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DIMENSION REDUCTION IN ELASTICITY

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Abstract. From the three-dimensional linear theory of elasticity, two- and one-dimensional descriptions are derived by involving the consistent-approximation approach. The pseudo-reduction technique yields well-known and higher-order theories for quasi two-dimensional and quasi one-dimensional structural members.

1. Introduction

Closed-form analytical solutions for boundary-value problems of the three-dimensional theory of elasticity are rather sparse. If the three-dimensional body under consideration (cf. Figure 1a) exhibits preferred directions, it is advisable to take advantage of this fact. If, e.g., one dimension, i.e., the thickness of the body, is much *smaller* than the other two dimensions, it might be possible to describe the mechanical response, i.e., displacements, strains and stresses by quantities defined, e.g., on the middle surface of the body and we may establish a quasi two-dimensional theory with a broader range of analytical solutions. If the middle surface is plane, we speak of plates and discs with the characteristic thickness dimension h being much smaller than the characteristic in-plane dimensions a, b, say,

$$h \ll a, b \tag{1.1}$$

(see Figure 1b). If the middle surface is curved, we speak of shells. In an engineering sense, much smaller means $h/a < 1/10$; however, $h/a < 1/5$ is often still acceptable.

If one characteristic dimension is much *larger* than the other two, the quantities of interest may be described on a line giving rise to a quasi one-dimensional theory with an even higher possibility of obtaining analytical results. If the (centre) line is straight, we speak of bars, beams or shafts. Then ℓ might be the characteristic length, whereas b and h the cross-sectional dimensions in width and height direction, respectively, with $\ell \gg b, h$ (cf. Figure 1c). If the centre line is curved in a plane, we speak of arches, if it is curved in space, we have, e.g. helices.

The well-known methodologies of dimension reduction in elasticity may be grouped into three categories:

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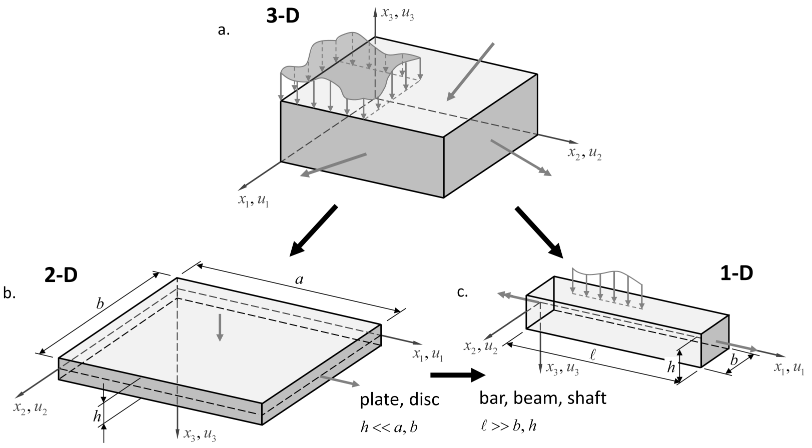


FIGURE 1. Three-dimensional body (a), quasi two-dimensional continuum (b) and quasi one-dimensional continuum (c)

- the engineering approach starts with the celebrated a priori assumptions, which “fall from heaven” for students getting acquainted with the theory for the first time. Only later, when they become more familiarized with the subject, they understand that the assumptions are experimentally motivated, based on physical grounds and on engineering intuition. In short, they are for plates and beams:
  - no stresses (strains) in thickness direction,
  - plane cross-sections remain plane,
  - cross-sections normal to the middle surface (centre line) in the undeformed state remain normal in the deformed state (normal hypothesis).
- the Cosserat approach is mathematically the most elegant method since it uses quantities which “live” on a surface (line). The other dimension(s) are described by a set of deformable directors, which are attached to each point of the surface (line). The disadvantage of the Cosserat theory lies in the fact that the derivation of the constitutive relations is not straight forward and not unambiguously possible.
- the consistent approach was introduced by Naghdi [14]. It starts with the development of the displacements in the “thin” direction(s) in Taylors series with subsequent integration with respect to the thinness direction(s). It will be used throughout the paper and described in more detail below.

There exists a vast amount of literature on the subject. It is not the aim of this paper to provide a thorough overview. We restrict ourselves merely to some references of our own work on the consistent approach, where further references may be found. For the engineering approach, any textbook on strength-of-materials may be consulted. An extensive literature overview on the Cosserat approach is compiled in Altenbach et al. [1].

In Chapter 2, we will treat the two-dimensional approach and in Chapter 3, the one-dimensional theory. We reach some conclusions in Chapter 4.

2. Consistent-approximation approach: 2-D

We assume a plane middle surface parameterized by Cartesian coordinates (x_1, x_2) with x_3 perpendicular to the plane, (see Figure 1b). The characteristic in-plane dimension is a and the characteristic out-of-plane dimension h. We introduce dimensionless coordinates ξ_i by

$$\xi_i = \frac{x_i}{a}$$

and develop the displacements u_j in thickness direction into Taylor series

$$u_j = a \sum_{r=0}^{\infty} {}^r u_j(\xi_1, \xi_2) \cdot \xi_3^r. \tag{2.1}$$

As a basis, we use monomials (MacLaurin-series expansion). Any other basis, e.g., Legendre polynomials or trigonometric functions are also applicable. The non-dimensional coefficients ${}^r u_j(\xi_1, \xi_2)$ are the unknowns of the problem. The upper left index (r) indicates the polynominal order of the series and is excluded from the summation convention, which is used otherwise throughout the paper with latin indices $i, j = 1, 2, 3$, Greek indices $\alpha, \beta = 1, 2$.

The components of the strain tensor ε_{ij} follow from the linearized kinematic relations

$$\varepsilon_{ij} = \frac{1}{2a}(u_{i,j} + u_{j,i})$$

$(u_{i,j} = \partial u_i / \partial \xi_j)$ and the strain-energy density is calculated by

$$U = \frac{1}{2a^2} E_{ijk\ell} u_{i,j} u_{k,\ell}$$

with the fourth-rank tensor of elasticity $E_{ijk\ell}$.

In the next step, we integrate the strain-energy density per unit of volume U over the thickness to obtain the strain-energy density per unit of mid-plane area $\bar{U}$ as

$$\bar{U} = \int_{-\frac{h}{2a}}^{+\frac{h}{2a}} U({}^r u_{i,j}(\xi_1, \xi_2), d\xi_3 = \bar{U}({}^r u_{i,j}(\xi_1, \xi_2)), \quad r = 1, 2, \ldots.$$

In a similar way, we obtain the potential of external (body-) forces f_1 and the tractions t_i by

$$\bar{V} = -\int_{-\frac{h}{2a}}^{+\frac{h}{2a}} f_i(\xi_i) u_i d\xi_3 - t_i u_i \Big|_{\xi_3 = -\frac{h}{2a}}^{\xi_3 = +\frac{h}{2a}}.$$

The elastic potential $\bar{\Pi}$ per unit of area is thus

$$\bar{\Pi} = \bar{U} + \bar{V} = \bar{\Pi}({}^r u_{i,j}(\xi_1, \xi_2), {}^s u_k(\xi_1, \xi_2))$$

and appears as polynominal in the *one* dimensionless characteristic plate parameter c^2

$$c^2 = \frac{h^2}{12\ell^2}, \tag{2.2}$$

which is a small quantity according to (1.1), in the form

$$\bar{\Pi} = Gh[(\cdot) + c^2(\cdot) + c^4(\cdot) + \dots]. \tag{2.3}$$

$G[Nm^{-2}]$ is a characteristic measure of the plate stiffness, e.g., the shear modules. The terms $(\cdot)$ in (2.3) turn out to be of the same order of magnitude.

Application of the Principle of Minimum Potential Energy leads to an *infinite* second-order *Partial* Differential Equation (PDE) System for the displacement coefficients (Navier-Lamé equations) and the von-Neumann-boundary conditions, whereas the Principle of Maximum Complementary Energy yields the Dirichlet-boundary conditions. Details of the procedure and the mathematical background are outlined in, e.g., Schneider et al. [**23**].

It has been shown that:

- the infinite PDE system is equivalent to the equations of linear 3-D elasticity [**23**]
- if the problem is symmetric with respect to the middle surface (geometry, material behaviour, loading), the two-dimensional problem decouples into two sub problems [**13**]
  - the disc problem (in-plane deformation)
  - the plate problem (out-of-plane deformation)
- if the elastic potential is truncated, e.g.,

$$\bar{\Pi} = Gh[(\cdot)c^0 + (\cdot)c^2 + \dots + (\cdot)c^{2N} + O^{2(N+1)}]. \tag{2.4}$$

  the error between an N^{th}-order theory (all terms of the order c^{2N} are considered in the elastic potential and all terms of the order $c^{2(N+1)}$ are neglected) and the exact solution $(N \to 0)$ is of order $c^{2(N+1)}$ [**22**].

To solve engineering problems, we need a *finite* PDE system and the question arises where to cut the series expansion (2.1). For a specific problem at hand, i.e., h and a given and c^2 calculated from (2.2), we have to choose an approximation order N according to the error estimate given in [**22**].

Then, in the PDE system, all terms up to the order of c^{2N} have to be considered. This is equivalent to the truncation of the elastic potential, cf. (2.4). In the next step, we reduce the number of unknown by an elimination process similar to the Gauss algorithm applied for algebraic-equation systems. In doing so, we obtain reduction PDEs and by introducing the reduction PDEs into the remaining main PDE(s), the order of differentiation is increased. During this process, we neglect higher-order terms, i.e., terms of the order $O(c^{2M})$, $M > N$, *also* during the reduction process. It has been shown that:

- the main PDE(s) with the reduction PDEs are equivalent to the original PDE system [**18**] (This is the meaning of the word "pseudo")
- the reduction is possible for any approximation order [**12**].

The Pseudo-Reduction Technique is explained in detail in [**18**] where peculiarities of the method are also discussed.

For simplicity, we restrict ourselves to homogeneous isotropic materials. Anisotropic materials have been treated in Schneider et al. [23], Schneider and Kienzler [21] and Kashtalyan et al. [4].

We assume further that the problem is symmetric with respect to the middle plane and concentrate on the plate problem, i.e., in-plane displacements u_α $\alpha = 1, 2$ are odd functions in ξ_3 and even functions in the out-of-plane displacement u_3.

For the sake of clarity, we choose with ${}^i w$ and ${}^j \Psi_\alpha$ more common symbols of plate theories

$$u_a = a({}^1\Psi_\alpha \xi_3 + {}^3\Psi_\alpha \xi_3^3 + {}^5\Psi_\alpha \xi_3^5 + \dots), \quad u_3 = a({}^0 w + {}^2 w \xi_3^2 + {}^4 w \xi_3^4 + \dots).$$

For elucidation, the PDE system is sketched schematically in Figure 2.

By colour-coding of the terms with different orders of c^2 it is seen that the PDE system is *ordered*, i.e., terms of the same order of magnitude are grouped in a diagonal manner. Further, it turns out that the PDE system is symmetric, e.g., the term in the $\delta^2 w$ row and ${}^3\Psi_1$ column is equal to the term in the $\delta^3\Psi_1$ row and

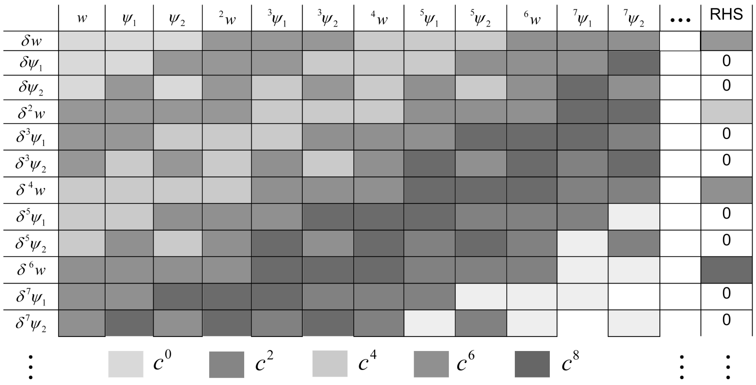


FIGURE 2. Structure of the Navier-Lamé equations for plates

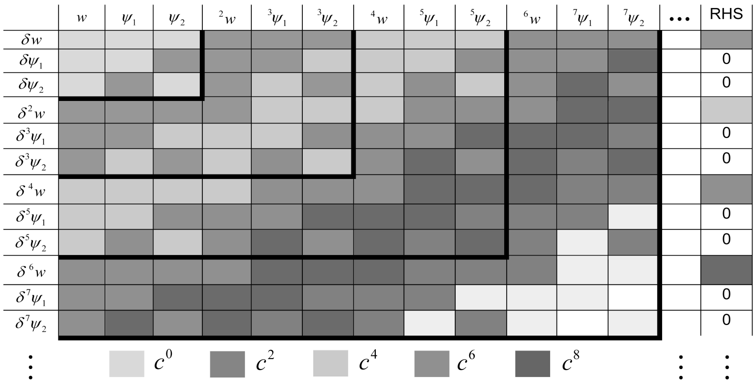


FIGURE 3. Different orders of magnitude of the PDE system

the ${}^2w$ column. PDE systems of different orders are indicated in Figure 3 by bold lines.

If we choose to treat a second-order approach, we have to take at least 9 displacement coefficients into account in order to consider *all* terms of $O(c^4)$, cf. Figure 4.

For the consistent-approximation approach, i.e., truncation of the energy, we only have to take into account those terms which are indicated in Figure 5.

Being more specific, the upper left 4x4 matrix of the PDE system is given explicitly in Figure 6.

The pseudo reduction for the plate problem has been performed and demonstrated in several papers [**5**, **6**, **8**, **21**, **23**]. Here, only the results are assembled.

Zeroth-order approximation. The external load $P(\xi_1,\xi_2)$ applied at the upper and lower faces, $\xi_3 = \pm h/(2a)$, turns out to be of $O(c^2)$, cf. [**5**]. Thus, the zeroth-order approximation admits only rigid-body motions

$$ {}^1\Psi_\alpha = \Psi_{\alpha 0}, \quad {}^0w = w_0 - \Psi_{\alpha 0}\xi_\alpha, \qquad w_0, \Psi_{\alpha 0} = \text{const.} $$

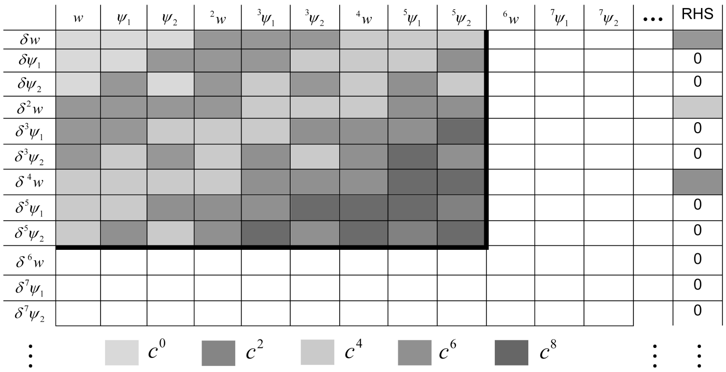


FIGURE 4. Second-order approximation: displacements

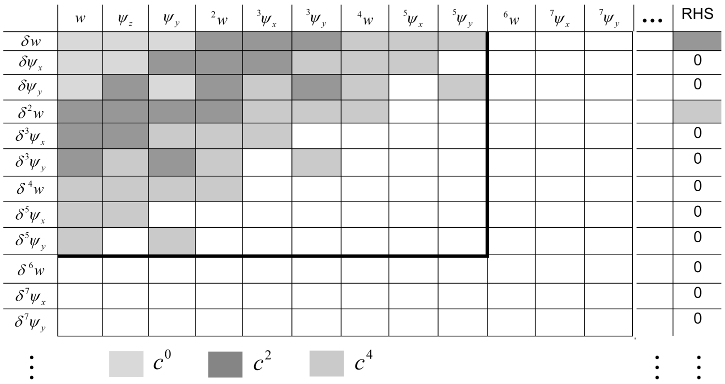
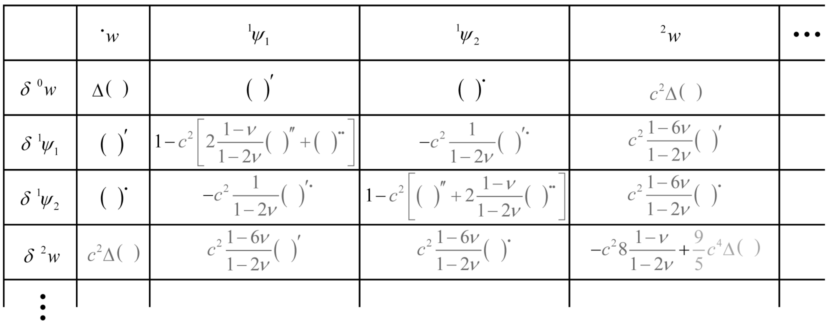


FIGURE 5. Second-order approximation: energy



	${}^{\cdot}w$	${}^{1}\psi_1$	${}^{1}\psi_2$	${}^{2}w$	$\cdots$
$\delta\,{}^{0}w$	$\Delta(\ )$	$(\ )^{\prime}$	$(\ )^{\cdot}$	$c^2\Delta(\ )$	
$\delta\,{}^{1}\psi_1$	$(\ )^{\prime}$	$1-c^2\left[2\frac{1-\nu}{1-2\nu}(\ )^{\prime\prime}+(\ )^{\cdot\cdot}\right]$	$-c^2\frac{1}{1-2\nu}(\ )^{\prime\cdot}$	$c^2\frac{1-6\nu}{1-2\nu}(\ )^{\prime}$	
$\delta\,{}^{1}\psi_2$	$(\ )^{\cdot}$	$-c^2\frac{1}{1-2\nu}(\ )^{\prime\cdot}$	$1-c^2\left[(\ )^{\prime\prime}+2\frac{1-\nu}{1-2\nu}(\ )^{\cdot\cdot}\right]$	$c^2\frac{1-6\nu}{1-2\nu}(\ )^{\cdot}$	
$\delta\,{}^{2}w$	$c^2\Delta(\ )$	$c^2\frac{1-6\nu}{1-2\nu}(\ )^{\prime}$	$c^2\frac{1-6\nu}{1-2\nu}(\ )^{\cdot}$	$-c^2 8\frac{1-\nu}{1-2\nu}+\frac{9}{5}c^4\Delta(\ )$	
$\vdots$					

FIGURE 6. The first matrix terms of the PDE system with $(\ )' = \partial(\ )/\partial\xi_1$, $(\ )^{\cdot} = \partial(\ )/\partial\xi_2$, Laplacian $\Delta = (\ )_{,\alpha\alpha} = (\ )'' + (\ )^{\cdot\cdot}$, Poisson's ration ν

If the plate is properly supported, w_0 and $\Psi_{\alpha 0}$ are zero.

First-order approximation. The classical Kirchhoff-plate theory turns out to be a consistent first-order approximation

$$K\Delta\Delta^0 w = a^3 P$$

with the external load P and the plate stiffness

$$K = \frac{Eh^3}{1-\nu^2}$$

(Young's modules E, Poisson's ratio ν), cf [5]. This includes also Kirchhoff's ersatz shear forces. The *a priori assumptions* of Kirchhoff's plate theory turn out to be *a posteriori results* in that the deviations from the hypotheses are of higher order, e.g.,

$${}^0\Psi_\alpha = {}^0 w_{,\alpha} + O(c^2), \quad {}^0\Psi_{2,1} - {}^0\Psi_{1,2} = \Psi = 0 + O(c^2).$$

The quantity Ψ is a measure of shear deformation and is needed as follows.

Second-order approximation. The second-order approximation delivers two main PDEs in $\tilde{w}$ and Ψ

$$K\Delta\Delta\tilde{w} = a^3\Big(1 - \frac{6}{2}\cdot\frac{2-\nu}{1-\nu}c^2\Delta\Big)P + O(c^6), \quad c^2\Big(1-\frac{6}{5}c^2\Delta\Big)\Psi = 0 + O(c^6). \tag{2.5}$$

$\tilde{w}$ is the energetic average of u_3, cf., e.g., Reissner [16, 17]

$$\tilde{w} = {}^0 w + \frac{3}{10}\cdot\frac{\nu}{1-\nu}c^2\Delta^0 w + O(c^4).$$

It may be mentioned that $P, \Psi, c^2\Delta\Psi$ and $K/(a^2 h)$ are of order $O(c^2)$. It may be mentioned further that, in deriving (2.5), neither a priori assumptions nor shear correction factors have been used.

In [9] and [19] it has been shown that, within the second-order approximation, our theory is equivalent to Reissner's [16, 17], Ambartsumyan's [2], Vekua's [24] and Zhilin's [25] plate theories. In order to improve these theories, e.g., a so-called third-order shear theory (Reddy [15]), a consistent third-order approach with *all terms* of order $O(c^6)$ included, should be used instead of adding single terms in the displacement ansatz in an unsystematic way or by assuming specific stress distributions in thickness direction a priori.

3. Consistent-approximation approach: 1-D

The procedure is quite similar as in the 2-D case. The McLaurin series expansion has to be taken in ξ_2- and in ξ_3-direction ($\xi_2 = x_2/\ell$, $\xi_3 = x_3/\ell$, characteristic length ℓ in x_1 direction)

$$u_i = \ell \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} {}^{rs}u_i(\xi_1)\xi_2^r \xi_3^s,$$

and the integration of the potential energy density Π per unit of volume has to be performed over the cross-sectional area cf. Figure 1c. The result is the elastic potential per unit of length $\bar{\bar{\Pi}}$. For a rectangular cross section $A = b \times h$ we obtain $(( )' = \partial( )/\partial\xi_1)$

$$\bar{\bar{\Pi}} = \int_{-\frac{h}{2\ell}}^{+\frac{h}{2\ell}} \int_{-\frac{b}{2\ell}}^{+\frac{b}{2\ell}} \Pi \, d\xi_2 \, d\xi_3 = \bar{\bar{\Pi}}({}^{rs}u'(\xi_1), {}^{pq}u(\xi_1))$$

which is a polynomial in *two* small dimensionless parameters

$$d^2 = \frac{b^2}{12\ell^2}, \quad c^2 = \frac{h^2}{12\ell^2}, \quad e^2 = c^2, \quad d^2$$

given by

$$\begin{aligned} \bar{\bar{\Pi}} &= GA[(\cdot) + (\cdot)c^2 + (\cdot)d^2 + (\cdot)c^4 + (\cdot)c^2d^2 + (\cdot)d^4 + \dots] \\ &= GA[(\cdot) + (\cdot)e^2 + (\cdot)e^4 + \dots]. \end{aligned}$$

The variational approach then leads to a system of *Ordinary* Differential Equations (ODE System) in the unknown displacement coefficients ${}^{rs}u_j(\xi_i)$ and the boundary conditions. The ODE system is, again, infinite, ordered, symmetric and of second order. The truncation of the energy at different orders $O(e^{2N})$ yields to a hierarchy of finite PDE systems with different orders of approximation.

It has been shown that, if the cross-section and the material behaviour is double-symmetric and the external load is splitted properly, the one dimensional problem decouples into four subproblems:

- bar in tension/compression,
- beam bending about the ξ_2-axis,
- beam bending about the ξ_3-axis,
- shaft in torsion,

cf., [20], where also details of the derivation and the mathematical background are thoroughly elaborated.

When we concentrate on rectangular cross-sections, we have in mind that the approach is also applicable to any double-symmetric, simply connected cross-section. For this, more general, case it would be useful to define the small dimensionless parameters by

$$c^2 = \frac{I_{22}}{\ell^2 A}, \quad d^2 = \frac{I_{33}}{\ell^2 A}$$

with I_{22} and I_{33} as the second-order moments of inertia with respect to the ξ_2- and ξ_3-axis, respectively, common in beam theory, and A as the cross-sectional area. It may be mentioned that the equations may also be derived by the reduction of the 2-D-theories to 1-D, cf. Figure 1.

Bar theory. Let us consider a bar with cross-section $b \times h$ and length ℓ loaded under forces per unit of length n in ξ_1-direction as depicted in Figure 7.

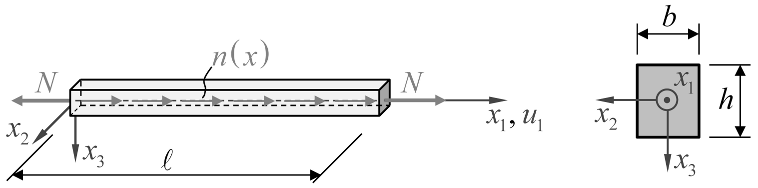


FIGURE 7. Bar loaded in length direction

Primes indicate differentiations with respect to the non-dimensional ξ_1-coordinate $(\ )' = \partial(\ )/\partial\xi_1$.

As outlined in [20], we have to consider even functions in ξ_2 and ξ_3 for u_1, odd functions in ξ_2 and even functions in ξ_3 for u_2 and for u_3 vice versa. Thus

$$\begin{aligned}
u_1 &= \ell({}^{00}u + {}^{20}u\xi_2^2 + {}^{02}u\xi_3^2 + {}^{40}u\xi_2^4 + {}^{22}u\xi_2^2\xi_3^2 + {}^{04}u\xi_3^4 + \dots),\\
u_2 &= \ell({}^{10}v\xi_2 + {}^{30}v\xi_2^3 + {}^{12}v\xi_2\xi_3^2 + {}^{50}v\xi_2^5 + {}^{32}v\xi_2^3\xi_3^2 + {}^{14}v\xi_2\xi_3^4 + \dots),\\
u_3 &= \ell({}^{01}w\xi_3 + {}^{21}w\xi_2^2\xi_3 + {}^{03}w\xi_3^3 + {}^{41}w\xi_2^4\xi_3 + {}^{23}w\xi_2^2\xi_3^3 + {}^{05}w\xi_3^5 + \dots).
\end{aligned}$$

The pseudo reduction together with the most general class of loading condition including body forces and tractions have been treated in [11]. The resulting main ODEs are given as follows.

Zeroth-order approximation. The approximation of zeroth order delivers the ordinary differential equation of the classical bar theory. With ${}^{00}u(\xi_1) = u(\xi_1)$ we obtain

$$\frac{EA}{\ell}u'' = -n + O(e^2).$$

(tension/compression stiffness EA [N], u dimensionless, normal load per unit of length n [Nm^{-1}]).

First-order approximation. For a normal load constant in ξ_2- and ξ_3-direction, we find that the energetic average $\tilde{u}$ is equal to u

$$\tilde{u} = u$$

and we obtain a refined bar theory as

$$\frac{EA}{\ell}\tilde{u}'' = -n + \frac{\nu}{2}(d^2 + c^2)n'' + O(c^4).$$

Details of the derivation, stress resultants and loading as well as boundary conditions are discussed in [11]. Higher-order bar theories are also possible but have not been dealt with so far.

Beam theory. We consider bending with respect to the ξ_2-direction as depicted in Figure 8. The beam has the same dimensions as the bar. Bending with respect to the ξ_3-axis may be treated accordingly.

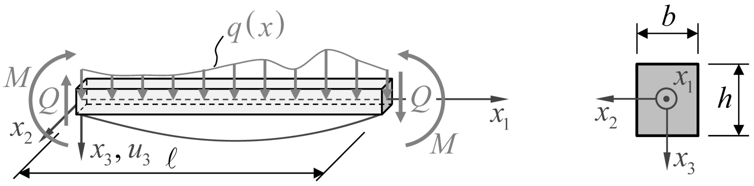


FIGURE 8. Beam in plane bending

For bending in the ξ_1, ξ_2-plane we have to admit the following displacement coefficients which are functions of ξ_1

$$\begin{aligned}
u_1 &= \ell({}^{01}u\xi_3 + {}^{03}u\xi_3^3 + {}^{21}u\xi_2^2\xi_3 + {}^{05}u\xi_3^5 + {}^{23}u\xi_2^2\xi_3^3 + {}^{41}u\xi_2^4\xi_3 + \dots),\\
u_2 &= \ell({}^{11}v\xi_2\xi_3 + {}^{13}v\xi_2\xi_3^3 + {}^{31}v\xi_2^3\xi_3 + \dots),\\
u_3 &= \ell({}^{00}w + {}^{20}w\xi_2^2 + {}^{02}w\xi_2^2 + {}^{40}w\xi_2^4 + {}^{22}w\xi_2^2\xi_3^2 + {}^{04}w\xi_3^4 + \dots),
\end{aligned}$$

cf. [20]. The pseudo reduction and stress and load resultants are shown in [20], and in more detail in [10].

Zeroth-order approximation. For convenience, we introduce the description ${}^{00}w = w$. As in plate theory, the zeroth-order approximation allows for rigid-body motions only

$$^{01}u = \Psi_0, \quad w = w_0 - \Psi_0\xi_1, \qquad w_0, \Psi_0 = \text{const}.$$

If rigid-body motions are constrained due to boundary conditions, w_0 and Ψ_0 are zero.

First-order approximation. Pseudo reduction delivers within the first-order approximation the differential equation of the classical Euler–Bernoulli-beam theory

$$\frac{EI}{\ell^3} w^{\mathrm{IV}} = q + O(c^4)$$

with the bending stiffness

$$EI = \frac{Ebh^3}{12}[\mathrm{Nm}^2]$$

(w dimensionless, q [Nm^{-1}]).

Second-order approximation. The energetic average $\tilde{w}$ of the transverse displacement w is given, in extension of Reissner's approach, as

$$\tilde{w} = w + \nu w''\left(\frac{3}{10}c^2 - \frac{1}{2}d^2 - \frac{2\nu}{1-\nu}d^2\frac{d^2}{5c^2+d^2}\right) + O(c^4), \tag{3.1}$$

cf. [10]. The second-order approximation delivers a Timoshenko–Ehrenfels-type-beam theory

$$\frac{EI}{\ell^3}\tilde{w}^{\mathrm{IV}} = q - \frac{6}{5}c^2(2+\nu)q'' + O(c^6). \tag{3.2}$$

The difference between (3.2) and the original Timoshenko–Ehrenfels theory [3] is that the bracket before q'' is there $(2+2\nu)$ instead of $2+\nu$. It can be shown that the difference has its origin in the neglection of the stresses σ_{33} in thickness direction. Therefore, it follows immediately that a shear-correction factor not involving Poisson's ratio cannot lead to a consistent second-order beam theory. It is interesting to note that, due to the definition of $\tilde{w}$ (3.1), the ODE (3.2) does not contain the aspect ratio b/h explicitly.

Shaft theory. For shaft theory, the remaining displacement coefficients are involved, cf. [7]

$$\begin{aligned}
u_1 &= \ell({}^{11}u\xi_2\xi_3 + {}^{31}u\xi_2^3\xi_3 + {}^{13}u\xi_2\xi_3^3 + {}^{51}u\xi_2^5\xi_3 + {}^{33}u\xi_2^3\xi_3^3 + {}^{15}u\xi_2\xi_3^5 + \dots),\\
u_2 &= \ell({}^{10}v\xi_3 + {}^{21}v\xi_2^2\xi_3 + {}^{03}v\xi_3^3 + {}^{41}v\xi_2^4\xi_3 + {}^{23}v\xi_2^2\xi_3^2 + {}^{05}v\xi_3^5 + \dots),\\
u_3 &= \ell({}^{10}w\xi_2 + {}^{30}w\xi_2^3 + {}^{12}w\xi_2\xi_3^2 + {}^{50}w\xi_2^5 + {}^{32}w\xi_2^3\xi_3^2 + {}^{14}w\xi_2\xi_3^4 + \dots).
\end{aligned}$$

The shaft under torsion loading is depicted in Figure 9. The characteristic dimensions are again, width b in ξ_2-direction, height h in ξ_3-direction and length ℓ in ξ_1-direction, with $\ell \ll b, h$.

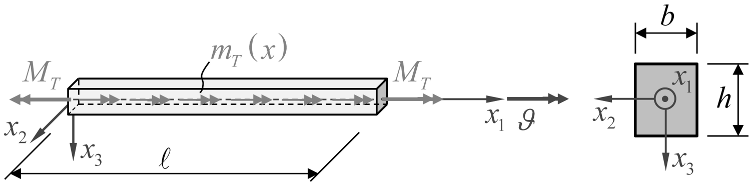


FIGURE 9. Shaft in torsion

In torsion theory, the displacement coefficients are not all independent. They rather have to obey compatibility equations as

$${}^{10}w = -{}^{01}v =: \vartheta(\xi_1), \quad \frac{c}{d}{}^{12}w = -\frac{d}{c}{}^{21}v =: \Theta(\xi_1), \quad \frac{d}{c}{}^{30}w = -\frac{c}{d}{}^{03}v =: \Psi(\xi_1) \quad \text{etc.}$$

Details will be found in [7] which is going to be published soon. Also, the pseudo reduction will be presented there in detail.

Zeroth-order approximation. The zeroth-order approximation allows for a rigid-body rotation of the shaft only

$$\vartheta = \vartheta_0 = \text{const.} + O(e^2)$$

with $e^2 = d^2, c^2$. The external torsional-moment load m_{T} per unit of length $[\mathrm{Nm\,m^{-1}}]$ is of order $O(e^2)$.

First-order approximation. The first-order approximation delivers a St. Venant-like theory involving the torsional stiffness GI_{T} $[\mathrm{Nm^2}]$

$$GI_{\mathrm{T}} = 4GA\ell^2\frac{d^2c^2}{d^2+c^2} = GA\ell^2 O(e^2).$$

The second-order ordinary differential equation is given by

$$GI_{\mathrm{T}}\frac{\vartheta''}{\ell^2} = -m_{\mathrm{T}} + O(e^4).$$

Differences to the classical St. Venant-torsion theory involving also the exact St. Venant-torsion stiffness $(GI_{\mathrm{T}})_{\mathrm{St.v.}}$ for a rectangular cross-section will be discussed in [7].

Second-order approximation. It is, again, appropriate to introduce an energetic average $\tilde{\vartheta}$ of the torsion angle ϑ. It is defined as

$$e^2\tilde{\vartheta} = e^2\Big(\vartheta + dc\Big(\Theta + \frac{9}{5}\Psi\Big)\Big) + O(e^6). \tag{3.3}$$

The warping stiffness EC_M [Nm^4] turns out to be

$$EC_M = \frac{EA\ell^4}{1+\nu}d^2c^2\frac{(d^2-c^2)^2}{(d^2+c^2)^2}d_M = EA\ell^4 O(e^4), \tag{3.4}$$

with

$$d_M = \frac{d^4 + c^4 - 2\nu\, d^2c^2}{(1-\nu)(d^4+c^4) + 2\nu\, d^2c^2} = O(e^0).$$

Note that $d_M = 1$, if $\nu = 0$

With (3.3) and (3.4), an ODE for the second-order torsion theory is derived as

$$EC_M\frac{\tilde{\vartheta}^{\mathrm{IV}}}{\ell^4} - GI_{\mathrm{T}}\frac{\tilde{\vartheta}''}{\ell^2} = m_{\mathrm{T}} + O(c^6).$$

Also, warping-moment loads per unit-of-length m_M [Nm^2 m^{-1}] may be taken into account similar as bending-moment loads per unit-of-length m_B [Nm m^{-1}] in beam theory. The differences and similarities of the second-order approximation to the classical warping-torsion theory will be discussed in [7].

4. Concluding remarks

The consistent- (or uniform-) approximation method leads, in combination with the pseudo-reduction technique, to hierarchies of theories of different approximation orders for structural members. The reduction from 3-D to 2-D yields classical, refined and even higher-order theories for discs and plates. The zeroth-order plate theory allows only for rigid-body motions, the first-order theory is identical with the classical Kirchhoff-plate theory, whereas the second-order approach recovers Reissner's plate theory. Higher-order plate theories may also be derived in a consistent way but are not dealt with in this paper. The reduction from 3-D to 1-D follows similar lines of reasoning. Hierarchies of bar, beam and shaft theories are derived and discussed.

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РЕДУКЦИЈА ДИМЕНЗИЈЕ У ЕЛАСТИЧНОСТИ

Резиме. Из тродимензионалне линеарне теорије еластичности, дводимензионални и једнодимензионални описи су изведени применом приступа конзистентне апроксимације. Техника псеудо-редукције даје добро познате теорије вишег реда за квази дводимензионалне и квази једнодимензионалне структурне елементе.

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