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EXPERIMENTAL STUDY OF TENSOR INVARIANTS OF HAMILTONIAN SYSTEMS

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Abstract. Typically, one considers the problem of finding the minimum number of invariants of a dynamical system sufficient for integrability. It can be also assumed that there are invariants not related to integrability that describe other properties of the dynamical system. We compute such tensor invariants for some integrable and non-integrable dynamical systems by using modern computer software and discuss their properties.

1. Introduction

A vector field X on the phase space M with coordinates $x = (x_1, \dots, x_n)$ defines a system of ordinary differential equations (dynamical system)

$$\frac{d}{dt}x = X(x_1, \dots, x_n), \tag{1.1}$$

depending on an evolution parameter t. Invariant $I(x)$ is a constant

$$I(x) = \mathrm{const}$$

along any solution of the system (1.1) including phase space functions (first integrals), vector and multivector fields (symmetry fields, hidden symmetries, Poisson structures), differential forms (symplectic forms, volume forms), recursion operators in bi-Hamiltonian geometry, invariant distributions in nonholonomic mechanics and control theory, Poincaré–Cartan absolute and relative invariants, etc.

For a smooth vector field X, tensor invariants T of the flow generated by X satisfy the equation

$$\mathcal{L}_X T = 0. \tag{1.2}$$

Here $\mathcal{L}_X T$ is a Lie derivative of the tensor field T along the vector field X. For this invariance equation (1.2), it is necessary to study the standard questions concerning the existence of solutions and their classification, the relations between

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different types of solutions, the applications of these invariants to study properties of trajectories $x(t)$, the construction of invariant-preserving integrators and so on.

In the case of scalar invariants, that is, first integrals of dynamical systems, all these questions are partly answered, see [1–10, 12] and references therein. For instance, if $f(x)$ is a scalar solution of (1.2), then the solutions $x(t)$ of (1.1) lie on the the integral manifold $f(x) = c$. On the other hand, in 1891, Poincaré provided a criterion by which analytic first integrals do not exist in analytic differential systems [13]. Indeed, if the analytic dynamical system (1.1) in $\mathbb{C}^n$ has a singularity at $x = 0$, i.e., $X(0) = 0$ and eigenvalues of the Jacobian matrix $DX(x)$ at $x = 0$ do not satisfy any $\mathbb{Z}^+$-resonant conditions, then the dynamical system has no analytic first integrals in a neighborhood of the origin.

Some well-known examples of the tensor solutions of (1.2) were also studied by Poincaré in [1]. For instance, if the divergence of the vector field X is equal to zero

$$\operatorname{div} X = \partial_1 X^1 + \partial_2 X^2 + \cdots + \partial_n X^n = 0, \tag{1.3}$$

then the completely antisymmetric unit tensor field Ω of type $(0, n)$

$$\Omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

and completely antisymmetric unit tensor field $\mathcal{E}$ of type $(n, 0)$

$$\mathcal{E} = \partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_n.$$

are solutions of the invariance equation (1.2) $\mathcal{L}_X \Omega = 0$ and $\mathcal{L}_X \mathcal{E} = 0$. In this case, the Poincaré recurrence theorem [14] states that a certain dynamical system (1.1) will, after a sufficiently long but finite time, return to a state arbitrarily close to its initial state.

The invariance of the differential form Ω was studied in 1838 by Liouville [15] as a theorem about the phase space distribution function, which for vector fields without divergence is the theorem about the conservation of the volume form

$$\int_V \Omega = \text{const}.$$

The invariance of the multivector field $\mathcal{E}$ is a key element of the Jacobi theory of functional determinants [16] developed during the period 1841–1845.

We obtain condition (1.3) by substituting the tensor fields Ω and $\mathcal{E}$ into the expression of the Lie derivative of the tensor field T of type (p, q)

$$(\mathcal{L}_X T)^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \sum_{k=1}^n X^k (\partial_k T^{i_1 \ldots i_p}_{j_1 \ldots j_q}) - \sum_{\ell=1}^n (\partial_\ell X^{i_1}) T^{\ell i_2 \ldots i_p}_{j_1 \ldots j_s} - \cdots - \sum_{\ell=1}^n (\partial_\ell X^{i_p}) T^{i_1 \ldots i_{p-1} \ell}_{j_1 \ldots j_s}$$
$$+ \sum_{m=1}^n (\partial_{j_1} X^m) T^{i_1 \ldots i_p}_{m j_2 \ldots j_q} + \cdots + \sum_{m=1}^n (\partial_{j_q} X^m) T^{i_1 \ldots i_p}_{j_1 \ldots j_{q-1} m},$$

where $\partial_\ell = \partial / \partial x^\ell$ is the partial derivative on the x^ℓ coordinate.

If $\operatorname{div} X = 0$ and there are invariant symmetries $X_1 = X, \ldots, X_m$ or first integrals $f_1, \ldots, f_m$ then tensor fields

$$\omega^{(i)} = \Omega \otimes X_i, \quad \omega^{(ij)} = \Omega \otimes X_i \otimes X_j, \quad \omega^{(ijk)} = \Omega \otimes X_i \otimes X_j \otimes X_k, \ldots$$

and

$$T^{(i)}=\mathcal{E}\otimes df_i,\quad T^{(ij)}=\mathcal{E}\otimes df_i\otimes df_j,\quad T^{(ijk)}=\mathcal{E}\otimes df_i\otimes df_j\otimes df_k,\dots$$

are also invariants of the flow of divergence free vector field X.

If $\operatorname{div} X=0$ and there are $n-1$ first integrals, the invariant vector field X is expressed through the functional determinant

$$X_i=\frac{\partial(x_i,f_1,\dots,f_{n-1})}{\partial(x_1,\dots,x_n)},\quad i=1,\dots,n,$$

see the mathematical exposition of the Jacobi theory of functional determinants in section 229 of the Vallée–Poussin lectures for engineers [19]. Similar tensor invariants exist for the vector field X satisfying the equation

$$\operatorname{div}(M(x)X)=\partial_1(M(x)X^1)+\partial_2(M(x)X^2)+\cdots+\partial_n(M(x)X^n)=0.$$

In Liouville theory, the function $M(x)$ is called an invariant measure, whereas in Jacobi theory, $M(x)$ is called a multiplier.

Dynamical systems with $n-1$ first integrals on n-dimensional manifolds and the topological properties of such systems are studied, particularly in the case when the integrals are multiple-valued and are pseudoperiodic functions on covering spaces in [20]. When the first integrals are single-valued functions, we have so-called degenerate or superintegrable systems [21].

The Lie derivative commutes with the exterior differentiation operation and satisfies the Leibnitz rule. It allows us to construct tensor invariants from a set of basic invariant tensor fields which either have a simpler functional dependence on the variables x, or have some special properties or physical interpretation. As an example, The Hamiltonian vector field X on a $2n$-dimensional symplectic manifold is defined by $\iota_X\omega=dH$ where ι is an interior product, ω is an invariant symplectic form and dH is an invariant 1-form constructed by differentiating the Hamilton function H, i.e., the scalar invariant of equation (1.2), which usually coincides with the mechanical energy of the dynamical system (1.1) [16–18].

The basic invariants H and ω define a family of invariant differential forms of type $(0,2k)$ and $(0,2k-1)$

$$\omega^{2k}=\wedge^k\omega\quad\text{and}\quad\omega^{2k-1}=\iota_X\omega_{2k},\qquad k=1,\dots n,$$

so that $\omega^{2n}=\Omega$ is an invariant volume form.

Similarly, for the Hamiltonian vector fields on the Poisson manifold $X=PdH$, where P is an invariant Poisson bivector independent on x, we can construct a family of invariant multivector fields of type $(2k,0)$ and $(2k-1,0)$

$$P^{2k}=\wedge^kP\quad\text{and}\quad P^{2k-1}=P^{2k}dH,\qquad k=1,\dots n.$$

The following general question arises in connection with the above: do systems of differential equations (1.1) admit non-trivial tensor invariants of arbitrary type (p,q) which can not be obtained from the basic invariants? The answer is in the affirmative, provided that we look at this problem locally, in a small neighbourhood of a non-singular point of the vector field X, see discussion in [8–10] for a generic case and in [4–6] for the Hamiltonian systems.

We aim to solve the invariance equation (1.2) directly by using a modern mathematical software and try to obtain its partial solutions in some restricted search search space, namely globally defined tensor fields on the entire state space. Following [22], computer computations can be used to explore the mathematical structure of the solutions of (1.2) and identify their properties and patterns. Examples are the integrable and non-integrable Hénon-Heiles systems, the non-integrable Volterra systems, the superintegrable Kepler problem and the integrable Toda lattice associated with the root system G_2.

1.1. Mathematical experiment. Below, we consider Hamiltonian systems defined by the following Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$$

and canonical Poisson bivector P, which in the matrix form reads as

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{so that} \quad X = PdH = \begin{pmatrix} p_1 \\ p_2 \\ -\frac{\partial V}{\partial q_1} \\ -\frac{\partial V}{\partial q_2} \end{pmatrix}.$$

The solutions of the invariance equation (1.2) are the mechanical energy H, the Poisson bivector P, the two completely antisymmetric tensor fields Ω and $\mathcal{E}$ and their derivatives.

Our aim is to find other globally defined solutions of (1.2). So, we will solve equation $\mathcal{L}_X P' = 0$ by using the following polynomial ansatz

$$P'^{ij} = \sum_{1\leqslant m\leqslant k\leqslant 2}^{2} a_{km}^{ij}(q) p_k p_m + \sum_{k=1}^{2} b_k^{ij}(q) p_k + c^{ij}(q). \tag{1.4}$$

for the entries of the bivector

$$P' = \sum P'^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad x = (q_1, q_2, p_1, p_2).$$

Here $a_{km}^{ij}(q), b_k^{ij}(q)$ and $c^{ij}(q)$ are functions of q_1 and q_2.

In coordinates $x = (q_1, q_2, p_1, p_2)$, the Lie derivative of the bivector P' along the vector field X is equal to

$$(L_X P')^{ij} = \sum_{k=1}^{2} \Big(X^k \frac{\partial P'^{ij}}{\partial x_k} - P'^{kj} \frac{\partial X^i}{\partial x_k} - P'^{ik} \frac{\partial X^j}{\partial x_k} \Big).$$

After the substitution of (1.4) into (1.2) we obtain 60 partial differential equations on 36 functions on q_1 and q_2. These equations were solved using various modern computer algebra systems for later comparison of the results obtained. The final results obtained in this way were verified analytically.

Substituting the same polynomial ansatz for entries

$$T^{ij\ell} = \sum_{k\geqslant m=1}^{2} a_{km}^{ij\ell}(q) p_k p_m + \sum_{k=1}^{2} b_k^{ij\ell}(q) p_k + c^{ij\ell}(q) \tag{1.5}$$

of the contravariant skew-symmetric tensor of valency three

$$T=\sum T^{ij\ell}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_\ell}$$

into the equation $\mathcal{L}_X T=0$

$$(L_X T)^{ij\ell}=\sum_{k=1}^{2}\Big(X^k\frac{\partial T^{ij\ell}}{\partial x_k}-T^{kj\ell}\frac{\partial X^i}{\partial x_k}-T^{ik\ell}\frac{\partial X^j}{\partial x_k}-T^{ijk}\frac{\partial X^\ell}{\partial x_k}\Big),$$

we obtain 81 partial differential equations for 60 functions of q_1 and q_2. We can also solve these equations using computer algebra systems. As an example, we present an answer for the Kepler problem.

2. Tensor invariants of integrable by Liouville systems

Let us consider Hamiltonian vector fields on a $2n$-dimensional symplectic manifold M endowed with coordinates $x=(q,p)$, so that

$$X=PdH,\quad P=\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}.$$

If the Hamiltonian vector field X has n functionally independent invariant first integrals

$$f_i(q_1,\ldots,q_n;p_1,\ldots,p_n,t)=c_i,\quad i=1,\ldots,n,$$

whose pairwise Poisson brackets are equal to zero

$$\{f_i,f_j\}=0,\quad i,j=1,\ldots,n$$

and

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(p_1,\ldots,p_n)}\neq 0.$$

So that, locally at least, we can express the p_i as functions of q_j, c_j, and t. It means that the differential form

$$\sum_{i=1}^{n}p_i(q,c,t)dq_i-H(q,p(q,c,t),t)dt$$

is the differential of a function

$$\mathcal{S}(q_1,\ldots,q_n;c_1,\ldots,c_n;t).$$

According to [23], it is a complete integral of the corresponding Hamilton–Jacobi equation

$$\frac{\partial\mathcal{S}}{\partial t}+H(q_1,\ldots,q_n,p_1,\ldots,p_n,t)=0,\quad\text{where}\quad p_i=\frac{\partial\mathcal{S}}{\partial q_i}.\tag{2.1}$$

A solution of a first-order partial differential equation with as many arbitrary constants as the number of independent variables is called a complete integral. Once a complete integral is found, a general solution can be constructed from it. Since a complete integral of a first-order partial differential equation can be found in quadratures, it completes the original Liouville proof of his theorem about integrability by quadratures [23].

2.1. Local invariant bivectors. Let us consider time-independent Hamiltonian systems. The Arnold–Liouville theorem [3] implies that almost all points of the symplectic manifold M are covered by a system of open toroidal domains

$$\mathcal{O}_c : \{x \in M,\ f_1(x) = c_1, \dots, f_n(x) = c_n,\ c_i \in \mathbb{R}\}$$

with the action-angle coordinates $I_1, \dots, I_n$ and $\theta_1, \dots, \theta_n$ so that

$$\dot{I}_j = 0, \quad \dot{\theta}_j = \frac{\partial H}{\partial I_j}, \quad j = 1, \dots, n. \tag{2.2}$$

and $\omega = \sum_{j=1}^n dI_j \wedge d\theta_j$. The action coordinates $I_1, \dots, I_n$ are defined in a ball

$$B_r : \sum_{j=1}^n (I_j - c_j)^2 < r^2,$$

while the angle coordinates $\theta_1, \dots, \theta_n$ run over a torus, in the compact case or over a toroidal cylinder if the integral manifold $I_j(x) = c_j$ is non-compact.

The completely integrable Hamiltonian system (2.2) is called non-degenerate if the Kolmogorov condition [21] for the Hessian matrix

$$\det \left\| \frac{\partial^2 H}{\partial I_i \partial I_j} \right\| \neq 0$$

holds almost everywhere. In this case, any first integral $f(I, \theta)$ is a function of the action variables only.

$$\frac{df}{dt} = 0 \quad \Rightarrow \quad f = f(I_1, \dots, I_n).$$

Using action-angle variables, we can describe all local solutions of equation (1.2) in the space of the symplectic forms and the corresponding Poisson bivectors.

Proposition 2.1. *In the neighbourhood of a compact toroidal domain, $\mathcal{O}_c$ closed 2-form ω' is invariant to the completely integrable non-degenerate Hamiltonian system* (2.2)*, i.e.*

$$d\omega' = 0, \quad L_X \omega' = 0,$$

if and only if it has the form

$$\omega' = \sum_{j=1}^n d\Big(\frac{\partial B(J)}{\partial J_j}\Big) \wedge d\theta_j - df_j(I) \wedge dI_j, \tag{2.3}$$

where $B(J_1, \dots, J_n)$ and $f(I_1, \dots, I_n)$ are arbitrary smooth functions of n arguments and functions $J_i = J_i(I_1, \dots, I_n)$ on action variables are equal to

$$J_i = \frac{\partial H}{\partial I_i}, \quad i = 1, \dots, n.$$

The 2-form ω' is non-degenerate if and only if the two non-degeneracy conditions hold

$$\det \left\| \frac{\partial^2 H}{\partial I_i \partial I_j} \right\| \neq 0, \quad \det \left\| \frac{\partial^2 B}{\partial J_i \partial J_j} \right\| \neq 0.$$

See all the details and similar theorem for the non-compact case in [4–6].

Using the Schouten–Nijenhuis bracket $[\![.,.]\!]$ between alternating multivector fields $P' = \omega'^{-1}$ and P, we immediately obtain the invariant trivector

$$T = [\![P, P']\!], \quad \text{such that} \quad \mathcal{L}_X T = 0.$$

In a generic case $T \neq 0$, two invariant Poisson structures P and $P' = \omega'^{-1}$ are incompatible in the Magri sense, i.e. the linear combination $P + \lambda P'$ is the invariant bivector which does not satisfy the Jacobi condition.

Summing up, the Bogoyavlenskij theorems reduce the rather difficult search for the invariant Poisson structures to the classical problem of constructing action-angle coordinates. When these coordinates are found, the formula (2.3) presents a continuous family of invariant symplectic and Poisson structures. Moreover, these theorems ensure that in this way we can obtain all invariant structures on the given symplectic manifold.

Of course, these theorems say nothing about global solutions of equation (1.2) since non-existence of the global action-angle coordinates, see discussion in [24].

2.2. Stäckel systems. One of the oldest examples of the action-angle variables was obtained by Stäckel in his dissertation [25]. We can use these action-angle variables to calculate local solutions of the invariance equation (1.2) using the Bogoyavlenskij construction (2.3) for non-degenerate Hamiltonian systems.

To construct a Stäckel family of integrable Hamiltonian systems, we take Darboux canonical variables $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ in which symplectic form and Poisson brackets read as

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j \quad \text{and} \quad \{p_j, q_k\} = \delta_{jk}.$$

Then, using nondegenerate $n \times n$ Stäckel matrix S, in which entries of the j-th column of S_{kj} depend only on the j-th coordinate q_j

$$\det S \neq 0, \quad \frac{\partial S_{kj}}{\partial q_m} = 0, \quad j \neq m,$$

inverse matrix $C = S^{-1}$ and arbitrary functions $V_j(q_j)$ we define n independent functions

$$I_k = \sum_{j=1}^{n} C_{jk}(p_j^2 + V_j(q_j)), \quad C = S^{-1}, \quad k = 1, \ldots, n \tag{2.4}$$

which are in involution

$$\{I_j, I_k\} = 0, \quad j, k = 1, \ldots, n.$$

These functions can be considered either as first integrals of n commuting Hamiltonian vector fields

$$X_j = P dI_j, \quad i = j, \ldots, n,$$

systems, action variables, since the connected surface of the level of these functions

$$\mathcal{O}_c = \{x \in M : I_j(x) = c_j, \ j = 1, \ldots, n\}$$

is diffeomorphic to n-dimensional real torus. The corresponding separated relations are

$$p_j^2 = \left(\frac{\partial \mathcal{S}}{\partial q_j}\right)^2 = \sum_{k=1}^{n} c_k S_{kj}(q_j) - V_j(q_j), \tag{2.5}$$

where $\mathcal{S}(q_1 \ldots, q_n; c_1, \ldots, c_n)$ is a complete integral of the Hamilton–Jacobi equation (2.1)

$$\mathcal{S}(q_1 \ldots, q_n; c_1, \ldots, c_n) = \sum_{j=1}^{n} \mathcal{S}_j,$$

for a given Hamiltonian system [25] where

$$\mathcal{S}_j(q_j, c_1, \ldots, c_n) = \int \sqrt{\sum_{k=1}^{n} c_k S_{kj}(q_j) - V_j(q_j)}\, dq_j.$$

Stäckel represented this torus as a direct product of n curves $\mathcal{C}_j$ on a plane with coordinates (x, y), which are given by the equations

$$\mathcal{C}_j : y^2 - F_j(x) = 0, \quad F_j(x) = \sum_{k=1}^{n} c_k S_{kj}(x) - V_j(x). \tag{2.6}$$

If this real torus is part of a complex algebraic torus, then the corresponding mechanical system is called an algebraic completely integrable system [26].

For example, if $\mathcal{C}_i = \mathcal{C}_j$, and n equations (2.5) define a divisor of points (p_i, q_i) on the symmetrized product of a hyperelliptic curve of genus n, then this special case of systems discovered by Stäckel in 1891 is sometimes called Jacobi-Mumford systems [27] or Mumford systems [28]. An explicit integration of the corresponding equations of motion, i.e., finding the functions $q_i(t)$ and $p_i(t)$, and a more detailed discussion of such private Stäckel-type systems, can be found in [29].

In the general case, according to the standard procedure of integration of the Hamilton-Jacobi equation by the method of separation of variables [30, 31], the functions $q_j(t_k, c_1, \ldots, c_n, \beta_1, \ldots, \beta_n)$ are found by means of quadrature reversal

$$\varphi_k = \sum_{j=1}^{n} \int^{q_j} \frac{S_{kj}(q)\, dq}{\sqrt{\sum_{k=1}^{n} c_k S_{kj}(q) - V_j(q)}} = t_k + \beta_k, \quad k = 1, \ldots, n, \tag{2.7}$$

where t_k are time variables corresponding to Hamilton functions $H_k = I_k$, and c_k and β_k are parameters determined by the initial conditions. In the case of finite motion, this motion will not be periodic in general, but only conditionally periodic [25]. If q_j^0 and q_j are the stopping points determined by the condition that the functions F_j (2.6) are equal to zero, then the periods θ_{jk} of the system motion are equal to

$$\theta_{kj} = \int_{q_j^0}^{q_j} \frac{S_{kj}(q)dq}{\sqrt{F_j(q)}}.$$

The sum of these periods (2.7), and in the most common special case this will be an Abelian sum, determines the variable angles φ_k for the action variables I_k (2.4) [25].

Substituting these action-angle variables into the Bogoyavlenskij definition (2.3) we obtain all possible invariant differential forms.

2.3. Global invariant bivectors for the Stäckel systems. If the Hamiltonian vector field X has n functionally independent invariant first integrals $f_1,\dots,f_n$, we can define a set of partial solutions of the equation (1.2) in the space of bivectors

$$P' = \sum a_{ij}(f_1,\dots,f_n) X_i\wedge X_j + b(f_1,\dots,f_n)P,\qquad X_i = Pdf_i, \tag{2.8}$$

where $a_{i,j,\dots,k}$ and b are some functions on the first integrals and P is a canonical Poisson bivector.

These partial solutions of equation (1.2) can be generalized if:

- there exist other first integrals of motion $g_1,\dots,g_k$ so that
$$\operatorname{rank}\frac{\partial(f_1,\dots,f_n;g_1,\dots,g_k)}{\partial(q_1,\dots,q_n;p_1,\dots,p_n)} = n+k;$$
- there exist other invariant bivectors $P_1,\dots,P_k$.

In the first case, we have a Kolmogorov-degenerate system (superintegrable or non-commutative integrable system) and the corresponding global invariant bivector reads as

$$P' = \sum a_{ij}(f_1,\dots,f_n,g_1,\dots,g_k) X_i\wedge X_j + b(f_1,\dots,f_n,g_1,\dots,g_k)P. \tag{2.9}$$

Here, X_i are now invariant vector fields associated with the non-commuting first integrals $f_1,\dots,f_n$ and $g_1,\dots,g_k$.

In the second case, we can add invariant bivectors $P_1,\dots,P_k$ to (2.8) and obtain another solution of equation (1.2)

$$\widetilde{P} = P' + \sum_{i=1}^{k} b_i(f_1,\dots,f_n)P_i.$$

Suppose that the complete integral of the stationary Hamilton–Jacobi equation $H=E$ admits a complete separation by coordinates and parameters

$$\mathcal{S}(q_1,\dots,q_n;c_1,\dots,c_n) = \sum_{j=1}^{n}\mathcal{S}_j(q_j,c_1,\dots,c_n) = \sum_{j=1}^{n}\mathcal{S}_j(q_j,c_j),$$

i.e. if Hamiltonian takes the following form in some Darboux coordinates (q,p)

$$H = I_1+\dots+I_n = \sum_{k=1}^{n}(p_k^2+V_k(q_k)),$$

then equation (1.2) has the following solution in the space of bivectors

$$P' = \sum_{j>i}^{n} A_{ij}(f_1,\dots,f_n)X_i\wedge X_j + \sum_{i=1}^{n} b_i(f_1,\dots,f_n)P_i,\qquad P_i = \frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}. \tag{2.10}$$

Here

$$X_i\wedge X_j = p_ip_j\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial q_j} - p_i\frac{\partial V_j}{\partial q_j}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_j} - p_j\frac{\partial V_i}{\partial q_i}\frac{\partial}{\partial q_j}\wedge\frac{\partial}{\partial p_i} + \frac{\partial V_i}{\partial q_i}\frac{\partial V_j}{\partial q_j}\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j},$$

and A_{ij}, b_j are arbitrary smooth functions of n first integrals $f_1, \ldots, f_n$. In the space of trivectors, the corresponding solution looks like

$$T = \sum_{k>j>i}^{n} A_{ijk}(f_1, \ldots, f_n) X_i \wedge X_j \wedge X_k + \sum_{i,j=1}^{n} b_{ij}(f_1, \ldots, f_n) P_i \wedge X_j .$$

Substituting the invariant bivector P' (2.10) into the Jacobi equation

$$[\![P', P']\!] = 0$$

we obtain a system of equations on the functions A_{ij} and b_j which can be solved. In a similar manner, we can attempt to obtain invariant trivectors T which satisfy the similar Jacobi conditions

$$[\![T, T]\!] = 0,$$

at the special choice of functions A_{ijk} and b_{ij}.

As an example, let us consider three integrable Hénon–Heiles systems which have a continuum of local Bogoyavlenskij invariants (2.3). The first Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + aq_1\Bigl(q_2^2 + \frac{q_1^2}{3}\Bigr) = \sum_{i=1}^{2}(P_i^2 + V_i(Q_i)),$$

allows complete separation in the coordinates $Q_{1,2} = q_1 \pm q_2$. The second Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + aq_1(q_2^2 + 2q_1^3)$$

admits separation simultaneously in the parabolic coordinates $u_{1,2} = q_1 \pm \sqrt{q_1^2 + q_2^2}$ and in their images after the Bäcklund transformation $u_{1.2} \to v_{1,2}$ obtained using the classical Abel's theorem [32]. The third Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + aq_1\Bigl(q_2^2 + \frac{16}{3}q_1^3\Bigr) = \sum_{i=1}^{2}(p_{v_i}^2 + V_i(v_i))$$

allows complete separation in the coordinates $v_{1,2}$. As a result, we have a global bi-Hamiltonian structure for the first and third complete separable Hamiltonians [33] and global non-Poisson invariant bivector (2.8) for the separable second Hamiltonian which belongs to a family of the Stäckel systems [25].

We do not have global Poisson bivectors compatible with the canonical bivector P for all the generic Stäckel systems [34]. Examples of superintegrable complete separable Stäckel systems may be found in [35]. In the following, using our simple mathematical experiment, we will compute complementary global invariants and discuss their “functional independence” for non-integrable Hénon-Heiles systems.

3. The Kepler problem

For degenerate in the Kolmogorov sense systems, there are global first integrals, which are functions on the both action and angle variables [36], and invariant bivectors of the form (2.9). The Kepler problem is one of the most fundamental problems in physics having such integrals of motion and, therefore, we take it as an example for our mathematical experiment.

Following Euler [37], we immediately move to consider orbit plane dynamics with the Cartesian coordinates $q_{1,2}$ so that the Hamiltonian H and the corresponding vector field X are

$$H=\frac{p_1^2+p_2^2}{2}-\frac{\kappa}{\sqrt{q_1^2+q_2^2}},\qquad (3.1)$$

$$X=p_1\frac{\partial}{\partial q_1}+p_2\frac{\partial}{\partial q_2}-\frac{\kappa}{(q_1^2+q_2^2)^{3/2}}\Big(q_1\frac{\partial}{\partial p_1}+q_2\frac{\partial}{\partial p_2}\Big).$$

This Hamiltonian commutes with the two components of the Laplace–Runge–Lenz vector

$$K_1=p_1(p_1q_2-p_2q_1)-\frac{\kappa q_2}{\sqrt{q_1^2+q_2^2}},\quad K_2=p_2(p_1q_2-p_2q_1)+\frac{\kappa q_1}{\sqrt{q_1^2+q_2^2}}$$

and the component of the orbital angular momentum $K_3=q_1p_2-q_2p_1$. According to Euler [37], the pair of first integrals H and K_1 (or K_2) has a Stäckel form in elliptic coordinates on the orbit plane and the existence of an additional independent first integral K_3 is a consequence of the Euler additional law on elliptic curves [38].

According to Jacobi [16], the pair of first integrals H and K_3 has a Stäckel form in polar coordinates and components of the Laplace–Runge–Lenz vector are derived using the Euler–Jacobi method of the last multiplier.

Action-angle variables can be computed using both elliptic and polar coordinates. In this paper, we will limit ourselves to the consideration of the action-angle variables obtained using polar coordinates.

3.1. Action-angle variables and invariant bivectors. Let us pass to the polar coordinates

$$q_1=r\cos\varphi,\quad q_2=r\sin\varphi$$

and the corresponding momenta

$$p_1=p_r\cos\varphi-\frac{p_\varphi\sin\varphi}{r},\quad p_2=p_r\sin\varphi+\frac{p_\varphi\cos\varphi}{r}$$

in which first integrals H and K_3^2 have the Stäckel form [25]

$$H=S_{11}^{-1}(p_r^2+V_1(r))+S_{21}^{-1}(p_\varphi^2+V_2(\varphi))=\frac{1}{2}\Big(p_r^2+\frac{p_\varphi^2}{r^2}\Big)-\frac{\kappa}{r},$$

$$K_3^2=S_{12}^{-1}(p_r^2+V_1(r))+S_{22}^{-1}(p_\varphi^2+V_2(\varphi))=p_\varphi^2,$$

where

$$S=\begin{pmatrix}2&0\\-r^{-2}&1\end{pmatrix},\quad V_1(r)=-2\kappa r,\quad V_2(\varphi)=0.$$

Next, for $H=h<0$ we introduce action-angle variables

$$I_\varphi=p_\varphi,\quad I_r=\frac{\kappa}{\sqrt{-2H}}-p_\varphi,\quad H=-\frac{\kappa^2}{2(I_r+I_\varphi)^2}$$

$$\theta_r=\arctan\left(\frac{rp_r\sqrt{2\kappa r-p_r^2r^2-p_\varphi^2}rp_r}{p_r^2r^2-\kappa r+p_\varphi^2}\right)-\frac{p_r\sqrt{2\kappa r-p_r^2r^2-p_\varphi^2)}}{\kappa},$$

$$\theta_\varphi=\theta_r+\varphi-\arcsin\left(\frac{\kappa r-p_\varphi^2}{\sqrt{p_\varphi^4+r(p_r^2r-2\kappa)p_\varphi^2+\kappa^2r^2}}\right)$$

and so-called Delauney elements [39]

$$I_1=I_\varphi,\quad I_2=I_r+I_\varphi,\quad \theta_1=\theta_\varphi-\theta_r,\quad \theta_2=\theta_r.$$

Invariant Poisson bivectors in the Delauney elements were obtained in [40] and then in [5,6]. In a similar way, we can get invariant Poisson bivectors in terms of the Poincaré elements, see discussion in [41].

Substituting these Delauney elements into the Bogoyavlenskij construction (2.3), we obtain a continuum of the local invariant Poisson bivectors which are compatible or non-compatible with the canonical Poisson bivector P. The main problem is to find global counterparts of these tensor invariants.

As an example, we present one pair of the invariant compatible Poisson bivectors

$$P=\frac{\partial}{\partial\theta_1}\wedge\frac{\partial}{\partial I_1}+\frac{\partial}{\partial\theta_2}\wedge\frac{\partial}{\partial I_2}\quad\text{and}\quad P'_I=I_1\frac{\partial}{\partial\theta_1}\wedge\frac{\partial}{\partial I_1}+I_2^{-2}\frac{\partial}{\partial\theta_2}\wedge\frac{\partial}{\partial I_2}$$

which are single valued functions of polar variables and the corresponding momenta

$$P=\frac{\partial}{\partial r}\wedge\frac{\partial}{\partial p_r}+\frac{\partial}{\partial\varphi}\wedge\frac{\partial}{\partial p_\varphi}$$

and

$$P'_I=\frac{2H}{\kappa^2}\frac{\partial}{\partial r}\wedge\frac{\partial}{\partial p_r}-p_\varphi\frac{\partial}{\partial\varphi}\wedge\frac{\partial}{\partial p_\varphi}-\frac{p_\varphi(\kappa^2p_\varphi+2H)}{\kappa^2r(\kappa^2+2Hp_\varphi^2)}\Big((p_\varphi^2-\kappa r)\frac{\partial}{\partial r}\wedge\frac{\partial}{\partial\varphi}+\frac{p_\varphi^2p_r}{r}\frac{\partial}{\partial\varphi}\wedge\frac{\partial}{\partial p_r}\Big).\tag{3.2}$$

The singularity in the Poisson bivector P'_I and in the corresponding recursion operator $N=P'P^{-1}$ has no physical meaning.

Using action-angle variables

$$J_{1,2}=\frac12(H\pm I_1),\quad \chi_{1,2}=\frac{\theta_2I_2^3}{\kappa^2}\pm\theta_1$$

we can rewrite Hamiltonian H (3.1) in the Fernandes form [42]

$$H=J_1+J_2$$

and define a bi-Hamiltonian structure for the Kepler problem associated with the following Poisson bivector

$$P'=\begin{pmatrix}0&0&J_1&0\\0&0&0&J_2\\-J_1&0&0&0\\0&-J_2&0&0\end{pmatrix},\quad H=\frac12\,\mathrm{tr}\,P'P^{-1}.$$

Recall that a bi-Hamiltonian system is defined by specifying two Hamiltonian functions H_1 and H_2 satisfying

$$X=PdH_1=P'dH_2,$$

with a diagonalizable recursion operator $N = P'P^{-1}$, having functionally independent real eigenvalues $\lambda_1, \ldots, \lambda_n$ [42].

In polar variables and momenta, this invariant Poisson bivector P' is a multi-valued function on the phase space. So, we can say that bi-Hamiltonian structure for the Kepler problem in the Fernandes sense exists only locally, i.e. in the neighbourhood of the open toroidal domains defined by the Arnold–Liouville theorem.

3.2. Mathematical experiment. Substituting polynomial ansatz (1.4) for entries of P' and the Kepler vector field X (3.1) into the equation $\mathcal{L}_X P' = 0$ and solving the resulting 60 partial differential equations, we obtain the following solution

$$P' = (a_1X_1 + a_2X_2)\wedge X_3 + (a_3H + a_4K_3^2 + a_5K_1 + a_6K_2 + a_7K + a_8)P + a_9\tilde{P}, \tag{3.3}$$

depending on nine parameters $a_i \in \mathbb{R}$. Here, $K_{1,2}$ are components of the Laplace–Runge–Lenz vector, K_3 is a component of the angular momentum vector, X_k are the corresponding invariant vector fields

$$X_1 = PdK_1, \quad X_2 = PdK_2, \quad X_3 = PdK_3,$$

and entries of the supplemental invariant bivector $\tilde{P}$ are

$$\tilde{P}^{12} = q_1p_2 - p_1q_2, \qquad \tilde{P}^{13} = -\frac{p_2^2}{2} + \frac{\kappa q_2^2}{(q_1^2+q_2^2)^{3/2}},$$

$$\tilde{P}^{14} = \frac{p_1p_2}{2} - \frac{\kappa q_1q_2}{(q_1^2+q_2^2)^{3/2}}, \qquad \tilde{P}^{23} = \frac{p_1p_2}{2} - \frac{\kappa q_1q_2}{(q_1^2+q_2^2)^{3/2}},$$

$$\tilde{P}^{24} = \frac{p_1^2}{2} + \frac{\kappa q_1^2}{(q_1^2+q_2^2)^{3/2}}, \qquad \tilde{P}^{34} = \frac{\kappa(p_1q_2 - p_2q_1)}{2(q_1^2+q_2^2)^{3/2}}.$$

Similar to the Hamilton–Jacobi equation, we can say that the equation

$$\mathcal{L}_X P' = 0$$

has a "complete integral" P' (3.3) depending on a sufficient number of arbitrary constants, which allows to get all the integrals of motion from invariant (1,1) vector field $N = P'P^{-1}$

$$\mathrm{tr}\ N = 2(a_9 - 2a_6)H + 2(a_2 - 2a_3)K_1 - 2(a_1 + a_4)K_2 - 4a_5K_3^2 - 4a_7K_3 - 4a_8.$$

In contrast with the standard recursion operator with vanishing Nijenhuis torsion, we obtain both commuting and non-commuting first integrals.

Substituting P' (3.3) into the Jacobi identity

$$[\![P', P']\!] = 0$$

and solving the resulting equation for parameters, we obtain four invariant Poisson bivectors $P'_1, \ldots, P'_4$, depending on parameters a, b

$$P'_1 = a(X_1 + iX_2)\wedge X_3 + bP, \qquad P'_2 = a(HP - 2\tilde{P}),$$

$$P'_3 = a(X_1\wedge X_3 - K_2P) + b(HP + \tilde{P}), \quad P'_4 = a(X_2\wedge X_3 + K_1P) + b(HP + \tilde{P}).$$

Only one of these Poisson bivectors is compatible with the canonical one

$$[\![P, P'_1]\!] = 0$$

where

$$P_1' - bP = e^{i\varphi}\begin{pmatrix} 0 & -p_\varphi & 0 & 0\\ p_\varphi & 0 & \frac{ip_\varphi^2}{r^2} & -ip_rp_\varphi - \kappa + \frac{p_\varphi^2}{r}\\ 0 & -\frac{ip_\varphi^2}{r^2} & 0 & 0\\ 0 & ip_rp_\varphi + \kappa - \frac{p_\varphi^2}{r} & 0 & 0 \end{pmatrix} \tag{3.4}$$

$$= \phi_1 d\theta_1 \wedge dI_1 + \phi_2 d\theta_1 \wedge d\theta_2.$$

Here, $\phi_{1,2}$ are functions on the Delauney elements and the last term is missing in the known results at [5, 6, 40].

In a similar manner, let us consider the equation $\mathcal{L}_X T = 0$ in the space of trivectors or skew-symmetric tensors of type $(3,0)$ with entries which are polynomials of second order in the momenta (1.5). Solving the corresponding system of partial differential equations, we obtain a solution depending on eight parameters $a_i \in \mathbb{R}$

$$T = \big((a_1K_3 + a_2)X + (a_3K_3 + a_4)X_1 + (a_5K_3 + a_6)X_2 \\ + (a_7K_3 + a_8 - a_3K_1 - a_5K_2)X_3\big) \wedge P.$$

This invariant trivector differs from the known invariant trivectors

$$T_i = \mathcal{E}dK_i \quad \text{and} \quad T_H = \mathcal{E}dH,$$

obtained from the Jacobi invariant $\mathcal{E}$.

Summing up, we failed to generalize the known bi-Hamiltonian structure for the Kepler system existing in the neighbourhood of the Liouville tori to the whole phase space. Nevertheless, there exists a well-defined global solution P' (3.1) of the equation $\mathcal{L}_X\, P' = 0$ that generates all noncommutative integrals of motion.

There are global Poisson bivectors P_I' (3.2) and P_1' (3.4) that satisfy the following equations

$$\mathcal{L}_X\, P' = 0, \quad (P' - FP)\, dH = 0, \quad \text{with} \quad P' \neq FP, \tag{3.5}$$

where $F = H$ and $F = b$, respectively. In the generic case, F (3.5) is a function on all the integrals of motion. Below, we will prove that similar solutions exist for other integrable and non-integrable Hamiltonian systems.

4. Open and periodic Toda lattices

Let us consider the following Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + \exp(q_1/\sqrt{3}) + \exp(-\sqrt{3}/2q_1 + q_2/2) + \alpha\exp(-q_2),$$

describing the open Toda lattice ($\alpha = 0$) or the so-called periodic Toda lattice ($\alpha = 1$) associated with the root system G_2. The second integral of motion is a polynomial of sixth order in the momenta, see [43].

Using the polynomial ansatz (1.4) for the entries of P' and the vector field $X = PdH$, we can directly solve the equation $\mathcal{L}_X P' = 0$ and obtain the following solution

$$P' = (a_1H + a_2)P + a_3(\alpha - 1)\tilde{P}, \quad a_k \in \mathbb{R}. \tag{4.1}$$

A supplemental invariant bivector $\tilde{P}$ exists only for the open Toda lattice and satisfies the equation

$$(\tilde{P} - HP)dH = 0 \tag{4.2}$$

similar to the Kepler problem equation (3.5). The entries of this supplemental solution $\tilde{P}$ are equal to

$$\begin{aligned}
\tilde{P}^{12} &= \sqrt{3}p_2 - 5p_1,\\
\tilde{P}^{13} &= \frac{p_2^2}{2} + \frac{5\exp(-\sqrt{3}/2q_1 + q_2/2)}{2},\\
\tilde{P}^{14} &= -\frac{p_1p_2}{2} - \frac{\sqrt{3}\exp(-\sqrt{3}/2q_1 + q_2/2)}{2},\\
\tilde{P}^{23} &= -\frac{p_1p_2}{2} - \frac{5\sqrt{3}\exp(q_1/\sqrt{3})}{3} + \frac{5\sqrt{3}\exp(-\sqrt{3}/2q_1 + q_2/2)}{2},\\
\tilde{P}^{24} &= \frac{p_1^2}{2} - \frac{3\exp(-\sqrt{3}/2q_1 + q_2/2)}{2} + \exp(q_1/\sqrt{3}),\\
\tilde{P}^{34} &= -\frac{\exp(-\sqrt{3}/2q_1 + q_2/2)}{4}p_1\\
&\quad + \frac{\sqrt{3}\big(2\exp(q_1/\sqrt{3}) - 3\exp(-\sqrt{3}/2q_1 + q_2/2)\big)}{12}p_2.
\end{aligned}$$

Unlike the Kepler problem, this solution P' (4.1) does not satisfy the Jacobi identity for any values of parameters a_k and generates only one integral of motion.

The most interesting result of the experiment is the fact that the non-trivial solution $\tilde{P} \neq HP$ of the equations

$$\mathcal{L}_X \tilde{P} = 0, \quad (\tilde{P} - HP)\,dH = 0$$

in the space of inhomogeneous polynomials of second order in the momenta (1.4) exists for an open Toda lattice ($\alpha = 0$) and not for a periodic lattice ($\alpha = 1$).

5. Non-integrable Hamiltonian systems

The above non-trivial solutions of equations (3.5) and (4.2)

$$\mathcal{L}_X P' = 0, \quad (P' - FP)\,dH = 0, \quad \text{with} \quad P' \neq FP,$$

for the superintegrable Kepler problem and integrable Toda lattice can be related to the Hamiltonian vector field transformation

$$X \to FX, \quad F = F(H)$$

and the corresponding time transformation $t \to \tau$ in the dynamical system (1.1)

$$dt \to F d\tau.$$

This transformation could be related to the Poincaré–Cartan form $pdq - Hdt$, which is a relative integral invariant [1, 2] in Hamiltonian mechanics. However, we are only able to construct non-trivial solutions of equations (3.5) for a few Hamiltonian systems.

As an example, let us consider Hamiltonian systems defined by the Hamiltonian

$$H=\tfrac{1}{2}(p_1^2+p_2^2)+V(q_1,q_2)$$

and solve $\mathcal{L}_X P'=0$ by using polynomial ansatz

$$P'^{ij}=\sum_{k\geqslant m=1}^{2} a^{ij}_{km}(q)p_kp_m+\sum_{k=1}^{2} b^{ij}_k(q)p_k+c^{ij}(q).$$

For a generic potential $V(q_1,q_2)$, the solution of equations (3.5) at $F=H$ depends on two parameters $a_{1,2}\in\mathbb{R}$

$$P'=(a_1H+a_2)P.$$

As an example, see the potential

$$V=aq_1^3+bq_1q_2,\quad a,b\in\mathbb{R},$$

In some special cases, the solution of equation (3.5) at $F=H$ depends on three or more parameters. For instance, if

$$V(q_1,q_2)=q_1^M(aq_2^N+bq_1^N),$$

then the solution depends on three parameters

$$P'=(a_1H+a_2)P+a_3\tilde{P}, \tag{5.1}$$

where

$$\tilde{P}^{12}=\frac{4(p_1q_2-p_2q_1)}{N+M},\quad \tilde{P}^{13}=p_1^2-p_2^2-2q_1^M\Big(\frac{a(N-M)q_2^N}{N+M}-bq_1^N\Big)$$
$$\tilde{P}^{14}=2p_1p_2+\frac{4aKq_2^{N-1}q_1^{M+1}}{N+M},\quad \tilde{P}^{23}=2p_1p_2+4q_1^{M-1}q_2\Big(\frac{aMq_2^N}{N+M}+bq_1^N\Big),$$
$$\tilde{P}^{24}=-\tilde{P}_{13},\quad \tilde{P}^{34}=2aKq_1^Mq_2^{N-1}p_1-2q_1^{M-1}(aMq_2^N+b(N+M)q_1^N)p_2.$$

Solving the corresponding Jacobi equation $[\![P',P']\!]=0$ with respect to the parameters a_k, we obtain two non-trivial invariant Poisson bivectors $P'_{1,2}$ at

$$a_2=0,\quad a_3=\frac{a_1}{2}\quad\text{and}\quad a_2=0,\quad a_3=\frac{N+M}{4}\,a_1.$$

If $a_1=1/2$ in the first case similar to the Kepler problem, we have

$$(P'_1-H\,P)dH=0,\quad\text{at}\quad P'_1\neq H\,P.$$

Thus, we can conclude that the invariant Poisson bivector P'_1 is associated with the change of time $dt\to Hd\tau$ and, therefore, the two incompatible Poisson bivectors P and P'_1 are "dependent" in a broad sense. In a similar way, we can consider the second invariant bivector P'_2.

Using the Schouten–Nijenhuis brackets between bivectors P and P' (5.1), we immediately obtain an invariant trivector for this non-integrable Hamiltonian system so that

$$T=a(H)[\![P,P']\!]+b(H)[\![P',P']\!]=c(H)P\wedge X+d(H)P'\wedge X,$$

Here, a,b,c and d are functions of the Hamiltonian H.

6. Conclusion

Global invariants of dynamical systems are usually associated with symmetries [1–3, 10]. In this paper, we discuss a simple mathematical experiment that allows us to find a class of Hamiltonian systems for which there exist additional tensor invariants that are "independent" of symmetries and well-known basic invariants.

For instance, the complementary invariant bivector P' (5.1) exists for the non-integrable Hénon-Heiles system with potential

$$V = q_1(aq_2^2 + bq_1^2), \quad a, b \in \mathbb{R}$$

and not for a system with potential $V = q_1(aq_1^2 + bq_2)$. In a similar way, the complementary invariant bivector P' (4.1) exists for an open Toda lattice and not for a periodic lattice associated with the root system G_2.

Of course, here we focus only on invariant bivectors with special entries that are inhomogeneous polynomials of second order in the momenta (1.4). For tensor invariants, we are not aware of an analog of Poincaré's theorem on the relation of complementary analytic first integrals and resonances for analytic dynamical systems.

For Hamiltonian systems on Poisson manifolds, the solutions of equation (1.2) $\mathcal{L}_X T = 0$ are more diverse. In particular, these solutions generate polynomial brackets on the low-dimensional Lie algebras and we can try to classify them in a way similar to the Cartan classification of the linear brackets [44].

After submitting the manuscript to the journal *Theoretical and Applied Mechanics*, the new Kozlov paper was published [11], where a new insight into tensor invariants is given. The experimental search of tensor invariants was also continued, see preprints [45–47]. We thank the reviewer for the work done to improve this text and for the opportunity to provide these references in the final version.

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ЕКСПЕРИМЕНТАЛНО ПРОУЧАВАЊЕ ТЕНЗОРСКИХ ИНВАРИЈАНТИ ХАМИЛТОНОВИХ СИСТЕМА

Резиме. Уобичајено се разматра проблем проналажења минималног броја инваријанти динамичког система довољног за интеграбилност. Такође се може претпоставити да постоје инваријанте које нису повезане са интеграбилношћу, а које описују друга својства динамичког система. Израчунавамо такве тензорске инваријанте за неке интеграбилне и неинтеграбилне динамичке системе користећи савремени рачунарски софтвер и разматрамо њихове особине.

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