

CLASSIFICATION OF THE GEODESIC CURVES OF A SUB-RIEMANNIAN LR SYSTEM ON THE HEISENBERG GROUP IN DIMENSION 5

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ABSTRACT. We study sub-Riemannian structures obtained by the restriction of the standard left-invariant Riemannian metric to a right-invariant completely nonholonomic distribution on the second Heisenberg group. The corresponding Hamiltonian system is completely integrable and we study its solutions in this paper. We obtain a complete classification of the geodesic curves by studying the reduced Hamiltonian system. Moreover, we compute $r(t)$ as the first step of the quadrature in the reduced system in the hyper-spherical coordinates and present numerical examples that illustrate the theoretical results.

1. Introduction

Sub-Riemannian geometry can be described as a geometry with restrictions. Motion is only possible in certain directions (which vary from point to point). These constraints are defined by the distribution in which the length is measured. If the distribution has a certain 'nice' property, every point in space is reachable. There are several recent papers presenting new integrable sub-Riemannian geodesic flows on Lie groups and homogeneous spaces, such as [4, 7–9, 13]. In this paper, we will study one such integrable sub-Riemannian geodesic flow, found in [13]. In general, sub-Riemannian geometry on Lie groups is studied by considering metric and the distribution, both of which are left-invariant. Less studied are situations where the metric is left-invariant and the distribution is right-invariant, the so-called LR system. Such a situation leads to strongly non-linear dynamics. LR systems have been already studied in nonholonomic mechanics. The basic example is a motion of a rigid body around a fixed point in the presence of a nonholonomic constraint: the projection of the angular velocity of the body onto a fixed axis in the space is equal to zero [6, 17]. LR systems are also studied in sub-Riemannian framework [1, 11, 16].

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In [16] a sub-Riemannian LR system on a three-dimensional Heisenberg group is studied and it is found that such a system is both integrable and associated with a certain magnetic field, while in [1] geodesics of a sub-Riemannian LR system on the first Heisenberg group are classified. In [13] first results on sub-Riemannian LR systems in dimensions larger than three are given. In particular, the integrability of sub-Riemannian LR systems in the general Heisenberg group H_{2n+1} is proved and the equivalence to motion in the presence of a higher-dimensional magnetic field is found. If $n > 2$, one has integrability in the non-commutative sense, while for $n = 1$ and $n = 2$, i.e. in dimensions three and five, systems are integrable in the classical, commutative sense.

The integrability of sub-Riemannian LR systems on the Heisenberg group has been the subject of previous investigations. Motivated by these results, we turn our attention to the study of geodesic lines, whose analysis provides a natural continuation of this research and contributes to a more comprehensive understanding of the underlying sub-Riemannian geometry. We classify the sub-Riemannian geodesics in dimension 5 and we study the reduced Hamiltonian system in hyperspherical coordinates. These coordinates allow us to classify geodesics according to the properties of $r(t)$.

The paper is organized as follows. Section 2 reviews the necessary background on sub-Riemannian geometry, the five-dimensional Heisenberg group, and elliptic integrals, while also introducing the notation used throughout the paper. In Section 3, we revisit the integrability of a sub-Riemannian LR system on H_{2n+1} , previously established in [13] for Heisenberg groups of arbitrary dimension. We derive the corresponding reduced Hamiltonian system and present its first integrals. In Section 4, a classification of the trajectories of the reduced Hamiltonian system in hyper-spherical coordinates is obtained. This classification extends naturally to the classification of the geodesic curves in the second Heisenberg group, and is the main result of the paper. We conclude this section by deriving explicit formulae for the first step of the quadrature integration, $r = r(t)$, and by presenting numerical examples that illustrate the theoretical results.

2. Preliminaries

In this section, we introduce the notation as well as some well-known facts of sub-Riemannian geometry, most of which can be found, for example, in [2, 10, 12, 15, 16], Heisenberg group, and Elliptic integrals.

2.1. Sub-Riemannian geometry. Let Q be a manifold of dimension n , $\mathcal{D}$ a distribution of constant rank m of the tangent bundle TQ and $\langle \cdot, \cdot \rangle_{sR}(q)$ a smoothly varying positive definite bilinear form acting as an inner product on this distribution. A sub-Riemannian structure on Q is a pair $(\mathcal{D}, \langle \cdot, \cdot \rangle_{sR})$. If at each point $q \in Q$ vector fields of $\mathcal{D}_q$ together with all their Lie brackets span the entire $T_q Q$, we say that this distribution is completely nonholonomic (or bracket-generating).

A horizontal (or admissible) curve is a curve whose tangent vector lies in the distribution. More precisely, the absolutely continuous curve $\gamma: [0, 1] \rightarrow Q$ is horizontal if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost every $t \in [0, 1]$. For horizontal γ we define its length

as $\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$, where $\|\dot{\gamma}\| = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_{sR}}$. In this case, the sub-Riemannian or Carnot-Carathéodory distance $dist_{CC}(x, y) = \inf \ell(\gamma)$ is defined, and the infimum is taken over all admissible curves γ connecting x and y . If there is no such curve, the Carnot-Carathéodory distance is infinite. For distributions with bracket-generating property, the theorems of Chow [5] and Rashievskii [14] guarantee that there is always an admissible curve connecting any two points.

For a given $\langle \cdot, \cdot \rangle_{sR}(q)$, the linear mapping $g(q): T_q^*M \rightarrow \mathcal{D}_q \subset T_qM$ can be defined as follows: $g(q)\lambda$ is a unique vector such that

$$\langle g(q)\lambda, Y \rangle_{sR}(q) = \lambda(Y(q)), \quad \forall Y \in \mathcal{D}_q,$$

with the usual pairing of vectors and covectors right-hand side. The symmetric tensor g^{ij} is called Carnot-Carathéodory metric tensor. It is positive semidefinite and degenerate.

For the pairing $\lambda(X(q))$, $q \in Q$ between the fixed vector field X and the covector λ we write $P_X(q, \lambda)$. The function $P_X: T^*M \rightarrow \mathbb{R}$ is called the momentum function. Let $X_1, \dots, X_m$ be a local basis of vector fields of the distribution, in coordinates $X_a = \sum_j X_a^j(q) \frac{\partial}{\partial x_j}$. The momentum functions are given by $P_{X_a} = \sum_j X_a^j(q) \lambda_j$. Let $g_{ab}(q)$ be the matrix of the inner products of the vectors X_a and $g^{ab}(q)$ their inverse matrix. Then $g^{ab}(q)$ is an $m \times m$ matrix-valued function (where m is the rank of the distribution) defined in an open set of Q . The sub-Riemannian Hamiltonian function can be written as

$$H(q, \lambda) = \frac{1}{2} \sum_{a,b} g^{ab}(q) P_{X_a}(q, \lambda) P_{X_b}(q, \lambda).$$

With $\{f, g\} = X_g(f)$ we denote the Poisson bracket of $f, g \in T^*Q$, where X_g is the Hamiltonian vector field associated with g . In canonical coordinates, the Poisson bracket is given by:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_i} \frac{\partial g}{\partial q_i}.$$

The corresponding Hamilton equations on the cotangent bundle are

$$(2.1) \quad \dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = \{\lambda_i, H\} = -\frac{\partial H}{\partial q_i}.$$

The projections of solutions of the Hamiltonian system (2.1) are locally length-minimising curves, which are referred to as normal geodesics (see [15, 16]). In contrast to Riemannian geometry, in sub-Riemannian geometry, we have locally minimising curves that do not arise as solutions of the Hamiltonian system, abnormal geodesics. The horizontal distribution in the Heisenberg group is the contact distribution, and all geodesics are solutions of the Hamiltonian system [2, Proposition 4.38].

2.2. Heisenberg group H_5 . The second Heisenberg group H_5 is a five-dimensional two-step nilpotent Lie group constructed on the base manifold $\mathbb{R}^4 \oplus \mathbb{R}$ by multiplication

$$(u, \zeta) \cdot (v, \chi) := (u + v, \zeta + \chi + \omega(u, v)).$$

$$\omega(u, v) = u^T J v, \quad \text{where } J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This group can also be seen as the group of matrices of the form

$$H_5 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_1, x_2, y_1, y_2, z \in \mathbb{R} \right\},$$

with respect to the matrix multiplication. The corresponding Lie algebra $\mathfrak{h}_5$ is spanned by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ and $\frac{\partial}{\partial z}$. The commutator relations that are not zero are

$$(2.2) \quad \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \right] = \left[\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right] = \frac{\partial}{\partial z}.$$

This representation is more convenient for our study and is used in this paper.

2.3. Elliptic integrals. In the following section, elliptic integrals will be used, therefore, we introduce the notation here. For more about elliptic integrals and elliptic functions we refer to extensive literature (e.g. [3, 18]). With F and E we denote elliptic integrals of the first and second kind respectively:

$$F(\phi, k) = \int_0^\phi \frac{dv}{\sqrt{1 - k^2 \sin^2 v}} = u = \int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \text{sn}^{-1}(y, k),$$

where $y = \sin \phi$, and

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 v} dv = \int_0^u \text{dn}^2 u_1 du_1 = \int_0^y \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt = E(\text{am } u, k),$$

where $\phi = \text{am } u$ is the amplitude. The parameter $0 \leq k \leq 1$ is the modulus and $k' = \sqrt{1 - k^2}$ is the complementary modulus. Twelve elliptic Jacobian functions are denoted by $pq(u, k)$, where $p, q \in \{s, c, d, n\}$, e.g. elliptic sine is $\text{sn}(u, k)$, elliptic cosine is $\text{cn}(u, k)$ and delta amplitude is $\text{dn}(u, k)$.

The exponentiation of the Jacobian function with -1 should always be understood as inversion. The abbreviation convention $E(u)$ should be understood as $E(u) = E(\text{am } u, k)$ if k is clear.

3. Integrability in H_5

The group H_5 acts on itself by left translations L_g : $L_g(h) = gh$ and by right translations R_g : $R_g(h) = hg$. Denote by $\mathcal{D}_0$ the linear space spanned by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$ at the identity. The left-invariant distribution generated by $\mathcal{D}_0$ consists of 4-planes $\mathcal{D}_L = L_{g*}\mathcal{D}_0$, where:

$$L_{g*}\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial x_k}, \quad L_{g*}\left(\frac{\partial}{\partial y_k}\right) = \frac{\partial}{\partial y_k} + x_k \frac{\partial}{\partial z}, \quad k \in \{1, 2\} \quad \text{and} \quad L_{g*}\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z},$$

while the right-invariant distribution generated by $\mathcal{D}_0$ is $\mathcal{D}_R = R_{g*}\mathcal{D}_0$, where:

$$R_{g*}\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial z}, \quad R_{g*}\left(\frac{\partial}{\partial y_k}\right) = \frac{\partial}{\partial y_k}, \quad k \in \{1, 2\} \quad \text{and} \quad R_{g*}\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}.$$

We note that H_5 is a contact manifold and $\mathcal{D}_R$ is a contact distribution. From the relation (2.2) it follows that this distribution is bracket-generating.

In [13] the geodesic flow of the sub-Riemannian metric, which corresponds to the left-invariant metric in H_5 and the right-invariant distribution $\mathcal{D}_R$, is considered and its integrability is proved. The left-invariant standard Riemannian metric in the coordinates (x_1, x_2, y_1, y_2, z) is:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_1^2 + 1 & x_1 x_2 & -x_1 \\ 0 & 0 & x_1 x_2 & x_2^2 + 1 & -x_2 \\ 0 & 0 & -x_1 & -x_2 & 1 \end{pmatrix}.$$

It induces the sub-Riemannian metric on the right-invariant distribution. The geodesic flow, which corresponds to the left-invariant sub-Riemannian metric and the horizontal distribution $\mathcal{D}_R$, forms the Hamiltonian system on T^*H_5 with the Hamiltonian function:

$$\begin{aligned} H(q, \lambda) = & \frac{1}{2(x_1^2 + x_2^2 + y_1^2 + y_2^2 + 1)} ((x_1^2 + x_2^2 + y_2^2 + 1)\lambda_1^2 + (x_1^2 + x_2^2 + y_1^2 + 1)\lambda_2^2 \\ & + (x_2^2 + y_1^2 + y_2^2 + 1)\lambda_3^2 + (x_1^2 + y_1^2 + y_2^2 + 1)\lambda_4^2 + (x_1^2 + x_2^2 + 1)(y_1^2 + y_2^2)\lambda_5^2 \\ & + 2(y_1^2 + y_2^2)(x_1\lambda_3 + x_2\lambda_4)\lambda_5 + 2y_1(-y_2\lambda_2 + x_1\lambda_3 + x_2\lambda_4 + (x_1^2 + x_2^2 + 1)\lambda_5)\lambda_1 \\ & - 2x_1x_2\lambda_3\lambda_4 + 2y_2\lambda_2(x_1\lambda_3 + x_2\lambda_4 + (x_1^2 + x_2^2 + 1)\lambda_5)). \end{aligned}$$

The Hamiltonian system takes the form:

$$\begin{aligned} \dot{x}_1 = \{x_1, H\} &= \frac{(1 + x_1^2 + x_2^2 + y_2^2)\lambda_1 + y_1(-y_2\lambda_2 + x_1\lambda_3 + x_2\lambda_4 + (1 + x_1^2 + x_2^2)\lambda_5)}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}, \\ \dot{x}_2 = \{x_2, H\} &= \frac{-y_1y_2\lambda_1(1 + x_1^2 + x_2^2)\lambda_2 + y_1^2\lambda_2 + y_2(x_1\lambda_3 + x_2\lambda_4 + (1 + x_1^2 + x_2^2)\lambda_5)}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}, \\ \dot{y}_1 = \{y_1, H\} &= \frac{(1 + x_1^2 + x_2^2 + y_2^2)\lambda_3 + x_1(y_1\lambda_1 - x_2\lambda_4 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5))}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}, \\ \dot{y}_2 = \{y_2, H\} &= \frac{(1 + x_1^2 + x_2^2 + y_2^2)\lambda_4 + x_2(y_1\lambda_1 - x_1\lambda_3 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5))}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}, \\ \dot{z} = \{z, H\} &= \frac{(1 + x_1^2 + x_2^2)y_1\lambda_1 + (1 + x_1^2 + x_2^2)y_2\lambda_2 + (y_1^2 + y_2^2)(x_1\lambda_3 + x_2\lambda_4) + (1 + x_1^2 + x_2^2)(y_1^2 + y_2^2)\lambda_5}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}, \\ (3.1) \quad \dot{\lambda}_1 = \{\lambda_1, H\} &= -\frac{1}{(1 + x_1^2 + x_2^2 + y_1^2 + y_2^2)^2} (y_1\lambda_1 - x_1\lambda_3 - x_2\lambda_4 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5)) \cdot \\ & \quad ((1 + x_2^2 + y_1^2 + y_2^2)\lambda_3 + x_1(y_1\lambda_1 - x_2\lambda_4 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5))), \\ \dot{\lambda}_2 = \{\lambda_2, H\} &= -\frac{1}{(1 + x_1^2 + x_2^2 + y_1^2 + y_2^2)^2} (y_1\lambda_1 - x_1\lambda_3 - x_2\lambda_4 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5)) \cdot \\ & \quad ((1 + x_1^2 + y_1^2 + y_2^2)\lambda_4 + x_2(y_1\lambda_1 - x_1\lambda_3 + y_1^2\lambda_5 + y_2(\lambda_2 + y_2\lambda_5))), \end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_3 &= \{\lambda_3, H\} = \frac{1}{(1+x_1^2+x_2^2+y_1^2+y_2^2)^2} (y_1\lambda_1+x_4\lambda_2 - x_1\lambda_3 - x_2\lambda_4 - (1+x_1^2+x_2^2)\lambda_5) \cdot \\
&\quad ((1+x_1^2+x_2^2+y_2^2)\lambda_1+x_3(-y_2\lambda_2+x_1\lambda_3+x_2\lambda_4+(1+x_1^2+x_2^2)\lambda_5)), \\
\dot{\lambda}_4 &= \{\lambda_4, H\} = \frac{1}{(1+x_1^2+x_2^2+y_1^2+y_2^2)^2} (y_1\lambda_1+x_4\lambda_2 - x_1\lambda_3 - x_2\lambda_4 - (1+x_1^2+x_2^2)\lambda_5) \cdot \\
&\quad (-y_1y_2\lambda_1+(1+x_1^2+x_2^2)\lambda_2+y_1^2\lambda_2+y_2(x_1\lambda_3+x_2\lambda_4+(1+x_1^2+x_2^2)\lambda_5)), \\
\dot{\lambda}_5 &= \{z, H\} = 0.
\end{aligned}$$

Here, the z -coordinate is cyclic, and we can restrict this flow to the level set $\lambda_5 = C_0$ and project this restriction onto a 4-dimensional hyperplane (x_1, x_2, y_1, y_2) . The new system is defined on 8-dimensional symplectic manifold, denoted as $\mathcal{M}_{C_0}$, diffeomorphic to the cotangent bundle of the 4-dimensional hyperplane with the coordinates (x_1, x_2, y_1, y_2) . On the manifold $\mathcal{M}_{C_0}$ we introduce the new variables:

$$p_1 = \lambda_1 + C_0 y_1, \quad p_2 = \lambda_2 + C_0 y_2, \quad p_3 = \lambda_3, \quad p_4 = \lambda_4,$$

to express the reduced Hamiltonian function in the form:

$$H_{C_0}(q, p) = \frac{1}{2} \left(p_1^2 + p_2^2 + p_3^2 + p_4^2 - \frac{(p_3 x_1 + p_4 x_2 - p_1 y_1 - p_2 y_2)^2}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2} \right).$$

The Poisson structure on the manifold $\mathcal{M}_{C_0}$ is no longer canonical for $C_0 \neq 0$:

$$\begin{aligned}
\{x_k, p_j\} &= \{y_k, p_{j+2}\} = \delta_{kj}, \quad \{p_k, p_{j+2}\} = \delta_{kj} C_0, \\
\{x_k, y_j\} &= \{x_k, p_{j+2}\} = \{y_k, p_j\} = \{p_j, p_k\} = \{p_{j+2}, p_{k+2}\} = 0,
\end{aligned}$$

for $j, k = 1, 2$. Set $W = \frac{(p_3 x_1 + p_4 x_2 - p_1 y_1 - p_2 y_2)}{1 + x_1^2 + x_2^2 + y_1^2 + y_2^2}$, the corresponding Hamiltonian system is:

$$\begin{aligned}
(3.2) \quad \dot{x}_1 &= \{x_1, H_{C_0}\} = p_1 + y_1 W, & \dot{x}_2 &= \{x_2, H_{C_0}\} = p_2 + y_2 W, \\
\dot{y}_1 &= \{y_1, H_{C_0}\} = p_3 - x_1 W, & \dot{y}_2 &= \{y_2, H_{C_0}\} = p_4 - x_2 W, \\
\dot{p}_1 &= \{p_1, H_{C_0}\} = (C_0 + W)(p_3 - x_1 W), \\
\dot{p}_2 &= \{p_2, H_{C_0}\} = (C_0 + W)(p_4 - x_2 W), \\
\dot{p}_3 &= \{p_3, H_{C_0}\} = -(C_0 + W)(p_1 + y_1 W), \\
\dot{p}_4 &= \{p_4, H_{C_0}\} = -(C_0 + W)(p_2 + y_2 W).
\end{aligned}$$

Change of coordinates to hyper-spherical coordinates $(r, \theta_1, \theta_2, \theta_3)$:

$x_1 = r \cos \theta_1 \cos \theta_2$, $y_1 = r \cos \theta_1 \sin \theta_2$, $x_2 = r \sin \theta_1 \cos \theta_3$, $y_2 = r \sin \theta_1 \sin \theta_3$, with $r \geq 0$, $\theta_1 \in (0, \frac{\pi}{2})$, $\theta_2, \theta_3 \in [0, 2\pi)$, and the transformation of the momenta given by:

$$\begin{aligned}
p_r &= \frac{1}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} (p_1 x_1 + p_2 x_2 + p_3 y_1 + p_4 y_2) \\
p_{\theta_1} &= (p_2 x_2 + p_4 y_2) \sqrt{\frac{x_1^2 + y_1^2}{x_2^2 + y_2^2}} - (p_1 x_1 + p_3 y_1) \sqrt{\frac{x_2^2 + y_2^2}{x_1^2 + y_1^2}} \\
p_{\theta_2} &= p_3 x_1 - p_1 y_1 \\
p_{\theta_3} &= p_4 x_2 - p_2 y_2,
\end{aligned}$$

leads to a Hamiltonian function for the reduced system:

$$(3.3) \quad \begin{aligned} \tilde{H}_{C_0}(r, \theta_1, \theta_2, \theta_3, p_r, p_{\theta_1}, p_{\theta_2}, p_{\theta_3}) \\ = \frac{1}{2} \left[p_r^2 + \frac{1}{r^2} p_{\theta_1}^2 + \frac{1}{r^2} \left(\frac{p_{\theta_2}^2}{\cos^2 \theta_1} + \frac{p_{\theta_3}^2}{\sin^2 \theta_1} \right) - \frac{1}{1+r^2} (p_{\theta_2} + p_{\theta_3})^2 \right]. \end{aligned}$$

The Poisson structure on the manifold $\mathcal{M}_{C_0}$ in the hyper-spherical coordinates is:

$$\begin{aligned} \{r, p_r\} &= 1, \quad \{r, p_{\theta_k}\} = \{\theta_j, p_r\} = 0, \quad \{\theta_j, p_{\theta_k}\} = \delta_{jk}, \quad j, k = 1, 2, 3, \\ \{p_r, p_{\theta_1}\} &= 0, \quad \{p_r, p_{\theta_2}\} = rC_0 \cos^2 \theta_1, \quad \{p_r, p_{\theta_3}\} = rC_0 \sin^2 \theta_1, \\ \{p_{\theta_1}, p_{\theta_2}\} &= -\{p_{\theta_1}, p_{\theta_3}\} = -r^2 C_0 \sin \theta_1 \cos \theta_1, \quad \{p_{\theta_2}, p_{\theta_3}\} = 0. \end{aligned}$$

The above structure is non canonical for $C_0 \neq 0$ and corresponds to the Hamiltonian system, which describes the motion of a charged particle in a constant magnetic field.

The Hamilton equations are given by

$$\begin{aligned} \dot{r} &= \{r, \tilde{H}_{C_0}\} = p_r, \quad \dot{\theta}_1 = \{\theta_1, \tilde{H}_{C_0}\} = \frac{p_{\theta_1}}{r^2}, \\ \dot{\theta}_2 &= \{\theta_2, \tilde{H}_{C_0}\} = \frac{p_{\theta_2}}{r^2 \cos^2 \theta_1} - \frac{p_{\theta_2} + p_{\theta_3}}{1+r^2}, \\ \dot{\theta}_3 &= \{\theta_3, \tilde{H}_{C_0}\} = \frac{p_{\theta_3}}{r^2 \sin^2 \theta_1} - \frac{p_{\theta_2} + p_{\theta_3}}{1+r^2}, \\ (3.4) \quad \dot{p}_r &= \{p_r, \tilde{H}_{C_0}\} = \frac{1}{r^3} p_{\theta_1}^2 + \frac{p_{\theta_2} + p_{\theta_3}}{r(1+r^2)^2} ((1+r^2)C_0 - r^2(p_{\theta_2} + p_{\theta_3})) + \frac{1}{r^3} \left(\frac{p_{\theta_2}}{\cos^2 \theta_1} + \frac{p_{\theta_3}}{\sin^2 \theta_1} \right), \\ \dot{p}_{\theta_1} &= \{p_{\theta_1}, \tilde{H}_{C_0}\} = \frac{1}{r^2} \left(p_{\theta_3} \left(r^2 C_0 + \frac{p_{\theta_3}}{\sin^2 \theta_1} \right) \cot \theta_1 - p_{\theta_2} \left(r^2 C_0 + \frac{p_{\theta_2}}{\cos^2 \theta_1} \right) \tan \theta_1 \right), \\ \dot{p}_{\theta_2} &= \{p_{\theta_2}, \tilde{H}_{C_0}\} = -C_0 \cos \theta_1 (r p_r \cos \theta_1 - p_{\theta_1} \sin \theta_1), \\ \dot{p}_{\theta_3} &= \{p_{\theta_3}, \tilde{H}_{C_0}\} = -C_0 \sin \theta_1 (r p_r \sin \theta_1 + p_{\theta_1} \cos \theta_1). \end{aligned}$$

The flow has four first integrals:

$$(3.5) \quad \begin{aligned} \tilde{I}_1 &= p_{\theta_1}^2 + p_{\theta_2}^2 \cos^{-2} \theta_1 + p_{\theta_3}^2 \sin^{-2} \theta_1 - (p_{\theta_2} + p_{\theta_3})^2, \\ \tilde{I}_2 &= p_{\theta_2} + C_0 \frac{r^2}{2} \cos^2 \theta_1, \quad \tilde{I}_3 = p_{\theta_3} + C_0 \frac{r^2}{2} \sin^2 \theta_1, \quad \tilde{I}_4 = \tilde{H}_{C_0}. \end{aligned}$$

These integrals are involutive and functionally independent almost everywhere, which implies the integrability of the reduced system in the classical sense. In the coordinates $(x, y, p) = (x_1, x_2, y_1, y_2, p_1, p_2, p_3, p_4)$ on the manifold $\mathcal{M}_{C_0}$, the first integrals (3.5) take the form:

$$(3.6) \quad \begin{aligned} I_1 &= (y_2 p_1 - x_1 p_4 + x_2 p_3 - y_1 p_2)^2 + (x_1 p_2 - x_2 p_1 - y_1 p_4 + y_2 p_3)^2, \\ I_2 &= p_3 x_1 - p_1 y_1 + \frac{C_0}{2} (x_1^2 + y_1^2), \quad I_3 = p_4 x_2 - p_2 y_2 + \frac{C_0}{2} (x_2^2 + y_2^2), \\ I_4 &= H_{C_0}. \end{aligned}$$

The main objective of this paper is to analyse the solutions of the reduced Hamiltonian system (3.4).

REMARK 3.1. It should be noted that $r = 0$ is merely coordinate singularity of the reduced Hamiltonian system in the hyper-spherical coordinates. Indeed, from

the first integrals (3.6) in the coordinates (x, y, p) it follows that the solutions of the reduced Hamiltonian system through $(0, 0, 0, 0)$ are characterized by $I_1 = I_2 = I_3 = 0$.

REMARK 3.2. Solutions of the reduced Hamiltonian system (3.4) naturally lift to solutions of the Hamiltonian system (3.1), thus yielding sub-Riemannian geodesics in H_5 . The coordinate $z(t)$ is reconstructed by integrating the corresponding equation in (3.1)

$$(3.7) \quad \dot{z} = \dot{x}_1 y_1 + \dot{x}_2 y_2.$$

Geometrical meaning to the evolution of $z(t)$ along sub-Riemannian geodesic line $\gamma: I \rightarrow H_5$ starting from $P_0 = \gamma(0) = (x_1(0), x_2(0), y_1(0), y_2(0), z(0))$ can be given together with corresponding formula. First, for $i \in \{1, 2\}$, let $\pi_i: H_5 \rightarrow \mathbb{R}^2$ be projection to subspace (x_i, y_i) and let $(x_i(0), y_i(0)) = \pi_i(P_0)$ be projected starting point. Let $A_i(t)$ be the signed sectorial area swept out by position vector from $\pi_i(\gamma(0))$ to $\pi_i(\gamma(t))$, i.e.

$$(3.8) \quad A_i(t) = \frac{1}{2} \int_0^t (x_i(s) \dot{y}_i(s) - \dot{x}_i(s) y_i(s)) ds.$$

From identity

$$\frac{d}{dt}(x_i y_i) = \dot{x}_i y_i + x_i \dot{y}_i$$

we have

$$(3.9) \quad x_i(t) y_i(t) - x_i(0) y_i(0) = \int_0^t (\dot{x}_i(s) y_i(s) + x_i(s) \dot{y}_i(s)) ds.$$

From (3.8) and (3.9) it follows that

$$\int_0^t \dot{x}_i(s) y_i(s) ds = \frac{1}{2} (x_i(t) y_i(t) - x_i(0) y_i(0)) - A_i(t).$$

This yields formulae for $z(t)$ of the form:

$$z(t) = z(0) + \frac{1}{2} (x_1(t) y_1(t) + x_2(t) y_2(t) - x_1(0) y_1(0) - x_2(0) y_2(0)) - A_1(t) - A_2(t).$$

We illustrate the behaviour of the solutions through several numerical examples presented at the end of the paper.

4. Classification of geodesics

We are going to describe solutions that belong to the 4-dimensional invariant manifold. Since $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ and $\tilde{I}_4$, defined in (3.5), are first integrals, there are constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$ such that

$$\tilde{I}_1 = C_1, \quad \tilde{I}_2 = C_2, \quad \tilde{I}_3 = C_3, \quad \tilde{I}_4 = C_4.$$

The values of the first integrals determine the initial points from which the corresponding geodesics start; in particular, for the geodesic through the origin all values C_1, C_2, C_3 vanish. In the analysis that follows, we study the first-integral

constants in their generic form, thereby encompassing all possible choices of their values, including the special case in which they all vanish.

With (3.5) we derive

$$(4.1) \quad \begin{aligned} p_{\theta_2} + p_{\theta_3} &= C_2 + C_3 - C_0 \frac{r^2}{2}, \\ \frac{C_1}{r^2} &= \frac{1}{r^2} (p_{\theta_1}^2 + p_{\theta_2}^2 \cos^{-2} \theta_1 + p_{\theta_3}^2 \sin^{-2} \theta_1 - (p_{\theta_2} + p_{\theta_3})^2), \end{aligned}$$

By further using the conservation laws and the expressions (3.3), (3.5) and (4.1), we can reduce the problem to a problem with one degree of freedom for the variable r :

$$(4.2) \quad 2C_4 = \dot{r}^2 + \frac{1}{r^2} C_1 + \frac{1}{r^2(1+r^2)} (C_2 + C_3 - C_0 \frac{r^2}{2})^2.$$

For $C_4 \neq 0$, equation (4.2) in a more compact form is:

$$\dot{r}^2 = 2C_4 \frac{f(r)}{r^2(1+r^2)},$$

where $f(r)$ is a polynomial of degree 4, in the general case:

$$(4.3) \quad \begin{aligned} f(r) &= Ar^4 + Br^2 + C, \\ A &= 1 - \frac{C_0^2}{8C_4}, \quad B = 1 - \frac{C_1}{2C_4} + \frac{(C_2 + C_3)C_0}{2C_4}, \quad C = -\frac{C_1 + (C_2 + C_3)^2}{2C_4}. \end{aligned}$$

We can restrict the discussion to the case where the solutions are parametrised by the arc length parameter, which is equivalent to $C_4 = \frac{1}{2}$ according to the Cauchy-Schwarz integral inequality between length and energy function, equivalent to $C_4 = \frac{1}{2}$. This simplifies the problem to:

$$(4.4) \quad \begin{aligned} \dot{r}^2 &= \frac{f(r)}{r^2(1+r^2)}, \quad \text{i.e.} \quad \dot{r} = \pm \frac{\sqrt{f(r)}}{r\sqrt{1+r^2}}, \\ A &= 1 - \frac{C_0^2}{4}, \quad B = 1 - C_1 + (C_2 + C_3)C_0, \quad C = -C_1 - (C_2 + C_3)^2. \end{aligned}$$

With (3.5) and the Hamilton's equations we obtain:

$$(4.5) \quad \begin{aligned} C_1 &= r^4 \dot{\theta}_1^2 + \frac{C_2^2}{\cos^2 \theta_1} + \frac{C_3^2}{\sin^2 \theta_1} - (C_2 + C_3)^2, \\ \dot{\theta}_1 &= \pm \frac{1}{r^2} \sqrt{C_1 + (C_2 + C_3)^2 - \frac{C_2^2}{\cos^2 \theta_1} - \frac{C_3^2}{\sin^2 \theta_1}}. \end{aligned}$$

The conservation laws lead to the following equations for the remaining coordinates:

$$(4.6) \quad \begin{aligned} \dot{\theta}_2 &= \frac{C_2}{r^2 \cos^2 \theta_1} - \frac{C_2 + C_3}{1 + r^2} - \frac{C_0}{2(1 + r^2)}, \\ \dot{\theta}_3 &= \frac{C_3}{r^2 \sin^2 \theta_1} - \frac{C_2 + C_3}{1 + r^2} - \frac{C_0}{2(1 + r^2)}. \end{aligned}$$

PROPOSITION 4.1. *The magnetic geodesic flow (3.4) is solvable by quadratures.*

PROOF. The integration of the entire problem can be carried out in four steps. First, integrate inverted (4.4) to implicitly obtain $r = r(t)$:

$$(4.7) \quad \dot{t} = \pm \frac{r\sqrt{r^2+1}}{\sqrt{f(r)}},$$

$$t + t_0 = \pm \int \frac{r\sqrt{r^2+1}}{\sqrt{f(r)}} dr, \quad t_0 \in \mathbb{R}.$$

After the initial choice of sign, the expression on the right-hand side of (4.7) has a constant sign, which means that $t(r)$ is strictly monotonic and, therefore, allows inversion.

The chain rule $\frac{d\theta_1}{dt} = \frac{d\theta_1}{dr} \frac{dr}{dt}$ allows the integration (4.5) to implicitly define $\theta_1(r)$. It should be noted that we can obtain $\theta_1(r)$ without knowing $r(t)$, together with the previous step $\theta_1(t) = \theta_1(r(t))$:

$$\int \frac{d\theta_1}{\sqrt{C_1 + (C_2 + C_3)^2 - \frac{C_2^2}{\cos^2 \theta_1} - \frac{C_3^2}{\sin^2 \theta_1}}} = \pm \int \frac{\sqrt{1+r^2}}{r\sqrt{f(r)}} dr.$$

Using the chain rules $\frac{d\theta_2}{dt} = \frac{d\theta_2}{dr} \frac{dr}{dt}$ and $\frac{d\theta_3}{dt} = \frac{d\theta_3}{dr} \frac{dr}{dt}$, the equations (4.6) and the results of the previous steps, the remaining coordinate functions are recovered by :

$$\theta_2 + \theta_2^0 = \int \frac{r\sqrt{r^2+1}}{\sqrt{f(r)}} \left(\frac{C_2}{r^2 \cos^2 \theta_1(r)} - \frac{C_2 + C_3 + \frac{1}{2}C_0}{1+r^2} \right) dr, \quad \theta_2^0 \in \mathbb{R};$$

$$\theta_3 + \theta_3^0 = \int \frac{r\sqrt{r^2+1}}{\sqrt{f(r)}} \left(\frac{C_3}{r^2 \sin^2 \theta_1(r)} - \frac{C_2 + C_3 + \frac{1}{2}C_0}{1+r^2} \right) dr, \quad \theta_3^0 \in \mathbb{R}.$$

Note that the second step is necessary for the last two and that we can obtain $\theta_i = \theta_i(r)$, $i \in \{2, 3\}$ without integrating the first step, while to define $\theta_1 = \theta_1(t)$ the integration of (4.4) is necessary. $\square$

The following statements will be used later.

LEMMA 4.1. $C_1 = p_{\theta_1}^2 + p_{\theta_2}^2 \cos^{-2} \theta_1 + p_{\theta_3}^2 \sin^{-2} \theta_1 - (p_{\theta_2} + p_{\theta_3})^2 \geq 0$.

PROOF. Statement of lemma follows from the inequality

$$(4.8) \quad \frac{x^2}{\cos^2 \theta} + \frac{y^2}{\sin^2 \theta} \geq (x + y)^2$$

applied to $x = p_{\theta_2}, y = p_{\theta_3}$ and $\theta = \theta_1$. To prove inequality (4.8) we first notice that for x and y with different signs, inequality is trivial. For $\text{sgn}(x) = \text{sgn}(y)$, we define $f: (0, \frac{\pi}{2}) \rightarrow \mathbb{R}, f(\theta) = \frac{x^2}{\cos^2 \theta} + \frac{y^2}{\sin^2 \theta}$ and find the minima for f . Since $f'(\theta) = 2 \sin \theta \cos \theta (\frac{x^2}{\cos^4 \theta} - \frac{y^2}{\sin^4 \theta})$, the minimum at $\theta_{\min}$ is such that $\tan^4 \theta_{\min} = \frac{y^2}{x^2}$. Now (4.8) is equivalent to

$$f(\theta_{\min}) \geq (x + y)^2 \Leftrightarrow x^2 \tan^2(\theta_{\min}) + y^2 \cotan^2(\theta_{\min}) \geq 2xy \Leftrightarrow x^2 \frac{y}{x} + y^2 \frac{x}{y} \geq 2xy. \quad \square$$

COROLLARY 4.1. The parameter C is non-positive, that is, $C = -C_1 - (C_2 + C_3)^2 \leq 0$. Furthermore, $C = 0$ implies $B = 1$.

REMARK 4.1. We note here that it is not possible for $f(r)$ to be a zero polynomial, which means that for $C_4 \neq 0$ $r(t)$ is never a constant function. Moreover, $A \leq 1$.

Let us notice that if $C_4 = 0$ in (4.2), motion degenerates to a point. This assertion follows from the following: Using Lemma 4.1 with $0 = C_4 = \tilde{H}_C$ implies that all the moments $p_r, p_{\theta_1}, p_{\theta_2}$ and p_{θ_3} are equal to zero. Now, the only solution to (3.4) are constants. In the sequel we assume $C_4 \neq 0$.

We fix values of the first integrals by: $\tilde{I}_1 = C_1, \tilde{I}_2 = C_2, \tilde{I}_3 = C_3, \tilde{I}_4 = C_4$ and formulate our main result as follows.

THEOREM 4.1. *The reduced Hamiltonian system (3.4) has trajectories of two types:*

- (i) *Trajectories of the first type satisfy $r(t) \geq r_0$ for all t , where r_0 is the largest non-negative root of (4.3).*
- (ii) *Trajectories of the second type satisfy $r_1 \leq r(t) \leq r_2$ for all t , where $r_1 < r_2$ are the positive roots of (4.3).*

REMARK 4.2. From (4.4) it can be observed that motion is possible only in regions where $f(r) > 0$ (see Figure 1 and Figure 3). This implies a strong dependence between the constants of motion $C_k, k \in \{1, 2, 3, 4\}$, and the set of admissible points. More precisely, given constants C_k , the motion is confined to certain regions and, consequently, not all initial conditions are admissible; conversely, the choice of initial conditions restricts the possible values of the constants of motion. For example, for geodesics passing through the origin holds $C_1 = C_2 = C_3 = 0$.

Theorem 4.1 provides a lower bound for the radial distance along a trajectory; however, this bound depends on the constants of motion and, in general, may be arbitrarily small.

From the Remark 3.2 it follows that the classification in Theorem 4.1 extends naturally to the classification of the original Hamiltonian system (3.1) in H_5 .

COROLLARY 4.2. *Every geodesic of the reduced system (3.2) lifts to a geodesic of the system (3.1) via the reconstruction equation (3.7).*

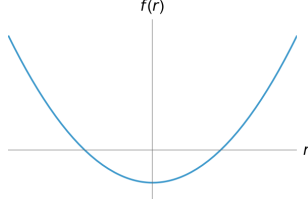
In the following section, we prove Theorem 4.1.

4.1. Case: $A = 0$. In this case, $C_0 = 2$ and $f(r)$ simplify to:

$$f(r) = Br^2 + C.$$

Since $f(r) \geq 0$, only $B > 0$ is possible in this case (shown in Figure 1) and motion is only possible in the region $r > r_0 = \sqrt{-\frac{C}{B}}$. Now, we continue with (4.3) to implicitly find $r(t)$:

$$\begin{aligned} \dot{r} &= \pm \frac{\sqrt{Br^2 + C}}{r\sqrt{1 + r^2}} \\ t &= \pm \frac{1}{\sqrt{B}} \int \frac{r\sqrt{1 + r^2} dr}{\sqrt{r^2 + C/B}}. \end{aligned}$$

FIGURE 1. $B > 0$, $C \leq 0$.

With $a = -\frac{C}{B} \geq 0$, the substitutions $1+r^2 = z^2$ and $p = \frac{z}{\sqrt{a+1}}$ we solve the integral on the right-hand side as follows:

$$\begin{aligned}
 \pm \frac{1}{\sqrt{B}} \int \frac{\sqrt{1+r^2} r dr}{\sqrt{r^2 + C/B}} &= \pm \frac{1}{\sqrt{B}} \int \frac{z^2 dz}{\sqrt{z^2 - (a+1)}} \\
 &= \pm \frac{1}{\sqrt{B}} \left(\int \sqrt{z^2 - (a+1)} dz + (a+1) \int \frac{dz}{\sqrt{z^2 - (a+1)}} \right) \\
 &= \pm \frac{a+1}{\sqrt{B}} \left(\int \sqrt{p^2 - 1} dp + \int \frac{dp}{\sqrt{p^2 - 1}} \right) \\
 &= \pm \frac{a+1}{\sqrt{B}} \left(\frac{1}{2} p \sqrt{p^2 - 1} + \frac{1}{2} \ln(p + \sqrt{p^2 - 1}) \right) \\
 &= \pm \frac{1+a}{2\sqrt{B}} \left(\sqrt{\frac{1+r^2}{1+a}} \sqrt{\frac{r^2-a}{1+a}} + \ln \frac{\sqrt{1+r^2} + \sqrt{r^2-a}}{\sqrt{1+a}} \right).
 \end{aligned}$$

Finally:

$$t = t_0 \pm \frac{1}{2\sqrt{B}} \left(\sqrt{(1+r^2)(r^2-a)} + (1+a) \ln \left(\frac{\sqrt{1+r^2} + \sqrt{r^2-a}}{\sqrt{1+a}} \right) \right).$$

EXAMPLE 4.1. Let us consider an example where $A = 0$, $B = -C$ and plot $r(t)$ and $\theta_1(t)$ (see Figure 2).

4.2. Case: $A \neq 0$. For the case $A \neq 0$ we have two sub-cases in which many situations are eliminated thanks to Lemma 4.1. The cases of interest, which will be explained later, are shown in Figure 3.

4.2.1. *Case:* $A > 0$. If we look at the discriminant $D = B^2 - 4AC$ of the quadratic equation $Ax^2 + Bx + C$, we see that it is always non-negative, since $A > 0$ and $C \leq 0$. This means that we can write $f(r) = A(r^2 - \bar{\alpha})(r^2 - \bar{\beta})$, with $C = \frac{\bar{\alpha}\bar{\beta}}{A} \leq 0$, and that we have exactly two real roots $\bar{\alpha} \geq 0$ and $\bar{\beta} \leq 0$. The function $f(r) = A(r + \alpha)(r - \alpha)(r^2 + \beta^2)$ is non-negative for $r \geq \alpha$. We distinguish between two cases: $C = 0$ and $C < 0$.

In the first case, the equation (4.4) reduces to:

$$(4.9) \quad \dot{t}^2 = \frac{1+r^2}{Ar^2+1}.$$

Note that if $C_0 = 0$, i.e. $A = 1$ we have simple form for $t(r)$, namely $t(r) = t_0 \pm r$.

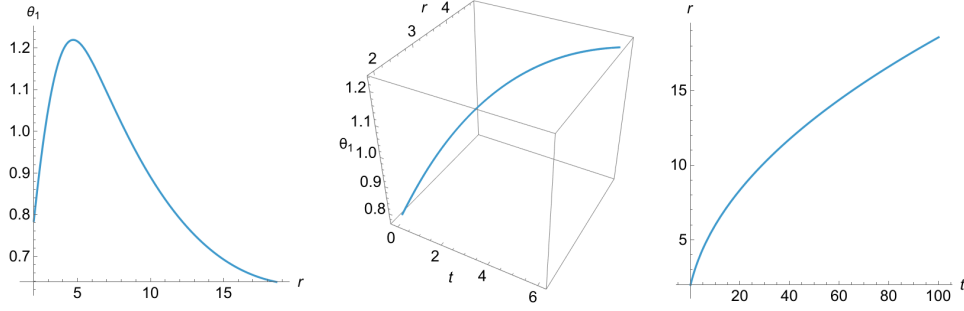


FIGURE 2. Plots of $(r(t), \theta_1(t))$ (left), $(t, r(t), \theta_1(t))$ (centre) and $(t, r(t))$ (right) for $f(r) = \frac{9}{2}r^2 - \frac{9}{2}$.

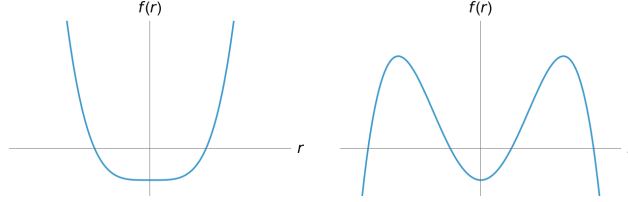


FIGURE 3. Case $A > 0$ (on the left) and Case $A < 0$ (on the right).

In general case, $C_0 \neq 0$, we integrate (4.9) as

$$t = t_0 \pm \int \frac{1 + r^2}{\sqrt{(1 + r^2)(Ar^2 + 1)}} dr,$$

which solves to

$$t = t_0 \pm F(\phi, k) \pm \frac{1}{A}(\text{dn } u \text{ sc } u - E(u)), \quad \phi = \text{am } u, \text{sc } u = r, k = \sqrt{1 - A^2}.$$

In this case, θ_1 is constant function, and θ_2 and θ_3 are linear functions as stated in the Corollary 4.4.

In the second case, if $C < 0$, we solve the equation:

$$(4.10) \quad \dot{t}^2 = \frac{r^2(1 + r^2)}{A(r + \alpha)(r - \alpha)(r^2 + \beta^2)}.$$

First, we must separately consider the case where $\beta = 1$. The solution of (4.10) is:

$$t(r) = t_0 \pm \sqrt{\frac{r^2 - \alpha^2}{A}}.$$

If $\beta \neq 1$ substitution $1 + r^2 = z^2$ reduces the problem to an elliptic integral of the second kind.

$$t = \pm \frac{1}{\sqrt{A}} \int \frac{r\sqrt{1 + r^2} dr}{\sqrt{(r + \alpha)(r - \alpha)(r^2 + \beta^2)}} = \pm \frac{1}{\sqrt{A}} \int \frac{z^2 dz}{\sqrt{(z^2 - (\alpha^2 + 1))(\beta^2 - 1 + z^2)}}.$$

We distinguish between cases where $\beta^2 > 1$ and $\beta^2 < 1$.

In the case of $\beta^2 < 1$ integral is further solved with substitution $\frac{z}{\sqrt{\alpha^2+1}} = \text{ns } u$ (see [3, p. 192]) as:

$$\begin{aligned} t &= \pm \frac{1}{\sqrt{A}} \int \frac{z^2 dz}{\sqrt{(z^2 - (\alpha^2 + 1))(\beta^2 - 1 + z^2)}} \\ &= \pm \frac{1}{\sqrt{A}} \int \frac{\sqrt{\alpha^2 + 1} \left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 d\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)}{\sqrt{\left(\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 - 1\right)\left(\frac{\beta^2 - 1}{\alpha^2 + 1} + \left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2\right)}} \\ &= \pm \frac{\sqrt{\alpha^2 + 1}}{\sqrt{A}} \int \text{ns}^2 u du + t_0 = \pm \frac{\sqrt{\alpha^2 + 1}}{\sqrt{A}} (u - E(u) - \text{dn } u \text{ cs } u) + t_0, \end{aligned}$$

where $\text{ns } u = \sqrt{\frac{r^2+1}{\alpha^2+1}}$, $t_0 \in \mathbb{R}$.

In the case of $\beta^2 > 1$, the integration is performed with the substitution $\frac{z}{\sqrt{\alpha^2+1}} = \text{nc } u$ (see [3, p. 193]) as:

$$\begin{aligned} t &= \pm \frac{1}{\sqrt{A}} \int \frac{z^2 dz}{\sqrt{(z^2 - (\alpha^2 + 1))(\beta^2 - 1 + z^2)}} \\ &= \pm \frac{1}{\sqrt{A(\alpha^2 + \beta^2)}} \int \frac{(\alpha^2 + 1) \left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 d\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)}{\sqrt{\left(\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 - 1\right)\left(\frac{\beta^2 - 1}{\alpha^2 + \beta^2} + \frac{\alpha^2 + 1}{\alpha^2 + \beta^2} \left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2\right)}} \\ &= t_0 \pm \frac{\alpha^2 + 1}{\sqrt{A(\alpha^2 + \beta^2)}} \int \text{nc}^2 u du = t_0 \pm \frac{\alpha^2 + 1}{k'^2 \sqrt{A(\alpha^2 + \beta^2)}} (k'^2 u - E(u) + \text{dn } u \text{ sc } u), \end{aligned}$$

where $\text{nc } u = \sqrt{\frac{r^2+1}{\alpha^2+1}}$, $k'^2 = \frac{\alpha^2+1}{\alpha^2+\beta^2}$, $t_0 \in \mathbb{R}$.

Finally, it is worth investigating the asymptotic behavior of the trajectories of the first type when $r \rightarrow +\infty$. From (4.5) and (4.6) it can be seen that we have a very simple form of the functions θ_i , $i \in \{1, 2, 3\}$.

COROLLARY 4.3. *If $A \geq 0$, when $r \rightarrow +\infty$, θ_1 tend to a constant function and θ_2 and θ_3 tend to linear functions in t .*

4.2.2. Case: $A < 0$. Here, too, we have two different cases.

In the first, similar to the previous consideration, we find that the function $f(r)$ has four real roots: α , $-\alpha$, β and $-\beta$. Without loss of generality, we can assume that $\alpha < \beta$. The function $f(r)$ has the form $f(r) = A(r - \alpha)(r + \alpha)(r - \beta)(r + \beta)$ and is non-negative in the interval $[\alpha, \beta]$. We solve the equation:

$$t^2 = \frac{r^2(1 + r^2)}{A(r + \alpha)(r - \alpha)(r + \beta)(r - \beta)}, \quad r \in (\alpha, \beta).$$

Substituting $1 + r^2 = z^2$ reduces the problem to an elliptic integral of the second kind, and with subsequent substitution $z = \sqrt{\alpha^2 + 1} \text{nd } u$ the solution is obtained

by the elliptic function E :

$$\begin{aligned}
 t &= \pm \frac{1}{\sqrt{|A|}} \int \frac{r\sqrt{1+r^2}dr}{\sqrt{(r+\alpha)(r-\alpha)(\beta+r)(\beta-r)}} \\
 &= \pm \frac{1}{\sqrt{|A|}} \int \frac{z^2 dz}{\sqrt{(z^2 - (\alpha^2 + 1))(\beta^2 + 1 - z^2)}} \\
 &= \pm \frac{\alpha^2 + 1}{\sqrt{|A|(\beta^2 + 1)}} \int \frac{\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 d\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)}{\sqrt{\left(\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2 - 1\right)\left(1 - \frac{\alpha^2 + 1}{\beta^2 + 1}\left(\frac{z}{\sqrt{\alpha^2 + 1}}\right)^2\right)}} \\
 &= \pm \frac{\alpha^2 + 1}{\sqrt{|A|(\beta^2 + 1)}} \int \text{nd}^2 u du = \pm \frac{\alpha^2 + 1}{k'^2 \sqrt{|A|(\beta^2 + 1)}} (E(u) - k^2 \text{sn } u \text{cd } u) + c_0,
 \end{aligned}$$

where $\text{nd } u = \sqrt{\frac{r^2 + 1}{\alpha^2 + 1}}$, $k^2 = \frac{\alpha^2 + 1}{\beta^2 + 1}$, $c_0 \in \mathbb{R}$.

The second case is that there are no real roots for the function $f(r)$. In this case, however, the function is negative in its domain, so we have to discard it.

This concludes the consideration and proves the Theorem 4.1. It also proves the following theorem.

THEOREM 4.2. *The first step for quadrature integration, i.e. the formulas for $r = r(t)$, are defined locally (implicitly or by inverting elliptic integrals) as follows:*

a) If $A = 0$:

$$t = t_0 \pm \frac{1+a}{\sqrt{B}} \left(\frac{1}{2} \sqrt{\frac{1+r^2}{1+a}} \sqrt{\frac{r^2-a}{1+a}} + \frac{1}{2} \ln \left(\sqrt{\frac{1+r^2}{1+a}} + \sqrt{\frac{r^2-a}{1+a}} \right) \right), \quad a = -\frac{C}{B}.$$

b) If $0 < A \leq 1, C = 0$:

$$t = t_0 \pm F(\phi, k) \pm \frac{1}{A} (\text{dn } u \text{sc } u - E(u)), \quad \phi = \text{am } u, \text{sc } u = r, k = \sqrt{1 - A^2}.$$

c) If $0 < A \leq 1, C < 0$ and $\pm i$ are roots of $f(r)$:

$$t(r) = t_0 \pm \sqrt{\frac{r^2 - \alpha^2}{A}}.$$

d) If $0 < A \leq 1, C < 0$ and $\beta^2 < 1$, where β is complex, and $\pm \alpha$ are real roots of $f(r)$:

$$t = t_0 \pm \frac{\sqrt{\alpha^2 + 1}}{\sqrt{A}} (u - E(u) - \text{dn } u \text{cs } u).$$

e) If $0 < A \leq 1, C < 0$ and $\beta^2 > 1$, where β is complex root, and $\pm \alpha$ are real roots of $f(r)$:

$$t = t_0 \pm \frac{\alpha^2 + 1}{k'^2 \sqrt{A(\alpha^2 + \beta^2)}} (k'^2 u - E(u) + \text{dn } u \text{sc } u), \text{nc } u = \sqrt{\frac{r^2 + 1}{\alpha^2 + 1}}, k'^2 = \frac{\alpha^2 + 1}{\alpha^2 + \beta^2}.$$

f) If $A < 0$ and $\pm \alpha, \pm \beta$ are real roots of $f(r)$:

$$t = t_0 \pm \frac{\alpha^2 + 1}{k'^2 \sqrt{|A|(\beta^2 + 1)}} (E(u) - k^2 \text{sn } u \text{cd } u), \text{nd } u = \sqrt{\frac{r^2 + 1}{\alpha^2 + 1}}, k^2 = \frac{\alpha^2 + 1}{\beta^2 + 1}.$$

Let us consider the case where $C = 0$, which contains geodesics passing through coordinate origin. We have a similar situation as in Corollary 4.3.

COROLLARY 4.4. *If $C = 0$, θ_1 is a constant function and θ_2 and θ_3 are linear functions in t .*

We conclude the paper with numerical examples of the sub-Riemannian geodesic curves of the original system on H_5 .

EXAMPLE 4.2. In this example, numerically obtained sub-Riemannian geodesic curves passing through coordinate origin are visualized for both types of trajectories from Theorem 4.1 (see Figure 4 and Figure 5). Corresponding function $f(r)$ from (4.3) is of the form $f(r) = -r^4 + r^2$ for geodesic in Figure 4 and of the form $f(r) = \frac{1}{2}r^4 + r^2$ for geodesic in Figure 5. The left and right panels of figures show the dependence of the z -coordinate and the radius $r = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}$ on t . The centre panels show the projection of the sub-Riemannian geodesic onto the (x_1, y_1, z) subspace.

In the previous example with a geodesic line through origin it was not apparent that projection of a geodesic line on the (x_1, x_2, y_1, y_2) subspace is bounded both below and above. The following example demonstrates that property.

EXAMPLE 4.3. Here, we visualize geodesic line through point $(0, -\sqrt{2}, -\sqrt{2}, 0, 0)$ on level set $C_0 = 2\sqrt{2}$ and with values of the first integrals as: $C_1 = 1, C_2 = \sqrt{2}, C_3 = \sqrt{2}$ and $C_4 = \frac{1}{2}$, resulting in $f(r) = -r^4 + 8r^2 - 9$. This example illustrates trajectories of type (ii) from Theorem 4.1 and they satisfy $4 - \sqrt{7} \leq r^2(t) \leq 4 + \sqrt{7}$ (see Figure 6). To visualize objects in five-dimensional real space, we interpret the coordinate pairs (x_1, y_1) and (x_2, y_2) as complex numbers z_1 and z_2 , respectively,

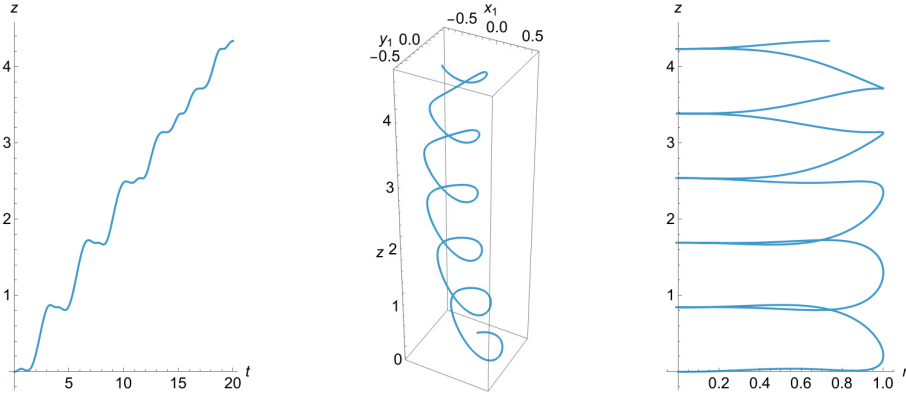


FIGURE 4. Plots of $(t, z(t))$ (left), $(r(t), z(t))$ (right) and projection onto the (x_1, y_1, z) subspace (centre) of a sub-Riemannian geodesic of type (ii) passing through origin.

and represent their moduli along the first two coordinates of the plot, while retaining the fifth coordinate as the third axis. The resulting three-dimensional plot, therefore, has coordinates $(\|z_1\|, \|z_2\|, z)$.

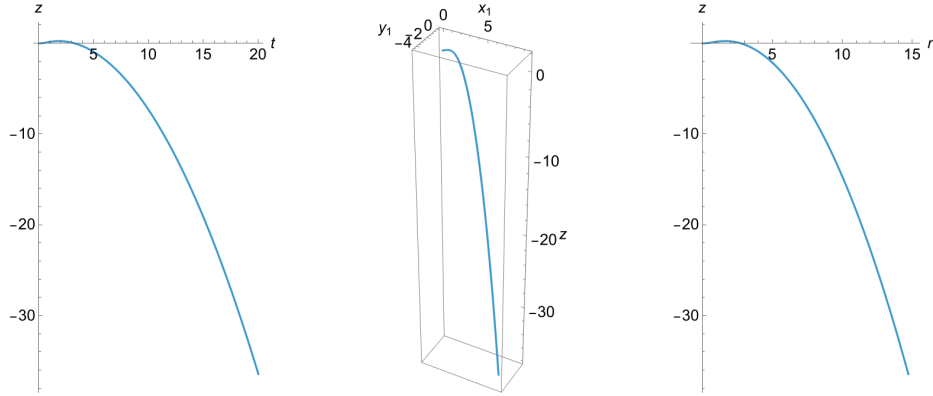


FIGURE 5. Plots of $(t, z(t))$ (left), $(r(t), z(t))$ (right) and projection onto the (x_1, y_1, z) subspace (centre) of sub-Riemannian geodesic of type (i) passing through origin.

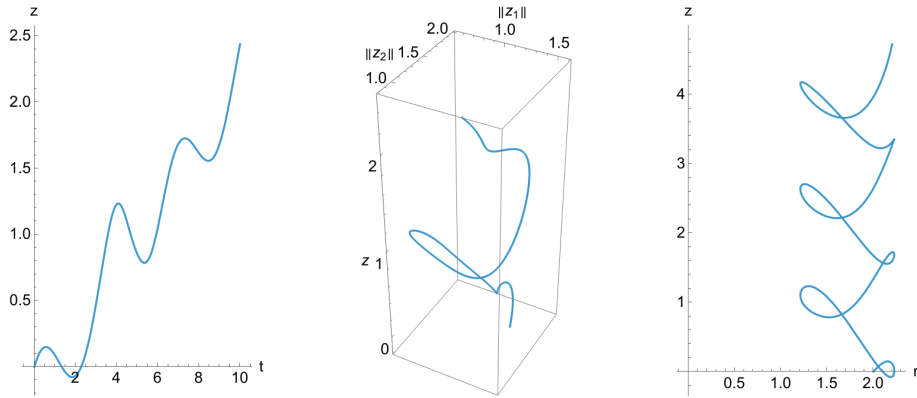


FIGURE 6. Plots of $(t, z(t))$ (left), $(r(t), z(t))$ (right) and $(\|z_1(t)\|, \|z_2(t)\|, z(t))$ (centre) of a sub-Riemannian geodesic passing through point $(0, -\sqrt{2}, -\sqrt{2}, 0, 0)$.

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