$\mathbb{Z}_2$-HOMOLOGY OF THE ORBIT SPACES $G_{n,2}/T^n$

Vladimir Ivanović and Svjetlana Terzić

Abstract. We present computation of the $\mathbb{Z}_2$-homology groups of the orbit space $X_n = G_{n,2}/T^n$ for the canonical action of the compact torus T^n on a complex Grassmann manifold $G_{n,2}$. Our starting point is the model (U_n, p_n) for X_n constructed by Buchstaber–Terzić in [1], where $U_n = \Delta_{n,2} \times \mathcal{F}_n$ for a hypersimplex $\Delta_{n,2}$ and an universal space of parameters $\mathcal{F}_n$, and $p_n \colon U_n \to X_n$ is a continuous projection. The basic input in the construction of this model is the result, proved by the same authors, which states that X_n can be represented as the disjoint union of spaces $\{C_\omega \times F_\omega\}$ together with the continuous projections $p_\omega \colon \mathcal{F}_n \to F_\omega$. Here $\mathbb{C}_\omega$ are the chambers in the hypersimplex $\Delta_{n,2}$ which correspond to its decomposition given by all possible intersections of matroids, that is admissible polytopes. The spaces F_ω are the orbit spaces of $\hat{\mu}^{-1}(C_\omega)$ by the canonical action of the algebraic torus $(\mathbb{C}^*)^n$, where $\hat{\mu} \colon G_{n,2}/T^n \to \Delta_{n,2}$ is the map induced by the standard moment map $\mu \colon G_{n,2} \to \Delta_{n,2}$.

The notion of the universal space of parameters is defined by Buchstaber–Terzić in [**2**] for general T^k-action on a smooth manifold M^{2n}. The universal space of parameters $\mathcal{F}_n$ for T^n-action on $G_{n,2}$ is studied in detail in [**1**]. They proved that $\mathcal{F}_n$ is diffeomorphic to the moduli space $\mathcal{M}_{0,n}$ of stable n-pointed genus zero curves. We exploit the results from Keel in [**6**] and Ceyhan in [**4**] on generators of homology groups for $\mathcal{M}_{0,n}$ and express them in terms of the objects of the stratifications of $\mathcal{F}_n$ which are incorporated in the model (U_n, p_n).

In the result we deduce that the homology groups for $\mathcal{F}_n$ are spanned by the divisors outgrowing in the compactification of F_n to $\mathcal{F}_n$, where $F_n = W_n/(\mathbb{C}^*)^n$ for the main stratum W_n of $G_{n,2}$. Moreover, for any F_ω being compactification of F_n, we show that the homology groups of F_ω are spanned as well by the divisors outgrowing in the compactification of F_n to F_ω.

We recover the computation of homology groups with $\mathbb{Z}_2$ coefficients for X_5 by the method different from those of Buchstaber-Terzić in [**3**] and Suess in [**7**]. In addition, we compute the homology groups with $\mathbb{Z}_2$-coefficients for X_6 which are, up to our knowledge not known. The space X_6 is an example of complexity 3 torus action.

In general, the complexity of the study of the orbit spaces M^{2n}/T^k and their homology structure shows up to follow the complexity of torus action. The homology of quasitoric manifolds M^{2n}/T^n, which belong to the class of manifolds with complexity zero torus action, is determined by the combinatorics of the moment polytope P^k. We believe that the results which we obtain describing inductively the structure of cycles in X_n may lead

University of Montenegro, Podgorica, Montenegro.

to successful application of the presented method for explicit computation of homology groups for X_n with $\mathbb{Z}_2$-coefficients for higher n as well.

References

1. V. M. Buchstaber, S. Terzić, *A resolution of singularities for the orbit spaces $G_{n,2}/T^n$*, Trudy Mat. Inst. Steklova **317** (2022), 27–63.
2. V. M. Buchstaber, S. Terzić, *The foundations of $(2n, k)$-manifolds*, Sbornik: Mathematics **210**(4) (2019), 41–86.
3. V. M. Buchstaber, S. Terzić, *Topology and geometry of the canonical action of T^4 on the complex Grassmannian $G_{4,2}$ and the complex projective space $\mathbb{C}P^5$*, Mosc. Math. J. **16**(2) (2016), 237–273.
4. Ö. Ceyhan, *Chow groups of the moduli spaces of weighted pointed stable curves of genus zero*, Adv. Math. **221**(6) (2009), 1964–1978.
5. V. Ivanović, S. Terzić, *$\mathbb{Z}_2$-homology of the orbit spaces $G_{n,2}/T^n$*, (2024), eprint: preprint (math.AT).
6. S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Am. Math. Soc. **330**(2) (1992), 545–574.
7. H. Suess, *Toric topology of the Grassmannian of planes in $\mathbb{C}^5$ and the Del Pezzo surface of degree 5*, Mosc. Math. J. **21**(3) (2020), 639–652.