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ON THE EXISTENCE OF GEODESIC VECTOR FIELDS ON CLOSED SURFACES

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ABSTRACT. We construct an example of a Riemannian metric on the 2-torus such that its universal cover does not admit global Riemann normal coordinates.

1. Introduction

DEFINITION 1.1. We call a vector field v = v(x) on a Riemannian manifold (M^n, g) geodesic, if its length is identically 1 and if $\nabla_v^g v = 0$, where ∇^g is the Levi-Civita connection of g.

Clearly, a vector field is geodesic if and only if any orbit of its flow is an arclength parameterised geodesic.

EXAMPLE 1.1. For the metric

(1.1)
$$g = (dx^{1})^{2} + \sum_{i,j=2}^{n} g_{ij}(x) dx^{i} dx^{j}$$

the vector field $\frac{\partial}{\partial x^1}$ is geodesic.

In dimension two the formula (1.1) reads

(1.2)
$$g = dx^2 + f(x, y)dy^2.$$

Coordinates such that the metric has the form (1.2) are called *Riemann normal (or geodesic normal or semi-geodesic)* coordinates. It is known [6] that, in dimension two, for any geodesic vector field there exists a local coordinate system (x, y) such that the metric has the form (1.2) and the vector field is $\frac{\partial}{\partial x}$. The goal of this paper is to construct an example of a Riemannian two-torus (T^2, g) such that its universal cover $(\mathbb{R}^2, \tilde{g})$, where \tilde{g} denotes the lift of g, has no geodesic vector field. Any sufficiently small C^2 -perturbation of this metric has the same property. The example can be easily generalised to closed surfaces of negative Euler characteristic.

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We have two reasons for studying the problem. The first one is related to the very recent paper [7] studying conformal product structures on Kähler manifolds. [7, Corollary 4.6] guarantees the existence of a geodesic vector field on compact Kähler manifold of real dimension $n \ge 4$ carrying an orientable conformal product structure with non-identically zero Lee form. [7, Proposition 4.7] uses the results of the present paper to show the existence of direct product compact Kähler metrics with no orientable conformal product structure with non-identically zero Lee form.

Another reason comes from the theory of integrable geodesic flows on closed surfaces. [2, Theorem 1.6] implies that for any Riemannian 2-torus (T^2, g) such that the geodesic flow is integrable and the integral satisfies \aleph -condition (see [2, Definition 1.3]), there exists a geodesic vector field on the universal cover $(\mathbb{R}^2, \tilde{g})$, where \tilde{g} denotes the lift of g. Our example is an "easy to construct" example of \aleph -nonintegrable geodesic flow. Recall that though generic geodesic flow is not integrable, proving that a geodesic flow is nonintegrable or constructing an example of an nonintegrable geodesic flow is not an easy task (see e.g. [4, §10] and [3, §3]).

Examples using a similar idea have already appeared in the literature; see, for instance, [1, pp. 46–47] and [5, p. 11]. In these papers, the authors primarily focused on minimal geodesic laminations. Their examples demonstrated the nonexistence of a smooth torus in T^*T^2 that is invariant under the geodesic flow and has the property that every trajectory of the geodesic flow lying on this torus projects to a minimal geodesic.

The examples from [1, 5] are also sufficient for constructing non- \aleph -integrable geodesic flows, see the second reason above. However, it is worth noting that in the first one, which relates to the results of [7], the minimality condition is not essential.

2. Example and proof of nonexistence of geodesic vector field

Take the standard sphere with the standard metric. Next, take a small $\varepsilon > 0$ and change the topology of the manifold in the ε -neighborhood of the south pole by gluing a handle in the neighborhood. The metric outside the neighborhood is not changed, the metric in the modified neighborhood can be chosen arbitrary such that the obtained metric on the two-torus is smooth (see Fig. 1).

We consider the universal cover \mathbb{R}^2 and denote by \tilde{g} the lift of the metric. Let us show that $(\mathbb{R}^2, \tilde{g})$ does not admit a geodesic vector field. We assume it does, denote the geodesic vector field by v, and find a contradiction.

In order to do it, consider the circle of radius 2ε around the north pole of the initial sphere and consider one of its lifts $C_{2\varepsilon}(N_0) = \partial B_{2\varepsilon}(N_0)$. Let us show that our geodesic vector field v is necessary transversal to it. Arguing by the method of contradiction, assume there exists a point where the vector field v is tangent to the circle. Consider the geodesic γ starting from this point in the direction of v. This geodesic, and also geodesics close to γ , do not enter the "light gray" region where we changed the sphere. Therefore, any geodesic γ_1 starting from a nearby point in the direction of our vector field intersects γ , as any two geodesics on the sphere intersect each other. This gives a contradiction, since velocity vectors of both geodesics at the point of intersection should be v.



FIGURE 1. The torus made of the sphere: the dark-gray part is the 2ε -ball around the north pole. The surgery was made in the light-gray part.

Thus, our vector field is transversal to the circle at every point. Then, the index of the restriction of v to $B_{2\varepsilon}(N_0)$ is nonzero. But the index must be zero since v is never zero. The contradiction proves the nonexistence of a geodesic vector field.

Note also that in the proof we used the following properties of the standard metric of the sphere only:

- 1. Every arc-length parameterised geodesic starting at a point of the 2ε circle and tangent to it does not reach the ε -neighborhood of the south pole within the time 2π .
- 2. Any two arc-length parameterised geodesics $\gamma_1, \gamma_2 \colon (0, 2\pi) \to S^2$ always intersect.

These properties are fulfilled for any sufficiently small perturbation, in the C^2 -topology, of the standard metric of the sphere. This implies that one can construct such an example in the real-analytic category. Moreover, by attaching more than one handle in the "light gray" region (see Fig. 1), one can construct an example of a closed Riemannian surface of arbitrary negative Euler characteristic such that the universal cover does not admit a geodesic vector field.

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О ПОСТОЈАЊУ ГЕОДЕЗИЈСКИХ ВЕКТОРСКИХ ПОЉА НА ЗАТВОРЕНИМ ПОВРШИМА

РЕЗИМЕ. Конструишемо пример Риманове метрике на 2-торусу тако да њено универзално наткривање не допушта глобалне Риманове нормалне координате.

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