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ON GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR COMPRESSIBLE SELF-GRAVITATING FLUIDS WITH UNBOUNDED DOMAINS

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Abstract. This paper examines to the global existence of weak solutions for compressible self-gravitating fluids in three-dimensional unbounded domain with a compact Lipschitz boundary, assuming a total finite fluid mass. We prove that there exists a globally defined weak solution that satisfies the energy inequality in differential form.

1. Introduction

Self-gravitating fluids have numerous applications in nuclear fluid theory and astrophysics. The behavior of self-gravitating fluids is an essential topic in astrophysics. In fact, the basic structure of the medium that exists between stars, as well as the development and evolution of cosmic structures, are founded on it. Compressible self-gravitating mechanics of fluids is important in both the formation of stars and the formation of an astronomical medium fractal structures [30]. The motion of a self-gravitating fluid can be investigated using hydrodynamic equations. Here, we examine a set of nonlinear partial differential equations that characterize the compressible self-gravitating fluid. Our objective is to investigate the global existence of weak solutions with an unbounded domain by examining the system of equations, such as:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} + \nabla p(\rho) = \rho \nabla \Psi, \\ \operatorname{div} \mathbf{T} = \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{v} \end{cases} \tag{1.1}$$

where the velocity vector is represented by $\mathbf{v}$ and the pressure is represented by $p(\rho) = \rho^\gamma$, such that $\gamma > \frac{3}{2}$. Operators such as Δ, ∇, and div are employed in

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relation to the variable $x \in \mathbf{R}^3$. The coefficient of viscosity λ, μ satisfies

$$\mu + \lambda \geqslant 0, \quad \text{and} \quad \mu > 0.$$

Moreover, the Newtonian gravitational potential $\Psi = \Psi(t, x)$ satisfies

$$\Delta\Psi = -4\pi g\Big(\rho - \frac{1}{|\Omega|}\int_\Omega \rho dx\Big), \tag{1.2}$$

assuming that ρ has sufficient regularity. Because of the unbounded nature of the underlying physical domain, the system is enhanced with additional conditions such as

$$\lim_{|x|\to\infty} \mathbf{v}(x,t) = 0, \quad \lim_{|x|\to\infty} \rho(x,t) = 0, \quad \lim_{|x|\to\infty} \Psi(x,t) = 0.$$

The initial and boundary data are specified by

$$\begin{aligned} &\rho(0,x) = \rho_0 \geqslant 0, && \Psi(0,x) = \Psi_0 && \text{and} \ \ \rho\mathbf{v}(0,x) = m_0, \\ &\mathbf{v}|_{\partial\Omega\times\mathbf{R}^+} = 0, && \frac{\partial\Psi}{\partial r} = 0 && \text{on} \ \ \partial\Omega\times\mathbf{R}^+. \end{aligned} \tag{1.3}$$

where r is the unit outward normal vector. This system physically explains the motion of a viscous isentropic compressible gas flow under the influence of self-gravitational force. The Euler–Poisson equation may be used to describe such a fluid and various mathematical investigations have been conducted on the Euler-Poisson equation such as [2, 7, 31]. Regarding that, we note that Ducomet and Feireisl [9] were among the first to examine the global existence of theof compressible self-gravitating fluids that are governed by Navier–Stokes–Poisson equations (NSPE). In addition, in [8] the authors demonstrated that as $\epsilon \to 0$ in the interval of time the 3D weak solutions of the NSPE converge to the 2D strong solution where these strong solutions exist.

To date, there exists an extensive body of literature regarding the energy equality of weak solutions for incompressible and compressible Navier–Stokes equations. Concerning the incompressible equations, it is noteworthy to mention that Lions [15] obtained energy equality of the Lions-Shinbrot type criteria on the velocity. Next, Shinbrot [25] proved the the energy equality for the Navier–Stokes equations regardless of the dimensions with some specified spaces for the weak solutions achieved by Serrin [24]. Similarly, energy equality under different constraints and restrictions was obtained by Da Veiga [5] and Yu [27]. Later on, to deal with boundary effects, Yu [29] improved the result proved by Shinbrot by implying the Besov regularity on the velocity for bounded domain. Similarly, without needing additional requirements of the velocity vector on the boundary, Nguyen [22] achieved the boundary effects.

Furthermore, energy equality for compressible fluids has been examined in greater detail by other authors such as Yu [28] who proved that the energy equality for compressible fluids for every $t > 0$, under some specific conditions and energy, may be conserved with constant viscosities. Later on, by applying global mollification, this result was improved by Chen [4]. The energy equality criterion was established with respect to velocity as well as its gradient by Liang [14] and Wang

[26]. We will follow [6] to prove the required result, where the authors proved the energy inequality of compressible fluids.

In addition, there is extensive literature describing the global existence of weak solutions in multiple dimensions such as 2-D and 3-D, whether the fluid is compressible or incompressible. We review some previous results, including those of Lions [17] and Novotny [23], in order to alleviate the constraints on the theory of compressible viscous fluids. They demonstrated a well-known result regarding the global existence of weak solutions by incorporating an isentropic fluid. This was achieved under the additional condition that $\gamma \geqslant \frac{3}{2}$ and $\gamma > \frac{9}{5}$ in 2D and 3D, respectively. The result of Lions for global existence in three dimensions was modified by Feireisl [11] for $\gamma > \frac{3}{2}$. In their study, Jiang and Zhang [13] examined the global existence of weak solutions to compressible fluids of isentropic type for $\gamma > 1$, with the additional assumption of symmetries on the initial data. The existence of weak solutions to compressible non-Newtonian fluids was demonstrated by Mamontov [18, 19] by taking into account multi-dimensionality. Zhikov and Pastukhova [32] demonstrated the global existence of weak solutions for multi-dimensional problems by examining the initial boundary value problem of non-Newtonian compressible fluids under certain constraints on the initial datum. We refer [10, 12, 16, 20] for more results.

We study the energy inequality for a fluid that is both compressible and self-gravitating motivated by the importance of these types of fluids, which have numerous applications in physics, astronomy, geology, seismology and oceanography. In recent years, mathematicians have focused intensively on studying non-linear fluids, mostly from the perspective of differential equation theory. The non-linearity may appear in many different aspects and experimental research has revealed that, due to the sensitive nature of the measurements in these regions, the effects of determining post-glacial sea level near the polar ice caps significantly differ when applying a flat Earth model that neglects self-gravity compared with a sphere-shaped Earth where self-gravity is taken into account. This illustrates the manner in which results can change dramatically when self-gravity is ignored. Therefore, in order to get a thorough understanding and enhance applications across various industries it is essential to examine the flow behavior of self-gravitating fluids.

In this research work, we prove the energy inequality in differential form by considering the weak solution of a self-gravitating fluid with a compact Lipschitz boundary in an unbounded three-dimensional domain and assuming that the total fluid mass is finite.

The remainder of the paper is organized as follows. Section 2 presents the primary result and defines a weak solution to the problem. Further, in Section 3, we demonstrate the primary result and the supplementary results that are necessary for the proof of our main result.

2. Main Results

The current section aims to state the main result and outline the weak solution of the problem (1.1)–(1.3). Prior to presenting the major result, it is imperative to acknowledge that our weak solution would satisfy the natural energy estimations.

From a physics perspective, an acceptable weak solution must adhere to the principles of mass, energy, and momentum conservation in terms of their distribution. With these essential criteria in consideration, we define our weak solution as:

DEFINITION 2.1. For a pre-chosen $T > 0$, the pair of functions $(\rho, \mathbf{v}, \Psi)$ is said to be a global weak solution of (1.1)–(1.3) on the interval of time $(0, T)$, such that

- $\rho \in (L^\infty(0, T; L^\gamma(\Omega)) \cap (C([0, T]; L^1(\Omega))),$

  $$\mathbf{v} \in L^2(0, T; W_0^{1,2}(\Omega)), \quad \sqrt{\varrho}\mathbf{v} \in L^\infty([0, T]; L^2(\Omega)), \quad \frac{|m_0|^2}{\rho_0} \in L^1(\Omega).$$
- Equation $(1.1)_1$ termed as continuity equation holds in a distributive sense
  $$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \text{ in } \mathcal{D}'((0, T) \times \mathbf{R}^3).$$
  The total mass is invariant such that
  $$\int_\Omega \rho dx = \int_\Omega \rho_0 dx = \mathcal{M}_0. \tag{2.1}$$
- Equation $(1.1)_1$ is satisfied in the renormalized form in $\mathcal{D}'((0, T) \times \Omega)$. Further, we have the following result for any $\xi \in C^1(\mathbb{R})$
  $$(\xi(\rho))_t + \operatorname{div}[\xi(\rho)\mathbf{v}] + (\xi'(\rho)\rho - \xi(\rho)) \operatorname{div} \mathbf{v} = 0.$$
- Any weak solution provided in Definition 2.1 will satisfy
  $$\rho \in C(0, T; L_w^\gamma(\Omega)), \quad \rho\mathbf{v} \in C\big(0, T; L_w^{\frac{2\gamma}{\gamma+1}}(\Omega)\big).$$
- Moreover, the weak solutions $(\rho, \mathbf{v}, \Psi)$ are constructed to satisfy the integral form of energy inequality, such that
  $$\mathcal{E}(t) + \int_0^t \int_\Omega (|\nabla \mathbf{v}|^2 + (\mu + \lambda)|\operatorname{div} \mathbf{v}|^2) dx\, dt \leqslant \mathcal{E}_0,$$
  with
  $$\mathcal{E}(t) = \int_\Omega \Big(\frac{1}{2}\rho|\mathbf{v}|^2(t) + \frac{\rho^\gamma(t)}{\gamma - 1} - \frac{1}{8\pi g}|\nabla\Psi|^2 \Big) dx$$
  and
  $$\mathcal{E}_0 = \int_\Omega \Big(\frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} - \frac{1}{8\pi g}|\nabla\Psi_0|^2 \Big) dx.$$

We emphasize that every classical solution described in (1.1)–(1.3) can be a weak solution and that any weak solution can solve the problem (1.1)–(1.3) with sufficient regularity in the classical sense. This is done to ensure that our defined weak solution given in Definition 2.1 is accurate. Next, the following theorem outlines the most significant result of this work.

THEOREM 2.1. *Assuming that Ω is an unbounded domain of Lipschitz boundary, let $(\rho_0, \mathbf{v}_0, \Psi_0)$ be the initial data satisfying (1.1). Then, the problem described by (1.1)–(1.3) with $T > 0$ has at least one solution $(\rho, \mathbf{v}, \Psi)$, as given in Definition (2.1), whose associated energy satisfies*

$$\frac{d}{dt}\mathcal{E}(t) + \int_\Omega (|\nabla \mathbf{v}|^2 + (\mu + \lambda)|\operatorname{div} \mathbf{v}|^2) dx \leqslant 0, \tag{2.2}$$

$$\operatorname{ess}\lim\sup_{t\to 0^+} \mathcal{E}(t) \leqslant \mathcal{E}_0, \quad in \ \mathcal{D}'(0,T).$$

Remark 2.1. In accordance with all the presumptions of Theorem 2.1, we have

$$\mathcal{E}(t) + \int_{\acute{z}}^{t}\int_{\Omega}(|\nabla \mathbf{v}|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{v}|^2)dx\,dt \leqslant \mathcal{E}(\acute{z}),$$

for a.e. $0 \leqslant t, \acute{z} \leqslant T$, $\acute{z} \leqslant t$ with the inclusion of $\acute{z} = 0$.

Remark 2.2. In our earlier work [21], we studied the existence of global compact attractors and complete bounded trajectories for a class of compressible magnetohydrodynamic (MHD) systems. That analysis was conducted in bounded domains and focused on the long-time asymptotic behavior of solutions. In contrast, the present paper addresses the global existence of weak solutions for compressible self-gravitating fluids in unbounded domains, which introduces additional mathematical difficulties, particularly related to the lack of compactness and the behavior of solutions at infinity. Despite the differences in physical models and domain settings, both studies contribute to the broader theory of compressible flows with complex interactions.

In addition, B_r will be used to symbolize a 3-D open ball having radius r and centered at the origin satisfying the assumptions that

$$\mathbf{R}^3 \setminus \Omega \subset B_1 \quad \text{with} \quad \Omega_m = \Omega \cap B_m, \quad \text{for} \ \ m \in \mathbf{N}.$$

3. Proof of Theorem 2.1

This part focuses on the proof of the major result of this research work. Based on the results obtained in [1] and [9], the global weak solution $(\rho_m, \mathbf{v}_m, \Psi_m)$ to the problem (1.1)–(1.3) exists in $(0,T)\times\Omega_m$ with the initial datum $(\rho_{0,m}, m_{0,m}, \Psi_{0,m})$ where the boundary condition is specified by

$$\mathbf{v}_m|_{\partial\Omega_m} = 0,$$

and its related energy function

$$\mathcal{E}_m(t) = \int_{\Omega_m}\Big(\frac{1}{2}\rho_m|\mathbf{v}_m|^2(t) + \frac{\rho_m^{\gamma}(t)}{\gamma-1} - \frac{1}{8\pi g}|\nabla\Psi_m|^2\Big)dx,$$

satisfies the differential form of energy inequality

$$\frac{d}{dt}\mathcal{E}_m(t) + \int_{\Omega_m}(|\nabla\mathbf{v}_m|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{v}_m|^2)dx \leqslant 0, \tag{3.1}$$

in $\mathcal{D}'(0,T)$ with $(\rho_m, \mathbf{v}_m)$ be zero extended outside $(0,T)\times\Omega_m$ and the density satisfy that

$$\int_{\Omega_m}\rho_m dx = \int_{\Omega_m}\rho_{0,m}dx \leqslant \int_{\Omega}\rho_0 dx = \mathcal{M}_0.$$

In addition, the following estimations hold

$$\begin{aligned} &\|\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))} \leqslant \mathrm{c}_1(\mathcal{E}_0),\\ &\|\sqrt{\rho_m}\mathbf{v}_m\|_{L^\infty(0,T;L^2(\Omega_m))} \leqslant \mathrm{c}_2(\mathcal{E}_0),\\ &\|\Psi_m\|_{L^2((0,T)\times\Omega_m)} \leqslant \mathrm{c}_3(\mathcal{E}_0, m), \end{aligned} \tag{3.2}$$

$$\int_0^T \int_{\Omega_m} (|\nabla \mathbf{v}_m|^2 + |\nabla \Psi_m|^2) dx\, dt \leqslant \mathrm{c}_4(\mathcal{E}_0),$$

where $\mathrm{c}_l > 0$ with $l = 1, 2, 3, 4$ are constants that depend solely on the energy function $\mathcal{E}_0$, and not on the parameter m. In addition, we get

$$\|\mathbf{T}(\nabla \mathbf{v}_m)\| \leqslant \mathrm{c}(\mathcal{E}_0).$$

Next, to show that ρ_m is uniformly bounded in $L^{\gamma+\Theta}((0,T)\times\Omega_m)$ with $\Theta = \Theta(\gamma) > 0$, we require the result given below.

3.1. A Linear Bounded Operator $\mathcal{B}$.

We list below several characteristics of the operator $\mathcal{B}$, which was introduced by Bogovskiî [3]. The operator $\mathcal{B}$ is regarded as the problem's solution

$$\begin{cases} \operatorname{div} g = k \\ k \in L^x(S), \quad \text{with} \quad S \subset \mathbf{R}^3 \text{ is a Lipschitz bounded domain.} \end{cases} \tag{3.3}$$

Lemma 3.1 ([3,23]). *Let $\mathcal{B}$ be a solution for auxiliary problem (3.3), satisfying the properties:*

- $\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3]\colon \left\{ k \in L^x(S) \middle| \int_S k\, dx = 0 \right\} \to W_0^{1,x}(S)$, *and its boundedness is given by* $\|\mathcal{B}\{k\}\|_{W_0^{1,x}(S)} \leqslant C_1(x,S)\|k\|_{L^x(S)}$, *with* $x \in (1,\infty)$.
- *Also,* $g = \mathcal{B}\{k\}$ *satisfies* $\operatorname{div} g = k$ *a.e. in* S, *and* $\|\mathcal{B}(k)\|_{L^y(S)} \leqslant C_2(y,S)\|(k)\|_{L^y(S)}$ *with* $y \in (1,\infty)$.
- *Next, for* $B_r = S$, C_1 *and* C_2 *do not depend on the radius* r, *so that* $\mathrm{C}_1(x) = \mathrm{C}_1(x, B_r)$ *and* $\mathrm{C}_2(y) = \mathrm{C}_2(y, B_r)$.

Lemma 3.2. *Let $\Theta = \Theta(\gamma) > 0$, then*

$$\int_0^T \int_{\Omega_m} \rho_m^{\gamma+\Theta} dx\, dt \leqslant \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T)$$

holds for $\mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T) > 0$, so that C is not dependent on the parameter m.

Proof. Let for any fixed $0 < \Theta < \frac{\gamma}{3}$, based on (2.1), to be appropriately chosen subsequently, we have

$$\begin{aligned} \int_0^T \int_{\Omega_m} \rho_m^{\gamma+\Theta} dx\, dt &= \int_0^T \int_{x\in\Omega_m:\rho_m\leqslant 1} \rho_m^{\gamma+\Theta} dx\, dt + \int_0^T \int_{x\in\Omega_m:\rho_m\geqslant 1} \rho_m^{\gamma+\Theta} dx\, dt \\ &\leqslant \mathcal{M}_0 T + \int_0^T \int_{\Omega_1} \rho_m^{\gamma+\Theta} dx\, dt + \int_0^T \int_{x\in\Omega_m\setminus\Omega_1:\rho_m\geqslant 1} \rho_m^{\gamma+\Theta} dx\, dt. \end{aligned} \tag{3.4}$$

By applying Lemma 3.1, the second integral in (3.4) implies that

$$\int_0^T \int_{\Omega_1} \rho_m^{\gamma+\Theta}(t,x) dx\, dt \leqslant \mathrm{C}(T, \mathcal{M}_0, \Omega_1).$$

Thus, it only remains to deal with the integral $\int_0^T \int_{x\in\Omega_m\setminus\Omega_1:\rho_m\geqslant 1} \rho_m^{\gamma+\Theta}(t,x)dx\,dt$. For this, let $\hbar \in C^2[0,\infty)$ satisfy

$$\hbar(\tilde r) = \begin{cases} 0 & \text{when } \ \tilde r \leqslant \frac{1}{2}, \\ \hbar'(\tilde r) \geqslant 0 & \text{when } \ \frac{1}{2} < \tilde r < 1, \\ \hbar(\tilde r) = \tilde r^{\Theta} & \text{when } \ \tilde r \geqslant 1. \end{cases}$$

Applying (3.2), we have

$$\sup_{t\in[0,T]} \|\hbar(\rho_m)(t)\|_{L^p(\Omega_m)} \leqslant \mathrm{C}(\mathcal{M}_0,\mathcal{E}_0), \quad p\in\Big[1,\frac{\gamma}{\Theta}\Big]. \tag{3.5}$$

In the view of Lemma 3.1 and taking $\mathcal{B}_m = \mathcal{B}_{B_m}$ be the ball B_m associated with the Bogovskiî operator. Next, let

$$\overline{\hbar(\rho_m)} = \hbar(\rho_m) - \frac{1}{B_m}\int_{B_m} \hbar(\rho_m)dx$$

and once again taking into account Lemma 3.1, along with (3.5), we get

$$\sup_{t\in[0,T]} \|\nabla\mathcal{B}_m \overline{\hbar(\rho_m)(t)}\|_{L^p(B_m)} \leqslant \mathrm{C}(\mathcal{M}_0,\mathcal{E}_0), \quad p\in\Big[1,\frac{\gamma}{\Theta}\Big], \tag{3.6}$$

$$\sup_{t\in[0,T]} \|\mathcal{B}_m \overline{\hbar(\rho_m)(t)}\|_{L^q(B_m)} \leqslant \mathrm{C}(\mathcal{M}_0,\mathcal{E}_0), \qquad q\in\Big[\frac{3}{2},\infty\Big]. \tag{3.7}$$

Furthermore, by taking $\varpi(t)\zeta(x)\mathcal{B}\big(\overline{\hbar(\rho_m)}\big)$ as test function in momentum equation $(1.1)_2$ with $\varpi \in C_c^\infty(0,T)$ and ζ satisfying the properties

$$\begin{cases} 0\leqslant \zeta \leqslant 1, & \\ \zeta \equiv 0 & \text{on } \partial\Omega, \\ \zeta \equiv 1 & \text{on } \mathbf{R}^3\setminus\Omega_1. \end{cases}$$

Thus, we get

$$\int_0^T \varpi \int_{\Omega_m} \zeta\rho_m^{\gamma}\hbar(\rho_m)dx\,dt = \sum_{i=1}^{9} I_i,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{|B_m|}\int_0^T \varpi \int_{B_m} \hbar(\rho_m)dx \int_{\Omega_m} \zeta\rho_m^{\gamma}dx\,dt,\\
I_2 &= -\int_0^T \varpi \int_{\Omega_m} \rho^{\gamma}\nabla\cdot\zeta\cdot\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)dx\,dt,\\
I_3 &= \int_0^T \varpi \int_{\Omega_m} \zeta\mathbf{T} : \nabla\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)dx\,dt,\\
I_4 &= \int_0^T \varpi \int_{\Omega_m} \nabla\zeta\cdot\mathbf{T}\cdot\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)dx\,dt,\\
I_5 &= -\int_0^T \varpi \int_{\Omega_m} \zeta(\rho_m\mathbf{v}_m\otimes\mathbf{v}_m) : \nabla\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)dx\,dt,
\end{aligned}$$

$$
\begin{aligned}
I_6 &= -\int_0^T \varpi \int_{\Omega_m} \nabla\zeta \cdot (\rho_m \mathbf{v}_m \otimes \mathbf{v}_m) \cdot \mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big) dx\, dt,\\
I_7 &= \int_0^T \varpi \int_{\Omega_m} \nabla\zeta \cdot (\rho_m \nabla\Psi_m) \cdot \mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big) dx\, dt,\\
I_8 &= \int_0^T \varpi \int_{\Omega_m} \nabla\zeta \cdot (\rho_m \nabla\Psi_m) : \nabla\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big) dx\, dt,\\
I_9 &= -\int_0^T \varpi' \int_{\Omega_m} \zeta\rho_m \mathbf{v}_m \cdot \mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big) dx\, dt,\\
I_{10} &= \int_0^T \varpi \int_{\Omega_m} \zeta\rho_m \mathbf{v}_m \cdot \mathcal{B}_m(\mathrm{div}(\hbar(\rho_m)\mathbf{v}_m)) dx\, dt,\\
I_{11} &= \int_0^T \varpi \int_{\Omega_m} \zeta\rho_m \mathbf{v}_m \cdot \mathcal{B}_m\big(\overline{b(\rho_m)}\big) dx\, dt,
\end{aligned}
$$

where

$$
b(\rho_m) = (\hbar'(\rho_m)\rho_m - \hbar(\rho_m))\mathrm{div}\mathbf{v}_m, \quad \overline{b(\rho_m)} = b(\rho_m) - \frac{1}{|B_m|}\int_{B_m} b(\rho_m) dx.
$$

Applying (3.5)–(3.7), we get

$$
\begin{aligned}
|I_1| &\leqslant \|\zeta\rho_m^\gamma\|_{L^\infty(0,T;L^1(\Omega_m))} \|\hbar(\rho_m)\|_{L^1((0,T)\times\Omega_m)} \|\varpi\|_{L^\infty(0,T)}\\
&\leqslant \|\varpi\|_{L^\infty(0,T)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T),\\
|I_2| &\leqslant \|\rho_m^\gamma \nabla\zeta\|_{L^\infty(0,T;L^1(\Omega_m))} \big\|\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(0,T;L^\infty(\Omega_m))} \|\varpi\|_{L^\infty(0,T)}\\
&\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)} \|\varpi\|_{L^\infty(0,T)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T).
\end{aligned}
$$

Next,

$$
\begin{aligned}
|I_3| &\leqslant \|\varpi\|_{L^\infty(0,T)} \|\zeta\mathbf{T}\|_{L^2(0,T;L^2(\Omega_m))} \big\|\nabla\mathcal{B}_m\big(\overline{b(\rho_m)}\big)\big\|_{L^2(0,T;L^2(\Omega_m))}\\
&\leqslant \|\varpi\|_{L^\infty(0,T)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T),\\
|I_4| &\leqslant \|\varpi\|_{L^\infty(0,T)} \|\mathbf{T}\nabla\zeta\|_{L^\infty(0,T;L^1(\Omega_m))} \big\|\mathcal{B}_m\big(\overline{b(\rho_m)}\big)\big\|_{L^2(0,T;L^2(\Omega_m))}\\
&\leqslant \|\varpi\|_{L^\infty(0,T)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T) \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}.
\end{aligned}
$$

By taking $0 < \Theta < (2\gamma - 3)/3$, we have

$$
\begin{aligned}
|I_5| &\leqslant \|\varpi\|_{L^\infty(0,T)} \|\zeta\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))} \|\nabla\mathbf{v}_m\|^2_{L^2((0,T)\times\Omega_m)} \big\|\nabla\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega_m))}\\
&\leqslant \|\varpi\|_{L^\infty(0,T)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T),
\end{aligned}
$$

$$
\begin{aligned}
|I_6| &\leqslant \|\rho_m\nabla\zeta\|_{L^\infty(0,T;L^\gamma(\Omega_m))} \|\nabla\mathbf{v}_m\|^2_{L^2((0,T)\times\Omega_m)} \|\varpi\|_{L^\infty(0,T)} \big\|\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(0,T;L^{\frac{2\gamma-3}{3\gamma}}(\Omega_m))}\\
&\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)} \mathrm{C}(\mathcal{E}_0, \mathcal{M}_0, T) \|\varpi\|_{L^\infty(0,T)},
\end{aligned}
$$

Similarly, for $\big|I_7\big|$ using the Sobolev embedding theorem, Hölder inequality, Lemma 3.1 and (1.2), for $\gamma \in \big(\frac{5}{3}, 3\big]$, one can get

$$\begin{aligned}|I_7| &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\int_0^T \|\rho\|_{L^\gamma(\Omega)}\|\nabla\Psi\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\big\|\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(\Omega)}dt\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,)\int_0^T \|\rho\|_{L^\gamma(\Omega)}\|\nabla\Psi\|_{W^{1,\frac{3\gamma}{4\gamma-3}}(\Omega)}dt\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0)\int_0^T \|\rho\|_{L^\gamma(\Omega)}\|\rho\|_{L^{\frac{3\gamma}{4\gamma-3}}(\Omega)}dt.\end{aligned}$$

If $\gamma \in \big(\frac{5}{3}, 3\big)$ then by applying the interpolation inequality we get

$$\|\rho\|_{L^{\frac{3\gamma}{4\gamma-3}}(\Omega)} \leqslant \|\rho\|^{\Theta}_{L^1(\Omega)}\|\rho\|^{1-\Theta}_{L^\gamma(\Omega)}, \quad \Theta = \frac{4\gamma-6}{3\gamma-3}.$$

Taking into account this inequality, we infer that

$$\begin{aligned}|I_7| &\leqslant \mathrm{C}\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}(\mathcal{E}_0,\mathcal{M}_0,T)\sup_{t\in[0,T]}\|\rho\|_{L^{\frac{2\gamma}{3\gamma-3}}(\Omega)}\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}(\mathcal{E}_0,\mathcal{M}_0,T)\sup_{t\in[0,T]}\|\rho\|^{\gamma}_{L^\gamma(\Omega)}.\end{aligned}$$

Similarly, for $\gamma = 3$

$$\begin{aligned}|I_7| &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)\sup_{t\in[0,T]}\|\rho\|_{L^\gamma(\Omega)}\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).\end{aligned}$$

In case, the $\gamma > 3$, then

$$\begin{aligned}|I_7| &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\int_0^T \|\rho\|_{L^1(\Omega)}\|\nabla\Psi\|_{L^\infty(\Omega)}\big\|\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(\Omega)}dt\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)\int_0^T \|\Psi\|_{W^{1,\gamma}(\Omega)}dt\\ &\leqslant \|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).\end{aligned}$$

In the same way, for $|I_8|$, we have

$$|I_8| \leqslant \|\varpi\|_{L^\infty(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}.$$

Furthermore,

$$\begin{aligned}|I_9| &\leqslant \|\varpi'\|_{L^1(0,T)}\|\sqrt{\rho_m}\mathbf{v}_m\|_{L^\infty(0,T;L^2(\Omega_m))}\|\zeta\sqrt{\rho_m}\|_{L^\infty(0,T;L^{2\gamma}(\Omega_m))}\big\|\mathcal{B}_m\big(\overline{\hbar(\rho_m)}\big)\big\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega_m))}\\ &\leqslant \|\varpi'\|_{L^1(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).\end{aligned}$$

By using Lemma 3.1, we infer that

$$|I_{10}| \leqslant c_2\|\varpi\|_{L^\infty(0,T)}\|\zeta\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))}\int_0^T \|\mathbf{v}_m\|_{L^6(\Omega_m)}\|\hbar(\rho_m)\mathbf{v}_m\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_m)}dt.$$

Since

$$\|\hbar(\rho_m)\mathbf{v}_m(t)\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_m)} \leqslant \|\mathbf{v}_m(t)\|_{L^6(\Omega_m)}\|\hbar(\rho_m)(t)\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega_m)},$$

taking into account these estimates along with (3.6), we get

$$
\begin{aligned}
I_{10}| &\leqslant c_2\|\varpi\|_{L^\infty(0,T)}\|\zeta\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))}\|\nabla\mathbf{v}_m\|^2_{L^2((0,T)\times\Omega_m)}\|\hbar(\rho_m)\|_{L^\infty(0,T;L^{\frac{2\gamma-3}{3\gamma}}(\Omega_m))}dt \\
&\leqslant \|\varpi'\|_{L^1(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).
\end{aligned}
$$

Next, for I_{11}, we consider two cases. When $\gamma<6$, it yields that

$$
|I_{11}| \leqslant \|\varpi\|_{L^\infty(0,T)}\|\zeta\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))}\int_0^T \|\mathbf{v}_m(t)\|_{L^6(\Omega_m)}\big\|\mathcal{B}_m\overline{b(\rho_m)}(t)\big\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_m)}dt,
$$

with

$$
\begin{aligned}
\big\|\mathcal{B}_m\overline{b(\rho_m)}(t)\big\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_m)} &\leqslant \|b(\rho_m)(t)\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_m)} \\
&\leqslant \|\nabla\mathbf{v}_m(t)\|_{L^2(\Omega_m)}\|\hbar'(\rho_m)\rho_m-\hbar(\rho_m)(t)\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega_m)},
\end{aligned}
$$

implies that

$$
\begin{aligned}
|I_{11}| \leqslant{}& \|\varpi\|_{L^\infty(0,T)}\|\nabla\mathbf{v}_m\|^2_{L^2(0,T;L^2(\Omega_m))}\|\zeta\rho_m\|_{L^\infty(0,T;L^\gamma(\Omega_m))}\|\hbar'(\rho_m)\rho_m \\
&-\hbar(\rho_m)(t)\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}})} \leqslant \|\varpi'\|_{L^1(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).
\end{aligned}
$$

In addition, for the case when $\gamma>6$

$$
|I_{11}| \leqslant \|\varpi\|_{L^\infty(0,T)}\|\zeta\rho_m\|_{L^\infty(0,T;L^3(\Omega_m))}\int_0^T \|\mathbf{v}_m\|_{L^6(\Omega_m)}\big\|\mathcal{B}_m\overline{b(\rho_m)}(t)\big\|_{L^2(\Omega_m)},
$$

with

$$
\big\|\mathcal{B}_m\overline{b(\rho_m)}(t)\big\|_{L^2(\Omega_m)} \leqslant \|\nabla\mathbf{v}_m(t)\|_{L^2(\Omega_m)}\|\hbar'(\rho_m)\rho_m-\hbar(\rho_m)(t)\|_{L^3(\Omega_m)},
$$

thus,

$$
|I_{11}| \leqslant \|\varpi'\|_{L^1(0,T)}\mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T).
$$

Summing up the estimations $I_1,...,I_{11}$, we get

$$
\begin{aligned}
&\int_0^T\varpi\int_{\Omega_m}\zeta\rho_m^\gamma\hbar(\rho_m)dx\,dt \\
&\quad\leqslant \mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)\big[\|\varpi\|_{L^\infty(0,T)}+\|\varpi'\|_{L^1(0,T)}+\|\varpi\|_{L^\infty(0,T)}\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}\big].
\end{aligned}
$$

Furthermore, applying the classical approximation argument implies that

$$
\int_0^T\varpi\int_{\Omega_m}\zeta\rho_m^\gamma\hbar(\rho_m)dx\,dt \leqslant \mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)[1+\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}],
$$

and

$$
\begin{aligned}
\int_0^T\varpi\int_{\{x\in\Omega_m:\rho_m>1\}}\zeta\rho_m^{\gamma+\Theta}dx\,dt &\leqslant \int_0^T\varpi\int_{\Omega_m}\zeta\rho_m^\gamma\hbar(\rho_m)dx\,dt \\
&\leqslant \mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)[1+\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}].
\end{aligned}
$$

Next, using the property of ζ, implies that

$$
\begin{aligned}
\int_0^T\varpi\int_{\{x\in\Omega_m\setminus\Omega_1:\rho_m>1\}}\zeta\rho_m^{\gamma+\Theta}dx\,dt &\leqslant \int_0^T\varpi\int_{\{x\in\Omega_m:\rho_m>1\}}\zeta\rho_m^\gamma dx\,dt \\
&\leqslant \mathrm{C}(\mathcal{E}_0,\mathcal{M}_0,T)[1+\|\nabla\zeta\|_{L^\infty(\mathbf{R}^3)}].
\end{aligned}
$$

This completes the desired proof. □

Taking into account the convergence results of [1] and [9], there exists a subsequence $(\rho_m, \mathbf{v_m})$ that converges to a weak solution $(\rho, \mathbf{v})$ in $(0,T)\times\Omega$, with

$$\nabla \mathbf{v}_m \longrightarrow \nabla \mathbf{v} \text{ in } L^2_{\text{weak}}((0,T)\times\Omega),$$

next, for the fixed ball B_r, we have

$$\begin{aligned} \rho_m|\mathbf{v}_m|^2 &\longrightarrow \rho|\mathbf{v}|^2 && \text{in } L^\infty_{\text{weak}}(0,T;L^{\frac{2\gamma}{2\gamma+2}}(B_r)),\\ |\nabla\Psi_m|^2 &\longrightarrow |\nabla\Psi|^2 && \text{in } L^2_{\text{weak}}(0,T;L^2(B_r)),\\ |\rho_m|^\gamma &\longrightarrow |\rho|^\gamma && \text{in } L^\gamma_{\text{weak}}((0,T)\times B_r). \end{aligned} \tag{3.8}$$

Through the use of weak lower semicontinuity of convex functionals, as

$$\begin{aligned} \int_0^T \varpi\int_\Omega |\nabla\mathbf{v}|^2 dx\,dt &\leqslant \lim\inf_{m\to\infty}\int_0^T\varpi\int_\Omega|\nabla\mathbf{v}_m|^2dx\,dt,\\ \int_0^T \varpi\int_\Omega |\operatorname{div}\mathbf{v}|^2 &\leqslant \lim\inf_{m\to\infty}\int_0^T\varpi\int_\Omega|\operatorname{div}\mathbf{v}_m|^2,\\ \int_0^T \varpi\int_\Omega |\nabla\Psi|^2 &\leqslant \lim\inf_{m\to\infty}\int_0^T\varpi\int_\Omega|\nabla\Psi_m|^2, \end{aligned}$$

for every $\varpi\in\mathcal{D}(0,T)$, and by using the estimation obtained in Lemma 3.2, we are allowed to take the limit of the term, such as

$$\int_{\Omega_m}(|\mu\nabla\mathbf{v}_m|^2+(\mu+\lambda)|\operatorname{div}\mathbf{v}_m|^2+|\nabla\Psi_m|^2)dx.$$

Furthermore, we must demonstrate that for the positive ε, there exists $p=p(\varepsilon)>0$, regardless of the dependence of the parameter m, satisfying

$$\int_0^T\int_{\{x\in\Omega:|x|\geqslant p\}}\rho_m^\gamma dx\,dt\leqslant\varepsilon, \tag{3.9}$$

$$\int_0^T\int_{\{x\in\Omega:|x|\geqslant p\}}\rho_m|\mathbf{v}_m|^2 dx\,dt\leqslant\varepsilon, \tag{3.10}$$

$$\int_0^T\int_{\{x\in\Omega:|x|\geqslant p\}}|\nabla\Psi_m|^2 dx\,dt\leqslant\varepsilon.$$

By applying (2.1), (2.2) and $p_m(t)\to\infty$, we get

$$\lim_{m\to\infty}\int_{|x|=p_m}(\rho_m|\mathbf{v}_m|^2+|\nabla\Psi_m|^2+\rho_m^\gamma)ds=0,$$

that implies the estimations (3.9) and (3.10). Subsequently, the use of (3.8)–(3.10) allow us to pass the limit $m\to\infty$ in the other leftover terms of (3.1). Thus, Theorem 2.1 is proved.

4. Conclusion

The energy inequality of weak solutions for compressible self-gravitating fluids is for the first time examined in this research study, and it is established that at least one of the weak solutions, as described in Definition 2.1, satisfies the differential form of energy inequality, as claimed in Theorem 2.1.

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О ГЛОБАЛНОМ ПОСТОЈАЊУ СЛАБИХ РЕШЕЊА ЗА СТИШЉИВЕ САМОГРАВИТИРАЈУЋЕ ФЛУИДЕ НА НЕОГРАНИЧЕНИМ ДОМЕНИМА

Резиме. Овај рад испитује глобално постојање слабих решења за стишљиве самогравитирајуће флуиде у тродимензионалном неограниченом домену са компактном Липшицовом границом, под претпоставком укупне коначне масе флуида. Доказујемо да постоји глобално дефинисано слабо решење које задовољава енергетску неједнакост у диференцијалном облику.

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