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POLYNOMIALS ORTHOGONAL ON THE SEMICIRCLE (*)

1. Introduction.

Given an inner product (.,.) on the space of polynomials, one calls $\{\pi_k\}$ a system of (monic) orthogonal polynomials if

(1.1) $$\pi_k(t) = t^k + \text{terms of lower degree}, \quad k = 0, 1, 2, \ldots,$$

(1.2) $$(\pi_k, \pi_l) \begin{cases} = 0 & \text{if} \quad k \neq l, \\ \neq 0 & \text{if} \quad k = l, \end{cases} \quad k, l = 0, 1, 2, \ldots .$$

The most common type of orthogonality is with respect to a positive measure $d\lambda$ on the real line $\mathbb{R}$, i.e.,

(1.3) $$(p, q) = \int_{\mathbb{R}} p(t)\, q(t)\, d\lambda(t), \quad d\lambda(t) \geq 0,$$

where the measure $d\lambda$ may have bounded or unbounded support. Classical examples are the polynomials of Legendre, Jacobi, Laguerre and Hermite. Polynomials orthogonal on the unit circle, with

(1.4) $$(p, q) = \int_{-\pi}^{\pi} p(e^{i\vartheta})\, \overline{q(e^{i\vartheta})}\, d\sigma(\vartheta), \quad d\sigma(\vartheta) \geq 0,$$

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have been introduced and studied by Szegö [6]. Orthogonal polynomials on curves and domains are also used occasionally. Here we briefly discuss a new type of orthogonality – orthogonality on the semicircle – with the inner product given by

$$(1.5) \qquad (p\,,q)=\int_0^\pi p\,(e^{i\vartheta})\,q\,(e^{i\vartheta})\,d\sigma\,(\vartheta) \quad , \qquad d\sigma\,(\vartheta)\geqslant 0\ .$$

Note that the second factor is not conjugated, so that the inner product (1.5) is not Hermitian. We consider only the simplest measure, $d\sigma(\vartheta)=d\vartheta$ (constant weight function). In this case, the orthogonal polynomials not only exist uniquely, but also exhibit a number of interesting properties: we present some of these here without proof. Full details will be given elsewhere.

We remark that integration over the full circle in (1.5) would be inappropriate, since the inner product would be equal to $2\pi p\,(0)\,q\,(0)$, and orthogonality in the sense of (1.2) could not be achieved. One could, however, consider integration over an arbitrary circular arc.

2. Existence and uniqueness.

The inner product in (1.5) defines the moment functional

$$(2.1) \qquad \mathcal{L}\ z^k=\mu_k,\ \mu_k=(1\,,z^k)=\int_0^\pi e^{ik\vartheta}\,d\vartheta=\begin{cases}\pi\ , & k=0\ ,\\ 2i/k\ , & k\ \text{odd}\ ,\\ 0\ , & k\neq 0\ \text{even}\end{cases}$$

The polynomials of interest are orthogonal with respect to this functional. They are known to exist uniquely if the moment sequence $\{\mu_k\}$ is *quasi-definite*, i.e., if $\Delta_n\neq 0$ for all $n\geqslant 1$, where

$$(2.2) \qquad \Delta_n=\det\begin{bmatrix}\mu_0 & \mu_1 & \cdots & \mu_{n-1}\\ \mu_1 & \mu_2 & \cdots & \mu_n\\ \cdots & \cdots & \cdots & \cdots\\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2}\end{bmatrix}, \qquad n=1,2,3,\ldots$$

(see, e.g., [1, Ch. 1, Thms. 3.1 and 3.2]).

Theorem 2.1. *The moment sequence* $\{\mu_k\}$ *in* (2.1) *is quasi-definite; indeed,* $\Delta_n > 0$ *for all* $n \geq 1$.

The proof is by explicit computation of the determinants in (2.2). Theorem 2.1 implies $(\pi_k, \pi_k) = \Delta_{k+1}/\Delta_k > 0$, all $k \geq 1$.

3. Three-term recurrence relation.

The fact that a three-term recurrence relation exists follows from the property $(zp, q) = (p, zq)$ of the inner product (1.5) (see [3, Thm. 2]). When $d\sigma(\vartheta) = d\vartheta$, as we have assumed, the recursion coefficients are purely imaginary and positive, respectively. We have, indeed,

Theorem 3.1. *The (monic) polynomials* $\{\pi_k\}$ *orthogonal with respect to the inner product* (1.5), *with* $d\sigma(\vartheta) = d\vartheta$, *satisfy*

$$\pi_{k+1}(z) = (z - i\alpha_k)\,\pi_k(z) - \beta_k\,\pi_{k-1}(z) \;,\; k = 0, 1, 2, \ldots, \qquad \pi_{-1}(z) = 0 \;,\; \pi_0(z) = 1 \;, \tag{3.1}$$

where

$$\alpha_0 = \vartheta_0 \;,\; \alpha_k = \vartheta_k - \vartheta_{k-1} \;,\; \beta_k = \vartheta_{k-1}^2 \;,\; k \geq 1 \;, \tag{3.2}$$

and ϑ_k *is given by*

$$\vartheta_k = \frac{2}{2k+1}\left[\frac{\Gamma((k+2)/2)}{\Gamma((k+1)/2)}\right]^2, \quad k \geq 0 \;. \tag{3.3}$$

The proof follows from well-known expressions for $i\alpha_k$ and β_k in terms of the determinants (2.2) and similar determinants without the penultimate column.

Comparison of coefficients in (3.1) yields

Corollary 3.1. $\pi_k(z) = z^k - i\vartheta_{k-1}\, z^{k-1} + \ldots ,\; k \geq 1$.

Corollary 3.2. $\|\pi_k\| = \left[\int_0^{\pi} [\pi_k(e^{i\vartheta})]^2\, d\vartheta\right]^{\frac{1}{2}} = \dfrac{2^{2k}\, k! \left[\Gamma\left(\frac{k+1}{2}\right)\right]^2}{\sqrt{\pi}\,(2k)!}$

This follows from $\|\pi_k\|^2 = \beta_0 \beta_1 \dots \beta_k$, where $\beta_0 = \pi$, together with (3.2) and (3.3). Stirling's formula in (3.3), and the relations in (3.2), moreover yield

Corollary 3.3. $\alpha_k \to 0$, $\beta_k \to \frac{1}{4}$ *as* $k \to \infty$.

The recursion coefficients thus exhibit the asymptotic behavior familiar from Szegö's class of orthogonal polynomials on the interval $[-1,1]$ (see [7, Eqs. (12.7.4) and (12.7.6)]) .

4. Connection with Legendre polynomials and differential equation.

Our polynomials $\{\pi_k\}$ are quasi-orthogonal in the sense of M. Riesz, involving a complex coefficient. This is the content of the next theorem.

Theorem 4.1. *Let* $\{\hat{P}_k\}$ *denote the monic Legendre polynomials. Then*

$$\pi_n(z) = \hat{P}_n(z) - i\vartheta_{n-1} \hat{P}_{n-1}(z) \ , \quad n \geqslant 1 \ , \tag{4.1}$$

where ϑ_k *is given by* (3.3).

Theorem 4.1 is easily established by expanding π_n in (monic) Legendre polynomials and evaluating the expansion coefficients by Cauchy's theorem.

From Legendre's differential equation satisfied by $\hat{P}_{n-1}$, and the property

$$(z^2 - 1) \frac{d}{dz} [\omega(z) \hat{P}_{n-1}(z)] = (2n-1)\omega(z)\pi_n(z) \ , \tag{4.2}$$

where $\omega(z) = (z-1)^{(n/2)-(n-\frac{1}{2})i\vartheta_{n-1}} (z+1)^{(n/2)+(n-\frac{1}{2})i\vartheta_{n-1}}$, it is possible to derive the linear second order differential equation

$$\begin{aligned}(1-z^2)[n^2 - (2n-1)^2 \vartheta_{n-1}^2 - 2n(2n-1)zi\vartheta_{n-1}]\pi_n'' \\ - 2[(n^2 - (2n-1)^2 \vartheta_{n-1}^2) z - n(2n-1)(z^2+1)i\vartheta_{n-1}]\pi_n' \\ + n[(n+1)n^2 - (n-1)(2n-1)^2 \vartheta_{n-1}^2 - 2(2n-1)n^2 zi\vartheta_{n-1}]\pi_n = 0 \ .\end{aligned} \tag{4.3}$$

The equation (4.3) has four regular singular points at $1, -1, z_0, \infty$, where z_0 is located on the negative imaginary axis and depends on n.

5. Zeros of $\pi_n(z)$.

By an elementary argument it can be shown that the zeros of π_n are located symmetrically with respect to the imaginary axis. A more technical argument, involving Rouché's theorem applied to (4.1), and a result of A. Giroux [4], establishes

Theorem 5.1. *All zeros of* π_n *are contained in the upper unit half disc* $D_+ = \{z \in \mathbb{C} : |z| < 1$ *and* $\operatorname{Im} z > 0\}$.

The proof actually shows more: The zeros are contained in the intersection of D_+ and the strip $|\operatorname{Re} z| \leq \xi_n$, where ξ_n is the largest zero of the Legendre polynomial P_n. Numerical evidence further suggests that the imaginary part of every zero of π_n is $\leq 2/\pi$ if $n \geq 1$, and $\leq .315076\ldots$, the unique positive root of $t^3 - (8/5\pi)\,t^2 + (3/5)\,t - (8/15\pi) = 0$, if $n \geq 2$.

Since the zeros are regular points of the differential equation (4.3), they are necessarily simple.

From the well-known fact that the zeros ζ_ν of π_n are eigenvalues of the (tridiagonal symmetric, but complex) Jacobi matrix associated with the recurrence (3.1), it follows by a similarity transformation that $\eta_\nu = -i\zeta_\nu$ are the eigenvalues of the real non ~~skew~~-symmetric tridiagonal matrix

$$
(5.1) \qquad \begin{bmatrix}
\alpha_0 & \vartheta_0 & & & & 0 \\
-\vartheta_0 & \alpha_1 & \vartheta_1 & & & \\
 & -\vartheta_1 & \alpha_2 & \cdot & & \\
 & & \cdot & \cdot & \cdot & \\
 & & & \cdot & \cdot & \vartheta_{n-2} \\
0 & & & & -\vartheta_{n-2} & \alpha_{n-1}
\end{bmatrix},
$$

where α_k and ϑ_k are given in (3.2), (3.3). The η_ν, and therefore the zeros ζ_ν, can thus be computed by standard eigenvalue routines (for example, the EISPACK routine HQR; see [5, p. 240]).

6. Applications.

One possible use of our polynomials (which in fact provided the motivation for studying them) is in connection with evaluating Cauchy principal value integrals. If f is analytic in the closed upper unit half disc D_+, then Cauchy's theorem applied to $\int_\Gamma f(z)\,dz/z$, where Γ is the boundary of D_+ with the origin deleted by a small upper semicircle, yields

$$\tag{6.1} ⨍_{-1}^{1} \frac{f(t)}{t}\,dt = i\left\{\pi f(0) - \int_0^\pi f(e^{i\vartheta})\,d\vartheta\right\},$$

the integral on the left being a Cauchy principal value integral. If, in addition, f is real on the real line, then

$$\tag{6.2} ⨍_{-1}^{1} \frac{f(t)}{t}\,dt = \operatorname{Im}\int_0^\pi f(e^{i\vartheta})\,d\vartheta\ ,\quad f(\mathbb{R}) \subset \mathbb{R}\,.$$

The integral on the right can now be evaluated by a (complex) Gauss-Christoffel quadrature formula

$$\tag{6.3} \int_0^\pi f(e^{i\vartheta})\,d\vartheta = \sum_{\nu=1}^{n} \sigma_\nu\, f(\zeta_\nu) + R_n(f)\ ,$$

where $R_n(f) = 0$ if f is a polynomial of degree $2n-1$. This requires the nodes ζ_ν to be the zeros of π_n. The weights σ_ν can be found by solving the system of linear algebraic equations

$$\tag{6.4} V_n\,\sigma = \pi e_1\ ,\quad \sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]^T\ ,$$

where V_n is the matrix of eigenvectors of the real matrix (5.1) (whose eigenvalues are $\eta_\nu = -i\zeta_\nu$), normalized to have the first component equal to 1, and e_1 is the first coordinate vector. A simple linear algebra argument shows that to a real eigenvalue η_ν there corresponds a real weight σ_ν, and to a pair of conjugate complex eigenvalues a pair of conjugate complex weights. The EISPACK routine HQR2 [5, p. 248] can be used to compute the matrix V_n, and the LINPACK routines CGECO, CGESL [2, Ch.1] to solve the system (6.4).

More general Cauchy principal value integrals $⨍_{-1}^{1} f(t)\,dt\,/\,(t-x)$,

$-1 < x < 1$, can be computed by first mapping x to the origin by a linear fractional transformation and then applying (6.1).

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