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FIRST STEPS TOWARDS THE AVERAGING WITH RESPECT TO A PART OF THE COORDINATES

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Abstract. The problem of averaging on an infinite time interval is considered. The classical results on averaging proved by N.N. Bogoluybov are generalized to the case in which only a part of the coordinates in the phase space remains close to the equilibrium position of the averaged system. We call this the averaging with respect to a part of the coordinates. The results are based on some topological ideas combined with the standard theorem on averaging on a finite time interval.

1. Introduction

The classical averaging theory consists of two main results proved by N. N. Bogolyubov [1]: the theorem on averaging over a finite time interval and the theorem on averaging over an infinite time interval. Let us briefly recall the main results of these theorems.

Let us have a system of ordinary differential equations

$$\dot{x} = v(x) + w(x, kt), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $k \in \mathbb{R}$, $k > 0$; $v\colon \mathbb{R}^n \to \mathbb{R}^n$ and $w\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are smooth functions, the function w is $(2\pi/k)$-periodic in t.

We additionally assume that the time average of the function w equals zero

$$\frac{1}{2\pi}\int_0^{2\pi} w(x,t)\,dt = 0.$$

If k is a positive large number, then the function w is rapidly oscillating and its average value equals zero. In the papers of N.N. Bogolyubov the above system is presented in the so-called standard form

$$\frac{dx}{d\tau} = x' = \varepsilon(v(x) + w(x,\tau)),$$

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where $\varepsilon = 1/k$ is the small parameter and $\tau = t/\varepsilon$. However, we will not use this form in our considerations.

System (1.1) is the original system and

$$\dot{x} = v(x) \tag{1.2}$$

is the averaged one.

Let us now present the main theorem of the classical theory of averaging developed by N.N. Bogolyubov. This result contains sufficient conditions for the closeness of solutions of systems (1.1) and (1.2) on a finite time interval.

THEOREM 1.1. *Let x_0 be an initial condition, t_0 be an initial time, and $[t_0, t_0 + T]$ be a finite time interval. Let us assume that the solution $x(t; t_0, x_0)$ is defined on $[t_0, t_0 + T]$. Then for any $\varepsilon > 0$ there exists K, such that for any $k > K$ for all $t \in [t_0, t_0 + T]$ the following inequality holds*

$$\|x_{\text{averaged}}(t; t_0, x_0) - x(t; t_0, x_0)\| < \varepsilon.$$

Here $x_{\text{averaged}}(t; t_0, x_0)$ is the solution of the averaged system, $x(t; t_0, x_0)$ is the solution of the original system; $x_{\text{averaged}}(t_0; t_0, x_0) = x(t_0; t_0, x_0) = x_0$.

The second basic result is the theorem on averaging on an infinite time interval. We present this theorem for the case in which the right hand side is periodic.

THEOREM 1.2. *Let $x = 0$ be a non-degenerate equilibrium of the averaged system: $v(0) = 0$ and*

$$\det \frac{\partial v}{\partial x}(0) \neq 0.$$

For any $\varepsilon > 0$ there exists K such that for any $k > K$ there exists a point x_0 and for all t

$$\|x(t; t_0, x_0)\| < \varepsilon.$$

Moreover, the solution $x(t; t_0, x_0)$ is $(2\pi/k)$-periodic.

In other words, for k sufficiently large there exists a solution of the original system which is arbitrarily close to the equilibrium solution of the averaged solution.

In our paper, we are going to present sufficient conditions for the existence of a solution of the original system such that only a part of the coordinates in the phase space remains close to the equilibrium. In some sense, this is analogues to the stability with respect to a part of the variables developed by V.V. Rumyantsev [2,3].

2. Main result

Let us now have the following system

$$\dot{x} = v_x(x) + w_x(x, y, kt), \quad \dot{y} = v_y(y) + w_y(x, y, kt). \tag{2.1}$$

Here $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, $k \in \mathbb{R}$, $v_x : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $v_y : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$, $w_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R} \to \mathbb{R}^{n_x}$, $w_y : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R} \to \mathbb{R}^{n_y}$. The functions w_x and w_y are 2π-periodic with respect to the last variable: $w_x(x, y, kt + 2\pi) = w_x(x, y, kt)$, $w_x(x, y, kt + 2\pi) = w_x(x, y, kt)$ for all x, y, t. All functions here and below are assumed to be C^∞-smooth.

Let the set W_x be defined by some function $F_x\colon \mathbb{R}^{n_x} \to \mathbb{R}$:

$$W_x = \{x \in \mathbb{R}^{n_x} : F_x(x) \leqslant 0\}.$$

Everywhere below we assume that W_x is a connected set and that on its boundary ∂W_x, defined by the equation $F_x(x) = 0$, the condition $dF_x \neq 0$ is satisfied, i.e. the boundary ∂W_x is a smooth submanifold of $\mathbb{R}^{n_x}$ of codimension one.

Definition 2.1. We say that $x \in \partial W_x$ is an egress point with respect to the system $\dot{x} = v_x(x)$ if $dF_x(v_x) > 0$ at x. The set of all egress points is denoted by $W_x^+ \subset \partial W_x$.

Definition 2.2. We say that the solution of $\dot{x} = v(x)$ enters W_x at $x \in \partial W_x$ if $dF_x(v_x) < 0$ at x. The set of all such points is denoted by $W_x^- \subset \partial W_x$.

Definition 2.3. We say that the vector field $v_x(x)$ is externally tangent to ∂W_x at x if $dF_x(v_x(x)) = 0$ and additionally

$$\sum_{i,j=1}^{n_x} \Big(\frac{\partial F_x^2}{\partial x_i \partial x_j} v_x^i v_x^j + \frac{\partial F_x}{\partial x_i} \frac{\partial v_x^i}{\partial x_j} v_x^j \Big) > 0.$$

The set of all points in which the vector field $v_x(x)$ is externally tangent to ∂W_x will be denoted by W_x^{0+}.

Remark 2.1. Similar definitions hold for the system $\dot{y} = v_y(y)$. In this case, we denote the corresponding sets by W_y, W_y^-, W_y^+, and W_y^{0+}.

Remark 2.2. As an example of the above three types of points one can consider the system on the plane

$$\dot{x}_1 = x_1,$$
$$\dot{x}_2 = -x_2.$$

The set W_x is the unit circle defined by the inequality $F_x(x) = x_1^2 + x_2^2 - 1 \leqslant 0$. The points of the boundary $x_1^2 + x_2^2 - 1 = 0$ for which $x_1^2 > x_2^2$ are the egress points; if $x_1^2 < x_2^2$ then the trajectories of solutions enter W_x; if $x_1^2 = x_2^2$ then the trajectory is externally tangent to the boundary ∂W_x at these four points. A schematic representation of the vector field of this system is shown in Figure 1.

The following statement is essentially the main result of the paper.

Theorem 2.1. *Let us have two compact sets $W_x \subset \mathbb{R}^{n_x}$ and $W_y \subset \mathbb{R}^{n_y}$ such that $\partial W_y = W_y^-$, $\partial W_x = W_x^+ \cup W_x^- \cup W_x^{0+}$ and W_x^+ has at least two connected components. Then there exists $(x_0, y_0) \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that the trajectory of solution of (2.1) starting at (x_0, y_0) remains in $W_x \times W_y$ for all $t \geqslant 0$, provided that k is large.*

Proof. The general idea of the proof can be outlined as follows. First, we will consider a modification of the system (2.1) and proof the statement for this system. Then we will show that for large k the obtained solution cannot reach the subset of $\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ in which we modify our system. Therefore, this solution exists in the original system, provided that k is large.

Let us consider the following system, which will be called the modified system.

$$\begin{aligned} \dot{x} &= v_x(x) + \sigma_x(x) w_x(x, y, kt), \\ \dot{y} &= v_y(y) + \sigma_y(y) w_y(x, y, kt). \end{aligned} \tag{2.2}$$

Here, $\sigma_x \colon \mathbb{R}^{n_x} \to \mathbb{R}$, $\sigma_y \colon \mathbb{R}^{n_y} \to \mathbb{R}$. These functions are defined as follows. Let $\sigma \colon \mathbb{R} \to \mathbb{R}$ be the constant function (which equals unity) minus a standard bell-shaped function:

$$\sigma(r) = \begin{cases} 1 - e^{\frac{(r/\delta)^2}{(r/\delta)^2 - 1}}, & \text{if } -\delta \leqslant r \leqslant \delta, \\ 1, & \text{if } |r| > \delta. \end{cases}$$

Therefore, σ equals unity outside the δ-neighborhood of zero. Here $\delta > 0$ is a parameter.

Let $O_\delta(\partial W_x)$ be the δ-neighborhood of the boundary ∂W_x. $O_\delta(\partial W_x)$ is a tabular neighborhood of ∂W_x provided that $\delta > 0$ is small (below we assume that this property holds).

Finally, we define $\sigma_x(x)$ as a composition

$$\sigma_x(x) = \sigma(\operatorname{dist}(x, \partial W_x)),$$

where $\operatorname{dist}(x, \partial W_x)$ is the usual Euclidean distance between the point x and the boundary. Similarly, $\sigma_y(y) = \sigma(\operatorname{dist}(y, \partial W_y))$.

Let us show that for any fixed $\delta > 0$, $k > 0$ there exists a point $(x_0, y_0) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that the solution of the modified system (2.2) starting at (x_0, y_0) never leaves $W_x \times W_y$.

We will prove by contradiction. For the brevity of notations we will denote the pair (x, y) by z. Therefore, by $z_{\text{mod}}(t) = z_{\text{mod}}(t; t_0, z_0)$ we will denote the solution $(x_{\text{mod}}(t; t_0, x_0), y_{\text{mod}}(t; t_0, y_0))$ of (2.2) satisfying the initial condition $z_{\text{mod}}(t_0; t_0, z_0) = z_0$ (in other words, we have $(x_{\text{mod}}(t_0; t_0, x_0), y_{\text{mod}}(t_0; t_0, y_0)) = (x_0, y_0)$).

Let $\tau \colon W_x \times W_y \to \mathbb{R} \cup \{+\infty\}$ be the following function:

$$\tau(z_0) = \sup\{t' \geqslant 0 \colon z_{\text{mod}}(t; t_0, z_0) \subset (W_x \setminus \partial W_x) \times (W_y \setminus \partial W_y) \text{ for all } t \in [0, t')\}.$$

In other words, $\tau(z_0)$ defines the time required to reach the boundary of the set $W_x \times W_y$ for the solution starting at z_0. For instance, for any $z_0 \in \operatorname{int}(W_x \times W_y)$ we have $\tau(z_0) > 0$. For all $z_0 \in \partial(W_x \times W_y)$ we put $\tau(z_0) = 0$.

Let us consider a curve $\Gamma \colon [0,1] \to W_x \times W_y$ satisfying the following properties. Let $\Gamma_x(s)$ and $\Gamma_y(s)$ be the functions which define Γ: $\Gamma_x \colon [0,1] \to W_x$ and $\Gamma_y \colon [0,1] \to W_y$. Therefore, $\Gamma(s) = (\Gamma_x(s), \Gamma_y(s))$. Let $\Gamma_x(s)$ defines an arbitrary path without self-intersections such that $\Gamma_x(0) \in W_x^+$, $\Gamma_x(1) \in W_x^+$, and $\Gamma_x(s) \in \operatorname{int}(W_x)$ for all $s \in (0,1)$; $\Gamma_y(s) = y_0 \in \operatorname{int}(W_y)$, where y_0 is an arbitrary interior point in W_y and $\Gamma_x(0)$, $\Gamma_x(1)$ belong to two different connected components of W_x^+.

It is not hard to prove that the map τ is continuous. This follows from the assumption that $\partial W_x = W_x^+ \cup W_x^- \cup W_x^{0+}$, i.e. the vector field v_x is either transversal to ∂W_x or externally tangent to the boundary (see Figure 1).

For any point $z_0 = (x_0, y_0) \in \Gamma$ we assume that $\tau(z_0) < \infty$. Therefore, for the initial point point $z_0 \in \Gamma$ we have some point $z_{\mathrm{mod}}(\tau(z_0); 0, z_0) \in \partial(W_x \times W_y)$ which belongs to the boundary of the considered set. Moreover, $x_{\mathrm{mod}}(\tau(z_0); 0, z_0) \in W_x^+$ (here we use that $\partial W_y = W_y^-$). Since the function $z_{\mathrm{mod}}(t; t_0, z_0)$ is continuous in all variables, then the map

$$z_0 \mapsto z_{\mathrm{mod}}(\tau(z_0); 0, z_0)$$

is also continuous as a composition of two continuous functions. We will denote this map by $\Sigma\colon \Gamma \to \partial(W_x \times W_y)$. Let C be the connected component of W_x^+ such that $\Gamma_x(0) \in C$. Let us define the following projection $\mathrm{Pr}\colon W_x^+ \times W_y \to \{\Gamma_x(0), \Gamma_x(1)\}$:

$$\mathrm{Pr}(x, y) = \begin{cases} (\Gamma_x(0), y_0), & \text{if } \ x \in C, \\ (\Gamma_x(1), y_0), & \text{if } \ x \in W_x^+ \setminus C. \end{cases}$$

This map is also continuous. Finally, the map $\mathrm{Pr}(\Sigma)\colon \Gamma \to \{\Gamma_x(0), \Gamma_x(1)\}$ is continuous. We have constructed a continuous map from Γ to its boundary. Therefore, our assumption that $\tau(z_0) < \infty$ for all $z_0 \in \Gamma$ cannot be true and we have at least one $z_0 \in \Gamma$ such that $\tau(z_0) = +\infty$. In other words, the corresponding trajectory never leaves $W_x \times W_y$.

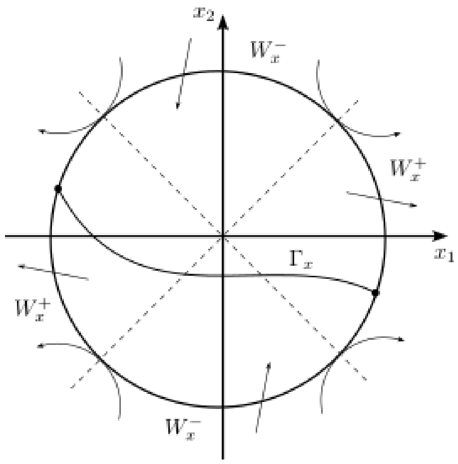

FIGURE 1. A schematic representation of the dynamics in W_x. The dynamics in W_y is trivial in the sense that the y-component of the solutions cannot leave W_y.

Now, we will show that for $\delta > 0$ sufficiently small and $k > 0$ sufficiently large, the above solution also exists in the original system.

Let us consider the averaged system

$$\dot{x} = v_x(x), \quad \dot{y} = v_y(y). \tag{2.3}$$

By $z_{\mathrm{averaged}}(t) = z_{\mathrm{averaged}}(t; t_0, z_0)$ we will denote the solution $(x_{\mathrm{averaged}}(t; t_0, x_0), y_{\mathrm{averaged}}(t; t_0, y_0))$ of (2.3) satisfying the initial condition $z_{\mathrm{averaged}}(t_0; t_0, z_0) = z_0$ (in other words, we have $(x_{\mathrm{averaged}}(t_0; t_0, x_0), y_{\mathrm{averaged}}(t_0; t_0, y_0)) = (x_0, y_0)$).

Let $x_0 \in \partial W_x$ and $y_0 \in W_y$. We will consider three cases. First, if $x_0 \in W_x^+$, $y_0 \in W_y$, then for any $t_0 \in [0, 2\pi]$ there exists $\Delta t \in [0, 1]$ such that $\mathrm{dist}(W_x \times W_y, z_{\text{averaged}}(t_0+\Delta t; t_0, z_0)) > 0$ (actually, it holds for all sufficiently small $\Delta t > 0$).

Indeed, the Taylor expansion of $x_{\text{averaged}}(t_0 + t; t_0, z_0)$ has the following form

$$x_{\text{averaged}}(t_0 + t; t_0, z_0) = x_0 + v_x(x_0)t + O(t^2).$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} F_x(x_{\text{averaged}}(t_0 + t; t_0, z_0)) &= F_x(x_0) + dF_x(v_x(x_0))t + O(t^2) \\ &= dF_x(v_x(x_0))t + O(t^2) > 0 \end{aligned}$$

for all sufficiently small $t > 0$.

From the continuous dependence on the initial conditions, we obtain that for any $t_0 \in [0, 2\pi]$ and any $x_0 \in W_x^+$, $y_0 \in W_y$ there exist two positive numbers $\rho = \rho(z_0, t_0) > 0$, $d = \rho(z_0, t_0) > 0$ such that for any initial conditions $\hat{t}_0, \hat{z}_0$ satisfying

$$\|z_0 - \hat{z}_0\|^2 + |t_0 - \hat{t}_0|^2 < \rho^2$$

we have

$$\mathrm{dist}(z_{\text{averaged}}(\hat{t}_0 + \Delta t; \hat{t}_0, \hat{z}_0), W_x \times W_y) > d$$

for some $\Delta t = \Delta t(z_0, t_0) \in [0, 1]$. Here and below the norm $\|z_0 - \hat{z}_0\|$ is the usual Euclidean distance in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$.

Similarly, we obtain $\rho = \rho(z_0, t_0) > 0$, $d = \rho(z_0, t_0) > 0$ with the same properties for any $t_0 \in [0, 2\pi]$ and any $x_0 \in W_x^{0+}$, $y_0 \in W_y$. In this case, we have to consider the Taylor expantion of the solutions up to the terms of the second order in time:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} F_x(x_{\text{averaged}}(t_0 + t; t_0, z_0)) &= F_x(x_0) + dF_x(v_x(x_0))t \\ &+ \frac{1}{2}\sum_{i,j=1}^{n_x} \Big(\frac{\partial F_x^2}{\partial x_i \partial x_j} v_x^i(x_0) v_x^j(x_0) + \frac{\partial F_x}{\partial x_i}\frac{\partial v_x^i}{\partial x_j} v_x^j(x_0)\Big)t^2 + O(t^3) > 0 \end{aligned}$$

If $x_0 \in W_x^{0+}$, $y_0 \in W_y$, then we again have d and ρ, yet the corresponding solutions leave $W_x \times W_y$ in the reversed time:

$$\mathrm{dist}(z_{\text{averaged}}(\hat{t}_0 - \Delta t; \hat{t}_0, \hat{z}_0), \partial(W_x \times W_y)) > d$$

for some $\Delta t \in [0, 1]$.

Similarly, for any $t_0 \in [0, 2\pi]$ and any $x_0 \in W_x$, $y_0 \in W^- = \partial W_y$ we have $\rho = \rho(z_0, t_0) > 0$, $d = \rho(z_0, t_0) > 0$ such that for any initial conditions $\hat{t}_0, \hat{z}_0$ satisfying

$$\|z_0 - \hat{z}_0\|^2 + |t_0 - \hat{t}_0|^2 < \rho^2$$

we have

$$\mathrm{dist}(z_{\text{averaged}}(\hat{t}_0 - \Delta t; \hat{t}_0, \hat{z}_0), W_x \times W_y) > d$$

for some $\Delta t \in [0, 1]$.

Therefore, for any point in the direct product $\partial(W_x \times W_y) \times [0, 2\pi] \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}$ we have an open ball $B(z_0, t_0)$ of radius $\rho(z_0, t_0)$. Let us consider a finite covering of $\partial(W_x \times W_y) \times [0, 2\pi]$ by $\{B_i\}_{i=1}^N$. Here $B_i = B(z_0, t_0)$ for some $(z_0, t_0) \in \partial(W_x \times W_y) \times [0, 2\pi]$. For any $B_i = B(z_0, t_0)$ we also have the corresponding $d_i = d(z_0, t_0)$.

Let $\delta > 0$ be such a number that $O_\delta(\partial(W_x \times W_y)) \times [0, 2\pi] \subset \bigcup_{i=1}^N B_i$. From the theorem on averaging on a finite time interval there exists $K > 0$ such that for any $k > K$ and any $(z_0, t_0) \in \bigcup_{i=1}^N B_i$ we have

$$\mathrm{dist}(z_{\mathrm{averaged}}(t_0 + t; t_0, z_0), z_{\mathrm{mod}}(t_0 + t; t_0, z_0)) < D/2,$$

for all $t \in [-1, 1]$. Here

$$D = \min_{1 \leqslant i \leqslant N} d_i.$$

Therefore, if $z_{\mathrm{mod}}(t) \in W_x \times W_y$, then $z_{\mathrm{mod}}(t)$ is also a solution of the original system. □

Corollary 2.1. *Let us assume that for any $\varepsilon > 0$ there are two compact sets $W_x \subset \mathbb{R}^{n_x}$, $W_x \subset O_\varepsilon(0)$ and $W_y \subset \mathbb{R}^{n_y}$ such that $\partial W_y = W_y^-$, $\partial W_x = W_x^+ \cup W_x^- \cup W_x^{0+}$ and W_x^+ has at least two connected components. Then, for any $\varepsilon > 0$ there exists $z_0 = (x_0, y_0) \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that for the solution $(x(t; 0, z_0), y(t; 0, z_0))$ of (2.1) starting at (x_0, y_0) it holds that $x(t; 0, z_0) \in O_\varepsilon(0)$ for all $t \geqslant 0$, provided that k is large.*

3. Mechanical example

As an example, let us consider a planar inverted pendulum and a point moving along a horizontal circle. We assume that the pendulum and the massive point interact and the magnitude of the interaction force is a rapidly oscillating function of time. To be more precise, let l be the length of the rigid rod of the pendulum. Let $Oxyz$ be an inertial coordinate system. By $x_{\mathrm{pend}}, y_{\mathrm{pend}}, z_{\mathrm{pend}}$ we denote the corresponding coordinates of the point mass of the pendulum located at the end of the rod.

$$x_{\mathrm{pend}} = 0, \quad y_{\mathrm{pend}} = l \sin\varphi, \quad z_{\mathrm{pend}} = l \cos\varphi.$$

Here φ is the angle between the rod and the vertical direction Oz. Let us also have a horizontal circle of radius r centered at the origin. Let ψ be a local coordinate on this circle. The coordinates $x_{\mathrm{point}}, y_{\mathrm{point}}, z_{\mathrm{point}}$ of the point mass on this circle can be written as usual:

$$x_{\mathrm{point}} = r \cos\psi, \quad y_{\mathrm{point}} = r \sin\psi, \quad z_{\mathrm{point}} = 0.$$

In addition, we assume that the magnitude of the force of interaction has the form $F(\varphi, \psi) \sin kt$ and the point mass on a circle is subject to a force of viscous friction.

The Lagrange equations have the following form

$$m_{\mathrm{pend}} l^2 \ddot{\varphi} = m_{\mathrm{pend}} l g \sin\varphi + Q_\varphi(\varphi, \psi) \sin kt,$$
$$m_{\mathrm{point}} r^2 \ddot{\psi} = -\mu \dot{\psi} + Q_\psi(\varphi, \psi) \sin kt.$$

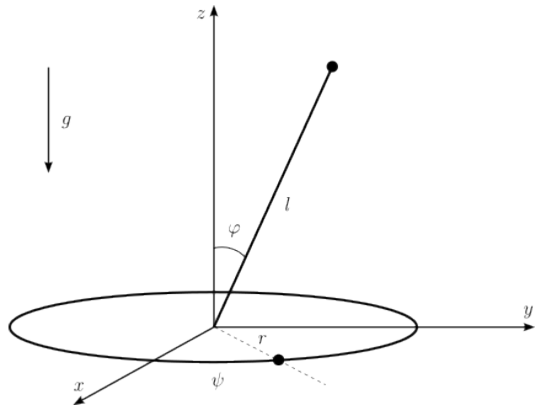

FIGURE 2. An inverted planar pendulum and a point mass moving along a horizontal circle.

Here $\mu > 0$ is the coefficient of viscous friction; $Q_\varphi(\varphi, \psi)$ and $Q_\psi(\varphi, \psi)$ can be obtained explicitly as functions of $F(\varphi, \psi)$. The averaged system has the following form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{g}{l}\sin x_1, \quad \dot{y}_1 = y_2, \quad \dot{y}_2 = -\frac{\mu}{m_{\text{point}} r^2} y_2.$$

Here $x_1 = \varphi$ and $y_1 = \psi$. Let

$$F_x = x_1^2 + x_2^2 - \delta^2.$$

Since the upper equilibrium of the pendulum is a saddle point, then F_x defines W_x such that $\partial W_x = W_x^+ \cup W_x^- \cup W_x^{0+}$. Let $F_y = y_2^2 - 1$. Formally, the set where $F_y \geqslant 0$ is not compact in $\mathbb{R}^2$. Yet, y_1 is an angular variable, so this set is $\mathbb{S}^1 \times [-1, 1]$ and compact. From the theorem we have that for any $\delta > 0$ there exists a solution of the original system such that for this solution $\varphi^2(t) + \dot{\varphi}^2(t) < \delta$ for all $t \geqslant 0$, provided that k is large.

Let us note that the above holds even if we add some additional force acting on the point mass on the circle. Namely, let the second equation in the original system has the form

$$m_{\text{point}} r^2 \ddot{\psi} = -\mu\dot{\psi} + Q_\psi(\varphi, \psi)\sin kt + f(\varphi, \psi).$$

In the averaged system we obtain the following equation

$$\dot{y}_2 = -\frac{\mu}{m_{\text{point}} r^2} y_2 + \frac{f(y_1)}{m_{\text{point}} r^2}.$$

The function $f(y_1)$ is bounded. Therefore, for $F_y = y_2^2 - a^2$ we have

$$\frac{d}{dt}F_y = 2y_2\dot{y}_2 = 2\Big(-\frac{\mu}{m_{\text{point}} r^2} y_2^2 + \frac{f(y_1)}{m_{\text{point}} r^2} y_2\Big) < 0,$$

provided that $y_2 = \pm a$, $a > 0$ is sufficiently large. Finally, the theorem can be applied for $F_x = x_1^2 + x_2^2 - \delta^2$ and $F_y = y_2^2 - a^2$. Moreover, it can be shown that this result also holds for $f = f(x_1, y_1)$.

4. Conclusion

The results presented in this paper are a continuation of a series of papers in which classical results of the Bogolyubov averaging method are proved using topological methods [4–8]. More precisely, the general scheme of reasoning is as follows:

(1) Instead of the original system, a modified system is considered.
(2) For it, using fixed point theorems or other topological theorems, the existence of a solution with the required properties is proved.
(3) Using the finite time averaging theorem, we show that the obtained solution cannot reach the region in which we modify the original system.

This approach, firstly, clarifies the geometry of the averaging method. In particular, it explains the difference between the averaging in the periodic and almost periodic cases [9]. Secondly, it allows us to generalize classical results to the case of degenerate (in the sense of linear theory) equilibrium positions. To be more precise, it has been shown that the applicability of the theorem on averaging on an infinite time interval does not depend on the properties of the linear part of the averaged system at the equilibrium, but on some qualitative properties of the averaged vector field. In particular, the systems $\dot{x}_1 = x_1$, $\dot{x}_2 = -x_2$ and $\dot{x}_1 = x_1^3$, $\dot{x}_2 = -x_2^3$ are identical from this point of view.

In this paper, we use the topological Ważewski method [10–12], which allows us to prove the existence of solutions that do not leave the considered regions of the extended phase space. Unlike fixed point theorems (which are not applicable to the problem under consideration), the use of the Ważewski method does not require the periodicity of the right hand side (in our case, the function w_x is periodic in time, but we cannot guarantee that the function $y(t)$ will also be periodic in time).

Note that the Ważewski method has been previously used to prove the existence of non-falling solutions for an inverted pendulum on a moving base (the Whitney pendulum) [13], as well as to prove the existence of non-falling solutions for the Kapitza–Whitney pendulum [5]. One of the most important areas of further research is the search for interesting mechanical problems to which the developed averaging method could be applied.

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References

1. N. N. Bogoliubov, Y. A. Mitropolski, *Asymptotic Methods in the Theory of Non-linear Oscillations*, CRC Press, 1961.
2. V. V. Rumiantsev, A. S. Oziraner, *Stability and Stabilization of Motion with Respect to Part of the Variables*, Moscow, Nauka, 1987.
3. A. Oziraner, V. Rumiantsev, *The method of Liapunov functions in the stability problem for motion with respect to a part of the variables*, J. Appl. Math. Mech. **36**(2) (1972), 341–362.
4. I. Polekhin, *Topological considerations and the method of averaging: A connection between local and global results*, in: *2020 International Conference Nonlinearity, Information and Robotics (NIR)*, IEEE, 2020, 1–4.
5. I. Yu. Polekhin, *The method of averaging for the Kapitza–Whitney pendulum*, Regul. Chaotic Dyn. **25**(4) (2020), 401–410.

6. I. Yu. Polekhin, *The spherical Kapitza–Whitney pendulum*, Regul. Chaotic Dyn. **27**(1) (2022), 65–76.
7. I. Polekhin, *Asymptotically stable non-falling solutions of the Kapitza–Whitney pendulum*, Meccanica **58**(7) (2023), 1205–1215.
8. I. Yu. Polekhin, *A topological–analytical method for proving averaging theorems on an infinite time interval in a degenerate case*, Proc. Steklov Inst. Math. **322**(1) (2023), 188–197.
9. V. Burd, *Method of Averaging for Differential Equations on an Infinite Interval: Theory and Applications*, Chapman and Hall/CRC, 2007.
10. T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Pol. Math. **20** (1947), 279.
11. P. Hartman, *Ordinary Differential Equations*, SIAM, 2002.
12. R. Srzednicki, *Wazewski method and Conley index*, in: A. Cañada, P. Drábek, A. Fonda (eds.), *Handbook of Differential Equations*, Elsevier, 2004, 591–684.
13. I. Yu. Polekhin, *Examples of topological approach to the problem of inverted pendulum with moving pivot point*, Russ. J. Nonlinear Dyn. **10**(4) (2014), 465–472.

ПРВИ КОРАЦИ КА УСРЕДЊАВАЊУ У ОДНОСУ НА ДЕО КООРДИНАТА

Резиме. Разматра се проблем усредњавања на бесконачном временском интервалу. Класични резултати о усредњавању које је доказао Н. Н. Богољубов су уопштени на случај у коме само део координата у фазном простору остаје близу равнотежног положаја усредњеног система. Ово називамо усредњавањем у односу на део координата. Резултати су засновани на неким тополошким идејама комбинованим са стандардном теоремом о усредњавању на коначном временском интервалу.

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