

Generalizations of a Formula due to Kummer with Applications

Yong Sup Kim^a, Gradimir V. Milovanović^{b,c}, Xiaoxia Wang^d, Arjun Kumar Rathie^e

^aDepartment of Mathematics Education, Wonkwang University, Iksan 570-749, Korea

^bSerbian Academy of Sciences and Arts, 11000 Belgrade, Serbia

^cUniversity of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia

^dDepartment of Mathematics, Shanghai University, Shanghai, 200444, P. R. China

^eDepartment of Mathematics, Vedant College of Engineering and Technology,
Rajasthan Technical University, Rajasthan State, India

Abstract. The aim of this research paper is to obtain explicit expressions of

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b \pm \ell + 1) \end{matrix}; \frac{1+x}{2} \right]$$

in the most general case for any $\ell = 0, 1, 2, \dots$. For $\ell = 0$, we have the well known, interesting and useful formula due to Kummer which was proved independently by Ramanujan. The results presented here are obtained with the help of known generalizations of Gauss's second summation theorem for the series ${}_2F_1(\frac{1}{2})$, which were given earlier by Rakha and Rathie [Integral Transforms Spec. Func. **22** (11) (2011), 823–840]. The results are further utilized to obtain new hypergeometric identities by using beta integral method developed by Krattenthaler & Rao [J. Comput. Appl. Math. **160** (2003), 159–173]. Several interesting results due to Ramanujan, Choi, et. al. and Krattenthaler & Rao follow special cases of our main findings.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by (cf. [20])

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}; z \right] &= {}_pF_q[\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{z^n}{n!} \end{aligned} \quad (1)$$

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Email addresses: yspkim@wonkwang.ac.kr (Yong Sup Kim), gvm@mi.sanu.ac.rs (Gradimir V. Milovanović), xiaoxiaawang@shu.edu.cn (Xiaoxia Wang), arjunkumarrathie@gmail.com (Arjun Kumar Rathie)

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) and defined, for any complex number α , by

$$(\alpha)_n = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + n - 1), & n \in \mathbb{N}; \\ 1, & n = 0. \end{cases} \tag{2}$$

Using the fundamental function relation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad (n \in \mathbb{N} \cup \{0\}), \tag{3}$$

where Γ is the well known Gamma function. For convergence condition and properties of ${}_pF_q$, we refer [1, 20].

It should be remarked here that whenever hypergeometric functions reduce to Gamma functions, the results are very important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role in the theory of generalized hypergeometric series. In fact, in recent years, a good deal of progress has been made in the direction of generalizing the above mentioned classical summation theorem. For these, we refer the papers [8–10, 13–17, 21, 22]. However, in our present investigation, we shall use the following generalizations of classical Gauss’s second summation theorem obtained earlier by Rakha and Rathie [21].

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b + \ell + 1) \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \times \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(-1)^i \Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} \tag{4}$$

for $\ell = 0, 1, 2, \dots$, and

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b - \ell + 1) \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})} \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} \tag{5}$$

for $\ell = 0, 1, 2, \dots$. Recently, an alternative summation formula for

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b + \ell + 1) \end{matrix} ; \frac{1}{2} \right],$$

when ℓ is an arbitrary integer has been given in [18].

It is interesting to observe here that in (4) or (5), if we take $\ell = 0$, we recover the classical Gauss’s second summation theorem viz.

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b + 1) \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}. \tag{6}$$

It is not out place to mention here that the result (5) is contained in [3, Entry (8.1.1.130), p. 582] and also by other means in [19, Entry (7.3.7.2), p. 414].

One of the interesting formula due to Kummer (cf. [12, Entry 72, p. 81]) is

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b + 1) \end{matrix} ; \frac{1}{2}(1 + x) \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b \\ \frac{1}{2} \end{matrix} ; x^2 \right] + 2x \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2} \\ \frac{3}{2} \end{matrix} ; x^2 \right]. \tag{7}$$

Further, in (7), if we change x to $-x$, we get

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}(1-x) \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b \\ \frac{1}{2} \end{matrix} ; x^2 \right] \\
 &\quad - 2x \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2} \\ \frac{3}{2} \end{matrix} ; x^2 \right].
 \end{aligned} \tag{8}$$

Now, adding and subtracting (7) and (8), we respectively have

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}(1+x) \right] + {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}(1-x) \right] \\
 = \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b \\ \frac{1}{2} \end{matrix} ; x^2 \right]
 \end{aligned} \tag{9}$$

and

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}(1+x) \right] - {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}(1-x) \right] \\
 = 4x \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2} \\ \frac{3}{2} \end{matrix} ; x^2 \right].
 \end{aligned} \tag{10}$$

It is not out of place to mention here that the results (9) and (10) are recorded in Goursat [7, p. 115].

Also, it is interesting to mention here that the formula (7) was independently proved by Ramanujan [2, p. 64, Entry 21] and is also recorded in [6, p. 64, Entry 21].

Very interesting special cases of Kummer's formula (7) and (8), when $a = b = \frac{1}{2}$ would yield the following results:

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1+x) \right] = \mu {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2} \end{matrix} ; x^2 \right] + \eta x {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2} \end{matrix} ; x^2 \right] \tag{11}$$

and

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1-x) \right] = \mu {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2} \end{matrix} ; x^2 \right] - \eta x {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2} \end{matrix} ; x^2 \right], \tag{12}$$

where the coefficients μ and η are given by

$$\mu = \frac{\Gamma(\frac{1}{2})}{[\Gamma(\frac{3}{4})]^2} \quad \text{and} \quad \eta = \frac{[\Gamma(\frac{3}{4})]^2}{[\Gamma(\frac{1}{2})]^3}.$$

The formula (11) is due to Ramanujan [2, p. 96, Entry 34 (i)] and is re-derived by Berndt [2] by using Gauss's second summation theorem [6].

Further, adding and subtracting (11) and (12), we respectively get

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1+x) \right] + {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1-x) \right] = 2\mu {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2} \end{matrix} ; x^2 \right] \tag{13}$$

and

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1+x) \right] - {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; \frac{1}{2}(1-x) \right] = 2\eta x {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2} \end{matrix} ; x^2 \right]. \tag{14}$$

In 2011, Choi, et al. [4] have generalized Kummer’s formula (7) and obtained the explicit expressions of

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b + \ell + 1) \end{matrix} ; \frac{1}{2}(1 + x) \right] \tag{15}$$

for $\ell = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. They have obtained the result (15) by employing generalizations of Gauss’s second summation theorem given earlier by Lavoie, et al. [15]. In the same paper [4], they have also considered a few very interesting special cases closely related to (7) and (11).

In addition to this, the beta function $B(\alpha, \beta)$ is defined by the first integral and known to be evaluated as the second one as follows:

$$B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt & (\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0), \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{16}$$

Recently, Krattenthaler & Rao [11] made a systematic use of so-called *beta integral method*, a method of deriving *new* hypergeometric identities from *old* ones by mainly using the beta integral in (16) based on the Mathematica Package HYP, to illustrate several interesting identities for the hypergeometric series and Kampé de Fériet series. Among others, in the same paper [11], from (8), using beta integral method, Krattenthaler & Rao [11] have obtained the following very interesting hypergeometric identity:

$${}_3F_2 \left[\begin{matrix} a, b, e - c \\ \frac{1}{2}(a + b + 1), e \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} {}_4F_3 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}c + \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e + \frac{1}{2} \end{matrix} ; 1 \right] - \frac{2c}{e} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} {}_4F_3 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}, \frac{1}{2}c + \frac{1}{2}, \frac{1}{2}c + 1 \\ \frac{3}{2}, \frac{1}{2}e + \frac{1}{2}, \frac{1}{2}e + 1 \end{matrix} ; 1 \right]. \tag{17}$$

In 2012, Choi et al. [5] generalized (17) and obtained a few results closely related to it.

The aim of this research paper is to obtain explicit expressions of

$${}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a + b \pm \ell + 1) \end{matrix} ; \frac{1 + x}{2} \right]$$

in the most general case for any $\ell = 0, 1, 2, \dots$. In order to put the results in the most general case, two master formulas have been constructed. These results are further utilized to obtain new hypergeometric identities. The results are derived with the help of generalizations of Gauss’s second summation theorem recently obtained by Rakha and Rathie [21]. Several interesting results due to Ramanujan [2], Choi, et al. [4] and Krattenthaler & Rao [11] follow special cases of our main findings. The results established in this paper are simple, interesting, easily established and may be useful (potentially).

2. Main Results

The generalizations of Kummer’s formula (7) are given in Theorems 1 and 2 and the generalizations of Krattenthaler & Rao’s identity (17) are given in Theorems 3 and 4 below.

In the sequel, with \mathbb{U} we denote the open unit disk, i.e.,

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Theorem 2.1. The first generalization of Kummer’s formula (7) which is given here holds true:

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1+x}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2})} \\
 &\times \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i)}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. + ax \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1, \frac{1}{2}b+\frac{1}{2}i+\frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1 \end{matrix}; x^2 \right] \right\} \quad (18)
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$.

Proof. In order to prove this theorem, we denote the left hand side of (18) by \mathbf{S} . Then upon expressing the ${}_2F_1$ as an infinite series given by the definition (1) and using the binomial theorem, we have

$$\mathbf{S} = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(\frac{1}{2}(a+b+\ell+1))_k 2^k k!} \sum_{j=0}^k \binom{k}{j} x^j = \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{(a)_k(b)_k}{(\frac{1}{2}(a+b+\ell+1))_k 2^k} \cdot \frac{x^j}{j!(k-j)!},$$

which, in light of the known formula for double series [20]

$$\sum_{k=0}^{\infty} \sum_{j=0}^k \Omega_{j,k} \frac{x^j}{j!} \cdot \frac{y^k}{(k-j)!} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=j}^{\infty} \Omega_{j,k} \frac{y^k}{(k-j)!},$$

yields

$$\mathbf{S} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=j}^{\infty} \frac{(a)_k(b)_k}{(\frac{1}{2}(a+b+\ell+1))_k (k-j)! 2^k}. \quad (19)$$

Upon setting $k \rightarrow k+j$ in (19) and using the familiar identity

$$(\lambda)_{j+k} = (\lambda)_j (\lambda+j)_k,$$

we get, after a little simplification,

$$\mathbf{S} = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j x^j}{j! (\frac{1}{2}(a+b+\ell+1))_j 2^j} \sum_{k=0}^{\infty} \frac{(a+j)_k (b+j)_k}{(\frac{1}{2}(a+b+\ell+1)+j)_k k! 2^k}.$$

Summing up the inner series with the definition (1), we have

$$\mathbf{S} = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j x^j}{j! (\frac{1}{2}(a+b+\ell+1))_j 2^j} {}_2F_1 \left[\begin{matrix} a+j, b+j \\ \frac{1}{2}(a+b+\ell+1)+j \end{matrix}; \frac{1}{2} \right]. \quad (20)$$

Separating the last series in (20) into even and odd powers of x , we thus obtain

$$\begin{aligned}
 \mathbf{S} &= \sum_{j=0}^{\infty} \frac{(a)_{2j}(b)_{2j} x^{2j}}{(2j)! (\frac{1}{2}(a+b+\ell+1))_{2j} 2^{2j}} {}_2F_1 \left[\begin{matrix} a+2j, b+2j \\ \frac{1}{2}(a+b+4j+\ell+1) \end{matrix}; \frac{1}{2} \right] \\
 &\quad + \sum_{j=0}^{\infty} \frac{(a)_{2j+1}(b)_{2j+1} x^{2j+1}}{(2j+1)! (\frac{1}{2}(a+b+\ell+1))_{2j+1} 2^{2j+1}} {}_2F_1 \left[\begin{matrix} a+2j+1, b+2j+1 \\ \frac{1}{2}(a+b+4j+\ell+3) \end{matrix}; \frac{1}{2} \right]. \quad (21)
 \end{aligned}$$

Finally, by applying the generalization of Gauss’s second summation formula (4) to each ${}_2F_1$ appearing on the right-hand side of (21) and making use of the following familiar identities

$$(\lambda)_{2j} = 2^{2j} \left(\frac{1}{2}\lambda\right)_j \left(\frac{1}{2}\lambda + \frac{1}{2}\right)_j \quad \text{and} \quad (\lambda)_{2j+1} = 2^{2j} \lambda \left(\frac{1}{2}\lambda + \frac{1}{2}\right)_j \left(\frac{1}{2}\lambda + 1\right)_j \quad (j \in \mathbb{N}_0),$$

as well as

$$(2j)! = 2^{2j} \left(\frac{1}{2}\right)_j j! \quad \text{and} \quad (2j + 1)! = 2^{2j} \left(\frac{3}{2}\right)_j j! \quad (j \in \mathbb{N}_0),$$

we are led, after some further simplification, to the right-hand side of the assertion (18) of Theorem 1. This completes the proof of Theorem 1. \square

In exactly the same manner, we can establish our main second result (22) given in the following theorem. In this case, of course, we use the generalization of Gauss’s second summation theorem (5).

Theorem 2.2. *The second generalization of Kummer’s formula (7) which is given here holds true:*

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1+x}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})} \\ &\times \sum_{i=0}^{\ell} \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2} \end{matrix}; x^2 \right] \right. \\ &\quad \left. + ax \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1, \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1 \end{matrix}; x^2 \right] \right\} \end{aligned} \quad (22)$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$.

Theorem 2.3. *The first generalization of Krattenthaler & Rao’s identity (17) which is given here holds true:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, b, e-c \\ \frac{1}{2}(a+b+\ell+1), e \end{matrix}; \frac{1}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \\ &\times \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}c, \frac{1}{2}c + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e + \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2} \end{matrix}; 1 \right] \right. \\ &\quad \left. - \frac{ac}{e} \cdot \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1, \frac{1}{2}c + \frac{1}{2}, \frac{1}{2}c + 1, \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}e + \frac{1}{2}, \frac{1}{2}e + 1, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1 \end{matrix}; 1 \right] \right\} \end{aligned} \quad (23)$$

for $\ell = 0, 1, 2, \dots$.

Proof. In order to prove Theorem 2.3, we proceed as follows. First of all, replace x by $-x$ in (18), we have

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1-x}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \\ &\times \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2} \end{matrix}; x^2 \right] \right. \\ &\quad \left. - ax \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1, \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1 \end{matrix}; x^2 \right] \right\} \end{aligned} \quad (24)$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$. Now, multiplying the left-hand side of (24) by $x^{c-1}(1-x)^{e-c-1}$ and integrating the resulting equation with respect to x from 0 to 1 and denoting it by I_1 , we have

$$I_1 = \int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix} ; \frac{1-x}{2} \right] dx.$$

Further, expressing the involved ${}_2F_1$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series involved in the process in the interval $(0, 1)$), we get

$$I_1 = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(\frac{1}{2}(a+b+\ell+1))_n 2^n n!} \int_0^1 x^{c-1}(1-x)^{e-c+n-1} dx.$$

Evaluating the integral using (16), after little simplification, and then summing up the series, we get

$$I_1 = \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)} {}_3F_2 \left[\begin{matrix} a, b, e-c \\ \frac{1}{2}(a+b+\ell+1), e \end{matrix} ; \frac{1}{2} \right]. \tag{25}$$

Similarly, multiplying the right-hand side of (24) by $x^{c-1}(1-x)^{e-c-1}$ and integrating the resulting equation with respect to x from 0 to 1 and denoting it by I_2 , we have

$$I_2 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \\ \times \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}a)_j(\frac{1}{2}a + \frac{1}{2})_j(\frac{1}{2}b + \frac{1}{2}i)_j}{(\frac{1}{2})_j(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})_j j!} \int_0^1 x^{c+2j-1}(1-x)^{e-c-1} dx \right. \\ \left. - a \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}a + \frac{1}{2})_j(\frac{1}{2}a + 1)_j(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})_j}{(\frac{3}{2})_j(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)_j j!} \int_0^1 x^{c+2j}(1-x)^{e-c-1} dx \right\}.$$

Evaluating the integral using (16),

$$\int_0^1 x^{c+2j-1}(1-x)^{e-c-1} dx = \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)} \cdot \frac{(c)_{2j}}{(e)_{2j}}$$

and

$$\int_0^1 x^{c+2j}(1-x)^{e-c-1} dx = \frac{c}{e} \cdot \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)} \cdot \frac{(c+1)_{2j}}{(e+1)_{2j}},$$

and using the result

$$(\lambda)_{2j} = 2^{2j} \left(\frac{\lambda}{2}\right)_j \left(\frac{\lambda+1}{2}\right)_j \quad (j \in \mathbb{N}_0),$$

after little simplification and then summing up both of the series, we finally obtain

$$I_2 = \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \\ \times \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i)}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2})} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}c, \frac{1}{2}c + \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e + \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + \frac{1}{2} \end{matrix} ; 1 \right] \right. \\ \left. - \frac{ac}{e} \cdot \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1, \frac{1}{2}c + \frac{1}{2}, \frac{1}{2}c + 1, \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}e + \frac{1}{2}, \frac{1}{2}e + 1, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1 \end{matrix} ; 1 \right] \right\}. \tag{26}$$

Finally, equating (25) and (26), we get the desired result (23).

This completes the proof of this theorem. \square

Starting from the following identity, obtained by changing x to $-x$ in (22),

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1-x}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \\
 &\times \sum_{i=0}^{\ell} \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i)}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. - ax \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1, \frac{1}{2}b+\frac{1}{2}i+\frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1 \end{matrix}; x^2 \right] \right\}, \tag{27}
 \end{aligned}$$

which holds for $x \in \mathbb{U}$ and each $\ell = 0, 1, 2, \dots$, we can in exactly the same manner as in the proof of Theorem 2.3 to prove the following result:

Theorem 2.4. *The second generalization of Krattenthaler & Rao’s identity (17) which is given here holds true:*

$$\begin{aligned}
 {}_3F_2 \left[\begin{matrix} a, b, e-c \\ \frac{1}{2}(a+b-\ell+1), e \end{matrix}; \frac{1}{2} \right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \\
 &\times \sum_{i=0}^{\ell} \binom{\ell}{i} \left\{ \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i)}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2})} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2} \end{matrix}; 1 \right] \right. \\
 &\quad \left. - \frac{ac}{e} \cdot \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1)} {}_5F_4 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}b+\frac{1}{2}i+\frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}e+1, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1 \end{matrix}; 1 \right] \right\} \tag{28}
 \end{aligned}$$

for $\ell = 0, 1, 2, \dots$

3. Special Cases and Consequences

In this section, we shall mention interesting special cases and consequences.

1° In our main Theorem 2.1 or 2.2, if we take $\ell = 0$, we immediately get Kummer’s formula (7).

2° In Theorem 2.1, if we take $\ell = 1, 2, 3, 4, 5$, we get results recorded in [4].

3° In Theorem 2.2, if we take $\ell = 1, 2, 3, 4, 5$, we get results also recorded in [4].

4° (a) If we add and subtract the results (18) and (24), we respectively get

$$\begin{aligned}
 &{}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1+x}{2} \right] + {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1-x}{2} \right] \\
 &= \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2})} \\
 &\quad \times \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(-1)^i \Gamma(\frac{1}{2}b+\frac{1}{2}i)}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2} \end{matrix}; x^2 \right] \tag{29}
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$, and

$$\begin{aligned}
 & {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1+x}{2} \right] - {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+\ell+1) \end{matrix}; \frac{1-x}{2} \right] \\
 &= \frac{2ax \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2}) \Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}b) \Gamma(\frac{1}{2}b + \frac{1}{2}) \Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\ell + \frac{1}{2})} \\
 &\quad \times \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(-1)^i \Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1, \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a + \frac{1}{2}i - \frac{1}{2}\ell + 1 \end{matrix}; x^2 \right] \tag{30}
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$

Form (29) and (30), we observe that (a) For $\ell = 0$, the results (29) and (30) reduce to (9) and (10) respectively, recorded in Goursat [7, p. 115].

Also, for $\ell = 0$ and $a = b = \frac{1}{2}$, the results (29) and (30) reduce to (13) and (14) respectively.

(b) For $\ell = 1$, the results (29) and (30) reduce respectively to

$$\begin{aligned}
 & {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+2) \end{matrix}; \frac{1+x}{2} \right] + {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+2) \end{matrix}; \frac{1-x}{2} \right] \\
 &= \frac{4}{(a-b)} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}a + \frac{1}{2}b + 1) \left\{ \frac{1}{\Gamma(\frac{1}{2}a) \Gamma(\frac{1}{2}b + \frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b \\ \frac{1}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. - \frac{1}{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{1}{2}b)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b + \frac{1}{2} \\ \frac{1}{2} \end{matrix}; x^2 \right] \right\} \tag{31}
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$, and

$$\begin{aligned}
 & {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+2) \end{matrix}; \frac{1+x}{2} \right] - {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+2) \end{matrix}; \frac{1-x}{2} \right] \\
 &= \frac{2}{(a-b)} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}a + \frac{1}{2}b + 1) \left\{ \frac{abx}{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{1}{2}b + 1)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + 1, \frac{1}{2}b + \frac{1}{2} \\ \frac{3}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. - \frac{abx}{\Gamma(\frac{1}{2}a + 1) \Gamma(\frac{1}{2}b + \frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b + 1 \\ \frac{3}{2} \end{matrix}; x^2 \right] \right\} \tag{32}
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$

Clearly, the results (31) and (32) are closely related to Goursat's (9) and (10).

Also, for $\ell = 1, a = \frac{1}{2}$ and $b = \frac{3}{2}$, the results (31) and (32) reduce respectively to

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}; \frac{1}{2}(1+x) \right] + {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}; \frac{1}{2}(1-x) \right] = 4\mu {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{5}{4} \\ \frac{1}{2} \end{matrix}; x^2 \right] - 8\eta {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ \frac{1}{2} \end{matrix}; x^2 \right] \tag{33}$$

for $x \in \mathbb{U}$, and

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}; \frac{1}{2}(1+x) \right] - {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}; \frac{1}{2}(1-x) \right] = 12\eta x {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{7}{4} \\ \frac{3}{2} \end{matrix}; x^2 \right] - 2\mu x {}_2F_1 \left[\begin{matrix} \frac{5}{4}, \frac{5}{4} \\ \frac{3}{2} \end{matrix}; x^2 \right] \tag{34}$$

for $x \in \mathbb{U}$. Clearly, the results (33) and (34) are closely related to the results (11) and (12). Thus, the results (29) and (30) can be regarded as the generalizations of Goursat’s results (9) and (10), respectively.

5° If we add and subtract the results (22) and (27), we respectively get

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1+x}{2} \right] + {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1-x}{2} \right] &= \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \\
 &\times \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i)}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2})} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}i \\ \frac{1}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+\frac{1}{2} \end{matrix}; x^2 \right] \quad (35)
 \end{aligned}$$

for $x \in \mathbb{U}$, $\ell = 0, 1, 2, \dots$, and

$$\begin{aligned}
 {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1+x}{2} \right] - {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b-\ell+1) \end{matrix}; \frac{1-x}{2} \right] &= 2ax \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \\
 &\times \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1)} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1, \frac{1}{2}b+\frac{1}{2}i+\frac{1}{2} \\ \frac{3}{2}, \frac{1}{2}a+\frac{1}{2}i-\frac{1}{2}\ell+1 \end{matrix}; x^2 \right] \quad (36)
 \end{aligned}$$

for $x \in \mathbb{U}$ and $\ell = 0, 1, 2, \dots$.

From the results (35) and (36), we observe that

(a) For $\ell = 0$, the results (35) and (36) reduce to (9) and (10), respectively recorded in Goursat [7, p. 115]. Also for $\ell = 0$ and $a = b = \frac{1}{2}$, the results (35) and (36) reduce to (13) and (14), respectively.

(b) For $\ell = 1$, the results (35) and (36) reduce to

$$\begin{aligned}
 &{}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b) \end{matrix}; \frac{1+x}{2} \right] + {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b) \end{matrix}; \frac{1-x}{2} \right] \\
 &= 2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b) \left\{ \frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. + \frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b \\ \frac{1}{2} \end{matrix}; x^2 \right] \right\} \quad (37)
 \end{aligned}$$

for $x \in \mathbb{U}$ and

$$\begin{aligned}
 &{}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b) \end{matrix}; \frac{1+x}{2} \right] - {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b) \end{matrix}; \frac{1-x}{2} \right] \\
 &= abx \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b) \left\{ \frac{1}{\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}b+\frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+1 \\ \frac{3}{2} \end{matrix}; x^2 \right] \right. \\
 &\quad \left. + \frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+1)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a+1, \frac{1}{2}b+\frac{1}{2} \\ \frac{3}{2} \end{matrix}; x^2 \right] \right\} \quad (38)
 \end{aligned}$$

for $x \in \mathbb{U}$.

Clearly, the results (37) and (38) are closely related to Goursat’s results (9) and (10).

Also, for $a = \frac{1}{2}$ and $b = \frac{3}{2}$, the results (37) and (38) reduce respectively to

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix}; \frac{1}{2}(1+x) \right] + {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix}; \frac{1}{2}(1-x) \right] = 2\mu {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{5}{4} \\ \frac{1}{2} \end{matrix}; x^2 \right] + 4\eta {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ \frac{1}{2} \end{matrix}; x^2 \right] \quad (39)$$

for $x \in \mathbb{U}$ and

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix}; \frac{1}{2}(1+x) \right] - {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix}; \frac{1}{2}(1-x) \right] = \mu x {}_2F_1 \left[\begin{matrix} \frac{5}{4}, \frac{5}{4} \\ \frac{3}{2} \end{matrix}; x^2 \right] + 6\eta x {}_2F_1 \left[\begin{matrix} \frac{3}{4}, \frac{7}{4} \\ \frac{3}{2} \end{matrix}; x^2 \right] \quad (40)$$

for $x \in \mathbb{U}$.

Clearly, the results (39) and (40) are closely related to the results (11) and (12). Thus, the results (35) and (36) can be regarded as the generalizations of Goursat's results (9) and (10) respectively.

6° In Theorem 2.3 or 2.4, if we set $\ell = 0$, we get, after some simplification, the identity (17) due to Krattenthaler & Rao [11]. Thus, the results (23) and (28) can be regarded as the generalizations of the result (17).

Similarly, a large number of results from our main formulas can be obtained.

4. Concluding Remarks

In this paper, we have provided the generalizations of Kummer's formula (7) (and in particular Ramanujan's formula (11)), and formula due to Krattenthaler & Rao's (17). We conclude the paper by remarking that following the same procedure presented in the paper, further extensions of Kummer's formula (7) and Krattenthaler & Rao's formula (17) in the form

$${}_3F_2 \left[\begin{matrix} a, b, d+1 \\ \frac{1}{2}(a+b+3 \pm \ell), d \end{matrix}; \frac{1+x}{2} \right] \quad \text{and} \quad {}_4F_3 \left[\begin{matrix} a, b, d+1, e-c \\ \frac{1}{2}(a+b+3 \pm \ell), d, e \end{matrix}; \frac{1}{2} \right],$$

each for $\ell = 0, 1, 2, \dots$ in the most general case are under investigation and will be published soon.

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