\mathfrak{p}(t), Ad_{h^{-1}(t}\mathbf{L}_\mathfrak{k}(t)] = [A - L_\mathfrak{p}(t), L_\mathfrak{k}(t)],\end{aligned}$$

and

$$\begin{aligned}\frac{dL_\mathfrak{k}}{dt} &= \frac{d}{dt}Ad_{h^{-1}(t)}(\mathbf{L}_\mathfrak{k}(t)) = Ad_{h^{-1}(t)}[\mathbf{A}, \mathbf{L}_\mathfrak{p}(t)] = [Ad_{h^{-1}(t)}\mathbf{A}, Ad_{h^{-1}(t)}(\mathbf{L}_\mathfrak{p}]\\ &= [A - Ad_{h^{-1}(t)}(\mathbf{L}_\mathfrak{p}(t)), Ad_{h^{-1}(t)}(\mathbf{L}_\mathfrak{p}(t))] = [A, L_\mathfrak{p}(t)].\end{aligned}$$ □

The converse also holds as this proposition demonstrates.

PROPOSITION 5.2. *Suppose that* $(\tilde{g}(t), L_\mathfrak{p}(t), L_\mathfrak{k}(t))$ *is an extremal curve of the affine Hamiltonian* $\mathcal{H} = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle A, L_\mathfrak{p}\rangle$. *Then*

$$\mathbf{g}(t) = ((\tilde{g}(t)h^{-1}(t), p(t)), \qquad \frac{dp}{dt} = \vec{\mathbf{A}}(o) + \vec{\mathbf{L}}_\mathfrak{p}(t)(o), \qquad \frac{dh}{dt} = h(t)(L_\mathfrak{k}(t))$$
$$\mathbf{L}_\mathfrak{p}(t) = Ad_{h(t)}(L_\mathfrak{p}(t)), \qquad \mathbf{L}_\mathfrak{k}(t) = Ad_{h(t)}(L_\mathfrak{k}(t)), \qquad \mathbf{A} = Ad_{h(t)}(A - L_\mathfrak{p}(t))$$

is an extremal curve of the rolling Hamiltonian $H = \frac{1}{2}\langle \mathbf{A} + \mathbf{L}_\mathfrak{p}, \mathbf{A} + \mathbf{L}_\mathfrak{p}\rangle$.

PROOF. The proof is essentially the same as in the previous proposition. □

6. Integrability

An $n\times n$ matrix equation $\frac{dL}{dt} = [M(t), L(t)]$ is called a Lax equation, and (M, L) is called Lax pair (recall our convention that $L, M] = ML - LM$). If (M, L) is a Lax pair then the spectrum of $L(t)$ is constant. The proof is simple: $g(t)L(t)g^{-1}(t)$ is a constant matrix Λ , for any solution of $\frac{dg}{dt} = g(t)M(t)$ in the general linear group $GL_n(R)$. Hence the spectrum of Λ is equal to the spectrum of $L(t)$.

The Poisson equation of any left-invariant Hamiltonian H on a semi-simple Lie algebra $\mathfrak{g}$ can be represented as $\frac{dL}{dt} = [dH, L]$ and therefore the eigenvalues of $L(t)$ are constants of motion, hence may be regarded as the conservation laws on $\mathfrak{g}$. A function h on a Poisson space is said to be invariant if $\{h, f\} = 0$ for any function f. On semi-simple Lie algebras any spectral function is invariant. In particular $\{\phi_k(L) = Tr(L^k), k = 1, 2, \dots\}$ form a family of invariant functions. Again the proof is simple: $\frac{d}{dt}Tr(L^k) = Tr([M, L]L^{k-1}) = Tr(ML^k - L^kM) = 0$.

In some situations a Lax equation $\frac{dL}{dt} = [M(t), L(t)]$ extends to the Lax equation $\frac{dL_\lambda}{dt} = [M_\lambda(t), L_\lambda(t)]$ with a parameter λ. Then a discrete spectrum of $L(t)$ is replaced by a continuous spectrum of $L_\lambda(t)$ which results in additional constants of motion. In the case of spheres rolling on their tangent planes, J. Zimmerman in his PhD thesis (2002 University of Toronto) discovered an extension of the Lax equation which he called *isospectral* [7, 30]. Remarkably, Zimmerman's extension is valid for any symmetric semi-Riemannian space rolling on its tangent plane, for the same reasons as in the case of rolling spheres. In fact, if $X_0(t) = A + L_\mathfrak{p}(t), X_1(t) = L_\mathfrak{k}(t), X_2(t) = -A, X_3 = 0$, then the Poisson equations may be written as

$$\frac{dX_i}{dt} = [X_0(t), X_{i+1}(t)], i = 0, 1, 2.$$

This equation is invariant under a dilational change $X_i \to \lambda^{i-1} X_i$. It then follows that

$$L_\lambda = \sum_{i=0}^{3} \lambda^i X_i = L_\mathfrak{p}(t) + \lambda L_\mathfrak{k}(t) + (1 - \lambda^2)A$$

satisfies the equation

$$\frac{dL_\lambda}{dt} = [M_\lambda(t), L_\lambda(t)], \quad M_\lambda(t) = \frac{1}{\lambda}(A + L_\mathfrak{p}(t)).$$

Therefore, the spectrum of $L_\lambda(t) = L_\mathfrak{p}(t) + \lambda L_\mathfrak{k}(t) + (1 - \lambda^2)A$ is constant. We will refer to L_λ as the spectral curve for H. Of course, the above implies that the Poisson system associated with the affine-quadratic Hamiltonian also admits an isospectral representation. To be more specific, revert to the notations of Proposition 5.1 with the bold-face variables corresponding to the rolling extremals. Then,

$$\begin{aligned}
\mathbf{L}_\lambda &= \mathbf{L}_\mathfrak{p}(t) + \lambda \mathbf{L}_\mathfrak{k}(t) + (1 - \lambda^2)\mathbf{A} \\
&= Ad_{h(t)} L_\mathfrak{p} + \lambda Ad_{h(t)} L_\mathfrak{k} + (1 - \lambda^2) Ad_{h(t)}(A - L_\mathfrak{p}) \\
&= Ad_{h(t)}(\lambda^2 L_\mathfrak{p} + \lambda L_\mathfrak{k} + (1 - \lambda^2)A) = Ad_{h(t)} L_\lambda,
\end{aligned}$$

where $L_\lambda = \lambda^2 L_\mathfrak{p} + \lambda L_\mathfrak{k} + (1 - \lambda^2)A$. It follows that

$$\begin{aligned}
\frac{d\mathbf{L}_\lambda}{dt} &= \frac{d}{dt}(Ad_{h(t)}(L_\lambda) = Ad_{h(t)}[L_\lambda, L_\mathfrak{k}] + Ad_{h(t)} \frac{dL_\lambda}{dt} \\
&= \Big[\frac{1}{\lambda}(\mathbf{A} + \mathbf{L}_\mathfrak{p}), \mathbf{L}_\lambda\Big] = Ad_{h(t)}\Big[\frac{1}{\lambda}A, L_\lambda\Big].
\end{aligned}$$

Therefore,

$$\frac{dL_\lambda}{dt} = [L_\mathfrak{k}, L_\lambda] + \Big[\frac{1}{\lambda}A, L_\lambda\Big] = \Big[\frac{1}{\lambda}A + L_\mathfrak{k}, L_\lambda\Big].$$

To be consistent with my earlier publications replace λ by $-\frac{1}{\lambda}$ to get

$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], \tag{6.1}$$

where $M_\lambda = L_\mathfrak{k} - \lambda A$, and $L_\lambda = L_\mathfrak{p} - \lambda L_\mathfrak{k} + (\lambda^2 - 1)A$. Equation (6.1) agrees with the isospectral representation in [27] (obtained by other means).

Each spectral curve L_λ defines a family of functions

$$\mathcal{F} = \{\phi_\lambda^{(k)}(L) = Tr(L_\lambda^k), k = 1, 2, \dots\}.$$

It follows that

$$Tr(L_\lambda^2) = Tr(L_\mathfrak{p} + A)^2 + \lambda^2 Tr(-AL_\mathfrak{p} + L_\mathfrak{k}^2) + \lambda^4 Tr(A^2),$$

after replacing A by $-A$. Therefore both $Tr(L_\mathfrak{p} + A)^2$ and $Tr(L_\mathfrak{k}^2 - AL_\mathfrak{p})$ belong to $\mathcal{F}$. Since these functions are scalar multiple of the Hamiltonians H and $\mathcal{H}$, we get that both Hamiltonians H and $\mathcal{H}$ are in the family $\mathcal{F}$. Each function in $\mathcal{F}$ is an integral of motion for both Hamiltonian systems. But, equally remarkably, any two functions in $\mathcal{F}$ Poisson commute (first proved by [31]). Thus, $\mathcal{F}$ is an involutive family of functions in $\mathfrak{g}$. This family of functions also figures prominently in the writings of Semenov Tian-Shansky [33] but in a completely different context. In addition to the family $\mathcal{F}$, $\mathcal{F}_A = \{\langle\langle L, X\rangle\rangle : X \in \mathfrak{k}, [X, A] = 0\}$ is another family of functions whose elements Poisson commute with each member of the family $\mathcal{F}$ (see [32], and [27, p. 164]).

The following proposition sums up the essential integrability implications.

Proposition 6.1. *Each function in $\mathcal{I} = \mathcal{F} \cup \mathcal{F}_A$ is an integral of motion for both $\mathcal{H} = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle A, L_\mathfrak{k}\rangle$ and $H = \frac{1}{2}\|\mathbf{A} + \mathbf{L}_\mathfrak{p}\|^2$. In the case that A is regular, then $\mathcal{I}$ is also complete, in the sense that it contains a subfamily $\mathcal{I}_0$ that is Liouville integrable on each coadjoint orbit in $\mathfrak{g}$ [27, p. 164–165].*

To say that a family of functions $\mathcal{F}$ on a symplectic manifold M is Liouville integrable means that $\mathcal{F}$ is involutive, i.e., $\{f, h\} = 0$ for any f and h in $\mathcal{F}$, and maximal, in the sense, that the Hamiltonian vector fields $\{\vec{f} : f \in \mathcal{F}\}$ span the tangent space of M at each point $p \in M$. An element A in $\mathfrak{p}$ is said to be regular if the set of elements in $\mathfrak{p}$ that commute with A is an abelian subalgebra in $\mathfrak{p}$, i.e., A is an element of a maximal abelian subalgebra in $\mathfrak{p}$.

Corollary 6.1. *Both $\mathcal{H} = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle A, L_\mathfrak{k}\rangle$and $H = \frac{1}{2}\|\mathbf{A} + \mathbf{L}_\mathfrak{p}\|^2$ are completely integrable on each coadjoint orbit in $\mathfrak{g}$ when A is regular.*

So, in principle, both Hamiltonian systems $\vec{H}$ and $\vec{\mathcal{H}}$ are solvable in terms of the action-angle variables. But, the path to these variables is known only in a few exceptional cases, so finding explicit solutions remains an open problem in general. In some special cases, the known solutions suggest deep connections between rolling and the geometry of symmetric spaces and the problems of mechanics. We will now return to the spheres and the hyperboloids to be more specific about these connections.

6.1. The elastic problem on spheres and hyperboloids. On spaces of constant curvature the problem of minimizing $\frac{1}{2}\int_0^T \kappa^2(s)\, ds$ subject to the given boundary conditions, stated below, is intertwined with the rolling problem. This problem, called the curvature problem, or the elastic problem consists of finding a continuously differentiable curve $p(t)$ in M in an interval $[0, T]$, with its tangent vector $\dot{p}(t)$ of unit length and its covariant derivative bounded and measurable in $[0, T]$ that satisfies fixed tangential directions $\dot{p}(0) = v_0, v_0 \in T_{p(0)}M$ and $\dot{p}(T) =$

$v_1, v_1 \in T_{p(T)}M$ along which the integral $\frac{1}{2}\int_0^T \kappa^2(s)\,ds$ minimal (critical) among all other curves that satisfy the same boundary conditions. Here $\kappa(t) = \|\frac{dD_{p(t)}}{dt}(\dot{p}(t))\|$, where $\frac{dD_{p(t)}}{dt}$ denotes the covariant derivative along $p(t)$. The integral $\frac{1}{2}\int_0^T \kappa^2(s)\,ds$ is known as the elastic energy of the curve $p(t)$ [5]. Curves $p(t)$ defined on some interval $[0,T]$ are called elastic if for each $t \in (0,T)$ there exits an interval $[t_0,t_1] \subset [0,T]$ over which the elastic energy of $p(t)$ is minimal (critical) relative to the boundary conditions $\dot{p}(t_0)$ and $\dot{p}(t_1)$ [27].

On symmetric semi-Riemannian manifolds $M = G/K$, the curvature problem can be lifted to the unit tangent bundle of the isometry group G, and it is this lifted version of the problem that will be of interest for this paper. We recall that each curve $p(t)$ in M is the projection of a horizontal curve $g(t)$ in G, i.e., $p(t) = \phi_{g(t)}(o)$. Then $\frac{dg}{dt} = g(t)U(t)$ for some curve $U(t) \in \mathfrak{p}$. It follows that the tangent vector $\dot{p}(t)$ is the projection of the manifold $V = \{(g(t)h, Ad_h(U(t))), h \in K\}$ in the left-trivialization $G \times \mathfrak{g}$ of the tangent bundle of G. In this representation the unit tangent bundle is represented by $G \times S_{\mathfrak{p}}$ where $S_{\mathfrak{p}} = \{\Lambda \in \mathfrak{p} : \langle\langle \Lambda, \Lambda\rangle\rangle = 1\}$. The lifted curvature problem consists of finding a curve $(g(t), \Lambda(t))$ in $G \times S_{\mathfrak{p}}$, a solution of

$$\frac{dg}{dt} = g(t)\Lambda(t), \quad \frac{d\Lambda}{dt} = U(t), \quad \langle U(t), \Lambda(t)\rangle = 0,$$

that originates in the manifold $V_0 = \{(g_0h, Ad_{h^{-1}}\Lambda_0), h \in K, \Lambda_0 \in S_{\mathfrak{p}}\}$ at $t = 0$ and terminates at the manifold $V_1 = \{(g_1h, Ad_{h^{-1}}\Lambda_1) : h \in K, \Lambda_1 \in S_{\mathfrak{p}}\}$ at $t = T$ for which the energy of transfer $\frac{1}{2}\int_0^T \|U(s)\|^2\,ds$ is minimal (critical). If $(g(t), \Lambda(t))$ is any solution of the above system then the projected curve $p(t) = \phi_{g(t)}(o)$ satisfies

$$\dot{p}(t) = d_{g(t)}\phi_{g(t)}\Lambda(t)(o), \quad \frac{D_{p(t)}}{dt}(\dot{p}(t)) = d_{g(t)}\phi_{g(t)}U(t)(o),$$

and the tangential boundary conditions

$$\dot{p}(0) = d_{g(0)}\phi_{(g(0)}\Lambda_0, \quad \dot{p}(T) = d_{g(T)}\phi_{(g(T)}\Lambda_1.$$

It is easy to show that a curve $p(t)$ is elastic if and only if it is the projection of a solution of the lifted curvature problem on a fixed interval $[0,T]$.

The solutions of the lifted curvature problem are the projections of the Hamiltonian system associated to the Hamilonian $\mathbf{H}$ in the cotangent bundle of $G \times S_{\mathfrak{p}}$ obtained through the Maximum principle properly modified to account for the constraints as outlined in [27, Chapter 11]. After the cotangent bundle of $G \times S_{\mathfrak{p}}$ is identified with the tangent bundle via the natural inner product in $\mathfrak{g}$, the configuration space for the Hamiltonian system can be represented by the quadruple (g, L, Λ, X) with L and X the tangent vectors at (g, Λ) subject to the constraints $\|\Lambda\|^2 - 1 = 0, \langle\langle \Lambda, X\rangle\rangle = 0$. Then $L = P + Q$ with $P \in \mathfrak{p}$ and $Q \in \mathfrak{k}$. In these variables the curvature Hamiltonian is given by

$$\mathbf{H} = \frac{1}{2}\|X\|^2 + \langle\langle \Lambda, P\rangle\rangle + \lambda_1 G_1, \quad \lambda_1 = \frac{1}{2}(\|X\|^2 - \langle\langle P, \Lambda\rangle\rangle),$$

and the associated Hamiltonian equations are given by

$$\frac{dg}{dt} = g\Lambda(t), \quad \frac{dP}{dt} = [\Lambda, Q], \quad \frac{dQ}{dt} = [\Lambda, P]$$

$$\frac{d\Lambda}{dt} = X(t), \quad \frac{dX}{dt} = -P - (\|X\|^2 - \langle\langle P, \Lambda\rangle\rangle)\Lambda,$$

subject to the transversality condition $Q(t) + [\Lambda(t), X(t)] = 0$ [27, p. 354–355]. The transversality condition can be incorporated into the above equations to yield an equivalent system

$$\begin{aligned} &\frac{dg}{dt} = g\Lambda(t), \quad && \frac{dX}{dt} = -P - (\|X\|^2 - \langle\langle P, \Lambda\rangle\rangle)\Lambda, \\ &\frac{d\Lambda}{dt} = X(t), && \frac{dP}{dt} = -[\Lambda, [\Lambda, X]], \quad \frac{dQ}{dt} = [\Lambda, P]. \end{aligned} \tag{6.2}$$

We will now confine our attention to the spaces of constant curvature, with a particular interest on the connections between the rolling problems and the elastic curves reported in [7]. Let us first recall the basic facts. If P is the plane spanned by the unit vectors $\vec{U}(o)$ and $\vec{V}(o)$ in T_oM such that $\langle\langle \vec{U}(o), \vec{V}(o)\rangle\rangle = 0$, then the sectional curvature $k(P)$ is given by

$$k(P) = \langle\langle [[U, V], U], V\rangle\rangle,$$

[21, Proposition 26, 11]. Then M is the space of constant curvature if the sectional curvature $k(P)$ is independent of the choice of the plane P. It is known that the spheres $S^n_\epsilon(\rho)$ are the only simply connected spaces of non-zero constant curvature [24]. In fact,

$$k(P) = \langle\langle [[U, V], U], V\rangle\rangle_\epsilon = \frac{\epsilon}{\rho^2}$$

for $U = u \wedge_\epsilon o, V = v \wedge_\epsilon o$ that satisfy $\|U\|_\epsilon = \|V\|_\epsilon = 1$ and $\langle\langle U, V\rangle\rangle_\epsilon = 0$. Hence, $k = \frac{\epsilon}{\rho^2}$ on $S^n_\epsilon(\rho)$.

On spaces of constant curvature $-[\Lambda, [\Lambda, X]] = kX$, hence extremal equations (6.2) simplify

$$\begin{aligned} &\frac{dg}{dt} = g\Lambda(t), \quad && \frac{d\Lambda}{dt} = X(t), \quad \frac{dX}{dt} = -P - (\|X\|^2 - \langle P, \Lambda\rangle)\Lambda, \\ &\frac{dP}{dt} = kX, && \frac{dQ}{dt} = [\Lambda, P]. \end{aligned}$$

It follows that $k\frac{d\Lambda}{dt} - \frac{dP}{dt} = 0$, and therefore, $k\Lambda - P = kA$ for some constant element A in $\mathfrak{p}$. The transversality condition $Q + [\Lambda, X] = 0$ can be recast as $0 = [\Lambda, Q] + [\Lambda, [\Lambda, X]] = [\Lambda, Q] - kX$. These observations can be incorporated in the preceding equations to get

$$\begin{aligned} \frac{dg}{dt} &= g(t)\Lambda(t) = g(t)\frac{1}{k}(kA + P), \\ \frac{dP}{dt} &= kX = [\Lambda, Q] = \frac{1}{k}[kA + P, Q], \\ \frac{dQ}{dt} &= [\Lambda, P] = \frac{1}{k}[kA + P, P] = \frac{1}{k}[kA, P]. \end{aligned}$$

We will now make use of the following isospectral integrals of motion associated with the preceding rolling problem extracted from the functions $f_{2,\lambda} = Tr(L^2_\lambda)$ and

$f_{4,\lambda} = Tr(L_\lambda^4)$

$$I_0 = 2H = \|A + L_\mathfrak{p}\|^2, I_1 = \|L_\mathfrak{p}\|^2 + \epsilon\|L_\mathfrak{k}\|^2,$$
$$I_2 = |k|\|L_\mathfrak{k}\|^2\|L_\mathfrak{p}\|^2 - \|[L_\mathfrak{p}, L_\mathfrak{k}]\|^2 + |k|(\|L_\mathfrak{k}\|^4 - \|L_\mathfrak{k}^2\|^2),$$
$$I_4 = |k|\|L_\mathfrak{k}\|^2\|A + L_\mathfrak{p}\|^2 - \|[A + L_\mathfrak{p}, L_\mathfrak{k}]\|^2.$$

These integrals of motion are the rescaled variants of the integrals of motion in [7] after the metric $\langle A, B\rangle = -\frac{\epsilon}{2}Tr(AB)$ in [7] is rescaled to $-\frac{\epsilon\rho^2}{2}Tr(AB)$ (the metric in this paper is a scalar multiple of the metric used in [7]).

Proposition 6.2. *Rolling geodesics that are the projections of the extremal curves on $H = \frac{1}{2}$ and $I_4 = 0$ project on the elastic curves in $S_\epsilon^n(\rho)$. Conversely, each elastic curve in $S_\epsilon^n(\rho)$ is the projection of such an extremal curve.*

Proof. Each elastic curve on $S_\epsilon^n(\rho)$ is the projection of an extremal curve, a solution of

$$\frac{dg}{dt} = g(t)\Lambda(t) = g(t)\frac{1}{k}(kA + P),$$
$$\frac{dP}{dt} = kX = [\Lambda, Q] = \frac{1}{k}[kA + P, Q],$$
$$\frac{dQ}{dt} = [\Lambda, P] = \frac{1}{k}[kA + P, P] = \frac{1}{k}[kA, P],$$

as shown above. If we now identify $\frac{1}{k}P$ with $L_\mathfrak{p}$, and $\frac{1}{k}Q$ with $L_\mathfrak{k}$, then

$$\Lambda = A + \frac{1}{k}P = A + L_\mathfrak{p}, \quad \text{and} \quad L_\mathfrak{k} = \frac{1}{k}Q = -\frac{1}{k}[A + L_\mathfrak{p}, X].$$

In these variables the above equations are given by

$$\frac{dL_\mathfrak{k}}{dt} = [A, L_\mathfrak{p}], \quad \frac{dL_\mathfrak{p}}{dt} = [A + L_\mathfrak{p}, L_\mathfrak{k}].$$

Hence, they agree with the extremal equations for the rolling geodesics. Since $\|A + L_\mathfrak{p}\| = 1$ the first constraint is satisfied. To verify the second constraint note that $L_\mathfrak{k} = -\frac{1}{k}[A + L_\mathfrak{p}, X]$, and therefore

$$\|L_\mathfrak{k}\|^2 = \frac{1}{k^2}\|[A + L_\mathfrak{p}, X]\|^2 = \frac{1}{k^2}\langle[[A + L_\mathfrak{p}, X], A + L_\mathfrak{p}], X\rangle = \frac{1}{k}\|X\|^2,$$

and $\|[A + L_\mathfrak{p}, L_\mathfrak{k}]\|^2 = \frac{1}{k^2}\|[A + L_\mathfrak{p}, [A + L_\mathfrak{p}, X]]\|^2 = \|X\|^2$. Therefore,

$$I_4 = \|L_\mathfrak{k}\|^2|k|\|A + L_\mathfrak{p}\|^2 - \|[A + L_\mathfrak{p}, L_\mathfrak{k}]\|^2 = \|X\|^2 - \|X\|^2 = 0.$$

To prove the converse assume that $g(t), p(t), A, L_\mathfrak{k}(t), L_\mathfrak{p}(t)$ is a rolling extremal curve on $I_4 = 0$. As a geodesic it satisfies $H = \frac{1}{2}$, or $\|A+L_\mathfrak{p}\| = 1$. We need to show that $L_\mathfrak{k}(t) = [A + L_\mathfrak{p}(t), X(t)]$ for some $X(t) \in \mathfrak{p}$ such that $\langle X(t), A + L_\mathfrak{p}(t)\rangle = 0$.

Let

$$\Lambda(t) = A + L_\mathfrak{p}(t), \quad \mathfrak{p}_{\Lambda(t)}^\perp = \{X(t) \in \mathfrak{p} : \langle X(t), \Lambda(t)\rangle = 0\},$$
$$\mathfrak{k}_{\Lambda(t)} = \{Q(t) \in \mathfrak{k} : [Q(t), \Lambda(t)] = 0\}, \quad \mathfrak{k}_{\Lambda(t)}^\perp = \{Q \in \mathfrak{k}; : \langle Q, \mathfrak{k}_\Lambda\rangle = 0\}.$$

Then $\Lambda(t) = \lambda(t) \wedge_\epsilon a$, $(\lambda(t), a)_\epsilon = 0$ for some vector $\lambda(t) \in R^{n+1}$. It then follows that $\mathfrak{p}^\perp_{\Lambda(t)} = \{u(t) \wedge_\epsilon a : (u(t), a)_\epsilon = (\lambda(t), u(t))_\epsilon = 0\}$, and $\mathfrak{k}^\perp_{\Lambda(t)} = \{\lambda(t) \wedge_\epsilon u(t) : (u(t), \lambda(t))_\epsilon = (a, u(t))_\epsilon = 0\}$.

Hence, $\dim(\mathfrak{p}^\perp_{\Lambda(t)}) = \dim(\mathfrak{k}^\perp_\Lambda)$. The mapping $F(t)X = ad\Lambda(t)(X)$, for $X \in \mathfrak{p}^\perp_{\Lambda(t)}$, satisfies $F(\mathfrak{p}^\perp_{\Lambda(t)}) \subseteq \mathfrak{k}^\perp_{\Lambda(t)}$, because $\langle [\Lambda, X], \mathfrak{k}_\Lambda \rangle = 0$. On spaces of non-zero constant curvature, the kernel of this mapping is zero because $ad\Lambda(t)X = 0$ implies that $0 = ad^2\Lambda(t)(X(t)) = -\epsilon\rho^2 X(t)$. Since $\mathfrak{p}^\perp_{\Lambda(t)}$ and $\mathfrak{k}^\perp_{\Lambda(t)}$ have the same dimension, F maps $\mathfrak{p}^\perp_{\Lambda(t)}$ onto $\mathfrak{k}^\perp_{\Lambda(t)}$. So every curve $L(t) \in \mathfrak{k}^\perp_{\Lambda(t)}$ is of the form $L(t) = [\Lambda(t), X(t)]$ for some $X(t) \in \mathfrak{p}$ perpendicular to $\Lambda(t)$.

It remains to show that $L_\mathfrak{k}(t)$ belongs to $\mathfrak{k}^\perp_{\Lambda(t)}$ when the rolling geodesic is on $I_4 = 0$, that is, when $\|L_\mathfrak{k}\|^2 = \frac{1}{|k|}\|[\Lambda(t), L_\mathfrak{k}(t)]\|^2$. Now assume that $L_\mathfrak{k}(t) = U_1(t) + U_2(t)$, $U_1(t) \in \mathfrak{k}_{\Lambda(t)}$ and $U_2(t) \in \mathfrak{k}^\perp_{\Lambda(t)}$. It follows from above that $U_2(t) = [\Lambda(t), X(t)]$, and therefore

$$\|U_2(t)\|^2 = \|[\Lambda(t), X(t)], [\Lambda(t), X(t)]\|^2 = |\langle ad^2\Lambda(t)(X), X(t)\rangle| = |k|\|X\|^2.$$

Hence,

$$\frac{1}{|k|}\|[\Lambda(t), U_1(t) + U_2(t)]\|^2 = \frac{1}{|k|}\|[\Lambda(t), U_2(t)]\|^2 = |k|\|X\|^2 = \|L_\mathfrak{k}\|^2.$$

But $\|L_\mathfrak{k}(t)\|^2 = \|U_1\|^2 + \|U_2(t)\|^2 = \|U_1(t)\|^2 + |k|\|X\|^2$, and therefore $U_1(t) = 0$. □

The following proposition characterizes elastic curves ([27], see also [34]).

Proposition 6.3. *Let $\kappa(t)$ and $\tau(t)$ denote the geodesic curvature and the torsion of the projection curve $p(t)$ associated with an extremal curve of the curvature problem. Then $\xi(t) = \kappa^2(t)$ is the solution of the following equation*

$$\left(\frac{d\xi}{dt}\right)^2 = -\xi^3 + 4(H - \epsilon)\xi^2 + 4(I_1 - H^2)\xi - 4I_2,$$

and $(\kappa^2(t)\tau(t))^2 = kI_2$. All other curvatures in the Serret-Frenet frame along $p(t)$ are zero.

Proof. We leave it to the reader to verify that $\|L_\mathfrak{k}\|^4 - \|L_\mathfrak{k}^2\|^2 = 0$ when $L_\mathfrak{k} = -\frac{1}{k}[A + L_\mathfrak{p}, X]$. In fact, the same is true for any $Q \in \mathfrak{k}$ of the form $Q = [U, V]$ for some mutually orthogonal matrices U, V in $\mathfrak{p}$.

Reverting to the notation used above, let $P(t) = kL_\mathfrak{p}(t)$, and $Q(t) = kL_\mathfrak{k}(t)$. Recall that

$$\|L_\mathfrak{k}\|^2 = \frac{1}{k^2}\|[A + L_\mathfrak{p}, X]\|^2 = \frac{1}{k^2}\langle [[A + L_\mathfrak{p}, X], A + L_\mathfrak{p}], X\rangle = \frac{1}{k}\|X\|^2,$$

hence $\|Q\|^2 = k\|X\|^2$. Also, recall that $\kappa^2 = \|X\|^2$ and that $2H = \frac{1}{2}\|X\|^2 + \langle\langle \Lambda, P\rangle\rangle$. Then,

$$\begin{aligned} I_2 k^2 &= k^2(\|L_\mathfrak{k}\|^2\|L_\mathfrak{p}\|^2 - \|[L_\mathfrak{p}, L_\mathfrak{k}]\|^2) \\ &= \|P\|^2\|X\|^2\| - \|[L_\mathfrak{p}, Q]\|^2 = \|P\|^2\|X\|^2 - \|[L_\mathfrak{p}, [A + L_\mathfrak{p}, X]]\|^2 \\ &= \|P\|^2\|X\|^2 - \langle A + L_\mathfrak{p}, L_\mathfrak{p}\rangle^2\|X\|^2 - \langle P, X\rangle^2, \end{aligned}$$

thanking to the formula in (4.4). In addition,

$$I_1k^2 = k^2(\|L_{\mathfrak{p}}\|^2 + \epsilon\|L_{\mathfrak{k}}\|^2) = \|P\|^2 + \epsilon\|Q\|^2 = \|P\|^2 + \epsilon k\|X\|^2.$$

Since $\kappa^2(t) = \|X\|^2$, $\frac{d\xi}{dt} = 2\langle X, \dot{X}\rangle = 2\langle X, P\rangle$. Therefore,

$$\begin{aligned}
\Big(\frac{d\xi}{dt}\Big)^2 &= 4\langle P, X\rangle^2 = 4(\|P\|^2\|X\|^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2\|X\|^2) - 4k^2I_2\\
&= 4\Big((I_1k^2 - \epsilon k\|X\|^2)\|X\|^2 - \Big(H - \frac{1}{2}\|X\|^2\Big)^2\|X\|^2\Big) - 4k^2I_2\\
&= 4(I_1k^2 - \epsilon k\xi)\xi - 4\Big(H - \frac{1}{2}\xi\Big)^2\xi - 4k^2I_2\\
&= -\xi^3 + 4(H - \epsilon k)\xi^2 + (I_1k^2 - H^2)\xi - 4k^2I_2.
\end{aligned}$$

As to the second part, let $T = A + L_{\mathfrak{p}}(t)$. Since $\|A + L_{\mathfrak{p}}(t)\| = 1$, $T(t)$ is a unit vector that projects onto the tangent vector $\dot{p}(t)$. Then

$$\frac{dT}{dt} = [A + L_{\mathfrak{p}}(t), L_{\mathfrak{k}}(t)] = \Big[A + L_{\mathfrak{p}}(t), -\Big[A + L_{\mathfrak{p}}(t), \frac{1}{k}X(t)\Big]\Big] = X(t).$$

Therefore $\frac{dT}{dt} = \kappa(t)N(t)$ where $N(t) = \frac{1}{\|X(t)\|}X(t)$ is a unit vector in $\mathfrak{p}$ that projects onto the unit normal $n(t)$ along $p(t)$. Continuing,

$$\begin{aligned}
\frac{dN}{dt} &= \frac{1}{\|X(t)}\|(-P - (\|X\|^2 - \langle\Lambda, P\rangle(A + L_{\mathfrak{p}})) - \frac{1}{\|X\|^2}\langle X, \dot{X}\rangle X)\\
&= -\|X\|(A + L_{\mathfrak{p}}) + \frac{1}{\|X\|}(-P + \langle A + L_{\mathfrak{p}}, P\rangle(A + L_{\mathfrak{p}})) + \frac{1}{\|X\|^2}\langle P, X\rangle X)\\
&= -\kappa(t)T(t) + Y(t),
\end{aligned}$$

where

$$Y(t) = \frac{1}{\|X\|}(-P(t) + \langle T(t), P(t)\rangle T(t)) + \frac{1}{\|X\|^2}\langle P(t), X(t)\rangle X.$$

Since $Y(t)$ is is orthogonal to $A + L_{\mathfrak{p}}$ and X, it is in the direction of the binormal vector $B(t)$. So, if we define $\tau(t) = \|Y(t)\|$ and $B(t) = \frac{1}{\|Y\|}Y$ then $\frac{dN}{dt} = \kappa(t)T(t) + \tau B(t)$ and $B(t)$ projects onto the binormal vector $b(t)$ along $p(t)$. Hence,

$$\|X\|^2\tau^2 = \|P\|^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2 - \frac{1}{\|X\|^2}\langle P, X\rangle^2,$$

or

$$|(\kappa^2\tau)^2 = \|X\|^4\tau^2 = \|P\|^2\|X\|^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2 - \langle P, X\rangle^2 = k^2I_2,$$

Evidently $\frac{dB}{dt}$ is in the linear span of $T(t), N(t), B(t)$, hence the Serret-Frenet frame along $p(t)$ terminates. □

Corollary 6.2. *Elastic curves in* $M_\epsilon = S^n_\epsilon(\rho)$ *are rolled on the elastic curves in the tangent space* $\hat{M} = T_oM$.

Proof. Since the geodesic curvature is preserved under rolling, the elastic curves in $S^n_\epsilon(\rho)$ are rolled on the elastic curves in $\widehat{M}$ relative to the Euclidean metric inherited from the metric on $\mathfrak{p}$. So, the statement follows from the rolling definition. □

This remarkable relation between elastic curves and rolling geodesics breaks down on spaces of non-constant Riemannian curvature, as it becomes evident when one compares equations (6.2) for the curvature problem to the equations (5.2) for the rolling problem. While the curvature equation seems particularly challenging beyond the spaces of constant curvature, the rolling extremal equations (5.3) remain integrable on all semi-Riemannian symmetric spaces according to Proposition 6.1, and should be "solvable" on some Abelian variety according to the general theory of integrable systems. However, as stated earlier, no such solution is known except in a few exceptional cases. We leave this challenge to an interested reader.

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КОТРЉАЈУЋЕ ГЕОДЕЗИЈСКЕ ЛИНИЈЕ НА СИМЕТРИЧНИМ ПСЕУДО-РИМАНОВИМ ПРОСТОРИМА

Резиме. Овај рад је настао из резултата добијених у области котрљања у нашем недавном раду написаним са Ф. Силвом Леитеом и И. Маркином, и ранијих радова о котрљању сфера написаних са Ј. Цимерманом. Показујемо да једначине котрљања повезане са симетричном псеудо-Римановом многострукошћу која се котрља по свом тангентном простору у издвојеној тачки многострукости у суштини имају исту структуру као једначине котрљања n-димензионалне сфере по хоризонталној хиперравни; то јест, показујемо да су једначине котрљања описане левоинваријантном дистрибуцијом $\mathcal{D}$ на Лијевој групи $\mathbf{G}$ са растом

$$\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = T\mathbf{G}.$$

То подсећа на раст $(2, 3, 5)$ за две сфере које се котрљају по хоризонталној равни. Затим, дефинишемо котрљајуће геодезијске линије на псеудо-Римановим многострукостима као проширења суб-Риманових геодезијских линија у Римановим симетричним просторима. Након тога показујемо да су котрљајуће геодезијске линије пројекције екстремних кривих, које су, донекле неочекивано, решења потпуно интеграбилног Хамилтоновог система у котангетном раслојењу конфигурационог простора. Коначно, илуструјемо теорију са неколико значајних примера.

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