THEORETICAL AND APPLIED MECHANICS
Article in Press (2026) DOI: https://doi.org/10.2298/TAM250425001L

WELL POSEDNESS AND STABILISATION FOR A FLEXIBLE MECHANICAL SYSTEM UNDER DISTRIBUTED DELAY

Billal Lekdim

Abstract. In this paper, we investigate the boundary control of flexible mechanical systems characterized by bending deformation, torsion deformation, and distributed delay. We establish the existence of solutions using the Faedo-Galerkin approach along with energy estimates. Under appropriate assumptions on the delay weight and the proposed control, we demonstrate the exponential stability of the solution via the Lyapunov method.

1. Introduction

This paper focuses on studying the vibration control problem concerning a class of flexible mechanical systems characterized by bending deformation and torsion deformation, considering the presence of distributed delay in the bending equation and the torsion equation. The dynamics model involves coupled vibration deformation [28], expressed as:

$$
(1.1)\quad \begin{cases}
my_{tt} = m_e\varphi_{tt} - k_b y_{txxxx} - E_b y_{xxxx} - \displaystyle\int_{\tau_1}^{\tau_2} \eta_b(s) y_t(x,t-s)ds, & \\
I\varphi_{tt} = m_e y_{tt} + k_a\varphi_{txx} + E_a\varphi_{xx} - \displaystyle\int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds, & x \in (0,L),\ t>0,\\
y(0,t) = y_x(0,t) = y_{xx}(L,t) = 0,\ \varphi(0,t) = 0, & \\
k_b y_{txxx}(L,t) + E_b y_{xxx}(L,t) + C_b(t) = 0, & \\
k_a\varphi_{tx}(L,t) + E_a\varphi_x(L,t) - C_a(t) = 0, & t>0,\\
y(x,0) = y^0(x),\ y_t(x,0) = y^1(x),\ y_t(x,-t) = y^3(x,t), & \\
\varphi(x,0) = \varphi^0(x),\ \varphi_t(x,0) = \varphi^1(x),\ \varphi_t(x,-t) = \varphi^3(x,t), & x \in (0,L)\times(0,\tau_2),
\end{cases}
$$

where y and φ represent the bending and torsion deformation. The subscripts in x and t denote partial derivatives. The parameters $m,\ ,m_e,\ k_b,\ E_b,\ I,\ k_a, E_a,\ \tau_1,\ \tau_2$ and L are positive constants related to the characteristics of mechanical systems,

2020 *Mathematics Subject Classification:* 35B35, 35B40, 93D23, 35R09.
Key words and phrases: exponential stability, well-posedness, boundary control, Faedo-Galerkin, Lyapunov method.

where $\tau_1 < \tau_2$. Additionally, η_b and η_a represent the weight of the delay, C_a and C_b for boundary control.

The boundary control of flexible systems has garnered significant attention in recent years due to its unique characteristics and extensive applications. The primary objective of controlling these systems is to achieve the desired stability rate with minimal internal dissipation (see, for example, [5–7,14–16,19,20]). Perturbations often arise from irregular material properties, external forces, or the presence of a delay factor. To achieve stability, researchers typically depend on the internal dissipation of the system. Boumaza and Boulaaras [4] established asymptotic stability results for systems utilizing internal frictional dissipation mechanisms. Al-Mahdi et al. [1] derived general decay results under the influence of nonlinear damping, demonstrating the system's adaptability to varying damping characteristics. In their paper [3], Al Mahdi et al. specifically relied on frictional dissipation to achieve stability, while in [2], they extended their analysis to systems incorporating viscoelastic dissipation, highlighting the versatility of such damping mechanisms to enhance system stability. However, when these internal dissipations are insufficient to achieve the desired rate of energy decay, boundary control presents a viable solution (we refer to [25,26,31]).

Previous research has established that the delay term can introduce disturbances and contribute to instability (see [29]). To mitigate the negative effects of delay, a viscous dissipation coefficient is relied upon [23,24,30]. Feng [10] analyzed a wave equation incorporating distributed delay, addressing its dynamic behavior under such influences. Gheraibia and Boumaza [13] extended this analysis to a wave equation featuring Balakrishnan-Taylor damping coupled with a non-linear delay, providing insight into the interaction of these factors. Meanwhile, Choucha et al. [6] focused on a coupled system that included both distributed delay and viscoelastic terms, highlighting the combined effects of delay and material memory on the stability and dynamics of the system. Kelleche and Tarar [17] studied the effect of distributed time delay on the stabilization of a Kirchhoff moving string. Related results can also be found in [11,18,21].

The system (1.1), excluding the delay term, has previously been studied in [12] using adaptive control. In addition, numerous control problems have been explored based on boundary control (see [28]).

To the best of our knowledge, there are no results in the literature that address the stability of system (1) with a delay distributed across both equations with two different weights. Additionally, we also mention the simultaneous appearance of the acceleration of bending and twist deflections in both equations. This complexity made our work particularly challenging, especially in proving well-posedness, as it necessitated setting the condition:

$$\min\{m, I\} > m_e. \tag{1.2}$$

The remainder of the paper is organized as follows: Section 2 introduces key assumptions and lemmas that are essential for proving our main result. Section 3 addresses the well-posedness of the problem. Section 4 is dedicated to starting and proving the stabilization result.

2. Preliminary

In this section, we will outline the assumptions and materials necessary to illustrate our main results.

(H1) : We assume that for $i \in \{a,b\}$, $\eta_i\colon \begin{bmatrix}\tau_1 & \tau_2\end{bmatrix} \to \mathbb{R}_+$ is a bounded function, such that

$$\int_{\tau_1}^{\tau_2} \eta_a(s)ds < \frac{k_a}{2L^2} \quad \text{and} \quad \int_{\tau_1}^{\tau_2} \eta_b(s)ds < \frac{k_b}{2L^4}.$$

which implies that there exists a positive constant c_a and c_b such that

$$\frac{k_a}{2L^2} - \int_{\tau_1}^{\tau_2} \Big(\eta_a(s) + \frac{c_a}{2}\Big)ds \geqslant 0 \quad \text{and} \quad \frac{k_b}{2L^4} - \int_{\tau_1}^{\tau_2} \Big(\eta_b(s) + \frac{c_b}{2}\Big)ds \geqslant 0.$$

The energy associated to system (1.1) is

$$e(t) = \frac{m}{2}\int_0^L y_t^2 dx + \frac{E_b}{2}\int_0^L y_{xx}^2 dx + \frac{I}{2}\int_0^L \varphi_t^2 dx + \frac{E_a}{2}\int_0^L \varphi_x^2 dx. \tag{2.1}$$

This expression represents the usual classical energy, where the first two terms account for kinetic energy, while the remaining terms account for potential energy.

To stabilize the system (1.1), we propose the control:

$$C_b(t) = -\alpha_b y_t(L,t) \quad \text{and} \quad C_a(t) = -\alpha_a \varphi_t(L,t), \tag{2.2}$$

where α_b and α_a are positive control gains.

LEMMA 2.1. *The energy (2.1) satisfies*

$$\begin{aligned} e'(t) = {} & m_e \int_0^L [y_{tt}\varphi_t + y_t\varphi_{tt}]dx + C_b(t)y_t(L,t) + C_a(t)\varphi_t(L,t) \\ & - k_b \int_0^L y_{txx}^2 dx - \int_0^L y_t(x,t)\int_{\tau_1}^{\tau_2} \eta_b(s)y_t(x,t-s)ds\,dx \\ & - k_a \int_0^L \varphi_{tx}^2 dx - \int_0^L \varphi_t(x,t)\int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds\,dx. \end{aligned} \tag{2.3}$$

PROOF. The combination of derivation of energy and the equations in (1.1) yields:

$$\begin{aligned} e'(t) = {} & \int_0^L y_t\Big[m_e\varphi_{tt} - k_b y_{txxxx} - E_b y_{xxxx} - \int_{\tau_1}^{\tau_2} \eta_b(s)y_t(x,t-s)ds\Big]dx \\ & + E_b\int_0^L y_{xx}y_{txx}dx + E_a\int_0^L \varphi_x\varphi_{tx}dx \\ & + \int_0^L \varphi_t\Big[m_e y_{tt} + k_a\varphi_{txx} + E_a\varphi_{xx} - \int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds\Big]dx. \end{aligned}$$

By integrating by parts, we find

$$e'(t) = m_e\int_0^L [y_{tt}\varphi_t + y_t\varphi_{tt}]dx - [k_b y_{txxx}(L,t) + E_b y_{xxx}(L,t)]y_t(L,t)$$

$$+ [k_a\varphi_{tx}(L,t) + E_a\varphi_x(L,t)]\varphi_t(L,t) - k_b\int_0^L y_{txx}^2 dx - k_a\int_0^L \varphi_{tx}^2 dx$$
$$- \int_0^L y_t \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(x,t-s)ds\,dx - \int_0^L \varphi_t \int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds\,dx.$$

In view of the boundary condition in (1.1), we get (2.3). □

To include the distributed delay term into the energy of the system (1.1), we introduce the modified energy as follows:

$$\begin{aligned}E(t) = {}& \frac{m}{2}\int_0^L y_t^2 dx + \frac{E_b}{2}\int_0^L y_{xx}^2 dx + \frac{I}{2}\int_0^L \varphi_t^2 dx + \frac{E_a}{2}\int_0^L \varphi_x^2 dx \\ &+ \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} s[\eta_b(s)+c_b]\int_0^1 (y_t(x,t-ps))^2 dp\,ds\,dx \\ &+ \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} s[\eta_a(s)+c_a]\int_0^1 (\varphi_t(x,t-ps))^2 dp\,ds\,dx.\end{aligned} \tag{2.4}$$

Lemma 2.2. *There exists a positive constant a_1 such that*

$$\begin{aligned}E'(t) \leqslant {}& m_e\int_0^L [y_{tt}\varphi_t + y_t\varphi_{tt}]dx + C_b(t)y_t(L,t) + C_a(t)\varphi_t(L,t) \\ &- \frac{k_b}{2}\int_0^L y_{txx}^2 dx - \left[\frac{k_b}{2L^4} - \int_{\tau_1}^{\tau_2}(\eta_b(s) + \frac{c_b}{2})ds\right]\int_0^L y_t^2 dx \\ &- \frac{k_a}{2}\int_0^L \varphi_{tx}^2 dx - \left[\frac{k_a}{2L^2} - \int_{\tau_1}^{\tau_2}(\eta_a(s) + \frac{c_a}{2})ds\right]\int_0^L \varphi_t^2 dx \\ &- \frac{c_b}{2}\int_0^L\int_{\tau_1}^{\tau_2}(y_t(x,t-s))^2 ds\,dx - \frac{c_a}{2}\int_0^L\int_{\tau_1}^{\tau_2}(\varphi_t(x,t-s))^2 ds\,dx.\end{aligned} \tag{2.5}$$

Proof. According to Lemma (2.1), the modified energy $E(t)$ satisfies

$$\begin{aligned}E'(t) = {}& m_e\int_0^L [y_{tt}\varphi_t + y_t\varphi_{tt}]dx + C_b(t)y_t(L,t) + C_a(t)\varphi_t(L,t) \\ &- k_b\int_0^L y_{txx}^2 dx - \int_0^L y_t(x,t)\int_{\tau_1}^{\tau_2}\eta_b(s)y_t(x,t-s)ds\,dx \\ &- k_a\int_0^L \varphi_{tx}^2 dx - \int_0^L \varphi_t(x,t)\int_{\tau_1}^{\tau_2}\eta_a(s)\varphi_t(x,t-s)ds\,dx \\ &+ \int_0^L\int_{\tau_1}^{\tau_2} s[\eta_b(s)+c_b]\int_0^1 y_t(x,t-ps)y_{tt}(x,t-ps)dp\,ds\,dx \\ &+ \int_0^L\int_{\tau_1}^{\tau_2} s[\eta_b(s)+c_a]\int_0^1 \varphi_t(x,t-ps)\varphi_{tt}(x,t-ps)dp\,ds\,dx.\end{aligned} \tag{2.6}$$

Applying Young's inequality to the fifth and seventh terms of the equation (2.6), we obtain

$$\int_0^L y_t(x,t)\int_{\tau_1}^{\tau_1}\eta_b(s)y_t(x,t-s)ds\,dx$$
$$\leqslant \frac{1}{2}\left(\int_{\tau_1}^{\tau_1}\eta_b(s)ds\right)\int_0^L (y_t(x,t))^2dx+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_1}\eta_b(s)(y_t(x,t-s))^2ds\,dx$$

and

$$\int_0^L \varphi_t(x,t)\int_{\tau_1}^{\tau_1}\eta_a(s)\varphi_t(x,t-s)ds\,dx$$
$$\leqslant \frac{1}{2}\left(\int_{\tau_1}^{\tau_1}\eta_a(s)ds\right)\int_0^L (\varphi_t(x,t))^2dx+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_1}\eta_a(s)(\varphi_t(x,t-s))^2ds\,dx.$$

For the last two terms, we can simplify as follows

$$\begin{aligned}\int_0^1 y_t(x,t-ps)y_{tt}(x,t-ps)dp &= -s^{-1}\int_0^1 y_t(x,t-ps)y_{tp}(x,t-ps)dp\\ &= -\frac{s^{-1}}{2}\left[(y_t(x,t-s))^2-(y_t(x,t))^2\right]\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \varphi_t(x,t-ps)\varphi_{tt}(x,t-ps)dp &= -s^{-1}\int_0^1 \varphi_t(x,t-ps)\varphi_{tp}(x,t-ps)dp\\ &= -\frac{s^{-1}}{2}\left[(\varphi_t(x,t-s))^2-(\varphi_t(x,t))^2\right].\end{aligned} \tag{2.7}$$

By combining results (2.6)–(2.7), we get

$$\begin{aligned}E'(t) \leqslant{}& m_e\int_0^L[y_{tt}\varphi_t+y_t\varphi_{tt}]dx+C_b(t)y_t(L,t)+C_a(t)\varphi_t(L,t)\\ &-k_b\int_0^L y_{txx}^2dx-k_a\int_0^L\varphi_{tx}^2dx\\ &+\int_{\tau_1}^{\tau_2}[\eta_b(s)+c_b/2]ds\int_0^L y_t^2dx+\int_{\tau_1}^{\tau_2}[\eta_a(s)+c_a/2]ds\int_0^L\varphi_t^2dx\\ &-\frac{c_b}{2}\int_0^L\int_{\tau_1}^{\tau_2}(y_t(x,t-s))^2ds\,dx-\frac{c_a}{2}\int_0^L\int_{\tau_1}^{\tau_2}(\varphi_t(x,t-s))^2ds\,dx,\end{aligned}$$

The proof is concluded by substituting the proposed control (2.2) and applying Poincare inequality. □

3. Well posedness

Here, to establish the well-posedness result using Faedo-Galerkin method, we will need to change the following variable:

$$\begin{cases} z(x,p,t,s)=y_t(x,t-ps), & (0,L)\times(0,1)\times\mathbb{R}_+\times(\tau_1,\tau_2),\\ \phi(x,p,t,s)=\varphi_t(x,t-ps), & (0,L)\times(0,1)\times\mathbb{R}_+\times(\tau_1,\tau_2),\end{cases}$$

Subsequently, problem (1.1) becomes

$$
(3.1)\quad \begin{cases}
my_{tt}-m_e\varphi_{tt}+k_by_{txxxx}+E_by_{xxxx}+\int_{\tau_1}^{\tau_2}\eta_b(s)y_t(x,t-s)ds=0,\\
I\varphi_{tt}-m_ey_{tt}-k_a\varphi_{txx}-E_a\varphi_{xx}+\int_{\tau_1}^{\tau_2}\eta_a(s)\varphi_t(x,t-s)ds=0,\\
z_t(x,p,t,s)+s^{-1}z_p(x,p,t,s)=0,\\
\phi_t(x,p,t,s)+s^{-1}\phi_p(x,p,t,s)=0,\\
y(0,t)=y_x(0,t)=y_{xx}(L,t)=0,\quad \varphi(0,t)=0,\\
k_by_{txxx}(L,t)+E_by_{xxx}(L,t)+C_b(t)=0,\\
k_a\varphi_{tx}(L,t)+E_a\varphi_x(L,t)-C_a(t)=0,\\
y(x,0)=y^0(x),\quad y_t(x,0)=y^1(x),\quad z(x,p,0,s)=y^3(x,ps)=z_0,\\
\varphi(x,0)=\varphi^0(x),\quad \varphi_t(x,0)=\varphi^1(x),\quad \phi(x,p,0,s)=\varphi^3(x,ps)=\phi_0,
\end{cases}
$$

for all $(x,p,t,s)\in(0,L)\times(0,1)\times\mathbb{R}_+\times(\tau_1,\tau_2)$.

We introduce the Hilbert spaces

$$U^1=\{f\in H^2(0,L),\ f(0)=f'(0)=0\},\quad U^2=\{f\in U^1\cap H^4(0,L),\ f''(L)=0\},$$

$$V^i=\{f\in H^i(0,L)\ f(0)=0\},\quad Z^i=L^2(0,L;H^i(0,1)),\qquad i=1,2.$$

The existence and uniqueness result is stated as follows.

THEOREM 3.1. *Assume that* (**H1**) *hold. Then given* $(y_0,\varphi_0,z_0,\phi_0)\in U^2\times V^2\times Z^2\times Z^1$ *and* $(y_1,\varphi_1)\in U^1\times V^1$, *there exists a unique weak solution* (y,φ,z,ϕ) *of the problem* (3.1) *such that*

$$(y,\varphi)\in W^{2,\infty}(0,T;U^1)\times W^{2,\infty}(0,T;V^1),$$

$$z,\phi\in W^{1,\infty}(0,T;L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))).$$

PROOF. We employ the Galerkin method to construct a solution. Let $T>0$ be fixed, and denote $W_n=\text{span}\{w_1,w_2,\dots,w_n\}$ and $V_n=\text{span}\{v_1,v_2,\dots,v_n\}$, where the sets $\{w_i\}_{i\in\mathbb{N}}$ and $\{v_i\}_{i\in\mathbb{N}}$ form bases for U^2 and V^2, respectively. Then, we give the sequence $\{u_i(x,p)\}_{0\leqslant i\leqslant n}$ and $\{\chi_i(x,p)\}_{0\leqslant i\leqslant n}$ as follows:

$$u_i(x,0)=w_i(x)\quad\text{and}\quad \chi_i(x,0)=v_i(x).$$

Next, we extend $u_i(x,0)$ by $u_i(x,p)$ and $\chi_i(x,0)$ by $\chi_i(x,p)$ over $L^2((0,L)\times(0,1))$ and denote by $Z_n^2=\text{span}\{u_1,u_2,\dots,u_n\}$ and $Z_n^1=\text{span}\{\chi_1,\chi_2,\dots,chi_n\}$. where the sets $\{u_i\}_{i\in\mathbb{N}}$ and $\{\chi_i\}_{i\in\mathbb{N}}$ form bases for Z^2 and Z^1, respectively.

We search the approximate solutions

$$y^n(x,t)=\sum_{i=1}^n w_i(x)f_i^n(t),\quad z^n(x,p,t)=\sum_{i=1}^n u_i(x,p)h_i^n(t),$$

$$\varphi^n(x,t)=\sum_{i=1}^n v_i(x)g_i^n(t),\quad \phi^n(x,p,t)=\sum_{i=1}^n \chi_i(x,p)k_i^n(t).$$

That satisfy the following system:

$$(3.2)\qquad \begin{cases} (my^n_{tt} - m_e y^n_{tt}, w_j) + (k_b y^n_{txx} + E_b y^n_{xx}, w_{jxx}) \\ + \int_{\tau_1}^{\tau_2} \eta_b(s)(z^n(x,1,t), w_j)ds + \alpha_b y^n_t(L) w_j = 0, \\ (I\varphi^n_{tt} - m_e y^n_{tt}, v_j) + (k_a \varphi^n_{tx} + E_a \varphi^n_x, v_{jx}) \\ + \int_{\tau_1}^{\tau_2} \eta_a(s)(\phi(x,1,t), v_j)ds + \alpha_a \varphi^n_t(L) v_j = 0, \\ (z^n_t + s^{-1} z^n_p, u_j) = 0, \\ (\phi^n_t + s^{-1}\phi^n_p, \chi_j) = 0, \quad 1 \leqslant j \leqslant n, \end{cases}$$

with initial conditions $(y^n(0), \varphi^n(0), z^n(0), \phi^n(0)) = (y^n_0, \varphi^n_0, z^n_0, \phi^n_0) \to (y_0, \varphi_0, z_0, \phi_0)$ in $U^2 \times V^2 \times Z^2 \times Z^1$ and $(y^n_t(0), \varphi^n_t(0)) = (y^n_1, \varphi^n_1) \to (y^n_1, \varphi^n_1)$ in $U^1 \times V^1$.

According to the standard theory of ordinary differential equations, the finite dimensional problem (3.2) has solution $f^n_i(t)$, $g^n_i(t)$, $h^n_i(t)$ and $k^n_i(t)$ on $[0, t_n)$. The a priori estimates that follow imply that $t_n = T$.

First estimate: Multiplying the first two equations in (3.2) by $(f^n_j)'$ and $(g^n_j)'$, respectively, then summing respect to j, we get

$$(3.3)\qquad \frac{1}{2}\frac{d}{dt}\int_0^L (m(y^n_t)^2 + E_b(y^n_{xx})^2 + I(\varphi^n_t)^2 + E_a(\varphi^n_{xx})^2)dx \\ + k_b \int_0^L (y^n_{txx})^2 dx + k_a \int_0^L (\varphi^n_{tx})^2 dx + \alpha_b (y^n_t(L))^2 + \alpha_a(\varphi^n_t(L))^2 \\ + \int_{\tau_1}^{\tau_2} \eta_b(s)(z^n(x,1,t), y^n_t)ds + \int_{\tau_1}^{\tau_2} \eta_a(s)(\phi(x,1,t), \varphi^n_t)ds \\ = m_e \frac{d}{dt}\int_0^L y^n_t \varphi^n_t dx.$$

Now, multiplying the last two equations in (3.2) by $(\eta_a(s) + c_a)h^n_j$ and $(\eta_b(s) + c_b))k^n_j$, respectively, and summing respect to j, then integrating over $(0,1)\times(\tau_1, \tau_2)$, we obtain

$$(3.4)\qquad \frac{1}{2}\frac{d}{dt}\int_0^L \left[\int_{\tau_1}^{\tau_2} (\eta_b(s) + c_b)\int_0^1 (z^n)^2 dp\, ds + \int_{\tau_1}^{\tau_2} (\eta_a(s) + c_a)\int_0^1 (\phi^n)^2 dp\, ds\right] dx \\ = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} (\eta_b(s) + c_b) ds\right)\int_0^L (y^n_t)^2 dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\eta_b(s) + c_b)(z^n(1,s))^2 ds\, dx \\ + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} (\eta_a(s) + c_a) ds\right)\int_0^L (\varphi^n_t)^2 dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\eta_a(s) + c_a)(\phi^n(1,s))^2 ds\, dx.$$

By summing (3.3) and (3.4), and by applying Young's inequality to the last two terms from the left side of equation, we find

$$\frac{d}{dt}E^n(t) + k_b \int_0^L (y^n_{txx})^2 dx + k_a \int_0^L (\varphi^n_{tx})^2 dx + \alpha_b (y^n_t(L))^2 + \alpha_a(\varphi^n_t(L))^2 \\ + \frac{c_b}{2}\int_0^L \int_{\tau_1}^{\tau_2} (z^n(1,s))^2 ds\, dx + \frac{c_a}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\phi^n(1,s))^2 ds\, dx$$

$$\leqslant \left(\int_{\tau_1}^{\tau_2}(\eta_b(s)+\frac{c_b}{2})ds\right)\int_0^L (y_t^n)^2dx$$
$$+\left(\int_{\tau_1}^{\tau_2}(\eta_a(s)+\frac{c_a}{2})ds\right)\int_0^L (\varphi_t^n)^2dx\, m_e\frac{d}{dt}\int_0^L y_t^n\varphi_t^n dx,$$

where E^n is the energy given by (2.4), for the solutions $(y^n, \varphi^n, z^n, \phi^n)$.

Then Integrating over $(0, t)$, we obtain

$$E^n(t)+\int_0^t k_b\int_0^L (y_{txx}^n)^2dx+k_a\int_0^L(\varphi_{tx}^n)^2dx+\alpha_b(y_t^n(L))^2+\alpha_a(\varphi_t^n(L))^2dt$$
$$+\frac{c_b}{2}\int_0^t\int_0^L\int_{\tau_1}^{\tau_2}(z^n(1,s))^2ds\,dx\,dt+\frac{c_a}{2}\int_0^t\int_0^L\int_{\tau_1}^{\tau_2}(\phi^n(1,s))^2ds\,dx\,dt$$
$$\leqslant\left(\int_{\tau_1}^{\tau_2}(\eta_b(s)+\frac{c_b}{2})ds\right)\int_0^t\int_0^L(y_t^n)^2dx+\left(\int_{\tau_1}^{\tau_2}(\eta_a(s)+\frac{c_a}{2})ds\right)\int_0^t\int_0^L(\varphi_t^n)^2dx$$
$$+m_e\int_0^L y_t^n\varphi_t^n dx-m_e\int_0^L y_1^n\varphi_1^n dx+E^n(0)$$

After simplifying this expression and taking it into account (1.2), we can arrive at

$$E^n(t)+\int_0^t\int_0^L((y_{txx}^n)^2+(\varphi_{tx}^n)^2)dx\,dt\leqslant d_1,$$

Where d_1 is a positive constant that depends only on the initial dates.

Second estimate: First, we estimate $y_{tt}(t=0)$ and $\varphi_{tt}(t=0)$. Similarly to the first estimation, we multiply the first two equations in (3.2) by $(f_j^n)''$ and $(g_j^n)''$, respectively and summing respect to j, then taking $t=0$ we obtain

$$(m-m_e)\int_0^L(y_{tt}^n(0))^2dx+(I-m_e)\int_0^L(\varphi_{tt}^n(0))^2dx$$
$$\leqslant-\left(k_by_{1txxxx}^n+E_by_{0xxxx}^n+\int_{\tau_1}^{\tau_2}\eta_b(s)z_0^nds,y_{tt}^n(0)\right)$$
$$+\left(k_a\varphi_{1txx}^n+E_a\varphi_{0xx}^n-\int_{\tau_1}^{\tau_2}\eta_a(s)\phi_0^nds,\varphi_{tt}^n\right)$$

Since the initial data is smooth enough, from Young's inequality, we obtain

$$y_{tt}^n(0),\ \varphi_{tt}^n(0)\in L^2(0,L).$$

Deriving the system (3.2) with respect to t, then multiplying by $(f_j^n)''$, $(g_j^n)''$, $(\eta_a(s)+c_a)(h_j^n)'$ and $(\eta_b(s)+c_b))(k_j^n)'$ respectively and summing with respect to j, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L\left(m(y_{tt}^n)^2+E_b(y_{txx}^n)^2+I(\varphi_{tt}^n)^2+E_a(\varphi_{tx}^n)^2\right)dx+k_b\int_0^L(y_{ttxx}^n)^2dx \tag{3.5}$$
$$+k_a\int_0^L(\varphi_{ttx}^n)^2dx+\alpha_by_t^ny_{tt}^n(L)+\alpha_a\varphi_t^n\varphi_{tt}^n(L)+\int_{\tau_1}^{\tau_2}\eta_b(s)(z_t^n(x,1,t),y_{tt}^n)ds$$

$$+\int_{\tau_1}^{\tau_2}\eta_a(s)(\phi_t^n(x,1,t),\varphi_{tt}^n)ds = m_e\frac{d}{dt}\int_0^L y_{tt}^n\varphi_{tt}^n dx$$

and

$$\begin{aligned}(3.6)\quad &\frac{1}{2}\frac{d}{dt}\int_0^L\left[\int_{\tau_1}^{\tau_2}(\eta_b(s)+c_b)\int_0^1(z_t^n)^2dp\,ds+\int_{\tau_1}^{\tau_2}(\eta_a(s)+c_a)\int_0^1(\phi_t^n)^2dp\,ds\right]dx\\ &=\frac{1}{2}\int_{\tau_1}^{\tau_2}(\eta_b(s)+c_b)ds\int_0^L(y_{tt}^n)^2dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}(\eta_b(s)+c_b)(z_t^n(1,s))^2ds\,dx\\ &\quad+\frac{1}{2}\int_{\tau_1}^{\tau_2}(\eta_a(s)+c_a)ds\int_0^L(\varphi_{tt}^n)^2dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}(\eta_a(s)+c_a)(\phi_t^n(1,s))^2ds\,dx\end{aligned}$$

Taking the sum of (3.5) and (3.6), and integrating over $(0,t)$, then following the same steps as in first estimation we find

$$\begin{aligned}&\frac{1}{2}\int_0^L\big(m(y_{tt}^n)^2+E_b(y_{txx}^n)^2+I(\varphi_{tt}^n)^2+E_a(\varphi_{tx}^n)^2\big)dx\\ &\quad\frac{1}{2}\int_0^L\left[\int_{\tau_1}^{\tau_2}(\eta_b(s)+c_b)\int_0^1(z_t^n)^2dp\,ds+\int_{\tau_1}^{\tau_2}(\eta_a(s)+c_a)\int_0^1(\phi_t^n)^2dp\,ds\right]dx\\ &\qquad+\int_0^t\int_0^L k_b(y_{ttxx}^n)^2+k_a(\varphi_{ttx}^n)^2dx\,dt\\ &\leqslant\int_0^t\left(\int_{\tau_1}^{\tau_2}(\eta_b(s)+\frac{c_b}{2})ds\right)\int_0^L(y_{tt}^n)^2dx+\left(\int_{\tau_1}^{\tau_2}(\eta_a(s)+\frac{c_a}{2})ds\right)\int_0^L(\varphi_{tt}^n)^2dx\,dt\\ &\qquad+m_e\int_0^L y_{tt}^n\varphi_{tt}^n dx-\int_0^t\alpha_b y_t^n y_{tt}^n(L)-\alpha_a\varphi_t^n\varphi_{tt}^n(L)dt+d_2\end{aligned}$$

where $d_2=d_2(y_0^n,\varphi_0^n,z_0^n,\phi_0^n,y_1^n,\varphi_1^n,y_{tt}^n(0),\varphi_{tt}^n(0))$.

Using Young and Poincare's inequalities, and taking the first estimate into account, we conclude

$$\begin{aligned}\alpha_b y_t^n y_{tt}^n(L)-\alpha_a\varphi_t^n\varphi_{tt}^n(L)&\leqslant\frac{\alpha_b^2L^3}{4k_b}(y_t^n(L))^2+\frac{k_b}{L^3}y_{tt}^n(L)+\frac{\alpha_a^2L}{4k_a}(\varphi_t^n(L))^2+\frac{k_a}{L}\varphi_{tt}^n(L)\\ &\leqslant d_4+\int_0^L k_b(y_{ttxx}^n)^2+k_a(\varphi_{ttx}^n)^2dx.\end{aligned}$$

By combining the aforementioned last two outcomes, we obtain

$$\begin{aligned}&\int_0^L\big((m-m_e)(y_{tt}^n)^2+E_b(y_{txx}^n)^2+(I-m_e)(\varphi_{tt}^n)^2+E_a(\varphi_{tx}^n)^2\big)dx\\ &\quad\int_0^L\int_{\tau_1}^{\tau_2}(\eta_b(s)+c_b)\int_0^1(z_t^n)^2dp\,ds\,dx+\int_0^L\int_{\tau_1}^{\tau_2}(\eta_a(s)+c_a)\int_0^1(\phi_t^n)^2dp\,ds\,dx\\ &\qquad+4\int_0^t\int_0^L k_b(y_{ttxx}^n)^2+k_a(\varphi_{ttx}^n)^2dx\,dt\\ &\qquad\qquad\leqslant d_5\int_0^t\int_0^L(y_{tt}^n)^2dx+(\varphi_{tt}^n)^2dx\,dt+d_5.\end{aligned}$$

Let's apply Gronwall's lemma on the last inequality. We estimate

$$\int_0^L (y_{tt}^n)^2 dx + \int_0^L (y_{txx}^n)^2 dx + \int_0^L (\varphi_{tt}^n)^2 dx + \int_0^L (\varphi_{tx}^n)^2 dx \\ + \int_0^t \int_0^L (y_{ttxx}^n)^2 dx\, dt + \int_0^L \int_{\tau_1}^{\tau_2} (\eta_b(s) + c_b) \int_0^1 (z_t^n)^2 dp\, ds\, dx \\ + \int_0^t \int_0^L (\varphi_{ttx}^n)^2 dx\, dt + \int_0^L \int_{\tau_1}^{\tau_2} (\eta_a(s) + c_a) \int_0^1 (\phi_t^n)^2 dp\, ds\, dx \leqslant d_6,$$

where d_6 is a positive constant independent of n.

From the first and second estimates we conclude that

$$\begin{aligned} y^n, y_t^n \quad & \text{are bounded in } L^\infty(0,T;U^1), \\ y_{tt}^n \quad & \text{is bounded in } L^\infty(0,T;L^2(0,L)) \cap L^\infty(0,T;U^1), \\ \varphi^n, \varphi_t^n \quad & \text{are bounded in } L^\infty(0,T;V^1), \\ \varphi_{tt}^n \quad & \text{is bounded in } L^\infty(0,T;L^2(0,L)) \cap L^\infty(0,T;V^1), \\ z^n, z_t^n \quad & \text{are bounded in } L^\infty\big(0,T;L^2\big((0,L)\times(0,1)\times(\tau_1,\tau_2)\big)\big), \\ \phi^n, \phi_t^n \quad & \text{are bounded in } L^\infty\big(0,T;L^2\big((0,L)\times(0,1)\times(\tau_1,\tau_2)\big)\big). \end{aligned}$$

We can obtain subsequences of (y^n), (φ^n), (z^n) and (ϕ^n), which will still be denoted as (y^n), (φ^n), (z^n) and (ϕ^n), respectively, such that

$$\begin{aligned} y^n \rightharpoonup y,\ y_t^n \rightharpoonup y_t \text{ weak star in} \quad & L^\infty(0,T;U^1), \\ y_{tt}^n \rightharpoonup y_{tt} \text{ weak star in} \quad & L^\infty(0,T;L^2(0,L)) \cap L^\infty(0,T;U^1), \\ \varphi^n \rightharpoonup \varphi,\ \varphi_t^n \rightharpoonup \varphi_t \text{ weak star in} \quad & L^\infty(0,T;V^1), \\ \varphi_{tt}^n \rightharpoonup \varphi_{tt} \text{ weak star in} \quad & L^\infty(0,T;L^2(0,L)) \cap L^\infty(0,T;V^1), \\ z^n \rightharpoonup z,\ z_t^n \rightharpoonup z_t \text{ weak star in} \quad & L^\infty\big(0,T;L^2\big((0,L)\times(0,1)\times(\tau_1,\tau_2)\big)\big), \\ \phi^n \rightharpoonup \phi,\ \phi_t^n \rightharpoonup \phi_t \text{ weak star in} \quad & L^\infty\big(0,T;L^2\big((0,L)\times(0,1)\times(\tau_1,\tau_2)\big)\big). \end{aligned}$$

We can now take the limit in the approximate problem (3.2) to obtain a weak solution to the problem (3.1) (see [**27**, **32**]).

For the sake of uniqueness, we will use the standard approach by assuming the existence of two distinct solutions. By following the same procedure used in the second estimation, we demonstrate that the two solutions are identical, thereby proving that the problem (3.1) has a unique solution. See [**8**, **9**, **22**]. □

4. Stability

Drawing from Lemma 2.2 and the suggested control (2.2), we are unable to deduce the decay of energy. Consequently, we will employ the Lyapunov method and introduce the following Lyapunov function:

$$\mathcal{L}(t) = E(t) + W(t) + \gamma V(t),$$

where

$$W(t)=\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_b(s)+c_b]\int_0^1 e^{-2ps}(y_t(x,t-ps))^2dp\,ds\,dx+\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_a(s)+c_a]\int_0^1 e^{-2ps}(\varphi_t(x,t-ps))^2dp\,ds\,dx,$$

$$V(t)=\int_0^L(my_t-m_e\varphi_t)ydx+\int_0^L(I\varphi_t-m_ey_t)\varphi dx+\frac{k_b}{2}\int_0^L y_{xx}^2dx+\frac{k_a}{2}\int_0^L\varphi_x^2dx-\frac{m_e}{\gamma}\int_0^L y_t\varphi_t dx$$

and γ is a positive constant.

Proposition 4.1. *There exist two positive constants a and b, such that*

$$aE(t)\leqslant a(E(t)+W(t))\leqslant\mathcal{L}(t)\leqslant b(E(t)+W(t)),\quad\forall t\geqslant 0. \tag{4.1}$$

Proof. Utilizing Young's inequality on the functional $V(t)$ and under the assumption that γ is sufficiently small, we readily obtain relation (4.1). □

Theorem 4.1. *Assume that* (**H1**) *holds, then the energy $E(t)$ of system* (1.1) *satisfies*

$$E(t)\leqslant\delta e^{-\lambda t},\quad t\geqslant 0,$$

where δ and λ are positive constants.

Proof. The differentiation of $W(t)$ yields

$$\begin{aligned}W'(t)&=\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_b(s)+c_b]\int_0^1 2e^{-2ps}y_t(x,t-ps)y_{tt}(x,t-ps)dp\,ds\,dx\\&\quad+\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_a(s)+c_a]\int_0^1 2e^{-2ps}\varphi_t(x,t-ps)\varphi_{tt}(x,t-ps)dp\,ds\,dx\\&=-\int_0^L\int_{\tau_1}^{\tau_2}[\eta_b(s)+c_b]\int_0^1 e^{-2ps}\frac{\partial}{\partial p}(y_t(x,t-ps))^2dp\,ds\,dx\\&\quad-\int_0^L\int_{\tau_1}^{\tau_2}[\eta_a(s)+c_a]\int_0^1 e^{-2ps}\frac{\partial}{\partial p}(\varphi_t(x,t-ps))^2dp\,ds\,dx\\&=\int_{\tau_1}^{\tau_2}[\eta_b(s)+c_b]ds\int_0^L y_t^2dx+\int_{\tau_1}^{\tau_2}[\eta_a(s)+c_a]ds\int_0^L\varphi_t^2dx\\&\quad-\int_0^L\int_{\tau_1}^{\tau_2}[\eta_b(s)+c_b]e^{-2s}(y_t(x,t-s))^2ds\,dx\\&\quad-\int_0^L\int_{\tau_1}^{\tau_2}[\eta_a(s)+c_a]e^{-2s}(\varphi_t(x,t-s))^2ds\,dx-2W(t)\\&\leqslant\int_{\tau_1}^{\tau_2}[\eta_b(s)+c_b]ds\int_0^L y_t^2dx+\int_{\tau_1}^{\tau_2}[\eta_a(s)+c_a]ds\int_0^L\varphi_t^2dx\\&\quad-\frac{e^{-2\tau_2}}{\tau_2}\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_b(s)+c_b](y_t(x,t-s))^2ds\,dx\\&\quad-\frac{e^{-2\tau_2}}{\tau_2}\int_0^L\int_{\tau_1}^{\tau_2}s[\eta_a(s)+c_a](\varphi_t(x,t-s))^2ds\,dx-2W(t).\end{aligned}\tag{4.2}$$

Now, let's derive the function $V(t)$ and utilize the equations in (1.1) to find

$$\begin{aligned}V'(t) = &-\int_0^L (k_b y_{txxxx} + E_b y_{xxxx} + \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(x,t-s)ds) y\,dx \\ &+ \int_0^L (k_a \varphi_{txx} + E_a \varphi_{xx} - \int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds)\varphi\,dx \\ &+ \int_0^L (m y_t - m_e \varphi_t) y_t dx + \int_0^L (I\varphi_t - m_e y_t)\varphi_t dx \\ &+ k_b \int_0^L y_{xx} y_{txx} dx + k_a \int_0^L \varphi_x \varphi_{tx} dx - \frac{m_e}{\gamma}\int_0^L (y_{tt}\varphi_t + y_t \varphi_{tt})dx.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}V'(t) = &-E_b \int_0^L y_{xx}^2 dx - E_a \int_0^L \varphi_x^2 dx + m\int_0^L y_t^2 dx + I\int_0^L \varphi_t^2 dx \\ &- \frac{m_e}{\gamma}\int_0^L (y_{tt}\varphi_t + y_t\varphi_{tt})dx - \int_0^L y \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(x,t-s)ds\,dx \\ &- \int_0^L \varphi \int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds\,dx - 2m_e \int_0^L \varphi_t y_t dx \\ &+ C_b(t) y(L,t) + C_a(t)\varphi(L,t).\end{aligned} \tag{4.3}$$

To estimate terms with varying signs, we can employ Young and Poincare's inequalities as follows

$$\begin{aligned}&\int_0^L y(x,t)\int_{\tau_1}^{\tau_2} \eta_b(s) y_t(x,t-s)ds\,dx \\ &\qquad \leqslant \left(\int_{\tau_1}^{\tau_2} \eta_b(s)ds\right)^2 \frac{L^4\sigma}{2}\int_0^L y_{xx}^2 dx + \frac{1}{2\sigma}\int_0^L \int_{\tau_1}^{\tau_2} (y_t(x,t-s))^2 ds\,dx,\end{aligned}$$

$$\begin{aligned}&\int_0^L \varphi(x,t)\int_{\tau_1}^{\tau_2} \eta_a(s)\varphi_t(x,t-s)ds\,dx \\ &\qquad \leqslant \left(\int_{\tau_1}^{\tau_2} \eta_a(s)ds\right)^2 \frac{L^2\sigma}{2}\int_0^L \varphi_x^2 dx + \frac{1}{2\sigma}\int_0^L \int_{\tau_1}^{\tau_2} (\varphi_t(x,t-s))^2 ds\,dx,\end{aligned}$$

$$2m_e \int_0^L \varphi_t y_t dx \leqslant m_e \int_0^L y_t^2 dx + m_e \int_0^L \varphi_t^2 dx$$

and

$$\begin{aligned}C_b(t)y(L,t) + C_a(t)\varphi(L,t) &= \alpha_b y_t(L,t) y(L,t) + \alpha_a \varphi_t(L,t)\varphi(L,t) \\ &\leqslant \frac{\alpha_b L^3 \sigma}{2}\int_0^L y_{xx}^2 dx + \frac{\alpha_b}{2\sigma} y_t^2(L,t) \\ &\quad + \frac{\alpha_a L\sigma}{2}\int_0^L \varphi_x^2 dx + \frac{\alpha_a}{2\sigma}\varphi_t^2(L,t),\end{aligned} \tag{4.4}$$

where $\sigma > 0$ constant.

By combining (4.3)-(4.4), we obtain

$$\begin{aligned} V'(t) \leqslant & -\Big(E_b - \Big(\int_{\tau_1}^{\tau_2} \eta_b(s)ds\Big)^2 \frac{L^4\sigma}{2} - \frac{\alpha_b L^3 \sigma}{2}\Big) \int_0^L y_{xx}^2 dx + (m+m_e)\int_0^L y_t^2 dx \\ & - \Big(E_a - \Big(\int_{\tau_1}^{\tau_2} \eta_a(s)ds\Big)^2 \frac{L^2\sigma}{2} - \frac{\alpha_a L \sigma}{2}\Big) \int_0^L \varphi_x^2 dx + (I+m_e)\int_0^L \varphi_t^2 dx \\ & - \frac{m_e}{\gamma}\int_0^L (y_{tt}\varphi_t + y_t\varphi_{tt})dx + \frac{1}{2\sigma}\int_0^L\int_{\tau_1}^{\tau_2} (y_t(x,t-s))^2 ds\, dx \\ & + \frac{1}{2\sigma}\int_0^L\int_{\tau_1}^{\tau_2} (\varphi_t(x,t-s))^2 ds\, dx + \frac{\alpha_b}{2\sigma} y_t^2(L,t) + \frac{\alpha_a}{2\sigma}\varphi_t^2(L,t). \end{aligned}$$

Utilizing the estimates (2.5), (4.2), and (4.5), we obtain

$$\begin{aligned} \mathcal{L}'(t) = & \,\alpha_b\Big(1 - \frac{\gamma}{2\sigma}\Big) y_t^2(L,t) + \alpha_a\Big(1-\frac{\gamma}{2\sigma}\Big)\varphi_t^2(L,t) \\ & - \frac{k_b}{2}\int_0^L y_{txx}^2 dx - \Big(E_b - \Big(\int_{\tau_1}^{\tau_2}\eta_b(s)ds\Big)^2\frac{L^4\sigma}{2} - \frac{\alpha_b L^3\sigma}{2}\Big)\gamma\int_0^L y_{xx}^2 dx \\ & - \Big[\frac{k_b}{2L^4} - \int_{\tau_1}^{\tau_2}(2\eta_b(s) + \frac{3c_b}{2})ds - (m+m_e)\gamma\Big]\int_0^L y_t^2 dx \\ & - \frac{k_a}{2}\int_0^L \varphi_{tx}^2 dx - \Big(E_a - \Big(\int_{\tau_1}^{\tau_2}\eta_a(s)ds\Big)^2\frac{L^2\sigma}{2} - \frac{\alpha_a L\sigma}{2}\Big)\gamma\int_0^L \varphi_x^2 dx \\ & - \Big[\frac{k_a}{2L^2} - \int_{\tau_1}^{\tau_2}(2\eta_a(s) + \frac{3c_a}{2})ds - (I+m_e)\gamma\Big]\int_0^L \varphi_t^2 dx \\ & - \Big[\frac{c_b}{2} - \frac{\gamma}{2\sigma}\Big]\int_0^L\int_{\tau_1}^{\tau_2}(y_t(x,t-s))^2 ds\, dx \\ & - \Big[\frac{c_a}{2} - \frac{\gamma}{2\sigma}\Big]\int_0^L\int_{\tau_1}^{\tau_2}(\varphi_t(x,t-s))^2 ds\, dx \\ & - \frac{e^{-2\tau_2}}{\tau_2}\int_0^L\int_{\tau_1}^{\tau_2} s[\eta_b(s)+c_b](y_t(x,t-s))^2 ds\, dx \\ & - \frac{e^{-2\tau_2}}{\tau_2}\int_0^L\int_{\tau_1}^{\tau_2} s[\eta_a(s)+c_a](\varphi_t(x,t-s))^2 ds\, dx - 2W(t). \end{aligned} \tag{4.5}$$

We have the flexibility to choose extremely small values for the constants β, α, and δ, guaranteeing that the coefficients in the previous relation become negative. Then, we deduce that

$$\mathcal{L}'(t) \leqslant -c(E(t) + W(t)), \quad \forall t \geqslant 0.$$

Applying the equivalence relation (4.1), we get

$$\mathcal{L}'(t) \leqslant -\lambda \mathcal{L}(t), \quad \forall t \geqslant 0,$$

where $\lambda = \frac{c}{b}$. By multiplying both sides of the above inequality by $e^{\lambda t}$, we obtain

$$\frac{d}{dt}\{\mathcal{L}(t)e^{\lambda t}\} \leqslant 0, \quad \forall t \geqslant 0.$$

Now, by integrating this inequality over $(0, t)$, we find

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)e^{-\lambda t}, \quad \forall t \geqslant 0.$$

Once again, by employing the equivalence proposition 4.1, we have

$$E(t) \leqslant \delta e^{-\lambda t}, \quad t \geqslant 0,$$

where $\delta = \frac{\mathcal{L}(0)}{a}$. □

Acknowledgments. The authors would like to thank the reviewers and express their gratitude to DGRSDT.

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ДОБРО ПОСТАВЉАЊЕ И СТАБИЛИЗАЦИЈА ФЛЕКСИБИЛНОГ МЕХАНИЧКОГ СИСТЕМА СА РАСПОДЕЉЕНИМ КАШЊЕЊЕМ

Резиме. У овом раду истражујемо гранично управљање флексибилним механичким системима које карактерише деформација савијања, торзиона деформација и расподељено кашњење. Утврђујемо постојање решења користећи Фаедо-Галеркинов приступ заједно са проценама енергије. Под одговарајућим претпоставкама о тежини кашњења и предложеном управљању, показујемо експоненцијалну стабилност решења путем Љапуновљеве методе.

Faculty of Exact Sciences and Computer Science
University of Djelfa
Djelfa
Algeria
SD laboratory
Faculty of Mathematics
University of Science and Technology Houari Boumediene
Bab Ezzouar, Algiers
Algeria
b.lekdim@univ-djelfa.dz
https://orcid.org/0000-0002-9710-9883

(Received 25.04.2025)
(Revised 06.01.2026)
(Available online 02.02.2026)