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OPTIMAL SYNTHESIS FOR AN INTERMEDIATE VEHICLE MODEL WITH STATE CONSTRAINTS

E. V. Malykh and O. Yu. Cherkasov

Abstract. The problem of optimal thrust programming for an intermediate vehicle model is considered. The motion occurs in a vertical plane under a uniform gravitational field, quadratic resistance friction, and thrust force. The control variables are the angle of attack and the thrust force. Phase constraints are imposed on the trajectory inclination angle. It is assumed that the total fuel consumption for thrust control is negligible compared to the vehicle mass, the fuel mass variation does not affect the center of mass dynamics, and a change of the lifting force does not affect the drag force. The region in the space of initial variables for which the problem is solvable is determined, and an optimal synthesis is constructed. It is established that within this domain, the thrust can be maximum, intermediate, or zero. The number and sequence of trajectory arcs with corresponding thrust values and the number of exits to state constrains are determined.

1. Introduction

One of the the earliest problems concerning the optimization of the shape of the trajectory of a point in a vertical plane under the action of gravity is the Brachistochrone problem [1]. In 1696, Johann Bernoulli formulated the following problem in the journal *Acta Eruditorum*: to find the shape of a curve along which a material point moving in a vertical plane under the action of gravity alone will move from one given point to another given point in the minimum amount of time.

The classical formulation of the Brachistochrone problem assumed the potential nature of the forces acting on the point. Subsequent research expanded the problem formulation to include various friction forces and thrust. Various approaches to solving the Brachistochrone problem with Coulomb friction are presented in [2–7]. The case incorporating both Coulomb and viscous friction was studied in [8], where viscous friction was assumed to be nonlinear with respect to velocity, and dry friction nonlinear with respect to normal pressure force. In [9, 10], the Brachistochrone problem with non-linear resistance was investigated, necessary optimality

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conditions were derived, and a qualitative analysis of the optimal trajectories was conducted. The problem with constant thrust force and linear viscous friction was analyzed in [11]. The case of quasi-constant thrust force was examined in [12]. Qualitative properties of trajectories in the presence of non-linear resistance and constant thrust force were established in [13]. In these studies, the point mass was assumed to be constant.

The presence of state constraints significantly complicates the study of optimal control problems. The main theoretical results in this field were obtained in the in [14, 15] and were subsequently developed further, for example, in [16, 17]. To solve such problems, numerical methods based on the necessary optimality conditions or the method of penalty functions are used to take these constraints into account. The fact that no constructive and reliable algorithm has been developed for solving these problems makes every successfully solved problem of this kind particularly valuable [18]. An effective solution can be designed if the structure of the optimal trajectory, the number of times constraints are reached and their sequence are determined. In [19], the case of state constraints on the trajectory inclination angle was studied. In [20], the constraints were represented as linear inequalities of phase coordinates in the vertical plane. In [21, 22], problems with constraints on the reaction force of a support curve and the curvature of the trajectory were considered, respectively.

The Brachistochrone problem with variable thrust, which acts as a control, and fuel consumption penalty was considered in [23]. For variable mass particles, it was analyzed in [24] under the action of Coulomb friction and in [25] under the action of viscous friction. The optimization of the trajectory shape of the center of mass of an aircraft in the vertical plane with variable mass, depending on a specified thrust law, was considered in [26]. This study introduced the concept of an intermediate vehicle model, where the angle of inclination of the trajectory was taken as control. Within this model, it is assumed that the lifting force required for trajectory shape variation does not affect the drag force. This assumption is valid for sufficiently small angles of attack. In this setup, the lift force in the problem of optimizing the trajectory shape of a point mass model acts as the reaction force of the support curve in the Brachistochrone problem for a point mass.

A significant step in the development of optimal control theory was the problem of programming thrust along the trajectory, in particular, the maximization of the altitude of a rocket with a given amount of fuel, formulated by R. Goddard in 1919 [27]. This became one of the first problems requiring optimal thrust programming taking into account changes in the mass of the aircraft.

The solution to Goddard's problem was obtained, for example, in [28], where the author used methods of variational calculus. Two special cases, namely, one with a linear dependence of resistance on speed, and the other with a quadratic dependence on speed, were considered in [29, 30]. It was established that the optimal program for changing the thrust consists, as a rule, of an arc of maximum thrust, then an intermediate thrust and ends with a zero thrust arcs. Significant advances in understanding the structure of singular control were achieved by Kelley [31], who developed a necessary condition for singular controls, known as the generalized Legendre-Clebsch condition. Jacobson [32] discovered an additional necessary condition, distinct from Kelley's condition. It was shown in [33] that these two necessary conditions, in general, are insufficient for optimality. In [34], the problem of optimal fuel consumption and thrust direction programming for vertical plane motion without air resistance was solved. For solving the problem, the method of the first variation of functional, also known as the Lagrange multiplier method for control problems, was formulated and applied.

Studies [35, 36] revealed the possibility of a second maximum thrust arc appearing at the end of the intermediate thrust arc under certain boundary conditions. Miele [37–39] was the first to extend previous results to the case with flight time constraints and indicated the possibility of more complex sequences of trajectory arcs, including those for more sophisticated drag models. In [40, 41], an investigation of Goddard's problem with a general drag model is presented. The three-dimensional case is examined in [40]. In [41], it was demonstrated that, under certain conditions, optimal control may exhibit a more complex switching structure, primarily due to the possibility of a second maximum thrust arc following a singular arc.

In [42, 43], a two-dimensional Goddard problem was investigated for various specified laws of changing the angle of inclination of the trajectory. The problem of simultaneously controlling the trajectory inclination angle and thrust programming was investigated in [44] for the case of linear viscous drag. In [45], these results were generalized for arbitrary drag depending solely on velocity.

In this paper, the problem of maximizing horizontal flight range in the case of quadratic viscous drag is investigated. The optimal synthesis of thrust and attack angle controls is analytically constructed for a certain region of initial variables, taking into account the presence of phase constraints on the trajectory inclination angle. Unlike [24, 25], we consider thrust force as a control variable alongside angle of attack control, and unlike [26, 42, 43], the angle of attack is treated as a control variable alongside thrust force control. It is assumed, as in [11–13], that the change in the amount of fuel does not affect the dynamics of the center of mass of the vehicle, but the fuel available for thrust control is fixed. The amount of fuel is considered negligible compared to the mass of the vehicle. Additionally, phase constraints on the trajectory inclination angle are imposed, as demonstrated in [19]. The numerical solution to the problem with simultaneous control of the angle of attack and the thrust force is presented in [43]. In [46], it was shown that the lunar landing site selection problem reduces to the Brachistochrone problem with constraints on the trajectory inclination angle.

The paper is organized as follows. Section 2 contains the problem statement and the reduction of the problem with state constraints to a problem with constraints on the control variable. In Section 3, the maximum principle is applied and the optimal control problem is reduced to a boundary value problem (BVP). Section 4 presents the rigorous construction of the optimal control synthesis with simultaneous control of the angle of attack and the thrust force. In Section 5, the numerical solution of the BVP is presented, demonstrating the analytical results of the constructed optimal synthesis.

2. Problem statement

The motion of a material point in a vertical plane under a uniform gravitational field, quadratic resistance, and thrust force is considered. The objective is to determine the trajectory shape that maximizes the horizontal coordinate while moving from a given initial state over a fixed time interval with a specified amount of fuel. The mathematical model of a rigid body's center of mass motion in the atmosphere, incorporating the assumption of thrust directed along the trajectory, has been employed in various studies (see, for example, [26, 42]).

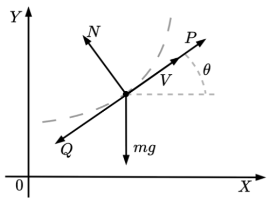

FIGURE 1. Acting forces and variables.

The equations of motion have the following form [26, 42]:

$$\begin{cases} \frac{dX}{d\tau} = V\cos\theta, \\ \frac{dY}{d\tau} = V\sin\theta, \\ \frac{dV}{d\tau} = -k_v\frac{V^2}{m} + c\frac{U}{m} - g\sin\theta, \\ \frac{d\theta}{d\tau} = k_\theta\frac{\varphi V}{m} - g\frac{\cos\theta}{V}, \\ \frac{dM}{d\tau} = -U, \end{cases} \tag{2.1}$$

where X, Y denote the horizontal and vertical coordinates of the particle, respectively, V is velocity modulus, m is mass of the particle, M is the amount of fuel, and U is the thrust force control, which satisfies the inequality $0 \leqslant U(\tau) \leqslant \bar{U}$, where $\bar{U}$ is the maximum thrust value. The angle of attack φ is the second control variable, for which there are no constraints. The trajectory inclination angle θ is subject to phase constraints of the form $\underline{\theta} \leqslant \theta(\tau) \leqslant \bar{\theta}$, where $\underline{\theta}, \bar{\theta}$ are given constants. Additionally, g denotes the acceleration of gravity, c is the exhaust velocity of the gas flow, k_v and k_θ are aerodynamic coefficients. The differentiation of the equations of motion is with respect to the dimensional time τ. We assume the motion occurs within altitudes where air density can be considered constant, thereby neglecting the dependence of drag force on altitude. The forces acting are shown in Fig. 1, where Q is the resistance force, $Q = k_vV^2$, P is the propulsive power, $P = cU$, N is the aerodynamic force, $N = k_\theta\varphi V^2$.

The boundary conditions for the equations of system (2.1) are specified as follows:

$$\begin{aligned} &X(0) = X_0, \quad Y(0) = Y_0, \qquad\quad Y(\tau_k) = Y_T, \\ &V(0) = V_0, \quad M(0) = M_0 > 0, \quad M(\tau_k) = 0 \end{aligned} \tag{2.2}$$

at the given terminal time τ_k. The values $X(\tau_k)$, $V(\tau_k)$, $\theta(0)$ and $\theta(\tau_k)$ are assumed to be free.

The objective of the control is to minimize the following functional:

$$J = -X(\tau_k) \;\to\; \min_{\varphi,\,U}. \tag{2.3}$$

For the subsequent analysis, we introduce dimensionless variables using the following formulas:

$$X = \tilde{X}x, \quad Y = \tilde{Y}y, \quad U = \tilde{U}u, \quad V = \tilde{V}v,$$
$$\tau = \tilde{\tau}t, \quad M = \tilde{M}\mu, \quad k_v' = k_v/m, \quad c' = c/m,$$

where $\tilde{X} = {c'}^2/g$, $\tilde{Y} = {c'}^2/g$, $\tilde{U} = k_v'c'$, $\tilde{V} = c'$, $\tilde{\tau} = c'/g$, $\tilde{M} = k_v'{c'}^2/g$ are the scales of the horizontal and vertical coordinates of a point, fuel consumption rate, point velocity, time, and the amount of fuel, respectively, x, y are dimensionless horizontal and vertical coordinates of a point, u is dimensionless fuel consumption rate, v is dimensionless velocity modulus, t is dimensionless time, μ is dimensionless specified amount of fuel. Hereafter, derivatives with respect to the dimensionless time t will be indicated by a dot.

The dimensionless equations of motion (2.1) can be rewritten as

$$\begin{cases} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \\ \dot{v} &= -v^2 + u - \sin\theta, \\ \dot{\mu} &= -u, \\ \dot{\theta} &= \gamma\varphi v - \dfrac{\cos\theta}{v}, \end{cases} \tag{2.4}$$

where $\gamma = k_\theta/k_v$. In this mathematical model, one of the control variables is the angle of attack φ. Following the method presented in [47], it is possible to perform a system reduction and transition to controlling the trajectory inclination angle θ. This method is valid for cases where the control variable appears in only one equation of the system and no constraints are imposed on this control variable. This approach has been used, for example, in [19] for the Brachistochrone problem, in [26] for the flight optimization problem, in [48] for the Zermelo's navigation problem, in [49, 50] for pursuit-evasion problems, and in [51] for modeling the movements of a person swinging on a swing. If phase constraints are imposed on the trajectory inclination angle θ, then, after system reduction, it is possible to reduce a problem with phase constraints to a problem with constraints on the control variable.

Since the control φ only appears in the equation for the trajectory inclination angle and there are no boundary conditions for the variable θ, a reduction can be made by discarding this equation. In the remaining system, u and θ are regarded as the control variables.

The following constraints are imposed on these control variables:

$$\underline{\theta} \leqslant \theta \leqslant \bar{\theta}, \quad 0 \leqslant u \leqslant \bar{u}, \tag{2.5}$$

where $\underline{\theta}$, $\bar{\theta}$, and $\bar{u}$ are given constants.

The objective of the control is to minimize the following functional:

$$J = -x(T) \;\rightarrow\; \min_{\theta,\, u}. \tag{2.6}$$

along the trajectories of the dynamic system:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{v} = -v^2 + u - \sin\theta, \\ \dot{\mu} = -u \end{cases} \tag{2.7}$$

using the control variables θ and u, subject to the constraints (2.5) and the boundary conditions:

$$\begin{aligned} &x(0) = x_0, \quad y(0) = y_0, \qquad\quad y(T) = y_T, \\ &v(0) = v_0, \quad \mu(0) = \mu_0 > 0, \quad \mu(T) = 0 \end{aligned} \tag{2.8}$$

at the given terminal time T. The values $x(T)$ and $v(T)$ are assumed to be free.

3. Necessary condition of optimality

To analyze the given problem, Pontryagin's Maximum Principle [52] is applied. The Hamilton–Pontryagin function for the problem (2.5)–(2.8) has the following form:

$$H = v\cos\theta\,\psi_x + v\sin\theta\,\psi_y + (-v^2 + u - \sin\theta)\,\psi_v + (-u)\,\psi_\mu. \tag{3.1}$$

In problems with a fixed process end time, in general, $H(t) = C \neq 0$, where C denotes an unknown constant.

The conjugate equations for system (2.7) are as follows:

$$\begin{cases} \dot{\psi}_x = 0, \\ \dot{\psi}_y = 0, \\ \dot{\psi}_v = -\cos\theta\,\psi_x - \sin\theta\,\psi_y + 2v\,\psi_v, \\ \dot{\psi}_\mu = 0. \end{cases} \tag{3.2}$$

From the transversality conditions, the final values for the conjugate variables are determined:

$$\psi_x(T) = 1, \quad \psi_y(T) = \alpha, \quad \psi_v(T) = 0, \quad \psi_\mu(T) = \beta, \tag{3.3}$$

where α, β are unknown constants.

From (3.2) and (3.3), it follows that on the optimal trajectory $\psi_x(t) = 1$, $\psi_y(t) = \alpha$, $\psi_\mu(t) = \beta \quad \forall t \in [0,\, T]$.

Maximizing function H with respect to the control θ yields:

$$\frac{\partial H}{\partial \theta} = -v\sin\theta + \alpha v\cos\theta - \cos\theta\,\psi_v = 0 \;\Rightarrow\; \psi_v = (\alpha\cos\theta - \sin\theta)\,\frac{v}{\cos\theta}. \tag{3.4}$$

Taking into account (3.4), we obtain

$$\frac{\partial^2 H}{\partial \theta^2} = -v\cos\theta - \alpha v\sin\theta + \sin\theta\,\psi_v = -v\frac{1}{\cos\theta} < 0 \;\Rightarrow\; \cos\theta > 0. \tag{3.5}$$

A trivial conclusion follows from inequality (3.5): to maximize the range, it is necessary to move toward increasing the range.

In accordance with (3.4), the extremal control θ is a function of the same smoothness class as the conjugate variable ψ_v. We obtain the differential equation for the control θ by substituting ψ_v from (3.4) into the third equation of system (3.2)

$$\dot{\theta} = (1 + (3v^2 - u)(\sin\theta - \alpha\cos\theta))\frac{\cos\theta}{v}. \tag{3.6}$$

Let us denote the solution of equation (3.6) by θ^*. From (3.3) and (3.4), the terminal condition for equation (3.6) follows as

$$\theta(T) = -\arctan\left(\frac{\psi_v}{v} - \alpha\right)\Big|_{t=T} = \arctan(\alpha). \tag{3.7}$$

Given the constraints on the angle θ specified in (2.5), a case can occur where the extremal law (3.6) can only be satisfied by surpassing the upper or lower constraints on the angle. On such a segment of the trajectory, it is necessary to treat the constraint as an equality and consider $\theta^* = \underline{\theta}$ or $\theta^* = \bar{\theta}$. Then the extremal control of the angle θ can be presented in the following form:

$$\theta^{\text{extr}}(t) = \begin{cases} \theta^*, & \underline{\theta} \leqslant \theta^* \leqslant \bar{\theta}, \\ \underline{\theta}, & \theta^* < \underline{\theta}, \\ \bar{\theta}, & \theta^* > \bar{\theta}. \end{cases} \tag{3.8}$$

The terminal condition for system (3.8) can be expressed as:

$$\theta^{\text{extr}}(T) = \begin{cases} \arctan(\alpha), & \underline{\theta} \leqslant \arctan(\alpha) \leqslant \bar{\theta}, \\ \underline{\theta}, & \arctan(\alpha) < \underline{\theta}, \\ \bar{\theta}, & \arctan(\alpha) > \bar{\theta}. \end{cases} \tag{3.9}$$

To maximize the Hamilton–Pontryagin function with respect to the control u, we represent (3.1) in the form:

$$H = H_0 + H_1 u, \tag{3.10}$$

where

$$H_0 = v\cos\theta + \alpha v\sin\theta + (-v^2 - \sin\theta)\,\psi_v, \quad H_1 = \psi_v - \psi_\mu = \psi_v - \beta.$$

The function H_1 is called the switching function.

The cases $H_1 > 0$ and $H_1 < 0$ correspond to the regular arcs of the extremal trajectory. If there exists an interval $\sigma \subset [0, T]$, such that for any $t \in \sigma$ the switching function is identically zero, i.e., $H_1 \equiv 0$, then the corresponding arc is singular [20, 31]. The control u_s denotes a singular control.

Maximizing function H with respect to the control u leads to the following control logic:

$$u^{\text{extr}} = \begin{cases} \bar{u}, & H_1 > 0, \\ u_s, & H_1 \equiv 0, \\ 0, & H_1 < 0. \end{cases} \tag{3.11}$$

The following relationships are fulfilled simultaneously on a singular arc:

$$H_0 \equiv C, \quad H_1 \equiv 0, \quad \frac{dH_1}{dt} \equiv 0, \quad \frac{d^2H_1}{dt^2} \equiv 0.$$

If these equations are linearly independent, then from the equation $d^2H_1/dt^2 \equiv 0$, we can obtain a formula for singular control. In such a case, the control is called first-order singular.

Differentiating relation $H_1 = \psi_v - \beta$ with respect to time along solutions of systems (2.7), (3.2), and taking into account (3.4), we obtain the equation:

$$\dot{H}_1 = \dot{\psi}_v = -\cos\theta - \alpha\sin\theta + 2v\psi_v = 0, \quad \forall t \in \sigma \subset [0, T]. \tag{3.12}$$

Differentiating relation (3.12) with respect to time along the solutions of systems (2.7) and (3.2), and taking into account (3.6),

$$\ddot{H}_1 = \dot{v}\Big((\sin\theta - \alpha\cos\theta)\frac{\cos\theta}{v} - \frac{2v}{\cos\theta}\Big)(\sin\theta - \alpha\cos\theta) = 0.$$

It follows that on the singular arc $\dot{v} = 0$, or

$$u_s = v^2 + \sin\theta. \tag{3.13}$$

For this control, Kelley's optimality condition [31]

$$\Big((\sin\theta - \alpha\cos\theta)\frac{\cos\theta}{v} - \frac{2v}{\cos\theta}\Big)(\sin\theta - \alpha\cos\theta) \geqslant 0. \tag{3.14}$$

Since the control u in problem (2.5)–(2.8) is classified as first-order singular, the coupling between singular and non-singular arcs must be discontinuous [53].

By substituting (3.12) into (3.6) and taking into account (3.4), we find that on the singular arc $\dot{\theta} = 0$.

To determine the moment of transition to the singular arc of the trajectory, we derive the formula for the singular surface. The system of equations, consisting of

$$H_0 = C, \quad \dot{H}_1 = 0 \tag{3.15}$$

is uniquely solvable with respect to two unknown variables v and θ. Consequently, the singular surface is a line in the space (θ, v, μ) and proves to be challenging for analysis because of the unknown constants C and β. For analysis, it is more convenient to express ψ_v from formulas (3.4) and (3.12) and set them equal:

$$\frac{2v^2}{\cos\theta} = \frac{\cos\theta + \alpha\sin\theta}{\alpha\cos\theta - \sin\theta}. \tag{3.16}$$

The resulting relation will henceforth be referred to as the formula of the set of singular surfaces G

$$G = \Big\{v, \theta : \frac{2v^2}{\cos\theta} = \frac{\cos\theta + \alpha\sin\theta}{\alpha\cos\theta - \sin\theta}\Big\}. \tag{3.17}$$

Eq. (3.16) describes the entire set of points obtained from solving system (3.15) with different unknown constants C and β.

Thus, the motion on the singular arc is steady-state, where the velocity and trajectory inclination angle remain constant.

4. Optimal synthesis

As a result of applying the maximum principle, the optimal control problem (2.5)–(2.8) is reduced to a boundary value problem (BVP) (2.7), (2.8), (3.8), and (3.9), which does not contain conjugate variables. The thrust control is determined according to the rule (3.11).

Assume that the following inequality is fulfilled

$$\frac{2v^2}{\cos\theta} < \frac{\cos\theta + \alpha\sin\theta}{\alpha\cos\theta - \sin\theta}. \tag{4.1}$$

In this case, the point in the plane (θ, v) lies to the right of G (see Fig. 2), and from (3.12), it follows $\dot{H}_1 < 0$. Thus, the maximum principle is satisfied only by switching from the control $u = \bar{u}$ to the control $u = 0$. If the point lies to the left of G (see Fig. 2), i.e., the inequality

$$\frac{2v^2}{\cos\theta} > \frac{\cos\theta + \alpha\sin\theta}{\alpha\cos\theta - \sin\theta} \tag{4.2}$$

is fulfilled, then $\dot{H}_1 > 0$ and the maximum principle is satisfied only by switching from the control $u = 0$ to the control $u = \bar{u}$.

The problem is to design optimal synthesis of the thrust, i.e., to find out the thrust as a function of the initial variables of system (2.7), taking into account the trajectory inclination angle constraints (2.5).

4.1. The case without phase constraints.

ASSUMPTION 4.1. The motion is considered in the domain of the plane (θ, v), where singular control (3.13) is admissible, i.e., $u_s \in [0, \bar{u}]$.

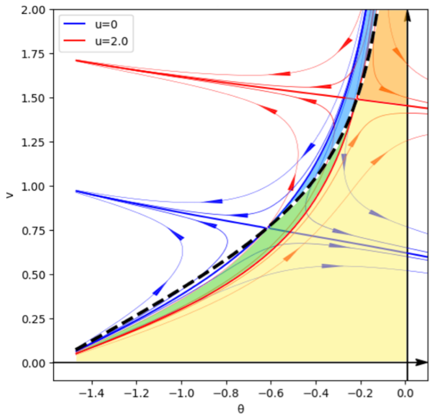

FIGURE 2. Phase portrait of system (4.3) for different constant thrust u.

Fig. 2 represents the phase trajectories for the following system:

$$\begin{cases} \dot{v} = -v^2 + u - \sin\theta, \\ \dot{\theta} = (1 + (3v^2 - u)(\sin\theta - \alpha\cos\theta))\frac{\cos\theta}{v}, \end{cases} \tag{4.3}$$

in the case of a free vertical coordinate $y(T)$ ($\alpha = 0, \theta(T) = 0$). A detailed analysis of system (4.3) is provided, for example, in [9]. It can be proved that the curve G, represented by a black dashed line, intersects the saddle points for different thrust values u. The phase trajectories are depicted by blue lines for $u = 0$ and red lines for $u = \bar{u} = 2$. The areas shaded in green, blue, yellow, and orange define the solvability domain of the problem, allowing the terminal condition (3.7) to be reached. Assumption 4.1 is not satisfied if the entry to the singular surface occurs above the saddle point for the system of equations (4.3) for $u = \bar{u}$. When $\alpha \neq 0$, the phase portrait of system (4.3) is topologically equivalent to that of the case where $\alpha = 0$.

Statement 4.1. If the initial point in plane (θ, v) lies to the right of G (Fig. 2), i.e., inequality (4.1) is satisfied, then $u(0) = \bar{u}$.

Proof. Suppose (4.1) is satisfied at $t = 0$ and let $u(0) = 0$. According to the optimality conditions, switching from control $u = 0$ to control $u = \bar{u}$ is non-optimal. It can be seen from Fig. 2 that trajectories lie to the right of the vertical separatrix of the saddle point; thus, reaching the terminal condition (3.7) occurs to the right of G and does not intersect it. Throughout the trajectory, $u = 0$, and by the end of the process, the terminal fuel condition $\mu(T) = \mu_T$ cannot be reached. Therefore, it must hold that $u(0) = \bar{u}$. □

Statement 4.2. If at the initial moment the point in the plane (θ, v) lies to the right of G, then the motion with $u = \bar{u}$ continues until a singular surface is reached or until the fuel is completely burned out.

Proof. To the right of G, the optimality conditions are satisfied only by switching from control $u = \bar{u}$ to control $u = 0$. Let the terminal fuel condition (reaching $\mu(t) = \mu_T$) be fulfilled before reaching G. It is the necessary to switch from $u = \bar{u}$ to $u = 0$ and proceed with it until condition (3.7) is met. If the final terminal condition is not satisfied when reaching G, then, similar to the proof of Statement 4.1, switching to zero thrust is non-optimal, as the terminal fuel condition remains unmet. Therefore, the motion with $u = \bar{u}$ continues until a singular surface is reached. □

Statement 4.3. If during the motion the point reaches the surface G, i.e., equality (3.16) is satisfied, the motion on a singular arc continues until the fuel is completely burned out.

Proof. Suppose the trajectory begins with an arc of maximum thrust $u = \bar{u}$ and, during motion, intersects G to the left of it. Then the motion continues with $u = \bar{u}$ until the fuel is completely burned out. As a result, to the left of G, switching from $u = \bar{u}$ to $u = 0$ occurs, which is non-optimal according to (4.2). Therefore, the point cannot intersect G. Once it is reached, switch to the intermediate thrust $u = u_s$ and continue with it until the fuel is completely burned out. In the case where the trajectory begins with an arc of zero thrust $u = 0$ and intersects G to the right of it, the proof is similar. □

STATEMENT 4.4. If the initial point in the plane (θ, v) lies to the right of G, then:

a) If the fuel is completely burned out before reaching G, the optimal trajectory of the problem (2.5)–(2.8) consists of two arcs: the maximum thrust arc $u(t) = \bar{u}$ at the beginning and the zero thrust arc $u(t) = 0$ at the end. (This area is shaded in yellow and orange in Fig. 2.)

b) If G is reached before the fuel is completely burned out, the optimal trajectory of the problem (2.5)–(2.8) includes three arcs: the maximum thrust arc $u(t) = \bar{u}$ at the beginning, then the intermediate thrust arc $u(t) = u_s$, and the zero thrust arc $u(t) = 0$ at the end. (This area is shaded in green in Fig. 2.)

PROOF. Suppose the trajectory begins with $u = \bar{u}$ and reaches the surface G. Then, according to Statement 4.3, once G is reached, switch to intermediate thrust $u = u_s$ and continue with it until the fuel is completely burned out. The motion continues until the fuel is completely burned out, and then the thrust remains at zero $u = 0$ until the boundary condition for the trajectory inclination angle (3.7) is met. □

STATEMENT 4.5. If the initial point in the plane (θ, v) lies to the left of G (Fig. 2), i.e., inequality (4.2) is satisfied, then the optimal trajectory of the problem (2.5)–(2.8) consists of three arcs: the zero thrust arc $u(t) = 0$ at the beginning, then the intermediate thrust arc $u(t) = u_s$, and the zero thrust arc $u(t) = 0$ at the end. (This area is shaded in blue in Fig. 2.)

The proof is similar to the proofs of Statements 4.1, 4.2, and 4.4.

STATEMENT 4.6. In the absence of resistance case, the optimal trajectory of the problem (2.5)–(2.8) consists of two arcs: the maximum thrust arc $u(t) = \bar{u}$ at the beginning and the zero thrust arc $u(t) = 0$ at the end.

PROOF. Without resistance, from (3.12) we have $\dot{H}_1 = -\cos\theta - \alpha\sin\theta < 0$. Therefore, there is no arc with intermediate thrust, and only switching from $u = \bar{u}$ to $u = 0$ satisfies the necessary optimality conditions. □

4.2. The case with phase constraints. Next, we consider cases where the angle θ (2.4) reaches its constraints during motion. Note that while moving along G, $\dot{\theta} = 0$ is satisfied. Therefore, the θ-constraint can only be reached during motion with $u = \bar{u}$ or $u = 0$ and cannot happen while moving along G with $u = u_s$. The number of exits to the θ constraint during motion with the corresponding thrust u can be determined by analysing (3.1), (3.4). Substituting (3.4) into (3.1), the first integral has the form

$$f(v) = (\alpha\cos\theta - \sin\theta)v^3 - \big(u(\alpha\cos\theta - \sin\theta) + 1\big)v + (C + u\beta)\cos\theta = 0. \tag{4.4}$$

The solutions of equation (4.4) are the phase curves of system (4.3). By specifying the parameter C, a specific phase curve of system (4.3) can be obtained. When the constraint $\theta = \underline{\theta}$ or $\theta = \bar{\theta}$ is specified, the number of exits to this constraint matches the number of roots obtained when solving equation (4.4) with respect to

v. By analyzing all possible values of the parameter C, the maximum number of exits to the θ constraint can be determined.

Consider (4.4) to determine the maximum number of exits to the θ constraint for $u = 0$ and $u = \bar{u}$. For $v > 0$, from $f'(v) = 0$, it follows that (4.4) has at most one extremum, $v = \sqrt{u + (\alpha \cos\theta - \sin\theta)^{-1}}$. Since the motion occurs in the domain $\theta(t) < \theta(T) = \arctan(\alpha)$, it follows that $f''(v) = 6v(\sin\theta - \alpha\cos\theta) < 0$. Therefore, the extremum is a maximum. When θ is constant, equation (4.4) yields at most two positive roots v. Therefore, the optimal trajectory reaches each constraint at most two times with $u = 0$ and $u = \bar{u}$.

If the trajectory reaches G while moving along the phase constraint $\theta = \underline{\theta}$ or $\theta = \bar{\theta}$, then according to Statement 4.3, it is necessary to switch to the intermediate thrust $u = u_s$ at that moment. Otherwise, the necessary optimality conditions for thrust control will not be satisfied.

STATEMENT 4.7. If the initial point moves with $u(0) = \bar{u}$, and the velocity during motion is less than the velocity at the saddle point for system (4.3) with $u(t) = \bar{u}$ (see Fig. 2, green and yellow areas), then the upper phase constraint $\theta = \bar{\theta}$ can be reached at most once before the fuel is completely burned out.

PROOF. On the surface G, $\dot{v} = 0$ and $\dot{\theta} = 0$ are satisfied. In the domain to the right of G, where the velocity during motion is less than at the saddle point for system (4.3) with $u(t) = \bar{u}$, it follows that $\dot{\theta} > 0$. Thus, the upper phase constraint $\theta = \bar{\theta}$ can be reached at most once before the fuel is completely burned out. □

STATEMENT 4.8. If the initial point moves with $u(0) = 0$ (see Fig. 2, blue area), then the lower phase constraint $\theta = \underline{\theta}$ can be reached at most once before the fuel is completely burned out.

PROOF. On the surface G, $\dot{v} = 0$ and $\dot{\theta} = 0$ are satisfied. Therefore, to the left of G, $\dot{\theta} < 0$ is satisfied. Thus, the lower phase constraint $\theta = \underline{\theta}$ can be reached at most once before the fuel is completely burned out. □

REMARK 4.1. If at the end of the process $\theta(T) = \arctan(a) > \bar{\theta}$, then $\theta(T) = \bar{\theta}$. In this case, it is necessary to solve a direct maximization problem on the set of phase trajectories leading to the terminal state (3.9), within a fixed time interval T, to find $\theta(0) = \theta_0$ that minimizes the functional (2.6). If $\theta(0) = \underline{\theta}$, it is also necessary to determine the time interval during which the initial arc of motion occurs at the lower angle constraint $\theta = \underline{\theta}$.

After solving the optimal control problem for the reduced system, the control law for the attack angle φ can be obtained from equation (3.6) and the last equation of system (2.4):

$$\varphi = (2 + (3v^2 - u)(\sin\theta - \alpha\cos\theta))\frac{\cos\theta}{v^2}. \tag{4.5}$$

5. Numerical simulation

This section presents the results of the numerical solution of the BVP for system (2.5), (2.7), (2.8), (3.8), and (3.9) with thrust control (3.11) and (3.13). The results provided are intended to demonstrate the analytical conclusions of Section 4. The simulation is conducted using dimensionless variables. The boundary values for the variables and other parameters were selected to obtain the desired control structures. For considered cases, the following initial conditions are assumed at the initial time: $x(0) = 0$, $y(0) = 0$. The control of the angle of attack was determined in accordance with (4.5), where $\gamma = 30$. Let us consider the following cases:

A) The initial point lies to the left of G. The set of boundary values and constants is as follows: $v(0) = 0.55$, $\mu(0) = 0.75$, $T = 2$, $\bar{u} = 1$, $\underline{\theta} = -0.6$, $\bar{\theta} = 0.025$. The value of y_T is considered free, therefore, in accordance with (3.3), $\alpha = 0$. The simulation results are presented in Fig. 3. The structure of the extremal thrust is as follows: $u = \bar{u} \to u = u_s \to u = 0$. The structure of the trajectory inclination angle includes one transition along the lower boundary $\theta = \underline{\theta}$.

B) The initial point lies to the right of G. The set of boundary values and constants is as follows: $v(0) = 1.3$, $\mu(0) = 1$, $T = 3.1$, $\bar{u} = 1$, $\underline{\theta} = -0.5$, $\bar{\theta} = -0.025$. The value of y_T is determined, and in accordance with (3.3), we assume $\theta(T) = \arctan(\alpha) = -0.1$ to demonstrate the desired behavior of the point. In this case, $y_T = -1.373$. The simulation results are presented in Fig. 4. The structure of the extremal thrust is as follows: $u = 0 \to u = u_s \to u = 0$. The structure of the trajectory inclination angle includes one transition along the lower boundary $\theta = \underline{\theta}$, during which the point transitions to a singular surface.

C) The initial point lies to the right of G. The set of boundary values and constants is as follows: $v(0) = 0.65$, $\mu(0) = 0.5$, $T = 3$, $\bar{u} = 1$, $\underline{\theta} = -0.545$, $\bar{\theta} = -0.53$. y_T is considered free, $\alpha = 0$. The simulation results are presented in Fig. 5. The structure of the extremal thrust is as follows: $u = \bar{u} \to u = u_s \to u = 0$. The structure of the trajectory inclination angle includes two transitions to each boundary. The transition to the singular surface occurs when moving along the upper boundary $\theta = \bar{\theta}$.

In Figs. 3, 4, and 5, the following notation are used: plot (a) illustrates motion in the plane (x, y); plot (b) represents motion in the plane (θ, v); plot (c) depicts the change in the trajectory inclination angle θ over time t; plot (d) depicts the change in the control of the angle of attack; plot (e) presents the change in the amount of fuel μ over time t; plot (f) shows the change in the thrust control u over time t. In the (θ, v) plane, red dashed lines indicate the separatrices of the saddle point of system (4.3) when $u = \bar{u}$, and blue dashed lines when $u = 0$. The surface G is indicated with an orange line, black dashed lines denote phase boundaries θ, and red dots indicate thrust control switching moments. In the (u, t) plane, the red dashed line presents the maximum thrust value $u = \bar{u}$. In Figs. 4b and 4c, purple dashed lines indicate the condition $\theta(T) = \arctan(\alpha) \neq 0$.

In Cases A and B, the BVP was solved using the shooting method. In Case A, the objective is to identify T_0, the period when the motion occurs along the lower angle boundary $\theta = \underline{\theta}$, so that the condition $\theta(T) = 0 < \bar{\theta}$ was satisfied. In Case C, the condition $\theta(T) = 0 > \bar{\theta}$ was met, so the problem was solved by directly maximizing the functional (2.6) over the set of trajectories (2.5), (2.7), (2.8), (3.8), and (3.9) utilizing the thrust control law (3.11) and (3.13) to determine the only unknown parameter T_0.

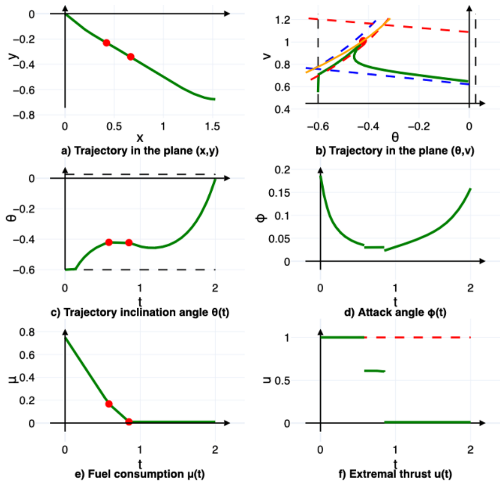

FIGURE 3. Simulation results for (Case A).

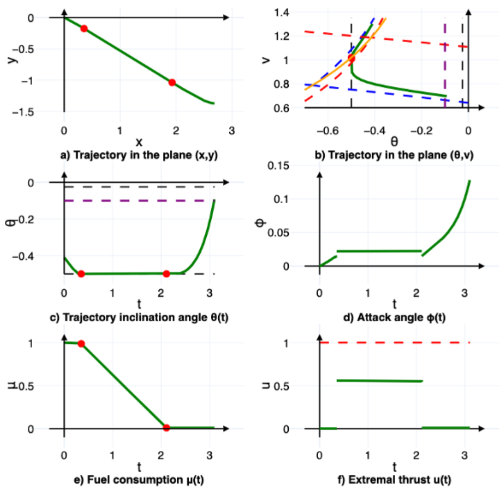

FIGURE 4. Simulation results for (Case B).

The parameters obtained from the numerical simulation are summarized in a table. T_0 indicates the moment of exit from the lower angle boundary $\theta = \underline{\theta}$, T_1 and T_2 are the moments of entering and exiting the singular surface, and u_s is the constant value of the intermediate thrust on the singular arc.

Kelley's optimality condition (3.14) was verified for the provided simulation results. In Figs. 3f, 4f, and 5f, the corresponding control jump according to the coupling conditions can be clearly observed.

The simulation results demonstrate that the angle of attack φ (4.5) for the extremal trajectories lies within an acceptable range.

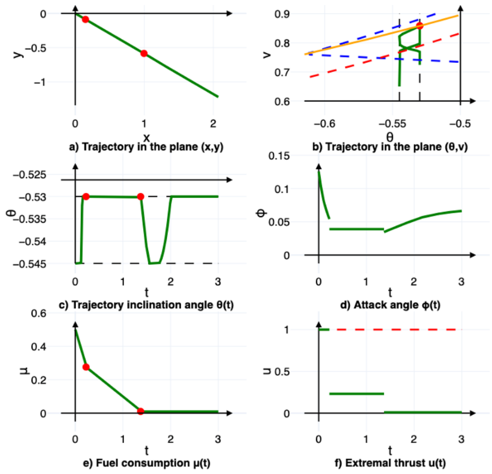

FIGURE 5. Simulation results for (Case C).

TABLE 1. Numerical Simulation Results.

Set	T_0	T_1	T_2	T_{k}	u_{s}
A	0.133175	0.583017	0.849988	2	0.609219
B	–	0.350055	2.109974	3.1	0.558164
C	0.119721	0.225032	1.369964	3	0.231484

6. Conclusion

The problem of maximizing horizontal range within an intermediate model of an aircraft, where the fuel mass is negligible compared to the aircraft mass, has been studied. The motion occurs in a vertical plane under a uniform gravitational field and quadratic resistance friction. The controls are the angle of attack and the thrust force. Phase constraints were imposed on the trajectory angle. In the study of this problem, the reduction was done, enabling the transition from a problem with phase constraints to a problem with constraints on the control variable. This allowed for an analytical investigation and the construction of the optimal synthesis in the entire domain where the problem is solvable. After reduction, the trajectory inclination angle was considered as control instead of the attack angle. A law for changing the angle of attack, allowing the realization of the determined trajectories, has been obtained. It was shown that, despite the phase constraints on the trajectory inclination angle, the thrust control law qualitatively corresponds to the classical solution to the Goddard problem. In the presence of resistance, areas of variables are indicated for which only the following combinations of arcs are possible: maximum-zero, maximum-intermediate-zero, and zero-intermediate-zero.

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СИНТЕЗА ОПТИМАЛНОГ УПРАВЉАЊА ЛЕТЕЛИЦЕ СА ОГРАНИЧЕЊИМА СТАЊА

Резиме. Разматра се проблем одређивања оптималног управљања летелице. Кретање се одвија у вертикалној равни у хомогеном гравитационом пољу, са квадратним отпором средине и силом потиска. Управљачке величине су нападни угао и сила потиска. Фазна ограничења су наметнута на угао нагиба путање. Претпоставља се да је укупна потрошња масе горива за управљање потиском занемарљива у поређењу са масом возила, промена масе горива не утиче на динамику центра масе, а промена силе потиска не утиче на силу отпора. Одређена је област у простору почетних промењљивих стања за коју је проблем решив и урађена је синтеза оптималног управљања. Утврђено је да унутар ове области потисак може бити максималан, променљив или нула. Одређен је број и редослед делова путање са одговарајућим вредностима потиска и број излазака на границу фазних ограничења.

Lomonosov Moscow State University
Moscow
Russia
wyvling@gmail.com
https://orcid.org/0000-0002-5081-0584

SMBU
Shenzhen
PRC
oyuche@yandex.ru
https://orcid.org/0000-0003-1435-1541

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