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BLOW-UP PHENOMENA FOR A DAMPED WAVE EQUATION WITH LOGARITHMIC SOURCE TERM AND VARIABLE-EXPONENTS

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Abstract. This paper investigates the wave equation with variable-exponent nonlinearity and logarithmic source term, given by the following:

$$u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2} \ln |u|,$$

where a and b are positive constants, and the functions $m(\cdot)$ and $p(\cdot)$ satisfy certain required conditions. Using the energy method and several inequality techniques, we establish a finite-time global nonexistence result for specific solutions with positive initial energy, under appropriate conditions. This type of equation has significant applications in various fields, including fluid dynamics, electrorheological fluids, quantum mechanics, nuclear physics, optics, and geophysics.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $a, b > 0$ are constants. We consider the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2} \ln |u|, & \text{in } \Omega \times (0, T) \\ u(x, t) = 0, & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω satisfying

$$\begin{aligned} 2 \leqslant \varphi_1 \leqslant \varphi(x) \leqslant \varphi_2 < \frac{2n}{n-2}, \qquad & \text{for} \quad n \geqslant 3, \\ 2 \leqslant \varphi_1 \leqslant \varphi(x) \leqslant \varphi_2 < +\infty, \qquad & \text{for} \quad n \leqslant 2, \end{aligned} \tag{1.2}$$

with

$$\varphi_1 := \operatorname{ess\,inf}_{x\in\Omega} \varphi(x), \quad \varphi_2 := \operatorname{ess\,sup}_{x\in\Omega} \varphi(x),$$

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and the log-Hölder continuity condition:

$$|\varphi(x) - \varphi(y)| \leqslant -\frac{A}{\log|x-y|}, \quad \text{for a.e. } x, y \in \Omega,\ |x-y| < \delta, \tag{1.3}$$

with $0 < \delta < 1$ and $A > 0$.

In the case of constant nonlinearity, considerable effort has been dedicated to studying problems associated with (1.1). For instance, concerning the nonlinearly damped wave equation, the following equation:

$$u_{tt} - \Delta u + a u_t |u_t|^m = b u |u|^p, \quad \text{in } \Omega \times (0,T),\ m,\, p \geqslant 0.$$

has been extensively studied and many results concerning global existence and nonexistence have been proved. Precisely, we know that the nonlinear source term $bu|u|^p (p > 0)$ causes finite time blow up of solutions with negative initial energy in the absence of the damping term $au_t|u_t|^m$ (see [5,17]). In the absence of the source term, the damping term assures global existence for arbitrary initial data (see [13, 16]). Levine was the first to examine the interaction between the damping and the source terms (see [17,25]). He addressed the case where $m = 2$ and demonstrated the finite-time blow-up of solutions with negative initial energy. Georgiev and Todorova [10] extended Levine's result to the case $m > 2$ by employing a distinct approach. Levine et al.[18] expanded the previous results to unbounded domains, demonstrating that any solution with negative initial energy blows up in finite time if $p > m$. Similarly, Messaoudi [20] established the finite-time blow-up of negative-initial-energy solutions under the same condition $p > m$.

Recently, significant attention has been given to the study of mathematical nonlinear models involving hyperbolic, parabolic, and elliptic equations with variable exponents of nonlinearity. These models have been used to describe various physical phenomena, such as the flow of electro-rheological fluids, fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes in porous media, and image processing. Further details on these applications are available in [1–4, 8, 15, 19]. However, the literature on equations with variable exponents of nonlinearity remains relatively sparse. Messaoudi et al.[21] investigated the following nonlinear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

where $a, b \geqslant 0$ are constants and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Using the Faedo–Galerkin method, the authors established the existence of a unique weak solution under appropriate conditions on the variable exponents m and p. Furthermore, the finite-time blow-up of solutions was established, and a two-dimensional numerical example was presented to demonstrate this blow-up behavior. Rahmoune [24] studied the problem (1.1) with $a = b = 1$. Combining Banach's fixed point theorem and Faedo-Galerkin techniques, the author proved the well-posedeness theorem and investigated the global nonexistence of solution with negative initial energy. This type of problems with logarithmic source term arises in many branches of physics such as nuclear physics, optics, geophysics and quantum field theory. More details on these problems can be found in [6, 7, 11, 12, 14, 22].

In this article, we extend the work in [24] to the cases with positive initial energy. This paper is organized into two sections, in addition to the Introduction. In Section 2, we recall the definitions of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and Sobolev spaces $W^{k,p(\cdot)}(\Omega)$, along with some of their properties, followed by a well-posedness theorem. In Section 3, we state and prove our blow-up theorem for certain solutions with positive initial energy.

2. Mathematical background

In this section, we provide some preliminary pieces of information and facts about Lebesgue and Sobolev spaces with variable exponents. Let $p\colon \Omega \subset \mathbb{R}^n \to [1, \infty]$ be a measurable function.We denote by $L^{p(\cdot)}(\Omega)$ all the real measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\int_\Omega |\lambda u(x)|^{p(x)} dx < \infty$, for some $\lambda > 0$.

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ equipped with the following Luxemburg-type norm

$$\|u\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\},$$

is a Banach space.

The Banach variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

In addition, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. As well as in the classical Sobolev spaces, $W^{-1,p'(\cdot)}(\Omega)$ is defined as the dual of $W_0^{1,p(\cdot)}(\Omega)$, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Lemma 2.1 (Poincaré's inequality [9]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p(\cdot)$ satisfies (1.3). Then*

$$\|u\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)}, \quad \text{for all } \; u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant depends on Ω, p_1, and p_2. As a direct result, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.

Lemma 2.2 ([9]). *If $p\colon \Omega \to [1, \infty)$ is a measurable function and*

$$2 \leqslant p_1 \leqslant p(x) \leqslant p_2 \leqslant \frac{2n}{n-2}, \quad n > 2,$$

$$2 \leqslant p_1 \leqslant p(x) \leqslant p_2 < +\infty, \quad n \leqslant 2.$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

LEMMA 2.3 (Young's inequality [9]). *Let $h, i, j \geqslant 1$ be measurable functions defined on Ω such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \quad \text{for a.e.} \quad y \in \Omega.$$

Then for all $a, b \geqslant 0$,

$$\frac{(ab)^{h(.)}}{h(.)} \leqslant \frac{(a)^{i(.)}}{i(.)} + \frac{(b)^{j(.)}}{j(.)}.$$

LEMMA 2.4 (Hölder's Inequality [9]). *Let $h, i, j \geqslant 1$ be measurable functions defined on Ω such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \quad \text{for a.e.} \quad y \in \Omega.$$

If $f \in L^{i(\cdot)}(\Omega)$ and $g \in L^{j(\cdot)}(\Omega)$, then $fg \in L^{h(\cdot)}(\Omega)$ and

$$\|fg\|_h \leqslant 2\|f\|_i\|g\|_j.$$

DEFINITION 2.1 (Weak solution). $u = u(x,t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T)$, if $u \in L^\infty(0,T; H_0^1(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ and satisfies the problem (1.1) in the weak sense, i.e.

$$(u_{tt}, v)_2 + (\nabla u, \nabla v)_2 + (u_t|u_t|^{m(\cdot)-2}, v)_2 = (u|u|^{p(\cdot)-2}\ln|u|, v)_2,$$

for any $v \in H_0^1(\Omega)$, $t \in (0,T)$, where $u(0,x) = u_0(x) \in H_0^1(\Omega)$, $u_t(x,0) = u_1(x) \in L^2(\Omega)$, and $(\cdot,\cdot)_2$ means the inner product $(\cdot,\cdot)_{L^2(\Omega)}$.

By using Faedo–Galerkin method with the help of Banach's fixed point theorem, Rahmoune [24] proved the following well-posedness theorem:

THEOREM 2.1. *Let $m(\cdot)$ and $p(\cdot)$ satisfy (1.2) and (1.3). Moreover, $p(\cdot)$ satisfies*

$$2 < p_1 \leqslant p(x) \leqslant p_2 < 2\frac{n-1}{n-2}, \quad \text{for} \quad n \geqslant 3,$$
$$2 < p_1 \leqslant p(x) \leqslant p_2 < +\infty, \quad \text{for} \quad n \leqslant 2.$$

Then, for any given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, it exists $T > 0$ and a unique solution u of the problem (1.1) on $(0,T)$ such that

$$u \in C((0,T), H_0^1(\Omega)) \cap C^1((0,T), L^2(\Omega))$$
$$u_{tt} \in L^2((0,T), H^{-1}(\Omega)).$$

LEMMA 2.5. *Let u be the solution of (1.1), then the modified energy functional $E(t)$ associated with the problem (1.1), defined as*

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \int_\Omega \frac{b}{p^2(x)}|u|^{p(x)}\mathrm{d}x - \int_\Omega \frac{b}{p(x)}|u|^{p(x)}\ln|u|\mathrm{d}x, \tag{2.1}$$

satisfies

$$E'(t) = -a\int_\Omega |u_t|^{m(x)}\mathrm{d}x \leqslant 0, \quad a.e \;\; t \in [0,T). \tag{2.2}$$

PROOF. Multiplying equation (1.1) by u_t and integrating over Ω easily we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\nabla u(t)\|_2^2+\int_\Omega \frac{b}{p^2(x)}|u|^{p(x)}\mathrm{d}x-\int_\Omega \frac{b}{p(x)}|u|^{p(x)}\ln|u|\mathrm{d}x\right\}$$
$$=-a\int_\Omega |u_t|^{m(x)}\mathrm{d}x. \qquad \square$$

3. Blowup result

In this section, we demonstrate the blow-up for specific solutions with positive initial energy. To state and establish our result, let B be the optimal constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^\sigma(\Omega)$ with

$$p_2<\sigma\leqslant\sigma_{\max},\quad \sigma_{\max}:=\begin{cases}\frac{2n}{n-2} & \text{if } n\geqslant 3\\ +\infty & \text{if } n\leqslant 2\end{cases},$$

and determine

$$\delta_1=\Big(\frac{ep_1(\sigma-p_2)}{b\sigma}\Big)^{\frac{2}{\sigma-2}}(B)^{\frac{-2\sigma}{\sigma-2}},\quad E_1=\Big(\frac{1}{2}-\frac{1}{(1-\eta)p_1}\Big)\delta_1,$$

$$H(t)=E_1-E(t), \tag{3.1}$$

$$L(t)=H^{1-\lambda}(t)+\varepsilon\int_\Omega u\,u_t(x,t)\mathrm{d}x,$$

where $\varepsilon>0$ and $0<\lambda<1$ will be determined later. We now present some useful lemmas, where a generic positive constant is denoted by C.

LEMMA 3.1 ([21]). *Suppose the conditions of Lemma 2.2 hold. Then there exists a positive $C>1$, depending on Ω only, such that*

$$\|u\|_{p_1}^s\leqslant \mathrm{C}(\|\nabla u\|_2^2+\|u\|_{p_1}^{p_1}),$$

for any $u\in H_0^1(\Omega)$ and $2\leqslant s\leqslant p_1$.

LEMMA 3.2 ([21]). *Let u be the solution of (1.1) and assume that*

$$\begin{aligned}&2\leqslant m_1\leqslant m(x)\leqslant m_2<p_1\leqslant p(x)\leqslant p_2<\frac{2n}{n-2},\quad \text{for } n\geqslant 3,\\ &2\leqslant m_1\leqslant m(x)\leqslant m_2<p_1\leqslant p(x)\leqslant p_2<+\infty,\quad \text{for } n\leqslant 2.\end{aligned} \tag{3.2}$$

holds. Then,

$$\int_\Omega |u|^{m(x)}\mathrm{d}x\leqslant C(\|u\|_{p_1}^{m_1}+\|u\|_{p_1}^{m_2}).$$

LEMMA 3.3. *For any $u\in H_0^1(\Omega)$, there exist two positive constants $C_1,C_2>0$ depending on Ω only such that*

$$\int_\Omega |u|^{p(x)}\ln|u|\mathrm{d}x\leqslant C_1\|\nabla u\|_2^\sigma+C_2,\quad t\geqslant 0.$$

PROOF. We set

$$\Omega_+ = \{x \in \Omega \mid |u| > 1\} \ \text{ and } \ \Omega_- = \{x \in \Omega \mid |u| \leqslant 1\}$$

Using the properties of the logarithmic function, we obtain

$$\begin{aligned}\int_\Omega |u|^{p(x)} \ln|u| \mathrm{d}x &= \int_{\Omega_+} |u|^{p(x)} \ln|u| \mathrm{d}x + \int_{\Omega_-} |u|^{p(x)} \ln|u| \mathrm{d}x \\ &\leqslant \frac{1}{e(\sigma - p_2)} \int_{\Omega_+} |u|^\sigma \mathrm{d}x + \frac{|\Omega|}{ep_1} \\ &\leqslant \frac{1}{e(\sigma - p_2)} \|u\|_\sigma^\sigma + \frac{|\Omega|}{ep_1} \leqslant \frac{B^\sigma}{e(\sigma - p_2)} \|\nabla u\|_2^\sigma + \frac{|\Omega|}{ep_1}. \qquad \square\end{aligned}$$

LEMMA 3.4 ([23]). *Let u be the solution of (1.1). If*

$$E(0) < E_1 \text{ and } \|\nabla u_0\|_2^2 \geqslant \delta_1,$$

then there exists $\delta_2 > \delta_1$ such that

$$\|\nabla u(t)\|_2^2 \geqslant \delta_2, t \geqslant 0. \tag{3.3}$$

LEMMA 3.5. *Let the assumptions in Lemma 3.4 be satisfied, then we have*

$$0 < H(0) \leqslant H(t) \leqslant \frac{bC_1}{p_1} \|\nabla u\|_2^\sigma + \frac{bC_2}{p_1}.$$

PROOF. Using (2.1), (2.2) and (3.1), we obtain

$$\begin{aligned}0 < H(0) \leqslant H(t) \leqslant E_1 - &\left[\frac{1}{2}\int_\Omega u_t^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega \frac{b}{p^2(x)} |u|^{p(x)} \mathrm{d}x\right] \\ &+ \int_\Omega \frac{b}{p(x)} |u|^{p(x)} \ln|u| \mathrm{d}x\end{aligned}$$

and, from (3.3), we get

$$\begin{aligned}E_1 - &\left[\frac{1}{2}\int_\Omega u_t^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega \frac{b}{p^2(x)} |u|^{p(x)} \mathrm{d}x\right] \\ &\leqslant E_1 - \frac{1}{2}\|\nabla u\|_2^2 \leqslant E_1 - \frac{\delta_2}{2} \leqslant E_1 - \frac{\delta_1}{2} \leqslant -\frac{\delta_1}{(1-\eta)p_1} < 0, \quad \forall t \geqslant 0.\end{aligned}$$

Hence

$$0 < H(0) \leqslant H(t) \leqslant \frac{bC_1}{p_1} \|\nabla u\|_2^\sigma + \frac{bC_2}{p_1}, \quad \forall t \geqslant 0,$$

by virtue of Lemma 3.3. $\square$

The following theorem presents the primary result of this article:

THEOREM 3.1. *Let the conditions of Lemma 3.4 and (3.2) be satisfied. Then, the solution of problem (1.1) given by Theorem 2.1 blows up in finite time.*

PROOF. We define the auxiliary function as follows:

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u\, u_t(x,t)\mathrm{d}x, \tag{3.4}$$

where $\varepsilon > 0$ small enough to be chosen later and

$$0 < \alpha \leqslant \min\Big\{\frac{p_1 - 2}{2p_1}, \frac{2(p_1 - m_2)}{\sigma p_1(m_2 - 1)}\Big\}. \tag{3.5}$$

Utilizing equation (1.1) and differentiating the auxiliary function (3.4), we obtain the result

$$\begin{aligned} L'(t) = {} & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega [u_t^2 - |\nabla u|^2]\mathrm{d}x \\ & + \varepsilon b \int_\Omega |u|^{p(x)} \ln|u|\mathrm{d}x - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2}\mathrm{d}x. \end{aligned} \tag{3.6}$$

The result of subtracting and adding $\varepsilon(1-\eta)p_1 H(t)$ to the right-hand side of (3.6) is

$$\begin{aligned} L'(t) \geqslant {} & (1-\alpha)H^{1-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1 H(t) + \varepsilon b\eta \int_\Omega |u|^{p(x)} \ln|u|\mathrm{d}x \\ & + \frac{\varepsilon b p_1(1-\eta)}{p_2^2} \int_\Omega |u|^{p(x)}\mathrm{d}x + \varepsilon\Big(\frac{(1-\eta)p_1}{2} + 1\Big)\|u_t\|_2^2 + \varepsilon\Big(\frac{(1-\eta)p_1}{2} - 1\Big)\|\nabla u\|_2^2 \\ & - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2}\mathrm{d}x - \varepsilon(1-\eta)p_1 E_1. \end{aligned}$$

From Lemma 3.5, we deduce

$$\frac{p_1}{p_2^2} \int_\Omega |u|^{p(x)}\mathrm{d}x \leqslant \int_\Omega |u|^{p(x)} \ln|u|\mathrm{d}x,$$

and from it, we have

$$\begin{aligned} L'(t) \geqslant {} & (1-\alpha)H^{1-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1 H(t) + \frac{\varepsilon b p_1}{p_2^2} \int_\Omega |u|^{p(x)}\mathrm{d}x \\ & + \varepsilon\Big(\frac{(1-\eta)p_1}{2} + 1\Big)\|u_t\|_2^2 + \varepsilon\Big(\frac{(1-\eta)p_1}{2} - 1\Big)\|\nabla u\|_2^2 \\ & - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2}\mathrm{d}x - \varepsilon(1-\eta)p_1 E_1. \end{aligned}$$

It implies that for $0 < \eta < \frac{p_1-2}{p_1}$,

$$\begin{aligned} L'(t) \geqslant {} & (1-\alpha)H^{1-\alpha}(t)H'(t) + \varepsilon\beta[H(t) + \|u\|_{p_1,\Omega_+}^{p_1} \\ & + \|u_t\|_2^2 + \|\nabla u\|_2^2] - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2}\mathrm{d}x, \end{aligned} \tag{3.7}$$

where $\beta = \min\big\{(1-\eta)p_1, \frac{bp_1}{p_2^2}, \frac{(1-\eta)p_1}{2} + 1, \big(\frac{(1-\eta)p_1}{2} - 1\big)\frac{\delta_2 - \delta_1}{\delta_2}\big\} > 0$ and $\|u\|_{p_1,\Omega_+}^{p_1} := \int_{\Omega_+} |u|^{p_1}\mathrm{d}x$. We estimate the last term in (3.7) using Young's inequality as follows:

$$\int_\Omega |u_t|^{m(x)-1}|u|\mathrm{d}x \leqslant \frac{1}{m_1} \int_\Omega \theta^{m(x)}|u|^{m(x)}\mathrm{d}x + \frac{m_2 - 1}{m_2} \int_\Omega \theta^{-\frac{m(x)}{m(x)-1}}|u_t|^m\mathrm{d}x, \quad \forall\theta > 0.$$

With θ selected, so that

$$\theta^{-\frac{m(x)}{m(x)-1}} = k \cdot H^{-\alpha}(t), \quad k > 0,$$

we obtain

$$\int_\Omega |u_t|^{m(x)-1}|u|\mathrm{d}x \leqslant \frac{1}{m_1}\int_\Omega k^{1-m(x)}|u|^m H^{\alpha(m(x)-1)}(t)\mathrm{d}x + \frac{(m_2-1)k}{am_2}H^{-\alpha}(t)H'(t). \tag{3.8}$$

Combining (3.7) with (3.8) yields

$$\begin{aligned} L'(t) \geqslant & \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k\Big]H^{-\alpha}(t)H'(t) - \varepsilon\frac{k^{1-m_1}a}{m_1}H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x \\ & + \varepsilon\beta[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2]. \end{aligned} \tag{3.9}$$

Using Lemma 3.2, Lemma 3.3 and Lemma 3.1 to obtain

$$\begin{aligned} H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x &\leqslant C(C_1\|\nabla u\|_2^\sigma + C_2)^{\alpha(m_2-1)}[\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2}] \\ &\leqslant C[\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2} + \|\nabla u\|_2^{\sigma\alpha(m_2-1)}\|u\|_{p_1}^{m_1} + \|\nabla u\|_2^{\sigma\alpha(m_2-1)}\|u\|_{p_1}^{m_2}] \\ &\leqslant C[\|\nabla u\|_2^{\frac{p_1\sigma\alpha(m_2-1)}{p_1-m_1}} + \|\nabla u\|_2^{\frac{p_1\sigma\alpha(m_2-1)}{p_1-m_2}} + \|\nabla u\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1}] \end{aligned} \tag{3.10}$$

thanks to Young's inequality. We now apply the well-known algebraic inequality that follows:

$$z^\tau \leqslant z + 1 \leqslant \Big(1 + \frac{1}{d}\Big)(z + d), \quad \forall z \geqslant 0,\ 0 < \tau \leqslant 1,\ d \geqslant 0,$$

for $z = \|\nabla u\|_2^2$, $d = H(0)$ and $\tau_1 = \frac{p_1\sigma\alpha(m_2-1)}{2(p_1-m_1)}\big(\tau_2 = \frac{p_1\sigma\alpha(m_2-1)}{2(p_1-m_2)}\big)$, we get

$$\|\nabla u\|_2^{\frac{p_1\sigma\alpha(m_2-1)}{p_1-m_1}} + \|\nabla u\|_2^{\frac{p_1\sigma\alpha(m_2-1)}{p_1-m_2}} \leqslant C[\|\nabla u\|_2^2 + H(t)]. \tag{3.11}$$

From (3.10) and (3.11), we arrive at

$$H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x \leqslant C(\|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2 + H(t)). \tag{3.12}$$

(3.9) and (3.12) together produce

$$\begin{aligned} L'(t) \geqslant & \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k\Big]H^{-\alpha}(t)H'(t) \\ & + \varepsilon\Big(\beta - \frac{ak^{1-m_1}}{m_1}C\Big)[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2]. \end{aligned} \tag{3.13}$$

Here, we choose k to be big enough that

$$\gamma = \beta - \frac{a \cdot k^{1-m_1}}{m_1}C > 0.$$

Once k is fixed, we find ε small enough to guarantee

$$(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k \geqslant 0 \quad \text{and} \quad L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)\mathrm{d}x > 0,$$

Thus, (3.13) takes the form

$$L'(t) \geqslant \gamma\varepsilon[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2]. \tag{3.14}$$

Consequently, we have

$$L(t) \geqslant L(0) > 0, \quad \text{for all} \ \ t \geqslant 0.$$

However, from (3.4), we derive:

$$L^{\frac{1}{1-\alpha}}(t) = \left[H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\mathrm{d}x\right]^{\frac{1}{1-\alpha}} \leqslant 2^{\frac{1}{1-\alpha}} \left[H(t) + \varepsilon \left(\int_\Omega u u_t(x,t)\mathrm{d}x\right)^{\frac{1}{1-\alpha}}\right]. \tag{3.15}$$

We observe that

$$\left|\int_\Omega u u_t(x,t)\mathrm{d}x\right| \leqslant \|u\|_2\|u_t\|_2 \leqslant C\|u\|_{p_1}\|u_t\|_2,$$

which implies

$$\left|\int_\Omega u u_t(x,t)\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leqslant C\|u\|_{p_1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Young's inequality once more offers

$$\left|\int_\Omega u u_t(x,t)\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leqslant C\left[\|u\|_{p_1}^{\frac{2}{1-2\alpha}} + \|u_t\|_2^2\right], \quad \text{for} \ \ \frac{1-2\alpha}{2(1-\alpha)} + \frac{1}{2(1-\alpha)} = 1.$$

Recalling (3.5) then applying Lemma 3.1, we find

$$\left|\int_\Omega u u_t(x,t)\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leqslant C[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2]. \tag{3.16}$$

By inserting (3.16) into (3.15), we obtain

$$L^{\frac{1}{1-\alpha}}(t) \leqslant \varepsilon C[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1} + \|\nabla u\|_2^2]. \tag{3.17}$$

From (3.14) and (3.17), we conclude that

$$L'(t) \geqslant \Lambda L^{\frac{1}{1-\alpha}}(t), \quad \text{for all} \ \ t \geqslant 0. \tag{3.18}$$

where Λ is a positive constant depending on $\Omega, u_{0,1}, m_{1,2}$ and $p_{1,2}$ only. When (3.18) is simply integrated over $(0,t)$, it produces

$$L^{\frac{\alpha}{1-\alpha}}(t) \geqslant \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\Lambda t}$$

Therefore, $L(t)$ blows up in finite time

$$T^* \leqslant \frac{1-\alpha}{\Lambda\alpha[L(0)]^{\frac{\alpha}{1-\alpha}}}.$$

The proof is complete. □

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ФЕНОМЕН РАЗДУВАВАЊА ЗА ЈЕДНАЧИНУ ПРИГУШЕНОГ ТАЛАСА СА ЛОГАРИТАМСКИМ ЧЛАНОМ ИЗВОРА И ПРОМЕНЉИВИМ ЕКСПОНЕНТИМА

Резиме. Овај рад истражује једначину таласа са нелинеарношћу променљивог експонента и логаритамским чланом извора дату са:

$$u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2} \ln |u|,$$

где су a и b позитивне константе, а функције $m(\cdot)$ и $p(\cdot)$ задовољавају одређене потребне услове. Користећи енергетски метод и неколико техника неједнакости, добијамо резултате глобалног непостојања у коначном времену за специфична решења са позитивном почетном енергијом. Овај тип једначине има значајне примене у различитим областима, укључујући динамику флуида, електрореолошке флуиде, квантну механику, нуклеарну физику, оптику и геофизику.

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